Graph Learning
1. Graph Structure

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2024 – 2025

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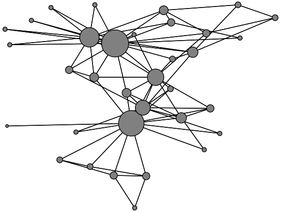
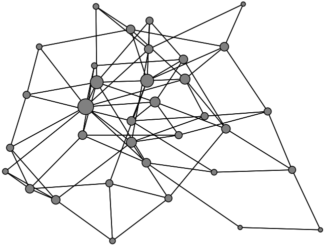
Motivation

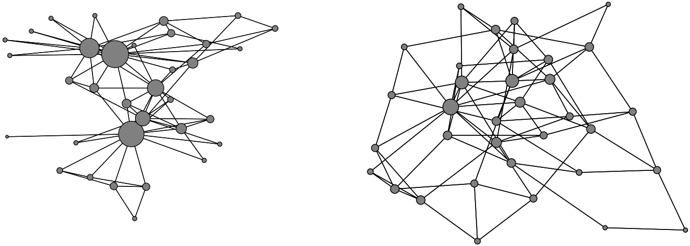
How do (large) **real graphs** look like?





Outline

1. **The friendship paradox**
2. Scale-free property
3. Small-world property
4. Clustering property



The friendship paradox

You have less friends than your friends have.

Sampling bias

Consider a graph of n nodes and m edges

Let D be the degree of a random node

Uniform sampling

$$\mathrm{E}(D) = \frac{2m}{n}$$

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Neighbor sampling

$$\mathrm{E}''(D) \geq \mathrm{E}(D)$$

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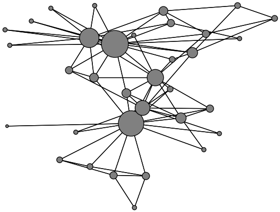
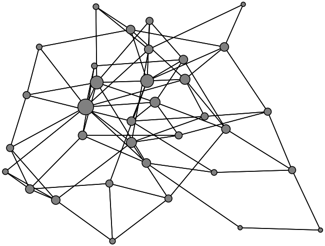




Power law

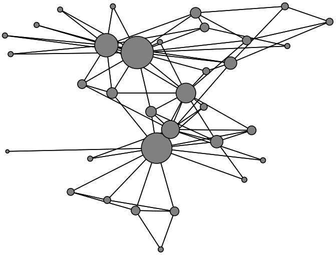
In real graphs, the degrees typically have a **power law** (= Zipf)

The degree D of a random node satisfies

$$\mathrm{P}(D \geq k) \approx \left(\frac{k_{\min}}{k}\right)^{\alpha}$$

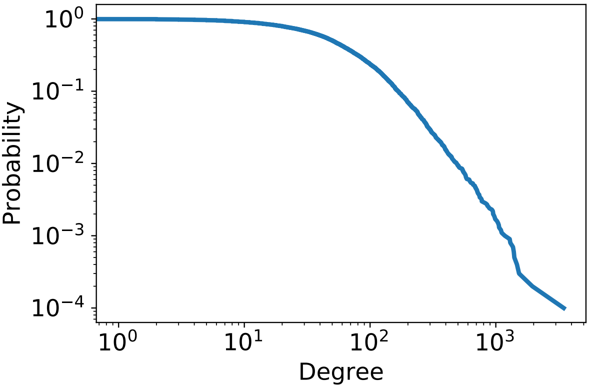
where

- $k_{\min}$ is the **minimum degree**
- α is the **exponent**, typically between 1 and 2



Example

In-degree distribution of Wikipedia Vitals
(10,011 nodes, 824,999 edges)



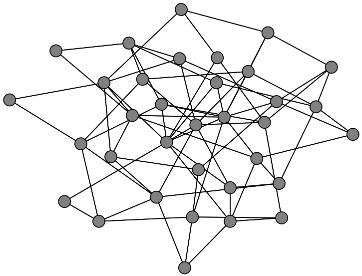
A random graph

Erdős-Rényi (1959)

