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From environments to representations—a mathematical theory of artificial perceptions ☆

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Abstract

Perception is the recognition of elements and events in the environment, usually through integration of sensory impressions. It is considered here as a broad, high-level, object centered, phenomenon which happens at and above the level of holistic recognition of objects and events, where semantics begin to play a role. We propose and develop a mathematical theory of artificial perceptions. A basic mathematical category is defined. Its objects are *perceptions*, consisting of *world elements, connotations*, and a three-valued *true, false, undefined* predicative correspondence between them. Morphisms describe paths between perceptions. This structure serves as premises for a mathematical theory. The theory provides rigorous tools of scrutiny that deal with fundamental issues of AI such as the diversity and embodiment of artificial perceptions. It extends and systematizes certain intuitive pre-theoretical conceptions about perception, about improving and/or completing an agent's perceptual grasp, about transition between various perceptions, etc. Mathematical tools and methods are used to formalize reasonable ways to go about producing a meaningful cognitive image of the environment from every perception.

Keywords: Mathematical foundations of AI; Artificial perception and cognition; Categorical formalizations in AI

1. Introduction

The science of Artificial Intelligence conceives and constructs autonomous intelligent artificial agents to bring about "intelligent" effects. Those effects are typically related to the environment of the artificial agent. Intelligence is marked by quick active

☆ There is no intention to deal with human perceptual or cognitive processes. Any anthropomorphisms or human analogs used in this study are for intuitive purposes only, to make the presentation more vivid and readable.

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perception and understanding. The essential core of autonomous cognitive behavior is thus the integration of sensing, perceiving and representing. High-level perception is the recognition of elements and events in the environment, usually through integration of sensory impressions. It is considered here as a broad, high-level, phenomenon, which resides higher than pixels on the screen or waveforms of sound. It is object centered and therefore it happens at and above the level of holistic recognition of objects, where semantics begin to play a role. The raw materials of high-level perception are connotations of wholesome recognized environmental entities. At that point perception needs to produce a logical, coherent and meaningful representation of the environment and to use it for various cognitive tasks.

In [48] Newell states that perception is an area which should definitely be covered by theories of cognition, since cognitive behavior is a function of the environment. In [50], Nilsson proposes to work towards what he calls *Habile Systems*: programs of general, humanlike, competence. The abilities of such systems should "... *include whatever is needed for an agent to get information about the environment in which it operates ... perceptual processing ... facilities for receiving, understanding, and generating communications ... ability to learn* ...". This direction was already suggested in [49] where Nilsson proposed to develop life long *Computer Individuals* who should have a continuing existence. They would have, among other things, a (constantly changing) model of the world, they would benefit from their experiences, and they would communicate with other agents, artificial or human. These agents could do anything that requires moving around in and sensing a real environment and manipulating that environment in some way. By having a continuing existence and learning from their own experience they would, in time, develop their own individual image of their environment, which draws upon their own sensory-motor-neural capabilities and their own experience. In planning such an agent, one cannot separate the sensory-motor-neural apparatus from cognition, from higher-level reasoning, and from the communication capabilities: all these functions should cooperate. The proposed mathematical theory of artificial perceptions and the tools it offers may constitute a step towards achieving the challenge of general basic artificial intelligence. The mathematical treatment provides a single context for the treatment of various perceptual and cognitive processes, including, among other things, transition and comparison between different perceptions, improving and completing a perceptual grasp, joint perceptions, and a bridge that integrates perception and higher reasoning (i.e., problem solving, decision making etc.). This warrants a unified theory, where the separate processes enhance one another rather then interfere with one another.

Artificial agents may collect information about their environment using their sensory-motor-neural apparatus. This information is then reflected inside the artificial agent in some way. The collecting of the information, together with its internal reflection, is conceived by this work as *artificial perception*. The raw internal reflection could then be arranged in any way to represent the agent's own *cognitive* image of its environment, the arrangement serving as the basis for further, higher level reasoning processes.

A mathematical structure is proposed which formalizes artificial perception. This proposed formalization provides standard mathematical tools of scrutiny, so that one can meticulously perform and analyze in the domain of artificial perceptions. The proposed system is general enough to formalize a wide spectrum of artificial perceptions. Any such perception usually has its own sensory-motor-neural mechanisms and its own method of internally reflecting the output which emerges from this apparatus into the agent's internal modules. The generality is achieved via a categorization which is able to accommodate any correspondence between an external environment and an internal reflection of it.

Many things seem to happen from the moment a phenomenon in the environment comes to being, through its perception by the agent, and until it is processed and eventually leaves its mark on the agent's cognition. The things which happen at the early stages of this chain of events are typically predetermined by innate mechanisms and equipment. These are formalized by the *perception predicate*. The agent is constructed to perceive *world elements* (w-elements for short) and to reflect them with *connotations* (the exact nature of these concepts will be explained soon). The connotations of every w-element are determined primarily by the built-in sensory-motor-neural apparatus. As an example from humans, the way we perceive colors, tastes, sounds, smells, textures and other outside phenomena is determined primarily by our neurophysiology, plus our experience and socio-cultural influences. For artificial agents, programming may stand for the latter ones in their broadest sense.

The later stages of the perceptual chain of processing feature, on the other hand, more apparent flexibility: the artificial agent now has the connotations of its w-elements, and is free to manipulate things internally. There would usually be room for a contingent course of action. That is where programmable, perceptual-cognitive processes come in. These are the processes with which this study is concerned. One asks questions such as: in this given perceptual situation, can one detect lawlike orders and patterns of connotations? Are there redundant connotations? What is the best way to arrange them for the benefit of future reasoning processes? Tools are proposed with which the agent may give its own, informed, solutions to these problems. It can systematically obtain a representation of the given situation. It creates its own cognitive image of its environment. The entire process is based only on information provided by the lower, embodied level.

There are debates as to how much of the intelligent, cognitive, processing lies below and/or above the symbolic, conscious level (see, for example, [34]). Our proposed formalization of perception is flexible enough to accommodate any location of the separation between lower- and higher-level processes. It presents an abstract concept of perception, and thus permits the introduction of either simple or more integrated connotations at the basic perception level.

All the definitions, constructions and results are operated within a mathematical system. This ensures a tidy treatment and thus communicates and introduces to the related domains tools of mathematical rigor and results that are meticulously stated. These tools could hopefully be used and further applied for research in these related domains.

2. Pre-theoretical and methodological considerations

This work presents and applies a mathematical system for the formalization of artificial perception and related cognitive processes. Mathematical systems are based on semantic primitives that are context independent. However, there is more than often an intuitive grounding for mathematical semantic primitives. This applies to this work where the formal system is grounded intuitively in a perceptual, cognitive environment. We first discuss these intuitions and their background.

2.1. Diversity of perceptions

We are dealing with an artificial agent that perceives its environment. Assume that it is placed next to a box, and that it has the sensory-motor-neural capabilities to determine the contents of the box. Someone asks the agent whether the box is full. Some of the following problems may arise:

- The agent may or may not have been programmed, according to its purposes, to consider a box full of air as full.
- It may or may not have been programmed to consider a half full box as full.
- What if the box is full of waste-paper, which could be disposed of. Is such a box considered full?
- The agent might have no "idea of fullness". It may or may not have other notions such as "filled" or "empty", but does not recognize "full".
- Finally, the agent may not perceive "a box", but rather six elements which form the sides of the box.

Indeed, perception and cognition do depend on the sensory-motor-neural apparatus of the agent, its history and experience, and on other capacities to perceive, to form mental images, and to organize them internally. Different individuals could break the same reality into different elements, and choose different uniformities as their properties. A partial list of AI-related works that touch on this issue is [7,30,35,38], to name just a few. Some extreme and intriguing examples of cases of human perceptions are given in [45]. AI has to deal with artificial agents that do not even share the same architecture. Their hardware is different, their sensory-motor-neural apparatus varies, and they are conceived and programmed for different purposes by different people who build their own conceptualizations into the system, each using his own encoding.

2.2. Categorization of perceptions

Given the diversity of artificial perceptions, the question is how one can account for such a diversity, yet at the same time formalize a theory of it. Our solution is in the categorization of perceptions.

The philosophical idea of categorization was introduced in the 18th century by Immanuel Kant, in his statement "*Grounding for the Metaphysics of Morals*" [31]. It is central to cognitive science (see [38]). A simple example is the category of cups. Cups can come in many shapes and forms, but they all have something in common. The agent then creates out of this "cuphood" a mental image of a cup, an abstract cup that does not match any particular cup from the category. The agent also has general motor actions for dealing with real instances of cups. It is generally agreed that humans are endowed with natural talents for categorization, whereas the task seems hard for artificial perceptions. In that case *we should be able to categorize artificial perceptions*: we should recognize the essence of artificial perception, and we should create a formal image of it. This formal, abstract, perception should not match any particular perception, yet we shall have the tools for dealing with all perceptions. In Kant's words, this should be done ... *entirely a priori, since here we do not enjoy the advantage of having its reality given in experience* ....

The abstract idea of a perception will be postulated as a mathematical construct which relates between phenomena outside the artificial agent, a set of *world elements* (w-elements for short), and reflections which are internal to the artificial agent, a set of *connotations*. World elements exist independently of any thought or perception. Anything which exists independent of the artificial agent, and could perhaps be discerned by it, is a legitimate w-element. Possible example w-elements are a face, a light blow of wind, the shadow of a smile, a slight shivering of voice, a tinge of smell or taste or color, etc. Not every sensory-motor-neural mechanism is able to discern every such outside phenomenon, and even if it does it may not attach the same connotations to them. Furthermore, different perceptions might break the same reality into different parts to serve as wholesome w-elements. Although the external environment has an objective existence, its division into w-elements is subjective.

The term *connotation* was chosen (rather than "attribute" or "property") to stress the affinity to the agent's own individual, personal experience, and to subtle distinctions. It is a meaning which is more than just a primary meaning. As an example, on top of the conventional term of "mother" may or may not come connotations such as "love", "comfort", and "warmth". Further more, connotations may be of a metonymic or metaphorical nature such as in "Necessity is the *mother* of invention". Connotations could also be, for example, iconic.

It is impossible to separate perception from the environment to which it relates. The idea of a cognitive *supraindividual* that includes its environment is elaborated in [29]. Every perception has its own set of w-elements, its own set of connotations, and its own predicative correspondence between the sets. They are given once the instance perception is fixed, in very much the same way that the details of the cup are accessible once a perception relates to a specific instance of a cup. The correspondence between w-elements and connotations is given as a two-place partial predicate.

Mathematical category theory started with [17]. It provides tools of scrutiny for stating results which can be used across a wide spectrum of mathematical domains and objects [1,13,28,43]. No specific knowledge of category theory is assumed for reading this paper. The required categorical concepts will be introduced whenever their actual applicability emerges from the context, providing an ad hoc justification for the formalism. It so happens that the categorical toolkit often allows a precise description of complex phenomena that are too complex to be grasped by a verbal description.

This work defines perceptions as a domain of mathematical discourse, where different perceptions represent different members of the category. Structural similarities among perceptions can be studied, yet leaving ample room for differences and variety. Indeed, the more general the setting, the less likely it is that the results will be profound. A combination of generality and depth is attained by gradually concentrating on more restricted subcategories of perceptions, thus identifying this part which is deep and proper to "better" perceptions and separating it from that part which is trivial. More specific results can be shown if discussion is restricted to a subset of "better" perceptions. Loosely, these perceptions can be qualified as those where the set of connotations is closed under Boolean operations.

2.3. Boolean algebra as a cognitive tool

Boolean algebra was first introduced by George Boole in his 1854 statement [12] *An Investigation of the Laws of Thought*. During the century that followed this first publication, the theory of Boolean algebras was developed both as a special kind of algebraic ring and as a generalization of the set-theoretical notion of a field of sets. Major contributions are due to Jevons, Peirce, Schroeder, Whitehead, Huntington, Tarski, and Stone (to name just a few).

There are debates as to the suitability of Boolean operations to model human cognition, especially at the lower, sensory-motor-neural level. The idea that Boolean algebra could be applied to express acts of conscious thinking is due to George Boole himself [8, pp. 433–447]. This work is, however, not committed to imitating human perceptual cognitive behavior. The categorical transition from basic artificial perceptions to Boolean artificial perceptions is conceived by this study to formalize a bridge between a lower, artificial sensory-motor-neural level (that could be based, for example, on a neural network), and higher artificial reasoning levels. As mentioned in the Introduction section, the mathematical model is flexible enough to accommodate any location of a boundary between an innate sub-symbolic lower level, and higher-level cognitive processes. Either very basic or more integrated connotations can be introduced at the level where perceptions finally label sensory-motor-neural outputs with symbols, and semantics begin to play a role.

The cognitive processes that are described in this work result in embodied cognitive structures that are cast as Boolean algebras. They are structures that are, among other things, interpretable as logical formulas. The dominant view in AI is that the knowledge content of high-level reasoning programs ought to be represented by data structures with this property [24].

2.4. Partiality of perception, three values of truth, nonmonotonicity

One of the assumptions that will be expressed by the formalization is that perception is not total. Recall the perception of the closed box from the beginning of Section 2. Assume now that it does discern it as a single w-element: "box", that it has the connotation "full", and that the predicative correspondence of "full" to w-elements in the world is adapted to the reader's choice.

- The agent may be unable to perceive whether the w-element "box" is "full" due to sensory-motor-neural deficiencies.
- In other cases the agent might have the required sensory-motor-neural capabilities, but it does not bother to use them because the question is irrelevant to its current purposes.
- In yet another case the agent might have knowledge of the contents of the box, but for current purposes it is more practical not to distinguish full boxes from others.
- There can also be a case where the box is only half full, and it is better to leave the question unanswered until, eventually, practical or other considerations will determine the box status as full or not.

In all these cases, and there may be others, it is desirable to leave perception undefined. For such reasons the p-predicate is partial: in some cases it gives no definite answer. Whether the "box" is "full" may be *true* (t), *false* (f), or *undefined* (u).

The proposed, non-classical, solution is to have a third truth value, *undefined* (u for short), and to define the predicate as total. The predicate then assumes the third truth value, u, whenever perception is undefined. This does not mean that we admit more than two genuine truth values. It rather captures the idea of a truth value gap, and provides us with a convenient designation for the undefined cases.

In the mathematical background there are a few three-valued logics [26,27,57], each of them with its intuitive interpretation. Two of them are relevant to us. Kleene's [36] intuitive interpretation is that the third truth value represents ignorance: there exists a truth value, only it is not exposed for some reason. This corresponds to the cases above where there is ignorance or irrelevance of the fullness of the box. It could also capture indifference or inattentiveness. Lukasiewicz's [41,42] intuitive interpretation is that the third truth value represents indeterminacy or future contingence. This corresponds to the case of the half full box where there is no decision yet, with the assumption that it might eventually be decided.

Intuitively, then, these are two different interpretations. Technically, though, the resulting logics are very similar. Both follow the principle that where one can determine the truth value, t or f, of a compound well-formed formula from its components, that *wff* should be assigned that truth value, regardless of whether or not certain of its components are undecided. So, for example, $A \vee B$ will be assigned the value t if one of A or B is assigned the value t, even if the value u is assigned to the other. The only formal difference between Lukasiewicz's and Kleene's connectives relates to the conditional and biconditional: under Lukasiewicz's interpretation the conditional $A \rightarrow B$ is assigned t when A and B are indeterminate. Consequently, his system, unlike Kleene's, preserves the law of identity: it is always t that $A \leftrightarrow A$.

From the interpretational point of view, we want our third truth value to capture both the ignorance/irrelevance case and the future contingence case. For other reasons we wanted to preserve the law of identity. Hence Lukasiewicz's three-valued logic is adopted here. Our interpretation is of a rather pragmatic nature: the undefined truth value might eventually become defined, t or f, but right now it is not. By the individuality of perceptions this is not a matter of a universal fact, so that the question whether or not it has already been decided in some transcendental way becomes meaningless and irrelevant to our purposes. Any instance of perception may give no answer on whether the "box" is "full". This is done for reasons "private" to the agent, with no need to give any account whatsoever about them. This is very much in the same way that there is no need to explain why a box is either "full" or not. The issue of *why perception is as it is* simply warrants no discussion.

Lukasiewicz's three-valued logic has an aspect of nonmonotonicity. Let us reconsider perception of the box. Suppose that it confirms about a set of boxes the following facts:

- *Any box that is perceived full is also perceived red.*
- *Any box that is perceived not red is also perceived not full.*

(By "not full" and "not red" boxes it is meant that the p-predicate actually yields the truth value *false* in answer to the relevant question, and not that the answer is undefined.) Suppose, in addition, that there is one box (or more) for which both redness and fullness are not defined by perception (the box is covered by a blanket). This is exactly a case where Lukasiewicz's logic will differ from Kleene's. In Kleene's logic, such a box will prevent perception from concluding that *a full box is always red*. However, by Lukasiewicz's logic which we have just adopted, perception will nonetheless come to that conclusion: *a full box is always red*. Indeed, some time in the future the blanket might be removed, so that perception will improve to the point where it definitely observes that this box is full but green. The conclusion will have to be retracted.

Reasoning is called nonmonotonic when the reasoning agent must withdraw a previously deduced conclusion in response to learning some new fact. Nonmonotonicity is often encountered in AI, and many efforts and achievements were recorded in this area of research ([46,47,52,53] to name just a few). The intelligent agent "jumps to conclusions" in spite of incomplete information. The incompleteness of the information is represented, in our case, by the third truth value, u. The agent might eventually use these perceptual conclusions for practical purposes (such as the advice to dump all boxes that are not red). This constitutes a risk. It is not impossible to imagine a natural situation where humans jump to such conclusions and take such risks. A typical reason for doing this is that the agent needs to come to as many plausible conclusions as possible in order to achieve something. Without taking the risk it might stay with nothing much to do. Insisting on absolute security with no risks may often be paralyzing: i.e., it might be impractical either to keep all boxes or to check all of them.

2.5. *Summary of Section* 2

The formalization of artificial perceptions in a mathematical system will be based on the following pre-theoretical intuitions:

- Perceptions differs from one artificial agent to another. There does not exist any perception which is "objectively" correct. All perceptions are legitimate.
- The necessary component of perceptions is the correspondence between a set of outside w-elements and a set of internal reflections (connotations). These sets and the correspondence between them are determined independently for every instance of artificial perception.
- In "better" perceptions the set of connotations is closed under Boolean connectives.
- Artificial perception is partial. It may not provide a definite answer to every question about its perception. The reasons for this may vary, but all the cases where perception is undefined are treated uniformly by Lukasiewicz's three-valued logic. It is assumed that eventually the undefined cases of perception might turn out to be defined either as true or as false.
- Due to the partiality of perceptions and its treatment by the three-valued logic, the agent may "jump to conclusions" that may eventually turn out to be incorrect. Such risks are taken for the sake of coming to as many plausible conclusions as possible in order to achieve something.

In commitment to the mathematical formalization, results will be inferred and concluded only from the formal premises using mathematical tools and methods. However, whenever a result is reached, it will be possible to examine it with regard to these pre-theoretical considerations, and to test it against existing theories and opinions about artificial perceptions and cognition.

3. Background and related research

This study does not directly carry forward an existing body of work. It tries to propose a new mathematical framework, where no such framework already exists, for a theory of artificial perceptions. It is, however, akin to several research paths. Methods and results from category theory, Boolean algebra and Lukasiewicz's three-valued logic will be applied, as explained above. These are the mathematical beaten tracks that we tread.

In AI this work falls in with other applications of mathematical methods for purposes of this domain [9,16]. The advantages of mathematical formalizations as analyzed, for example, in the introduction to [15] include clarity, precision, versatility, generalizability, testability, allowance to model complex phenomena that are far too complex to be grasped by a verbal description, and allowance to use results of a well-developed science.

Within mathematics, category theory seems suitable for purposes of AI. AI tries, in a sense, to approximate intelligence by creating particular models of artificial intelligence as well as foundations for a general account of such intelligence. In that context the following quotation from Lawvere [39] seems relevant: "even within mathematical experience, only that [category] theory has approximated a *particular* model of the general, sufficient as a foundation for a *general* account of all particulars". Lawvere further argues that category theory provides a guide to the complex, but very non-arbitrary constructions of the concepts and their interactions which grow out of the study of *any* serious object of study. There has not been, however, much AI related research utilizing mathematical category theory. A few examples are given in [10]. They include employing categorical terminology and tools for problem solving strategies [4], for program reformulation [40], and for representation engineering [59]. Except for the very use of the categorical infrastructure, these applications are different from the category of perceptions presented here. A recent revival of interest in category theory for computer science is demonstrated by the publication of several books such as [3,6,51,58]. Their emphasis is typically on categorical logic and semantics.

Another long research path that this work touches is the study of cognition. Cognitive studies have other motivations and goals, because they are typically interested in human cognition and in being empirically adequate from a psychological point of view. However, their track often coincides with that of AI. Concepts and processes of human intelligence have inherently been a source of inspiration for research in AI and the present one is no exception. (However, this work is not committed to being empirically adequate from a psychological point of view.) One formalism for perception and cognition that shares some common aspects with the descriptive features of the formalization in this study is the "conceptual spaces" framework by Gärdenfors [21–23]. It is applied by the cognitive architecture that is described in [14]. Conceptual spaces do not go, however, into the formalization of the variety of complex cognitive constructions and processes that is enabled here by the usage of the categorical toolkit. Marrying mathematical category theory with cognitive studies is also proposed in [44]. As for the use of Boolean algebra in the cognitive sciences, an example is [33] which formalizes semantics for natural language using Boolean algebra.

The issue of alternative viewpoints is shared with research in the area of ontology design. Different ontologies represent different perceptions. [20] agrees that achieving interoperability and sharing of independently created ontologies is a challenging task. Dealing with alternative viewpoints is also shared with research in user modeling [32]: systems that try to model their individual users also need to deal with the particular perception of each user (or users' class). Some of our examples (e.g., Example 9) will be related to that.

As we proceed with the definitions, constructions, and results of this study, associations of specific aspects with other research will be mentioned in the context of their presentation.

4. The formal concept of artificial perception

We postulate the abstract idea of a perception as a mathematical construct which relates between phenomena outside the artificial agent, a set of *world elements*, and reflections which are internal to the artificial agent, a set of *connotations*. Every perception has its own set of w-elements, its own set of connotations, and its own predicative correspondence between the sets.

Definition 1. A *perception machine* (*perception* for short) is a three-tuple $\langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ where:
- $\mathcal{E}$ and $\mathcal{I}$ are finite, disjoint sets.
- ϱ is a three-valued predicate $\varrho : \mathcal{E} \times \mathcal{I} \to \{\mathtt{t}, \mathtt{f}, \mathtt{u}\}$.

The set $\mathcal{E}$ represents the outside, objective, world which the machine perceives. Anything which exists independent of the artificial agent, and could perhaps be discerned by it, could be a legitimate element of $\mathcal{E}$ and hence an w-element. Example w-elements may be a sound, a light, a blow of wind, a vapor (smelly or not), a candy bar, etc. These example w-elements are typically discerned by the human sensory-motor-neural apparatus, but some artificial perceptions may be unable to discern them. They may, however, discern w-elements that are imperceptible for humans, such as certain kinds of radiation. Furthermore, different perceptions might break the same reality into different wholesome w-elements. An example was given in the introduction: wherever one perceives a single wholesome w-element "box", another may perceive an arrangement of six w-elements "board". For humans, a human face would usually be a single, wholesome w-element that is easily perceived. Whether this is also the case where machine perception is involved, is, however, not so clear. Hence, although we assume the external environment to have an objective existence, its division into w-elements depends on the specific perception. (This phenomenon, as related to humans, has been studied by gestalt psychology [18].)

The set $\mathcal{I}$ stands for the internal representation, the ontology, of w-elements. Its elements have a subjective existence dependent on the machine. Anything which may be stored and manipulated in the machine (words, symbols, icons, etc.) could be a legitimate element of $\mathcal{I}$ and hence a *connotation*. Example connotations may stand for the pitch and/or duration and/or timbre and/or volume of a sound, the brightness and/or hue and/or saturation of a light, etc. These example connotations typically represent attributes or properties that are measurable by humans, and hence considered "objective". However, "hot" and/or "dark" and/or "good" and/or "?!?!?" are legitimate connotations as well (the last one is not a typo). These are definitely not "objective", they depend on the specific perception.

