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In search of a "true" logic of knowledge: the nonmonotonic perspective

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Abstract

Modal logics are currently widely accepted as a suitable tool of knowledge representation, and the question what logics are better suited for representing knowledge is of particular importance. Usually, some axiom list is given, and arguments are presented justifying that suggested axioms agree with intuition. The question why the suggested axioms describe all the desired properties of knowledge remains answered only partially, by showing that the most obvious and popular additional axioms would violate the intuition.

We suggest the general paradigm of maximal logics and demonstrate how it can work for nonmonotonic modal logics. Technically, we prove that each of the modal logics KD45, SW5, S4F and S4.2 is the strongest modal logic among the logics generating the same nonmonotonic logic. These logics have already found important applications in knowledge representation, and the obtained results contribute to the explanation of this fact.

1. Introduction

The idea of using modal logics to capture semantics of knowledge and belief is due, probably, to Hintikka [7]. The idea was to interpret the necessitation modality L as "is believed" or "is known".

There has been much argument among philosophers around the question, what is the "true" logic of knowledge and/or belief be (see [12] for a comprehensive survey). Many different axioms and systems have been proposed, argued pro and contra. It is now clear that notions of knowledge and belief are important in fields such as artificial intelligence, economics, computer science. Reasoning about knowledge became subject

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of interdisciplinary conferences attracting prominent researchers in all the above fields. A survey of reasoning about knowledge, emphasizing its interdisciplinary character, and especially research of "applied epistemology" in computer science, can be found in [5]. Importance of epistemic notions for artificial intelligence was stressed and illustrated by McCarthy [17], McCarthy and Hayes [18].

While philosophers were mostly interested in the questions "what is the 'true' logic of knowledge and belief", the most important question of knowledge related research in AI is, how to apply logic of knowledge (and, more generally, logic at all) in concrete situations.

Robert Moore [23] argued that nonmonotonicity is a natural property of autoepistemic reasoning, that is, reasoning of a rational agent capable of reasoning about its own knowledge and belief. In fact we feel that logic of knowledge as it is discussed in philosophical literature is, in a sense, "autoepistemic". Its autoepistemic character becomes clear as we try to see how the nested modalities are interpreted. Say, the sentence $K(K\varphi \supset \varphi)$ means "the agent knows that if it knows φ then φ". That is, everything in scope of the outer K reflects the agent's reasoning, and embedded K's relate agent's self-reflection. The novelty of Moore's approach was that all the sentences ("axioms" and "theorems") were supposed to represent agent's belief rather than the real world. So the "axioms" are interpreted as "initial beliefs" of an agent, and theorems are supposed to represent final beliefs based on initial beliefs. So, roughly speaking, an external epistemic operator is always implicitly assumed. Following Moore's work, other nonmonotonic formalisms for knowledge and belief have been suggested (see, e.g., [14, 15].

We illustrate the natural nonmonotonicity of the logic of knowledge by the following simple example. Assume that p is some elementary statement, and assume that the initial belief set (or knowledge base) is empty. Then it is natural to assume that $\neg Kp$ is the autoepistemic consequence of $\emptyset$. We can write this fact as $\emptyset \vdash_{\mathrm{ae}} \neg Kp$. But now consider the initial belief set $\{p\}$. Then, of course, it is quite natural to assume $\{p\} \vdash_{\mathrm{ae}} Kp$, and $\{p\} \nvdash_{\mathrm{ae}} \neg Kp$: if p is in the knowledge set of an agent, then the self-reflecting agent should be able to conclude that it knows p.

Thus, the modal logic of knowledge/belief should be applied in some nonmonotonic way. Note however that philosophers were interested mostly in "pure" logics of knowledge and belief, the questions of "applied logics" remained uninvestigated. As long as we consider "pure logics", that is we do not have any nonlogical axioms specific to a given situation, the question of nonmonotonicity simply does not arise.

The basic idea of nonmonotonic versions of monotonic modal logics was suggested by McDermott and Doyle in [19, 20]. They did not state explicitly epistemic interpretation of modalities. However Moore's analysis [23] applies to McDermott and Doyle's logic too. An earlier disregard of McDermott and Doyle's approach was due, to the large extent, to the unsuccessful choice of the underlying monotonic modal logic—namely, it turned out that the popular logic S5 does not give any nonmonotonicity. However, it was proved in [30] that Moore's autoepistemic logic is, formally, a special case of McDermott and Doyle's logic, if we take the logic KD45 as an underlying monotonic basis. Subsequently, formal properties of McDermott and Doyle's logics were investigated in much detail [16, 30]. Epistemic analysis of McDermott's scheme and comparison with other approaches to the problem of formalizing autoepistemic reasoning was sketched in [28].

The present paper is an attempt to contribute to the problem "what is a good logic of knowledge" by solving some formal problems. We look at the problem from the nonmonotonic standpoint. First, let us note that we do not think that some logic is the "true" logic of knowledge: the very notions of knowledge and belief are too vague and too complicated. Moreover, it is sometimes reasonable and convenient to consider some logic as a suitable logic of knowledge for some particular applications, even realizing that its axioms do not correspond fully to intuitive meaning of knowledge (see, e.g., [5]). Our goal here is to give arguments that some of the logics are more reasonable than others, and some are in fact indistinguishable, if considered from the point of view of knowledge representation. Such results, we think, are also of philosophical importance, because if we show that two modal logics are indistinguishable from the epistemic standpoints, the possible discussion of "which of these two logics is better" becomes unnecessary.

Let us recall McDermott and Doyle's construction. Assume we have a (monotonic) modal logic $\mathcal{S}$ with the basic (necessitation) modality L interpreted as "is known" or "is believed".

Let $\mathcal{S}$ be any monotonic logic in $\mathcal{L}$. By an *$\mathcal{S}$-expansion of A* we mean a set T of formulas such that

$$T = \{\varphi\colon\ A \cup \{\neg L\psi\colon\ \psi \notin T\} \vdash_{\mathcal{S}} \varphi\}. \tag{1}$$

Intuitively, an $\mathcal{S}$-expansion may be understood as a knowledge set (or as a belief set) of a rational agent based on initial assumptions A. It is an analogue to the notion of a deductive closure of an axiom set. In usual monotonic setting, if two logics are (extensionally) different then they give raise to different deductive closures of the same axiom set (say, deductive closures of the empty set of axioms are always different for different logics). The situation for nonmonotonic logics is different. It may happen that two different monotonic modal logics $\mathcal{S}$ and $\mathcal{T}$ give rise to the same nonmonotonic modal logic in a sense that for each A and for each consistent set T, T is an $\mathcal{S}$-expansion of A if and only if T is a $\mathcal{T}$-expansion of A. In this case we will say that $\mathcal{S}$ and $\mathcal{T}$ are *in the same range.* [1]

The reason two monotonically different logics may generate the same expansions, is the presence of the negative introspection term $\{\neg L\psi\colon\ \psi \notin T\}$ in Eq. (1): say, if $\mathcal{S} \subset \mathcal{T}$, it may happen that all the axiom schemata of $\mathcal{T}$ are derivable from the negative introspection and $\mathcal{S}$. Clearly, in this case all consistent $\mathcal{T}$-expansions are $\mathcal{S}$-expansions, too.

The possibility for different monotonic modal logics to collapse into one nonmonotonic modal logic was first noticed in [16], and a few ranges were exhibited. For instance, all logics between the logic with the only axiom schema 5 ($\neg L\varphi \supset L\neg L\varphi$) and KD45 are in the same range (and coincide with the autoepistemic logic of Moore).

[1] The usage of the term "range" here is motivated by a result of McDermott [19] stating that if $\mathcal{S} \subseteq \mathcal{T}$, then each $\mathcal{S}$-expansion of A is also a $\mathcal{T}$-expansion of A; hence if $\mathcal{S} \subseteq \mathcal{T}$ are in the same range, then all the logics between $\mathcal{S}$ and $\mathcal{T}$ are in the same range, too.

Our point here is that, if two modal logics are in the same range, then they are indistinguishable as epistemic logics. Among the logics in the same range, the largest logic is of particular interest: it states explicitly all the general epistemological principles that hold in the logic, so it gives us the maximum available information about the logic. For instance, it follows from the results of [26] that KD45 is a maximal logic in its range: that is, there does not exist a logic in the same range which properly contains KD45. That is, all the general epistemic principles that follow from KD45 and negative introspection are already contained in KD45. Such result, in our opinion, is an additional justification for the special interest this logic has received recently in the AI research.

In this paper we strengthen this result. We prove that KD45 is not just maximal, but the largest in its range: each logic which generates the same nonmonotonic logic, is contained in KD45. So we rule out the possibility that some logic incomparable with KD45 could be "nonmonotonically indistinguishable" from KD45. In fact, we prove even stronger result: if a logic $\mathcal{S}$ is contained in S5 and has any theorem not derivable in KD45, then there is a theory A such that some $\mathcal{S}$-expansion of A is not a KD45-expansion of A. [2]

Similarly, the maximality of two more logics, S4.4 and S4F was proved in [25,28]; we strengthen these results by proving that these logics are largest in their ranges.

But our main motivation for writing this paper was the famous logic S4. The segment of logics between S4 and S5 has been especially attractive to epistemologists (see, e.g., [13]). S4 and S5 are the best known modal logics. It has been much argued that S4 is a true logic of knowledge. Moore [22] used S4 as a basis for his theory of knowledge and action. In [30] it was shown that using nonmonotonic S4 rather than Moore's autoepistemic logic (= nonmonotonic KD45) some intuitively undesirable properties of Moore's autoepistemic logic (mentioned in [10,24]) can be avoided.

On the other hand, Lenzen [13] argued that S4.2 rather than S4 is the "true" logic of knowledge. In this paper we show that S4 and S4.2 are in the same range, moreover, S4.2 is the *largest* logic in that range. This means that if we accept that logic of knowledge is nonmonotonic, then S4.2 and S4 do not differ. On the other hand, as S4.2 is the largest logic in this range, this means that Lenzen was right in stressing the special role of S4.2 as an epistemic logic: it does not imply any hidden general epistemic principles but those explicitly formulated in S4.2, and their logical consequences.

