# A condensed semantics for qualitative spatial reasoning about oriented straight line segments

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### A B S T R A C T

More than 15 years ago, a set of qualitative spatial relations between oriented straight line segments (dipoles) was suggested by Schlieder. However, it turned out to be difficult to establish a sound constraint calculus based on these relations. In this paper, we present the results of a new investigation into dipole constraint calculi which uses algebraic methods to derive sound results on the composition of relations of dipole calculi. This new method, which we call condensed semantics, is based on an abstract symbolic model of a specific fragment of our domain. It is based on the fact that qualitative dipole relations are invariant under orientation preserving affine transformations.

The dipole calculi allow for a straightforward representation of prototypical reasoning tasks for spatial agents. As an example, we show how to generate survey knowledge from local observations in a street network. The example illustrates the fast constraint-based reasoning capabilities of dipole calculi. We integrate our results into two reasoning tools which are publicly available.

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## 1. Introduction

Qualitative reasoning about space abstracts from the physical world and enables computers to make predictions about spatial relations, even when precise quantitative information is not available [4]. A *qualitative* representation provides mechanisms which characterize the essential properties of objects or configurations. In contrast, a *quantitative* representation establishes a measure in relation to a unit of measurement which must be generally available [12]. The constant and general availability of common measures is now self-evident. In history, however, there used to be a lot of measurement systems that were only standardized locally. If you said that a pole was six feet long, that pole would have been 150 cm long in the grand duchy of Hesse, but 300 cm in the duchy of Nassau. Even today several quantitative systems of measurements are used in the world, with the SI-system, the Imperial system and the United States Customary Units being the predominant ones. One need only recall the history of length measurement technologies to see that the more local relative measures, which are represented qualitatively,[1] can be managed by biological/epigenetic cognitive systems much more easily than absolute quantitative representations.

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[1] Compare for example the qualitative expression "one piece of material is longer than another" with the quantitative expression "this thing is two meters long."
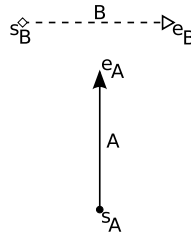
**Fig. 1.** Orientation between two dipoles.

Qualitative spatial calculi usually deal with elementary objects (e.g., regions, points) and qualitative relations between them (e.g., "included in," "adjacent," "to the left of"). This is the reason why qualitative descriptions are quite natural for people. The two main trends in qualitative spatial reasoning (QSR) are topological reasoning about regions [9,44,45,49,64] and positional (e.g., direction and distance) reasoning about point configurations.[2] Positional relations can refer to absolute (e.g., cardinal) directions [10,28,48] or to relative directions [25]. Relative position calculi based on points as basic entities are [3,13,23,35,54,65]. Most relative position calculi use ternary relations. In contrast cardinal directions are expressed as binary relations. Positional calculi can be related to the results of psycholinguistic research in the field of reference systems [25,37]. Human natural language spatial propositions often express relative spatial positions based on reference directions derived from the shape (and function) of one of the objects involved [25] (e.g., "The hill is to the left of the train"). This leads to binary relations between objects in which at least one of the objects has the feature of orientedness. For that reason, in our conception, orientedness is an important feature of natural objects. In a corresponding qualitative calculus it is necessary to use more complex basic entities than points. One option for building more complex basic entities is to use oriented line segments (see Fig. 1) as basic entities. In this abstraction we lose the specific shape of the object, but preserve the feature of orientedness. With this approach we can design relative position calculi in which directions are expressed as binary relations. The corresponding calculus, Schlieder's line segment calculus [53],[3] is the main topic of this paper. Oriented straight line segments (which were called *dipoles* by Moratz et al. [36]) may be specified by their start and end points.

Using dipoles as basic building blocks, more complex objects can be constructed (e.g., polylines, polygons) in a straight-forward manner (see Fig. 11). Therefore, dipoles can be used as the basic units in numerous applications. To give an example, line segments are central to edge-based image segmentation and grouping in computer vision. In addition, GIS systems often have line segments as basic entities [21]. Polylines are particularly interesting for representing paths in cognitive robotics [40] and can serve as the geometric basis of a mobile robot when autonomously mapping its working environment [61]. To sum up, dipole calculi are qualitative calculi that abstract from metric information. They focus on directional relations, but can also be used to express certain topological relations (see Section 2.7).

In the previous paragraphs, we discussed the *representation* of spatial knowledge. The central topic of this paper is the collection of *reasoning* mechanisms which are employed to make use of the represented initial knowledge to infer indirect knowledge. In qualitative spatial reasoning two main reasoning modes are used: conceptual neighborhood-based reasoning, and constraint-based reasoning about (static) spatial configurations. Conceptual neighborhood-based reasoning describes whether two spatial configurations of objects can be transformed into each other by small changes [11,15]. The conceptual neighborhood of a qualitative spatial relation is the set of relations into which a relation can be changed with minimal transformations, e.g., by continuous deformation. Such a transformation can be a movement of one object in the configuration in a short period of time. The movement of an agent can then be modeled qualitatively as a sequence of neighboring spatial relations which hold for adjacent time intervals.[4] Based on this qualitative representation of trajectories, neighborhood-based spatial reasoning can for example be used as a simple, abstract model of the navigation of a spatial agent.[5]

In constraint-based reasoning about spatial configurations, typically a partial initial knowledge of a scene is represented in terms of qualitative constraints between spatial objects. Implicit knowledge about spatial relations is then derived by constraint propagation.[6] Previous research has found that the mathematical notion of a *relation algebra* and related notions are well-suited for this kind of reasoning. In particular, in an arbitrary relation algebra, the well-known path consistency algorithm computes an algebraic closure of a given constraint network, and this approximates, and in some cases also decides, the consistency of the network in polynomial time. Intelligent backtracking techniques and the study of maximal tractable subclasses also allow of efficiently deciding networks involving disjunctions. Starting with Allen's interval algebra, this approach has been successfully applied to several qualitative constraint calculi, and is now supported by freely available

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[2] There is also some work about directions between regions [18,55].

[3] However, Schlieder's first presentation of dipole relations [53] does not mention composition of dipole relations. Moreover, he focuses on $\mathcal{DRA}_{lr}$ (with no more than two start or end points on the same straight line), which cannot be used for polylines, while his presentation of the finer calculi is sketchy and imprecise, leading to the wrong number, 63, of $\mathcal{DRA}_f$ relations. See Section 2.1 for the definition of these calculi.

[4] This was the reasoning used in the first investigation of dipole relations by Schlieder [53].

[5] For an application of neighborhood based reasoning of spatial agents, we refer the reader to the simulation model SAILAWAY [7].

[6] For an application of constraint-based reasoning for spatial agents, we refer the reader to the example in Section 2.5.

toolboxes [16,59]. Moreover, people have started to develop benchmark problem libraries [42] and have shown that this method performs quite well also when compared to other constraint reasoning techniques [60].

Since this approach is inherently based on composition tables, an essential prerequisite is the computation of such tables. For the dipole calculi, composition tables are quite large, and existing composition tables contain errors.[7] The main objective of this paper is the first computation of correct composition tables for all dipole calculi. The next sections of this paper present a detailed and technical description of dipole calculi and prove new results about algebraic features of these constraint calculi. In Section 2 we introduce base relation sets of the dipole calculi. Then we revisit the theory of constraint reasoning with algebraic closure. To use constraint reasoning in applications, our newly computed correct composition table can directly be fed into the SparQ [58,59] and GQR [16] tools and used for reasoning in dipole calculi. Using this, a concrete sample application of dipole calculi using a spatial reasoning toolbox is presented. Finally, an inconsistent but algebraically closed scenario is presented, showing the limits of the algebraic closure method.

Section 3 tackles the problem of obtaining a correct composition table for dipole calculi. It provides a condensed semantics for the dipole calculus. A condensed semantics, as we call it, provides spatial domain knowledge to the calculus in the form of an abstract symbolic model of a specific fragment of the spatial domain. In this model, possible configurations of very few of the basic spatial entities of a calculus are enumerated. In our case, we use orbits in the affine group $\mathbf{GA}(\mathbb{R}^2)$. This provides a useful abstraction for reasoning about qualitatively different configurations in the Euclidean plane, so that the mathematical reasoning can be rather directly transformed to a computer program checking the configurations. This avoids the sources of errors of previous approaches to composition table computation. We use affine geometry at a rather elementary level and appeal to pictures instead of complete analytic arguments, whenever it is easy to fill in the details—however, at key points in the argument, careful analytic treatments are provided. Further, we calculate the composition tables for the dipole calculi using the condensed semantics and we investigate properties of the composition. It should be noted that condensed semantics techniques can be applied to other calculi as well, for example, the $\mathcal{OPRA}$ calculus [34]. In general, condensed semantics can be summarized as the study of geometric transformations that leave the relations of the calculus at hand invariant, and then work with qualitative configurations, which essentially can be identified as orbits in the automorphism group of these transformations.

Our paper ends with a summary and conclusion and discussion of future work.

## 2. Representation of dipole relations and relation algebras

In this section, we first present a set of spatial relations between dipoles, then variants of this set of spatial relations. The next subsection gives a formal representation of dipole relations. Then a subsection presents the standard reasoning method with dipole relations. The last three subsections discuss application contexts for dipole reasoning and exhibit some of the limits of dipole reasoning.

### 2.1. Basic representation of dipole relations

The basic entities we use are dipoles, i.e., oriented line segments formed by a pair of two points, a start point and an end point. Dipoles are denoted by $A, B, C, \ldots$, start points by $\mathbf{s}_A$ and end points by $\mathbf{e}_A$, respectively (see Fig. 1). These dipoles are used for representing spatial objects with an intrinsic orientation. When examining different relations, the goal is to obtain a set $\mathcal{B}$ of jointly exhaustive and pairwise disjoint *atomic* or *base*-relations, such that exactly one relation holds between any two dipoles. The set of *general relations* $\mathcal{P}(\mathcal{B})$ is the powerset of the set of base relations. That is, a general relation is a set $R \subseteq \mathcal{B}$ of base relations. It is interpreted as the union of these base relations, and by abuse of notation, we will identify a general relation $R \subseteq \mathcal{B}$ with the union $\bigcup R$. General relations are used to express uncertainty about the relative position of dipoles. If these relations form an algebra which fulfills certain requirements, it is possible to apply standard constraint-based reasoning mechanisms that were originally developed for temporal reasoning [1] and that have also proved valuable for spatial reasoning (see Section 2.4).

So as to enable efficient reasoning, an attempt should be made to keep the number of different base relations relatively small. For this reason, we will restrict ourselves to using two-dimensional continuous space for now, in particular $\mathbb{R}^2$, and distinguish the location and orientation of different dipoles only according to a small set of seven different dipole–point relations. We distinguish between whether a point lies to the left, to the right, or at one of five qualitatively different locations on the straight line that passes through the corresponding dipole.[8] The corresponding regions are shown in Fig. 2.

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[7] There has been an explosion of new qualitative spatial calculi in the last decade. A recent diploma thesis supervised by us has found errors in the composition tables of several such calculi. But the composition table is the core of the standard reasoning mechanism (i.e. algebraic closure) for a calculus. A wrong composition table can easily lead to wrong conclusions in reasoning. While algebraic closure has its limits and is incomplete, we know that it is still sound, i.e. always gives a safe approximation of the space of possible geometric configurations. However, this soundness depends on a correct composition table. Developing techniques for correctly computing composition tables is therefore an important contribution. This means that the methods for computing composition tables should be subject to research. Moreover, the method we use also gives insight into the nature of spatial configurations, by identifying them with orbits in the affine group $\mathbf{GA}(\mathbb{R}^2)$.

[8] In his introduction of a set of qualitative spatial relations between oriented line segments, Schlieder [53] mainly focused on configurations in which no more than two end or start points were on the same straight line (e.g., all points were in general position). However, in many domains, we may wish to represent spatial arrangements in which more than two start or end points of dipoles are on a straight line.
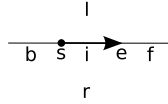
**Fig. 2.** Dipole–point relations (= $\mathcal{LR}$ relations).

A corresponding set of relations between three points was proposed by Ligozat [27] under the name flip-flop calculus and later extended to the $\mathcal{LR}$ calculus [54].[9]

Then these dipole–point relations describe cases where the point is: to the left of the dipole (l); to the right of the dipole (r); straight behind the dipole (b); at the start point of the dipole (s); inside the dipole (i); at the end of the dipole (e); or straight in front of the dipole (f). For example, in Fig. 1, $\mathbf{s}_B$ lies to the left of $A$, expressed as $A$ l $\mathbf{s}_B$. Using these seven possible relations between a dipole and a point, the relations between two dipoles may be specified according to the following four relationships:

$$A \text{ R}_1 \mathbf{s}_B \wedge A \text{ R}_2 \mathbf{e}_B \wedge B \text{ R}_3 \mathbf{s}_A \wedge B \text{ R}_4 \mathbf{e}_A,$$

where $\text{R}_i \in \{l, r, b, s, i, e, f\}$ with $1 \leqslant i \leqslant 4$. Theoretically, this gives us 2401 relations, out of which 72 relations are geometrically possible. They constitute the dipole calculus $\mathcal{DRA}_f$ (f stands for fine grained) and they are listed in Fig. 3. In the next subsection we present several versions of sets of dipole base relations also in an informal way. Then in Section 2.3 we define the dipole base relations in an algebraic way.

**Proposition 1.** *Allen's interval algebra can be embedded into* $\mathcal{DRA}_f$ *by the following mapping of base relations*[10]:

$$
\begin{array}{llll}
= & \mapsto & sese & \\
b & \mapsto & ffbb & \quad bi \mapsto bbff \\
m & \mapsto & efbs & \quad mi \mapsto bsef \\
o & \mapsto & ifbi & \quad oi \mapsto biif \\
d & \mapsto & bfii & \quad di \mapsto iibf \\
s & \mapsto & sfsi & \quad si \mapsto sisf \\
f & \mapsto & beie & \quad fi \mapsto iebe
\end{array}
$$

In cases stemming from the embedding of Allen's interval algebra, the dipoles lie on the same straight lines and have the same direction. $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ also contain 13 additional relations which correspond to the case with dipoles lying on a line but facing opposite directions.

### 2.2. Several versions of sets of dipole base relations

In their paper on customizing spatial and temporal calculi, Renz and Schmid [51] investigated different methods for deriving variants of a given calculus that have a granularity better-suited for certain tasks. One of these methods uses only a subset of the base relations as a new set of base relations. For example, Schlieder [53] introduced a set of base relations in which no more than two start or end points were on the same straight line. As a result, only a subset of the $\mathcal{DRA}_f$ base relations is used. We call $\mathcal{DRA}_{lr}$ a calculus based on these base relations (where lr stands for left/right). The following base relations are part of $\mathcal{DRA}_{lr}$: rrrr, rrll, llrr, llll, rrrl, rrlr, rlrr, rllr, rlll, lrrr, lrrl, lrll, llrl, lllr.

Moratz et al. [36] introduced an extension of $\mathcal{DRA}_{lr}$ which adds relations for representing polygons and polylines. In this extension, two start or end points can share an identical location. In this calculus three points at different locations still cannot belong to the same straight line. This subset of $\mathcal{DRA}_f$ was named $\mathcal{DRA}_c$ (c refers to *coarse*, f refers to *fine*). The set of 24 base relations of $\mathcal{DRA}_c$ extends the base relations of $\mathcal{DRA}_{lr}$ with the following relations: ells, errs, lere, rele, slsr, srsl, lsel, rser, sese, eses.

The method of using only a subset of base relations reduces the number of base relations. Conversely, other methods extend the number of base relations. For example, Dylla and Moratz [8] have observed that $\mathcal{DRA}_f$ may not be sufficient for robot navigation tasks, because the dipole configurations that are pooled in certain base relations are too diverse. Thus, the representation has been extended with additional orientation knowledge and a more fine-grained $\mathcal{DRA}_{fp}$ calculus with additional orientation distinctions has been derived. It has slightly more base relations.

