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Parallel belief revision: Revising by sets of formulas

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ABSTRACT

The area of *belief revision* studies how a rational agent may incorporate new information about a domain into its belief corpus. An agent is characterised by a belief state K, and receives a new item of information α which is to be included among its set of beliefs. Revision then is a function from a belief state and a formula to a new belief state.

We propose here a more general framework for belief revision, in which revision is a function from a belief state and a finite *set* of formulas to a new belief state. In particular, we distinguish revision by the set $\{\alpha, \beta\}$ from the set $\{\alpha \wedge \beta\}$. This seemingly innocuous change has significant ramifications with respect to *iterated belief revision*. A problem in approaches to iterated belief revision is that, after first revising by a formula and then by a formula that is inconsistent with the first formula, all information in the original formula is lost.

This problem is avoided here in that, in revising by a set of formulas S, the resulting belief state contains not just the information that members of S are believed to be true, but also the counterfactual supposition that if some members of S were later believed to be false, then the remaining members would nonetheless still be believed to be true. Thus if some members of S were in fact later believed to be false, then the other elements of S would still be believed to be true. Hence, we provide a more nuanced approach to belief revision. The general approach, which we call *parallel belief revision*, is independent of extant approaches to iterated revision. We present first a basic approach to parallel belief revision. Following this we combine the basic approach with an approach due to Jin and Thielscher for iterated revision. Postulates and semantic conditions characterising these approaches are given, and representation results provided. We conclude with a discussion of the possible ramifications of this approach in belief revision in general.

1. Introduction

An agent situated in a sufficiently complex domain will have only incomplete and possibly inaccurate information about that domain. Consequently, such an agent would be expected to receive new information about the domain which it would incorporate into its belief corpus. Since new information may conflict with the agent's accepted beliefs, the agent may also have to discard some of its beliefs before the new information can be consistently incorporated. *Belief revision* is the area of knowledge representation that addresses how an agent may incorporate new information about a domain into its belief corpus. It is generally accepted that there is no single best revision operator, and different agents may have different revision functions. However, revision functions are not arbitrary, but may be considered as being guided or characterised by various *rationality criteria*, expressed formally as a set of postulates. The original and best-known set of postulates is called the *AGM postulates* [1,16] named after the developers of this framework. As well, several formal constructions of revision functions

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have been proposed based, for example, on an ordering on sentences of the language or on an ordering on possible states of the world. Ideally, a set of postulates is linked with a formal construction by a *representation result*, showing that a revision function that satisfies a postulate set can be represented by the formal construction, and vice versa.

The foundations of AGM revision are well studied and well understood.[1] Subsequently, there has been a great deal of attention paid to *iterated belief revision*, which addresses logical relations among a sequence of revisions involving possibly-conflicting observations. While there has been much progress in the area of iterated belief revision, virtually all such work suffers from the following problem: if one revises by a formula and then by a formula that is inconsistent with this formula, then the agent's beliefs are exactly the same as if only the second revision had taken place.

For example, consider the situation where there was a party, but where you have no knowledge about whether Alice (a) or Bob (b) were there. You are subsequently informed by a reliable source that both Alice and Bob went to the party. This would correspond to a revision by $a \wedge b$, and your resulting belief state would be one in which you believe $a \wedge b$ to be true. You later learn that Alice in fact did not go to the party. Not only do you now accept $\neg a$, but in all major approaches to iterated belief revision, including [9,6,32,21], you no longer accept b either. While there may indeed be cases where it's reasonable to no longer believe Bob was at the party (for example perhaps Bob is Alice's spouse), this certainly shouldn't be a *required* outcome.

This example can be exaggerated to emphasise the point: Consider where an agent initially has no contingent beliefs, and so its beliefs are characterised by the set of tautologies. Next, a substantial body of knowledge, given by the conjunction $p_1 \wedge \cdots \wedge p_{10^{12}}$, is loaded into the agent's knowledge base. If we subsequently revise by, say, the negation of p_1, then all other knowledge is lost. That is, if the agent's original (tautological) beliefs were given by K and $*$ is the revision function, we would obtain:

$$\big(K * (p_1 \wedge \cdots \wedge p_{10^{12}})\big) * \neg p_1 \equiv K * \neg p_1. \tag{1}$$

Thus all other information is lost, except for the newly-negated item. Again, this is clearly too strong a condition to impose on every revision function in all circumstances.

We suggest that this problem is appropriately addressed not by modifying the foundations of belief revision, but rather by providing a more nuanced or expressive approach to revision. Specifically, we propose that the second argument of a revision function be generalised to be a *set* of formulas. This then distinguishes revision by a set of formulas from revision by the conjunction of that set of formulas. Consider again our Alice/Bob example, where again at the outset you have no beliefs about whether either of them attended a party or not, but you are subsequently informed that they both went to the party. Consequently, if you were now asked "Do you believe that Alice went to the party?", clearly you would answer in the affirmative. Assume further that you have no reason to believe that Alice and Bob know each other well, nor have been in contact; i.e. each individual's attendance is independent of the other's. If you were asked "If it were in fact the case that Alice did not go, would you still believe that Bob went?", then again you would answer in the affirmative. However, it can be noted that this last question is a *counterfactual* query, in that as far as you know the antecedent is false. We are not going to be concerned with counterfactuals per se in this paper; however, this does have implications for further revisions: If you were subsequently informed that *in fact* Alice did not go, then you should in turn continue to believe that Bob went. If, on the other hand, you had some reason to believe that Alice and Bob's attendance were linked – for example that they're a couple – then this would no longer apply.

The key point here is that we are treating the propositions A and B as separate items of information. Our central thesis is that revision by a conjunction and revision by the set of conjuncts should be treated differently. If a formula is taken as representing some item of information, then informally a conjunction represents a single item of information, while the corresponding set of conjuncts represents a collection of items of information. To be sure, the conjunction $\alpha \wedge \beta$ and the set $\{\alpha, \beta\}$ have the same logical content, in that they entail exactly the same formulas. Hence an agent's contingent beliefs should be the same regardless of whether a revision is by a conjunction or a corresponding set of formulas. However, as argued above, in revising by a set $\{\alpha, \beta\}$ the agent's resulting belief state should be such that, if there is no known connection between α and β, then if β were subsequently learned to be false, then α should still be believed to be true.

To this end, we develop an account of belief revision that we call *parallel belief revision* in which the second argument to a revision function is a finite set of formulas. Thus, if the agent's belief state is given by $\mathcal{K}$[2] and $*$ is a revision function, then we distinguish $\mathcal{K} * \{\alpha \wedge \beta\}$ from $\mathcal{K} * \{\alpha, \beta\}$. In the former, revision is by a single formula that happens to be expressed as a conjunction. If a subsequent revision contradicts this formula, then this formula is simply no longer believed. On the other hand, if the agent views α and β as independent, then it makes sense that α is believed in $\mathcal{K} * \{\alpha, \beta\} * \{\neg\beta\}$, since if one element of the input set is contradicted, this need not affect belief in other element. Essentially, for a revision $\mathcal{K} * \{\alpha, \beta\}$, the agent comes to believe not only that α and β are contingently true, but also counterfactual assertions such as *if* β were false then α would (where "reasonable") still be believed to be true. In terminology introduced in the next section, the agent's *belief state* or *epistemic state* is modified so that such counterfactuals are implicitly believed. Hence, continuing the above

[1] See [35] for a recent, comprehensive survey of revision in general.

[2] The distinction between the K of (1) and $\mathcal{K}$ is described more fully in the next section. Basically K is the agent's contingent beliefs concerning the domain; $\mathcal{K}$ is the agent's full epistemic state, containing not just K but also, for example, information about how the agent's beliefs would change if it were to learn a new piece of information.

example, if the agents were told that *in fact* β were false then it would (again with a caveat, "where reasonable") continue to believe that α was true. That is, all other things being equal, we would have that α is believed in $\mathcal{K} * \{\alpha, \beta\} * \{\neg\beta\}$.

In this paper, we develop approaches to parallel belief revision and show how the aforementioned problems are resolved. Notably, in our approach we obtain that under reasonable assumptions (for example, that α and β are not logically equivalent) that α is believed in $\mathcal{K} * \{\alpha, \beta\} * \{\neg\beta\}$. As well, it proves to be the case that parallel belief revision is independent of other accounts of iterated belief revision, in that it can be *combined* with extant approaches to belief revision. This then supports the assertion that our approach provides a more fine-grained account of revision, rather than providing an alternative to existing accounts. Consequently, we first describe the most basic approach to parallel revision, and then show how this approach can be combined with the approach to iterated revision of Jin and Thielscher [21].

The next section reviews the area of belief revision and further motivates our approach. Following this, we give an account of the most basic approach to parallel revision. We then combine this approach with that of Jin and Thielscher [21]. In each case, postulates characterising the revision function are given, a semantic account is provided, and a representation result is provided linking the postulates and semantic construction. We conclude with a discussion of wider implications of the approach to belief revision as a whole, and iterated revision in particular. Proofs of all formal results are given in Appendix A. An earlier version of this paper appeared in [10].

2. Background

2.1. Formal preliminaries

We assume a propositional language $\mathcal{L}$ generated from a finite set $\mathcal{P}$ of atomic propositions. The language is that of classical propositional logic, and with the classical consequence relation $\vdash$. Formulas are denoted by lower-case Greek letters $\alpha, \beta, \ldots$, while sets of formulas are denoted by upper case Roman letters, $A, B, S, \ldots$. The symbol $\top$ stands for some arbitrary tautology and $\bot$ stands for $\neg\top$. $Cn(A)$ is the set of logical consequences of A, that is $Cn(A) = \{\alpha \in \mathcal{L} \mid A \vdash \alpha\}$. For a (finite) set of formulas S, $\wedge S$ is the conjunction of members of S, $\vee S$ is the corresponding disjunction, and $\overline{S} = \{\neg\alpha \mid \alpha \in S\}$. Given two sets of formulas A and B, $A + B$ denotes the *expansion* of A by B, that is $A + B = Cn(A \cup B)$. Expansion of a set A by a formula β is defined analogously. Two sentences α and β are *logically equivalent*, written $\alpha \equiv \beta$, iff $\alpha \vdash \beta$ and $\beta \vdash \alpha$. This extends to sets of sentences by: $S_1 \equiv S_2$ iff $S_1 \vdash \alpha$ for every $\alpha \in S_2$ and $S_2 \vdash \beta$ for every $\beta \in S_1$. Thus in particular for any finite set of formulas S, we have $S \equiv \wedge S$.

A propositional *interpretation* (also referred to as a *possible world*) is a mapping from $\mathcal{P}$ to {true, false}. The set of all interpretations is denoted by $\Theta_{\mathcal{P}}$. A *model* of a sentence α is an interpretation w that makes α true according to the usual definition of truth, and is denoted by $w \models \alpha$. For $\mathcal{W} \subseteq \Theta_{\mathcal{P}}$, we also write $\mathcal{W} \models \alpha$ if $w \models \alpha$ for every $w \in \mathcal{W}$. For a set of sentences A, $Mod(A)$ is the set of all models of A. For simplicity, $Mod(\{\alpha\})$ is also written as $Mod(\alpha)$. Conversely, given a set of possible worlds $\mathcal{W} \subseteq \Theta_{\mathcal{P}}$, we denote by $\mathcal{T}(\mathcal{W})$ the set of sentences which are true in all elements of $\mathcal{W}$, that is $\mathcal{T}(\mathcal{W}) = \{\alpha \in \mathcal{L} \mid w \models \alpha \text{ for all } w \in \mathcal{W}\}$.

A total preorder $\preccurlyeq$ is a reflexive, transitive binary relation, such that either $w_1 \preccurlyeq w_2$ or $w_2 \preccurlyeq w_1$ for every w_1, w_2. The strict part of $\preccurlyeq$ is denoted by $\prec$, that is, $w_1 \preccurlyeq w_2$ and $w_2 \not\preccurlyeq w_1$. We use $w_1 = w_2$ to abbreviate $w_1 \preccurlyeq w_2$ and $w_2 \preccurlyeq w_1$.[3] Given a set S and total preorder $\preccurlyeq$ defined on members of S, we denote by $\min(S, \preccurlyeq)$ the set of minimal elements of S in $\preccurlyeq$.

2.2. Belief revision

In the original AGM theory, beliefs of an agent are represented by a *belief set*, that is, a set of formulas K such that $K = Cn(K)$. Belief revision is modelled as a function from a belief set K and a formula α to a belief set K' such that α is believed in K', i.e. $\alpha \in K'$. Since α may be inconsistent with K, and since it is desirable to maintain consistency if at all possible (i.e. if not $\vdash \neg\alpha$), then some formulas may need to dropped from K before α can be consistently added.

The AGM approach also addressed two other operators. The *expansion* of a belief set K by a formula α has already been defined in the previous subsection. In contrast to revision and expansion, where an agent gains information, in *belief contraction* the reasoner loses information. The contraction of a belief set K by a formula α, denoted $K \dot{-} \alpha$, is a belief set where $K \dot{-} \alpha \subseteq K$ and $\alpha \notin K \dot{-} \alpha$. So in a contraction by α, the agent loses its belief in α while not necessarily believing $\neg\alpha$. Since our focus in this paper is on revision, we do not consider contraction further, except briefly in Section 2.4, where we review set-based approaches to contraction that have been proposed in the literature.

An important assumption concerning belief revision is that it takes place in an inertial (or static) world, so that the input is with respect to the same, static world. However, various researchers have argued that, in order to address iterated belief revision, it is more appropriate to consider *belief states* (also called *epistemic states*) as objects of revision. A belief state $\mathcal{K}$ effectively encodes information regarding how the revision function itself changes under a revision.[4] The belief set corresponding to belief state $\mathcal{K}$ is denoted $Bel(\mathcal{K})$. Formally, a revision operator $*$ maps a belief state $\mathcal{K}$ and formula α to a revised belief state $\mathcal{K} * \alpha$. Then, in the spirit of [9], the AGM postulates for revision can be reformulated as follows:

[3] Relations in a total preorder will be subscripted with an epistemic state, described in the next subsection. In particular, for the last relation we will write $w_1 =_{\mathcal{K}} w_2$. Thus there is no confusion with equality, written $=$ as usual.

[4] This glosses over a number of issues on the nature of a revision function, which need not concern us here. See [36,32] for more on this issue.

($\mathcal{K} * 1$) $Bel(\mathcal{K} * \alpha) = Cn(Bel(\mathcal{K} * \alpha))$.
($\mathcal{K} * 2$) $\alpha \in Bel(\mathcal{K} * \alpha)$.
($\mathcal{K} * 3$) $Bel(\mathcal{K} * \alpha) \subseteq Bel(\mathcal{K}) + \alpha$.
($\mathcal{K} * 4$) If $\neg\alpha \notin Bel(\mathcal{K})$ then $Bel(\mathcal{K}) + \alpha \subseteq Bel(\mathcal{K} * \alpha)$.
($\mathcal{K} * 5$) $Bel(\mathcal{K} * \alpha)$ is inconsistent, only if $\vdash \neg\alpha$.
($\mathcal{K} * 6$) If $\alpha \equiv \beta$ then $Bel(\mathcal{K} * \alpha) \equiv Bel(\mathcal{K} * \beta)$.
($\mathcal{K} * 7$) $Bel(\mathcal{K} * (\alpha \wedge \beta)) \subseteq Bel(\mathcal{K} * \alpha) + \beta$.
($\mathcal{K} * 8$) If $\neg\beta \notin Bel(\mathcal{K} * \alpha)$ then $Bel(\mathcal{K} * \alpha) + \beta \subseteq Bel(\mathcal{K} * (\alpha \wedge \beta))$.

Thus, the result of revising $\mathcal{K}$ by α yields an epistemic state in which α is believed in the corresponding belief set (($\mathcal{K} * 1$), ($\mathcal{K} * 2$)); whenever the result is consistent, the revised belief set consists of the expansion of $Bel(\mathcal{K})$ by α (($\mathcal{K} * 3$), ($\mathcal{K} * 4$)); the only time that $Bel(\mathcal{K})$ is inconsistent is when α is inconsistent (($\mathcal{K} * 5$)); and revision is independent of the syntactic form of the formula for revision (($\mathcal{K} * 6$)). The first six postulates are referred to as the *basic* revision postulates. The last two postulates are called the *supplementary* postulates, and deal with the relation between revising by a conjunction and expansion: whenever consistent, revision by a conjunction corresponds to revision by one conjunct and expansion by the other. Motivation for these postulates can be found in [16,35]. The intent of these postulates is that they should hold for *any* rational belief revision function. We will call a revision operator an *AGM revision operator* if it satisfies the reformulated AGM postulates. Katsuno and Mendelzon [22] have shown that a necessary and sufficient condition for constructing an AGM revision operator is that any belief state $\mathcal{K}$ can induce, as its preferential information, a total preorder on the set of possible worlds.

Definition 1. A *faithful assignment* is a function that maps each belief state $\mathcal{K}$ to a total preorder $\preccurlyeq_{\mathcal{K}}$ on $\Theta_{\mathcal{P}}$ such that for any $w_1, w_2 \in \Theta_{\mathcal{P}}$:

1. If $w_1, w_2 \models Bel(\mathcal{K})$ then $w_1 =_{\mathcal{K}} w_2$.
2. If $w_1 \models Bel(\mathcal{K})$ and $w_2 \not\models Bel(\mathcal{K})$, then $w_1 \prec_{\mathcal{K}} w_2$.

The resulting total preorder is referred to as the *faithful ranking corresponding to*, or *induced by* $\mathcal{K}$. Intuitively, $w_1 \preccurlyeq_{\mathcal{K}} w_2$ if w_1 is at least as plausible as w_2, according to the agent. As enforced by the first condition in Definition 1, an agent's beliefs are characterised by the least worlds in the ordering.