The situation with S4 is especially intriguing in the following respect. It was proved in [16] that for each *finite* set of sentences, A, a consistent set T is an S4-expansion of A if and only if T is an S4F-expansion of A. It was also proved that S4-expansions coincide with S4F-expansions for sets A having only positive occurrences of the modality L, and for A's without nested modalities, having only negative occurrences of L. Thus, it is clear that examples of theories A demonstrating that nonmonotonic logics between S4 and S4F are different should be rather sophisticated (one such example was presented in [16]). The authors of [16] hoped, because of strong positive results, that there should be not too many "nonmonotonically different" logics between S4 and S4F. However, the

[2] We argue in the next section that only logics contained in S5 are of interest in the nonmonotonic context; however, for a more rigorous reader we prove that a logic not contained in S5 always has an expansion of an $\emptyset$ which cannot be an expansion for any logic contained in S5.

methods of this paper allow us to construct infinitely many logics between S4.2 and S4.3 generating different nonmonotonic logics. More precisely, we give the infinite sequence of logics $\{I_n\}_n$ such that for each n, S4.2 is properly contained in I_{n+1}, I_{n+1} is properly contained in I_n, I_1 is S4.3, and each I_n is the largest logic in its range (therefore, all I_n are different). Originally we were not going to consider all these logic, believing that they are irrelevant for knowledge representation (traditionally, S4.3 is considered as a tense logic with the operator "always in future"). However, recently Boutilier [1] found interesting applications of S4.3. Logic S4.3 also naturally appears in the work of Lamarre and Shoham [11] devoted to the investigation of relationships between different epistemic modalities, so we believe that it would be both interesting and useful to take a look at nonmonotonic S4.3. It is still an open question, what happens between S4.3 and S4F. By using methods of [16], it is not difficult to prove that all logics of finite depth contained in S4F [3] are in the same range of S4F. However, there are infinitely many logics of infinite depth between S4.3 and S4F (see [2]). Because all the extensions of S4.3 were effectively described by Fine [2], it should be not too difficult to find how many nonmonotonically different logics are between S4.3 and S4F.

The rest of the paper is organized as follows. Section 2 contains necessary preliminaries of modal logic and nonmonotonic modal logic. In particular, Section 2.2 contains description of semantics of nonmonotonic modal logics, previously published in [26]. In Section 3 we strengthen earlier results of [25,28] and show that KD45, SW5 and S4F are strongest logics in their ranges. Another purpose of that section is to demonstrate the main idea of the technique we use in Section 4 to prove that S4.2 is the strongest logic in the range of S4. The proof of the result for S4.2 is much more technically complicated than, say, for S4F, but the basic idea is the same. Section 5 presents a paradigm of maximal logics, explains significance of obtained technical results for the KR theory, suggests directions for further research.

Short proofs are presented in the main body of the paper, whereas longer proofs are presented in Appendix A. Finally, some technical results about logics between S4.2 and SW5, which we feel are of lesser conceptual significance, are presented in Appendix B.

2. Preliminaries

2.1. Modal logics

We deal in the paper with propositional modal logics only. We write L for the necessity operator. The dual to L, the possibility operator M, is considered as an abbreviation to $\neg L \neg$. We refer the reader to [8] for the basics of modal logics. All modal logics considered in the paper have two inference rules: modus ponens and necessitation $(\varphi / L\varphi)$. The modal axiom schemata of interest to us are:

[3] Roughly speaking, logics of finite depth n are logics whose axioms may be falsified on each Kripke frame consisting of $n+1$ consecutive clusters; say, S5 has depth 1, S4F and SW5 are of depth 2.

K	$L(\varphi \supset \psi) \supset (L\varphi \supset L\psi)$
4	$L\psi \supset LL\psi$
T	$L\psi \supset \psi$
D	$L\psi \supset M\psi$
5	$ML\psi \supset L\psi$
W5	$\psi \supset (ML\psi \supset L\psi)$
F	$(\varphi \wedge ML\psi) \supset L(M\varphi \vee \psi)$
2	$ML\psi \supset LM\psi$
3	$L(L\varphi \supset \psi) \vee L(L\psi \supset \varphi)$.

All logics discussed in the paper are *normal*, that is, contain the schema K. S5 is the logic based on schemata K, T, 4 and 5; S4 contains K, T and 4; T contains K and T; SW5 is S4 with the schema W5, S4F is S4 with F, S4.2 is S4 with the schema 2. In other cases, we will simply concatenate names of schemata a modal logic is based on (for example, KD45).

Logics SW5 and S4F are not widely known. However, these logics are rather old and have been known in classical modal logic for a long time under the names S4.4 and S4.3.2. Formal properties of these logics were investigated in detail by Segerberg [29]. Lenzen [13] investigated SW5 from the epistemological point of view and gave arguments in favor of SW5 as a logic of true belief. In [28] interesting applications of logic S4F to knowledge representation were found.

Let $\mathcal{S}$ be a modal logic, A be a set of formulas. We write $A \vdash_{\mathcal{S}} \psi$ to denote that formula ψ is derivable from A in $\mathcal{S}$. That means that there exist a sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that φ_n is ψ, and for each i, $1 \leqslant i \leqslant n$, at least one of the following conditions holds:

- $\varphi_i \in A$;
- φ_i is a propositional tautology;
- φ_i is an instance of a modal axiom schema of the logic $\mathcal{S}$;
- for some $j, k < i$, φ_k has the form $\varphi_j \supset \varphi_i$;
- φ_i has the form $L\varphi_k$ for some $k < i$.

We write $A \vdash \psi$ to denote that ψ is a propositional consequence of A (in the modal language).

We assume the notion of a Kripke model and basics of Kripke models to be known (see. e.g. [8]). Here we recall some notation. A *Kripke model* is a triple $\mathcal{M} = (M, R, V)$, where M is a nonempty set of *worlds*, R is a binary relation on M (called also the *accessibility relation*) and $\{V_\alpha\}_{\alpha \in M}$ is a family of propositional valuations. By $(\mathcal{M}, \alpha) \models \psi$ we denote that a formula ψ is true in the world $\alpha \in M$. Let us recall that $(\mathcal{M}, \alpha) \models L\psi$ if and only if for each $\beta \in M$ such that $\alpha R \beta$, $(\mathcal{M}, \beta) \models \psi$. We write $\mathcal{M} \models \psi$, if for each $\alpha \in M$, $(\mathcal{M}, \alpha) \models \psi$.

Let $\mathcal{K}$ be a class of Kripke models, $\mathcal{S}$ be a modal logic. We say that $\mathcal{S}$ *is determined by* $\mathcal{K}$, if for each formula φ of the modal language, $\vdash_{\mathcal{S}} \varphi$ if and only if for each $\mathcal{M} \in \mathcal{K}$, $\mathcal{M} \models \varphi$.

A Kripke model of the form $(M, M \times M, V)$ is called an S5-*model*. It is well known that S5 is determined by the class of all S5-models. We will denote this S5-model by (M, V), skipping the accessibility relation, which is uniquely determined by M.

A KD45-*model* is a Kripke model of the form $(\{\alpha\} \cup M, (\{\alpha\} \cup M) \times M, V)$, where M is nonempty (thus, if $\alpha \in M$, then a KD45-model is an S5-model). An SW5-*model* is a Kripke model of the form $(\{\alpha\} \cup M, R, V)$, where M is nonempty, and $\gamma R \beta$ if and only if $\beta \in M$ or $\gamma = \alpha$. Thus, a KD45-model consists of an S5-model and an irreflexive world below the S5-model; an SW5-model is an S5-model plus a reflexive world below the model. We will call the corresponding S5-model an *upper cluster* of the KD45-model (or SW5-model). It was proved in [29] that the logic KD45 (respectively SW5) is *determined* by class of all KD45-models (respectively SW5-models). Moreover, both KD45 and SW5 have the finite model property, that is, KD45 (SW5) is determined by the class of all finite KD45-models (respectively SW5-models).

A Kripke model (M, R, V) is an S4F-model, if M is a disjoint union of its subsets M_1 and M_2, and for each α and β, $\alpha R \beta$ if and only if $\alpha \in M_1$ or $\beta \in M_2$. Clearly, M_1 and M_2 are uniquely determined for an S4F-model; we call M_2 an *upper cluster*, and M_1 a *lower cluster* of the model. It was proved by Segerberg [29] that S4F is determined by the classes of all S4F-models, and of all finite S4F-models.

A model $\mathcal{M} = (M, R, V)$ is an S4.2-model, if R is a directed relation on M, that is, for each $\alpha, \beta \in M$ there is some $\gamma \in M$ such that $\alpha R \gamma$ and $\beta R \gamma$. Logic S4.2 is determined by classes of all directed frames, and of all finite directed frames [8,29]. If R is also connected, that is, for each $\alpha, \beta \in M$, $\alpha R \beta$ or $\beta R \alpha$, then $\mathcal{M}$ is also an S4.3-model. Again, S4.3 is determined by classes of all connected, and of all finite connected models.

Assume that (M, R, V) is a finite Kripke model with directed R. Consider the set

$$M_0 = \{\beta \in M\colon \forall \gamma \in M(\beta R \gamma \to \gamma R \beta)\}.$$

It is easy to see that M_0 is a final cluster (that is, for each $\alpha, \beta \in M_0$, $\alpha R \beta$; for each $\alpha \in M_0$, for each $\beta \in M$, $\beta R \alpha$; if $\alpha R \beta$ and $\alpha \in M_0$, then $\beta \in M_0$). Also, because M is finite, M_0 is nonempty (otherwise there would exist an infinite sequence of worlds $\{\alpha_i\}_i$ such that for each i, $\alpha_i R \alpha_{i+i}$ and not $\alpha_{i+1} R \alpha_i$; because R is transitive and reflexive, all the α_i must be different).

Thus, each finite directed (in particular, connected) model has the nonempty final cluster.

We will use also some stronger completeness results than just listed. Let $\mathcal{K}$ be a class of Kripke models. We will say that a modal logic $\mathcal{S}$ is *characterized* by $\mathcal{K}$, if for each set A of formulas, for each formula φ, $A \vdash_{\mathcal{S}} \varphi$ if and only if for each $\mathcal{M} \in K$, $\mathcal{M} \models A$ implies $\mathcal{M} \models \varphi$. It is well known that S4 is characterized by class of all transitive reflexive models, S4.2 is characterized by the class of all reflexive transitive directed models, S4.3 is characterized by the class of all connected models, S4F is characterized by the class of all S4F-models. Note, however, that, for instance, S4 is not characterized by the class of all finite transitive and reflexive models, although S4 is determined by this class of models.

If $\mathcal{K}$ is a class of frames, then we say that $\mathcal{S}$ is determined (characterized) by $\mathcal{K}$ if $\mathcal{S}$ is determined (characterized) by the class of all models based on frames in $\mathcal{K}$.