The large configuration space for the rrrr relation is visualized in Fig. 4. The other analogous relations which are extremely coarse are llrr, rrll and llll. In many applications, this unwanted coarseness of four relations can lead to problems.[11]

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[9] The $\mathcal{LR}$ calculus also features the relations dou and tri for both reference points or all points being equal, respectively. These cases are not possible for dipoles, since the start and end points cannot coincide by definition.

[10] Indeed, this yields a homomorphism of non-associative algebras.

[11] An investigation by Dylla and Moratz into the first cognitive robotics applications of dipole relations integrated in situation calculus [8] showed that the coarseness of $\mathcal{DRA}_f$ compared to $\mathcal{DRA}_{fp}$ would indeed lead to rather meandering paths for a spatial agent.
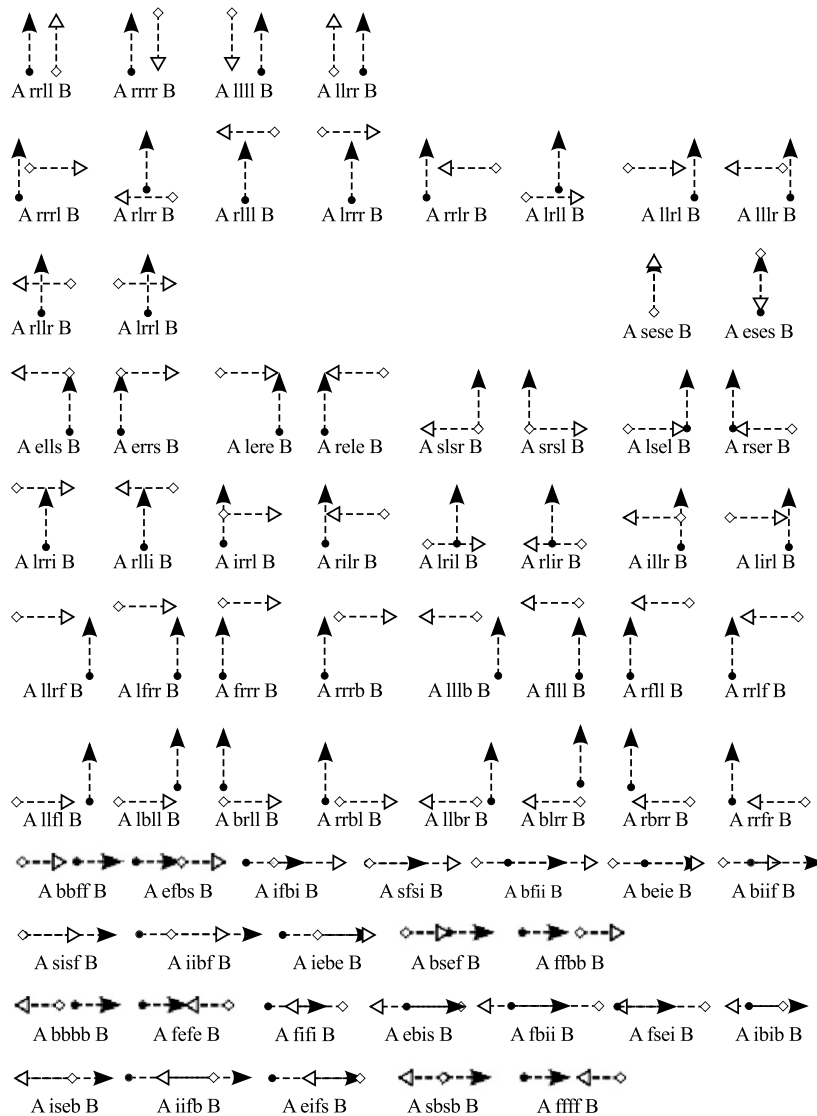
**Fig. 3.** The 72 atomic relations of the $\mathcal{DRA}_f$ calculus. In the dipole calculus, orthogonality is not defined, although the graphical representation may suggest this.
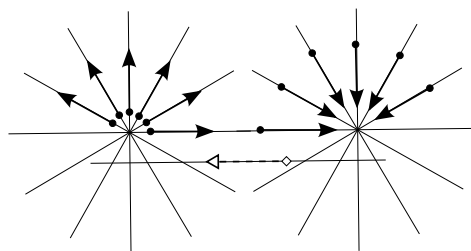


**Fig. 4.** Pairs of dipoles subsumed by the same relation.

Therefore, we introduce an additional qualitative feature by considering the angle spanned by the two dipoles. This gives us an important additional distinguishing feature with four distinctive values. These qualitative distinctions are parallelism (P) or anti-parallelism (A) and mathematically positive and negative angles between $A$ and $B$, leading to three refining relations for each of the four above-mentioned relations (Fig. 5).
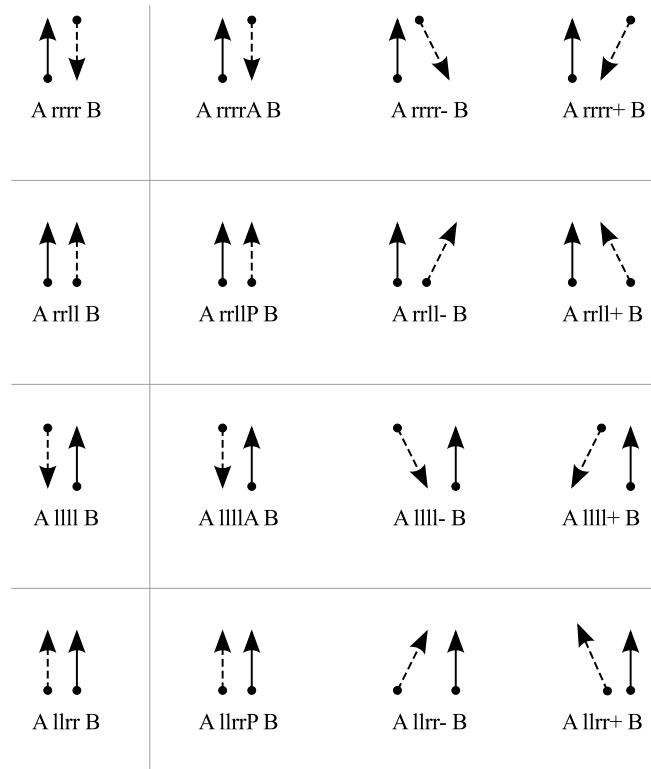
**Fig. 5.** Refined base relations in $\mathcal{DRA}_{fp}$.

We call this algebra $\mathcal{DRA}_{fp}$, as it is an extension of the fine-grained relation algebra $\mathcal{DRA}_f$ with additional distinguishing features due to "parallelism." For the relations different from rrrr, llrr rrll and llll, a '+' or '−', 'P' or 'A' respectively, is already determined by the original base relation and does not have to be mentioned explicitly. These base relations then have the same relation symbol as in $\mathcal{DRA}_f$.

The introduction of parallelism into dipole calculi not only has benefits in certain applications, the algebraic features also benefit from this extension (see Section 3.6).

### 2.3. Formal representation of dipole relations

The dipole relations have been introduced in an informal way in Section 2.1, but they can also be defined in an algebraic way. The following derivation of the semantics of the dipole calculi may seem a bit complicated to some readers, but we want to remain as compatible to the work begun by Moratz et al. in [36] as possible. Every dipole $D$ on the plane $\mathbb{R}^2$ is an ordered pair of two points $\mathbf{s}_D$ and $\mathbf{e}_D$ in the Euclidean plane, each of them being represented by its Cartesian coordinates $x$ and $y$, with $x, y \in \mathbb{R}$ and $\mathbf{s}_D \neq \mathbf{e}_D$.

$$D = (\mathbf{s}_D, \mathbf{e}_D), \quad \mathbf{s}_D = \big((\mathbf{s}_D)_x, (\mathbf{s}_D)_y\big), \quad \mathbf{e}_D = \big((\mathbf{e}_D)_x, (\mathbf{e}_D)_y\big)$$

The basic relations are then described by equations with the coordinates as variables. The set of solutions for a system of equations describes all the possible coordinates for these points. A first specification based on coordinates was presented in Moratz et al. [36]. Let us focus here on the $\mathcal{LR}$ relations first. Given a dipole $A$ and a point $p$, we want to determine whether $p$ lies to the right or to the left of $A$, or is collinear with $A$. We begin the derivation by constructing vectors $\vec{A}$ and $\vec{P}$ having base point $\mathbf{s}_A$ and pointing to $\mathbf{e}_A$ and $p$ respectively as

$$\vec{A} = \begin{pmatrix} (\mathbf{e}_A)_x - (\mathbf{s}_A)_x \\ (\mathbf{e}_A)_y - (\mathbf{s}_A)_y \end{pmatrix}, \qquad \vec{P} = \begin{pmatrix} p_x - (\mathbf{s}_A)_x \\ p_y - (\mathbf{s}_A)_y \end{pmatrix}$$

If we construe $\vec{A}$ and $\vec{P}$ as vectors in three-dimensional Euclidean space, located in the plane determined by $z = 0$, the vector product (Fig. 6)

$$\vec{P} \times \vec{A}$$
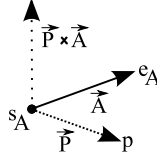
**Fig. 6.** Constructing $\vec{P} \times \vec{A}$.
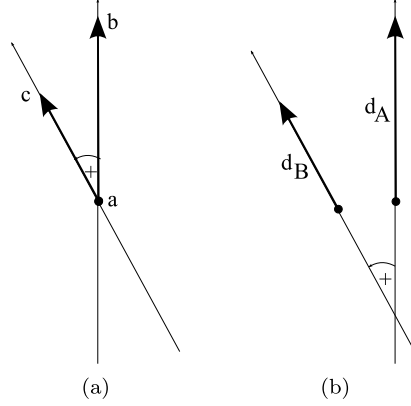


**Fig. 7.** Computing $\mathcal{LR}$-relation (a) and qualitative angle (b) via vectors.

points upwards ($z > 0$) iff $p$ is to the right of $A$, and downwards ($z < 0$) iff $p$ is to the left of $A$. The $z$ coordinate of $\vec{P} \times \vec{A}$ is given by

$$z = -p_x \cdot (\mathbf{s}_a)_y + (\mathbf{e}_a)_x \cdot (\mathbf{s}_a)_y + p_y \cdot (\mathbf{s}_a)_x - (\mathbf{e}_a)_y \cdot (\mathbf{s}_a)_x - p_y \cdot (\mathbf{e}_a)_x + p_x \cdot (\mathbf{e}_a)_y$$

If the vectors $\vec{A}$ and $\vec{P}$ are collinear ($z = 0$), we can use

$$\begin{aligned}
\eta = {}^t\vec{A} \cdot \vec{P} \\
= (\mathbf{s}_a)_y^2 + (\mathbf{s}_a)_x^2 - p_y \cdot (\mathbf{s}_a)_y - (\mathbf{e}_a)_y \cdot (\mathbf{s}_a)_y - p_x \cdot (\mathbf{s}_a)_x - (\mathbf{e}_a)_x \cdot (\mathbf{s}_a)_x \\
+ p_y \cdot (\mathbf{e}_a)_y + p_x \cdot (\mathbf{e}_a)_x
\end{aligned}$$

− this is positive if $\vec{A}$ and $\vec{P}$ point into the same direction, and negative if they point into opposite directions. Altogether, we can define the $\mathcal{LR}$ relations as:

$$R = \begin{cases}
l & \text{if } z < 0 \\
r & \text{if } z > 0 \\
b & \text{if } z = 0 \wedge \eta < 0 \\
s & \text{if } p = \mathbf{s}_a \\
i & \text{if } \begin{bmatrix} z = 0 \wedge \eta > 0 \\ \|\mathbf{e}_a - \mathbf{s}_a\| > \|p - \mathbf{s}_a\| \end{bmatrix} \\
e & \text{if } p = \mathbf{e}_a \\
f & \text{if } \begin{bmatrix} z = 0 \wedge \eta > 0 \\ \|\mathbf{e}_a - \mathbf{s}_a\| < \|p - \mathbf{s}_a\| \end{bmatrix}
\end{cases}$$

This description of the relations is particularly useful for reasoning tasks, e.g., Gröbner reasoning. Gröbner reasoning can be used to check the solvability of linear equalities over the complex numbers. For the `SparQ` toolbox [58] an extension of Gröbner reasoning has been defined and implemented that can check the solvability of nonlinear inequalities over the real numbers. Unfortunately Gröbner reasoning has doubly exponential running time. This particular extension is still work in progress, but preliminary results have been published in [63]. A follow-up article is to appear.

The semantics of the relations can also be expressed in a more convenient and more regular way using atan2 and the function angle based on it:

$$\text{angle}(\vec{A}, \vec{B}) := \text{atan2}(\vec{A}_x \cdot \vec{B}_y - \vec{A}_y \cdot \vec{B}_x, \vec{A}_x \cdot \vec{B}_x + \vec{A}_y \cdot \vec{B}_y)$$

To determine an $\mathcal{LR}$-relation ($a\ b\ R\ c$) with $a \neq b$, we compute the angle from the vector $\vec{ab}$ to the vector $\vec{ac}$ as depicted in Fig. 7(a). Now we just need to substitute the $\mathcal{LR}$-relations for their definitions by the case distinction:

$$R = \begin{cases} l & \text{if } 0 < \text{angle}(ab, ac) < \pi \\ r & \text{if } -\pi < \text{angle}(ab, ac) < 0 \\ b & \text{if } \text{angle}(ab, ac) = \pi \vee \text{angle}(ab, ac) = -\pi \\ s & \text{if } \text{angle}(ab, ac) = 0 \wedge \|\vec{ac}\| = 0 \\ i & \text{if } \text{angle}(ab, ac) = 0 \wedge \|\vec{ac}\| < \|\vec{ab}\| \\ e & \text{if } \text{angle}(ab, ac) = 0 \wedge \|\vec{ac}\| = \|\vec{ab}\| \\ f & \text{if } \text{angle}(ab, ac) = 0 \wedge \|\vec{ac}\| > \|\vec{ab}\| \end{cases}$$

The conditions for s and e can be rewritten as $a = c$ and $b = c$ respectively. Please note that the $\mathcal{LR}$ relations *Dou* and *Tri* (which express that the first two points coincide) cannot occur, since dipoles always have a non-zero length. To obtain a $\mathcal{DRA}_f$ relation, we determine all four $\mathcal{LR}$-relations by this operation and concatenate them in the manner described above.

For determining a $\mathcal{DRA}_{fp}$-relation, we additionally need to determine the angle of intersection of two dipoles $dp_A$ and $dp_B$. Again we consider them as vectors $\vec{dp}_A$ and $\vec{dp}_B$ as shown in Fig. 7(b) and we can get the quantitative angle via

$$\text{angle}(\vec{dp}_A, \vec{dp}_B)$$

by using the simple case-distinction:

$$R_5 = \begin{cases} + & \text{if } 0 < \text{angle}(\vec{dp}_A, \vec{dp}_B) < \pi \\ - & \text{if } -\pi < \text{angle}(\vec{dp}_A, \vec{dp}_B) < 0 \\ A & \text{if } \text{angle}(\vec{dp}_A, \vec{dp}_B) = \pi \vee \text{angle}(\vec{dp}_A, \vec{dp}_B) = -\pi \\ P & \text{if } \text{angle}(\vec{dp}_A, \vec{dp}_B) = 0 \end{cases}$$

## 2.4. Constraint reasoning with algebraic closure

The domain of the dipole calculi, the Euclidean plane, is infinite. Standard methods developed for finite domains generally do not apply to constraint reasoning over infinite domains. The theory of relation algebras and non-associative algebras [24,32] allows of a purely symbolic treatment of constraint satisfaction problems involving relations over infinite domains. The corresponding constraint reasoning techniques were originally introduced for temporal reasoning [1] and later proved to be valuable for spatial reasoning [22,49]. The central data for a qualitative calculus are given by:

- a set $\mathcal{B}$ of (symbolic names for) *base relations*, which are interpreted as relations over some domain, having the crucial properties of *pairwise disjointness* and *joint exhaustiveness* (a general relation is then simply a set of base relations);
- a table for the computation of the *converses* of relations;
- a table for the computation of the *compositions* of relations.

