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Probably approximately optimal satisficing strategies ★

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Abstract

A *satisficing search problem* consists of a set of probabilistic experiments to be performed in some order, seeking a satisfying configuration of successes and failures. The expected cost of the search depends both on the success probabilities of the individual experiments, and on the *search strategy*, which specifies the order in which the experiments are to be performed. A strategy that minimizes the expected cost is *optimal*. Earlier work has provided "optimizing functions" that compute optimal strategies for certain classes of search problems from the success probabilities of the individual experiments. We extend those results by providing a general model of such strategies, and an algorithm PAO that identifies an approximately optimal strategy when the probability values are not known. The algorithm first estimates the relevant probabilities from a number of trials of each undetermined experiment, and then uses these estimates, and the proper optimizing function, to identify a strategy whose cost is, with high probability, close to optimal. We also show that if the search problem can be formulated as an and-or tree, then the PAO algorithm can also "learn while doing", i.e. gather the necessary statistics while performing the search.

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1. Introduction

Consider the following situation: there are two reliable tests for deciding whether an individual has hepatitis; one involves a blood test and the other a liver biopsy. Assuming there can be false negatives but no false positives, there are two "strategies" a doctor can follow to obtain a diagnosis. Using strategy $\Theta_1 = \langle \texttt{blood}, \texttt{liver} \rangle$, he would first perform the blood test and conclude the patient has hepatitis if that test is positive. If not, he would then examine the patient's liver, and conclude his diagnosis based on the result of that biopsy. The doctor's other option, strategy $\Theta_2 = \langle \texttt{liver}, \texttt{blood} \rangle$, performs these tests in the other order—first the liver test and then, only if necessary, the blood test.

Which strategy is better? Our goal is a strategy that will perform well in practice. To quantify the measurement, we assume there is a distribution of patients that the doctor will be asked to evaluate. We can then define a strategy's *expected cost* as the average cost required to perform these tests, averaged over the distribution of anticipated patients. Assuming, for now, that these tests (`blood` and `liver`) have the same cost, strategy Θ_1 is clearly better if the probability of a positive blood test (p_B) is larger than the probability of a positive liver test (p_L); otherwise, strategy Θ_2 is preferable.

Earlier research on this decision making model has produced a number of "optimizing functions" that each identify a strategy optimal for a specific testing situation, given the success probability values of the relevant experiments [6,7,18,21,22]. A limitation of these techniques, however, is that the probability values are in practice typically not known *a priori*. This paper specifies the number of trials of each experiment that are required to obtain estimates of these probability values that are good enough to identify a *nearly-optimal* strategy, with *high confidence*. It also addresses the complexities of observing this many trials.

Section 2 below first generalizes from the doctor's situation to a general class of arbitrary "decision structures" and defines strategies, and optimal strategies, for these structures. Section 3 then specifies the PAO algorithm, a general process that uses a set of observed trials of each experiment to identify a strategy whose cost is, with high probability, approximately optimal. The algorithm presumes the existence of an optimizing function for the class of search structures considered. When dealing with certain search structures, notably and-or trees, the PAO algorithm can "learn while doing", i.e. gather the necessary statistics while solving relevant performance tasks.

An extended version of this paper, available as a technical report [11], discusses several variants and applications of the basic algorithm presented here.

2. Framework

2.1. Decision structures

The doctor's task presented in Section 1 is a simple example of a *satisficing search problem* (term due to Simon and Kadane [21]), as his goal is to find a single satisfactory configuration of events: in this case, an informative combination of test results. Other

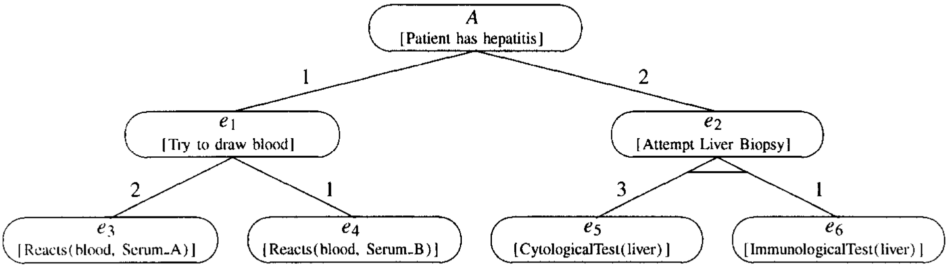


Fig. 1. An and-or tree representation of a decision structure G_1.

examples of such problems include, e.g., performing a sequence of tests to decide whether a product specimen is satisfactory [6], screening employment candidates for a position [6], competing for prizes at a quiz show [6], mining for gold buried in treasure chests [21], and performing inference in simple expert systems [10,22]. In general, such tasks may involve searching through general "decision structures", which can involve an arbitrary number of experiments, constrained by various precedence constraints.

And-or decision trees: More general versions of this diagnostic task can be represented by and-or decision trees, such as G_1 in Fig. 1. Here, the nodes $\{A, e_1, \ldots, e_6\}$ correspond to experiments, and the arcs encode the precedence relationships—e.g., the doctor cannot perform experiment e_3 (test whether the patient's blood reacts with a particular serum) until he has performed experiment e_1 (attempted to draw blood from the patient) and moreover, found that e_1 succeeds. The experiment associated with the A node is formally degenerate, i.e. it is guaranteed to succeed. The set of arcs descending from a given node can be either disjunctive, or conjunctive (here indicated by a horizontal line, e.g., connecting the e_2–e_5 arc to the e_2–e_6 arc). Hence, the graph in Fig. 1 states that the patient has hepatitis iff the condition $[e_1 \wedge (e_3 \vee e_4)] \vee [e_2 \wedge e_5 \wedge e_6]$ on the experiments holds. The number near each arc designates the cost of traversing that arc—hence it costs 1 unit to reduce the top e_0 node to the e_1 subgoal (draw blood), and 2 more units to further reduce e_1 to e_3 (test the blood against serum_A), and so forth. The incremental cost of performing each experiment is the sum of the costs of the *additional* arcs that must be traversed. (This cost specification means such trees cannot always be "collapsed" to simpler two-level trees.)

Beyond and-or trees: The general class of decision structures we shall consider is strictly more general than and-or trees. First, and-or trees can encode only simple formulae, which can include each experiment only once, and whose connections are only "and"s and "or"s. In general, we may want to express more complicated interrelationships of the experiments; e.g., the XOR of m experiments, or "at least 3 of 5 specified experiments". Second, and-or trees only permit relatively simple precedence relationships; in general, we may want to specify that an experiment can only be performed if some complicated boolean combination of other experiments has succeeded or failed. Third, and-or trees use a restricted form of cost function, in which the incremental cost of performing experiment e can depend only on which other experiments have been performed. In general, we may want the cost to depend also on whether e, and/or various prior experiments, have been successful. There are also situations which require yet more complicated ways of computing the incremental cost of performing a particular experiment; see the extended paper [11].

To accommodate these extensions, we define a more general class of "decision structures". A decision structure can involve an arbitrary set of experiments $W = \{e_i\}_{i=1}^n$, with general precedence constraints that can prevent an experiment from being performed until after certain other specified experiments have been performed with the specified (success or failure) result. The overall test result (e.g., whether the patient has hepatitis) can correspond to an arbitrary boolean combination of the successes and failures of any subset of these experiments, and the costs of performing a sequence of experiments can be given by an arbitrary nondecreasing function. This leads to the "decision structures" defined below.

Notation: Given two sequences, $\sigma = \langle\sigma_1, \ldots, \sigma_n\rangle$ and $\tau = \langle\tau_1, \ldots, \tau_m\rangle$, let $\sigma \cdot \tau$ refer to the sequence formed by concatenating σ and τ—i.e. $\sigma \cdot \tau = \langle\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m\rangle$. The definition is extended to the case where σ or τ are single elements in the obvious manner. A sequence σ is a *subsequence* of sequence τ, denoted $\sigma \sqsubseteq \tau$, if $\sigma_i = \tau_{h(i)}$ for all $i = 1, \ldots, n$, for some monotonically increasing function h. The empty sequence $\langle\rangle$ is trivially a subsequence of any sequence σ.

Definition 1 (*Decision structures*). A *decision structure* is a four-tuple $G = \langle W, F, R, c\rangle$ where

- $W = \{e_1, \ldots, e_n\}$ is a set of *experiments*.
- $F \subseteq [W \times \{+,-\}]^* \times W$ is a *precedence relation* that specifies which further experiments can be performed given the results of a previous sequence of experiments. (E.g., $F(\langle\rangle, e_1)$ means that e_1 may be performed initially; and $F(\langle\langle e_1+\rangle, \langle e_3-\rangle\rangle, e_2)$ means that experiment e_2 may be performed after e_1 has been performed and was successful, and then e_3 has been performed but was unsuccessful.)

The following conditions use the notion of a *legal labeled experiment sequence* (abbreviated "*lles*"). This is a sequence of the form $\langle\langle e_1 \pm_1\rangle, \ldots, \langle e_k \pm_k\rangle\rangle$, where each $e_i \in W$, each $\pm_i \in \{+,-\}$, and no $e \in W$ appears more than once. Furthermore, the sequence must satisfy the precedence constraints specified by the F relation: a sequence $\ell = \langle\langle e_1 \pm_1\rangle, \ldots, \langle e_k \pm_k\rangle\rangle$ is a lles only if $F(\langle\langle e_1 \pm_1\rangle, \ldots, \langle e_{m-1} \pm_{m-1}\rangle\rangle, e_m)$ holds for all $m = 1, \ldots, k$. The collection of all such sequences is denoted $\mathcal{LLES}(G)$.