Katsuno and Mendelzon then provide the following representation result, where $\mathcal{T}(W)$ is the set of formulas of classical logic true in W:

Theorem 1. *(See [22].) A revision operator $*$ satisfies Postulates ($\mathcal{K} * 1$)–($\mathcal{K} * 8$) iff there exists a faithful assignment that maps each belief set $\mathcal{K}$ to a total preorder $\preccurlyeq_{\mathcal{K}}$ such that*

$$\mathcal{K} * \phi = \mathcal{T}\big(\min\big(Mod(\phi), \preccurlyeq_{\mathcal{K}}\big)\big).$$

Thus for a belief state $\mathcal{K}$, the agent's beliefs following revision by a formula α are characterised by those possible worlds of ϕ that are most plausible according to the agent. A ranking function $\preccurlyeq_{\mathcal{K}}$ corresponding to belief state $\mathcal{K}$ can also be understood as specifying the (counterfactual) information of what the agent would believe after coming to believe some formula ϕ.

2.3. Iterated belief revision

The AGM postulates do not address properties of iterated belief revision. This can be seen by observing that, while Theorem 1 specifies what the agent's beliefs will be following a revision by formula ϕ, it has nothing to say about what the new ranking function $\preccurlyeq_{\mathcal{K}*\phi}$ should look like. As noted by [32], the only interesting result that follows from the AGM approach concerning iterated belief revision is the following:

$$\text{If } \neg\beta \notin Bel(\mathcal{K} * \alpha) \quad \text{then } Bel\big((\mathcal{K} * \alpha) * \beta\big) = Bel\big(\mathcal{K} * (\alpha \wedge \beta)\big).$$

This has led to the development of extensions of the AGM approach to address iterated revision; the best-known approach is that of Darwiche and Pearl [9] (DP). They propose the following postulates, adapted according to our notation:

(C1) If $\beta \vdash \alpha$, then $Bel((\mathcal{K} * \alpha) * \beta) = Bel(\mathcal{K} * \beta)$.
(C2) If $\beta \vdash \neg\alpha$, then $Bel((\mathcal{K} * \alpha) * \beta) = Bel(\mathcal{K} * \beta)$.
(C3) If $\alpha \in Bel(\mathcal{K} * \beta)$, then $\alpha \in Bel((\mathcal{K} * \alpha) * \beta)$.
(C4) If $\neg\alpha \notin Bel(\mathcal{K} * \beta)$, then $\neg\alpha \notin Bel((\mathcal{K} * \alpha) * \beta)$.

(C1) states that if an agent revises by a formula and then by a logically stronger formula then, with respect to its belief set, this is no different than simply revising by the stronger formula. The other postulates may be given similar informal readings. As with the AGM postulates, the intent is that these postulates should hold for *any* rational belief revision function. Darwiche and Pearl show that an AGM revision operator satisfies Postulates (C1)–(C4) iff the way it revises faithful rankings satisfies the (respective) conditions:

(CR1) If $w_1, w_2 \models \alpha$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ iff $w_1 \preccurlyeq_{\mathcal{K}*\alpha} w_2$.
(CR2) If $w_1, w_2 \not\models \alpha$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ iff $w_1 \preccurlyeq_{\mathcal{K}*\alpha} w_2$.
(CR3) If $w_1 \models \alpha$ and $w_2 \not\models \alpha$, then $w_1 \prec_{\mathcal{K}} w_2$ implies $w_1 \prec_{\mathcal{K}*\alpha} w_2$.
(CR4) If $w_1 \models \alpha$ and $w_2 \not\models \alpha$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ implies $w_1 \preccurlyeq_{\mathcal{K}*\alpha} w_2$.

These conditions are natural and appealing; moreover they appear to be intuitively very reasonable: When $\mathcal{K}$ is revised by α, Conditions (CR1) and (CR2) require that the relative ranking of any two α-worlds (resp. $\neg\alpha$-worlds) do not change. Conditions (CR3) and (CR4) require that if an α-world w_1 is (strictly) more plausible than a $\neg\alpha$-world w_2, then following revision by α, w_1 continues to be (strictly) more plausible than w_2.

The DP postulates have been criticised in two respects. On one hand, it has been suggested that they are too permissive, in that they support revision operators which allow arbitrary dependencies among the items of information which an agent acquires along the way. Consequently, Jin and Thielscher [21] have proposed the so-called postulate of independence[5]:

(Ind) If $\neg\alpha \notin Bel(\mathcal{K} * \beta)$ then $\alpha \in Bel((\mathcal{K} * \alpha) * \beta)$.

Thus, if a revision of $\mathcal{K}$ by β does not rule out α, then if $\mathcal{K}$ is first revised by α and then by β, α is believed in the resulting belief set. Postulate (Ind) strengthens both (C3) and (C4). Thus, the suggested set of postulates according to Jin and Thielscher [21] consists of (C1), (C2), and (Ind). They also give a necessary and sufficient condition for an AGM revision operator to satisfy (Ind):

(IndR) If $w_1 \models \alpha$ and $w_2 \models \neg\alpha$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ implies $w_1 \prec_{\mathcal{K}*\alpha} w_2$.

This condition is also natural: if α is true at w_1 and false at w_2, and if w_1 and w_2 are equally plausible, then after revising by α, w_1 is strictly more plausible than w_2. If $w_1 \prec_{\mathcal{K}} w_2$ then we simply have (CR3).

On the other hand, it can be argued that the DP postulates are too strong. In particular, Postulate (C2) has been noted by many researchers as producing the undesirable result described in Section 1 [28,23,11]. As a further example, consider a scenario proposed by Konieczny and Pino Pérez [23]:

Example 1. Suppose an electric circuit contains an adder and a multiplier. The atomic propositions a and m denote respectively that the adder and the multiplier are working. Initially we have no information about this circuit; and we then learn that the adder and the multiplier are working ($\alpha = a \wedge m$). Thereafter, someone tells us that the adder is actually not working ($\beta = \neg a$).

As argued in [23], there is no reason to "forget" that the multiplier is working; however by (C2) we *must* have $(K * \alpha) * \beta = K * \beta$, since $\beta \vdash \neg\alpha$. Hence, in this case (C2) appears to be too strong.

Intuitively, such examples are compelling. However, the case against (C2) isn't clear cut. First, it can be observed that many researchers (e.g., [28,23]), who are against (C2) are nonetheless quite happy with Postulate (C1). However, the semantic characterisation of Postulate (C2) (viz. (CR2)) seems as reasonable as that of (C1) (viz. (CR1)): If being informed about α does not change the relative plausibility of α-worlds, why should the relative ordering of $\neg\alpha$-worlds be changed? This idea is also articulated in [37], which argues that in a belief change involving α, the relative ordering between α-worlds remains unchanged, as it does between $\neg\alpha$-worlds.

As an informal defense of (C2), it can be observed that in Example 1 it is implicitly assumed that a and m are separate items of information. However, in the AGM approach, a simultaneous revision by a and m can only be represented by a conjunction, viz. $\mathcal{K} * (a \wedge b)$. What makes Example 1 credible is the fact that there is no apparent relation between being informed of the adder working and, at the same time, of the multiplier working. Hence learning $\neg a$ would seem to not influence belief in m. However, the example can be elaborated upon so that this isn't necessarily the case. Consider for example where we are told by someone that both the adder and multiplier are working, and then determine ourselves that the adder is not working. One might argue plausibly that the original source was suspect, and so it makes sense to give up *in toto* all information provided by that source. Thus plausibly in this case one might want to not believe that m was true.[6]

[5] Essentially the same system is discussed in [5], where it is called *admissable revision*.

[6] Another support for (C2) is that it is in fact the only DP postulate which puts additional constraints on the retention of propositional beliefs. To see this, let's consider so-called amnesic revision $*_a$:

$$K *_a \alpha = \begin{cases} K + \alpha & \text{if } K \nvdash \neg\alpha, \\ Cn(\alpha) & \text{otherwise.} \end{cases}$$

Note that radical as it is, amnesic revision satisfies the AGM postulates, (C1), (C3) and (C4), but violates (C2).

This discussion shows that a revision of a belief state $\mathcal{K}$ by formulas α and β can be interpreted in at least two different ways. In the first interpretation, the agent has been informed that α and β are true; to revise simultaneously by α and β, the best that can be done is to revise by their conjunction. In the second interpretation, the agent has been informed of an item of information, and this item of information has been expressed as a conjunction. Under the first interpretation, α and β are regarded as separate items of information; under this interpretation it is reasonable that (C2) not necessarily hold. Under the second interpretation, α and β are seen as components of an item of information; here it seems reasonable that (C2) *does* hold. Clearly, extant accounts of iterated belief revision are not sufficiently expressive to deal with both situations. Thus, Example 1 doesn't provide a counterexample to (C2), so much as it highlights the limitations of the expressibility of revision functions. What this suggests then is that AGM revision should be generalised so that both above-mentioned situations can be handled.

2.4. Conjunctions of formulas vs. sets of formulas

The preceding discussion suggests that $\mathcal{K} * (\alpha \wedge \beta)$ should be treated differently from $\mathcal{K} * \{\alpha, \beta\}$. The former case represents the situation in the AGM framework in which revision is by a formula, here comprised of two conjuncts. In the latter case, revision is by a set of formulas. An immediate effect of this distinction is that revision now becomes a function whose second argument is a set of formulas, rather than a single formula. Hence the above distinction is appropriately expressed as $\mathcal{K} * \{\alpha \wedge \beta\}$ vs. $\mathcal{K} * \{\alpha, \beta\}$.

This distinction between a set of formulas and their conjunction has been noted and explored elsewhere and under different guises. Perhaps the most direct recognition of this distinction is in [24]. There the comma that appears in an expression of a set of objects is referred to as "the forgotten connective". Their interests however concern reasoning under inconsistency, where one can plausibly make the argument that there are cases where a distinction between $\{a \wedge b \wedge \neg b\}$ and $\{a, b, \neg b\}$ is of value.

As well, the distinction between a set of formulas and their conjunction has cropped up in nonmonotonic reasoning, specifically with respect to nonmonotonic inference relations, conditional logics, and related systems (see [25,26,7] for monotonic systems of defaults, and [33,8,2,29] for nonmonotonic approaches). In these systems, a default "if α then normally β" can be written $\alpha \Rightarrow \beta$ in a conditional logic or $\alpha \mid\!\sim \beta$ in a nonmonotonic inference relation. There is a difficulty with such approaches, in that for defaults $\alpha \Rightarrow \beta_1$ and $\alpha \Rightarrow \beta_2$ if α is known to be contingently true while β_1 is false, one would still want to conclude β_2 by default. Similarly, if α is known to be contingently true, where $\models \alpha \Rightarrow \gamma$ and $\gamma \Rightarrow \beta$ is a default, one would want to conclude β by default in general. However these results are difficult to obtain. These issues have been addressed in several ways, but solutions in general have relied on how a default is represented. In [17], possible worlds are ranked according to the defaults that they violate. In the approach of [18], maximum entropy is used to essentially assert that things are as normal as possible. The *lexicographic closure* of a set of defaults [27,4] formalises the idea that in applying defaults, one prefers to violate a smaller number of defaults to violating a larger number. All these approaches are syntax dependent, in that they depend on how a set of defaults is represented. In particular, one may obtain different answers for the set of defaults $\{\alpha \Rightarrow \beta_1, \alpha \Rightarrow \beta_2\}$ as opposed to $\{\alpha \Rightarrow (\beta_1 \wedge \beta_2)\}$: if β_1 is false, then in the former case one may still conclude β_2 whereas in the latter case the default as a whole is inapplicable. So, roughly, the intuition underlying the lexicographic closure is that as many defaults are applied as consistently possible, and the results of this notion of maximum applicability will vary depending on how the defaults are expressed. The focus in the approach presented here is somewhat different: one revises by a set of formulas S, and this set is accepted. (In particular, if S is inconsistent then the agent falls into an inconsistent belief state.) If the agent subsequently learns that some elements of S are in fact false then, where consistent, the remaining elements of S are still believed to be true. So for two sets of formulas S and S' where $S \equiv S'$, the agent's beliefs will be the same in $\mathcal{K} * S$ and $\mathcal{K} * S'$, but may differ in subsequent revisions.

With respect to belief change, a *belief base* [20] is a set of formulas representing an agent's beliefs. Since a belief base is in general not deductively closed, it may be seen as having more structure than the corresponding belief set. Hence belief change with respect to a belief base may have differing results, depending on how the agent's beliefs are expressed. For example, [30] considers a scenario in which the agent's beliefs are represented by an *infobase* consisting of a finite sequence of formulas. Each formula in the infobase is assumed to be an explicit piece of information, obtained independently from the other formulas. For contraction, an ordering is specified over interpretations depending on the number of formulas in the infobase that they satisfy. Hence if two formulas in an infobase were replaced by their conjunction, one would expect quite different results. Somewhat similar intuitions are employed for *disjunctive maxi-adjustment* [3]; see also [39]. An *ordered knowledge base* is employed toward conflict resolution, for application to tasks such as belief change and information integration. An ordered knowledge base places a ranking on formulas, which can be seen as a compact representation of an ordinal conditional function [37]. Interpretations are ranked by the highest formula that they falsify (and given rank 0 if all formulas are satisfied). The disjunctive maxi-adjustment is shown to satisfy a lexicographic strategy, which is to say, essentially a maximal set of formulas for a given rank is selected. Since one is working at the level of formulas, again results depend on how the formulas are represented.

The preceding approaches to belief change consider a knowledge base as being comprised of a set of (possibly ranked) formulas. There has also been work in which the input for belief change is a set of formulas, rather than a single formula. Fuhrmann and Hansson [15] survey *multiple contraction*; in particular they propose *package contraction*, which is concerned with removing a set of formulas from a belief set. In the AGM approach, a contraction $K - \alpha$ yields a belief set K' that is a subset of K in which α is not believed (except in the case where α is a tautology). The standard construction for contraction is phrased in terms of *remainder sets*, or maximal subsets of K that fail to imply α. In a package contraction such as $K - \{\alpha, \beta\}$, the resulting belief set is one in which neither α nor β is believed. The notion of remainder set extends naturally in this case to maximal subsets of K that fail to imply either α or β. It can be noted that the package contraction $K - \{\alpha, \beta\}$ is distinct from contraction $K - (\alpha \wedge \beta)$ or $K - (\alpha \vee \beta)$. In the former case, it is possible that $\alpha \in K - (\alpha \wedge \beta)$ although this is not allowed in the corresponding package contraction. For the latter case, it is possible that $\alpha \vee \beta \in K - \{\alpha, \beta\}$, although clearly $\alpha \vee \beta \notin K - (\alpha \vee \beta)$. Fuhrmann and Hansson [15] also give a set of postulates for package contraction analogous to the set of basic postulates for AGM contraction, and they prove a representation result linking these postulates to the semantic construction.

The multiple contraction in [42] also studies how to contract a belief set so that it is consistent with a set of formulas. In this work, the authors propose supplementary postulates for multiple contraction and provide a representation result; they also consider the case where the set for contraction may be infinite. In other work, Fermé et al. [14] examine a construction for multiple contraction in a (nondeductively-closed) belief base, while Falappa et al. [13] address revision of an arbitrary set of formulas by a set of *explanations*, with application to argumentation systems. Last, Delgrande and Wassermann [12] consider package contraction where the underlying logic is that governing Horn clauses. Here package contraction is of greater importance than in the case of classical logic, since in classical logic one has the option of contracting by several formulas via contracting their disjunction. However, the disjunction of Horn clauses in general is not Horn, and so one requires package contraction to concurrently remove several formulas.

It can be noted that while these approaches to set-based contraction have a syntactic resemblance to the approach developed here, the emphasis is quite different. In a package or multiple contraction $K - S$, the resulting belief set in general is different from a contraction made up of some Boolean combination of members of S. On the other hand, none of these approaches address iterated operations. In contrast, in our approach it will be seen that the belief set resulting from $\mathcal{K} \otimes S$ and from $\mathcal{K} \otimes \{\wedge S\}$ will be the same. However the faithful rankings resulting from $\mathcal{K} \otimes S$ and $\mathcal{K} \otimes \{\wedge S\}$ will in general be quite different, and this will have significant ramifications for iterated revision.

There has also been work in revision by sets of formulas, in particular the *set revision* of [41] and *multiple revision* of [34]. There are two main differences between these approaches and our's. First, our focus is on iterated revision and, in particular, constraints that need to be imposed on an agent's underlying epistemic state in order to effect plausible revisions. Second, Zhang and Foo [41] and Peppas [34] primarily study infinite sets. In our approach, the focus is on the distinction between revising by a finite set of formulas and a corresponding conjunction of those formulas. Therefore, while set revision or multiple revision might be useful for investigating infinite nonmonotonic reasoning, arguably our approach is more suitable for modelling the evolution of an agent's belief state where, at least in a practical setting, an agent will not receive an infinite set as input. Finally, Nayak [31] anticipates some of the properties of parallel revision, in an approach where both the belief state and input are represented by epistemic entrenchment relations.