By using standard canonical model technique [8,19], it is straightforward to show that S4.2 is also characterized by class of all directed models with the final cluster, and S4.3 is characterized by class of all connected models with the final cluster.

2.2. Nonmonotonic modal logics and their semantics

The basic notion of nonmonotonic modal logic is the notion of an expansion of a given axiom set. Let $\mathcal{S}$ be any monotonic logic in $\mathcal{L}$. By an *$\mathcal{S}$-expansion of A* we mean a set T of formulas such that

$$T = \{\varphi \colon\ A \cup \{\neg L\psi \colon\ \psi \notin T\} \vdash_{\mathcal{S}} \varphi\}.$$

A formula is called *objective* if it does not contain any occurrence of L. We denote by T_0 the set of all objective formulas in T, by $Cn(T)$ the set of all tautological consequences of T, and by $Cn_0(T)$ we denote $(Cn(T))_0$.

We always assume logic $\mathcal{S}$ to contain the necessitation rule $(\varphi/L\varphi)$. Hence, each $\mathcal{S}$-expansion T is *stable*, which means that: T is closed under the propositional consequence; for any φ, if $\varphi \notin T$ then $\neg L\varphi \in T$; and for any φ, if $\varphi \in T$ then $L\varphi \in T$.

It was argued [6, 23] that a knowledge set or a belief set of a rational agent must be stable.

We now recall some important properties of expansions we use in subsequent sections.

Proposition 2.1 (McDermott [19]). *Let $\mathcal{S}$ and $\mathcal{T}$ be modal logics, let $\mathcal{S} \subseteq \mathcal{T} \subseteq$ S5. Then any $\mathcal{S}$-expansion of A is a $\mathcal{T}$-expansion of A too.*

For a set A of objective formulas, we denote by $St(A)$ the unique stable set whose objective part is $Cn_0(A)$. For finite A, $St(A)$ is decidable (see, e.g., [10]). We will use the following semantic characterization of $St(A)$ (described in [6, 21]. Let $\mathcal{M} = (M, U)$ be an S5-model such that its set of worlds M consists of *all* propositional valuations making all the formulas in A true, and for each $\alpha \in M$, $U_\alpha(p) = \alpha(p)$, for each propositional variable p. Then $St(A) = Th(\mathcal{M})$. The above model $\mathcal{M}$ will be called the *canonical* S5*-model for A*, or *the canonical* S5*-model of T*.

If for each theory A, for each T, T is a consistent $\mathcal{S}$-expansion of A if and only if T is a consistent $\mathcal{T}$-expansion of A, then we can say, that different monotonic modal logics, $\mathcal{S}$ and $\mathcal{T}$, determine the same nonmonotonic modal logic, or that "nonmonotonic $\mathcal{S}$ coincides with nonmonotonic $\mathcal{T}$". Clearly, if $\mathcal{S} \subseteq \mathcal{T} \subseteq$ S5, and $\mathcal{S}$ and $\mathcal{T}$ determine the same nonmonotonic logic, then each logic between $\mathcal{S}$ and $\mathcal{T}$ determines the same nonmonotonic logic. A *range of $\mathcal{S}$* is the collection of all modal logics which determine the same nonmonotonic logic as $\mathcal{S}$ does.

We will need in the next section the semantic characterization of $\mathcal{S}$-expansions given in [26], so called minimal model semantics.

A nonempty set $N \subseteq M$ is called a *final cone of the model* $\mathcal{M} = \langle M, R, V\rangle$ if:

(i) N is a *cone*, that is, for each $\alpha \in N$, for each β, $\alpha R\beta$ implies $\beta \in N$;

(ii) N is final, that is, for each $\alpha \in M$, for each $\beta \in N$, $\alpha R\beta$.

It follows from (ii), that a final cone is necessarily a cluster (because $N \subseteq M$), so we will call a final cone also a *final cluster*. Also, a Kripke model can have only one final cluster (assume N, K are two final clusters of $\mathcal{M}$; let $\alpha \in N$, $\beta \in K$; from (ii) for K we have $\alpha R\beta$, from (i) for N, we get $\beta \in N$). A model $\mathcal{N} = \langle N, R^*, V^*\rangle$, where R^* and V^* are the restrictions on N of R and V, respectively, is called a *final cluster model of $\mathcal{M}$*. We will sometimes write "final cluster" for denoting a final cluster model.

No ambiguity will appear.

The final cluster (and the corresponding cluster model) is *proper* if there is some $\alpha \in M \setminus N$ for each $\beta \in N$ there is a propositional variable p such that $V_\alpha(p) \neq V_\beta(p)$.

Let $\mathcal{M} = \langle M, R, V \rangle$ and $\mathcal{N} = \langle N, Q, W \rangle$ be two Kripke models. We say, that $\mathcal{M}$ *is preferred over* $\mathcal{N}$ if $\mathcal{N}$ is a proper final cluster model of $\mathcal{M}$. We write $\mathcal{M} \sqsubset \mathcal{N}$ to denote that $\mathcal{M}$ is preferred over $\mathcal{N}$. Clearly, if $\mathcal{M} \sqsubset \mathcal{N}$, then $\mathcal{N}$ is necessarily an S5-model.

The following obvious proposition explains why we need to distinguish proper final clusters.

Proposition 2.2. *Assume that $\mathcal{M}$ is a final cluster of $\mathcal{N}$, but not a proper final cluster of $\mathcal{N}$. Then for each modal formula φ, $\mathcal{M} \models \varphi$ if and only if $\mathcal{N} \models \varphi$.*

If we interpret the knowledge (or belief) set of an agent as the set of sentences true in each world of a model, then $\mathcal{M} \sqsubset \mathcal{N}$ implies that the knowledge set with respect to $\mathcal{M}$ is included in the one with respect to $\mathcal{N}$. Thus, intuitively, we prefer models of agent's knowledge, in which the agent knows less. If A is the initial knowledge set of an agent, we look for a minimal model according to our preference relation, where the agent still knows A. These informal considerations are formalized as follows.

If A is a set of formulas, then, as usual, a model for A is a Kripke model in any world of which all formulas of A are true. Let $\mathcal{K}$ be a class of Kripke models. Let $\mathcal{M}$ be an S5-model in $\mathcal{K}$, i.e. a model with the universal accessibility relation. We say, that $\mathcal{M}$ is a *$\mathcal{K}$-minimal model for A*, if $\mathcal{M}$ is a model for A, and there is no $\mathcal{N} \in \mathcal{K}$ such that $\mathcal{N}$ is a model for A and $\mathcal{N} \sqsubset \mathcal{M}$.

If $\mathcal{S}$ is any of the logics KD45, SW5, S4F, S4 or S4.2, by an $\mathcal{S}$-minimal model we mean a $\mathcal{K}$-minimal model, where $\mathcal{K}$ is a class of all $\mathcal{S}$-models. (So it is clear that S4-minimal models and S4.2-minimal models coincide).

Analogous notions may be introduced for other modal logics which have good Kripke characterizations.

Let $\mathcal{M} = \langle M, R, V \rangle$ be a Kripke model, let $M_0 \subseteq M$ be its final cluster. Let $\mathcal{N} = \langle N, U \rangle$ be a cluster. By the *cluster substitution* of $\mathcal{N}$ in $\mathcal{M}$ we mean the model $\langle (M \setminus M_0) \cup N, Q, W \rangle$, where for each $\alpha, \beta \in (M \setminus M_0) \cup N$, $\alpha Q \beta$ if and only if $\alpha R \beta$ or $\beta \in N$, W agrees with V on $M \setminus M_0$, and agrees with U on N. (In other words, we substitute the cluster $\mathcal{N}$ in place of M_0 into $\mathcal{M}$.)

Let $\mathcal{K}$ be a class of models. We say that $\mathcal{K}$ is *cluster closed* if $\mathcal{K}$ contains all clusters and for each $\mathcal{M} \in \mathcal{K}$, at least one of the following two conditions holds: the concatenation of $\mathcal{M}$ and each cluster belongs to $\mathcal{K}$, or $\mathcal{M}$ has the final cluster and for each cluster $\mathcal{N}$, the cluster substitution of $\mathcal{N}$ in $\mathcal{M}$ belongs to $\mathcal{K}$.

Theorem 2.3. *Let a normal modal logic $\mathcal{S}$ be contained in* S5 *and be characterized by a cluster closed class of models $\mathcal{K}$. Let T be any consistent stable set of formulas, $\mathcal{M}_T$ be a canonical* S5*-model of A. Then the following three conditions are equivalent:*

(1) *T is an $\mathcal{S}$-expansion of A;*

(2) *$\mathcal{M}_T$ is an $\mathcal{S}$-minimal model for A;*

(3) *for some* S5*-model $\mathcal{M}$, $T = \{\psi\colon\ \mathcal{M} \models \psi\}$ and $\mathcal{M}$ is $\mathcal{S}$-minimal for A.*

Clearly, Theorem 2.3 applies to logics S4, S4.2, S4F, S4.4, KD45, S5, S4.3 and many others.

McDermott [19] proved that for each logic $\mathcal{S}$ containing the necessitation rule all $\mathcal{S}$-expansions are closed under S5. We believe that this, along with the intuition, is a clear indication that only logics contained in S5 can be a reasonable basis for nonmonotonic reasoning, and consider only logics contained in S5. However, for a formally oriented reader, we can prove the following simple proposition asserting that a logic not contained in S5 (but containing the necessitation rule) can never be in one range with any logic contained in S5.

