Artificial Intelligence 232 (2016) 76–113

Contents lists available at ScienceDirect

Artificial Intelligence

www.elsevier.com/locate/artint

# Characteristic function games with restricted agent interactions: Core-stability and coalition structures

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ARTICLE INFO

*Article history:*
Received 15 January 2014
Received in revised form 14 December 2015
Accepted 17 December 2015
Available online 23 December 2015

*Keywords:*
Coalitional games
Solution concepts
Computational complexity
Treewidth
Marginal contribution networks

ABSTRACT

In many real-world settings, the structure of the environment constrains the formation of coalitions among agents. These settings can be represented by characteristic function games, also known as *coalitional games*, equipped with interaction graphs. An interaction graph determines the set of all feasible coalitions, in that a coalition $C$ can form only if the subgraph induced over the nodes/agents in $C$ is connected. Our work analyzes stability issues arising in such environments, by focusing on the *core* as a solution concept, and by considering the coalition structure viewpoint, that is, without assuming that the grand-coalition necessarily forms.
The complexity of the coalition structure core is studied over a variety of interaction graph structures of interest, including complete graphs, lines, cycles, trees, and nearly-acyclic graphs (formally, having bounded treewidth). The related stability concepts of the *least core* and the *cost of stability* are also studied. Results are derived for the setting of *compact* coalitional games, i.e., for games that are implicitly described via a compact encoding, and where simple calculations on this encoding are to be performed in order to compute the payoff associated with any coalition. Moreover, specific results are provided for compact games defined via *marginal contribution networks*, an expressive encoding mechanism that received considerable attention in the last few years.


## 1. Introduction

### 1.1. Coalitional games and interaction graphs

Cooperative game theory aims to provide models of cooperation among interacting agents. One of the prevalent classes of games used within this framework is the class of *characteristic function games*. A characteristic function game (CFG) is defined over a set $N = \{1, \dots, n\}$ of agents and is determined by a *payoff function* $v : 2^N \mapsto \mathbb{R}$, such that, for each *coalition* $C$, i.e., for any non-empty set $C \subseteq N$ of agents, the value $v(C)$ expresses the payoff that the members of $C$ can jointly achieve by cooperating among themselves [1,2]. The outcome is an *allocation*, i.e., a payoff vector $\boldsymbol{x} = \langle x_1, \dots, x_n \rangle \in \mathbb{R}^n$ assigning some payoff to each agent $i \in N$. Characteristic function games are also known as *coalitional games with transferable utility*, as it is assumed that the agents forming a coalition $C$ can distribute the payoff $v(C)$ among themselves in any way. The

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http://dx.doi.org/10.1016/j.artint.2015.12.005
0004-3702/

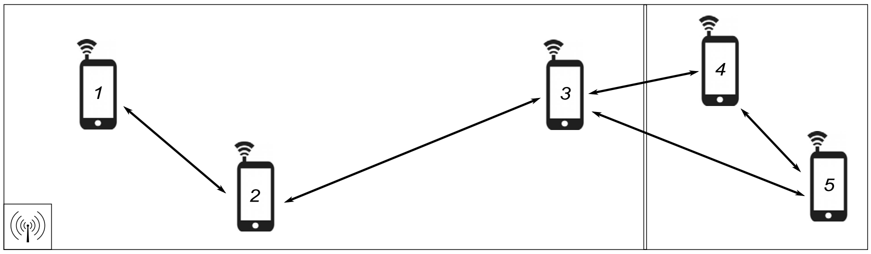


**Fig. 1.** Coalitional games in Example 1.1 and Example 1.3.

question of interest for such games, is to identify desirable (e.g., fair and stable) outcomes in terms of worth distributions, which are called *solution concepts*.

Characteristic function games provide a rich framework for understanding and reasoning about cooperative actions and have a wide spectrum of applications in different areas of research. Indeed, while they have been traditionally grounded in the game theory and economic literature, they have gained popularity in the context of multi-agent systems and artificial intelligence research as a means of studying interactions among autonomous agents (see, e.g., [3–9]). Moreover, they have recently attracted attention in engineering too, because of their use in the design of intelligent protocols and middleware algorithms for wireless communication networks [10–12]. As an example, we next illustrate a *wireless cooperative file sharing system*, where mobile subscribers cluster together by downloading (portions of) files of interest over long-range cellular links, and by exchanging them over a wireless ad-hoc network (see, e.g., [13]).

**Example 1.1.** Consider the setting with three mobile users (hence, $N = \{1, 2, 3\}$) that is illustrated in the left part of Fig. 1, and a function $v$ such that, for each coalition $C \subseteq N$, $v(C)$ is meant to express the payoff that users in $C$ can jointly achieve when clustering together and cooperating to download the given file of interest. In particular, assume that $v(\{1\}) = 10$, $v(\{2\}) = 5$, $v(\{3\}) = 4$, $v(\{1, 2\}) = 17$, $v(\{2, 3\}) = 10$, and $v(\{1, 2, 3\}) = 22$. Intuitively, the payoff of each coalition $C$ is meant to express the sum of the utilities that the agents in $C$ get when the file is downloaded minus the cost that they overall incur for the download, with this cost being proportional to the time required and to certain technological features, such as bandwidth and energy consumption.

For instance, when 1 and 2 cluster together, each of them can download only half of the file and then share the portion via the wireless connection. While doing so, each of them increases the throughput due to the better performance of the wireless network when compared to the cellular links, and reduces the overall downloading costs. This is reflected in the payoff function $v$, which is such that $v(\{1, 2\}) \geq v(\{1\}) + v(\{2\})$. Similarly, 2 and 3 can cluster together leading to the payoff $v(\{2, 3\}) \geq v(\{2\}) + v(\{3\})$.

In contrast, note that users 1 and 3 are outside of each other's transmission range, hence it is impossible for them to cluster together (and, in fact, the payoff function is not specified for this coalition). However, user 2 might still act as a bridge between them, so that when all users join together, the resulting payoff is $v(\{1, 2, 3\}) = 22$. Indeed, it is advantageous for the users to cluster all together, because $v(\{1, 2, 3\}) \geq v(\{1, 2\}) + v(\{3\})$, $v(\{1, 2, 3\}) \geq v(\{1\}) + v(\{2, 3\})$, and $v(\{1, 2, 3\}) \geq v(\{1\}) + v(\{2\}) + v(\{3\})$. While doing so, each user gets the payoff that can be achieved by downloading the whole file alone, and a surplus of $v(\{1, 2, 3\}) - (v(\{1\}) + v(\{2\}) + v(\{3\})) = 22 - (10 + 5 + 4) = 3$ still remains to be divided among them. $\lhd$

As the above example demonstrates, characteristic function games might be defined within an environment imposing restrictions on the formation of coalitions. Indeed, users 1 and 3 are outside each other's transmission range, and the coalition $\{1, 3\}$ cannot form.

In general, for reasons that might range from physical limitations and constraints to legal banishments, certain agents might not be allowed to form coalitions with certain others. Sensor networks, communication networks, or transportation networks, within which units are connected through bilateral links, provide natural settings for such classes of games. In many multiagent coordination settings, agents might be restricted to communicate or interact with only a subset of other agents in the environment, due to limited resources or existing physical barriers. Another example is provided by hierarchies of employees within an enterprise, where employees at the same level work together under the supervision of a boss, i.e., of an employee at the immediately higher level in the hierarchy. In all these settings, the environment can be seen to possess some *structure* that forbids the formation of certain coalitions. This can be formalized as an *interaction graph* $G = (N, E)$, an undirected graph, where agents are transparently viewed as nodes so that a coalition $C$ is *feasible*, i.e., allowed to form, only if the subgraph of $G$ induced over the nodes of $C$ is connected [14]. For instance, it is immediate to check that the graph shown in the left part of Fig. 1, is the interaction graph associated with the game of Example 1.1, where all coalitions are allowed to form but $\{1, 3\}$.

Note that when an interaction graph $G$ is the *complete graph* over $N$, then $G$ induces no structural restrictions and we are back to the basic setting where a game is completely specified by its payoff function (and all coalitions are allowed to form). Hence, the setting with interaction graphs generalizes the basic one.

### 1.2. Stability and coalition structures

The main solution concept adopted in the literature in order to deal with the problem of stability in characteristic function games (i.e., what are the incentives for the agents to stay in formed coalitions), is, arguably, the *core* [15,16].

In settings without any restriction on the possible interactions among the agents, the core is defined as the set of all allocations that are stable because there is no coalition having an incentive to deviate, that is, as the set

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(C) \geq v(C), \text{ for each coalition } C \subseteq N\},$$

where $x(C)$ is a shorthand for $\sum_{i \in C} x_i$. Note that it is implicitly assumed here that the *grand-coalition* of *all* players in $N$ forms and, accordingly, the solution concept suggests how the total payoff $v(N)$ can be divided among them in a way that is stable [2].

In the presence of an interaction graph, the core can be smoothly adapted so as to focus only on feasible coalitions [17–19], as illustrated below—formal definitions are provided in Section 2.

**Example 1.2.** The core of the characteristic function game defined in Example 1.1 consists of all allocations $\langle x_1, x_2, x_3 \rangle \in \mathbb{R}^3$ that are solutions to the following set of linear (in)equalities:

$$\begin{cases} x_1 + x_2 + x_3 = 22 \\ x_1 + x_2 \geq 17 \\ x_2 + x_3 \geq 10 \\ x_1 \geq 10 \\ x_2 \geq 5 \\ x_3 \geq 4 \end{cases}$$

Note that, in the example, the core contains infinitely many allocations (e.g., the allocation $\langle 12, 6, 4 \rangle$ is in the core). However, it is instructive to point out that the core of a game might as well be empty. For instance, the reader might check that this is the case for the modified game where $v(\{1,2\}) = 18$, $v(\{2,3\}) = 12$, and by assuming that the coalition $\{1,3\}$ is also allowed to form, with $v(\{1,3\}) = 17$. $\lhd$

So far we have considered only allocations of the value $v(N)$, i.e., we have assumed that the grand coalition will form in the possible outcomes of the game. The formation of the grand-coalition is indeed natural in many settings. This is for example the case for *superadditive* games, which means that whenever three feasible coalitions $S$, $T$, and $S \cup T$ are given, with $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$ holds. For instance, the reader can check that the game defined in Example 1.1 is superadditive. In a plethora of realistic application settings however, the emergence of the grand coalition is either not guaranteed, plainly impossible, or even perceivably harmful. This might happen because of issues ranging from normative considerations, to information (observability) imperfections, and to technological constraints (cf. [20]). Moreover, additional motivations emerge when the problem is looked at from an economics perspective [21]. Under such circumstances, it is very natural for the agents to organize themselves in a *coalition structure*, i.e., in a partition $\Pi$ of $N$, consisting of disjoint and exhaustive feasible coalitions.

**Example 1.3.** Consider a slight modification of the setting discussed in Example 1.1, where two additional mobile users join the environment as illustrated in the right part of Fig. 1. In particular, assume that $v(\{4\}) = v(\{5\}) = 5$, $v(\{4,5\}) = 11$, and $v(C \cup C') = v(C) + v(C') - 1$, for each pair of non-empty sets $C \subseteq \{1,2,3\}$ and $C' \subseteq \{4,5\}$ such that $C \cup C'$ is feasible (according to the interaction graph depicted in the figure). Intuitively, in this modified scenario, in order to cluster together the two new mobile users with the rest of the users, we have to pay some cost that is no longer compensated by the network capabilities of the wireless connection. In particular, we have $v(\{1,2,3,4,5\}) < v(\{1,2,3\}) + v(\{4,5\})$, i.e., the modified game is no longer superadditive. Therefore, the five users will likely organize themselves in the coalition structure $\Pi = \{\{1,2,3\}, \{4,5\}\}$, with the two coalitions $\{1,2,3\}$ and $\{4,5\}$ downloading independently the file of interest. $\lhd$

As demonstrated above, the *social welfare* of a coalition structure $\Pi$, i.e., the total available payoff $\sum_{C \in \Pi} v(C)$, might happen to be greater than the payoff associated with the grand-coalition. Whenever this is the case, the classic solution concepts are not appropriate. Indeed, agents will likely organize themselves in the coalition structure achieving the maximum social welfare, so that stable outcomes have to be characterized from the "coalition structure" perspective, as first suggested by [21]. According to this view, the *core* of a characteristic function game finds a counterpart in the *coalition structure core*. Intuitively, at the coalition structure core, if $\Pi$ is a coalition structure achieving the maximum social welfare, then it holds that, for each coalition $C$ in $\Pi$, the members of $C$ distribute the value $v(C)$ among themselves only.

Coming back to Example 1.3, it is immediate to check that the maximum social welfare in the game is achieved by the coalition structure $\Pi = \{\{1,2,3\}, \{4,5\}\}$. An allocation $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ in the coalition structure core must be such that $x_1 + x_2 + x_3 = 22$ and $x_4 + x_5 = 11$. That is, the members of $\{1,2,3\}$ and $\{4,5\}$ distribute the payoff $v(\{1,2,3\})$ and $v(\{4,5\})$, respectively, among themselves only. Moreover, as usual, the allocation is required to be stable, i.e., $x(C) \geq v(C)$, for each feasible coalition $C \subseteq \{1,2,3,4,5\}$. For instance, $\langle 12, 6, 4, 6, 5 \rangle$ is an allocation with these properties.

### 1.3. Game encodings and complexity issues

In order to fully specify a characteristic function game, we need to explicitly list all associations of coalitions with their payoffs. However, this requires exponential space w.r.t. the number of agents and it is often hardly applicable in practice. Motivated by this observation, *compact* game-encoding mechanisms have been proposed. Building on the seminal papers by Megiddo [22], by Kalai and Zemel [23], and by Deng and Papadimitriou [24], the idea is to equip a characteristic function game with a structure (e.g., a combinatorial structure, a graphical structure, or a logical theory) and to define a function that, given this structure and any set $C$ of agents, is able to return (usually in polynomial time) the payoff $v(C)$. As an example, in the games studied by Deng and Papadimitriou [24], the value $v(C)$ is determined by a weighted graph whose nodes are the agents and equals the sum of the weights of the edges in the subgraph induced over $C$. Another remarkable example of compact games that received considerable attention in the last few years is that of games encoded via *marginal contribution networks (MC-nets)*, proposed by Ieong and Shoham [25]. Marginal contribution networks provide a simple, yet very powerful method to specify characteristic function games. Indeed, any characteristic function game can be encoded in terms of a marginal contribution network, and exponentially more compact encodings can be defined in many practical cases.

While an explicit encoding makes the input sizes so large that computing the various solution concepts is trivially—and artificially—easy, this is no longer the case over compact encodings. Indeed, most of the research on computational aspects of coalitional games is focused on assessing the amount of resources needed to compute these concepts over different compact-game encodings, and several deep and useful results have already been achieved (see, for instance, [26,24,27,25,5], and [28] for a review).

In fact, studying the complexity of realistic cooperative game settings, and identifying tractable game instances by limiting some source of intractability is crucial for the design of general solution algorithms. As such, the problem of characterizing the precise computational complexity of coalitional games is a well-established topic of research in the artificial intelligence community. To give an idea of some practical consequences of complexity results, assume we would like to solve an **NP**-complete problem on a standard (deterministic) machine. The theory suggests that (unless $\mathbf{P} = \mathbf{NP}$) any algorithm has to explore the search space via some kind of backtracking mechanism, which in the worst case exhaustively generates all the potential solutions. For a problem that is $\mathbf{\Delta_2^P}$-hard, things are even worse as the backtracking mechanism might require to be invoked by a master algorithm polynomially many times. In fact, these problems might be solved by means of algorithms that make use of *SAT solvers*[1] to address the computation of the **NP**-oracle calls. Finally, note that any $\mathbf{\Sigma_2^P}$-complete problem exhibits two orthogonal sources of intractability. Then, we cannot design any *flat-backtracking* algorithm for our problem, since any algorithm with a search-space tree having a polynomial number of levels should solve a nested co-**NP**-hard problem to check whether a leaf node is a solution or not. In these cases, a practical solution might be given by encoding the problem of interest in some expressive logic formalisms, such as in *Answer Set Programming*, and to use the solvers[2] that have been developed for them.

### 1.4. Summary of contributions

With a few notable exceptions discussed in the following sections, complexity studies about coalitional games concerned so far mostly the setting where all coalitions are unconditionally allowed to form. Furthermore, most works tacitly assume that the goal is to distribute over the agents the total payoff available to the grand-coalition, even when games are not superadditive. Moving from this background, the goal of the paper is to provide a clear picture of the complexity of core-related problems over compact games under restricted cooperation, as specified by an interaction graph. In particular, we consider environments where agent interactions can be restricted according to a variety of interaction graph structures of interest, and we look at the computational issues from the coalition structure viewpoint, that is, without assuming that the grand-coalition necessarily forms. In our analysis, the input for any algorithmic problem includes (at least) the encoding of a game whose characteristic function $v$ is computable in time polynomial in the size of the given representation. As a special instance of this general setting, we also provide results that are specific for MC-nets. In more detail:

- ► We study how the structure of the underlying interaction graphs affects the complexity of checking whether a given outcome belongs to the coalition structure core (in short: CS-Core-Membership) and of deciding whether the core is non-empty (in short: CS-Core-NonEmptiness). We analyze these problems over lines, cycles, trees, nearly-acyclic graphs, formally, graphs having *bounded treewidth* [29] (in short: BTW), and complete graphs. In the results we have derived (also in those listed in the other points below), islands of tractability and membership results are identified for arbitrary compact games, whereas intractability results are shown to hold even on MC-nets. A summary of these results is reported in Table 1. The most technical proofs are those establishing the results about CS-Core-Membership and CS-Core-NonEmptiness over complete interactions graphs. Proofs for trees and graphs having bounded treewidth follow with suitable adaptations, whereas the analysis of lines and cycles is based on simple combinatorial properties.

[1] See http://www.satcompetition.org/.

[2] See https://www.mat.unical.it/aspcomp2014/.

**Table 1**
Results for arbitrary compact games. Hardness results are shown to hold even on MC-nets.

| Problem | Lines | Trees | Cycles | BTW | Complete |
|---|---|---|---|---|---|
| CS-Core-Membership | in **P**<br>6.1 | co-**NP**-c<br>[9]+5.4 | in **P**<br>6.3 | co-**NP**-c<br>[9]+5.4 | co-**NP**-c†<br>[9]+4.9 |
| CS-Core-NonEmptiness | in **P**<br>[30] | in **P**<br>[19] | in **P**<br>6.3 | $\mathbf{\Delta_2^P}$-c<br>4.1+5.3 | $\mathbf{\Delta_2^P}$-c<br>4.1+4.4 |
| CS-Core-Find | in **FP**<br>6.1 | **NP**-hard<br>5.9 | in **FP**<br>6.3 | **NP**-hard<br>from 5.3 | **NP**-hard<br>from 4.4 |
| CS-CoS | in **P**<br>[30] | in **P**<br>[19] | in **P**<br>6.3 | $\mathbf{\Delta_2^P}$-c<br>4.1+5.3 | $\mathbf{\Delta_2^P}$-c<br>4.1+4.4 |
| CS-LCV$^-$ | in **P**<br>6.1 | $\mathbf{\Delta_2^P}$-c<br>4.3+5.7 | in **P**<br>6.3 | $\mathbf{\Delta_2^P}$-c<br>4.3+5.7 | $\mathbf{\Delta_2^P}$-c<br>4.3+4.4 |
| CS-LCV$^+$ | in **P**<br>6.1 | in **P**<br>5.7 | in **NP**<br>6.3 | in $\mathbf{\Sigma_2^P}$, $\mathbf{\Delta_2^P}$-hard<br>4.3+4.4 | $\mathbf{\Sigma_2^P}$-c<br>4.3+4.6 |

Each entry is equipped either with the theorem number where the result is proven (or can be derived from), or with the reference to the work in the literature where a proof of the result can either be found or derived from with minor effort. For the completeness results (-c), such labels are given in the form M+H, with M and H being associated with the membership and the hardness result, respectively.

† The result has been independently proven in [31]. Our proof actually shows that hardness holds even when all rules are associated with non-negative values and at most one negated literal occurs in each rule (see Section 4.2). Therefore, it charts the boundary of tractability, given that, without negated literals, the resulting game encoding is known to be tractable.

- CS-Core-Membership and CS-Core-NonEmptiness are decision problems. In addition to them, we consider the problem of *computing* an allocation in the coalition structure core (in short: CS-Core-Find). Of course, whenever CS-Core-NonEmptiness is intractable, then CS-Core-Find is intractable, too. However, it turns out that CS-Core-Find remains intractable on trees (see again Table 1, where **FP** stands for the class of search problems solvable in polynomial time). This is rather interesting because the coalition structure core is always non-empty over trees [19]. In fact, the result was already known to hold over classes of games equipped with arbitrary functions computable in polynomial time [32], and here it is strengthened to games encoded as MC-nets. Tractability results are eventually established for lines and cycles.
- We consider the *cost of stability* [33], which is a core-related stability criterion defined as the minimum worth that a benevolent external party must supply in order for the game to have a non-empty core. We study the CS-CoS problem of deciding whether this minimum worth is at most a given threshold. While being a weaker concept of stability than the classical concept of core, it emerges that it is characterized by the same intrinsic computational complexity (see again Table 1). In particular, results for CS-CoS coincide with those for CS-Core-NonEmptiness.
- We consider another core-related stability criterion, that is, the *least core* [34], which is an approximate version of the core, with an additional penalty imposed for leaving the grand coalition. We define *a natural generalization of this concept to the case where coalition structures are taken into account*, and we study its (analytical) properties. From the computational viewpoint, we consider the problem CS-LCV, which asks whether a game can be made stable if a given penalty $\varepsilon$ is considered. Note that in the literature about the least core, the range of $\varepsilon \leq 0$ is also considered (i.e., incentives to leave the grand-coalition), with the aim of singling out the most stable subset of the core. Accordingly, towards a finer-grained analysis, we consider the variants CS-LCV$^-$ and CS-LCV$^+$, where we require $\varepsilon \leq 0$ and $\varepsilon \geq 0$, respectively. For $\varepsilon = 0$, the problem inherits all hardness results from CS-Core-NonEmptiness. However, more stringent and somehow surprising complexity results are exhibited, via a number of rather elaborate reductions even in the basic cases of lines and cycles. The analysis shows that CS-LCV$^+$ is intrinsically more complex than CS-LCV$^-$. Results are again in Table 1.
- We also analyze the complexity of superadditive games, for which the grand-coalition always forms (and the coalition structure core reduces to the usual concept of the core considered in the literature). Our results, reported in Table 2, demonstrate that superadditivity does not help too much to enlarge the islands of tractability. In fact, tractability results coincide with those discussed for arbitrary games. However, all hardness results are now actually completeness results for the class co-**NP**, rather than for the (likely larger) classes $\mathbf{\Delta_2^P}$ and $\mathbf{\Sigma_2^P}$.
- Finally, motivated by the observation that each node in a line or a cycle (over which the problems we considered are tractable) has *degree* at most 2, we embark on the classification of the complexity of core-related questions based on the maximum degree of the nodes. Results are summarized in Table 3. While all problems are tractable when the degree of any node is at most 2, they suddenly become intractable when the maximum degree is 3 or higher. Actually, for arbitrary games when the degree is at most 2, only an upper bound on the complexity of CS-LCV$^+$ is reported. In fact, these results are corollaries of those described in Table 1 and Table 2. However, to achieve tight bounds that hold even

**Table 2**
Results for superadditive compact games. The meaning of the symbols is as in Table 1.

| Problem | Lines | Trees | Cycles | BTW | Complete |
|---|---|---|---|---|---|
| CS-CORE-MEMBERSHIP | in **P**<br>6.1 | co-**NP**-c<br>[9]+5.4 | in **P**<br>6.3 | co-**NP**-c<br>[9]+5.4 | co-**NP**-c$^{\dagger}$<br>[9]+4.9 |
| CS-CORE-NONEMPTINESS | in **P**<br>[30] | in **P**<br>[19] | in **P**<br>6.3 | co-**NP**-c<br>[9]+5.6 | co-**NP**-c$^{\dagger}$<br>[9]+4.10 |
| CS-CORE-FIND | in **FP**<br>6.1 | in **FP**<br>cf. [19] | in **FP**<br>6.3 | **NP**-hard<br>from 5.6 | **NP**-hard<br>from 4.10 |
| CS-CoS | in **P**<br>[30] | in **P**<br>[19] | in **P**<br>6.3 | co-**NP**-c<br>4.12+5.6 | co-**NP**-c<br>4.12+4.10 |
| CS-LCV$^{-}$ | in **P**<br>6.1 | co-**NP**-c<br>5.12+5.11 | in **P**<br>6.3 | co-**NP**-c<br>4.12+5.6 | co-**NP**-c<br>4.12+4.10 |
| CS-LCV$^{+}$ | in **P**<br>6.1 | in **P**<br>5.7 | in **P**<br>6.5 | co-**NP**-c<br>4.12+5.6 | co-**NP**-c<br>4.12+4.10 |

**Table 3**
Complexity classification based on the degree of the nodes in the interaction graphs.

| Problem | Superadditive | | Arbitrary | |
|---|---|---|---|---|
| | Degree $\leq 2$ | Arbitrary$^{\ddagger}$ | Degree $\leq 2$ | Arbitrary$^{\ddagger}$ |
| CS-CORE-MEMBERSHIP | in **P** | co-**NP**-c | in **P** | co-**NP**-c |
| CS-CORE-NONEMPTINESS | in **P** | co-**NP**-c | in **P** | $\mathbf{\Delta_2^P}$-c |
| CS-CORE-FIND | in **FP** | **NP**-hard | in **FP** | **NP**-hard |
| CS-CoS | in **P** | co-**NP**-c | in **P** | $\mathbf{\Delta_2^P}$-c |
| CS-LCV$^{-}$ | in **P** | co-**NP**-c | in **P** | $\mathbf{\Delta_2^P}$-c |
| CS-LCV$^{+}$ | in **P** | co-**NP**-c | in **NP** | $\mathbf{\Sigma_2^P}$-c |

$^{\ddagger}$ Hardness results hold even for interaction graphs where each node has degree at most 3.

when each node has degree at most 3, a mechanism to encode any structurally-restricted game into an "equivalent" one enjoying this property is exhibited.

**Organization.** The rest of the paper is structured as follows. Section 2 formalizes the main concepts considered in our analysis, while Section 3 presents characterization results—useful for deriving the complexity results in the sequel. Results over complete interaction graphs and over graphs with bounded treewidth appear in Sections 4 and 5, respectively. Results over lines and cycles are shown in Section 6. Finally, Section 7 reviews related work, while conclusions and open research issues are discussed in Section 8. For the sake of readability, proofs of claims are reported in Appendix A.

## 2. Formal framework

In this section, we discuss the stability concepts considered in the paper and present the computational setting we adopt for the analysis of their complexity. To this end, we start the formalization by introducing some background and the relevant notation.

Let $G = (N, E)$ be an interaction graph over a set $N = \{1, \ldots, n\}$ of agents. We assume that $G$ is connected, so that there are no structural constraints on the formation of the grand-coalition (otherwise, our results apply to each of the connected components of $G$). The set of all coalitions that are *feasible* over $G$, i.e., the coalitions $C \subseteq N$, with $C \neq \emptyset$, such that the subgraph of $G$ induced over the nodes of $C$ is connected, is denoted by $\mathcal{F}(G)$. A *coalition structure* over $G$ is a *partition* of $N$, say $\Pi = \{C_1, \ldots, C_k\}$, into feasible and disjoint coalitions; that is, we have: $C_1 \cup \cdots \cup C_k = N$; $C_i \cap C_j = \emptyset$, for each pair $i, j \in \{1, \ldots, k\}$ with $i \neq j$; and $C_i \in \mathcal{F}(G)$, for each $i \in \{1, \ldots, k\}$. The set of all possible coalition structures over $G$ is hereinafter denoted by $\mathcal{CS}(G)$.

We consider games that are structurally restricted, in that they are equipped with interaction graphs. Intuitively, two agents can belong to the same coalition only if they are connected with a path in the interaction graph. Formally, if $v : 2^N \mapsto \mathbb{R}$ is a function assigning payoffs to the coalitions that can be built over $N$, where conventionally $v(\emptyset) = 0$, then the triple $\Gamma = \langle N, v, G\rangle$ is a *structurally-restricted* characteristic function game.

Now, assume that $\Gamma = \langle N, v, G\rangle$ is a given game, and let $\Pi$ be a coalition structure in $\mathcal{CS}(G)$. The *social welfare* of $\Pi$ in $\Gamma$ is the value $\sum_{C\in\Pi} v(C)$, denoted by $SW(\Pi)$. The maximum social welfare that can be attained in $\Gamma$ over all the possible coalition structures is denoted by $SW_{opt}(\Gamma)$. The set of all coalition structures $\Pi \in \mathcal{CS}(G)$ such that $SW(\Pi) = SW_{opt}(\Gamma)$ is denoted by CS-$opt(\Gamma)$.

Recall that for a payoff allocation $\boldsymbol{x} = \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$ to the agents, and for a coalition $C \subseteq N$, we denote by $x(C)$ the sum $\sum_{i \in C} x_i$. An allocation $\boldsymbol{x} \in \mathbb{R}^n$ is called an *imputation* for a coalition structure $\Pi$ in $\Gamma$ if it is *efficient*, i.e., $x(C) = v(C)$ for each $C \in \Pi$, and *individually rational*, i.e., $x_i \geq v(\{i\})$ for all $i \in N$. Note that if $\boldsymbol{x}$ is an imputation for $\Pi$, then $x(N) = SW(\Pi)$ holds. In words, imputations for $\Pi$ are allocations where each coalition in $\Pi$ distributes the available payoff among its members only, in a way that each agent gets at least the payoff it would get alone, without participating to the game. The set of all imputations for $\Pi$ is henceforth denoted by $\mathcal{I}(\Pi)$.

### *2.1. Stability concepts*

**The core.** The main stability concept considered in the literature is the core. In fact, we consider the notion modified by [17–19] in order to deal with the restrictions imposed by an underlying interaction graph, so that only feasible solutions are taken into account. Accordingly, if $\Gamma = \langle N, v, G \rangle$ is a structurally-restricted game with $|N| = n$, then its core is defined as the set $core(\Gamma) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(C) \geq v(C), \forall C \in \mathcal{F}(G)\}$.

