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Sampled-Data Control of Interval Systems with Guaranteed $\mathcal{H}_\infty$ Performance

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***Abstract*— Robustness and performance are important properties that a control system should present. In this paper, we consider linear systems with interval uncertainties and propose design techniques for robust controllers that ensure an upper bound for an associated $\mathcal{H}_\infty$ norm. Numerical examples point out the main features of the proposed methods.**

I. INTRODUCTION

In the last decades, a popular performance measure has been employed in optimal control theory: the $\mathcal{H}_\infty$ norm [1]. This importance is twofold: in frequency domain, the $\mathcal{H}_\infty$ norm represents the peak gain of the Bode plot of the system; in time domain, the $\mathcal{H}_\infty$ norm represents the worst-case $\mathcal{L}_2$ gain of the system. Through this performance index, several robust control problems have been tackled [2].

Usually, we deal with continuous-time systems expressed in terms of ordinary differential equations. However, with the dissemination of digital devices, it is interesting that the action control occurs in discrete time. In this way, we have a continuous-time system being controlled in discrete time, denominated as hybrid systems. Therefore, the final aim of this paper is to control a linear uncertain system of the form

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}x(t) + \begin{bmatrix}\mathbf{B_w} & \mathbf{B_u}\end{bmatrix}\begin{bmatrix} w(t_k) \\ u(t_k)\end{bmatrix}, \quad x(0)=0, \\ y(t) &= \mathbf{C}x(t) + \begin{bmatrix}\mathbf{D_w} & \mathbf{D_u}\end{bmatrix}\begin{bmatrix} w(t_k) \\ u(t_k)\end{bmatrix}, \end{aligned} \tag{1}$$

with the sampled-data state-feedback control signal

$$u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, \tag{2}$$

aiming at minimizing an upper bound $\mu > 0$ for the $\mathcal{L}_2$ gain

$$\mathcal{J}_\infty(K) = \sup_{w(t_k)\in\ell_2\setminus\{0\}} \frac{\int_0^\infty y(t)^\intercal y(t)\mathrm{d}t}{h\sum_{k\in\mathbb{N}} w(t_k)^\intercal w(t_k)}. \tag{3}$$

Note that w is also a piecewise constant signal, similar to u. In these equations, $x : \mathbb{T} \to \mathbb{R}^{n_x}$ is the state, $w : \mathbb{T} \to \mathbb{R}^{n_w}$ is the disturbance input, $u : \mathbb{T} \to \mathbb{R}^{n_u}$ is the control input and $y : \mathbb{T} \to \mathbb{R}^{n_y}$ is the output. The interval matrices $\mathbf{A} \in [\underline{A}, \overline{A}]$, $\mathbf{B_w} \in [\underline{B}_w, \overline{B}_w]$, $\mathbf{B_u} \in [\underline{B}_u, \overline{B}_u]$, $\mathbf{C} \in [\underline{C}, \overline{C}]$, $\mathbf{D_w} \in [\underline{D}_w, \overline{D}_w]$ and $\mathbf{D_u} \in [\underline{D}_u, \overline{D}_u]$ are uncertain. To this end, we must review the literature on interval systems and adapt classic results to cope with $\mathcal{H}_\infty$ optimization.

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A. Related Work

The $\mathcal{H}_\infty$ control problem based on state-space formulations has been considered by several scientists. For example, the optimal solution for precisely known problems can be determined by algebraic Riccati equations [1], [3], [4]. However, in many practical cases, the system may present uncertainties and, thus, other methods must be sought to ensure robustness to the control system [5]. The literature is rich in the area of robust control of interval and (the closely related) polytopic systems. Following the remarkable robust stability result provided by Kharitonov's Theorem [6], [7], classic references in the area of interval systems [8], [9], [10], [11] focused on devising simple tests and conditions for the stability of an interval dynamic system. Stability conditions for interval systems have been revisited and restated in the linear matrix inequality (LMI) [12] framework in [13], [14], in which other properties such as controllability are also investigated; these new results are based on Petersen's Lemma [15], [16], [17] for robust stability under norm-bounded uncertainties. These LMI conditions provide a simple and computationally efficient stability test for the interval system and can be extended to design stabilizing controllers, for instance. Even though these conditions were initially published as necessary and sufficient for quadratic stability and stabilizability of an uncertain system, it was later verified that they are, in fact, only sufficient; the necessity implication of Petersen's Lemma does not hold for more than one norm-bounded uncertainty [16], [17], [18], [19]. In the following papers [20], [21], [22], robust conditions $\mathcal{H}_\infty/\mathcal{H}_2$ have been proposed in terms of LMIs for continuous interval systems; as the results are based on the Petersen's Lemma these conditions are only sufficient.

