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A note on Trillas' CHC models ☆

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Abstract

Trillas et al. [E. Trillas, S. Cubillo, E. Castiñeira, On conjectures in orthocomplemented lattices, Artificial Intelligence 117 (2000) 255–275] recently proposed a mathematical model for *conjectures, hypotheses and consequences* (abbr. CHCs), and with this model we can execute certain mathematical reasoning and reformulate some important theorems in classical logic. We demonstrate that the orthomodular condition is not necessary for holding Watanabe's structure theorem of hypotheses, and indeed, in some orthocomplemented but not orthomodular lattices, this theorem is still valid. We use the CHC operators to describe the theorem of deduction, the theorem of contradiction and the Lindenbaum theorem of classical logic, and clarify their existence in the CHC models; a number of examples is presented. And we re-define the CHC operators in residuated lattices, and particularly reveal the essential differences between the CHC operators in orthocomplemented lattices and residuated lattices.

Keywords: Orthocomplemented lattices; Orthomodular lattices; Residuated lattices; Conjectures; Consequences; Quantum logic

1. Introduction

To some extent, the evolution of humankind and the progress of scientific research are the processes of making hypotheses, posing conjectures and then verifying or refuting them, after which one may put forward other guesses and propose new conjectures to be proved or disproved [14,24,25,31].

Roughly speaking, reasoning means that given some general knowledge together with some specific facts, certain consequences are deduced. Logical reasoning is a central issue in mathematics and AI, so developing an appropriate reasoning method and setting up some appropriate reasoning models for *conjectures, hypotheses, and consequences* (abbr. CHCs) are both intriguing and significant.

Recently, Trillas et al. [30] have established an interesting mathematical model for CHCs in orthocomplemented lattices. In their models (we call it the CHC models), the statements and propositions of human thinking are represented as those elements in an orthocomplemented lattice; they then defined several meaningful operators (we call

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them CHC operators), which act on each given set of premises, intuitively standing for the conjectures and hypotheses as well as the consequences of that set of premises. In addition, through studying such a model, they were able to present a clear classification of conjectures [30], and verify two structure's theorems of hypotheses, generalizing Watanabe's structure theorem to the CHC models [31]. As is known, residuated lattices (see, for example, [1,2,6,7, 10,13,15–17,19,22,23]) have close links with various important algebras (such as MV-algebras, Product algebras, and Gödel algebras) and branches of logic such as probabilistic logic [28] and linear logic [12], and are used as a basis of fuzzy logic [2,10,13,15–17,19,23]. Therefore, more recently, Ying and Wang [32] continued to investigate the CHC operators in the framework of residuated lattices and orthocomplemented lattices.

Orthocomplemented lattices are quite general algebraic structures, so, Trillas et al. [30] established a sufficiently extensive reasoning model in which we can mathematically describe CHCs. We now state the results of this paper in detail. In Section 2, we first introduce the CHC operators and some related properties, and, we then present three orthocomplemented but not orthomodular lattices such that the Watanabe's structure theorem of hypotheses (which was established in the framework of Boolean algebras) is demonstrated as still valid in one of the three lattices, but invalid in the others.

Implication operators are one of the principal connectives in logical calculi; therefore, in Section 3, implication operators are brought into the CHC models. We are concerned with the study of combining implication operators with these CHC operators in the framework of orthomodular lattices, and, using CHC operators to describe some generalized forms of several important theorems of classical logic, as well as further clarifying their existence. However, all conditional connectives one can reasonably introduce in quantum logic (orthomodular lattices) are, to a certain extent, anomalous [5], for they do not share most of the characteristic properties that the "material implication" in classical logic satisfies. A reasonably satisfactory implication is Sasaki hook, which contains some useful characteristics. It should be noticed that only in Boolean algebras it holds the equivalence between $a \wedge z \leqslant b$ and $z \leqslant a' \vee b$. In proper orthomodular lattices, $a \wedge z \leqslant b$ only has some maximal solutions, like Sasaki $(a' \vee (a \wedge b))$ or Dishkant $(b \vee (a' \wedge b'))$ hooks. The problem in CHC models seems to be less that of having an "implication" (in fact, a Boolean concept) and that of having a "conditional" $\rightarrow$, i.e., an operation verifying the inequality $a \wedge (a \rightarrow b) \leqslant b$, Modus Ponens. At this respect, one of the important subjects lies in the "ordering" property: $a \leqslant b$ iff $a \rightarrow b = 1$, verified by the Sasaki hook and Dishkant hook. Therefore, in this paper, Sasaki hook is mainly thought of as the implication operator in the CHC models.

In Section 3, we first offer some fundamental properties of the implication operator in the CHC models. It is worth indicating that Pineda et al. [27] showed that, in the sense of the CHC models, the *theorem of contradiction* holds in Boolean algebras. Here we demonstrate that in any given orthocomplemented lattice L, if the theorem of contradiction holds, then L must be an orthomodular lattice. Notwithstanding, we also show that, in some orthomodular lattices, the theorem may prove invalid. Therefore, in the CHC models, orthomodular law is a *necessary but not sufficient* condition for holding the theorem of contradiction.

Subsequently, using such a model, we present a characterization of the theorem of deduction of classical logic; briefly, the generalized form of the deduction theorem holds if and only if the lattice L of the underlying logic is a Boolean algebra. This also clarifies in some sense the non-existence of the deduction theorem in quantum logic. Furthermore, we give an instance verifying that in some orthomodular lattices, a form of the Lindenbaum theorem described by CHC operators does not hold, either, and we present a type of "weak Lindenbaum theorem" represented by these operators of CHCs.

In Section 4, we first recall some fundamentals of residuated lattices, and then define the CHC operators in residuated lattices. Particularly we concentrate on revealing some essential differences between CHCs in orthocomplemented lattices and those in residuated lattices. In effect, we offer some characterizations between the properties of CHC operators and those of residuated lattices themselves, which show some of the essential characteristics of CHC operators in residuated lattices. We discover that, in residuated lattices, the theorem of contradiction holds iff the residuated lattices satisfy the double negation ($x = x''$), weaker condition than Boolean algebras, since from [17] we know that a residuated lattice satisfying both the double negation and $x \wedge x' = 0$ reduces to a Boolean algebra. Still, we prove that Watanabe's structure theorem of hypotheses in the setting of MV-algebras (therefore, residuated lattices) does *not* hold, either. Finally, in Section 5, the main results of the paper are summarized, and some related issues are addressed.

In the interest of readability, we will make this paper sufficiently self-contained. To this end, we introduce related notions and notation in Section 2. On the other hand, to make the supremum and the infimum of P valid ($P \subseteq L$), the lattices L considered in this paper are complete.

2. Watanabe's structure theorem of hypotheses in the CHC models

First let us recall the definitions of orthocomplemented lattices and orthomodular lattices. An *orthocomplemented lattice* is an algebraic structure $\langle L, \leqslant, ', 0, 1\rangle$ where

(1) $\langle L, \leqslant, 0, 1\rangle$ is a bounded lattice with the least element 0 and the greatest element 1. In other words, (i) $\leqslant$ is a partial order relation on L (i.e. reflexive, antisymmetric, and transitive); and (ii) any pair of elements a, b has an infimum $a \wedge b$ and a supremum $a \vee b$ (i.e. for any c, $a \wedge b \leqslant a, b$, and if $c \leqslant a, b$ then $c \leqslant a \wedge b$; $a, b \leqslant a \vee b$, and if $a, b \leqslant c$ then $a \vee b \leqslant c$).
(2) The unary operation $' : L \to L$ called orthocomplement satisfies the following conditions:
   (i) $a \wedge a' = 0$ for any $a \in L$.
   (ii) $a = a''$ for any $a \in L$.
   (iii) $a \leqslant b$ implies $b' \leqslant a'$ for any $a, b \in L$.

From conditions (ii) and (iii) it follows readily that for any $a, b \in L$,

$$(a \wedge b)' = a' \vee b', \qquad (a \vee b)' = a' \wedge b'.$$

For the sake of completeness, we verify the first one, and the second is similar. Due to $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$, with (iii) we have $a' \leqslant (a \wedge b)'$ and $b' \leqslant (a \wedge b)'$. Thus, $a' \vee b' \leqslant (a \wedge b)'$. Contrarily, since $a' \leqslant a' \vee b'$ and $b' \leqslant a' \vee b'$, we have $(a' \vee b')' \leqslant a$ and $(a' \vee b')' \leqslant b$. Therefore, $(a' \vee b')' \leqslant a \wedge b$, which, together with (iii) and (ii), results in $(a \wedge b)' \leqslant a' \vee b'$.