The above notion implicitly assumes that the grand-coalition always forms. In particular, if $\boldsymbol{x}$ is in the core, then $\boldsymbol{x}$ is an imputation for $\{N\}$. By removing this assumption, we get a more general notion of the core where coalition structures are taken into account. We refer to this notion as the *coalition structure core* of $\Gamma$, formally defined as the set

$$\text{CS-}core(\Gamma) = \{(\Pi, \boldsymbol{x}) \mid \Pi \in \mathcal{CS}(G), \boldsymbol{x} \in \mathcal{I}(\Pi), \text{and } x(C) \geq v(C),\ \forall C \in \mathcal{F}(G)\}. \tag{1}$$

Note that Aumann and Dreze [21] proposed an extension for the standard notion of the core (i.e., when all coalitions are in principle feasible ones) where a coalition structure $\bar{\Pi}$ is fixed beforehand. In that setting, the question is whether an imputation for $\bar{\Pi}$ exists which is stable—formally, this amounts to use $\{\bar{\Pi}\}$ in place of $\mathcal{CS}(G)$ in Equation (1). Accordingly, the above definition of the coalition structure core can be viewed as the natural generalization of the proposal by Aumann and Dreze to the case where players might organize themselves in any arbitrary coalition structure taken from $\mathcal{CS}(G)$.

The following is a useful and intuitive property of the coalition structure core, which is in fact well-known for games where all coalitions are feasible ones. The adaptation to our setting is straightforward and the proof is reported for the sake of completeness only.

**Fact 2.1.** *Let $\Gamma = \langle N, v, G \rangle$ be a structurally-restricted characteristic function game, and let $(\Pi, \boldsymbol{x}) \in$ CS-$core(\Gamma)$. Then, $\Pi \in$ CS-$opt(\Gamma)$.*

**Proof.** Suppose $(\Pi, \boldsymbol{x}) \in$ CS-$core(\Gamma)$. For each coalition structure $\Pi' \in \mathcal{CS}(G)$ and for each coalition $C \in \Pi'$, which is actually a feasible one by the definition of a coalition structure, $x(C) \geq v(C)$ holds by (1). By the fact that coalitions in $\Pi'$ are disjoint and they exclusively cover $N$, we get $x(N) = \sum_{C \in \Pi'} x(C)$, and hence $x(N) \geq \sum_{C \in \Pi'} v(C) = SW(\Pi')$. That is, $x(N)$ is an upper bound on the social welfare that can be attained by any feasible coalition structure in $\Gamma$. Since $x(N) = SW(\Pi)$ holds because $\boldsymbol{x} \in \mathcal{I}(\Pi)$, we conclude that $x(N)$ coincides with $SW_{opt}(\Gamma)$. Hence, $x(N) = SW(\Pi) = SW_{opt}(\Gamma)$. □

According to the above result, the coalition structure core can be viewed as a method to distribute the maximum social welfare that can be achieved over all the possible coalition structures in a way that no feasible coalition has an incentive to deviate from it. Moreover, note that since any singleton coalition $\{i\}$, with $i \in N$, is a feasible one, then the individual rationality requirement over the possible imputations is implied by the fact that $x(C) \geq v(C)$ must hold at the coalition structure core, for each $C \in \mathcal{F}(G)$.

**Example 2.2.** The function $v$ and the interaction graph $G$ discussed in Example 1.3 give rise to the structurally-restricted characteristic function game $\Gamma = \langle N, v, G \rangle$, where $\mathcal{F}(G) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{3,5\}, \{4,5\}, \{1,2,3\}, \{2,3,4\}, \{2,3,5\}, \{3,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\}$. The coalition structure core of this game is not empty, as is witnessed by the pair $(\Pi, \boldsymbol{x})$ such that $\Pi = \{\{1,2,3\}, \{4,5\}\}$ and $\boldsymbol{x} = \langle 12, 6, 4, 6, 5 \rangle$. Indeed, note that $\Pi \in$ CS-$opt(\Gamma)$ holds and that $\boldsymbol{x}$ is an imputation for $\Pi$, because $x_1 + x_2 + x_3 = v(\{1,2,3\})$, $x_4 + x_5 = v(\{4,5\})$, and $x_i \geq v(\{i\})$, for each agent $i \in \{1,2,3,4,5\}$. Moreover, it can be checked that there is no feasible coalition $C \in \mathcal{F}(G)$ such that $x(C) < v(C)$. ◁

**The least core value.** As already discussed in the introduction, it might well be the case that the core of a given game is empty. This observation motivated a number of different proposals for weaker notions of stability, which are able to single out desirable outcomes in any possible characteristic function game. The *least core* [34] is a well-known solution concept of this kind, which provides an approximation of the core where an additional penalty is imposed for leaving the grand-coalition. Next, we present its definition by taking into account the feasibility of the coalitions, starting with the case where coalition structures are not dealt with.

For any structurally-restricted characteristic function game $\Gamma = \langle N, v, G \rangle$ and for any real number $\varepsilon$, let $\Gamma_{\text{-}\varepsilon} = \langle N, v_{\text{-}\varepsilon}, G \rangle$ be the game such that $v_{\text{-}\varepsilon}(C) = v(C) - \varepsilon$, for each $C \in \mathcal{F}(G)$, and $v_{\text{-}\varepsilon}(N) = v(N)$. The *least core* of $\Gamma$ is then the set of all allocations in $core(\Gamma_{\text{-}\varepsilon})$ such that $core(\Gamma_{\text{-}\varepsilon}) \neq \emptyset$, and $core(\Gamma_{\text{-}\bar{\varepsilon}}) = \emptyset$, for any $\bar{\varepsilon} < \varepsilon$.

**Example 2.3.** Consider the game $\Gamma' = \langle\{1,2\}, v', K_2\rangle$ where $v'(\{1\}) = 2$, $v'(\{2\}) = 2$, $v'(\{1,2\}) = 2$, and where $K_2$ is the complete graph over $\{1,2\}$. Note that structural restrictions are immaterial as all coalitions are feasible ones. Note also that $core(\Gamma') = \emptyset$.

Consider then the game $\Gamma'_{-1} = \langle\{1,2\}, v'_{-1}, G'\rangle$ where $v'_{-1}(\{1\}) = v'_{-1}(\{2\}) = 1$ and $v'_{-1}(\{1,2\}) = 2$, and note that $core(\Gamma'_{-1})$ contains only the allocation $\langle 1,1\rangle$ where the agents equally divide the payoff $v'(\{1,2\})$. Moreover, note that for any game $\Gamma'_{-\bar{\varepsilon}}$ such that $\bar{\varepsilon} < 1$, $core(\Gamma'_{-\bar{\varepsilon}})$ is empty. Hence, the least core of $\Gamma'$ is precisely the set $\{\langle 1,1\rangle\}$. $\lhd$

In the above example, the main reason for the emptiness of the core is that the grand-coalition would hardly form, because $v'(\{1\}) + v'(\{2\}) > v'(\{1,2\})$. Hence, the coalition structure core might be a more appropriate solution concept for such instances. Indeed, this concept would single out the non-empty set of outcomes CS-$core(\Gamma') = \{(\Pi, \boldsymbol{x})\}$ where $\Pi = \{\{1\},\{2\}\}$ and $\boldsymbol{x} = \langle 2,2\rangle$, thereby suggesting that the two agents will not join together, with each of them individually getting a payoff of 2.

At this point, the reader might wonder whether, unlike the core, the coalition structure core is always non-empty. As demonstrated by the game below, this is not the case.

**Example 2.4.** Consider the game $\Gamma'' = \langle\{1,2,3,4\}, v'', K_4\rangle$ over the complete interaction graph $K_4$, where the payoff function is such that $v''(C \cup \{4\}) = 0$, for each non-empty set $C \subseteq \{1,2,3\}$, $v''(\{1\}) = v''(\{2\}) = v''(\{3\}) = v''(\{4\}) = 1$, $v''(\{1,2\}) = v''(\{1,3\}) = v''(\{2,3\}) = 5$, and $v''(\{1,2,3\}) = 7$. Note that $\{\{1,2,3\},\{4\}\}$ is the only element in CS-$opt(\Gamma'')$. Hence, by Fact 2.1, elements in the coalition structure core have the form $(\{\{1,2,3\},\{4\}\}, \boldsymbol{x})$ where $\boldsymbol{x} = \langle x_1, x_2, x_3, x_4\rangle$ is such that:

$$\left\{\begin{array}{rcl} x_1+x_2+x_3 &=& 7\\ x_4 &=& 1\\ x_1 &\geq& 1\\ x_2 &\geq& 1\\ x_3 &\geq& 1\\ x_4 &\geq& 1 \end{array}\right. \qquad \begin{array}{rcl} x_1+x_2 &\geq& 5\\ x_1+x_3 &\geq& 5\\ x_2+x_3 &\geq& 5 \end{array} \qquad \begin{array}{rcl} x_1+x_2+x_3+x_4 &\geq& 0\\ x_1+x_2+x_4 &\geq& 0\\ x_1+x_3+x_4 &\geq& 0\\ x_2+x_3+x_4 &\geq& 0\\ x_1+x_4 &\geq& 0\\ x_2+x_4 &\geq& 0\\ x_3+x_4 &\geq& 0 \end{array}$$

The (in)equalities in the rightmost group are implied by the others, and can be discarded. In fact, the above set of (in)equalities is not satisfiable, because of the (in)equalities $x_1+x_2+x_3 = 7$, $x_1+x_2 \geq 5$, $x_1+x_3 \geq 5$, and $x_2+x_3 \geq 5$. Hence, CS-$core(\Gamma'') = \emptyset$. $\lhd$

Given the above observation, we now look at a counterpart of the least core over coalition structures, too. To this end, for any feasible coalition structure $\Pi = \{C_1, \ldots, C_m\}$ and for any real number $\varepsilon$, let $\Gamma_{\Pi,\text{-}\varepsilon} = \langle N, v_{\Pi,\text{-}\varepsilon}, G\rangle$ be the game with $v_{\Pi,\text{-}\varepsilon}(C_j) = v(C_j)$, for each coalition $C_j \in \Pi$; and $v_{\Pi,\text{-}\varepsilon}(C) = v(C) - \varepsilon$, for each feasible coalition $C$ not in $\Pi$. Moreover, let CS-$core_{\Pi,\text{-}\varepsilon}(\Gamma)$ be the set of all the pairs $(\Pi, \boldsymbol{x}) \in$ CS-$core(\Gamma_{\Pi,\text{-}\varepsilon})$. Note that, in general, a pair $(\Pi, \boldsymbol{x}) \in$ CS-$core(\Gamma_{\Pi,\text{-}\varepsilon})$ is not an imputation for the game $\Gamma$. In particular, consistently with the original definition of the least core by Maschler et al. [34], the penalty $\varepsilon$ is imposed on each feasible coalition including the singletons—hence, $\boldsymbol{x}$ is not necessarily individually rational in $\Gamma$.[3]

Hence, we now define the *least coalition structure core* of $\Gamma$ as the non-empty set $\bigcup_{\Pi \in \mathcal{CS}(G)}$ CS-$core_{\Pi,\text{-}\varepsilon}(\Gamma)$ such that $\bigcup_{\Pi \in \mathcal{CS}(G)}$ CS-$core_{\Pi,\text{-}\bar{\varepsilon}}(\Gamma) = \emptyset$, for any $\bar{\varepsilon} < \varepsilon$.

**Example 2.5.** Consider again the setting of Example 2.4, and let $\Pi = \{\{1,2,3\},\{4\}\}$. Then, consider the game $\Gamma''_{\Pi,\text{-}\varepsilon}$ and note that CS-$core_{\Pi,\text{-}\varepsilon}(\Gamma'')$ is determined by the solutions to the following set of (in)equalities:

$$\left\{\begin{array}{rcl} x_1+x_2+x_3 &=& 7\\ x_4 &=& 1\\ x_1 &\geq& 1-\varepsilon\\ x_2 &\geq& 1-\varepsilon\\ x_3 &\geq& 1-\varepsilon\\ x_1+x_2 &\geq& 5-\varepsilon\\ x_1+x_3 &\geq& 5-\varepsilon\\ x_2+x_3 &\geq& 5-\varepsilon \end{array}\right.$$

The minimum value of $\varepsilon$ for which the above (in)equalities are satisfiable is $\frac{1}{3}$. In particular, for this value, we have that $\langle \frac{7}{3}, \frac{7}{3}, \frac{7}{3}, 1\rangle$ is the only solution.

[3] The least core variant where individual rationality constraints are enforced might be appropriate for modelling certain application scenarios. The careful reader can check that most of our results apply to that setting also.

Consider now any feasible coalition structure $\bar{\Pi} \neq \Pi$ and the game $\Gamma''_{\bar{\Pi},\text{-}\bar{\varepsilon}}$. Let $(\bar{\Pi}, \bar{\boldsymbol{x}})$ be an imputation in CS-$core_{\bar{\Pi},\text{-}\bar{\varepsilon}}(\Gamma'')$. Since $\bar{\Pi} \neq \Pi$, we have $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 \geq 7 - \bar{\varepsilon}$ and $\bar{x}_4 \geq 1 - \bar{\varepsilon}$. Moreover, we know that $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4 = \sum_{\bar{C} \in \bar{\Pi}} v''(\bar{C})$ must hold. In particular, $\sum_{\bar{C} \in \bar{\Pi}} v''(\bar{C}) \leq 7$ holds since $\bar{\Pi} \neq \Pi$, and so $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4 \leq 7$. Then, we derive $\bar{\varepsilon} \geq \frac{1}{2} > \frac{1}{3}$, and we conclude that $(\{\{1,2,3\},\{4\}\}, \langle \frac{7}{3}, \frac{7}{3}, \frac{7}{3}, 1\rangle)$ is the only imputation in the least coalition structure core. $\triangleleft$

We point out that the notion of least coalition structure core illustrated above can be found in the work by Service [35]. This notion appears to be the natural counterpart of the least core over coalition structures. Indeed, the intuition underlying the original notion of the least core is to characterize the outcomes for which the grand-coalition is stable when deviations are charged an additional cost. Accordingly, the counterpart just characterizes the minimum cost that has to be charged to stabilize any coalition, hence not necessarily the one formed by all the agents.[4] In what follows, the number $\varepsilon$ determined by the least core (resp., least coalition structure core) of $\Gamma$ is called its *least core value* (resp., *least coalition structure core value*), and it is denoted by LCV($\Gamma$) (resp., CS-LCV($\Gamma$)). Note that, whenever the game has already a non-empty core (resp., coalition structure core), then this value might be even a negative one, thereby characterizing the "most stable" subset of the core (resp., coalition structure core).

**The cost of stability.** A criticism that applies to the notion of the least core is that it is not "constructive", in the sense that it does not practically tell us how a given unstable game can be made stable. This drawback is overcome by the *cost of stability* [33], which is the minimum worth that a benevolent external party must supply in order for the game to have a non-empty core.[5] Let us first formalize this concept by starting with the case where coalition structures are not allowed to form.

For any real number $\Delta \geq 0$, let $\Gamma_\Delta = \langle N, v_\Delta, G\rangle$ denote the structurally-restricted game such that $v_\Delta(C) = v(C)$, for each coalition $C \in \mathcal{F}(G)$, and $v_\Delta(N) = v(N) + \Delta$. The *cost of stability* of $\Gamma$ is the value $\text{CoS}(\Gamma) = \min\{\Delta \mid core(\Gamma_\Delta) \neq \emptyset \wedge \Delta \geq 0\}$.

**Example 2.6.** Consider the game $\Gamma' = \langle\{1,2\}, v', K_2\rangle$ presented in Example 2.3, and recall that $core(\Gamma') = \emptyset$. Its cost of stability is then given by the minimum value of $\Delta \geq 0$ such that the following set of linear (in)equalities admits a solution:

$$\begin{cases} x_1 + x_2 = 2 + \Delta \\ x_1 \geq 2 \\ x_2 \geq 2 \end{cases}$$

Therefore, we have that $\text{CoS}(\Gamma') = 2$. $\triangleleft$

When we allow coalition structures, the definition of the cost of stability can be modified accordingly. In particular, for any coalition structure $\Pi = \{C_1, \ldots, C_m\}$ and for any vector $\vec{\Delta} = \langle \Delta_1, \ldots, \Delta_m\rangle \in \mathbb{R}^m$, let $\Gamma_{\Pi,\vec{\Delta}} = \langle N, v_{\Pi,\vec{\Delta}}, G\rangle$ be the game with $v_{\Pi,\vec{\Delta}}(C) = v(C)$, for each coalition $C \in \mathcal{F}(G)$ such that $C \notin \Pi$; and $v_{\Pi,\vec{\Delta}}(C_j) = v(C_j) + \Delta_j$, for each coalition $C_j \in \Pi$. If $\Pi$ is clear from the context, then the above game is just denoted as $\Gamma_{\vec{\Delta}} = \langle N, v_{\vec{\Delta}}, G\rangle$ in order to simplify the notation. The cost of stability of $\Pi$ in $\Gamma$ is the value $\text{CS-CoS}(\Gamma, \Pi) = \min\{\sum_{j=1}^m \Delta_j \mid \exists \boldsymbol{x}$ such that $(\Pi, \boldsymbol{x}) \in$ CS-$core(\Gamma_{\vec{\Delta}})$ and $\forall j \in \{1, \ldots, m\}, \Delta_j \geq 0\}$. The *coalition structure cost of stability* of $\Gamma$, denoted by CS-CoS($\Gamma$), is the minimum value of CS-CoS($\Gamma, \Pi$) over all feasible coalition structures $\Pi \in \mathcal{CS}(G)$.

**Example 2.7.** Consider again the game $\Gamma'' = \langle\{1,2,3,4\}, v'', K_4\rangle$ discussed in Example 2.4, by recalling that CS-$core(\Gamma'') = \emptyset$. Let $\Pi$ denote the coalition structure $\{\{1,2,3\},\{4\}\}$, which attains the maximum social welfare. Then, $(\Pi, \boldsymbol{x})$ belongs to CS-$core(\Gamma_{\vec{\Delta}})$ if $\boldsymbol{x} = \langle x_1, x_2, x_3, x_4\rangle$ and $\vec{\Delta} = \langle \Delta_1, \Delta_2\rangle$, with $\Delta_1 \geq 0$ and $\Delta_2 \geq 0$, satisfy the following (in)equalities:

$$\begin{cases} x_1 + x_2 + x_3 = 7 + \Delta_1 & \\ x_4 = 1 + \Delta_2 & \\ x_1 \geq 1 & x_1 + x_2 \geq 5 \\ x_2 \geq 1 & x_1 + x_3 \geq 5 \\ x_3 \geq 1 & x_2 + x_3 \geq 5 \\ x_4 \geq 1 & \end{cases}$$

In particular, the minimum value of $\Delta_1 + \Delta_2$ for which the above (in)equalities have a solution is obtained for $\Delta_1 = \frac{1}{2}$ and $\Delta_2 = 0$, in which case $\boldsymbol{x} = \langle \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, 1\rangle$ witnesses the satisfiability of all (in)equalities. Therefore, $\text{CS-CoS}(\Gamma'', \Pi) = \frac{1}{2}$, and it is interesting to observe that this cost is actually paid to stabilize the coalition $\{1,2,3\}$.

By considering any other $\bar{\Pi} \neq \Pi$, one can check that $\text{CS-CoS}(\Gamma'', \bar{\Pi}) > \text{CS-CoS}(\Gamma'', \Pi)$, so that $\text{CS-CoS}(\Gamma'', \Pi)$ is the coalition structure cost of stability of $\Gamma''$. $\triangleleft$

[4] Different notions, which look at "approximations" of the coalition structure core from different perspectives, can be found in [36–38]. See, also, [37] for further notions related to the least core.

[5] By considering the cost of stability relative to the worth of the grand-coalition, we get the related notion of the *cost recovery ratio* (see, e.g., [39]).

*2.2. Computational setting*

As we have already pointed out, we focus on games with a compact encoding. Founding on the seminal papers by [22–24], several such encoding mechanisms have been studied. Although our work deals with general encodings, we will also consider as a special case of this general setting encodings via marginal contribution networks, as defined by Ieong and Shoham [25].

**Marginal contribution networks.** A marginal contribution network $M$ consists of a set $\{r_1, \ldots, r_n\}$ of rules involving a number of Boolean variables that represent the agents. For each $1 \leq i \leq n$, the rule $r_i$ has the form $\{pattern_i\} \rightarrow value_i$, where $pattern_i$ is a conjunction that may include both positive and negative literals, and $value_i$ is the additive contribution associated with this pattern.[6] A rule *applies* to a set $C$ of agents if all the agents whose literals occur positively in the pattern belong to $C$, and all the players whose literals occur negatively in the pattern do not belong to $C$. The value $v(C)$ in the game induced by $M$ is given by the sum of the values of all rules that apply to $C$. If no rule applies, then the value for the coalition is set to zero, by default.

Note that the encoding mechanism defines a payoff for each possible coalition—i.e., all of them are implicitly assumed to be feasible. In the context of structured environments, the specification of a CFG can be given by an MC-net plus an interaction graph describing the coalitions for which the values returned by the encoding are meaningful.

**Example 2.8.** Consider the following marginal contribution network $M$, over the Boolean variables in the set $\{a_1, a_2, a_3\}$:

$$\begin{cases} \{a_1 \wedge a_2\} & \rightarrow\ 2 \\ \{a_2 \wedge a_3\} & \rightarrow\ 1 \\ \{a_1\} & \rightarrow\ 10 \\ \{a_2\} & \rightarrow\ 5 \\ \{a_3\} & \rightarrow\ 4 \end{cases}$$

These rules define a game over a set $\{1, 2, 3\}$ of agents bijectively corresponding to the elements in $\{a_1, a_2, a_3\}$. In fact, as no confusion can arise, it is convenient to interchangeably view Boolean variables as agents, so that we can directly write $v(\{a_1\}) = 10$, $v(\{a_2\}) = 5$, $v(\{a_3\}) = 4$, $v(\{a_1, a_2\}) = 17$, $v(\{a_1, a_3\}) = 14$, $v(\{a_2, a_3\}) = 10$, and $v(\{a_1, a_2, a_3\}) = 22$. To see how these values are computed, observe for instance that $v(\{a_2, a_3\}) = 1 + 5 + 4$ is derived from the application of the rules $\{a_2 \wedge a_3\} \rightarrow 1$, $\{a_2\} \rightarrow 5$, and $\{a_3\} \rightarrow 4$. Moreover, all rules have to be applied to compute $v(\{a_1, a_2, a_3\})$.

Note that patterns do not contain negative literals in this example. Moreover, note that this marginal contribution network, combined with the interaction graph depicted in the left part of Fig. 1, fully specifies the game discussed in Example 1.1. In particular, the interaction graph tells us that the value $v(\{a_1, a_3\})$ defined by the encoding is not meaningful, because the corresponding coalition is not allowed to form. $\triangleleft$

Marginal contribution networks provide a simple, yet very powerful method to specify characteristic function games. Indeed, any characteristic function game can be encoded in terms of a marginal contribution network, even though in some cases exponentially many rules might be required [25].

Interestingly, in many practical cases, exponentially more compact encodings can be defined via marginal contribution networks. For a trivial example of this kind, think about the MC-net over a set $\{a_1, \ldots, a_n\}$ of agents consisting of the rule $\{a_i\} \rightarrow 1$, for each $a_i \in N$. This network just states that $v(C) = |C|$, for each coalition $C \subseteq N$. Therefore, it succinctly defines $2^n$ payoffs, in terms of $n$ rules only.

**Compact representations.** We recall from [9] a unifying model for compactly specified games, by extending it to deal with restrictions on the formation of coalitions. A compact representation $\mathcal{R}$ defines suitable encodings for a class of structurally-restricted characteristic function games, denoted by $\mathcal{C}(\mathcal{R})$. Formally, any representation $\mathcal{R}$ defines an encoding function $\xi^{\mathcal{R}}$ and a payoff function $v^{\mathcal{R}}$ such that, for any structurally-restricted characteristic function game $\Gamma \in \mathcal{C}(\mathcal{R})$, $\xi^{\mathcal{R}}(\Gamma)$ is the encoding of $\Gamma$, and $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C)$ is the payoff associated with the coalition $C$ according to $\Gamma$. In the following, we consider only *polynomial-time compact representations* (in short: **P**-representations), i.e., we assume that $v^{\mathcal{R}}$ is a polynomial-time computable function. Moreover, we assume that the game representation $\xi^{\mathcal{R}}(\Gamma)$ includes the interaction graph $G$ associated with $\Gamma$. This entails in particular that, for every coalition $C$, $|C| \leq ||\xi^{\mathcal{R}}(\Gamma)||$ holds.

Whenever a compact representation $\mathcal{R}$ is clear from the context or whenever its specification is irrelevant, we can just write $\Gamma$ instead of $\xi^{\mathcal{R}}(\Gamma)$, and $v(C)$ instead of $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C)$. While doing so, we are identifying the game with its actual representation (by furthermore ignoring the actual procedure underlying the computation of the payoff values). As an example, for the case of the marginal-contribution network compact-representation, henceforth denoted as ‘mcn’, any game $\Gamma$ is encoded in $\xi^{\mathsf{mcn}}(\Gamma)$ by a set of rules plus the underlying interaction graph, and the function $v^{\mathsf{mcn}}(\xi^{\mathsf{mcn}}(\Gamma), C)$ computes

[6] The encoding can be immediately generalized by defining patterns as arbitrary Boolean formulas over the agents. This is precisely the setting proposed and studied by Elkind et al. [40], where marginal contribution networks where patterns are constrained to be conjunctions of literals are called *basic*. In the paper, we always deal with basic patterns but, for the sake of simplicity and according to the terminology of Ieong and Shoham [25], we shall omit to explicitly specify that they are “basic”.

the payoff of $C$ as the sum of the values of those rules in $\xi^{\mathsf{mcn}}(\Gamma)$ that apply to coalition $C$. Moreover, as the encoding is understood, we can use the naming "marginal contribution network" to denote the game rather than—more formally—its encoding. Eventually, note that mcn is a **P**-representation.

**Expressive power.** Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be a pair of game representations. We say that $\mathcal{R}_2$ is *at least as expressive (and succinct) as* $\mathcal{R}_1$, denoted by $\mathcal{R}_1 \precsim_e \mathcal{R}_2$, if there exists a polynomial-time computable function $f$ that translates the encoding $\xi^{\mathcal{R}_1}(\Gamma)$ of $\Gamma$ in $\mathcal{R}_1$ into an encoding $\xi^{\mathcal{R}_2}(\Gamma)$ for the same game $\Gamma$ in $\mathcal{R}_2$. Note that if $\mathcal{R}_1 \precsim_e \mathcal{R}_2$ holds, then any hardness result for the complexity of reasoning problems over $\mathcal{R}_1$ immediately applies to $\mathcal{R}_2$, and any positive result for problems over $\mathcal{R}_2$ immediately applies to $\mathcal{R}_1$. Thus, expressiveness relations can be used to derive further complexity results after the complexity of some representation has been characterized.

**Problems studied.** In order to assess the amount of resources needed to reason about core-related concepts in structured environments, we provide answers to a number of questions, thus far unaddressed in the literature. In the analysis, we follow the classical setting where numerical computations deal with numbers given in the (irreducible) fractional form $p/q$, where $p$ (resp., $q$) is an integer (resp., natural number) encoded in binary. In particular, all parameters provided as inputs to the problems we study are assumed to be *rational numbers*, rather than arbitrary real numbers. For instance, the value associated with any rule of a marginal contribution network (and, hence, the payoff associated with any coalition) is a rational number. For models of computations tailored to work with real numbers (as well as with other fields), we refer the interested reader to [41].

Formally, we analyze the complexity of the following decision problems, all of them receiving as input a game $\Gamma = \langle N, v, G\rangle$, with $N = \{1, \dots, n\}$ (in fact, its encoding $\xi^{\mathcal{R}}(\Gamma)$ defined according to a **P**-representation $\mathcal{R}$), possibly with some further inputs.

CS-Core-Membership: Given a pair $(\Pi, \boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{Q}^n$, and $\Pi$ is a partition of $N$, does $(\Pi, \boldsymbol{x}) \in$ CS-*core*$(\Gamma)$?
CS-Core-NonEmptiness: Does CS-*core*$(\Gamma) \neq \emptyset$ hold?
CS-CoS: Given a rational number $\Delta \geq 0$, is CS-CoS$(\Gamma) \leq \Delta$?
CS-LCV: Given a rational number $\varepsilon$, is CS-LCV$(\Gamma) \leq \varepsilon$?

In particular, we also study the problems LCV$^-$ and LCV$^+$, which are two specializations of CS-LCV where $\varepsilon$ is constrained to be such that $\varepsilon \leq 0$ and $\varepsilon \geq 0$, respectively. Moreover, apart from the decision problems, we also study the following search problem:

CS-Core-Find: Find an element $(\Pi, \boldsymbol{x}) \in$ CS-*core*$(\Gamma)$ or decide that CS-*core*$(\Gamma) = \emptyset$.

Our analysis will be conducted on various classes of compact games identified by looking at the underlying interaction graphs. To this end, if $\mathcal{G}$ is a class of graphs and $\mathcal{C}(\mathcal{R})$ is a class of compact games, then we define $\mathcal{C}_{\mathcal{G}}(\mathcal{R})$ to be the subclass of all those games $\Gamma = \langle N, v, G\rangle \in \mathcal{C}(\mathcal{R})$ such that $G \in \mathcal{G}$ holds.

## 3. Characterizing the graph-restricted core with coalition structures

In this section, we illustrate analytical characterizations for the coalition structure core and for the related stability concepts discussed in Section 2.1. These characterizations will be useful in the sections that follow. A crucial role in the analysis is played by the concept of the *cohesive cover* of a game.

**Definition 3.1.** Let $\Gamma = \langle N, v, G\rangle$ be a structurally-restricted characteristic function game. The *cohesive cover* of $\Gamma$ is the game $\tilde{\Gamma} = \langle N, \tilde{v}, G\rangle$ such that $\tilde{v}(C) = v(C)$, for each feasible coalition $C \in \mathcal{F}(G) \setminus \{N\}$, and $\tilde{v}(N) = SW_{opt}(\Gamma)$. □

Note that $\tilde{\Gamma}$ is such that $\tilde{v}(N) \geq SW(\Pi)$, for each coalition structure $\Pi \in \mathcal{CS}(G)$. Hence, the grand-coalition always forms in $\tilde{\Gamma}$.