3. Parallel revision

3.1. Intuitions

We have argued that $\mathcal{K} * \{\alpha \wedge \beta\}$ should be treated differently from $\mathcal{K} * \{\alpha, \beta\}$. Hence the epistemic state resulting from $\mathcal{K} * \{\alpha \wedge \beta\}$ will in general be different from that resulting from $\mathcal{K} * \{\alpha, \beta\}$. However, the logical content of $\{\alpha \wedge \beta\}$ and $\{\alpha, \beta\}$ is the same, and so one might reasonably expect that the agent's beliefs following revision by either of these sets would be the same. Thus one might reasonably expect that

$$Bel\big(\mathcal{K} * \{\alpha, \beta\}\big) = Bel\big(\mathcal{K} * \{\alpha \wedge \beta\}\big).$$

On the right-hand side of the equality we revise by a single item of information, $\alpha \wedge \beta$. If β is later shown to be false, then so too is $\alpha \wedge \beta$, and it is reasonable that all original information (including α) may be lost. Hence, possibly $\alpha \notin Bel(\mathcal{K} * \{\alpha \wedge \beta\} * \{\neg\beta\})$. This argument doesn't apply to $Bel(\mathcal{K} * \{\alpha, \beta\})$, where we revise by a set consisting of two items of information. Thus if we later revise by $\neg\beta$, then one would expect that $\alpha \in Bel(\mathcal{K} * \{\alpha, \beta\} * \{\neg\beta\})$ where "reasonable".[7]

Semantically, this has the following ramifications. An agent's belief state (at least as far as revision is concerned) is modelled by a faithful ranking on possible worlds. In the faithful ranking that results from the revision $\mathcal{K} * \{\alpha, \beta\}$, we have that the least $\alpha \wedge \beta$ worlds are ranked lower than the least $\neg(\alpha \wedge \beta)$ worlds in $\preccurlyeq_{\mathcal{K}*\{\alpha,\beta\}}$. (This is a trivial consequence of the fact that the least $\alpha \wedge \beta$ worlds are minimal in $\preccurlyeq_{\mathcal{K}*\{\alpha,\beta\}}$.) The key intuition in parallel revision is that these considerations extend to subsets of the set of formulas for revision. Hence, in revising by $\{\alpha, \beta\}$, α and β are accepted as being true. However, implicit in the revision is the (counterfactual) notion that *if* β were found to not be true then α would still be held to be true. Semantically, this means that, among the $\neg\beta$ worlds, the least α worlds are ranked below the least $\neg\alpha$ worlds. This then would have the effect that if β were later determined to be false, thus necessitating a revision by $\neg\beta$,

[7] A case that is not "reasonable" is where β is of the form $\alpha \vee \gamma$. Then $\mathcal{K} * \{\alpha, \beta\}$ is the same as $\mathcal{K} * \{\alpha, \alpha \vee \gamma\}$, and clearly one requires that $\alpha \notin Bel(\mathcal{K} * \{\alpha, \alpha \vee \gamma\} * \{\neg(\alpha \vee \gamma)\})$.

then α would still be believed to be true. These considerations extend straightforwardly to arbitrary (but finite) sets of formulas. Thus, for a set of formulas S, after the revision $\mathcal{K} * S$ all elements of S will be believed. Implicit in the resulting ranking function is the counterfactual notion that if the members of some subset of S, say S', were found to be false, then $S \setminus S'$ would still be believed to be true. This then has the effect that in $\mathcal{K} * S * \overline{S'}$, members of $S \setminus S'$ will continue to be believed.

Another way of thinking of the underlying procedure is that in a revision $\mathcal{K} * S$, subsets of S implicitly define a *context* that provides additional structure in the resulting ranking function. That is, in $\mathcal{K} * S$, the minimal S worlds are ranked below all $\neg(\wedge S)$ worlds. For $S' \subset S$, in the "context" of $\overline{S'}$ the same considerations apply. Thus, in the restriction of the ranking function to $\overline{S'}$ worlds, the minimal $S \setminus S'$ worlds are ranked below all $\neg(\wedge(S \setminus S'))$ worlds.

Essentially then, for a revision $\mathcal{K} * S$, changes to the underlying ranking on worlds will depend not just on the set S, but also on subsets of S. The intuition is that, in revising by S, all elements of S are believed; if some members of S are subsequently disbelieved then, insofar as possible, the remaining members of S are still believed. In the next section we formalise this intuition. The approach is independent of previous approaches to iterated revision, in that it can be combined with an existing approach to iterated revision to yield a "parallel" hybrid of that approach. Consequently, in the following section we combine the basic approach with that of [21] to yield what we suggest is the appropriate general model for iterated belief revision.

3.2. The basic approach

This section describes the basic approach to parallel revision, in which new information for revision is represented by a finite set of formulas. The intuition is that each formula of the set represents an undecomposable (with regards to revision) piece of information. To distinguish this from standard belief revision, we denote a parallel revision operator by $\otimes$. Formally, $\otimes$ maps a belief state $\mathcal{K}$ and finite set of formulas S to a revised belief state $\mathcal{K} \otimes S$. We assume henceforth that the second argument to $\otimes$ is a *finite*, nonempty[8] set of formulas.

To begin, we adapt the AGM postulates for revision by a set of formulas. The following are analogous to postulates given in [34], adapted for belief states:

($\mathcal{K} \otimes 1$) $Cn(Bel(\mathcal{K} \otimes S)) = Bel(\mathcal{K} \otimes S)$.
($\mathcal{K} \otimes 2$) $S \subseteq Bel(\mathcal{K} \otimes S)$.
($\mathcal{K} \otimes 3$) $Bel(\mathcal{K} \otimes S) \subseteq Bel(\mathcal{K}) + S$.
($\mathcal{K} \otimes 4$) If $Bel(\mathcal{K}) \cup S$ is consistent, then $Bel(\mathcal{K}) + S \subseteq Bel(\mathcal{K} \otimes S)$.
($\mathcal{K} \otimes 5$) $Bel(\mathcal{K} \otimes S)$ is inconsistent iff S is inconsistent.
($\mathcal{K} \otimes 6$) If $S_1 \equiv S_2$, then $Bel(\mathcal{K} \otimes S_1) = Bel(\mathcal{K} \otimes S_2)$.
($\mathcal{K} \otimes 7$) $Bel(\mathcal{K} \otimes (S_1 \cup S_2)) \subseteq Bel(\mathcal{K} \otimes S_1) + S_2$.
($\mathcal{K} \otimes 8$) If $Bel(\mathcal{K} \otimes S_1) \cup S_2$ is consistent, then $Bel(\mathcal{K} \otimes S_1) + S_2 \subseteq Bel(\mathcal{K} \otimes (S_1 \cup S_2))$.

Note that ($\mathcal{K} \otimes 6$) yields $Bel(\mathcal{K} \otimes S) = Bel(\mathcal{K} \otimes \{\wedge S\})$. With a slight abuse of terminology, we will also refer to revision operators that satisfy the above postulates as AGM revision operators.

It has been shown in [34] that the representation theorem of Grove [19] can be generalised to revision by a set of formulas. Given that we deal with a finite language, the *systems of spheres* of Grove's construction are interdefinable with faithful rankings, and so the representation theorem of [22] can also be generalised:

> A revision operator $\otimes$ satisfies ($\mathcal{K} \otimes 1$)–($\mathcal{K} \otimes 8$) iff there exists a faithful ranking $\preccurlyeq_{\mathcal{K}}$ for an arbitrary belief state $\mathcal{K}$, such that for any set of sentences S:
>
> $$Bel(\mathcal{K} \otimes S) = \mathcal{T}\big(\min\big(Mod(S), \preccurlyeq_{\mathcal{K}}\big)\big). \tag{2}$$

For the basic approach to parallel revision, we give two postulates and semantic conditions that characterise the approach.

The following postulates characterise the basic approach to parallel revision:

($\mathrm{P}^{\otimes}$) For $S_1 \subset S$, where $S_1 \cup (\overline{S \setminus S_1}) \nvdash \bot$,

$$S_1 \subseteq Bel\big(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1})\big).$$

[8] This differs from [10], which allowed the empty set, but is in agreement with [34]. There are several reasons for this change. Foremost, it is not clear that revision by the empty set is a meaningful operation, since $\mathcal{K} * \emptyset$ would seem to have the informal interpretation of revising in the absence of a report of information. Second, it eases the technical development. Last, while revision by $\emptyset$ could be equated most naturally with revision by $\{\top\}$, the resulting revision $\mathcal{K} * \{\top\}$ isn't entirely trivial, since in the case that $Bel(\mathcal{K})$ is inconsistent ($\mathcal{K} * 5$) stipulates that the revision by the empty set is sufficient to extract an agent from the inconsistent belief state.

($S^\otimes$) For $S_1 \subset S$,

$$Bel\big(\mathcal{K} \otimes \big(S_1 \cup (\overline{S \setminus S_1})\big)\big) = Bel\big(\mathcal{K} \otimes S \otimes \big(S_1 \cup (\overline{S \setminus S_1})\big)\big).$$

($P^\otimes$) is a postulate of *success preservation*. It asserts that for a revision of $\mathcal{K}$ by S, a subset $S_1 \subset S$ is preserved in revising by the negations of members of $S \setminus S_1$ whenever it is consistent to do so. This reflects the intuition that, in revising by S and after which some members of S are subsequently disbelieved, then insofar as possible the remaining members of S are still believed. ($S^\otimes$) expresses the fact that for set S and subset $S_1 \subset S$, revising by $S_1 \cup (\overline{S \setminus S_1})$ yields the same beliefs as first revising by S and then by $S_1 \cup (\overline{S \setminus S_1})$. Thus it expresses a condition of *conservativism* with respect to iterated belief.

These postulates can be combined as follows:

($PP^\otimes$) Let $S_1 \subset S$ where $S_1 \cup (\overline{S \setminus S_1}) \nvdash \bot$.
Then $Bel(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1})) = Bel(\mathcal{K} \otimes (S_1 \cup (\overline{S \setminus S_1})))$.

Thus revising by a set S and then the negations of some members of S yields the same belief set as revising by the negations of these members in S together with the remaining members of S.

We obtain the following result:

Proposition 1. *Suppose $\otimes$ is a revision operator satisfying Postulates ($\mathcal{K} \otimes 1$)–($\mathcal{K} \otimes 8$). Then $\otimes$ satisfies* ($PP^\otimes$) *if and only if it satisfies* ($P^\otimes$) *and* ($S^\otimes$).

In order to justify the Postulates ($P^\otimes$) and ($S^\otimes$), we turn next to the corresponding conditions on faithful orderings. From a semantic point of view, consider the following condition on a faithful ranking $\preccurlyeq_{\mathcal{K} \otimes S}$ defined in terms of a faithful ranking $\preccurlyeq_{\mathcal{K}}$:

($P^\otimes R$) Let $S_1 \subset S$ where $S_1 \cup (\overline{S \setminus S_1}) \nvdash \bot$. Then

$$\min\big(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K} \otimes S}\big) \subseteq Mod(S_1).$$

($S^\otimes R$) Let $S_1 \subset S$. Then

$$\min\big(Mod\big(S_1 \cup (\overline{S \setminus S_1})\big), \preccurlyeq_{\mathcal{K}}\big) = \min\big(Mod\big(S_1 \cup (\overline{S \setminus S_1})\big), \preccurlyeq_{\mathcal{K} \otimes S}\big).$$

Unsurprisingly, these conditions can also be combined into the single condition:

($PP^\otimes R$) Let $S_1 \subset S$ where $S_1 \cup (\overline{S \setminus S_1}) \nvdash \bot$. Then

$$\min\big(Mod\big(S_1 \cup (\overline{S \setminus S_1})\big), \preccurlyeq_{\mathcal{K}}\big) = \min\big(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K} \otimes S}\big).$$

Thus, informally, for $S_1 \subset S$, the minimum $S_1 \cup (\overline{S \setminus S_1})$ worlds in a faithful ranking are just the minimum $(\overline{S \setminus S_1})$ worlds following revision by S. This in turn means that following revision by S, S_1 will be true at the least $(\overline{S \setminus S_1})$ worlds, as desired.

Proposition 2. *Suppose $\preccurlyeq_{\mathcal{K}}$ is a faithful ranking. Then $\preccurlyeq_{\mathcal{K}}$ satisfies* ($PP^\otimes R$) *if and only if it satisfies* ($P^\otimes R$) *and* ($S^\otimes R$).

For illustration, consider the following examples:

Example 2. Let $\mathcal{P} = \{a, b\}$, and let the agent's faithful ranking be given as follows:

$$\{a\bar{b}\} \prec_{\mathcal{K}} \{ab\} \prec_{\mathcal{K}} \{\bar{a}\bar{b}\} \prec_{\mathcal{K}} \{\bar{a}b\}.$$

Thus $Bel(\mathcal{K}) = Cn(a \wedge \neg b)$. There are three possible faithful rankings resulting from $\mathcal{K} \otimes \{a, b\} = \mathcal{K}'$:

$$\{ab\} \prec_{\mathcal{K}'} \{a\bar{b}\} \prec_{\mathcal{K}'} \{\bar{a}b\} \prec_{\mathcal{K}'} \{\bar{a}\bar{b}\},$$

$$\{ab\} \prec_{\mathcal{K}'} \{\bar{a}b\} \prec_{\mathcal{K}'} \{a\bar{b}\} \prec_{\mathcal{K}'} \{\bar{a}\bar{b}\},$$

$$\{ab\} \prec_{\mathcal{K}'} \{a\bar{b}, \bar{a}b\} \prec_{\mathcal{K}'} \{\bar{a}\bar{b}\}.$$

This example is very simple. In fact it can be observed in the example that, since the possible worlds correspond to subsets of $\{a, b\}$, the possible outcomes are independent of the initial belief state $\mathcal{K}$. The next example is a little more complicated.

Example 3. Let $\mathcal{P} = \{a, b, c\}$, and let the agent's faithful ranking be given as follows:

$$\{a\bar{b}c, a\bar{b}\bar{c}\} \prec_{\mathcal{K}} \{abc, \bar{a}bc, \bar{a}\bar{b}c\} \prec_{\mathcal{K}} \{ab\bar{c}, \bar{a}b\bar{c}, \bar{a}\bar{b}\bar{c}\}.$$

For $\mathcal{K} \otimes \{a, b\} = \mathcal{K}'$, three possible faithful rankings are as follows:

$$\{abc\} \prec_{\mathcal{K}'} \{a\bar{b}c, a\bar{b}\bar{c}\} \prec_{\mathcal{K}'} \{\bar{a}bc\} \prec_{\mathcal{K}'} \{\bar{a}b\bar{c}, \bar{a}\bar{b}c\} \prec_{\mathcal{K}'} \{ab\bar{c}, \bar{a}\bar{b}\bar{c}\},$$

$$\{abc\} \prec_{\mathcal{K}'} \{ab\bar{c}, a\bar{b}c, a\bar{b}\bar{c}, \bar{a}bc\} \prec_{\mathcal{K}'} \{\bar{a}b\bar{c}, \bar{a}\bar{b}c\} \prec_{\mathcal{K}'} \{\bar{a}\bar{b}\bar{c}\},$$

$$\{abc\} \prec_{\mathcal{K}'} \{ab\bar{c}\} \prec_{\mathcal{K}'} \{a\bar{b}c, a\bar{b}\bar{c}, \bar{a}bc\} \prec_{\mathcal{K}'} \{\bar{a}b\bar{c}\} \prec_{\mathcal{K}'} \{\bar{a}\bar{b}c\} \prec_{\mathcal{K}'} \{\bar{a}\bar{b}\bar{c}\}.$$

Thus $Bel(\mathcal{K}) = Cn(a \wedge \neg b)$ and $Bel(\mathcal{K}') = Cn(a \wedge b \wedge c)$. Two properties can be observed for each of the example rankings for $\mathcal{K}'$. First, for any $S_1 \subseteq S = \{a, b\}$, the minimum $S_1 \cup (\overline{S \setminus S_1})$ worlds are the same in $\mathcal{K}$ and $\mathcal{K}'$. Thus the minimum $\{a, b\}$ worlds in each case is given by $\{abc\}$, and the minimum $\{a, \neg b\}$ worlds is given by $\{a\bar{b}c, a\bar{b}\bar{c}\}$. Second, for any $S_2 \subset S_1 \subseteq S = \{a, b\}$, the minimum $S_1 \cup (\overline{S \setminus S_1})$ worlds are ranked strictly lower than the minimum $S_2 \cup (\overline{S \setminus S_2})$ worlds. Hence, the minimum $\{a, b\}$ worlds are ranked strictly lower than the minimum $\{a, \neg b\}$ and $\{\neg a, b\}$ worlds, and these latter sets are strictly lower than the minimum $\{\neg a, \neg b\}$ worlds.

Otherwise, for the three rankings given in the example, the first is not particularly special one way or another. The second, in a sense, is the most compact possible ranking, in that worlds are positioned as low in the ranking as possible. The third ranking has the property that for $S_2 \subset S_1 \subseteq S$, every $S_1 \cup (\overline{S \setminus S_1})$ world is ranked below every $S_2 \cup (\overline{S \setminus S_2})$ world. This last example can be seen as extending the approach of Nayak et al. [32] or Jin and Thielscher [21] to apply to subsets of a set for revision; we develop this latter point in Section 4 with regards to [21].

The next result expresses basic properties of faithful rankings that satisfy ($\mathrm{P}^{\otimes}\mathrm{R}$) and ($\mathrm{S}^{\otimes}\mathrm{R}$).

Proposition 3. *Let $\mathcal{K}$ be a belief state and let $\preccurlyeq_{\mathcal{K}}$ be the faithful ranking induced by $\mathcal{K}$. Let $\otimes$ revise faithful rankings corresponding to an AGM revision operator, and let $\preccurlyeq_{\mathcal{K}\otimes S}$ be a faithful ranking satisfying* ($\mathrm{P}^{\otimes}\mathrm{R}$) *and* ($\mathrm{S}^{\otimes}\mathrm{R}$).