Proposition 2.4. *Let $\mathcal{S}$ be any logic contained in* S5, *and $\mathcal{T}$ be any logic not contained in* S5. *Then there exists a consistent $\mathcal{S}$-expansion of the empty theory which is not a $\mathcal{T}$-expansion of $\emptyset$.*

Proof. All $\mathcal{T}$-expansions are closed both under $\mathcal{T}$ and S5, hence they are closed under the logic $\mathcal{C}$ which contains all the schemas of $\mathcal{S}$ and $\mathcal{T}$. $\mathcal{C}$ is a proper extension of S5. Segerberg [29] gave a complete description of all extensions of S5. Each such extension is a logic S_n, which is determined by the class of all S5-models containing no more than n worlds, for some finite n. Thus, for some n, $\mathcal{C}$ is S_n. But $St(\emptyset)$ is an $\mathcal{S}$-expansion of $\emptyset$ [30]. Let $p_1, \ldots, p_n$ be pairwise different propositional variables. Then $St(\emptyset)$ contains all the formulas $M(p_1 \wedge p_{i-1} \wedge \neg p_i \wedge p_{i+1} \wedge \cdots \wedge p_n)$ for each i, $0 \leqslant i \leqslant n$. But the conjunction of all these formulas is refutable in S_n, hence $St(\emptyset)$ cannot be closed under S_n. $\square$

Remark 2.5. We defined two logics to be "undistinguishable", if they have the same consistent expansions. What if we take inconsistent expansions into account, too? McDermott [19] noticed that if A has an inconsistent $\mathcal{S}$-expansion then it is the only $\mathcal{S}$-expansion of A, and A is inconsistent with $\mathcal{S}$ (that is, $A \vdash_{\mathcal{S}} \mathit{false}$). Hence if $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S}$ and $\mathcal{T}$ are in the same range, then the only possibility for $\mathcal{T}$ and $\mathcal{S}$ to have different expansions is if some A has no $\mathcal{S}$-expansions, and the only $\mathcal{T}$-expansion is inconsistent. Because the absence of expansions and the presence of the only inconsistent expansion are usually understood in nonmonotonic logic as nonmonotonic inconsistency, we can still consider $\mathcal{S}$ and $\mathcal{T}$ as nonmonotonically indistinguishable. But if we consider logics containing S4, then such subtleties are inessential at all. It is well known that a theory is consistent with S4 iff it is consistent with S5. Hence for logics containing S4 all our results concerning consistent expansions hold for arbitrary expansions too.

3. Maximality property of KD45, SW5 and S4F

We proved in [27] that the empty theory has only one KD45-expansion, and only one SW5-expansion, but for each extension $\mathcal{S}$ of KD45 or SW5, contained in S5, the empty theory has at least two $\mathcal{S}$-expansions. Similarly, for each logic $\mathcal{S}$ containing S4F and contained in S5, there is a theory I such that I has no nested modalities and no positive occurrences of L, and there is an $\mathcal{S}$-expansion of I, which is not an S4F-expansion of I [28]. It follows from these results that KD45, SW5 and S4F are maximal in their ranges. However results of [28] do not imply that, say, S4F is the largest logic in its range: they do not exclude the possibility of the existence of a logic incomparable with S4F, whose expansions coincide with S4F-expansions. The situation for KD45 or SW5 is the same. Moreover, methods of [27,28] cannot give the stronger result that the logics are largest in their ranges: we used the complete description of all extensions of S4F (or KD45) given by Segerberg in [29], and no effective description of all the logics incomparable with S4F (or KD45) is known by now.

Below we prove that KD45, SW5 and S4F are largest in their ranges. That means, for example, that each logic whose expansions coincide with S4F-expansions, is contained in S4F.

Theorem 3.1. *Let $\mathcal{S}$ be any logic contained in* S5 *and not contained in* KD45. *Then there exists a theory I such that some $\mathcal{S}$-expansion of I is not a* KD45*-expansion of I.*

Proof. Presented in Appendix A. □

Corollary 3.2. *If $\mathcal{S}$ is any logic in the same range as* KD45, *then each theorem of $\mathcal{S}$ is a theorem of* KD45.

Theorem 3.3. *Let $\mathcal{S}$ be any logic contained in* S5 *and not contained in* SW5. *Then there exists a theory I such that some $\mathcal{S}$-expansion of I is not an* SW5*-expansion of I.*

Proof. See Appendix A. □

Corollary 3.4. *If $\mathcal{S}$ is any logic in the same range as* SW5, *then each theorem of $\mathcal{S}$ is a theorem of* SW5.

Theorem 3.5. *Let $\mathcal{S}$ be any logic contained in* S5 *and not contained in* S4F. *Then there exists a theory I such that some $\mathcal{S}$-expansion of I is not an* S4F*-expansion of I.*

Proof. See Appendix A. □

Corollary 3.6. *If $\mathcal{S}$ is any logic in the same range as* S4F, *then each theorem of $\mathcal{S}$ is a theorem of* S4F.

The *depth* of a Kripke model (M, R, V) with the transitive accessibility relation is the largest number k such that there exists a sequence of worlds of the length k, $w_1, \dots, w_k$ such that for each $1 \leqslant i < k$, $w_i R w_{i+1}$, and not $w_{i+1} R w_i$. Logic $\mathcal{S}$ has depth k, if each underivable formula of $\mathcal{S}$ can be refuted on a Kripke model of depth k, and some theorem I of $\mathcal{S}$ can be refuted on a Kripke model of depth $k+1$. For instance, S5 is of finite depth 1; SW5 and S4F are of depth 2. Logic of finite depth is a logic which contains K4 and has depth k for some natural number k. Logics S4, S4.2 and S4.3 are of infinite depth. We show in Appendix B that all logics of finite depth which are contained in S4F are in the range of S4F. We also present for each k a (well-known) logic of depth k, so the range of S4F is quite large.

4. The range of S4

It is clear that if $\mathcal{M} \sqsubset \mathcal{N}$, and $\mathcal{M}$ is an S4-model, then $\mathcal{M}$ is also an S4.2-model, hence consistent S4-expansions and S4.2-expansions coincide. However, this fact is quite simple and can be established without any knowledge of minimal model semantics. Moreover, the scheme 2 always leaves a logic in the same range.

Proposition 4.1. *Let $\mathcal{S}$ be any modal logic contained in* S5. *Then the logic $\mathcal{T}$ obtained by extending $\mathcal{S}$ with the schema* 2 *is in the same range as $\mathcal{S}$.*

Proof. It is sufficient to show that each consistent $\mathcal{T}$-expansion of A is also an $\mathcal{S}$-expansion of A. Assume $T = Cn_{\mathcal{T}}(A \cup \neg L\psi\colon\ \psi \notin T)$. Because $\mathcal{S} \subseteq \mathcal{T}$, $Cn_{\mathcal{S}}(A \cup \{\neg L\psi\colon\ \psi \notin T\}) \subseteq T$. Thus, it remains to prove that each $\varphi \in T$ is derivable in $\mathcal{S}$ from $A \cup \{\neg L\psi\colon\ \psi \notin T\}$. For, it suffices to prove that each instance of schema 2 is derivable in $\mathcal{S}$ from $\{\neg L\psi\colon\ \psi \notin T\}$. Let $\neg L\neg L\varphi \supset L\neg L\neg\varphi$ be such an instance. If $\varphi \notin T$, then, using the necessitation rule, we have $L\neg L\varphi \in Cn_{\mathcal{S}}(\{\neg L\psi\colon\ \psi \notin T\})$, hence, by propositional logic, we have $\neg L\neg L\varphi \supset L\neg L\neg\varphi \in Cn_{\mathcal{S}}(A \cup \{\neg L\psi\colon\ \psi \notin T\})$.

Assume now that $\varphi \in T$. Then $\neg\varphi \notin T$, and again, using necessitation and propositional logic we obtain $\neg L\neg L\varphi \supset L\neg L\neg\varphi \in Cn_{\mathcal{S}}(A \cup \{\neg L\psi\colon\ \psi \notin T\})$. $\square$

Thus, S4.2 and S4 are in the same range. We prove that S4.2 is the largest logic in this range. The basic idea is the same as for S4F, but it requires essential modification: if we try to repeat the construction of the proof of Theorem 3.5, we will fail, because for general S4.2-models we will not have $\beta Q\alpha_0$ for each β not in the final cluster of $\mathcal{P}$, like we did for S4F-models. Moreover, it was proved in [16] that for all finite theories I, S4F-expansions and S4-expansions of I are the same. Thus, the theory I having an $\mathcal{S}$-expansion which is not an S4.2-expansion, must necessarily be infinite. We achieve the goal carefully splitting the formula $LF \supset (C \wedge r)$ of the proof of Theorem 3.5 into infinitely many formulas.

Theorem 4.2. *Let $\mathcal{S}$ be any modal logic which is contained in* S5, *but does not contain* S4.2. *Then there exists a theory A such that A has a consistent $\mathcal{S}$-expansion which is not an* S4.2-*expansion.*

Proof. See Appendix A. $\square$

Corollary 4.3. *Logic* S4.2 *is largest in the range of* S4.

The proof of Theorem 4.2 was constructive enough, so for a logic not contained in S4.2 we can construct an infinite theory which will distinguish the corresponding nonmonotonic logics. We give an example of applying this technique to show that nonmonotonic logic S4.3 differs from nonmonotonic S4.2 (and, hence, from nonmonotonic S4). S4.3 is very popular and serves as a basis for tense logics (the accessibility relation r is interpreted as "always in the future"). But it appears that S4.3 is also relevant to knowledge representation, and to nonmonotonic logics: Craig Boutilier [1] used S4.3-models as a basis for his proposal of a generalization of the autoepistemic logic so that it be possible to introduce gradation of belief. Thus, it might be worthy to locate correctly nonmonotonic S4.3.

Example 4.4. Consider an instance of the axiom of S4.3, $F = L(Lp \supset q)) \vee L(Lq \supset p)$, where p and q are different atoms. Consider the following Kripke model: $\mathcal{M} = \langle M, R, V\rangle$, where:

- $M = \{0, P, Q, 1\}$;
- $\alpha R \beta$ iff $\alpha = 0$ or $\beta = 1$;
- p is true in P and 1, and is false in Q and 0; q is true in Q and 1, and is false in P and 0.

Thus, $\{1\}$ is a final cluster, and $\mathcal{M}$ is an S4.2-model. Clearly, $F(p,q)$ is false in 0, so we can use F to obtain a theory having an S4.3-expansion which is not an S4.2-expansion, as we did in the proof of Theorem 4.2. In our case $K = \{1\}$, so formula $C(p,q)$ is just one disjunct, namely, $C(p,q) = p \wedge q$. Let $p_1, p_2, \dots$ and $q_1, q_2, \dots$ be infinite lists of different variables, r be different from all p_i and q_i. From the proof of Theorem 4.2, we obtain the following theory A having an S4.3-expansion which is not an S4.2-expansion:

$$A = \{(L(Lp_1 \supset q_1) \vee L(Lq_1 \supset p_1)) \supset r\}$$
$$\cup \bigcup \{L(L(Lp_{m+1} \supset q_{m+1}) \vee L(Lq_{m+1} \supset p_{m+1})) \supset (p_m \wedge q_m)\}_{1 \leqslant m < \infty}.$$

Remark 4.5. In [16] an example of a theory having an S4F-expansion, but having no S4-expansions, was constructed; in fact, the (infinite) theory constructed there has no S4.3-expansions, which is easy to check. So nonmonotonic S4.3 lies strictly between nonmonotonic S4.2 and S4.F. Note that for finite theories all three logics coincide.