- $R : \mathcal{LLES}(G) \to \{\mathcal{S}, \mathcal{F}, \mathcal{U}\}$ is the *result* function that specifies whether a given legal labeled experiment sequence renders the overall test successful or not; i.e. R maps each lles to one of $\{\mathcal{S}, \mathcal{F}, \mathcal{U}\}$ (for Success, Failure, and Undecided). We require R to be monotonic, in the sense that $R(\sigma) = \mathcal{S} \Rightarrow R(\sigma \cdot \tau) = \mathcal{S}$ and $R(\sigma) = \mathcal{F} \Rightarrow R(\sigma \cdot \tau) = \mathcal{F}$ whenever σ and $\sigma \cdot \tau$ are lles.
- $c : \mathcal{LLES}(G) \to \mathbb{R}_0^+$ is the *cost* function that maps each lles to its nonnegative real cost. It is required to be nondecreasing: $c(\sigma \cdot \tau) \geqslant c(\sigma)$ whenever σ and $\sigma \cdot \tau$ are lles.

We let $\mathcal{DS}$ refer to the class of all such decision structures.

To illustrate these definitions: the diagnostic tree of Fig. 1 can be encoded as a structure $G_1 = \langle\{e_1, \ldots, e_6\}, F_1, R_1, c_1\rangle$, where e.g. e_3 corresponds to "a patient's blood reacts with Serum_A" and e_5 corresponds to "a patient's liver has a positive cytological test". The F_1 relation, encoding the precedence relationships, includes $F_1(\langle\rangle, e_1)$ and $F_1(\langle\rangle, e_2)$ to mean that e_1 or e_2 can be performed initially; and $F_1(\langle\langle e_1+\rangle\rangle, e_3)$ but not $F_1(\langle\langle e_1-\rangle\rangle, e_3)$ (respectively $F_1(\langle\langle e_1+\rangle\rangle, e_4)$ but not $F_1(\langle\langle e_1-\rangle\rangle, e_4)$) to indicate that e_3 (respectively, e_4) can be performed if and only if e_1 has already succeeded. The R_1 function includes $R_1(\langle\langle e_1+\rangle, \langle e_3+\rangle\rangle) = \mathcal{S}$, $R_1(\langle\langle e_1+\rangle, \langle e_4+\rangle\rangle) = \mathcal{S}$, $R_1(\langle\langle e_2+\rangle, \langle e_5+\rangle, \langle e_6+\rangle\rangle) = \mathcal{S}$ and so forth, as well as $R_1(\langle\langle e_1-\rangle, \langle e_2-\rangle\rangle) = \mathcal{F}$, and $R_1(\langle\langle e_1-\rangle, \langle e_2+\rangle\rangle) = \mathcal{U}$, etc. The cost function c_1 encodes the incremental cost of performing any sequence of experiments: for instance, $c_1(\langle\langle e_1+\rangle, \langle e_3+\rangle, \langle e_4-\rangle\rangle) = 1+2+1=4$ and $c_1(\langle\langle e_1+\rangle, \langle e_3-\rangle, \langle e_2+\rangle, \langle e_6+\rangle\rangle) = 1+2+2+1=6$.

Notice that for standard graph-like decision structures, whenever an example becomes reachable, it stays so until performed, i.e. $F(\sigma, e)$ implies $F(\tau, e)$ whenever σ is a subsequence of τ and τ is a lles that does not include experiment e. For instance, both conditions $F_1(\langle\langle e_1-\rangle\rangle, e_2)$ and $F_1(\langle\langle e_1+\rangle\rangle, e_2)$ follow from the condition $F_1(\langle\rangle, e_2)$. When there is a unique minimal reachability condition for every experiment, we say that the structure is "tree-like". Formally, we define:

Definition 2 (*Tree-like decision structures*). A decision structure $G = \langle W, F, R, c\rangle$ is *tree-like* if we can identify with each experiment $e \in W$ a single minimal lles, denoted path(e), that encodes the necessary and sufficient conditions for reaching e; i.e.

$$\forall e \in W\ \exists \mathrm{path}(e) \in \mathcal{LLES}(G)\ \forall \sigma \in \mathcal{LLES}(G)\ F(\sigma, e) \iff [\mathrm{path}(e) \sqsubseteq \sigma].$$

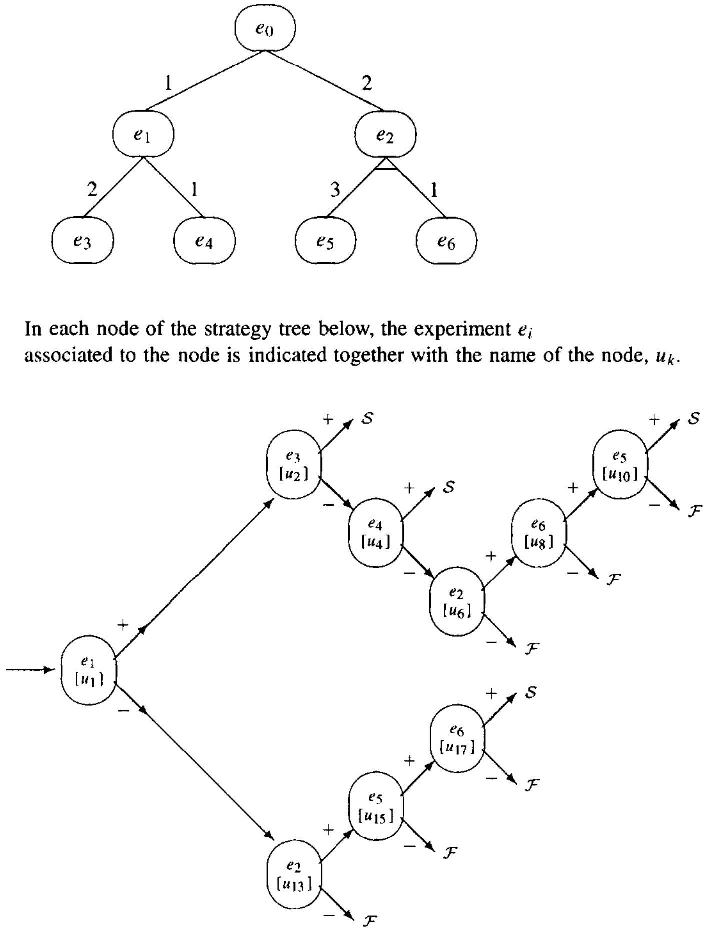
The class of all tree-like decision structures is denoted $\mathcal{TDS}$.

When the structure G represents an and-or decision tree, the lles path(e) corresponds to the unique path leading to the experiment e in the tree: e.g., as the path to e_4 in G_1 goes through e_0 and e_1, path$(e_4) = \langle\langle e_0+\rangle, \langle e_1+\rangle\rangle$.

2.2. Satisficing search strategies

A "search strategy" for a satisficing search problem specifies the order of traversal through the associated decision structure—in the sample application G_1 of Fig. 1, it tells the doctor when to perform which tests to determine whether the patient has hepatitis.

A strategy can be represented as a binary tree; for example the tree shown on the right side of Fig. 2 represents one possible strategy Θ_1 for the decision structure G_1. Each internal node in the strategy tree is labeled with an experiment that corresponds to some node in the decision structure. The strategy specifies the sequence of experiments to be performed in any given situation. For example, the Θ_1 strategy first performs the experiment e_1 associated with Θ_1's root u_1. If e_1 succeeds, Θ_1 then follows the $+$-labeled arc to the strategy subtree rooted in the e_3-labeled node, and performs e_3. If that test succeeds, Θ_1 advances up to the $\mathcal{S}$-labeled node, signifying that Θ_1 terminates with success. Alternatively, if e_3 fails, Θ_1 then follows the $-$-labeled arc, descending to the tree rooted in the e_4-labeled node, then performs e_4, and so forth. A general definition of this process is as follows:



In each node of the strategy tree below, the experiment e_i associated to the node is indicated together with the name of the node, u_k.

Fig. 2. Decision structure G_1 (above) and an associated strategy tree Θ_1 (below).

Definition 3 (*Search strategies*). A *strategy* for a decision structure $G = \langle W, F, R, c\rangle$ is a node- and arc-labeled binary tree $\Theta = \langle N, A, l_N, l_A\rangle$, where N is the set of nodes and $A \subseteq N \times N$ is the set of arcs connecting nodes to their descendants. The node-labeling l_N maps each internal node $n \in N$ in the tree to an experiment $e \in W$, and each leaf node to either $\mathcal{S}$ or $\mathcal{F}$. The arc-labeling l_A maps each arc $a \in A$ to either $+$ or $-$. Each internal node must have exactly two descending arcs, one labeled $+$ and the other $-$.

A *path* π in Θ is an alternating sequence of nodes and arcs leading from the root of the tree to a leaf—i.e. a sequence of the form $\pi = \langle n_1, a_{1,2}, n_2, a_{2,3}, \dots, a_{k-1,k}, n_k\rangle$ where each $n_i \in N$, and each $a_{i,i+1} = \langle n_i, n_{i+1}\rangle \in A$. Each such path has an associated labeled experiment sequence $l(\pi) = \langle\langle l_N(n_1) l_A(a_{1,2})\rangle, \dots, \langle l_N(n_{k-1}) l_A(a_{k-1,k})\rangle\rangle$. For Θ to be a proper strategy for G, the following conditions must be fulfilled by each path $\pi = \langle n_1, a_{1,2} \dots, n_k\rangle$ in Θ:

(1) $l(\pi) \in \mathcal{LLES}(G)$ (i.e. $l(\pi)$ must be a legal labeled experiment sequence);

(2) $l_N(n_k) = R(l(\pi)) \in \{\mathcal{S}, \mathcal{F}\}$ (i.e. the label of the final node must be either "success" or "failure"); and

(3) $R(l_{1,\dots,j}(\pi)) = \mathcal{U}$ for all $j < k$, where $l_{1,\dots,j}(\pi) = \langle\langle l_N(n_1) l_A(a_{1,2})\rangle, \dots, \langle l_N(n_j) l_A(a_{j,j+1})\rangle\rangle$ is the sequence of the first j elements of $l(\pi)$ (i.e. no proper prefix of a path can be conclusive).