1. *Let $S_1 \subset S$ be such that $S_1 \cup (\overline{S \setminus S_1})$ is consistent.*
   Then $\min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S}) = \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$.
2. *Let $S_2 \subset S_1 \subset S$ and $S_2 \nvdash S_1$.*
   If $w_1 \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$ and $w_2 \in \min(Mod(\overline{S \setminus S_2}), \preccurlyeq_{\mathcal{K}\otimes S})$ then $w_1 \prec_{\mathcal{K}\otimes S} w_2$.

The first part expresses the fundamental intuition underlying the approach: after revising by a set of formulas S, in the resulting faithful ordering restricted to $(\overline{S \setminus S_1})$ worlds, the least S_1 worlds will be ranked lower than any world in which S_1 isn't true. The second part expresses another fundamental property of the approach, that after revising by a set S, for $S_2 \subset S_1 \subset S$, the minimum $(\overline{S \setminus S_1})$ worlds are ranked below the minimum $(\overline{S \setminus S_2})$ worlds. Both parts refer to the faithful ranking corresponding to $\mathcal{K} \otimes S$. Together they can be thought of as expressing a relation among conditional (or counterfactual) beliefs; that is, after revising by S, the faithful ranking given by $\mathcal{K} \otimes S$ reflects the counterfactual assertion that if some members of S were determined to be false, the remaining elements of S would still be believed to be true. Hence if a revision by S were *in fact* followed by a revision wherein some members of S were asserted to be false, the remaining elements of S would be believed to be true.

We obtain the representation result:

Theorem 2. *Let $\otimes$ be a revision operator satisfying Postulates* ($\mathcal{K} \otimes 1$)–($\mathcal{K} \otimes 8$).

1. *$\otimes$ satisfies* ($\mathrm{P}^{\otimes}$) *iff it revises faithful rankings according to* ($\mathrm{P}^{\otimes}\mathrm{R}$).
2. *$\otimes$ satisfies* ($\mathrm{S}^{\otimes}$) *iff it revises faithful rankings according to* ($\mathrm{S}^{\otimes}\mathrm{R}$).

An AGM revision operator that satisfies Postulates ($\mathrm{P}^{\otimes}$) and ($\mathrm{S}^{\otimes}$) will be referred to as a (*basic*) *parallel revision operator*.

We have shown that in revising by a set of formulas S, and then revising by the negations of some subset S' of S, that the remaining members of S will continue to be believed. The next result shows that if we revise by a set of formulas S, and then revise where *some* formulas of a subset S' of S are false (but it is not necessarily known which), then the other formulas in S will continue to be believed. This result is more or less a corollary to the factoring result in AGM revision in the context of parallel revision.

Proposition 4. *Let $\otimes$ be a basic parallel revision operator and let $S_1 \subset S$.*
Then $S_1 \subseteq Bel(\mathcal{K} \otimes S \otimes \{\vee(\overline{S \setminus S_1})\})$.

Thus, if all we know is that some members of $S \setminus S_1$ are false, then S_1 will still be believed after revising by S followed by $\vee(\overline{S \setminus S_1})$.

Some examples will make the properties and ramifications of the approach clear. Throughout the following examples, $\mathcal{K}$ will be some belief state and S will be a set of formulas. α, β, γ will be logically independent formulas, that is, for $\phi_1, \phi_2 \in \{\alpha, \beta, \gamma\}$ if $\phi_1 \vdash \phi_2$ then $\phi_1 = \phi_2$.

The first example considers the situation where the elements of the set for revision are independent.

Example 4. Consider $\mathcal{K} \otimes S$ where $S = \{\alpha, \beta, \gamma\}$.

We get that in a faithful ranking resulting from the revision $\preccurlyeq_{\mathcal{K}\otimes S}$, the least $\{\alpha, \beta, \gamma\}$ worlds are strictly less than the least $\{\alpha, \beta, \neg\gamma\}$ worlds, which in turn are strictly less than the least $\{\alpha, \neg\beta, \neg\gamma\}$ worlds.

Consequently we obtain:

$$\alpha \wedge \gamma \in Bel\big(\mathcal{K} \otimes \{\alpha, \beta, \gamma\} \otimes \{\neg\beta\}\big),$$
$$\alpha \in Bel\big(\mathcal{K} \otimes \{\alpha, \beta, \gamma\} \otimes \{\neg\beta, \neg\gamma\}\big),$$
$$\alpha \in Bel\big(\mathcal{K} \otimes \{\alpha, \beta, \gamma\} \otimes \{\neg\beta \vee \neg\gamma\}\big).$$

The first two parts illustrate the basic property of the approach in the case of logically independent formulas: that revising by a set of formulas, then by the negation of some members of the set leaves the remaining elements still in the agent's belief set. The last part illustrates the result given in Proposition 4.

In the next example, the elements of the set for revision are not independent.

Example 5. Consider $\mathcal{K} \otimes S$ where $S = \{\alpha, \alpha \wedge \beta, \gamma\}$.

We obtain:

$$\alpha \wedge \gamma \in Bel\big(\mathcal{K} \otimes \{\alpha, \alpha \wedge \beta, \gamma\} \otimes \{\neg\beta\}\big),$$
$$\gamma \in Bel\big(\mathcal{K} \otimes \{\alpha, \alpha \wedge \beta, \gamma\} \otimes \{\neg\alpha\}\big).$$

On the other hand, there are resulting faithful rankings in which

$$\beta \notin Bel\big(\mathcal{K} \otimes \{\alpha, \alpha \wedge \beta, \gamma\} \otimes \{\neg\alpha\}\big).$$

In the first case, since β is believed to be false, then $\alpha \wedge \beta$ must certainly also be false. On the other hand, the other elements of the set, α and γ continue to be believed after revision by $\neg\beta$. Similarly in the second part, if α is false, then γ can continue to be believed.

The following two examples illustrate a very interesting phenomenon, that the approach can be used to express a preference over which formulas are accepted.

Example 6. Consider $\mathcal{K} \otimes S$ where $S = \{\alpha, \alpha \wedge \beta\}$.

With respect to the agent's contingent beliefs, revision by $\{\alpha, \alpha \wedge \beta\}$ is of course the same as revision by $\{\alpha \wedge \beta\}$.

However, we also obtain:

$$\alpha \in Bel\big(\mathcal{K} \otimes \{\alpha, \alpha \wedge \beta\} \otimes \{\neg\alpha \vee \neg\beta\}\big).$$

That is, in revising by $\{\alpha, \alpha \wedge \beta\}$, we effectively encode the preference that if one of α or β are to be subsequently given up, then β will be given up and α retained. In terms of faithful rankings, we have the following. After revising $\mathcal{K}$ by $S = \{\alpha, \alpha \wedge \beta\}$, at the minimum worlds in the resulting ranking, $\{\alpha, \alpha \wedge \beta\}$ will be true. For subsets S_1 of S, we must have that $S_1 \cup (\overline{S \setminus S_1})$ is consistent. This will be the case for $S_1 = \{\alpha\}$ and (PP$^{\otimes}$R) stipulates that the minimum $\neg\alpha \vee \neg\beta$ worlds in the ranking associated with $\mathcal{K} \otimes S$ is the same as the minimum $\{\alpha\} \cup \{(\overline{\alpha \wedge \beta})\} = \{\alpha \wedge \neg\beta\}$ worlds associated with $\mathcal{K}$. Hence in revising by S and then $\{\neg\alpha \vee \neg\beta\}$, we get that $\{\alpha \wedge \neg\beta\}$ is true in the resulting belief set.

Example 7. Consider $\mathcal{K} \otimes S$ where $S = \{\alpha, \alpha \vee \beta\}$.

Clearly, for atoms α, β, we don't generally obtain that $\beta \in Bel(\mathcal{K} \otimes \{\alpha, \alpha \vee \beta\})$, since the logical content of $\{\alpha, \alpha \vee \beta\}$ is equivalent to that of $\{\alpha\}$.

However, we do obtain:

$$\beta \in Bel\big(\mathcal{K} \otimes \{\alpha, \alpha \vee \beta\} \otimes \{\neg\alpha\}\big).$$

Thus, after revising by $\{\alpha, \alpha \vee \beta\}$ we don't necessarily believe that β is true; however we do believe that β is true on subsequently revising by $\{\neg\alpha\}$. This result, on reflection, is to be expected: In revising by a set, if one of the elements of the set is found later to be false then, where consistently possible, the remaining elements of the set would still be believed. Thus in revising by $\{\alpha, \alpha \vee \beta\}$, if α were subsequently determined to be false then the remaining element, viz. $\alpha \vee \beta$ would remain true. But since α is now believed false, this requires that β is now believed to be true. Thus in this case in revising by $\{\alpha, \alpha \vee \beta\}$, a preference is established between α and β, to the effect of "accept α, but if it is subsequently found to be false, accept β".

The next two examples deal with revising by a set of formulas where the set is inconsistent. In the first case, individual elements of the set are consistent; in the second, some member of the set is inconsistent. In both cases, we obtain desirable results in subsequent revisions. This illustrates that, even though revision by an inconsistent set is defined to yield an inconsistent belief set, the underlying faithful ranking nonetheless retains nontrivial information about the agent's belief state.

Example 8. Consider $\mathcal{K} \otimes S$ where $S = \{\alpha, \neg\alpha, \beta, \gamma\}$.

Obviously $Bel(\mathcal{K} \otimes \{\alpha, \neg\alpha, \beta, \gamma\})$ is inconsistent. However we obtain the following:

$$\beta \wedge \gamma \in Bel\big(\mathcal{K} \otimes \{\alpha, \neg\alpha, \beta, \gamma\} \otimes \{\alpha\}\big),$$
$$\perp \notin Bel\big(\mathcal{K} \otimes \{\alpha, \neg\alpha, \beta, \gamma\} \otimes \{\alpha\}\big),$$
$$\beta \wedge \gamma \in Bel\big(\mathcal{K} \otimes \{\alpha, \neg\alpha, \beta, \gamma\} \otimes \{\alpha \vee \neg\alpha\}\big),$$
$$\gamma \in Bel\big(\mathcal{K} \otimes \{\alpha, \neg\alpha, \beta, \gamma\} \otimes \{\neg\beta\}\big).$$

Analogous results obtain when an element of the set for revision is inconsistent.

Example 9. Consider $\mathcal{K} \otimes S$ where $S = \{\perp, \alpha, \beta\}$.

We obtain:

$$\beta \in Bel\big(\mathcal{K} \otimes \{\perp, \alpha, \beta\} \otimes \{\alpha\}\big),$$
$$\beta \in Bel\big(\mathcal{K} \otimes \{\perp, \alpha, \beta\} \otimes \{\neg\alpha\}\big),$$
$$\perp \notin Bel\big(\mathcal{K} \otimes \{\perp, \alpha, \beta\} \otimes \{\alpha\}\big).$$

4. Parallel revision and iterated revision

The basic approach only deals with limited situations where we first revise by a set of formulas then by the negations of some of these formulas. In this section, we extend the basic approach to deal with more general cases. We first show that the straightforward generalisation of the well-known iterated revision postulates are problematic and insufficient. Then, we present a postulate of *evidence retainment*, which offers an alternative that avoids these difficulties.

We start with the following generalisation of the DP postulates to sets of formulas, as suggested by [40]:

($C1^{\otimes}$) If $S_2 \vdash S_1$, then $Bel((\mathcal{K} \otimes S_1) \otimes S_2) = Bel(\mathcal{K} \otimes S_2)$.
($C2^{\otimes}$) If $S_1 \cup S_2$ is inconsistent, then $Bel((\mathcal{K} \otimes S_1) \otimes S_2) = Bel(\mathcal{K} \otimes S_2)$.
($C3^{\otimes}$) If $S_1 \subseteq Bel(\mathcal{K} \otimes S_2)$, then $S_1 \subseteq Bel((\mathcal{K} \otimes S_1) \otimes S_2)$.
($C4^{\otimes}$) If $S_1 \cup Bel(\mathcal{K} \otimes S_2)$ is consistent, then $S_1 \cup Bel((\mathcal{K} \otimes S_1) \otimes S_2)$ is also consistent.

We remark that, while ($C1^{\otimes}$), ($C3^{\otimes}$) and ($C4^{\otimes}$) still seem as reasonable as their counterparts, ($C2^{\otimes}$) is not desirable. First, previous criticisms of (C2) apply equally well to ($C2^{\otimes}$). Second, ($C2^{\otimes}$) is clearly inconsistent with ($P^{\otimes}$) in the presence of the (adapted to sets) AGM postulates. As a specific example, let α and β be logically independent formulas, and assume that $\neg\alpha \in Bel(\mathcal{K})$ and $\beta \notin Bel(\mathcal{K})$. Then ($C2^{\otimes}$) dictates that $Bel(\mathcal{K} \otimes \{\alpha, \beta\} \otimes \{\neg\beta\}) = Bel(\mathcal{K} \otimes \{\neg\beta\}) = Bel(\mathcal{K}) + \{\neg\beta\}$. Thus $\neg\alpha \in Bel(\mathcal{K} \otimes \{\alpha, \beta\} \otimes \{\neg\beta\})$. On the other hand, ($P^{\otimes}$) requires that $\{\alpha\} \subseteq Bel(\mathcal{K} \otimes \{\alpha, \beta\} \otimes \{\neg\beta\})$. Hence, we do not consider ($C2^{\otimes}$) further as a general postulate for parallel revision.

For reference, the semantical conditions for the DP postulates can be generalised as follows:

($C1^{\otimes}R$) If $w_1, w_2 \models S$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ iff $w_1 \preccurlyeq_{\mathcal{K} \otimes S} w_2$.
($C2^{\otimes}R$) If $w_1, w_2 \not\models S$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ iff $w_1 \preccurlyeq_{\mathcal{K} \otimes S} w_2$.
($C3^{\otimes}R$) If $w_1 \models S$ and $w_2 \not\models S$, then $w_1 \prec_{\mathcal{K}} w_2$ implies $w_1 \prec_{\mathcal{K} \otimes S} w_2$.
($C4^{\otimes}R$) If $w_1 \models S$ and $w_2 \not\models S$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ implies $w_1 \preccurlyeq_{\mathcal{K} \otimes S} w_2$.

To show ($C2^{\otimes}$) is undesirable from another perspective, one may argue that ($C2^{\otimes}R$) is overly strong: in the case where w_2 satisfies more sentences of S than w_1, it is perfectly reasonable that we might have $w_2 \prec_{\mathcal{K} \otimes S} w_1$ even if $w_1 \preccurlyeq_{\mathcal{K}} w_2$.

Similarly, we can also generalise the postulate of independence and its corresponding semantical condition:

(Ind$^{\otimes}$) If $S_1 \cup Bel(\mathcal{K} \otimes S_2)$ is consistent, then $S_1 \subseteq Bel((\mathcal{K} \otimes S_1) \otimes S_2)$.
(Ind$^{\otimes}$R) If $w_1 \models S$ and $w_2 \not\models S$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ implies $w_1 \prec_{\mathcal{K} \otimes S} w_2$.

Note that, among the above-mentioned postulates, (C2$^{\otimes}$) is the only one which deals with the case where S_1 and S_2 are jointly inconsistent. This suggests that we need some new postulates in order to address this situation. As already argued, it is too radical to give up all formulas of S_1 (as imposed by (C$^{\otimes}$2)) just because $S_1 \cup S_2$ is inconsistent. The question is, in revising by S_1 and then S_2 when $S_1 \cup S_2$ is inconsistent, what formulas in S_1 should be retained? Intuitively, a formula $\alpha \in S_1$ should be kept if there is no evidence (in S_1 and S_2) against α after learning S_2. To formalise this idea, we need the following definition:

Definition 2. Let S_1, S_2 be two sets of sentences. We denote by $S_1 || S_2$ the set of all subsets of S_1 that are consistent with S_2. That is $S_c \in S_1 || S_2$ iff:

1. $S_c \subseteq S_1$.
2. $S_c \cup S_2$ is consistent.

Formally, the fact that there exists evidence in S_1 against α after learning S_2 (given the original belief state $\mathcal{K}$) can be expressed as: $\exists S_c \in S_1 || S_2$ such that $\neg\alpha \in Bel(\mathcal{K} \otimes (S_c \cup S_2))$.

Based on these considerations, we obtain the so-called postulate of *evidence retainment*:

$$\text{If } \alpha \in S_1 \text{ and } \alpha \notin Bel\big((\mathcal{K} \otimes S_1) \otimes S_2\big), \quad \text{then } \exists S_c \in S_1 || S_2 \quad \text{such that} \quad \neg\alpha \in Bel\big(\mathcal{K} \otimes (S_c \cup S_2)\big).$$

This postulate is inspired by the postulate of *core retainment* [20], which says a formula α is removed from a belief set K by a contraction with β only if there is some evidence in K that shows that α contributes to the implication of β. Formally, core retainment is expressed as follows:

$$\text{If } \alpha \in K \text{ and } \alpha \notin K - \beta, \quad \text{then } \exists A \subseteq K \quad \text{such that} \quad A \nvdash \beta \text{ but } A \cup \alpha \vdash \beta.$$

The postulate of evidence retainment can be equivalently rephrased as follows:

(Ret$^{\otimes}$) If $\alpha \in S_1$, and for every $S_c \in S_1 || S_2$ where $S_2 \neq \emptyset$ we have $\neg\alpha \notin Bel(\mathcal{K} \otimes (S_c \cup S_2))$, then

$$\alpha \in Bel\big((\mathcal{K} \otimes S_1) \otimes S_2\big).$$

Recall Example 1 with $S_1 = \{a, m\}$ and $S_2 = \{\neg a\}$. Since $S_1 || S_2 = \{\{m\}\}$, Postulate (Ret$^{\otimes}$) implies that $(\mathcal{K} \otimes S_1) \otimes S_2 \vdash m$, which gives us the desired result. Note that, in case a and m make up a single piece of information (i.e. $S_1 = \{a \wedge m\}$), Postulate (Ret$^{\otimes}$) does not apply.