As described in Section 2.1, the set of *general relations* $\mathcal{P}(\mathcal{B})$ is the powerset of the set of base relations, and each such general relation is identified with its union. The converse and composition operations are easily extended from $\mathcal{B}$ to $\mathcal{P}(\mathcal{B})$. These data together generate a so-called non-associative algebra [29,32]. Then, the path consistency algorithm [33] and backtracking techniques [57] are the tools used to tackle the problem of consistency of constraint networks (and related problems). These algorithms have been implemented in two generic reasoning tools, GQR [16] and SparQ [59]. To integrate a new calculus into these tools, only a list of base relations and tables for compositions and converses need to be provided. Thereby, the qualitative reasoning facilities of these tools become available for this calculus. Since the compositions and converses of general relations can be reduced to compositions and converses of base relations, these tables only need to be given for base relations. Based on these tables, the tools provide a means to approximate the consistency of constraint networks, list all their atomic refinements, and more (see Section 2.5 for an application).[12]

If $b$ is a base relation, the converse $\{(x, y) \mid (y, x) \in b\}$ is often itself a base relation and is denoted by $b^{\smile}$.[13] In the dipole calculus, it is obvious that the converse of a relation can easily be computed by exchanging the first two and second two letters of the name of a relation, see Table 1. Since $(R^{\smile})^{\smile} = R$ in the $\mathcal{DRA}_f$ calculus, the entries in the table can be read from top to bottom as well as from bottom to top yielding correct converses. For the dipole calculus $\mathcal{DRA}_{fp}$ with additional orientation distinctions, converses can be obtained by adding the simple rule that '+' and '−' are exchanged, while 'P' and 'A' remain invariant.

Since base relations generally are not closed under composition, this operation is approximated by a *weak composition*:

$$b_1 \diamond b_2 = \bigcup \{b \in \mathcal{B} \mid (b_1 \circ b_2) \cap b \neq \emptyset\}$$

where $b_1 \circ b_2$ is given by the usual set theoretic composition

---

[12] With more information about a calculus, both of the tools can provide functionality that goes beyond simple qualitative reasoning for constraint calculi.

[13] In Freksa's double-cross calculus [12] the converses are not necessarily base relations, but for the calculi that we investigate this property holds.

**Table 1**
The converse ($\smile$) operation of $\mathcal{DRA}_f$ can be reduced to a simple permutation.

| $R$ | rrrr | rrrl | rrlr | rrll | rllr | rlll | llll | lrll |
|---|---|---|---|---|---|---|---|---|
| $R^{\smile}$ | rrrr | rlrr | lrrr | llrr | lrrl | llrl | llll | lllr |

| $R$ | eses | sese | ells | errs | lere | slsr | lrri | rlli |
|---|---|---|---|---|---|---|---|---|
| $R^{\smile}$ | eses | sese | lsel | rser | rele | srsl | rilr | lirl |

| $R$ | irrl | lril | llrf | lfrr | frrr | rrrb | lllb | flll |
|---|---|---|---|---|---|---|---|---|
| $R^{\smile}$ | rlir | illr | rfll | rrlf | rrfr | rbrr | lbll | llfl |

| $R$ | brll | rrbl | bbff | efbs | ifbi | sfsi | bfii | beie |
|---|---|---|---|---|---|---|---|---|
| $R^{\smile}$ | llbr | blrr | ffbb | bsef | biif | sisf | iibf | iebe |

| $R$ | bbbb | fefe | fifi | ebis | fbii | fsei | ibib | sbsb |
|---|---|---|---|---|---|---|---|---|
| $R^{\smile}$ | bbbb | fefe | fifi | iseb | iifb | eifs | ibib | sbsb |

| $R$ | ffff |
|---|---|
| $R^{\smile}$ | ffff |

$$R \circ S = \big\{ (x, z) \mid \exists y \, . \, (x, y) \in R, (y, z) \in S \big\}$$

Computing composition tables is non-trivial and will be the subject of Section 3. Generally, $b_1 \diamond b_2$ over-approximates the set-theoretic composition (i.e., $b_1 \diamond b_2 \supseteq b_1 \circ b_2$), while composition is said to be *strong* if approximation is exact, i.e., $b_1 \diamond b_2 = b_1 \circ b_2$. Strong composition has attracted some interest under the name *extensional composition* [2,26].

Note that neither $\mathcal{DRA}_{lr}$ nor $\mathcal{DRA}_c$ provide a jointly exhaustive set of base relations over the Euclidean plane. This leads to the lack of an identity relation in the case of $\mathcal{DRA}_{lr}$, and more severely, for both $\mathcal{DRA}_{lr}$ and $\mathcal{DRA}_c$, weak composition does not lead to an over-approximation (nor to an under-approximation)[14] of set-theoretic composition, because, e.g., ffbb is missing from the composition of llll with itself. This also means that we do not obtain a non-associative algebra for these calculi. By contrast, $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ provide jointly exhaustive and pairwise disjoint sets of base relations, and lead to non-associative algebras.

Let us now apply the relation-algebraic method to constraint reasoning. Given a qualitative calculus with set of base relations $\mathcal{B}$, a *constraint network* is a map $\nu : N \times N \to \mathcal{P}(\mathcal{B})$, where $N$ is a set of nodes (or variables) [29]. Individual constraints $\nu(X, Y) = R$ are written $XRY$, where $X$, and $Y$ are variables in $N$ and R is a relation in $\mathcal{P}(\mathcal{B})$. A constraint network $\nu : N \times N \to \mathcal{P}(\mathcal{B})$ is *atomic* or a *scenario* if each $\nu(X, Y)$ is a base relation.

Given a constraint network $\nu$, an important reasoning problem is to decide whether $\nu$ is *consistent*, i.e., whether there is an assignment of all variables of $\nu$ with dipoles such that all constraints are satisfied (a *solution*). We call this problem DSAT. DSAT is a constraint satisfaction problem (CSP) [31]. We rely on relation-algebraic methods to check consistency, namely the above mentioned path consistency algorithm. For non-associative algebras, the abstract composition of relations need not coincide with the (associative) set-theoretic composition. Hence, in this case, the standard path-consistency algorithm does not necessarily lead to path consistent networks, but only to algebraic closure, which is defined as follows [47]:

**Definition 2** (*Algebraic closure*). A constraint network over binary relations is called *algebraically closed* if for all variables $X_1, X_2, X_3$ and all relations $R_1, R_2, R_3$ the constraint relations

$$X_1 R_1 X_2, \qquad X_2 R_2 X_3, \qquad X_1 R_3 X_3$$

imply

$$R_3 \leqslant R_1 \diamond R_2.$$

Algebraic closure can be enforced by iterating

$$R_3 := R_3 \cap (R_1 \diamond R_2)$$

for $X_1 R_1 X_2$, $X_2 R_2 X_3$, $X_1 R_3 X_3$ until a fixed point is reached. Note that this procedure leaves the set of solutions of the constraint network invariant. This means that if the algebraic closure contains the empty relation, the original network is inconsistent.

However, in general, algebraic closure is only a one-sided approximation of consistency: if algebraic closure detects an inconsistency, then we are sure that the constraint network is inconsistent; however, algebraic closure may fail to detect some inconsistencies: an algebraically closed network is not necessarily consistent. For some calculi, like Allen's interval

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[14] Recall that generally, weak composition over-approximates composition, and exactly captures it in the case of strong composition.
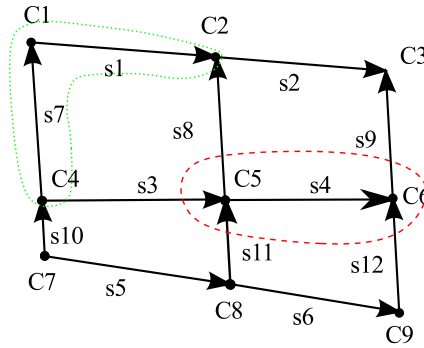
**Fig. 8.** A street network and two local observations.

algebra, algebraic closure is known to exactly decide the consistency of scenarios, but for others it does not, see [47], where it is also shown that this question is completely orthogonal to the question of whether the composition is strong. We will examine these questions for the dipole calculi in Sections 2.7 and 3.4 ff. below.

### 2.5. A sample application of algebraic closure reasoning with the dipole calculus

In this subsection, we want to demonstrate with an example why the dipole calculus is a useful qualitative model for directional information. Moreover, the example shows that composition-based reasoning is useful although it is incomplete. Our example uses the spatial knowledge expressed in $\mathcal{DRA}_{fp}$ for deductive reasoning based on constraint propagation (algebraic closure), resulting in the generation of useful indirect knowledge from partial observations in a spatial scenario. This is a direct application of the composition table which we generated based on our new condensed semantics for the dipole calculi (see Section 3).

In our sample application, a spatial agent (a simulated robot, cognitive simulation of a biological system, etc.) explores a spatial scenario. The agent collects local observations and wants to generate survey knowledge. Fig. 8 shows our spatial environment. It consists of a street network in which some streets continue straight after a crossing and some streets run parallel. These features are typical of real-world street networks. Spatial reasoning in our example uses constraint propagation (e.g., algebraic closure computation) to derive indirect constraints between the relative location of streets which are further apart than are the local observations between neighboring streets. The resulting survey knowledge can be used for several tasks including navigation tasks.

The environment is represented by streets $s_i$ and crossings $C_j$. The streets and crossings have unique names (e.g., $s_1, \ldots, s_{12}$, and $C_1, \ldots, C_9$ in the concrete example). The local observations are modeled in the following way, based on specific visibility rules (we want to simulate the prototypical features of visual perception): Both at each crossing and at each straight street segment we have an observation. At each crossing the agent observes the neighboring crossings. At the middle of each straight street segment the agent can observe the direction of the outgoing streets at the adjacent crossings (but not at their other ends). Two specific examples of observations are marked in Fig. 8. The observation "s1 errs s7" is marked at crossing C1. The observation "s8 rrllP s9" is marked at street s4.

These observations relate spatially neighboring streets to each other in a pairwise manner, using $\mathcal{DRA}_{fp}$ base relations. The agent has no additional knowledge about the specific environment. The spatial world knowledge of the agent is expressed in the converse and composition tables of $\mathcal{DRA}_{fp}$.

The following sequence of partial observations could be the result of a tour made by the spatial agent exploring the street network of our example (see Fig. 8):

```
Observations at crossings
C1: (s7 errs s1)
C2: (s1 efbs s2) (s8 errs s2) (s1 rele s8)
C3: (s2 rele s9)
C4: (s10 efbs s7) (s10 errs s3) (s7 srsl s3)
C5: (s3 efbs s4) (s11 efbs s8) (s11 errs s4) (s3 ells s8)
    (s3 rele s11) (s8 srsl s4)
C6: (s12 efbs s9) (s4 ells s9) (s4 rele s12)
C7: (s10 srsl s5)
C8: (s5 efbs s6) (s5 ells s11) (s11 srsl s6)
C9: (s6 ells s12)
```
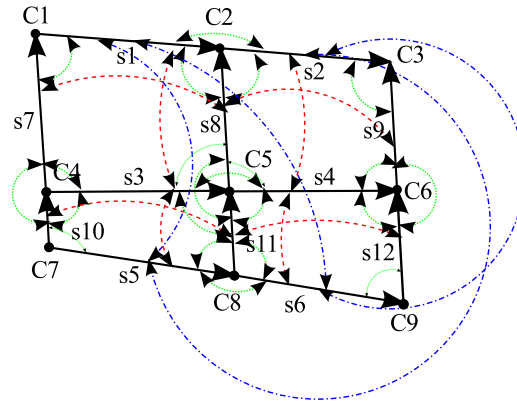
**Fig. 9.** All observation and resulting uncertainty marked with different colors.

```
Observations at streets
s1: (s7 rrllP s8)
s2: (s8 rrllP s9)
s3: (s10 rrllP s11)
s4: (s11 rrllP s12)
s8: (s3 llrr- s1)
s9: (s4 llrr- s2)
s10:(s3 rrll- s5)
s11:(s4 rrll- s6)
```

The result of the algebraic closure computation/constraint propagation is a refined network with the same solution set (the results are computed with the publicly available SparQ reasoning tool supplied with our newly computed $\mathcal{DRA}_{fp}$ composition table [58]). We have listed the results in the next Section 2.6. Three different models are the only remaining consistent interpretations (see Section 2.6 for a list of all the resulting data). The three different models agree on all but four relations. The solution set can be explained with the help of the diagram in Fig. 9. The input crossing observations are marked with green arrows, the input street observations are marked with red arrows. The result shows that for all street pairs which could not be observed directly, the algebraic closure algorithm deduces a strong constraint, i.e., precise information: typically, the resulting spatial relation between street pairs comprises just one $\mathcal{DRA}_{fp}$ base relation. The exception consists of four relations between streets in which the three models differ (marked with dashed blue arrows in Fig. 9). For these four relations, each model from the solution set agrees on the same $\mathcal{DRA}_f$ base relation for a given pair of dipoles, but the three consistent models differ on the finer granularity level of $\mathcal{DRA}_{fp}$ base relations. Since the refinement of one of these four underspecified relations into a single interpretation ($\mathcal{DRA}_{fp}$ base relation) as a logical consequence also assigns a single base relation to the other three relations, only three interpretations are valid models. The uncertainty/indeterminacy is the result of the specific street configuration in our example. The streets in a North–South direction are parallel, but the streets in an East–West direction are not parallel, resulting in fewer constraint composition results. However, the small solution set of consistent models agrees on most of the relative position relations between street pairs and the differences between the models are small. In our judgement, this means that the system has generated the relevant survey knowledge about the whole street network from local observations alone.

### 2.6. Computation for the street network application with the SparQ tool

In this section, we demonstrate how to use the publicly available SparQ QSR toolbox [58] to compute the algebraic closure by constraint propagation for the street network example from the previous Section 2.5. For successful relative position reasoning, the SparQ tool has to be supplied with our newly computed $\mathcal{DRA}_{fp}$ composition table which we generated based on our new condensed semantics for the dipole calculi (see Section 3).