An *orthomodular lattice* is an orthocomplemented lattice satisfying the following condition (3):

(3) For any $a, b \in L$, if $a \leqslant b$, then $b = a \vee (a' \wedge b)$.
By using the above condition (2), condition (3) can be equivalently stated as:
(3′) For any $a, b \in L$, if $a \leqslant b$, then $a = b \wedge (b' \vee a)$.

It is clear that orthomodularity represents a weak form of distributivity. Of course, distributivity implies orthomodularity but, actually, what orthomodular law states is the existence, when $a \leqslant b$, of the relative complement $b - a = a' \wedge b$, giving a' if $b = 1$. This existence allows, for example, the verification of the important property "$a \leqslant b \Rightarrow \text{prob}(a) \leqslant \text{prob}(b)$" for probabilities in orthomodular lattices and, of course, in Boolean algebras.

Let L be an orthocomplemented lattice, and for any $P \subseteq L$, $\bigvee P$ and $\bigwedge P$ stand for the supremum and the infimum of P, respectively. Denote

$$\mathbf{P}_0(L) = \left\{P \subseteq L \colon \bigwedge P \neq 0\right\} \subseteq \mathcal{P}(L); \qquad L_0 = L - \{0\} \subseteq L,$$

where $\mathcal{P}(L)$ denotes the power set of L, and for any $P \in \mathbf{P}_0(L)$, P represents a set of some noncontradictory premises, that is, for any $p_i, p_j \in P$, $p_i \nleqslant p_j'$. Trillas, Cubillo and Castiñeira [30] defined several interesting operators $\Phi_\vee$, $\Phi_\wedge$, $C_\vee$, $C_\wedge$, and $H_\wedge$ together with their intuitive implications as follows:

$$\Phi_\vee(P) = \left\{q \in L \colon \bigvee P \nleqslant q'\right\} \quad \text{(conjectures of } P\text{)},$$

$$\Phi_\wedge(P) = \left\{q \in L \colon \bigwedge P \nleqslant q'\right\} \quad \text{(strict conjectures of } P\text{)},$$

$$C_\vee(P) = \left\{q \in L \colon \bigvee P \leqslant q\right\} \quad \text{(loose consequences of } P\text{)},$$

$$C_\wedge(P) = \left\{q \in L \colon \bigwedge P \leqslant q\right\} \quad \text{(consequences of } P\text{)},$$

$$H_\wedge(P) = \left\{q \in L_0 \colon q \leqslant \bigwedge P\right\} \quad \text{(hypotheses of } P\text{)}.$$

Moreover, the following derived operators were given.

$$C(P)=\left\{q\in L\colon \bigwedge P\leqslant q\leqslant \bigvee P\right\}\quad \text{(restricted consequences of } P\text{)},$$

$$\Phi(P)=\left\{q\in L\colon \bigwedge P\nleqslant q', q'\nleqslant \bigvee P\right\}\quad \text{(strict and restricted conjectures of } P\text{)},$$

$$\Phi^*_\vee(P)=\Phi_\vee(P)-\Phi_\wedge(P)\quad \text{(loose conjectures of } P\text{)},$$

$$\Phi^*_\wedge(P)=\Phi_\wedge(P)-\big(C_\wedge(P)\cup H_\wedge(P)\big)\quad \text{(proper conjectures of } P\text{)},$$

$$H^*_\wedge(P)=H_\wedge(P)-\left\{\bigwedge P\right\},$$

$$H_\vee(P)=\left\{q\in L_0\colon q\leqslant \bigvee P\right\}\quad \text{(we call it loose hypotheses of } P\text{)}.$$

Denote $L_{01}=L-\{0,1\}$; $pNCq$ represents that p is *incomparable* with q, i.e., $p\nleqslant q$ and $q\nleqslant p$. For any $P\in \mathbf{P}_0(L)$, set $A_\wedge(P)=\{a\in L_0\colon (\bigwedge P)NCa, a\wedge(\bigwedge P)\neq 0\}$ and $B_\wedge(P)=\{a\in L_{01}\colon \bigwedge P\nleqslant a, a\wedge(\bigwedge P)\neq 0\}$. Obviously, $A_\wedge(P)\subseteq B_\wedge(P)$. Trillas, Cubillo, and Castiñeira [30] showed that (i) if L is an orthocomplemented lattice, then for $P\in\mathbf{P}_0(L)$,

$$H^*_\wedge(P)=\left(\bigwedge P\right)\wedge B_\wedge(P)\doteq\left\{q\wedge\left(\bigwedge P\right)\colon q\in B_\wedge(P)\right\};\tag{1}$$

(ii) if L is an orthomodular lattice, then for $P\in\mathbf{P}_0(L)$ and $\bigwedge P\neq 1$,

$$H^*_\wedge(P)=\left(\bigwedge P\right)\wedge A_\wedge(P)\doteq\left\{q\wedge\left(\bigwedge P\right)\colon q\in A_\wedge(P)\right\},\tag{2}$$

which generalize the Watanabe's structure theorem of hypotheses in the case of Boolean algebras [31].

It is worth pointing out that in Eq. (2), the condition $\bigwedge P\neq 1$ cannot be ignored; otherwise, Eq. (2) may not hold, since $\bigwedge P=1$ implies both $A_\wedge(P)=\emptyset$ and $H^*_\wedge(P)=L_{01}$.

Clearly, $(\bigwedge P)\wedge A_\wedge(P)\subseteq(\bigwedge P)\wedge B_\wedge(P)$ always holds. On the other hand, the orthomodular condition entails $(\bigwedge P)\wedge B_\wedge(P)\subseteq(\bigwedge P)\wedge A_\wedge(P)$. Nevertheless, when L is only required to be an orthocomplemented lattice, we will construct one example of orthocomplemented but not orthomodular lattice depicted in Fig. 2 (Example 3), in which $H^*_\wedge(P)\neq(\bigwedge P)\wedge A_\wedge(P)$ for some $P\in\mathbf{P}_0(L)$ with $\bigwedge P\neq 1$. A natural question to raise is whether the orthomodular law is necessary to guarantee $H^*_\wedge(P)=(\bigwedge P)\wedge A_\wedge(P)$? Here we also present an example (Example 1) to verify that $H^*_\wedge(P)=(\bigwedge P)\wedge A_\wedge(P)$ may be still valid in some orthocomplemented but not orthomodular lattices. Therefore, such a question can be answered negatively.

Example 1. There exists an orthocomplemented but not orthomodular lattice L such that in this lattice, for any $P\in\mathbf{P}_0(L)$ with $\bigwedge P\neq 1$, then

$$H^*_\wedge(P)=\left(\bigwedge P\right)\wedge A_\wedge(P)\doteq\left\{q\wedge\left(\bigwedge P\right)\colon q\in A_\wedge(P)\right\}.$$

Proof. Let orthocomplemented lattice $L=\{a,b,c,d,e,a',b',c',d',e',0,1\}$ be visualized in Fig. 1.

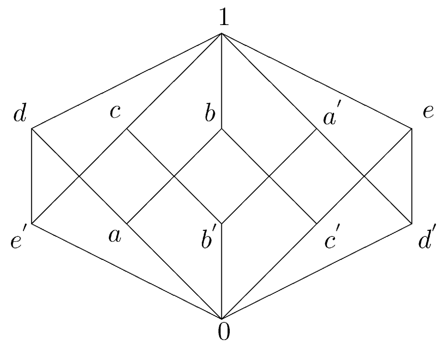

Fig. 1.