B. Contributions of this Paper

In this paper, we review the robust stability and stabilization conditions presented in [14] and extend these results to consider a guaranteed $\mathcal{H}_\infty$ performance that must be minimized. It is important to state that this robust control problem has already been addressed in the literature using polytopic models and affine Lyapunov functions. The main drawback presented by this approach is the exponential number of vertices: an interval matrix with m uncertain (independent) entries can be represented by a matrix polytope with 2^m vertices, which means more constraints and variables are required to assess its stability. Our conditions, albeit more conservative, only need one constraint and a polynomial number of variables and, thus, they can be solved more efficiently even for large systems. Moreover, our conditions can

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be promptly extended to consider time-varying uncertainties. For briefness and to simplify notation, we present merely time-invariant results. Finally, these design conditions are applied to the design of sampled-data controllers.

II. Mathematical Preliminaries

A. Notation

We now introduce some notation that shall be used throughout the paper. The sets of natural, real and non-negative real numbers are indicated by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$, respectively. The set of time instants is denoted by $\mathbb{T}$ and corresponds to $\mathbb{R}_+$ for continuous-time signals and to $\mathbb{N}$ in the discrete case. The set of real $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$ and the set of n-dimensional real column vectors is denoted by $\mathbb{R}^n$. For any matrix $X \in \mathbb{R}^{m\times n}$, $X^\mathsf{T} \in \mathbb{R}^{n\times m}$ denotes its transpose. Additionally, for any matrix $X = (x_{ij}) \in \mathbb{R}^{m\times n}$, $X > 0$ ($X \geq 0$) denotes that $x_{ij} > 0$ ($x_{ij} \geq 0$) for all i, j. For any symmetric matrix $X = X^\mathsf{T} \in \mathbb{R}^{n\times n}$, $X \succ 0$ ($X \succeq 0$) denotes that X is positive definite (semidefinite). The set $\mathbb{S}^n$ is formed by all symmetric definite matrices of order in $\mathbb{R}^{n\times n}$ and $\mathbb{S}^n_+$ is formed by all symmetric positive definite matrices of order in $\mathbb{R}^{n\times n}$. $\mathbf{He}(X)$ means the sum of a matrix X with its transport: $\mathbf{He} := X + X^\mathsf{T}$. For any two given matrices $\underline{X}, \overline{X} \in \mathbb{R}^{m\times n}$ such that $\underline{X} \leq \overline{X}$, we define the interval matrix $[\underline{X}, \overline{X}]$ as the set

$$[X] = \left[\underline{X}, \overline{X}\right] := \left\{X \in \mathbb{R}^{m\times n} \ : \ \underline{X} \leq X \leq \overline{X}\right\}, \quad (4)$$

whose *center* and *radius* are defined as $X_0 := \frac{1}{2}\underline{X} + \frac{1}{2}\overline{X}$ and as $\Delta X := \frac{1}{2}\overline{X} - \frac{1}{2}\underline{X}$, respectively. We may also write the matrix interval above as

$$[X] := \left\{X_0 + \sum_{i,j} e_i \delta x_{ij} f_j^\mathsf{T} \ : \ |\delta x_{ij}| \leq \Delta x_{ij}\right\}, \quad (5)$$

in which $\Delta x_{ij} = [\Delta X]_{ij}$, e_i and f_j are the i-th and the j-th columns of compatible identity matrices. Uncertain or arbitrary elements in a matrix interval are denoted in bold. Finally, to optimize the space in the description of symmetric matrices, we use the symbol $\star$ to denote a block whose symmetric correspondent is already described.

B. Interval System

We define in this session the time-invariant system in discrete time

$$\Sigma_d \ : \ \left\{ \begin{array}{rcl} x[k+1] & = & \mathbf{A}x[k] + \mathbf{B_w}w[k] + \mathbf{B_u}u[k], \\ y[k] & = & \mathbf{C}x[k] + \mathbf{D_w}w[k] + \mathbf{D_u}u[k], \end{array} \right. \quad (6)$$

which evolves from zero initial conditions and in which $x : \mathbb{T} \to \mathbb{R}^{n_x}$ is the state, $w : \mathbb{T} \to \mathbb{R}^{n_w}$ is the disturbance input, $u : \mathbb{T} \to \mathbb{R}^{n_u}$ is the control input and $y : \mathbb{T} \to \mathbb{R}^{n_y}$ is the output. In this article, Σ_d is said to be an *interval system*, meaning that their realization matrices $\mathbf{A} \in [A]$, $\mathbf{B_w} \in [B_w]$, $\mathbf{B_u} \in [B_u]$, $\mathbf{C} \in [C]$, $\mathbf{D_w} \in [D_w]$ and $\mathbf{D_u} \in [D_u]$ are not precisely known. For simplicity, define the set

$$\mathbb{X} = [A] \times [B_w] \times [B_u] \times [C] \times [D_w] \times [D_u], \quad (7)$$

which allows us to write the uncertainties in compact form $(\mathbf{A}, \mathbf{B_w}, \mathbf{B_u}, \mathbf{C}, \mathbf{D_w}, \mathbf{D_u}) \in \mathbb{X}$.