5. Discussion. The paradigm of maximal logics

There are infinitely many modal logics, but only few of them are considered as reasonable logics of knowledge and belief. Usually, arguments in favor of any particular logic consist in philosophical speculations about intuitive meaning of particular axioms. While intuitive considerations are important, we feel that it would be nice, to augment intuitions by exact mathematical results. A drawback of intuitive considerations based on axiomatizations of logics is the following. They usually give reasonable answers on the question "why these axioms?" by providing arguments, why axioms of a given system are intuitively acceptable. But they fail to provide a reasonable argument to the question, "why not more?". For instance, a typical argument in favor of the point that S4 is "the" logic of knowledge, is the following. All the modal axioms of S4 appear to be natural if we interpret the modality L as "is known". An S5-axiom $\neg L\varphi \supset L\neg L\varphi$ (negative introspection) does not correspond to the intuition (an agent can be convinced that φ holds when φ is in fact false; so s/he does not know φ, although is not aware of this). Thus, S4 is the logic of knowledge.

But there are infinitely many normal modal logics between S4 and S5! In fact, from a result of Jankov [9] it follows that the set of logics between S4 and S5 has the cardinality of the continuum. So how we can be sure that there is no logic between S4 and S5 satisfying our intuitions?

We suggest the following approach to the problem. Consider some property, $\mathcal{P}$, of modal logics such that:

- $\mathcal{P}$ is "natural" for a suggested interpretation of the modal operator, or is otherwise desirable;
- $\mathcal{P}$ is stable with respect to intervals of logics: if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two logics both possessing the property $\mathcal{P}$, then any logic between $\mathcal{S}_1$ and $\mathcal{S}_2$ also has the property $\mathcal{P}$.

Then, if we find a logic $\mathcal{S}$ which is otherwise acceptable, has the property $\mathcal{P}$ and is maximal with respect to $\mathcal{P}$, then we have a strong, and not an ad hoc style, argument that we have put as much as we could in $\mathcal{S}$: nothing can be added to $\mathcal{S}$. If, in addition, we can prove that $\mathcal{S}$ is the largest logic with the property $\mathcal{P}$, then we are even better off: $\mathcal{S}$ has no potential competitor for its role.

Of course, the above scheme may be applicable not only to logics of knowledge and belief, but in any situation, when we are in search for a "true" logic for any application.

There has been a reasonable argument that logics of knowledge and belief are nonmonotonic logics, and we focus our investigation on one of the most known and mostly advanced approaches to nonmonotonic modal logics, namely McDermott and Doyle's approach, which emphasizes the self-referential character of epistemic reasoning. This approach is convenient to such investigations when we are interested in properties of a logic as a member of a family of all possible logics. The reason is that this approach is technically very uniform: It assigns to *each possible* monotonic modal logic the corresponding nonmonotonic logic.

Several results has been previously obtained in this direction.

In [25] we proved that logics KD45 and SW5 are maximal logics $\mathcal{S}$ with the property that each objective theory has only one $\mathcal{S}$-expansion. This is a very natural property: it states that if our axioms do not say anything about knowledge, then the behavior of expansions should be the same as that of the usual monotonic consequence.

In [28] we proved that S4F is maximal with respect to the following property: if A consists only of formulas without nested modalities, and all the occurrences of L in A are negative, then A has only one $\mathcal{S}$-expansion T, and the objective part of T is the objective part of the deductive closure of A. This is a very important and desirable property: roughly, it says that nonmonotonic $\mathcal{S}$ does not admit so-called ungrounded expansions, like an expansion $St(p)$ of the theory $Lp \supset p$ in Moore's autoepistemic logic.

It follows from the above results that logics KD45, SW5 and S4F are maximal in their ranges. In the present paper we strengthened this result by proving that these logics are also largest in their ranges. This is only a partial success in the direction we outlined above: It would be interesting to show that they are largest not just in their ranges, but largest with respect to properties of the two previous paragraphs. We presented a technique for proving maximality results based on the minimal model semantics of nonmonotonic modal logics we developed earlier in [26], which is a significant advance in comparison with the technique used in [25,28]: although the results there looked general as they referred to *all* the logic containing, say, S4F (which are infinitely many), the proofs had a very ad hoc character, as they used the complete description of all extensions of logics in question.

The central technical result of the paper is that S4 is not a maximal logic in its range, but the logic S4.2 is. This is a formal support to an argument that S4.2 should replace S4 as a candidate for "the" logic of knowledge, a point expressed earlier in [13] based on purely philosophical considerations.

In view of our general program, it would be interesting to find a general property for which S4.2 would be maximal. Because for finite theories S4.2-expansions coincide with S4F-expansions, such a property must necessarily involve infinite theories.

Another open technical question we can suggest is the following. Our example of an infinite theory A which has an S4F-expansion, but has no S4-expansions, involves infinitely many propositional variables. We know that for finite theories S4F-expansions coincide with S4-expansions. But can we give such an example involving only finitely many propositional variables?

Finally, we would like to pose a problem concerning S4F which, although it lies beyond our outlined program of investigating ranges and maximal properties, is of importance for knowledge representation as part of general program of exhibiting reasonable epistemic logics.

Previous research, including the present paper, suggest that the following logics are of particular interest: S5, KD45, SW5, S4F and S4.2. Among these logics, S4F is the only one whose axioms do not look very natural under the epistemological understanding of modality: we cannot provide a reasonable argument in favor of the axiom F. On the other hand, recent work shows that nonmonotonic S4F is a truly remarkable logic (see [28] for a detailed investigation of that logic). It generalizes Moore's autoepistemic logic, default logic, Lin and Shoham's and Lifschitz's versions of logics for grounded (or minimal) knowledge. In many respects it is more natural then Moore's autoepistemic logic, or reflexive autoepistemic logic introduced in [25].

Logics KD45 and SW5 have an important advantage over other modal logics: the corresponding nonmonotonic logics admit description in purely propositional terms, without referring to any particular modal logics. Nonmonotonic KD45 coincides with Moore's autoepistemic logic (see [25]), and nonmonotonic SW5 coincides with the reflexive autoepistemic logic introduced in [25]. A theory T is a consistent KD45-expansion of A if and only if

$$T = Cn(A \cup \{\neg L\psi\colon\ \psi \notin T\} \cup \{L\psi\colon\ \psi \in T\}),$$

where Cn denotes usual propositional closure. Similarly, consistent SW5-expansions are described as solutions to the fixed point equation

$$T = Cn(A \cup \{\neg L\psi\colon\ \psi \notin T\} \cup \{L\psi \equiv \psi\colon\ \psi \in T\}).$$

It would be interesting to get a similar "purely propositional" description for the nonmonotonic S4F. Or to prove that such a description is impossible (this would require also an appropriate formal definition of what a "propositional description" is). In fact, recent work of Gottlob [4], who proved that in some precise sense, a syntactical translation of Reiter's default logic into Moore's autoepistemic logic is impossible, suggests that it is very likely that a nice propositional description of nonmonotonic S4F is impossible.

Concluding, we would like to express the hope that the idea of maximal logics can find applications outside the McDermott-Doyle scheme. Our goal, besides presenting some new results on nonmonotonic logics, was to show how such a paradigm can work.

Appendix A. Proofs

Theorem 3.1. *Let $\mathcal{S}$ be any logic contained in* S5 *and not contained in* KD45. *Then there exists a theory I such that some $\mathcal{S}$-expansion of I is not a* KD45*-expansion of I.*

Proof. Since $\mathcal{S}$ is not contained in KD45, there is a formula which is a theorem of $\mathcal{S}$ but not a theorem of KD45. Let $F = F(p_1, \dots, p_n)$ be such formula and let $p_1, \dots, p_n$ be the complete list of its propositional variables. By the completeness theorem for KD45, there is a finite KD45-model $\mathcal{M} = \langle \{\alpha_0\} \cup M, R, V\rangle$ with the upper cluster M, such that $\mathcal{M} \not\models F$. Because F is a theorem of S5, F is true in all the worlds in M, hence $\alpha_0 \notin M$ and $(\mathcal{M}, \alpha_0) \not\models F(p_1, \dots, p_n)$.

Now let $M = \{\beta_1, \dots, \beta_k\}$. For each $\beta \in M$, by p^β we will denote the variable p, if p is true in the world β, and its negation $\neg p$, if p is false in β. By P^β we denote the conjunction $p_1{}^\beta \wedge \cdots \wedge p_k{}^\beta$. Finally, let $C = C(p_1, \dots, p_n)$ be a disjunction $P^{\beta_1} \vee \cdots \vee P^{\beta_n}$. Let r be a propositional letter different from all the p_i's. Finally, we put

$$I = \{F(p_1, \dots, p_n) \supset (C(p_1, \dots, p_n) \wedge r)\}.$$

We shall prove that I satisfies the conclusion of the theorem, that is, there is an $\mathcal{S}$-expansion of I, which is not a KD45-expansion of I.

First, since F is a theorem of $\mathcal{S}$, I is equivalent in $\mathcal{S}$ to an objective theory $C \wedge r$. Therefore, according to a result of [30], the theory $T = St(C \wedge r)$ is an $\mathcal{S}$-expansion of I. Let $\mathcal{N} = \langle N, W\rangle$ be the canonical S5-model for $C \wedge r$. We show that $\mathcal{N}$ is a model of I, which is not KD45-minimal. Hence, the desired conclusion will follow by Theorem 2.3.

Formula $C \wedge r$ is true in $\mathcal{N}$, hence I is true, too. Now it remains to construct a KD45-model $\mathcal{P}$ such that $\mathcal{P} \sqsubset \mathcal{N}$, and $\mathcal{P} \models I$.

Clearly N consists of all the worlds where r is true and one of P_β for some $\beta \in M$ is true. Let us denote by N_β, for $\beta \in M$, the set of all worlds in N where P^β is true. Clearly, each N_β is nonempty, and $N = \bigcup\{N_\beta\colon\ \beta \in M\}$.

Consider now the KD45-model $\mathcal{P} = \langle \{\alpha_0\} \cup N, Q, U\rangle$, where $Q = (\{\alpha_0\} \cup N) \times N$, U coincides with W on N, and put $U_{\alpha_0}(p_i) = V_{\alpha_0}(p_i)$, $U_{\alpha_0}(r) = \mathit{false}$. We claim that for each formula φ containing only variables $p_1, \dots, p_n$ the following hold:

- for each $\beta \in M$, for each $\gamma \in N_\beta$, $(\mathcal{M}, \beta) \models \varphi$ if and only if $(\mathcal{P}, \gamma) \models \varphi$;
- $(\mathcal{M}, \alpha_0) \models \varphi$ if and only if $(\mathcal{P}, \alpha_0) \models \varphi$.

