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Manipulating the quota in weighted voting games ☆

Michael Zuckerman [a], Piotr Faliszewski [b], Yoram Bachrach [c], Edith Elkind [d,*]

[a] School of Computer Science and Engineering, The Hebrew University of Jerusalem, Israel
[b] AGH University of Science and Technology, Krakow, Poland
[c] Microsoft Research Ltd., Cambridge, United Kingdom
[d] Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore

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ABSTRACT

Weighted voting games provide a simple model of decision-making in human societies and multi-agent systems. Such games are described by a set of players, a list of players' weights, and a quota; a coalition of the players is said to be winning if the total weight of its members meets or exceeds the quota. The power of a player in a weighted voting game is traditionally identified with her Shapley–Shubik index or her Banzhaf index, two classic power measures that reflect the player's marginal contribution under different coalition formation scenarios. In this paper, we investigate by how much one can change a player's power, as measured by these indices, by modifying the quota. We give tight bounds on the changes in the individual player's power that can result from a change in quota. We then describe an efficient algorithm for determining whether there is a value of the quota that makes a given player a *dummy*, i.e., reduces her power (as measured by both indices) to 0. We also study how the choice of quota can affect the relative power of the players. Finally, we investigate scenarios where one's choice in setting the quota is constrained. We show that optimally choosing between two values of the quota is complete for the complexity class PP, which is believed to be significantly more powerful than NP. On the other hand, we empirically demonstrate that even small changes in quota can have a significant effect on a player's power.

1. Introduction

Cooperation and joint decision-making are key aspects of many interactions among self-interested agents. In such interactions, the collaborating agents may have different preferences, so they need a method to agree on a common course of action. One possible solution to this problem is to use a voting procedure, and select a plan that is supported by a majority of voters. This approach to decision-making is very common in human societies and can be naturally extended to multi-agent systems [12].

Under majority voting, all agents have the same power. However, treating all voters as equals is not always appropriate: some of the agents may be more important for the task at hand than others, or contribute a larger amount of resources to it. Similarly, in parliamentary voting, some of the legislators may represent a larger constituency, and therefore should be given more influence over the final outcome. This issue can be addressed by employing the machinery of *weighted voting games*. In such games, each agent is associated with a nonnegative weight, and a subset (coalition) of agents is deemed to

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* Corresponding author.
E-mail addresses: michez@cs.huji.ac.il (M. Zuckerman), faliszew@agh.edu.pl (P. Faliszewski), yorambac@gmail.com (Y. Bachrach), eelkind@ntu.edu.sg (E. Elkind).

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be winning if its weight meets or exceeds a given quota. The voter's weight reflects her relative importance in the decision-making process: more important voters are assigned a higher weight. The quota is typically set to be slightly greater than half of the total weight, but other values of quota (e.g., 2/3 of the total weight) are quite common as well.

Even though weights are intended to model the agents' relative importance, an agent's ability to influence the group decision is not always directly proportional to her weight. For example, if the quota is so high that the only winning coalition is the one that involves all agents, each agent can veto the decision, and hence all agents have equal power. Thus, to measure the power, instead of using agents' weights, one typically employs one of the so-called *power indices*. Perhaps the most prominent ones among them are the Shapley–Shubik index [41] and the Banzhaf index [11,6]. Intuitively, both of these indices measure the probability that a given agent is critical to a forming coalition, i.e., that the coalition would become winning if the agent joined in; the difference between these two power indices comes from different coalition formation models. Besides measuring the apriori voting power, the power indices can be used to share the payoff obtained by executing the task: a natural approach is to pay each agent in proportion to their voting power, i.e., their Shapley–Shubik index or their Banzhaf index. Also, in politics, power indices provide very useful information to lobbyists who need to decide how to allocate their contributions.

The importance of power indices makes them a natural target for manipulators, i.e., rogue parties that want to increase or decrease the voting power of a certain agent.[1] Now, accomplishing this goal by changing an agent's weight may require a substantial investment on the manipulator's part, such as, e.g., recruiting additional supporters of a political party. In contrast, it may be relatively easy to change the quota. Indeed, such changes are not unusual in political decision-making, and can be explained by the desire to build a consensus (if the quota is increased) or simplify the passage of bills (if the quota is decreased)—for instance, a recent move by Democratic members of the U.S. Senate to change the filibuster rules [22] can be viewed as an attempt to change the quota. Therefore, the entity that determines the format of the decision-making procedure (in what follows, we will refer to this entity as the *central authority*) might be able to change the quota without encountering substantial resistance. However, this seemingly innocent change may have very different effects on different voters, and therefore the central authority can use it to advance its own goals.

In some settings, the quota may have to be updated in response to other changes in the voting system, such as expansion of the system to include new players (as was the case, for instance, when the European Union expanded from 15 to 27 member states) or changes to players' weights (it is plausible that in the future the countries' weights in the EU Council may have to be updated to reflect the demographic changes). In such scenarios, the central authority would normally have some freedom in setting the quota and may pursue a variety of objectives when doing so; for a discussion of this issue in the context of European Union enlargement, see [28,30,32].

In this paper, we study the effect of quota changes on the agents' power, as measured by the Shapley–Shubik power index and by the Banzhaf power index. We first provide tight bounds on the change in voter's power that can be accomplished by modifying the quota. It turns out that there are settings where all voters except for the one with the maximum weight can have their voting power reduced to zero by an appropriate choice of the quota, i.e., the *ratio* between the voter's power before and after the change of quota can be unbounded; however, for both indices, we can obtain tight worst-case bounds on the *difference* between the values of the index before and after the change.

Having established that changing the quota may have a very significant effect on the agents' power, we focus on the algorithmic aspects of the manipulator's problem. The manipulator may want to either minimize or maximize the target player's power. We limit our attention to the former problem. In this case, the best that the manipulator can hope for is to make the target player a dummy, i.e., to ensure that this player's power (as measured by both indices) is 0. We show that the center can easily determine whether there is a quota value that accomplishes this. This result is somewhat surprising, since *checking* if a given agent is a dummy for a fixed value of the quota is well-known to be coNP-complete [37,10,35].

The ranking of the agents is sometimes more important than the exact power they possess: for instance, a party in parliament may have a better negotiating position if it is among the top three most powerful players. Therefore, we also study the problem of setting the quota so as to guarantee a particular relation (equality or inequality) between two agents' power-index values. We demonstrate that as long as two agents have different weights, the quota can be selected so that they have different voting power. A related issue that we consider is that of selecting the quota so as to ensure that all agents with different weights have different power-index values. We exhibit a family of weight vectors for which essentially any value of the quota has this property. In contrast, we show that if agents' weights grow fast enough, this goal cannot be achieved.

In many real-life settings, the center will only be able to change the quota by a relatively small amount, or choose among a few acceptable quota values. It is therefore interesting to ask if the manipulator can achieve his goals when his ability to change the quota is constrained. We provide a twofold answer to this question. First, we show that choosing the quota optimally from a given set is likely to be hard. Specifically, we prove that the problem of deciding which of the two given values of the quota is better for a particular agent is complete for the complexity class PP, which is believed to be more powerful than NP. However, if the manipulator's computational resources are not limited, he may be able to achieve his

[1] In voting theory literature, the term "manipulation" is reserved for voters' dishonest behavior, while the dishonest behavior by the election authorities is usually referred to as "control". However, in this paper we will use both terms interchangeably.

goals even if the range of available quotas is fairly small: we present experimental results showing that even changing the quota by up to 20% may have a noticeable effect on agents' power.

We remark that our work *does not* provide an algorithm for choosing a quota so as to maximize of minimize a given player's power. This question has been recently addressed by Zick et al. [43]; their paper builds on the conference version of our work. Zick et al. focus on the Shapley–Shubik power index, and show that the power of a player can be maximized by setting the quota to that player's weight; in contrast, a slightly higher quota is quite likely to minimize this player's power. More precisely, if a player's weight w is small relative to the weights of the other players, then $w + 1$ is often the worst possible value of the quota for this player; on the other hand, if w is relatively large, the player's power can often be minimized by setting the quota to 1 (in which case all players have the same power). Zick et al. also show that checking whether a given quota maximizes or minimizes a player's power is NP-hard, and provide a polynomial-time algorithm for deciding whether all players have equal power. Their results, together with the work presented in this paper, demonstrate that changing the quota is a subtle, but effective way to alter the distribution of power in a weighted voting system.

The rest of the paper is organized as follows. After reviewing the related work in Section 1.1 and presenting the necessary definitions in Section 2, in Section 3 we establish tight upper bounds on the changes in players' power that can be achieved by altering the quota. Section 4 describes an efficient algorithm for checking if there is a choice of quota that turns a given player into a dummy. Setting the quota so as to ensure that different players have different power is discussed in Section 5. In Section 6, we analyze the complexity of comparing a player's power for two different values of the quota. Section 7 presents our experimental results and Section 8 concludes.

1.1. Related work

A detailed study of many aspects of weighted voting games can be found in [20] and [42]. The Shapley value originated in a seminal paper of Shapley [40], who considered how to fairly allocate the utility gained by the grand coalition in a cooperative game. A subsequent paper of Shapley and Shubik [41] applied the Shapley value to weighted voting games, so this value is referred to as the *Shapley–Shubik power index* in this context. The first version of the Banzhaf power index was introduced by Banzhaf in [6]; an alternative definition was later proposed by Dubey and Shapley [11]. The definition given in [11] has a direct probabilistic interpretation, while for Banzhaf's original definition this is not the case. The index proposed in [6] is now known as the *normalized* Banzhaf index, as it rescales the players indices (as defined in [11]) so that they sum up to 1. Felsenthal and Machover [21] provide a persuasive argument against using the normalized version of the Banzhaf index; therefore, in this paper we use the definition of Dubey and Shapley.

Both power indices have been thoroughly studied and are considered standard tools [39]. Their practical applications include analyzing the voting structures of the European Union Council of Ministers and the IMF [33,29]. Computational complexity of power indices is also quite well understood: while computing both indices is #P-hard [37,10,35,16], they can be computed in polynomial time when all weights are at most polynomial in the number of players [34], and several papers (e.g., [19,4]) discuss ways to *approximate* them. Some of these algorithms work well in practice and thus justify the use of power indices as a practical way to estimate a player's influence.

The effect of the choice of quota on the players' power has been studied by Leech and Machover [30] in the context of power distribution in the European Union. However, Leech and Machover focus on the inverse problem: namely, for each value of the quota between 51% to 99% they determine the weights that ensure that the voting power of each country (as measured by power indices) is proportional to its population. Further, Leech and Machover only consider a specific voting scenario and do not investigate the algorithmic aspects of the quota selection problem.

More recently, designing weighted voting games with pre-specified values of power indices has been studied by a number of authors [3,18,1,9,26]. However, in these papers the game designer is assumed to be able to select both the weights and the quota, whereas we assume that the players' weights cannot be changed. Moreover, none of these papers provides a provably polynomial-time exact algorithm for the problem they study. In more detail, Aziz et al. [3] use a generating function approach, while Kurz [26] makes use of integer linear programming; for both methods, the running time is exponential in the worst case. The algorithm of de Keijzer et al. [9] is based on direct enumeration and is therefore exponential as well. Fatima et al. [18] provide an approximation algorithm. Finally, Alon and Edelman [1] focus on identifying vectors that can be approximated by normalized Banzhaf vectors (i.e., vectors of normalized Banzhaf indices) of weighted voting games. Another related question is whether one can build a game with a target power distribution by combining several weighted voting games: this issue was investigated by Faliszewski et al. [13].

A number of papers [2,27,38] consider manipulation by *voters* in weighted voting games, namely, splitting the weight between two or more identities, as well as merging and annexation. In contrast with our work, all these papers assume that, despite the changes in the number of players, the quota always remains fixed. Another form of manipulation that may be available to a voter in a weighted voting game is to declare a conflict with another voter, i.e., to refuse to be in the same coalition with him. Kilgour [25] demonstrates that such manipulation may increase the Shapley value of both voters; Brams [8] shows that this remains true for the Banzhaf index.

Computational aspects of various forms of dishonest behavior in voting with m alternatives received a lot of attention in recent years: see [15,17,14] for surveys of this stream of research. Specifically, this research considers *manipulation* (dishonest behavior by the voters), *control* (dishonest behavior by the election authority), and *bribery* (dishonest behavior by an outside party). This line of work, and, in particular, the papers devoted to control, provides motivation for our research, but results for the model with m alternatives cannot be directly applied to our setting.