C. Petersen's Lemma on Quadratic Stability

In this section, we present some relevant auxiliary results for the main theoretical developments of this article. We begin by the important matrix lemma stated by Ian Petersen:

Lemma 1 ([15], [16]). Let $Y \in \mathbb{S}^n$, $M \in \mathbb{R}^{n\times p}$, and $N \in \mathbb{R}^{q\times n}$ be given. The inequality

$$Y + M\Delta[k]N + N^\mathsf{T}\Delta[k]^\mathsf{T}M^\mathsf{T} \prec 0, \quad \forall k \in \mathbb{N}, \quad (8)$$

holds for all $\Delta : \mathbb{N} \to \mathbb{R}^{p\times q}$ such that $\Delta[k]^\mathsf{T}\Delta[k] \preceq I$ for all $k \in \mathbb{N}$ if, and only if, there exists $\epsilon > 0$ such that

$$Y + \epsilon MM^\mathsf{T} + \epsilon^{-1}N^\mathsf{T}N \prec 0. \quad (9)$$

Unfortunately, this remarkable result cannot be extended for several uncertain matrices without losing necessity; see [17] for a generalized necessary and sufficient result for two uncertain matrices. The same drawback exists in the S-procedure and, as there is a relationship between Petersen's Lemma and the S-procedure, this might be one way of justifying this fact [16]. As a consequence of this loss of necessity, the results developed in [13] and in [14], which made use of Petersen's Lemma for interval systems, are not necessary, as pointed out by [18]. Nevertheless, the conditions presented in both papers are very interesting as they are based on only one linear matrix inequality, which means they can tackle even systems with a relatively large number of uncertain parameters. As pointed out by [16], the conservativeness of such conditions is acceptable in practice.

Remark 2. As our main results are heavily dependent on Petersen's Lemma, the same design conditions presented in this paper can deal with interval time-varying uncertainties with no additional assumptions.

III. $\mathcal{H}_\infty$ Robust Control of Discrete-Time Interval Systems

In this section, we extend the robust control results presented in [14] for discrete-time systems. We also aim to include a performance bound in the design techniques, which means that not only stability is sought, but also some desired guaranteed norm. Hence, we seek to stabilize Σ_d by means of a robust state-feedback control law that minimizes an upper bound for

$$\|H_{wy}(z)\|_\infty = \sup_{w[k]\neq 0 \in \mathcal{L}_2} \frac{\sum_{k\in\mathbb{N}} \|y[k]\|_2}{\sum_{k\in\mathbb{N}} \|w[k]\|_2}. \quad (10)$$

To compute the controller which ensures the bound (10) consider the following Lemma.

Lemma 3. Let Σ_d be an interval system with $(\mathbf{A}, \mathbf{B_w}, \mathbf{B_u}, \mathbf{C}, \mathbf{D_w}, \mathbf{D_u}) \in \mathbb{X}$. There exist a quadratically stabilizing feedback gain $K \in \mathbb{R}^{n_u\times n_x}$ for Σ_d and a positive scalar $\mu > 0$ such that the performance defined in (3) verifies the bound

$$\|H_{wy}(z)\|_\infty < \gamma \quad (11)$$

for all $(\mathbf{A},\mathbf{B_w},\mathbf{B_u},\mathbf{C},\mathbf{D_w},\mathbf{D_u}) \in \mathbb{X}$ if, and only if, there exist matrices $X \in \mathbb{S}_+^{n_x}$, $G \in \mathbb{R}^{n_x \times n_x}$ and $L \in \mathbb{R}^{n_u \times n_x}$ such that

$$\begin{bmatrix} X & \star & \star & \star \\ G^\mathsf{T}\mathbf{A}^\mathsf{T} + L^\mathsf{T}\mathbf{B_u}^\mathsf{T} & \mathbf{He}(G) - X & \star & \star \\ 0 & \mathbf{C}G + \mathbf{D_u}L & I & \star \\ \mathbf{B_w}^\mathsf{T} & 0 & \mathbf{D_w}^\mathsf{T} & \mu I \end{bmatrix} \succ 0 \quad (12)$$

holds for all $(\mathbf{A},\mathbf{B_w},\mathbf{B_u},\mathbf{C},\mathbf{D_w},\mathbf{D_u}) \in \mathbb{X}$. Furthermore, the quadratically stabilizing state-feedback gain is given by $K = LG^{-1}$ and the upper bound for the $\mathcal{H}_\infty$ norm is $\gamma := \sqrt{\mu}$.