We let path(Θ) refer to the set of all such proper paths in the strategy Θ. We also let $\mathcal{SS}(G)$ refer to the set of all strategies defined for the decision structure G, and $\mathcal{SS}(\mathcal{DS}) = \{\mathcal{SS}(G) \mid G \in \mathcal{DS}\}$ refer to the class of all strategies for all decision structures.

For an illustration of these notions, see the strategy tree Θ_1 shown on the right side of Fig. 2. There are 10 paths in Θ_1, one corresponding to each leaf node (indicated by the letters $\mathcal{S}$ and $\mathcal{F}$ in the figure). For instance, one such path is $\pi_{16} = \langle u_1, a_{1,13}, u_{13}, a_{13,15}, u_{15}, a_{15,16}, u_{16}\rangle$, for which the corresponding lles is $l(\pi_{16}) = \langle\langle e_1 -\rangle, \langle e_2 +\rangle, \langle e_5 -\rangle\rangle$.

2.3. *Optimal strategies*

We wish to identify the best strategy for traversing a given decision structure, i.e. the strategy whose expected cost is minimal. As this depends on the success probabilities of the individual experiments, different strategies will be optimal for different distributions. To state this more precisely, we define:

Definition 4 (*Expected cost of a strategy*). Let Θ be a strategy for the decision structure $G = \langle W, F, R, c\rangle$, and $p : W \to [0,1]$ be a distribution function that maps each experiment to its success probability. The (*expected*) *cost* of strategy Θ relative to the distribution p, denoted $C_p(\Theta)$, is defined as the sum of the cost of each path in the strategy, weighted by its probability, i.e.

$$C_p(\Theta) = \sum_{\pi \in \text{path}(\Theta)} p(l(\pi)) \times c(l(\pi)).$$

Here the probability of a path π is defined as $p(l(\pi)) = \prod_{\langle e_i \pm_i\rangle \in l(\pi)} p^{\pm_i}(e_i)$, where $p^{\pm}(e)$ is $p(e)$ if $\pm$ equals $+$, and $1 - p(e)$ if $\pm$ equals $-$.

Definition 5 (*Optimizing functions*). An *optimizing function* for a class of decision structures $\mathcal{D} \subseteq \mathcal{DS}$ is a function OSS that maps any decision structure $G = \langle W, F, R, c\rangle \in \mathcal{D}$, together with a distribution $p \in [0,1]^W$, to a strategy in $\mathcal{SS}(G)$ whose cost is minimal. That is,

$$\forall G \in \mathcal{D}\ \forall p \in [0,1]^W\ \forall \Theta \in \mathcal{SS}(G)\ C_p(\mathrm{OSS}(G,p)) \leqslant C_p(\Theta).$$

For brevity, when the decision structure G is understood from the context, we often denote by Θ_p the optimal strategy OSS(G,p) provided by the optimizing function for a given distribution p.

While these definitions assume that the experiments are independent of each other, both these definitions and the theorems below could be extended to handle more complicated situations.

Since we are only dealing with finite decision structures, optimal strategies can always be found by exhaustive search. Of course, exhaustive search is in general impractical, and if we are dealing with decision structures with concise encodings, such as and-or trees, the optimal strategies may not even have polynomial-size representations.

Nevertheless, optimal strategies can be determined in polynomial time in many interesting special cases. Garey [6] provided an algorithm for finding the optimal search strategy when the constraints can be represented as a regular "or tree" (i.e. no conjunctive subgoals and no multiple predecessors are allowed; cf. also [22]). Simon and Kadane [21] later extended this algorithm to deal with directed acyclic graphs in the special case where success at any intermediate node implies global success. (In dag's where global success requires reaching a specified goal node, the problem is NP-hard [8].) It is currently not known whether optimal strategies can be found in polynomial time for and-or trees. Some partial results on this question exist: for instance, Natarajan [18] presents an efficient algorithm for finding optimal "depth-first" search strategies in this case, and Smith [22] provides an algorithm for finding optimal "serial strategies". In the more general case of and-or dag's, the problem is NP-hard even when all the success probabilities are 1 [20].

3. The PAO algorithm

Each of the above-mentioned optimization algorithms assumes that the precise success probabilities of the experiments are known, which of course is not the case in most real-life situations. The best one can do then is to estimate these probabilities by observing a set of trials of the experiments, and then use these estimates to compute a near-optimal strategy. A potential pitfall in this approach, however, is that the strategies computed by any of the above algorithms are very sensitive to errors in the probability estimates: small changes in the estimates may lead to drastically different strategies. Fortunately, even though the choice of the actual strategy is very sensitive to estimation errors, the *cost* of the strategy obtained is not. This realization is one of the main contributions of this paper, as it means that we can use our estimates to obtain a near-optimal strategy.

Below, we describe an algorithm PAO that can be used in conjunction with any optimizing function OSS. Section 3.1 first formally defines the task of finding approximately optimal strategies and outlines the algorithm. The following sections then discuss the technical issues in more detail. First, in Section 3.2 we compute the *sample complexity* of this task: how many samples of each experiment are needed to guarantee, with a high level of confidence, that a strategy based on the resulting estimates will be close to optimal.

Section 3.3 then addresses a second problem: guaranteeing that the PAO algorithm will be able to obtain a sufficient number of samples of each experiment. The main complication arises from the precedence constraints. For example, in the context of our diagnostic example (Fig. 1), the sample complexity analysis may suggest that the