It is direct to check that the above defined L is an orthocomplemented lattice, but it is not orthomodular, because in this lattice, $a \leqslant b$ but $b \neq a \vee (a' \wedge b) = a$ which violates the orthomodular condition. Furthermore, we can verify that for any $P \in \mathbf{P}_0(L)$, it holds that

$$H_\wedge^*(P) = \left(\bigwedge P\right) \wedge A_\wedge(P).$$

More specially, it suffices to consider the cases of $P \in \{\{d\}, \{c\}, \{b\}, \{a'\}, \{e\}\}$, because when $\bigwedge P \in \{e', a, b', c', d'\}$,

$$A_\wedge(P) = H_\wedge^*(P) = \emptyset.$$

Now, we have

$$\begin{aligned} &A_\wedge(\{d\}) = \{c, b\}; && A_\wedge(\{c\}) = \{d, a'\};\\ &A_\wedge(\{b\}) = \{d, e\}; && A_\wedge(\{a'\}) = \{c, e\};\\ &A_\wedge(\{e\}) = \{b, a'\}; && H_\wedge^*(\{d\}) = \{a, e'\};\\ &H_\wedge^*(\{c\}) = \{e', b'\}; && H_\wedge^*(\{b\}) = \{a, c'\};\\ &H_\wedge^*(\{a'\}) = \{b', d'\}; && H_\wedge^*(\{e\}) = \{c', d'\};\end{aligned}$$

and it is easy to check that

$$\begin{aligned} &d \wedge A_\wedge(\{d\}) = H_\wedge^*(\{d\});\\ &c \wedge A_\wedge(\{c\}) = H_\wedge^*(\{c\});\\ &b \wedge A_\wedge(\{b\}) = H_\wedge^*(\{c\});\\ &a' \wedge A_\wedge(\{a'\}) = H_\wedge^*(\{a'\}); \quad \text{and}\\ &e \wedge A_\wedge(\{e\}) = H_\wedge^*(\{e\}). \qquad \square\end{aligned}$$

Moreover, we can characterize $H_\vee^*(P) = \{q \in L_0\colon q \leqslant \bigvee P\} - \{\bigvee P\}$, which may be called the proper loose hypotheses of P. The main results are as follows:

Theorem 2.1. (1) *Let L be an orthocomplemented lattice. Then for $P \in \mathbf{P}_0(L)$,*

$$H_\vee^*(P) = \left\{q \wedge \left(\bigvee P\right)\colon q \in B_\vee(P)\right\} \doteq \left(\bigvee P\right) \wedge B_\vee(P),$$

where $B_\vee(P) = \{q \in L_{01}\colon (\bigvee P) \nleqslant q,\ q \wedge (\bigvee P) \neq 0\}$.

(2) *If L is an orthomodular lattice, then for $P \in \mathbf{P}_0(L)$, $\bigvee P \neq 1$,*

$$H_\vee^*(P) = \left(\bigvee P\right) \wedge A_\vee(P),$$

where $A_\vee(P) = \{q \in L_{01}\colon (\bigvee P) NC q,\ q \wedge (\bigvee P) \neq 0\}$.

Proof. (1) If $q \in H_\vee^*(P)$, then $q = (q \wedge (\bigvee P)) \wedge (\bigvee P)$ and $q \wedge (\bigvee P) \in B_\vee(P)$, which shows that $H_\vee^*(P) \subseteq (\bigvee P) \wedge B_\vee(P)$. On the other hand, $(\bigvee P) \wedge B_\vee(P) \subseteq H_\vee^*(P)$ is clear.

(2) First it is clear that $(\bigvee P) \wedge A_\vee(P) \subseteq H_\vee^*(P)$. On the other hand, if $q \in H_\vee^*(P)$, then $q \leqslant \bigvee P$ and $q \notin \{0, \bigvee P\}$, and thus $q = q \wedge (\bigvee P)$. We recall that the orthomodular condition says: for any $a, b \in L$, if $a \leqslant b$, then $b = a \vee (a' \wedge b)$. Therefore, for any $a, b \in L$, if $a \leqslant b$, then $b' \leqslant a'$ and, thus, with orthomodular condition we have $a' = b' \vee (b \wedge a')$, i.e., $a = b \wedge (b' \vee a)$. Here, since $q \wedge (\bigvee P) \leqslant \bigvee P$, in terms of the orthomodular condition just given we get

$$q \wedge \left(\bigvee P\right) = \left(\bigvee P\right) \wedge \left(\left(\bigvee P\right)' \vee \left(q \wedge \left(\bigvee P\right)\right)\right),$$

from which, together with $q = q \wedge (\bigvee P)$ we have

$$q = q \wedge \left(\bigvee P\right) = \left(\bigvee P\right) \wedge \left(\left(\bigvee P\right)' \vee \left(q \wedge \left(\bigvee P\right)\right)\right).$$

It is enough to verify that $(\bigvee P)' \vee (q \wedge (\bigvee P)) \in A_\vee(P)$. If not so, we have either $(\bigvee P)' \vee (q \wedge (\bigvee P)) \leqslant (\bigvee P)$ or $(\bigvee P) < (\bigvee P)' \vee (q \wedge (\bigvee P))$, where $a < b$ means that $a \leqslant b$ but $a \neq b$. For the former, i.e., $(\bigvee P)' \vee (q \wedge (\bigvee P)) \leqslant (\bigvee P)$, it follows that $(\bigvee P)' \leqslant (\bigvee P)' \vee (q \wedge (\bigvee P)) \leqslant \bigvee P$, and consequently, $\bigvee P = 1$, contradicting the assumption; but the later results in $q \wedge (\bigvee P) = (\bigvee P) \wedge ((\bigvee P)' \vee (q \wedge (\bigvee P))) = (\bigvee P)$, and thus $\bigvee P \leqslant q$ which is absurd since $q \in H_\vee^*(P)$. Therefore, it holds that $(\bigvee P)' \vee (q \wedge (\bigvee P)) \in A_\vee(P)$. This completes the proof of the theorem. □

Noticeably, the above method of proof is different from [30] by Trillas et al. and applies to their proof. As above, we naturally ask whether L being orthomodular lattice is necessary for preserving part (2) in Theorem 1. Indeed, the answer is negative, and we can verify that the orthocomplemented, but not orthomodular lattice presented in Example 1 validates this view. We further formulate it by Example 2 as follows.

Example 2. There exists an orthocomplemented but not orthomodular lattice L such that in this lattice, for any $P \in \mathbf{P}_0(L)$ with $\bigvee P \neq 1$, then

$$H_\vee^*(P) = \Big(\bigvee P\Big) \wedge A_\vee(P).$$

Proof. Let L be the orthocomplemented lattice from Example 1. That is, $L = \{a, b, c, d, e, a', b', c', d', e', 0, 1\}$ is depicted by Fig. 1. The remainder of the verification is also similar to Example 1, but in the interest of completeness, we briefly check it here. For any given $P \in \mathbf{P}_0(L)$ with $\bigvee P \neq 1$, we consider it by the following two scenarios:

- If $\bigvee P \in \{e', a, b', c', d'\}$, then $A_\vee(P) = H_\vee^*(P) = \emptyset$, and hence $H_\vee^*(P) = (\bigvee P) \wedge A_\vee(P)$.
- If $\bigvee P \in \{d, c, b, a', e\}$, then we further divide it into five cases: (i) $\bigvee P = d$; (ii) $\bigvee P = c$; (iii) $\bigvee P = b$; (iv) $\bigvee P = a'$; (v) $\bigvee P = e$. We only check the first case, since the others are completely analogous. If $\bigvee P = d$, then $P \in \{\{d\}, \{d, e'\}, \{d, a\}\}$. In any case of P, we have $A_\vee(P) = \{c, b\}$, $H_\vee^*(P) = \{e', a\}$, and therefore, $(\bigvee P) \wedge A_\vee(P) = \{e', a\} = H_\vee^*(P)$.

So far we have completed the proof. □

The further question raised naturally is whether there are some orthocomplemented but not orthomodular lattices such that neither $H_\wedge^*(P) = (\bigwedge P) \wedge A_\wedge(P)$ nor $H_\vee^*(P) = (\bigvee P) \wedge A_\vee(P)$ holds for some appropriate P. The answer is positive from the following example.

Example 3. There exist orthocomplemented but not orthomodular lattices L such that there are $P \in \mathbf{P}_0(L)$ with $\bigwedge P \neq 1$ and $\bigvee P \neq 1$, satisfying both $H_\wedge^*(P) \neq (\bigwedge P) \wedge A_\wedge(P)$ and $H_\vee^*(P) \neq (\bigvee P) \wedge A_\vee(P)$.

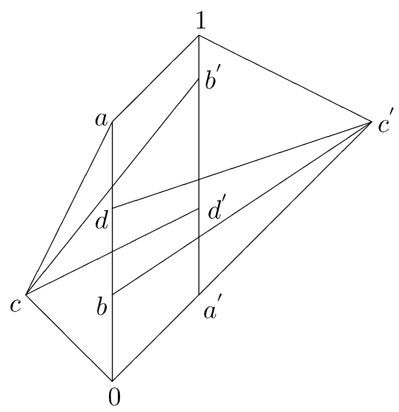

Fig. 2.