2. Preliminaries and notation

A *transferable utility game* $G=(I,v)$ is given by a finite set of *players* I and a function $v:2^I\to\mathbb{R}$; the function v is called the *characteristic function* of the game G. A *coalition* is a subset of players $S\subseteq I$; the set I of all players is called the *grand coalition*. A transferable utility game G is said to be *simple* if v only takes values in $\{0,1\}$, i.e., $v(S)\in\{0,1\}$ for every coalition $S\subseteq I$, and, moreover, $v(S)=1$ implies $v(S')=1$ for every coalition S' such that $S\subseteq S'$. In a simple game $G=(I,v)$, a coalition $S\subseteq I$ is said to be *winning* if $v(S)=1$ and *losing* otherwise. A *weighted voting game* is a simple game that can be described by a *weight vector* $\boldsymbol{w}=(w_1,\dots,w_n)\in\mathbb{R}^n_+$ and a *quota* $q\in\mathbb{R}_+$: a coalition S is winning if its total weight meets or exceeds the quota, i.e., $\sum_{i\in S}w_i\geqslant q$, and losing otherwise; we write $G=[I;\boldsymbol{w};q]$. When discussing weighted voting games, we use the terms “players”, “agents” and “voters” interchangeably.

Given a weighted voting game G, we say that an agent $i\in S$ is *pivotal* for a coalition S if $v(S)=1$ and $v(S\setminus\{i\})=0$; similarly, i *contributes* to S if $v(S)=0$, $v(S\cup\{i\})=1$. A player i is called a *dummy* if he does not contribute to any coalition, i.e., for every $S\subseteq I$ we have $v(S\cup\{i\})=v(S)$. Two players i and j are said to be *symmetric* if $v(S\cup\{i\})=v(S\cup\{j\})$ for every $S\subseteq I\setminus\{i,j\}$; note that if $w_i=w_j$ then i and j are symmetric, but the converse is not always true. We denote by $w(S)$ the total weight of a coalition S, i.e., $w(S)=\sum_{i\in S}w_i$. Unless explicitly specified otherwise, we assume that $0<w_1\leqslant\cdots\leqslant w_n$ and that $0<q\leqslant w(I)$. It is easy to see that this does not affect the generality of our results.

Though weighted voting games are usually defined for arbitrary positive real weights, it is well-known [42] that for every weighted voting game $G=[I;\boldsymbol{w};q]$ there exists a game $G'=[I;\boldsymbol{w}';q']$ with $\boldsymbol{w}'\in\mathbb{N}^n$, $q'\in\mathbb{N}$ such that for every $S\subseteq I$ it holds that $w(S)\geqslant q$ if and only if $w'(S)\geqslant q'$. Moreover, it can be assumed that each w'_i, $i=1,\dots,n$, and q' can be expressed using poly(n) bits. Thus, in what follows we assume that all weights and the quota are positive integers given in binary, i.e., each game with n players can be described using poly(n) bits.

2.1. Shapley–Shubik index and Banzhaf index

Both the Shapley–Shubik index and the Banzhaf index measure an agent's marginal contribution to possible coalitions. However, they differ in the underlying coalition formation scenarios: while the Shapley–Shubik index implicitly assumes that the agents join a coalition in random order, the Banzhaf index is based on the assumption that each agent decides whether to join a coalition independently at random. Both of these measures can be defined for arbitrary transferable utility games. However, in what follows we provide definitions that are tailored to weighted voting games.

Let $\Pi(I)$ be the set of all one-to-one mappings from I to I (i.e., the set of all permutations of I); an element of $\Pi(I)$ is denoted by π. Set $S_\pi(i)=\{j\mid\pi(j)<\pi(i)\}$: the set $S_\pi(i)$ consists of all predecessors of i in π. The *Shapley–Shubik index* of the i-th agent in a game $G=[I;\boldsymbol{w};q]$ is denoted by $\varphi_i(G)$ and is given by the following expression:

$$\varphi_i(G)=\frac{1}{n!}\sum_{\pi\in\Pi(I)}\bigl[v\bigl(S_\pi(i)\cup\{i\}\bigr)-v\bigl(S_\pi(i)\bigr)\bigr].\tag{1}$$

In words, the Shapley–Shubik power index counts the fraction of all orderings of the agents in which agent i is pivotal for the coalition formed by his predecessors and himself. We will occasionally abuse notation and say that an agent i is pivotal for a permutation π if it is pivotal for the coalition $S_\pi(i)\cup\{i\}$.

The Banzhaf index $\beta_i(G)$ of an agent i in a game $G=[I;\boldsymbol{w};q]$ is computed as follows:

$$\beta_i(G)=\frac{1}{2^{n-1}}\sum_{S:\,i\in S}\bigl[v(S)-v\bigl(S\setminus\{i\}\bigr)\bigr].\tag{2}$$

This index simply counts the fraction of coalitions for which agent i is pivotal.

Both of these indices have several useful properties that make them very convenient to work with. In particular, both of them have the *dummy player* property, which states that the value of the index for a given player is 0 if and only if he does not contribute to any coalition, and the *symmetry* property, which states that if two players are symmetric, then their indices are equal. Also, Shapley–Shubik index (but not the Banzhaf index) has the *normalization property*, which means that the sum of Shapley–Shubik indices of all players is equal to 1.[2] All of these properties are easy to verify from the definitions.

Both the number of coalitions in an n-player game and the number of permutations of n players grow exponentially with n. Therefore, computing each of the power indices directly from its definition would take superpolynomial time. Moreover, both the Shapley–Shubik index and the Banzhaf index are known to be #P-hard to compute [37,10,35,16].

To simplify notation, given a game $G=[I;\boldsymbol{w};q]$, we will sometimes write $\varphi_i(q)$ and $\beta_i(q)$ instead of $\varphi_i(G)$ and $\beta_i(G)$ if I and $\boldsymbol{w}$ are clear from the context.

[2] One can define a *normalized* version of the Banzhaf index by setting $\beta'_i(G)=\beta_i(G)/(\sum_{j\in I}\beta_j(G))$; indeed, this is the definition given in Banzhaf's original paper [6]. However, the resulting index does not admit a direct probabilistic interpretation; see [21] for a discussion.

We remark that for every weighted voting game with a set of players I and a weight vector $\boldsymbol{w} \in \mathbb{N}$, and every integer $q \in (0, w(I)]$ we have $\varphi_i(q) = \varphi_i(w(I)+1-q)$, $\beta_i(q) = \beta_i(w(I)+1-q)$ for all $i \in I$. Indeed, player i is pivotal for a coalition S in the game $[I; \boldsymbol{w}; q]$ if and only if it is pivotal for a coalition $(I \setminus S) \cup \{i\}$ in the game $[I; \boldsymbol{w}; w(I)+1-q]$. Similarly, i is pivotal for a permutation π in the game $[I; \boldsymbol{w}; q]$ if and only if it is pivotal for the permutation π' in the game $[I; \boldsymbol{w}; w(I)+1-q]$, where π' is obtained by reversing π. Observe also that $\max\{q, w(I)-q+1\} > w(I)/2$. The reason why this observation is useful is that in most realistic applications of weighted voting games the quota is usually required to be at least half of the total weight. On the other hand, examples with small values of q are sometimes easier to construct and describe. The argument above shows that we may focus on such examples, since the requirement $q > w(I)/2$ is easy to satisfy: any example with $q < w(I)/2$ can be transformed into one with $q > w(I)/2$.

3. Upper and lower bounds for a single player

We will start this section by showing that the center can significantly change the players' Shapley–Shubik and Banzhaf indices by manipulating the quota. We then proceed to quantify the worst case effects of this manipulation for all players. We will be interested both in the *ratios* of the player's powers for a given pair of quotas and in their *differences*.

Example 1. Consider a weighted voting game $G = [I; (1,2,3); 3]$. In this game, player 3 is pivotal for three coalitions (namely, $\{3\}$, $\{1,3\}$ and $\{2,3\}$) and for four permutations (namely, 312, 321, 132 and 231), so we have $\beta_3(G) = 3/4$, $\varphi_3(G) = 2/3$. Now change the quota to 1. In the resulting game $G' = [I; (1,2,3); 1]$, player 3 is only pivotal for the singleton coalition $\{3\}$, so we have $\beta_3(G') = 1/4$. Similarly, player 3 is only pivotal if it appears first in a permutation, so we have $\varphi_3(G') = 1/3$.

A natural bound on manipulator's influence is the worst-case ratio between a given player's values of the index in the two games corresponding to two different values of the quota. Unfortunately, as we will now show, this ratio can only be bounded for the largest player; for all other players, it might be possible to turn them into dummies.

Theorem 2. *Given a set of players I, $|I| = n$, there exists a weight vector $\boldsymbol{w}$, $0 < w_1 \leqslant \cdots \leqslant w_n$, and quotas $q, q' \leqslant w(I)$ such that for $i = 1, \ldots, n-1$, we have $\varphi_i(q) \neq 0$, $\varphi_i(q') = 0$ and $\beta_i(q) \neq 0$, $\beta_i(q') = 0$. On the other hand, for every weight vector $\boldsymbol{w}$ such that $0 < w_1 \leqslant \cdots \leqslant w_n$ and every $q, q' \leqslant w(I)$, we have $\varphi_n(q)/\varphi_n(q') \leqslant n$, $\beta_n(q)/\beta_n(q') \leqslant 2^{n-1}$, and these bounds are tight.*

Proof. Set $\boldsymbol{w} = (\underbrace{1, \ldots, 1}_{n-1}, n)$. In the game $G = [I; \boldsymbol{w}; 1]$ all players have equal power, so by symmetry we have $\varphi_i(1) = 1/n$ for $i = 1, \ldots, n$. Moreover, each player is pivotal for exactly one coalition, so we have $\beta_i(1) = 1/2^{n-1}$. On the other hand, in the game $G' = [I; \boldsymbol{w}; n]$, all the players except for the last one are dummies, so their Shapley–Shubik and Banzhaf indices are 0, and we have $\varphi_n(n) = 1$, $\beta_n(n) = 1$. Hence, $\varphi_n(n)/\varphi_n(1) = n$, $\beta_n(n)/\beta_n(1) = 2^{n-1}$.

To see that the ratio $\varphi_n(q)/\varphi_n(q')$ cannot exceed n, observe that for every n-player weighted voting game G it holds that $1/n \leqslant \varphi_n(G) \leqslant 1$, where both inequalities follow from the fact that $0 \leqslant \varphi_i(G) \leqslant \varphi_n(G)$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n \varphi_i(G) = 1$. Similarly, in any weighted voting game G we have $1/2^{n-1} \leqslant \beta_n(G) \leqslant 1$, so the ratio $\beta_n(q)/\beta_n(q')$ cannot exceed 2^{n-1}. □

The change of quota in the proof of Theorem 2 reduced some players' voting power to 0. What if we are only allowed to change the quota so that all index values remain positive? It turns out that even under this constraint a change in quota can reduce a player's Shapley–Shubik index and Banzhaf index by an exponential factor.

Example 3. Consider the weight vector $\boldsymbol{w} = (1, 2, 4, \ldots, 2^{n-1})$ and quotas $q = 2^k - 1$, $k = 1, \ldots, n$. For $i = 1, \ldots, n$, set $S_i = \{1, \ldots, i\}$. When $q = 2^{i-1} - 1$, player i is pivotal for every coalition of the form $X \cup \{i\}$, where X is a strict subset of S_{i-1}, i.e., for $2^{i-1} - 1$ coalitions. However, when $q = 2^i - 1$, i is only pivotal for a single coalition, namely, S_i. Thus, for $i = \lfloor n/2 \rfloor$, changing the quota from $2^{i-1} - 1$ to $2^i - 1$ lowers the Banzhaf index of player i by an exponential factor.

Similarly, for $q = 2^{i-1} - 1$, player i is pivotal for every permutation where it appears after a strict subset of S_{i-1}, i.e., for

$$\begin{aligned}
\sum_{k=0}^{i-2} \binom{i-1}{k} k!(n-k-1)! &= \sum_{k=1}^{i-1} \binom{i-1}{k-1} (k-1)!(n-k)! \\
&= \sum_{k=1}^{i-1} \frac{(i-1)!}{(i-k)!} (n-k)! \\
&= \sum_{k=1}^{i-1} \binom{n-k}{i-k} (i-1)!(n-i)!
\end{aligned}$$

permutations. In contrast, for $q = 2^i - 1$, player i is only pivotal if it appears after all players in S_{i-1}, i.e., for $(i-1)!(n-i)!$ permutations. For $i = \lfloor n/2 \rfloor$, the gap between these two quantities is exponential in n.

Since for the first $n-1$ players it is impossible to bound the worst-case ratio between their index values for two different quotas, we will now give tight bounds on the worst-case *difference* between a given player's index values in the corresponding games. We first present our result for the Shapley–Shubik index.