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Algorithm PAO($G : \mathcal{DS}$, $\varepsilon : \mathbb{R}^+$, $\delta : (0,1]$)
  $\hat{p} \leftarrow$ GS($G, \varepsilon, \delta$)
      /* GS may call oracle $\mathcal{O}$ a polynomial number of times */
  $\hat{\Theta} \leftarrow$ OSS($G, \hat{p}$)
  Return $\hat{\Theta}$
End PAO
```

Fig. 3. Outline of the PAO algorithm.

doctor needs to obtain 100 samples of the test "CytologicalTest(liver)". This is impossible if he is never able to perform a biopsy on any patients; i.e. if the experiment e_2 in structure G_1 never succeeds. In Section 3.3 we provide a solution to this problem for general "tree-like" decision structures, but also observe that the task is intractable in general.

3.1. *The* PAO *task*

A PAO problem instance consists of: a decision structure $G = \langle W, F, R, c\rangle \in \mathcal{DS}$; a bound on the allowed excess $\varepsilon \in \mathbb{R}^+$; and the required confidence $\delta \in (0,1]$. The algorithm also uses an oracle $\mathcal{O}$ that produces samples drawn at random from some fixed but unknown distribution.

For each instance, the PAO algorithm returns a strategy $\Theta_{pao} \in \mathcal{SS}(G)$, whose expected cost is, with high probability, close to optimal. Stated more precisely, let $\Theta_p = \mathrm{OSS}(G, p)$ be the optimal strategy for a true, but unknown, distribution p. Then with probability at least $1 - \delta$, the cost of the strategy Θ_{pao} is no more than ε higher than the cost of this optimal strategy, i.e.

$$\Pr\left[C_p(\Theta_{pao}) \leqslant C_p(\Theta_p) + \varepsilon\right] \geqslant 1 - \delta.$$

We split the PAO task into two subtasks: subroutine GS, which gathers the relevant statistics, and OSS, which uses those statistics to produce an appropriate strategy; see Fig. 3.

The GS subroutine takes as input the decision structure G and the parameters ε and δ; it computes how many samples are required, and makes the specified number of calls to the oracle $\mathcal{O}$ (specified below) to obtain them. The subroutine produces a vector of probability estimates, $\hat{p} = \langle \hat{p}_1, \ldots, \hat{p}_n\rangle$, where each $\hat{p}_i$ is the estimate for the success probability of the ith experiment $e_i \in W$. (To simplify our description, we are assuming that we do not know *a priori* the success probabilities of any of the experiments. If we happen to know some of the values, we can simply use those values directly, and not bother with the estimation.) The PAO algorithm then concludes by running an appropriate optimizing function OSS on these estimated probabilities $\hat{p}$, instead of the unknown true values. We concentrate here on the sample-gathering part of the PAO algorithm, GS; for the OSS functions, we rely on the ones provided by earlier researchers.

3.2. Sample complexity

We first analyze the sample complexity of the PAO task in the simple case where we can always perform the experiments whose success probabilities we need to estimate. Here we assume access to an oracle $\mathcal{O}$ that will, upon request, produce a sample κ_j from the population, together with its complete labeling $\mathcal{L}(\kappa_j) = \langle \ell_1^j, \dots, \ell_n^j \rangle$, where ℓ_i^j is 1 if κ_j passes experiment $e_i \in W$, and 0 otherwise. The GS routine performs a number M (specified below) calls to this $\mathcal{O}$ oracle, and returns a vector of probability estimates $\hat{p} = \langle \hat{p}_1, \dots, \hat{p}_n \rangle$, where each $\hat{p}_i = 1/M \sum_{j=1}^{n} \ell_i^j$; the OSS optimizing function then uses these values. We prove (Corollary A.1 in Appendix A) that the cost of the strategy $\Theta_{pao} = \text{OSS}(G, \hat{p})$ is within ε of the optimal, with reliability at least $1 - \delta$, whenever

$$M = \left\lceil 2 \left(\frac{nC}{\varepsilon} \right)^2 \ln \frac{2n}{\delta} \right\rceil, \tag{1}$$

where C is the worst-case cost of performing any sequence of experiments in W.

In fact, we can improve on the constant C somewhat. Let us denote by $D(e)$ the maximal cost of any concluding sequence of experiments beginning with experiment e. Formally:

Definition 6. Let $G = \langle W, F, R, c \rangle$ be a decision structure. For each experiment $e \in W$ we define

$$D(e) = \max\{c(\alpha \cdot \langle e\pm \rangle \cdot \beta) - c(\alpha) \mid \alpha \cdot \langle e\pm \rangle \cdot \beta \in \mathcal{LCES}(G)\}.$$

(The ± above indicates that the max should range over both + and − values.)

We can then let $C = \max_{e \in W}\{D(e)\}$ be the maximum remainder cost starting with any experiment $e \in W$. These $D(e)$ values are quite easy to compute when the underlying decision structure is an and-or tree: here, $D(e) = C_{tot} - c(\text{path}(e))$, where C_{tot} is the sum of the costs of all of the tree's arcs and $c(\text{path}(e))$ is the cost of path(e), the unique path in the and-or tree that leads from the root to experiment e; see Definition 2. (For instance, in the tree G_1 of Fig. 2 we have $C_{tot} = 10$ and $c(\text{path}(e_4)) = 2$, so $D(e_4) = 10 - 2 = 8$.)

The sample complexity bound (1) is derived in Appendix A as Corollary A.1 of a more general result that also takes into account the difficulty of labeling the samples (i.e. performing the experiments; see below). To very briefly outline the proof for this simple case: we first prove that after M samples, we are at least $1 - \delta$ confident that each probability estimate $\hat{p}_i$ is within $\varepsilon/2nC$ of the correct value p_i; we then show, based directly on the definition of the cost of a strategy and independent of which optimizing function is used, that this precision of the probability estimates suffices to guarantee that the cost of the obtained strategy is within ε of the optimal, i.e. that

$$\left[\forall i \ |p_i - \hat{p}_i| \leqslant \frac{\varepsilon}{2nC} \right] \Rightarrow |C_p(\Theta_{\hat{p}}) - C_p(\Theta_p)| \leqslant \varepsilon.$$

3.3. "Learning while doing" in tree-like structures

The simple PAO algorithm presented above assumes that the oracle $\mathcal{O}$ produces a complete labeling for each sample, i.e. it returns a complete vector $\mathcal{L}(\kappa) = \langle \ell_1, \ldots, \ell_n \rangle \in \{0,1\}^n$ on each query. In practical situations, however, such an oracle will typically not be available. Instead, the learning system must collect the statistics it needs (i.e. the individual component ℓ_i values of $\mathcal{L}(\kappa)$) while watching a performance element perform its task, over a sufficiently large set of samples. In the context of our diagnostic example, the learning module would observe the doctor as he examines patients, recording how many of these patients pass the various tests. After gathering enough information, the learner would compute the approximately optimal strategy Θ_{pao}, instruct the doctor to use this Θ_{pao} strategy, and terminate itself.[1] We view this as a "learning while doing" protocol [16], as the overall system is performing useful work during the learning phase (here, examining patients).

From now on, we assume our oracle $\mathcal{O}$, when queried, provides only an unlabeled sample (e.g., a patient κ), rather than the full labelings of that sample, $\mathcal{L}(\kappa)$. In order to determine the value of any label ℓ_i on sample κ, the GS subroutine must then actually "reach" and perform experiment e_i on κ.

Computing these ℓ_i values is problematic when there are intermediate experiments: for instance, in the case of our decision structure G_1, the doctor cannot immediately determine whether a patient's blood will react to serum_A; he must first be able to draw blood from the patient. Hence, our learning system will be unable to estimate the probability of event "`blood reacts to serum_A`" if the doctor is never able to extract blood (i.e. if Pr[Draw blood] = 0).

Fortunately, there is a way around this problem. The critical observation is the following: let $\rho(e_i)$ be the probability of "reaching" an experiment e_i during the execution of a strategy. (This notion is defined formally below.) If $\rho(e_i)$ is very small, we will be unlikely to reach e_i and hence to obtain samples of this experiment. However, the smaller the value of $\rho(e_i)$, the less sensitive the cost of the optimal strategy is to the value of the success probability $p(e_i)$, which means that we also *need* fewer samples of e_i. In the limit, if there is *no* chance of reaching e_i (i.e. $\rho(e_i) = 0$), then we will also need no samples of it (i.e. OSS can produce an optimal strategy even if $|\hat{p}_i - p_i| = 1$).

Definition 7. Let $G = \langle W, F, R, c \rangle \in \mathcal{DS}$ be a decision structure, and $p : W \rightarrow [0,1]$ a distribution function that maps each experiment to its success probability. For any strategy $\Theta = \langle N, A, l_N, l_A \rangle \in \mathcal{SS}(G)$, and any experiment $e \in W$, let $\rho(e, \Theta)$ be the probability that Θ will reach e, i.e.

$$\rho(e, \Theta) = \sum_{n: l_N(n) = e} p(l(\pi(n))),$$

[1] We are still considering only "one-shot learning", in which the learner sets the strategy only once, after the learning phase. We are not considering ways of modifying the strategy gradually over time to become incrementally better; but see [9]. Also, this issue differs from the "Exploration-Exploitation" trade-off discussed in the context of the Bandit problem (cf. [3,17]) as we are not concerned with minimizing the cumulative cost of the learning and performance systems together, over an infinite sequence of samples.

where the sum is over nodes n in the strategy Θ labeled with e, $\pi(n)$ is the path in Θ that leads to n, and the probability of this path $p(l(\pi(n)))$ is as defined above in Definition 4. Finally, let $\rho(e) = \max\{\rho(e,\Theta) \mid \Theta \in SS(G)\}$.

The formula for $\rho(e)$ reduces to a particularly simple form when the decision structure G is tree-like. In this case $\rho(e) = \prod_{i=1}^{k} p^{\pm_i}(f_i)$, where path$(e) = \langle\langle f_1 \pm_1\rangle, \ldots, \langle f_k \pm_k\rangle\rangle$ is the unique path that leads to e in G. For instance, in the G_1 decision structure of Fig. 1, we have path$(e_4) = \langle\langle e_0, +\rangle, \langle e_1, +\rangle\rangle$, and so $\rho(e_4) = p(e_0) \times p(e_1)$.

Now let $\Theta_p = \mathrm{OSS}(G,p)$ be the actual optimal strategy based on the unknown correct probability vector $p = \langle p_1, \ldots, p_n\rangle$, and $\Theta_{\hat{p}} = \mathrm{OSS}(G,\hat{p})$ be the strategy that our PAO algorithm will produce, based on the estimates the GS subroutine has obtained, $\hat{p} = \langle \hat{p}_1, \ldots, \hat{p}_n\rangle$. We wish to bound the cost difference $C_p(\Theta_{\hat{p}}) - C_p(\Theta_p)$. The following lemma shows that, in place of obtaining precise estimates of the probabilities p_i, we need only ensure that the product $\rho(e_i) \times |p_i - \hat{p}_i|$ is small for each e_i.[2]

Lemma 8. *Let $G = \langle W, F, R, c\rangle$ be a decision structure with $|W| = n$ experiments. Let $p = \langle p_1, \ldots, p_n\rangle$ be a vector of success probabilities for W, and $\hat{p} = \langle \hat{p}_1, \ldots, \hat{p}_n\rangle$ a vector of their estimates. Let the optimal search strategy for G with respect to p be $\Theta_p = \mathrm{OSS}(G,p)$, and let $\Theta_{\hat{p}} = \mathrm{OSS}(G,\hat{p})$ be the strategy based on the estimated probabilities. Then*

$$C_p(\Theta_{\hat{p}}) - C_p(\Theta_p) \leqslant 2\sum_{i=1}^{n} D(e_i) \times \rho(e_i) \times |p_i - \hat{p}_i|.$$

A further complication now arises from the fact that the $\rho(e_i)$ values actually depend on the unknown true distribution p. Fortunately, we can also approximate these values as we are obtaining the estimates of the p_i. In essence, we need only "aim for e_i" a certain number of times: each time we reach e_i, we improve our estimate of p_i (i.e. reduce the $|\hat{p}_i - p_i|$ "error bars") and each time our path to e_i is blocked, we can, with confidence, reduce the value of $\rho(e_i)$.

The rest of this subsection first shows how to estimate the products $\rho(e_i) \times |p_i - \hat{p}_i|$ in tree-like decision structures, then discusses the difficulties in computing near-optimal strategies in more general structures.

Dealing with tree-like decision structures: Given an experiment e in a tree-like decision structure G, recall that path$(e) = \langle\langle e_1 \pm_1\rangle, \langle e_2 \pm_2\rangle, \ldots, \langle e_k \pm_k\rangle\rangle$ is the unique minimal lles that determines when e can be performed. We say that a strategy $\Theta \in SS(G)$ is a *direct strategy* for e if it contains this lles as an initial segment, in the sense that the root of Θ is labeled with the experiment e_1, and its $\pm_1$-labeled arc (i.e. the $+$-labeled arc if $\pm_1$ equals $+$, and the $-$-labeled arc if $\pm_1$ equals $-$) descends to a node labeled with the experiment e_2, and the $\pm_2$-labeled arc from that node descends to a node labeled with e_3, and so on, down to a node labeled e_k, whose $\pm_k$-labeled arc leads to a node labeled with e. We denote the class of direct strategies for an experiment e by $SS(e)$. As an example, the strategy Θ_1 shown in Fig. 2 goes directly to e_1 and hence $\Theta_1 \in SS(e_1)$;

[2] Appendix A contains the proofs for all lemmata, theorems and corollaries presented in the text.