Proof. We present one example here. Let lattice $L = \{a, b, c, d, a', b', c', d', 1, 0\}$ be visualized in Fig. 2. Then it is straightforward to check that L constructed is an orthocomplemented but not orthomodular lattice. Firstly, it satisfies the conditions of orthocomplemented lattices defined above. Secondly, it does not satisfy orthomodular law, because, $b \leqslant d$, but $d \neq b \vee (b' \wedge d) = b$. Thirdly, if we take $P = \{a\}$, then $H^*_\wedge(P) = H^*_\vee(P) = \{d, b, c\}$, and $A_\wedge(P) = A_\vee(P) = \{b', c', d'\}$. Therefore, we have both $H^*_\wedge(P) \neq \{c, d\} = (\bigwedge P) \wedge A_\wedge(P)$ and $H^*_\vee(P) \neq \{c, d\} = (\bigvee P) \wedge A_\vee(P)$. □

3. A number of logic theorems in the CHC models

Quantum logic is usually defined as orthomodular lattices [26], in which one of the most important issues is the possibility of defining reasonable implication connectives. Indeed, it has been verified [18] that there are exactly five definitions of implications $\rightarrow$ that satisfy the primitive implication condition:

$$a \leqslant b \quad \text{if and only if} \quad a \rightarrow b = 1.$$

We might point out here that in classical logic (Boolean), these five implication operators are equivalent to "material implication" denoted by $\rightarrow_0$, defined as $a \rightarrow_0 b = a' \vee b$.

Among these comparatively reasonable five implications, Sasaki hook $\rightarrow_Q$ defined as $a \rightarrow_Q b = a' \vee (a \wedge b)$ is the unique one satisfying condition [11,29]: There exists a binary operation defined as $a \,\&\, b = b \wedge (a \vee b')$ such that for any $a, b, c \in L$, $a \,\&\, b \leqslant c$ if and only if $a \leqslant b \rightarrow_Q c$, which is similar to the form of deduction theorem of the first-order logic. The following proposition not only shows that Sasaki hook has an intrinsic connection with orthomodular condition, but is also useful for our discussion.

Proposition 3.1. (*Mittelstaedt [21]*) *Let $L = \langle L, \leqslant, ', 0, 1\rangle$ be an orthocomplemented lattice.*

(1) *L is orthomodular iff there exists binary operation $S(a, b)$ such that the following conditions are satisfied.*
   (i) *$a \wedge S(a, b) \leqslant b$;*
   (ii) *$a \wedge c \leqslant b$ implies $a' \vee (a \wedge c) \leqslant S(a, b)$.*
   The operation $S(a, b)$ satisfying the above conditions is unique, namely, $S(a, b) = a' \vee (a \wedge b)$ is exactly Sasaki hook.
(2) *L is a Boolean algebra iff the above conditions are satisfied by material implication, i.e., $S(a, b) = a \rightarrow_0 b = a' \vee b$.*

Therefore, we choose Sasaki hook $\rightarrow_Q$ as our implication connective, simply denoted by $\rightarrow$. First we consider the connections of the implication operator to conjecture operators.

Proposition 3.2. *Let L be an orthocomplemented lattice. If $P \in \mathbf{P}_0(L)$, then*:

(1) *If $a \in C_\wedge(P)$ and $a \rightarrow b \in \Phi_\wedge(P)$, then $\{a, b, a \wedge b\} \subseteq \Phi_\wedge(P)$.*
(2) *If $a \wedge b \in \Phi_\wedge(P)$, then $a \rightarrow b \in \Phi_\wedge(P)$.*

Proof. (1) If $a \in C_\wedge(P)$ and $a \rightarrow b \in \Phi_\wedge(P)$, then $\bigwedge P \leqslant a$ and $\bigwedge P \nleqslant (a \rightarrow b)' = a \wedge (a \wedge b)'$. If $\bigwedge P \leqslant (a \wedge b)'$, then $\bigwedge P \leqslant a \wedge (a \wedge b)'$, which is contradictory. So, $\bigwedge P \nleqslant (a \wedge b)'$, i.e., $a \wedge b \in \Phi_\wedge(P)$. Since $a \wedge b \leqslant a$, and $a \wedge b \leqslant b$, it follows that $a, b \in \Phi_\wedge(P)$.

(2) If $a \wedge b \in \Phi_\wedge(P)$ and $a \rightarrow b \notin \Phi_\wedge(P)$, then $\bigwedge P \nleqslant (a \wedge b)'$ and $\bigwedge P \leqslant a \wedge (a \wedge b)' \leqslant (a \wedge b)'$, which is absurd. Therefore, (2) holds, and the proposition is proved. □

Proposition 3.3. *Let L be an orthocomplemented lattice. If $P \in \mathbf{P}_0(L)$, then*:

(1) *If $a \in C_\vee(P)$ and $a \rightarrow b \in \Phi_\vee(P)$, then $\{a, b, a \wedge b\} \subseteq \Phi_\vee(P)$.*
(2) *If $a \wedge b \in \Phi_\vee(P)$, then $a \rightarrow b \in \Phi_\vee(P)$.*

Proof. The proof is similar to that of Proposition 3.2. □

Implication operator is also closely related to consequence operators.

Proposition 3.4. *Let $P \in \mathbf{P}_0(L)$. Then*:

(1) *If L is an orthocomplemented lattice, then $b \in C_\vee(P \cup \{a\})$ if and only if $b \in C_\vee(P)$ and $a \leqslant b$.*
(2) *If L is an orthomodular lattice, then*:
  (i) *If $a \to b \in C_\wedge(P)$, then $a \wedge b$ and $b \in C_\wedge(P \cup \{a\})$.*
  (ii) *$(\bigwedge P) \mathbin{\&} a \leqslant b$ iff $a \to b \in C_\wedge(P)$, where $a \mathbin{\&} b = b \wedge (a \vee b')$, as indicated above.*

Proof. The proof of (1) is immediate. (2) (i) If $a \to b \in C_\wedge(P)$, then $\bigwedge P \leqslant a' \vee (a \wedge b)$. With orthomodular law, $(\bigwedge P) \wedge a \leqslant a \wedge (a' \vee (a \wedge b)) = a \wedge b$ and thus $a \wedge b \in C_\wedge(P \cup \{a\})$. Also, $b \in C_\wedge(P \cup \{a\})$, since $a \wedge b \leqslant b$. (ii) It follows from $(\bigwedge P) \mathbin{\&} a \leqslant b$ iff $\bigwedge P \leqslant a \to b$ [11,29]. Therefore (ii) holds, and the proof is completed. □

With the consequence operator $C_\wedge$, the *theorem of contradiction* of classical logic may be reformulated in the CHC model as follows: For an orthocomplemented lattice L, and $P \in \mathbf{P}_0(L)$,

$$\text{(C)} \quad \forall a \in L\colon\ a \in C_\wedge(P) \Longleftrightarrow a' \wedge \Big(\bigwedge P\Big) = 0.$$

Now we may naturally ask whether it holds in orthologic and quantum logic, i.e., when L is an orthocomplemented or orthomodular lattice. Firstly, we verify that, in any given orthocomplemented lattice L, the theorem of contradiction holding implies that L must be *orthomodular*.

Theorem 3.5. *Let L be an orthocomplemented lattice. If the theorem of contradiction expressed by* (C) *in L holds, then L must be an orthomodular lattice.*

Proof. We prove by contradiction. Assume that L is *not* orthomodular. Then with the orthomodular condition there exist $a, b \in L$ such that $a \leqslant b$ but $b \neq a \vee (a' \wedge b)$, that is, $a \vee (a' \wedge b) < b$ since $a \vee (a' \wedge b) \leqslant b$ holds by $a \leqslant b$, where, here and in the paper, "$e_1 < e_2$" means that $e_1 \leqslant e_2$ but $e_1 \neq e_2$. In addition, we note $b \in L_{01}$, i.e., $b \neq 0, 1$; otherwise, $a \vee (a' \wedge b) = b$. Set $x = a \vee (a' \wedge b)$. Then $x < b$ and

$$b \wedge x' = b \wedge \big(a \vee (a' \wedge b)\big)' = b \wedge \big(a' \wedge (a' \wedge b)'\big) = (b \wedge a') \wedge (a' \wedge b)' = 0,$$

from which, together with the theorem of contradiction described by condition (C) (by taking $P = \{b\}$), we obtain $b \leqslant x$, that is also contradiction with $x < b$ given above. Therefore L must be an orthomodular lattice. □

A direct corollary from Theorem 3.5 is as follows.