We prove these claims by induction on φ. The induction basis follows immediately from the definition of $\mathcal{P}$. The only nontrivial case in the induction step is if $\varphi = L\psi$.

Assume that $\beta \in M$, $\gamma \in N_\beta$, $(\mathcal{M},\beta) \models L\psi$. Then for each $\eta \in M$, $(\mathcal{M},\eta) \models \psi$. Hence, by the induction hypothesis, for each $\eta \in M$, for each $\delta \in N_\eta$, we have $(\mathcal{P},\delta) \models \psi$. Because each $\delta \in N$ belongs to some N_γ, we have for each $\delta \in N$, $(\mathcal{P},\delta) \models \psi$, hence $(\mathcal{P},\gamma) \models L\psi$.

Conversely, assume that $\gamma \in N_\beta$, $(\mathcal{M},\beta) \models L\psi$. Consider an arbitrary $\delta \in M$. Because N_δ is nonempty, we have some $\eta \in N_\delta$. We have $\gamma Q \eta$, hence $(\mathcal{P},\eta) \models \psi$. By the induction hypothesis we have $(\mathcal{M},\delta) \models \psi$. Because δ was arbitrary in $\mathcal{M}$, we have $(\mathcal{M},\beta) \models L\psi$.

Assume now that $(\mathcal{M},\alpha_0) \models L\psi$. Then for each $\beta \in M$, $(\mathcal{M},\beta) \models \psi$. Repeating our above considerations, we obtain that for each $\gamma \in N$, $(\mathcal{P},\gamma) \models \psi$, hence $(\mathcal{P},\alpha_0) \models L\psi$.

Conversely, if $(\mathcal{P},\alpha_0) \models L\psi$, then for each $\delta \in N$, $(\mathcal{P},\delta) \models \psi$. Again, repeating the considerations we presented for the first clause, we get for each $\beta \in M$, $(\mathcal{M},\beta) \models \psi$, hence $(\mathcal{M},\alpha_0) \models L\psi$. Thus, our claim is proved.

Because F contains no propositional letters other then $p_1,\ldots,p_n$, we have

$$(\mathcal{P},\alpha_0) \not\models F. \tag{A.1}$$

Hence we have $(\mathcal{P},\alpha_0) \models F \supset (C \wedge r)$. Also, $\mathcal{P} \sqsubset \mathcal{N}$, because r is true everywhere in $\mathcal{N}$, and is false in α_0. Thus, $\mathcal{N}$ is a model of I which is not KD45-minimal, therefore, $T = Th(\mathcal{N})$ is not a KD45-expansion of $\{F \supset (C \wedge r)\}$. $\square$

Theorem 3.3. *Let $\mathcal{S}$ be any logic contained in* S5 *and not contained in* SW5. *Then there exists a theory I such that some $\mathcal{S}$-expansion of I is not an* SW5*-expansion of I.*

Proof. The proof is the same as that of Theorem 3.1. The only difference is that as F we should take a formula derivable in $\mathcal{S}$ but not in SW5, and as a model—an SW5-model falsifying F. $\square$

Theorem 3.5. *Let $\mathcal{S}$ be any logic contained in* S5 *and not contained in* S4F. *Then there exists a theory I such that some $\mathcal{S}$-expansion of I is not an* S4F*-expansion of I.*

Proof. Here we need a small modification to the proof of Theorem 3.1. Since $\mathcal{S}$ is not contained in S4F, there is a formula which is a theorem of $\mathcal{S}$ but not a theorem of S4F. Let F be such formula and $p_1,\ldots,p_n$ the complete list of its propositional variables. By the completeness theorem for S4F, there is a finite S4F-model $\mathcal{M} = \langle M,R,V\rangle$ and a world $\alpha_0 \in M$ such that $(\mathcal{M},\alpha_0) \not\models F(p_1,\ldots,p_n)$. Let M_0 and M_1 be the lower and upper clusters of $\mathcal{M}$, respectively. That is, $M = M_0 \cup M_1$, M_0 and M_1 are disjoint, and for each α and β, $\alpha R \beta$ iff $\alpha \in M_0$ or $\beta \in M_1$.

Because $\mathcal{S}$ is contained in S5, we have $(\mathcal{M},\beta) \models F$ for each $\beta \in M_1$. Hence

$$\alpha_0 \in M_0.$$

Now let $M_1 = \{\beta_1,\ldots,\beta_k\}$. We define formula $C = C(p_1,\ldots,p_n)$ exactly as we did in the proof of Theorem 3.1, using M_1 in place of M. Let r be a propositional letter different from all the p_i's. We put:

$$I = \{LF(p_1, \ldots, p_n) \supset (C(p_1, \ldots, p_n) \wedge r)\}.$$

(Note that we would fail if we took the formula $F \supset (C \wedge r)$ like we did in two previous theorems, because then we would be unable to prove the truth of I in the worlds of the lower cluster, different from α_0.)

We shall prove that I satisfies the conclusion of the theorem, that is there is an $\mathcal{S}$-expansion of I, which is not an S4F-expansion of I. First, since F is a theorem of S, I is equivalent in $\mathcal{S}$ to an objective theory $C \wedge r$. Therefore, according to a result of [30], the theory $T = St(C \wedge r)$ is an $\mathcal{S}$-expansion of I. We will prove that T is not an S4F-expansion of I by using Theorem 2.3.

Let $\mathcal{N} = \langle N, W\rangle$ be a canonical S5-model for $C \wedge r$. Let $\mathcal{P} = \langle M_0 \cup N, Q, U\rangle$ be a Kripke model such that $\alpha Q \beta$ iff $\alpha \in M_0$ or $\beta \in N$, U coincides with W on M, $U_\beta(p_i) = V_\beta(p_i)$, and $U_{\alpha_0}(r) = \mathit{false}$. Like in the proof of Theorem 3.1, we can show that for each formula φ consisting of variables $p_1, \ldots, p_n$ only, for each $\alpha \in M_0$, $(\mathcal{M}, \alpha) \models \varphi$ if and only if $(\mathcal{P}, \alpha) \models \varphi$. Hence we obtain that $(\mathcal{P}, \alpha_0) \models \neg F$, and, since for each $\beta \in M_0$, $\beta Q \alpha_0$, we have for each $\beta \in M_0$, $(\mathcal{P}, \beta) \models \neg LF$, hence $(\mathcal{P}, \beta) \models LF \supset (C \wedge r)$. And because we have $\mathcal{N} \models C \wedge r$, we obtain $\mathcal{P} \models I$.

Also, $\mathcal{N}$ is a final cluster of $\mathcal{P}$, and, since r is true in $\mathcal{N}$ and false in $\alpha_0 \in M_0$, $\mathcal{N}$ is a proper final cluster of $\mathcal{P}$. Thus, $\mathcal{N}$ is a model of I which is not an S4F-minimal model of I, hence $T = Th(\mathcal{N})$ is not an S4F-expansion of I. □

Theorem 4.2. *Let $\mathcal{S}$ be any modal logic which is contained in* S5, *but does not contain* S4.2. *Then there exists a theory A such that A has a consistent $\mathcal{S}$-expansion which is not an* S4.2*-expansion.*

Proof. Assume that there is a formula $F = F(p_1, \ldots, p_n)$ which is a theorem of $\mathcal{S}$, but not a theorem of S4.2. Then there exists a finite S4.2-model $\mathcal{M} = \langle M, R, V\rangle$ and $\alpha_0 \in M$ such that $(\mathcal{M}, \alpha_0) \not\models F(p_1, \ldots, p_n)$. Because M is finite and directed, $\mathcal{M}$ has a final cluster. Let K be a final cluster of $\mathcal{M}$, $P = M \setminus K$. That means that M is a disjoint union of K and P, for each $\alpha \in M$, for each $\beta \in K$, $\alpha R \beta$, and for each $\beta \in K$, for each $\gamma \in M$, $\beta R \gamma$ implies $\gamma \in K$.

Clearly, $\alpha_0 \notin K$. (Because F is a theorem of S5, we have $\beta \models F$ for each $\beta \in K$.)

Let $K = \{\beta_1, \ldots, \beta_k\}$. For each j, $1 \leqslant j \leqslant k$, and for each $\beta \in K$, let $P_j{}^\beta$ be p_j, if $(\mathcal{M}, \beta) \models p_j$, and let $P_j{}^\beta$ be $\neg p_j$ otherwise. Let P^β be a conjunction $P_1{}^\beta \wedge \cdots \wedge P_n{}^\beta$. By $C = C(p_1, \ldots, p_n)$ we denote the disjunction of all formulas P^β for all $\beta \in K$.

Now we are in a position to give an example of a theory A. Let $\{p_j{}^l\}_{1 \leqslant j \leqslant n, 1 \leqslant l < \infty}$ be pairwise distinct propositional letters, let r be different from all $p_j{}^l$. Let A be an infinite theory, consisting of the formula $F(p_1{}^1, \ldots, p_n{}^1) \supset r$, and of all the formulas

$$LF(p_1{}^{m+1}, \ldots, p_n{}^{m+1}) \supset C(p_1{}^m, \ldots, p_n{}^m),$$

for all $m \geqslant 1$.

We assume that there are countably many propositional variables in our language. Therefore, we can assume, without loss of generality, that the $p_i{}^j$ and r are all the variables of the language.

We claim that A satisfies the desired properties. First, by the definition of a logic, the set of theorems of $\mathcal{S}$ is closed under substitutions for variables, and we consider only logics containing the necessitation rule. Hence, all the formulas $LF(p_1{}^m, \ldots, p_n{}^m)$ are theorems of $\mathcal{S}$. Hence, in $\mathcal{S}$, A is equivalent to the objective theory O consisting of r and all the formulas $C(p_1{}^m, \ldots, p_n^m)$, $m \geqslant 1$. Hence, $T = St(O)$ is an $\mathcal{S}$-expansion of A. We will prove that T is not an S4.2-expansion of A, using the minimal model semantics.