**Example 3.2.** Consider the setting of Example 2.4 and the game $\Gamma'' = \langle \{1,2,3,4\}, v'', K_4\rangle$, by recalling that $\Pi = \{\{1,2,3\},\{4\}\}$ is the only coalition structure that attains the maximum social welfare, with $SW(\Pi) = 8$. Its cohesive cover $\tilde{\Gamma}'' = \langle \{1,2,3,4\}, \tilde{v}'', K_4\rangle$ is then such that $\tilde{v}''(\{1,2,3,4\}) = 8 > v''(\{1,2,3,4\}) = 0$, while $\tilde{v}''(C) = v''(C)$, for each coalition $C \subset \{1,2,3,4\}$. Therefore, we have that $\tilde{v}''(C \cup \{4\}) = 0$, for each non-empty set $C \subset \{1,2,3\}$, $\tilde{v}''(\{1\}) = \tilde{v}''(\{2\}) = \tilde{v}''(\{3\}) = \tilde{v}''(\{4\}) = 1$, $\tilde{v}''(\{1,2\}) = \tilde{v}''(\{1,3\}) = \tilde{v}''(\{2,3\}) = 5$, and $\tilde{v}''(\{1,2,3\}) = 7$. ◁

The notion of cohesive cover is directly inspired by the definition of *cohesive games* [2], and it has been firstly formulated by Greenberg and Weber [30]. In fact, a game $\Gamma$ is cohesive whenever it coincides with its cohesive cover $\tilde{\Gamma}$.[7] A similar no-

[7] This is the natural extension of the concept of cohesive games defined in the classic setting where the interaction graph is a complete graph, and hence where any coalition is a feasible one [2].

tion is the *superadditive cover* discussed in [21], which is the superadditive game defined by the payoff function $\hat{v}$ such that, for each coalition $C \subseteq N$, $\hat{v}(C)$ is the maximum social welfare that can be attained over all the possible coalition structures built by agents in $C$. Interestingly, it is known that the superadditive cover can be used to characterize core-related solution concepts when coalition structures are allowed [21], but this is not useful to derive tight bounds in our complexity analysis, since building the superadditive cover requires to compute the social welfare over exponentially many coalitions.

Actually, similar characterizations are also known (or can be trivially derived from known results in the literature) when we just "cover" the grand-coalition (by the cohesive cover $\tilde{\Gamma}$) rather than all the possible coalitions (as in the superadditive cover). Such characterizations for the cohesive cover are next illustrated.

### 3.1. CS-CoS *and* CS-*core characterizations*

Firstly, we consider a characterization of the coalition structure cost of stability in terms of the standard cost of stability.

**Fact 3.3.** *For any structurally-restricted characteristic function game* $\Gamma$, CS-CoS$(\Gamma) =$ CoS$(\tilde{\Gamma})$. *In particular, if* $\Gamma$ *is cohesive, then* CS-CoS$(\Gamma) =$ CoS$(\Gamma)$.

Note that, in the standard setting where all coalitions can form, the characterization is well-known, and it was explicitly stated by Meir et al. [39] for the superadditive cover (more precisely for the *subadditive cover* of cost-sharing games). The proof relies on results by Aumann and Dreze [21], and smoothly applies to the cohesive cover, since it just uses the property that the payoff associated with the grand-coalition in the given cover is the maximum social welfare of the original game.

In fact, a detailed proof of the characterization in the setting where all coalitions are feasible can be furthermore found in the work by Greco et al. [36]. Inspection of that proof reveals that the arguments apply even when some coalitions are not allowed to form (so Fact 3.3 is immediate). Indeed, this is not surprising because any structurally-restricted game can be transformed into an "equivalent" standard game (in that core constraints defining the coalition structure core and the coalition structure cost of stability are the same), by just assigning to each infeasible coalition any value that is at most the sum of the values of all connected sub-components (see also [14]).

**Example 3.4.** Consider the game $\tilde{\Gamma}'' = \langle \{1,2,3,4\}, \tilde{v}'', K_4\rangle$ defined in Example 3.2, and recall from Example 2.7 that CS-CoS$(\Gamma'') = \frac{1}{2}$.

Then, based on Fact 3.3, we can derive that CoS$(\tilde{\Gamma}'') = \frac{1}{2}$. Indeed, it can be checked that $\Delta = \frac{1}{2}$ is the minimum value for which the following set of linear (in)equalities, characterizing the core of $\tilde{\Gamma}''_{\Delta}$, admits a solution:

$$\left\{\begin{array}{rclrcl} x_1+x_2+x_3+x_4 &=& 8+\Delta & & & \\ x_1 &\geq& 1 & x_1+x_2 &\geq& 5 \\ x_2 &\geq& 1 & x_1+x_3 &\geq& 5 \\ x_3 &\geq& 1 & x_2+x_3 &\geq& 5 \\ x_4 &\geq& 1 & x_1+x_2+x_3 &\geq& 7 \end{array}\right.$$

In particular, for $\Delta = \frac{1}{2}$, the allocation $\langle x_1, x_2, x_3, x_4\rangle = \langle \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, 1\rangle$ is a solution. ◁

We now move to illustrate a characterization of the coalition structure core based on the core of the cohesive covers. Again, the result is known for superadditive covers [21] where all coalitions are allowed to form and the proof immediately applies to cohesive covers. The extension to structurally-restricted games is trivial.

**Fact 3.5.** *For any structurally-restricted characteristic function game* $\Gamma$, CS-*core*$(\Gamma) =$ CS-*opt*$(\Gamma) \times$ *core*$(\tilde{\Gamma})$. *In particular, if* $\Gamma$ *is cohesive, then*

(1) CS-*core*$(\Gamma) =$ CS-*opt*$(\Gamma) \times$ *core*$(\Gamma)$*;*
(2) CS-*core*$(\Gamma) = \emptyset$ *if, and only if, core*$(\Gamma) = \emptyset$*;*
(3) $\boldsymbol{x} \in$ *core*$(\Gamma)$ *if, and only if,* $(\{N\}, \boldsymbol{x}) \in$ CS-*core*$(\Gamma)$.

**Example 3.6.** Consider the game $\Gamma''' = \langle \{1,2,3,4\}, v''', K_4\rangle$ where $v''' = v''_{-1}$, with $v''$ being the payoff function discussed in Example 2.4. So, we have that: $v'''(\{1,2,3,4\}) = 0$; $v'''(C \cup \{4\}) = 0 - 1$, for each non-empty set $C \subset \{1,2,3\}$; $v'''(\{1\}) = v'''(\{2\}) = v'''(\{3\}) = v'''(\{4\}) = 1 - 1$; $v'''(\{1,2\}) = v'''(\{1,3\}) = v'''(\{2,3\}) = 5 - 1$; and $v'''(\{1,2,3\}) = 7 - 1$. Let $\tilde{\Gamma}''' = \langle \{1,2,3,4\}, \tilde{v}''', K_4\rangle$ denote the cohesive cover of $\Gamma'''$, and note that $\tilde{v}'''(\{1,2,3,4\}) = 6$ and that $\Pi = \{\{1,2,3\},\{4\}\}$ is the only coalition structure in CS-*opt*$(\Gamma''')$.

Then, by Fact 3.5, we will conclude that CS-*core*$(\Gamma''') = \{(\Pi, \langle 2,2,2,0\rangle)\}$, after checking that $\langle x_1, x_2, x_3, x_4\rangle = \langle 2,2,2,0\rangle$ is the only solution to the following set of (in)equalities, defining the core of $\tilde{\Gamma}'''$:

$$\begin{cases} x_1+x_2+x_3+x_4 = v''_{-1}(\{1,2,3\})+v''_{-1}(\{4\}) = (7-1)+(1-1)=6 \\ x_1 \geq v''_{-1}(\{1\}) = 1-1=0 \\ x_2 \geq v''_{-1}(\{2\}) = 1-1=0 \\ x_3 \geq v''_{-1}(\{3\}) = 1-1=0 \\ x_4 \geq v''_{-1}(\{4\}) = 1-1=0 \\ x_1+x_2 \geq v''_{-1}(\{1,2\}) = 5-1=4 \\ x_1+x_3 \geq v''_{-1}(\{1,3\}) = 5-1=4 \\ x_2+x_3 \geq v''_{-1}(\{2,3\}) = 5-1=4 \end{cases}$$

◁

### 3.2. CS-LCV *characterization*

Let us now analyze the least core value. In contrast to the case of the coalition structure cost of stability, it is not possible to end up with a simple (and in fact useful) closed-form characterization over arbitrary games. In particular, it is not the case that in general CS-LCV$(\Gamma)$ = LCV$(\tilde{\Gamma})$.

**Example 3.7.** Consider the game $\Gamma'' = \langle\{1,2,3,4\}, v'', K_4\rangle$ discussed in Example 2.4 and Example 2.5, and recall from there that CS-LCV$(\Gamma'') = \frac{1}{3}$.

Consider then the cohesive cover $\tilde{\Gamma}'' = \langle\{1,2,3,4\}, \tilde{v}'', K_4\rangle$ defined in Example 3.2, and note that the value LCV$(\tilde{\Gamma}'')$ is the minimum value of $\varepsilon$ for which the following set of (in)equalities admits a solution:

$$\begin{cases} x_1+x_2+x_3+x_4 = 8 & & x_1+x_2+x_4 \geq 0-\varepsilon \\ x_1 \geq 1-\varepsilon & x_1+x_2 \geq 5-\varepsilon & x_1+x_3+x_4 \geq 0-\varepsilon \\ x_2 \geq 1-\varepsilon & x_1+x_3 \geq 5-\varepsilon & x_2+x_3+x_4 \geq 0-\varepsilon \\ x_3 \geq 1-\varepsilon & x_2+x_3 \geq 5-\varepsilon & x_1+x_4 \geq 0-\varepsilon \\ x_4 \geq 1-\varepsilon & x_1+x_2+x_3 \geq 7-\varepsilon & x_2+x_4 \geq 0-\varepsilon \\ & & x_3+x_4 \geq 0-\varepsilon \end{cases}$$

Note that for $\varepsilon = \frac{1}{5}$, the payoff vector $\langle x_1,x_2,x_3,x_4\rangle = \langle\frac{12}{5},\frac{12}{5},\frac{12}{5},\frac{4}{5}\rangle$ is a solution. Therefore, LCV$(\tilde{\Gamma}'')$ < CS-LCV$(\Gamma'')$. ◁

In fact, two properties that will be useful in our subsequent analysis are stated below.

**Lemma 3.8.** *Let $\Gamma = \langle N, v, G\rangle$ be a game, and let $\Pi$ be a feasible coalition structure. If $(\Pi, \boldsymbol{x})$ is in* CS-*core*$_{\Pi,\text{-}\varepsilon}(\Gamma)$ *with $\varepsilon < 0$, then*

(1) $\Pi \in$ CS-*opt*$(\Gamma)$*;*
(2) *there is no $\Pi' \in$* CS-*opt*$(\Gamma)$ *such that $\Pi' \neq \Pi$.*

**Proof.** Assume that $(\Pi, \boldsymbol{x})$ is in CS-*core*$_{\Pi,\text{-}\varepsilon}(\Gamma)$. By Fact 3.5, we know that $\Pi \in$ CS-*opt*$(\Gamma_{\Pi,\text{-}\varepsilon})$. Moreover, by construction of $v_{\Pi,\text{-}\varepsilon}$, $\sum_{C\in\Pi} v_{\Pi,\text{-}\varepsilon}(C) = \sum_{C\in\Pi} v(C)$ whereas $v_{\Pi,\text{-}\varepsilon}(C') = v(C') - \varepsilon > v(C')$ holds, for each feasible coalition $C' \notin \Pi$. Hence, if $\Pi'$ is a feasible coalition structure in $\mathcal{CS}(G)$ with $\Pi' \neq \Pi$, then we get that $\sum_{C\in\Pi} v(C) = \sum_{C\in\Pi} v_{\Pi,\text{-}\varepsilon}(C) \geq \sum_{C'\in\Pi'} v_{\Pi,\text{-}\varepsilon}(C') > \sum_{C'\in\Pi'} v(C')$. It follows that $\Pi \in$ CS-*opt*$(\Gamma)$. Moreover, the same chain of (in)equalities shows that any feasible coalition structure $\Pi' \neq \Pi$ cannot be optimal, since the strict inequality $\sum_{C\in\Pi} v(C) > \sum_{C'\in\Pi'} v(C')$ holds. Therefore, we derive that $\Pi' \notin$ CS-*opt*$(\Gamma)$, for any feasible coalition structure $\Pi' \neq \Pi$. □

**Lemma 3.9.** *Let $\Gamma = \langle N, v, G\rangle$ be a game, and let $\varepsilon < 0$. Then, the least coalition structure core value of $\Gamma$ is at most $\varepsilon$ if, and only if, (i) $|$*CS-*opt*$(\Gamma)| = 1$*, and (ii) $\Pi \in$* CS-*opt*$(\Gamma)$ *implies $\Pi \in$* CS-*opt*$(\Gamma_{\Pi,\text{-}\varepsilon})$ *and* CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon}) \neq \emptyset$*.*

**Proof.** Assume there is an element $(\Pi, \boldsymbol{x}) \in$ CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon})$. Then, CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon}) \neq \emptyset$ trivially holds. Moreover, we can apply Lemma 3.8 and derive that *(i)* holds, with $\Pi$ being in particular in CS-*opt*$(\Gamma)$. Then, by Fact 3.5, we know that $\Pi \in$ CS-*opt*$(\Gamma_{\Pi,\text{-}\varepsilon})$ holds, and hence *(ii)* holds, too. For the converse, assume that *(i)* and *(ii)* hold. Let $\Pi$ be the unique coalition structure in CS-*opt*$(\Gamma)$. By *(ii)* and Fact 3.5, an element of the form $(\Pi, \boldsymbol{x})$ is in CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon})$. □

However, when moving to cohesive games it is possible to end up with the counterpart of Fact 3.3. The result is basically immediate, but the proof is reported in detail as it might be helpful to understand how the properties (and the analysis) of the least coalition structure core value are affected by its algebraic sign.

**Theorem 3.10.** *Let $\Gamma = \langle N, v, G\rangle$ be a cohesive game. Then,* CS-LCV$(\Gamma)$ = LCV$(\Gamma)$.

**Proof.** We shall show that, for any real number $\varepsilon$, CS-LCV($\Gamma$) $\leq \varepsilon$ if, and only if, LCV($\Gamma$) $\leq \varepsilon$, from which the claim follows. We distinguish two cases.

*(Case $\varepsilon \geq 0$)* Assume that CS-LCV($\Gamma$) $\leq \varepsilon$. So, there is a pair $(\Pi, \boldsymbol{x}) \in$ CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon})$. By the definition of the coalition structure core, we derive that: $x(C) = v(C)$, $\forall C \in \Pi$; and, $x(C) \geq v(C) - \varepsilon$, $\forall C \in \mathcal{F}(G) \setminus \Pi$. In particular, since $\varepsilon \geq 0$, we derive that $x(C) \geq v(C) - \varepsilon$, for each $C \in \mathcal{F}(G)$. Moreover, $x(N) = \sum_{C\in\Pi} x(C) = \sum_{C\in\Pi} v(C) \leq v(N)$ holds, where the inequality follows because $\Gamma$ is cohesive. That is, the payoff vector $\boldsymbol{x}$ witnesses that the core of the game $\Gamma_{\text{-}\varepsilon}$ is not empty. For instance, an element in *core*$(\Gamma_{\text{-}\varepsilon})$ is $\boldsymbol{x}'$ with $x'_i = x_i$, for each $i \in N \setminus \{1\}$, and $x'_1 = x_1 + (v(N) - x(N))$. Therefore, LCV($\Gamma$) $\leq \varepsilon$.

For the converse, assume that *core*$(\Gamma_{\text{-}\varepsilon})$ is not empty, and let $\boldsymbol{x}$ be an element in *core*$(\Gamma_{\text{-}\varepsilon})$. Given that $\Gamma$ is cohesive and that $\varepsilon \geq 0$, we derive that $\{N\}$ belongs to CS-*opt*$(\Gamma_{\text{-}\varepsilon})$ and $\Gamma_{\text{-}\varepsilon} = \tilde{\Gamma}_{\text{-}\varepsilon}$. Moreover, note that $\Gamma_{\text{-}\varepsilon} = \Gamma_{\{N\},\text{-}\varepsilon}$. Therefore, by Fact 3.5, these expressions entail that $(\{N\}, \boldsymbol{x})$ belongs to CS-*core*$(\Gamma_{\{N\},\text{-}\varepsilon})$. Hence, CS-LCV($\Gamma$) $\leq \varepsilon$.

*(Case $\varepsilon < 0$)* Assume that $(\Pi, \boldsymbol{x}) \in$ CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon})$, so that CS-LCV($\Gamma$) $\leq \varepsilon$. By Lemma 3.8 and since $\Gamma$ is cohesive, we derive that $\Pi = \{N\}$. Thus, $\Gamma_{\Pi,\text{-}\varepsilon} = \Gamma_{\text{-}\varepsilon}$ holds, and we immediately get that $\boldsymbol{x}$ belongs to *core*$(\Gamma_{\text{-}\varepsilon})$. That is, LCV($\Gamma$) $\leq \varepsilon$.

For the converse, assume that LCV($\Gamma$) $\leq \varepsilon$, and in particular that $\boldsymbol{x}$ is an element in *core*$(\Gamma_{\text{-}\varepsilon})$. First, it can be trivially observed that $\{N\}$ belongs to CS-*opt*$(\Gamma_{\text{-}\varepsilon})$. Indeed, by contradiction, if there is a feasible coalition structure $\Pi \neq \{N\}$ such that $\sum_{C\in\Pi} v_{\text{-}\varepsilon}(C) > v_{\text{-}\varepsilon}(N) = v(N)$, then we derive that $x(N) = \sum_{C\in\Pi} x(C) \geq \sum_{C\in\Pi} v_{\text{-}\varepsilon}(C) > v(N)$, which is impossible. By Fact 3.5, we hence get that $(\{N\}, \boldsymbol{x})$ belongs to CS-*core*$(\Gamma_{\text{-}\varepsilon})$. Since $\Gamma_{\text{-}\varepsilon} = \Gamma_{\{N\},\text{-}\varepsilon}$, we derive CS-LCV($\Gamma$) $\leq \varepsilon$. □

## 4. Computing stable configurations: games over complete graphs

Given the characterization results of the previous section, we can now start our analysis by considering games defined over the class of complete interaction graphs, so that any coalition is a feasible one. Let $K_n$ denote the complete graph on $n$ nodes, and let $\mathbf{K}$ be the class of all such complete graphs. Note that this is the classic setting considered in the literature. However, the picture emerging from earlier works is far from being complete. Indeed, very often, such works do not consider solution concepts specifically designed for games with coalition structures, but they tacitly assume that the goal is to distribute the total payoff available to the grand-coalition. Even further, several complexity results have been obtained either for very specific classes of games or for specific encodings (i.e., **P**-representations). Here, we provide a more complete and finer-grained picture, by studying characteristic functions that lead to either superadditive or non-superadditive games. In particular, note that while the hardness results we shall illustrate are specific for games defined over complete interaction graphs, membership results are instead more general and they will be shown to hold over any class of interaction graphs.

### 4.1. Arbitrary games

The complexity of solution concepts in the presence of coalition structures has been investigated by [42], which is up to now our main source of knowledge about this setting. The authors considered *weighted voting games* as a specific compact **P**-representation scheme, and they observed that CS-Core-NonEmptiness and CS-Core-Membership are **NP**-hard and co-**NP**-complete problems over them, respectively. Since these problems are feasible in polynomial time over weighted voting games without considering coalition structures, the results evidence that coalition structures provide an additional source of complexity to solution concepts. Note that an "upper bound" (i.e., a non-trivial membership result in some complexity class) for the non-emptiness problem was missing in [42], and looked for. Moreover, no analysis has been conducted for core-related concepts (different from the core itself), and over arbitrary classes of compact games.

In the rest of the section, we reconsider these problems, by providing membership results that hold on any **P**-representation $\mathcal{R}$ and even on core-related solution concepts. Note that the characterizations discussed in Section 3 greatly simplify the analysis.

We start by observing that the problems CS-CoS and CS-Core-NonEmptiness belong to the complexity class $\mathbf{\Delta_2^P}$ consisting of all decision problems that can be solved in polynomial time by exploiting **NP** *oracles*, i.e., non-deterministic Turing machines whose invocations are assumed to have unitary cost. The counterpart of $\mathbf{\Delta_2^P}$ over search problems, rather than decision ones, is denoted by $\mathbf{F\Delta_2^P}$. Note that $\mathbf{NP} \subseteq \mathbf{\Delta_2^P}$ and co-$\mathbf{NP} \subseteq \mathbf{\Delta_2^P}$ hold, with the inclusions being believed to be strict.[8]

**Theorem 4.1.** *Let $\mathcal{R}$ be any compact* **P***-representation, and let $\mathcal{G}$ be any class of graphs (e.g., $\mathcal{G} = \mathbf{K}$). On the class $\mathcal{C}_{\mathcal{G}}(\mathcal{R})$, the problems* CS-CoS *and* CS-Core-NonEmptiness *belong to* $\mathbf{\Delta_2^P}$.

**Proof.** Let us focus on the problem CS-CoS. We are given a game encoding $\xi^{\mathcal{R}}(\Gamma)$, and a rational number $\Delta \geq 0$, and we have to decide whether CS-CoS($\Gamma$) $\leq \Delta$.

By Fact 3.3, we know that CS-CoS($\Gamma$) = CoS($\tilde{\Gamma}$). Moreover, CoS($\tilde{\Gamma}$) $\leq \Delta$ holds if, and only if, the core of $\tilde{\Gamma}_{\Delta}$ is not empty. Since checking the non-emptiness of the core (when all coalitions are feasible ones) is in co-**NP** [44], the proof

[8] For expanding on concepts about complexity theory, the reader is referred to [43].

basically reduces to show that the value $SW_{opt}(\Gamma)$, whose knowledge is required to define the payoff associated with the grand-coalition in $\tilde{\Gamma}_\Delta$, can be computed in polynomial time by exploiting an **NP** oracle. Further technical ingredients will be eventually put in place in order to reduce the setting where some coalitions are not feasible to the setting where all coalitions are allowed to form, and in order to properly deal with arbitrary compact **P**-representations.

Let $\tilde{\Gamma}_\Delta$ be the game $\langle N, \tilde{v}_\Delta, G\rangle$ with $G \in \mathcal{G}$, and define $\Gamma' = \langle N, v', K_n\rangle$, with $N = \{1,\dots,n\}$, as the game where $v'(C) = \tilde{v}_\Delta(C)$, for each coalition in $\mathcal{F}(G)$ (in particular note that $\{i\} \in \mathcal{F}(G)$ holds, for each $i \in N$); and $v'(C) = \sum_{i\in C} v(\{i\})$, for each coalition $C \notin \mathcal{F}(G)$. Note that $K_n$ over which $\Gamma'$ is defined denotes a complete interaction graph; and, moreover, the core of $\tilde{\Gamma}_\Delta$ is not empty if, and only if, the core of $\Gamma'$ is not empty. Indeed, it is immediate to check that $core(\Gamma') \subseteq core(\tilde{\Gamma}_\Delta)$, given that $\mathcal{F}(G) \subseteq \mathcal{F}(K_n)$ and that the payoff functions of the two games coincide over coalitions in $\mathcal{F}(G)$. Moreover, note that if $\boldsymbol{x} \in core(\tilde{\Gamma}_\Delta)$, then $x_i \geq v(\{i\}) = v'(\{i\})$ holds, for each $i \in N$, because $\{i\}$ is in $\mathcal{F}(G)$. Hence, given a coalition $C \notin \mathcal{F}(G)$, we have that $x(C) = \sum_{i\in C} x_i \geq \sum_{i\in C} v(\{i\}) = v'(C)$.

In the light of the above observation, the strategy is to first build (an encoding for) the game $\Gamma'$. To this end, consider the representation $\mathcal{R}'$ defined as follows. For each game $\Gamma \in \mathcal{C}_\mathcal{G}(\mathcal{R})$ and for each rational number $\Delta \geq 0$, $\mathcal{C}(\mathcal{R}')$ contains the game $\Gamma'$ such that $\xi^{\mathcal{R}'}(\tilde{\Gamma}_\Delta)$ consists of the encoding of $\Gamma$ plus (a standard encoding for) the values $SW_{opt}(\Gamma)$ and $\Delta$. No further game is in $\mathcal{C}(\mathcal{R}')$. Moreover, for each game $\Gamma' \in \mathcal{C}(\mathcal{R}')$, we define $v^{\mathcal{R}'}(\xi^{\mathcal{R}'}(\Gamma'), C) = v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C)$, for each coalition $C \subset N$ with $C \in \mathcal{F}(G)$; $v^{\mathcal{R}'}(\xi^{\mathcal{R}'}(\Gamma'), C) = \sum_{i\in C} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), \{i\})$, for each coalition $C \subset N$ with $C \notin \mathcal{F}(G)$; and $v^{\mathcal{R}'}(\xi^{\mathcal{R}'}(\Gamma'), N) = SW_{opt}(\Gamma) + \Delta$ (in fact, $N$ is in $\mathcal{F}(G)$, because we assume that interaction graphs are connected). Note that $\mathcal{R}'$ is a **P**-representation, because $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), N)$ can be computed in polynomial time given the encoding $\xi^{\mathcal{R}}(\Gamma)$, which is a part of the encoding $\xi^{\mathcal{R}'}(\Gamma')$, and because $SW_{opt}(\Gamma)$ and $\Delta$ are also part of this encoding. However, in order to build (the encoding for) $\Gamma'$, we need to compute the value $SW_{opt}(\Gamma)$, which is not immediately available in the input $\xi^{\mathcal{R}}(\Gamma)$. We now claim that the computation of $SW_{opt}(\Gamma)$ is feasible in $\mathbf{F\Delta_2^P}$—recall that, for the sake of readability, proofs of claims are reported in Appendix A.

**Claim 4.2.** *Let $\mathcal{R}$ be any compact* **P***-representation, and let $\mathcal{G}$ be any class of graphs. On the class $\mathcal{C}_\mathcal{G}(\mathcal{R})$, computing the maximum social welfare and an optimal coalition structure is feasible in* $\mathbf{F\Delta_2^P}$.

So, we know that by using an **NP** oracle, we can build in polynomial time the (encoding for the) game $\Gamma'$, which belongs to the class $\mathcal{C}(\mathcal{R}')$ with $\mathcal{R}'$ being a **P**-representation. Eventually, it just remains to be checked whether $core(\Gamma')$ is empty. For this latter check, we notice that we are now considering the standard notion of core, and a class of compact games $\mathcal{C}(\mathcal{R}')$ where all coalitions are feasible ones. In this setting, the non-emptiness of the core is known to be feasible in co-**NP** [44]. Thus, by means of a further call to an oracle checking whether $core(\tilde{\Gamma}_\Delta)$ is empty, we are able to overall solve CS-CoS in $\mathbf{\Delta_2^P}$. Given this result, the fact that CS-Core-NonEmptiness is in $\mathbf{\Delta_2^P}$ is immediate, as the problem just reduces to CS-CoS by considering the cost $\Delta = 0$. □

We now move to the problems related to the least coalition structure core. In the statement below, we refer to the complexity class $\mathbf{\Sigma_2^P}$ consisting of all the decision problems that can be solved in polynomial time by a non-deterministic Turing machine using **NP** oracles with unitary cost.

**Theorem 4.3.** *Let $\mathcal{R}$ be any compact* **P***-representation, and let $\mathcal{G}$ be any class of graphs (e.g., $\mathcal{G} = \mathbf{K}$). On the class $\mathcal{C}_\mathcal{G}(\mathcal{R})$, the problems* CS-LCV *and* CS-LCV$^-$ *belong to $\mathbf{\Sigma_2^P}$ and $\mathbf{\Delta_2^P}$, respectively.*

**Proof.** Consider first the general problem CS-LCV. Given a rational number $\varepsilon$, we have to decide whether there is a feasible coalition structure $\Pi$ such that there is a pair of the form $(\Pi, \boldsymbol{x})$ in CS-$core(\Gamma_{\Pi,\text{-}\varepsilon})$. The problem can be solved by a non-deterministic Turing machine that first guesses (in **NP**) a feasible coalition structure $\Pi$. Given $\Pi$ and $\varepsilon$, we observe that $\Gamma_{\Pi,\text{-}\varepsilon}$ can be specified in compact form, via a **P**-representation, say $\mathcal{R}''$, by using the approach detailed in the proof of Theorem 4.1. Thus, the machine deals with a game specified in compact form and, according to Fact 3.5, it has then to check that:

(1) $\Pi \in$ CS-$opt(\Gamma_{\Pi,\text{-}\varepsilon})$;
(2) $core(\tilde{\Gamma}_{\Pi,\text{-}\varepsilon}) \neq \emptyset$.

Concerning (1), we consider the problem of checking the complementary condition that $\Pi$ is not optimal for $\Gamma_{\Pi,\text{-}\varepsilon}$. This problem can be solved by an **NP** oracle that guesses a feasible coalition structure $\bar{\Pi}$ and then checks that $v_{\Pi,\text{-}\varepsilon}(\bar{\Pi}) > v_{\Pi,\text{-}\varepsilon}(\Pi)$ holds.

Now, observe that if condition (1) holds, then the machine can proceed to build the game $\tilde{\Gamma}_{\Pi,\text{-}\varepsilon}$ by just noticing that the value associated with the grand-coalition is precisely $SW_{opt}(\Gamma_{\Pi,\text{-}\varepsilon}) = v_{\Pi,\text{-}\varepsilon}(\Pi)$. Therefore, (2) is reduced to checking whether the core of a given game (specified in compact form) is not empty. By following the same line of reasoning as in the proof of Theorem 4.1, we conclude that this task is feasible in co-**NP** (and hence we can use an oracle solving the complementary problem).

By putting it all together, we have then shown that CS-LCV can be solved in polynomial time by a non-deterministic Turing machine using **NP** oracles. Hence, the problem belongs to $\boldsymbol{\Sigma}_2^{\mathbf{P}}$.

Let us now move to the problem CS-LCV$^-$. The case $\varepsilon = 0$ trivially reduces to CS-Core-NonEmptiness, and hence can be solved in polynomial time with the use of **NP** oracles. We show that this also holds for the case $\varepsilon < 0$. Consider an algorithm that firstly computes the value $SW_{opt}(\Gamma)$ and a coalition structure $\Pi$ witnessing that $\sum_{C\in\Pi} v(C) = SW_{opt}(\Gamma)$, which is feasible in $\mathbf{F}\boldsymbol{\Delta}_2^{\mathbf{P}}$ by Claim 4.2.