Proof. The proof can be found in the following paper [23]. $\square$

The conditions stated in this lemma involve linear matrix inequality that depend on all realization matrices in $\mathbb{X}$. The following theorem provides a set of computationally viable conditions for the robust control design problem.

Theorem 4. Let Σ_d be an interval system with $(\mathbf{A},\mathbf{B_w},\mathbf{B_u},\mathbf{C},\mathbf{D_w},\mathbf{D_u}) \in \mathbb{X}$ and let the scalar $\mu > 0$ be given. There exists a quadratically stabilizing feedback gain $K \in \mathbb{R}^{n_u \times n_x}$ for Σ_d such that the $\mathcal{H}_\infty$ performance defined in (10) verifies the bound

$$\|H_{wy}(z)\|_\infty < \sqrt{\mu} \quad (13)$$

for all $(\mathbf{A},\mathbf{B_w},\mathbf{B_u},\mathbf{C},\mathbf{D_w},\mathbf{D_u}) \in \mathbb{X}$ if there exists matrices $X \in \mathbb{S}_+^{n_x}$, $G \in \mathbb{R}^{n_x \times n_x}$ and $L \in \mathbb{R}^{n_u \times n_x}$ and positive scalars α_{ij}, $i,j \in \{1,\cdots,n_x\}$, $\beta_{\mathrm{w}ij}$, $i \in \{1,\cdots,n_x\}$ and $j \in \{1,\cdots,n_w\}$, $\beta_{\mathrm{u}ij}$, $i \in \{1,\cdots,n_x\}$ and $j \in \{1,\cdots,n_u\}$, γ_{ij}, $i \in \{1,\cdots,n_y\}$ and $j \in \{1,\cdots,n_x\}$, $\varepsilon_{\mathrm{w}ij}$, $i \in \{1,\cdots,n_y\}$ and $j \in \{1,\cdots,n_w\}$, $\varepsilon_{\mathrm{u}ij}$, $i \in \{1,\cdots,n_y\}$ and $j \in \{1,\cdots,n_u\}$, such that the linear matrix inequalities

$$\begin{bmatrix} X - S & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ G^\mathsf{T}A_0^\mathsf{T} + L^\mathsf{T}B_{u0}^\mathsf{T} & \mathbf{He}(G) - X & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & C_0G + D_{u0}L & I - T & \star & \star & \star & \star & \star & \star & \star \\ B_{w0}^\mathsf{T} & 0 & D_{w0}^\mathsf{T} & \mu I & \star & \star & \star & \star & \star & \star \\ 0 & \mathcal{G}_a & 0 & 0 & \mathcal{A} & \star & \star & \star & \star & \star \\ 0 & \mathcal{G}_c & 0 & 0 & 0 & \Gamma & \star & \star & \star & \star \\ 0 & \mathcal{L}_b & 0 & 0 & 0 & 0 & \mathcal{B}_u & \star & \star & \star \\ 0 & \mathcal{L}_d & 0 & 0 & 0 & 0 & 0 & \mathcal{E}_u & \star & \star \\ 0 & 0 & 0 & \mathcal{I}_b & 0 & 0 & 0 & 0 & \mathcal{B}_w & \star \\ 0 & 0 & 0 & \mathcal{I}_d & 0 & 0 & 0 & 0 & 0 & \mathcal{E}_w \end{bmatrix} \succ 0, \quad (14)$$

in which

$$S = \Big(\sum_{i,j} \Delta a_{ij}^2 \alpha_{ij} + \sum_{i,j} \Delta b_{w_{ij}}^2 \beta_{w_{ij}} + \sum_{i,j} \Delta b_{u_{ij}}^2 \beta_{u_{ij}}\Big) e_i e_i^\mathsf{T} \quad (15)$$

$$T = \Big(\sum_{i,j} \Delta c_{ij}^2 \gamma_{ij} + \sum_{i,j} \Delta d_{w_{ij}}^2 \varepsilon_{w_{ij}} + \sum_{i,j} \Delta d_{u_{ij}}^2 \varepsilon_{u_{ij}}\Big) g_i g_i^\mathsf{T}, \quad (16)$$