The three-valued predicate ϱ is the *perception predicate* (p-predicate for short) which relates w-elements and connotations, the connection between the environment and internal representations. The terminology for the various ϱ values will be the following:

- $\varrho(\mathrm{w}, \alpha) = \mathtt{t}$, it will be said that w *has connotation* α.
- $\varrho(\mathrm{w}, \alpha) = \mathtt{f}$, it will be said that w *lacks connotation* α.
- $\varrho(\mathrm{w}, \alpha) = \mathtt{u}$, it will be said that w *may either have or lack this connotation*, the "or" is evidently exclusive. This undefined value might eventually become defined but right now it is not.

The perception, and the values of ϱ in particular, is part of the definition of an agent, given data. This is supposed to capture the intuition that subsymbolic, early perceptual processing is innate to the agent and its architecture. The emergence of higher-level perception from the sensory-motor-neural apparatus depends on this apparatus itself, the agent's function and internal organization, its gestalt perception, mental imagery, etc. Connotations that are alphabetic strings do not necessarily follow their dictionary definitions (if they have any). A smelly invisible vapor may, for instance, have the connotation "pink". This may depend on the agent's own individual architecture, programming and experience. Likewise, the issue of *why* the p-predicate has any one of the three values at a certain point simply warrants no discussion. As an example, the undefined $\mathtt{u}$ value of the p-predicate may be due to ignorance, irrelevance, future contingence or other reasons. From the philosophical point of view, these possible reasons are quite different one from the other. In our context, however, the actual reason for a specific $\mathtt{u}$ value, or whether or not it is already "decided" in some transcendental way, is irrelevant.

4.1. Example perceptions

Our example environment will be a bookstore environment, where books are the w-elements. Agents who "enter" the store have different perceptions of this environment, varying with their topics of interest, budget restrictions, goals and reasons for "entering" the store, etc.

Example 2. Let $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ be a "catalog" perception where: $\mathcal{I} = \{$*science, fiction, art, travel, children, cookbooks*, ..., *title*1, *title*2, ..., *authorname*1, *authorname*2, ..., *publisher*1, *publiser*2, ..., *paperback, hardcover, colorplates, leatherbound, topten, reduced*, ..., *ISBN*1, *ISBN*2, ..., *pages*1, *pages*2, ..., *price*1, *price*2, ..., *edition*1, *edition*2, ...$\}$. For all books w in $\mathcal{E}$ and for all connotations $\alpha \in \mathcal{I}$, $\varrho(\mathrm{w}, \alpha) = \mathtt{t}$ if and only if w has that connotation by $\mathcal{P}$. It is $\mathtt{f}$ if and only if w does not have that connotation by $\mathcal{P}$. It is $\mathtt{u}$ if $\mathcal{P}$ does not offer any perception of that connotation. In this example (and in other typical cases as well) the connotations can be subdivided into "families" (such as topic connotations, title connotations, etc.) and the number of different connotations can be very large. [1] However, they all share the same status as elements of $\mathcal{I}$. Gärdenfors [21–23] offers to make a distinction among different "quality dimensions" that make the "conceptual space". A distinction in this spirit will follow naturally from our later Boolean constructions in Section 9.1.

Example 3. Let $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ be a "customer" perception where: $\mathcal{I} = \{$*topic-of-interest, favorite-author, not-interesting, got-it-already, buy-it, good-price, thick, heard-of-it, makes-a-nice-present, in-bad-shape, hmm,* ... $\}$. For all books w in $\mathcal{E}$, ϱ(w,*favorite-author*) $=$ t if and only if w is written by a favorite author of that perception, etc. The values of ϱ are decided individually for every customer perception. In a typical case many p-predicate values are u, since most books are closed and lying on the shelves.

We terminate with two perceptions of a more abstract nature.

Definition 4. Let $\mathcal{E}$ be an environment. The *universal perception of* $\mathcal{E}$ is $\mathcal{U}_{\mathcal{E}} = \langle \mathcal{E}, 2^{\mathcal{E}}, \varepsilon \rangle$ where:
- the set of connotations, $2^{\mathcal{E}}$, is the field of all subsets of $\mathcal{E}$;
- for all $\mathrm{w} \in \mathcal{E}$ and for all $A \subset \mathcal{E}$, $\varepsilon(\mathrm{w}, A) = \mathrm{t}$ if and only if $\mathrm{w} \in A$, otherwise $\varepsilon(\mathrm{w}, A) = \mathrm{f}$.

The universal perception of $\mathcal{E}$ thus has a totally two-valued p-predicate. For any subset of books in the bookstore example, for instance, it has a unique connotation that describes it accurately.

Definition 5. Let $\mathcal{E}$ be an environment. In the *empty perception* of $\mathcal{E}$ the set of connotations is empty. The p-predicate is, of course, degenerate. $\mathcal{P}_\emptyset = \langle \mathcal{E}, \emptyset, \varrho_\emptyset \rangle$.

An empty perception cannot relate to its environment $\mathcal{E}$.

4.2. Perception morphisms

Given the variety and individuality of perceptions as above, one needs a way to bridge, if possible, the differences between different perceptions. Perception is also known to be a dynamic, or "fluid" phenomenon. It changes all the time and one needs channels for the flow of change. *Perception morphisms* (p-morphisms for short) are going to serve as a formal tool for this purpose. Suppose that $\mathcal{P}_1 = \langle \mathcal{E}_1, \mathcal{I}_1, \varrho_1 \rangle$ and $\mathcal{P}_2 = \langle \mathcal{E}_2, \mathcal{I}_2, \varrho_2 \rangle$ are two perceptions. We are going to consider cases where the environment is the same for both perceptions: $\mathcal{E}_1 = \mathcal{E}_2$, and hence designated simply $\mathcal{E}$. Since $\mathcal{E}$ is fixed, we shall omit the first component from the definition of perceptions: $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ is a short designation for $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$. (The cases of paths between perceptions with different environments are going to be treated in a separate study.)

A p-morphism from $\mathcal{P}_1$ to $\mathcal{P}_2$ will be defined as a set mapping of the connotations. However, this "translation" between connotations should "make sense": the essence of

[1] At any point there is, however, only a finite number of active connotations.

connotations as meaningful representations of the outside world should be maintained. One thus needs to define some "structure preservation" condition on the mapping. The formal definition follows:

Definition 6. Let $\mathcal{E}$ be an environment. Consider two perceptions, $\mathcal{P}_1 = \langle \mathcal{I}_1, \varrho_1 \rangle$ and $\mathcal{P}_2 = \langle \mathcal{I}_2, \varrho_2 \rangle$. $h : \mathcal{P}_1 \to \mathcal{P}_2$ is a *perception morphism* (p-morphism for short) if the two following conditions hold:

(i) h is a set mapping of the connotations $h : \mathcal{I}_1 \to \mathcal{I}_2$.

(ii) h is *no-blur*: for all $\mathrm{w} \in \mathcal{E}$, and for all the domain connotations $\alpha \in \mathcal{I}_1$, whenever $\varrho_1(\mathrm{w}, \alpha) \neq \mathrm{u}$ then $\varrho_2(\mathrm{w}, h(\alpha)) = \varrho_1(\mathrm{w}, \alpha)$.

The definite (t/f) values of the p-predicate are preserved by p-morphisms.

4.3. Examples of perception morphisms

The following examples illustrate the flexibility of p-morphisms, (often called "arrows") and their ability to bridge between different perceptions (whenever such a bridge is possible). We return to the "bookstore" example environment of books, to the "catalog" perception of Example 2 and the various "customer" perceptions of Example 3.

Example 7. Consider two "bookstore" perceptions: $\mathcal{P}_1 = \langle \mathcal{I}_1, \varrho_1 \rangle$ and $\mathcal{P}_2 = \langle \mathcal{I}_2, \varrho_2 \rangle$ where $\mathcal{I}_i = \{interesting, not\text{-}interesting\}$, with "opposite" tastes—for all books w in $\mathcal{E}$:

$$\varrho_1(\mathrm{w},\ interesting) = \varrho_2(\mathrm{w},\ not\text{-}interesting),$$

$$\varrho_1(\mathrm{w},\ not\text{-}interesting) = \varrho_2(\mathrm{w},\ interesting).$$

h is defined by:

$$(interesting \overset{h}{\mapsto} not\text{-}interesting), \qquad (not\text{-}interesting \overset{h}{\mapsto} interesting).$$

- It is easy to see that h is no-blur and hence a p-morphism by Definition 6. As a matter of fact, it is a *rigid* case of a p-morphism: no-blur requires the equality $\varrho_2(\mathrm{w}, h(\alpha)) = \varrho_1(\mathrm{w}, \alpha)$ to hold only in the definite cases where $\varrho_1(\mathrm{w}, \alpha) \neq \mathrm{u}$.
- h is one-to-one and onto. Moreover, h^{-1} is also no-blur, so that h has an inverse p-morphism and hence the perceptions $\mathcal{P}_1$ and $\mathcal{P}_2$ are *isomorphic*.

Example 8. Consider two "catalog" perceptions of $\mathcal{E}$, $\mathcal{P}_1 = \langle \mathcal{I}_1, \varrho_1 \rangle$ and $\mathcal{P}_2 = \langle \mathcal{I}_2, \varrho_2 \rangle$, and a p-morphism $h : \mathcal{P}_1 \to \mathcal{P}_2$.

- h may generalize perception in that several domain topics map to a single codomain topic in a many-to-one manner, e.g., $(math, physics \overset{h}{\mapsto} science)$.
- In some cases h could feature a simple translation, for instance if the domain prices are in US dollars, while the codomain's prices are in ECU:

$$\big(price\text{-}i(USdollars) \overset{h}{\mapsto} price\text{-}i(ECU)\big).$$

- The domain perception may not specify certain connotations, such as either *topten* or *leatherbound*, so that these connotations in the codomain do not have sources under h, and hence h features an "expansion".

– In some cases the mapping may be simple, e.g., (*topten* $\overset{h}{\mapsto}$ *topten*), however, *topten* may be undefined for the domain perception $\varrho_1(\mathrm{w}, \textit{topten}) = \mathtt{u}$ but $\varrho_2(\mathrm{w}, \textit{topten}) = \mathtt{t}$ is defined, so that there is some "unblurring".

Example 9. Let $\mathcal{P}_1 = \langle \mathcal{I}_1, \varrho_1 \rangle$ be a "catalog" perception as in Example 2, and let $\mathcal{P}_2 = \langle \mathcal{I}_2, \varrho_2 \rangle$ be a "customer" perception as in Example 3. Let $h : \mathcal{P}_2 \to \mathcal{P}_1$ be a p-morphism based on the mapping:

(*art, travel* $\overset{h}{\mapsto}$ *topic-of-interest*),

(all other topics... $\overset{h}{\mapsto}$ *not-interesting*),

(prices less than 25... $\overset{h}{\mapsto}$ *good-price*),

(*titleN* $\overset{h}{\mapsto}$ *heard-of-it*),

(pages more than 400... $\overset{h}{\mapsto}$ *thick*),

(*titleM* $\overset{h}{\mapsto}$ *got-it-already*),

(*topten, reduced* $\overset{h}{\mapsto}$ *hmm*),

(*leatherbound, colorplates* $\overset{h}{\mapsto}$ *makes-a-nice-present*),

for all other connotations: $(\alpha \overset{h}{\mapsto} \textit{blabla-}\alpha)$.

– The essence of no-blur: books that are *art* and books that are *travel* for the catalog perception are *topic-of-interest* for $\mathcal{P}_2$. Books that do not cost less than 25 are not *good-price* for $\mathcal{P}_2$, etc. However, it may be that, for some book w, $\varrho_1(\mathrm{w}, \textit{title}7) = \mathtt{u}$, but $\varrho_2(\mathrm{w}, \textit{got-it-already}) = \mathtt{t}$.
– Many connotations of $\mathcal{P}_1$, such as *ISBN, publisher*, etc., map to the respective *blabla* for $\mathcal{P}_2$. The intuition of the *blabla* connotations is that they have no significance for $\mathcal{P}_2$, although they are perceived by it. (Technically, h has to be defined on all the source connotations.)
– h is not onto: some $\mathcal{P}_2$ connotations (such as *buy-it*) do not have an h source connotation in $\mathcal{P}_1$. Loosely, they are not captured by $\mathcal{P}_1$ and h. A few possible explanations:
  - The connotation *buy-it* is a function of a combination of catalog connotations, and we do not have a way to express it, yet.
  - The customer's favorite authors are simply not in the authorlist of the catalog perception.
  - The catalog perception is unable to perceive that a book is *in-bad-shape*, while the customer's perception is able to perceive this.

If the catalog perception serves the store owner, and he also has knowledge of the p-morphism h, then h may be considered as a "customer model" that may be used to better serve the customer. In a context where individual customers are abstracted as individual perceptions, an arrow such as h would constitute the core of a "model" of the customer. Customer modeling can be viewed as a special case of user modeling, also mentioned at the end of Section 3.

Example 10. Let $\mathcal{P}_1 = \langle \mathcal{I}, \varrho_1 \rangle$ be a bookstore perception, and let $\mathcal{P}_2 = \langle \mathcal{I}, \varrho_2 \rangle$ be a perception with the same set of connotations. Let $h : \mathcal{P}_1 \to \mathcal{P}_2$ be a p-morphism based on the identity mapping. By the no-blur property of the p-morphism h, $\mathcal{P}_2$ is, in the general case, an *improvement* of $\mathcal{P}_1$: there may be books with imperceptible connotations for the domain perception, yet these connotations are definitely perceived by the target, improved, perception. In this case h formalizes improvement of perception. (e.g., the books are now open so that more things can be perceived about them). The term *improvement* will designate either the improving p-morphism h, the target perception $\mathcal{P}_2$, or the target p-predicate ϱ_2. Since h is the identity on connotations these concepts uniquely imply one another. Note that h is one-to-one and onto, yet it is not necessarily an isomorphism: if there is some "unblurring" of perception, then h does not have an inverse p-morphism.

Example 11. Let $\mathcal{P}$ be yet another perception that perceives books using connotations that stand for the size of the book ($x \times y \times z$ centimeters), its font and the quality of the paper. It is probably impossible to construct a p-morphism from or into any of the example perceptions above.

We terminate with two special arrows:

Example 12. Let $\mathcal{U}_\mathcal{E} = \langle \mathcal{E}, 2^\mathcal{E}, \varepsilon \rangle$ be the universal perception of $\mathcal{E}$, as in Definition 4, then for every perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, there exists a morphism $h : \mathcal{P} \to \mathcal{U}_\mathcal{E}$ as follows:
- Define $\widehat{\mathcal{P}} = \langle \mathcal{I}, \widehat{\varrho} \rangle$ to be a totally two-valued perception where $\widehat{\varrho}(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \alpha)$ if and only if $\varrho(\mathrm{w}, \alpha) \neq \mathtt{u}$. This can be achieved from $\mathcal{P}$ by an arbitrary choice of a definite truth value whenever $\varrho(\mathrm{w}, \alpha) = \mathtt{u}$. The identity mapping on $\mathcal{I}$ defines a p-morphism $\widehat{h} : \mathcal{P} \to \widehat{\mathcal{P}}$: it is no-blur by definition of $\widehat{\varrho}$. As a matter of fact, it is a special case of an improvement morphism (as in Example 10): it is a *total improvement*.
- There exists a *natural p-morphism* $\eta : \widehat{\mathcal{P}} \to \mathcal{U}_\mathcal{E}$, which is defined by: $\eta(\alpha) = \{\mathrm{w} \in \mathcal{E} \,\|\, \widehat{\varrho}(w, \alpha) = \mathtt{t}\}$. It is easy to see that η defines a rigid, and hence no-blur p-morphism.
- It is also easy to see that the composite mapping $h = \widehat{h} \circ \eta$ defines a p-morphism from $\mathcal{P}$ to $\mathcal{U}_\mathcal{E}$. (As a matter of fact, a composition of p-morphisms is always a p-morphism.)
- $\widehat{h}$, and hence also h, is not unique. Its definition introduces two possibilities for every $\mathtt{u}$ value of $\mathcal{P}$: $\mathtt{t}$ or $\mathtt{f}$.

Example 13. Let $\mathcal{P}_\emptyset = \langle \mathcal{E}, \emptyset, \varrho_\emptyset \rangle$ be the empty perception of $\mathcal{E}$ (as in Definition 5). For every perception $\mathcal{P}$ of $\mathcal{E}$, there exists a (unique) morphism $h : \mathcal{P}_\emptyset \to \mathcal{P}$. It is based on the empty mapping of connotations, and it emptily stands the no-blur condition for p-morphisms.

4.4. *Categorical formalization of perception machines*

Having defined perceptions and perception morphisms, we would like to define the *category of perceptions* as a basis for a mathematical theory of artificial perceptions. (In the same manner the infrastructure for group theory is provided by defining groups, group homomorphisms, and the category of groups.)

All perception machines $\langle\mathcal{E},\mathcal{I},\varrho\rangle$ with the same first component $\mathcal{E}$ will be regarded as a collection, soon to be formalized as a mathematical category.

Definition 14. $\mathcal{P}rc_{\mathcal{E}}$, *perceptions of* $\mathcal{E}$, is the collection of all objects of the form $\langle\mathcal{E},\mathcal{I},\varrho\rangle$, called *perceptions*, where $\mathcal{E}$, $\mathcal{I}$, ϱ are as in Definition 1. Since $\mathcal{E}$ already appears in the designation $\mathcal{P}rc_{\mathcal{E}}$, we use $\langle\mathcal{I},\varrho\rangle$ instead of $\langle\mathcal{E},\mathcal{I},\varrho\rangle$.

There are various possible sets $\mathcal{E}$ of w-elements, so we are actually defining a family of collections. Definition 14 does not discard $\mathcal{E}$, the environment. It rather raises it one level higher in the hierarchy. Inside $\mathcal{P}rc_{\mathcal{E}}$ all perceptions refer to the same environment (such as the bookstore environment), so that it becomes redundant in the specification of single perceptions.

Defining $\mathcal{P}rc_{\mathcal{E}}$ as a mathematical category provides infrastructure from a well-developed science: category theory. The definition of a category requires that:

- One is given a set of *objects*.
- Given any pair of objects $\mathcal{P}$, $\mathcal{Q}$, one has a collection of *morphisms* $f:\mathcal{P}\to\mathcal{Q}$ from $\mathcal{P}$ to $\mathcal{Q}$. Given a morphism such as f, $\mathcal{P}$ is the *domain* of f, and $\mathcal{Q}$ is the *codomain* of f.
- Morphisms should be closed under composition: given two morphisms $f:\mathcal{P}\to\mathcal{Q}$ and $g:\mathcal{Q}\to\mathcal{R}$, where the codomain of f is the same as the domain of g, one may form their *composite*, $f\circ g$, which is a morphism: $f\circ g:\mathcal{P}\to\mathcal{R}$, such that $f\circ g(a)=g(f(a))$ (i.e., apply f, then g).
- Composition should be associative: $f\circ g\circ h=(f\circ g)\circ h=f\circ(g\circ h)$.
- For every object $\mathcal{P}$ there should be an identity morphism $Id_{\mathcal{P}}:\mathcal{P}\to\mathcal{P}$.
- The identity morphism should be the (left and right) unit element of composition: for every $f:\mathcal{P}\to\mathcal{Q}$, $Id_{\mathcal{P}}\circ f=f=f\circ Id_{\mathcal{Q}}$.

In our context the objects are perceptions $\mathcal{P}$, $\mathcal{Q}$, ... and morphisms are p-morphisms. The remaining requirements can be easily settled, since composition of p-morphisms is defined by set composition of the mappings, and the identity p-morphism is defined by the identity mapping.

Lemma 15. *$\mathcal{P}rc_{\mathcal{E}}$, together with Definition* 6 *of morphisms* (*composition of morphisms and the identity morphism are defined at the level of set mappings*) *is a category.*

The construction and formalization of all perceptual cognitive processes will be trimmed in terms of these very few primitives that category theory provides for the study of artificial perceptions: perception, p-morphism, domain perception and codomain perception of a p-morphism, and composition of p-morphisms. This predicts theoretical as well as applicational tidiness.

A discussion of the basic mathematical properties of the category of perceptions, as well as more example applications, is provided by [2]. Already at this point example applications of basic categorical notions can be provided:

Example 16. Consider Examples 8 and 9. If there is a p-morphism $h:\mathcal{P}_1\to\mathcal{P}_2$ translating between two "catalog" perceptions, and there is a p-morphism $g:\mathcal{P}_2\to\mathcal{P}_3$ that models a "customer" perception $\mathcal{P}_3$ in terms of the catalog perception $\mathcal{P}_2$, then a composite p-morphism $h \circ g$ would neatly model the customer's perception $\mathcal{P}_3$ in terms of the catalog perception $\mathcal{P}_1$. This opens the way, for instance, to a formalization of cooperation between two systems where each one models its own customers.

A perception should be able, among other things, to preserve its autonomy within a society of other perceptions. The variety can occur between several distinct agents, or within one single agent. Some standard categorical tools are capable of formalizing forms of joint perceptions, with varying degrees of trust and partnership. They are elaborated in [2].

Example 17. Maximal trust using coproducts—a coproduct of a family of perceptions is their "least expanded common expansion": an expansion of perception to include the perceptions of the other participants as well. Injecting morphisms are the formal tool that puts them together. This kind of joint perceptions could be useful in any one of the many cases where there is more than one possible perception of a given environment. It provides a neat formal way to go about joining them.

Partnership in coproduct perceptions may be enhanced by merging connotations that are shared by different perceptions (common sense connotations). This can be formally done by a proper pushout.

Minimal trust using products—a product of a family of perceptions is their "least blurred common blur". Example cases where such product perceptions may be useful are cases where points of disagreement have to be blurred. Projecting morphisms are the formal tool that filters out separate aspects of the joint perceptions.

Pullbacks are capable of formally restricting the product perceptions to the desired subset of connotations that feature definite or possible future agreement. This formalizes minimal trust partnerships that concentrate on similarities between the participants.

4.5. Summary of Section 4

The domain of discourse of perception machines was formally defined and categorized. This provides a well-known mathematical environment within which one can scrutinize artificial perceptions. In the sequel this formalization will be justified by showing that tools provided by category theory are useful and meaningful to the study of artificial perceptions and related cognitive processes. The scrutiny will pay by leading us to more insights and to practical constructions and results: it is a means rather than an end by itself.

5. A natural structure of perceptions

By merely looking at the examples it can already be observed that a connotation is more than just an arbitrary entity which stands all by itself. No matter how a connotation is internally represented, once it is tied with the environment $\mathcal{E}$ via the perception ϱ, there are certain lawlike orders and patterns which can be observed about it: one connotation may suggest another, some connotations come always together while others never do, and there are other possible connections as well. All these interlacing connections suggest that

$\mathcal{I}$ has an inner structure, induced by $\mathcal{E}$ and ϱ. This is suggestive of more structure in the objects of the category $\mathcal{P}rc_{\mathcal{E}}$: perception of the environment seems to introduce a structural element into the internal representation.

5.1. Synonyms

Two connotations may be indistinguishable in that they stand for the same perception values. If they could be merged, it could mean a useful extraction of the perceptual essence out of the set of connotations, with no duplications and redundancies. This formalizes a cognitive process of generalization: forming a general term from particulars.

Definition 18. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ be a perception, and let $\alpha, \beta \in \mathcal{I}$. α and β are *synonyms* (or *ϱ-synonyms*), denoted $\alpha \simeq \beta$, if for all w in $\mathcal{E}$ $\varrho(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \beta)$. [2]

It is obvious that:

Proposition 19. *$\simeq$ is an equivalence relation.*

The quotient set $\mathcal{I}/\simeq$, whose elements are the distinct synonymity equivalence classes, will be designated $\mathcal{I}^*$. The predicate ϱ is of course well-defined on $\mathcal{E} \times \mathcal{I}^*$ as well, and the resulting perception will be designated $\mathcal{P}^* = \langle \mathcal{I}^*, \varrho \rangle$.

A change in the set of connotations, from $\mathcal{I}$ to $\mathcal{I}^*$, captures a cognitive process: the internalization of synonymity between connotations. It is neatly and easily formalized by one of our categorical primitives: a p-morphism. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and $\mathcal{P}^* = \langle \mathcal{I}^*, \varrho \rangle$. Define the mapping into the quotient set: $\mathcal{M}_{\mathcal{P}} : \mathcal{I} \to \mathcal{I}^* : \alpha \mapsto [\alpha]$, where $[\alpha]$ designates the class of all synonyms of α. Obviously:

Proposition 20. *$\mathcal{M}_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}^*$ is a rigid p-morphism.* (Rigid *was explained in Example* 7.)