The local street configuration observations by the spatial agent are listed in Section 2.5. The direct translation of these logical propositions into a SparQ spatial reasoning command looks as follows[15]:

```
sparq constraint-reasoning dra-fp path-consistency "( (s7 errs s1) (s1 efbs s2) (s8
errs s2) (s1 rele s8) (s2 rele s9) (s10 efbs s7) (s10 errs s3) (s7 srsl s3) (s3 efbs
s4) (s11 efbs s8) (s11 errs s4) (s3 ells s8) (s3 rele s11) (s8 srsl s4) (s12 efbs s9)
```

---

[15] For technical details of SparQ we refer the reader to the SparQ manual [58].

```
(s4 ells s9) (s4 rele s12) (s10 srsl s5) (s5 efbs s6) (s5 ells s11) (s11 srsl s6) (s6
ells s12) (s7 rrllP s8) (s8 rrllP s9) (s10 rrllP s11) (s11 rrllP s12) (s3 llrr- s1) (s4
llrr- s2) (s3 rrll- s5) (s4 rrll- s6) )"^16
```

The result of this reasoning command is a refined network with the same solution set derived by the application of the algebraic closure/constraint propagation algorithm (see Section 2.4). SparQ omits the converses for a more compact presentation.

```
   Modified network.
((S5 (EFBS) S6)(S12 (LSEL) S6)(S12 (LLFL) S5)(S11 (SRSL) S6)(S11 (LSEL) S5)(S11
(RRLLP) S12) (S4 (RRLL-) S6)(S4 (RRLL-) S5)(S4 (RELE) S12)(S4 (RSER) S11)(S3 (RRLL-)
S6)(S3 (RRLL-) S5) (S3 (RFLL) S12)(S3 (RELE) S11)(S3 (EFBS) S4)(S10 (RRBL) S6)(S10
(SRSL) S5)(S10 (RRLLP) S12) (S10 (RRLLP) S11)(S10 (RRRB) S4)(S10 (ERRS) S3)(S9 (LBLL)
S6)(S9 (LLLL-) S5)(S9 (BSEF) S12) (S9 (LLRRP) S11)(S9 (LSEL) S4)(S9 (LLFL) S3)(S9
(LLRRP) S10)(S8 (BRLL) S6)(S8 (LBLL) S5) (S8 (RRLLP) S12)(S8 (BSEF) S11)(S8 (SRSL)
S4)(S8 (LSEL) S3)(S8 (LLRRP) S10)(S8 (RRLLP) S9) (S2 (RRLL+ RRLL- RRLLP) S6)(S2
(RRLL+ RRLL- RRLLP) S5)(S2 (RRLF) S12)(S2 (RRFR) S11)(S2 (RRLL+) S4) (S2 (RRLL+)
S3)(S2 (RRRR+) S10)(S2 (RELE) S9)(S2 (RSER) S8)(S1 (RRLL+ RRLL- RRLLP) S6) (S1
(RRLL+ RRLL- RRLLP) S5)(S1 (RRLL+) S12)(S1 (RRLF) S11)(S1 (RRLL+) S4)(S1 (RRLL+) S3)
(S1 (RRFR) S10)(S1 (RFLL) S9)(S1 (RELE) S8)(S1 (EFBS) S2)(S7 (RRLL-) S6)(S7 (BRLL) S5)
(S7 (RRLLP) S12)(S7 (RRLLP) S11)(S7 (RRBL) S4)(S7 (SRSL) S3)(S7 (BSEF) S10)(S7 (RRLLP)
S9) (S7 (RRLLP) S8)(S7 (RRRB) S2)(S7 (ERRS) S1))
```

SparQ can output all path-consistent scenarios (i.e., constraint networks in base relations). For this constraint network, only three slightly different path consistent scenarios exist. They differ only in the following three relation subsets:

1. `(S2 (RRLLP) S6)(S2 (RRLLP) S5) (S1 (RRLLP) S6)(S1 (RRLLP) S5)`
2. `(S2 (RRLL-) S6)(S2 (RRLL-) S5) (S1 (RRLL-) S6)(S1 (RRLL-) S5)`
3. `(S2 (RRLL+) S6)(S2 (RRLL+) S5) (S1 (RRLL+) S6)(S1 (RRLL+) S5)`

All the other relations were already assigned a single base relation in the refined network which is shown above as a result of the application of the algebraic closure algorithm. This result can be visualized with a diagram and can be interpreted in terms of the goals of the reasoning task (see Section 2.5).

## 2.7. Limits of algebraic closure

We now consider the question of whether algebraic closure decides consistency. This has been open for $\mathcal{DRA}_f$. Recall from Section 2.4 that the set of constraints between all dipoles at hand is called a *constraint network*. If no constraint between two dipoles is given, we agree that they are in the universal relation. Further recall that by a *scenario*, we denote a constraint network in which all constraints are base relations.[17]

With the help of the embedding of the interval algebra into $\mathcal{DRA}_f$ (see Proposition 1), we can show that algebraic closure decides the consistency of $\mathcal{DRA}_f$ scenarios that only involve images of relations of the interval algebra. Moreover, for calculi such as RCC8 [46], the interval algebra [41], etc., (maximal) *tractable subsets* (see [50]) have been determined, i.e., sets of relations for which algebraic closure decides the consistency also of non-atomic constraint networks involving these relations. We then also obtain that algebraic closure in $\mathcal{DRA}_f$ decides the consistency of any constraint network involving (the image of) a maximal tractable subset of the interval algebra only. Similar remarks apply to $\mathcal{DRA}_{fp}$.

However, the situation changes if we move to the full calculus. The scenario consistency problem for the $\mathcal{DRA}_f$ calculus is already NP-hard, see [62], and hence algebraic closure (which is polynomial) does not decide scenario consistency in this case (assuming $P \neq NP$). This means that there are essentially no tractable subsets.

To illustrate the failure of algebraic closure to decide consistency, we now construct constraint networks which are geometrically unrealizable but still algebraically closed. This also gives some insight into the calculi: note that for the $\mathcal{LR}$ calculus, such counterexamples can be extremely simple [30], which is not the case here. We obtain such a counterexample by constructing constraint networks that are consistent and algebraically closed, and then we will change a relation in such a way that they remain algebraically closed but become inconsistent. We follow the approach of [52] in using a simple geometric shape for which scenarios exist, where algebraic closure fails to decide consistency. In our case, the basic shape is a convex hexagon (see Fig. 10).

---

[16] SparQ does not accept line breaks which we have inserted here for better readability. All the data for this sample application including the new composition table can be obtained from the URL http://www.informatik.uni-bremen.de/~till/Oslsa.tar.gz (which also provides the composition table and other data for the GQR reasoning tool https://sfbtr8.informatik.uni-freiburg.de/R4LogoSpace/Resources/).

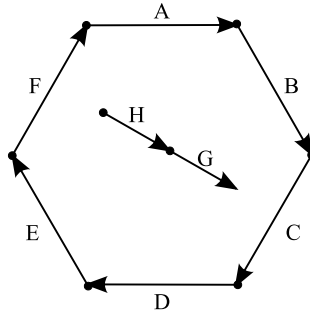[17] In this case, a base relation between every pair of distinct dipoles has to be provided.

**Fig. 10.** Convex hexagon.

First we will show that algebraic closure does not decide consistency for $\mathcal{DRA}_f$. Consider a convex hexagon consisting of the dipoles $A$, $B$, $C$, $D$, $E$ and $F$. Such an object is described as

(*A* errs *B*)(*B* errs *C*)(*C* errs *D*)(*D* errs *E*)(*E* errs *F*)(*F* errs *A*)(*F* rrrr *C*)

(*A* rrrr *D*)(*B* rrrr *E*)(*A* rrrr *C*)(*F* rrrr *D*)(*B* rrrr *D*)(*A* rrrr *E*)(*C* rrrr *E*)

(*B* rrrr *F*)

where the relations rrrr make sure that none of the dipoles intersect and together with the components $r$ of the relations errs ensure convexity, since they enforce an angle between 0 and $\pi$ between the respective first and second dipole, i.e., the endpoint of consecutive dipoles always lies to the right of the preceding dipole. Such an object is given in Fig. 10. Any object inside the hexagon lies to the right of all the dipoles, otherwise it is on the border or outside. To this scenario we add dipoles $G$ and $H$ inside the hexagon

(*F* rrrl *H*)(*C* rrlr *G*)(*H* efbs *G*)

that are collinear and such that the endpoint of $H$ is the startpoint of $G$. This gives us the constraint network

(*A* errs *B*)(*B* errs *C*)(*C* errs *D*)(*D* errs *E*)(*E* errs *F*)(*F* errs *A*)(*F* rrrr *C*)

(*A* rrrr *D*)(*B* rrrr *E*)(*A* rrrr *C*)(*F* rrrr *D*)(*B* rrrr *D*)(*A* rrrr *E*)(*C* rrrr *E*)

(*B* rrrr *F*)(*F* rrrl *H*)(*C* rrlr *G*)(*H* efbs *G*)

We construct an inconsistency by postulating that $H$ (i.e., its start- and endpoint) lies to the left of $E$, meaning that it lies outside the hexagon by introducing the constraint (*E* llrr *H*). By applying algebraic closure, we get a refinement of our network that does not contain the empty set:

(*H* efbs *G*)(*F* rrrl *G*)(*F* rrrl *H*)(*E* llrr *G*)(*E* llrr *H*)(*E* errs *F*)

(*D* (llrr rrrr) *G*)(*D* (llrr rrrr) *H*)(*D* rrrr *F*)(*D* errs *E*)(*C* rrlr *G*)

(*C* rrlr *H*)(*C* rrrr *F*)(*C* rrrr *E*)(*C* errs *D*)(*B* (llll rrll) *G*)

(*B* (llll rrll) *H*)(*B* rrrr *F*)(*B* rrrr *E*)(*B* rrrr *D*)(*B* errs *C*)

(*A* (llll rrll) *G*)(*A* (llll rrll) *H*)(*A* rser *F*)(*A* rrrr *E*)

(*A* rrrr *D*)(*A* rrrr *C*)(*A* errs *B*)

But $H$ has to lie to the left of $E$, meaning outside the convex hexagon and inside of it at the same time. This is impossible in the Euclidean plane. In fact, we can construct similar inconsistencies for several dipoles, just check the above constraint network for the relation llrr. Unfortunately algebraic closure with $\mathcal{DRA}_f$ does not decide consistency.

The constraint network can be extended to a $\mathcal{DRA}_{fp}$ constraint network in a straightforward manner by replacing rrrr by {rrrr+, rrrr-, rrrrA} and llrr by {llrr+, llrr-, llrrP}. Algebraic closure with $\mathcal{DRA}_{fp}$ then detects the inconsistency in the network. We drop the constraint (*E* (llrr+, llrr-, llrr P) *H*) and observe that the relation between $E$ and $H$ is refined to

$$\bigl( E \,(\text{rrrr+ rrrr- rrrrA}) \, H \bigr).$$

This constraint has no component that demands $H$'s being outside of the hexagon, as in the $\mathcal{DRA}_f$ case.

We have found an example that shows that algebraic closure for $\mathcal{DRA}_{fp}$ finds inconsistencies in constraint networks where it fails for $\mathcal{DRA}_f$. This leads to the question: Does algebraic closure decide consistency for $\mathcal{DRA}_{fp}$? We can give a negative answer also to this question.
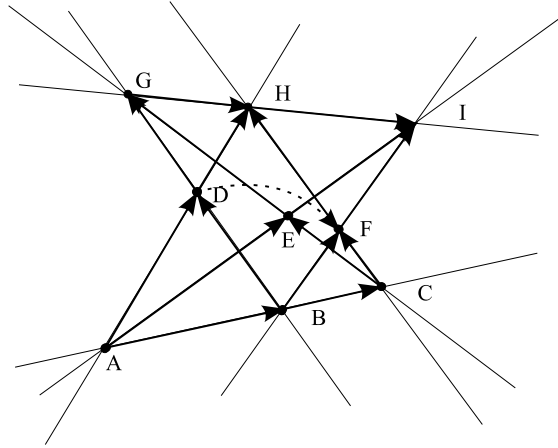
**Fig. 11.** Construction of the counterexample as a Pappus configuration.

To construct a counterexample, we begin with a point configuration with nine points $A, B, \ldots, I$ as in Fig. 11. This configuration corresponds to a Pappus configuration [5]. A Pappus configuration has nine points and nine straight lines. Eight collinearities of point triples: $GHI$, $ABC$, $ADH$, $AEI$, $BDG$, $BFI$, $CEG$, $CFH$ enforce the collinearity of the ninth point triple $DEF$ (by Pappus' Hexagon Theorem [5]). We can reconstruct this arrangement with dipoles and add an inconsistency with Pappus' Hexagon Theorem which is not detected with the algebraic closure for $\mathcal{DRA}_{fp}$.

The configuration from Fig. 11 can be described as a constraint network in the following way:

$(GH$ efbs $HI)(AB$ efbs $BC)$ $(AD$ efbs $DH)(AE$ efbs $EI)$ $(BD$ efbs $DG)$
$(BF$ efbs $FI)$ $(CE$ efbs $EG)$ $(CF$ efbs $FH)(DG$ errs $GH)(DG$ rele $EG)$
$(FI$ lere $HI)$ $(FI$ lere $EI)$ $(AD$ srsl $AE)$ $(AD$ srsl $AB)$ $(CF$ lsel $BC)$
$(CF$ slsr $CE)$ $(GH$ rele $DH)(GH$ rele $FH)$ $(AB$ ells $BF)$ $(AB$ ells $BD)$
$(AD$ ells $DG)(AD$ rele $BD)$ $(CF$ errs $FI)$ $(CF$ lere $BF)$ $(AE$ rele $CE)$

a dipole $XY$ in this description is the dipole from point $X$ to point $Y$. We observe that by Pappus' Hexagon Theorem the points $D$, $E$ and $F$ are collinear. We now add a constraint

$(AE$ (lrrr lrrl) $DF)$

that states that the carrier lines of $AE$ and $DF$ intersect between $A$ and $E$ or in front of $AE$, but not in $E$. But since $D$, $E$ and $F$ are collinear, the only possible intersection point is $E$, a contradiction. Any scenario based on this constraint network cannot be consistent. But applying algebraic closure with $\mathcal{DRA}_{fp}$ yields a refinement and dozens of possible scenarios, e.g.,

$(FH$ rser $DF)(CF$ rele $DF)$ $(CF$ efbs $FH)$ $(EG$ rrrr+ $DF)$
$(EG$ rrll- $FH)$ $(EG$ brll $CF)$ $(CE$ rrlr $DF)$ $(CE$ rrbl $FH)$
$(CE$ srsl $CF)$ $(CE$ efbs $EG)$ $(FI$ rser $DF)$ $(FI$ slsr $FH)$
$(FI$ rser $CF)$ $(FI$ llrr+ $EG)$ $(FI$ rlrr $CE)$ $(BF$ rele $DF)$
$(BF$ ells $FH)$ $(BF$ rele $CF)$ $(BF$ lllr $EG)$ $(BF$ rllr $CE)$
$(BF$ efbs $FI)$ $(DG$ rrrr+ $DF)(DG$ rrllP $FH)$ $(DG$ rrllP $CF)$
$(DG$ rele $EG)$ $(DG$ rrlf $CE)$ $(DG$ rrll- $FI)$ $(DG$ brll $BF)$
$(BD$ rrlr $DF)$ $(BD$ rrllP $FH)$ $(BD$ rrllP $CF)$ $(BD$ rfll $EG)$
$(BD$ rrll+ $CE)$ $(BD$ rrbl $FI)$ $(BD$ srsl $BF)$ $(BD$ efbs $DG)$
$(EI$ lrrr $DF)$ $(EI$ rllr $FH)$ $(EI$ rrlr $CF)$ $(EI$ slsr $EG)$
$(EI$ rser $CE)$ $(EI$ rele $FI)$ $(EI$ rrlf $BF)$ $(EI$ llrr+ $DG)$
$(EI$ rlrr $BD)$ $(AE$ lrrr $DF)$ $(AE$ rlll $FH)$ $(AE$ rrll+ $CF)$
$(AE$ ells $EG)$ $(AE$ rele $CE)$ $(AE$ rfll $FI)$ $(AE$ rrll+ $BF)$
$(AE$ llllr $DG)$ $(AE$ rllr $BD)$ $(AE$ efbs $EI)$ $(DH$ rrrrA $DF)$
$(DH$ rele $FH)(DH$ rrlf $CF)$ $(DH$ rll $EG)$ $(DH$ rrlr $CE)$
$(DH$ rrll+ $FI)$ $(DH$ rrll+ $BF)$ $(DH$ slsr $DG)$ $(DH$ rser $BD)$
$(DH$ rrll- $EI)$ $(DH$ brll $AE)$ $(AD$ rrrrA $DF)(AD$ rfll $FH)$
$(AD$ rrll+ $CF)(AD$ rlll $EG)$ $(AD$ rrll+ $CE)$ $(AD$ rrll+ $FI)$
$(AD$ rrll+ $BF)(AD$ ells $DG)$ $(AD$ rele $BD)$ $(AD$ rrbl $EI)$