Now, because of Lemma 3.8 and since we know that $\Pi \in$ CS-$opt(\Gamma)$ holds, it eventually remains to show that there is a pair of the form $(\Pi, \boldsymbol{x})$ in CS-$core(\Gamma_{\Pi,-\varepsilon})$. The task can be faced as discussed in the points (1) and (2) above, and hence again in polynomial time with the help of an **NP** oracle. It follows that CS-LCV$^-$ is in $\boldsymbol{\Delta}_2^{\mathbf{P}}$. $\square$

We next show that the above results are tight over games with complete interaction graphs, even on the class of games defined via marginal contribution networks.

Hardness results are established by reductions that refer to Boolean formulas. The following notation is hereinafter used with them. For any Boolean formula $\Phi$, we denote by $vars(\Phi)$ the set of all its variables. For any set $S$, we denote by $\sigma(S)$ the truth assignment where $X \in vars(\Phi)$ is true if, and only if, $X$ occurs in $S$. Moreover, we denote by $\sigma(S) \models \Phi$ the fact that $\sigma(S)$ satisfies $\Phi$, i.e., that $\sigma(S)$ is a satisfying truth assignment for $\Phi$. A literal is either a variable $X$ or a negated variable $\neg X$. For a variable $X$ (resp., negated variable $\neg X$), we denote by $\bar{X}$ (resp., $\overline{\neg X}$) the literal $\neg X$ (resp., $X$).

We start with the hardness results for $\boldsymbol{\Delta}_2^{\mathbf{P}}$.

**Theorem 4.4.** *On the class $\mathcal{C}_{\mathbf{K}}$(mcn) of games encoded via marginal contribution networks, the problems* CS-CoS, CS-Core-NonEmptiness, *and* CS-LCV$^-$ *are* $\boldsymbol{\Delta}_2^{\mathbf{P}}$*-hard.*

**Proof.** Let $\Phi$ be a *satisfiable* Boolean formula over the variables $a_1, \ldots, a_n$. In particular, assume that $\Phi$ is in conjunctive normal form (CNF), i.e., $\Phi = c_1 \wedge \cdots \wedge c_m$, where each clause $c_j$, $j \in \{1, \ldots, m\}$, is a disjunction of literals $y_{j,1} \vee \cdots \vee y_{j,m_j}$. Moreover, assume a variable ordering such that $a_i$ is less significant than $a_j$ if, and only if, $i < j$. This ordering induces a lexicographical ordering over truth assignments for $\Phi$. Deciding whether $a_1$ is true in the lexicographically maximal satisfying assignment $\sigma^*$ of $\Phi$ is $\boldsymbol{\Delta}_2^{\mathbf{P}}$-complete [45].

Based on $\Phi$, we build in polynomial time a structurally-restricted characteristic function game $\Gamma(\Phi) \in \mathcal{C}_{\mathbf{K}}$(mcn) whose encoding $\xi^{\text{mcn}}(\Gamma(\Phi))$ is defined by the MC-net over the variables/agents in $N = vars(\Phi) \cup \{a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}\}$ with the rules:

$$\left\{\begin{array}{rcl}
\{a_{n+4} \wedge \neg a_{n+1} \wedge \neg a_{n+2} \wedge \neg a_{n+3} \wedge a_i\} & \rightarrow & 2^i, \forall i \in \{1, \ldots, n\} \\
\{a_{n+4} \wedge a_{n+1} \wedge a_{n+2} \wedge a_{n+3} \wedge a_i\} & \rightarrow & 2^i, \forall i \in \{1, \ldots, n\} \\
\{a_{n+4} \wedge a_{n+1} \wedge a_{n+2} \wedge a_{n+3} \wedge a_1\} & \rightarrow & 1 \\
\{\neg a_{n+4} \wedge a_{n+1} \wedge a_{n+2} \wedge a_{n+3} \wedge \bigwedge_{i=1}^n \neg a_i\} & \rightarrow & 2/3 \\
\{\neg a_{n+4} \wedge \neg a_{n+1} \wedge a_{n+2} \wedge a_{n+3} \wedge \bigwedge_{i=1}^n \neg a_i\} & \rightarrow & 2/3 \\
\{\neg a_{n+4} \wedge a_{n+1} \wedge \neg a_{n+2} \wedge a_{n+3} \wedge \bigwedge_{i=1}^n \neg a_i\} & \rightarrow & 2/3 \\
\{\neg a_{n+4} \wedge a_{n+1} \wedge a_{n+2} \wedge \neg a_{n+3} \wedge \bigwedge_{i=1}^n \neg a_i\} & \rightarrow & 2/3 \\
\{a_{n+4} \wedge a_{n+1} \wedge a_{n+2} \wedge a_{n+3} \wedge \neg a_1\} & \rightarrow & -2 - \sum_{i=1}^n 2^i \\
\{a_{n+4} \wedge \neg a_{n+1} \wedge \neg a_{n+2} \wedge \neg a_{n+3} \wedge \bar{y}_{j,1} \wedge \cdots \wedge \bar{y}_{j,m_j}\} & \rightarrow & -2 - \sum_{i=1}^n 2^i, \forall j \in \{1, \ldots, m\} \\
\{a_{n+4} \wedge a_{n+1} \wedge a_{n+2} \wedge a_{n+3} \wedge \bar{y}_{j,1} \wedge \cdots \wedge \bar{y}_{j,m_j}\} & \rightarrow & -2 - \sum_{i=1}^n 2^i, \forall j \in \{1, \ldots, m\}
\end{array}\right.$$

Note that, according to the definition of the payoff function $v^{\text{mcn}}$ for marginal contribution networks, $v^{\text{mcn}}(\xi^{\text{mcn}}(\Gamma(\Phi)), C) \leq \sum_{i=1}^n 2^i + 1$ holds, for each coalition $C \subseteq N$. Hence, whenever one of the above three last (groups of) rules applies to a given $C$, its associated payoff is a negative value. In particular, note that the two last (groups of) rules can apply to a coalition $C$ only if the truth assignment associated with $C$ does not satisfy one of the $m$ clauses in $\Phi$, i.e., only if $\sigma(C \setminus \{a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}\}) \models \Phi$ does not hold. With the above observations in place, one can check that the payoff function, henceforth denoted as $v$ given that the representation scheme is understood, is such that:

$$v(C) = \begin{cases}
w(S), & \text{if } C = \{a_{n+4}\} \cup S,\ S \subseteq \{a_1, \ldots, a_n\}, \text{ and } \sigma(S) \models \Phi \\
w(S) + 1, & \text{if } C = \{a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}\} \cup S,\ \sigma(S) \models \Phi, \text{ and } a_1 \in S \\
2/3, & \text{if } C \subseteq \{a_{n+1}, a_{n+2}, a_{n+3}\} \text{ and } |C| \geq 2 \\
0, & \text{if } |C| = 1, \text{ with } C \neq \{a_{n+4}\} \\
\leq 0, & \text{otherwise,}
\end{cases}$$

where $w(S) = \sum_{a_i \in S} 2^i$. In particular, note that $v(C) = 0$, for each $C \subseteq \{a_1, \ldots, a_n\}$.

Let $S^* \subseteq \{a_1, \ldots, a_n\}$ be the coalition such that $\sigma(S^*) = \sigma^*$. Given the above expression for the payoff function, the following two properties are easily seen to hold.

**Property 4.4.1** Assume that $a_1$ evaluates true in the lexicographically maximal satisfying assignment of $\Phi$. Then, the coalition structure $\Pi^*_t$ including $S^* \cup \{a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}\}$, plus each agent in $\{a_1, \dots, a_n\} \setminus S^*$ as a singleton is in CS-*opt*$(\Gamma(\Phi))$.
**Property 4.4.2** Assume that $a_1$ evaluates false in the lexicographically maximal satisfying assignment of $\Phi$. Then, the coalition structure $\Pi^*_f$ including $S^* \cup \{a_{n+4}\}$, $\{a_{n+1}, a_{n+2}, a_{n+3}\}$, and each agent in $\{a_1, \dots, a_n\} \setminus S^*$ as a singleton is an optimal coalition structure, i.e., it is in CS-*opt*$(\Gamma(\Phi))$.

Let $\tilde{\Gamma}$ be the cohesive cover of $\Gamma(\Phi)$ defined over the function $\tilde{v}$. By Fact 3.5, we known that CS-*core*$(\Gamma(\Phi)) \neq \emptyset$ if, and only if, *core*$(\tilde{\Gamma}) \neq \emptyset$. So, we prove that CS-Core-NonEmptiness is $\mathbf{\Delta_2^P}$-hard by showing that *core*$(\tilde{\Gamma}) \neq \emptyset$ if, and only if, $a_1 \in S^*$:

($\Leftarrow$) Assume that $a_1 \in S^*$, so that $\Pi^*_t$ is an optimal coalition structure (cf. Property 4.4.1). In particular, we have $SW(\Pi^*_t) = w(S^*) + 1$ and $\tilde{v}(N) = w(S^*) + 1$ holds. Consider then the allocation $\boldsymbol{x}$ such that $x_{n+4} = w(S^*)$, $x_{n+1} = x_{n+2} = x_{n+3} = 1/3$, and $x_i = 0$, for any other agent. Note that $x(N) = \tilde{v}(N)$. Moreover, we claim that $\boldsymbol{x}$ belongs to the core of $\tilde{\Gamma}$. Indeed, for each coalition $C \subset N$ such that $a_{n+4} \in C$ and $|\{a_{n+1}, a_{n+2}, a_{n+3}\} \cap C| \leq 2$, $x(C) \geq w(S^*) \geq v(C) = \tilde{v}(C)$; for each coalition $C \subset N$ such that $a_{n+4} \in C$ and $C \supseteq \{a_{n+1}, a_{n+2}, a_{n+3}\}$, $x(C) \geq w(S^*) + 1 \geq v(C) = \tilde{v}(C)$; for each coalition $C \subset N$ such that $\{a_{n+1}, a_{n+2}, a_{n+3}\} \supseteq C$ and $|C| \geq 2$, $x(C) = |C| \times 1/3 \geq 2/3 \geq v(C) = \tilde{v}(C)$; and for each remaining coalition $C$, $x(C) \geq 0 \geq v(C) = \tilde{v}(C)$.
($\Rightarrow$) Assume that $a_1 \notin S^*$, so that $\Pi^*_f$ is an optimal coalition structure (cf. Property 4.4.2). In particular, we have $SW(\Pi^*_f) = w(S^*) + 2/3$ and $\tilde{v}(N) = w(S^*) + 2/3$. Assume, for the sake of contradiction, that $\boldsymbol{x}$ is an allocation in the core of $\tilde{\Gamma}$. Then, we have $x(N) = w(S^*) + 2/3$ and $x(S^* \cup \{a_{n+4}\}) \geq w(S^*)$. Therefore, $x_{n+1} + x_{n+2} + x_{n+3} \leq 2/3$, which is impossible since $x(C) \geq 2/3 = \tilde{v}(C) = v(C)$ must hold, for each $C \subseteq \{a_{n+1}, a_{n+2}, a_{n+3}\}$ such that $|C| \geq 2$.

To conclude, we observe that for any given game $\Gamma$, checking CS-CoS$(\Gamma) \leq 0$ and CS-LCV$(\Gamma) \leq 0$ are both equivalent to check that CS-*core*$(\Gamma) \neq \emptyset$. Therefore, $\mathbf{\Delta_2^P}$-hardness for CS-CoS and CS-LCV follows by the above result. □

In the above proof, it is instructive to observe that the source of complexity of the basic CS-Core-NonEmptiness problem is related to the need of reasoning about optimal coalition structures (cf. Property 4.4.1 and Property 4.4.2). In fact, by inspecting the proof, we can observe that the optimal social welfare in the game $\Gamma(\Phi)$ is an integer value (having the form $w(S^*) + 1$) if, and only if, $a_1$ is in $S^*$. This immediately implies that the bound provided in Claim 4.2 is tight, as we formalize below.

**Theorem 4.5.** *On the class $\mathcal{C}_{\mathbf{K}}$(mcn) of games encoded via marginal contribution networks, computing the maximum social welfare is $\mathbf{\Delta_2^P}$-hard.*

The picture of the hardness results corresponding to the membership results in Theorem 4.3 is completed below. Recall that CS-LCV$^+$ is the restriction of CS-LCV where the rational number $\varepsilon$ is constrained to be non-negative. Formally, given $\varepsilon \geq 0$, CS-LCV$^+$ asks whether there is a feasible coalition structure $\Pi$ such that CS-*core*$(\Gamma_{\Pi,-\varepsilon})$ contains an imputation of the form $(\Pi, \boldsymbol{x})$. To analyze this problem, note that even if a feasible coalition structure $\Pi$ was given in advance, checking whether CS-*core*$(\Gamma_{\Pi,-\varepsilon})$ is not empty would be still co-**NP**-hard (this easily follows from the co-**NP**-hardness of checking whether the core is not empty for games specified as marginal contribution networks [25]). Of course, $\Pi$ is not actually provided, and has to be identified out of all possible feasible coalition structures (or we have to decide that no feasible coalition structure of that kind exists). This provides a basic source of **NP**-hardness on top of which the above co-**NP**-hard task is defined, leading to a $\mathbf{\Sigma_2^P}$-hard problem. In fact, we have observed that whenever $\varepsilon \leq 0$ holds, the two sources of complexity do not interplay (cf. proof of Theorem 4.3), so that more favorable complexity results have been obtained.

**Theorem 4.6.** *On the class $\mathcal{C}_{\mathbf{K}}$(mcn) of games encoded via marginal contribution networks,* CS-LCV$^+$ *(and, hence,* CS-LCV*) is $\mathbf{\Sigma_2^P}$-hard.*

**Proof.** Let $\Phi = c_1 \wedge \cdots \wedge c_m$ be a Boolean (CNF) formula over the variables $a_1, \dots, a_n, b_1, \dots, b_\ell$, where $c_j = y_{j,1} \vee \cdots \vee y_{j,k_j}$, for each $j \in \{1, \dots, m\}$. Consider the problem $\exists\forall$CNF-UNSAT of deciding whether there is an assignment $\sigma^a$ over $N_\exists = \{a_1, \dots, a_n\}$ (the set of "existentially" quantified variables) such that, for each assignment $\sigma^b$ over $N_\forall = \{b_1, \dots, b_\ell\}$ (the set of "universally" quantified variables), $\sigma^a \uplus \sigma^b$ does not satisfy $\Phi$, where $\sigma^a \uplus \sigma^b$ is the truth assignment for the whole set of variables in $\Phi$ agreeing with $\sigma^a$ and $\sigma^b$ on their respective domains. This is a well-known $\mathbf{\Sigma_2^P}$-complete problem—in fact, the complementary problem of deciding whether there is no such an assignment is the prototypical hard problem for the complementary complexity class $\mathbf{\Pi_2^P}$ [43].

We shall show that $\exists\forall$CNF-UNSAT can be reduced to CS-LCV$^+$. In particular, we shall restrict ourselves to certain kinds of Boolean functions only. To state the restriction more formally, for any truth assignment $\sigma$, let $\bar{\sigma}$ denote the truth assignment that is complementary w.r.t. $\sigma$, i.e., $\bar{\sigma}(X)$ is true if, and only if, $\sigma(X)$ is false.

**Claim 4.7.** $\exists\forall$CNF-UNSAT *is $\mathbf{\Sigma_2^P}$-hard even restricted to formulas $\Psi$ such that, for each pair of assignments $\sigma^a$ and $\sigma^b$ over the existentially and universally quantified variables of $\Psi$, respectively, $\sigma^a \uplus \sigma^b \models \Psi$ if, and only if, $\sigma^a \uplus \bar{\sigma}^b \models \Psi$.*

According to the above result, the formula $\Phi$ provided as input over the variables in the sets $N_\exists$ and $N_\forall$ is assumed, w.l.o.g., to be such that: $\sigma^a \uplus \sigma^b \models \Phi$ if, and only if, $\sigma^a \uplus \bar{\sigma}^b \models \Phi$. Based on $\Phi$ (which will be omitted from the notation, as it is understood), we define a game over the set $N = \bigcup_{i=1}^{n}\{a_i, \hat{a}_i\} \cup \bigcup_{j=1}^{\ell}\{b_j, \hat{b}_j\} \cup \{r\}$ of agents[9] as follows.

For any set $C \subseteq N$ of agents, we say that $C$ is $\exists$-*consistent* (resp., $\forall$-*consistent*) if $|C \cap \{a_i, \hat{a}_i\}| = 1$ holds, for each $i \in \{1, \dots, n\}$ (resp., $|C \cap \{b_i, \hat{b}_i\}| = 1$ holds, for each $i \in \{1, \dots, \ell\}$). Moreover, we denote by $\hat{C}$ the set $\{\hat{z} \mid z \in C\} \cup \{z \mid \hat{z} \in C\}$. Let $\varepsilon \geq 0$ be a rational number whose value will be later specified. Then, consider the structurally-restricted characteristic function game $\Gamma^{\exists\forall} = \langle N, v, K_{|N|}\rangle$ such that $N = \{r\} \cup N_\exists \cup \hat{N}_\exists \cup N_\forall \cup \hat{N}_\forall$ and where $v$ satisfies the following:

$$v(C) = \begin{cases} 1, & \text{if } r \in C, C \cap (N_\forall \cup \hat{N}_\forall) = \emptyset, \text{ and } C \text{ is } \exists\text{-consistent} \\ \varepsilon + \varepsilon/n, & \text{if } r \notin C, C \text{ is } \exists\text{- and } \forall\text{-consistent and}, \sigma(C) \models \Phi \\ 0, & \text{if } r \notin C, C \cap (N_\exists \cup \hat{N}_\exists) = \emptyset, \text{ and } C \supseteq N_\forall \cup \hat{N}_\forall \\ 0, & \text{if } r \notin C, C \cap (N_\forall \cup \hat{N}_\forall) = \emptyset, \text{ and } C \text{ is } \exists\text{-consistent} \\ \leq -|N|, & \text{otherwise} \end{cases}$$

**Claim 4.8.** *Assume that $\varepsilon = \frac{1}{3}$. Then,* CS-*opt*$(\Gamma^{\exists\forall}_{\Pi,\text{-}\varepsilon}) = \{\{A \cup \{r\}, \hat{A}, N_\forall \cup \hat{N}_\forall\} \mid A \subseteq N_\exists \cup \hat{N}_\exists$ *is* $\exists$-*consistent*$\}$.

Our goal is now to show that it is $\mathbf{\Sigma_2^P}$-hard to decide whether the least coalition structure core value of $\Gamma^{\exists\forall}$ is $\varepsilon = \frac{1}{3}$ at most. To this end, recall that CS-LCV$(\Gamma) \leq \varepsilon$ holds if, and only if, there is a pair $(\Pi, \boldsymbol{x})$ in CS-*core*$(\Gamma^{\exists\forall}_{\Pi,\text{-}\varepsilon})$. Given Claim 4.8 and Fact 3.5, we can restrict ourselves to coalition structures $\Pi_A = \{A \cup \{r\}, \hat{A}, N_\forall \cup \hat{N}_\forall\}$, where $A \subseteq N_\exists \cup \hat{N}_\exists$ is $\exists$-consistent. In particular, we show the desired result by proving that there is a $\exists$-consistent set $A \subseteq N_\exists \cup \hat{N}_\exists$ with CS-*core*$(\Gamma^{\exists\forall}_{\Pi_A,\text{-}\varepsilon}) \neq \emptyset$ if, and only if, there is an assignment $\sigma^a$ over $N_\exists$ such that, for each assignment $\sigma^b$ over $N_\forall$, $\sigma^a \uplus \sigma^b \not\models \Phi$.

($\Leftarrow$) Assume there is an assignment $\sigma^a$ over $N_\exists$ such that, for each assignment $\sigma^b$ over $N_\forall$, $\sigma^a \uplus \sigma^b \not\models \Phi$. Let $A \subseteq N_\exists \cup \hat{N}_\exists$ be the $\exists$-consistent set such that $\sigma(\hat{A}) = \sigma^a$—recall that, for any set $S$, we denote by $\sigma(S)$ the truth assignment where $X \in vars(\Phi)$ is true if, and only if, $X$ occurs in $S$. Moreover, consider the payoff vector $\boldsymbol{x}$ such that: $x(\{r\}) = 1 - \varepsilon$; $x(\{z\}) = \varepsilon/n$, for each $z \in A$; and $x(\{z\}) = 0$, for each $z \in N \setminus (A \cup \{r\})$. We now show that $(\Pi_A, \boldsymbol{x})$ belongs to CS-*core*$(\Gamma^{\exists\forall}_{\Pi_A,\text{-}\varepsilon})$. To this end, observe first that the following inequalities hold: $x(A \cup \{r\}) = x(\{r\}) + \sum_{z\in A} x(\{z\}) = 1 = v(A \cup \{r\}) = v_{\Pi_A,\text{-}\varepsilon}(A \cup \{r\})$; $x(\hat{A}) = 0 = v(\hat{A}) = v_{\Pi_A,\text{-}\varepsilon}(\hat{A})$; and $x(N_\forall \cup \hat{N}_\forall) = 0 = v(N_\forall \cup \hat{N}_\forall) = v_{\Pi_A,\text{-}\varepsilon}(N_\forall \cup \hat{N}_\forall)$. Moreover, $x(\{z\}) \geq 0 \geq v(\{z\}) \geq v_{\Pi_A,\text{-}\varepsilon}(\{z\})$ holds, for each $z \in N$. Hence, $\boldsymbol{x}$ is an imputation for $\Pi_A$. It remains to show that $\boldsymbol{x}$ is stable, in that $x(C) \geq v_{\Pi_A,\text{-}\varepsilon}(C)$ holds, for each $C \subseteq N$. Since $x(C) \geq 0$, we focus below on the two only possible kinds of coalitions $C$ such that $v_{\Pi_A,\text{-}\varepsilon}(C) > 0$.

(1) Consider first a coalition $C$ such that $r \in C$, $C \cap (N_\forall \cup \hat{N}_\forall) = \emptyset$, and with $C$ being $\exists$-consistent. If $C = A \cup \{r\}$, then we already know that $x(C) = v_{\Pi_A,\text{-}\varepsilon}(C)$. If $C \neq A \cup \{r\}$ holds, then $v_{\Pi_A,\text{-}\varepsilon}(C) = 1 - \varepsilon$. In fact, $x(\{r\}) = 1 - \varepsilon$ and hence $x(C) \geq x(\{r\}) = 1 - \varepsilon = v_{\Pi_A,\text{-}\varepsilon}(C)$.

(2) Consider now any coalition $C$ such that $r \notin C$ and such that $C$ is both $\exists$-consistent and $\forall$-consistent. Note that $v_{\Pi_A,\text{-}\varepsilon}(C) = v(C) - \varepsilon$ and, hence, that $v_{\Pi_A,\text{-}\varepsilon}(C) > 0$ (in particular, $v_{\Pi_A,\text{-}\varepsilon}(C) = \varepsilon/n$) if, and only if, $\sigma(C) \models \Phi$. We have now two sub-cases. If $C \cap A \neq \emptyset$, then $x(C) \geq \varepsilon/n \geq v_{\Pi_A,\text{-}\varepsilon}(C)$ trivially holds. So, assume that $C \cap A = \emptyset$. In this case, $x(C) = 0$ holds and $\hat{A} \subseteq C$. Now recall that $\sigma(\hat{A}) = \sigma^a$ holds, by construction of $\hat{A}$, and define $\sigma^b = \sigma(C \cap (N_\forall \cup \hat{N}_\forall))$. Then, observe that $\sigma(C) = \sigma^a \uplus \sigma^b$ and we know by hypothesis that $\sigma^a \uplus \sigma^b \not\models \Phi$. Therefore, $v_{\Pi_A,\text{-}\varepsilon}(C) < 0$ and we have $x(C) = 0 > v_{\Pi_A,\text{-}\varepsilon}(C)$.

($\Rightarrow$) Consider an element in CS-*core*$(\Gamma^{\exists\forall}_{\Pi_A,\text{-}\varepsilon})$, and recall that it must have the form $(\Pi_A, \boldsymbol{x})$. By definition of the coalition structure core, we know that $x(A \cup \{r\}) = 1$, $x(\hat{A}) = 0$, and $x(N_\forall \cup \hat{N}_\forall) = 0$. Moreover, $x(C) \geq v_{\Pi_A,\text{-}\varepsilon}(C) = v(C) - \varepsilon$ holds, for each coalition $C$. Let $B \subseteq N_\forall \cup \hat{N}_\forall$ be any $\forall$-consistent set. Then, we get: $x(\hat{A} \cup B) = x(B) \geq v(\hat{A} \cup B) - \varepsilon$ and $x(\hat{A} \cup \hat{B}) = x(\hat{B}) \geq v(\hat{A} \cup \hat{B}) - \varepsilon$. Assume now, for the sake of contradiction, that $\sigma(\hat{A}) \uplus \sigma(B) \models \Phi$. Because of our assumption based on Claim 4.7, we also know that $\sigma(\hat{A}) \uplus \sigma(\hat{B}) \models \Phi$. Therefore, we have $x(B) \geq \varepsilon/n$ and $x(\hat{B}) \geq \varepsilon/n$, which is impossible since $x(B \cup \hat{B}) = x(N_\forall \cup \hat{N}_\forall) = 0$. By noticing that $\sigma(B)$ ranges over all possible truth assignments for the variables in $N_\forall$, $\sigma^a = \sigma(\hat{A})$ therefore witnesses that for each assignment $\sigma^b$ over $N_\forall$, $\sigma^a \uplus \sigma^b \not\models \Phi$.

The last step of the proof is to show that the payoff function $v$ associated with the game $\Gamma^{\exists\forall}$ can be actually encoded (in polynomial time) in terms of a marginal contribution network. To this end, consider first the following (sets of) rules, where $M$ is a sufficiently large positive value (e.g., greater than $2 \times |N|$):

[9] As usual, each agent is intended as being univocally mapped to a natural number, so that $N = \{1, \dots, 2 \times n + 2 \times \ell + 1\}$ holds in this case. In fact, in order to help the intuition in some of the proofs where agents belong to different conceptual classes, we transparently identify the agents with their "names", rather than with their associated indices.

$$R_1 = \begin{cases} \{r \wedge b_i\} \rightarrow -M, & \forall i \in \{1,\dots,\ell\} \\ \{r \wedge \hat{b}_i\} \rightarrow -M, & \forall i \in \{1,\dots,\ell\} \\ \{r \wedge \neg b_1 \wedge \cdots \wedge \neg b_\ell \wedge \neg \hat{b}_1 \wedge \cdots \wedge \neg \hat{b}_\ell \wedge a_i \wedge \hat{a}_i\} \rightarrow -M, & \forall i \in \{1,\dots,n\} \\ \{r \wedge \neg b_1 \wedge \cdots \wedge \neg b_\ell \wedge \neg \hat{b}_1 \wedge \cdots \wedge \neg \hat{b}_\ell \wedge \neg a_i \wedge \neg \hat{a}_i\} \rightarrow -M, & \forall i \in \{1,\dots,n\} \end{cases}$$

$$R_2 = \begin{cases} \{\neg r \wedge b_1 \wedge \cdots \wedge b_\ell \wedge \hat{b}_1 \wedge \cdots \wedge \hat{b}_\ell \wedge a_i\} \rightarrow -M, & \forall i \in \{1,\dots,n\} \\ \{\neg r \wedge b_1 \wedge \cdots \wedge b_\ell \wedge \hat{b}_1 \wedge \cdots \wedge \hat{b}_\ell \wedge \hat{a}_i\} \rightarrow -M, & \forall i \in \{1,\dots,n\} \end{cases}$$

$$R_3 = \begin{cases} \{\neg r \wedge \neg Z \wedge a_i \wedge \hat{a}_i\} \rightarrow -M, & \forall i \in \{1,\dots,n\}, \\ & \forall Z \in (N_\forall \cup \hat{N}_\forall), \\ \{\neg r \wedge \neg Z \wedge \neg a_i \wedge \neg \hat{a}_i\} \rightarrow -M, & \forall i \in \{1,\dots,n\}, \\ & \forall Z \in (N_\forall \cup \hat{N}_\forall), \end{cases}$$

$$R_4 = \begin{cases} \{\neg r \wedge \neg Z \wedge Z' \wedge b_i \wedge \hat{b}_i\} \rightarrow -M, & \forall i \in \{1,\dots,\ell\}, \\ & \forall Z \in (N_\forall \cup \hat{N}_\forall), \\ & \forall Z' \in (N_\forall \cup \hat{N}_\forall) \\ \{\neg r \wedge \neg Z \wedge Z' \wedge \neg b_i \wedge \neg \hat{b}_i\} \rightarrow -M, & \forall i \in \{1,\dots,\ell\}, \\ & \forall Z \in (N_\forall \cup \hat{N}_\forall), \\ & \forall Z' \in (N_\forall \cup \hat{N}_\forall) \end{cases}$$

$$R_5 = \begin{cases} \{\neg r \wedge Z \wedge W \wedge \bar{y}_{j,1} \wedge \cdots \wedge \bar{y}_{j,k_j}\} \rightarrow -M, & \forall j \in \{1,\dots,m\}, \\ & \forall Z \in (N_\forall \cup \hat{N}_\forall), \\ & \forall W \in (N_\exists \cup \hat{N}_\exists) \end{cases}$$

Observe that whenever some rules apply to any coalition $C \subseteq N$, then the payoff function $v_R$ induced by them is such that $v_R(C) \leq -M$ holds. In particular, $v_R(C) = 0$ if, and only if, none of these rules applies. When $r$ is in $C$, then no rule applies whenever $C \cap (N_\forall \cup \hat{N}_\forall) = \emptyset$ and $C$ is $\exists$-consistent, because of $R_1$. Assume then that $r \notin C$ holds, and note that no rule applies whenever all the above conditions hold:

- $C \supseteq (N_\forall \cup \hat{N}_\forall)$ and $C \cap (N_\exists \cup \hat{N}_\exists) = \emptyset$; or $C \not\supseteq (N_\forall \cup \hat{N}_\forall)$; (by $R_2$)
- $C \not\supseteq (N_\forall \cup \hat{N}_\forall)$ and $C$ is $\exists$-consistent; or $C \supseteq (N_\forall \cup \hat{N}_\forall)$; (by $R_3$)
- $C \not\supseteq (N_\forall \cup \hat{N}_\forall)$, $C \cap (N_\forall \cup \hat{N}_\forall) \neq \emptyset$, and $C$ is $\forall$-consistent; or $C \supseteq (N_\forall \cup \hat{N}_\forall)$; or $C \cap (N_\forall \cup \hat{N}_\forall) = \emptyset$; (by $R_4$)
- $C \cap (N_\exists \cup \hat{N}_\exists) \neq \emptyset$, $C \cap (N_\forall \cup \hat{N}_\forall) \neq \emptyset$, and $\sigma(C) \models \Phi$; or $C \cap (N_\exists \cup \hat{N}_\exists) = \emptyset$; or $C \cap (N_\forall \cup \hat{N}_\forall) = \emptyset$ (by $R_5$)

By putting it all together and by simple Boolean manipulations, we derive that none of the above rule applies to a coalition $C$ if, and only if, we are in one of the scenarios for which the payoff function $v$ returns a non-negative value. That is, for each $C \subseteq N$, $v_R(C) = 0$ (resp., $v_R(C) \leq -M$) if, and only if, $v(C) \geq 0$ (resp., $v(C) < 0$). Given this property, the function $v$ can be encoded via the rules in $R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5$ plus the following set of rules taking care of the coalitions where $v$ gets positive values:

$$R_6 = \begin{cases} \{r\} \rightarrow 1 \\ \{\neg r \wedge a_1 \wedge b_1\} \rightarrow \varepsilon + \varepsilon/n \\ \{\neg r \wedge a_1 \wedge \hat{b}_1\} \rightarrow \varepsilon + \varepsilon/n \\ \{\neg r \wedge \hat{a}_1 \wedge b_1\} \rightarrow \varepsilon + \varepsilon/n \\ \{\neg r \wedge \hat{a}_1 \wedge \hat{b}_1\} \rightarrow \varepsilon + \varepsilon/n \end{cases}$$

In fact, note that for each coalition $C$ such that $v_R(C) = 0$, precisely one of the above rules can apply. On the other hand, more than one rule can apply to some coalition $C$ such that $v_R(C) \leq -M$. However, in this case, we only would like to guarantee that $v(C) \leq -|N|$ holds. Indeed, this immediately follows by the choice of $M$ in the construction, and given that each rule in $R_6$ additively contributes 1 at most (for $\varepsilon = \frac{1}{3}$). $\square$

### 4.2. Superadditive games

Let us move to analyze the complexity of superadditive games. The complexity of core-related concepts for these games has been investigated by [5], where the authors showed that deciding whether the core is not empty is an **NP**-complete problem, even when restricted to superadditive games based on *synergies among coalitions*. However, as they explicitly pointed out, even computing the payoff associated with some given coalition is **NP**-hard in the proposed encoding, i.e., the proposed setting does not fit our definition of a **P**-representation scheme. Therefore, the intractability of the core comes with no surprise there, and we still did not know so far whether core-related problems remain intractable over superadditive games whose payoff functions can moreover be computed in polynomial-time. Below, we provide an answer to this open question.