To give a formal justification for (Ret$^{\otimes}$), we will show a representation theorem.

Definition 3. Let S be a set of sentences and w a possible world. Then $S|w$ denotes the set of elements of S which are true in w, i.e., $S|w = \{\alpha \in S \mid w \models \alpha\}$.

The following theorem gives a necessary and sufficient semantical condition for (Ret$^{\otimes}$):

Theorem 3. *Suppose $\otimes$ is a parallel revision operator satisfying Postulates $(\mathcal{K} \otimes 1)$–$(\mathcal{K} \otimes 8)$. Then $\otimes$ satisfies* (Ret$^{\otimes}$) *iff it revises faithful rankings in the following manner*:

(Ret$^{\otimes}$R) *If $S|w_2 \subset S|w_1$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ implies that $w_1 \prec_{\mathcal{K} \otimes S} w_2$.*

Arguably, (Ret$^{\otimes}$R) is very natural and intuitive. It essentially says: if w_1 confirms more new information (in S) than w_2, and w_1 is at least as plausible as w_2, then w_1 becomes more plausible than w_2 after revising by S. It is not difficult to see that (Ret$^{\otimes}$) implies (Ind$^{\otimes}$).

Proposition 5. *Suppose $\otimes$ is a parallel revision operator satisfying Postulates $(\mathcal{K} \otimes 1)$–$(\mathcal{K} \otimes 8)$. Then $\otimes$ satisfies* (Ret$^{\otimes}$) *only if it satisfies* (Ind$^{\otimes}$).

For the effect of (Ret$^{\otimes}$), consider the following example.

Example 10. Let $\otimes$ be a revision operator satisfying Postulates $(\mathcal{K}\otimes 1)$–$(\mathcal{K}\otimes 8)$, $(K\otimes P)$, $(K\otimes S)$, and $(\mathrm{Ret}^{\otimes})$. Let α, β, and γ be logically independent formulas.

We obtain:

$$\text{If } \neg\alpha \notin Bel\big(\mathcal{K}\otimes\{\neg\beta,\gamma\}\big) \quad \text{then } \alpha \in Bel\big(\mathcal{K}\otimes\{\alpha,\beta\}\otimes\{\neg\beta,\gamma\}\big).$$

On the other hand, there is a revision operator $\otimes$ satisfying Postulates $(\mathcal{K}\otimes 1)$–$(\mathcal{K}\otimes 8)$, $(K\otimes P)$, and $(K\otimes S)$, but not $(\mathrm{Ret}^{\otimes})$, such that

$$\neg\alpha \notin Bel\big(\mathcal{K}\otimes\{\neg\beta,\gamma\}\big) \quad \text{and} \quad \alpha \notin Bel\big(\mathcal{K}\otimes\{\alpha,\beta\}\otimes\{\neg\beta,\gamma\}\big).$$

The examples of the previous section are of the form $\mathcal{K}\otimes S\otimes T$ where members of T were denials of elements of S. In the basic approach to parallel revision, it is possible to have $\alpha \notin Bel(\mathcal{K}\otimes\{\alpha,\beta\}\otimes\{\neg\beta,\gamma\})$ because, after the revision $\mathcal{K}\otimes\{\alpha,\beta\}$, the minimal $\neg\beta$ worlds will also have α be true, as expected. However, the minimal $\neg\beta$ worlds may also happen to have $\neg\gamma$ also be true; the basic approach places no constraints on the minimal $\neg\beta\wedge\gamma$ worlds, and at these worlds it is quite possible that α not be true. On the other hand, $(\mathrm{Ret}^{\otimes})$ guarantees that α is believed following the revisions by $\{\alpha,\beta\}$ and $\{\neg\beta,\gamma\}$, provided $\neg\alpha$ is not believed in a revision by $\{\neg\beta,\gamma\}$. Informally, in revising by $\{\alpha,\beta\}$, we have that α will be believed, and in the subsequent revision $\{\neg\beta,\gamma\}$ if there is no reason to disbelieve α.

Based on similar considerations, we present two additional postulates which also seem quite intuitive, and which naturally extend $(\mathrm{C3}^{\otimes})$ and $(\mathrm{C4}^{\otimes})$.

$(\mathrm{PC3}^{\otimes})$ If for every $S_c \in S_1||S_2$ where $S_2 \neq \emptyset$ we have that $\alpha \in Bel(\mathcal{K}\otimes(S_c\cup S_2))$, then $\alpha \in Bel((\mathcal{K}\otimes S_1)\otimes S_2)$.
$(\mathrm{PC4}^{\otimes})$ If for every $S_c \in S_1||S_2$ where $S_2 \neq \emptyset$ we have that $\neg\alpha \notin Bel(\mathcal{K}\otimes(S_c\cup S_2))$, then $\neg\alpha \notin Bel((\mathcal{K}\otimes S_1)\otimes S_2)$.

Essentially, $(\mathrm{PC3}^{\otimes})$ says if all evidence in S_1 supports α after learning S_2, then α must be believed; $(\mathrm{PC4}^{\otimes})$ says if no evidence in S_1 is against α, then there is no reason to believe $\neg\alpha$. We present a representation theorem for $(\mathrm{PC3}^{\otimes})$ and $(\mathrm{PC4}^{\otimes})$ as the formal justification.

Theorem 4. *Suppose $\otimes$ is a parallel revision operator satisfying Postulates $(\mathcal{K}\otimes 1)$–$(\mathcal{K}\otimes 8)$. Then $\otimes$ satisfies $(\mathrm{PC3}^{\otimes})$ and $(\mathrm{PC4}^{\otimes})$ iff it revises faithful rankings in the following manner*:

$(\mathrm{PC3}^{\otimes}\mathrm{R})$ *If $S|w_2 \subseteq S|w_1$, then $w_1 \prec_{\mathcal{K}} w_2$ implies $w_1 \prec_{\mathcal{K}\otimes S} w_2$.*
$(\mathrm{PC4}^{\otimes}\mathrm{R})$ *If $S|w_2 \subseteq S|w_1$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ implies $w_1 \preccurlyeq_{\mathcal{K}\otimes S} w_2$.*

It can be observed that $(\mathrm{PC3}^{\otimes}\mathrm{R})$ and $(\mathrm{PC4}^{\otimes}\mathrm{R})$ extend $(\mathrm{C3}^{\otimes}\mathrm{R})$ and $(\mathrm{C4}^{\otimes}\mathrm{R})$, respectively.

Proposition 6. *Suppose $\otimes$ is a parallel revision operator satisfying Postulates $(\mathcal{K}\otimes 1)$–$(\mathcal{K}\otimes 8)$. Then $\otimes$ satisfies $(\mathrm{PC3}^{\otimes})$ only if it satisfies $(\mathrm{C3}^{\otimes})$; and $\otimes$ satisfies $(\mathrm{PC4}^{\otimes})$ only if it satisfies $(\mathrm{C4}^{\otimes})$.*

Moreover, the semantical conditions of $(\mathrm{PC3}^{\otimes}\mathrm{R})$ and $(\mathrm{PC4}^{\otimes}\mathrm{R})$ require that the relative ordering of two possible worlds remain unchanged, provided they satisfy the same subset of the new information.

Proposition 7. $(\mathrm{PC3}^{\otimes}\mathrm{R})$ *and* $(\mathrm{PC3}^{\otimes}\mathrm{R})$ *imply the following semantical condition*:

$$\textit{If } S|w_2 = S|w_1, \quad \textit{then } w_1 \preccurlyeq_K w_2 \quad \textit{iff} \quad w_1 \preccurlyeq_{K\otimes S} w_2.$$

It is not difficult to see that $(\mathrm{PC3}^{\otimes})$ and $(\mathrm{PC4}^{\otimes})$ together imply $(\mathrm{C1}^{\otimes})$ and $(\mathcal{K}\otimes S)$.

Proposition 8. *Suppose $\otimes$ is a parallel revision operator satisfying Postulates $(\mathcal{K}\otimes 1)$–$(\mathcal{K}\otimes 8)$. If $\otimes$ satisfies $(\mathrm{PC3}^{\otimes})$ and $(\mathrm{PC4}^{\otimes})$ then it also satisfies $(\mathrm{C1}^{\otimes})$ and $(\mathcal{K}\otimes S)$.*

As well, $(\mathrm{P}^{\otimes})$ does not follow from $(\mathrm{Ret}^{\otimes})$, $(\mathrm{PC3}^{\otimes})$, and $(\mathrm{PC4}^{\otimes})$; we give an example in terms of the corresponding semantical conditions.[9] Consider the language over propositional atoms a and b, and the faithful ranking $\preccurlyeq_{\mathcal{K}}$ given by:

$$\{\overline{ab}\} \prec_{\mathcal{K}} \{ab\} \prec_{\mathcal{K}} \{\bar{a}b\} \prec_{\mathcal{K}} \{a\bar{b}\}.$$

Assume that after revising by $S=\{a,b\}$ the faithful ranking $\preccurlyeq_{\mathcal{K}\otimes S}$ is given by:

$$\{ab\} \prec_{\mathcal{K}\otimes S} \{\overline{ab}\} \prec_{\mathcal{K}\otimes S} \{\bar{a}b\} \prec_{\mathcal{K}\otimes S} \{a\bar{b}\}.$$

It can be verified that $\preccurlyeq_{\mathcal{K}\otimes S}$ satisfies $(\mathrm{Ret}^{\otimes}\mathrm{R})$, $(\mathrm{PC3}^{\otimes}\mathrm{R})$, and $(\mathrm{PC4}^{\otimes}\mathrm{R})$, but not $(\mathrm{P}^{\otimes}\mathrm{R})$.

[9] We are indebted to a reviewer for this example.

We conclude with a result that further illustrates the relation between the basic parallel revision postulates and those for iterated revision. It has already been noted that ($P^{\otimes}$), and so ($PP^{\otimes}$), does not follow from ($Ret^{\otimes}$). However, we can show that a weakened version of ($PP^{\otimes}$) does follow from ($Ret^{\otimes}$). Consider the following weaker versions of ($PP^{\otimes}$) and ($PP^{\otimes}R$):

($PP^{\otimes'}$) Let $S_1 \subset S$ where $S_1 \cup Bel(\mathcal{K} \otimes (\overline{S \setminus S_1}))$ is consistent. Then

$$Bel\big(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1})\big) = Bel\big(\mathcal{K} \otimes \big(S_1 \cup (\overline{S \setminus S_1})\big)\big).$$

($PP^{\otimes}R'$) Let $S_1 \subset S$ where $S_1 \cup Bel(\mathcal{K} \otimes (\overline{S \setminus S_1}))$ is consistent. Then

$$\min\big(Mod\big(S_1 \cup (\overline{S \setminus S_1})\big), \preccurlyeq_{\mathcal{K}}\big) = \min(Mod\big(\overline{S \setminus S_1}\big), \preccurlyeq_{\mathcal{K}\otimes S}).$$

It is not difficult to see that ($Ret^{\otimes}$), ($PC3^{\otimes}$) and ($PC4^{\otimes}$) together imply ($PP^{\otimes'}$).

Proposition 9. *Suppose $\otimes$ is a parallel revision operator satisfying Postulates* ($\mathcal{K} \otimes 1$)–($\mathcal{K} \otimes 8$). *If $\otimes$ satisfies* ($Ret^{\otimes}$), ($PC3^{\otimes}$) *and* ($PC4^{\otimes}$) *then it also satisfies* ($PP^{\otimes'}$).

Based on the above development, we suggest a general parallel revision operator should satisfy the AGM postulates (extended to sets), ($P^{\otimes}$), ($Ret^{\otimes}$), ($PC3^{\otimes}$), and ($PC4^{\otimes}$).

5. OCF-based parallel revision

Up to this point, we have considered those properties that parallel revision, regarded a mathematical function, should satisfy. We now present a concrete parallel revision operator which satisfies all the proposed postulates. The operator is based on Spohn's proposal of *ordinal conditional functions* [37].

Originally, an ordinal conditional function (OCF) was defined as a mapping κ from the set of possible worlds $\Theta_{\mathcal{P}}$ to the class of ordinals such that some world was assigned the value 0. An OCF provides one concrete form of a belief state. As in [38], for the sake of simplicity, we take the signature of an OCF κ as $\Theta_{\mathcal{P}} \to \mathbb{N}$, where $\kappa(w)$ is called the *rank* of w. Intuitively, the rank of a world represents its degree of plausibility. The lower a world's rank, the more plausible that world is. A formula α is in the belief set $Bel(\kappa)$ just if every world of rank 0 is a model of α; that is:

$$Mod\big(Bel(\kappa)\big) = \big\{w \bigm| \kappa(w) = 0\big\}.$$

The corresponding faithful ranking can be defined as follows:

$$w_1 \preccurlyeq_{\kappa} w_2 \quad \text{iff} \quad \kappa(w_1) \leqslant \kappa(w_2). \tag{3}$$

Given an OCF κ, we extend this function to a ranking on sentences (or sets of sentences) as follows:

$$\kappa(\mu) = \begin{cases} \infty & \text{if } \vdash \neg\mu, \\ \min\{\kappa(w) \mid w \models \mu\} & \text{otherwise} . \end{cases}$$

Put in words, the rank of a sentence is the lowest rank of a world in which the sentence holds.

In what follows we give a concrete parallel revision function in terms of an OCF. That is, for a given OCF κ and finite set of formulas S, we define a revised OCF $\kappa \otimes S$; the corresponding faithful ranking can then be obtained via (3).

Consider for reference the semantic conditions for parallel revision:

($P^{\otimes}R$) Let $S_1 \subset S$ where $S_1 \cup (\overline{S \setminus S_1}) \nvdash \bot$. Then $\min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S}) \subseteq Mod(S_1)$.
($Ret^{\otimes}R$) If $S|w_2 \subset S|w_1$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ implies that $w_1 \prec_{\mathcal{K}\otimes S} w_2$.
($PC3^{\otimes}R$) If $S|w_2 \subseteq S|w_1$, then $w_1 \prec_{\mathcal{K}} w_2$ implies $w_1 \prec_{\mathcal{K}\otimes S} w_2$.
($PC4^{\otimes}R$) If $S|w_2 \subseteq S|w_1$, then $w_1 \preccurlyeq_{\mathcal{K}} w_2$ implies $w_1 \preccurlyeq_{\mathcal{K}\otimes S} w_2$.

We will use the following notation: Let S be a finite set of formulas, and let $S_1 \subseteq S$.

$$C_s(S_1) = S_1 \cup (\overline{S \setminus S_1}),$$
$$\min(\mu, \kappa) = \big\{w \bigm| w \models \mu, \text{ and for every } w' \text{ where } \kappa\big(w'\big) < \kappa(w), w' \not\models \mu\big\}.$$

Mnemonically, $C_s(S_1)$ is the "completion" of S_1 with respect to S: If $\alpha \in S \setminus S_1$ then $\neg\alpha \in C_s(S_1)$. $\min(\mu, \kappa)$ is the set of least μ worlds with respect to κ. We also use $\min(S, \kappa)$ for a set of formulas S to denote the κ-least set of S-worlds.

The construction of $\kappa \otimes S$ is given in terms of a recurrence relation. The definition is admittedly somewhat complicated, although each part of the definition reflects a basic intuition concerning postulates of parallel OCF revision. Consequently, after presenting the definition, we discuss intuitions underlying the parts of the definition, and then provide a formal statement that shows that the definition indeed specifies a general parallel revision operator.

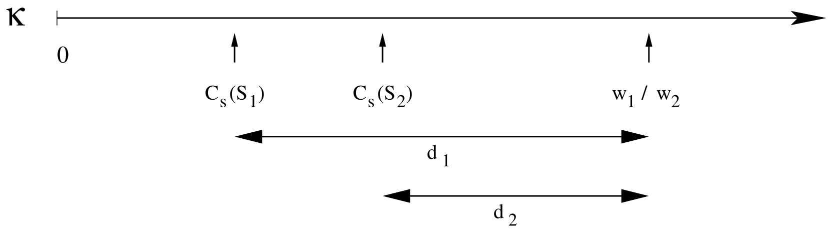

Fig. 1. Enforcing (Ret$^\otimes$R).

Definition 4. Let κ be an ordinal conditional function over $\Theta_{\mathcal{P}}$ and let S be a finite satisfiable set of sentences. Define the parallel OCF revision of κ by S, $\kappa \otimes S$, by

1. If $w \in \min(S, \kappa)$ then $(\kappa \otimes S)(w) = 0$.
2. Assume that $(\kappa \otimes S)(w_1)$ has been assigned for every $w_1 \in \Theta_{\mathcal{P}}$ where $w_1 \in \min(C_s(S_1), \kappa)$, and where $S_1 \subseteq S$ and $|S| - |S_1| < i$ for $i > 0$.
   Let $w_2 \in \min(C_s(S_2), \kappa)$ where $S_2 \subset S$ and $|S| - |S_2| = i$. Then

$$(\kappa \otimes S)(w_2) = \max\bigl\{(\kappa \otimes S)\bigl(C_s\bigl(S_1'\bigr)\bigr), \tag{4}$$

$$(\kappa \otimes S)\bigl(C_s\bigl(S_1'\bigr)\bigr) + \kappa\bigl(C_s(S_2)\bigr) - \kappa\bigl(C_s\bigl(S_1'\bigr)\bigr) \tag{5}$$

$$\bigm| S_2 \subset S_1' \subseteq S \text{ and } \bigl|S_1'\bigr| + 1 = |S_2|\bigr\} + 1.$$

3. For $w \notin \min(C_s(S|w), \kappa)$,

$$(\kappa \otimes S)(w) = (\kappa \otimes S)\bigl(C_s(S|w)\bigr) + \kappa(w) - \kappa\bigl(C_s(S|w)\bigr).$$

Each part of the definition applies to a particular set of worlds with respect to κ. The first condition in the definition applies to minimum S worlds in κ, and ensures that these worlds are given rank 0 in $\kappa \otimes S$; the remaining conditions implicitly assign a rank greater than 0 to all other worlds. The second part of the definition assigns ranks in $\kappa \otimes S$ to minimum (in κ) $C_s(S_1)$ worlds for every $S_1 \subset S$. This part is phrased iteratively, working from larger subsets of S to smaller. The third part of the definition assigns ranks in $\kappa \otimes S$ to all remaining worlds.