| | | | |
|---|---|---|---|
| (*AD* srsl *AE*) | (*AD* efbs *DH*) | (*BC* llll- *DF*) | (*BC* lllb *FH*) |
| (*BC* ells *CF*) | (*BC* lllb *EG*) | (*BC* ells *CE*) | (*BC* llbr *FI*) |
| (*BC* slsr *BF*) | (*BC* llbr *DG*) | (*BC* slsr *BD*) | (*BC* llrr+ *EI*) |
| (*BC* blrr *AE*) | (*BC* llrr+ *DH*) | (*BC* blrr *AD*) | (*AB* llrl *DF*) |
| (*AB* llll+ *FH*) | (*AB* flll *CF*) | (*AB* llll+ *EG*) | (*AB* flll *CE*) |
| (*AB* lllb *FI*) | (*AB* ells *BF*) | (*AB* lllb *DG*) | (*AB* ells *BD*) |
| (*AB* llbr *EI*) | (*AB* slsr *AE*) | (*AB* llbr *DH*) | (*AB* slsr *AD*) |
| (*AB* efbs *BC*) | (*HI* lrrr *DF*) | (*HI* rser *FH*) | (*HI* rrfr *CF*) |
| (*HI* rbrr *EG*) | (*HI* rrrr+ *CE*) | (*HI* rele *FI*) | (*HI* rrlf *BF*) |
| (*HI* rbrr *DG*) | (*HI* rrrr+ *BD*) | (*HI* rele *EI*) | (*HI* rrlf *AE*) |
| (*HI* rser *DH*) | (*HI* rrfr *AD*) | (*HI* rrllP *BC*) | (*HI* rrllP *AB*) |
| (*GH* lrrr *DF*) | (*GH* rele *FH*) | (*GH* rrlf *CF*) | (*GH* rser *EG*) |
| (*GH* rrfr *CE*) | (*GH* rfll *FI*) | (*GH* rrll+ *BF*) | (*GH* rser *DG*) |
| (*GH* rrfr *BD*) | (*GH* rfll *EI*) | (*GH* rrll+ *AE*) | (*GH* rele *DH*) |
| (*GH* rrlf *AD*) | (*GH* rrllP *BC*) | (*GH* rrllP *AB*) | (*GH* efbs *HI*) |

For $\mathcal{DRA}_{fp}$ also, algebraic closure does not decide consistency even for scenarios. This counterexample can also be used for $\mathcal{DRA}_f$, but the above is simpler and shows differences in the reasoning effectiveness of algebraic closure for $\mathcal{DRA}_{fp}$ and $\mathcal{DRA}_f$.

## 3. A condensed semantics for the dipole calculus

Since the domains of most spatial calculi are infinite (e.g., the Euclidean plane), it is impossible just to enumerate all possible configurations relative to the composition operation when deriving a composition table.[18] Hence, the question remains how a composition table can be computed in an effective and automatic way. To start, we tried generating the composition table of $\mathcal{DRA}_f$ directly, using the resulting quadratic inequalities as described in [36] and derived exhaustively on p. 2105. However, it turned out that it is infeasible to base the reasoning on these inequalities, even with the aid of interactive theorem provers such as Isabelle/HOL [43] and HOL-light [20] (the latter is dedicated to proving facts about real numbers) and Gröbner base reasoners.[19] This infeasibility is probably related to the above-mentioned NP-hardness of the consistency problem for dipole base relations.

Therefore, we developed a qualitative abstraction instead which we call *condensed semantics*. It provides a level of abstraction from the metrics of the underlying space. We observe the Euclidean plane with respect to all possible line configurations that are distinguishable within the $\mathcal{DRA}$ calculi.

From a more formal point of view, a key insight is that two configurations are qualitatively different if they cannot be transformed into each other by maps that keep that part of the spatial structure invariant that is essential for the calculus. In our case, these maps are the (orientation preserving) affine bijections. A set of configurations that can be transformed into each other by appropriate maps is an *orbit* of a suitable automorphism group. Here, we use primarily the affine group $\mathbf{GA}(\mathbb{R}^2)$ and show in detail how this leads to qualitatively different spatial configurations. The results of this analysis can be mapped onto an efficient method for computing the composition tables for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$.

### 3.1. Seven qualitatively different configurations

For the binary composition operation of $\mathcal{DRA}$ calculi, we have to consider all qualitatively different configurations of three lines. In order to formalize "qualitatively different configurations," we regard the $\mathcal{DRA}$ calculus as a first-order structure, with the Euclidean plane as its domain, together with all the base relations. Let us start with having a look at the automorphism groups for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$.

**Definition 3.** An affine map $f$ from the Euclidean plane to itself is given by a $2 \times 2$ transformation matrix $A$ and a translation vector $(b_x, b_y)$ such that

---

[18] It can be shown that the exhaustive inspection of a finite number of configurations in a finite grid would suffice to compute the composition table of the dipole calculi. The size of the grid needs to be double-exponential in the number of points [17], and therefore the number of grids to consider is triple-exponential. This is practically infeasible: even for three points, already $2^{2^{23}} \approx 10^{77}$ grids would need to be inspected.

[19] For the computation of the $\mathcal{DRA}_c$ composition table reported in [36], Gröbner base reasoning needed to be complemented by a grid method. In general, the research history of QSR about dipoles shows that it is necessary to use methods that yield more reliable results. The dipole composition on which we focus in this section involves configurations of three dipoles. However, even the much simpler question about a complete list of distinguishable dipole base relations characterized by certain properties (e.g., dipole to point relations) is not trivial. This question can be answered by configurations of just two dipoles and how to list them exhaustively. Deriving manually the 72 base relations of $\mathcal{DRA}_f$, or the 80 base relations of $\mathcal{DRA}_{fp}$, is an error-prone procedure. For this reason, the manually derived sets of base relations for the finer-grained dipole calculi described in [36,53], as well as the composition tables, contained errors.

$$f(x, y) = A \begin{pmatrix} x \\ y \end{pmatrix} + (b_x, b_y)$$

$f$ is a bijection iff $\det(A)$ is non-zero, and $f$ is orientation preserving iff $\det(A)$ is positive.

**Proposition 4.** *All orientation preserving affine bijections are $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ automorphisms.*

(In [39], the converse is also shown.)

**Proof.** It suffices to show that orientation preserving affine bijections preserve the $\mathcal{LR}$ relations. Now, any orientation preserving affine bijection is a product of translations, rotations, scalings and shears. It is straightforward to see that these mappings preserve the $\mathcal{LR}$ relations. □

Automorphisms and their compositions form a group which acts on the set of points (and tuples of points, lines, etc.) by function application. Recall that if a group $G$ acts on a set, an *orbit* consists of the set reachable from a fixed element by performing the action of all group elements: $G x = \{ f(x) \mid f \in G \}$. The importance of this notion is the following:

Two configurations which are qualitatively different belong to disjoint orbits of the automorphism group.

Note that while this is related to the theory of line arrangements [19], we here work in a slightly different setting. First, the theory of line arrangements uses a weaker notion of isomorphism than we do. Second, work about line arrangements mostly uses the projective plane where there are only two configurations of three lines, instead of the Euclidean plane where parallelism is possible. Here we are only interested in the Euclidean plane and have to distinguish the cases where two or more lines are parallel or even identical. The reason is that, e.g., $\mathcal{DRA}_{fp}$ distinguishes between $A$ llrrP $B$ ($A$ and $B$ point into the same direction and have distinct parallel carrier lines) and $A$ and $B$ being in some Allen relation ($A$ and $B$ point into the same direction and have the same carrier line). Third, we also consider triples of lines (later on), not just sets of three lines.

Further note that Cristani's 2DSLA calculus [6], which can be used to reason about sets of lines, is too coarse for our purposes: our orbits (1) and (2) introduced below cannot be distinguished in 2DSLA.

We start with configurations consisting of from one up to three lines in the Euclidean plane, i.e., we consider the orbits of all sets $\{l_1, l_2, l_3\}$ where $l_1$, $l_2$ and $l_3$ are not necessarily distinct. We consider two such configurations to be isomorphic if they can be mapped into each other by an affine bijection. That is, we work with orbits in the group of *all* affine bijections (and not just the orientation preserving ones—orientations will come in at a later stage). This group is usually called the affine group of $\mathbb{R}^2$ and denoted by **GA**($\mathbb{R}^2$).

A line in the Euclidean plane is given by the set of all points $(x, y)$ for which $y = mx + b$. Given three lines $y = m_i x + b_i$ ($i = 1, 2, 3$), we list their orbits by giving a defining property. In each case, it is fairly obvious that the defining property is preserved by affine bijections. Moreover, in each case, we show a *transformation property*, namely that given two instances of the defining properties, the first instance can be transformed into the second by an affine bijection. Together, this means that the defining property exactly specifies an orbit. The transformation property often follows from the following basic facts about affine bijections, see [14]:

1. An affine frame [14] is for an affine space what a basis is for a vector space; in particular, any point of an affine space is a unique affine combination of points from the frame. An affine frame for an $n$-dimensional affine space consists of $n + 1$ points; in particular, an affine frame for the Euclidean plane is a point triple in general position. The importance of this notion in the present context is the following: An affine bijection is uniquely determined by its action on an affine frame, the result of which is given by another affine frame. Hence, given any two point triples in the Euclidean plane in general position, there is a unique affine bijection mapping the first point triple to the second.
2. Affine maps transform lines into lines.
3. Affine maps preserve parallelism of lines.

That is, it suffices to show that an instance of the defining property is determined by three points in general position and drawing lines and parallel lines.

We will consider the intersection of line $i$ with line $j$ ($i \neq j \in \{1, 2, 3\}$). This is given by the system of equations

$$\{y = m_i x + b_i, \ y = m_j x + b_j\}$$

This does not cover the case of the line $x = 0$; however, without loss of generality, we can assume that this case does not occur: we always can apply an appropriate affine bijection mapping the three lines away from the line $x = 0$.

For $m_i \neq m_j$, the above system of equations has a unique solution:

$$x = -\frac{b_i - b_j}{m_i - m_j}, \qquad y = \frac{m_i b_j - m_j b_i}{m_i - m_j}$$

For $m_i = m_j$, there is either is no solution ($b_i \neq b_j$; the lines are parallel), or there are infinitely many solutions ($b_i = b_j$; the lines are identical).

We can now distinguish seven cases:

1. All $m_i$ are distinct and the three systems of equations $\{y = m_i x + b_i, \ y = m_j x + b_j\}$ ($i \neq j \in \{1, 2, 3\}$) yield three different solutions. Geometrically, this means that all three lines intersect with three different intersection points. The transformation property follows from the fact that the three intersection points determine the configuration. In the theory of line arrangements, this is called a *simple arrangement* [19].
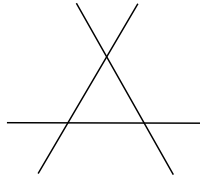
2. All $m_i$ are distinct and at least two of the three systems of equations $\{y = m_i x + b_i, \ y = m_j x + b_j\}$ ($i \neq j \in \{1, 2, 3\}$) have a common solution. Then, obviously, the single solution is common to all three equation systems. Geometrically, this means that all three lines intersect at the same point. In the theory of line arrangements, this is called a *trivial arrangement* [19].
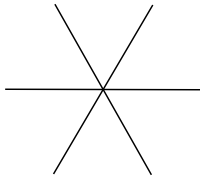
Take this point and a second point on one of the lines. By drawing parallels through this second point, we obtain two more points, one on each of the other two lines, such that the four points form a parallelogram. The transformation property now follows from the fact that any two non-degenerate parallelograms can be transformed into each other by an affine bijection.
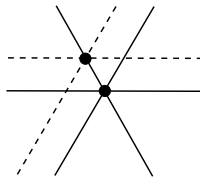
3. $m_i = m_j \neq m_k$ and $b_i \neq b_j$ for distinct $i, j, k \in \{1, 2, 3\}$. Geometrically, this means that two lines are parallel but not coincident, and the third line intersects them. Such a configuration is determined by three points: the points of intersection, plus a further point on one of the parallel lines. Hence, the transformation property follows.
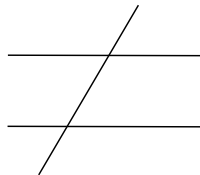
4. $m_i = m_j \neq m_k$ and $b_i = b_j$ for distinct $i, j, k \in \{1, 2, 3\}$. Geometrically, this means that two lines are equal and the third one intersects them. Again, such a configuration is determined by three points: the intersection point plus a further point on each of the (two) different lines. Hence, the transformation property follows.
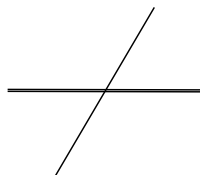
5. All $m_i$ are equal, but the $b_i$ are distinct. Geometrically, this means that all three lines are parallel, but not coincident. We cannot show the transformation property here, which means that this case comprises several orbits. Actually, we get one orbit for each distance ratio

$$\frac{b_1 - b_2}{b_1 - b_3}$$

An affine bijection

$$f(x, y) = A\begin{pmatrix} x \\ y \end{pmatrix} + (b_x, b_y)$$

transforms a line $y = mx + b$ to $y = m'x + b'$, with $b' = c_1(m)b + c_2(m)$, where $c_1$ and $c_2$ depend nonlinearly on $m$. However, since $m = m_1 = m_2 = m_3$, this nonlinearity does not matter. This means that

$$\frac{b'_1 - b'_2}{b'_1 - b'_3} = \frac{c_1(m)b_1 - c_1(m)b_2}{c_1(m)b_1 - c_1(m)b_3} = \frac{b_1 - b_2}{b_1 - b_3}$$

i.e., the distance ratio is invariant under affine bijections (which is well-known in affine geometry). Given a fixed distance ratio, we can show the transformation property: three points suffice to determine two parallel lines, and the position of the third parallel line is then determined by the distance ratio. For a distance ratio 1, this configuration looks as follows:
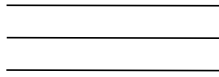
Actually, for the qualitative relations between dipoles placed on parallel lines, their distance ratio does not matter. Hence, we will ignore distance ratios when computing the composition table below, and the fact that we get infinitely many orbits does not matter.

6. All $m_i$ are equal and two of the $b_i$ are equal but different from the third. Geometrically, this means that two lines are coincident, and the third one is parallel but not coincident. Such a configuration is determined by three points: two points on the coincident lines and a third point on the third line. Hence, the transformation property follows.

7. All $m_i$ are equal, and the $b_i$ are equal as well. This means that all three lines are equal. The transformation property is obvious.

Since we have exhaustively distinguished the various possible cases based on relations between the $m_i$ and $b_i$ parameters, this describes all possible orbits of three lines under the action of the group of affine bijections. Although we get infinitely many orbits for case (5), in contexts where the distance ratio introduced in case (5) does not matter, we will speak of seven qualitatively different configurations, and it is understood that the infinitely many orbits for case (5) are conceptually combined into one equivalence class of configurations.

Recall that we have considered *sets* of (up to) three lines. If we consider *triples* of lines instead, cases (3) to (6) split up into three sub-cases, because they feature distinguishable lines. We then get 15 different configurations, which we name 1, 2, 3a, 3b, 3c, 4a, 4b, 4c, 5a, 5b, 5c, 6a, 6b, 6c and 7. While 5a, 5b and 5c correspond to case (5) above and therefore comprise infinitely many orbits, the remaining configurations comprise a single orbit.