For readers interested in the mathematical-categorical context, a p-morphism is a coequalizer if and only if it merges synonyms only [2].

Example 21. In Example 3 the following could be synonyms:

(*big-red-sticker*$\simeq$*good-price*).

5.2. Subsumptions

The idea behind synonyms is now relaxed. Instead of two connotations representing exactly the same w-elements, one appears to represent a *subset* of the w-elements

[2] In its original understanding synonymity is a relation that holds between linguistic labels, and not between connotations. Our reading of this term, as well as other terms such as "connotation" is, indeed, inspired by their original readings. However, there is no obligation to their exact traditional understanding.

represented by the other. The family of w-elements with connotation β has, as "sub-family" the w-elements with connotation α. Taken the other way round, w-elements with connotation α seem to always have connotation β as well. This means that connotation α seems to *subsume* or imply connotation β.

Definition 22. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ be a perception, and let $\alpha, \beta \in \mathcal{I}$. α *subsumes* β, denoted $\alpha \trianglelefteq \beta$, if for all w in $\mathcal{E}$, $\varrho(\mathrm{w}, \alpha) = \mathrm{t} \Rightarrow \varrho(\mathrm{w}, \beta) = \mathrm{t}$ and $\varrho(\mathrm{w}, \beta) = \mathrm{f} \Rightarrow \varrho(\mathrm{w}, \alpha) = \mathrm{f}$.

The condition of Definition 22 is designated $\forall \mathrm{w} \in \mathcal{E}\ \varrho(\mathrm{w}, \alpha) \overset{\text{Luk}}{\longrightarrow} \varrho(\mathrm{w}, \beta)$. In the absence of u values, $\overset{\text{Luk}}{\longrightarrow}$ becomes the classical two-valued material implication.

Proposition 23. *The $\trianglelefteq$ relation is a quasi-order in $\mathcal{I}$.*

The passage from the set of connotations $\mathcal{I}$ to the set of equivalence classes of synonyms $\mathcal{I}^*$ was shown to be a rigid p-morphism of $\langle \mathcal{I}, \varrho \rangle$ onto $\langle \mathcal{I}^*, \varrho \rangle$. Namely, it rigidly preserves the p-predicate ϱ. Hence subsumptions between elements of $\mathcal{I}^*$ can be defined in exactly the same way as subsumptions between connotations of $\mathcal{I}$. Moreover:

Proposition 24. *The subsumption relation $\trianglelefteq$ is a partial ordering on $\mathcal{I}^*$.*

Example 25. In the *universal perception of $\mathcal{E}$* (see Definition 4), let $A, B \in 2^{\mathcal{E}}$ be subsets of $\mathcal{E}$. Then $A \trianglelefteq B$ *if and only if* $A \subseteq B$.

Example 26. In Example 2 the following subsumptions may be observed (depending on the specific perception):

(*Asimov* $\trianglelefteq$ *science-fiction*) (*art* $\trianglelefteq$ *colorplates*).

The definition of $\overset{\text{Luk}}{\longrightarrow}$, $\simeq$, $\trianglelefteq$ are inspired by the definition that Lukasiewicz gave to the biconditional and to the conditional (respectively) in his 3-valued logic. The choice of this logic was discussed in Section 2. By preserving the law of identity as in Lukasiewicz's logic, any connotation both subsumes itself and is synonym to itself. Without having $\mathrm{u} \leftrightarrow \mathrm{u}$, connotations that are undefined for some w-elements (i.e., $\varrho(\mathrm{w}, \alpha) = \mathrm{u}$) could not have been their own synonyms.

Synonyms and subsumptions need not be universal. In Example 21, for instance, it is indeed not a universal truth that reduced price books are marked with big red stickers. The practical possibility to treat worlds with special features and patterns is another aspect of the subjectivity and flexibility of perceptions. Artificial perceptions "browse" in their environments and detect synonyms and subsumptions.

5.3. *Nonmonotonicity of the relations*

If synonyms and subsumptions are to be regarded as structure in the category of perceptions, then this structure might naturally be expected to be preserved by p-morphisms. However, it can be easily seen that this is generally not the case: let $\alpha \simeq \beta$

in a given perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$. Let $\mathrm{w} \in \mathcal{E}$ be such that $\varrho(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \beta) = \mathrm{u}$. Let $h : \mathcal{P} \to \mathcal{Q}$ be a p-morphism, where $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$. By Definition 6 of p-morphisms it may be that $\tau(\mathrm{w}, h(\alpha)) \neq \tau(\mathrm{w}, h(\beta))$. In that case $h(\alpha) \not\simeq h(\beta)$. Similarly, if $\alpha \trianglelefteq \beta$, it may be that $\tau(\mathrm{w}, h(\alpha)) = \mathrm{t}$ but $\tau(\mathrm{w}, h(\beta)) \neq \mathrm{t}$. In that case $h(\alpha) \ntrianglelefteq h(\beta)$.

Hence, when a p-morphism is applied, either one of the relations $\simeq$ and $\trianglelefteq$ may not be fully preserved. This phenomenon will be henceforth called the *nonmonotonicity* of these relations. It was also discussed in the context of our pre-theoretical considerations (Section 2.4).

The example cases used above are such that for some $\mathrm{w} \in \mathcal{E}$ it so happens that $\varrho(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \beta) = \mathrm{u}$. It can be easily verified from the definitions of $\simeq$ and $\trianglelefteq$ that these are the only ones that represent possible nonmonotonicity under a p-morphism.

The issue of monotonicity of improvements (see Example 10) and total improvements (see Example 12) of perceptions is central to this study. The "open-minded" consideration of all possible improvements and total improvements generally means nonmonotonicity. On the other hand, restricting the discussion only to monotone improvements and monotone total improvements is less "open-minded", but there are advantages in their stability. Whenever a perception is (totally) improved, one of the first questions that will be asked is whether the (total) improvement is monotone or not.

Example 27. The f-*total improvement*, designated ϱ^{f}, of a perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, takes every u value of perception to an f value: for all pairs (w, α) in $\mathcal{E} \times \mathcal{I}$, $\varrho^{\mathrm{f}}(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \alpha)$ if $\varrho(\mathrm{w}, \alpha) \neq \mathrm{u}$, otherwise $\varrho^{\mathrm{f}}(\mathrm{w}, \alpha) = \mathrm{f}$. This improvement is monotone. It represents an acceptable "default" strategy.

Example 28. The t-*total improvement*, designated ϱ^{t}, of a perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, takes every u value of perception to a t value: for all pairs (w, α) in $\mathcal{E} \times \mathcal{I}$, $\varrho^{\mathrm{t}}(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \alpha)$ if $\varrho(\mathrm{w}, \alpha) \neq \mathrm{u}$, otherwise $\varrho^{\mathrm{t}}(\mathrm{w}, \alpha) = \mathrm{t}$. This improvement is monotone as well.

The idea of monotone total improvements has a conservative flavor. However, there is quite a variety within monotone total improvements, and anything which is undefined could still turn to be either true or false. (As demonstrated, for example, by the f- and t-total improvements.)

To be able to restrict ourselves to monotone p-morphisms only (including, of course, monotone improvements and total improvements), a subcategory is defined within the category of perceptions. (A subcategory consists of (subsets of) objects and morphisms from the category, such that composition and identities in the subcategory coincide with those of the category.)

Definition 29. $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Mon}}$, the *monotone subcategory of perceptions*, consists of:
- all the perceptions of $\mathcal{P}rc_{\mathcal{E}}$,
- *monotone p-morphisms*, $h : \mathcal{P} \to \mathcal{Q}$, such that $\alpha \trianglelefteq \beta \Rightarrow h(\alpha) \trianglelefteq h(\beta)$.

Proposition 30. *$\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Mon}}$ is a subcategory of $\mathcal{P}rc_{\mathcal{E}}$.*

Example 31. Following Example 21 of synonyms, there may be an improving p-morphism (see Example 10) into an improved "customer" perception of the bookstore where some books with *big-red-stickers* are not *good-price*. In that case the improvement p-morphism is not in $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Mon}}$.

5.4. Summary of Section 5

In the quest for structure in the set of connotations one may detect lawlike orders and patterns. Two relations were defined and exemplified: synonyms and subsumptions. Subsumptions define a quasi-order on any perception, and in *perceptions, where all synonyms are merged, one even gets as much as a partial order. Synonyms and subsumptions are not necessarily preserved by p-morphisms. For that reason the monotone subcategory of perceptions was defined, which constitutes all perceptions, but only the p-morphisms that preserve the two relations.

6. Boolean perceptions

The previous section gave us a lead to the structure of perceptions: perceptions are naturally endowed with a quasi-order, and *perceptions (where synonyms are merged) are similarly endowed with a partial order. The structure of a lattice thus comes to mind. Links in the lattice may capture relationships which are implicit in a perception. Since the purpose of this study is to have agents possess very structured and expressive perceptions, let us continue with this suggestive idea even further. Let us see what happens if a perception has a set of connotations which has the structure of a complemented and distributive lattice, namely a Boolean algebra. This is, in a sense, the most structured form of a partially ordered set. Indeed, lattices can alternatively be defined in terms of the two operations $\vee$ and $\wedge$. On the intuitive level, it is not unnatural to expect that an intelligent artificial agent should have a perception of connotations that are Boolean combinations of other connotations. In Example 3 it would be natural for perceptions to have connotations such as (*topic-of-interest*$\wedge$*good-price*), or (*not-interesting*$\vee$*got-it-already*), or ($\neg$*thick*).

It is remembered, however, that a quasi-ordered set is still far from being a Boolean algebra, and perceptions, as defined so far, do not generally have such a complex structure. The gap should somehow be closed. We first define and study perceptions with a Boolean set of connotations. Later, we shall examine various ways of marrying the concreteness of basic perceptions with the powers of abstraction of the Boolean structure.

6.1. Permits

If the set of connotations is closed under Boolean connectives, then an arbitrary p-predicate (as allowed until now) might violate a certain sense of adequacy to the external world. Whereas our original definition of perceptions had every connotation perceived independently of the other connotations, the new construct calls for some kind of dependence between the p-predicate values of different connotations that are dependent in the Boolean sense. As an example, if a w-element has both connotations α and β, then an agent claiming that this w-element lacks $\alpha \wedge \beta$ could hardly be described as "intelligent". Technically, this would disgrace the qualification of ϱ as a *predicate*.

Consider the universal perception of Definition 4. Its set of connotations $2^{\mathcal{E}}$ is, of course, a Boolean algebra with the set-theoretical operations of conjunction, disjunction and complementation. It is easy to see that the universal p-predicate is "well-behaved". As an example, if a w-element in $\mathcal{E}$ has both connotations $A, B \subset \mathcal{E}$ then this w-element is an element of both subsets A and B. w is thus an element of both their union $A \cup B$ and their intersection $A \cap B$, and hence it has the connotation $A \vee B$ as well as $A \wedge B$, as expected.

In the case of a total, two valued, p-predicate there is a known classical manner in which a two-valued predicate should be defined on a Boolean algebra. In the present case, however, the p-predicate is three-valued. We have to find a sensible way to embed a three-valued predicate in a Boolean algebra. This will be first done in a global, categorical manner, without surgery into specific w-elements, connotations or predicates.

The definition of Boolean perceptions will "test" these perceptions against the universal perception that was just shown to "behave well". Such a "test" is neatly cast as a p-morphism into the universal perception. By Example 12, p-morphisms from any perception $\mathcal{P}$ into the universal perception always exist:

- Let $\widehat{h}: \mathcal{P} \to \widehat{\mathcal{P}}$ be a total improvement of $\mathcal{P}$.
- Define the natural p-morphism η from $\widehat{\mathcal{P}}$ into $\mathcal{U}_{\mathcal{E}}$:

$$\eta(\alpha) = \{\mathrm{w} \in \mathcal{E} \,\|\, \widehat{\varrho}(w, \alpha) = \mathtt{t}\}.$$

- $h = \widehat{h} \circ \eta$ defines a p-morphism from $\mathcal{P}$ to $\mathcal{U}_{\mathcal{E}}$.

h is not unique. For $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ with a Boolean algebra of connotations to be a b-perception, it will be necessary that at least one of these p-morphisms should be based on a mapping that is a Boolean homomorphism.

Definition 32. Let $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ be a perception such that $\mathcal{B}$ is a Boolean algebra. A *permit* of $\mathcal{C}$, if it exists, is a total improvement of $\mathcal{C}$ that yields a natural morphism into $\mathcal{U}_{\mathcal{E}}$ which is a Boolean homomorphism of the connotations.

As with improvements in general (see Example 10), we shall use the word *permit* to designate either the improving p-morphism or, alternatively, the target total p-predicate. These two concepts uniquely imply one another.

Example 33. There are "seemingly well-behaved" total improvements that are not permits. Two of them are the $\mathtt{f}$-total and the $\mathtt{t}$-total improvements (Examples 27 and 28). In the case where $\sigma(\mathrm{w}, \beta) = \sigma(\mathrm{w}, \neg\beta) = \mathtt{u}$, any permit should have one of the values $\mathtt{t}$ and the other $\mathtt{f}$, while these improvements assign the same value to both.

An improvement (Example 10) consists of assigning definite values to some of the cases where the p-predicate has had a $\mathtt{u}$ value. It is thus very similar to *a partial formation of a "possible world"* [37]. In case of a total improvement (as in Example 12), there is an assignment of definite values to *all* those cases, and it is thus similar to *a total formation of a possible world*. If ϱ represents a partial perception of the environment, then a total improvement $\widehat{\varrho}$ is a total description of one possible perception for $\langle \mathcal{I}, \varrho \rangle$.

A permit thus indicates a *possible perception* with regard to the incomplete information represented by the b-perception $\langle \mathcal{B}, \sigma \rangle$. This possible perception has the additional property that it is "sensible", in that it can be naturally mapped in a Boolean way into the universal perception. It thus "arranges" the world $\mathcal{E}$ in a sensible way.

6.2. Definition and categorical status of Boolean perceptions

Definition 34. Let a perception $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ be such that $\mathcal{B}$ is a Boolean algebra. $\mathcal{C}$ is a *Boolean perception* b-perception for short (σ is a *Boolean perception predicate*, b-p-predicate for short) if:

(i) it has a permit as in Definition 32;

(ii) *closure* of the b-p-predicate: let $\mathcal{V}_\mathcal{C}$ be the (nonempty) set of permits of $\mathcal{C}$, then, for all $\mathrm{w} \in \mathcal{E}$ and for all $\beta \in \mathcal{B}$:

$$\sigma(\mathrm{w}, \beta) = \begin{cases} \mathrm{t} & \text{if } \forall \widehat{\sigma} \in \mathcal{V}_\mathcal{C},\ \widehat{\sigma}(\mathrm{w}, \beta) = \mathrm{t}, \\ \mathrm{f} & \text{if } \forall \widehat{\sigma} \in \mathcal{V}_\mathcal{C},\ \widehat{\sigma}(\mathrm{w}, \beta) = \mathrm{f}, \\ \mathrm{u} & \text{otherwise.} \end{cases}$$

It is easy to see that the p-predicate σ is uniquely determined by the permits.

Example 35. $\mathcal{U}_\mathcal{E}$ itself is a b-perception with the identity as its unique permit.

By Definition 32 permits are based on Boolean homomorphisms into the universal perception. Since Boolean homomorphisms always map top to top and bottom to bottom, then:

Corollary 36. *If* $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ *is a b-perception, then* $\sigma(\mathrm{w}, \top) = \mathrm{t}$, $\sigma(\mathrm{w}, \bot) = \mathrm{f}$.

B-perceptions are perceptions by definition, hence they constitute a subset of the objects in the category $\mathcal{P}rc_\mathcal{E}$. This subset will be designated $\mathcal{P}rc_\mathcal{E}^{\mathrm{bl}}$, and its elements are of the form $\langle \mathcal{B}, \sigma \rangle$ (to make a notational difference from $\langle \mathcal{I}, \varrho \rangle$). To define $\mathcal{P}rc_\mathcal{E}^{\mathrm{bl}}$ as a subcategory one first has to establish the p-morphisms which could be applied within this subcategory. The expected, natural, requirement is that the additional, i.e., Boolean, structure be preserved.

Definition 37. Let $\mathcal{C}_1 = \langle \mathcal{B}_1, \sigma_1 \rangle$ and $\mathcal{C}_2 = \langle \mathcal{B}_2, \sigma_2 \rangle$ be b-perceptions. A p-morphism $f : \mathcal{C}_1 \to \mathcal{C}_2$ is a *Boolean perception morphism*, b-p-morphism for short, if the mapping $f : \mathcal{B}_1 \to \mathcal{B}_2$ is a Boolean homomorphism.

Example 38. In Example 9, if the perceptions involved were Boolean, then a b-p-morphism between them would open the possibility for a more complex model of the customer such as:

$(\mathit{authornameN} \vee \mathit{authornameM} \overset{h}{\mapsto} \mathit{hmm})$,

$(\mathit{art} \wedge \mathit{leather\text{-}bound} \wedge \neg \mathit{thick} \overset{h}{\mapsto} \mathit{makes\text{-}a\text{-}perfect\text{-}present} \wedge \mathit{buy\text{-}it})$.

Example 39. Consider color blindness. Let $\mathcal{E}$ be any environment of colorful w-elements, let $\mathcal{P}_1$ be a "red-blind" perception, and let $\mathcal{P}_2$ be a "normal" color perception. A p-morphism $h : \mathcal{P}_1 \to \mathcal{P}_2$ needs to use Boolean combinations:

$$(\mathit{yellowgreen} \overset{h}{\mapsto} \mathit{yellow} \vee \mathit{green}), \qquad (\mathit{cyanwhite} \overset{h}{\mapsto} \mathit{cyan} \vee \mathit{white}),$$
$$(\mathit{bluemagenta} \overset{h}{\mapsto} \mathit{blue} \vee \mathit{magenta}), \qquad (\mathit{blackred} \overset{h}{\mapsto} \mathit{black} \vee \mathit{red}).$$

Since Boolean homomorphisms are closed under composition, then:

Lemma 40. *$\mathcal{P}rc_{\mathcal{E}}^{\mathrm{bl}}$ with b-p-morphisms is a subcategory of $\mathcal{P}rc_{\mathcal{E}}$.*

6.3. Equivalent characterizations of Boolean perceptions

Definition 34 is an external, categorical one. One might want to fathom its details and consequences, and, in particular, to show that the natural expectations from a b-p-predicate are fulfilled. The main "internal" result is Lemma 41 below. It provides a formulation of a necessary and sufficient condition for the recognition of b-perceptions "from inside". The technical details and proofs that lead to this lemma are given in Appendix A.1.

Lemma 41. *A perception $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ such that the set $\mathcal{B}$ of connotations is a Boolean algebra is a b-perception if and only if for all $\mathrm{w} \in \mathcal{E}$:*
(i) *The set $\{\beta \in \mathcal{B} \| \sigma(\mathrm{w}, \beta) = \mathtt{t}\}$ is a filter.*
(ii) *The set $\{\beta \in \mathcal{B} \| \sigma(\mathrm{w}, \beta) = \mathtt{f}\}$ is an ideal.*
(iii) *The above filter and ideal are dual one to the other: for all β in $\mathcal{B}$, $\sigma(\mathrm{w}, \beta) = \mathtt{t}$ if and only if $\sigma(\mathrm{w}, \neg\beta) = \mathtt{f}$.*

The permits of a b-perception are defined by all the possible maximal ideals (and dual maximal filters) that include the ideal (and dual filter) of Lemma 41. Classical two-valued predicates on a Boolean algebra are characterized by such a division into a maximal ideal and a dual maximal filter. This is, in a certain sense, the point where the present study meets the classical theory that admits only total descriptions and two truth values. Loosely: b-perceptions are neither total nor two-valued, but they have the potential of evolving into total two-valued perceptions.

6.4. Computing Boolean perception predicates

Any agent that uses a b-perception for practical purposes will eventually deal with perception of specific w-elements and their connotations. It might need the value of $\sigma(\mathrm{w}, \beta)$ for some $\mathrm{w} \in \mathcal{E}$ and a connotation $\beta \in \mathcal{B}$. While the categorical approach gives the whole discussion its formal support, it would not be practical for an agent to deal with possible perceptions, ideals, and filters, every time it needs the value of some $\sigma(\mathrm{w}, \beta)$. A deductive apparatus is needed to guide the computation of the values of the p-predicate directly from the basic perception. Ideals are closed under disjunction and under subsumption from below, while filters are closed under conjunction and under

Table 1
Negation in Boolean perceptions

$\sigma(\mathrm{w},\alpha)$	$\sigma(\mathrm{w},\neg\alpha)$
t	f
f	t
u	u

Table 2
Disjunction in Boolean perceptions, $\sigma(\mathrm{w},\alpha\vee\beta)$

	$\sigma(\mathrm{w},\beta)$	t	f	u
$\sigma(\mathrm{w},\alpha)$				
t		t	t	t
f		t	f	u
u		t	u	$\begin{cases} \text{t} & \text{if } (\neg\alpha)\leqslant\beta \text{ (also } (\neg\beta)\leqslant\alpha \text{ and } \alpha\vee\beta=\top) \\ \text{u} & \text{otherwise} \end{cases}$

Table 3
Conjunction in Boolean perceptions, $\sigma(\mathrm{w},\alpha\wedge\beta)$

	$\sigma(\mathrm{w},\beta)$	t	f	u
$\sigma(\mathrm{w},\alpha)$				
t		t	f	u
f		f	f	f
u		u	f	$\begin{cases} \text{f} & \text{if } \alpha\leqslant(\neg\beta) \text{ (also } \beta\leqslant(\neg\alpha) \text{ and } \alpha\wedge\beta=\bot) \\ \text{u} & \text{otherwise} \end{cases}$

subsumption from above. This gives us the insight we needed into the behavior of b-p-predicates, and provides us with truth tables:

Lemma 42. *Let $C=\langle\mathcal{B},\sigma\rangle$ be a b-perception. The truth tables for the b-p-predicate σ are given by Tables* 1–3. (*In these tables $\leqslant$ designates the conventional Boolean partial order, defined by the Boolean algebraic* Law of Consistency.)

The proof is given in Appendix A.1. It is noted that the truth tables are not an arbitrary choice of some three-valued logic, but rather a result of the global categorical structure.

6.5. Summary of Section 6

The category of perceptions has a subcategory of b-perceptions where the sets of connotations are Boolean algebras and the p-predicates are restricted accordingly. An agent with b-perception has an adequate perception of Boolean combinations of connotations. B-perceptions can be characterized in more than one way:

- Their Definition 34 in terms of possible total perceptions (permits) is of a category theoretical, global, nature.
- B-perceptions can be characterized with the necessary and sufficient Boolean algebraic conditions of Lemma 41, using proper ideals and their dual proper filters.

Lemma 42 provides a deductive apparatus that may be algorithmically applied for the computation of specific values of the b-p-predicate. These truth tables show that the categorical definition yields a p-predicate that is "Boolean adequate" in a certain common sense of the term.

7. Free generation of Boolean perceptions

A general perception has a quasi-ordered set of connotations (Proposition 23), but this is still far from being a Boolean algebra as required for the perceptions of the Boolean subcategory presented above. Closing this gap means finding a way to somehow form a Boolean version of any perception, and thus marry the concreteness of basic perceptions with the powers of abstraction of the Boolean structure.

For simple perceptions the connection between w-elements and connotations is innate. It emerges from the sensory-motor-neural apparatus, and accepted as it is. The main property of an agent with this perception is the very fact that it has a direct perception of the environment. For b-perceptions, on the other hand, there is already some kind of structure imposed on perception. Not every predicate will do, and the predicate is subject to an adequacy condition which stems from abstract arguments about the meaning of the connectives "or", "and", "not". These connectives do not have an objective existence in the environment. They are defined in an abstract way, creations of symbolic processing. In this sense b-perceptions have a somewhat abstractive flavor, a flavor one expects from cognitive perceptions.

The following step is to try and combine the advantages of simple perceptions with those of b-perceptions. An agent could, hopefully, relate to its environment via direct perception as in $\mathcal{P}rc_{\mathcal{E}}$, and at the same time process its basic perception in an abstractive way as in $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{bl}}$. This could be a step towards a certain intuition about artificial cognition.

The most general Boolean generation over a given perception is introduced and studied, then evaluated.