Intuitively the various postulates are obtained as follows. Since the set of 0-ranked worlds in $\kappa \otimes S$ is the same as the minimum S worlds in κ, and since every world is assigned a rank in $\kappa \otimes S$, it follows that $\kappa \otimes S$ defines a faithful ranking. Thus, via [22] we have an AGM revision operator. (P$^\otimes$R) is obtained by requiring, for $S_2 \subset S_1 \subseteq S$ where $|S_2| + 1 = |S_1|$, that

$$(\kappa \otimes S)\bigl(C_s(S_2)\bigr) \geqslant (\kappa \otimes S)\bigl(C_s(S_1)\bigr) + 1.$$

That is, in $\kappa \otimes S$ the rank of the least $C_s(S_2)$ worlds is greater than the rank of the least $C_s(S_1)$ worlds. This is taken care of by (4) in the definition. The general case, where $|S_2| + i = |S_1|$ for $i > 0$ follows trivially by transitivity of $\geqslant$. (PC3$^\otimes$R) and (PC4$^\otimes$R) are obtained by a condition similar to that in [9], adjusted for subsets of a set of formulas for revision: For worlds w_1 and w_2 where we have $S_1|w_1 = S_1|w_2$ for $S_1 \subseteq S$, we require that the difference in rankings between w_1 and w_2 will be the same in $\kappa \otimes S$ as in κ. Condition 3 of the definition ensures that this is the case. (Ret$^\otimes$R) is trickier; refer to Fig. 1, where we have that $\kappa(w_1) = \kappa(w_2)$, and assume that $w_1 \in Mod(C_s(S_1))$ and $w_2 \in Mod(C_s(S_2))$. The minimum $C_s(S_1)$ and $C_s(S_2)$ worlds are indicated by arrows on the κ ranking. A potential problem arises, in that to this point there is nothing to prevent

$$(\kappa \otimes S)\bigl(C_s(S_2)\bigr) - (\kappa \otimes S)\bigl(C_s(S_1)\bigr) < \kappa\bigl(C_s(S_2)\bigr) - \kappa\bigl(C_s(S_1)\bigr).$$

Our constraints for (PC3$^\otimes$R) and (PC4$^\otimes$R) require that the respective distances d_1 and d_2 be the same in κ and $\kappa \otimes S$, and this would yield that

$$(\kappa \otimes S)(w_2) - (\kappa \otimes S)(w_1) < 0.$$

This in turn violates (Ret$^\otimes$R). Condition (5) in the definition ensures that this doesn't occur, and so that (Ret$^\otimes$R) is satisfied.

To establish formally that the definition indeed stipulates a parallel OCF revision function, we first state several small results that identify pertinent facts concerning the definition.

First, the definition yields an AGM revision function.

Lemma 1. *Let κ be an OCF and let $\kappa \otimes S$ be given by Definition 4. Then $\kappa \otimes S$ defines a faithful ranking.*

The next lemma shows that the definition satisfies the basic parallel revision postulate.

Lemma 2. *Let κ be an OCF, let $\kappa \otimes S$ be given by Definition 4, and let $S_2 \subset S_1 \subseteq S$. Then $(\kappa \otimes S)(C_s(S_1)) < (\kappa \otimes S)(C_s(S_2))$.*

As well, worlds that satisfy exactly the same elements of a set S retain their relative ranking before and after revision.

Lemma 3. *Let κ be an OCF, let $\kappa \otimes S$ be given by Definition 4, and let $S|w_1 = S|w_2$. Then $(\kappa \otimes S)(w_1) - (\kappa \otimes S)(w_2) = \kappa(w_1) - \kappa(w_2)$.*

We obtain the following result.

Theorem 5. *Parallel OCF revision satisfies the extended AGM postulates,* ($\mathrm{P}^{\otimes}$), ($\mathrm{Ret}^{\otimes}$), ($\mathrm{PC3}^{\otimes}$), *and* ($\mathrm{PC4}^{\otimes}$).

6. Conclusion

In this paper, we have developed an account of *parallel belief revision*, in which the second argument to a revision function is a finite set of formulas. Each formula of the set represents an individual item of information. Thus $\mathcal{K} * \{\alpha, \beta\}$ specifies a revision of $\mathcal{K}$ by two formulas, while $\mathcal{K} * \{\alpha \wedge \beta\}$ specifies a revision of $\mathcal{K}$ by a single formula that happens to be expressed as a conjunction. The intention is that, following revision by a set of formulas, if a subsequent revision is in conflict with some members of the original set, then belief in the other elements of that set is retained. Thus, in revising by $\{\alpha, \beta\}$ and then by $\{\neg\beta\}$, then, if α and β are independent, α continues to be believed in the resulting belief state. This is not necessarily the case in revising by $\{\alpha \wedge \beta\}$ and then by $\{\neg\beta\}$. Informally, a revision $\mathcal{K} * \{\alpha, \beta\}$ can be seen as yielding not just the (contingent) incorporation of α and β among the beliefs of $\mathcal{K}$, but also incorporating a counterfactual assumption that if one of α or β was subsequently believed to be false, then the agent would still believe the other formula to be true (provided there is no positive logical dependence between α and β).

We presented two accounts of parallel belief revision. First, we consider a basic approach, in which minimal conditions for revising by a set of formulas are developed. Two postulates are proposed, along with corresponding semantic conditions, and a representation result is given. Semantically we require that in a revision by a set of formulas S, in the associated faithful ordering on worlds, for $S_1 \subset S$, at the least $\overline{S \setminus S_1}$ worlds we also have that S_1 is true. As a consequence, problems associated with the DP Postulate (C2) are sidestepped. Second, we develop a "preferred" account of iterated parallel revision, consisting of an additional three new postulates. This is carried out by extending the approach of Jin and Thielscher [21] for iterated revision to deal with sets of formulas. Again, corresponding semantic conditions are given and a representation result derived. Last, Section 5 provides a concrete construction of a parallel revision operator.

Our account of parallel revision is intended as an extension of the AGM approach. In particular, revising by an inconsistent set of formulas yields an inconsistent belief set. For future work, an obvious and interesting extension is to examine revision in the case of an inconsistent set of formulas. As the examples at the end of Section 3 indicate, there is information that may be gleaned in revising by an inconsistent set of formulas, provided some of the elements of the set are consistent. Thus, given some "reasonable" means of extracting consistent information from a set S, say $\Delta(S)$, one could express revision as follows:

$$Bel(\mathcal{K} \otimes S) = Bel\big(\mathcal{K} \otimes (\Delta S)\big).$$

Hence in this case, if β were consistent, one would expect that $Bel(\mathcal{K} \otimes \{\alpha, \neg\alpha, \beta\})$ would also be consistent and entail β, while $Bel(\mathcal{K} \otimes \{\alpha \wedge \neg\alpha \wedge \beta\})$ would of course be inconsistent. In this way, one might obtain consistent revisions in some cases where the input is inconsistent.

Last, as indicated in the examples in the basic approach, parallel revision may be used to encode preferences over formulas with respect to revision. A second, intriguing direction for future research is to further explore this phenomenon, to determine to what extent the present approach may be used to express a general notion of preference over formulas in revision.

Appendix A. Proofs of Theorems

Notation. For $\mathcal{W} \subseteq \Theta_{\mathcal{P}}$, $form(\mathcal{W})$ is a formula such that $\mathcal{W} = Mod(form(\mathcal{W}))$. Since we assume a finite underlying language, such a formula is guaranteed to exist. Most often we will have $|\mathcal{W}| = 2$, and to avoid an overabundance of brackets we will abuse notation and write e.g. $\mathcal{K} \otimes \{form(\{w_1, w_2\})\}$ as $\mathcal{K} \otimes form(w_1, w_2)$.

In analogy to $Bel(\mathcal{K})$ standing for the set of sentences comprising the belief set of $\mathcal{K}$ (and to simplify notation), $Mod(\mathcal{K})$ will be the set of models of the belief set of $\mathcal{K}$. I.e. $Mod(\mathcal{K})$ is defined as $Mod(Bel(\mathcal{K}))$.

Proof of Proposition 1. Assume that $S_1 \subset S$ and that $S_1 \cup (\overline{S \setminus S_1})$ is consistent.

1. (a) ($\mathrm{PP}^{\otimes}$) implies ($\mathrm{P}^{\otimes}$):
   From the success postulate we have that $S_1 \subseteq Bel(\mathcal{K} \otimes (S_1 \cup (\overline{S \setminus S_1})))$. ($\mathrm{PP}^{\otimes}$) asserts that $Bel(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1})) = Bel(\mathcal{K} \otimes (S_1 \cup (\overline{S \setminus S_1})))$, and so $S_1 \subseteq Bel(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1}))$.

(b) ($PP^{\otimes}$) implies ($S^{\otimes}$):
($PP^{\otimes}$) states that $Bel(\mathcal{K} \otimes (S_1 \cup (\overline{S \setminus S_1}))) = Bel(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1}))$. Since $S_1 \subseteq Bel(\mathcal{K} \otimes (S_1 \cup (\overline{S \setminus S_1})))$ so also $S_1 \subseteq Bel(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1}))$. Via ($\mathcal{K} \otimes 3$) and ($\mathcal{K} \otimes 4$) we obtain that $Bel(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1})) = Bel(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1})) \cup S_1 = Bel(\mathcal{K} \otimes S \otimes (S_1 \cup (\overline{S \setminus S_1})))$ from which we get $Bel(\mathcal{K} \otimes (S_1 \cup (\overline{S \setminus S_1}))) = Bel(\mathcal{K} \otimes S \otimes (S_1 \cup (\overline{S \setminus S_1})))$.

2. ($P^{\otimes}$) and ($S^{\otimes}$) imply ($PP^{\otimes}$):
From ($\mathcal{K} \otimes 3$), ($\mathcal{K} \otimes 4$), and ($P^{\otimes}$), we get that $Bel(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1})) = Bel(\mathcal{K} \otimes S \otimes (S_1 \cup (\overline{S \setminus S_1})))$. From ($S^{\otimes}$) we have in turn that $Bel(\mathcal{K} \otimes S \otimes (S_1 \cup (\overline{S \setminus S_1}))) = Bel(\mathcal{K} \otimes (S_1 \cup (\overline{S \setminus S_1})))$, from which we obtain ($PP^{\otimes}$). □

Proof of Proposition 2. The proof is very similar to that of Proposition 1. Assume that $S_1 \subset S$ and that $S_1 \cup (\overline{S \setminus S_1})$ is consistent.

1. (a) ($PP^{\otimes}R$) implies ($P^{\otimes}R$):
We have that $\min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}}) \subseteq Mod(S_1)$. ($PP^{\otimes}R$) then implies that $\min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S}) \subseteq Mod(S_1)$.

1.1. ($PP^{\otimes}R$) implies ($S^{\otimes}R$):
($PP^{\otimes}R$) states that $\min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}}) = \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$. Since $\min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}}) \subseteq Mod(S_1)$, so $\min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S}) \subseteq Mod(S_1)$. Consequently, we obtain that $\min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S}) = \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$ from which we get $\min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}}) = \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$.

2. ($P^{\otimes}R$) and ($S^{\otimes}R$) imply ($PP^{\otimes}R$):
From ($P^{\otimes}R$), we can show that $\min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S}) = \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$.
From ($S^{\otimes}R$) we have $\min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S}) = \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}})$, from which we obtain ($PP^{\otimes}R$). □

Proof of Proposition 3. Let $\mathcal{K}$ be a belief state and let $\preccurlyeq_{\mathcal{K}}$ be the faithful ranking induced by the faithful assignment induced by $\mathcal{K}$. Let $\preccurlyeq_{\mathcal{K}\otimes S}$ satisfy ($P^{\otimes}R$) and ($S^{\otimes}R$). Then $\preccurlyeq_{\mathcal{K}\otimes S}$ satisfies ($PP^{\otimes}R$) by the preceding result.

1. Let $S_1 \subset S$ where $S_1 \cup (\overline{S \setminus S_1})$ is consistent.
Let $w \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$. By ($PP^{\otimes}R$), $w \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}})$, whence $w \in Mod(S_1)$, and so $w \models S_1$.
Since $w \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$ and $w \models S_1$, and since $\preccurlyeq_{\mathcal{K}\otimes S}$ is a faithful ranking, it follows that $w \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$.
Conversely, assume $w \notin \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$. If $w \not\models \overline{S \setminus S_1}$ or $w \not\models S_1$, then trivially $w \notin \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$. So assume that $w \models \overline{S \setminus S_1}$ and $w \models S_1$.
Towards a contradiction assume that $w \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$. Then since $w \notin \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$, for any $w' \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$ we have $w' \prec_{\mathcal{K}\otimes S} w$.
However, ($PP^{\otimes}R$) implies that $w' \models S_1$, and this together with $w' \prec_{\mathcal{K}\otimes S} w$ contradicts $w \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$.
Hence $w \notin \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$, which was to be shown.
2. Assume $S_2 \subset S_1 \subset S$ and $S_2 \nvdash S_1$.
Let $w_1 \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$ and $w_2 \in \min(Mod(\overline{S \setminus S_2}), \preccurlyeq_{\mathcal{K}\otimes S})$. (1)
Since $S_2 \subset S_1$ we have that $(S \setminus S_1) \subset (S \setminus S_2)$, and so $(\overline{S \setminus S_1}) \subset (\overline{S \setminus S_2})$. From this together with (1), and since $\preccurlyeq_{\mathcal{K}\otimes S}$ is a faithful ranking, it follows that $w_1 \preccurlyeq_{\mathcal{K}\otimes S} w_2$.
Since $w_1 \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$, we obtain via ($PP^{\otimes}R$) that $w_1 \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}})$, and so $w_1 \models S_1$.
By the same argument with respect to w_2 and S_2 we get that $w_2 \models S_2$.
Consider $\phi \in (S_1 \setminus S_2)$. We have that $w_2 \models \neg\phi$ since $w_2 \models \overline{S \setminus S_2}$ and $\phi \in S \setminus S_2$.
As well, $\phi \in S_1$.
From Part 1 of the proposition, we showed that $\min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S}) = \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$.
Thus, since $w_2 \models \neg\phi$ and $\phi \in S_1$, so $w_2 \notin \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$, and so $w_2 \notin \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$.
We have already shown that $w_1 \preccurlyeq_{\mathcal{K}\otimes S} w_2$. However, since $w_1 \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$, $w_2 \models \overline{S \setminus S_1}$, and $w_2 \notin \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$, it follows that $w_1 \prec_{\mathcal{K}\otimes S} w_2$. □

Proof of Theorem 2. It is more compact to show each direction for the two conditions, rather than showing both directions for each postulate.

Construction to Postulates: Let $\preccurlyeq$ be a faithful ranking induced by an underlying epistemic state, and define $\otimes$ according to (2), i.e.

$$Bel(\mathcal{K} \otimes S) = \mathcal{T}\big(\min\big(Mod(S), \preccurlyeq_{\mathcal{K}}\big)\big).$$

Katsuno and Mendelzon [22] show that $\otimes$ satisfies ($\mathcal{K} \otimes 1$)–($\mathcal{K} \otimes 8$).

1. Let $\preccurlyeq$ satisfy ($P^{\otimes}R$). To show that $\otimes$ satisfies $P^{\otimes}$, let $S_1 \subset S$ where $S_1 \cup (\overline{S \setminus S_1}) \nvdash \bot$. Thus $Mod(S_1 \cup (\overline{S \setminus S_1})) \neq \emptyset$, and so $Mod(\overline{S \setminus S_1}) \neq \emptyset$.
Let $w_1 \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$.

From Proposition 3.1 we get that $w_1 \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$. Hence $w_1 \in Mod(S_1)$.
This means that for any $w \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$ that $w \models S_1$.
Hence $\min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S}) \subseteq Mod(S_1)$ or $S_1 \subseteq Bel(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1}))$ via (2), which was to be shown.

2. Let $\preccurlyeq$ satisfy ($S^{\otimes}$R). If $S_1 \cup (\overline{S \setminus S_1}) \vdash \perp$ then $S^{\otimes}$ is trivially satisfied.
So assume that $S_1 \cup (\overline{S \setminus S_1}) \nvdash \perp$, and let $w \in Mod(\mathcal{K} \otimes (S_1 \cup (\overline{S \setminus S_1})))$.
Thus in the associated faithful ranking we have $w \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}})$.
Via ($S^{\otimes}$R) this means that $w \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$.
But then in terms of our defined revision operator this means that $w \in Mod(\mathcal{K} \otimes S \otimes (S_1 \cup (\overline{S \setminus S_1})))$.
Consequently, we have that $Mod(\mathcal{K} \otimes (S_1 \cup (\overline{S \setminus S_1}))) \subseteq Mod(\mathcal{K} \otimes S \otimes (S_1 \cup (\overline{S \setminus S_1})))$.
The reverse containment follows by noting that each step above is in fact an if-and-only-if.
We obtain that

$$Mod\big(\mathcal{K} \otimes \big(S_1 \cup (\overline{S \setminus S_1})\big)\big) = Mod\big(\mathcal{K} \otimes S \otimes \big(S_1 \cup (\overline{S \setminus S_1})\big)\big),$$

whence

$$Bel\big(\mathcal{K} \otimes \big(S_1 \cup (\overline{S \setminus S_1})\big)\big) = Bel\big(\mathcal{K} \otimes S \otimes \big(S_1 \cup (\overline{S \setminus S_1})\big)\big).$$

Postulates to Construction: Define, for every $w_1, w_2 \in \Theta_{\mathcal{P}}$, $w_1 \preccurlyeq_{\mathcal{K}} w_2$ iff $w_1 \in Mod(\mathcal{K} \otimes form(w_1, w_2))$. We have from [22] that $\preccurlyeq_{\mathcal{K}}$ (and so of course $\preccurlyeq_{\mathcal{K}\otimes S}$) is a total preorder that captures one-shot AGM revision.