$$\mathcal{G}_a = \underbrace{\begin{bmatrix} G^\mathsf{T} & \cdots & G^\mathsf{T} \end{bmatrix}}_{n_x}^\mathsf{T}, \quad \mathcal{G}_c = \underbrace{\begin{bmatrix} G^\mathsf{T} & \cdots & G^\mathsf{T} \end{bmatrix}}_{n_y}^\mathsf{T}, \quad (17)$$

$$\mathcal{L}_b = \underbrace{\begin{bmatrix} L^\mathsf{T} & \cdots & L^\mathsf{T} \end{bmatrix}}_{n_x}^\mathsf{T}, \quad \mathcal{L}_d = \underbrace{\begin{bmatrix} L^\mathsf{T} & \cdots & L^\mathsf{T} \end{bmatrix}}_{n_y}^\mathsf{T}, \quad (18)$$

$$\mathcal{I}_b = \underbrace{\begin{bmatrix} I_{n_w} & \cdots & I_{n_w} \end{bmatrix}}_{n_x}^\mathsf{T}, \quad \mathcal{I}_d = \underbrace{\begin{bmatrix} I_{n_w} & \cdots & I_{n_w} \end{bmatrix}}_{n_y}^\mathsf{T}, \quad (19)$$

$$\mathcal{A} = \mathbf{diag}\big(\alpha_{11},\cdots,\alpha_{1n_x},\cdots,\alpha_{n_x1},\cdots,\alpha_{n_xn_x}\big), \quad (20)$$

$$\Gamma = \mathbf{diag}\big(\gamma_{11},\cdots,\gamma_{1n_x},\cdots,\gamma_{n_y1},\cdots,\gamma_{n_yn_x}\big), \quad (21)$$

$$\mathcal{B}_w = \mathbf{diag}\big(\beta_{\mathrm{w}11},\cdots,\beta_{\mathrm{w}1n_w},\cdots,\beta_{\mathrm{w}n_x1},\cdots,\beta_{\mathrm{w}n_xn_w}\big), \quad (22)$$

$$\mathcal{B}_u = \mathbf{diag}\big(\beta_{\mathrm{u}11},\cdots,\beta_{\mathrm{u}1n_u},\cdots,\beta_{\mathrm{u}n_x1},\cdots,\beta_{\mathrm{u}n_xn_u}\big), \quad (23)$$

$$\mathcal{E}_w = \mathbf{diag}\big(\varepsilon_{\mathrm{w}11},\cdots,\varepsilon_{\mathrm{w}1n_w},\cdots,\varepsilon_{\mathrm{w}n_y1},\cdots,\varepsilon_{\mathrm{w}n_yn_w}\big), \quad (24)$$

$$\mathcal{E}_u = \mathbf{diag}\big(\varepsilon_{\mathrm{u}11},\cdots,\varepsilon_{\mathrm{u}1n_u},\cdots,\varepsilon_{\mathrm{u}n_y1},\cdots,\varepsilon_{\mathrm{u}n_yn_u}\big), \quad (25)$$

hold; vectors e_i and g_i are the i-th columns of identity matrices of compatible dimensions. The matrices A_0, B_{w0}, B_{u0}, C_0, D_{w0} and D_{u0} are the center matrices of $\mathbb{X}$. Furthermore, the quadratically stabilizing state-feedback gain is given by $K = LG^{-1}$.

Proof. First, we apply Schur Complement and some algebraic manipulations on (14). By Petersen's Lemma (Lemma 1), we can rewrite it in interval form as (12), implying that $u = Kx$, with $K = LG^{-1}$ makes Σ_d quadratically stable and verifies the guaranteed performance $\|H_{wy}(z)\|_\infty < \sqrt{\mu}$. For more details, the papers [13], [14] present methods similar to those used in this theorem. $\square$

Example 5. In this example, we consider the example given in [14]. In this case, we take the discrete-time uncertain system Σ_d with

$$[A] = \begin{bmatrix} [0.5000 \;\; 1.0000] & [0.2499 \;\; 0.5000] & [0.1666 \;\; 0.3333] & [0.1249 \;\; 0.2501] \\ [0.2499 \;\; 0.5000] & [0.1666 \;\; 0.3333] & [0.1249 \;\; 0.2501] & [0.1000 \;\; 0.2001] \\ [0.1666 \;\; 0.3333] & [0.1249 \;\; 0.2501] & [0.1000 \;\; 0.2001] & [0.0832 \;\; 0.1677] \\ [0.1249 \;\; 0.2501] & [0.1000 \;\; 0.2001] & [0.0832 \;\; 0.1667] & [0.0713 \;\; 0.1430] \end{bmatrix},$$

$$[B_w] = [B_u] = \begin{bmatrix} [1.0000 \;\; 1.5000] \\ [1.0000 \;\; 1.5000] \\ [1.0000 \;\; 1.5000] \\ [1.0000 \;\; 1.5000] \end{bmatrix}.$$