Theorem 4. *For a set of players I, $|I| = n$, every weight vector $\boldsymbol{w}$, $0 < w_1 \leqslant \cdots \leqslant w_n$, and every pair of quotas $q, q' \leqslant w(I)$, for $i = 1, \ldots, n-1$ the absolute difference $|\varphi_i(q) - \varphi_i(q')|$ does not exceed $1/(n-i+1)$ and this bound is tight. For player n, we have $|\varphi_n(q) - \varphi_n(q')| \leqslant 1 - 1/n$, and this bound is tight.*

Proof. Consider an arbitrary weight vector $\boldsymbol{w}$ that satisfies $0 < w_1 \leqslant \cdots \leqslant w_n$ and a player i, $1 \leqslant i < n$. We have $\varphi_i(I; \boldsymbol{w}; q') \geqslant 0$ for every $q' \in (0, w(I)]$. On the other hand, the monotonicity of the Shapley–Shubik index implies $\varphi_i(I; \boldsymbol{w}; q) \leqslant \varphi_j(I; \boldsymbol{w}; q)$ for every $j > i$ and every $q \in (0, w(I)]$. As $\sum_{k=i}^{n} \varphi_k(I; \boldsymbol{w}; q) \leqslant 1$, we have $\varphi_i(I; \boldsymbol{w}; q) \leqslant 1/(n-i+1)$. Thus, we obtain $|\varphi_i(I; \boldsymbol{w}; q) - \varphi_i(I; \boldsymbol{w}; q')| \leqslant 1/(n-i+1)$. For player n, we have $\varphi_n(I; \boldsymbol{w}; q') \geqslant 1/n$, $\varphi_n(I; \boldsymbol{w}; q) \leqslant 1$, so $|\varphi_n(I; \boldsymbol{w}; q) - \varphi_n(I; \boldsymbol{w}; q')| \leqslant 1 - 1/n$.

To see that these bounds are tight, set $\boldsymbol{w} = (1, 2, 4, \ldots, 2^{n-1})$. In the game $[I; \boldsymbol{w}; 2^k]$, where $k \in \{1, \ldots, n-1\}$, the first k players are dummies, and the last $n-k$ players have equal power, $1/(n-k)$. Hence, for $i = 1, \ldots, n-1$, by changing the quota from 2^i to 2^{i-1}, we change the Shapley–Shubik index of the i-th player from 0 to $1/(n-i+1)$, as required. For player n, changing the quota from 2^{n-1} to 1 changes n's Shapley–Shubik index from 1 to $1/n$, yielding the difference $1 - 1/n$. $\square$

For the Banzhaf index, the proof is somewhat more difficult.

Theorem 5. *For a set of players I, $|I| = n$, every weight vector $\boldsymbol{w}$, $0 < w_1 \leqslant \cdots \leqslant w_n$, and every pair of quotas $q, q' \leqslant w(I)$, for $i = 1, \ldots, n-1$ the absolute difference $|\beta_i(q) - \beta_i(q')|$ can be at most $\binom{n-i}{\lfloor \frac{n-i}{2} \rfloor} \cdot 2^{i-n}$ and this bound is tight. For player n, we have $|\beta_n(q) - \beta_n(q')| \leqslant 1 - 1/2^{n-1}$ and this bound is tight.*

Proof. We consider the case $i < n$ first. To build up intuition, we will first describe a family of games in which our bounds are achieved; subsequently, we will prove that these bounds hold for every weighted voting game. Let $I = \{1, \ldots, n\}$ be a set of players, fix $i \in \{1, \ldots, n-1\}$, and let $(\underbrace{1, \ldots, 1}_{i-1}, i, \underbrace{2i, \ldots, 2i}_{n-i})$ be the vector of the players' weights. Set $q = 2i \cdot \lfloor \frac{n-i}{2} \rfloor + i$, and $q' = 2i$. For quota q, agent i contributes to a coalition exactly if this coalition contains $\lfloor \frac{n-i}{2} \rfloor$ players of weight $2i$ and any number of players of weight 1. There are $\binom{n-i}{\lfloor \frac{n-i}{2} \rfloor} \cdot 2^{i-1}$ such coalitions and thus $\beta_i(q) = \binom{n-i}{\lfloor \frac{n-i}{2} \rfloor} \cdot 2^{i-1}/2^{n-1}$. Since $\beta_i(q') = 0$, we have

$$\beta_i(q) - \beta_i(q') = \binom{n-i}{\lfloor \frac{n-i}{2} \rfloor} \cdot 2^{i-n}.$$

Now, consider an arbitrary weighted voting game $[I; \boldsymbol{w}; q]$ with $I = \{1, \ldots, n\}$, $w_1 \leqslant \cdots \leqslant w_n$, and $0 < q \leqslant w(I)$. Fix a player $i \in \{1, \ldots, n-1\}$, and let $X = \{1, \ldots, i-1\}$ and $Y = \{i+1, \ldots, n\}$. Let $S \subseteq 2^I$ be the set of all the coalitions that player i is pivotal for. We will now argue that

$$|S| \leqslant \binom{n-i}{\lfloor \frac{n-i}{2} \rfloor} \cdot 2^{i-1}.$$

Since $\beta_i(q) = |S|/2^{n-1}$ and $\beta_i(q') \geqslant 0$ for every $q' \in (0, w(I)]$, this proves the theorem for $i < n$.

Pick $Z_1, Z_2 \in S$ so that $Z_1 \neq Z_2$ and $Z_1 \cap Y \supseteq Z_2 \cap Y$. We claim that $Z_1 \cap X \neq Z_2 \cap X$. Indeed, suppose for the sake of contradiction that $Z_1 \cap X = Z_2 \cap X$. As $Z_1 \neq Z_2$, it follows that $Z_2 \cap Y$ is a strict subset of $Z_1 \cap Y$. On the other hand, $w_j \geqslant w_i$ for all $j \in Y$, and hence

$$q > \sum_{j \in Z_1 \setminus \{i\}} w_j \geqslant \sum_{j \in Z_2 \setminus \{i\}} w_j + w_i \geqslant q,$$

a contradiction.

Now, recall that a collection of sets $\{Q_1, Q_2, \ldots, Q_\ell\}$ is called a *chain* if $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_\ell$. By Sperner's theorem (see, e.g., [31]), there exists a partition of Y into at most $\binom{|Y|}{\lfloor \frac{|Y|}{2} \rfloor}$ chains. Let $\mathcal{P}$ be some such partition. $\mathcal{P}$ induces a partition of S: for every $Z_1, Z_2 \in S$, we write $Z_1 \sim Z_2$ if $Z_1 \cap Y$ and $Z_2 \cap Y$ belong to the same chain in $\mathcal{P}$. It is not hard to see that $\sim$ is an equivalence relation on S. By the argument above, the size of each equivalence class with respect to $\sim$ does not exceed $2^{|X|}$. Hence,

$$|S| \leqslant \binom{|Y|}{\lfloor \frac{|Y|}{2} \rfloor} \cdot 2^{|X|} = \binom{n-i}{\lfloor \frac{n-i}{2} \rfloor} \cdot 2^{i-1},$$

as required.

For player n, we have argued that $1/2^{n-1} \leqslant \beta_i(G) \leqslant 1$ for every n-player game G. Thus, for every weight vector $\boldsymbol{w} \in \mathbb{R}^n_+$ with $w_1 \leqslant \cdots \leqslant w_n$ and every pair of games $G = [I; \boldsymbol{w}; q]$ and $G' = [I; \boldsymbol{w}; q']$ we have $|\beta_n(G) - \beta_n(G')| \leqslant 1 - 1/2^{n-1}$. Also, we have seen that for the weight vector $\boldsymbol{w} = (1, \ldots, 1, n)$ and games $G = [I; \boldsymbol{w}; n]$ and $G' = [I; \boldsymbol{w}; 1]$ we have $\beta_n(G) - \beta_n(G') = 1 - 1/2^{n-1}$. □

4. Turning a player into a dummy

Turning a given player into a dummy is a very natural goal for a central authority that strongly dislikes a particular agent, e.g., an election authority that wants to ensure that a certain party has no influence in the parliament. One might expect this problem to be computationally difficult: indeed, it is well-known that checking whether a given player is a dummy is coNP-complete [37,10,35]. However, it turns out that we can efficiently determine if there exists a value of the quota that turns a given player into a dummy. That is, if the center's goal is to ensure that a certain player has no power, finding a "good" quota is easier than checking if a given quota is "good" (assuming P $\neq$ NP).

Definition 6. Given a weight vector $\boldsymbol{w} = (w_1, \ldots, w_n)$ such that $0 < w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$ and a weight w, we say that w is *essential* for $\boldsymbol{w}$ if $\sum_{i=1}^{t-1} w_i \geqslant w_t - w$ for all $1 \leqslant t \leqslant n$.

Example 7. Suppose that $w_i = i$ for $i = 1, \ldots, n$. Then any positive integer value of w is essential for $\boldsymbol{w} = (w_1, \ldots, w_n)$: for $t = 1$ we have $w \geqslant w_1$ and hence $\sum_{i=1}^{0} w_i = 0 \geqslant w_1 - w$, and for $t \geqslant 2$ we have $\sum_{i=1}^{t-1} w_i \geqslant t - 1 \geqslant w_t - w$.

In contrast, suppose that $w_i = 3^i$ for $i = 1, \ldots, n$. Then $\sum_{i=1}^{t-1} w_i = \frac{1}{2}(3^t - 3) < 3^t$. Thus, w is essential for $\boldsymbol{w} = (w_1, \ldots, w_n)$ only if it is sufficiently large, i.e., $w \geqslant \frac{1}{2}(3^n + 3)$.

The next theorem justifies using the term "essential" in Definition 6: A player whose weight is essential for the vector of weights of the remaining players is never a dummy, irrespective of the choice of the quota value for the game. (Before proceeding, the reader may want to convince herself that this is true for the weighted voting games considered in Example 7.)

Theorem 8. *Let $\boldsymbol{w} = (w_1, \ldots, w_n)$ be a vector of weights such that $0 < w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$. A weight w is essential for $\boldsymbol{w}$ if and only if for every $q \in [1, w + w(I)]$ player $n + 1$ is not a dummy in the game $G(q) = [I'; \boldsymbol{w}'; q]$, where $I' = \{1, \ldots, n, n + 1\}$ and $\boldsymbol{w}' = (w_1, \ldots, w_n, w)$.*

Proof. Suppose first that w is not essential for $\boldsymbol{w}$, i.e., $w + \sum_{i=1}^{t-1} w_i < w_t$ for some $t \in \{1, \ldots, n\}$. Set $q = w_t$. In the game $G(q)$, a coalition is winning if and only if it includes a player $s \in \{t, \ldots, n\}$, i.e., all players $1, \ldots, t - 1, n + 1$ are dummies in $G(q)$.

Conversely, suppose that w is essential for $\boldsymbol{w}$. Fix a $t \in \{1, \ldots, n\}$, and let $(S^t_1, \ldots, S^t_{2^t})$ be a list of all subsets of $\{1, \ldots, t\}$, ordered by their weight (from the smallest to the largest). We will now show that this list is sufficiently "dense".

Lemma 9. *For every two adjacent sets S^t_i and S^t_{i+1} in this ordering it holds that $w(S^t_{i+1}) - w(S^t_i) \leqslant w$.*

Before we prove Lemma 9, let us show that it implies our theorem. Fix an arbitrary quota $q \in [1, w + w(I)]$. If $q \leqslant w$ or $q > w(S^n_{2^n})$, then clearly player $n + 1$ is not a dummy in $G(q)$. Now, suppose that $w < q \leqslant w(S^n_{2^n})$. Since w is essential for $\boldsymbol{w}$, we have $w_1 \leqslant w$ and hence $w(S^n_1) = w_1 < q$. Combining the inequalities $w(S^n_{2^n}) \geqslant q$ and $w(S^n_1) < q$, we conclude that there exists some $i \in \{1, \ldots, 2^n - 1\}$ such that $w(S^n_i) < q$, $v(S^n_{i+1}) \geqslant q$. By Lemma 9 we have $w(S^n_i) + w \geqslant w(S^n_{i+1}) \geqslant q$. This means that player $n + 1$ is pivotal for $S^n_i \cup \{n + 1\}$ in $G(q)$, which is exactly what we need to show. It remains to prove Lemma 9.

Proof of Lemma 9. The lemma is proved by induction on t. Specifically, we prove that for all $t = 1, \ldots, n$ and all $x \in [0, \sum_{i=1}^{t} w_i]$ there exists a subset $S \subseteq \{1, \ldots, t\}$ such that $w(S) - w \leqslant x \leqslant w(S)$.

For $t = 1$, consider any $x \in [0, w_1]$. Since w is essential for $\boldsymbol{w}$, we have $w \geqslant w_1$, and hence $w_1 - w \leqslant x \leqslant w_1$, so we can set $S = \{1\}$.