1. We need to show that, given this definition and Postulate ($P^{\otimes}$) that condition ($P^{\otimes}$R) holds.
Let $S_1 \subset S$ where $S_1 \cup (\overline{S \setminus S_1}) \nvdash \perp$. Let $w \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S})$.
Since $\preccurlyeq_{\mathcal{K}\otimes S}$ is a faithful ranking, this means that $w \in Mod(\mathcal{K} \otimes S \otimes (\overline{S \setminus S_1}))$.
By ($P^{\otimes}$) we obtain that $w \models S_1$, from which it follows that $\min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}\otimes S}) \subseteq Mod(S_1)$.
2. We need to show that, given the initial definition and Postulate ($S^{\otimes}$) that condition ($S^{\otimes}$R) holds.
Let $S_1 \subset S$. If $S_1 \cup (\overline{S \setminus S_1}) \vdash \perp$, then ($S^{\otimes}$R) holds trivially.
So assume that $S_1 \cup (\overline{S \setminus S_1}) \nvdash \perp$, and let $w \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}})$.
Thus $w \in Mod(\mathcal{K} \otimes (S_1 \cup (\overline{S \setminus S_1})))$.
Therefore by ($S^{\otimes}$) we obtain that $w \in Mod(\mathcal{K} \otimes S \otimes (S_1 \cup (\overline{S \setminus S_1})))$.
Consequently $w \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$.
This shows that $\min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}}) \subseteq \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}\otimes S})$.
The reverse containment, viz.

$$\min\big(Mod\big(S_1 \cup (\overline{S \setminus S_1})\big), \preccurlyeq_{\mathcal{K}\otimes S}\big) \subseteq \min\big(Mod\big(S_1 \cup (\overline{S \setminus S_1})\big), \preccurlyeq_{\mathcal{K}}\big)$$

follows by observing that each step above is in fact an if-and-only-if. □

Proof of Proposition 4. The proof is by induction on $|S \setminus S_1|$.

If $|S \setminus S_1| = 1$ then for $S \setminus S_1 = \{\phi\}$ we are to show that $S_1 \subseteq Bel(\mathcal{K} \otimes S \otimes \{\neg\phi\})$. But this is just an instance of ($P^{\otimes}$).

If $|S \setminus S_1| = 2$ then the result follows from the factoring result in AGM revision. That is, let $S \setminus S_1 = \{\phi, \psi\}$. Then the AGM factoring result yields

$$Bel\big(\mathcal{K} \otimes S \otimes \{\neg\phi \vee \neg\psi\}\big) = \begin{cases} Bel(\mathcal{K} \otimes S \otimes \{\neg\phi\}) & \text{or} \\ Bel(\mathcal{K} \otimes S \otimes \{\neg\psi\}) & \text{or} \\ Bel(\mathcal{K} \otimes S \otimes \{\neg\phi\}) \cap Bel(\mathcal{K} \otimes S \otimes \{\neg\psi\}). & \end{cases}$$

We have already noted that $S \setminus \{\phi\} \subseteq Bel(\mathcal{K} \otimes S \otimes \{\neg\phi\})$, from which it trivially follows that $S \setminus \{\phi, \psi\} \subseteq Bel(\mathcal{K} \otimes S \otimes \{\neg\phi\})$, and analogously $S \setminus \{\phi, \psi\} \subseteq Bel(\mathcal{K} \otimes S \otimes \{\neg\psi\})$. It follows also that $S \setminus \{\phi, \psi\} \subseteq Bel(\mathcal{K} \otimes S \otimes \{\neg\phi\}) \cap Bel(\mathcal{K} \otimes S \otimes \{\neg\psi\})$, from which our result follows using the factoring result.

The general case with $|S \setminus S_1| > 2$ follows by a straightforward induction, again using the AGM factoring result. We omit the details. □

Lemma 4. *Let S, S' be two sets of sentences and w_1, w_2 two possible worlds, such that $Mod(S') = \{w_1, w_2\}$ and $S|w_2 \subseteq S|w_1$. Then for all $A \subseteq S$ such that $A \cup S'$ is consistent, we have $w_1 \models A$.*

Proof. Suppose there exists $A \subseteq S$ such that, $A \cup S'$ is consistent and $w_1 \not\models A$. Since $Mod(S') = \{w_1, w_2\}$, it follows that $w_2 \models A$. From $S|w_2 \subseteq S|w_1$, we have $A \supseteq S|w_1$. This contradicts $w_1 \not\models A$. □

Proof of Theorem 3.

($\Rightarrow$) Assume $S|w_2 \subset S|w_1$, $w_1 \preccurlyeq_{\mathcal{K}} w_2$ and $w_2 \preccurlyeq_{\mathcal{K}\otimes S} w_1$. Let S' be a set of sentences such that, $Mod(S') = \{w_1, w_2\}$. Let $\alpha \in S$ be a sentence such that, $w_1 \models \alpha$ and $w_2 \not\models \alpha$. From Lemma 4, it follows that $\forall S_c \in S||S'$ that $w_1 \models S_c$. Thus $w_1 \models S_c \cup S'$. From $w_1 \preccurlyeq_{\mathcal{K}} w_2$, it follows that $w_1 \models Bel(\mathcal{K} \otimes (S_c \cup S'))$. Therefore, $Bel(\mathcal{K} \otimes (S_c \cup S')) \nvdash \neg\alpha$. Then ($Ret^{\otimes}$) implies that $Bel((\mathcal{K} \otimes S) \otimes S') \vdash \alpha$, which contradicts $w_2 \preccurlyeq_{\mathcal{K}\otimes S} w_1$.

($\Leftarrow$) Assume $\alpha \in S_1, \forall S_c \in S_1||S_2$ that we have $Bel(\mathcal{K} \otimes (S_c \cup S_2)) \nvdash \neg\alpha$ and $Bel((\mathcal{K} \otimes S_1) \otimes S_2) \nvdash \alpha$. Let w_2 be a possible world such that $w_2 \models Bel((\mathcal{K} \otimes S_1) \otimes S_2)$ and $w_2 \not\models \alpha$. Let $S_c = S_1|w_2$. Obviously, $S_c \in S_1||S_2$. Thus $Bel(\mathcal{K} \otimes (S_c \cup S_2)) \nvdash \neg\alpha$. Let w_1 be a possible world such that $w_1 \models Bel(\mathcal{K} \otimes (S_c \cup S_2))$ and $w_1 \models \alpha$. It follows immediately that $w_1 \preccurlyeq_{\mathcal{K}} w_2$. Since $w_1 \models S_c$, $w_1 \models \alpha$ and $w_2 \not\models \alpha$, we have $S_1|w_2 \subset S_1|w_1$. Thus (Ret$^{\otimes}$R) implies $w_1 \prec_{\mathcal{K} \otimes S_1} w_2$. This contradicts $w_2 \models Bel((\mathcal{K} \otimes S_1) \otimes S_2)$. □

Proof of Proposition 5. It suffices to show that (Ret$^{\otimes}$R) implies (Ind$^{\otimes}$R). Assume $w_1 = S$, $w_2 \not\models S$ and $w_1 \preccurlyeq_{\mathcal{K}} w_2$. Obviously, $S|w_2 \subset S|w_1$. From (Ret$^{\otimes}$R), it follows that $w_1 \prec_{\mathcal{K} \otimes S} w_2$. □

Proof of Theorem 4.

1. ($\Rightarrow$) Assume $S|w_2 \subseteq S|w_1$, $w_1 \prec_{\mathcal{K}} w_2$ and $w_2 \preccurlyeq_{\mathcal{K} \otimes S} w_1$. Let S' be a set of sentences such that $Mod(S') = \{w_1, w_2\}$. From Lemma 4, it follows that $\forall S_c \in S||S'$ we have $w_1 \models S_c$. Thus $Mod(\mathcal{K} \otimes (S_c \cup S')) = \{w_1\}$, since $w_1 \prec_{\mathcal{K}} w_2$. Let α be a sentence such that $w_1 \models \alpha$ and $w_2 \not\models \alpha$. It follows that $\forall S_c \in S||S'$ we have $Bel(\mathcal{K} \otimes (S_c \cup S')) \vdash \alpha$. Then (PC3$^{\otimes}$) implies that $Bel((\mathcal{K} \otimes S) \otimes S') \vdash \alpha$. This contradicts $w_2 \preccurlyeq_{\mathcal{K} \otimes S} w_1$ and $w_2 \not\models \alpha$.

   ($\Leftarrow$) Assume $\forall S_c \in S_1||S_2$ we have $Bel(\mathcal{K} \otimes (S_c \cup S_2)) \vdash \alpha$, and $Bel((\mathcal{K} \otimes S_1) \otimes S_2) \nvdash \alpha$. Let w_2 be a possible world such that $w_2 \models Bel((\mathcal{K} \otimes S_1) \otimes S_2)$ and $w_2 \not\models \alpha$. Let $S_c = S_1|w_2$. Obviously, $S_c \in S_1||S_2$. Thus $Bel(\mathcal{K} \otimes (S_c \cup S_2)) \vdash \alpha$. Let w_1 be a possible world such that $w_1 \models Bel(\mathcal{K} \otimes (S_c \cup S_2))$. It is easy to see that $w_1 \prec_{\mathcal{K}} w_2$ and $S_c|w_2 \subseteq S_c|w_1$. Then (PC3$^{\otimes}$R) implies $w_1 \prec_{\mathcal{K} \otimes S_1} w_2$. This contradicts $w_2 \models Bel((\mathcal{K} \otimes S_1) \otimes S_2)$.
2. ($\Rightarrow$) Assume $S|w_2 \subseteq S|w_1$, $w_1 \preccurlyeq_{\mathcal{K}} w_2$ and $w_2 \prec_{\mathcal{K} \otimes S} w_1$. Let S' be a set of sentences such that $Mod(S') = \{w_1, w_2\}$. From Lemma 4, it follows that $\forall S_c \in S||S'$ we have that $w_1 \models S_c$. Thus $w_1 \models Bel(\mathcal{K} \otimes (S_c \cup S'))$, since $w_1 \preccurlyeq_{\mathcal{K}} w_2$. Let α be a sentence such that $w_1 \models \alpha$ and $w_2 \not\models \alpha$. It follows that $Bel(\mathcal{K} \otimes (S_c \cup S')) \nvdash \neg\alpha$. Then (PC4$^{\otimes}$) implies that $Bel((\mathcal{K} \otimes S) \otimes S') \nvdash \neg\alpha$. This contradicts $w_2 \prec_{\mathcal{K} \otimes S} w_1$ and $w_2 \not\models \alpha$.

   ($\Leftarrow$) Assume $\forall S_c \in S_1||S_2$ we have that $Bel(\mathcal{K} \otimes (S_c \cup S_2)) \nvdash \neg\alpha$, and $Bel((\mathcal{K} \otimes S_1) \otimes S_2) \vdash \neg\alpha$. Let w_2 be a possible world such that $w_2 \models Bel((\mathcal{K} \otimes S_1) \otimes S_2)$. Let $S_c = S_1|w_2$. Obviously, $S_c \in S_1||S_2$. Thus $Bel(\mathcal{K} \otimes (S_c \cup S_2)) \nvdash \neg\alpha$. Let w_1 be a possible world such that $w_1 \models Bel(\mathcal{K} \otimes (S_c \cup S_2))$ and $w_1 \models \alpha$. It is easy to see that $w_1 \preccurlyeq_{\mathcal{K}} w_2$ and $S_1|w_2 \subseteq S_1|w_1$. Then (PC4$^{\otimes}$R) implies that $w_1 \preccurlyeq_{\mathcal{K} \otimes S_1} w_2$. Thus $w_1 \models Bel((\mathcal{K} \otimes S_1) \otimes S_2)$, which contradicts $Bel((\mathcal{K} \otimes S_1) \otimes S_2) \vdash \neg\alpha$. □

Proof of Proposition 7. Assume $S|w_2 = S|w_1$. Suppose $w_1 \preccurlyeq_{\mathcal{K}} w_2$. Then (PC4$^{\otimes}$R) implies $w_1 \preccurlyeq_{K \otimes S} w_2$. Suppose $w_1 \preccurlyeq_{K \otimes S} w_2$. It follows from (PC3$^{\otimes}$R) that $w_2 \not\prec_{\mathcal{K}} w_1$. Thus $w_1 \preccurlyeq_{\mathcal{K}} w_2$, since $\preccurlyeq_{\mathcal{K}}$ is total. □

Proof of Proposition 9. It suffices to show that (Ret$^{\otimes}$R), (PC3$^{\otimes}$R) and (PC4$^{\otimes}$R) imply (PP$^{\otimes}$R$'$). Let $S_1 \subset S$ where S_1 is consistent with $Bel(\mathcal{K} \otimes (\overline{S \setminus S_1}))$. Thus there is a possible world w such that $w \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}})$ and $w \models S_1$. We need to show that $\min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}}) = \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K} \otimes S})$.

- $\subseteq$ Suppose there is a possible world w_1 such that $w_1 \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}})$ and $w_1 \notin \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K} \otimes S})$. This implies that there exists another possible world $w_2 \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K} \otimes S})$ and $w_2 \prec_{\mathcal{K} \otimes S} w_1$. Since $w_2 \in Mod(\overline{S \setminus S_1})$ and $w \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}})$, we have $w \preccurlyeq_{\mathcal{K}} w_2$. As $w \models S_1 \cup (\overline{S \setminus S_1})$ and $w_2 \models (\overline{S \setminus S_1})$, it is obvious that $S|w_2 \subseteq S|w$. It follows from (PC4$^{\otimes}$R) that $w \preccurlyeq_{\mathcal{K} \otimes S} w_2$. On the other hand, since $w_1 \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}})$ and $w \in Mod(S_1 \cup (\overline{S \setminus S_1}))$ we have $w_1 \preccurlyeq_{\mathcal{K}} w$. It follows from (PC4$^{\otimes}$R) that $w_1 \preccurlyeq_{\mathcal{K} \otimes S} w$ as $S|w_1 = S|w$. This contradicts $w \preccurlyeq_{\mathcal{K} \otimes S} w_2$ and $w_2 \prec_{\mathcal{K} \otimes S} w_1$.
- $\supseteq$ Suppose there is a possible world w_1 such that $w_1 \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K} \otimes S})$ and $w_1 \notin \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}})$. Since $w \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K}})$ and $w_1 \in Mod(\overline{S \setminus S_1})$, we have $w \preccurlyeq_{\mathcal{K}} w_2$. Now we show that $w_1 \models S_1$. Suppose $w_1 \not\models S_1$. Since $w \models S_1$, we have $S|w_1 \subset S|w$. It follows from (Ret$^{\otimes}$R), $w \prec_{\mathcal{K} \otimes S} w_1$. This contradicts $w_1 \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K} \otimes S})$. Thus we have $w_1 \models S_1$. It implies that there exists another possible world w_2 such that $w_2 \in \min(Mod(S_1 \cup (\overline{S \setminus S_1})), \preccurlyeq_{\mathcal{K}})$ and $w_2 \prec_{\mathcal{K}} w_1$. Since $S|w_1 = S|w_2$, it follows from (PC3$^{\otimes}$R) that $w_2 \prec_{\mathcal{K} \otimes S} w_1$. This contradicts $w_1 \in \min(Mod(\overline{S \setminus S_1}), \preccurlyeq_{\mathcal{K} \otimes S})$. □

Proof of Lemma 1. It is straightforward to verify that $\kappa \otimes S$ is a total function on $\Theta_{\mathcal{P}}$. Thus $\kappa \otimes S$ defines a total preorder over $\Theta_{\mathcal{P}}$.

If $w \in \min(S, \kappa)$ then, by the first condition in Definition 4, $(\kappa \otimes S)(w) = 0$. Similarly, if $w \notin \min(S, \kappa)$, then it can be seen that $(\kappa \otimes S)(w) \neq 0$, as follows.

If $w \notin \min(S, \kappa)$ and $w \models S$ then $(\kappa \otimes S)(w) \neq 0$ by Condition 3 in Definition 4.

If $w \in \min(C_s(S_1), \kappa)$ for some $S_1 \subset S$, then it can be observed from Condition 2 in Definition 4 that $(\kappa \otimes S)(w) \geqslant 1$.

If $w \notin \min(C_s(S_1), \kappa)$ for any $S_1 \subset S$ then, since $(\kappa \otimes S)(S_1) \geqslant 1$, it can be observed from Condition 3 in Definition 4 that $(\kappa \otimes S)(w) \geqslant 1$.

Hence $\kappa \otimes S$ satisfies the conditions for a faithful ranking. □

Proof of Lemma 2. Since by Lemma 1, $\kappa \otimes S$ defines a faithful ranking, the lemma holds for $S_1 = S$. For the induction hypothesis, assume that the claim holds for all sets $S_1 \subseteq S$ where $|S| - |S_1| < j$ for some $j \geqslant 0$.