The realization matrices C, D_w and D_u are given by

$$\underline{C} = \overline{C} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \; \underline{D}_w = \overline{D}_w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; \underline{D}_u = \overline{D}_u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

When we apply the conditions of the Theorem 4, we obtain the robust state feedback gain

$$K = \begin{bmatrix} -0.3800 & -0.2193 & -0.1573 & -0.1235 \end{bmatrix}.$$

TABLE I

COMPARATIVE TABLE OF CLOSED-LOOP PERFORMANCE ACHIEVED BY TWO DESIGN TECHNIQUES.

	Central	Interval
Guaranteed Norm	–	10.1986
Worst Case	9.8102	7.9318
Average Norm	3.7647	3.7067
Std. Deviation	0.9766	0.6943

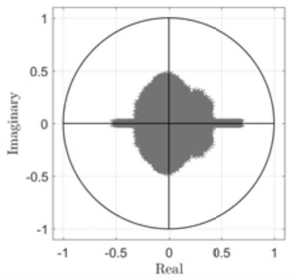

Fig. 1. Distribution of the closed-loop poles for the interval-based controller.

To devise this feedback gain, 72 inequality lines and 85 variables were necessary. The overall computation time was of 0.9849s [1].

The robust controller provided by the polytopic approach presented in [25] is a natural candidate for comparison. However, due to the combinatorial nature of this design technique, which requires around 16 millions of linear matrix inequalities, the optimization procedure did not finish within 72 hours of computation. We also compute the central controller, by using the nominal matrices from the set $\mathbb{X}$ and a discrete-time Riccati equation. This nominal gain is given by

$$K = \begin{bmatrix} -0.3377 & -0.2023 & -0.1478 & -0.1174 \end{bmatrix}.$$

These controllers were validated by a Monte Carlo procedure. In this example, we generate around 65 thousand possible realizations and compute the closed-loop performance. These simulation results are compiled in Table I. The robust control approach presents a best worst-case performance than the nominal controller. The distribution of closed-loop poles for the interval approach is depicted in Figure 1. As the distribution of closed-loop poles for the nominal gain was similar to the one yielded by the interval gain, then we omit it in this paper.

[1] All the code in this paper was run on a Dell computer with Windows 10 64 bits operating system, Intel Xeon processor X5570, 2.93 GHz, and 12GB memory. Also, interval operations and functions are carried out with INTLAB [24], which is a computational package developed for reliable computing that runs on MATLAB.

IV. Application: Sampled-Data Linear Quadratic Robust Control

Let us now briefly discuss how the main ideas devised in this paper can be applied to design sampled-data state-feedback robust controllers for sampled-data systems. Our aim is to solve the problem presented in the Introduction of this paper, which focuses on devising a control law u of the form (2) that ensures robust stability to (1) whilst simultaneously minimizing an upper bound for (3).

Our first step is to generalize the *equivalent discretization procedure* presented in [26], [27], which embeds the hybrid sampled-data nature of the original problem into an equivalent discrete-time model, to make them cope with interval systems. For this, we obtain a discrete-time interval realization for (1). The matrices set $\mathbb{X}$ can be obtained by the choice of matrices that guarantee equivalent discretization as defined in [26], [27] with the augmented matrices $B = [B_u \quad B_w]$ and $D = [D_u \quad D_w]$, allowing us to calculate

$$\|H(z)\|_\infty = \sup_{w[k]\neq 0 \in \mathcal{L}_2} \frac{1}{\sqrt{h}} \frac{\sum_{k\in\mathbb{N}} \|y[k]\|_2}{\sum_{k\in\mathbb{N}} \|w[k]\|_2} \tag{26}$$

as a standard $\mathcal{H}_\infty$ problem [27]. In order to compute the intervals $[A_d]$ and $[B_d]$ matrices, we may use the scaling and squaring method presented in [28] such that

$$e^{h[\mathcal{A}]} = \exp\left(h \begin{bmatrix} [A] & [B] \\ 0 & 0 \end{bmatrix}\right) \in \begin{bmatrix} [A_d] & [B_d] \\ 0 & I \end{bmatrix} \tag{27}$$

which can be partitioned according to the dimensions of the inputs u_k and w_k. These computations, thus, yield the discrete-time interval system

$$x[k+1] = \mathbf{A}_d x[k] + \mathbf{B}_{\mathrm{w}d} w[k] + \mathbf{B}_{\mathrm{u}d} u[k], \quad x[0] = x_0, \tag{28}$$