```
Algorithm GS($G$: $\mathcal{TDS}$, $\varepsilon$: $\mathbb{R}^+$, $\delta$: $(0,1]$)
  ForEach $e \in W$ do
    Find some $\Theta_e \in \mathcal{SS}(e)$
    $tot(e) \leftarrow 0$
    $suc(e) \leftarrow 0$
    $m(e) \leftarrow \begin{cases} \left\lceil 2 \left(\frac{nD(e)}{\varepsilon}\right)^2 \ln\frac{2n}{\delta} \right\rceil & \text{if } F(\langle\rangle, e) \\ \left\lceil 2\left(1+\frac{\varepsilon}{nD(e)}\right) \left(\frac{nD(e)}{\varepsilon}\right)^2 \ln\frac{4n}{\delta} \right\rceil & \text{if } \neg F(\langle\rangle, e) \end{cases}$
  End ForEach
  While $\exists e$ such that $m(e) > 0$ do
    Get sample $\kappa$ from oracle $\mathcal{O}$
    Execute strategy $\Theta_e$ on sample $\kappa$
      After performing each experiment $e_j$:
        $tot(e_j) \leftarrow tot(e_j) + 1$
        $m(e_j) \leftarrow m(e_j) - 1$
        If $e_j$ succeeds: $suc(e_j) \leftarrow suc(e_j) + 1$
        If $e_j$'s result (success or failure) means
          $e$ cannot be reached: $m(e) \leftarrow m(e) - 1$
  End While
  ForEach $e_i \in W$ do
    $\hat{p}(e_i) \leftarrow \begin{cases} \frac{suc(e_i)}{tot(e_i)} & \text{if } tot(e_i) > 0 \\ \frac{1}{2} & \text{otherwise} \end{cases}$
  End ForEach
  Return $\hat{p} = \langle \hat{p}(e_1), \ldots, \hat{p}(e_n) \rangle$
End GS
```

Fig. 4. A GS algorithm for tree-like decision structures.

it also contains the direct route to e_3 as an initial segment, and hence $\Theta_1 \in \mathcal{SS}(e_3)$. On the other hand, the strategy "digresses" to consider e_3 before e_4, and so $\Theta_1 \notin \mathcal{SS}(e_4)$; similarly $\Theta_1 \notin \mathcal{SS}(e_2)$ as Θ_1 considers e_1 before e_2.

The GS algorithm shown in Fig. 4 can deal with any tree-like decision structure G. The algorithm first identifies a direct strategy $\Theta_e \in \mathcal{SS}(e)$ for each $e \in W$. (There can in general be many such strategies, performing different experiments outside their common initial path to e; this paper does not consider how to choose between the alternatives. Nor does it consider the cost of identifying any of these strategies, except to observe that for e.g. and-or trees they can be constructed quite efficiently, directly from the tree structure.) After selecting this set of strategies, GS associates three counters with each experiment, $tot(e_i)$, $suc(e_i)$ and $m(e_i)$, that will record, respectively, the number of times experiment e_i has been performed, the number of times e_i succeeded, and the number of attempts that remain to be performed. As it processes the instances, GS updates each of these counters by: incrementing $tot(e_i)$ each time GS performs experiment e; incrementing $suc(e_i)$ each time the experiment e succeeds; and decrementing $m(e_i)$ each time GS has attempted to reach experiment e_i either by performing e_i, or by using the strategy Θ_{e_i} but failing to reach e_i.

The remaining challenge is to identify when to use which strategy. (Clearly GS will not, in general, be able to observe enough trials of the different experiments if it uses the same strategy throughout.) On each sample, GS first identifies the needy experiments, i.e. those e's for which $m(e) > 0$. If there are none, then GS has collected enough samples, and so can terminate, returning the obtained vector of estimates $\hat{p}$, which PAO will pass to the OSS algorithm. Otherwise, GS selects one of the needy experiments e, and executes the associated strategy Θ_e.

Notice that GS decrements at least one $m(e)$ counter on each sample, viz. the one associated with the experiment e to which it is currently aiming. Hence, after at most $\sum_{e \in W} m_0(e)$ samples (where the $m_0(e)$ are the initial values of the counters), all of the $m(e)$ counters will be zero and GS will terminate; it therefore requires only a polynomial number of samples. (The algorithm may of course use far fewer samples, as most Θ_e strategies will reduce the $m(e_j)$ values for several different experiments e_j. GS can also be changed to decrease the counters of all experiments e' that are deemed unreachable in the process of following Θ_e.) The following lemma characterizes the behavior of the algorithm:

Lemma 9. *Let $G = \langle W, F, R, c\rangle$ be a tree-like decision structure with $|W| = n$ experiments, and let $p = \langle p_1, \ldots, p_n\rangle$ be a vector of success probabilities for the experiments. Furthermore, let $\varepsilon, \delta > 0$ be any given constants, and let $\hat{p} = \langle \hat{p}_1, \ldots, \hat{p}_n\rangle$ be a vector of probability estimates computed by the GS algorithm of Fig. 4. Then*

$$\forall e_i \in W \ \Pr\left[D(e_i) \times \rho(e_i) \times |p_i - \hat{p}_i| \geqslant \frac{\varepsilon}{2n}\right] \leqslant \frac{\delta}{n}.$$

(While our analysis uses the $\rho(e_i)$ values, notice that the GS algorithm never actually computes them.) Combining the results of Lemmas 8 and 9, we obtain the following theorem:

Theorem 10. *Let $G = \langle W, F, R, c\rangle$ be a tree-like decision structure with $|W| = n$ experiments, and let $p = \langle p_1, \ldots, p_n\rangle$ be a vector of success probabilities for the experiments. Furthermore, let $\varepsilon, \delta > 0$ be any given constants, and let $\Theta_{pao} = \text{PAO}(G, \varepsilon, \delta)$ be the strategy produced by the PAO algorithm using the GS subroutine of Fig. 4. Then, with probability at least $1 - \delta$, $C_p(\Theta_{pao}) - C_p(\Theta_p) \leqslant \varepsilon$, where $\Theta_p = \text{OSS}(G, p)$ is the optimal strategy for probability vector p.*

Beyond tree-like decision structures: While the specific GS algorithm presented above applies only to tree-like decision structures, there can be other related algorithms that can learn strategies for other decision structures. The main challenge is in estimating $\rho(e_i)$, as required to bound the product $\rho(e_i) \times |p_i - \hat{p}_i|$, which is complicated by the fact that there can be many distinct ways of reaching an experiment in a general decision structure.

To address this task, recall from Definition 7 that $\rho(e)$ is the *maximum* probability of reaching the experiment e, where the maximum is taken over all possible strategies. We can always approximate this value by first estimating $\rho(e,\Theta)$ for every possible strategy Θ, and then taking the maximum of these values: if each estimate $\hat{\rho}(e,\Theta)$ is within ε of $\rho(e,\Theta)$ with probability at least $1-\delta/|\mathcal{SS}(G)|$, then the value $\hat{\rho}(e)=\max\{\hat{\rho}(e,\Theta) \mid \Theta \in \mathcal{SS}(G)\}$ will be within ε of $\rho(e)=\max\{\rho(e,\Theta) \mid \Theta \in \mathcal{SS}(G)\}$ with probability at least $1-\delta$. Even though the number of strategies in $\mathcal{SS}(G)$ for a given decision structure G can be exponential in the size of G, there can be ways of exploiting the structure of G, and hence of $\mathcal{SS}(G)$, to limit the number of $\hat{\rho}(e,\Theta)$ values that need to be considered. From this point of view, the GS algorithm for tree-like structures is based on the observation that for any tree-like structure G, the direct strategies $\Theta_e \in \mathcal{SS}(e)$ necessarily yield the largest values of $\rho(e,\Theta_e)$ for any experiment e. In fact, one can use the same GS algorithm whenever it is possible to identify each experiment e with a strategy Θ_e for which $\rho(e,\Theta_e)=\rho(e)$.