The next split appears at the point when we consider qualitatively different configurations of triples of unoriented lines with respect to *orientation preserving* affine bijections. An affine map $f(x, y) = A\begin{pmatrix} x \\ y \end{pmatrix} + (b_x, b_y)$ is orientation preserving if $\det(A)$ is positive. In the above arguments, we now have to consider oriented affine frames. Let us call an affine frame $(p_1, p_2, p_3)$ positively $(+)$ oriented, if the angle $\angle p_1 p_2 p_3$ is positive, otherwise, it is negatively $(-)$ oriented. Two given affine frames with the same orientation determine a unique orientation preserving affine bijection transforming the first one into the second. Thus, the orientation of the affine frame matters, and hence cases 1 and 2 above are split into two sub-cases each. For all the other cases, we have the freedom to choose the affine frames so that their orientations coincide. In the end, we get 17 different orbits of triples of oriented lines: 1+, 1-, 2+, 2-, 3a, 3b, 3c, 4a, 4b, 4c, 5a, 5b, 5c, 6a, 6b, 6c and 7. They are shown in Fig. 12.

The structure of the orbits already gives us some insight into the nature of the dipole calculus. The fact that sub-case (1) corresponds to one orbit means that neither angles nor ratios of angles can be measured in the dipole calculus. By way of contrast, the presence of infinitely many orbits in sub-case (5) means that ratios of distances in a specific direction
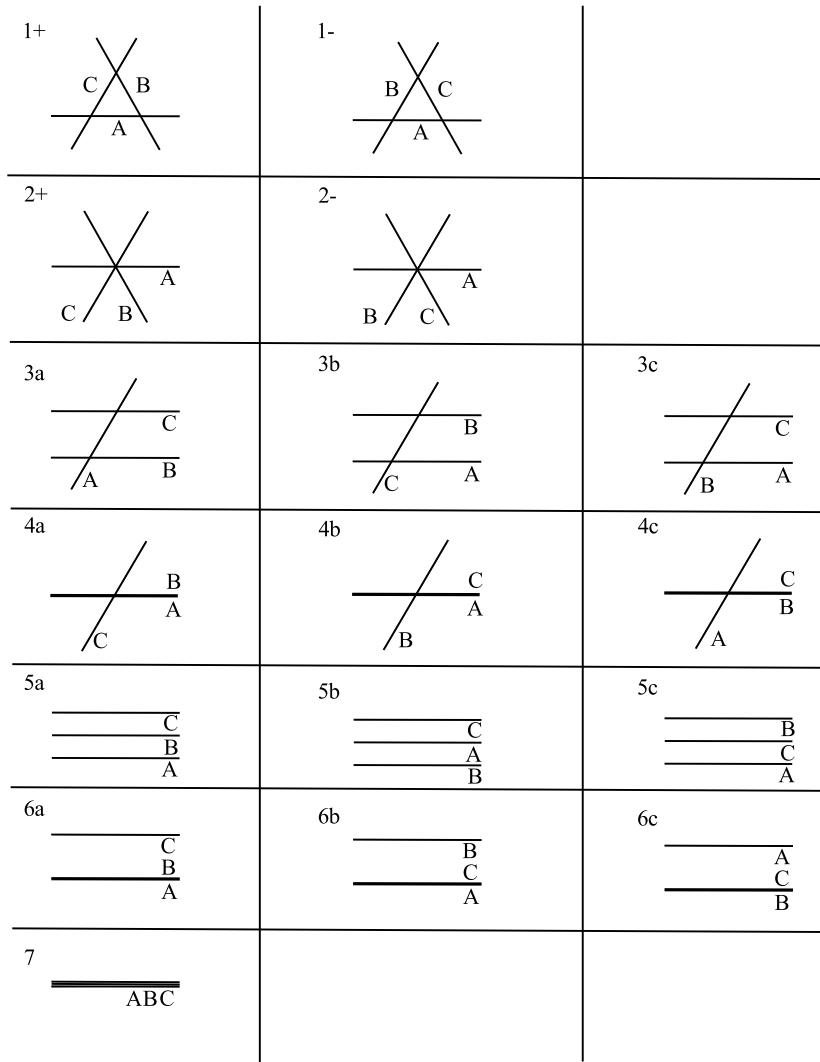
**Fig. 12.** The 17 qualitatively different configurations of triples of oriented lines w.r.t. orientation preserving affine bijections.

(not distances themselves) *can* be measured in the dipole calculus. Indeed, in $\mathcal{DRA}_{fp}$, it is even possible to replicate a given distance arbitrarily many times, as indicated in Fig. 13. That is, $\mathcal{DRA}_{fp}$ can be used to generate a one-dimensional coordinate system. Note however that, due to the lack of well-defined angles, a two-dimensional coordinate system cannot be constructed. The ability to "count" in the $\mathcal{DRA}_{fp}$ calculus stems from the existence of relations able to capture the feature of parallelism. Consider a sequence of parallelograms described in $\mathcal{DRA}_{fp}$ as

$(A_i$ ells $B_i)$   $(A_i$ llrrP $C_i)$  $(A_i$ slsr $D_i)$  $(A_i$ lere $E_i)$
$(B_i$ lere $C_i)$   $(B_i$ llrrP $D_i)$  $(B_i$ lsel $E_i)$
$(C_i$ rser $D_i)$   $(C_i$ srsl $E_i)$
$(D_i$ errs $E_i)$.

Such a sequence is depicted in Fig. 13. The counting can be established by replicating such parallelograms by adding relations

$(B_i$ sese $D_{i+1})$

which claim that $B_i$ and $D_{i+1}$ of two consecutive parallelograms coincide. Such parallelograms can be constructed with all relations describing parallelism. The construction for anti-parallelism is a little more involved, in this case sequences of two parallelograms will be replicated.
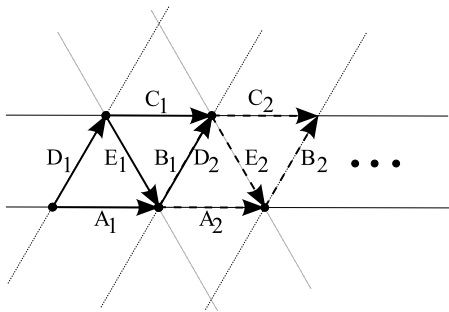
**Fig. 13.** Example for replication of a given distance in $\mathcal{DRA}_{fp}$.
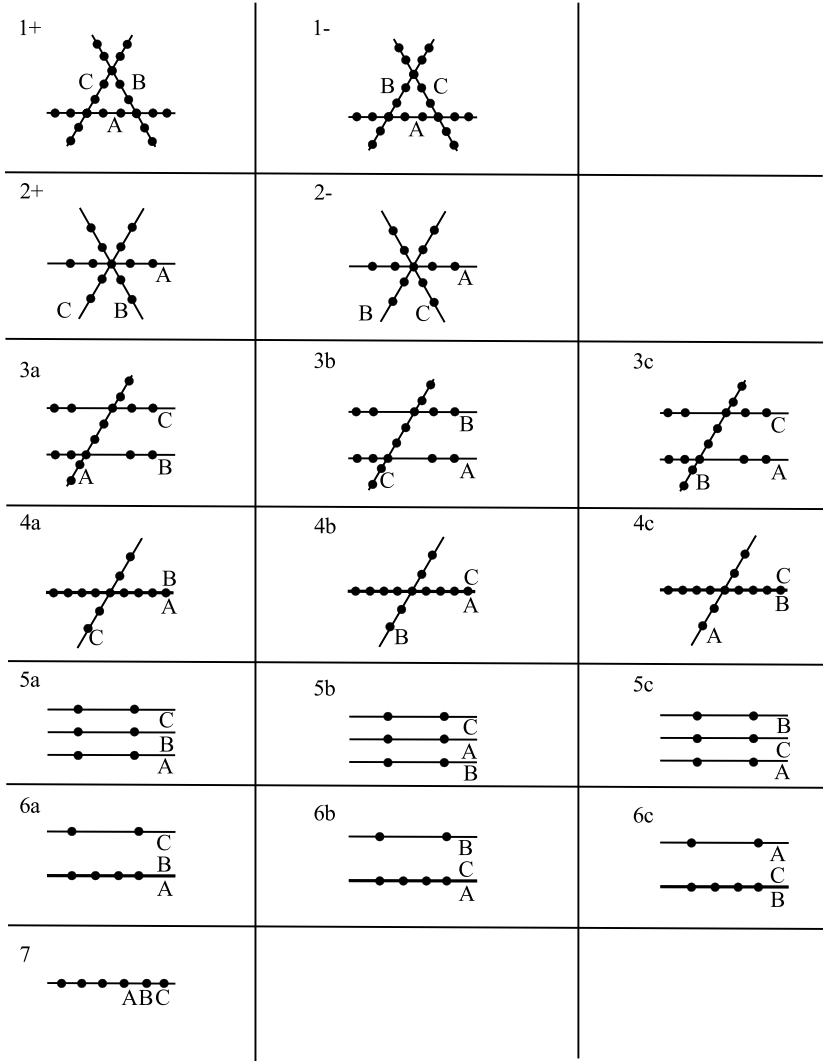


**Fig. 14.** The 17 canonical configurations.

### 3.2. Computing the composition table with condensed semantics

The purpose of condensed semantics is to provide a way of computing composition tables in a finite way. Therefore, we need to reduce the infinite space of possible dipole configurations to a finite one. For each of the 17 oriented orbits in Fig. 12 we introduce a *canonical configuration* in the Euclidean plane (depicted in Fig. 14), i.e., a configuration with a suitable number of positions for the start and end points of the dipoles on each line that suffice to compute the composition

table. The number of points needed is a function of the unoriented orbits, we call these points (that are displayed in Fig. 14) *prototypical points*. We call a configuration from Fig. 14 with three assigned dipoles a *prototypical configuration*. The computation of the composition table needs the orientation in order to be exhaustive.

---

**Algorithm 1** Composition table

---

Computation of composition table for $\mathcal{DRA}$

1: *Conf* := the set of prototypical configurations from Fig. 14
2: $R := \emptyset$
3: **for all** configurations $c \in Conf$ **do**
4:   **for all** dipoles $d_A = (\mathbf{s}_A, \mathbf{e}_A)$ of different prototypical points on $A$ in $c$ **do**
5:     **for all** dipoles $d_B = (\mathbf{s}_B, \mathbf{e}_B)$ of different prototypical points on $B$ in $c$ **do**
6:       **for all** dipoles $d_C = (\mathbf{s}_C, \mathbf{e}_C)$ of different prototypical points on $C$ in $c$ **do**
7:         compute the relations $d_A \, R_1 \, d_B$, $d_B \, R_2 \, d_C$, $d_A \, R_3 \, d_C$ by the formula on p. 2106 and add the triple $(R_1, R_2, R_3)$ to $R$
8:       **end for**
9:     **end for**
10:   **end for**
11: **end for**
12: collect the triples in $R$ in such a way that there is exactly one entry for every $R_1$ and $R_2$ having the union of all $R_3$ as third component

---

The algorithm for computing the composition table is given in Algorithm 1. We place the configurations of Fig. 14 into an arbitrary orthogonal coordinate system. Each configuration provides a finite number of prototypical points with specific coordinates, which serve as start and end points of prototypical dipoles. For each triple of such prototypical dipoles we compute the $\mathcal{DRA}$-relations using the atan2-method as described in Section 2.3. Each triple that is obtained in this way corresponds to an entry in the composition table.

A program has been implemented in Java that uses Algorithm 1 and on a notebook with an Intel Core 2 T7200 with 1.5 Gbyte of RAM, the computation of the composition tables for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ takes approximately 14 s.

*3.3. Soundness and completeness of composition*

By the soundness of Algorithm 1 we mean that the computed composition table contains enough entries to make it over-approximate geometric reality (i.e., no false conclusions can be drawn by qualitative reasoning). Conversely, completeness means that there are not more entries than necessary, that is, the composition table does not lead to overly weak conclusions. (However note that even in case of completeness it still can be the case that *algebraic closure* leads to overly weak conclusions, e.g., inconsistencies are not detected, see Section 2.7.)

More specifically, completeness means that Algorithm 1 outputs only triples of dipole relations that are geometrically realizable, while soundness means that it outputs all such triples.[20] Soundness and completeness together imply that prototypical dipole triples are representative for all dipole triples, at least for what concerns dipole relations.

**Proposition 5.** *Algorithm* 1 *is complete.*

**Proof.** Easy, since the triples of dipole relations are generated from prototypical dipole triples, which provide geometric realizations. □

Showing the soundness of Algorithm 1 is more involved. We need to identify a lower bound of points that is needed on our oriented orbits in Fig. 12 with respect to the $\mathcal{DRA}$ semantics. We can identify those lower bounds for intersecting and collinear lines separately.

In a first step, we consider collinear lines. For soundness of the construction, we need to show that for two or three dipoles on the same line there is a lower bound for the number of prototypical points needed to distinguish between the possible $\mathcal{DRA}$ relations on a line.

Consider any configuration of collinear $n \in \{2, 3\}$ dipoles $A$, $B$ (and $C$). We use an order induced by $\mathbf{e}_A < \mathbf{s}_A$ on the line, i.e., if $B$ points into the same direction as $A$, we have $\mathbf{e}_B < \mathbf{s}_B$, otherwise $\mathbf{s}_B < \mathbf{e}_B$, and the same for $C$. This construction reflects the fact that dipoles always have non-zero length.

We translate the 13 Allen relations and the "opposite" Allen relations componentwise into our order for two dipoles $A$ and $B$:

$$A \, b \_{\_\_} \, B \mapsto \mathbf{s}_A < \mathbf{s}_B \wedge \mathbf{e}_A < \mathbf{s}_B$$
$$A \, s \_{\_\_} \, B \mapsto \mathbf{s}_A = \mathbf{s}_B \wedge \mathbf{e}_A < \mathbf{s}_B$$

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[20] Actually, the algorithm will output many triples more than once; these duplicates could be filtered out.
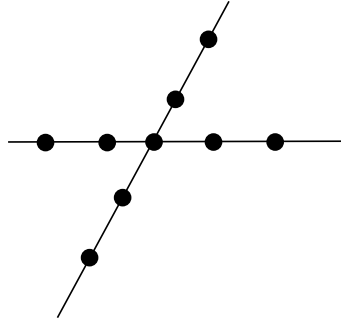
**Fig. 15.** Intersecting carrier lines.

$$A\,i\,{}_{---}\,B \mapsto \mathbf{s}_B < \mathbf{s}_A \wedge \mathbf{e}_A < \mathbf{s}_B$$
$$A\,e\,{}_{---}\,B \mapsto \mathbf{s}_B < \mathbf{s}_A \wedge \mathbf{e}_A = \mathbf{s}_B$$
$$A\,f\,{}_{---}\,B \mapsto \mathbf{s}_B < \mathbf{s}_A \wedge \mathbf{s}_B < \mathbf{e}_A$$

and likewise for the other components of the relations in question.

**Example 6.** Consider the relation ($A$ bbff $B$). Since both dipoles point into the same direction, we can derive $\mathbf{e}_A < \mathbf{s}_A \wedge \mathbf{e}_B < \mathbf{s}_B$. Now, we apply the translation rules for each component:

$$\mathbf{s}_A < \mathbf{s}_B \ \wedge\ \mathbf{e}_A < \mathbf{s}_B \ \wedge$$
$$\mathbf{s}_A < \mathbf{e}_B \ \wedge\ \mathbf{e}_A < \mathbf{e}_B \ \wedge$$
$$\mathbf{s}_A < \mathbf{e}_B \ \wedge\ \mathbf{s}_A < \mathbf{s}_B \ \wedge$$
$$\mathbf{e}_A < \mathbf{s}_B \ \wedge\ \mathbf{e}_A < \mathbf{e}_B$$

we observe that the overall inequalities can be simplified to

$$\mathbf{e}_A < \mathbf{s}_A \ \wedge\ \mathbf{e}_B < \mathbf{s}_B \ \wedge$$
$$\mathbf{s}_A < \mathbf{s}_B \ \wedge\ \mathbf{e}_A < \mathbf{s}_B \ \wedge$$
$$\mathbf{s}_A < \mathbf{e}_B \ \wedge\ \mathbf{e}_A < \mathbf{e}_B$$

By transitivity of $<$, we can derive $\mathbf{e}_A < \mathbf{s}_A < \mathbf{e}_B < \mathbf{s}_B$. Hence we need at least four points in the plane to realize this dipole relation.