Corollary 3.6. *Let L be an orthocomplemented but not orthomodular lattice. Then the theorem of contradiction expressed by* (C) *in L does not hold.*

Naturally we may ask whether L being orthomodular infers that the theorem of contradiction holds. Indeed, this implication may be not true by the following example, i.e., we present an orthomodular lattice (Fig. 3), such that the above theorem of contradiction may not hold.

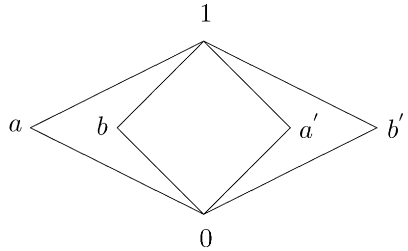

Fig. 3.

Example 4. There is an orthomodular lattice such that the theorem of contradiction expressed by (C) does not hold.

Proof. Let orthomodular lattice $L = \{a, b, a', b', 1, 0\}$ be of a "Chinese lantern" form drawn in Fig. 3. Then it is clear that $a' \wedge b = 0$ but $a \nleqslant b$. That is, both $a \notin C_\wedge(\{b\})$ and $a' \wedge b = 0$ hold, which invalidates (C) in this lattice. □

Remark. From Theorem 3.5 and Example 4 we have seen that orthomodularity is only a *necessary but not sufficient* condition for holding the theorem of contradiction. However, in Boolean algebras, Pineda et al. [27] proved the following result.

Theorem 3.7. *[27] If L is a complete Boolean algebra, then the theorem of contradiction expressed by* (C) *holds.*

Next we deal with the theorem of deduction in the CHC models, which may be represented as follows: For orthocomplemented lattice L, and for any $P \in \mathbf{P}_0(L)$ and any $a, b \in L$,

$$b \in C_\wedge\big(P \cup \{a\}\big) \quad \text{iff} \quad a \to b \in C_\wedge(P). \tag{3}$$

Indeed, from the following theorem it follows that the theorem of deduction in the CHC model may not hold.

Theorem 3.8. *Let L be an orthocomplemented lattice. Then the following statements are equivalent.*

(1) *L is a Boolean algebra.*
(2) *For any $P \in \mathbf{P}_0(L)$ and any $a, b \in L$: $b \in C_\wedge(P \cup \{a\})$ iff $a \to b \in C_\wedge(P)$.*

Proof. (1) inferring (2) is direct. With regard to (2) deducing (1), by means of Proposition 3.1(2) it suffices to show that for any $a, b \in L$, $a \wedge (a' \vee b) \leqslant b$. If either a or b is 0, it clearly holds. In general, we utilize the condition (2) repeatedly. Since $a \wedge b \leqslant b$, by the condition (2) we have $b \leqslant a \to b$. Moreover, due to $a' \leqslant a' \vee (a \wedge b) = a \to b$, we obtain $a' \vee b \leqslant a \to b$. Again, by means of the condition (2), it follows that $a \wedge (a' \vee b) \leqslant b$. □

Another important negative result in quantum logic is that the "Lindenbaum theorem" does not hold, which was verified by Dalla Chiara [4]. According to Lindenbaum theorem of classical logic, any noncontradictory set X of sentences can be extended to a complete set (a complete set is a noncontradictory set satisfying that for any sentences α, either α or the negation of α is one of its consequences). In the CHC models, Lindenbaum theorem may be expressed as:

Let L be an orthocomplemented lattice. For any $P \in \mathbf{P}_0(L)$, there is $Q \in \mathbf{P}_0(L)$ such that $P \subseteq Q$ and for any $q \in L$, either $q \in C_\wedge(Q)$ or *$q' \in C_\wedge(Q)$.*

Here, by means of an orthomodular lattice (Greechie lattice $\mathcal{G}_{12}$) represented in Fig. 4, we can demonstrate that this theorem in this orthomodular lattice does not hold.

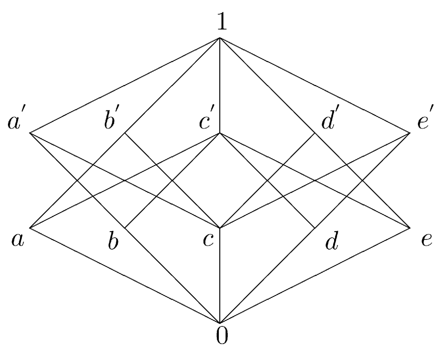

Fig. 4.

In such a lattice, set $P = \{a\}$ for example. Then for any $Q \subseteq L$, with $P \subseteq Q$ and $\bigwedge Q \neq 0$, it entails that

$$Q \in \big\{\{a\}, \{a, 1\}, \{a, b'\}, \{a, c'\}, \{a, b', c'\}, \{a, b', 1\}, \{a, c', 1\}, \{a, b', c', 1\}\big\},$$

and thus $\bigwedge Q = a$, so it follows that neither d nor d' belongs to $C_\wedge(Q)$; and neither e nor e' belongs to $C_\wedge(Q)$, either. As a result, it invalidates the Lindenbaum theorem in this lattice. That is to say, Lindenbaum theorem may not hold in quantum logic.

Indeed, in orthomodular lattice L ("Chinese lantern") depicted in Fig. 3, the above theorem does not hold, either. Take $P = \{a\}$ for example. Then for any $Q \subseteq L$, with $P \subseteq Q$ and $\bigwedge Q \neq 0$, it entails that $Q \in \{\{a\}, \{a, 1\}\}$. However, neither b nor b' belongs to $C_\wedge(Q)$.

Notwithstanding this, we may offer a kind of "weak Lindenbaum theorem". First, we define the compatibility between two sets.

Definition 3.1. Let $P_1, P_2 \in \mathbf{P}_0(L)$. P_1 is called compatible with P_2, denoted by $P_1 C P_2$, if and only if for each $q \in L$ such that if $\bigwedge P_1 \leqslant q$, then $\bigwedge P_2 \nleqslant q'$.

Remark. Clearly, (1) if $P_1 \cup P_2 \in \mathbf{P}_0(L)$, then $P_1 C P_2$; and (2) $P_1 C P_2$ implies $P_2 C P_1$ and vice versa.

Proposition 3.9. *Let $P_1, P_2 \in \mathbf{P}_0(L)$.*

(i) *If both $\bigwedge P_1$ and $\bigwedge P_2$ are atomic, and $P_1 C P_2$, then $\Phi_\wedge(P_1) = \Phi_\wedge(P_2)$, where an element p is called atomic if for any $q \in L$, either $p \leqslant q$ or $p \leqslant q'$ holds.*
(ii) *If either $\Phi_\wedge(P_1) \subseteq \Phi_\wedge(P_2)$ or $\Phi_\wedge(P_2) \subseteq \Phi_\wedge(P_1)$, then $P_1 C P_2$.*

Proof. (i) Suppose $q \in \Phi_\wedge(P_1)$, then $\bigwedge P_1 \nleqslant q'$. Since $\bigwedge P_1$ is atomic, it holds that $\bigwedge P_1 \leqslant q$. With $P_1 C P_2$, we have $\bigwedge P_2 \nleqslant q'$, i.e., $q \in \Phi_\wedge(P_2)$. Therefore $\Phi_\wedge(P_1) \subseteq \Phi_\wedge(P_2)$. On the other hand, $\Phi_\wedge(P_2) \subseteq \Phi_\wedge(P_1)$ can be similarly verified.

(ii) Let $\Phi_\wedge(P_1) \subseteq \Phi_\wedge(P_2)$. If $\bigwedge P_2 \leqslant q$, then $\bigwedge P_1 \leqslant q$, and thus $\bigwedge P_1 \nleqslant q'$ since $P_1 \in \mathbf{P}_0(L)$. This verifies $P_2 C P_1$. If $\Phi_\wedge(P_2) \subseteq \Phi_\wedge(P_1)$, then we can analogously show that $P_2 C P_1$ and thus $P_1 C P_2$. This proof is completed. □

Now we present the so-called "weak Lindenbaum theorem".