Now, suppose that the lemma holds for $t - 1$. We will show that it is also true for t. Fix an $x \in [0, \sum_{i=1}^{t} w_i]$. We consider three cases:

1. $x \leqslant w_t - w$. In this case, since w is essential for $\boldsymbol{w}$, we have $x \leqslant \sum_{i=1}^{t-1} w_i$, so by the inductive hypothesis there exists some $S \subseteq \{1, \ldots, t - 1\}$ such that $w(S) - w \leqslant x \leqslant w(S)$. Since S is also a subset of $\{1, \ldots, t\}$, we are done.
2. $w_t - w < x \leqslant w_t$. We can set $S = \{t\}$.
3. $w_t < x \leqslant \sum_{i=1}^{t} w_i$. We have $0 < x - w_t \leqslant \sum_{i=1}^{t-1} w_i$, so by the inductive hypothesis there exists a set $S' \subseteq \{1, \ldots, t - 1\}$ such that $w(S') - w \leqslant x - w_t \leqslant w(S')$. Take $S = S' \cup \{t\}$.

This completes the proof of the inductive step. Thus, the lemma is proved. □

We have already argued that Lemma 9 implies Theorem 8. Hence, the proof is complete. □

Theorem 8 yields a simple algorithm for finding a quota that makes a specific agent a dummy player.

Theorem 10. *There exists a polynomial-time algorithm that, given a weight vector $\boldsymbol{w} = (w_1, \dots, w_n)$ and $i \in \{1, \dots, n\}$, decides whether there exists a quota q, $q \in [1, \dots, \sum_{i=1}^{n} w_i]$, such that player i is a dummy in the game $[\{1, \dots, n\}; \boldsymbol{w}; q]$, and, if so, outputs such a quota.*

Proof. First, we sort the set $W' = \{w_1, \dots, w_n\} \setminus \{w_i\}$ in non-decreasing order; this can be done in $O(n \log n)$ steps. Let $\boldsymbol{w}' = (w'_1, \dots, w'_{n-1})$ be the resulting sorted list; the elements of $\boldsymbol{w}'$ satisfy $w'_1 \leqslant \dots \leqslant w'_{n-1}$. By Theorem 8, it remains to check whether w_i is essential for $\boldsymbol{w}'$. A straightforward implementation of this check requires $O(n^2)$ arithmetic operations; this running time can be improved to $O(n)$ by observing that we can obtain the $(t+1)$-st sum $\sum_{j=1}^{t} w'_j$ from the t-th sum $\sum_{j=1}^{t-1} w'_j$ using a single addition.

Now, if w_i is essential for $\boldsymbol{w}'$, Theorem 8 implies that i cannot be made a dummy. Otherwise, there exists a $t \in \{1, \dots, n\}$ such that $w_i + \sum_{j=1}^{t-1} w'_j < w'_t$; as argued in the proof of Theorem 8, setting $q = w'_t$ ensures that i is a dummy in $[\{1, \dots, n\}; \boldsymbol{w}; q]$. □

Using Theorem 10, we can easily find a quota that minimizes the Banzhaf index of an agent (we remark, however, that this approach does not work for the Shapley–Shubik index).

Theorem 11. *There exists a polynomial-time algorithm that, given a weight vector $(w_1, \dots, w_n)$ and a player i, finds a value of the quota that minimizes the Banzhaf index of i.*

Proof. We first use the algorithm given in the proof of Theorem 10 to check if there is a quota that makes i a dummy player, and if so, return this quota. Otherwise, we return $q = 1 + \sum_{j=2}^{n} w_j$. Under q, the Banzhaf index of i is $1/2^{n-1}$, since the only coalition it is pivotal for is the grand coalition. □

We remark that the approach to minimizing a player's Banzhaf index that is suggested by Theorem 11 is not necessarily practical: the quota value $q = 1 + \sum_{j=2}^{n} w_j$ may be considered to be unacceptably large, and, moreover, it equalizes all players' power. An interesting question is how to choose a value of q from a given interval (say, between 50% and 75% of the total weight) so as to minimize/maximize the player's power. This question appears to be considerably more difficult and presents a promising direction for future work; some relevant empirical results can be found in Section 7.

5. Altering the relative power of two or more players

So far, we have considered the effects that a change of quota can have on the power of a single player, both in absolute and in relative terms. This focus is justified when the manipulator can be assumed to be interested in increasing or decreasing the influence of a given player, irrespective of how it affects the other players. However, the manipulator may also want to alter the *relative* power of two players i and j. For instance, suppose that $w_i < w_j$, and the center prefers player i to player j. From the monotonicity properties of both indices, it follows that for every value of the quota q we have $\varphi_i(q) \leqslant \varphi_j(q)$, $\beta_i(q) \leqslant \beta_j(q)$. Hence, the best that the center may hope for is to find a value of the quota q that satisfies $\varphi_i(q) = \varphi_j(q)$ or $\beta_i(q) = \beta_j(q)$. Conversely, if the center prefers player j to player i, it may try to choose the quota so that $\varphi_j(q) > \varphi_i(q)$ (respectively, $\beta_j(q) > \beta_i(q)$). It turns out that both of these objectives are easy to accomplish. On the other hand, choosing a quota so that *all* players have different power is more difficult.

Throughout this section, we use the following notation: given a weighted voting game $G = [I; \boldsymbol{w}; q]$, two players $i, j \in I$, and a set $S \subseteq I$ such that $i \in S$, $j \notin S$, we write $S^{i,j} = (S \setminus \{i\}) \cup \{j\}$. Similarly, given a permutation $\pi \in \Pi(I)$, we denote by $\pi^{i,j}$ the permutation obtained from π by transposing i and j.

The following proposition is the basis of many proofs in this section.

Proposition 12. *Consider a weighted voting game $G = [I; \boldsymbol{w}; q]$ and two players i, j with $w_i \leqslant w_j$. We have $\beta_i(q) < \beta_j(q)$ if and only if there exists a set $S \subseteq I \setminus \{j\}$ such that $i \in S$, i is not pivotal for S, but j is pivotal for $S^{i,j}$ (or, equivalently, if the set $S' = S \setminus \{i\}$ satisfies $w(S' \cup \{i\}) < q \leqslant w(S' \cup \{j\})$). Similarly, $\varphi_i(q) < \varphi_j(q)$ if and only if there exists a permutation $\pi \in \Pi(I)$ such that i is not pivotal for π, but j is pivotal for $\pi^{i,j}$.*

Before we proceed to the proof of Proposition 12, we present a small illustrative example. Let $I = \{1, 2, 3, 4\}$, $\boldsymbol{w} = (2, 3, 5, 8)$ and $q = 11$, and consider the game $G = [I; \boldsymbol{w}; q]$. Let $i = 3$, $j = 4$, $S = \{1, 2, 3\}$. We have $w(S) < q$, so i is not pivotal for S. However, we have $w(S^{i,j}) = 13$, $w(S^{i,j} \setminus \{j\}) = 5$, so j is pivotal for S. Similarly, consider a permutation π given by $\pi(\ell) = \ell$. Player i is not pivotal for π, but player j is pivotal for the permutation $\pi^{i,j}$, which is given by $\pi^{i,j}(1) = 1$, $\pi^{i,j}(2) = 2$, $\pi^{i,j}(3) = 4$, $\pi^{i,j}(4) = 3$.

Proof. Consider first the Banzhaf index. Let $\mathcal{S} = \{S \subseteq I \mid i \in S\}$ be the set of all coalitions that contain player i. Set $\mathcal{S}_1 = \{S \in \mathcal{S} \mid j \notin S\}$ and $\mathcal{S}_2 = \mathcal{S} \setminus \mathcal{S}_1$. Let $f(S) = S^{i,j}$ if $S \in \mathcal{S}_1$ and $f(S) = S$ if $S \in \mathcal{S}_2$.

We claim that f is injective on $\mathcal{S}$. Indeed, if $S, S' \in \mathcal{S}_1$ or $S, S' \in \mathcal{S}_2$ then clearly $S \neq S'$ implies $f(S) \neq f(S')$. On the other hand, if $S \in \mathcal{S}_1$, $S' \in \mathcal{S}_2$, then $i \notin f(S)$, $i \in f(S')$, so $f(S) \neq f(S')$ in this case as well.

Now, consider any $S \in \mathcal{S}$ such that i is pivotal for S. We claim that j is pivotal for $f(S)$. Indeed, if $S \in \mathcal{S}_1$, we have $w(f(S) \setminus \{j\}) = w(S) - w_i < q$, $w(f(S)) = w(S) - w_i + w_j \geqslant w(S) \geqslant q$, and if $S \in \mathcal{S}_2$, we have $w(f(S) \setminus \{j\}) = w(S) - w_j \leqslant w(S) - w_i < q$, $w(f(S)) = w(S) \geqslant q$.

Thus, each set that i is pivotal for corresponds to a distinct set that j is pivotal for. Therefore, $\beta_i(q) < \beta_j(q)$ if and only if we can find a set in $\mathcal{S}$ such that i is not pivotal for S, but j is pivotal for $f(S)$. It remains to show that we can pick this set in $\mathcal{S}_1$. To see this, observe that if i is not pivotal for $S \in \mathcal{S}_2$, but j is pivotal for S, we have $w(S) - w_i \geqslant q$, $w(S) - w_i - w_j < w(S) - w_j < q$, i.e., j is also pivotal for the set $S \setminus \{i\} \in \mathcal{S}_1$.

For the Shapley–Shubik index, the proof is similar. Suppose that i is pivotal for a permutation $\pi \in \Pi(I)$. It is easy to see that j is pivotal for $\pi^{i,j}$. Indeed, if $j \in S_\pi(i)$, we have $i \in S_{\pi^{i,j}}(j)$ and $w(S_{\pi^{i,j}}(j)) \leqslant w(S_\pi(i)) < q$, $w(S_{\pi^{i,j}}(j) \cup \{j\}) = w(S_\pi(i) \cup \{i\}) \geqslant q$. On the other hand, if $j \notin S_\pi(i)$, we have $S_{\pi^{i,j}}(j) = S_\pi(i)$ and hence $w(S_{\pi^{i,j}}(j)) < q$, while $w(S_{\pi^{i,j}}(j) \cup \{j\}) \geqslant w(S_\pi(i) \cup \{i\}) \geqslant q$. Thus, the mapping $g(\pi) = \pi^{i,j}$ is injective and maps any permutation that i is pivotal for to a permutation that j is pivotal for. Therefore, $\varphi_i(q) < \varphi_j(q)$ is and only if there is a permutation π such that i is not pivotal for π, but j is pivotal for $\pi^{i,j}$. □

Proposition 12 has a number of useful consequences. First, Corollary 13 allows us to restrict our attention to the Banzhaf power index throughout this section.

Corollary 13. *Given a weighted voting game $G = [I; \boldsymbol{w}; q]$ and two players $i, j \in I$ we have $\varphi_i(G) = \varphi_j(G)$ if and only if $\beta_i(G) = \beta_j(G)$.*

Proof. It $w_i = w_j$, our claim is obviously true. Thus, assume without loss of generality that $w_j > w_i$. If $\beta_j(G) > \beta_i(G)$, then by Proposition 12 there exists a set S such that $i \in S$, i is not pivotal for S, but j is pivotal for $S^{i,j}$. Consider a permutation π that places elements of $S \setminus \{i\}$ first, followed by i, followed by j. It is not hard to see that i is not pivotal for π, but j is pivotal for $\pi^{i,j}$, so the claim follows.

Conversely, suppose that $\varphi_j(G) > \varphi_i(G)$. By Proposition 12, there exists a permutation π such that i is not pivotal for π, but j is pivotal for $\pi^{i,j}$. If i precedes j in π, set $S = S_\pi(i) \cup \{i\}$. Clearly, i is not pivotal for S. On the other hand, $S^{i,j} = S_\pi(i) \cup \{j\}$, so j is pivotal for $S^{i,j}$, and by Proposition 12 we have $\beta_j(G) > \beta_i(G)$. If i appears after j in π, set $S = (S_\pi(i) \setminus \{j\}) \cup \{i\}$. Since j is pivotal for $\pi^{i,j}$, we have $w(S) = w_{\pi^{i,j}}(j) < q$. Hence, i is not pivotal for S. Further, we have $w(S \cup \{j\}) = w_{\pi^{i,j}}(j) \cup \{j\} \geqslant q$. Since i is not pivotal for π, this implies $w(S^{i,j}) = w(S_\pi(i)) \geqslant q$. On the other hand, $w(S^{i,j} \setminus \{j\}) = w(S) < q$. Hence, by Proposition 12 we have $\beta_j(G) > \beta_i(G)$. □

Further, Proposition 12 enables us to determine the complexity of comparing the power indices of two players in the same game.

Theorem 14. *Given a weighted voting game $G = [I; \boldsymbol{w}; q]$ and two players $i, j \in I$, the problem of deciding whether $\beta_j(G) > \beta_i(G)$ is* NP-*complete.*

Proof. It is not hard to see that this problem is NP-hard. Indeed, we have already mentioned that the problem of checking whether a given player i in a game $G = [I; \boldsymbol{w}; q]$ is a dummy is coNP-complete [37,10,35]. We will now give a reduction from the complement of this problem to our problem.