7.1. Definition of free Boolean perceptions

Trying to integrate Boolean features into a simple perception naturally means that the set of connotations will have to somehow be closed under Boolean operations. At the same time one wants to preserve perception of the generating connotations. The categorical environment provides a neat formulation of that: If $\langle \mathcal{I}, \varrho \rangle \in \mathcal{P}rc_{\mathcal{E}}$ is a perception, one is looking for a *generating p-morphism*: $\xi : \langle \mathcal{I}, \varrho \rangle \to \langle \mathcal{B}, \sigma \rangle$ such that

- $\xi(\mathcal{I})$ is a set of generators for the Boolean algebra $\mathcal{B}$. (This guarantees closure under Boolean operations.)
- ξ is a rigid p-morphism. (This guarantees that perception of the generating connotations is preserved.)

$$\begin{tikzcd} \mathcal{I} \arrow[rr, hook] \arrow[dr, "f"'] & & \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}} \arrow[dl, "\exists! h"] \\ & \mathcal{B} & \end{tikzcd}$$

Fig. 1. The free Boolean algebra over $\mathcal{I}$.

$\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ could, hopefully, serve as a b-perception for the agent that was, so far, equipped with $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ only.

The simplest and most general way to close $\mathcal{I}$ under Boolean operations is to take this set of original connotations as *free generators*. The free Boolean algebra over $\mathcal{I}$ will be designated $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$. Its main property (see Fig. 1) is that for any Boolean algebra $\mathcal{B}$ and for any mapping $f : \mathcal{I} \to \mathcal{B}$ there exists a unique extension h of f which is a Boolean homomorphism $h : \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}} \to \mathcal{B}$. One is thus looking for a Boolean generation of the form

$$\xi^{\mathrm{fr}} : \langle \mathcal{I}, \varrho \rangle \to \langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}, \varrho^{\mathrm{fr}} \rangle$$

where $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$ is freely generated by $\mathcal{I}$, ξ^{fr} is the inclusion map of generators, and it remains to define ϱ^{fr}.

For $\langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}, \varrho^{\mathrm{fr}} \rangle$ to be Boolean, by Definition 34, it needs to have a permit. Obviously, any such permit would be an extension of some total improvement of ϱ since $\mathcal{I} \subset \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$. On the other hand, by the freedom of $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$:

Lemma 43. *Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ be a perception. Let $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$ be the free Boolean algebra generated over $\mathcal{I}$. Every total improvement $\widehat{\varrho}$ of ϱ can be uniquely extended into a total b-p-predicate*:

$$\widehat{\varrho}^{\mathrm{bl}} : \mathcal{E} \times \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}} \to \{\mathtt{t}, \mathtt{f}\}.$$

It follows that the Boolean extensions of the total improvements of the generating perception $\mathcal{P}$ are exactly all the "candidate" permits of a Boolean generation based on the free Boolean algebra $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$ of connotations. For the sake of generality it is desirable that *all of them* should be permits. The p-predicate ϱ^{fr} will be thus defined to accommodate all these possible permits (the designation $\widehat{\varrho}^{\mathrm{bl}}$ is used as in Lemma 43).

Definition 44. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ be a perception. The *free b-perception over $\mathcal{P}$*, designated $\mathcal{C}^{\mathrm{fr}} = \langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}, \varrho^{\mathrm{fr}} \rangle$, is defined:

- $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$, the set of connotations, is the free Boolean algebra generated over $\mathcal{I}$.
- Let $\mathcal{V}_{\mathcal{P}}$ be the set of total improvements of $\mathcal{P}$. ϱ^{fr}, the *free b-p-predicate*, is defined, for all $\mathrm{w} \in \mathcal{E}$ and for all $\beta \in \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$, by:

$$\varrho^{\mathrm{fr}}(\mathrm{w}, \beta) = \begin{cases} \mathtt{t} & \text{if } \forall \widehat{\varrho} \in \mathcal{V}_{\mathcal{P}},\ \widehat{\varrho}^{\mathrm{bl}}(\mathrm{w}, \beta) = \mathtt{t}, \\ \mathtt{f} & \text{if } \forall \widehat{\varrho} \in \mathcal{V}_{\mathcal{P}},\ \widehat{\varrho}^{\mathrm{bl}}(\mathrm{w}, \beta) = \mathtt{f}, \\ \mathtt{u} & \text{otherwise.} \end{cases}$$

By Definition 44 that uses *all* total improvements $\widehat{\mathcal{Q}}$ as a basis for permits, one can conclude that ξ^{fr} does what it was expected to do:

Corollary 45. *$\xi^{\mathrm{fr}}(\mathcal{I})$ is a set of generators for the Boolean algebra $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$, and ξ^{fr} is a rigid p-morphism.*

Example 46. Let $\mathcal{P}$ be a bookstore perception as in Section 4.1. The free b-perception over $\mathcal{P}$ would have connotations that consist of all possible Boolean combinations of the generating connotations, with b-p-predicate values that are computed by the truth tables of Lemma 42.

7.2. Free Boolean generation as a functor

The generation of a b-perception over any given perception as defined above is a mapping from the category of perceptions into the Boolean subcategory:

$$\mathcal{G}^{\mathrm{fr}} : \mathcal{P}rc_{\mathcal{E}} \to \mathcal{P}rc^{\mathrm{bl}}_{\mathcal{E}} : \mathcal{P} \mapsto \mathcal{C}^{\mathrm{fr}}.$$

It comes together with the generating morphism $\xi^{\mathrm{fr}} : \mathcal{P} \to \mathcal{C}^{\mathrm{fr}}$. A few things are to be expected from the $\mathcal{G}^{\mathrm{fr}}$ mapping, if it is supposed to generate a b-perception over any given simple perception in a "methodical" manner:

(i) If two perceptions were able to communicate as simple perceptions via a p-morphism $f : \mathcal{P} \to \mathcal{Q}$, then this communication should be preserved by Boolean generation: $\mathcal{G}^{\mathrm{fr}}$ should provide an extension of f which is a b-p-morphism:

$$\mathcal{G}^{\mathrm{fr}}(f) : \mathcal{G}^{\mathrm{fr}}(\mathcal{P}) \to \mathcal{G}^{\mathrm{fr}}(\mathcal{Q}),$$

and the diagram in Fig. 2 should be commutative. The only definition that could do this is the following: let $h : \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}} \to \mathcal{B}^{\mathrm{fr}}_{\mathcal{J}}$ be the unique extension of the mapping

$$\mathcal{I} \to \mathcal{B}^{\mathrm{fr}}_{\mathcal{J}} : \alpha \mapsto f(\alpha)$$

into a Boolean homomorphism such that $\forall \alpha \in \mathcal{I}\ h(\alpha) = f(\alpha)$. Define $\mathcal{G}^{\mathrm{fr}}(f)$ by $\beta \mapsto h(\beta)$. (We still have to show that this is a b-p-morphism.)

(ii) The provided communication between the Boolean generations should preserve compositions:

$$\mathcal{G}^{\mathrm{fr}}(f \circ g) = \mathcal{G}^{\mathrm{fr}}(f) \circ \mathcal{G}^{\mathrm{fr}}(g).$$

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "f"] \arrow[d, "\xi^{\mathrm{fr}}"'] & \mathcal{Q} \arrow[d, "\xi^{\mathrm{fr}}"] \\
\mathcal{G}^{\mathrm{fr}}(\mathcal{P}) \arrow[r, "\mathcal{G}^{\mathrm{fr}}(f)"] & \mathcal{G}^{\mathrm{fr}}(\mathcal{Q})
\end{tikzcd}$$

Fig. 2. Boolean perception generation with morphisms.

The categorical framework provides well developed concepts for the above: $\mathcal{G}^{\mathrm{fr}}$ needs to be a *functor*, and ξ^{fr} should be a *natural transformation* from the identity functor on $\mathcal{P}rc_{\mathcal{E}}$ to the functor $\mathcal{G}^{\mathrm{fr}}$. An immediate example result of this demand is that $\mathcal{G}^{\mathrm{fr}}$ should generate isomorphic b-perceptions over isomorphic perceptions—a very plausible expectation. The functor is thus a formal guarantee that agents are generating b-perceptions in a consistent, methodical way. This is an instance where categorization of perceptions provides us with tools of scrutiny that capture certain pre-theoretical intuitions about cognitive perceptions.

Lemma 47. *Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$, and let $f : \mathcal{P} \to \mathcal{Q}$ be a p-morphism. Let $\mathcal{G}^{\mathrm{fr}}(\mathcal{P}) = \langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}, \varrho^{\mathrm{fr}} \rangle$, and $\mathcal{G}^{\mathrm{fr}}(\mathcal{Q}) = \langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{J}}, \tau^{\mathrm{fr}} \rangle$. Let $h : \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}} \to \mathcal{B}^{\mathrm{fr}}_{\mathcal{J}}$ be the unique extension of the mapping*

$$\mathcal{I} \to \mathcal{B}^{\mathrm{fr}}_{\mathcal{J}} : \alpha \mapsto f(\alpha)$$

into a Boolean homomorphism such that $\forall \alpha \in \mathcal{I}\ h(\alpha) = f(\alpha)$. Then the mapping

$$\mathcal{G}^{\mathrm{fr}}(f) : \mathcal{G}^{\mathrm{fr}}(\mathcal{P}) \to \mathcal{G}^{\mathrm{fr}}(\mathcal{Q}),$$

defined by: $\beta \mapsto h(\beta)$, is a b-p-morphism such that the diagram of Fig. 2 is commutative.

The proof is given in Appendix A.2. The essence of the proof is "rooting" the diagram and all involved perceptions into the universal perception, using permit arrows (i.e., possible perceptions). Permits and the universal perception constitute the foundations of b-perceptions and hence a basis for all proofs.

It also follows from the definition of $\mathcal{G}^{\mathrm{fr}}(f)$ that $\mathcal{G}^{\mathrm{fr}}$ preserves compositions and identities, and hence:

Corollary 48. *ξ^{fr} is a natural transformation from the identity functor on $\mathcal{P}rc_{\mathcal{E}}$ to the functor $\mathcal{G}^{\mathrm{fr}}$.*

7.3. Freedom of the functor

There are additional things to expect from the generating functor $\mathcal{G}^{\mathrm{fr}}$ that are easily categorized. One of them is *generality*. In categorical terms, this neatly translates to a *free* generation. The following lemma uses a standard category theoretical characterization of freedom, as illustrated by Fig. 3.

Lemma 49. *Let $\mathcal{P}$ be a perception. Let*

$$\mathcal{G}^{\mathrm{fr}} : \mathcal{P}rc_{\mathcal{E}} \to \mathcal{P}rc^{\mathrm{bl}}_{\mathcal{E}} : \mathcal{P} \mapsto \mathcal{C}^{\mathrm{fr}}$$

be as in Definition 44, and let $\xi^{\mathrm{fr}} : \mathcal{P} \to \mathcal{G}^{\mathrm{fr}}(\mathcal{P})$ be the rigid inclusion of generators. Then for any b-perception $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ and any p-morphism $f : \mathcal{P} \to \mathcal{C}$ there exists a unique b-p-morphism $\psi : \mathcal{G}^{\mathrm{fr}}(\mathcal{P}) \to \mathcal{C}$ such that $\xi^{\mathrm{fr}} \circ \psi = f$.

Proof outline. A natural transformation such as ξ^{fr} always defines a free functor: the definition of ψ is similar in nature to the definition of $\mathcal{G}^{\mathrm{fr}}(f)$, and the proof is similar to that of Lemma 47, with $\mathcal{C}$ replacing $\mathcal{G}^{\mathrm{fr}}(\mathcal{Q})$ and ψ replacing h. $\square$

$$
\begin{tikzcd}
\mathcal{P} = \langle \mathcal{I}, \varrho \rangle \arrow[rr, hook, "\xi^{\mathrm{fr}}"] \arrow[dr, "f"'] & & \mathcal{G}^{\mathrm{fr}}(\mathcal{P}) = \langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}, \varrho^{\mathrm{fr}} \rangle \arrow[dl, "\exists! \psi"] \\
& \mathcal{C} &
\end{tikzcd}
$$

Fig. 3. Freedom of Boolean perception generation.

7.4. Computing free Boolean perception predicates

The values of the p-predicate are innate and fixed for the generating perceptions. When it comes to the Boolean closure the agent may use the truth tables of Lemma 42. These truth tables provide a deductive apparatus which may guide the computation of the values of the p-predicate directly from the generating perception. Since $\mathcal{I}$ is a set of generators for $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$, the computation will eventually "bottom out" at the generating perception $\langle \mathcal{I}, \varrho \rangle$.

In this context we distinguish between two kinds of connotations in $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$:

Definition 50. Let β be a connotation in $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$.
- If $\beta \in \mathcal{I}$ then it is a *simple* or *generating* connotation.
- If $\beta \notin \mathcal{I}$ then it is a *complex* or *derived* connotation.

In free Boolean generation the subset of complex connotations is, of course, disjoint from the subset of simple connotations. Since $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$ is generated over $\mathcal{I}$, any $\beta \in \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$ is equal to a Boolean expression with simple connotations as its atomic expressions. The agent can compute $\varrho^{\mathrm{fr}}(\mathrm{w}, \beta)$, starting from its immediate perception of the simple connotations that make β and using the truth tables.

At the end of this study, in Section 10, there is a discussion of some methodological fallout, where simple versus complex connotations are discussed in Section 10.1. Simple connotations are assumed to be closest to, and readily recognized by, the sub-symbolic sensory-motor-neural apparatus of an artificial agent, without further procedure. If α is a generating connotation, then the value of $\varrho(\mathrm{w}, \alpha)$ emerges without need of a deductive apparatus (this "emergence" could be based, for example, on a neural network). Higher level artificial perception of derived connotations in the Boolean closure is achieved with due recourse to the deductive apparatus.[3]

More computational effort would be needed as the expression gets more complex: the answer is expected to be as complex as the question. An algorithmic implementation of the process should also detect general Boolean $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$ dependencies within the expression, to deal with the lower right entries of the disjunction and conjunction truth Tables 2 and 3. It will be rewarded with more definite values.

[3] The reader is reminded that whether or not Boolean perception provides a suitable model of human perception is not an issue of this study.

7.5. Summary of Section 7

Free Boolean generation provides a rigorous mathematical description of a methodical cognitive transition from basic perceptions to b-perceptions. In addition to basic sensory-motor-neural perception, a capability of abstraction is captured by an adequate perception of Boolean combinations of connotations. An agent that performs this process may claim and show for fact that it is, among other things, *methodical* (the natural functor), *open minded*, and *general* (freedom of the generation and of $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$). This cognitive transition has some good features, but it is maybe somewhat too free, too general. The following sections consist of further attempts for a methodical Boolean generation with, perhaps, better features.

8. Validity and completeness in Boolean perceptions

The perceptual order relations of subsumptions and synonyms gave rise to the idea that the set of connotations is a lattice. B-perceptions were defined to follow that idea, and structure a set of connotations as a complemented distributive lattice: a Boolean algebra. Given a b-perception $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$, it is thus natural to check whether the Boolean algebra captures the intended meaning. One needs to compare two order relations within the carrier set $\mathcal{B}$ of connotations.

- The *Boolean* partial order, denoted $\leqslant$. It is a formal construct that comes with the Boolean algebra, defined by the Boolean algebraic *Law of Consistency*:

  $$\underline{\alpha \leqslant \beta} \text{ if and only if } \underline{\alpha \wedge \neg\beta = \bot} \text{ if and only if } \underline{\neg\alpha \vee \beta = \top}.$$

- The *perceptual* quasi-order, denoted $\trianglelefteq$, and set by Definition 22:

  $$\underline{\alpha \trianglelefteq \beta} \text{ if and only if } \underline{\forall \mathrm{w} \in \mathcal{E}\ \sigma(\mathrm{w}, \alpha) \xrightarrow{\text{Luk}} \sigma(\mathrm{w}, \beta)}.$$

  This order relation emerges from the agent's perception. It describes perceptible patterns and was extensively discussed in Section 5.2.

In this context of comparison between $\trianglelefteq$ and $\leqslant$ within b-perceptions, we define:

Definition 51. A b-perception $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ is *valid* if, for all $\alpha, \beta \in \mathcal{B}$, $\alpha \leqslant \beta \Rightarrow \alpha \trianglelefteq \beta$.

In a valid b-perception all the Boolean subsumptions reflect perceptible σ-subsumptions. To justify the Boolean structure, b-perceptions should be, at least, valid: patterns reflected by the structure should be supported by perception. This was the motivation for b-perceptions (see Section 6).

Example 52. The universal perception $\mathcal{U}_{\mathcal{E}}$, is valid: for all $A, B \subset \mathcal{E}$, $A \subset B \Rightarrow A \trianglelefteq B$.

The corresponding concept of *completeness* comes immediately to mind, with the converse implication: $\alpha \trianglelefteq \beta \Rightarrow \alpha \leqslant \beta$. In a complete b-perception all perceptible patterns should be reflected by the Boolean structure. This also captures a certain intuition about the Boolean structure serving as a "mental image" that is generated by perception.

Before embarking on the actual definition of completeness in b-perceptions, we need to restrict the definition of $\trianglelefteq$ for these perceptions. This extra care is warranted by the three-valued context. Assume that, in a b-perception $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$, there are two connotations, $\alpha, \beta \in \mathcal{B}$, such that, for all $\mathrm{w} \in \mathcal{E}$, $\sigma(\mathrm{w}, \alpha) = \sigma(\mathrm{w}, \beta) = \mathrm{u}$. It follows that α and β are synonyms, also with their negations, and all the following σ-subsumptions hold:

$$\alpha \trianglelefteq \beta, \quad \alpha \trianglelefteq \neg\beta, \quad \neg\alpha \trianglelefteq \beta, \quad \neg\alpha \trianglelefteq \neg\beta.$$

If one substitutes all of them for the Boolean partial order $\leqslant$, one gets a degenerate Boolean algebra: $\bot = \top$. This is, of course, undesirable. The most one might want to do in this situation is merge the synonyms α and β into one connotation (see Section 5.1).

It is also easy to verify that this kind of problem occurs only in the mentioned situation where $\forall \mathrm{w} \in \mathcal{E}\ \sigma(\mathrm{w}, \alpha) = \sigma(\mathrm{w}, \beta) = \mathrm{u}$.

Definition 53. Let $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ be a b-perception, and let $\alpha, \beta \in \mathcal{B}$. Then $\alpha \trianglelefteq \beta$ if:

- $\forall \mathrm{w} \in \mathcal{E}\ \sigma(\mathrm{w}, \alpha) \xrightarrow{\text{Luk}} \sigma(\mathrm{w}, \beta)$ as in Definition 22.
- There exists some $\mathrm{w} \in \mathcal{E}$ such that either $\sigma(\mathrm{w}, \alpha) \neq \mathrm{u}$ or $\sigma(\mathrm{w}, \beta) \neq \mathrm{u}$.

We are now ready to define:

Definition 54. A b-perception $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ is *complete* if, for all $\alpha, \beta \in \mathcal{B}$, $\alpha \trianglelefteq \beta \Rightarrow \alpha \leqslant \beta$.

Example 55. The universal perception $\mathcal{U}_{\mathcal{E}}$ is complete: for all $A, B \subset \mathcal{E}$, $A \trianglelefteq B \Rightarrow A \subset B$. This follows directly from its Definition (4).

Having defined validity and completeness for b-perceptions, we proceed to examine where they hold. Validity of all b-perceptions is based on the validity of their permits:

Proposition 56. *Let $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ be a b-perception, let $\alpha \leqslant \beta$, and let $\widehat{\sigma}$ be any one of its permits. Then, for all $\mathrm{w} \in \mathcal{E}$, $\widehat{\sigma}(\mathrm{w}, \alpha) \xrightarrow{\text{Luk}} \widehat{\sigma}(\mathrm{w}, \beta)$.*

Proof. By the Law of Consistency and by the natural $\eta : \widehat{\mathcal{C}} \to \mathcal{U}_{\mathcal{E}}$ being a Boolean homomorphism one gets (overline designates set complementation):

$$\emptyset = \eta(\bot) = \eta(\alpha \wedge \neg\beta) = \eta(\alpha) \cap \overline{\eta(\beta)}.$$

It follows that for no $\mathrm{w} \in \mathcal{E}$ could it be that both $\mathrm{w} \in \eta(\alpha)$ and $\mathrm{w} \in \overline{\eta(\beta)}$ at the same time. Hence, for all $\mathrm{w} \in \mathcal{E}$ and for all permits $\widehat{\sigma}$, $\widehat{\sigma}(\mathrm{w}, \alpha) \xrightarrow{\text{Luk}} \widehat{\sigma}(\mathrm{w}, \beta)$. □

By the closure condition of Definition 34 it follows that, for all w in $\mathcal{E}$, also $\sigma(\mathrm{w}, \alpha) \xrightarrow{\text{Luk}} \sigma(\mathrm{w}, \beta)$, and hence:

Corollary 57. *B-perceptions are valid.*

Example 58. By Corollary 57 free Boolean generation of Section 7 is valid: for all $\alpha, \beta \in \mathcal{B}^{\text{fr}}_{\mathcal{I}}$, $\alpha \leqslant \beta \Rightarrow \alpha \trianglelefteq \beta$.

As might have been expected, completeness is scarcer than validity. It turns out that there is a connection between completeness of a b-perception and the monotonicity of its permits.

Proposition 59. *In a complete b-perception all permits are monotone.*

Proof. Let $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ be a complete b-perception, and let $\widehat{\sigma}$ be any one of its permits. By completeness $\alpha \trianglelefteq \beta$ implies $\alpha \leqslant \beta$, and by Proposition 56 this implies that, for all $\mathrm{w} \in \mathcal{E}$,

$$\widehat{\sigma}(\mathrm{w}, \alpha) \xrightarrow{\text{Luk}} \widehat{\sigma}(\mathrm{w}, \beta). \qquad \square$$

To show incompleteness of a general free Boolean generation, recall that by generating the free Boolean algebra over the set $\mathcal{I}$ of connotations, we ignored ϱ-subsumptions between them. These subsumptions are not reflected by the Boolean structure. The ϱ-subsumption relation in $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ is a subset of the σ-subsumption relation in $\mathcal{C}^{\mathrm{fr}} = \langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}, \varrho^{\mathrm{fr}} \rangle$ (by rigidity of the generating morphism ξ^{fr}). In Section 5.3 it was shown that, in the general case, $\mathcal{P}$ has total improvements $\widehat{\varrho}$ that are nonmonotone. They extend to permits of $\mathcal{C}^{\mathrm{fr}}$ that do not preserve $\trianglelefteq$ of $\mathcal{P}$, and hence they are nonmonotone permits of $\mathcal{C}^{\mathrm{fr}}$. It is concluded that:

Lemma 60. *Free Boolean generation is, in the general case, incomplete.*

We have thus observed two disjoint sources for the perceptual order relation $\trianglelefteq$ in a free Boolean generation:

- By validity in $\mathcal{C}^{\mathrm{fr}}$, $\alpha \leqslant \beta$ implies that $\alpha \trianglelefteq \beta$. These subsumptions are based on abstract "logical speculations". They are monotone in the sense that they will hold with any permit (i.e., any possible improvement of perception).
- Observed lawlike patterns, the ϱ-subsumptions that hold between the generating connotations. They are based on perceptual observations of "facts".[4] They are the cause of nonmonotonicity of some permits of $\mathcal{C}^{\mathrm{fr}}$, and also the cause of incompleteness of $\mathcal{C}^{\mathrm{fr}}$.

Complete b-perceptions constitute a subset, designated $\mathcal{P}rc_{\mathcal{E}}^{\text{bl-cmp}}$, of the objects of $\mathcal{P}rc_{\mathcal{E}}$ and of $\mathcal{P}rc_{\mathcal{E}}^{\text{bl}}$. It is easy to see that:

Lemma 61. *$\mathcal{P}rc_{\mathcal{E}}^{\text{bl-cmp}}$ with b-p-morphisms is a (full[5]) subcategory of the subcategory $\mathcal{P}rc_{\mathcal{E}}^{\text{bl}}$ of b-perceptions.*

It was just shown that $\mathcal{P}rc_{\mathcal{E}}^{\text{bl}} \neq \mathcal{P}rc_{\mathcal{E}}^{\text{bl-cmp}}$. On the other hand, by Example 55, $\mathcal{P}rc_{\mathcal{E}}^{\text{bl-cmp}}$ is not empty: the universal perception is a complete b-perception.