In the following, if $\mathcal{C}_{\mathcal{G}}(\mathcal{R})$ is a class of structurally-restricted characteristic function games, then we denote by $\mathcal{C}^{sa}_{\mathcal{G}}(\mathcal{R})$ the subclass of all superadditive games $\Gamma = \langle N, v, G\rangle$ in $\mathcal{C}_{\mathcal{G}}(\mathcal{R})$—recall that a game is superadditive if whenever three feasible coalitions $S, T, S \cup T \in \mathcal{F}(G)$ are given with $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$ holds. In fact, throughout this section, we limit our attention to the case of complete interaction graphs, hence to $\mathcal{C}^{sa}_{\mathbf{K}}(\mathcal{R})$.

We start by studying the complexity of the CS-Core-Membership problem, which has not been addressed so far in our analysis. We again focus on games encoded via marginal contribution networks.

**Theorem 4.9.** *On the class $\mathcal{C}^{sa}_{\mathbf{K}}(\mathsf{mcn})$ (hence, on $\mathcal{C}_{\mathbf{K}}(\mathsf{mcn})$) of games encoded via marginal contribution networks,* CS-Core-Membership *is* co-**NP***-hard even if each rule contains at most one negated literal/agent and if the associated values are non-negative.*

**Proof.** Let $\Phi = c_1 \wedge \cdots \wedge c_m$ be a Boolean formula in conjunctive normal form over the variables $a_1, \ldots, a_n$, such that each clause $c_j$, with $j \in \{1, \ldots, m\}$, is a disjunction of three literals having the form $y_{j,1} \vee y_{j,2} \vee y_{j,3}$. A *not-all-equal* (satisfying) truth assignment $\sigma$ for $\Phi$ is such that, for each clause $c_j$, at least one of its literals evaluates true in it and one of its literals evaluates false in it. Deciding whether there exists a not-all-equal truth assignment is **NP**-hard, and it is well-known and easy to see that the problem remains intractable in its *monotone* variant, i.e., when $\Phi$ does not contain negated variables. In particular, the **NP**-hardness of the monotone variant when each clause contains *at most* three variables (and, w.l.o.g., at least two variables) follows by Schaefer's dichotomy theorem [46]. Moreover, observe that "yes" instances are preserved if each clause $c_j = y_{j,1} \vee y_{j,2}$ including two variables is replaced by the conjunction of the clauses $c^1_j = y_{j,1} \vee y_{j,2} \vee \alpha_j$, $c^2_j = y_{j,1} \vee y_{j,2} \vee \beta_j$, $c^3_j = y_{j,1} \vee y_{j,2} \vee \gamma_j$, and $c^4_j = \alpha_j \vee \beta_j \vee \gamma_j$, with $\alpha_j$, $\beta_j$, and $\gamma_j$ being fresh variables.

In the following, for a truth assignment $\sigma$, we denote by $\gamma_\sigma(\Phi)$ the number of clauses of $\Phi$ where at least one of its literals evaluates true in $\sigma$ and one of its literals evaluates false in $\sigma$. Note that $\gamma_\sigma(\Phi) = m$ holds if, and only if, $\sigma$ is a not-all-equal satisfying truth assignment. Moreover, note that in the monotone setting we are considering, the assignments $\sigma(\{\})$ and $\sigma(\{a_1, \ldots, a_n\})$ are not not-all-equal satisfying ones.

Based on $\Phi$, we build a structurally-restricted characteristic function game $\Gamma_{nae}(\Phi) \in \mathcal{C}_{\mathbf{K}}(\mathsf{mcn})$ whose encoding $\xi^{\mathsf{mcn}}(\Gamma_{nae}(\Phi))$ consists of the MC-net $M_{nae}(\Phi)$ defined over the variables/agents in $N = vars(\Phi) \cup \{a_{n+1}\}$ and including the following rules:

$$\begin{cases} \{a_{n+1} \wedge a_i\} & \rightarrow \; 1, \forall i \in \{1, \ldots, n\} \\ \{a_{n+1} \wedge a_1 \wedge \cdots \wedge a_n\} & \rightarrow \; 1 - 1/m - \gamma_{\sigma(\{a_1,\ldots,a_n\})}(\Phi)/m \\ \{a_{n+1} \wedge \bar{y}_{j,1} \wedge y_{j,2}\} & \rightarrow \; 1/(2 \times m), \forall j \in \{1, \ldots, m\} \\ \{a_{n+1} \wedge \bar{y}_{j,1} \wedge y_{j,3}\} & \rightarrow \; 1/(2 \times m), \forall j \in \{1, \ldots, m\} \\ \{a_{n+1} \wedge \bar{y}_{j,2} \wedge y_{j,1}\} & \rightarrow \; 1/(2 \times m), \forall j \in \{1, \ldots, m\} \\ \{a_{n+1} \wedge \bar{y}_{j,2} \wedge y_{j,3}\} & \rightarrow \; 1/(2 \times m), \forall j \in \{1, \ldots, m\} \\ \{a_{n+1} \wedge \bar{y}_{j,3} \wedge y_{j,1}\} & \rightarrow \; 1/(2 \times m), \forall j \in \{1, \ldots, m\} \\ \{a_{n+1} \wedge \bar{y}_{j,3} \wedge y_{j,2}\} & \rightarrow \; 1/(2 \times m), \forall j \in \{1, \ldots, m\} \end{cases}$$

Since the formula $\Phi$ does not contain negated literals, each of the above rules contains at most one negated literal/agent. Moreover, since $\sigma(\{a_1, \ldots, a_n\})$ is not a not-all-equal satisfying assignment, the values associated with each rule are non-negative.

Let us now analyze the function induced by these rules. Note that rules belonging to the last six groups of rules reported above apply to a coalition $C$ only if $a_{n+1} \in C$ and $\sigma(C \setminus \{a_{n+1}\})$ encodes a not-all-equal truth assignment for some clauses. In particular, precisely $2 \times \gamma_{\sigma(C\setminus\{a_{n+1}\})}(\Phi)$ rules of this kind apply to any coalition $C$ with $a_{n+1} \in C$. Indeed, observe that if $\sigma(C \setminus \{a_{n+1}\})$ encodes (resp., does not encode) a not-all-equal truth assignment for $c_j$, then exactly two of the rules associated with the index $j$ apply (resp., none of the rules associated with the index $j$ applies). Hence, the payoff function $v$ induced by this marginal contribution network is:

$$v(C) = \begin{cases} n + 1 - 1/m, & \text{if } C = N \\ |S| + \gamma_{\sigma(S)}(\Phi)/m, & \text{if } C = \{a_{n+1}\} \cup S, \text{ and } S \subset \{a_1, \ldots, a_n\} \\ 0, & \text{otherwise} \end{cases}$$

Consider now the payoff vector $\boldsymbol{x}$ such that $x_i = 1$, for each $i \in \{1, \ldots, n\}$, and $x_{n+1} = 1 - 1/m$. Note that $x(N) = v(N)$. We claim that $(\{N\}, \boldsymbol{x})$ belongs to CS-*core*$(\Gamma_{nae}(\Phi))$ if, and only if, there is *no* not-all-equal satisfying truth assignment. This reduces the complement of an **NP**-complete problem to CS-Core-Membership, hence showing its co-**NP**-hardness. To prove this claim, observe first that, since $v(N) \geq v(C) \geq 0$, for each coalition $C \subseteq N$, and since $v(T) = 0$, for each coalition such that $a_{n+1} \notin T$, it is immediate to check that $\Gamma_{nae}(\Phi)$ is a cohesive game, and that the maximum social welfare is attained by the grand-coalition. Therefore, we can apply Fact 3.5 stating that, for any payoff vector $\boldsymbol{x}$, $\boldsymbol{x} \in core(\Gamma_{nae}(\Phi))$ if, and only if, $(\{N\}, \boldsymbol{x}) \in \text{CS-}core(\Gamma_{nae}(\Phi))$.

Hence, it only remains to show that $\boldsymbol{x} \in core(\Gamma_{nae}(\Phi))$ if, and only if, $\Phi$ has no not-all-equal satisfying truth assignments:

($\Leftarrow$) Assume there is no not-all-equal satisfying truth assignment. Then, for each coalition $C \subseteq N$ with $C = \{a_{n+1}\} \cup S$, it holds that $v(C) \leq |S| + 1 - 1/m$, because $\gamma_{\sigma(S)}(\Phi) \leq m - 1$. Moreover, for a coalition $C$ of this kind, $x(C) = x(S) + x_{n+1} = |S| + 1 - 1/m$, i.e., $x(C) \geq v(C)$. Consider then any coalition $C$ such that $a_{n+1} \notin C$. In this case, we trivially have that $x(C) \geq 0 = v(C)$. By putting the two cases together, we have shown that there is no coalition wishing to deviate from $\boldsymbol{x}$, that is, $\boldsymbol{x}$ is in $core(\Gamma_{nae}(\Phi))$.

($\Rightarrow$) Assume $S^* \subseteq \{a_1, \ldots, a_n\}$ is such that $\sigma(S^*)$ is a not-all-equal satisfying truth assignment. Note that $\sigma(\bar{S}^*)$ is a not-all-equal satisfying truth assignment too, where $\bar{S}^* = \{a_1, \ldots, a_n\} \setminus S^*$. Since $\sigma(\{\})$ is not a not-all-equal satisfying assignment, this means that $S^* \neq \emptyset$ and $S^* \subset \{a_1, \ldots, a_n\}$. Then, we have $v(S^* \cup \{a_{n+1}\}) + v(\bar{S}^* \cup \{a_{n+1}\}) = (|S^*| + 1) + (|\bar{S}^*| + 1) = n + 2$. Let now $\boldsymbol{x}'$ be an allocation in $core(\Gamma_{nae}(\Phi))$. Therefore, we have that $x'(S^*) + x'_{n+1} \geq v(S^* \cup \{a_{n+1}\})$ and $x'(\bar{S}^*) + x'_{n+1} \geq v(\bar{S}^* \cup \{a_{n+1}\})$. By summing the two inequalities, we derive $x'(N) + x'_{n+1} \geq n + 2$. Moreover, observe that $\boldsymbol{x}'$ must be such that $x'(N) = v(N) = n + 1 - 1/m$. So, for any allocation $\boldsymbol{x}'$ in the core, we have derived that $x'_{n+1} \geq 1 + 1/m$ must hold. This entails that the allocation $\boldsymbol{x}$, which is in particular such that $x_{n+1} = 1 - 1/m$, does not belong to $core(\Gamma_{nae}(\Phi))$.

In order to conclude the proof, we just show that the game $\Gamma_{nae}(\Phi)$ is superadditive. Let $S$ and $T$ be disjoint coalitions. Note that if $S \cup T = N$, then $v(S \cup T) \geq v(S) + v(T)$ holds, since we have observed that $\Gamma_{nae}(\Phi)$ is cohesive. Accordingly, assume that $S \cup T \subset N$, and consider the following expression for the payoff associated with $S \cup T$:

$$v(S \cup T) = \begin{cases} |S| + |T| + \gamma_{\sigma(S \cup T)}(\Phi)/m, & \text{if } a_{n+1} \in S \cup T \\ 0, & \text{if } a_{n+1} \notin S \cup T \end{cases}$$

Note that if $a_{n+1} \notin S \cup T$, then $v(S) = v(T) = 0$ and we trivially get that $v(S \cup T) = v(S) + v(T) = 0$. It remains to consider the case where $a_{n+1} \in S \cup T$. Assume, w.l.o.g., that $a_{n+1} \in S$. Hence, we have $v(T) = 0$ and $v(S) = |S| + \gamma_{\sigma(S)}(\Phi)/m$. We have to show that $v(S \cup T) = |S| + |T| + \gamma_{\sigma(S \cup T)}(\Phi)/m \geq |S| + \gamma_{\sigma(S)}(\Phi)/m$, i.e., that $|T| + \gamma_{\sigma(S \cup T)}(\Phi)/m \geq \gamma_{\sigma(S)}(\Phi)/m$. Clearly, for $|T| = 0$, the inequality trivially holds (as an equality). Otherwise, the property follows from the fact that $|T| \geq 1$, $\gamma_{\sigma(S \cup T)}(\Phi) \geq 0$, and $\gamma_{\sigma(S)}(\Phi) \leq m$, by definition of the game. □

Note that, independently from our work, the fact that CS-Core-Membership is co-**NP**-hard over marginal contribution networks has been proven by Li and Conitzer [31]. In their work, a class of marginal contribution networks is exhibited for the hardness reduction where each rule, however, might contain more than just one negated literal—but with its associated value being still non-negative. In fact, Li and Conitzer actually observed that when no negated literals occur at all, marginal contribution networks reduce to *hypergraph games*, so that tractability of CS-Core-Membership in the presence of non-negative values only follows from the results in [24]; instead, our result (stating the hardness even when each rule contains at most one negated literal/agent) traces the boundary of tractability and emphasizes the role played by negated literals in the specification of the network.

Let us now turn to the problems CS-CoS, CS-Core-NonEmptiness, and CS-LCV over superadditive games. In this case, we can establish more favorable complexity results than for the general case, as these problems are no longer $\boldsymbol{\Delta_2^P}$-complete. However, they are still intractable, so that again superadditivity does not help very much.

**Theorem 4.10.** *On the class $\mathcal{C}_{\mathbf{K}}^{sa}$(mcn) of games encoded via marginal contribution networks, the problems* CS-CoS, CS-Core-NonEmptiness, CS-LCV$^-$, CS-LCV$^+$, *and* CS-LCV *are* co-**NP**-*hard even if each rule contains at most one negated literal/agent and if the associated values are non-negative.*

**Proof.** The proof of co-**NP**-hardness of CS-Core-NonEmptiness over marginal contribution networks is based on an adaptation of the proof of Theorem 4.9. Given a Boolean formula $\Phi = c_1 \wedge \cdots \wedge c_m$ in conjunctive normal form over the variables $a_1, \ldots, a_n$ and such that no clause contains a negated variable, we build in polynomial time the marginal contribution network $M'_{nae}(\Phi)$ consisting of the rules in $M_{nae}(\Phi)$ (defined in that proof) plus the following rule

$$\{\neg a_{n+1} \wedge a_1 \wedge \cdots \wedge a_n\} \rightarrow n,$$

which in fact applies only to the coalition $C = \{a_1, \ldots, a_n\}$. Then, it is immediate to check that the game $\Gamma'_{nae}(\Phi)$, with associated payoff function $v'$, induced by $M'_{nae}(\Phi)$ is such that: $v'(C) = v(C)$, for each coalition $C \subseteq N = \{a_1, \ldots, a_n, a_{n+1}\}$ with $C \neq \{a_1, \ldots, a_n\}$; and $v'(\{a_1, \ldots, a_n\}) = n > v(\{a_1, \ldots, a_n\}) = 0$, where $v$ is the payoff function associated with the game $\Gamma_{nae}(\Phi)$.

First, we observe that $\Gamma'_{nae}(\Phi)$ is superadditive. In fact, since $\Gamma_{nae}(\Phi)$ is superadditive and given the construction of $v'$, we have only to check that the following conditions hold:

(1) $v'(N) \geq v'(\{a_1, \ldots, a_n\}) + v'(\{a_{n+1}\})$. Indeed, note that $v'(\{a_1, \ldots, a_n\}) = n$ and $v'(\{a_{n+1}\}) = v(\{a_{n+1}\}) = |\{\}| + \gamma_{\sigma(\{\})}(\Phi)/m \leq 1 - 1/m$, whereas $v'(N) = v(N) = n + 1 - 1/m$. In particular, note that $\gamma_{\sigma(\{\})}(\Phi) \leq m - 1$, because $\sigma(\{\})$ is not a not-all-equal satisfying assignment.
(2) $v'(\{a_1, \ldots, a_n\}) \geq v'(S) + v'(T)$, for each pair of disjoint coalitions $S$ and $T$ such that $S \cup T = \{a_1, \ldots, a_n\}$. Indeed, $v'(\{a_1, \ldots, a_n\}) = n$, while $v'(S) = v(S) = 0$ and $v'(T) = v(T) = 0$, because $a_{n+1} \notin S \cup T$.

Now, since $\Gamma'_{nae}(\Phi)$ is superadditive (and therefore cohesive), we can apply Fact 3.5 in order to conclude that CS-*core*$(\Gamma'_{nae}(\Phi)) = \emptyset$ if, and only if, *core*$(\Gamma'_{nae}(\Phi)) = \emptyset$. The co-**NP**-hardness of CS-Core-NonEmptiness eventually follows, by claiming that *core*$(\Gamma'_{nae}(\Phi)) \neq \emptyset$ if, and only if, $\Phi$ has no not-all-equal satisfying truth assignment:

($\Leftarrow$) Assume there is no not-all-equal satisfying truth assignment, and consider the payoff vector $\boldsymbol{x}$ such that $x_i = 1$, for each $i \in \{1, \ldots, n\}$, and $x_{n+1} = 1 - 1/m$. We claim that $\boldsymbol{x} \in$ *core*$(\Gamma'_{nae}(\Phi))$. Indeed, we first note that $x(N) = v(N) = v'(N)$. Moreover, for each coalition $C \subseteq N$ with $C = \{a_{n+1}\} \cup S$, it holds that $v(C) \leq |S| + 1 - 1/m$. Therefore, in this case, $x(C) = x(S) + x_{n+1} = |S| + 1 - 1/m \geq v(C) = v'(C)$. Consider then any coalition $C$ such that $a_{n+1} \notin C$ and $C \neq \{a_1, \ldots, a_n\}$. We trivially have that $x(C) \geq 0 = v(C) = v'(C)$. Finally, we have $x(\{a_1, \ldots, a_n\}) = n = v'(\{a_1, \ldots, a_n\})$. By putting all these cases together, we have shown that there is no coalition having an incentive to deviate from $\boldsymbol{x}$. Hence, $\boldsymbol{x}$ is in *core*$(\Gamma'_{nae}(\Phi))$.
($\Rightarrow$) Assume that $S^* \subseteq \{a_1, \ldots, a_n\}$ is such that $\sigma(S^*)$ is a not-all-equal satisfying truth assignment. For the sake of contradiction, assume that $\boldsymbol{x}$ is an allocation in *core*$(\Gamma'_{nae}(\Phi))$. By the same line of reasoning as in the proof of Theorem 4.9, we get that $x_{n+1} \geq 1 + 1/m$. Moreover, it holds that $x_1 + \cdots + x_n \geq v'(\{a_1, \ldots, a_n\}) = n$. Thus, $x_1 + \cdots + x_n + x_{n+1} \geq n + 1 + 1/m$, which is impossible since $x(N)$ must coincide with $v'(N) = v(N) = n + 1 - 1/m$.

To conclude, we observe that the co-**NP**-hardness of CS-Core-NonEmptiness entails that CS-CoS, CS-LCV$^-$, CS-LCV$^+$, and CS-LCV are also co-**NP**-hard over superadditive games encoded via marginal contribution networks. In particular, recall that CS-LCV$^-$ and CS-LCV$^+$ are specializations of CS-LCV, where the penalty $\varepsilon$ provided as input is constrained to be such that $\varepsilon \leq 0$ and $\varepsilon \geq 0$, respectively. Therefore, in both cases, by fixing $\varepsilon = 0$, we are back to the problem of deciding whether the coalition structure core is not empty. □

Similarly to the case of Theorem 4.9, the above result evidences the role played by negated literals in the marginal contribution networks, because in their absence CS-Core-NonEmptiness is known to be solvable in polynomial time, whenever the values associated with the rules are non-negative [24,31]. In particular, note that earlier results on the co-**NP**-hardness of CS-Core-NonEmptiness either use negative values (cf. [24]) or are based on rules containing many occurrences of negated literals [31].

We conclude the section by stating the corresponding membership results, and by generalizing the analysis conducted so far to any compact representation scheme that is at least as expressive as MC-nets. To this end, we first state the following property.

**Lemma 4.11.** *Let $\mathcal{R}$ be any compact* **P***-representation such that* mcn $\precsim_e \mathcal{R}$*. Assume that a problem* P *is* **C***-hard, with* **C** $\in$ $\{$**NP**, co-**NP**, $\boldsymbol{\Delta}_2^{\mathbf{P}}$, $\boldsymbol{\Sigma}_2^{\mathbf{P}}\}$*, on a class $\mathcal{X}$ of games encoded as input in terms of marginal contribution networks. Then,* P *is* **C***-hard on $\mathcal{X}$ even when its games are encoded according to the representation scheme $\mathcal{R}$.*

**Proof.** From the **C**-hardness for marginal contribution networks on the class $\mathcal{X}$, for any problem $\Upsilon$ in **C** there is a polynomial-time reduction $f_1$ to the problem P on the class $\mathcal{X}$. Moreover, recall that mcn $\precsim_e \mathcal{R}$ means that there exists a polynomial-time function $f_2$ that translates any encoding $\xi^{\mathsf{mcn}}(\Gamma)$ into an encoding $f_2(\xi^{\mathsf{mcn}}(\Gamma))$ for $\Gamma$ in $\mathcal{R}$. Therefore, the composition of $f_1$ and $f_2$ is a polynomial-time reduction from any problem in **C** to the problem P for games in $\mathcal{X}$. □

Note that the results derived so far and the above lemma immediately lead to the hardness result stated below. Concerning the membership results (that have not been proven in Section 4.1), we point out that their proofs are routine and they are reported for the sake of completeness, only. In fact, for the CS-CoS problem, the characterization derived in Section 3 is rather useful and simplify the analysis.

**Theorem 4.12.** *Let $\mathcal{R}$ be any compact* **P***-representation.*

- *On the class $\mathcal{C}_{\mathbf{K}}(\mathcal{R})$,* CS-CoS, CS-Core-NonEmptiness*, and* CS-LCV$^-$ *belong to* $\boldsymbol{\Delta}_2^{\mathbf{P}}$*;* CS-LCV$^+$ *and* CS-LCV *belong to* $\boldsymbol{\Sigma}_2^{\mathbf{P}}$*;* CS-Core-Membership *belongs to* co-**NP**.
- *On the class $\mathcal{C}_{\mathbf{K}}^{sa}(\mathcal{R})$,* CS-Core-Membership, CS-CoS, CS-Core-NonEmptiness, CS-LCV$^-$, CS-LCV$^+$*, and* CS-LCV *belongs to* co-**NP**.

*Moreover, if $\mathcal{R}$ is such that* mcn $\precsim_e \mathcal{R}$*, then the corresponding hardness results hold.*

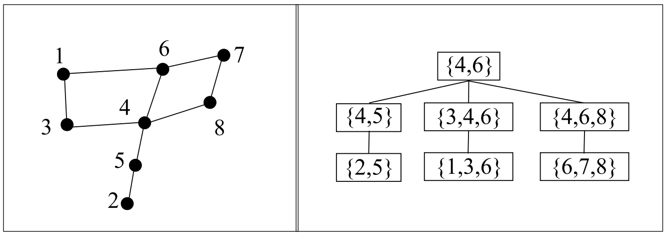


**Fig. 2.** A graph $G$ and a tree decomposition for it.

**Proof.** Results for CS-CoS, CS-Core-NonEmptiness, CS-LCV$^-$, CS-LCV$^+$, and CS-LCV on $\mathcal{C}_{\mathbf{K}}(\mathcal{R})$ are either already stated in Section 4.1 (membership results) or derive from them via Lemma 4.11 and the assumption that $\mathsf{mcn} \precsim_e \mathcal{R}$ (hardness results). Similarly, hardness results for the problems on superadditive games follow by Lemma 4.11, by the results in Section 4.2, and by the assumption that $\mathsf{mcn} \precsim_e \mathcal{R}$. Moreover, the proof of the membership of CS-Core-Membership in co-**NP** over the class $\mathcal{C}_{\mathbf{K}}(\mathcal{R})$ (and, hence, over $\mathcal{C}^{sa}_{\mathbf{K}}(\mathcal{R})$) is a simple adaptation of the result derived in the setting where all coalitions are allowed to form [9]—see the proof of Theorem 4.1.

Then, consider the remaining results to be shown on the class $\mathcal{C}^{sa}_{\mathbf{K}}(\mathcal{R})$. Since we are dealing with superadditive (and hence cohesive) games, we can apply Fact 3.5. Therefore, for any game $\Gamma \in \mathcal{C}^{sa}_{\mathbf{K}}(\mathcal{R})$, CS-*core*$(\Gamma) = \emptyset$ if, and only if, *core*$(\Gamma) = \emptyset$. Hence, membership in co-**NP** for the problem CS-Core-Membership is again easily seen. Similarly, in order to decide whether CS-CoS$(\Gamma) \leq \Delta$, we have to check whether CS-*core*$(\Gamma_\Delta) \neq \emptyset$. In fact, whenever $\Gamma$ is superadditive, $\Gamma_\Delta$ is superadditive, too. Moreover, an encoding for $\Gamma_\Delta$ can be built in polynomial time (cf. proof of Theorem 4.1), and so CS-CoS is again in co-**NP**. Finally, for cohesive games, we recall from Theorem 3.10 that CS-LCV$(\Gamma) =$ LCV$(\Gamma)$. Hence, deciding whether the coalition structure least core value of $\Gamma$ is $\varepsilon$ at most is reduced to checking whether *core*$(\Gamma_{\text{-}\varepsilon}) \neq \emptyset$. Again, an encoding for $\Gamma_{\text{-}\varepsilon}$ can be built in polynomial time, and the fact that the core is not empty can be checked in co-**NP**. So, CS-LCV$^-$, CS-LCV$^+$, and CS-LCV belongs to co-**NP**. $\square$

## 5. Games with nearly-acyclic interaction graphs

Many **NP**-hard problems arising in different application areas are known to be efficiently solvable when restricted to instances whose underlying structures can be modeled via *acyclic* graphs. Indeed, for such restricted classes of instances, we can often compute solutions efficiently by means of dynamic programming. However, as a matter of fact, (graphical) structures arising from real applications are in most relevant cases not properly acyclic. Yet, they are often not very intricate and exhibit some rather limited degree of cyclicity, which suffices to retain most of the nice properties of acyclic instances. The investigation of graph properties that are best suited to identify nearly-acyclic graphs/hypergraphs has led to the definition of a number of so-called *structural decomposition methods* (see, e.g., [47,48] and the references therein).

### *5.1. Basic notions*

There are different possible notions to measure how far a graph is from a tree, that is, to measure its degree of cyclicity or, dually, its tree-likeness. Among them, the treewidth [29] is the most powerful one, as it is able to extend the nice computational properties of trees to the largest possible classes of graphs, in many applications from different fields. In fact, useful tractability results for core-related concepts in coalitional games over bounded treewidth graphs have been derived in the literature, such as, for instance, in the context of transferable utility planning games [49], graph games and marginal contribution networks [9,25]. However, the question of whether core-related concepts over bounded treewidth games *with coalition structures* are tractable was not addressed so far. We embark on this study and we start by recalling the formal definition of treewidth, and its basic properties.

**Definition 5.1.** (See [29].) A *tree decomposition* of a graph $G = (N, E)$ is a pair $\langle T, \chi \rangle$, where $T = (V, F)$ is a tree, and $\chi$ is a labeling function assigning to each vertex $p \in V$ a set of vertices $\chi(p) \subseteq N$, such that the following three conditions are satisfied: (1) for each node $b$ of $G$, there exists $p \in V$ such that $b \in \chi(p)$; (2) for each edge $(b, d) \in E$, there exists $p \in V$ such that $\{b, d\} \subseteq \chi(p)$; and (3) for each node $b$ of $G$, the set $\{p \in V \mid b \in \chi(p)\}$ induces a connected subtree of $T$. The *width* of $\langle T, \chi \rangle$ is the number $\max_{p \in V}(|\chi(p)| - 1)$. The *treewidth* of $G$, denoted by $tw(G)$, is the minimum width over all its tree decompositions. $\square$

Note that the notion of treewidth is a true generalization of graph acyclicity: A graph $G$ is acyclic if, and only if, $tw(G) = 1$. For example, the graph $G$ in Fig. 2 is cyclic and its treewidth is 2, as it is witnessed by the width-2 tree decomposition depicted in the same figure.