Given a game $G = [I; \boldsymbol{w}; q]$ with $|I| = n$ and a player $i \in G$, we construct a new game $G' = [I'; \boldsymbol{w}'; q']$ as follows. We set $I' = I \cup \{n+1\}$, $w'_j = 2w_j$ for $j = 1, \ldots, n$, $w'_{n+1} = 1$, and $q' = 2q$. We will now argue that i is not a dummy in G if and only if $\beta_i(G') > \beta_{n+1}(G')$.

It is clear that player $n+1$ is a dummy in G' and hence $\beta_{n+1}(G') = 0$. To complete the proof, it remains to argue that i is not a dummy in G if and only if $\beta_i(G') > 0$, i.e., i is not a dummy in G'.

Indeed, if i is pivotal for a coalition S in G, he is also pivotal for S in G'. Conversely, suppose that i is pivotal for a coalition S' in G'. If $n+1 \notin S'$, then clearly i is pivotal for S' in G as well. On the other hand, if $n+1 \in S'$, then $w'(S')$ is an odd number. Since q' is even, this means that $w'(S') \geqslant q'+1$ and hence $w'(S' \setminus \{n+1\}) \geqslant q'$. Further, $w'(S' \setminus \{i\}) < q'$ implies $w'((S' \setminus \{n+1\}) \setminus \{i\}) < q'$. Thus, i is pivotal for the coalition $S' \setminus \{n+1\}$ in G'. Since this coalition does not contain $n+1$, by the argument above i is pivotal for $S' \setminus \{n+1\}$ in G. This completes our hardness proof.

To see that our problem is in NP, we make use of Proposition 12. Consider a game $G = [I; \boldsymbol{w}; q]$ and two players $i, j \in I$. If $w_i = w_j$, then $\beta_j(G) = \beta_i(G)$. Now, suppose that $w_j > w_i$. By Proposition 12, to check that $\beta_j(G) > \beta_i(G)$, it suffices to guess a set $S' \in I \setminus \{i, j\}$ such that $w(S' \cup \{i\}) < q \leqslant w(S' \cup \{j\})$. □

Thus, comparing two players' power indices in the same game is considerably easier (assuming $P \neq NP$) than computing a player's index (which is known to be #P-hard), or comparing the power indices of two players in different games (see [16] and Section 6).

Another implication of Proposition 12 is that it is easy to select a quota so as to ensure that two players with different weights have different power.

Corollary 15. *Consider a set of players $I = \{1, \dots, n\}$ and a vector of weights $\boldsymbol{w} = (w_1, \dots, w_n)$ that satisfies $w_1 \leqslant \dots \leqslant w_n$. For each player j there is a quota value q such that for each player i with $w_i < w_j$ it holds that $\beta_i(q) < \beta_j(q)$. On the other hand, there is a quota value q' such that for every pair of players $i, j \in I$ it holds that $\beta_i(q') = \beta_j(q')$.*

Proof. Consider players $i, j \in I$ with $w_i < w_j$ and set $q = w_j$. We claim that $\beta_j(q) > \beta_i(q)$. Indeed, player j is pivotal for $\{j\}$, but player i is not pivotal for $\{j\}^{i,j} = \{i\}$, so our claim follows by Proposition 12. To prove our second claim, set $q' = w_2 + \dots + w_n + 1$. Then each player i is pivotal for exactly one coalition, namely, the grand coalition. Hence, Banzhaf indices of all players are equal. □

In practice, ensuring that i and j have different voting power is not always sufficient: the center may want to set the quota so as to maximize the difference between the power indices of the two players, i.e., find a value of q in $\arg\max_q |\beta_i(q) - \beta_j(q)|$. However, it seems likely that finding such a quota is computationally hard; proving this and/or providing an (approximation) algorithm for this problem is an interesting direction for future work.

Corollary 15 demonstrates that the center can set the quota so that all players have the same power. However, the center may also have the opposite goal, i.e., it may want to find a quota such that all players with different weights have different Shapley–Shubik indices (or Banzhaf indices). This choice can be motivated by fairness, i.e., a desire that a player with a larger weight has strictly more influence than a player with a smaller weight. Formally, we say that q is a *separating quota* for a weight vector $\boldsymbol{w}$ with respect to the Banzhaf index (respectively, the Shapley–Shubik index) if for every pair of players $i, j \in I$ with $w_i \neq w_j$ it holds that $\beta_i(q) \neq \beta_j(q)$ (respectively, $\varphi_i(q) \neq \varphi_j(q)$). Note that by Corollary 13 a quota is separating for the Banzhaf index if and only if it is separating for the Shapley–Shubik index; thus, in what follows, we will simply talk about a *separating quota* without mentioning the underlying index.

Example 16. Consider a weighted voting game with $I = \{1, 2, 3, 4\}$, $\boldsymbol{w} = (1, 1, 2, 2)$. Suppose first that $q = 3$. Clearly, player 1 is pivotal for coalitions $\{1, 3\}$ and $\{1, 4\}$, while player 3 is pivotal for coalitions $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$ and $\{3, 4\}$, so $\beta_1(q) = 1/4$, $\beta_3(q) = 1/2$, and, by symmetry, $\beta_2(q) = \beta_1(q) = 1/4$, $\beta_4(q) = \beta_3(q) = 1/2$. Thus, for $q = 3$ and all $i, j \in I$, if $w_i < w_j$ then $\beta_i(q) < \beta_j(q)$. However, for $q' = 6$ or $q' = 1$, we have $\beta_i(q') = 1/8$, for $i = 1, \dots, 4$. Thus, $q = 3$ is a separating quota for the weight vector $\boldsymbol{w}$, but $q = 6$ and $q = 1$ are not.

For many weighted voting games, finding a separating quota is not difficult. As an illustration, we will now prove that for every game with the weight vector of the form $\boldsymbol{w} = (1, \dots, n)$ for $n \geqslant 20$ any quota between n and $w(I) - n + 1$ is separating. (It is easy to see that if $q \leqslant n - 1$ then players $n - 1$ and n have equal power and thus such a quota is not separating; the same holds for quotas larger than $w(I) - n + 1$.)

Proposition 17. *Let $n \geqslant 20$, and set $I = \{1, \dots, n\}$, $\boldsymbol{w} = (1, \dots, n)$. Then for every $q \in \{n, \dots, \frac{n(n+1)}{2} + 1 - n\}$, all players in the game $[I; \boldsymbol{w}; q]$ have different Banzhaf indices.*

Proof. We have argued that we can assume without loss of generality that $q \leqslant \lceil w(I)/2 \rceil$. Thus, it suffices to show that $\beta_{i-1}(q) \neq \beta_i(q)$ for all $i = 2, \dots, n$ and all $q = n, \dots, \lceil \frac{n(n+1)}{4} \rceil$. Fix some $q \in \{n, \dots, \lceil \frac{n(n+1)}{4} \rceil\}$ and $i \in \{2, \dots, n\}$.

By Proposition 12, it suffices to construct a set S such that $i \in S$, $i - 1 \notin S$, and $w(S) = q$. There exist nonnegative integers a, b such that $q - i = a(n + 1) + b$, where $a \leqslant \frac{n}{4}$, $0 \leqslant b \leqslant n$. Observe that there are $\lceil \frac{n}{2} \rceil - 1$ pairs of players (j, k) such that $w_j + w_k = n + 1$, namely, $(1, n), (2, n - 1), \dots, (\lceil \frac{n}{2} \rceil - 1, \lfloor \frac{n}{2} \rfloor + 2)$. We will construct the set S by picking a such pairs and adding player i as well as one or two extra players of total weight b. We have to be careful in our selection process, as we have to make sure that none of the selected pairs contains i, $i - 1$, or the extra players needed at the last step; however, we have at most 4 players to avoid, and we need to select $a \leqslant \frac{n}{4}$ pairs out of $\lceil \frac{n}{2} \rceil - 1$, so as long as $n \geqslant 20$ this can always be achieved. We have the following cases to consider:

1. $b = 0$. In this case, we simply select a pairs that do not contain i or $i - 1$, and then add i.
2. $b \neq i - 1, i$. In this case, we select a pairs that do not contain i, $i - 1$ or b, and add i and b.
3. $b \in \{i - 1, i\}$ and $b \geqslant 5$. In this case, we select a pairs that do not contain i, $i - 1$, 2 or $b - 2$, and add i, 2 and $b - 2$.
4. $b \in \{i - 1, i\}$ and $b \leqslant 4$. In this case, we select $a - 1$ pairs that do not contain i, $i - 1$, 5 or $n + b - 5$, and add i, 5 and $n + b - 5$.

In all cases, we have successfully constructed a set S with the required properties, so we are done. □

Given the proof of Proposition 17, one may conjecture that all weighted voting games that have sufficiently many players with distinct weights admit a separating quota. However, it turns out that this is not the case. We first need the following definition.

Definition 18. A sequence of positive numbers $(w_1, \ldots, w_n)$ is called *super-increasing* if we have $\sum_{j=1}^{k-1} w_j < w_k$ for all $k = 2, \ldots, n$.

Recall that if two players i and j are symmetric in a game G, i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for every set $S \subseteq I \setminus \{i, j\}$, then $\beta_i(G) = \beta_j(G)$. We will now prove that for every super-increasing weight vector of length at least 3, for every value of the quota at least two of the first three players are symmetric, and hence there is no separating quota for every such weight vector.

Lemma 19. *For every game $G = [I; \boldsymbol{w}; q]$ with $|I| \geqslant 3$ and a super-increasing vector of weights $\boldsymbol{w} = (w_1, \ldots, w_n)$, it holds that either players 1 and 2 are symmetric, or players 2 and 3 are symmetric.*

Proof. We prove the lemma by induction on the number of agents n. For $n = 3$, we have three cases to consider:

1. $q \leqslant w_2$. In this case, we have $v(\{2\}) = v(\{3\}) = 1$, $v(\{1, 2\}) = v(\{1, 3\}) = 1$, so players 2 and 3 are symmetric.
2. $w_2 < q \leqslant w_3 + w_1$. In this case, we have $v(\{1\}) = v(\{2\}) = 0$, $v(\{1, 3\}) = v(\{2, 3\}) = 1$, so players 1 and 2 are symmetric.
3. $w_3 + w_1 < q$. In this case, we have $v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = 0$, so players 2 and 3 are symmetric.

For the inductive step, we assume that the claim is correct for $n - 1$ and prove it for n. Consider a game $G = [I; \boldsymbol{w}; q]$ with $I = \{1, \ldots, n\}$ and a super-increasing sequence of weights $\boldsymbol{w} = (w_1, \ldots, w_n)$.

Suppose first that $q \leqslant w_n$. Set $I' = I \setminus \{n\}$, $\boldsymbol{w}' = (w_1, \ldots, w_{n-1})$, and let $G' = (I'; \boldsymbol{w}'; q)$. By the inductive hypothesis, there exist two players $i, j \in \{1, 2, 3\}$ that are symmetric in G'. We claim that i and j are also symmetric in G. Indeed, consider an arbitrary coalition $S \subseteq I \setminus \{i, j\}$. If $n \notin S$, then $S \subseteq I' \setminus \{i, j\}$ and $v(S \cup \{i\}) = v(S \cup \{j\})$ by the inductive hypothesis. On the other hand, if $n \in S$, then $v(S \cup \{i\}) = v(S \cup \{j\}) = 1$, since $q \leqslant w_n$.

Now suppose that $w_n < q \leqslant w(I)$. Set $I' = I \setminus \{n\}$, $\boldsymbol{w}' = (w_1, \ldots, w_{n-1})$, and let $G' = (I'; \boldsymbol{w}'; q - w_n)$. By the inductive hypothesis, there exist two players $i, j \in \{1, 2, 3\}$ that are symmetric in G'. We claim that i and j are also symmetric in G. Indeed, consider an arbitrary coalition $S \subseteq I \setminus \{i, j\}$. Since $\boldsymbol{w}$ is a super-increasing sequence, player n belongs to each winning coalition in G. Thus, if $n \notin S$, then $v(S \cup \{i\}) = v(S \cup \{j\}) = 0$. On the other hand, suppose that $n \in S$, and let $S' = S \setminus \{n\}$. We have $S' \subseteq I' \setminus \{i, j\}$, so by the inductive hypothesis $v_{G'}(S' \cup \{i\}) = v_{G'}(S' \cup \{j\})$. Clearly, a coalition T is winning in G' if and only if $T \cup \{n\}$ is winning in G. Thus, we have $v_G(S \cup \{i\}) = v_G(S \cup \{j\})$ in this case as well. This completes the proof of the inductive step. Thus, the lemma is proved. □

Lemma 19 immediately implies the following result.