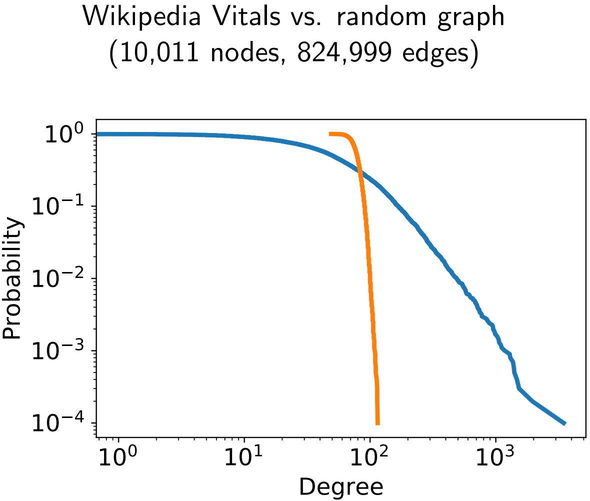
Consider n nodes, with pairs connected with probability p

Adjacency matrix = symmetric matrix with

$$A_{ij} \sim \text{Bernoulli}(p) \text{ for } i < j$$



Example

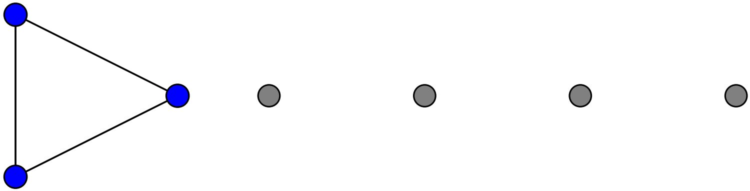

Wikipedia Vitals vs. random graph
(10,011 nodes, 824,999 edges)
Probability
Degree
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^0
10^1
10^2
10^3

Why the power law?

Barabasi-Albert model (1999)

Start from a clique of d nodes (with $d \geq 1$)

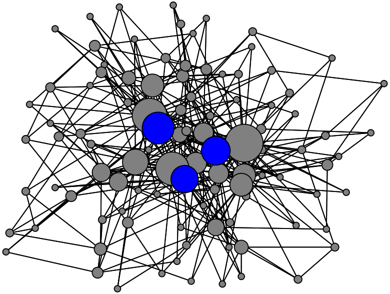
Add new nodes one at a time, each of degree d and with **preferential attachment**



“rich get richer”

Example

Graph generated from the **Barabasi-Albert model** ($n = 100, d = 3$)



The scale-free property

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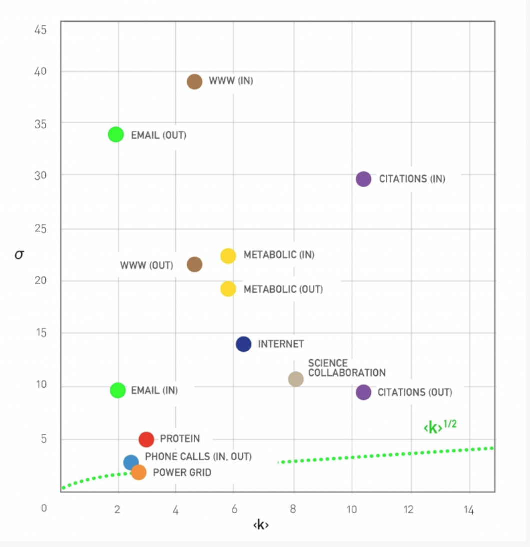
For a typical power exponent $\alpha \in (1, 2]$, we have:

$$\mathrm{E}(D) < \infty$$

$$\mathrm{var}(D) = \infty$$

$\rightarrow$ The average degree is **not** informative!