Theorem 3.10. *Let $P \in \mathbf{P}_0(L)$. If $q \in \Phi_\wedge(P)$, then there is $Q \in \mathbf{P}_0(L)$, such that $Q C P$ and $q \in C_\wedge(Q)$.*

Proof. Take $Q = \{q\}$. First, clearly $Q \in \mathbf{P}_0(L)$ and $q \in C_\wedge(Q)$, since $\bigwedge Q = q \in \Phi_\wedge(P)$. If $q \leqslant r$, then $\bigwedge P \nleqslant r'$ holds; otherwise, $\bigwedge P \leqslant r' \leqslant q'$, which contradicts $q \in \Phi_\wedge(P)$. Therefore, $Q C P$ holds and the proof is completed. □

4. The CHCs in residuated lattices

Residuated lattices were introduced in 1924 [20] and further investigated in the late 1930s by algebraists [6,7], but the study has been revived recently as a study of algebraic structures for fuzzy logics and other non-classical logics (for example, see [10,13,15,17,19,22,23] and the references therein). A residuated lattice is an algebra $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$, satisfying:

(i) $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded lattice with the least element 0 and the greatest element 1;
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, that is, $\otimes$ is a commutative and associative operation with the identity $a \otimes 1 = a$;
(iii) $\langle \otimes, \rightarrow\rangle$ satisfies: for all $a, b, c \in L$, $a \otimes b \leqslant c$ if and only if $a \leqslant b \rightarrow c$.

From axioms (i), (ii), (iii) it readily follows that $\otimes$ is isotone in both arguments, and $\rightarrow$ is antitone in the first and isotone in the second variable.

We briefly recall other algebras closely related to residuated lattices. For the details, we refer to, for example, [1,2, 10,13,15,17–19,22] and the references therein. A special residuated lattice is the Boolean algebra for classical logic; other typical examples of residuated lattices include the real unit interval [0,1] equipped with the most important three continuous t-norms on [0,1]: Lukasiewicz t-norm, Product t-norm, Gödel t-norm. The three t-norms and their associated residua correspond to the most significant fuzzy logics: Lukasiewicz logic, Product logic, and Gödel logic, respectively. The MV-algebras, the Product algebras, and the Gödel algebras constitute the algebraic models for these three types of logics, respectively. The class of BL-algebras contains the MV-algebras, the Product algebras, and the Gödel algebras (Gödel algebra = linear Heyting algebras = L-algebras).

In the interest of readability, we further review the relationships between residuated lattices and other important algebras. For residuated lattice $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$, $\mathcal{L}$ satisfies the *prelinearity axiom* [13,15], i.e. L is an MTL algebra [9], if and only if for all $a, b \in L$, $(a \to b) \vee (b \to a) = 1$ holds; $\mathcal{L}$ is called *divisible* [15] iff for all $a, b \in L$, $a \wedge b = a \otimes (a \to b)$. $\mathcal{L}$ satisfies the *Double Negation* [15] iff for any $a \in L$, $a = (a \to 0) \to 0$ holds, where, here and in the sequel, we denote $a' = a \to 0$, and thus, $a'' = (a \to 0) \to 0$. If for any $a, b \in L$, $a \otimes b = a \wedge b$, then $\mathcal{L}$ reduces to a Heyting algebra [1]; if $\mathcal{L}$ is divisible and satisfies the prelinearity axiom, then $\mathcal{L}$ is a BL-algebra [13,16]; if $\mathcal{L}$ is a BL-algebra and satisfies the double negation, then $\mathcal{L}$ is an MV-algebra [2,13,15,16]. A product algebra [13,16] is a BL-algebra, satisfying that for all elements a, b, c in the lattice, both $c'' \leqslant ((a \otimes c) \to (b \otimes c)) \to (a \to b)$ and $a \wedge a' = 0$ hold. A Gödel algebra [13,16] is a BL-algebra that satisfies $a \otimes a = a$ for each a in the lattice. Finally, a Boolean algebra is a residuated lattice that is both a Heyting algebra and an MV-algebra.

In addition, it is worth pointing out that, *if the residuated lattice $\mathcal{L}$ satisfies the law of double negation and the condition*: $a \wedge a' = 0$ *for all* $a \in L$, *then $\mathcal{L}$ is a Boolean algebra* (see [17], Proposition 2.35).

Now we set out dealing with the CHC models in residuated lattices. Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$ be a complete residuated lattice. Denote

$$L^{(0)} = \{q \in L\colon\ q \nleqslant q'\} = \{q \in L\colon\ q \otimes q \neq 0\} \subseteq L$$

whose elements are not self-contradictory, where $q' = q \to 0$. Obviously, $q_1 \in L^{(0)}$ together with $q_1 \leqslant q_2$ implies $q_2 \in L^{(0)}$. Define the family of all sets of premises:

$$\mathcal{P}_0(L) = \Big\{P \subseteq L\colon\ \bigwedge P \in L^{(0)}\Big\} = \Big\{P \subseteq L\colon\ \Big(\bigwedge P\Big) \otimes \Big(\bigwedge P\Big) \neq 0\Big\} \subseteq \mathcal{P}(L).$$

Therefore, with $(\bigwedge P) \otimes (\bigwedge P) \leqslant (\bigwedge P) \wedge (\bigwedge P)$, we have $\mathcal{P}_0(L) \subseteq \mathbf{P}_0(L)$, but they may be unequal. Now, for any $P \in \mathcal{P}_0(L)$ and any $p_i, p_j \in P$, since $(\bigwedge P) \otimes (\bigwedge P) \leqslant p_i \otimes p_j$, we have $p_i \otimes p_j \neq 0$; equivalently, $p_i \nleqslant p_j \to 0$, i.e., $p_i \nleqslant p'_j$. But $\bigwedge P \neq 0$ cannot derive $p_i \nleqslant p'_j$ for any $p_i, p_j \in P$. This is why we define $\mathcal{P}_0(L)$ as the family of sets of premises. As we know, the elements of CHCs are not self-contradictory, and therefore, the CHC operators may be suitably defined as follows. For any $P \in \mathcal{P}_0(L)$,

$$\Phi_\vee(P) = \Big\{q \in L^{(0)}\colon\ \bigvee P \nleqslant q'\Big\},$$

$$\Phi_\wedge(P) = \Big\{q \in L^{(0)}\colon\ \bigwedge P \nleqslant q'\Big\},$$

$$C_\vee(P) = \Big\{q \in L\colon\ \bigvee P \leqslant q\Big\},$$

$$C_\wedge(P) = \Big\{q \in L\colon\ \bigwedge P \leqslant q\Big\},$$

$$H_\wedge(P) = \Big\{q \in L^{(0)}\colon\ q \leqslant \bigwedge P\Big\}.$$

It is worth remarking that the restriction of $q \in L^{(0)}$ in the above definitions of $\Phi_\vee(P)$, $\Phi_\wedge(P)$ and $H_\wedge(P)$ is needed, because neither $\bigvee P \nleqslant q'$, $\bigwedge P \nleqslant q'$ nor $q \leqslant \bigwedge P$ can derive $q \in L^{(0)}$. On the other hand, it is easy to show that

$$C_\vee(P) \subseteq C_\wedge(P) \subseteq L^{(0)},$$

by using $P \in \mathcal{P}_0(L)$ and $(\bigwedge P) \otimes (\bigwedge P) \leqslant (\bigvee P) \otimes (\bigvee P)$.

Because for any p, q and $r \in L$, $p \leqslant q \to r$ iff $p \otimes q \leqslant r$, we have

$$\Phi_\vee(P) = \Big\{q \in L^{(0)}\colon\ \Big(\bigvee P\Big) \otimes q \neq 0\Big\},$$

$$\Phi_\wedge(P) = \Big\{q \in L^{(0)}\colon\ \Big(\bigwedge P\Big) \otimes q \neq 0\Big\}.$$

The operators of CHCs in residuated lattices include most of those basic properties present in the framework of orthocomplemented lattices, but there are some essential differences. As we know, orthocomplemented lattices satisfy orthocomplemented law (i.e., $x = x''$ and $x \wedge x' = 0$), whereas residuated lattices may not satisfy the Double Negation (i.e., $x'' = x$, where $x' = x \to 0$). On the other hand, though orthocomplemented lattices and residuated lattices may not satisfy distributive law, in residuated lattices, there exist properties: $a \otimes (\bigvee_i x_i) = \bigvee_i (a \otimes x_i)$, $(\bigvee_i x_i) \to a = \bigwedge_i (x_i \to a)$, and $a \to (\bigwedge_i x_i) = \bigwedge_i (a \to x_i)$, if $\bigvee_i x_i$ and $\bigwedge_i x_i$ exist. Therefore they may render some intrinsic distinctions between the CHCs in these lattices. First we describe some of the intrinsic attributes of the CHC operators in residuated lattices. As mentioned above, if L is a complete Boolean algebra, then

$$\forall a \in L\colon\ a \in C_\wedge(P) \Longleftrightarrow a' \wedge \left(\bigwedge P\right) = 0.$$

However, this result may not hold in the framework of residuated lattices. We offer the following proposition.