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[4] Perception follows, in this context, the scientific enquiry principle *Hypotheses non fingo* [11, p. 261].

[5] *Full* means that all b-p-morphisms between perceptions are "inherited" from $\mathcal{P}rc_{\mathcal{E}}^{\text{bl}}$, so that $\mathcal{P}rc_{\mathcal{E}}^{\text{bl-cmp}}$ is fully determined by its collection of perceptions.

8.1. Completion of Boolean perceptions

Consider an incomplete b-perception $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$. "Internalization" of all observed σ-subsumptions $\alpha \trianglelefteq \beta$ means actually changing the structure of the Boolean lattice of connotations, "moving things around" in order that $\alpha \leqslant \beta$ should hold as well. The situation may be figuratively compared to the situation of an analyst who internalizes all the experimental lab results. That done, he may lean back in his armchair, close his eyes, and figure out all the rest. (Eventually he may open one eye to ask for additional experimental results, namely improved generating perception.)

We shall now use the categorical framework and Boolean tools for a methodical modification of any b-perception (and free Boolean generation in particular) so that it should not only be valid, but complete as well: the artificial agent will thus have a rigorous tool for the internalization of its perceptual observations and the creation of a meaningful cognitive image of its environment.

Let $\mathcal{C}$ be a b-perception. In the categorical, arrowed, context one is looking for a b-p-morphism $\xi^{\mathrm{cmp}} : \mathcal{C} \to \overline{\mathcal{C}}$, such that $\overline{\mathcal{C}} = \langle \overline{\mathcal{B}}, \overline{\sigma} \rangle$ is complete: whenever $\alpha \trianglelefteq \beta$ holds then $\alpha \leqslant \beta$ holds as well. $\overline{\mathcal{C}}$ could, hopefully, replace $\mathcal{C}$ as the agent's b-perception. A few natural requirements are:

- In the case where $\mathcal{C}$ is already complete ξ^{cmp} should be the identity.
- ξ^{cmp} should be σ-monotone: the idea is to preserve the σ-subsumptions, not to discard them.
- ξ^{cmp} should introduce a minimal modification of $\mathcal{C}$: no change except for that which is needed for completeness.

There are standard Boolean concepts (related to the Boolean algebra of connotations $\mathcal{B}$) that can neatly do what we want:

- The set of elements: $\mathcal{S} = \{\alpha \wedge \neg\beta \,\|\, \alpha \trianglelefteq \beta\}$.
- The ideal $\triangle$ that is generated by $\mathcal{S}$, $\triangle = \{\beta \in \mathcal{B} \| \beta \leqslant \bigvee_{s \in \mathcal{S}} s\}$.
- The quotient algebra: $\overline{\mathcal{B}} = \mathcal{B}/\triangle$.
- The natural Boolean homomorphism into the quotient algebra:

  $$\xi^{\mathrm{cmp}} : \mathcal{B} \to \overline{\mathcal{B}} : \beta \mapsto [\beta].$$

  By definition of the natural homomorphism, $[\alpha] = [\beta]$ if and only if $\alpha \cong \beta$. This "congruence modulo $\triangle$" means that $\alpha \wedge \neg\beta \in \triangle$ and $\beta \wedge \neg\alpha \in \triangle$. In particular, $[\bot] = [\beta]$ if and only if $\beta \in \triangle$.

It is easy to see from the definitions of $\mathcal{S}$, $\triangle$, and $\overline{\mathcal{B}}$ that:

- σ-subsumptions are integrated into the Boolean algebra $\overline{\mathcal{B}}$: $\alpha \trianglelefteq \beta$ implies a Boolean subsumption in $\overline{\mathcal{B}}$: $[\alpha] \leqslant [\beta]$.
- If $\mathcal{C}$ is already complete then $\mathcal{S} = \triangle = \{\bot\}$, and hence ξ^{cmp} is the identity.
- If $\overline{\mathcal{B}}$ turns to be a set of connotations for a b-perception then, by validity, ξ^{cmp} will define a σ-monotone p-morphism.
- ξ^{cmp} introduces the minimal necessary modification of $\mathcal{B}$, because $\triangle$ is the smallest ideal that includes $\mathcal{S}$.

$\overline{\mathcal{B}}$ is thus the candidate set of connotations for $\overline{\mathcal{C}}$ (in Appendix A.3, Lemma A.4, it is shown that $\overline{\mathcal{B}}$ is not a degenerate Boolean algebra). ξ^{cmp} is the candidate for the b-p-morphism onto $\overline{\mathcal{C}}$. It remains to properly define a p-predicate for $\overline{\mathcal{C}}$. This will be done in the obviously expected manner:

Definition 62. Let $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ be a b-perception. Its *completed perception*, $\overline{\mathcal{C}} = \langle \overline{\mathcal{B}}, \overline{\sigma} \rangle$, consists of:
- $\overline{\mathcal{B}}$ as defined above is the Boolean set of connotations;
- the p-predicate $\overline{\sigma} : \mathcal{E} \times \overline{\mathcal{B}} \to \{\mathtt{t}, \mathtt{f}, \mathtt{u}\}$, where:

$$\overline{\sigma}(\mathrm{w}, [\beta]) = \begin{cases} \mathtt{t} & \text{if } \exists \alpha \cong \beta \text{ such that } \sigma(\mathrm{w}, \alpha) = \mathtt{t}, \\ \mathtt{f} & \text{if } \exists \alpha \cong \beta \text{ such that } \sigma(\mathrm{w}, \alpha) = \mathtt{f}, \\ \mathtt{u} & \text{otherwise.} \end{cases}$$

In Appendix A.3, Proposition A.6, it is shown that there is no conflict in the definition.

To establish the legitimacy of the construction, we show in Appendix A.3, Proposition A.7 through Corollary A.13, that:
- $\overline{\mathcal{C}}$ has permits. Its permits are, exactly, all the monotone permits of $\mathcal{C}$ (it is also shown that $\mathcal{C}$ does have monotone permits).
- The p-predicate $\overline{\sigma}$ answers the closure condition for b-perceptions, from Definition 34.
- ξ^{cmp} defines a b-p-morphism.

The main result of this section (proven in Appendix A.3) is:

Lemma 63. *Let $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ be a b-perception. Its completed perception $\overline{\mathcal{C}} = \langle \overline{\mathcal{B}}, \overline{\sigma} \rangle$ of Definition* 62 *is a complete b-perception.*

It is also shown (Corollary A.14) that the $\overline{\sigma}$ subsumptions are exactly the σ-subsumptions: no new subsumptions are added.

Example 64. Consider a completion of the free b-perception that is generated over the perception of Example 3. If (*in-bad-shape* $\trianglelefteq_{\rho^{\mathrm{fr}}}$ $\neg$*buy-it*), then in the completed perception one gets also (*in-bad-shape* $\leqslant$ $\neg$*buy-it*), as part of the Boolean structure. Intuitively, this subsumption is "mentally internalized".

8.2. *Completion as a free functor*

As argued in Section 7.2, a general and methodical cognitive transition is best formalized by a free functor and a natural transformation. The transition from a b-perception to a complete b-perception (as just described) will thus be formalized by a free functor ($\mathcal{G}^{\mathrm{cmp}}$) into $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{bl-cmp}}$, $\mathcal{G}^{\mathrm{cmp}}(\mathcal{C}) = \overline{\mathcal{C}}$, and we will show that ξ^{cmp} is a natural transformation from the identity functor on the domain subcategory to the functor $\mathcal{G}^{\mathrm{cmp}}$. In particular, if two b-perceptions were able to communicate using a b-p-morphism $f : \mathcal{C} \to \mathcal{C}'$, then $\mathcal{G}^{\mathrm{cmp}}$ should provide a b-p-morphism for communication between the completed perceptions:

$$\mathcal{G}^{\mathrm{cmp}}(f) : \mathcal{G}^{\mathrm{cmp}}(\mathcal{C}) \to \mathcal{G}^{\mathrm{cmp}}(\mathcal{C}').$$

Such that the diagram in Fig. 4 should be commutative. However, the results of the former Section 8.1 predict that only monotone b-p-morphisms f could fit into that diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f\,(\text{monotone})"] \arrow[d, "\xi^{\text{cmp}}"'] & \mathcal{C}' \arrow[d, "\xi^{\text{cmp}}"] \\
\mathcal{G}^{\text{cmp}}(\mathcal{C}) \arrow[r, "\mathcal{G}^{\text{cmp}}(f)"] & \mathcal{G}^{\text{cmp}}(\mathcal{C}')
\end{tikzcd}$$

Fig. 4. Boolean perception completion with morphisms.

In the general case, there may be b-p-morphisms in $\mathcal{P}rc^{\text{bl}}_{\mathcal{E}}$ that are nonmonotone. This may be exemplified in the context of free b-perception generation, with which we are familiar from Section 7: assume that, in a given perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, $\alpha \trianglelefteq \beta$. Assume further that $\mathcal{P}$ communicates, using $f : \mathcal{P} \to \mathcal{Q}$, with a perception $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$, but $f(\alpha) \ntrianglelefteq f(\beta)$, so that f is non-ϱ-monotone. Consider the respective free Boolean generations $\mathcal{G}^{\text{fr}}(\mathcal{P})$ and $\mathcal{G}^{\text{fr}}(\mathcal{Q})$. In $\mathcal{G}^{\text{fr}}(\mathcal{P})$, $\alpha \trianglelefteq \beta$ holds, but $\alpha \nleqslant \beta$ because α and β are free generators. In $\mathcal{G}^{\text{fr}}(\mathcal{Q})$, $f(\alpha) \ntrianglelefteq f(\beta)$, because $f(\alpha)$ and $f(\beta)$ are free generators and because $f(\alpha) \ntrianglelefteq f(\beta)$. f can be extended to a b-p-morphism between the respective free Boolean generations:

$$\mathcal{G}^{\text{fr}}(f) : \mathcal{G}^{\text{fr}}(\mathcal{P}) \to \mathcal{G}^{\text{fr}}(\mathcal{Q}),$$

and this extension preserves $f(\mathcal{P})$ (see Fig. 2), so that

$$\mathcal{G}^{\text{fr}}(f)(\alpha) \ntrianglelefteq \mathcal{G}^{\text{fr}}(f)(\beta).$$

Let us now "complete" the b-perceptions $\mathcal{G}^{\text{fr}}(\mathcal{P})$ and $\mathcal{G}^{\text{fr}}(\mathcal{Q})$, into $\overline{\mathcal{G}^{\text{fr}}(\mathcal{P})}$ and $\overline{\mathcal{G}^{\text{fr}}(\mathcal{Q})}$, respectively. $\mathcal{G}^{\text{fr}}(f)$ cannot be properly extended to a b-p-morphism between these complete b-perceptions, since now $\alpha \leqslant \beta$ but $f(\alpha) \nleqslant f(\beta)$.

Example 65. Consider two "bookstore" "customer" perceptions as in Example 3. Assume that, in $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, (*buy-it* $\trianglelefteq$ *heard-of-it*). Assume that $\mathcal{P}$ communicates, using $(\alpha \overset{h}{\mapsto} \alpha)$, with a perception $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$, but $(h(\textit{buy-it}) \ntrianglelefteq h(\textit{heard-of-it}))$, so that h is non-ϱ-monotone. The story in everyday words: $\mathcal{P}$ buys any book only after having heard of it. $\mathcal{P}$ is not certain whether he has heard about a specific book, and therefore he is also not certain whether he wants to buy that book. $\mathcal{Q}$ does not have the same rules, says "so what!" and is ready to buy the book in spite of not having heard of it. They are having a row because of that. They usually communicate well, but some rules of $\mathcal{P}$ keep causing trouble between them. h can be extended to a b-p-morphism between the respective free Boolean generations, but it cannot be properly extended to a b-p-morphism between the *complete* b-perceptions. $\mathcal{P}$ and $\mathcal{Q}$ could pursue "casual" communication only, using their basic apparatus (namely h), or free Boolean generation (namely $\mathcal{G}^{\text{fr}}(h)$) at most (e.g., they can talk about the weather—basic perception with, perhaps, some heady but neutral logical speculations). Introduction of "own" lawlike structures by way of completeness ruins the communication. The anthropomorphism should not be misleading. Humans communicate for a multitude of motives. We are capable of containing, even enjoying, this kind of trouble. Artificial perceptions are conceived for practical purposes, so that nonmonotonicity cannot be ignored.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\xi^{\mathrm{cmp}}"] \arrow[dr, "f"', "\text{(monotone)}"'] & & \mathcal{G}^{\mathrm{cmp}}(\mathcal{C}) \arrow[dl, "\exists!\psi"] \\ & \mathcal{C}' & \end{tikzcd}$$

Fig. 5. Freedom of Boolean perception completion.

We are thus going to define a functor $\mathcal{G}^{\mathrm{cmp}}$ for the completion of b-perceptions on a subcategory, designated $\mathcal{P}rc_{\mathcal{E}}^{\text{bl-mon}}$, that includes *all b-perceptions* but *only monotone b-p-morphisms*. $\mathcal{G}^{\mathrm{cmp}}$ will be a free functor, and ξ^{cmp} will be a natural transformation from the identity functor on $\mathcal{P}rc_{\mathcal{E}}^{\text{bl-mon}}$ to $\mathcal{G}^{\mathrm{cmp}}$:

Lemma 66. *Let $\mathcal{C}$ be a b-perception. Let*

$$\mathcal{G}^{\mathrm{cmp}} : \mathcal{P}rc_{\mathcal{E}}^{\text{bl-mon}} \to \mathcal{P}rc_{\mathcal{E}}^{\text{bl-cmp}} : \mathcal{C} \mapsto \overline{\mathcal{C}}, \quad \xi^{\mathrm{cmp}} : \mathcal{C} \to \mathcal{G}^{\mathrm{cmp}}(\mathcal{C})$$

be as in Definition 62. Then for any other complete b-perception $\mathcal{C}'$ and for any monotone b-p-morphism $f : \mathcal{C} \to \mathcal{C}'$, there exists a unique b-p-morphism $\psi : \overline{\mathcal{C}} \to \mathcal{C}'$ such that $\xi^{\mathrm{cmp}} \circ \psi = f$ (see Fig. 5).

Proof. Define ψ by $[\beta] \mapsto f(\beta)$. Clearly, $\xi^{\mathrm{cmp}} \circ \psi = f$, and the definition is unique since ξ^{cmp} is onto. ψ is a b-p-morphism by f being one, and by showing that, for all $\alpha, \beta \in \mathcal{B}$, $\alpha \cong \beta$ implies that $f(\alpha) = f(\beta)$. The proof is similar to that of Proposition A.8 of Appendix A, with the Boolean monotone f replacing $\widehat{\sigma}$, completeness of $\mathcal{C}'$ replacing that of the universal perception, and $\perp$ replacing $\emptyset$. $\square$

8.3. Summary of Section 8

Artificial agents with b-perceptions may perform a methodical cognitive transition to complete b-perceptions. The transition consists of a complete internalization of perceptually observed patterns into the Boolean structure. The process is formalized by a free functor from the subcategory of b-perceptions with monotone b-p-morphisms into the subcategory of complete b-perceptions. However, b-perceptions with nonmonotone communication cannot pursue this communication after the transition.

9. Sketching complete Boolean perceptions

In the last two sections two free generations were introduced. The first free functor, $\mathcal{G}^{\mathrm{fr}}$, generates a b-perception over any perception. The second one, $\mathcal{G}^{\mathrm{cmp}}$, generates a complete b-perception over any b-perception. If one considers them as simple mappings between sets of perceptions, then the set of b-perceptions is both the codomain of $\mathcal{G}^{\mathrm{fr}}$ and the domain of $\mathcal{G}^{\mathrm{cmp}}$, so that the mappings can be composed:

$$\mathcal{G}^{\mathrm{fr}} \circ \mathcal{G}^{\mathrm{cmp}} : \mathcal{P}rc_{\mathcal{E}} \to \mathcal{P}rc_{\mathcal{E}}^{\text{bl-cmp}}.$$

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "?"] \arrow[d, "\xi^{\mathrm{fr}}"'] & \mathcal{Q} \arrow[d, "\xi^{\mathrm{fr}}"] \\
\mathcal{G}^{\mathrm{fr}}(\mathcal{P}) \arrow[r, "f\,(\text{monotone})"] \arrow[d, "\xi^{\mathrm{cmp}}"'] & \mathcal{G}^{\mathrm{fr}}(\mathcal{Q}) \arrow[d, "\xi^{\mathrm{cmp}}"] \\
\mathcal{G}^{\mathrm{fr}} \circ \mathcal{G}^{\mathrm{cmp}}(\mathcal{P}) \arrow[r, "\mathcal{G}^{\mathrm{cmp}}(f)"] & \mathcal{G}^{\mathrm{fr}} \circ \mathcal{G}^{\mathrm{cmp}}(\mathcal{Q})
\end{tikzcd}$$

Fig. 6. Composite generation with morphisms.

This is, however, not a legitimate composition of functors, since the domain $\mathcal{P}rc_{\mathcal{E}}^{\text{bl-mon}}$ of $\mathcal{G}^{\mathrm{cmp}}$ is a restriction of the codomain $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{bl}}$ of $\mathcal{G}^{\mathrm{fr}}$, and $\mathcal{G}^{\mathrm{cmp}}$ is undefined for the nonmonotone morphisms of $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{bl}}$. The composite mapping can be applied to any perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$. This would yield the perception

$$\mathcal{G}^{\mathrm{fr}} \circ \mathcal{G}^{\mathrm{cmp}}(\mathcal{P}) = \overline{\mathcal{C}^{\mathrm{fr}}}$$

that is valid and complete. The benefits of such a perception to an artificial intelligent agent were discussed in the previous section. We would like to study under what conditions the composite mapping does define a free functor. Such a functor, if it exists, would define a *free complete b-perceptions generation*. By Section 8 one has to eliminate from $\mathcal{P}rc_{\mathcal{E}}$ the p-morphisms that yield, under $\mathcal{G}^{\mathrm{fr}}$, nonmonotone morphisms in $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{bl}}$ (see Fig. 6). In other words, certain communication paths between the generating perceptions will have to be eliminated for complete b-perception generation. Total improvements are also cases of p-morphisms, and some of them will have to be eliminated as well. (Indeed, we already know that the resulting complete b-perceptions are less general than free Boolean generation in that they have less permits.) We ask what exactly is the price in open-mindedness, namely in total improvements of $\mathcal{P}$ and in communication paths. The non-ϱ-monotone p-morphisms (and total improvements) of $\mathcal{P}$ will obviously have to go: they extend to non-ϱ^{fr}-monotone b-p-morphisms of $\mathcal{C}^{\mathrm{fr}}$. It seems, however, that the price is higher than that:

Example 67. Let $\alpha, \beta \in \mathcal{I}$ be two generating connotations such that, using truth tables of ϱ^{fr} as in Section 6.4, for all $\mathrm{w} \in \mathcal{E}$,

$$\varrho(\mathrm{w}, \alpha) = \varrho^{\mathrm{fr}}(\mathrm{w}, \alpha) = \varrho^{\mathrm{fr}}(\mathrm{w}, \neg\beta).$$

Clearly, α and $\neg\beta$ are ϱ^{fr}-synonyms, as well as β and $\neg\alpha$. (In a bookstore perception these could be $\alpha = $ *paperback* and $\beta = $ *hardcover*.) Assume now that, for some $\mathrm{w} \in \mathcal{E}$, $\varrho(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \beta) = \mathrm{u}$. The f-total improvement (of Example 27) will definitely map neither pair of synonyms into the same universal perception connotation. This total improvement is thus excluded from the domain of $\mathcal{G}^{\mathrm{cmp}}$, because it is not a ϱ^{fr}-monotone permit of $\mathcal{C}^{\mathrm{fr}}$, although it is ϱ-monotone!

In Section 8 we have observed two disjoint sources for the perceptual order relation $\trianglelefteq_{\rho^{\mathrm{fr}}}$: the Boolean $\leqslant$ and the observed $\trianglelefteq$. However, the last example is neither a free Boolean subsumption, nor a ϱ-synonym or subsumption. There could be "links of a third kind", traces of some Boolean structure in generating perceptions. Complete Boolean generation forces us to discard p-morphisms that are not committed to that structure.

9.1. Boolean sketches

Boolean sketches are formal structures that capture traces of Boolean structure in generating perceptions. It is a useful tool for definition and understanding of complete Boolean generation. Let $\mathcal{K}$ be a set. We designate by $\mathcal{K}^{BE}$ the *set of all Boolean expressions over* $\mathcal{K}$. Clearly, elements of $\mathcal{K}^{BE}$ can be identified with elements of any Boolean algebra $\mathcal{B}$ that includes $\mathcal{K}$, and, in particular, with elements of the free Boolean algebra over $\mathcal{K}$, designated $\mathcal{B}^{\mathrm{fr}}_{\mathcal{K}}$. A Boolean Sketch consists of a set $\mathcal{K}$, together with a quasi-order on $\mathcal{K}^{BE}$ that extends the usual Boolean partial order. Formally:

Definition 68. A *Boolean sketch* is a pair $\langle \mathcal{K}, R\rangle$ where:
- $\mathcal{K}$ is a set.
- R is a quasi-order on $\mathcal{K}^{BE}$.
- R has a *Boolean property*: let $\leqslant_{\mathrm{bl-fr}}$ designate the Boolean partial order on $\mathcal{B}^{\mathrm{fr}}_{\mathcal{K}}$, then for all Boolean expressions $e_1, e_2 \in \mathcal{K}^{BE}$, $e_1 \leqslant_{\mathrm{bl-fr}} e_2$ implies that $e_1 R e_2$.

Hence the smallest quasi-order R for $\langle \mathcal{K}, R\rangle$ is $R = \leqslant_{\mathrm{bl-fr}}$.

Example 69. Every Boolean algebra $\mathcal{B}$ is a Boolean sketch $\langle \mathcal{B}, \leqslant\rangle$.

Example 70. If $A \subset \mathcal{B}$ is any subset of the elements of a Boolean algebra $\mathcal{B}$ then $\langle A, \leqslant\rangle$ is a Boolean sketch.

The following example is the Boolean sketch that we need:

Example 71. Let $\mathcal{P} = \langle \mathcal{I}, \varrho\rangle$ be a perception, and let $\trianglelefteq_{\rho^{\mathrm{fr}}}$ be the perceptual order relation of the free b-perception $\mathcal{C}^{\mathrm{fr}} = \langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}, \varrho^{\mathrm{fr}}\rangle$, then $\langle \mathcal{I}, \trianglelefteq_{\rho^{\mathrm{fr}}}\rangle$ is a Boolean sketch. $\trianglelefteq_{\rho^{\mathrm{fr}}}$ has the required Boolean property by validity of $\mathcal{C}^{\mathrm{fr}}$.

Clearly, $\trianglelefteq$ of $\langle \mathcal{I}, \varrho\rangle$ is included in $\trianglelefteq_{\rho^{\mathrm{fr}}}$, but there are other sketch subsumptions and synonyms, as well as other relationships in $\langle \mathcal{I}, \trianglelefteq_{\rho^{\mathrm{fr}}}\rangle$. We designate, for all $x, y \in \mathcal{K}^{BE}$:
- x *subsumes* y if xRy.
- x, y are *synonyms* if xRy and yRx.
- x, y are *disjoints* if $xR\neg y$ and $yR\neg x$.
- x, y are *complements* if $\neg xRy$ and $\neg yRx$.
- x, y are *antonyms* if they are both disjoints and complements.

Clearly, in $\langle \mathcal{I}, \trianglelefteq_{\rho^{\mathrm{fr}}}\rangle$, if x, y are atomic expressions and x subsumes y, then one gets the familiar ϱ-subsumption, and the same goes for synonyms. Disjoints, complements, and antonyms cannot be expressed with ϱ-synonyms and subsumptions. It is easy to see that, for all expressions $x, y \in \mathcal{I}$:

- They are disjoints if and only if for all $\mathrm{w} \in \mathcal{E}$:

$$\varrho(\mathrm{w}, x) = \mathtt{t} \Rightarrow \varrho(\mathrm{w}, y) = \mathtt{f} \quad \text{and} \quad \varrho(\mathrm{w}, y) = \mathtt{t} \Rightarrow \varrho(\mathrm{w}, x) = \mathtt{f}.$$

- They are complements if and only if for all $\mathrm{w} \in \mathcal{E}$:

$$\varrho(\mathrm{w}, x) = \mathtt{f} \Rightarrow \varrho(\mathrm{w}, y) = \mathtt{t} \quad \text{and} \quad \varrho(\mathrm{w}, y) = \mathtt{f} \Rightarrow \varrho(\mathrm{w}, x) = \mathtt{t}.$$

- They are antonyms if and only if for all $\mathrm{w} \in \mathcal{E}$:

$$\varrho(\mathrm{w}, x) = \mathtt{t} \Leftrightarrow \varrho(\mathrm{w}, y) = \mathtt{f} \quad \text{and} \quad \varrho(\mathrm{w}, y) = \mathtt{t} \Leftrightarrow \varrho(\mathrm{w}, x) = \mathtt{f}.$$

The Boolean sketch structure is thus capable of capturing "links of the third kind" as suspected before.