Scale-free graphs



Source: Barabasi, Network Science, 2016

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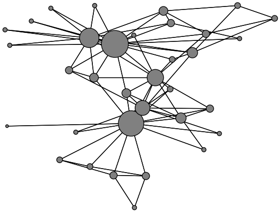

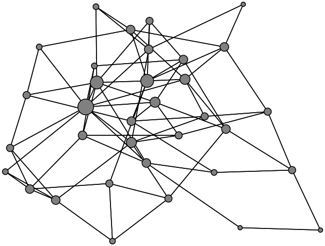

Small world

Which fraction of the articles are **accessible in** k **clicks** from Plato on Wikipedia?

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Using Wikipedia Vitals (10,011 articles):

# clicks	# nodes	proportion
1	392	4%
2	5866	59%
3	9939	99%
4	9990	99.8%

The six degrees of separation

- First mention in *Chains*, a short story by Karinthy in 1929

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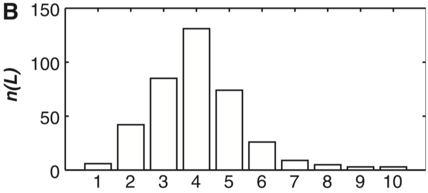
Emails

Dodds, Muhamad, Watts 2003

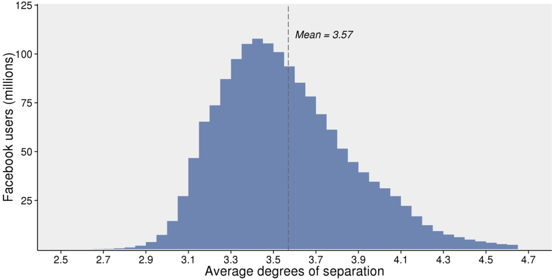
- 18 target people from all over the world
- 24,163 volunteers
- 384 successful chains
  Length of successful chains



Facebook

Bhagat, Burke, Diuk, Filiz, Edunov 2016

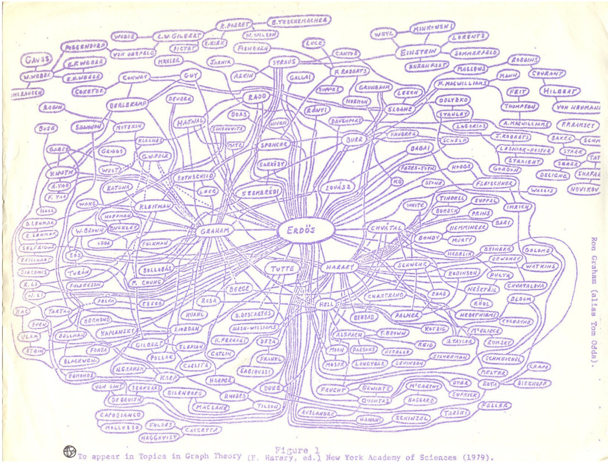
- Based on the 1.6 billion people active on Facebook
- Compute the average path length to any other people



The 3.5 degrees of separation of Facebook

Erdős number

- Graph of co-authors of scientific papers
- Distance to Erdős (1913-1996)