Let $\mathcal{N} = \langle N, W \rangle$ be a canonical S5-model for O (that means, in particular, that $T = Th(\mathcal{N})$). Each world α of $\mathcal{N}$ has the following property: for each natural number m, there exists $\beta_m \in M$, such that for each j, $1 \leqslant j \leqslant n$, $W_\alpha(p_j{}^m) = V_{\beta_m}(p_j)$ (this property is implied by the fact that the formula $C(p_1{}^m, \ldots, p_n{}^m)$ is true in $\mathcal{N}$). Conversely, because N consists of all the worlds where all formulas of O are true, for each sequence of worlds of $\mathcal{M}$, $\{\beta_m\}_m$, there exists a world $\alpha \in N$ such that for each j, $1 \leqslant j \leqslant n$,

$$W_\alpha(p_j{}^m) = V_{\beta_m}(p_j). \tag{A.2}$$

Because r is true in all the worlds of $\mathcal{N}$, and all the propositional variables of our language are $p_i{}^m$ and r, we can, without loss of generality, assume that N consists of all the infinite sequences $\alpha = \{\beta_m\}_m$ of worlds of M, and the valuation W is determined by (A.2) for $p_j{}^m$, and $W_\alpha(r)$ is *true* for each $\alpha \in N$.

In other words, α treats $p_j{}^m$ like the mth component of α treats p_j.

Because each member of A is an implication whose successor is in O, and $\mathcal{N}$ is a model of O, we have

$$\mathcal{N} \models A. \tag{A.3}$$

Thus, we have that $\mathcal{N}$ is an S5-model of A, and T is the theory of $\mathcal{N}$. To complete the proof that T is not an S4.2-expansion of A, it remains to show that $\mathcal{N}$ is not S4.2-minimal for A, that is, to construct an S4.2-model $\mathcal{P}$ such that $\mathcal{P} \models A$ and $\mathcal{P} \sqsubset \mathcal{N}$.

We construct the model $\mathcal{P} = \langle B, Q, U \rangle$ as follows. First, remember that our initial model $\mathcal{M}$ consists of two parts: the final cluster K and the rest P. For each natural number m, let P_m denote the set $\{m\} \times P$. Thus, the P_m are disjoint copies of P. Let B be the disjoint union of N and all P_m, $m = 1, 2, \ldots$. Now, define the accessibility relation Q as follows. Put $\gamma Q \beta$ for all $\beta \in N, \gamma \in B$. For $\beta, \gamma \in P$, put $\langle m, \beta \rangle Q \langle n, \gamma \rangle$ if and only if $m < n$ or ($m = n$ and $\beta R \gamma$).

In other worlds, N is the final cluster of $\mathcal{P}$, and we put the infinite sequence $\{P_i\}_i$ of copies of P, "converging" to the final cluster $\mathcal{N}$. Now, define the valuation U. For $\beta \in N$, we put U_β to coincide with W_β.

Let us fix an arbitrary $\beta_0 \in K$. Let $\alpha = \langle m, \beta \rangle$. We define $U_\alpha(p_i{}^m) = V_\beta(p_i)$, and for all $l \neq m$, we define $U_\alpha(p_i{}^l) = V_{\beta_0}(p_i)$. Finally, we put $W_{\langle 1, \alpha_0 \rangle}(r) = \mathit{false}$, and $W_\beta(r) = \mathit{true}$ for all β different from $\langle 1, \alpha_0 \rangle$.

A formula r is false in $\langle 1, \alpha_0 \rangle$ and true everywhere in N. Therefore, $\mathcal{N}$ is a proper final cluster of $\mathcal{P}$. Hence, to complete the proof of the theorem, it remains to prove that $\mathcal{P} \models A$.

Claim A.1. *Let $\varphi(p_1,\dots,p_n)$ be any formula constructed from variables $p_1,\dots,p_n$. Then:*

(i) *For each $\alpha = \{\alpha_j\}_j \in N$, for each m, $\mathcal{P},\alpha \models \varphi(p_1{}^m,\dots,p_n{}^m)$ if and only if $(\mathcal{M},\alpha_m) \models \varphi(p_1,\dots,p_n)$;*

(ii) *for each m, for each $l > m$, for each $\gamma \in P_l$, $(\mathcal{P},\gamma) \models \varphi(p_1{}^m,\dots,p_n{}^m)$ if and only if $(\mathcal{M},\beta_0) \models \varphi(p_1,\dots,p_n)$.*

(iii) *For each m, for each $\gamma \in P$, $(\mathcal{P},\langle m,\gamma\rangle) \models \varphi(p_1{}^m,\dots,p_n{}^m)$ if and only if $(\mathcal{M},\gamma) \models \varphi(p_1,\dots,p_n)$.*

Proof. All the items are proved by induction on the complexity of φ.

(i) The induction basis follows immediately from the definition of the model $\mathcal{P}$. The only nontrivial case in the induction step is if φ is $L\psi(p_1{}^m,\dots,p_n{}^m)$. Assume $(\mathcal{P},\alpha) \models L\psi(p_1{}^m,\dots,p_n{}^m)$. Then for each $\beta \in M$, for the world $\gamma = \{\gamma_m\}_m$ such that for each m $\gamma_m = \beta$, we have $(\mathcal{P},\gamma) \models \psi(p_1{}^m,\dots,p_n{}^m)$. Hence, by the induction hypothesis, we have for each $\beta \in M$, $(\mathcal{M},\beta) \models \psi(p_1,\dots,p_n)$, hence $(\mathcal{M},\alpha_m) \models L\psi(p_1,\dots,p_n)$.

Conversely, assume $(\mathcal{M},\alpha_m) \models L\psi(p_1,\dots,p_n)$. Then for each $\gamma \in M$, $(\mathcal{M},\gamma) \models \psi(p_1,\dots,p_n)$. But for each $\beta \in N$ there is some $\gamma \in M$ such that $\beta_m = \gamma$, hence, applying the induction hypothesis, we obtain for each $\beta \in N$, $(\mathcal{P},\beta) \models \psi(p_1{}^m,\dots,p_n{}^m)$, hence $(\mathcal{P},\alpha) \models L\psi(p_1{}^m,\dots,p_n{}^m)$.

(ii) The induction basis follows from the definition (for all $l \neq m$, values of $p_i{}^m$ in all the worlds of P_l coincide with values of p_i in β_0). In the induction step the only nontrivial case is if the main logical symbol of φ is L.

Assume $(\mathcal{P},\gamma) \models L\psi(p_1{}^m,\dots,p_n{}^m)$. Then we have for each $\alpha \in N$ $(\mathcal{P},\alpha) \models \psi(p_1{}^m,\dots,p_n{}^m)$, hence by (i), for each $\alpha \in M$, $(\mathcal{M},\alpha) \models \psi(p_1,\dots,p_n)$, hence $(\mathcal{M},\beta_0) \models L\psi(p_1,\dots,p_n)$.

Conversely, assume $(\mathcal{M},\beta_0) \models L\psi(p_1,\dots,p_n)$, $\gamma \in P_l$, $l > m$. Assume $\gamma Q\delta$. Then either $\delta \in N$, or $\delta \in P_j$ for some $j \geqslant l$. In the former case by (i), and in the latter case by the induction hypothesis we obtain $(\mathcal{P},\delta) \models \psi(p_1{}^m,\dots,p_n{}^m)$, hence $(\mathcal{P},\gamma) \models L\psi(p_1{}^m,\dots,p_n{}^m)$.

(iii) Again the induction basis follows from the definition, and the only nontrivial case is if φ begins with L.

Assume $(\mathcal{P},\langle m,\gamma\rangle) \models L\psi(p_1{}^m,\dots,p_n{}^m)$ and prove $(\mathcal{M},\gamma) \models L\psi(p_1,\dots,p_n)$. Assume $\gamma R\delta$. Either $\delta \in N$, or $\delta \in P$ (in which case also $\langle m,\gamma\rangle Q\langle m,\delta\rangle$). In the former case by (i), and in the latter case by the induction hypothesis we have $(\mathcal{M},\delta) \models \psi(p_1,\dots,p_n)$, hence $(\mathcal{M},\gamma) \models L\psi(p_1,\dots,p_n)$.

Conversely, assume $(\mathcal{M},\gamma) \models L\psi(p_1,\dots,p_n)$, $\gamma \in P$. Let $\langle m,\gamma\rangle Q\delta$. Then one of three cases is possible:

(1) $\delta \in N$;

(2) for some $l \geqslant m$, $\delta \in P_l$;

(3) for some $\eta \in P$ such that $\gamma R\eta$, $\delta = \langle m,\eta\rangle$.

If (1) holds, we obtain $(\mathcal{P},\delta) \models \psi(p_1{}^m,\dots,p_n{}^m)$ from (i). Assume that (2) holds. We have $\gamma R\beta_0$, hence $(\mathcal{M},\beta_0) \models \psi(p_1,\dots,p_n)$, hence by (ii), $(\mathcal{P},\delta) \models \psi(p_1{}^m,\dots,p_n{}^m)$. If (3) holds, then we have $(\mathcal{P},\delta) \models \psi(p_1{}^m,\dots,p_n{}^m)$ by the induction hypothesis. Thus, we obtain $(\mathcal{P},\gamma) \models L\psi(p_1{}^m,\dots,p_n{}^m)$. □

Proof of Theorem 4.2 (*Continued*). From the claim we conclude that, for each m,

$$(\mathcal{P}, \langle m, \alpha_0 \rangle) \models \neg F(p_1{}^m, \ldots, p_n{}^m). \tag{A.4}$$

Now we show that for each world α of $\mathcal{P}$, $(\mathcal{P}, \alpha) \models A$.

Consider all the members of A. Because in all the worlds β different from $\langle 1, \alpha_0 \rangle$, r is true, we have also $F(p_1{}^1, \ldots, p_n{}^1) \supset r$ is true in all these worlds. On the other hand, according to (A.4), $F(p_1{}^1, \ldots, p_n{}^1)$ is false in $\langle 1, \alpha_0 \rangle$. Thus we have $\mathcal{P} \models F(p_1{}^1, \ldots, p_n{}^1) \supset r$.