in which $\mathbf{A}_d \in [A_d]$, $\mathbf{B}_{\mathrm{w}d} \in [B_{\mathrm{w}d}]$ and $\mathbf{B}_{\mathrm{u}d} \in [B_{\mathrm{u}d}]$. Indeed, using a self-validating numeric quadrature procedure presented in [29], we compute intervals $[C_d]$ and $[D_d]$ such that

$$\int_0^h e^{\tau[\mathcal{A}]^\mathsf{T}} \begin{bmatrix} [C]^\mathsf{T} \\ [D]^\mathsf{T} \end{bmatrix} \begin{bmatrix} [C]^\mathsf{T} \\ [D]^\mathsf{T} \end{bmatrix}^\mathsf{T} e^{\tau[\mathcal{A}]} \mathrm{d}\tau \in \begin{bmatrix} [C_d]^\mathsf{T} \\ [D_d]^\mathsf{T} \end{bmatrix} \begin{bmatrix} [C_d]^\mathsf{T} \\ [D_d]^\mathsf{T} \end{bmatrix}^\mathsf{T}. \tag{29}$$

Then, using Cholesky's factorization, we obtain the matrices $[C_d]$ and $[D_d]$. A SVD-based decomposition can also be used whenever the Cholesky computation fails due to singularity of some matrix in (29). Also, $[D_{\mathrm{u}d}]$ and $[D_{\mathrm{w}d}]$ can be obtained from the partition of $[D_d]$ according to the dimensions of the inputs u_k and w_k.

After computing the matrices of the discretized equivalent system, we compute a robust sampled-data controller (2) that optimizes (3) for (1) using Theorem 4 for the equivalent system which provides state feedback gain $K = LG^{-1}$ and the bound $\mathcal{J}_\infty(K) < \mu/h$.

Note that these techniques are based on well established numerical procedures and, therefore, can be efficiently computed.

Example 6. Consider the uncertain system (1) with realization matrices taken from Example 2 of [30]:

$$[A] = \begin{bmatrix} 1 \pm 0.1 & -1 \pm 0.1 \\ 0 \pm 0.5 & 4 \pm 0.2 \end{bmatrix}, \quad B_w = B_u = \begin{bmatrix} 1.0000 \\ -1.0000 \end{bmatrix}.$$

The realization matrices C, D_w and D_u are given by

$$\underline{C} = \overline{C} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \ \underline{D}_w = \overline{D}_w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \underline{D}_u = \overline{D}_u = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

using a sampled-data control law of the form (2) with sampling period of $h = 0.1$s.

We begin by computing the intervals $[A_d]$, $[B_{w_d}]$ and $[B_{u_d}]$ using the scaling and squaring method, with both parameters set to 10, obtaining:

$$[A_d] = \begin{bmatrix} [\ \ 1.0937 & 1.1167] & [-0.1440 & -0.1138] \\ [-0.0066 & 0.0066] & [\ \ 1.4613 & 1.5224] \end{bmatrix},$$

$$[B_{w_d}] = [B_{u_d}] = \begin{bmatrix} [\ \ 0.1098 & 0.1124] \\ [-0.1246 & -0.1213] \end{bmatrix}.$$

We also compute the intervals $[C_d]$, $[D_{w_d}]$ and $[D_{u_d}]$ following the procedure discussed above:

$$[C_d] = \begin{bmatrix} [0.3309 & 0.3346] & [-0.0228 & -0.0154] \\ [0.0000 & 0.0000] & [\ \ 0.3864 & 0.3966] \\ [0.0000 & 0.0000] & [\ \ 0.0000 & 0.0000] \\ [0.0000 & 0.0000] & [\ \ 0.0000 & 0.0000] \end{bmatrix},$$

$$[D_{w_d}] = \begin{bmatrix} [\ \ 0.0167 & 0.0177] \\ [-0.0207 & -0.0185] \\ [\ \ 0.0002 & 0.0008] \\ [\ \ 0.3163 & 0.3167] \end{bmatrix}, [D_{u_d}] = \begin{bmatrix} [\ \ 0.0167 & 0.0177] \\ [-0.0207 & -0.0185] \\ [\ \ 0.3163 & 0.3167] \\ [\ \ 0.0000 & 0.0000] \end{bmatrix}.$$