Example 72. In a bookstore perception, the following patterns could exist:
- *edition*1 *subsumes* ¬*paperback*.
- *paperback* and ¬*hardcover* are *synonyms*, and hence *paperback* and *hardcover* are *antonyms*.
- For $N \neq M$ *priceN* and *priceM* are *disjoints*. This is actually an observation that a book can have at most one price in the given environment.
- *ISBNn* and $\bigvee\{ISBNk\}_{k \neq n}$ are *complements*. This is actually an observation that a book must have some *ISBNi* in the given environment.
- *publisherN* and $\bigvee\{publisherK\}_{K \neq N}$ are *antonyms*. This is actually an observation that a book must have exactly one publisher in the given environment.

Indeed, this is only an example of one possible perception with a specific environment. There may be environments that feature, for instance, books that are co-published. In that case the last item above would not hold. As explained before, perceptions are not meant to reflect patterns that are necessarily "universal".

In Example 2 "families" of connotations were observed (such as topic connotations, title connotations, etc.). The formulation of disjoints, complements and antonyms (and other Boolean patterns) as above intrinsically defines these families, similar in spirit to the "quality dimensions" of a "conceptual space" suggested by Gärdenfors [21–23]. These "families" will be naturally integrated into the Boolean structure.

Mappings between Boolean sketches that preserve the sketch structure will be a useful concept for complete Boolean generation. To be able to formulate the preservation of the sketch structure, one first needs to define the meaning of a set mapping when applied to the Boolean expressions over the set. This will be done in the obvious way: Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a set mapping. We define $f : \mathcal{K}_1^{BE} \to \mathcal{K}_2^{BE}$. By replacing every atom $a \in \mathcal{K}_1$ in the domain expressions by $f(a) \in \mathcal{K}_2$ in the target expression. This is formally done by induction on the structure of the expression: if $e = a \in \mathcal{K}_1$ then $f(e) = f(a)$, if $e = \neg e'$ then $f(e) = \neg f(e')$, etc., ...

Definition 73. Let $\langle \mathcal{K}_1, R_1 \rangle$ and $\langle \mathcal{K}_2, R_2 \rangle$ be two Boolean sketches. A set mapping $f : \mathcal{K}_1 \to \mathcal{K}_2$ is a *Boolean Sketch Morphism* $f : \langle \mathcal{K}_1, R_1 \rangle \to \langle \mathcal{K}_2, R_2 \rangle$ if, for all $e_1, e_2 \in \mathcal{K}_1^{BE}$, $e_1 R_1 e_2$ implies that $f(e_1) R_2 f(e_2)$.

Example 74. Boolean algebras are a special case of Boolean sketches (as in Example 69). In that case Boolean homomorphisms between them are Boolean sketch morphisms.

Example 75. Consider a Boolean sketch as in Example 70. The restriction of a Boolean homomorphism on the relevant subalgebra that is generated by A is a Boolean sketch morphism.

The sketch morphisms that we are after are arrows between Boolean sketches $\langle \mathcal{I}, \trianglelefteq_{\rho^{\mathrm{fr}}} \rangle$ as in Example 71. The following lemma follows directly from the definitions:

Lemma 76. *Let $f : \mathcal{P} \to \mathcal{Q}$ be a p-morphism ($\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$). Let $\langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}, \varrho^{\mathrm{fr}} \rangle$ and $\langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{J}}, \tau^{\mathrm{fr}} \rangle$ be the corresponding free Boolean generations. Then*

$$f : \langle \mathcal{I}, \trianglelefteq_{\rho^{\mathrm{fr}}} \rangle \to \langle \mathcal{J}, \trianglelefteq_{\tau^{\mathrm{fr}}} \rangle$$

is a Boolean sketch morphism if and only if

$$\mathcal{G}^{\mathrm{fr}}(f) : \mathcal{G}^{\mathrm{fr}}(\mathcal{P}) \to \mathcal{G}^{\mathrm{fr}}(\mathcal{Q})$$

is a ϱ^{fr}-monotone b-p-morphism.

The connection of Boolean sketches to the issues of this section is obvious now. Lemma 76 provides the required information about the p-morphisms that are able to "survive" a natural transformation that is based on the composite mapping $\xi^{\mathrm{fr}} \circ \xi^{\mathrm{cmp}}$: the ones that are sketch morphisms. Establishing the formal categorical framework for that will lead to a better understanding of the completion process.

Lemma 77. *Boolean sketches with sketch morphisms (composition and the identity sketch morphisms are defined at the set level) form a category.*

Definition 78. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$. $f : \mathcal{P} \to \mathcal{Q}$ is a *sketch structured p-morphism* if the set mapping f defines a Boolean sketch morphism

$$f : \langle \mathcal{I}, \trianglelefteq_{\rho^{\mathrm{fr}}} \rangle \to \langle \mathcal{J}, \trianglelefteq_{\tau^{\mathrm{fr}}} \rangle.$$

Example 79. Sketch structured p-morphisms and (total) improvements are monotone. However, there are monotone such arrows that are not sketch structured. For instance, the `f`-total improvement and the `t`-total improvement of Examples 27 and 28 do not necessarily preserve sketch complements, disjoints, or antonyms. The counter examples are, as in Examples 33 and 67, based on cases where α and β are complements, disjoints, or antonyms, but some $\mathrm{w} \in \mathcal{E}$ is such that $\varrho(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \beta) = \mathrm{u}$.

To restrict oneself to consideration of perceptions with sketch structured morphisms only, another subcategory of perceptions is introduced.

Definition 80. $\mathcal{P}rc^{\mathrm{Sk}}_{\mathcal{E}}$, the *sketch structured subcategory of perceptions*, consists of:
- all the perceptions of $\mathcal{P}rc_{\mathcal{E}}$;
- sketch structured p-morphisms only.

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "f\,\text{(sketch structured)}"] \arrow[d, "\xi^{\mathrm{fr}} \circ \xi^{\mathrm{cmp}}"'] & \mathcal{Q} \arrow[d, "\xi^{\mathrm{fr}} \circ \xi^{\mathrm{cmp}}"] \\
\mathcal{G}^{\mathrm{fr\text{-}cmp}}(\mathcal{P}) \arrow[r, "\mathcal{G}^{\mathrm{fr\text{-}cmp}}(f)"] & \mathcal{G}^{\mathrm{fr\text{-}cmp}}(\mathcal{Q})
\end{tikzcd}$$

Fig. 7. Complete Boolean perception generation with morphisms.

Obviously, $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Sk}}$ is a subcategory of $\mathcal{P}rc_{\mathcal{E}}$ and of $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Mon}}$.

Example 81. Consider the sketch structured subcategory of bookstore perceptions. If a perception $\mathcal{P}$ features the sketch antonyms

paperback and *hardcover*

then all the sketch structured p-morphisms h from this perception should preserve that pattern:

h(*paperback*) and h(*hardcover*) should be sketch antonyms in $h(\mathcal{P})$.

We are now finally ready to define the functor that generates a complete b-perception over any given perception, and to show its natural transformation and freedom properties (by Lemma 76).

Corollary 82. *Define the functor* $\mathcal{G}^{\mathrm{fr\text{-}cmp}} : \mathcal{P}rc_{\mathcal{E}}^{\mathrm{Sk}} \to \mathcal{P}rc_{\mathcal{E}}^{\mathrm{bl\text{-}cmp}}$ *by*:

$$\mathcal{G}^{\mathrm{fr\text{-}cmp}}(\mathcal{P}) = \mathcal{G}^{\mathrm{fr}} \circ \mathcal{G}^{\mathrm{cmp}}(\mathcal{P}), \qquad \mathcal{G}^{\mathrm{fr\text{-}cmp}}(f) = \mathcal{G}^{\mathrm{fr}} \circ \mathcal{G}^{\mathrm{cmp}}(f),$$

then $\xi^{\mathrm{fr}} \circ \xi^{\mathrm{cmp}} : \langle \mathcal{I}, \varrho \rangle \to \overline{\mathcal{C}^{\mathrm{fr}}}$ *is a natural transformation from the identity functor on* $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Sk}}$ *to the functor* $\mathcal{G}^{\mathrm{fr\text{-}cmp}}$, $\mathcal{G}^{\mathrm{fr\text{-}cmp}}$ *is free, and communications are preserved in a way that the diagram of Fig. 7 is commutative.*

9.2. An internal view of free complete Boolean generation

Boolean sketches and the sketch structured subcategory of perceptions have enabled the definition of a free generation $\mathcal{G}^{\mathrm{fr\text{-}cmp}}$ of a complete b-perception over any given perception. This is the global, categorical framework. It provides an external characterization of

$$\mathcal{G}^{\mathrm{fr\text{-}cmp}}(\mathcal{P}) = \langle \overline{\mathcal{B}_{\mathcal{I}}^{\mathrm{fr}}}, \overline{\varrho^{\mathrm{fr}}} \rangle.$$

Having defined Boolean sketches, one can further acquire characterizations of $\overline{\mathcal{B}_{\mathcal{I}}^{\mathrm{fr}}}$ and $\overline{\varrho^{\mathrm{fr}}}$ in terms of the generating perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, rather than in the general terms of Definition 62.

We start with the characterization of the Boolean algebra of connotations for free complete Boolean generation. Given a Boolean sketch $\langle \mathcal{K}, R \rangle$, the quasi-order R (with its

$$\begin{tikzcd}
\langle \mathcal{K}, R\rangle \arrow[rr, "\xi^{\mathrm{Sk}}"] \arrow[dr, "f"', "\text{(sketch structured)}"'] & & \mathcal{B}^{\mathrm{Sk}}_{\mathcal{K}} \arrow[dl, "\exists! \psi"] \\
& \mathcal{B} &
\end{tikzcd}$$

Fig. 8. The free sketch structured Boolean algebra.

"Boolean property") provides a "sketchy" information about the structure of some Boolean algebra over $\mathcal{K}$. Sketch synonyms and subsumptions, as well as complements, disjoints, and antonyms should all be built into this Boolean algebra. By the standard category-theoretical procedure, one is looking for a Boolean algebra $\mathcal{B}^{\mathrm{Sk}}_{\mathcal{K}}$ and a sketch morphism $\xi^{\mathrm{Sk}} : \langle \mathcal{K}, R\rangle \to \mathcal{B}^{\mathrm{Sk}}_{\mathcal{K}}$.

Lemma 83 (See Fig. 8). *Let $\langle \mathcal{K}, R\rangle$ be a Boolean sketch. In $\mathcal{B}^{\mathrm{fr}}_{\mathcal{K}}$ (the free Boolean algebra over $\mathcal{K}$), let Δ_R be the ideal that is generated by the set $S = \{e_1 \wedge \neg e_2 \| e_1 R e_2\}$, let $\mathcal{B}^{\mathrm{Sk}}_{\mathcal{K}} = \mathcal{B}^{\mathrm{fr}}_{\mathcal{K}} / \Delta_R$ be the quotient Boolean algebra, and let $\xi^{\mathrm{Sk}} : \mathcal{B}^{\mathrm{fr}}_{\mathcal{K}} \to \mathcal{B}^{\mathrm{Sk}}_{\mathcal{K}}$ be the natural Boolean homomorphism from the Boolean algebra onto its quotient algebra. Then for every other Boolean algebra $\mathcal{B}$ and every sketch morphism $f : \langle \mathcal{K}, R\rangle \to \mathcal{B}$ there exists a unique Boolean homomorphism $\psi : \mathcal{B}^{\mathrm{Sk}}_{\mathcal{K}} \to \mathcal{B}$ that is a homomorphic extension of f: for all $\alpha \in \mathcal{K}$, $\xi^{\mathrm{Sk}} \circ \psi(\alpha) = f(\alpha)$. $\mathcal{B}^{\mathrm{Sk}}_{\mathcal{K}}$ will be called* the free sketch structured Boolean algebra over $\langle \mathcal{K}, R\rangle$.

Proof outline. The proof is similar to that of Lemma 66 in Section 8.2, using the freedom of $\mathcal{B}^{\mathrm{fr}}_{\mathcal{K}}$, and showing that connotations that are congruent modulo Δ_R are mapped by the sketch morphism f to the same element of $\mathcal{B}$. $\square$

Replacing the general $\langle \mathcal{K}, R\rangle$ by $\langle \mathcal{I}, \trianglelefteq_{\rho^{\mathrm{fr}}}\rangle$, one gets the free sketch structured Boolean algebra over $\langle \mathcal{I}, \trianglelefteq_{\rho^{\mathrm{fr}}}\rangle$, designated $\mathcal{B}^{\mathrm{Sk}}_{\mathcal{I}}$. On the other hand, in the general Boolean construction of Section 8.1, one may replace $\langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}, \varrho^{\mathrm{fr}}\rangle$ for $\mathcal{C} = \langle \mathcal{B}, \sigma\rangle$. In that case Δ_R above replaces Δ and one gets the following characterizations, in terms of the generating perception $\mathcal{P}$:

$$\overline{\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}} = \mathcal{B}^{\mathrm{Sk}}_{\mathcal{I}}, \quad \xi^{\mathrm{cmp}} = \xi^{\mathrm{Sk}}.$$

It remains to fathom the permits of free complete b-perceptions and their p-predicates. The categorical Definition 62 provides theoretical support but not much insight into $\overline{\varrho^{\mathrm{fr}}}$. By substituting the universal perception $\mathcal{U}_{\mathcal{E}}$ instead of the general $\mathcal{C}$ in the freedom Corollary 82 of $\mathcal{G}^{\mathrm{fr\text{-}cmp}}$, one can see that the permits of the free complete b-perception over $\mathcal{P} = \langle \mathcal{I}, \varrho\rangle$ are exactly the Boolean extensions of the sketch structured total improvements of $\mathcal{P}$. It was shown in Section 8.1 that $\mathcal{G}^{\mathrm{fr\text{-}cmp}}(\mathcal{P})$ has permits, and hence every $\mathcal{P}$ has sketch structured total improvements.

The internal view of free complete b-perception generation can thus be summarized by the following characterization. (A comparison with Definition 44 of free Boolean generation is recommended.)

Lemma 84. *Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ be a perception, then the free complete b-perception over $\mathcal{P}$, $\mathcal{G}^{\text{fr-cmp}}(\mathcal{P}) = \langle \mathcal{B}^{\text{Sk}}_{\mathcal{I}}, \varrho^{\text{cmp}} \rangle$, is such that:*

- *$\mathcal{B}^{\text{Sk}}_{\mathcal{I}}$, the set of connotations, is the free Boolean algebra generated over the Boolean sketch $\langle \mathcal{I}, \trianglelefteq_{\rho^{\text{fr}}} \rangle$.*
- *Let $\mathcal{V}^{\text{Sk}}_{\mathcal{P}}$ be the set of all sketch structured total improvements of $\mathcal{P}$, then ϱ^{cmp}, the* free complete b-p-predicate, *is defined, for all $\text{w} \in \mathcal{E}$ and for all $\beta \in \mathcal{B}^{\text{Sk}}_{\mathcal{I}}$, by:*

$$\varrho^{\text{cmp}}(\text{w}, \beta) = \begin{cases} \mathtt{t} & \text{if } \forall \widehat{\varrho} \in \mathcal{V}^{\text{Sk}}_{\mathcal{P}},\ \widehat{\varrho}^{\text{bl}}(\text{w}, \beta) = \mathtt{t}, \\ \mathtt{f} & \text{if } \forall \widehat{\varrho} \in \mathcal{V}^{\text{Sk}}_{\mathcal{P}},\ \widehat{\varrho}^{\text{bl}}(\text{w}, \beta) = \mathtt{f}, \\ \mathtt{u} & \text{otherwise}. \end{cases}$$

- *The generating morphism is $\xi^{\text{Sk}}|_{\mathcal{I}} : \langle \mathcal{I}, \varrho \rangle \hookrightarrow \langle \mathcal{B}^{\text{Sk}}_{\mathcal{I}}, \varrho^{\text{cmp}} \rangle$.*

Indeed, $\overline{\varrho^{\text{fr}}} = \varrho^{\text{cmp}}$. The characterizations introduced by Lemma 84 are in terms of the generating perception $\mathcal{P}$.

9.3. Boolean generations: evaluation and tying of ends

The artificial agent, endowed with a basic perception and willing to make a methodical cognitive transition to a b-perception, now has a choice between free Boolean generation and free complete Boolean generation. This flavor of "self-awareness" is enhanced by the categorical framework that allows a rigorous comparison between the two. Whatever the choice, it can be argued, possibly using the agent's own data. Both processes are natural transformations and both are general.

The most obvious differences between the two generations are along a tradeoff line between open-mindedness on one side and completeness on the other side: free generation is more open-minded in that it retains all communication paths (i.e., p-morphisms) with other perceptions, and in that it does not rule out any future possible improvement of its perception. For that purpose it ignores all perceptually observed patterns that it might have been able to notice. Free complete generation is less open-minded: to gain completeness of its cognitive Boolean image of the environment it stops non-structured communications with other perceptions (loosely: "those that do not agree with its conjectures that are based on perceptually observed patterns"), and it rules out some future improvements of its perception (loosely: "those that would defy its conjectures that are based on perceptually observed patterns"). The elimination of the non sketch structured p-morphisms might not be a loss after all. Actually, one gets a means of distinction between "deeper" and "shallower" communications (see Example 65). Sketch structured p-morphisms communicate between perceptions that possess, in a certain sense, similar cognitive inner images, and this communication enhances the similarity. In a case where more than one p-morphism could communicate between two perceptions, it is clear that a sketch structured one, if it exists, should be "preferred".

Another issue for comparison is combinatorial. Free Boolean generation creates a Boolean set of connotations with 2^{2^n} elements for n generating connotations. It is the largest possible Boolean closure. In particular, if the generating perception happens to be already Boolean, free Boolean generation is unable to "sense" that and it leads to a combinatorial disaster. Free complete Boolean generation, on the other hand, is sensitive both to Boolean structure and to completeness. Whatever traces of Boolean structure the generating perception has, they are built into the Boolean closure. In particular, if the generating perception is already Boolean, it will only be completed, and if it is already a complete b-perception then free complete Boolean generation is the identity. The minimization of the number of connotations is maximal: there are no distinct synonyms and hence no redundancy of connotations in free complete Boolean.

The above "smallest Boolean closure" feature of free complete Boolean generation has other effects. The transition from the totally free Boolean closure to a smaller quotient algebra means, inevitably, a many-to-one b-p-morphism. Distinct connotations in the free Boolean closure $\mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}$ are merged. This may, among other things, involve some unblurring of perception. In this context a few questions may be raised:

- The generating morphism ξ^{fr} of free Boolean generation was shown to rigidly preserve the generating perception. Is this also true of the generating morphism ξ^{Sk} of free complete Boolean generation?
- In free Boolean generation there is a clear distinction between simple and complex connotations: the generating connotations are perceived with no need of computation, while derived, connotations are Boolean combinations of simple connotations and need computation to be perceived. This distinction may be lost with free complete b-perceptions. Simple connotations might get merged with complex ones.
- How do these modifications to perceptions affect the computation of the free complete p-predicate?

The answer to these questions is best illustrated by an example:

Example 85. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ be a perception, where $\alpha, \beta, \gamma \in \mathcal{I}$ are connotations. Let $(\beta \trianglelefteq \gamma)$, and let $(\alpha \trianglelefteq_{\varrho^{\mathrm{fr}}} \beta \wedge \neg\gamma)$. In particular, it is possible that, for some $\mathrm{w} \in \mathcal{E}$, both $\varrho(\mathrm{w}, \beta) = \varrho(\mathrm{w}, \gamma) = \mathrm{u}$. In that case $\varrho(\mathrm{w}, \alpha)$ is either $\mathtt{f}$ or u, and hence also $\varrho^{\mathrm{fr}}(\mathrm{w}, \alpha)$. However, by the definitions of Section 8, not only $(\beta \wedge \neg\gamma \in \Delta)$, but also $(\alpha \wedge \neg(\beta \wedge \neg\gamma) \in \Delta)$. It follows that, for all $\mathrm{w} \in \mathcal{E}$, $\varrho^{\mathrm{cmp}}(\mathrm{w}, \beta \wedge \neg\gamma) = \mathtt{f}$, as well as $\varrho^{\mathrm{cmp}}(\mathrm{w}\,\alpha \wedge \neg(\beta \wedge \neg\gamma)) = \mathtt{f}$, and hence it must be that $\varrho^{\mathrm{cmp}}(\mathrm{w}, \alpha) = \mathtt{f}$. It is concluded that p-predicate values involving a simple connotation α might have to be unblurred, α might get merged with the bottom connotation $\perp$. This should, of course, affect computation of the b-p-predicate for derived connotations that have α as an atom. $\neg\alpha$ is, of course, merged with the top connotation $\top$ and the unblurring is from u to $\mathtt{t}$.

For a more specific example, let $\mathcal{P}$ be the "bookstore catalog" perception of Example 2. Let

$$(\beta = \textit{travelguide}), \qquad (\gamma = \textit{maps}), \qquad (\alpha = \textit{no-map-travelguide}).$$

If *travelguide* subsumes *maps*, and *no-map-travelguide* subsumes (*travelguide* $\wedge\neg$*maps*), then a complete b-perception with these subsumptions would not consider the possibility of a w-element with the connotation *no-map-travelguide*, even if the book is closed and hence its basic apparatus does not definitely rule out the option.

One might say that by freely generating a complete b-perception, the agent internalizes the Boolean sketch subsumptions to a point where it affects its basic perception.

In Section 8.1 we compared complete b-perception to the situation of an analyst who internalizes all the experimental lab results (namely perceptual observations), leans back in his armchair, closes his eyes, and figures out all the rest, eventually opening an eye to query the lab again. The unblurring of the generating perceptions by free complete Boolean generation may be compared to a situation where our analyst realizes that although its perception is undefined at a certain point, $\varrho(\mathrm{w}, \alpha) = \mathrm{u}$, for all he knows, it must be that w should have (lack) α, or else "he has it all wrong" (by Section 2.4 his conjectures are based, after all, on partial perception and nonmonotonic logic). In scientific research terminology, if $\langle \mathcal{B}^{\mathrm{Sk}}_{\mathcal{I}}, \varrho^{\mathrm{cmp}} \rangle$ is a "theory", then an experiment where the definite value of $\varrho(\mathrm{w}, \alpha)$ is tested is one possible experiment for the verification of the theory. This is one of the cases where our analyst (i.e., the higher reasoning module) might want to query the lab (i.e., the sensory-motor-neural module) for improved perception. If he gets the *unexpected* value for $\varrho(\mathrm{w}, \alpha)$ he will open the other eye, too.

These intuitive considerations can be neatly formalized by a *closure* of the sketch structured subcategory of perceptions. Definition 80 of that subcategory will now have another version with a closure condition (just like that of Definition 34 of b-perceptions):

Definition 86. $\overline{\mathcal{P}rc^{\mathrm{Sk}}_{\mathcal{E}}}$, the *closed sketch structured subcategory of perceptions*, consists of

- perceptions $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ of $\mathcal{P}rc_{\mathcal{E}}$ such that, if $\mathcal{V}^{\mathrm{Sk}}_{\mathcal{P}}$ is the set of all sketch structured total improvements of $\mathcal{P}$, then for all $\mathrm{w} \in \mathcal{E}$ and for all $\alpha \in \mathcal{I}$,

$$\varrho(\mathrm{w}, \alpha) = \begin{cases} \mathrm{t} & \text{if } \forall \widehat{\varrho} \in \mathcal{V}^{\mathrm{Sk}}_{\mathcal{P}},\ \widehat{\varrho}(\mathrm{w}, \beta) = \mathrm{t}, \\ \mathrm{f} & \text{if } \forall \widehat{\varrho} \in \mathcal{V}^{\mathrm{Sk}}_{\mathcal{P}},\ \widehat{\varrho}(\mathrm{w}, \beta) = \mathrm{f}, \\ \mathrm{u} & \text{otherwise}; \end{cases}$$

- sketch structured p-morphisms only.