By an easy induction, we can show

**Lemma 7.** *For n collinear dipoles in the Euclidean plane, $2 \cdot n$ points that can be the start and end points of the dipoles suffice to constitute all possible $\mathcal{DRA}$ relations between those dipoles.*

**Corollary 8.** *Realizing the relations between* 1 (2, 3) *collinear dipoles in the planes requires* 2 (4, 6) *prototypical points in the plane.*

After having considered the number of prototypical points needed for collinear dipoles, we need to do the same for dipoles with intersecting carrier lines. For this purpose we need to consider the semantics of the $\mathcal{DRA}$ relations. The only case in which a point can lie on both intersecting lines is when it is positioned on the point of intersection. This is the only case where in this scenario relations can have a component from $b$, $s$, $i$, $e$, $f$, since these relations require one dipole's being collinear with the start or end point of the other dipole. So we need to place a prototypical point onto the point of intersection. On each line on each side of the point of intersection, the rules for collinear lines are applied. Fig. 15 shows the case for no collinear lines.

**Lemma 9.** *Transforming a scenario of dipoles along an orientation preserving affine transformation preserves the $\mathcal{DRA}_f$ ($\mathcal{DRA}_{fp}$) relations.*

**Proof.** This follows directly from Proposition 4. □

For the soundness proof, we need some preparatory lemmas.

**Lemma 10.** *Transforming a scenario in three dipoles along an orientation preserving affine transformation preserves betweenness of points.*
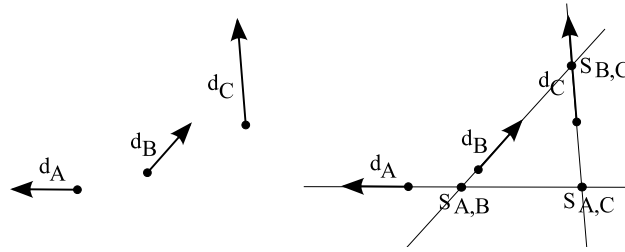
**Fig. 16.** Introduction of carrier lines.

**Proof.** This follows from the proof of Proposition 4 for the $\mathcal{LR}$-relation $i$. $\square$

Analogously to the qualitative angle in $\mathcal{DRA}_{fp}$, we define a *qualitative orientation* between two dipoles.

**Definition 11.** Given two non-parallel dipoles $A$ and $B$, we say that the *qualitative orientation* from $A$ to $B$ is $+$ if the angle from $A$ to $B$ is positive, otherwise it is $-$.

**Lemma 12.** *Given a fixed dipole $B$ and a fixed intersection point $S_{AB}$, the relation $A\,R\,B$ is determined by betweenness and equality among $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$, and the qualitative orientation between $B$ and $A$ provided that $R$ does not involve parallelism or anti-parallelism.*

**Proof.** Let $A\,R\,B$, and let $A'$ be such that $S_{AB} = S_{A'B}$ and that betweenness and equality among $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$, and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, S_{AB}\}$ are the same, and the qualitative orientations from $B$ to $A$ and from $B$ to $A'$ are also the same. We introduce carrier rays for the dipoles called $l_A$, $l_{A'}$ and $l_B$. Without loss of generality, the rays point in the same direction as the dipoles, and hence reflect the qualitative orientation. The rays $l_A$ and $l_{A'}$ are divided into three segments by $S_{AB}$ and $S_{A'B}$ respectively. For $l_A$ these are the segments with points $x <_r S_{AB}$, $x =_r S_{AB}$ and $S_{AB} <_r x$, and $l_{A'}$ is segmented in the same way, where we call the order $<_{r'}$. The relation $R$ can be decomposed into the four $\mathcal{LR}$-relations:

$$(A\,R_1\,\mathbf{s}_B)\ (A\,R_2\,\mathbf{e}_B)\ (B\,R_3\,\mathbf{s}_A)\ (B\,R_4\,\mathbf{e}_A).$$

First we will consider the relations $R_3$ and $R_4$. By definition of the $\mathcal{LR}$-relations, the relations between $B$ and $\mathbf{s}_A$ or $\mathbf{e}_A$ change if the respective point is moved into a different segment, but since the betweenness and equality among $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$ are the same and the qualitative orientations also coincide, if $\mathbf{s}_A <_r S_{AB}$, so is $\mathbf{s}_{A'} <_r S_{A'B}$ and the same for the other segments and $\mathbf{e}_A$ and $\mathbf{e}_{A'}$. Hence, we obtain that $R_3 = R_3'$ and $R_4 = R_4'$. For $R_1$ and $R_2$, we use a similar argument with the roles of $A'$ or $A$ and $B$ swapped. $\square$

**Proposition 13.** *Algorithm 1 is sound.*

**Proof.** We will first give this proof for $\mathcal{DRA}_{fp}$ since soundness for $\mathcal{DRA}_f$ follows from soundness for $\mathcal{DRA}_{fp}$ by uniting particular relations. Given any triple of dipoles $(d_A, d_B, d_C)$ in the Euclidean plane, we inspect their carrier lines $(A, B, C)$ and the intersection points of the latter to identify their oriented orbit from Fig. 12. As an example consider the configuration of dipoles in Fig. 16 on the left-hand side. The configuration on the right-hand side shows the carrier lines and we can identify three different points of intersection. Together with the orientation of the lines, we see that this configuration lies in orbit $1-$.

We can identify the relations $R_{AB}$, $R_{BC}$ and $R_{AC}$ in that scenario. By $S_{XY}$ we denote the point of intersection of the carrier lines $X$ and $Y$. We call the lines in the corresponding configuration from Fig. 14 $A'$, $B'$ and $C'$, and we will find respective dipoles $d_{A'}$, $d_{B'}$ and $d_{C'}$ based on the prototypical points. We will show that the dipole relations for $d_A$, $d_B$, $d_C$ and $d_{A'}$, $d_{B'}$, $d_{C'}$ are the same.

Note that on collinear lines the number of minimally needed points per section has been shown in Corollary 8. In any of the following cases of the proof, we need to consider all possible choices of points.

$1+$ & $1-$)   In this orbit, three distinct points of intersection exist, denoted by $S_{AB}$, $S_{AC}$, $S_{BC}$ for the dipole configuration and $S_{A'B'}$, $S_{A'C'}$, $S_{B'C'}$ for the prototypical configuration. Since both triples $A$, $B$, $C$ and $A'$, $B'$, $C'$ are in the same oriented orbit, there is an orientation preserving affine bijection $h$ between them, mapping $A$, $B$, $C$, $S_{AB}$, $S_{AC}$ and $S_{BC}$ to their primed variants. By Lemma 10 the point sets $\{\mathbf{s}_A, \mathbf{e}_A, S_{AC}\}$ and $\{\mathbf{s}_A, \mathbf{e}_A, S_{A'C'}\}$ are ordered in corresponding ways, and so are all other all other point sets involving the start and end points of dipoles and an intersection point. The points of the dipoles $d_A$, $d_B$ and $d_C$ are not necessarily mapped onto $d_{A'}$, $d_{B'}$ and $d_{C'}$, but the order between the start and end points and points of intersection is the same. By Lemma 12, only order and qualitative orientation has an influence on the $\mathcal{DRA}_{fp}$ relations at hand, so the mapped start and end points can be just moved onto the ones $d_{A'}$, $d_{B'}$ and $d_{C'}$ without changing the $\mathcal{DRA}_{fp}$ relations.

2+ & 2−)  In this case all points of intersection coincide. This is the only difference between cases 1+ and 1−, but the argument stays the same.

3*a* & 3*b* & 3*c*)  In this case we have parallel lines. First we consider case 3a. Here, the line $A$ is intersected by the parallel lines $B$ and $C$. Since $A$, $B$ and $C$ are in the same oriented orbit as $A'$, $B'$ and $C'$, there is an orientation preserving affine transformation between them. By case 3 on p. 2115, choose the transformation $h$ in such a way that it takes the affine frame $\{x_b, S_{AC}, S_{AB}\}$ to $\{x_{b'}, S_{A'C'}, S_{A'B'}\}$. The point $x_b$ is chosen as $\mathbf{s}_b$ if $\mathbf{s}_b \neq S_{AB}$ and $\mathbf{e}_b$ otherwise. $x_{b'}$ is defined analogously. By Lemma 10 betweenness of $\{\mathbf{s}_A, \mathbf{e}_A, S_{AC}\}$ and $\{\mathbf{s}_C, \mathbf{e}_C, S_{AC}\}$ is preserved by $h$. The preservation of betweenness of the respective point-triples implies two possible orders. We introduce an order as in the proof of Lemma 12. The points $\mathbf{s}_X$ and $\mathbf{e}_X$ are not necessarily mapped onto $\mathbf{s}_{X'}$ and $\mathbf{e}_{X'}$ but the order with respect to the intersection points is the same. So the points in the image of $h$ can be moved onto the respective prototypical points without affecting the dipole relations. On the parallel lines, the relation is preserved, since the direction of the dipoles is preserved (by the order with respect to the point of intersection). The preservation of betweenness is true for the triples $\{\mathbf{s}_A, \mathbf{e}_A, S_{AC}\}$, $\{\mathbf{s}_C, \mathbf{e}_C, S_{AC}\}$, $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$ and $\{\mathbf{s}_B, \mathbf{e}_B, S_{AB}\}$. By the preservation of the betweenness and orientation, the relations with respect to $d_A$ are also the same. The argument for 3b and 3c is analogous.

4*a* & 4*b* & 4*c*)  First we consider case 4a. We only have one point of intersection $S_{AB} = S_{AC} = S_{BC} = S$. There is an orientation preserving affine transformation $h$ that takes the scenario to the instance shown in Fig. 12, since both configurations are in the same orbit. We need to consider several triples of points on the line: $\{S, \mathbf{s}_A, \mathbf{e}_A\}$, $\{S, \mathbf{s}_B, \mathbf{e}_B\}$, $\{S, \mathbf{s}_A, \mathbf{s}_B\}$, $\{S, \mathbf{s}_A, \mathbf{e}_B\}$, $\{S, \mathbf{s}_B, \mathbf{e}_A\}$, and $\{S, \mathbf{e}_B, \mathbf{e}_A\}$ as well as $\{S, \mathbf{s}_C, \mathbf{e}_C\}$. By Lemma 10 the betweenness of the triples is preserved under $h$. From this betweenness of the triples and the orientation, we construct an order between the points. Since there are two possibilities to establish an order from the betweenness of a triple, we need to construct the order as in the proof of Lemma 12. If the dipoles on the coinciding line point into different directions, we can still construct an overall compatible order, by inverting one of the orders induced by $A$ or $B$, which of them we invert is arbitrary. With this mapping we can determine a setup of prototypical points on this configuration and move the mapped points onto them. This does not change the involved $\mathcal{DRA}_{fp}$ relations, since the order of the points is preserved in this operation. Since the order of points on the line is not changed, we get in both cases the same relation and for the intersection we set the same relation by Lemma 12. The cases 4b and 4c are proved analogously.

5*a* & 5*b* & 5*c*)  We do not have any points of intersection in this case. Without loss of generality we assume the ratio of the distances between the parallel lines to be 1. First, we have a look at case 5a. We intersect the lines $A$, $B$ and $C$ with an additional line orthogonal to $A$ in such a way that no intersection points are equal to any of the start or end points of any dipole. We call the points of intersection $S_A$, $S_B$ and $S_C$. There is an orientation preserving affine transformation to the instance of 5a given in Fig. 12, since both configurations are in the same orbit. The order of the triples $\{S_X, \mathbf{s}_X, \mathbf{e}_X\}$ with $X \in \{A, B, C\}$ is preserved. Again we can move the points in the image of $h$ to the prototypical points and get in both cases the same relations, since they just depend on the order of the points. The cases 5b and 5c are treated analogously.

6*a* & 6*b* & 6*c*)  We start with case 6a. We intersect the lines with a new one that is orthogonal to $A$ and intersects all carrier lines in such a way that all points on $A$, $B$ and $C$ are on the same side of the new line. We call the points of intersection $S_A$, $S_B$ and $S_C$. Again there is an orientation preserving affine transformation to the representative of the orbit. The respective orders of the start and endpoints of the dipoles and points of intersection are preserved. And the start and endpoints of the dipoles can be moved to the respective prototypical points without changing the dipole relation. As in case 5a the relations stay the same. Cases 6b and 6c are treated analogously.

7)  We intersect the lines with a new one orthogonally (and do the same with the representative of the orbit) in such a way that the point is intersection is different from all start and end points of the dipoles and call this point of intersection $S$. There is an orientation preserving affine transformation $h$ that maps the configuration of the orbit. There are several triples of points on the line we need to consider: $\{S, \mathbf{s}_A, \mathbf{e}_A\}$, $\{S, \mathbf{s}_B, \mathbf{e}_B\}$, $\{S, \mathbf{s}_C, \mathbf{e}_C\}$, $\{S, \mathbf{s}_A, \mathbf{s}_B\}$, $\{S, \mathbf{s}_A, \mathbf{s}_C\}$, $\{S, \mathbf{s}_A, \mathbf{e}_B\}$, $\{S, \mathbf{s}_A, \mathbf{e}_C\}$, $\{S, \mathbf{s}_B, \mathbf{s}_C\}$, $\{S, \mathbf{s}_B, \mathbf{e}_C\}$, $\{S, \mathbf{e}_B, \mathbf{s}_C\}$, $\{S, \mathbf{e}_B, \mathbf{e}_C\}$, whose order, which is constructed as in the proof of Lemma 12. As in step 4a & 4b & 4c we can make the orders compatible, if dipoles point into different directions. From the above list, we can infer that six prototypical points are needed to compute all dipole relations between three collinear dipoles. Again we can move the mapped points onto the prototypical ones without any harm, since only the relative ordering of the points matters by the definition of the $\mathcal{DRA}_{fp}$ relations.

For $\mathcal{DRA}_f$, we just take the union of the refined relations, i.e., we use the mapping

$$\{LLLL+, LLLL-, LLLLA\} \mapsto LLLL$$

$$\{LLRR+, LLRR-, LLRRP\} \mapsto LLRR$$

$$\{RRLL+, RRLL-, RRLLP\} \mapsto RRLL$$

$$\{RRRR+, RRRR-, RRRRA\} \mapsto RRRR. \qquad \square$$
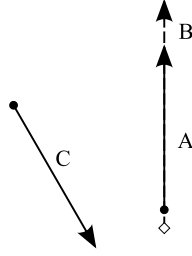
**Fig. 17.** $\mathcal{DRA}_f$ weak composition.

### 3.4. Algebraic properties of composition

We now investigate several properties of the composition tables for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$. For both tables the properties

$$id^{\smile} = id$$
$$\left(R^{\smile}\right)^{\smile} = R$$
$$id \circ R = R$$
$$R \circ id = R$$
$$(R_1 \circ R_2)^{\smile} = R_2^{\smile} \circ R_1^{\smile}$$
$$R_1^{\smile} \in R_2 \circ R_3 \quad \Longleftrightarrow \quad R_3^{\smile} \in R_1 \circ R_2$$

hold with $R$, $R_1$, $R_2$, $R_3$ being any base relations and $id$ the identical relation. These properties can be automatically tested by the GQR and SparQ qualitative reasoners. The other properties for a non-associative algebra follow trivially. Furthermore, we have tested the associativity of the composition. For $\mathcal{DRA}_f$, we have 373,248 triples of relations to consider of which 71,424 are not associative. So the composition of 19.14% of all possible triples of relations is not associative,[21] e.g., associativity is violated in the compositions:

$$(rrrl \diamond rrrl) \diamond llrl \neq rrrl \diamond (rrrl \diamond llrl).$$

For $\mathcal{DRA}_{fp}$ all 512,000 triples of base relations are associative w.r.t. composition. Hence $\mathcal{DRA}_{fp}$ is a relation algebra.