This is not always straightforward. The extended paper [11] includes an algorithm that uses dynamic programming techniques to sequentially estimate the probabilities of each "layer" of certain types of decision structures. Unfortunately, the following general result shows that the computational complexity of any such algorithm is likely to be exponential in the number of experiments.

Theorem 11. *Assume* RP $\neq$ NP. (RP *is the class of problems solvable by probabilistic polynomial time algorithms with one-way error, cf.* [12].) *Then there is no probabilistic polynomial time algorithm, and consequently no deterministic polynomial time algorithm that, given a decision structure* $G=\langle W,F,R,c\rangle$, *an experiment* $e \in W$, *a distribution function* $p : W \longrightarrow [0,1]$, *and parameters* $\varepsilon,\delta > 0$, *can estimate the value* $\rho(e)$ *to within* ε *with probability at least* $1-\delta$.

4. Conclusion

The results presented in this paper have been motivated by, and extend, various other lines of research. The underlying objective of finding a provably good search strategy comes from the work on optimal satisficing search strategies [2,6,18,21,22]. Each of these earlier papers considered some specifically defined class of decision structures and, moreover, required the user to supply precise success probability values for the experiments. Our work extends this body of research in three ways. First, we have defined a general framework of "decision structures" and "search strategies", which encompasses and generalizes the models used before. Second, we have analyzed, in this very general setting, the sensitivity of optimal search strategies to errors in the probability estimates. Third, we have provided an efficient algorithm for finding good estimates of the probability values in the case of tree-like decision structures, and proved that (unless RP = NP) there can be no efficient algorithm for this task for general structures.

Our approach also resembles the work on speed-up learning (including both "explanation-based learning" [5,14,15] and "chunking" [13]), as it uses previous solutions to suggest a way of improving the speed of a performance system. Most speed-up learning systems, however, use only a single example to suggest an improvement; we extend those works by showing how to use a *set* of samples and by describing, furthermore, the exact number of samples required. Also, while most speed-up learning systems are based on purely heuristic considerations, we use mathematically sound techniques to guarantee that our new strategies will be close to optimal, with provably high probability.

Finally, this work derives many of its mathematical methods, as well as its title, from the field of "probably approximately correct learning" [23]. We hope to have enriched this field by providing an application of the PAC framework outside of its traditional setting of concept learning.

Appendix A. Proofs

This appendix contains the proofs of the results mentioned in the body of the paper.

Lemma 8. *Let $G = \langle W, F, R, c\rangle$ be a decision structure with $|W| = n$ experiments. Let $p = \langle p_1, \ldots, p_n\rangle$ be a vector of success probabilities for G, and $\hat{p} = \langle \hat{p}_1, \ldots, \hat{p}_n\rangle$ a vector of their estimates. Let the optimal search strategy for G with respect to p be $\Theta_p = \mathrm{OSS}(G, p)$, and let $\Theta_{\hat{p}} = \mathrm{OSS}(G, \hat{p})$ be the strategy based on the estimated probabilities. Then*

$$C_p(\Theta_{\hat{p}}) - C_p(\Theta_p) \leqslant 2\sum_{i=1}^{n} D(e_i) \times \rho(e_i) \times |p_i - \hat{p}_i|.$$

Proof. Given the vectors p and $\hat{p}$, let $p^{(i)}$ denote the vector $\langle \hat{p}_1, \ldots, \hat{p}_i, p_{i+1}, \ldots, p_n\rangle$, and as special cases, $p^{(0)} = p$ and $p^{(n)} = \hat{p}$. We shall prove below that for any strategy Θ for G, and for every $i = 0, \ldots, n-1$,

$$\left|C_{p^{(i)}}(\Theta) - C_{p^{(i+1)}}(\Theta)\right| \leqslant D(e_i) \times \rho(e_i) \times |p_i - \hat{p}_i|, \tag{A.1}$$

which implies that

$$\begin{aligned} &|C_p(\Theta) - C_{\hat{p}}(\Theta)| \\ &\quad\leqslant \sum_{i=1}^{n} \left|C_{p^{(i)}}(\Theta) - C_{p^{(i-1)}}(\Theta)\right| \leqslant \sum_{i=1}^{n} D(e_i) \times \rho(e_i) \times |\hat{p}_i - p_i|. \end{aligned}$$

Applying this bound to the strategies Θ_p and $\Theta_{\hat{p}}$, and noting that by optimality $C_{\hat{p}}(\Theta_{\hat{p}}) - C_{\hat{p}}(\Theta_p) \leqslant 0$, then yields the desired result:

$$\begin{aligned} &C_p(\Theta_{\hat{p}}) - C_p(\Theta_p) \\ &\quad= [C_p(\Theta_{\hat{p}}) - C_{\hat{p}}(\Theta_{\hat{p}})] + [C_{\hat{p}}(\Theta_{\hat{p}}) - C_{\hat{p}}(\Theta_p)] + [C_{\hat{p}}(\Theta_p) - C_p(\Theta_p)] \\ &\quad\leqslant \left[\sum_{i=1}^{n} D(e_i) \times \rho(e_i) \times |p_i - \hat{p}_i|\right] + 0 + \left[\sum_{i=1}^{n} D(e_i) \times \rho(e_i) \times |\hat{p}_i - p_i|\right] \\ &\quad= 2\sum_{i=1}^{n} D(e_i) \times \rho(e_i) \times |\hat{p}_i - p_i|. \end{aligned}$$

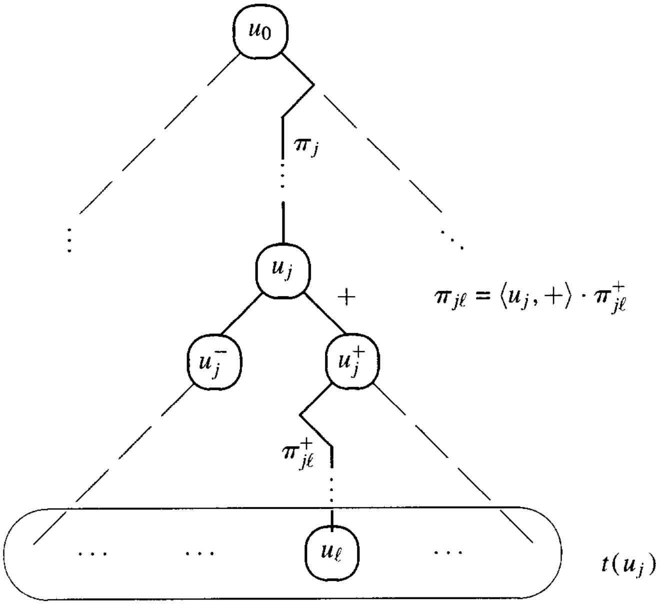

Fig. A.1. Illustration of notation.

In proving inequality (A.1), we shall make use of the following notation (cf. Fig. A.1): given any node u_j in a strategy Θ, let π_j denote the path leading from the root of Θ to u_j. For any such path π_j, we denote the associated cost $c(l(\pi_j))$ briefly by $c(\pi_j)$. (Here, we have extended the $l(\cdot)$ function to partial sequences: u_j, the final entry in π_j, does not have to be a leaf node in the strategy tree.) We also extend the cost function to incomplete paths in the strategy tree by defining

$$\begin{aligned}&c(\langle u_i, a_{i,i+1}, u_{i+1}, \dots, u_k\rangle)\\&\quad = c(\langle u_0, a_{0,1}, u_1, \dots, u_i, a_{i,i+1}, u_{i+1}, \dots, u_k\rangle) - c(\langle u_0, a_{0,1}, u_1, \dots, u_i\rangle),\end{aligned}$$

where u_0 is the root node of the strategy tree and $\langle u_0, a_{0,1}, u_1, \dots, a_{k-1,k}, u_k\rangle$ is any connected path through the tree; furthermore, for single nodes we define $c(u_i) = c(\langle u_i\rangle)$.

Let u_j^+ be the +-descendant of node u_j, and let $t(u_j)$ denote the set of leaf nodes below u_j. For each $u_l \in t(u_j)$, let π_{jl} denote the path from u_j to u_l, and if $u_l \in t(u_j^+)$, let π_{jl}^+ denote the path from node u_j^+ to u_l. Let $C_p(\Theta_j^+)$ denote the "expected cost of the +-subtree of u_j":

$$C_p(\Theta_j^+) = c(u_j^+) + \sum_{u_l \in t(u_j^+)} p(\pi_{jl}^+)c(\pi_{jl}^+).$$

Analogous definitions hold for u_j^-, $t(u_j^-)$, $p(\pi_{jl}^-)$, and $C_p(\Theta_j^-)$.

For a given experiment $e_i \in W$, let $N(e_i) = \{u_j\}_{j=1,\dots,k}$ be the set nodes in Θ labeled with e_i. Let $\Theta \mid e_i$ be the subtree within Θ consisting of the paths from the root of Θ through a node in $N(e_i)$ down to a leaf; and let $\Theta \mid \bar{e}_i$ be the subtree consisting of the other paths. Notice that $C_p(\Theta) = C_p(\Theta \mid e_i) + C_p(\Theta \mid \bar{e}_i)$. We may partition the paths in Θ^+ according to which of the u_j nodes each passes through (recall that in any strategy Θ, an experiment e can occur at most once on any path from the root to a leaf node), and use this representation to obtain a very simple formula expressing the influences of e_i's success probability p_i on the function $C_p(\Theta \mid e_i)$ (and hence on $C_p(\Theta)$):

$$\begin{aligned}
&C_p(\Theta \mid e_i) \\
&\quad = \sum_{\pi \in \text{path}(\Theta^{+})} p(\pi) c(\pi) = \sum_{j=1}^{k} \sum_{u_l \in t(u_j)} p(\pi_l) c(\pi_l) \\