Remark 87. $\mathcal{P}$ has to be unblurred for a certain pair (w, α) if, and only if, all sketch structured total improvements of $\mathcal{P}$ and hence all ϱ^{fr}monotone permits of $\mathcal{C}^{\mathrm{fr}}$, unblur the p-predicate at this point. In terms of Appendix A.3, this could happen if, and only if, either α or $\neg\alpha$ are elements of Δ_0. It follows that such an unblurring depends only on α (and not on w). This could be meaningful for algorithmic implementations.

For a known price in "open-mindedness" (some p-predicates have to be modified: the agent "closes its mind" to certain possibilities), the closed version $\overline{\mathcal{P}rc^{\mathrm{Sk}}_{\mathcal{E}}}$ of the sketch structured subcategory $\mathcal{P}rc^{\mathrm{Sk}}_{\mathcal{E}}$ settles that:

- For all $\mathcal{P} \in \overline{\mathcal{P}rc^{\mathrm{Sk}}_{\mathcal{E}}}$, the p-morphism $\xi^{\mathrm{Sk}} : \mathcal{P} \to \mathcal{G}^{\text{fr-cmp}}(\mathcal{P})$ is rigid.
- The computation of ϱ^{cmp} using truth tables similar to those of Section 6.4 still "bottoms-out" at the generating perception level. (Remark 87 should be useful for algorithmic implementations.)

Free complete b-perceptions are indeed more complex than free Boolean perceptions. One may compute the free complete b-p-predicate ϱ^{cmp}, using truth tables similar to those of Section 6.4. Detection of Boolean dependencies in $\mathcal{B}^{\mathrm{Sk}}_{\mathcal{I}}$ (the lower right entries of the disjunction and conjunction truth tables of Lemma 42) will be rewarded, as always, by more definite values. Only now there are more such dependencies (compared to free

Boolean generation), so that there is both more to look for, as well as more to be gained. This should not be surprising: as more efforts are being invested in the agent's internal cognitive model, the agent should be expected to come up with more answers to more complex questions.

As some distinctions between simple and complex connotations fade away, it may be that, for some generating connotation $\alpha \in \mathcal{I}$, and for some non-atomic expression $e \in \mathcal{K}^{BE}$, α and e are ϱ^{fr}-synonyms, and hence they are merged in $\langle \mathcal{B}_{\mathcal{I}}^{\mathrm{Sk}}, \varrho^{\mathrm{cmp}} \rangle$. In the case where a very complex connotation is merged with a simple one, later computational efforts could be reduced. Consider, for instance, a case where w is a pretzel, *pretzelshape* is a simple connotation, holistically recognized by the sensory-motor-neural apparatus, and β is a complex formal description of a pretzel shape. $\varrho(\mathrm{w}, \mathit{pretzelshape}) = \mathtt{t}$ should be immediate. $\varrho^{\mathrm{fr}}(\mathrm{w}, \beta)$ necessitates computation, but $\varrho^{\mathrm{cmp}}(\mathrm{w}, \beta)$ is immediate because a complete b-perception means that the derived formal description of a pretzel shape has been internalized: $\beta = \mathit{pretzelshape}$ holds in the Boolean algebra of connotations $\mathcal{B}_{\mathcal{I}}^{\mathrm{Sk}}$.

9.4. Summary of Section 9

Free complete Boolean generation provides a rigorous mathematical description of a direct methodical cognitive transition from basic perceptions to a valid and complete inner image of the environment. In addition to the features of the more general free Boolean generation, an agent that performs this process may claim and show for fact that its own perceptual observations contribute enough interesting material on top of the general Boolean speculations, so that it can completely rely on the Boolean algebra of connotations for all it knows. The mathematical framework allows for a detailed comparison between the more general free Boolean generation and this generation.

Artificial perceptions observe and, consequently, create an internal image of their environment. An agent with a free b-perception will never jump to a conclusion, on the other hand, its perception is very general. It is not going to feature neither mistakes, nor novel observations. It might be somewhat clumsy due to the huge size of its set of connotations. An agent with a free complete b-perception has invested lots of effort in its cognitive image of the environment and it is expected to come up with some interesting, novel observations. On the other hand, some of them may be "far-fetched", because the agent jumped to conclusions. Such an agent may also feature some welcome "shortcuts" in its cognitive perception.

10. Methodological fallout and some AI perspectives

10.1. Intermediate Boolean generations and embodied perception

The two canonical Boolean generations represent two extremes. A relation R, where $\leqslant_{\mathrm{bl-fr}} \subset R \subset \trianglelefteq_{\varrho^{\mathrm{fr}}}$, could determine another Boolean sketch and ideal ($\perp \subset \triangle^R \subset \triangle$, where $\triangle$ is as in Section 8.1). For free Boolean generation it is the case that $R = \leqslant_{\mathrm{bl-fr}}$, and for free complete Boolean generation $R = \trianglelefteq_{\varrho^{\mathrm{fr}}}$. In a typical case an agent's perception is, probably, somewhere between a free b-perception and a free complete b-perception. In a typical situation the agent has computed and internalized only some of the subsumptions between its connotations. Boolean sketch subsumptions may get arbitrarily complex and hard to handle. For humans it may sometimes take a lifetime of intense contemplation and expertise to internalize all perceptually observed patterns, and their logical consequences, even in a restricted, specialized, professional environment $\mathcal{E}$. Besides complexity, there may be other reasons for the preference of a specific, "intermediate" R. A subset of subsumptions $\{\alpha \trianglelefteq \beta\}$ may be, for instance, supported by more positive definite values of the p-predicate (i.e., more w in $\mathcal{E}$ such that $\varrho(\mathrm{w}, \alpha) = \mathrm{t}$ and also $\varrho(\mathrm{w}, \beta) = \mathrm{t}$).

Out of the entire collection of intermediate Boolean generations, one merits special attention. *Free monotone Boolean generation* is defined for $R = \leqslant_{\mathrm{bl-fr}} \cup \trianglelefteq$. In that case the Boolean set of connotations is the free Boolean algebra that is generated over $\mathcal{I}$ with the quasi order $\trianglelefteq$. (Alternatively: the free Boolean algebra that is generated over $\mathcal{I}^*$ with the partial order $\trianglelefteq$.) The set of permits for that b-perception consists exactly of the Boolean extensions of monotone total improvements of the generating perception, and this generation is free over the monotone subcategory $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Mon}}$.

Free monotone generation merits special attention because of the special role that generating connotations play in perception. From the Boolean theoretic point of view there are many alternative subsets of connotations in the Boolean closure that could serve as generators. Some of them might even seem easier to work with then others. Assume, for example, a set of *free* generators for $\mathcal{B}_{\mathcal{I}}^{\mathrm{Sk}}$, the sketch structured Boolean algebra of connotations. If one started from such a set of connotations as the generating set, then free generation would be the same as free complete generation. However, we let the agent start from "its own" set of generators $\mathcal{I}$, the set of simple connotations. This set is assumed to be perception specific, so that the categorical treatment must provide for an arbitrary set of generating connotations. This approach is based on the background assumption that perception is subjective and embodied. Generating connotations are assumed to be innate to the agent's architecture, hard-wired in its sensory-motor-neural apparatus. They have an integrity of their own, serving as the most immediate and natural means of relating to its environment. Starting from the set of free generators to $\mathcal{B}_{\mathcal{I}}^{\mathrm{Sk}}$, for example, would have meant that all the sketch subsumptions are innate to the agent's own architecture and sensory-motor-neural apparatus. This is not always a reasonable assumption. Considering that different environments feature different patterns, this perception will not be able to adapt itself easily to some new environments.

The analog in human perception are *basic level categories* (sometimes also called *natural properties* or *natural kinds*) extensively elaborated in [38]. They were isolated by empirical studies as a significant level of human interaction with the external environment. They are the easiest to learn, remember, and use. They are characterized, among other things, by fast identification, single mental images, shortest lexemes, and overall perceived shape (gestalt perception). It is at this level that humans easily distinguish tigers from elephants. (One level down things are more difficult. It is harder to distinguish one species of giraffe from another.) Analogous to the ease of cognitive processing of basic level human categories, simple connotations are assumed to be readily recognized by the sensory-motor-neural apparatus of an artificial agent, without further procedure. The single mental image and shortest lexemes ideas are analogous to the fact that α is an atomic connotation expression. This gestalt perception is assumed to be embodied in the architecture of the agent. Perception of derived connotations in the Boolean closure is achieved with due recourse to the deductive apparatus. This calls for an algorithmic treatment which might involve access and retrieval procedures: a resource consuming cognitive effort.

It is reasonable to let the agent's internal imagery be organized at the natural level of its generating connotations. This is why the categorical treatment assumes a perception specific set of connotational generators. There is nothing in generating connotations that gives them an objective status external to the agent. Reasoning and making inferences using Boolean perceptions may be figuratively described as moving along the sloping lines of the lattice graph. In that case the generating connotations are like glittering signposts that facilitate navigation. They are a form of representation of the *embodiment* of perception.

In free monotone Boolean generation only subsumptions and synonyms between generating connotations are internalized. It should be expected that these are observed first, like an easy path between two familiar signposts. Boolean sketch subsumptions, on the other hand, necessitate complex Boolean connotational expressions. Ockham's razor is also in favor of simpler patterns.

Another support for the preference of subsumptions and synonyms between simple connotations also comes from arguments about inductive inference and the problem of *projectibility*, explained in [25]. Inferences between derived Boolean connotational expressions are not only harder to arrive at, they often seem far-fetched, counterintuitive, *nonprojectible*.

10.2. Constraints and imagination in Boolean perception

Subsumptions that are internalized by b-perceptions are actually *constraints*. Free Boolean generation features no constraints (except the obvious Boolean ones). Free complete Boolean generation could be regarded as the result of the cognitive internalization of constraints. The essence of generating the quotient Boolean algebra over an ideal is that the connotations of the ideal are "beamed down" to the bottom $\perp$ of the quotient algebra, they are perceived as impossible, and hence *negative constraints*. Dually, the negations of these connotations, which constitute the respective dual filter, are "beamed up" to the top $\top$, and perceived as *positive constraints*.

The generation of complete b-perceptions, with its restriction of the set of possible total improvements, may have a conservative flavor, yet one can show that it is still capable of some imagination and abstraction. There may be connotations that do not have a definite positive example but they do not generate negative constraints. Dually, their negations do not have a definite negative example, yet they do not generate positive constraints. Figuratively, free complete Boolean generation can *conceive* a situation (formally: has a permit) where a w-element has (lacks) a connotation that has no current definite positive (negative) example.

To show the above, one needs to find generating connotations α,β such that:

- There exists some $\mathrm{w}' \in \mathcal{E}$ where either $\varrho(\mathrm{w}',\alpha) = \mathtt{t}$ and $\varrho(\mathrm{w}',\beta) = \mathtt{u}$, or $\varrho(\mathrm{w}',\alpha) = \mathtt{u}$ and $\varrho(\mathrm{w}',\beta) = \mathtt{f}$. In that case the subsumption $\alpha \trianglelefteq \beta$ does not hold, neither do $\alpha \trianglelefteq_{\rho^{\mathrm{fr}}} \beta$ and $\alpha \trianglelefteq_{\rho^{\mathrm{cmp}}} \beta$. However:

- There exists no $w \in \mathcal{E}$ such that $\varrho(w, \alpha) = t$ and $\varrho(w, \beta) = f$. In that case it is possible for α to subsume β in some future improvement of that perception, there is no counterexample.

In that case:

- $\varrho^{cmp}(w', \alpha \wedge \neg\beta) = u$, so that the connotation $\alpha \wedge \neg\beta$ is not a negative constraint: $\alpha \wedge \neg\beta \neq \bot$.
- For no $w \in \mathcal{E}$ does the connotation $\alpha \wedge \neg\beta$ hold. $\forall w \in \mathcal{E}$ $\varrho^{fr}(w, \alpha \wedge \neg\beta) \neq t$.

Connotations like $\alpha \wedge \neg\beta$ above show that free complete Boolean generation is still capable of *imagination*. Such connotations have no definite positive example, yet they do not generate negative constraints. Dually, their negations are connotations with no definite negative example, yet they do not generate positive constraints.

Example 88. In our bookstore environment, let (α = *children*) and (β = *bigprint*), then no w-element is definitely both *children* and $\neg$*bigprint*, but such a book is conceivable.

In Section 5.2 a similarity was shown between Lukasiewicz's three-valued conditionals and our definition of subsumptions (and synonyms): $\alpha \trianglelefteq \beta$ if and only if for all $w \in \mathcal{E}$ $\varrho(w, \alpha) \to \varrho(w, \beta)$ is t by Lukasiewicz's three-valued conditional. One case, $t \to f$, where this does not hold yields an f value, while the other two cases ($t \to i$ and $i \to f$) yield an indefinite value (i). It turns out that this is exactly the distinction that was made above. The second case that yields an f value was eliminated while the cases that yield an indefinite value were the ones that demonstrate imagination. These are, actually, examples of *(con)notations without denotations*, showing that free complete generation leaves room for *abstraction*: an inner representation that goes beyond things which are actually perceived. The distinction between conceivable and inconceivable is the existence of a suitable permit. Of course, if a perception is totally two-valued, then it is its one and only permit: total perceptions leave no room for imagination.

10.3. Perception morphisms revisited

Perceptions vary across agents, modules, time, situations, goals, interests, etc. The main tool of comparison and transition between them are p-morphisms. P-morphisms provide a versatile tool that is able to capture a variety of cognitive processes:

- Perhaps the most obvious use of a p-morphism is to "translate" between different perceptions of the same environment $\mathcal{E}$, as shown in our "bookstore" examples. In this case the nature of the mapping (set-isomorphism, one-to-one, onto, Boolean, impossible etc.) carries meticulous information about how close these perceptions are. The extent of the modification that is introduced by a p-morphism is proportional to the extent of the change that has caused it. In an AI environment these p-morphisms could be used to:
  - Communicate and compare between distinct agents.
  - Communicate and compare between different modules of the same agent. Different modules may use different representations for their different purposes. If the reasoning module, for instance, was planned independently of the sensory-motor-neural module, it may well be that the former assumes a perception that is different

from the agent's own generation. A suitable p-morphism would have to be used to bridge between them. The nature of the mapping, if at all possible, carries meticulous information about how well they fit.

  - The essence of some learning, discovery and other creative cognitive processes is in finding the most suitable representation (i.e., set of connotations) for a given environment and goals. The shift in perception, if possible, should be best formalized by a p-morphism.

- The current study was mostly dedicated to the construction of p-morphisms that capture high-level representation formation: starting from a basic perception, organize and shape a structured representation that can be further used for high-level cognitive processes. The study further provided tools to determine where and when certain constructions might cost a communication.
- A lifelong autonomous agent is naturally expected to constantly improve its perception and learn more about its environment. This improving change of perception within the same set of connotations is also easily captured by the family of "improving" p-morphisms.
- One of the aspects of cognitive behavior is the ability to preserve an individual perception within a society of other perceptions. In Example 17 it was shown how categorical notions provide us with convenient tools of scrutiny to formalize several forms of joint perceptions.

10.4. Learning and knowledge acquisition

The prime idea in the category of perceptions is that the agent could organize its internal representation relying only on its perception. There might be cases, however, where "outside advice" could save time or other computational resources. The question is exactly how and where such outside advice could be used in our context. For a given perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ there are two families of "facts":

- *Primary facts.* These are essentially the values of the p-predicate for all w-elements and generating connotations.
- *Derived facts*, which consist of material that is derived from the primary facts:
  - Synonyms, subsumptions, and Boolean sketch structure. These can be attained by forms of inductive learning.
  - Values of the p-predicate for derived connotations, which may be computed from the primary p-predicate using the deductive apparatus from Section 6.4.

Technically speaking, any subset of facts could be directly entered into the agent's memory.

The insertion of *derived facts* typically saves the time of the agent's own processes. The agent could, theoretically at least, reach that fact all by itself. This kind of "learning by being told" has as human analog one's communication with the heritage of its culture, its community's accumulative knowledge. Nobody after Newton is expected to rediscover the law of gravity all by himself, one is simply being taught about it. One thus starts with all available knowledge explicitly at hand, and can use its resources to acquire some genuine new knowledge. Once a fact is internalized, there is no obvious way of telling how it got there. There seem to be no methodological problem with the agent acquiring any subset of these facts "by being told". However, there is the usual word of warning: in view of the subjectivity of perceptions, entering any fact concerning w-elements and connotations presupposes that the interpretation of these entities is shared by the "learner" and the "teacher". If this is not the case, an appropriate p-morphism should be used. In [19] it is explained how communication is the exchange of representations, while meanings are created within the individual. Even with a p-morphism, one has to *trust* that the two sides share meanings.

The direct insertion of *primary facts* is more problematic. "Telling" an agent the value of $\varrho(\mathrm{w}, \alpha)$ is bypassing perception. As opposed to the insertion of a derived fact, one could not always claim that the agent could, not even theoretically, reach that fact all by itself. If the agent has the definite perception of the fact $\varrho(\mathrm{w}, \alpha)$, then it does not need "to be told" about it. If the agent's perception has $\varrho(\mathrm{w}, \alpha) = \mathrm{u}$, it could perhaps sometimes be claimed that the insertion of a definite value saves efforts from the agent's sensory-motor-neural apparatus: the agent could, for example, have checked inside the book for the edition number, but to save efforts this information could be inserted. Not all u values of the p-predicate, however, are of this nature.

If a perception does not include either w or α, then $\varrho(\mathrm{w}, \alpha)$ does not mean much. Faking "perception" this way brings us into Searle's Chinese room [54]. The facts could be entered into memory and retrieved at any time, but they are not grounded by perception in the environment.

11. Conclusion

It is generally accepted that true understanding can only be gained by actually experiencing the world and thereby developing an internal representation of it. Parallel to experiencing its environment, an intelligent artificial perception should also need to "contemplate" about its experience. It would thus infer and internalize facts that are needed for a valid (and possibly complete) internal representation.

Almost forty years ago, Bar-Hillel was the first to point out the world modeling process that should go on in the mind of agents. In his case it was supposed to guide understanding of natural language. Bar-Hillel wrote:

> A translation machine should not only be supplied with a dictionary but also with a universal encyclopedia. This is surely utterly chimerical and hardly deserves any further discussion .... We know ... facts by inferences which we are able to perform ... instantaneously, and it is clear that they are not, in any serious sense, stored in our memory. Though one could envisage that a machine would be capable of performing the same inferences, there exists so far no serious proposal for a scheme that would make a machine perform such inferences in the same or similar circumstances under which an intelligent human being would perform them. [5, pp. 160–161]

The theory of artificial perceptions suggests that, theoretically, such *encyclopaedias* could perhaps be schemed and organized, individually for every perception and environment, as Boolean perceptions generated over basic sensory-motor-neural perceptions. Boolean algebras are, in a certain sense, bimodal: they have the (somewhat tedious) aspect of symbolic processing using the Boolean connectives, and, on the other hand, the non-symbolic iconographic aspect of their lattice graphs with sloping lines (offering eventual shortcuts like "ladders and ropes"). Facts and inferences could be internalized by the lattice structures themselves, rather than conventionally stored in memories. Perceptive-cognitive and reasoning processes would then perform by sliding along gratings of Boolean trellises. Environments should train upon these trellises with tendrils provided by the generating perceptions and clinging at the nodes of simple connotations.

The theory of artificial perceptions also provides general, categorical, tools for creating particular *encyclopaedias*, as well as foundations for a general account of these structures and of relationships between them.

12. Future research

This is a theoretical study that proposes a foundational mathematical "unified standard" for AI artifacts with perceptions, for their cognitive behavior, and for dealing with them. Within the theoretical framework the ongoing study is concerned with the study of cognitive processes that involve more then a single environment at a time.

Although a few implementational considerations were touched in the context of the computation of Boolean perception predicates, the theory still calls for more research on complexity and implementational issues.

The categorical approach is inherently "top–down". Given the theory, future research could now take a "bottom-up" approach. One may test the practical applicability of the theory by:

- designing architectures for AI artifacts in terms of the proposed theory;
- analyzing existing artifacts in terms of the proposed theory.

One may start with the construction or analysis of basic artificial perceptions with simple environments: w-elements, connotations, and the perception predicate. One may proceed to design uniform ducts between these perceptions using perception morphisms, then further categorical and Boolean constructs that capture the various cognitive processes, training on these basic perceptions.

Applying the terminology of this study to the environment of AI research itself, future research concerning particular perceptive-cognitive AI artifacts is invited to conceive, design, and analyze its own w-elements (namely these artifacts) with connotations that are the primitives of this theory of artificial perceptions: w-element, connotation, perception predicate, the categorical primitives (morphism, domain, codomain, composition) and the Boolean primitives. This relatively small number of primitives predicts the possibility of tidily structured implementations with a reduced component set, where components are reusable across a broad spectrum of cognitive activity.

Appendix A. Technicalities are a necessary evil

A rigorous mathematical theory warrants tidiness and neat formulations. The price is, sometimes, a tedious proof of various technicalities. Once they are taken care of, one could usually forget about them (one has, however, gained insights into the structure). The material in this appendix would have interfered with the flow of the presentation, so it was gathered here. No new premises or concepts are introduced in this appendix.

A.1. Technicalities for Boolean perceptions

For Section 6, the following considerations are needed:

- Given a perception $\langle \mathcal{B}, \sigma \rangle$ every $\mathrm{w} \in \mathcal{E}$ naturally defines a three-valued mapping: $\sigma_{\mathrm{w}} : \mathcal{B} \to \{\mathtt{t}, \mathtt{f}, \mathtt{u}\} : \beta \mapsto \sigma(\mathrm{w}, \beta)$. For the current purpose it is convenient to regard σ_{w} as a *partial* two-valued mapping, such that σ_{w} is undefined for β if and only if $\sigma(\mathrm{w}, \beta) = \mathtt{u}$. Whenever the perception $\langle \mathcal{B}, \sigma \rangle$ is total, then, for all $\mathrm{w} \in \mathcal{E}$, σ_{w} is a total two-valued mapping.
- $\{\mathtt{t}, \mathtt{f}\}$ could be regarded as the two-element Boolean algebra which consists of a "top" $\mathtt{t}$ and a "bottom" $\mathtt{f}$ only. In that case, a total $\sigma_{\mathrm{w}} : \mathcal{B} \to \{\mathtt{t}, \mathtt{f}\}$ is either a two-valued Boolean homomorphism or it is not. (A *predicate* defined on a Boolean algebra is traditionally expected to be such a two-valued homomorphism.) Similarly, a partial $\sigma_{\mathrm{w}} : \mathcal{B} \to \{\mathtt{t}, \mathtt{f}\}$ could either be extended to a two-valued Boolean homomorphism, or it could not.

A few known Boolean algebraic results are needed as well. The first one (see, for example, [55]) is due to a natural bijective correspondence between maximal ideals, maximal filters, and two-valued homomorphisms.

> A mapping $h : \mathcal{B} \to \{\mathtt{t}, \mathtt{f}\}$ is a two-valued homomorphism if and only if the set $\nabla = \{\beta \in \mathcal{B} \| h(\beta) = t\}$ is a maximal filter, and the set $\Delta = \{\beta \in \mathcal{B} \| h(\beta) = f\}$ is then the dual maximal ideal.

Extensions to two-valued homomorphisms, together with the implied partition into a maximal ideal and its dual maximal filter, are going to serve as an alternative touchstone for b-perceptions. Based on the definitions of σ_{w} above, and of η, the natural morphism of Example 12, it is easy to show that:

Lemma A.1. *Let $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ be a total perception where $\mathcal{B}$ is a Boolean algebra. Then the natural p-morphism (of Example 12)*

$$\eta : \mathcal{C} \to \mathcal{U}_{\mathcal{E}} : \beta \mapsto \{\mathrm{w} \in \mathcal{E} \| \sigma(\mathrm{w}, \beta) = \mathtt{t}\}$$

is Boolean if and only if, for every $\mathrm{w} \in \mathcal{E}$, $\sigma_{\mathrm{w}} : \mathcal{B} \to \{\mathtt{t}, \mathtt{f}\}$ is a two-valued homomorphism.