All the other members of A are of the form

$$LF(p_1{}^{m+1}, \ldots, p_n{}^{m+1}) \supset C(p_1{}^m, \ldots, p_n{}^m).$$

Let us denote this formula by φ_m. If $\alpha \in N$, then the successor of φ_m is true in α, hence φ_m is true in α. If $\alpha \in P_k$ where $k > m$, then again, according to (ii) of the claim, $(\mathcal{P}, \alpha) \models C(p_1{}^m, \ldots, p_n{}^m)$, hence $(\mathcal{P}, \alpha) \models \varphi_m$. Assume that $\alpha \in P_k$, where $k \leqslant m$. Then we have $\alpha Q \langle (m+1), \alpha_0 \rangle$, and, by (A.4), $(\mathcal{P}, \langle (m+1), \alpha_0 \rangle) \models \neg F(p_1{}^{m+1}, \ldots, p_n{}^{m+1})$. Hence $(\mathcal{P}, \alpha) \models \neg LF(p_1{}^{m+1}, \ldots, p_n{}^{m+1})$, hence $(\mathcal{P}, \alpha) \models \varphi_m$. Thus, we have $\mathcal{P} \models A$, and, because $\mathcal{P}$ is an S4.2-model, $\mathcal{N}$ is not an S4.2-minimal model of A. □

Appendix B. Between S4.2 and SW5

In this section we take a closer look at the segment [S4.2, SW5]. In particular, we show that S4.3 is the largest logic in its range. Also we present infinitely many different logics between S4.2 and S4.4 which are largest in their ranges (and therefore, different). All these logics belong to the family of so-called logics of finite width.

As for the segment [S4.3, S4F], the situation remains unclear. We show that all logics of finite depth contained in S4F are in the same range as S4F. However, there are infinitely, many logics of infinite depth between S4.3 and S4F. We do not know by now if they are nonmonotonically different, or coincide with S4F, or whatever. But because all the extensions of S4.3 are effectively described by Kit Fine [2], we hope that these questions should not be too difficult.

Careful examination of the proof of Theorem 4.2 allows us to separate the properties of a logic and its characterizing class of Kripke frames which are really used in the proof. We will list these properties and exhibit infinitely many logics with these properties.

First, let us introduce some notation. Assume that $\mathcal{F} = (M, R)$ is a Kripke frame, and K be the final cluster of $\mathcal{F}$. By $\mathcal{F}^-$ we will denote the result of removing F from $\mathcal{F}$: $\mathcal{F}^- = (M \setminus K, R \setminus M \times K)$. By $\mathcal{F}^*$ we denote the frame which is a result of concatenating infinitely many copies of $\mathcal{F}$. Formally, let ω be the set of natural numbers. Then $\mathcal{F}^* = \langle \omega \times M, Q \rangle$, where $\langle m, \alpha \rangle Q \langle n, \beta \rangle$ if and only if $m < n$ or $m = n$ and $\alpha R \beta$. By $\mathcal{F}\hat{}\mathcal{G}$ we denote the concatenation of frames $\mathcal{F}$ and $\mathcal{G}$.

Now, it is easy to formulate conditions on logic $\mathcal{S}$ which were really used in the proof of Theorem 4.2.

Theorem B.1. *Assume that $\mathcal{S}$ is a logic contained in* S5, *and $\mathcal{C}$ and $\mathcal{D}$ are classes of frames. Assume that the following hold:*

- *all frames from $\mathcal{C}$ and $\mathcal{D}$ are transitive, reflexive and have a final cluster;*
- *$\mathcal{D}$ is a class of finite frames;*
- *$\mathcal{S}$ is determined by $\mathcal{D}$;*
- *$\mathcal{S}$ is cluster closed;*
- *$\mathcal{S}$ is characterized by $\mathcal{C}$;*
- *for each $\mathcal{F} \in \mathcal{D}$, and for each cluster K, $(\mathcal{F}^{-})^{*\wedge}K \in \mathcal{C}$.*

Then for each logic $\mathcal{T}$ which is contained in S5 *but not contained in $\mathcal{S}$, there is a theory I such that some $\mathcal{T}$-expansion of I is not an $\mathcal{S}$-expansion of I.*

Proof. Careful examination of the proof of Theorem 4.2 shows that, in fact, this, more general, proposition was proved. □

Now it is clear that S4.3 meets the conditions of the theorem: we can take the class of all finite connected frames as $\mathcal{D}$, and the class of all connected frames with the final cluster as $\mathcal{C}$.

Now consider, for each n, the logic J_n which is the result of adding to S4.2 the following axiom schema:

$$\bigwedge_{i=0}^{n} M\varphi_i \supset \bigvee_{0\leqslant i\neq j\leqslant n} M(\varphi_i \wedge M\varphi_j).$$

Intuitively, this schema asserts that there cannot be more than n pairwise incomparable worlds, visible from any world.

A frame (M, R) is said to be *of width n* if there are no worlds $\alpha_1, \ldots, \alpha_{n+1}$ such that for each different i and j, neither $\alpha_i R\alpha_j$, nor $\alpha_j R\alpha_i$. By using standard methods of canonical model and the Lemmon filtration (see [3,6,29]), it is straightforward to prove that J_n is characterized by class of all frames of width n with the final cluster, and is determined by the subclass of all finite frames of this class. Also, it is clear that J_1 coincides with S4.3, and that each I_{n+1} is properly included in I_n. Also, if $\mathcal{F}$ is of width n, then, clearly, $\mathcal{F}^{-*}$ is also of width n, hence all the conditions of theorem B.1 apply to J_n. We summarize all these observations in a proposition.

Proposition B.2. *There is a sequence of logics $\{J_n\}_n$ such that for each n,* S4.2 $\subset J_n$, *J_{n+1} is properly contained in J_n, J_1 is* S4.3, *and each J_i is the largest logic in its range.*

There is another well-known class of logics, namely logics of finite depth. A transitive and reflexive frame (M, R) *has depth at most n* , if there is no sequence $\alpha_1, \ldots, \alpha_{n+1}$ such that for each i, $1 \leqslant i \leqslant n$, $\alpha_i R\alpha_{i+1}$, but not $\alpha_{i+1} R\alpha_i$. Let I_n be a logic determined by the class of all frames of depth at most n. All these logics are different and can be axiomatized [29]. Also, using the methods of [8,19,29], it is easy to show that each I_n is also *characterized* by the class of all frames of depth n. Also, logics $I.2_n$, which are obtained from I_n by adding schema 2, are characterized by classes of frames of depth n with the final cluster. It was proved in [16] that all logics between S4 and S4F have the same expansions for finite theories. The careful analysis of the proof shows, that it is also good for any logic I_n, $n \geqslant 2$, but without restriction for finite theories. Using minimal model semantics, even simpler proof can be given.

Proposition B.3. *For each $n \geqslant 2$, for each A and each T, T is an I_n-expansion of A if and only if T is an* S4F*-expansion of A.*

Proof. According to Proposition 4.1, nonmonotonic I_n coincides with nonmonotonic $I.2_n$ which is I_n with schema 2, so it suffices to prove the proposition for $I.2_n$ in place of I_n. $I.2_n$ is characterized by the class of all frames of depth n which have the final cluster. This class is cluster closed, so we can apply the minimal model semantics.

Clearly, $I.2_n$ is a subsystem of S4F (because all S4F-frames are of depth 2), hence each $I.2_n$-expansion of A is also an S4F-expansion of A. Conversely, assume that T is an S4F-expansion of A. Let us prove that T is an $I.2_n$-expansion, too. Let $\mathcal{M} = (M, V)$ be the canonical S5-model for T. It suffices to prove that $\mathcal{M}$ is K_n-minimal for A, where K_n is a class of all models of depth at most n. Assume that $\mathcal{N} = (N, R, U)$ is a Kripke model of depth at most n, that $\mathcal{N} \sqsubset \mathcal{M}$. Then for some $\alpha \in N$, V_α is different from all V_β for all $\beta \in M$. For each $\beta \in N$, by $rk(\beta)$ (rank of β) we will denote the maximal number m such that there exists a sequence of worlds $\beta_1, \dots, \beta_m$ such that for each i, $\beta_i R \beta_{i+1}$, but not $\beta_{i+1} R \beta_i$. Since $\mathcal{N}$ is of depth n, $rk(\beta)$ is defined and does not exceed n for each $\beta \in N$. Now let m be the smallest number such that for some α of rank m, V_α differs from all V_γ for all $\gamma \in M$, and α is some world of rank m with this property. Let $M_\alpha = \{\gamma \in N\colon\ \alpha R \gamma \wedge \gamma R \alpha\}$. Consider the S4F-model $\mathcal{M}_\alpha = \langle M_\alpha \cup M, Q, U\rangle$, where $Q = (M_\alpha \times (M_\alpha \cup M)) \cup (M \times M)$. It follows from the choice of α and from Proposition 2.2 that for each $\beta \in N \setminus (M_\alpha \cup M)$, such that $\alpha R \beta$, for each formula η, $(\mathcal{M}, \beta) \models L\eta$ if and only if $\mathcal{M} \models L\eta$. Using this observation, it is straightforward to prove by induction on φ that for each φ, for each $\beta \in M_\alpha$, $(\mathcal{N}, \beta) \models \varphi$ if and only if $(\mathcal{M}_\alpha, \beta) \models \varphi$. Hence we obtain $\mathcal{M}_\alpha \models A$, and $(\mathcal{M}_\alpha, \alpha) \models \neg\psi$, that is, $\mathcal{M}_\alpha \sqsubset \mathcal{M}$, hence $\mathcal{M}$ is not an S4F-minimal model of A, which contradicts our assumption that T is an S4F-expansion of A. So we assumed that T is not an $I.2_n$-expansion of A and got a contradiction, hence T is an $I.2_n$-expansion of A. $\square$

Thus, the whole bunch of logics of finite depth, contained in S4F, is in the same range. Unfortunately, this gives us no final answer to the question, how many different modal logics are between S4.3 and S4F, because among such logics there are infinitely many logics of infinite depth. Fine [2] gives an effective description of these logics in terms of Kripke frames which determine the logics. Unfortunately, these classes seemingly do not characterize corresponding logics, and do not meet conditions of Theorem B.1, so methods of this paper are not applicable directly to that case.

Interestingly, the situation for logics of depth 2 (that is, those between SW5 and S4F) is quite different: there are infinitely many nonmonotonically different logics among them.

Proposition B.4. *Let* $\mathrm{SW5}_m$ *be a logic determined by a class of all* S4F*-frames consisting of two clusters, where the lower cluster has at most m worlds. Then for each*

m, SW_{m+1} is properly contained in SW_m and there is a theory A_m such that some SW_m-expansion of S_m is not an SW_{m+1}-expansion of A_m.

The proof is essentially the same as the proof of Theorem 3.5. The complete proof is, however, rather long, because to apply minimal model semantics we must show that $SW5_m$ is not only determined, but also characterized by the class of all S4F-frames where the cardinality of the lower cluster does not exceed n. This can be done by using the canonical model method, see [8].

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