Theorem 20. *For every game $G = [I; \boldsymbol{w}; q]$ with $|I| \geqslant 3$ and a super-increasing vector of weights $\boldsymbol{w} = (w_1, \ldots, w_n)$, either $\beta_1(q) = \beta_2(q)$, or $\beta_2(q) = \beta_3(q)$. Consequently, there is no separating quota for $\boldsymbol{w}$.*

Given Proposition 17 and Theorem 20, it is natural to ask if one can efficiently determine whether a given weight vector admits a separating quota. Proposition 12 implies that this problem is in NP for both indices. Indeed, given a quota q and a collection of sets $S_2, \ldots, S_n$, one can easily check whether for all values of $i \in \{2, \ldots, n\}$ such that $w_{i-1} < w_i$ it holds that (1) i is pivotal for S_i and (2) $i - 1$ is not pivotal for $S_i^{i,i-1}$. However, it seems unlikely that this problem is in P; we propose this question as a topic for future work.

6. Comparing two values of the quota

We have argued that when the center can choose any quota that she likes, some of the associated computational problems (e.g., turning a player into a dummy) become easy. However, in real-life scenarios the center may be restricted in the choice of quota: For example, the center might be able to modify the quota only very slightly or have a choice of only several quota values. We will now show that the problem of deciding which of two given quotas is more favorable to a particular player is computationally hard, even if the quotas differ only by 1.

Definition 21. Given a power index f, the Quota_f problem is defined as follows. We are given a set of players I, $|I| = n$, a vector of weights $\boldsymbol{w} = (w_1, \ldots, w_n)$, two quota values, q' and q'', and an index $i \in I$. Let $G' = [I; \boldsymbol{w}; q']$, $G'' = [I; \boldsymbol{w}; q'']$. The goal is to decide whether $f_i(G') > f_i(G'')$.

The notion of hardness that we will make use of is PP-hardness. The class PP (see, e.g., [36]) captures the notion of probabilistic polynomial-time computation. The idea is that one can look at nondeterministic computations in terms of probabilistic ones: Given an NP machine (a nondeterministic polynomial-time Turing machine) N, at each computation step we can toss a coin to choose the next move uniformly at random from the set of possible ones, as defined by N's transition relation. Thus, we can naturally define the probability of the event that N accepts a string x. Formally, we say that a language L belongs to PP if there exists an NP machine N such that $x \in L$ if and only if the probability that N accepts x is at least $\frac{1}{2}$.

PP is a surprisingly powerful class. For example, NP $\subseteq$ PP, and, moreover, PP contains the class Θ_2^p (also known as $\mathrm{P}^{\mathrm{NP[log]}}$) of all decision problems that can be solved via parallel access to NP (see [7]). Used as an oracle, PP is essentially as powerful as #P [5]; in fact, #P can be viewed as a functional counterpart of PP.[3]

There are many natural PP-complete problems. In particular, Faliszewski and Hemaspaandra [16] recently studied the following one.

Definition 22. (See [16].) Given a power index f, the PowerCompare$_f$ problem is defined as follows. We are given two weighted voting games, G' and G'', a player i in G', and a player j in G''. The goal is to decide whether $f_i(G') > f_j(G'')$.

Faliszewski and Hemaspaandra show that this problem is PP-complete both for the Shapley–Shubik power index and for the Banzhaf power index. Effectively, they give a reduction from Sat-Compare, the problem of deciding, given two propositional formulas, x and y, if #Sat(x) > #Sat(y), where #Sat(x) is the function that takes as input a propositional formula x and returns the number of satisfying truth assignments for x.[4]

Faliszewski and Hemaspaandra's proof proceeds by giving a reduction from Sat-Compare to SubsetSum-Compare. Recall that an instance of Subset Sum (see [23]) is a sequence of nonnegative integers $[x_1, \ldots, x_m; t_x]$; a solution to this instance is a subset of indices $S \subseteq \{1, \ldots, m\}$ such that $\sum_{i \in S} x_i = t_x$. #SubsetSum(X) is a #P function that takes as input an instance of Subset Sum and returns the number of solutions to that instance. SubsetSum-Compare is defined similarly to Sat-Compare, i.e., it compares the number of solutions to two instances of this problem.

As Quota$_f$ is a simple special case of PowerCompare$_f$, the result of Faliszewski and Hemaspaandra immediately implies that Quota$_f$ is in PP both for $f = \varphi$ and for $f = \beta$. To show that Quota$_f$ is PP-hard for $f \in \{\varphi, \beta\}$, rather than using the result of Faliszewski and Hemaspaandra as a black box, we make use of a technical lemma proved in their paper, which provides a reduction from Sat-Compare to SubsetSum-Compare that has several useful properties. We then show that an instance of SubsetSum-Compare output by this reduction can be transformed into an instance of Quota$_f$ for $f \in \{\varphi, \beta\}$, so that a "yes"-instance of the former problem becomes a "yes"-instance of the latter problem and vice versa.

The following lemma is a corollary to the reduction used in [16].

Lemma 23. *(See [16].) Given two propositional formulas, x and y, one can compute in polynomial time two instances of the* Subset Sum *problem,* $X = [x_1, \ldots, x_m, t_x]$ *and* $Y = [y_1, \ldots, y_m, t_y]$ *such that* #Sat(x) = #SubsetSum(X), #Sat(y) = #SubsetSum(Y). *In addition, there is a nonnegative integer k such that*: (1) *any solution to X has exactly k elements, and* (2) *any solution to Y has exactly k elements.*

We are now ready to prove the main result of this section.

Theorem 24. Quota$_\varphi$ *and* Quota$_\beta$ *are* PP-*complete. This holds even if we stipulate that* $|q' - q''| = 1$.

Proof. Membership in PP is clear, and thus we focus on proving PP-hardness. We give the proof for φ; for β, we can use the same reduction and its proof of correctness is analogous (and easier).

We give a reduction from Sat-Compare. Let x and y be two propositional formulas. We first compute two instances of SubsetSum, $X = [x_1, \ldots, x_m; t_x]$ and $Y = [y_1, \ldots, y_m; t_y]$, as described in Lemma 23. We have #Sat(x) = #SubsetSum(X), #Sat(y) = #SubsetSum(Y), and for every solution S' to X and every solution S'' to Y it holds that $|S'| = |S''| = k$ for some integer k.

Let K be the smallest power of 2 greater than $16 \cdot (\sum_{i=1}^m x_i + t_x) + 1$. We form a sequence of weights $\boldsymbol{w} = [w_1, \ldots, w_{2m+2}, w_{2m+3}]$ as follows:

1. For each $i = 1, \ldots, m$, we set $w_i = 16x_i$, $w_{m+i} = Ky_i$.
2. We set $w_{2m+1} = Kt_y + 4$, $w_{2m+2} = 16t_x + 5$, $w_{2m+3} = 1$.

[3] Briefly put, a function f belongs to #P if there is an NP machine N such that for each input string x, N has exactly $f(x)$ accepting computation paths on input x. We point the readers to [36] for more details.

[4] Strictly speaking, Faliszewski and Hemaspaandra used problem X3C (see [23]) and its counting variant #X3C. However, there exist parsimonious reductions (i.e., reductions that preserve the number of solutions) between #Sat and #X3C: the reduction from #Sat to # X3C is given in [24], and the other reduction is standard. Therefore, the results and lemmas of Faliszewski and Hemaspaandra can be phrased in terms of #Sat as well.

We set $q = Kt_y + 16t_x + 5$. Our reduction outputs a set of players $I = \{1, \ldots, 2m + 3\}$, vector of weights $\boldsymbol{w}$, quota values $q' = q$ and $q'' = q + 1$, and player $i = 2m + 3$. It is easy to see that the reduction works in polynomial time, and it remains to show that it is correct.

Let us form two games, $G' = [I, \boldsymbol{w}, q']$ and $G'' = [I, \boldsymbol{w}, q'']$. Our games have three special players, $p = 2m + 3$ (the player whose power index we are interested in), $f' = 2m + 1$ (filler player for G'), and $f'' = 2m + 2$ (filler player for G''). We claim that $\varphi_p(G') > \varphi_p(G'')$ if and only if #SubsetSum(X) > #SubsetSum(Y), or, equivalently, #Sat(x) > #Sat(y).

Let us first consider $\varphi_p(G')$ and a permutation π for which p is pivotal. Since $w_p = 1$, we have $w(S_\pi(p)) = Kt_y + 16t_x + 4$. Thus, it is easy to see that in π player p is preceded by f', but not by f'': otherwise, it would not be the case that $w(S_\pi(p)) = 4 \bmod 16$. We have $w(S_\pi(p) \setminus \{f'\}) = 16t_x$, and it is easy to see that there is a one-to-one correspondence between $S_\pi(p) \setminus \{f'\}$ and a subset $A \subseteq \{1, \ldots, m\}$ such that $\sum_{i \in A} x_i = t_x$. Thus, we have

$$\varphi_p\big(G'\big) = \frac{k!(2m + 2 - k)!}{(2m + 3)!} \#\textsc{SubsetSum}(X) = \frac{k!(2m + 2 - k)!}{(2m + 3)!} \#\textsc{Sat}(x).$$

Now, consider $\varphi_p(G'')$. Let π be a permutation of I such that p is pivotal for π. This means that the players preceding p have total weight $Kt_y + 16t_x + 5$. As in the previous paragraph, it is easy to see that in π player p is preceded by f'', but not by f': otherwise, it would not be the case that $w(S_\pi(p)) = 5 \bmod 16$. Hence, there is a one-to-one correspondence between $S_\pi(p) \setminus \{f''\}$ and a subset $B \subseteq \{1, \ldots, m\}$ such that $\sum_{i \in B} y_i = t_y$. Thus, we have

$$\varphi_p\big(G''\big) = \frac{k!(2m + 2 - k)!}{(2m + 3)!} \#\textsc{SubsetSum}(Y) = \frac{k!(2m + 2 - k)!}{(2m + 3)!} \#\textsc{Sat}(y).$$

As a result, $\varphi_p(G') > \varphi_p(G'')$ if and only if #Sat(x) > #Sat(y). This proves that our reduction is correct. □

We remark that this hardness result shows that computational complexity can be a barrier to manipulation by the central authority, as it implies that it will be difficult for the center to choose the quota so as to obtain the desired result. Moreover, as PP is a more powerful complexity class than NP, and our problem is complete for it, the manipulators will not be able to use the existing heuristics for problems in NP. However, PP-hardness does not necessarily imply that the problem is hard *on average*; determining whether manipulating the quota is hard in this sense is an interesting open problem. We would also like to remark that, even though power indices themselves are hard to compute, a hardness of manipulation result is still significant: power indices reflect the distribution of power among the agents, and the center may want to manipulate this distribution even if it cannot compute it.

On the flip side, it is known [34] that both the Shapley–Shubik and the Banzhaf index are easy to compute if the weights are polynomially bounded (or, equivalently, given in unary). Clearly, these algorithms can be used to solve Quota$_\varphi$ and Quota$_\beta$, as we can directly compute the values of a player's power index for both quotas, and choose the quota that gives us a better outcome. Hence, computational complexity alone does not provide adequate protection from this form of manipulation, and other approaches are needed.

7. Small perturbations: an empirical analysis

In the previous section, we considered the situation where the center has to choose between two permissible values of the quota. However, it may also happen that the center can choose from a much larger set of quotas, namely, all quotas that differ from the current one by a small amount. For example, consider weighted voting in political decision-making bodies [33]. It may be fairly easy to change the quota from 60% of the votes to anywhere between 51% and 66% of the votes, but changing the quota to 80% of all votes would be considerably more difficult. Analyzing this form of manipulation is a challenging problem, and we were not able to derive analytical bounds on the manipulator's power in this setting. Therefore, in this section we provide an empirical study of the effects of the *maximal magnitude of the perturbation* on the center's ability to change the power of a target player.

We first briefly describe our simulation system, the game construction and the power index calculations, and then present the empirical results obtained.

7.1. Simulation system and settings

Our simulation system creates weighted voting games by first choosing the number of players in the game, uniformly at random from a given interval of positive integers. Then, the system draws the weight of each player independently from $N(\mu, \sigma^2)$ (the normal distribution with mean μ and variance σ^2). To speed up calculations, the weights are rounded to the nearest integer. Given a weight vector $\boldsymbol{w} = (w_1, \ldots, w_n)$, we set $w = \sum_{i=1}^n w_i$. The quota for the game is chosen uniformly at random between 0 and w and rounded to the nearest integer.

In our experiments, we have used a mean of $\mu = 200$ and standard deviation $\sigma = 30$. The number of players was chosen uniformly at random from the set $\{6, 7, \ldots, 25\}$, and the target player is chosen uniformly among all players in the game. Having generated a game G, we construct a number of perturbed games $G'_1, \ldots, G'_k$ (each with a different quota), and test whether (and by how much) the power of a target agent has increased in the perturbed game. Such tests require computing the power index of the target player in the original game and the perturbed games. Since power indices are hard to compute, we have applied the approximation method of Bachrach et al. [4], and used the Shapley–Shubik power index in our experiments.