Proposition 4.1. *Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$ be a complete residuated lattice. Then the following two statements are equivalent.*

(1) *For any $q \in L$, $q = q''$.*
(2) *For any $P \in \mathcal{P}(L)$ and any $q \in L$, $q \in C_\wedge(P)$ iff $q' \otimes (\bigwedge P) = 0$.*

Proof. Firstly we know $q \leqslant q''$ and $q \otimes q' = q' \otimes q = 0$ for all $q \in L$. (1) $\Rightarrow$ (2) is clear, since (2) can be equivalently described as: for any $P \in \mathcal{P}(L)$ and any $q \in L$, $\bigwedge P \leqslant q$ iff $\bigwedge P \leqslant (q' \to 0) = q''$. (2) $\Rightarrow$ (1): For any $q \in L$, by taking $P = \{q''\}$, then, $q' \otimes (\bigwedge P) = 0$ and, in terms of (2) we have $q'' \leqslant q$, which, together with $q \leqslant q''$, results in $q = q''$, i.e., (1) holds. □

Since $L^{(0)} \subseteq L$ and $\mathcal{P}_0 \subseteq \mathcal{P}(L)$, we have the following corollary.

Corollary 4.2. *Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$ be a complete residuated lattice. Then the following two statements are equivalent.*

(1) *For any $q \in L^{(0)}$, $q = q''$.*
(2) *For any $P \in \mathcal{P}_0(L)$ and any $q \in L^{(0)}$, $q \in C_\wedge(P)$ iff $q' \otimes (\bigwedge P) = 0$.*

Proof. The details are similar to Proposition 4.1, but for completeness, we still give the procedure. (1) $\Rightarrow$ (2): This is easy to prove. (2) $\Rightarrow$ (1): Since for any $q \in L$, $q \otimes (q \to 0) \leqslant 0$; equivalently, $q \leqslant q''$, we have that $q \in L^{(0)}$ implies $q'' \in L^{(0)}$. Note that if for any $P \in \mathcal{P}_0(L)$, $\bigwedge P \leqslant q_1$ iff $\bigwedge P \leqslant q_2$ where $q_1, q_2 \in L^{(0)}$, then $q_1 = q_2$. As we know, (2) can be equivalently expressed as: $\bigwedge P \leqslant q$ iff $\bigwedge P \leqslant (q' \to 0)$ for any $P \in \mathcal{P}_0(L)$ and any $q \in L^{(0)}$, which results in $q = (q' \to 0) = q''$ for any $q \in L^{(0)}$. Therefore, this proof is completed. □

Corollary 4.2 gives, in residuated lattices, the equivalence between the theorem of contradiction and the double negation. However, in the setting of CHC models, whose elements are not self-contradictory, therefore, the elements are restricted in $L^{(0)}$. We recall that if a residuated lattice L satisfies the double negation and $x \wedge x' = 0$ for all $x \in L$, then L is a Boolean algebra [17]. Therefore, we first proved Proposition 4.1 above, by considering all elements in L, instead of in $L^{(0)}$ only.

Remark. In Section 3, we proved that, in any orthocomplemented lattice L, if the theorem of contradiction holds, then L must be orthomodular, but, contrarily, even if L is an orthomodular lattice, the theorem of contradiction may not hold. We may naturally ask, in orthocomplemented lattices, what suitable condition, together with orthomodularity, is equivalent to the theorem of contradiction. Here, in residuated lattices, we have answered this question by showing that the double negation condition and the theorem of contradiction are equivalent (Proposition 4.1). We knew that, in Boolean algebras, the theorem of contradiction always holds, but the premise condition—Boolean algebras, is stronger than that of residuated lattices together with the double negation, because, as indicated above, a residuated lattice L satisfying the double negation and $x \wedge x' = 0$ for all $x \in L$ reduces to a Boolean algebra.

Since MV-algebras satisfy the law of double negation, the theorem of contradiction holds in the framework of MV-algebras. Therefore, by means of Proposition 4.1 we have the following corollary, where, to be consistent, we still use the representation form of residuated lattices.

Corollary 4.3. *Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$ be a complete MV-algebra ($\langle L, \vee, \wedge, 0, 1\rangle$ is a complete lattice). Then, for any $P \in \mathcal{P}(L)$ and any $q \in L$, $q \in C_\wedge(P)$ iff $q' \otimes (\bigwedge P) = 0$.*

Next we further deal with the CHC models in residuated lattices. Here we need an operator δ introduced in Ref. [27], which is defined as follows: for any operator F on $\mathcal{P}(L)$ that represents the set of all subsets of L, $\delta F = c \circ F \circ {\prime}$, where $\prime$ and c represents orthocomplemented operation and complement operation, respectively (for example, for any $P \subseteq L$, $P' = \{p'\colon\ p \in P\}$ and $P^c = L - P$), then in orthocomplemented lattices, the statements (2) and (3) in the following Corollary 4.5 hold naturally. However, in residuated lattices, we first have:

Proposition 4.4. *Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$ be a complete residuated lattice. Then the following three statements are equivalent.*

(1) *For any $p \in L$, $p'' = p$.*
(2) *For any $P \in \mathcal{P}(L)$, $\delta\Phi_\wedge^{(0)}(P) = H_\vee^{(0)}(P) \doteq \{q \in L\colon\ q \leqslant \bigvee P\}$.*
(3) *For any $P \in \mathcal{P}(L)$, $\delta\Phi_\vee^{(0)}(P) = H_\wedge^{(0)}(P) \doteq \{q \in L\colon\ q \leqslant \bigwedge P\}$.*

Here $\Phi_\wedge^{(0)}(P)$ and $\Phi_\vee^{(0)}(P)$, to be distinguished from $\Phi_\wedge(P)$ and $\Phi_\vee(P)$, are defined as follows:

$$\Phi_\wedge^{(0)}(P) = \Big\{q \in L\colon\ q \leqslant \bigwedge P\Big\}; \qquad \Phi_\vee^{(0)}(P) = \Big\{q \in L\colon\ q \leqslant \bigvee P\Big\}.$$

Proof. It suffices to verify that (1) is equivalent to (2) and (3), respectively. We only demonstrate the equivalence between (1) and (2), because the other is analogous. First note that for any $P \in \mathcal{P}(L)$, and any $p \in P$, $\bigwedge P \in L$ and $p \in L$ always hold. If (1) holds, then

$$\begin{aligned}
\delta\Phi_\wedge(P) &= \big(\Phi_\wedge(P')\big)^c \\
&= \Big\{q \in L\colon\ \bigwedge P' \nleqslant q'\Big\}^c \\
&= \Big\{q \in L\colon\ \bigwedge P' \leqslant q'\Big\} \\
&= \Big\{q \in L\colon\ q \otimes \bigwedge P' = 0\Big\} \\
&= \Big\{q \in L\colon\ q \leqslant \bigwedge P' \to 0\Big\} \\
&= \Big\{q \in L\colon\ q \leqslant \bigwedge \{p'\colon\ p \in P\} \to 0\Big\} \\
&= \Big\{q \in L\colon\ q \leqslant \bigvee_{p \in P} (p' \to 0)\Big\} \quad \Big(\text{with } \bigwedge_i a_i \to 0 = \bigvee_i (a_i \to 0)\Big) \\
&= \Big\{q \in L\colon\ q \leqslant \bigvee_{p \in P} p''\Big\} \\
&= \Big\{q \in L\colon\ q \leqslant \bigvee P\Big\} \quad \big(\text{with the condition (1)}\big) \\
&= H_\vee(P).
\end{aligned}$$

On the other hand, let (2) hold. For any $p \in L$, take $P = \{p\}$. Clearly, also $P \in \mathcal{P}(L)$, and

$$\begin{aligned}
\delta\Phi_\wedge(P) &= \big(\Phi_\wedge(P')\big)^c \\
&= \{q \in L\colon\ p' \nleqslant q'\}^c
\end{aligned}$$

$$
\begin{aligned}
&= \{q \in L\colon\ p' \leqslant q'\} \\
&= \{q \in L\colon\ p' \leqslant q \to 0\} \\
&= \{q \in L\colon\ p' \otimes q \leqslant 0\} \\
&= \{q \in L\colon\ q \leqslant p' \to 0\} \\
&= \{q \in L\colon\ q \leqslant p''\}.
\end{aligned}
$$

Since $\delta\Phi_\wedge(P) = H_\vee(P) = \{q \in L\colon\ q \leqslant p\}$ and $p \leqslant p''$, it holds that $p'' = p$. □

Since $L^{(0)} \subseteq L$ and $\mathcal{P}_0 \subseteq \mathcal{P}(L)$, we have the following corollary.