The Bacon number

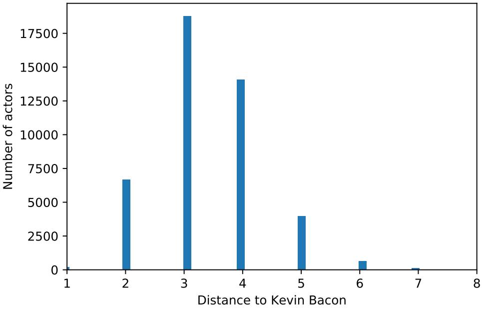
See The Oracle of Bacon

- Originated from an interview of Kevin Bacon by Premiere Magazine in 1994

The Bacon number

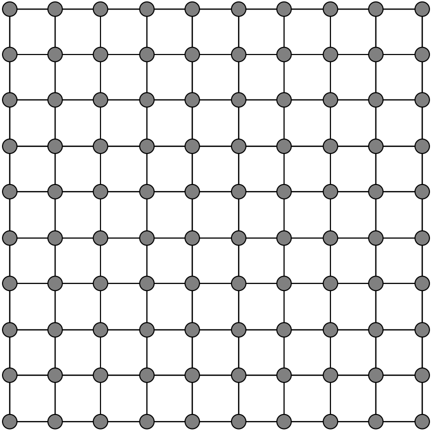
See The Oracle of Bacon

- Originated from an interview of Kevin Bacon by Premiere Magazine in 1994
- Graph of co-starring in movies

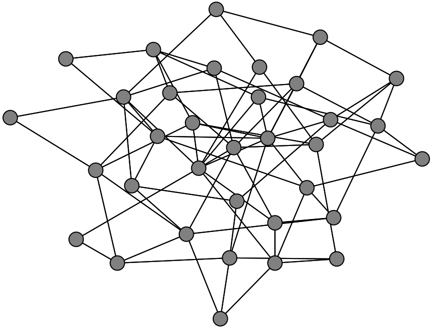


Results from YAGO database (44,586 actors)

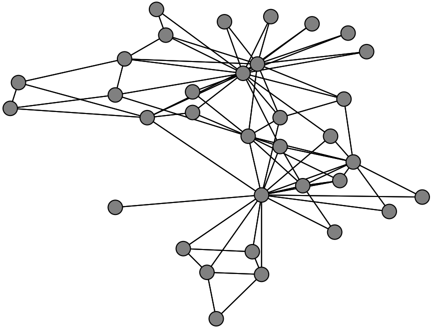
Planar graphs



Random graphs



Power-law graphs



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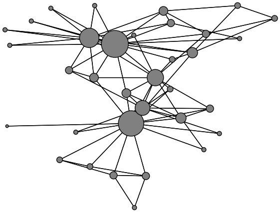
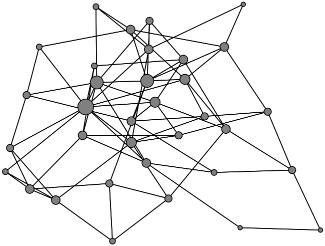
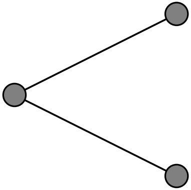




Clustering coefficient



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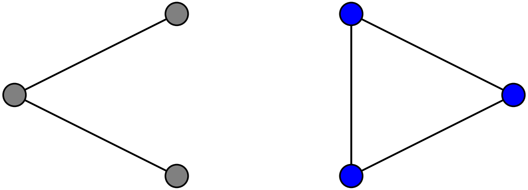

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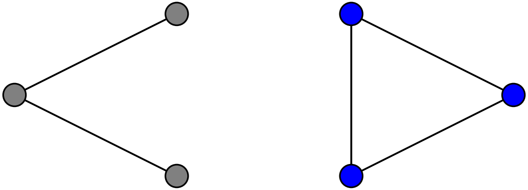
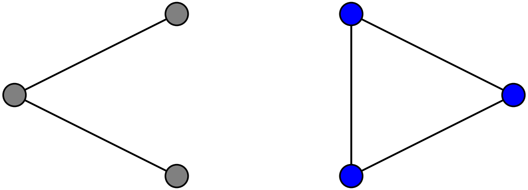


Fraction of **closed** triangles:

$$C = \frac{3\ \#\text{triangles}}{\sum_i \binom{d_i}{2}}$$

Clustering coefficient



Graph	C
Karate Club	0.26
Les Miserables	0.50
Openflights	0.25
WikiVitals	0.21

Summary

Graph structure

1. **Friendship paradox** (sampling)
2. **Power law** (degrees)
3. **Small world** (distances)
4. **Clustering** (triangles)