To determine the treewidth of a graph $G$ is **NP**-hard. However, for each fixed natural number $k$, checking whether $tw(G) \leq k$, and if so, computing a tree decomposition for $G$ of optimal width, is achievable in linear time [50], and was recently shown to be achievable in logarithmic space [51]. Therefore, it is sensible to focus our analysis on the class **TW**-$k$ of those interaction graphs having treewidth bounded by some fixed natural number $k$. Accordingly, if $\mathcal{C}(\mathcal{R})$ is the class induced by the representation $\mathcal{R}$, then $\mathcal{C}_{\mathbf{TW}\text{-}k}(\mathcal{R})$ is the subclass of all games $\Gamma = \langle N, v, G\rangle$ such that $tw(G) \leq k$.

*5.2. Results on graphs with bounded treewidth (and trees)*

As in Section 4, we start the analysis by considering arbitrary payoff functions. Our first result is a useful technical property. The result is a variant of those discussed in Section 3.1, and trivially follows by the arguments illustrated there.

**Fact 5.2.** *Let $\Gamma = \langle N, v, G\rangle$ and $\Gamma' = \langle N, v, G'\rangle$ be two structurally-restricted characteristic function games such that: $v(\{i\}) \geq 0$, for each $i \in N$; $\mathcal{F}(G') \subseteq \mathcal{F}(G)$; and $v(C) \leq 0$, for each coalition $C \in \mathcal{F}(G) \setminus \mathcal{F}(G')$. Then,*

$$\text{CS-}core(\Gamma') = \{(\Pi, \boldsymbol{x}) \in \text{CS-}core(\Gamma) \mid C \in \mathcal{F}(G'), \forall C \in \Pi\}.$$

The property is now used to adapt the proof of the hardness result stated in Theorem 4.4 in order to deal with an interaction graph having bounded treewidth.

Actually, the fact that our reasoning problems remain intractable over bounded treewidth interaction graphs is not surprising given that computing the maximum social welfare is **NP**-hard, even over *(i)* trees and over *(ii) simple* and *monotone* games[10] with interaction graphs having treewidth 2 [32]. However, from a technical viewpoint, the results in [32] are specific for the computation of the maximum social welfare and, therefore, do not imply the result below. Moreover, our result is provided for a different complexity class ($\mathbf{\Delta_2^P}$) and for a specific representation scheme (marginal contribution networks). Similar considerations apply to the subsequent results in this section.

**Theorem 5.3.** *Let $\mathcal{R}$ be any compact* **P***-representation such that* mcn $\precsim_e \mathcal{R}$*, and let $k > 0$ be a fixed constant. On the class $\mathcal{C}_{\mathbf{TW}\text{-}k}(\mathcal{R})$,* CS-CoS *and* CS-Core-NonEmptiness *are $\mathbf{\Delta_2^P}$-complete. Hardness holds even for $k = 2$.*

**Proof.** Membership results in $\mathbf{\Delta_2^P}$ follow from Theorem 4.1. For the hardness results, consider the proof of Theorem 4.4, where it is shown that the problems CS-CoS, CS-Core-NonEmptiness, and CS-LCV$^-$ are $\mathbf{\Delta_2^P}$-complete on the class $\mathcal{C}_{\mathbf{K}}(\text{mcn})$. Recall that we are given a Boolean formula $\Phi$, and we build a game $\Gamma(\Phi) \in \mathcal{C}_{\mathbf{K}}(\text{mcn})$ whose associated payoff function $v$ is such that:

$$v(C) = \begin{cases} w(S), & \text{if } C = \{a_{n+4}\} \cup S,\ S \subseteq \{a_1, \ldots, a_n\}, \text{ and } \sigma(S) \models \Phi \\ w(S) + 1, & \text{if } C = \{a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}\} \cup S,\ \sigma(S) \models \Phi, \text{ and } a_1 \in S \\ 2/3, & \text{if } C \subseteq \{a_{n+1}, a_{n+2}, a_{n+3}\} \text{ and } |C| \geq 2 \\ 0, & \text{if } |C| = 1, \text{ with } C \neq \{a_{n+4}\} \\ \leq 0, & \text{otherwise} \end{cases}$$

In particular, recall that we have shown that CS-$core(\Gamma(\Phi)) \neq \emptyset$ if, and only if, $a_1$ is true in the lexicographically maximal satisfying assignment of $\Phi$.

First, we can assume, w.l.o.g., that the assignment $\sigma(\{\})$ is satisfying. Indeed, given any Boolean function $\Phi = c_1 \wedge \cdots \wedge c_m$ in CNF over the variables $a_1, \ldots, a_n$, we can build a Boolean function $\Phi' = c'_1 \wedge \cdots \wedge c'_m$ over the variables $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_m\}$ such that $c'_i = c_i \vee \neg b_i$, for each $i \in \{1, \ldots, m\}$. Moreover, we assume that variables in $\{b_1, \ldots, b_m\}$ always precede all variables in $\{a_1, \ldots, a_n\}$ in the given ordering. Therefore, $a_1$ is true in the lexicographically maximal satisfying assignment of $\Phi$ if, and only if, $a_1$ is true in the lexicographically maximal satisfying assignment of $\Phi'$. In fact, by construction, $\Phi'$ is satisfied by the assignment where all variables are false.

Consider then the interaction graph $G(\Phi)$ depicted in the left part of Fig. 3. Note that the graph is cyclic, and it has treewidth 2, as it is witnessed by the tree decomposition depicted on the right of the same figure.

Let $C$ be a coalition that is not allowed to form according to $G(\Phi)$, i.e., with $C \notin \mathcal{F}(G(\Phi))$. We can distinguish the following cases: (1) $C \cap \{a_1, \ldots, a_n\} \neq \emptyset$, and $a_{n+4} \notin C$; (2) $a_{n+4} \in C$, $C \cap \{a_{n+1}, a_{n+2}\} \neq \emptyset$, and $a_{n+3} \notin C$. In both cases, we have that $v(C) \leq 0$. Moreover, observe that $v(C) \geq 0$, for each coalition $C$ with $|C| = 1$. In particular, observe that $v(\{a_{n+4}\}) = w(\{\}) = 0$, as $\sigma(\{\}) \models \Phi$. Thus, we are in a position to apply Fact 5.2 in order to derive:

$$\text{CS-}core(\langle N, v, G(\Phi)\rangle) = \{(\Pi, \boldsymbol{x}) \in \text{CS-}core(\Gamma(\Phi)) \mid C \in \mathcal{F}(G(\Phi)), \forall C \in \Pi\}.$$

Now, we claim that CS-$core(\langle N, v, G(\Phi)\rangle) = \emptyset \Leftrightarrow$ CS-$core(\Gamma(\Phi)) = \emptyset$. Indeed, the '$\Leftarrow$'-part trivially follows from the above expression. To show that the '$\Rightarrow$'-part holds too, consider an element $(\Pi, \boldsymbol{x})$ in CS-$core(\Gamma(\Phi))$. Because of Fact 3.5, we already know that CS-$core(\Gamma(\Phi)) =$ CS-$opt(\Gamma(\Phi)) \times core(\tilde{\Gamma}(\Phi))$. Therefore, we have just to show that there is a coalition

[10] A game is monotone if $v(C) \geq v(C')$ holds, for each pair of feasible coalitions $C$ and $C'$ with $C \supseteq C'$. A game is simple if $v(C) \in \{0, 1\}$ holds, for each feasible coalition $C$.

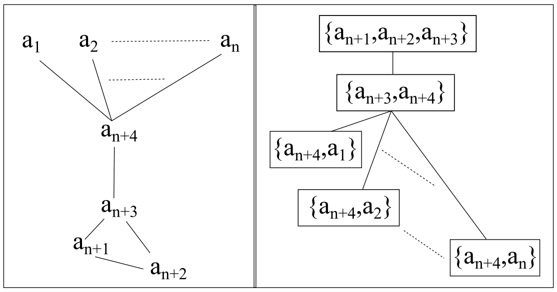


**Fig. 3.** Interaction graph in the proof of Theorem 5.3, and a width-2 tree decomposition.

structure $\Pi^* \in$ CS-*opt*$(\Gamma(\Phi))$ such that $C \in \mathcal{F}(G(\Phi))$, for each coalition $C \in \Pi^*$. Indeed, in this case, we would have that $(\Pi^*, \boldsymbol{x})$ is in CS-*core*$(\langle N, v, G(\Phi)\rangle)$. In fact, since CS-*core*$(\Gamma(\Phi)) \neq \emptyset$, we have that $a_1$ is true in the lexicographically maximal satisfying assignment of $\Phi$. By Property 4.4.1, $\Pi^*_t$ is then an optimal coalition structure, where $\Pi^*_t$ includes the coalition $S^* \cup \{a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}\}$, plus each agent in $\{a_1, \ldots, a_n\} \setminus S^*$ as a singleton—and we are done, as each coalition in $\Pi^*_t$ is in $\mathcal{F}(G(\Phi))$.

From the above claim, it follows that CS-*core*$(\langle N, v, G(\Phi)\rangle) \neq \emptyset$ if, and only if, $a_1$ is true in the lexicographically maximal satisfying assignment of $\Phi$, which proves the $\boldsymbol{\Delta_2^P}$-hardness of CS-Core-NonEmptiness on the class $\mathcal{C}_{\mathbf{TW}\text{-}2}(\mathsf{mcn})$. By Lemma 4.11, the same hardness result holds on $\mathcal{C}_{\mathbf{TW}\text{-}2}(\mathcal{R})$, for any compact **P**-representation $\mathcal{R}$ such that $\mathsf{mcn} \precsim_e \mathcal{R}$. Finally, we observe that hardness results for CS-Core-NonEmptiness trivially entail the corresponding hardness result for CS-CoS. $\square$

We stress here that the hardness results above are given over the class $\mathcal{C}_{\mathbf{TW}\text{-}k}(\mathsf{mcn})$ with $k = 2$, and hence apply to any class $\mathcal{C}_{\mathbf{TW}\text{-}k'}(\mathsf{mcn})$, with $k' \geq k$. Indeed, by the definition of these classes, we have that $\mathcal{C}_{\mathbf{TW}\text{-}k}(\mathsf{mcn}) \subseteq \mathcal{C}_{\mathbf{TW}\text{-}k'}(\mathsf{mcn})$. However, these results do not imply the corresponding hardness results over $\mathcal{C}_{\mathbf{K}}(\mathsf{mcn})$, i.e., Theorem 5.3 does not entail Theorem 4.4.[11] Similar considerations apply to all results in the section.

Now, based on the reduction in the proof of Theorem 4.9, we prove that CS-Core-Membership remains co-**NP**-complete over games equipped with interaction graphs having bounded treewidth, in fact with trees.

**Theorem 5.4.** *Let $\mathcal{R}$ be any compact* **P**-*representation, and let $k > 0$ be a constant. On the class $\mathcal{C}_{\mathbf{TW}\text{-}k}(\mathcal{R})$,* CS-Core-Membership *belongs to* co-**NP**. *Moreover, if $\mathcal{R}$ is such that $\mathsf{mcn} \precsim_e \mathcal{R}$, then* CS-Core-Membership *is* co-**NP**-*hard, even on the subclass $\mathcal{C}^{sa}_{\mathbf{TW}\text{-}k}(\mathcal{R})$ of superadditive games with $k = 1$.*

**Proof.** The membership result is easily seen to hold, and it follows by inspection in the proof of Theorem 4.12. For the hardness result, consider the proof of Theorem 4.9, where it is shown that the problem CS-Core-Membership is co-**NP**-complete over $\mathcal{C}^{sa}_{\mathbf{K}}(\mathsf{mcn})$. Recall that we are given a Boolean function $\Phi$ and we build a (marginal contribution network inducing a) game $\Gamma_{nae}(\Phi)$ over the set $N = \{a_1, \ldots, a_n, a_{n+1}\}$ of agents and whose associated payoff function $v$ is such that:

$$v(C) = \begin{cases} n + 1 - 1/m, & \text{if } C = N \\ |S| + \gamma_{\sigma(S)}(\Phi)/m, & \text{if } C = \{a_{n+1}\} \cup S, \text{ and } S \subset \{a_1, \ldots, a_n\} \\ 0, & \text{otherwise} \end{cases}$$

In particular, recall that we have exhibited a payoff vector $\boldsymbol{x}$ such that $(\{N\}, \boldsymbol{x})$ belongs to CS-*core*$(\Gamma_{nae}(\Phi))$ if, and only if, there is *no* not-all-equal satisfying truth assignment.

Consider the interaction graph $G_{nae}(\Phi)$ where agent $a_{n+1}$ is connected with all other agents, and where no further edge occurs. Therefore, $G_{nae}(\Phi)$ is actually a tree. Note also that a coalition $C \subseteq N$ with $C \notin \mathcal{F}(G_{nae}(\Phi))$ is such that $a_{n+1} \notin C$. In fact, for this coalition $C$, it is the case that $v(C) = 0$ holds. Thus, we are in the position of applying Fact 5.2 in order to derive:

$$\text{CS-}core(\langle N, v, G_{nae}(\Phi)\rangle) = \{(\Pi, \boldsymbol{x}) \in \text{CS-}core(\Gamma_{nae}(\Phi)) \mid C \in \mathcal{F}(G_{nae}(\Phi)), \forall C \in \Pi\}.$$

Since $N$ is a feasible coalition in $G_{nae}(\Phi)$, we immediately conclude that $(\{N\}, \boldsymbol{x})$ belongs to CS-*core*$(\Gamma_{nae}(\Phi))$ if, and only if, $(\{N\}, \boldsymbol{x})$ belongs to CS-*core*$(\langle N, v, G_{nae}(\Phi)\rangle)$. This proves that CS-Core-Membership is co-**NP**-complete on $\mathcal{C}_{\mathbf{TW}\text{-}1}(\mathsf{mcn})$ and, hence, by Lemma 4.11 on the class $\mathcal{C}_{\mathbf{TW}\text{-}1}(\mathcal{R})$, for each $\mathcal{R}$ such that $\mathsf{mcn} \precsim_e \mathcal{R}$.

[11] As we have discussed in Section 3, we could take any game (in $\mathcal{C}_{\mathbf{TW}\text{-}2}(\mathsf{mcn})$) and translate it to a game defined over a complete interaction graph, by assigning to each infeasible coalition a value that is at most the sum of the values of all connected sub-components. Combining this ingredient with Theorem 5.3, we could derive Theorem 4.4 if the translation were feasible in polynomial time. In fact, the feasibility of implementing the translation by adding polynomial many rules to the original marginal contribution network is not immediate and would have still to be proven, in general.

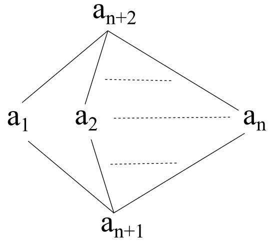


**Fig. 4.** Interaction graph in the proof of Lemma 5.5.

In order to complete the proof, we have to show that the game $\langle N, v, G_{nae}(\Phi)\rangle$ is superadditive. To this end, recall that $\Gamma_{nae}(\Phi)$ is superadditive. Therefore, if $S, T, S \cup T$ are three coalitions in $\mathcal{F}(G_{nae}(\Phi))$ with $S \cap T = \emptyset$, then we have $v(S \cup T) \geq v(S) + v(T)$, because these coalitions are also feasible in the game $\Gamma_{nae}(\Phi)$ which is defined over a complete interaction graph. $\square$

Let us now move to analyze CS-CoS, CS-Core-NonEmptiness, and CS-LCV over superadditive games (whose associated interaction graphs have bounded treewidth). In this case, we need a more sophisticated construction.

**Lemma 5.5.** *Let $h^n : \mathbb{R}^n \to \mathbb{R}^{n+2}$ be such that $h^n(\langle x_1, \ldots, x_n\rangle) = \langle x_1, \ldots, x_n, 0, 0\rangle$. Let $\Gamma = \langle N, v, K_n\rangle \in \mathcal{C}^{sa}_{\mathbf{K}}(\mathsf{mcn})$ be a structurally-restricted superadditive game, with $N = \{1, \ldots, n\}$. Then, we can build in polynomial time w.r.t. $||\xi^{\mathsf{mcn}}(\Gamma)||$ (the encoding of) a superadditive game $\Gamma' = \langle N \cup \{n+1, n+2\}, v', G'\rangle \in \mathcal{C}^{sa}_{\mathbf{TW}\text{-}2}(\mathsf{mcn})$ such that:*

- *if $(\{N\}, \boldsymbol{x}) \in$ CS-core$(\Gamma)$, then $(\{N \cup \{n+1, n+2\}\}, h^n(\boldsymbol{x})) \in$ CS-core$(\Gamma')$;*
- *if $(\{N \cup \{n+1, n+2\}\}, \boldsymbol{x}') \in$ CS-core$(\Gamma')$, then $x'_{n+1} = x'_{n+2} = 0$ and the vector $\boldsymbol{x}$ with $\boldsymbol{x}' = h^n(\boldsymbol{x})$ is such that $(\{N\}, \boldsymbol{x}) \in$ CS-core$(\Gamma)$.*

**Proof.** Let $\Gamma = \langle N, v, K_n\rangle \in \mathcal{C}^{sa}_{\mathbf{K}}(\mathsf{mcn})$ be a game encoded as a marginal contribution network $M$. Consider the game $\Gamma' = \langle N \cup \{n+1, n+2\}, v', G'\rangle$ whose payoff function $v'$ is encoded in terms of the same marginal contribution network $M$. Therefore, $v'(\{n+1\}) = v'(\{n+2\}) = 0$, and $v'(S') = v(S' \setminus \{n+1, n+2\})$, for each feasible coalition $S' \subseteq N \cup \{n+1, n+2\}$ with $S' \cap N \neq \emptyset$. In particular, assume that the interaction graph $G'$ is such that $n+1$ and $n+2$ are connected to all the remaining agents, and no further edges are in $G'$. The graph is depicted in Fig. 4. Note that $tw(G') = 2$.

First, we claim that $\Gamma'$ is superadditive, whenever $\Gamma$ is superadditive. Consider any two disjoint feasible coalitions $S'$ and $T'$ such that $S' \cup T'$ is feasible, too. In the case where $S' \cap N \neq \emptyset$ and $T' \cap N \neq \emptyset$, we have $v'(S' \cup T') = v(S' \cup T' \setminus \{n+1, n+2\}) \geq v(S' \setminus \{n+1, n+2\}) + v(T' \setminus \{n+1, n+2\}) = v'(S') + v'(T')$. In fact, note that $v'(S' \cup T') = v(S' \cup T' \setminus \{n+1, n+2\}) \geq v(S' \setminus \{n+1, n+2\}) + v(T' \setminus \{n+1, n+2\})$ follows from the fact that $\Gamma$ is superadditive and any coalition is feasible in $K_n$, i.e., according to it. To complete the analysis, consider the case where $T' \subseteq \{n+1, n+2\}$ (and $S' \cap N \neq \emptyset$). In this case, we have $v'(S' \cup T') = v(S' \cup T' \setminus \{n+1, n+2\}) = v(S' \setminus \{n+1, n+2\}) = v'(S')$. Again, we have $v'(S' \cup T') \geq v'(S') + v'(T')$, because $v'(T') = 0$.

Now, let $(\{N\}, \boldsymbol{x})$ be an element in CS-core$(\Gamma)$, and consider the vector $\boldsymbol{x}' = h^n(\boldsymbol{x})$ such that $x'_i = x_i$, for each $i \in N$, while $x'_{n+1} = x'_{n+2} = 0$. We claim that $(\{N \cup \{n+1, n+2\}\}, \boldsymbol{x}')$ is in CS-core$(\Gamma')$. Indeed, note first that $x'(N \cup \{n+1, n+2\}) = x(N) = v(N) = v'(N \cup \{n+1, n+2\})$. Moreover, for each $S' \subseteq N \cup \{n+1, n+2\}$ with $S' \cap N \neq \emptyset$, $x'(S') = x(S' \setminus \{n+1, n+2\}) \geq v(S' \setminus \{n+1, n+2\}) = v'(S')$.

In order to complete the proof, consider any element $(N \cup \{n+1, n+2\}, \boldsymbol{x}')$ in CS-core$(\Gamma')$. We have $x'(N \cup \{n+1, n+2\}) = v'(N \cup \{n+1, n+2\}) = v(N)$, $x'(N \cup \{n+1\}) \geq v'(N \cup \{n+1\}) = v(N)$ and $x'(N \cup \{n+2\}) \geq v'(N \cup \{n+2\}) = v(N)$. Therefore, $x'_{n+1} = x'_{n+2} = 0$. Consider then the vector $\boldsymbol{x}$ such that $\boldsymbol{x}' = h(\boldsymbol{x})$, and note that $(\{N\}, x)$ is in CS-core$(\Gamma)$. Indeed, $x(N) = x'(N \cup \{n+1, n+2\}) = v'(N \cup \{n+1, n+2\}) = v(N)$. Moreover, for each non-empty $S \subseteq N$, $x(S) = x'(S \cup \{n+1, n+2\}) \geq v'(S \cup \{n+1, n+2\}) = v(S)$. In particular, note that, in the above relationships, $S \cup \{n+1, n+2\}$ is guaranteed to induce a connected subgraph over $G'$. Therefore, this coalition is feasible and we are guaranteed that $x'(S \cup \{n+1, n+2\}) \geq v'(S \cup \{n+1, n+2\})$, because $(\{N \cup \{n+1, n+2\}\}, \boldsymbol{x}')$ is in CS-core$(\Gamma')$. $\square$

By combining the lemma with Fact 3.5 and Theorem 4.12, we get the following (where proof of the membership results are trivial adaptations of those given for games over complete interaction graphs).

**Corollary 5.6.** *Let $\mathcal{R}$ be any compact **P**-representation, and let $k$ be a fixed constant. On the class $\mathcal{C}^{sa}_{\mathbf{TW}\text{-}k}(\mathcal{R})$, CS-CoS, CS-Core-NonEmptiness, CS-LCV$^-$, CS-LCV$^+$, and CS-LCV belong to co-**NP**. Moreover, if $\mathcal{R}$ is such that $\mathsf{mcn} \precsim_e \mathcal{R}$, then all problems are co-**NP**-complete, even for $k = 2$.*

Observe that, concerning the problems CS-CoS and CS-Core-NonEmptiness, there is no hope to extend Corollary 5.6 and Theorem 5.3 and show that hardness holds even on trees (i.e., when treewidth equals one), because we know from [19] that the core is non-empty over trees, even for non-superadditive games. However, the scenario is more interesting when turning to CS-LCV and its variants.

**Theorem 5.7.** *Let $\mathcal{R}$ be any compact* **P**-*representation. On the class $\mathcal{C}_{\mathbf{TW}\text{-}1}(\mathcal{R})$,* CS-LCV$^+$ *is in* **P**. *Moreover, if $\mathcal{R}$ is such that* mcn $\precsim_e \mathcal{R}$, *then* CS-LCV$^-$ *and* CS-LCV *are $\mathbf{\Delta_2^P}$-complete.*

**Proof.** Let us consider the problem CS-LCV$^+$. We are given a rational number $\varepsilon \geq 0$ and a game $\Gamma = \langle N, v, G\rangle$. Let $\Pi$ be an optimal coalition structure in CS-*opt*$(\Gamma)$, and observe that $\Pi$ belongs to CS-*opt*$(\Gamma_{\Pi,\text{-}\varepsilon})$, too. Moreover, recall from [19] that CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon}) \neq \emptyset$. Hence, by Fact 3.5, a pair of the form $(\Pi, \boldsymbol{x})$ is guaranteed to occur in CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon})$. That is, CS-LCV$^+$ is immaterial over trees.

Let us now consider the problem CS-LCV$^-$, where the rational number $\varepsilon$ provided as input is such that $\varepsilon \leq 0$. Membership in $\mathbf{\Delta_2^P}$ is established in Theorem 4.3. To prove the hardness result, consider the UO-TSP problem defined as follows. We are given a complete directed graph $(\{a_1, \ldots, a_n\}, E)$ over $n$ nodes, where each edge $e_{i,j} = (a_i, a_j)$, with $a_i \neq a_j$ is associated with a positive *integer* weight $w(e_{i,j})$. A valid tour is a set $T$ of $n$ edges inducing a directed cycle in the graph (touching each node exactly once). The weight of $T$ is the value $\sum_{e_{i,j}\in T} w(e_{i,j})$. An optimal tour is one with the minimum weight over all possible valid tours. The traveling salesman problem (TSP) is the well-known **NP**-hard problem asking to computing an optimal tour. Instead, UO-TSP is the $\mathbf{\Delta_2^P}$-hard decision problem asking whether there is exactly one optimal tour [52].

Based on the graph $(\{a_1, \ldots, a_n\}, E)$ and the given weighting function $w$, we build a game $\Gamma = \langle N, v, G\rangle$ where $N = \{e_{i,j,k} \mid i, j, k \in \{1, \ldots, n\}\} \cup \{t\}$ is the set of the agents. Moreover, $G$ is a tree where $t$ is connected with all other agents, and no further edge occurs in it. By letting $W = \max_{e_{i,j}\in E} w(e_{i,j}) + 1$, the payoff function $v$ is encoded as a marginal contribution network $M$ such that:

$$\begin{cases}
r_1 : \{t \wedge e_{i,j,k}\} \rightarrow W - w(e_{i,j}), & \forall e_{i,j} \in E, \forall k \in \{1, \ldots, n\} \\
r_2 : \{t \wedge \bigwedge_{j\neq 1} \neg e_{1,j,1}\} \rightarrow -n^2 \times W & \\
r_3 : \{t \wedge \bigwedge_{i\neq 1} \neg e_{i,1,n}\} \rightarrow -n^2 \times W & \\
r_4 : \{t \wedge e_{i,j,k} \wedge \bigwedge_h \neg e_{j,h,k+1}\} \rightarrow -n^2 \times W, & \forall e_{i,j} \in E, \forall k \in \{1, \ldots, n-1\} \\
r_5 : \{t \wedge e_{i,j,k} \wedge e_{i,j',k'}\} \rightarrow -n^2 \times W, & \forall i, j, j', k, k' \in \{1, \ldots, n\}, \text{ with } j \neq j' \\
r_6 : \{t \wedge e_{i,j,k} \wedge e_{i',j,k'}\} \rightarrow -n^2 \times W, & \forall i, i', j, k, k' \in \{1, \ldots, n\}, \text{ with } i \neq i' \\
r_7 : \{t \wedge e_{i,j,k} \wedge e_{i',j',k}\} \rightarrow -n^2 \times W, & \forall i, i', j, j', k \in \{1, \ldots, n\}, \text{ with } i \neq i' \text{ or } j \neq j'
\end{cases}$$

For each feasible coalition $C$, let $T(C) = \{e_{i,j} \mid \exists e_{i,j,k} \in C\}$ be the set of edges obtained from $C$ by "stripping off" the third index. Then, consider the following properties establishing a precise correspondence between valid tours and feasible coalitions:

- Let $C$ be a feasible coalition with $v(C) > 0$. We claim that $T(C)$ is a valid tour. Indeed, $t$ is in $C$ and $|C| > 1$ (because of $r_1$), and no rule with a negative value applies. So, by $r_2$, an agent of the form $e_{1,j,1}$ is in $C$, with $j \neq 1$. Then, by repeated applications of $r_4$, we build a sequence $e_{1,j,1} = e_{1,j_1,1}, e_{j_1,j_2,2}, \ldots, e_{j_{n-1},j_n,n}$ of agents that are in $C$. Moreover, by $r_3$, an agent of the form $e_{i,1,n}$ is in $C$. By $r_7$, it hence holds that $j_n = 1$. Eventually, by $r_5$ and $r_6$, $\{e_{1,j_1,1}, e_{j_1,j_2,2}, \ldots, e_{j_{n-1},j_n,n}\}$ is a valid tour, and combined with $r_7$ we conclude that no further agent occurs in $C$. That is, this tour coincides with $T(C)$.
- Assume that $T$ is a valid tour. Then, we show that there is a feasible coalition $C$ with $t \in C$ and such that $v(C) > 0$ and $T(C) = T$. Indeed, if $T$ is a tour, then its edges can be ordered starting with the one outgoing from node $a_1$. So edges in $T$ can be listed as follows: $e_{1,j_1}, e_{j_1,j_2}, \ldots, e_{j_{n-1},1,n}$. The required coalition is then $C = \{t, e_{1,j_1,1}, e_{j_1,j_2,2}, \ldots, e_{j_{n-1},1,n}\}$.

In particular, note that in the above correspondence $v(C) = n \times W - \sum_{e_{i,j}\in T(C)} w(e_{i,j})$. Therefore, optimal coalitions one-to-one correspond with optimal tours. This leads to the following, whose proof generalizes the correspondence to optimal coalition structures.

**Claim 5.8.** *Let $\mathcal{R}$ be any compact* **P**-*representation such that* mcn $\precsim_e \mathcal{R}$. *On the class $\mathcal{C}_{\mathbf{TW}\text{-}1}(\mathcal{R})$, computing the maximum social welfare is* **NP**-hard. *In particular, a reduction can be exhibited where optimal coalition structures one-to-one correspond to optimal tours for a* TSP *instance provided as input.*

Let now $\varepsilon < 0$ be a value sufficiently close to 0, e.g., $\varepsilon = -\frac{\min_{e_{i,j}} w(e_{i,j})}{n^2 \times W}$. Recall from Lemma 3.9 that the least coalition structure core value of $\Gamma$ is at most $\varepsilon$ if, and only if, *(i)* $|\text{CS-}opt(\Gamma)| = 1$, and *(ii)* $\Pi \in$ CS-*opt*$(\Gamma)$ implies $\Pi \in$ CS-*opt*$(\Gamma_{\Pi,\text{-}\varepsilon})$ and CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon}) \neq \emptyset$. In fact, for the given value of $\varepsilon$, it is immediate to check that if *(i)* holds and $\Pi$ is the only feasible coalition structure in CS-*opt*$(\Gamma)$, then $\Pi \in$ CS-*opt*$(\Gamma_{\Pi,\text{-}\varepsilon})$ holds. Then, by Fact 3.5 and the result in [19], we would be guaranteed that a pair of the form $(\Pi, \boldsymbol{x})$ occurs in CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon})$ (that is, *(ii)* also holds). So, we have shown that CS-LCV$^-$ just amounts to deciding whether condition *(i)* holds, i.e., whether there is exactly one optimal tour (cf. Claim 5.8). Hardness is shown for marginal contribution networks, and by Lemma 4.11 it extends to more expressive game encodings. □

Note that the **NP**-hardness of computing the maximum social welfare over trees has already been established in our earlier work [53], and independently in [32]. Hence, Claim 5.8 can be viewed as strengthening the hardness result over games encoded as marginal contribution networks (in addition to being an important technical ingredient to show the above $\mathbf{\Delta_2^P}$-hardness). By Fact 3.5, the following is an immediate corollary.