### 3.5. $\mathcal{DRA}_f$ composition is weak

The failure of $\mathcal{DRA}_f$ to be associative implies that its composition is weak. We will still prove directly that $\mathcal{DRA}_f$ has weak composition by giving an example, since this is more illustrative:

**Proposition 14.** *The composition of $\mathcal{DRA}_f$ is weak.*

**Proof.** Consider the $\mathcal{DRA}_f$ composition $A$ bfii $B \diamond B$ lllb $C \mapsto A$ llll $C$. We show that there are dipoles $A$ and $C$ such that there is no dipole $B$ which reflects the composition. Consider dipoles $A$ and $C$ as shown in Fig. 17. We observe that they are in the $\mathcal{DRA}_{fp}$ relation llll- with the dipole $C$ pointing towards the line $l_A$ dipole $A$ lies on. Because of $A$ bfii $B$, dipole $B$ has to lie on $l_A$. But, since $l_C$, the carrier line of $C$, is a straight line and lines $l_A$ and $l_B$ lie in front of $C$ with respect to the direction of the dipole, the endpoint of $B$ cannot lie behind $C$. $\quad\square$

$\mathcal{DRA}_{fp}$ behaves differently, as shown in the next section.

### 3.6. Strong composition

We are now going to prove that $\mathcal{DRA}_{fp}$ has strong composition. The following lemma will be crucial; note that it does *not* hold for $\mathcal{DRA}_f$.
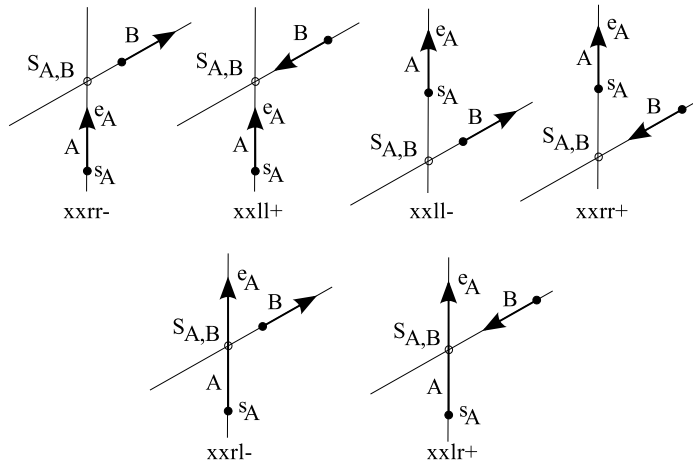
**Lemma 15.** *For $\mathcal{DRA}_{fp}$ base relations R not involving parallelism or anti-parallelism, betweenness and equality among $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$ for given dipoles $A$ R $B$ are independent of the choice of A and B, hence uniquely determined by R alone.*

---

[21] In the masters thesis of one of our students, a detailed analysis of a specific non-associative dipole configuration is presented [38].

**Proof.** Let $R = r_1 r_2 r_3 r_4 r_5$, where $r_5 \in \{+, -\}$ even if $r_5$ is omitted in the standard notation. Note that the assumption $r_5 \in \{+, -\}$ implies that $S_{AB}$ is defined. If $r_3 \in \{b, s, i, e, f\}$, $\mathbf{e}_A \neq \mathbf{s}_A = S_{AB}$, and there is no betweenness. Analogously, $\mathbf{s}_A \neq \mathbf{e}_A = S_{AB}$ if $r_4 \in \{b, s, i, e, f\}$. The remaining possibilities for $r_3 r_4 r_5$ are:

1. ll+, rr-: in these cases, $\mathbf{e}_A$ is between $\mathbf{s}_A$ and $S_{AB}$;
2. ll-, rr+: in these cases, $\mathbf{s}_A$ is between $\mathbf{e}_A$ and $S_{AB}$;
3. rl-, lr+: in these cases, $S_{AB}$ is between $\mathbf{s}_A$ and $\mathbf{e}_A$.

Note that cases 1 and 2 cannot be distinguished in $\mathcal{DRA}_f$. In particular, the pictures for xxll+ and xxll- lead to the same $\mathcal{DRA}_f$ relation xxll, but for xxll+, $\mathbf{e}_A$ is between $\mathbf{s}_A$ and $S_{AB}$, while for xxll-, $\mathbf{s}_A$ is between $\mathbf{e}_A$ and $S_{AB}$.



**Corollary 16.** *Let $R$ be a $\mathcal{DRA}_{fp}$ base relation not involving parallelism or anti-parallelism. Let $A\,R\,B$ and $A'\,R\,B'$. Then, the map $\{\mathbf{s}_A \mapsto \mathbf{s}_{A'}; \mathbf{e}_A \mapsto \mathbf{e}_{A'}; S_{AB} \mapsto S_{A'B'}\}$ preserves betweenness and equality.*

**Theorem 17.** *Composition in $\mathcal{DRA}_{fp}$ is strong.*

**Proof.** Obviously, strong composition $\circ$ is contained in weak composition $\diamond$. To show the converse, let $r_{ac} \in r_{ab} \diamond r_{bc}$ be an entry in the composition table, with $r_{ac}$, $r_{ab}$ and $r_{bc}$ base relations. We need to show that $r_{ac} \in r_{ab} \circ r_{bc}$, i.e., that for any given dipoles $A$ and $C$ with $A r_{ac} C$, there exists a dipole $B$ with $A r_{ab} B$ and $B r_{bc} C$.

Since $r_{ac} \in r_{ab} \diamond r_{bc}$, by definition of weak composition, we know that there are dipoles $A'$, $B'$ and $C'$ with $A' r_{ab} B'$, $B' r_{bc} C'$ and $A' r_{ac} C'$. Given dipoles $X$ and $Y$, let $S_{XY}$ denote the point of intersection of the lines carrying $X$ and $Y$; it is only defined if $X$ and $Y$ are not parallel. Consider now the three lines carrying $A'$, $B'$ and $C'$, respectively. According to the results of Section 3.1, for the configuration of these three lines, there are seventeen qualitatively different cases 1+, 1-, 2+, 2-, 3a, 3b, 3c, 4a, 4b, 4c, 5a, 5b, 5c, 6a, 6b, 6c and 7:

1. We consider cases 1+ and 1- simultaneously. Recall that all line configurations in the orbit 1+ have the same orientation, and the same holds for 1-. The three points of intersection $S_{A'B'}$, $S_{B'C'}$ and $S_{A'C'}$ exist and are different. Since $A r_{ac} C$ and $A' r_{ac} C'$, by Lemma 15, the point sets $\{\mathbf{s}_A, \mathbf{e}_A, S_{AC}\}$ and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, S_{A'C'}\}$ are ordered in corresponding ways on their lines. Hence, it is possible to choose a point $S_{AB}$ on the carrier line of $A$ in such a way that the point sets $\{\mathbf{s}_A, \mathbf{e}_A, S_{AC}, S_{AB}\}$ and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, S_{A'C'}, S_{A'B'}\}$ are ordered in corresponding ways on their lines. In a similar way (interchanging $A$ and $C$), $S_{BC}$ can be chosen.
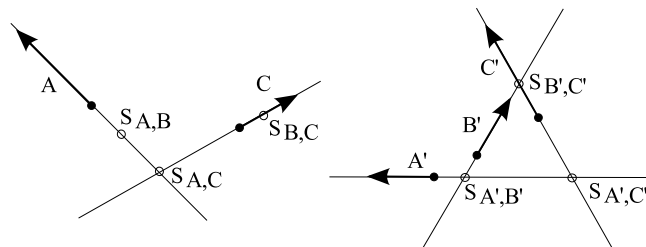


Since both $\{S_{AB}, S_{AC}, S_{BC}\}$ and $\{S_{A'B'}, S_{A'C'}, S_{B'C'}\}$ are affine frames, there is a unique affine bijection $h : \mathbb{R}^2 \to \mathbb{R}^2$ with $h(S_{A'B'}) = S_{AB}$, $h(S_{A'C'}) = S_{AC}$ and $h(S_{B'C'}) = S_{BC}$. Since all line configurations in the orbit 1+ have the same

orientation (and the same holds for 1-), $h$ preserves orientation. Thus by Proposition 4 the $\mathcal{DRA}_{fp}$ relations are also preserved along $h$. Hence, by choosing $B = h(B')$, we get $h(A')r_{ab}B$ and $Br_{bc}h(C')$. Since the point sets $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$ and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, S_{A'B'}\}$ are ordered in corresponding ways on their lines and $h$ is an affine bijection, also $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$ and $\{h(\mathbf{s}_{A'}), h(\mathbf{e}_{A'}), S_{AB}\}$ are ordered in corresponding ways on their lines, and moreover the qualitative orientation for $A$ to $B$ is the same as that from $A'$ to $B$. Since also $S_{AB} = S_{A'B}$, by Lemma 12, from $h(A')r_{ab}B$ we thus get $Ar_{ab}B$. A similar argument shows that $Br_{bc}C$.

2. We prove cases 2+ and 2- simultaneously. The three intersection points $S_{A'B'}$, $S_{B'C'}$ and $S_{A'C'}$ exist and coincide, i.e., $S_{A'B'} = S_{B'C'} = S_{A'C'} =: S'$. Let $S := S_{AC}$. Let $x_A$ be $\mathbf{s}_A$ and $x_{A'}$ be $\mathbf{s}_{A'}$ if $\mathbf{s}_A \neq S$ (and therefore $\mathbf{s}_{A'} \neq S'$), otherwise, let $x_A$ be $\mathbf{e}_A$ and $x_{A'}$ be $\mathbf{e}_{A'}$. $x_C$ and $x_{C'}$ are chosen in a similar way. Since both $\{S, x_A, x_C\}$ and $\{S', x_{A'}, x_{C'}\}$ are affine frames, there is a unique affine bijection $h : \mathbb{R}^2 \to \mathbb{R}^2$ with $h(S') = S$, $h(x_{A'}) = x_A$ and $h(x_{C'}) = x_C$. The rest of the argument is similar to case (1).

3. (Two lines are parallel and intersect with the third one.) In the sequel, we will just specify how two affine frames are chosen; the rest of the argument (as well as the choice of points on the unprimed side in such a way that qualitative relations are preserved) is then similar to the previous cases.
   Subcases (3a), (3b): The lines carrying $A$ and $C$ intersect. Choose $x_A$ and $x_{A'}$ as in case (2), and chose an appropriate point $S_{BC}$. Then use the affine frames $\{x_A, S_{AC}, S_{BC}\}$ and $\{x_{A'}, S_{A'C'}, S_{B'C'}\}$.
   Subcase (3c): The lines carrying $A$ and $C$ are parallel. Choose appropriate points $S_{AB}$ and $S_{BC}$ and use the affine frames $\{\mathbf{s}_A, S_{AB}, S_{BC}\}$ and $\{\mathbf{s}_{A'}, S_{A'B'}, S_{B'C'}\}$.

4. (Two lines are identical and intersect with the third one.)
   Subcases (4a) and (4b): The lines carrying $A$ and $C$ intersect. Choose $x_A$, $x_{A'}$, $x_C$ and $x_{C'}$ as in case (2) and use the affine frames $\{S_{AC}, x_A, x_C\}$ and $\{S_{A'C'}, x_{A'}, x_{C'}\}$.
   Subcase (4c): The lines carrying $A$ and $C$ are identical. This means that $S_{A'B'} = S_{A'C'} =: S'$. Choose an appropriate point $S$ and $x_A$, $x_{A'}$ as in case (2). Moreover, in a similar way, choose $x_{B'} \neq S'$, and then some corresponding $x_B$ being in the same $\mathcal{LR}$-relation to $A$ as $x_{B'}$ has to $A'$. Then use the affine frames $\{S, x_A, x_B\}$ and $\{S, x_{A'}, x_{B'}\}$.

5. (All three lines are distinct and parallel.) Subcases (5a), (5b) and (5c) can all be treated in the same way: Use the affine frames $\{\mathbf{s}_A, \mathbf{e}_A, \mathbf{s}_C\}$ and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, \mathbf{s}_{C'}\}$. Note that the distance ratios may need to adjusted by a non-affine transformation which however preserves the dipole relations.

6. (Two lines are identical and are parallel to the third one.)
   Subcases (6a) and (6b): The lines carrying $A$ and $C$ are parallel. Proceed as in case (5).
   Subcase (6c): The lines carrying $A$ and $C$ are identical. Choose some $\mathbf{s}_B$ in the same $\mathcal{LR}$-relation to $A$ as $\mathbf{s}_{B'}$ is to $A'$. Then use the affine frames $\{\mathbf{s}_A, \mathbf{e}_A, \mathbf{s}_B\}$ and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, \mathbf{s}_{B'}\}$.

7. (All three lines are identical.) For this case, the result follows from the fact that Allen's interval algebra has strong composition (refer to [47]). $\quad\square$

## 4. Summary and conclusion

We have presented different variants of qualitative spatial reasoning calculi about oriented straight line segments which we call dipoles. These spatial calculi provide a basis for representing and reasoning about qualitative position information in intrinsic reference systems.

We have computed the composition table for dipole calculi by a new method based on the algebraic semantics of the dipole relations. We have used what we have called a condensed semantics which uses the orbits of the affine group $\mathbf{GA}(\mathbb{R}^2)$ to provide an abstract notion of qualitative configuration of lines. This can be used to compute the composition table in a computer-assisted way, relying on prototypical dipole configurations that are located on lines as given by a qualitative configuration represented by an orbit.

This has been the first computation of the composition table for $\mathcal{DRA}_{fp}$. So far, the only composition tables for $\mathcal{DRA}_c$ and $\mathcal{DRA}_f$ that exist contain many errors [56]. We also have analyzed the algebraic features of the various dipole calculi. We have proved that $\mathcal{DRA}_{fp}$ has strong composition. This is an interesting result because in this case an application-motivated calculus extension has benefits for the algebraic features of this calculus extension.

We have demonstrated a prototypical application of reasoning about qualitative position information in relative reference systems. In this scenario about cognitive spatial agents and qualitative map building, coarse locally perceived street configuration information has to be integrated by constraint propagation in order to get survey knowledge. The well-known path consistency method (the more precise term in the present context is algebraic closure) which is implemented with standard QSR tools can make use of our new dipole calculus composition table and compute the desired result in polynomial time. Such concrete but generalizable application scenarios for relative position calculi are more important after the recent result by Wolter and Lee [62] which shows that relative position calculi are intractable even in base relations. For this reason, it is necessary to gain experience as to in which application contexts the unavoidably approximate reasoning is effective and produces relevant inference results. With our street network example, we have a test case which puts an emphasis on deriving implicit knowledge as the output of qualitative spatial reasoning based on observed data. This is a prototypical application scenario which in the future can also be applied to other relative position calculi.

Since the observed data in the case of error-free perception leads to consistent input constraints, the general consistency problem can be avoided: we instead rely on logical consequence. Now both problems are intractable and need to be ap-

proximated using algebraic closure; however, in our setting, the losses due to approximation are less harmful, since we do not risk working with inconsistent scenarios.

Our future work will address the question of how in general the quality of approximations for relative position reasoning can also be assessed with quantitative measures. An important open question is whether the problem of the consistency of constraint networks can be better approximated in polynomial time than through the algebraic closure algorithm. Concerning exponential time algorithms for consistency, our condensed semantics may be generalized to constraint networks of arbitrary size, using a suitable method for determining the possible line arrangements. Another part of our future QSR research will apply our new condensed semantics method to other calculi.

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