The algorithm of Bachrach et al. estimates the power indices and returns a result which is *probably approximately correct*. Formally, given a game in which a player's true power index is ϕ, and given a target accuracy level ϵ and confidence level δ, the algorithm returns an approximation $\widehat{\phi}$ such that with probability at least $1-\delta$ we have $|\phi-\widehat{\phi}| \leqslant \epsilon$ (i.e. the result is likely to be close to the correct value). To achieve a confidence level δ and accuracy level ϵ, this algorithm chooses $s = \frac{\ln \frac{2}{\delta}}{2\epsilon^2}$ random permutations of the agents, and measures the fraction of these permutations where the agent is pivotal. Thus, the total running time is logarithmic in the confidence δ and quadratic in the accuracy ϵ, so the approach is tractable even for high accuracy and confidence. We have used $\delta = 0.00001$ and $\epsilon = 0.001$, so the power was estimated very accurately.

Our simulation system was written using C# and Microsoft SQL database. Every single generated game required computing many power indices—one per each quota tried. Due to the massive amount of simulated games and the number of computed power indices, we have used a computer cluster with 250 cores for our experiments.

We will now describe our experimental setup in more detail. Our goal is to understand how the manipulator's ability to change a player's power depends on the permitted changes to the quota. The manipulator may be constrained in two ways: (1) he may be allowed to raise the quota, but not to lower it, or vice versa, and (2) he may be allowed to only change the quota by a certain amount (say, 20%). Any such set of constraints determines an interval of permissible quotas. Given such an interval, we try to identify the best value of the quota from the manipulator's perspective; of course, the answer depends on whether the manipulator wants to increase or decrease the target player's power. To present the results, we group them according to the manipulator's intentions (helping or hurting a player) and the change direction (raising or lowering the quota); given these choices, we graph the change in the target player's power that can be accomplished by the manipulator as a function of the permissible quota range.

We will describe our algorithm for the case where we are only allowed to lower the quota; the algorithm for raising the quota is similar. The most important parameter in our experiments is the *maximum allowed perturbation magnitude* θ. For a given value of θ and a weighted voting game $[I; \boldsymbol{w}; q]$, we are interested in the changes in target player's power that can be achieved by choosing the quota q' so that $0 \leqslant \frac{q-q'}{q} \leqslant \theta$ (for quota-raising manipulation, we consider quotas q' that satisfy $0 \leqslant \frac{q'-q}{q} \leqslant \theta$). We employ a very simple algorithm to search for a good quota in this range. Given an additional parameter c, our algorithm determines the minimal quota $q_b = q(1-\theta)$ that the center may set, and simply tries c different possible quotas in constant intervals between q_b and q. In other words, denoting $d = \frac{q\theta}{c}$, the algorithm tries the quotas $Q = \{q_b, q_b+d, q_b+2d, \ldots, q_b+(c-1)d\}$. For each quota $q' \in Q$, it approximates the power index of the target player using the method of Bachrach et al. [4], and keeps track of the optimal manipulation found. We remark that our algorithm only tries some of the possible values of the quota, so it may fail to find an optimal manipulation in the allowed quota range. Thus, it may underestimate the power of the manipulator. However, our experiments show that even this very simple algorithm is usually successful in finding a beneficial manipulation.

7.2. Empirical results

Our empirical results considered many randomly constructed games. For each such game, we use the algorithm described above to find the optimal manipulation for different maximal perturbation magnitudes. The experiments described below consider both raising and lowering the quota, and both increasing and decreasing the power of a target player.

The following results consider the relationship between the maximal allowed perturbation magnitude θ, and the optimal change in the power of a target player. For each game, we have considered different values of θ, and used the algorithm described above to find the optimal quota manipulation in the allowed range. The following figures show the relation between the value of θ and the proportional change in power, averaged across many experiments. Note that we have considered the *proportional* change in the target agent's power (rather than the power change in absolute terms).

Fig. 1 shows the average power increase that can be achieved by lowering the quota. It shows that on average (for the games generated as described in Section 7.1), even moderate values of θ such as $\theta = 20\%$ can result in an increase of 15% in an agent's power. On the other hand, the curve saturates quite quickly, and for $\theta = 70\%$ the power of the target agent can only be increased by 25%. Despite an increase in the curve's slope starting at $\theta = 70\%$, even for very large values of θ, which allow decreasing the quota to almost zero, on average it is only possible to increase an agents power by roughly 40%.

Similarly, Fig. 2 shows the average power decrease that can be achieved by lowering the quota. The shape of the curve is similar to that in Fig. 1. Again, even for perturbation magnitudes of almost 100%, which can decrease the quota to almost zero, on average it is only possible to decrease an agent's power by about 40%.

Figs. 3 and 4 consider *raising* the quota to change the power of a target agent. Fig. 4 shows the average power change achieved when raising the quota to *increase* the target agent's power. The shape of the curve shows that while small quota perturbations may affect the target player's power relatively strongly, the curve saturates quickly. In particular, although setting $\theta = 25\%$ allows increasing the power by over 15%, setting $\theta = 100\%$ yields an average power increase of only 20%. Note that for the case of increasing the quota, it is possible to use perturbation magnitudes that exceed 100%. However, we do not allow quotas that exceed the total weight. Thus, results for high perturbation magnitudes should be treated more carefully. For instance, if the graph shows that setting $\theta = 400\%$ allows increasing an agent's power by 35% on average, this

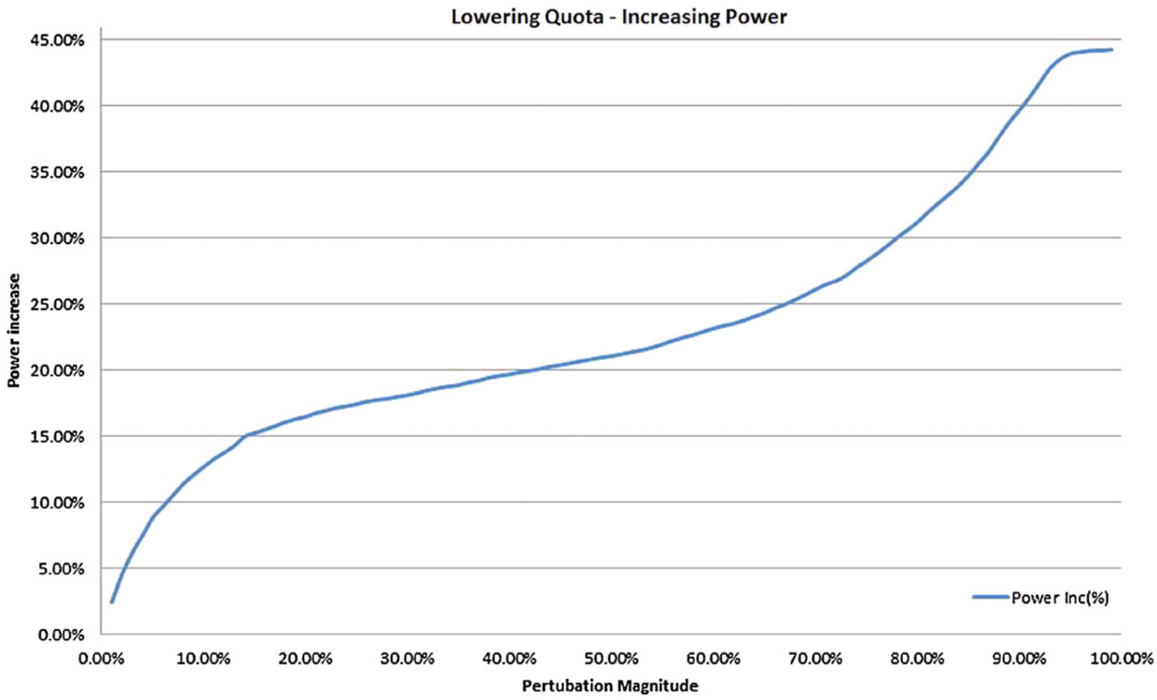

Fig. 1. Average achieved power increase for different perturbation magnitudes (lowering the quota).

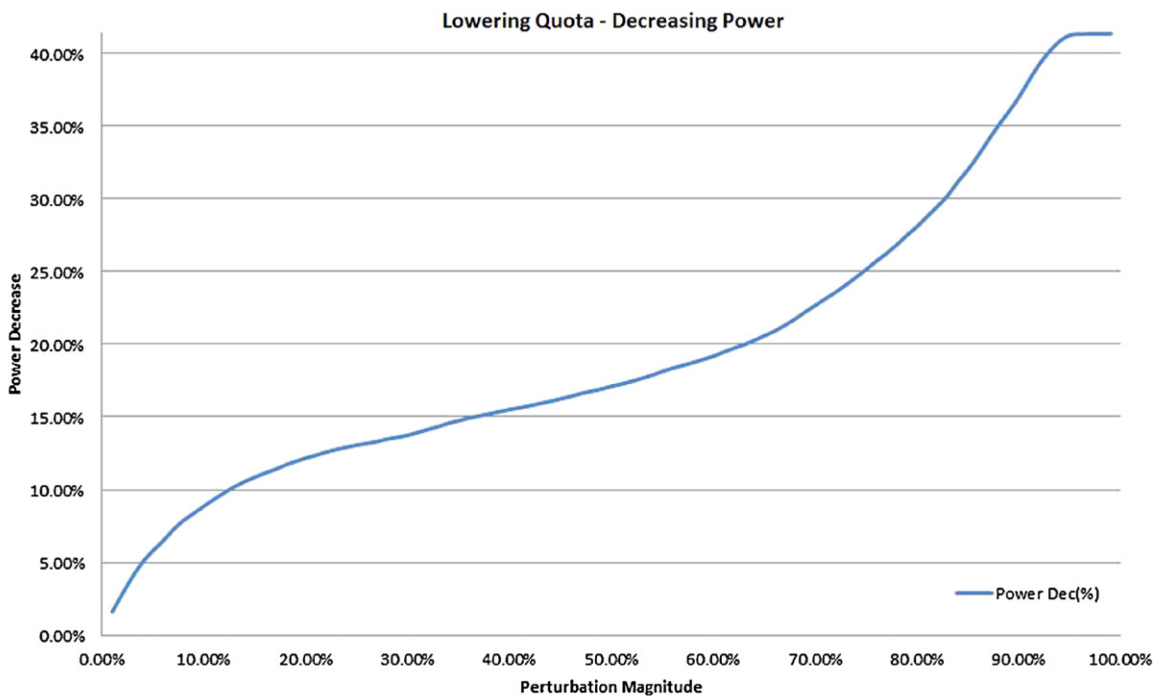

Fig. 2. Average achieved power decrease for different perturbation magnitudes (lowering the quota).

means that this is the average for all games where such a perturbation would still result in a quota that does not exceed the total weight. In other words, for each value of $\theta > 100\%$ the results are displayed contingent on the quota not exceeding the sum of the weights.

Fig. 4 considers increasing the quota to *decrease* the target agent's power. Similarly to Fig. 3, the curve saturates quickly.

Fig. 5 considers the effect of the number of agents in a randomly generated weighted voting game on the achieved power change (for different values of θ). It considers lowering the quota in order to increase the power of the target agent

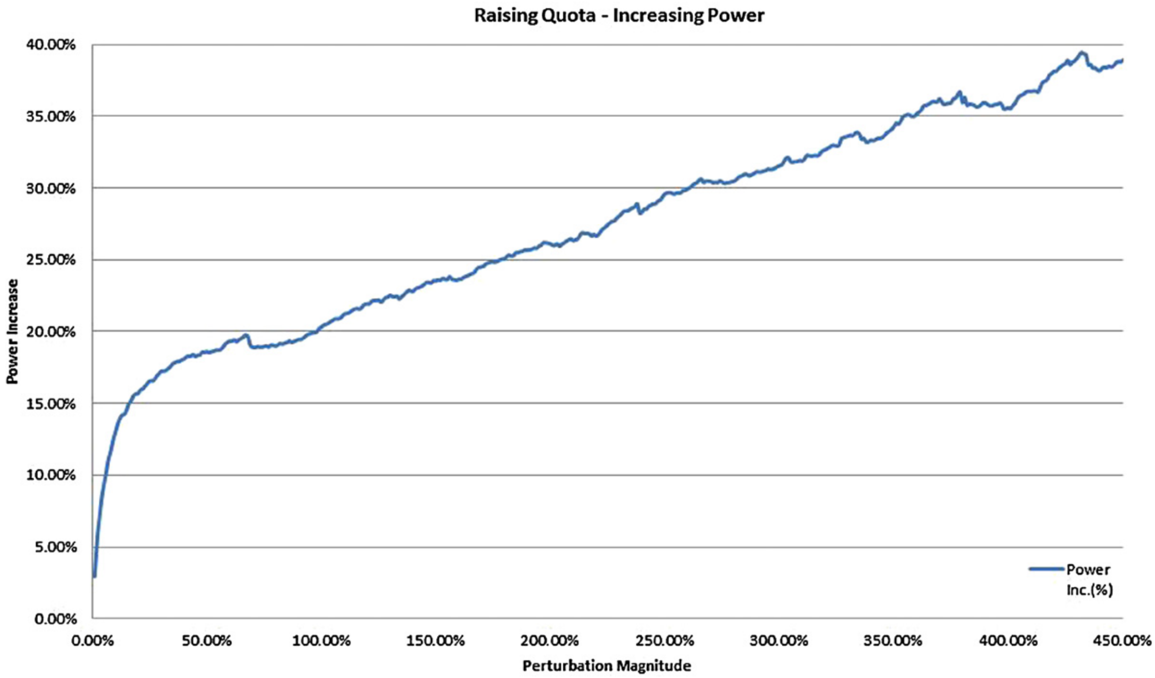

Fig. 3. Average achieved power increase for different perturbation magnitudes (raising the quota).