Let $S_2 \subset S_1$ where $|S| - |S_2| = j$, and let $w_2 \in (\kappa \otimes S)(C_s(S_2))$. The result follows immediately from the second condition in Definition 4, since $(\kappa \otimes S)(C_s(S_2))$ can be seen to be greater than $(\kappa \otimes S)(C_s(S_1'))$ for any S_1' where $|S_1'| + 1 = |S_2|$. $\square$

Proof of Lemma 3. Let $S' = C_s(S|w_2)$ $(= C_s(S|w_1))$. There are four cases:

1. If $w_1, w_2 \in \min(S', \kappa)$, then $\kappa(w_1) = \kappa(w_2)$, and Condition 2 in Definition 4 implies that $(\kappa \otimes S)(w_1) = (\kappa \otimes S)(w_2)$.
2. If $w_1 \in \min(S', \kappa)$, $w_2 \notin \min(S', \kappa)$ then substituting into the equation in Condition 3 of Definition 4 we get:

$$(\kappa \otimes S)(w_2) = (\kappa \otimes S)\big(C_s(S|w_2)\big) + \kappa(w_2) - \kappa\big(C_s(S|w_2)\big)$$

or

$$(\kappa \otimes S)(w_2) = (\kappa \otimes S)(w_1) + \kappa(w_2) - \kappa(w_1).$$

Rearranging terms we get:

$$(\kappa \otimes S)(w_2) - (\kappa \otimes S)(w_1) = \kappa(w_2) - \kappa(w_1).$$

3. The same argument establishes the result for $w_1 \notin \min(S', \kappa)$, $w_2 \in \min(S', \kappa)$.
4. If $w_1 \notin \min(S', \kappa)$, $w_2 \notin \min(S', \kappa)$ then two instances of Condition 3 of Definition 4 give

$$(\kappa \otimes S)(w_1) = (\kappa \otimes S)\big(S'\big) + \kappa(w_1) - \kappa\big(S'\big), \tag{6}$$
$$(\kappa \otimes S)(w_2) = (\kappa \otimes S)\big(S'\big) + \kappa(w_2) - \kappa\big(S'\big). \tag{7}$$

Subtracting (7) from (6) yields:

$$(\kappa \otimes S)(w_1) - (\kappa \otimes S)(w_2) = \kappa(w_1) - \kappa(w_2). \qquad \square$$

Proof of Theorem 5. It suffices to show that $\kappa \otimes S$ is a faithful ranking that satisfies the semantic conditions ($\mathrm{P}^{\otimes}\mathrm{R}$), ($\mathrm{Ret}^{\otimes}\mathrm{R}$), ($\mathrm{PC3}^{\otimes}\mathrm{R}$), and ($\mathrm{PC4}^{\otimes}\mathrm{R}$).

- From Lemma 1 we have that $\kappa \otimes S$ defines a faithful ranking. By the representation theorem of [22] (extended to sets), $\kappa \otimes S$ satisfies the extended AGM postulates.
- For ($\mathrm{P}^{\otimes}\mathrm{R}$), assume that $S' \subset S$ where $S' \cup (\overline{S \setminus S'}) \nvdash \bot$, and let $w \in \min((\overline{S \setminus S'}), \kappa \otimes S)$.
Assume toward a contradiction that $w \notin Mod(S')$. So we have for some $S'' \subset S'$ that $w \models S''$ and $w \models \overline{S \setminus S''}$. Since $w \in \min((\overline{S \setminus S'}), \kappa \otimes S)$, $(\overline{S \setminus S'}) \subset (\overline{S \setminus S''})$, and $w \models \overline{S \setminus S''}$, this means that $w \in \min((\overline{S \setminus S''}), \kappa \otimes S)$. Since $w \models S''$ this also means that $w \in \min(S'' \cup (\overline{S \setminus S''}), \kappa \otimes S)$ or $w \in \min(C_s(S''), \kappa \otimes S)$. We also have by assumption that $w \in \min((\overline{S \setminus S'}), \kappa \otimes S)$, and so $(\kappa \otimes S)(w) \leqslant (\kappa \otimes S)(C_s(S'))$.
But this in turn implies that $(\kappa \otimes S)(C_s(S'')) \leqslant (\kappa \otimes S)(C_s(S'))$ where $S'' \subset S'$, contradicting Lemma 2. Consequently we must have that $w \in Mod(S')$.
- For [($\mathrm{PC3}^{\otimes}\mathrm{R}$)] and [($\mathrm{PC4}^{\otimes}\mathrm{R}$)], consider where $S|w_2 = S|w_1$; the case $S|w_2 \subset S|w_1$ is implied by ($\mathrm{Ret}^{\otimes}\mathrm{R}$), covered in the next item.
Lemma 3 states that $(\kappa \otimes S)(w_1) - (\kappa \otimes S)(w_2) = \kappa(w_1) - \kappa(w_2)$, which immediately implies [($\mathrm{PC3}^{\otimes}\mathrm{R}$)] and [($\mathrm{PC4}^{\otimes}\mathrm{R}$)].
- For ($\mathrm{Ret}^{\otimes}\mathrm{R}$), assume that $S|w_2 \subset S|w_1$ and that $\kappa(w_1) \leqslant \kappa(w_2)$; we must show that $(\kappa \otimes S)(w_1) < (\kappa \otimes S)(w_2)$.
Let $S_1 = S|w_1$ and $S_2 = S|w_2$.
If we can show that the result holds for $|S_2| + 1 = |S_1|$ then by transitivity of $\leqslant$ the result holds trivially for $|S_2| + i = |S_1|$ for $i \geqslant 1$. So assume further that $|S_2| + 1 = |S_1|$.
There are three cases:
  1. $w_1 \in \min(C_s(S_1), \kappa)$ and $w_2 \in \min(C_s(S_2), \kappa)$.
  From Condition 2 of Definition 4 we obtain that

$$\begin{aligned}(\kappa \otimes S)(w_2) &= \max\big\{(\kappa \otimes S)\big(C_s\big(S_1'\big)\big), (\kappa \otimes S)\big(C_s\big(S_1'\big)\big) + \kappa\big(C_s(S_2)\big) - \kappa\big(C_s\big(S_1'\big)\big) \bigm| S_1' \subseteq S \\ &\qquad \text{and } \big|S_1'\big| + 1 = |S_2|\big\} + 1 \\ &\geqslant \max\big\{(\kappa \otimes S)\big(C_s(S_1)\big), (\kappa \otimes S)\big(C_s(S_1)\big) + \kappa\big(C_s(S_2)\big) - \kappa\big(C_s(S_1)\big)\big\} + 1 \\ &= \max\big\{(\kappa \otimes S)(w_1), (\kappa \otimes S)(w_1) + \kappa(w_2) - \kappa(w_1)\big\} + 1 \\ &= (\kappa \otimes S)(w_1) + \kappa(w_2) - \kappa(w_1) + 1.\end{aligned}$$

  The last step comes from the fact that $\kappa(w_1) \leqslant \kappa(w_2)$ by assumption. Hence

$$(\kappa \otimes S)(w_2) - (\kappa \otimes S)(w_1) \geqslant \kappa(w_2) - \kappa(w_1) + 1 > \kappa(w_2) - \kappa(w_1),$$

  which establishes the result.

2. $w_1 \notin \min(C_s(S_1), \kappa)$ and $w_2 \in \min(C_s(S_2), \kappa)$.
   Let $w_1' \in \min(C_s(S_1), \kappa)$. Since $S|w_1 = S|w_1'$, Lemma 3 implies that $(\kappa \otimes S)(w_1) - (\kappa \otimes S)(w_1') = \kappa(w_1) - \kappa(w_1')$ where in addition we have $\kappa(w_1) - \kappa(w_1') > 0$.
   Rearranging terms we get

$$(\kappa \otimes S)(w_1') - \kappa(w_1') = (\kappa \otimes S)(w_1) - \kappa(w_1). \tag{8}$$

   From the previous case, above, we have that

$$(\kappa \otimes S)(w_2) - (\kappa \otimes S)(w_1') > \kappa(w_2) - \kappa(w_1')$$

   or

$$(\kappa \otimes S)(w_2) > (\kappa \otimes S)(w_1') + \kappa(w_2) - \kappa(w_1').$$

   Substituting (8) into this inequality yields

$$(\kappa \otimes S)(w_2) > (\kappa \otimes S)(w_1) + \kappa(w_2) - \kappa(w_1)$$

   or

$$(\kappa \otimes S)(w_2) - (\kappa \otimes S)(w_1) > \kappa(w_2) - \kappa(w_1)$$

   which establishes the result.
3. $w_2 \notin \min(C_s(S_2), \kappa)$.
   Let $w_1' \in \min(C_s(S_1), \kappa)$ and $w_2' \in \min(C_s(S_2), \kappa)$.
   From Lemma 3 we have that

$$(\kappa \otimes S)(w_1') - (\kappa \otimes S)(w_1) = \kappa(w_1') - \kappa(w_1),$$
$$(\kappa \otimes S)(w_2') - (\kappa \otimes S)(w_2) = \kappa(w_2') - \kappa(w_2).$$

   Rearranging terms yields:

$$(\kappa \otimes S)(w_1') - \kappa(w_1') = (\kappa \otimes S)(w_1) - \kappa(w_1), \tag{9}$$
$$(\kappa \otimes S)(w_2') - \kappa(w_2') = (\kappa \otimes S)(w_2) - \kappa(w_2). \tag{10}$$

   From the first case, above, we have:

$$(\kappa \otimes S)(w_2') - (\kappa \otimes S)(w_1') > \kappa(w_2') - \kappa(w_1')$$

   or

$$(\kappa \otimes S)(w_2') - \kappa(w_2') > (\kappa \otimes S)(w_1') - \kappa(w_1').$$

   Substituting (9) and (10) into the above gives

$$(\kappa \otimes S)(w_2) - \kappa(w_2) > (\kappa \otimes S)(w_1) - \kappa(w_1)$$

   and rearranging terms gives

$$(\kappa \otimes S)(w_2) - (\kappa \otimes S)(w_1) > \kappa(w_2) - \kappa(w_1),$$

   which establishes the result. □

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References

[1] C.E. Alchourrón, P. Gärdenfors, D. Makinson, On the logic of theory change: Partial meet functions for contraction and revision, Journal of Symbolic Logic 50 (2) (1985) 510–530.
[2] S. Benferhat, D. Dubois, H. Prade, Representing default rules in possibilistic logic, in: Proceedings of the Third International Conference on the Principles of Knowledge Representation and Reasoning, Cambridge, MA, October 1992, pp. 673–684.
[3] S. Benferhat, S. Kaci, D. Le Berre, M.-A. Williams, Weakening conflicting information for iterated revision and knowledge integration, Artificial Intelligence 153 (2004) 339–371.
[4] S. Benferhat, C. Cayrol, D. Dubois, J. Lang, H. Prade, Inconsistency management and prioritized syntax-based entailment, in: Proceedings of the International Joint Conference on Artificial Intelligence, Chambéry, France, 1993, pp. 640–645.
[5] R. Booth, T.A. Meyer, Admissible and restrained revision, Journal of Artificial Intelligence Research 26 (2006) 127–151.
[6] C. Boutilier, Revision sequences and nested conditionals, in: Proceedings of the International Joint Conference on Artificial Intelligence, 1993, pp. 519–531.
[7] C. Boutilier, Conditional logics of normality: A modal approach, Artificial Intelligence 68 (1) (1994) 87–154.
[8] C. Boutilier, Unifying default reasoning and belief revision in a modal framework, Artificial Intelligence 68 (1) (1994) 33–85.
[9] A. Darwiche, J. Pearl, On the logic of iterated belief revision, Artificial Intelligence 89 (1997) 1–29.
[10] J. Delgrande, Y. Jin, Parallel belief revision, in: Proceedings of the AAAI National Conference on Artificial Intelligence, 2008, pp. 317–322.
[11] J. Delgrande, T. Schaub, A consistency-based approach for belief change, Artificial Intelligence 151 (1–2) (2003) 1–41.
[12] J. Delgrande, R. Wassermann, Horn clause contraction functions: Belief set and belief base approaches, in: F. Lin, U. Sattler (Eds.), Proceedings of the Twelfth International Conference on the Principles of Knowledge Representation and Reasoning, AAAI Press, Toronto, 2010, pp. 143–152.
[13] M.A. Falappa, G. Kern-Isberner, G.R. Simari, Explanations, belief revision and defeasible reasoning, Artificial Intelligence 141 (1–2) (2002) 1–28.
[14] E.L. Fermé, K. Saez, P. Sanz, Multiple kernel contraction, Studia Logica 73 (2) (2003) 183–195.
[15] A. Fuhrmann, S.O. Hansson, A survey of multiple contractions, Journal of Logic, Language, and Information 3 (1994) 39–76.
[16] P. Gärdenfors, Knowledge in Flux: Modelling the Dynamics of Epistemic States, MIT Press, Cambridge, MA, 1988.
[17] H. Geffner, J. Pearl, Conditional entailment: Bridging two approaches to default reasoning, Artificial Intelligence 53 (2–3) (1992) 209–244.
[18] M. Goldszmidt, P. Morris, J. Pearl, A maximum entropy approach to nonmonotonic reasoning, in: Proceedings of the AAAI National Conference on Artificial Intelligence, Boston, MA, 1990.
[19] A. Grove, Two modellings for theory change, Journal of Philosophical Logic 17 (1988) 157–170.
[20] S.O. Hansson, A Textbook of Belief Dynamics, Applied Logic Series, Kluwer Academic Publishers, 1999.
[21] Y. Jin, M. Thielscher, Iterated belief revision, revised, Artificial Intelligence 171 (1) (2007) 1–18.
[22] H. Katsuno, A.O. Mendelzon, Propositional knowledge base revision and minimal change, Artificial Intelligence 52 (3) (1991) 263–294.
[23] S. Konieczny, R. Pino Pérez, A framework for iterated revision, Journal of Applied Non-Classical Logics 10 (3–4) (2000) 339–367.
[24] S. Konieczny, J. Lang, P. Marquis, Reasoning under inconsistency: The forgotten connective, in: Proceedings of the International Joint Conference on Artificial Intelligence, Edinburgh, 2005, pp. 484–489.
[25] S. Kraus, D. Lehmann, M. Magidor, Nonmonotonic reasoning, preferential models and cumulative logics, Artificial Intelligence 44 (1–2) (1990) 167–207.
[26] P. Lamarre, S4 as the conditional logic of nonmonotonicity, in: Proceedings of the Second International Conference on the Principles of Knowledge Representation and Reasoning, Cambridge, MA, April 1991, pp. 357–367.
[27] D. Lehmann, Another perspective on default reasoning, Annals of Mathematics and Artificial Intelligence 15 (1) (1995) 61–82.
[28] D. Lehmann, Belief revision, revised, in: Proceedings of the International Joint Conference on Artificial Intelligence, Morgan Kaufmann Publishers, Montréal, 1995, pp. 1534–1540.
[29] D. Lehmann, M. Magidor, What does a conditional knowledge base entail? Artificial Intelligence 55 (1) (1992) 1–60.
[30] T. Meyer, Basic infobase change, Studia Logica 67 (2) (2001) 215–242.
[31] A.C. Nayak, Iterated belief change based on epistemic entrenchment, Erkenntnis 41 (1994) 353–390.
[32] A.C. Nayak, M. Pagnucco, P. Peppas, Dynamic belief revision operators, Artificial Intelligence 146 (2) (2003) 193–228.
[33] J. Pearl, System Z: A natural ordering of defaults with tractable applications to nonmonotonic reasoning, in: R. Parikh (Ed.), Proceedings of the Third Conference on Theoretical Aspects of Reasoning About Knowledge, Morgan Kaufmann Publishers, Pacific Grove, CA, 1990, pp. 121–135.
[34] P. Peppas, The limit assumption and multiple revision, Journal of Logic and Computation 14 (3) (2004) 355–371.
[35] P. Peppas, Belief revision, in: F. van Harmelen, V. Lifschitz, B. Porter (Eds.), Handbook of Knowledge Representation, Elsevier Science, San Diego, USA, 2008, pp. 317–359.
[36] H. Rott, Change, Choice and Inference – A Study of Belief Revision and Nonmonotonic Reasoning, Clarendon Press, Oxford, 2001.
[37] W. Spohn, Ordinal conditional functions: A dynamic theory of epistemic states, in: W.L. Harper, B. Skyrms (Eds.), Causation in Decision, Belief Change, and Statistics, vol. II, Kluwer Academic Publishers, 1988, pp. 105–134.
[38] W. Spohn, A reason for explanation: Explanations provide stable reasons, in: W. Spohn, et al. (Eds.), Existence and Explanation, Kluwer Academic Publishers, 1991, pp. 165–196.
[39] M.-A. Williams, Transmutations of knowledge systems, in: J. Doyle, P. Torasso, E. Sandewall (Eds.), Proceedings of the Fourth International Conference on the Principles of Knowledge Representation and Reasoning, Bonn, Germany, May 1994, pp. 619–629.
[40] D. Zhang, Properties of iterated multiple belief revision, in: V. Lifschitz, I. Niemelä (Eds.), Proceedings of the Conference on Logic Programming and Nonmonotonic Reasoning, Springer, 2004, pp. 314–325.
[41] D. Zhang, N. Foo, Infinitary belief revision, Journal of Philosophical Logic 30 (6) (2001) 525–570.
[42] D. Zhang, S. Chen, W. Zhu, Z. Chen, Representation theorems for multiple belief changes, in: Proceedings of the International Joint Conference on Artificial Intelligence, Morgan Kaufmann Publishers, Nagoya, Japan, August 1997, pp. 89–94.