These realization matrices yield the discrete-time equivalent system. Using the design conditions of Theorem 4 with the equivalent system, we obtain the robust gain

$$K = \begin{bmatrix} 10.8005 & 24.0549 \end{bmatrix}.$$

This design procedure required 35 lines of inequalities and 34 optimization variables and took 0.3328s to terminate. We once again use the conditions in [25] to design another robust gain, which is given by

$$K = \begin{bmatrix} 9.7260 & 22.8904 \end{bmatrix}.$$

This alternative method required $235, 9298$ lines of inequalities and $786, 439$ decision variables to run and took around 4 hours to finish. In this example, the central controller is obtaining by solving the sampled-data $\mathcal{H}_\infty$ problem for the central continuous-time matrices, following [27]. This central gain is given by

$$K = \begin{bmatrix} 5.4467 & 18.1557 \end{bmatrix}.$$

These sampled-data controllers were also validated by a Monte Carlo procedure. In this example, we generate around 50 thousand possible realizations and compute the closed-loop performance. These analyze results are compiled in Table II. The central controller presents an unstable system, resulting in a possibly unstable closed-loop system.The interval approach presents a guaranteed $\mathcal{H}_\infty$ performance around 5% higher than the polytopic technique. In terms of worst case and of average performance, the interval-based controller is slightly – around 3% in the worst case – outperformed by the polytopic-based controller, as expected, since the interval-based design can cope even with time varying uncertainties. Note that this small sampled-data design example already shows that the interval-based design procedure can deal with larger systems whereas the polytopic-based design method can easily be impractical for systems with order higher than two.

TABLE II

COMPARATIVE TABLE OF CLOSED-LOOP $\mathcal{H}_\infty$ PERFORMANCE ANALYZE ACHIEVED BY INTERVAL AND POLYTOPIC TECHNIQUES.

	Interval	**Polytopic**
Guaranteed Norm	2.5656	2.4486
Worst Case	2.5162	2.4476
Average Norm	2.2487	2.2078
Std. Deviation	0.1132	0.1027

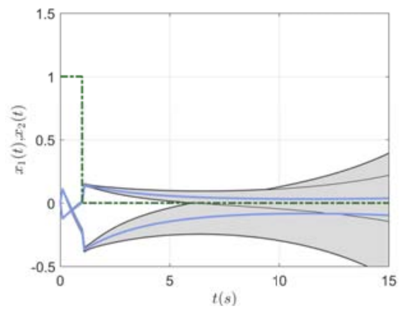

Fig. 2. Closed-loop response to step input of 1 second using the central controller.

We also performed the simulation of the three gains, the signal of the exogenous input was a step with 1 second of duration. Figures 2, 3 and 4 show the temporal response of the simulations. The response of the central approach is unstable, while interval and polytopic techniques stabilize the systems with a guaranteed $\mathcal{H}_\infty$ norm. The Table III shows the performance obtained for the closed-loop response to step input of 1 second using the interval and polytopic controllers, the performances obtained were bounded by guaranteed $\mathcal{H}_\infty$ norm.

TABLE III

COMPARATIVE TABLE OF CLOSED-LOOP $\mathcal{H}_\infty$ PERFORMANCE SIMULATION ACHIEVED BY INTERVAL AND POLYTOPIC APPROACHES.

	Interval	**Polytopic**
Worst Case	1.7642	1.8029
Average Norm	1.7094	1.7286
Std. Deviation	0.0118	0.0145

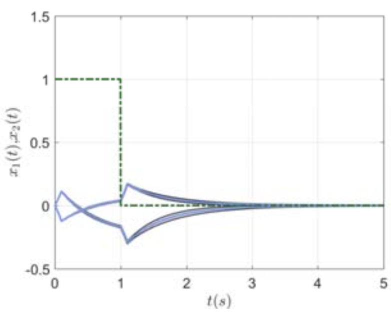

Fig. 3. Closed-loop response to step input of 1 second using the interval-based controller.

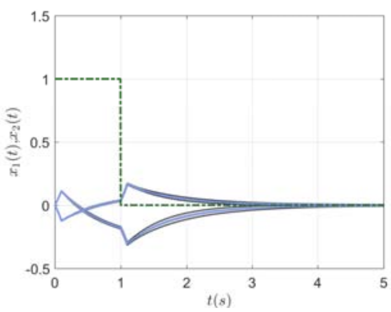

Fig. 4. Closed-loop response to step input of 1 second using the polytopic-based controller.

V. Conclusion

In this paper, we provide novel robust control design techniques for interval systems. We present results that ensure stability and guaranteed $\mathcal{H}_\infty$ performance for discrete-time interval systems. As an application, we consider a linear quadratic sampled-data control design problem for interval systems. As the numerical examples point out, the design techniques presented in this paper can deal with systems that present a relatively large number of uncertain parameters, which cannot be tackled by polytopic-based approaches.

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