&\quad = \sum_{j=1}^{k} \left[\sum_{u_l \in t(u_j^{+})} p(\pi_l) c(\pi_l) + \sum_{u_l \in t(u_j^{-})} p(\pi_l) c(\pi_l) \right] \\
&\quad = \sum_{j=1}^{k} \left[\sum_{u_l \in t(u_j^{+})} p(\pi_j) p_i p(\pi_{jl}^{+}) \Big(c(\pi_j) + c(u_j^{+}) + c(\pi_{jl}^{+}) \Big) \right. \\
&\qquad\qquad \left. + \sum_{u_l \in t(u_j^{-})} p(\pi_j)(1 - p_i) p(\pi_{jl}^{-}) \Big(c(\pi_j) + c(u_j^{-}) + c(\pi_{jl}^{-}) \Big) \right] \\
&\quad = \sum_{j=1}^{k} p(\pi_j) \left[p_i \sum_{u_l \in t(u_j^{+})} p(\pi_{jl}^{+}) \Big(c(\pi_j) + c(u_j^{+}) + c(\pi_{jl}^{+}) \Big) \right. \\
&\qquad\qquad \left. + (1 - p_i) \sum_{u_l \in t(u_j^{-})} p(\pi_{jl}^{-}) (c(\pi_j) + c(u_j^{-}) + c(\pi_{jl}^{-})) \right] \\
&\quad = \sum_{j=1}^{k} p(\pi_j) \left[p_i \left((c(\pi_j) + c(u_j^{+})) \sum_{u_l \in t(u_j^{+})} p(\pi_{jl}^{+}) + \sum_{j=1}^{k} p(\pi_{jl}^{+}) c(\pi_{jl}^{+}) \right) \right. \\
&\qquad\qquad + (1 - p_i) \left((c(\pi_j) + c(u_j^{-})) \sum_{u_l \in t(u_j^{-})} p(\pi_{jl}^{-}) \right. \\
&\qquad\qquad\qquad \left. \left. + \sum_{j=1}^{k} p(\pi_{jl}^{-}) c(\pi_{jl}^{-}) \right) \right] \\
&\quad = \sum_{j=1}^{k} p(\pi_j) \left[p_i \left(c(\pi_j) + c(u_j^{+}) + \sum_{j=1}^{k} p(\pi_{jl}^{+}) c(\pi_{jl}^{+}) \right) \right. \\
&\qquad\qquad \left. + (1 - p_i) \left(c(\pi_j) + c(u_j^{-}) + \sum_{j=1}^{k} p(\pi_{jl}^{-}) c(\pi_{jl}^{-}) \right) \right]
\end{aligned}$$

$$= \sum_{j=1}^{k} p(\pi_j) \Big[p_i(c(\pi_j) + C_p(\Theta_j^+)) + (1-p_i)(c(\pi_j) + C_p(\Theta_j^-)) \Big]$$

$$= \sum_{j=1}^{k} p(\pi_j) \Big(c(\pi_j) + p_i C_p(\Theta_j^+) + (1-p_i) C_p(\Theta_j^-) \Big).$$

(Near the end of the proof we have simplified the formulas using the fact that a complete system of elementary probabilities sums to 1: in this case

$$\sum_{u_l \in t(u_j^+)} p(\pi_{jl}^+) = \sum_{u_l \in t(u_j^-)} p(\pi_{jl}^-) = 1.)$$

An analogous formula can be derived for the cost function $C_{\hat{p}_i}(\Theta \mid e_i)$, where $\hat{p}_i = \langle p_1, \ldots, p_{i-1}, \hat{p}_i, p_{i+1}, \ldots, p_n \rangle$ is the probability vector that differs from p only in the ith value. Using now the facts that $C_p(\Theta \mid \bar{e}_i) = C_{\hat{p}_i}(\Theta \mid \bar{e}_i)$ and $C_p(\Theta_j^+) = C_{\hat{p}_i}(\Theta_j^+)$, as none of the substrategies $\Theta \mid \bar{e}_i$, Θ_j^+, $j \neq i$, involve the experiment e_i, we obtain the following bound:

$$\begin{aligned}
|C_p(\Theta) - C_{\hat{p}_i}(\Theta)| &= \Big| \big(C_p(\Theta \mid e_i) + C_p(\Theta \mid \bar{e}_i)\big) - \big(C_{\hat{p}_i}(\Theta \mid e_i) + C_{\hat{p}_i}(\Theta \mid \bar{e}_i)\big) \Big| \\
&= \Big| \big(C_p(\Theta \mid e_i) - C_{\hat{p}_i}(\Theta \mid e_i)\big) + \big(C_p(\Theta \mid \bar{e}_i) - C_{\hat{p}_i}(\Theta \mid \bar{e}_i)\big) \Big| \\
&= \left| \sum_{j=1}^{k} p(\pi_j) \big((p_i - \hat{p}_i) C_p(\Theta_j^+) + (\hat{p}_i - p_i) C_p(\Theta_j^-) \big) + 0 \right| \\
&= \left| \sum_{j=1}^{k} p(\pi_j)(p_i - \hat{p}_i) \big(C_p(\Theta_j^+) - C_p(\Theta_j^-) \big) \right| \\
&\leqslant \sum_{j=1}^{k} p(\pi_j) \times |p_i - \hat{p}_i| \times \max_{l(u_j)=e_i} |C_p(\Theta_j^+) - C_p(\Theta_j^-)| \\
&\leqslant \rho(e_i) \times |p_i - \hat{p}_i| \times \max_{l(u_j)=e_i} \max \big\{ C_p(\Theta_j^+), C_p(\Theta_j^-) \big\}.
\end{aligned}$$

The last line of the calculation uses the facts that $C_p(\cdot) \geqslant 0$ and that $\sum_{j=1}^{k} p(\pi_j) = \rho(e_i, \Theta)$ is the probability that this strategy Θ will reach e_i, and hence is bounded by $\rho(e_i) = \max\{\rho(e_i, \Theta) \mid \Theta \in \mathcal{SS}(G)\}$.

All that remains is to show that the value $\max_{l(u_j)=e_i}\{C_p(\Theta_j^+), C_p(\Theta_j^-)\}$ is bounded by $D(e_i)$. To see this, consider any u_j such that $l(u_j) = e_i$. Then

$$\begin{aligned}
C_p(\Theta_j^+) &= c(u_j^+) + \sum_{u_l \in t(u_j^+)} p(\pi_{jl}^+) c(\pi_{jl}^+) \\
&\leqslant c(u_j^+) + \Big(\max_{u_l \in t(u_j^+)} c(\pi_{jl}^+) \Big) \sum_{u_l \in t(u_j^+)} p(\pi_{jl}^+)
\end{aligned}$$

$$
\begin{aligned}
&= c(u_j^+) + \max_{u_l \in t(u_j^+)} c(\pi_{jl}^+)\\
&= \max_{u_l \in t(u_j^+)} c(\pi_{jl})\\
&\leqslant D(e_i).
\end{aligned}
$$

Similarly, $C_p(\Theta_j^-) \leqslant \max_{u_l \in t(u_j^-)} c(\pi_{jl}) \leqslant D(e_i)$. □

Lemma 9. *Let $G = \langle W, F, R, c\rangle$ be a tree-like decision structure with $|W| = n$ experiments, and let $p = \langle p_1, \ldots, p_n\rangle$ be a vector of success probabilities. Furthermore, let $\varepsilon, \delta > 0$ be any given constants, and let $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)$ be a vector of probability estimates computed by the* GS *algorithm of Fig.* 4. *Then*

$$
\forall e_i \in W \ \Pr\left[D(e_i) \times \rho(e_i) \times |p_i - \hat{p}_i| \geqslant \frac{\varepsilon}{2n}\right] \leqslant \frac{\delta}{n}. \tag{A.2}
$$

Proof. We use Hoeffding's inequality, which is a simple form of Chernoff bounds [1,4]: let $\{X_i\}$ be a set of independent, identically distributed random Bernoulli variables, whose common mean is μ. Let $\bar{S}^{(M)} = 1/M \sum_{i=1}^{M} X_i$ be the sample mean after taking M samples. Then

$$
\Pr[\bar{S}^{(M)} > \mu + \lambda] < \mathrm{e}^{-2M\lambda^2}, \quad \Pr[\bar{S}^{(M)} > \mu - \lambda] < \mathrm{e}^{-2M\lambda^2}. \tag{A.3}
$$

To prove that inequality (A.2) holds for each experiment $e_i \in W$, we consider two cases, depending on whether the experiment can be performed initially or not. If so (i.e. if $F(\langle\rangle, e_i)$ and consequently $\rho(e_i) = 1$), then the GS algorithm will perform at least

$$
m(e_i) = \left\lceil 2\left(\frac{nD(e_i)}{\varepsilon}\right)^2 \ln\frac{2n}{\delta}\right\rceil \tag{A.4}
$$

trials of e_i. As the samples are drawn at random from a fixed distribution, we can use inequality (A.3): after $m(e_i)$ samples,

$$
\begin{aligned}
\Pr\left[\rho(e_i) \times |\hat{p}_i - p_i| \geqslant \frac{\varepsilon}{2nD(e_i)}\right] &= \Pr\left[|\hat{p}_i - p_i| \geqslant \frac{\varepsilon}{2nD(e_i)}\right]\\
&\leqslant 2\exp\left(-2m(e_i)\left(\frac{\varepsilon}{2nD(e_i)}\right)^2\right)\\
&\leqslant \frac{\delta}{n}.
\end{aligned}
$$

Now consider an experiment e_i that is not immediately reachable (i.e., such that $F(\langle\rangle, e_i)$ does not hold). Here, the GS algorithm will attempt to reach e_i along a direct path a total of M times, where

$$
M \geqslant \left\lceil 2\left(1 + \frac{\varepsilon}{nD(e_i)}\right)\left(\frac{nD(e_i)}{\varepsilon}\right)^2 \ln\frac{4n}{\delta}\right\rceil. \tag{A.5}
$$

Of these, GS reaches and performs e_i some number k times; and does not reach e_i the remaining $M-k$ times. Denote by $j(k)$ the value of the product $\rho(e_i) \times |\hat{p}_i - p_i|$ assuming that GS succeeds exactly k times. It suffices to show that for any value of k, $j(k) \leqslant \varepsilon/(2nD(e_i))$ with probability at least $1-\delta/n$.

Using the fact that $\hat{\rho}(e_i) = k/M$ is the estimated value of $\rho(e_i)$ here (as GS has succeeded k times out of M) together with inequality (A.3), we know that with probability at least $1-\delta/2n$,

$$\rho(e_i) \leqslant \hat{\rho}(e_i) + \sqrt{\frac{1}{2M}\ln\frac{2n}{\delta}},$$

and with probability at least $1-\delta/2n$,

$$|\hat{p}_i - p_i| \leqslant \sqrt{\frac{1}{2k}\ln\frac{4n}{\delta}}.$$

Of course, each of these terms tops off at 1 (since $\rho(e_i) \leqslant 1$ and $|\hat{p}_i - p_i| \leqslant 1$). We therefore define