Corollary 4.5. *Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$ be a complete residuated lattice. Then the following three statements are equivalent.*

(1) *For any $p \in L^{(0)}$, $p'' = p$.*
(2) *For any $P \in \mathcal{P}_0(L)$, $\delta\Phi_\wedge(P) = H_\vee(P) \doteq \{q \in L^{(0)}\colon\ q \leqslant \bigvee P\}$.*
(3) *For any $P \in \mathcal{P}_0(L)$, $\delta\Phi_\vee(P) = H_\wedge(P)$.*

Proof. Actually, the proof is analogous to that of Proposition 4.4, only by substituting $\mathcal{P}_0(L)$ and $L^{(0)}$ for $\mathcal{P}(L)$ and L, respectively. We therefore leave the details out. □

Remark. As mentioned above, in orthocomplemented lattices, statements (2) and (3) in Proposition 4.4 and Corollary 4.5 hold naturally. But, in residuated lattices, they are equivalent to the double negation condition, which, however, is still weaker than that of Boolean algebras.

From Proposition 4.4 and the double negation of MV-algebras it follows the following corollary.

Corollary 4.6. *Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$ be a complete MV-algebra. Then*:

(1) *For any $P \in \mathcal{P}(L)$, $\delta\Phi^{(0)}_\wedge(P) = H^{(0)}_\vee(P) \doteq \{q \in L\colon\ q \leqslant \bigvee P\}$.*
(2) *For any $P \in \mathcal{P}(L)$, $\delta\Phi^{(0)}_\vee(P) = H^{(0)}_\wedge(P) \doteq \{q \in L\colon\ q \leqslant \bigwedge P\}$.*

Finally, we deal with the structure's theorem of hypotheses in residuated lattices.

Example 5. The structure's theorem of hypotheses in the framework of residuated lattices (also, MV-algebras) may not hold; that is to say, there exist residuated lattice (also, MV-algebra) L and $P \in \mathcal{P}_0(L)$ such that

$$H^*_\wedge(P) \neq \Big(\bigwedge P\Big) \wedge A_\wedge(P) \doteq \Big\{\Big(\bigwedge P\Big) \wedge q\colon\ q \in A_\wedge(P)\Big\},$$

where $A_\wedge(P) = \{q \in L^{(0)}\colon\ q \wedge (\bigwedge P) \in L^{(0)}, qNC(\bigwedge P)\}$, $H^*_\wedge(P) = \{q \in L^{(0)}\colon\ q \leqslant \bigwedge P, q \neq \bigwedge P\}$.

Proof. Let $\mathcal{L}$ be a complete residuated lattice (exactly, an MV algebra) of [0,1]-valued functions over some nonempty set X, that is, $\mathcal{L} = \langle [0,1]^X, \vee, \wedge, \otimes, \to, \mathbf{0}, \mathbf{1}\rangle$, where $\vee, \wedge, \otimes, \to, \mathbf{0}, \mathbf{1}$ are defined in the following: for each two functions $f, g \in [0,1]^X$ we put, for all $x \in X$,

$$
\begin{aligned}
&(f \vee g)(x) = \max\big(f(x), g(x)\big), \qquad (f \otimes g)(x) = f(x) \otimes g(x), \\
&(f \wedge g)(x) = \min\big(f(x), g(x)\big), \qquad (f \to g)(x) = f(x) \to g(x),
\end{aligned}
$$

where $\otimes$ is Lukasiewicz t-norm and $\to$ the corresponding residuum, namely,

$$f(x) \otimes g(x) = \max\big(0, f(x) + g(x) - 1\big), \qquad f(x) \to g(x) = \min\big(1, 1 - f(x) + g(x)\big),$$

and let $\mathbf{0}$, $\mathbf{1}$ be two constant functions taking values 0 and 1, respectively.

Let $P=\{\mathbf{a}\}$, where $\mathbf{a}$ is a constant function on X and $\mathbf{a}(x)=a$ for any $x\in X$, $\frac{1}{2}<a<1$. Clearly, $\mathbf{a}\in\mathcal{P}_0(L)$, where $L=[0,1]^X$. Suppose $\mathbf{b}$ is also a constant function on X, satisfying $\mathbf{b}(x)=b$ for any $x\in X$, and $\frac{1}{2}<b<a<1$. Then it is clear that $\mathbf{b}\in H^*_\wedge(P)$; and for any function $f\in A_\wedge(P)$, we claim $\mathbf{b}\neq f\wedge\mathbf{a}$. If not so, then $\mathbf{b}=f\wedge\mathbf{a}$ implies $f(x)=b$ for any $x\in X$, i.e., $f=\mathbf{b}$ and consequently, $f\leqslant\mathbf{a}=\bigwedge P$, which contradicts $f\in A_\wedge(P)$. So, our claim is verified, and therefore $\mathbf{b}\notin(\bigwedge P)\wedge A_\wedge(P)$. The proof is completed. □

Remark. However, in residuated lattices, $(\bigwedge P)\wedge A_\wedge(P)\subseteq H^*_\wedge(P)$ always holds. Furthermore, set $B_\wedge(P)=\{q\in L^{(0)}\colon \bigwedge P\nleqslant q, q\wedge(\bigwedge P)\in L^{(0)}\}$. Then it is easy to check that

$$H^*_\wedge(P)=\Big(\bigwedge P\Big)\wedge B_\wedge(P).$$

Remark. In Boolean algebras, $C_\wedge$ is the largest Tarsk's consequence operator and $\Phi_\wedge$, in some sense, is the smallest expansive and anti-monotonic operator [27]. However, the two results do not generally hold in the framework of residuated lattices (see [3]).

5. Concluding remarks

The CHCs operators proposed newly in the framework of orthocomplemented lattices provide a new arena for performing mathematical reasoning and reformulating those pivot theorems in classical logic. In this paper we have clarified some important issues associated with certain CHC operators and have presented some characterizations in the CHC models. The main points are summed up as follows: (1) We have verified that the orthomodular law is not the necessary condition for characterizing the structure's theorem of hypotheses. (2) Connections between implication operators and CHC operators have been investigated. We have used these CHC operators to describe a generalized form of the theorem of contradiction, the deduction theorem, and the Lindenbaum theorem of classical logic. In particular, we proved that the theorem of contradiction holding infers that the underlying lattices must be orthomodular. On the other hand, we have demonstrated that, however, in some orthomodular lattices, the theorem of contradiction does *not* holds. (3) We have re-defined the CHC operators in residuated lattices, and particularly discovered some essential differences between CHCs in orthocomplemented lattices and those in residuated lattices, showing that (i) the theorem of contradiction holds iff the residuated lattice under consideration satisfies the double negation, and (ii) the structure's theorem of hypotheses in the framework of MV-algebras (therefore, residuated lattices) may not hold.

An interesting issue emerges. Usually, it is under certain logic that we try to validate or invalidate a proposition or a theorem, and to apply reasoning. However, we may ask, what is the weakest, or the sufficient and necessary, logic for holding a theorem or a proposition? This is a significant problem that may motivate us to consider further these important results in AI and mathematics.

Finally, it may be worth mentioning that dealing with the CHC models in other important algebras (such as BL-algebras [13,16], MV-algebras [2,13,15,16], Product algebras [13,16], Gödel algebras [13,16], Heyting algebras [1,16], and quantum algebras [8]) is also an intriguing issue worthy of further consideration.

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