**Corollary 5.9.** *Let $\mathcal{R}$ be any compact* **P***-representation such that* mcn $\precsim_e \mathcal{R}$*. On the class $\mathcal{C}_{\mathbf{TW}\text{-}1}(\mathcal{R})$,* CS-Core-Find *is* **NP***-hard.*

The above result is given for games that are not superadditive. In fact, for games defined on trees, Demange [19] proposed a procedure that computes an element in the core, which does not run in polynomial time for non-superadditive games. However, it can be easily seen by inspection that it does so for superadditive ones (see also [53,54]).

**Fact 5.10.** *(See cf. [19].) Let $\mathcal{R}$ be any compact* **P***-representation. On the class $\mathcal{C}^{sa}_{\mathbf{TW}\text{-}1}(\mathcal{R})$,* CS-Core-Find *is in* **FP**.

We now leave the section by establishing the counterpart (for CS-LCV$^-$ and CS-LCV) of Theorem 5.7 over superadditive games.

**Theorem 5.11.** *Let $\mathcal{R}$ be any compact* **P***-representation. On the class $\mathcal{C}^{sa}_{\mathbf{TW}\text{-}1}(\mathcal{R})$,* CS-LCV$^-$ *and* CS-LCV *are* co-**NP***-complete. Moreover, if $\mathcal{R}$ is such that* mcn $\precsim_e \mathcal{R}$*, then the corresponding hardness results hold.*

**Proof.** We start the proof with a simple, yet quite useful characterization.

**Claim 5.12.** *Let $\mathcal{R}$ be any compact* **P***-representation, let $\Gamma \in \mathcal{C}^{sa}_{\mathbf{TW}\text{-}1}(\mathcal{R})$, and let $\varepsilon \leq 0$. Then,* CS-LCV$(\Gamma) \leq \varepsilon$ *holds if, and only if, $\{N\} \in$* CS-*opt*$(\Gamma_{\text{-}\varepsilon})$*. Moreover,* CS-LCV$^-$ *is in* co-**NP**.

Observe now that CS-LCV is feasible in co-**NP**, by Theorem 5.7 and the claim above. Thus, let us focus on the hardness result for CS-LCV$^-$.

Let $G = (N, E)$ be an undirected graph over the set $N = \{a_1, \dots, a_n\}$ of nodes, and let $k$ be a natural number. A set $V \subseteq N$ is a vertex cover if, for each edge $e \in E$, $V$ contains at least one of the two endpoints of $e$. Deciding whether $G$ admits a vertex cover $V$ with $|V| \leq k$ is a well-known **NP**-hard problem [55].

Note that the problem remains **NP**-hard, even if the given threshold is less than half of the number of nodes. Indeed, given $G$, we can build a graph $G_k$ by adding $2 \times (k+1)$ fresh nodes with one of them being connected with the remaining $2 \times (k+1) - 1$ nodes. Then, $G$ has a vertex cover with size at most $k$ if, and only if, $G_k$ has a vertex cover with size at most $k+1$. Moreover, $G_k$ is clearly defined over $n + 2 \times (k+1)$ nodes and, hence, $k+1 < \frac{n+2\times(k+1)}{2}$ holds.

Based on $G$, we build a game $\Gamma = \langle N \cup \{a_{n+1}\}, v, G' \rangle \in \mathcal{C}_{\mathbf{TW}\text{-}1}(\text{mcn})$ as follows. The graph $G'$ is a tree where $a_{n+1}$ is connected with all other nodes, and no further edge occurs in it. The game is encoded as a marginal contribution network $M$ such that:

$$\begin{cases} \{a_{n+1} \wedge a_i\} & \rightarrow \ |\{e \in E \mid a_i \in e\}|, \forall i \in \{1, \dots, n\} \\ \{a_{n+1} \wedge a_i \wedge a_j\} & \rightarrow \ -1, \forall \{a_i, a_j\} \in E \\ \{a_{n+1} \wedge a_1 \cdots \wedge a_n\} & \rightarrow \ 1 \end{cases}$$

Thus, the payoff function $v$ determined by these rules is such that, for each feasible coalition $C \in \mathcal{F}(G')$,

$$v(C) = \begin{cases} |E| + 1, & \text{if } C = \{a_1, \dots, a_n, a_{n+1}\} \\ |\{e \in E \mid e \cap C \neq \emptyset\}|, & \text{if } a_{n+1} \in C \text{ and } C \neq \{a_1, \dots, a_n, a_{n+1}\} \\ 0, & \text{otherwise} \end{cases}$$

Observe that $v(C) = 0$ holds, for each coalition $C$ such that $a_{n+1} \notin C$. Moreover, for each pair of coalitions $C$ and $C'$ with $a_{n+1} \in C \cap C'$ and $C' \supseteq C$, it clearly holds that $v(C') \geq v(C)$. Therefore, the game is superadditive.

Then, consider the value $\varepsilon = -\frac{1}{n-k}$. We claim that $\{N\} \notin$ CS-*opt*$(\Gamma_{\text{-}\varepsilon})$ if, and only if, $G$ has a vertex cover with size at most $k$.

($\Leftarrow$) Assume that $V$ is a vertex cover with $|V| \leq k$. Consider the coalition $C = V \cup \{a_{n+1}\}$ and note that $v(C) = |E|$ while $v(N) = |E| + 1$. In fact, $C \neq N$ holds. Consider then the feasible coalition structure $\Pi$ including $C$ and the remaining $n - |V|$ nodes as singletons. In the game $\Gamma_{\text{-}\varepsilon}$, we clearly have that $v_{\text{-}\varepsilon}(N) = v(N) = |E| + 1$, while the social welfare of $\Pi$ is $v_{\text{-}\varepsilon}(C) - (n - |V|) \times \varepsilon = |E| + (n - |V| + 1) \times \frac{1}{n-k} > |E| + 1 = v(N)$. That is, $\{N\} \notin$ CS-*opt*$(\Gamma_{\text{-}\varepsilon})$.

($\Rightarrow$) Assume there is no vertex cover $V$ with $|V| \leq k$. Consider the game $\Gamma_{\text{-}\varepsilon}$ where $v_{\text{-}\varepsilon}(N) = v(N) = |E| + 1$, and let us show that $\{N\} \in$ CS-*opt*$(\Gamma_{\text{-}\varepsilon})$ holds. To this end, consider any feasible coalition structure $\Pi \neq \{N\}$. Given the interaction graph $G'$, $\Pi$ consists of a coalition $C \neq N$ such that $a_{n+1} \in C$ and of all the remaining agents included as singletons. Thus, the social welfare of $\Pi$ is $v_{\text{-}\varepsilon}(\Pi) = v_{\text{-}\varepsilon}(C) - (n + 1 - |C|) \times \varepsilon = v(C) + \frac{n+2-|C|}{n-k}$. Let us distinguish three cases.

– Assume that $|C| = 1$. Then, $C = \{a_{n+1}\}$ and $v(C) = 0$. Therefore, $v_{\text{-}\varepsilon}(\Pi) = v(C) + \frac{n+2-|C|}{n-k} = \frac{n+1}{n-k} < 2 + \frac{2}{n}$, with the latter inequality following by the fact that we can assume that $k < n/2$. This entails that $v_{\text{-}\varepsilon}(\Pi) < |E| + 1$ (w.l.o.g., we can assume of course that $|E| > 3$ holds).
– Assume that $1 < |C| \leq k+1$. Then, $C \setminus \{a_{n+1}\}$ is not a vertex cover, and hence $v(C) \leq |E| - 1$ holds. Then, $v_{\text{-}\varepsilon}(\Pi) = v(C) + \frac{n+2-|C|}{n-k} \leq |E| + \frac{n+2-|C|}{n-k} - 1 = |E| + \frac{k+2-|C|}{n-k} \leq |E| + \frac{k}{n-k} < |E| + 1$, again with the latter inequality following by the fact that $k < n/2$.
– Assume that $|C| > k+1$, with $C \neq N$. In this case, it might happen that $v(C) = |E|$. In particular, $v_{\text{-}\varepsilon}(\Pi) \leq |E| + \frac{n+2-|C|}{n-k} \leq |E| + \frac{n-k}{n-k} = |E| + 1$.

In all the possible cases, we have derived that $v_{\text{-}\varepsilon}(\Pi) \leq |E| + 1 = v_{\text{-}\varepsilon}(N) = v(N)$. Therefore, we conclude that $\{N\} \in$ CS-*opt*$(\Gamma_{\text{-}\varepsilon})$.

By Claim 5.12, the above entails that the vertex cover problem has been reduced to the complement of CS-LCV$^-$, hence showing that this latter problem is co-**NP**-hard over games encoded via marginal contribution networks. By Lemma 4.11, hardness extends to more expressive game encodings. □

## 6. Lines, cycles, and degree-based classification

In this section, we analyze the complexity of core-related questions over classes of games whose interaction graphs are lines and cycles. Indeed, these are special classes of graphs having bounded treewidth and, therefore, there is a possibility that, by exploiting their peculiarities, tractability results can be established. Moreover, these results are interesting from an applications point of view, as many computer or sensor networks exhibit linear or ring topologies.

Eventually, since these kinds of graphs share the property that each node has at most two *neighbors*, results for them motivate the subsequent classification of the complexity of core-related questions based on the maximum degree in the interaction graph.

### *6.1. Lines and cycles*

Let **L** and **C** denote the classes of all graphs that are lines and cycles, respectively. We start by observing that our core-related problems are tractable over **L**.

**Theorem 6.1.** *Let $\mathcal{R}$ be any compact* **P***-representation. On the class $\mathcal{C}_{\mathbf{L}}(\mathcal{R})$,* CS-Core-Membership, CS-CoS, CS-Core-NonEmptiness, CS-LCV$^-$, CS-LCV$^+$, *and* CS-LCV *are in* **P***. Moreover,* CS-Core-Find *are in* **FP**.

**Proof.** CS-Core-NonEmptiness and CS-CoS are immaterial over lines, as the coalition structure core is not empty in this setting [30].

Consider then the CS-Core-Membership problem, which given a game $\Gamma = \langle N, v, G\rangle$ and a pair $(\Pi, \boldsymbol{x})$, asks whether $(\Pi, \boldsymbol{x})$ is in CS-*core*$(\Gamma)$. Note that checking whether $\boldsymbol{x}$ is an imputation for $\Pi$ is trivially feasible in polynomial time, whenever $\Gamma$ is given according to a **P**-representation scheme. Therefore, the basic computational challenge is to check whether $x(C) \geq v(C)$ holds, for each feasible coalition $C \in \mathcal{F}(G)$. In general, this is a hard task, because there might be exponentially many feasible coalitions. However, if $G \in \mathbf{L}$ holds, then any feasible coalition in $\mathcal{F}(G)$ is completely determined by two (possibly coinciding) endpoints. Thus, $|\mathcal{F}(G)| \leq \binom{|N|}{2} + \binom{|N|}{1}$ holds and, consequently, CS-Core-Membership is feasible in polynomial time.

Concerning the computation problem CS-Core-Find, observe that, whenever an optimal coalition structure $\Pi \in$ CS-*opt*$(\Gamma)$ is given, then CS-*core*$(\Gamma)$ is completely determined by a system of linear inequalities, because of Fact 3.5. Moreover, we have just observed that when the underlying interaction graph is a line, we have to deal with polynomially many inequalities only. Therefore, by well-known results in linear programming [56,57], CS-Core-Find can be shown to be in **FP**, provided we can compute in polynomial time $\Pi$. Below, we show that this is actually possible.

**Claim 6.2.** *Let $\mathcal{R}$ be any compact* **P***-representation. On the class $\mathcal{C}_{\mathbf{L}}(\mathcal{R})$, computing the maximum social welfare (and a feasible coalition structure with this value) is in* **FP**.

Finally, consider the problems CS-LCV$^+$, CS-LCV$^-$, and CS-LCV. By Theorem 5.7, CS-LCV$^+$ is feasible in polynomial time. To conclude, we need to consider CS-LCV$^-$ that, given a game $\Gamma \in \mathcal{C}_{\mathbf{L}}(\mathcal{R})$ and a rational number $\varepsilon \leq 0$, asks to decide whether there is a pair $(\Pi, \boldsymbol{x})$ in CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon})$. To address this problem, actually with $\varepsilon < 0$ (as the case $\varepsilon = 0$ reduces to CS-Core-NonEmptiness), we recall the characterization of Lemma 3.9: the least coalition structure core value of $\Gamma$ is at most $\varepsilon$ if, and only if, *(i)* $|$CS-*opt*$(\Gamma)| = 1$, and *(ii)* $\Pi \in$ CS-*opt*$(\Gamma)$ implies $\Pi \in$ CS-*opt*$(\Gamma_{\Pi,\text{-}\varepsilon})$ and CS-*core*$(\Gamma_{\Pi,\text{-}\varepsilon}) \neq \emptyset$. Therefore, we need just to show that *(i)* and *(ii)* can be checked in polynomial time over lines.

Indeed, by Claim 6.2, the optimal social welfare can be computed in polynomial time, in fact, together with a feasible coalition structure $\Pi \in$ CS-*opt*$(\Gamma)$. Then, for each coalition $C \in \Pi$, consider the game $\Gamma_C = \langle N, v_C, G\rangle$ coinciding with $\Gamma$, but where the value associated with $C$ is now $v_C(C) = -M$, with $M \geq 0$ being a large enough positive rational number, so that CS-*opt*$(\Gamma_C)$ includes a coalition structure with $C$ only if there is no other possible coalition structure at all. Clearly enough, $SW_{opt}(\Gamma_C)$ can be again computed in polynomial time by Claim 6.2. If there is some coalition $C \in \Pi$ such that $SW_{opt}(\Gamma_C) = SW_{opt}(\Gamma)$, then we have immediately derived that *(i)* does not hold. Otherwise, we have shown that to get the maximum social welfare in $\Gamma$, it is mandatory to include all the coalitions in $\Pi$, so that *(i)* holds.

In this latter case, it remains to be checked whether *(ii)* holds, too. But, this now just requires to build the game $\Gamma_{\Pi,\text{-}\varepsilon}$ and to compute again the maximum social welfare, by eventually checking that $SW_{opt}(\Gamma) = SW_{opt}(\Gamma_{\Pi,\text{-}\varepsilon})$. Moreover, it requires to answer CS-Core-NonEmptiness (which is in fact immaterial) over $\Gamma_{\Pi,\text{-}\varepsilon}$. Therefore, the whole computation is feasible in polynomial time—game encodings can be built in polynomial time, as discussed in the proof of Theorem 4.1. □

Similarly, moving from lines to cycles, we are again able to show tractability of a number of core-related questions. This is more interesting, since the presence of cycles can create instances with an empty core, even in superadditive games [38]. Moreover, a given cycle admits exponentially many coalition structures.

**Theorem 6.3.** *Let $\mathcal{R}$ be any compact* **P***-representation. On the class $\mathcal{C}_{\mathbf{C}}(\mathcal{R})$,* CS-Core-Membership, CS-CoS, CS-Core-NonEmptiness, CS-LCV$^-$*, and* CS-LCV *are in* **P***, while* CS-LCV$^+$ *belongs to* **NP***. Moreover,* CS-Core-Find *is in* **FP**.

**Proof.** Consider first the CS-Core-Membership problem. Similarly to the proof of Theorem 6.1, it is sufficient to show that over cycles polynomially many coalitions can form, only. Indeed, if $\Gamma = \langle N, v, G\rangle$ is a game with $G \in \mathbf{C}$, then any feasible coalition is determined by two (possibly coinciding) endpoints and the direction (clockwise or counterclockwise), so that there are at most $2 \times \binom{|N|}{2} + \binom{|N|}{1}$ such coalitions.

Let us now focus on the computation problem CS-Core-Find, and note that after the above observation we are precisely in the same position as in the proof of Theorem 6.1. So, we just show that we can compute an optimal coalition structure in polynomial time.

**Claim 6.4.** *Let $\mathcal{R}$ be any compact* **P***-representation. On the class $\mathcal{C}_{\mathbf{C}}(\mathcal{R})$, computing the maximum social welfare (and a feasible coalition structure with this value) is in* **FP**.

By the tractability of CS-Core-Find, it follows that CS-Core-NonEmptiness is in **P**. Consider now CS-CoS. In this case, we have to decide whether CS-*core*$(\Gamma_\Delta)$ is not empty, where $\Delta \geq 0$ is a rational number additionally given as input. Therefore, we are back to CS-Core-NonEmptiness on a game defined on a cycle, and the result follows since an encoding for $\Gamma_\Delta$ can be built in polynomial time (cf. Theorem 4.1).

Consider now CS-LCV$^-$. In this case, we can use Lemma 3.9 and precisely the same argument as in the proof of Theorem 6.1, but with the tractability of computing the maximum social welfare now provided by Claim 6.4 (instead of Claim 6.2), and with the tractability of the CS-Core-NonEmptiness problem following by the above arguments (instead of being trivial).

Finally, we show that CS-LCV$^+$ belongs to **NP**. The result follows by inspecting the proof of Theorem 4.3. Indeed, recall that the problem can be solved by a non-deterministic Turing machine that first guesses a feasible coalition structure $\Pi$ and then, given $\Pi$ and $\varepsilon$, checks that: (1) $\Pi \in$ CS-*opt*$(\Gamma_{\Pi,\text{-}\varepsilon})$; (2) *core*$(\tilde{\Gamma}_{\Pi,\text{-}\varepsilon}) \neq \emptyset$. In fact, we know that $\Gamma_{\Pi,\text{-}\varepsilon}$ can be specified in compact form. Hence, (1) can be checked in polynomial time because of Claim 6.4. Similarly, (2) can be checked in polynomial time, since we have already shown that CS-Core-NonEmptiness is in **P** over games defined on cycles. □

We next refine the above result for CS-LCV$^+$, by focusing on superadditive games.

**Theorem 6.5.** *Let $\mathcal{R}$ be any compact* **P***-representation. On the class $\mathcal{C}^{sa}_{\mathbf{C}}(\mathcal{R})$,* CS-LCV$^+$ *is in* **P**.

**Proof.** Let $\Gamma = \langle N, v, G\rangle$ be a superadditive game with $G \in \mathbf{C}$ and let $\varepsilon$ be the penalty provided as input to CS-LCV$^+$. From Theorem 3.10, we know that CS-LCV$(\Gamma)$ = LCV$(\Gamma)$. Therefore, CS-LCV$(\Gamma) \leq \varepsilon$ holds if, and only if, *core*$(\Gamma_{\text{-}\varepsilon})$ is not empty. In fact, $\Gamma_{\text{-}\varepsilon}$ is a game defined on a cycle which can be specified in compact form. The result then follows because *core*$(\Gamma_{\text{-}\varepsilon}) \neq \emptyset$ can be checked in polynomial time (by the same arguments as in the proof of Theorem 6.3). □

### 6.2. A degree-based look at complexity results

A unifying feature of the classes **C** and **L** of interaction graphs is that each node has degree at most 2. Moreover, over connected graphs the union $\mathbf{C} \cup \mathbf{L}$ precisely coincides with the set of all graphs having this property. Therefore, if $\mathbf{D}[\leq 2]$ denotes the class of all interactions graphs where each node has at most two adjacent nodes, then the following is derived from the results provided in this section.

**Corollary 6.6.** *Let $\mathcal{R}$ be any compact* **P***-representation. On the class $\mathcal{C}_{\mathbf{D}[\leq 2]}(\mathcal{R})$,* CS-CoS, CS-Core-Membership, CS-Core-NonEmptiness*, and* CS-LCV$^-$ *are in* **P***, while* CS-Core-Find *is in* **FP***. Moreover, on the class $\mathcal{C}_{\mathbf{D}[\leq 2]}(\mathcal{R})$ (resp., $\mathcal{C}^{sa}_{\mathbf{D}[\leq 2]}(\mathcal{R})$),* CS-LCV$^+$ *and* CS-LCV *are in* **NP** *(resp.,* **P***).*

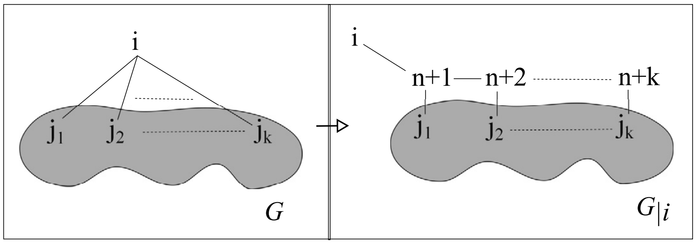


**Fig. 5.** Construction in Section 6.2.

The question hence emerges about whether the result is tight, i.e., whether interaction graphs whose nodes have degree at most 3 already suffice to encode intractable instances of core-related questions. We next show that this is the case. The line of the proof is to show that any game can be 'simulated' by means of a game where each agent has degree at most 3. Below, we incrementally present the construction. Prior to our result, it was known that a game without structural restrictions can be simulated by a structurally-restricted game where each agent has degree 6 at most (and by preserving the fact that the worth function is superadditive and simple) [54].

Let $\Gamma = \langle N, v, G\rangle$ be a structurally-restricted characteristic function game with $N = \{1, \ldots, n\}$, and let $i \in N$ be a node such that the set of its adjacent nodes is $\{j_1, \ldots, j_k\}$ with $k > 3$. Consider then the interaction graph $G_{|i}$ built over a set $N_{|i} = N \cup \{n+1, \ldots, n+k\}$ of agents as follows (and depicted in Fig. 5). There is an edge connecting agent $i$ with agent $n+1$. For each $h \in \{1, \ldots, k-1\}$, there is an edge connecting agent $n+h$ with $n+h+1$ and an edge connecting agent $n+h$ with $j_h$. There is an edge connecting agent $n+k$ with $j_k$. Each edge in $G$ such that $i$ is not one of its endpoints is in $G_{|i}$. Finally, no further edge is in $G_{|i}$.

Consider now the game $\Gamma_{|i} = \langle N_{|i}, v_{|i}, G_{|i}\rangle$ where $v_{|i}$ is such that, for each feasible coalition $C \in \mathcal{F}(G_{|i})$,

$$v_{|i}(C) = \begin{cases} -R \times (|N \setminus C| + 1), & \text{if } \{i, n+1, \ldots, n+k\} \cap C \neq \emptyset \text{ and } |\{i, n+1, \ldots, n+k\} \cap C| \neq k+1 \\ v(C \cap N) & \text{otherwise, i.e., either } \{i, n+1, \ldots, n+k\} \subseteq C \text{ or } C \subseteq N \setminus \{i\} \end{cases}$$

where $R$ is set to $n+k$ times an upper bound on the maximum absolute value over the payoffs associated with the various coalitions. Intuitively, $R$ is set in a way that it is definitively advantageous for the agents in $\{i, n+1, \ldots, n+k\}$ to stay all together in the same coalition. Then, for any coalition $C \subseteq N$, we denote by $C_{|i}$ the coalition $C$ itself if $i \notin C$, and the coalition $C \cup \{n+1, \ldots, n+k\}$ if $i \in C$ holds. Similarly, for any coalition structure $\Pi$ over $G$, we denote by $\Pi_{|i}$ the coalition structure where each element $C \in \Pi$ is replaced by $C_{|i}$. The following is immediate by construction.

**Fact 6.7.** *Assume that $\Pi^*$ is a coalition structure in* CS-*opt*$(\Gamma_{|i})$. *Then, there is a coalition structure $\Pi$ over $G$ such that $\Pi^* = \Pi_{|i}$.*

With the above notation and result in place, we can show the following technical result relating the coalition structure cores of $\Gamma$ and $\Gamma_{|i}$.

**Lemma 6.8.** *The following statements hold on the games $\Gamma$ and $\Gamma_{|i}$.*

(A) *If $(\Pi, \boldsymbol{x})$ is in* CS-*core*$(\Gamma)$, *then $(\Pi_{|i}, \boldsymbol{x}^*)$ is in* CS-*core*$(\Gamma_{|i})$ *where $\boldsymbol{x}^*$ is such that $x^*_j = x_j$, for each $j \in N$; and $x^*_j = 0$, for each $j \in \{n+1, \ldots, n+k\}$.*
(B) *Let $(\Pi^*, \boldsymbol{x}^*)$ be in* CS-*core*$(\Gamma_{|i})$, *and let $\boldsymbol{x}$ be such that $x_i = x^*_i + x^*(\{n+1, \ldots, n+k\})$ and $x_j = x^*_j$, for each $j \in N \setminus \{i\}$. Then, $(\Pi, \boldsymbol{x})$ is in* CS-*core*$(\Gamma)$, *where $\Pi_{|i} = \Pi^*$.*

**Proof.** (A) Assume that $(\Pi, \boldsymbol{x})$ is an element in CS-*core*$(\Gamma)$. Since $\boldsymbol{x}$ is an imputation for $\Pi$, we have that $x(C) = v(C)$ holds, for each $C \in \Pi$, and $x_j \geq v(\{j\})$ holds, for each $j \in N$. Then, consider the corresponding coalition $C_{|i} \in \Pi_{|i}$ and note that $x^*(C_{|i}) = x(C) = v(C) = v_{|i}(C)$. Moreover, for each $j \in \{n+1, \ldots, n+k\}$, $x^*_j = 0 \geq -R \geq v_{|i}(\{j\})$. Therefore, $\boldsymbol{x}^*$ is an imputation for $\Pi_{|i}$ in the game $\Gamma_{|i}$. To conclude the proof, we have to show that $\boldsymbol{x}^*$ is stable. To this end, consider a feasible coalition $C \in \mathcal{F}(G_{|i})$. In the case where $\{i, n+1, \ldots, n+k\} \cap C \neq \emptyset$ and $|\{i, n+1, \ldots, n+k\} \cap C| \neq k+1$, it holds that $x^*(C) = x(C \cap N) = \sum_{j \in C \cap N} x_j \geq \sum_{j \in C \cap N} v(\{j\}) \geq -R \geq v_{|i}(C)$. Instead, in the case where either $\{i, n+1, \ldots, n+k\} \subseteq C$ or $C \subseteq N \setminus \{i\}$, we have that $x^*(C) = x(C \cap N) \geq v(C \cap N) = v_{|i}(C)$, where $x(C \cap N) \geq v(C \cap N)$ follows by the fact that $(\Pi, \boldsymbol{x})$ is an element in CS-*core*$(\Gamma)$ and by observing that $C \cap N$ is feasible over $G$ in this case. By putting it all together, we conclude that $(\Pi_{|i}, \boldsymbol{x}^*)$ is in CS-*core*$(\Gamma_{|i})$.

(B) Let $(\Pi^*, \boldsymbol{x}^*)$ be in CS-*core*$(\Gamma_{|i})$. First, observe that by Fact 6.7 (combined with Fact 2.1) there is a coalition structure $\Pi$ over $G$ such that $\Pi^* = \Pi_{|i}$. Then, consider the payoff vector $\boldsymbol{x}$ such that $x_i = x^*_i + x^*(\{n+1, \ldots, n+k\})$; $x_j = x^*_j$, for each $j \in N \setminus \{i\}$. Note that $x_i = x^*_i + x^*(\{n+1, \ldots, n+k\}) = x^*(\{i, n+1, \ldots, n+k\}) \geq v_{|i}(\{i, n+1, \ldots, n+k\}) = v(\{i\})$, and $x_j = x^*_j \geq v_{|i}(\{j\}) = v(\{j\})$ for each $j \in N \setminus \{i\}$. That is, $\boldsymbol{x}$ is an imputation for $\Pi$ in the game $\Gamma$. We now conclude by showing that $\boldsymbol{x}$ is stable. Indeed, for each coalition $C$ that is feasible over $G$, note that $x(C) = x^*(C) \geq v_{|i}(C) = v(C)$ holds, whenever $i \notin C$. Instead, in the case where $i$ is in $C$, $x(C) = x^*(C \cup \{n+1, \ldots, n+k\}) \geq v_{|i}(C \cup \{n+1, \ldots, n+k\}) = v(C)$. Therefore, $(\Pi, \boldsymbol{x})$ is in CS-*core*$(\Gamma)$. $\square$

Moreover, we can show that the transformation preserves superadditivity.

**Theorem 6.9.** *If $\Gamma$ is superadditive, then $\Gamma_{|i}$ is superadditive, too.*

**Proof.** Assume that $\Gamma$ is superadditive and let $S$ and $T$ be two feasible disjoint coalitions over $N_{|i}$ such that $S \cup T$ is feasible, too. If $v_{|i}(S) = v(S \cap N)$ and $v_{|i}(T) = v(T \cap N)$, then $v_{|i}(S \cup T) = v((S \cup T) \cap N)$ holds and, hence, $v_{|i}(S \cup T) \geq v(S \cap N) + v(T \cap N) = v_{|i}(S) + v_{|i}(T)$. If $v_{|i}(S) \neq v(S \cap N)$, $v_{|i}(T) \neq v(S \cap N)$, and $v_{|i}(S \cup T) = v((S \cup T) \cap N)$, then given the choice of $R$ we are trivially guaranteed that $v_{|i}(S \cup T) \geq v_{|i}(S) + v_{|i}(T)$. If $v_{|i}(S) \neq v(S \cap N)$, $v_{|i}(T) \neq v(T \cap N)$, and $v_{|i}(S \cup T) \neq v((S \cup T) \cap N)$, then we have $v_{|i}(S \cup T) = -R \times (|N \setminus (S \cup T)| + 1) \geq -R \times (|N \setminus S| + 1) - R \times (|N \setminus T| + 1) = v_{|i}(S) + v_{|i}(T)$. Consider then the remaining case where $v_{|i}(S) = v(S \cap N)$ and $v_{|i}(T) \neq v(T \cap N)$. Note that this entails that $v_{|i}(S \cup T) \neq v((S \cup T) \cap N)$. Thus, $v_{|i}(S \cup T) = -R \times (|N \setminus (S \cup T)| + 1) \geq -R \times (|N \setminus T| + 1 - 1) = v_{|i}(T) + R \geq v_{|i}(T) + v(S \cap N) = v_{|i}(T) + v_{|i}(S)$. $\square$

These results are the crucial ingredients to prove the following, where $\mathbf{D}[\leq 3]$ denotes the class of all interaction graphs whose nodes have degree at most 3.