$$g(k) = \min\left\{1, \frac{k}{M} + \sqrt{\frac{1}{2M}\ln\frac{2n}{\delta}}\right\} \times \min\left\{1, \sqrt{\frac{1}{2k}\ln\frac{4n}{\delta}}\right\},$$

and observe that $j(k) \leqslant g(k)$ with probability at least $(1-\delta/2n)^2 \geqslant 1-\delta/n$.

We now need to bound the largest possible value of $g(k)$. First, note that $g(k)$ can be bounded by

$$g(k) \leqslant \begin{cases} \dfrac{k}{M} + \sqrt{\dfrac{1}{2M}\ln\dfrac{2n}{\delta}} & \text{when } 0 \leqslant k \leqslant k_0, \\ \left(\dfrac{k}{M} + \sqrt{\dfrac{1}{2M}\ln\dfrac{2n}{\delta}}\right) \times \sqrt{\dfrac{1}{2k}\ln\dfrac{4n}{\delta}} & \text{when } k_0 \leqslant k \leqslant M, \end{cases}$$

where $k_0 = 1/2\ln(4n/\delta)$. (The two expressions for $g(k)$ have the same value at $k = k_0$.) As the first expression has a positive first derivative with respect to k, it is largest at its largest allowed value of k, viz. k_0. Using first and second derivatives, we see that the second expression is upwards concave on the interval $k = [k_0, M]$, and thus its value is maximal at either $k = k_0$ or $k = M$.

Hence, the largest value of $g(k)$ is bounded by the value of the second expression at either $k = k_0$ or $k = M$, i.e. by the larger of the values

$$g(k_0) = \frac{1}{2M}\ln\frac{4n}{\delta} + \sqrt{\frac{1}{2M}\ln\frac{2n}{\delta}},$$

$$g(M) = \frac{1}{2M}\sqrt{\ln\frac{4n}{\delta}\ln\frac{2n}{\delta}} + \sqrt{\frac{1}{2M}\ln\frac{4n}{\delta}}.$$

By inspection, both of these values, and hence all values of $g(k)$, are below

$$g_M = \frac{1}{2M}\ln\frac{4n}{\delta} + \sqrt{\frac{1}{2M}\ln\frac{4n}{\delta}} = \left(\frac{1}{2} + \sqrt{\frac{1}{2M}\ln\frac{4n}{\delta}}\right)^2 - \frac{1}{4}.$$

As $p(e_i) \times |\hat{p}_i - p_i| = j(k) \leqslant g_M$, we need only find an M sufficiently large so that $g_M \leqslant \varepsilon/(2nD(e_i))$. Solving $g_M = \varepsilon/(2nD(e_i))$ for M yields

$$M = \left(\sqrt{\frac{2\varepsilon}{nD(e_i)} + 1} - 1 \right)^{-2} \ln \frac{4n}{\delta}.$$

To see that the $m(e_i)$ value from Eq. (A.5) is larger than this M (and hence the corresponding g_M value will be yet smaller), just observe that $(\sqrt{2\gamma + 1} - 1)^{-2} \leqslant (1 + \gamma)/(\gamma^2)$ holds for any $\gamma > 0$, so in particular for $\gamma = \varepsilon/(nD(e_i))$. Hence, after taking at least $m(e_i)$ samples, we can be confident that the product $p(e_i) \times |\hat{p}_i - p_i|$ is sufficiently small, as desired.

Notice that in the typical situation where ε is small relative to $D(e)$, the $m(e_i)$ value obtained here is only slightly larger than the value obtained from Eq. (A.4). □

Theorem 10. *Let $G = \langle W, F, R, c\rangle$ be a tree-like decision structure with $|W| = n$ experiments, and let $p = \langle p_1, \ldots, p_n\rangle$ be a vector of success probabilities for the experiments. Furthermore, let $\varepsilon, \delta > 0$ be any given constants, and let $\Theta_{pao} = \text{PAO}(G, \varepsilon, \delta)$ be the strategy produced by the* PAO *algorithm using the GS subroutine of Fig. 4. Then, with probability at least $1 - \delta$, $C_p(\Theta_{pao}) - C_p(\Theta_p) \leqslant \varepsilon$, where $\Theta_p = \text{OSS}(G, p)$ is the optimal strategy for probability vector p.*

Proof. Let $\hat{p} = \langle \hat{p}_1, \ldots, \hat{p}_n\rangle$ be the vector of probability estimates produced by the GS subroutine. By Lemma 9 each of the products $D(e_i) \times p(e_i) \times |p_i - \hat{p}_i|$ is upper bounded by the value $\varepsilon/2n$ with probability at least $1 - \delta/n$. Hence, the probability that they are *all* less than $\varepsilon/2n$ is at least $1 - \delta$. The theorem's claim follows from this by Lemma 8. □

Corollary A.1. *Let $\Theta_{pao} = \text{PAO}(G, \varepsilon, \delta)$ be the result of the* PAO *algorithm, where $G = \langle W, F, R, c\rangle$ is any decision structure in $\mathcal{DS}$, and $\varepsilon, \delta > 0$ are given constants. Then, with probability at least $1 - \delta$, $C_p(\Theta_{pao}) - C_p(\Theta_p) \leqslant \varepsilon$ where $\Theta_p = \text{OSS}(G, p)$ is the optimal strategy, based on the correct probability vector p.*

Proof. This result follows immediately from the proof of Theorem 10, using only the easy first part of Lemma 9, and the observation that $C \geqslant D(e_i)$ guarantees that the value M in Eq. (1) is larger than the value $m(e_i)$ in Eq. (A.4). □

Theorem 11. *Assume* RP $\neq$ NP. *Then there is no probabilistic polynomial time algorithm that, given a decision structure $G = \langle W, F, R, c\rangle$, an experiment $e \in W$, a distribution function $p : W \to [0, 1]$, and parameters $\varepsilon, \delta > 0$, can estimate the value $p(e)$ to within ε with probability at least $1 - \delta$.*

Proof. Assume to the contrary that such an algorithm exists for some fixed values of $\varepsilon, \delta > 0$; say $\varepsilon = 1/3 = \delta$. We show that this algorithm could also be used to decide the satisfiability of boolean formulas (SAT) with reliability $1 - \delta$.[3] As the SAT problem is NP-complete, it would follow by standard arguments that RP = NP.

Let φ be a boolean formula over the variables $x_1, \ldots, x_n$. We show how to construct a corresponding decision structure $G_\varphi = \langle W_\varphi, F_\varphi, R_\varphi, c_\varphi \rangle$, such that the formula φ has (respectively does not have) a satisfying assignment to its variables if and only if the value $\rho(g)$ for a specific experiment g in W_φ is 1 (respectively 0). We could thus decide the satisfiability of formula φ, with reliability $1 - \delta$, by running our hypothetical algorithm on structure G_φ and experiment g, and checking whether the estimate it provides for $\rho(g)$ is greater than $1 - \varepsilon$ or less than ε.

The structure G_φ has $2n + 1$ experiments $W_\varphi = \{e_1, \bar{e}_1, \ldots, e_n, \bar{e}_n, g\}$. The precedence relation F_φ permits exactly one of e_1 or $\bar{e}_1$ to be performed initially, then exactly one of e_2 or $\bar{e}_2$, and so on; in general permitting exactly one of e_k or $\bar{e}_k$ as the kth experiment. (That is, $F_\varphi(\alpha, e_{k+1})$ and $F_\varphi(\alpha, \bar{e}_{k+1})$ both hold iff α is of the form $\langle \langle \tilde{e}_1 + \rangle, \langle \tilde{e}_2 + \rangle, \ldots, \langle \tilde{e}_k + \rangle \rangle$, where each $\tilde{e}_i$ is either e_i or $\bar{e}_i$.) Now each complete sequence of the $\tilde{e}$-type experiments, $\alpha = \langle \langle \tilde{e}_1 + \rangle, \langle \tilde{e}_2 + \rangle, \ldots, \langle \tilde{e}_n + \rangle \rangle$, can be identified with a truth assignment to the variables $x_1, \ldots, x_n$, whereby x_i is true (respectively false) if and only if $\tilde{e}_i = e_i$ (respectively $\tilde{e}_i = \bar{e}_i$) in α. The precedence relation F finally specifies that experiment g can be performed (i.e. $F_\varphi(\alpha, g)$ holds) if and only if this truth assignment satisfies the formula φ.

Consider then the trivial probability distribution that assigns success probability 1 to all experiments, and recall that $\rho(g)$ is the maximum probability of reaching experiment g using any strategy. Given the precedence constraints specified above, it is clear that there exists a strategy Θ for reaching g if and only if there exists a satisfying truth assignment for φ, and any such strategy will have $\rho(g, \Theta) = 1$. Hence $\rho(g) = 1$ if and only if φ is satisfiable, and otherwise $\rho(g) = 0$. □

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