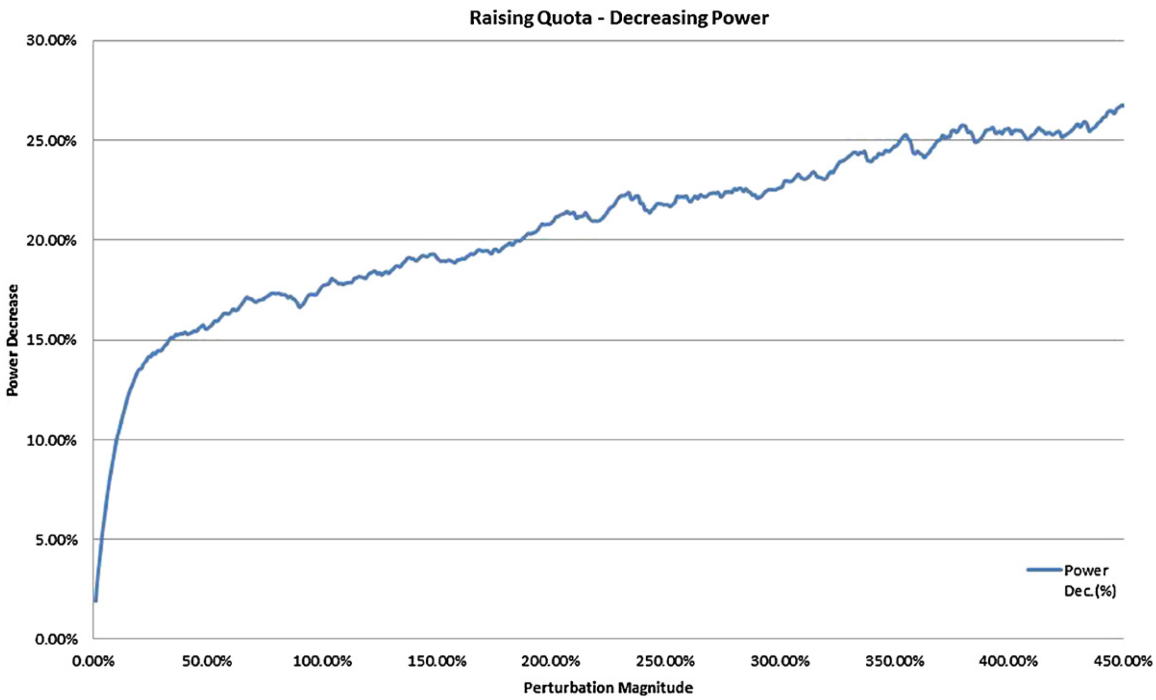

Fig. 4. Average achieved power decrease for different perturbation magnitudes (raising the quota).

(similarly to Fig. 1), for different numbers of agents in the generated game. The results show that although the general shape of the curve is quite similar across different numbers of players, quota manipulations are more effective when there are fewer players in the game. For example, for $\theta = 70\%$, on average our algorithm achieves a 40% power increase when the number of players is between 6 and 10, but only achieves a 10% power increase on average when there are 21 to 25 players. One possible explanation for this is that when there are more players in the game, it is more likely that there are several "competing" players with weights similar to the target player's weight, who also gain from quota manipulations.

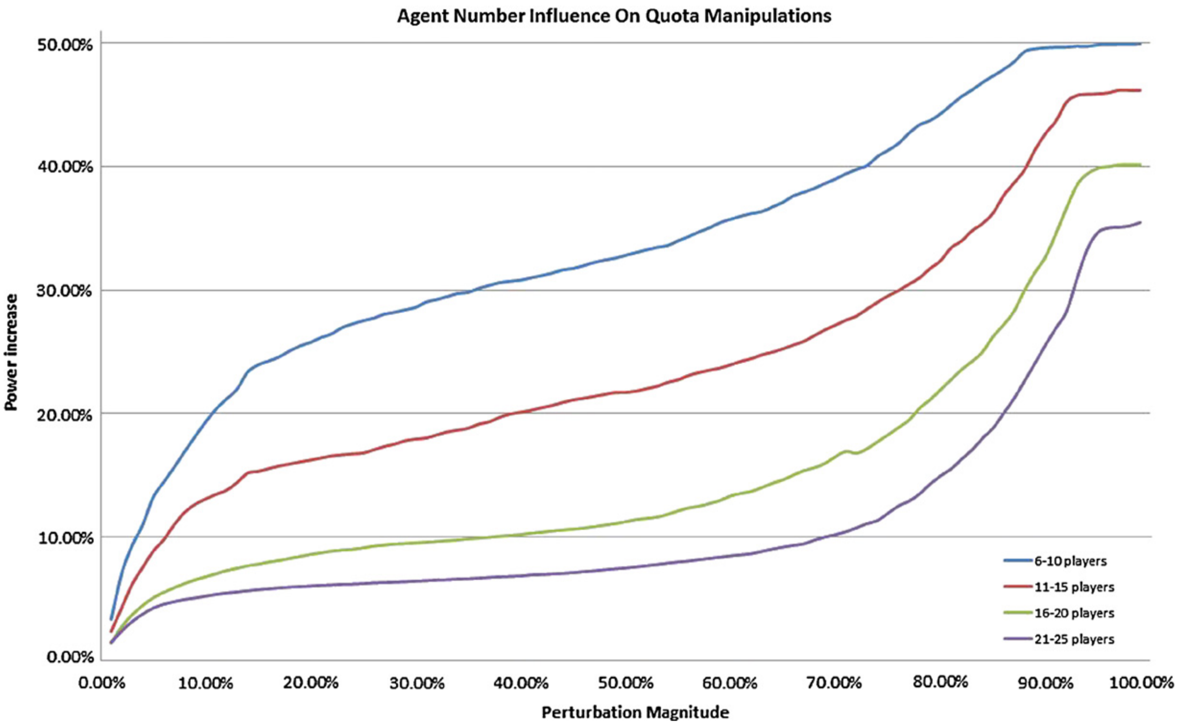

Fig. 5. Effect of the number of agents on the achieved power change.

Fig. 6 shows a histogram of the achieved power increase across the experiments. The histogram shows which power increase values are more likely to be achieved by our algorithm (for a specific value of θ). To generate the figure, we have used $\theta = 50\%$. We have partitioned the experiments into buckets according to the optimal proportional power change uncovered by our algorithm. The first bucket contained experiments where the power change was between 5% and 6%, the second bucket contained experiments where the power change was between 6% and 7% and so on, until the last bucket, which contained experiments where the power change was between 84% and 85%. We have then counted the number of experiments in each bucket, n_i being the number of experiments for bucket i, and normalized by $\sum_i n_i$, the total number of experiments, so the frequency of bucket j is $f_j = \frac{n_j}{\sum_i n_i}$. The X-axis shows the buckets and the Y-axis shows the frequencies of these buckets. Fig. 6 shows that games where the optimal power increase (achievable by our algorithm for $\theta = 50\%$) is low are more common than games where the power increase is high. This indicates that even significant changes of the quota, such as $\theta = 50\%$ are not very likely to trigger massive changes in a target agent power. Yet, our results show that certainly players can gain (or lose) nonnegligible amounts of power by altering the quota. This holds even under significant restrictions on the magnitude of the quota perturbation, and even when using a very simple algorithm to find quota manipulations. Thus, such manipulations present a real danger for practical applications of weighted voting.

8. Conclusion

We have considered quota manipulations in weighted voting games, i.e., situations where the central authority sets the game's quota to suit its purposes. We have argued that the central authority can affect the agents' power significantly by choosing a suitable quota and quantified the possible effect of such manipulations. We have given an efficient procedure for testing whether there exists a quota that makes a given player a dummy. Further, we have discussed the problem of finding a quota that ensures that some or all players have different power. Also, we have shown that checking which of two possible quota values makes a certain agent more powerful is PP-complete.

We have also provided empirical results regarding quota manipulations in situations where the center is only allowed to perturb the quota within certain limits. We have quantified the expected change in power the center can achieve using a simple quota manipulation algorithm for various possible manipulation magnitudes. Also, we have examined the effect of the number of players on the changes in players' power achievable by quota manipulations.

Several directions remain open for further research. The most immediate one is deriving analytical bounds on the manipulator's power in settings where the manipulator has to select the quota from a certain range. Another interesting question is how should the manipulator pick a quota in order to maximize the difference between two players' power. More broadly, since manipulations by changing the quota are possible in weighted voting games, what measures can be taken against such manipulations? Are there restricted domains where there is a polynomial-time algorithm for checking which quota

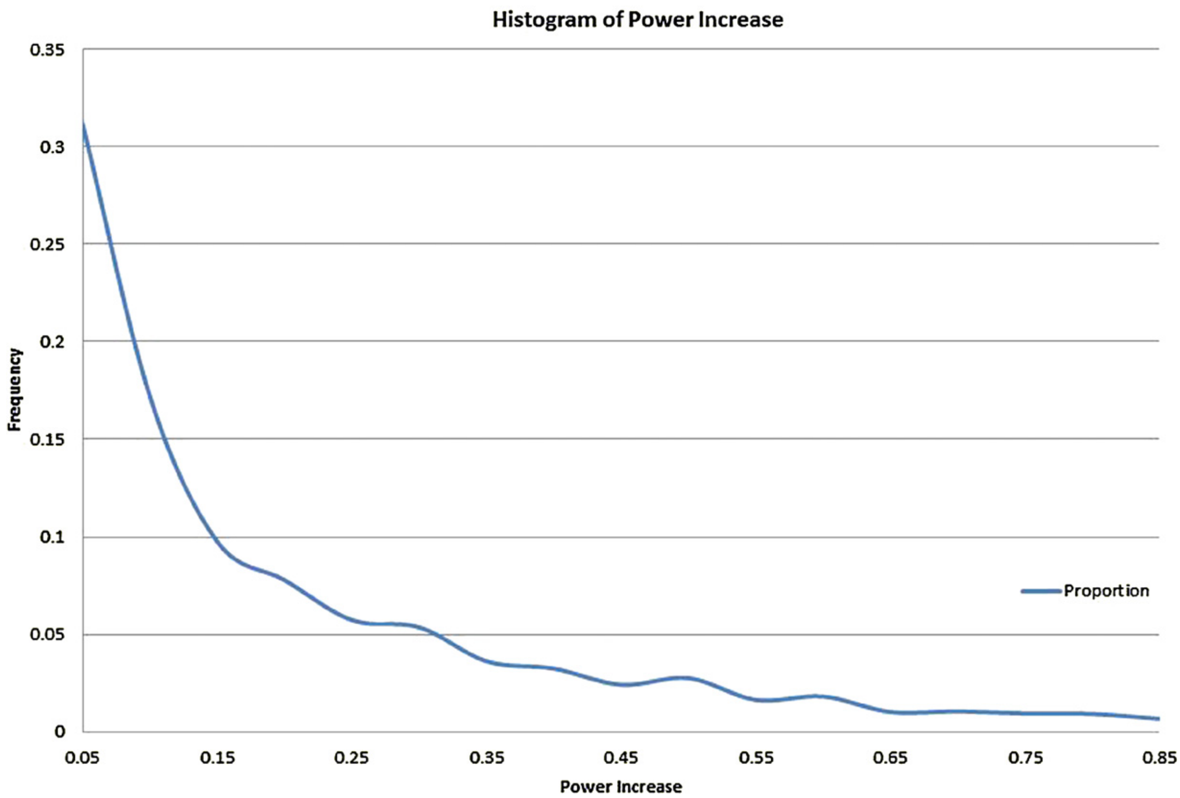

Fig. 6. Histogram of achieved power increase.

makes a certain agent more powerful than another agent? Are there other interesting domains where such manipulations are possible?

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