Since a permit defines a total perception which answers the conditions of Lemma A.1, then, using it and the known Boolean results that were quoted before one gets:

Corollary A.2. *The following three conditions are equivalent:*

(i) *$\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ is a b-perception.*

(ii) *For every $\mathrm{w} \in \mathcal{E}$ the two-valued partial mapping σ_{w} could be extended to a two-valued homomorphism.*

(iii) *For every $\mathrm{w} \in \mathcal{E}$, $\mathcal{B}$ can be divided into a maximal filter ∇_{w} and a dual maximal ideal Δ_{w} such that, for all β in $\mathcal{B}$, $\sigma(\mathrm{w}, \beta) = t \Rightarrow \beta \in \nabla_{\mathrm{w}}$ and $\sigma(\mathrm{w}, \beta) = f \Rightarrow \beta \in \Delta_{\mathrm{w}}$. In that case a permit $\widehat{\sigma}$ of $\mathcal{C}$ is defined, for all $\mathrm{w} \in \mathcal{E}$ and for all $\beta \in \mathcal{B}$, in the following way: if $\beta \in \nabla_{\mathrm{w}}$ then $\widehat{\sigma}(\mathrm{w}, \beta) = \mathtt{t}$ and if $\beta \in \Delta_{\mathrm{w}}$ then $\widehat{\sigma}(\mathrm{w}, \beta) = \mathtt{f}$.*

We shall designate by $\Delta_{\mathrm{w}}^{\widehat{\sigma}}$ the maximal ideal associated with w and the permit $\widehat{\sigma}$, and its dual maximal filter by $\nabla_{\mathrm{w}}^{\widehat{\sigma}}$.

In a certain sense, Corollary A.2 is the point where the present study meets the classical theory that admits only total descriptions and two truth values. Loosely: b-perceptions are neither total nor two-valued, but they have the potential of evolving into two-valued perceptions.

The necessary and sufficient Boolean characterization of Corollary A.2 provides us with further tools to understand b-p-predicates. Consider $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$, and the set $\mathcal{V}$ of all its permits. For all $\mathrm{w} \in \mathcal{E}$ consider the following intersections:

$$\nabla_{\mathrm{w}} = \bigcap_{\widehat{\sigma} \in \mathcal{V}} \nabla_{\mathrm{w}}^{\widehat{\sigma}}, \quad \Delta_{\mathrm{w}} = \bigcap_{\widehat{\sigma} \in \mathcal{V}} \Delta_{\mathrm{w}}^{\widehat{\sigma}}.$$

As an intersection of maximal filters, ∇_{w} is a proper filter. Similarly, Δ_{w} is a proper ideal since it is an intersection of maximal ideals. Moreover, they are dual one to the other. Furthermore, by the last item of Corollary A.2 and by the second condition of Definition 34, one gets:

Lemma A.3. *For all* w *in* $\mathcal{E}$:

$$\nabla_{\mathrm{w}} = \{\beta \in \mathcal{B} \| \sigma(\mathrm{w}, \beta) = \mathtt{t}\} \quad \textit{and} \quad \Delta_{\mathrm{w}} = \{\beta \in \mathcal{B} \| \sigma(\mathrm{w}, \beta) = \mathtt{f}\}.$$

Definition 34 provided an external, categorical, definition of b-perceptions. Lemma A.3 enables us to formulate a necessary and sufficient condition for the recognition of b-perceptions "from inside":

We now prove:

Lemma 41. *A perception $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ such that the set $\mathcal{B}$ of connotations is a Boolean algebra is a b-perception if and only if for all* $\mathrm{w} \in \mathcal{E}$:

(i) *The set $\{\beta \in \mathcal{B} \| \sigma(\mathrm{w}, \beta) = \mathtt{t}\}$ is a filter.*
(ii) *The set $\{\beta \in \mathcal{B} \| \sigma(\mathrm{w}, \beta) = \mathtt{f}\}$ is an ideal.*
(iii) *The above filter and ideal are dual one to the other: for all β in $\mathcal{B}$, $\sigma(\mathrm{w}, \beta) = \mathtt{t}$ if and only if $\sigma(\mathrm{w}, \neg\beta) = \mathtt{f}$.*

Proof. The condition is necessary by Lemma A.3. It is sufficient by the following Boolean results which are part of Stone's fundamental representation theorem of [56]:

- For every proper ideal Δ (proper filter ∇) there exists a maximal ideal (maximal filter) containing Δ (containing ∇).
- For every proper ideal Δ (proper filter ∇) there exists a two-valued homomorphism h such that $h(\beta) = \mathtt{f}$ for all $\beta \in \Delta$ ($h(\beta) = \mathtt{t}$ for all $\beta \in \nabla$). □

Algorithm 1. One may generate a permit for a given b-perception, using a method that is based on the general Boolean algebraic construction of the maximal ideal (filter) which is used for a proof of Stone's representation theorem (see, for example, [55]). The essence of that construction is that, for every $\mathrm{w} \in \mathcal{E}$, and every ordering $\{\beta_i\}_{i=1,n}$ on the connotations, one builds an increasing sequence $\{\Delta_i\}_{i=0,n}$ of proper ideals, where $\Delta_0 = \Delta_{\mathrm{w}}$ and Δ_n is a maximal ideal containing Δ_{w}. Every ideal Δ_i represents a Boolean improvement σ_i of σ, where, for all $\beta \in \Delta_i$, $\sigma_i(\mathrm{w}, \beta) = \mathtt{f}$ and $\sigma_i(\mathrm{w}, \neg\beta) = \mathtt{t}$, otherwise $\sigma_i(\mathrm{w}, \beta) = \mathtt{u}$. σ_n is thus a permit of $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$. Proof is by induction as for the cited construction.

We now prove:

Lemma 42. *The truth tables for the b-p-predicate σ are given by Tables* 1–3.

Proof outline. Lemma 41 provides proofs to all cases where one is able to come up with a definite ($\mathtt{t}$ or $\mathtt{f}$) value for σ. To prove the remaining cases, where the value of σ should be $\mathtt{u}$, one needs to go back to the original Definition 34. An example is the following proof that if $\sigma(\mathrm{w},\alpha)=\sigma(\mathrm{w},\beta)=\mathtt{u}$, then $\sigma(\mathrm{w},\alpha\vee\beta)=\mathtt{u}$ (unless $\neg\alpha\leqslant\beta$, in which case it is $\mathtt{t}$):

> Since Δ_{w} is an ideal, it cannot be that $\sigma(\mathrm{w},\alpha)=\sigma(\mathrm{w},\beta)=\mathtt{u}$ and $\sigma(\mathrm{w},\alpha\vee\beta)=\mathtt{f}$. The value of $\sigma(\mathrm{w},\alpha\vee\beta)$ is thus either $\mathtt{t}$ or $\mathtt{u}$. We show that it is $\mathtt{t}$ if and only if $\neg\alpha\leqslant\beta$. The "if" part holds since, in that case, $\alpha\vee\beta=\top$ (see Corollary 36). To show the "only if" part, let $\sigma(\mathrm{w},\alpha)=\sigma(\mathrm{w},\beta)=\mathtt{u}$, and assume negatively that $\sigma(\mathrm{w},\alpha\vee\beta)=\mathtt{t}$. Since σ is a b-p-predicate it follows that for all permits $\widehat{\sigma}(\mathrm{w},\alpha\vee\beta)=\mathtt{t}$. Recall the maximal ideal $\Delta_{\mathrm{w}}^{\widehat{\sigma}}$ and the maximal filter $\nabla_{\mathrm{w}}^{\widehat{\sigma}}$ of Corollary A.2. By their maximality, whenever β is an element of $\Delta_{\mathrm{w}}^{\widehat{\sigma}}$, then α must be an element of $\nabla_{\mathrm{w}}^{\widehat{\sigma}}$, and hence $\neg\alpha$ must be an element of $\Delta_{\mathrm{w}}^{\widehat{\sigma}}$. This necessarily implies (by $\Delta_{\mathrm{w}}^{\widehat{\sigma}}$ being an ideal) that $\neg\alpha\leqslant\beta$.

Argumentations of a similar nature can be used to prove that:
- Let $\sigma(\mathrm{w},\alpha)=\sigma(\mathrm{w},\beta)=\mathtt{u}$, then $\sigma(\mathrm{w},\alpha\wedge\beta)=\mathtt{u}$ (unless $\alpha\leqslant\neg\beta$, in which case it is $\mathtt{f}$).
- Let $\sigma(\mathrm{w},\alpha)=\mathtt{f}$ and $\sigma(\mathrm{w},\beta)=\mathtt{u}$, then $\sigma(\mathrm{w},\alpha\vee\beta)=\mathtt{u}$.
- Let $\sigma(\mathrm{w},\alpha)=\mathtt{t}$ and $\sigma(\mathrm{w},\beta)=\mathtt{u}$, then $\sigma(\mathrm{w},\alpha\wedge\beta)=\mathtt{u}$. □

A.2. Technicalities for free generation of Boolean perceptions

For Section 7 we provide the proof of Lemma 47 that free Boolean generation preserves morphisms:

Lemma 47. *Let $\mathcal{P}=\langle\mathcal{I},\varrho\rangle$ and $\mathcal{Q}=\langle\mathcal{J},\tau\rangle$, and let $f:\mathcal{P}\to\mathcal{Q}$ be a p-morphism. Let $\mathcal{G}^{\mathrm{fr}}(\mathcal{P})=\langle\mathcal{B}_{\mathcal{I}}^{\mathrm{fr}},\varrho^{\mathrm{fr}}\rangle$, and $\mathcal{G}^{\mathrm{fr}}(\mathcal{Q})=\langle\mathcal{B}_{\mathcal{J}}^{\mathrm{fr}},\tau^{\mathrm{fr}}\rangle$. Let $h:\mathcal{B}_{\mathcal{I}}^{\mathrm{fr}}\to\mathcal{B}_{\mathcal{J}}^{\mathrm{fr}}$ be the unique extension of the mapping*

$$\mathcal{I}\to\mathcal{B}_{\mathcal{J}}^{\mathrm{fr}}:\alpha\mapsto f(\alpha)$$

into a Boolean homomorphism such that $\forall\alpha\in\mathcal{I}\ h(\alpha)=f(\alpha)$. Then the mapping

$$\mathcal{G}^{\mathrm{fr}}(f):\mathcal{G}^{\mathrm{fr}}(\mathcal{P})\to\mathcal{G}^{\mathrm{fr}}(\mathcal{Q}),$$

defined by: $\beta\mapsto h(\beta)$, is a b-p-morphism such that the diagram of Fig. 2 is commutative.

Proof. For the proof we shall consider the "inverse pyramid" diagram of Fig. A.1. The diagram of Fig. 2 is its "top cover". It is first observed, using no-blur of f, that any total improvement $\widehat{\tau}$ of $\mathcal{Q}$ implies a total improvement $\widehat{\varrho}$ of $\mathcal{P}$: $\forall\alpha\in\mathcal{I}\ \widehat{\varrho}(\mathrm{w},\alpha)=\widehat{\tau}(\mathrm{w},f(\alpha))$. These total improvements can be uniquely extended, respectively, into a permit $\widehat{\tau}^{\mathrm{bl}}$ of $\mathcal{G}^{\mathrm{fr}}(\mathcal{Q})$ and a permit $\widehat{\varrho}^{\mathrm{bl}}$ of $\mathcal{G}^{\mathrm{fr}}(\mathcal{P})$ (Lemma 43). Moreover, since these permits agree on the

$$\begin{tikzcd}
 & \mathcal{P} = \langle \mathcal{I}, \varrho \rangle \arrow[rr, "f"] \arrow[dl, "\xi^{\mathrm{fr}}"'] \arrow[drrr, "f"] \arrow[dr, "\widehat{\varrho}"] & & \mathcal{Q} = \langle \mathcal{J}, \tau \rangle \arrow[dr, "\xi^{\mathrm{fr}}"] \arrow[dl, "\widehat{\tau}"'] & \\
\mathcal{G}^{\mathrm{fr}}(\mathcal{P}) = \langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{I}}, \varrho^{\mathrm{fr}} \rangle \arrow[rrrr, "h"] \arrow[drr, "\widehat{\varrho}^{\mathrm{bl}}"'] & & \mathcal{U}_{\mathcal{E}} & & \mathcal{G}^{\mathrm{fr}}(\mathcal{Q}) = \langle \mathcal{B}^{\mathrm{fr}}_{\mathcal{J}}, \tau^{\mathrm{fr}} \rangle \arrow[dll, "\widehat{\tau}^{\mathrm{bl}}"] \\
 & & & &
\end{tikzcd}$$

Fig. A.1. "Inverse pyramid" diagram.

set of generators $\mathcal{I}$, then also, for all β in $\mathcal{B}$, $\widehat{\varrho}^{\mathrm{bl}}(\mathrm{w}, \beta) = \widehat{\tau}^{\mathrm{bl}}(\mathrm{w}, h(\beta))$. It follows that the permits of $\mathcal{G}^{\mathrm{fr}}(\mathcal{Q})$ define a subset of the permits of $\mathcal{G}^{\mathrm{fr}}(\mathcal{P})$. Assume now that $\varrho^{\mathrm{fr}}(\mathrm{w}, \beta) \neq \mathrm{u}$. By definition of ϱ^{fr}, for any permit $\widehat{\varrho}^{\mathrm{bl}}$ of $\mathcal{G}^{\mathrm{fr}}(\mathcal{P})$, $\widehat{\varrho}^{\mathrm{bl}}(\mathrm{w}, \beta) = \varrho^{\mathrm{fr}}(\mathrm{w}, \beta) \neq \mathrm{u}$. This holds, in particular, for the permits that are defined, as above, by the permits of $\mathcal{G}^{\mathrm{fr}}(\mathcal{Q})$. It follows that for any permit $\widehat{\tau}^{\mathrm{bl}}$, $\widehat{\tau}^{\mathrm{bl}}(\mathrm{w}, h(\beta)) = \varrho^{\mathrm{fr}}(\mathrm{w}, \beta) \neq \mathrm{u}$, and this holds for τ^{fr} as well: $\tau^{\mathrm{fr}}(\mathrm{w}, h(\beta)) = \varrho^{\mathrm{fr}}(\mathrm{w}, \beta) \neq \mathrm{u}$, so that $\mathcal{G}^{\mathrm{fr}}(f)$ is no-blur and hence a unique, b-p-morphism. It is hence easy to see that the "inverse pyramid" diagram of Fig. A.1 is commutative. □

A.3. Technicalities for completion of Boolean perceptions

The rest of this technical appendix is needed for Section 8.1. Definition 62 of the completed perception is quite straightforward, but it has to be technically justified. We first establish that $\overline{\mathcal{B}}$ is a not a degenerate Boolean algebra:

Lemma A.4. *$\triangle$ is a proper ideal, $\triangle \neq \mathcal{B}$.*

Proof. We are going to show that $\triangle = \mathcal{B}$ implies that one of the elements of $\mathcal{S}$ is generated by an "illegal" σ-subsumption. First observe that:

– If $\triangle = \mathcal{B}$, then $\bigvee_{s \in \mathcal{S}} s = \top$.

– By definition of $\mathcal{S}$ and of $\trianglelefteq$ it follows that, for all $s \in \mathcal{S}$ and for all $\mathrm{w} \in \mathcal{E}$, $\sigma(\mathrm{w}, s) \neq \mathtt{t}$.

It follows from the truth table for disjunction in b-perceptions (Table 2) that there is some Boolean dependence in the set $\mathcal{S}$: there is a subset $X \subset \mathcal{S}$, of at least two elements, such that, although for all its subsets $Y \subset X$ $\forall \mathrm{w} \in \mathcal{E}$ $\sigma(\mathrm{w}, \bigvee_{s \in Y} s) \neq \mathtt{t}$, $\bigvee_{s \in X} s = \top$. Let $X = Y \cup \{s'\}$. If $\bigvee_{s \in Y} s \vee s' = \top$, then, $\neg \bigvee_{s \in Y} s \leqslant s'$, and $\neg s' \leqslant \bigvee_{s \in Y} s$. By validity of $\mathcal{C}$ the above Boolean subsumptions imply the corresponding σ-subsumptions, and thus $\forall \mathrm{w} \in \mathcal{E}$ $\sigma(\mathrm{w}, s') = \mathrm{u}$. By definition of $\mathcal{S}$, $s' = \alpha \wedge \neg \beta$ for some $\alpha \trianglelefteq \beta$, so, for all $\mathrm{w} \in \mathcal{E}$, $\sigma(\mathrm{w} \alpha) = \sigma(\mathrm{w}, \beta) = \mathrm{u}$. This is a contradiction to Definition 53. □

Corollary A.5. *For all* $\mathrm{w} \in \mathcal{E}$, $\sigma(\mathrm{w}, \bigvee_{s \in \mathcal{S}} s) \neq \mathrm{t}$.

We verify now that ξ^{cmp} could be a b-p-morphism onto $\overline{\mathcal{C}}$.

Proposition A.6. *Let* $\mathcal{C} = \langle \mathcal{B}, \sigma \rangle$ *be a b-perception. Whenever* $\alpha \cong \beta$, *then it cannot be that one of* $\sigma(\mathrm{w}, \alpha)$, $\sigma(\mathrm{w}, \beta)$ *is* t *and the other one is* f.

Proof. It follows by Corollary A.5 and validity of $\mathcal{C}$ that, for all $\mathrm{w} \in \mathcal{E}$ and for all connotations $\beta \in \Delta$, $\sigma(\mathrm{w}, \beta) \neq \mathrm{t}$. If, for some $\mathrm{w} \in \mathcal{E}$, one of $\sigma(\mathrm{w}, \alpha), \sigma(\mathrm{w}, \beta)$ is t and the other one is f, then either $\sigma(\mathrm{w}, \alpha \wedge \neg\beta) = \mathrm{t}$, or $\sigma(\mathrm{w}, \beta \wedge \neg\alpha) = \mathrm{t}$, so that $\alpha \ncong \beta$. □

It remains to establish that $\overline{\mathcal{C}}$ is a b-perception. It is first shown that a permit of $\overline{\mathcal{C}}$, if it exists, is monotone (as one should expect by Proposition 59). Existence is shown afterwards.

Proposition A.7. *Let* τ *be a permit of* $\overline{\mathcal{C}}$, *then a* σ*-subsumption,* $\alpha \trianglelefteq \beta$, *implies a* τ *subsumption,* $[\alpha] \trianglelefteq [\beta]$.

Proof. By construction of $\overline{\mathcal{B}}$ a σ-subsumption, $\alpha \trianglelefteq \beta$, implies a Boolean subsumption, $[\alpha] \leqslant [\beta]$. A permit τ defines a total b-p-predicate, and, by validity, the Boolean subsumption implies the perceptual τ subsumption $[\alpha] \trianglelefteq [\beta]$. □

τ, if it exists, induces a monotone permit, $\tau' = \xi^{\mathrm{cmp}} \circ \tau$ of $\mathcal{C}$. We are going to show that all the monotone permits of $\mathcal{C}$ are so induced. As always in this category of perceptions, the universal perception, $\mathcal{U}_{\mathcal{E}}$, is the primary tool for bootstrapping technicalities.

Proposition A.8. *Let* $\widehat{\sigma}$ *be a monotone permit of* $\mathcal{C}$, *then* $\alpha \cong \beta$ *implies that* α *and* β *are* $\widehat{\sigma}$*-synonyms.*

Proof. As a monotone permit, $\widehat{\sigma}$ defines a monotone b-p-morphism into the universal perception which is complete (by Example 55). It follows that connotations in Δ are mapped to the bottom of $\mathcal{U}_{\mathcal{E}}$ (namely to the $\emptyset$ connotation), and the proposition follows. □

Let $\widehat{\sigma}$ be a monotone permit of $\mathcal{C}$. By the last proposition $\widehat{\sigma}$ is also a permit of $\overline{\mathcal{C}}$, and hence:

Corollary A.9. *If* $\mathcal{C}$ *has a monotone permit, then so does* $\overline{\mathcal{C}}$.

We need to show that every b-perception has a monotone permit. A construction of a general permit for a b-perception was described in Algorithm 1 of Appendix A.1. It can be easily modified to produce monotone permits.

Algorithm 2. This is similar to Algorithm 1, except that one starts from an ideal Δ_0 which is generated by both Δ_{w} and Δ.

Lemma A.10. *All the ideals $\{\Delta_i\}_{i=0,n}$ of Algorithm 2 are proper and the corresponding p-predicates σ_i are monotone Boolean improvements of σ.*

Proof outline. First show that Δ_0 is proper. Δ_{w} is proper because $\mathcal{C}$ is a b-perception and Δ is proper by Lemma A.4. It remains to show that, for all $\beta \in \Delta$, $\neg\beta \notin \Delta_{\mathrm{w}}$. Let $\beta \in \Delta$, then $\beta \cong \bot$. By Proposition A.6, for all $\mathrm{w} \in \mathcal{E}$, $\sigma(\mathrm{w}, \beta) \neq \mathtt{t}$, and hence $\sigma(\mathrm{w}, \neg\beta) \neq \mathtt{f}$. It follows that $\neg\beta \notin \Delta_{\mathrm{w}}$. This completes the proof that Δ_0 is proper.

We show now that σ_0 is a monotone Boolean improvement of σ: it is an improvement since, by Proposition A.6, for all $\mathrm{w} \in \mathcal{E}$, $\sigma(\mathrm{w}, \beta) \neq \mathtt{t}$. It is Boolean because Δ_0 is proper. Lastly, to show that it is monotone, let $\alpha \trianglelefteq \beta$, so that $\alpha \wedge \neg\beta \in \mathcal{S} \subset \Delta \subset \Delta_0$. It follows that $\sigma_0(\mathrm{w}, \alpha \wedge \neg\beta) = \mathtt{f}$, and in that case

$$\sigma_0(\mathrm{w}, \alpha) \xrightarrow{\text{Luk}} \sigma_0(\mathrm{w}, \beta).$$

The general induction step is similar to Algorithm 1. □

Corollary A.11. *$\overline{\mathcal{C}}$ has a (monotone) permit.*

It remains to show the closure condition of Definition 34 for $\overline{\mathcal{C}}$:

Proposition A.12. *Let $\overline{\sigma}(\mathrm{w}, [\beta]) = \mathtt{u}$. Then there exists some monotone permit, $\widehat{\sigma}_1$, of $\mathcal{C}$, where $\widehat{\sigma}_1(\mathrm{w}\,[\beta]) = \mathtt{f}$, and another permit, $\widehat{\sigma}_2$, where $\widehat{\sigma}_2(\mathrm{w}\,[\beta]) = \mathtt{t}$.*

Proof outline. Assume negatively that, for all monotone permits $\widehat{\sigma}$, $\widehat{\sigma}(\mathrm{w}, \beta) = \mathtt{f}(\mathtt{t})$, and hence $\beta \in \Delta_0$ $(\neg\beta \in \Delta_0)$. It can neither be that $\beta \in \Delta_{\mathrm{w}}$ $(\neg\beta \in \Delta_{\mathrm{w}})$, nor that $\beta \in \Delta$ $(\neg\beta \in \Delta)$. We then use the closure property of $\mathcal{C}$ and the definition of $\overline{\sigma}$ to contradict the remaining possibility that, for some $\gamma \in \Delta_{\mathrm{w}}$ and some $\delta \in \Delta$, $\beta = \gamma \vee \delta$ $(\neg\beta = \gamma \vee \delta)$. □

Corollary A.13. *$\overline{\mathcal{C}}$ is a b-perception, and the set of its permits consists of all the monotone permits of $\mathcal{C}$.*

Finally we can prove:

Lemma 63. *A completed perception is a complete b-perception.*

Proof. $[\alpha] \trianglelefteq [\beta]$ is a $\overline{\sigma}$ subsumption if, and only if, for all monotone permits $\widehat{\sigma}$ of $\mathcal{C}$, $\alpha \trianglelefteq \beta$. This holds if and only if $\alpha \trianglelefteq \beta$ is a σ-subsumption. In that case, by definition of $\overline{\mathcal{B}}$, $[\alpha] \leqslant [\beta]$. □

Corollary A.14.

$$\underline{[\alpha] \overset{\overline{\mathcal{B}}}{\leqslant} [\beta]} \Leftrightarrow \underline{[\alpha] \overset{\overline{\sigma}}{\trianglelefteq} [\beta]} \Leftrightarrow \underline{[\alpha] \overset{\sigma}{\trianglelefteq} [\beta]}.$$

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