**Theorem 6.10.** *Let $\mathcal{R}$ be any compact* **P***-representation such that* mcn $\precsim_e \mathcal{R}$*. On the class $\mathcal{C}^{sa}_{\mathbf{D}[\leq 3]}(\mathcal{R})$,* CS-Core-Membership, CS-CoS, CS-Core-NonEmptiness, CS-LCV$^-$*, and* CS-LCV$^+$ *are* co-**NP***-complete, whereas* CS-Core-Find *is* **NP***-hard. On the class $\mathcal{C}_{\mathbf{D}[\leq 3]}(\mathcal{R})$,* CS-CoS, CS-Core-NonEmptiness*, and* CS-LCV$^-$ *are $\mathbf{\Delta}_2^{\mathbf{P}}$-complete,* CS-LCV$^+$ *is $\mathbf{\Sigma}_2^{\mathbf{P}}$-complete, whereas* CS-Core-Membership *is* co-**NP***-complete.*

**Proof.** Let $\Gamma$ be a game where $i \in N$ is a node such that the set of its adjacent nodes is $\{j_1, \ldots, j_k\}$ with $k > 3$, and consider the game $\Gamma_{|i}$. By Lemma 6.8, CS-Core-NonEmptiness can be equivalently answered over $\Gamma_{|i}$ in place of $\Gamma$. However, $G_{|i}$ is an interaction graph where all the fresh nodes have degree bounded by 3, all nodes in $G$ but $i$ keep their original degree, and node $i$ has degree 1. Therefore, the number of nodes whose degree is greater than 3 decreases when moving from $\Gamma$ to $\Gamma_{|i}$. If $\Gamma_{|i}$ contains no node whose degree is greater than 3, then we have concluded. Otherwise, we can apply the transformation on $\Gamma_{|i}$, by considering a node $j$ whose degree is greater than 3.

The above line of reasoning leads to a sequence of at most $|N|$ games converging to a game $\Gamma' \in \mathcal{C}_{\mathbf{D}[\leq 3]}$. Moreover, Lemma 6.8 transitively applies to all these games, so that CS-*core*$(\Gamma) = \emptyset$ if, and only if, CS-*core*$(\Gamma') = \emptyset$. Similarly, given the definition of $R$, the proof of Lemma 6.8 can be smoothly adapted to show that the least coalition structure core value is preserved in the transformation. And, finally, by Theorem 6.9, if $\Gamma$ is superadditive, then $\Gamma'$ is superadditive, too.

Now, we claim that $\Gamma_{|i}$ can be encoded in terms of a marginal contribution network $M_{|i}$, provided that $\Gamma$ is already given as a marginal contribution network $M$. Let *none* denote the conjunction $\neg a_i \wedge \neg a_{n+1} \wedge \cdots \wedge \neg a_{n+k}$ and let *all* denote the conjunction $a_i \wedge a_{n+1} \wedge \cdots \wedge a_{n+k}$. Then, for each rule having the form $\{pattern\} \rightarrow value$ in $M$, the novel encoding $M_{|i}$ includes the rules $\{pattern \wedge none\} \rightarrow value$ and $\{pattern \wedge all\} \rightarrow value$. Moreover, $M_{|i}$ includes the rules $\{\neg a_h\} \rightarrow -R$, $\{\neg a_h \wedge none\} \rightarrow R$, and $\{\neg a_h \wedge all\} \rightarrow R$, for each agent $a_h \in N$, plus the rules $\{all\} \rightarrow R$, $\{none\} \rightarrow R$, and $\{\} \rightarrow -R$. No further rule is in $M_{|i}$. Again, by repeatedly applying this transformation, we end up in polynomial time with a marginal contribution network $M'$ encoding the game $\Gamma'$. This shows that any hardness result for the problems CS-CoS, CS-Core-NonEmptiness (hence, Core-Find), and CS-LCV still holds over the subclass of those games encoded via marginal contribution networks and whose interaction graphs belong to $\mathbf{D}[\leq 3]$. The results in the statement about these problems then follow by our results in Section 4.

Finally, note that the problem CS-Core-Membership on input a pair $(\Pi, \boldsymbol{x})$ can be equivalently formulated as the problem of checking whether the pair $(\Pi_{|i}, \boldsymbol{x}^*)$ defined in Lemma 6.8.(A) belongs to the coalition structure core of $\Gamma_{|i}$. In fact, the equivalence follows by both points in that lemma. By iteratively applying the transformation, the problem can be equivalently formulated as the problem of checking whether a pair $(\Pi', \boldsymbol{x}')$, which can be built in polynomial time, belongs to the coalition structure core of the game $\Gamma'$. Thus, all hardness results about CS-Core-Membership hold over games encoded via marginal contribution networks and whose interaction graphs belong to $\mathbf{D}[\leq 3]$. Again, the results in the statement about CS-Core-Membership follow by our results in Section 4. $\square$

## 7. Related work

The model considered in this paper, where coalitional games are equipped with interaction graphs constraining the formation of coalitions among the agents, goes back to the seventies [14], but has recently attracted more attention in the literature because of its natural applicability in real-world domains. For instance, algorithms to compute the *Shapley value* and related concepts in this model have been studied by [58,59] (see also the references therein), with applications to the analysis of terrorist networks. Moreover, task allocation problems arising over agents interconnected in a social environment have been studied by [60]. Within this setting, our goal has been instead to characterize the complexity of core-related questions, which are largely unexplored in the literature.

Recently, complexity issues over social networks have been studied by Sless et al. [61]. However, in their model all coalitions are allowed to form (with the network being only used to provide values to the coalitions, in the spirit of graph games [24]). Moreover, coalition structures are restricted to those consisting of $k$ coalitions only, with $k$ being the number of tasks that a central authority is asking the agents to execute. In fact, works that are closely related to our research either focused on the specific case where interaction graphs are trees, or analyzed the setting of complete interaction graphs, hence reducing to the standard setting where all coalitions are allowed to form. Concerning the former case, we observe that in [17] and [18] it is shown that if a game is superadditive and the graph is a tree, then the core is non-empty. However, their existential proof, based on Scarf's lemma [62], does not provide an efficient algorithmic construction of a core element. The follow-up work by Demange [19] not only guarantees that the coalition structure core is non-empty for trees, for both superadditive and non-superadditive games, but it also proposes a procedure that computes an element in the core. Although Demange's algorithm does not run in polynomial time for non-superadditive games, it can be seen that it does so for superadditive ones. In fact, this is the best one can hope for, given that computing an element in the coalition structure core is known to be **NP**-hard, even over *(i)* trees [53,32], and over *(ii)* simple and monotone games with interaction graphs having treewidth 2 (cf. [32]). Our results evidence that hardness holds in the same setting, but even on marginal contribution networks. Moreover, in addition to the computation problem, we have considered the decision problems CS-Core-NonEmptiness and CS-Core-Membership, by providing completeness results for the classes co-**NP** and $\mathbf{\Delta_2^P}$.

Moving to the standard setting where all coalitions are allowed to form, several works in the literature can be found where the complexity of core-related questions is addressed (see, e.g., [26,24,27,25,5,28,9]). However, these works tacitly assume that the goal is to distribute the total payoff available to the grand-coalition, and they do not take into account the possibility of forming coalition structures. In fact, our main source of knowledge for the complexity of the coalition structure core is the work by Elkind et al. [42], where **NP**-hardness results for CS-Core-NonEmptiness and CS-Core-Membership problems over weighted voting games are illustrated. In the paper, we have provided non-trivial upper bounds that apply to any game that is given in compact representation (and hence apply to weighted voting games, too), and we have shown the corresponding hardness results over games encoded as marginal contribution networks.

Closely related to our research are also complexity studies where the traditional concept of the core is adopted (again with all coalitions being allowed to form), but where the focus is on games that are superadditive. Indeed, in this setting, the coalition structure core reduces to the (standard) core. With this respect, we refer the reader to the work by Conitzer and Sandholm [5] on games based on synergies among coalitions, by stressing however that this work does not fit our definition of **P**-representation, as even computing the payoff associated with some given coalition is **NP**-hard there. Instead, it is worth noticing that a study of the complexity of the problems CS-Core-NonEmptiness and CS-Core-Membership over superadditive marginal contribution networks has been recently carried out by Li and Conitzer [31], in fact independently of our work. The authors considered an extension of marginal contribution networks where the Boolean connective '$\vee$' can be used (in addition to '$\wedge$' and '$\neg$'), and charted the picture of complexity results as the allowed connectives vary. For the case where all values associated with the rules are non-negative, they observed that without disjunction and negation the problems are tractable, but they become co-**NP**-hard when negation is additionally taken into account. Our results stated in Theorem 4.9 and Theorem 4.10 further strengthened this finding, as they evidence that intractability suddenly emerges as soon as even just one occurrence of '$\neg$' is allowed in each rule.

Note that, in addition to the problems CS-Core-NonEmptiness and CS-Core-Membership, we have considered other core-related questions involving approximations of the core, namely the problems CS-CoS and CS-LCV (with its variants CS-LCV$^-$ and CS-LCV$^+$). The complexity analysis conducted over them does not have counterparts in the existing literature. However, one should point out that the cost of stability has been studied in depth from an analytical viewpoint, even in its general formulation over coalition structures, and in the presence of interaction graphs (see [54,38,32,39]). In particular, the cost of stability has been connected to the treewidth of the interaction graph in [32,54]. Connections with other parameters of the games and of the interaction graphs have been studied by Meir et al. [38], who also provided results for a least core notion, which nevertheless differs from ours in that it imposes a penalty for leaving the grand-coalition (rather than for an arbitrary coalition structure to be rendered stable). Finally, bounds on the cost of stability for general, subadditive, and anonymous games are given in [39], where the special case of facility games is also discussed.

Interestingly, as a side result of our analysis, we have shown the tractability of the problem of computing an optimal coalition structure over lines and cycles. This *(optimal) coalition structure generation (CSG) problem* has attracted much attention in the literature, because of its applications in the design and analysis of multi agent systems (see, e.g., [63–67]). The framework discussed by Voice et al. [68], in particular, is very close to ours. Indeed, they studied a model for coalition formation where feasible coalitions are also identified on the basis of an underlying interaction graph. However, their results are for payoff functions that are *independent of disconnected members*—that is, two agents have no effect on each other's marginal contribution to their vertex separator. In addition, several tractability results are known in the CSG literature for specific representation schemes and for specific kinds of interaction graphs [66,68,69]. A number of useful results are known for games encoded in terms of marginal contribution networks [70–72]. However, it must be observed that our results are independent of the specific representation scheme used (provided that payoff functions are computable in polynomial time), while being specific for interaction graphs that are lines and cycles.

Finally, we note that our work is originating from two earlier papers, namely [53] and [36]. Nevertheless, we stress that the results reported in this paper significantly extend and generalize those discussed in [53,36]. In particular, the analysis of the computational issues arising with lines and cycles (excluding the problems related to the cost of stability and to the least core) appears in a preliminary form in [53]. The work of [36] focuses on deriving complexity results for complete interaction graphs and for bounded treewidth graphs. An *oracle setting* has been considered in [36], which ignores any issues related to the representation of the payoff functions, and even the space needed for the encodings and the time necessary to build them in the reductions. In our paper, completely different machineries have therefore been employed in order to show that the results hold even on marginal contributions networks. Finally, we also note that the classification of the complexity based on the degree of the nodes and the analysis of the problems CS-LCV$^-$ and CS-LCV$^+$ (together with the characterizations for the least coalition structure core) are entirely novel contributions.

## 8. Conclusions

We analyzed stability issues in *compact* coalitional games—i.e., games described via a compact encoding, and equipped with interaction graphs constraining the formation of coalitions among agents according to the model introduced by Myerson [14]. We studied the complexity of interesting computational problems in this setting over various (interaction) graph structures, and we presented a classification based on the degree of the nodes. In particular, *positive* complexity results have been obtained for graph structures, such as trees and cycles, that form the backbone of networks found in the real world. Our results can serve as indications for the tractability of respective real world coalitional game settings, and could help to design practical solution algorithms.

Our results are summarized in Table 1, Table 2, and Table 3. All the entries are either tractability results or hardness results (over marginal contribution networks) for which the corresponding membership result is given (over arbitrary **P**-representations). A completeness result is missing only for the CS-LCV$^+$ problem on arbitrary games over cycles and bounded treewidth interaction graphs. Other avenues of research naturally include the study of further classes of interaction graphs (for instance, of classes of graphs having bounded *vinewidth*, a notion generalizing treewidth and recently introduced by [73] to analyze the cost of stability), and the study of hypergraph-based representations and restrictions (see, e.g., [74]). Moreover, it would be interesting to study alternative stability concepts and payoff allocation schemes, such as the *kernel*, the *Shapley value*, and the *nucleolus* [28,75]. In particular, it would be relevant to assess whether solution concepts can be singled out which somewhat satisfy the core constraints and are computationally tractable even if applied over arbitrary interactions graphs. Yet another avenue for research is the study of approximation algorithms to deal with the problems analyzed in this paper, which would possibly match the inapproximability results that trivially follow from the reductions we exploited to prove the hardness results. Finally, we also envisage to concretely link our theoretical results to real-world applications. For instance, *virtual power plants* corresponding to coalitions of electricity consumers or (potentially heterogeneous) alternative energy producers, present natural paradigms where the grouping of agents is restricted by technical or legal considerations [76].

## Acknowledgements

We are grateful with the anonymous referees for their useful suggestions that helped us to improve the quality of the paper. They pointed out a work [31] published during the review period, where co-**NP**-hardness proofs for CS-Core-Membership and CS-Core-NonEmptiness over MC-nets have been exhibited—stimulated by this reference, we further improved the original proofs of Theorem 4.9 and Theorem 4.10, where hardness results were shown to hold for rules containing at most 3 negative literals.

We also thank E. Malizia, L. Palopoli, F. Scarcello, and N.R. Jennings for their permission to adapt co-authored material for this article.

G. Greco was also supported by a Kurt Gödel Research Fellowship, awarded by the Kurt Gödel Society.

E. Markakis was also supported by the European Union (European Social Fund) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF)—Research Funding Program: THALES.

## Appendix A. Proofs of the claims

**Claim 4.2.** *Let $\mathcal{R}$ be any compact* **P***-representation, and let $\mathcal{G}$ be any class of graphs. On the class $\mathcal{C}_{\mathcal{G}}(\mathcal{R})$, computing the maximum social welfare and an optimal coalition structure is feasible in* $\mathbf{F\Delta_2^P}$.

**Proof.** We exhibit an algorithm that works in two steps.

First, let $n$ be the number of bits used to represent the encoding $\xi^{\mathcal{R}}(\Gamma)$, and consider an oracle that receives as input a natural number $h > 0$, and checks whether there is a feasible coalition structure $\Pi$, such that the size required to encode (in the standard fractional form) the rational number $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C)$ is at least $h$. Since $\mathcal{R}$ is a **P**-representation, the encoding for the social welfare of any feasible coalition structure requires polynomially many bits. Therefore, the oracle can just guess (in **NP**) the coalition structure $\Pi$, in order to subsequently check in polynomial time that the desired conditions hold. This oracle is firstly invoked over the input $h = n^k$, with $k = 1$. If the oracle succeeded to find a coalition structure of this kind, then it is invoked over $h = n^k$, with $k = 2$, and so on for increasingly large values of $k$. Eventually, after a constant number of invocations, the process converges to a value $k^*$ such that $n^{k^*}$ is an upper bound on the size required to encode the payoff associated with any feasible coalition structure. In the following, let $m = n^{k^*}$ and $M = 2^m$.

For the second step, we start by asking an oracle whether there is a feasible coalition structure $\Pi$ such that $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C) \geq 0$. Let us analyze the scenario where this is actually the case, so that $SW_{opt}(\Gamma)$ is a value such that $0 \leq SW_{opt}(\Gamma) \leq M$. In order to compute $SW_{opt}(\Gamma)$, one can use well-known methods to search for rational numbers encoded in the fractional form $p/q$ (see [77], and the references therein). More formally, observe that the value $\alpha^* = SW_{opt}(\Gamma) \times 2 \times M^2$ is such that $0 \leq \alpha^* \leq 2 \times M^3$, and let us compute the integer value $\mu$ such that $\mu \leq \alpha^* < \mu + 1$. This task can be easily accomplished by means of a binary search using an oracle receiving as input an integer $y$ with $0 \leq y \leq 2 \times M^3$, and checking whether there is a feasible coalition structure $\Pi$ such that $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C) \times 2 \times M^2 \geq y$. In fact, the search space is exponential in the size of the representation of the game, but the number of calls made to the oracle is polynomial (w.r.t. $m$) because we have to focus on integer values only and because we are using a binary search (with $M = 2^m$). Therefore, after polynomially many **NP** oracle calls, we have computed the integer value $\mu$, with $0 \leq \mu \leq 2 \times M^3$, such that

$$\frac{\mu}{2 \times M^2} \leq SW_{opt}(\Gamma) < \frac{\mu + 1}{2 \times M^2}.$$

Now, assume that $\Pi^*$ is an optimal coalition structure and that $\Pi \neq \Pi^*$ is a feasible coalition structure such that $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C) \geq \frac{\mu}{2\times M^2}$. We claim that $\Pi$ is optimal, too. Indeed, let $p/q$ and $p^*/q^*$ be the irreducible fractional forms for the values $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C)$ and $\sum_{C\in\Pi^*} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C)$, respectively. Assume by contradiction that $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C) < \sum_{C\in\Pi^*} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C)$. Then, $\frac{p^*}{q^*} - \frac{p}{q} = \frac{p^*\times q - p\times q^*}{q^*\times q} \geq \frac{1}{q^*\times q} \geq \frac{1}{M^2}$. Therefore, $\sum_{C\in\Pi^*} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C) \geq \frac{1}{M^2} + \sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C) \geq \frac{1}{M^2} + \frac{\mu}{2\times M^2} > \frac{\mu+1}{2\times M^2}$, which is impossible.

In light of the above observation, we now aim to compute an arbitrary coalition structure $\Pi$ such that $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C) \geq \frac{\mu}{2\times M^2}$, since we are guaranteed that for any coalition of this kind, $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C) = SW_{opt}(\Gamma)$ holds. To this end, we use another **NP** oracle defined as follows. The oracle receives as input a set $\{S_1, \ldots, S_m, S_{m+1}\}$ of disjoint coalitions and decides whether there is a feasible coalition structure $\bar{\Pi} \supseteq \{S_1, \ldots, S_m, S^*_{m+1}\}$ with $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C) \geq \frac{\mu}{2\times M^2}$ and where $S^*_{m+1} \supseteq S_{m+1}$. The oracle is invoked starting with $m = 0$. At each step, $S_{m+1}$ is initially defined as the empty set, and it is progressively enlarged by including agents that are not covered so far, ending up with a maximal set $S_{m+1}$ over which the oracle still returns a positive answer. At this point, $m$ is incremented and the process is repeated, until all agents are covered in the current set of disjoint coalitions. By the non-deterministic nature of the computation, we are immediately guaranteed that the set computed in this way, say $\Pi$, is an optimal coalition structure for $\Gamma$.

Eventually, now that an optimal coalition structure $\Pi$ is at hand, its associated value $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C)$ can be explicitly computed—note that, so far, we just exploited the knowledge of a lower bound for it. Overall, we have shown that the whole computation, for the case where $SW_{opt}(\Gamma) \geq 0$ holds, is feasible in $\mathbf{F\Delta_2^P}$.

In order to complete the proof, we need now to consider the case where, at the beginning of the second phase, we realize that there is no feasible coalition structure $\Pi$ such that $\sum_{C\in\Pi} v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C) \geq 0$. This means that $SW_{opt}(\Gamma) < 0$ and the same line of reasoning as above can be applied, by just inverting the sign of the payoff function and by considering a minimization problem in place of a maximization one. $\square$

**Claim 4.7.** $\exists\forall$CNF-UNSAT *is* $\mathbf{\Sigma_2^P}$*-hard even restricted to formulas* $\Psi$ *such that, for each pair of assignments* $\sigma^a$ *and* $\sigma^b$ *over the existentially and universally quantified variables of* $\Psi$*, respectively,* $\sigma^a \uplus \sigma^b \models \Psi$ *if, and only if,* $\sigma^a \uplus \bar{\sigma}^b \models \Psi$.

**Proof.** Assume that $\Phi = c_1 \wedge \cdots \wedge c_m$ is a Boolean formula in conjunctive normal form over the sets $N_\exists$ and $N_\forall$ of existentially and universally quantified variables, respectively. Consider the formula $\Psi = c'_1 \wedge \cdots \wedge c'_m \wedge c''_1 \wedge \cdots \wedge c''_m$ such that $c'_i = y_{j,1} \vee \cdots \vee y_{j,k_j} \vee P$ and $c''_i = h(y_{j,1}) \vee \cdots \vee h(y_{j,k_j}) \vee \neg P$, for each $j \in \{1, \ldots, m\}$, where $P$ is a fresh universally quantified variable and $h$ is a function such that $h(Z)$ returns the literal $Z$ (resp., the complement $\bar{Z}$) whenever $Z$ is built over a variable in $N_\exists$ (resp., $N_\forall$). Let $N'_\forall = N_\forall \cup \{P\}$. By construction, we immediately observe that for each pair of assignments $\sigma^a$ and $\sigma^b$ over $N_\exists$ and $N'_\forall$, respectively, $\sigma^a \uplus \sigma^b \models \Psi$ if, and only if, $\sigma^a \uplus \bar{\sigma}^b \models \Psi$. We now show that $\Psi$ is "equivalent" to $\Phi$.

Let $\sigma^a$ be any truth assignment over $N_\exists$. Then, the following statements are equivalent: (1) for each assignment $\sigma^b$ over $N'_\forall$, $\sigma^a \uplus \sigma^b \not\models \Psi$; (2) for each assignment $\sigma^b_\Phi$ over $N_\forall$, $\sigma^a \uplus \sigma^b_\Phi \not\models \Phi$. The fact that (1) implies (2) is immediate, by just considering the assignments $\sigma^b$ such that $\sigma^b(P)$ is false. To show that the converse holds, assume that $\sigma^b$ is such that $\sigma^a \uplus \sigma^b \models \Psi$, so (1) does not hold. In the case where $\sigma^b(P)$ is false, then the restriction of $\sigma^b$ over the variables in $N_\forall$, say $\sigma^b_\Phi$, is such that $\sigma^a \uplus \sigma^b_\Phi \models \Phi$. Instead, in the case where $\sigma^b(P)$ is true, then $\bar{\sigma}^b_\Phi$ is such that $\sigma^a \uplus \bar{\sigma}^b_\Phi \models \Phi$. In both cases, we get that (2) does not hold. $\square$

**Claim 4.8.** *Assume that* $\varepsilon = \frac{1}{3}$*. Then,* CS-*opt*$(\Gamma^{\exists\forall}_{\Pi,-\varepsilon}) = \{\{A \cup \{r\}, \hat{A}, N_\forall \cup \hat{N}_\forall\} \mid A \subseteq N_\exists \cup \hat{N}_\exists$ *is* $\exists$*-consistent*$\}$.

**Proof.** Let $\Pi_A$ be the coalition structure such that $\Pi_A = \{A \cup \{r\}, \hat{A}, N_\forall \cup \hat{N}_\forall\}$, where $A \subseteq N_\exists \cup \hat{N}_\exists$ is $\exists$-consistent. Note that $v(A \cup \{r\}) + v(\hat{A}) + v(N_\forall \cup \hat{N}_\forall) = 1 + 0 + 0 = 1$. In fact, $\Pi_A$ belongs to CS-$opt(\Gamma^{\exists\forall})$. In particular, note that $v(\hat{A}) = 0$, because $\hat{A}$ is $\exists$-consistent, $r \notin \hat{A}$, and $\hat{A} \subseteq N_\exists \cup \hat{N}_\exists$. Since we are assuming $\varepsilon \geq 0$, then $\Pi_A$ belongs to CS-$opt(\Gamma^{\exists\forall}_{\Pi_A,\text{-}\varepsilon})$, too. Consider now any coalition structure $\Pi$ such that $\Pi \neq \Pi_{A'}$ holds, for each set $A' \subseteq N_\exists \cup \hat{N}_\exists$ which is $\exists$-consistent. Note that $\sum_{C \in \Pi} v(C) < 0$ holds, by definition of the payoff function $v$. Hence, $\sum_{C \in \Pi} v_{\Pi,\text{-}\varepsilon}(C) = \sum_{C \in \Pi} v(C) < 0$. Instead, $v_{\Pi,\text{-}\varepsilon}(A \cup \{r\}) + v_{\Pi,\text{-}\varepsilon}(\hat{A}) + v_{\Pi,\text{-}\varepsilon}(N_\forall \cup \hat{N}_\forall) \geq 1 - \varepsilon + 0 - \varepsilon + 0 - \varepsilon = 1 - \frac{3}{3} = 0$ holds. That is, $\Pi$ is not optimal in $\Gamma^{\exists\forall}_{\Pi,\text{-}\varepsilon}$. $\square$

**Claim 5.8.** *Let $\mathcal{R}$ be any compact* **P***-representation such that* mcn $\precsim_e \mathcal{R}$*. On the class $\mathcal{C}_{\mathbf{TW}\text{-}1}(\mathcal{R})$, computing the maximum social welfare is* **NP***-hard. In particular, a reduction can be exhibited where optimal coalition structures one-to-one correspond to optimal tours for a* TSP *instance provided as input.*

**Proof.** By definition of the marginal contribution network $M$, each coalition $C$ with $t \notin C$ is such that $v(C) = 0$. Therefore, if $\Pi \in$ CS-$opt(\Gamma)$, then $\Pi$ includes an optimal coalition $C$, whose corresponding optimal tour is $T(C)$. All other agents occur in $\Pi$ as singletons, as there is no way to form coalitions consisting of more than two agents in absence of $t$. Eventually, we get $SW(\Pi) = n \times W - \sum_{e_{i,j} \in T(C)} w(e_{i,j})$. On the other hand, it is immediate that if $T$ is an optimal tour, then there is an optimal coalition structure $\Pi$ including $C$ and all other agents as singletons. As optimal coalition structures correspond to optimal tours, their computation is **NP**-hard, given the hardness of the traveling salesmen problem. $\square$

**Claim 5.12.** *Let $\mathcal{R}$ be any compact* **P***-representation, let $\Gamma \in \mathcal{C}^{sa}_{\mathbf{TW}\text{-}1}(\mathcal{R})$, and let $\varepsilon \leq 0$. Then,* CS-LCV$(\Gamma) \leq \varepsilon$ *holds if, and only if, $\{N\} \in$ CS-$opt(\Gamma_{\text{-}\varepsilon})$. Moreover,* CS-LCV$^-$ *is in* co-**NP**.

**Proof.** Since the coalition structure core of $\Gamma$ is always non-empty [19], by Fact 3.5 CS-LCV$(\Gamma) \leq \varepsilon$ holds if, and only if, there is a feasible coalition structure $\Pi$ such that $\Pi \in$ CS-$opt(\Gamma_{\Pi,\text{-}\varepsilon})$. However, because $\Gamma$ is superadditive and $\varepsilon \leq 0$, the check can be restricted to the grand-coalition $\{N\}$. Moreover, concerning the complexity of CS-LCV$^-$, we can just focus on the complementary problem of deciding whether CS-LCV$(\Gamma) > \varepsilon$, and observe that it can be solved in **NP** by guessing a feasible coalition structure $\bar{\Pi}$ and by checking that $\sum_{\bar{C} \in \bar{\Pi}} v_{\text{-}\varepsilon}(\bar{C}) > v(N)$ holds. $\square$

**Claim 6.2.** *Let $\mathcal{R}$ be any compact* **P***-representation. On the class $\mathcal{C}_{\mathbf{L}}(\mathcal{R})$, computing the maximum social welfare (and a feasible coalition structure with this value) is in* **FP**.

**Proof.** Let $\Gamma = \langle N, v, G\rangle$ be a game where $G$ is a line, with $N = \{1, \ldots, n\}$. To keep notation simple and without loss of generality, assume that, for each $j \in \{1, \ldots, n-1\}$, there is an edge in $G$ connecting $j$ and $j+1$, and that there is no further edge in $G$. Consider the following recursive definition of the values $sw_j$, for each $j \in \{0, \ldots, n\}$:

$$\begin{cases} sw_0 = & 0; \\ sw_j = & \max\{v(\{i, \ldots, j\}) + sw_{i-1} \mid i \in \{1, \ldots, j\}\}, \text{for each } j \in \{1, \ldots, n\}. \end{cases}$$

Moreover, for each $j \in \{1, \ldots, n\}$, let $\Gamma_j = \langle\{1, \ldots, j\}, v_j, G_j\rangle$ denote the game where $G_j$ is the subgraph of $G$ induced over the nodes $\{1, \ldots, j\}$ (therefore a line with endpoints 1 and $j$), and where $v_j$ is such that $v_j(C) = v(C)$, for each coalition $C \in \mathcal{F}(G_j)$. By keeping this notation, it is immediate to check that for each $j \in \{1, \ldots, n\}$, $sw_j = SW_{opt}(\Gamma_j)$ holds. Therefore, $SW_{opt}(\Gamma)$ coincides with the value $sw_{|N|}$. By implementing the above definition via dynamic programming, it immediately follows that $SW_{opt}(\Gamma)$ and an optimal coalition structure $\Pi$ with $\sum_{C \in \Pi} v(C) = SW_{opt}(\Gamma)$ can be computed in polynomial time when $\Gamma$ is given according to any **P**-representation. $\square$

**Claim 6.4.** *Let $\mathcal{R}$ be any compact* **P***-representation. On the class $\mathcal{C}_{\mathbf{C}}(\mathcal{R})$, computing the maximum social welfare (and a feasible coalition structure with this value) is in* **FP**.

**Proof.** Assume that $\Gamma = \langle N, v, G\rangle$ is a game where $G$ is a cycle with $|N| = n$, and let $L_1, \ldots, L_n$ be the $n$ lines that can be obtained by removing exactly one edge from $G$. Then, an optimal coalition structure is either the grand-coalition $N$, or the best coalition structure over the optimal coalition structures of $L_1, \ldots, L_n$. Since $SW_{opt}(L_i)$ can be computed in polynomial time (with a witness for it), for each $i \in \{1, \ldots, n\}$ (cf. Claim 6.2), the result is established. $\square$

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