Available online at www.sciencedirect.com

ScienceDirect

Artificial Intelligence 172 (2008) 1195–1218

Artificial Intelligence

www.elsevier.com/locate/artint

The measurement of ranks and the laws of iterated contraction

Matthias Hild [a], Wolfgang Spohn [b,*,1]

[a] Darden Graduate School of Business Administration, PO Box 6550, Charlottesville, VA 22906, USA
[b] Department of Philosophy, University of Konstanz, 78457 Konstanz, Germany

Received 2 February 2007; received in revised form 28 December 2007; accepted 14 March 2008

Available online 20 March 2008

Abstract

Ranking theory delivers an account of iterated contraction; each ranking function induces a specific iterated contraction behavior. The paper shows how to reconstruct a ranking function from its iterated contraction behavior uniquely up to multiplicative constant and thus how to measure ranks on a ratio scale. Thereby, it also shows how to completely axiomatize that behavior. The complete set of laws of iterated contraction it specifies amend the laws hitherto discussed in the literature.

Keywords: Ranking theory; Belief revision theory; Difference measurement; Contraction; Iterated contraction

1. Introduction

Ranking theory, as first presented in Spohn ([30, Section 5.3] and [31]) is well known to offer a complete model of the dynamics of belief, i.e., it allows to state an arbitrarily iterable rule of belief change. By contrast, AGM belief revision theory, as summarized by Gärdenfors [13], founders at the problem of iterated belief change, as observed in Spohn ([30, Section 5.2] and [31, Section 3]), because it violates the principle of categorical matching, as Gärdenfors, Rott [14, p. 37] called it later on. Both theories agree, though, on single belief changes.

There is a price to pay for the greater strength of ranking theory; it makes substantial use of numerical degrees of (dis-)belief. While one can well see how the dynamics of belief works on the basis of these degrees, one may wonder about the meaning of these degrees; they look arbitrary and seem to lack intuitive access (unlike subjective probabilities, for instance). By contrast, AGM belief revision theory, in order to justify its revision postulates, only appeals to entrenchment orderings, an ordinal and intuitively well grasped notion.

This difference does not weigh much for those interested in computing, but for the more philosophically minded—recall that both, AGM and ranking theorizing, originated in philosophy—there remains a problem. What do numerical ranks mean? Where exactly is the difference between two numerically different, but ordinally equivalent ranking functions? Just in vague feelings concerning the strength of belief? This would certainly be a poor answer.

* Corresponding author.
E-mail addresses: matthias@hild.org (M. Hild), wolfgang.spohn@uni-konstanz.de (W. Spohn).

1 We are indebted to three anonymous referees. Their extensive reviews helped improving this paper considerably.

0004-3702/$ – see front matter
doi:10.1016/j.artint.2008.03.002

Is there really an objection? This is debatable. Historically, though, this kind of objection has played a most important role. Cardinal utility became acceptable only after von Neumann, Morgenstern [34, Chapter 3] proved that preferences conforming to certain axioms determine cardinal utilities on an interval scale. Thus, the cardinal concept turned out to be definable by, or reducible to, the ordinal concept; one cannot accept the one and reject the other. Ranks likewise are psychological magnitudes, and hence it appears legitimate, at least from an operationalistic point of view, to demand a measurement theory for them, too.

Perhaps, though, the concern is not operationalism, but rather logic. Customarily, any logical calculus is ennobled by a correctness and completeness, i.e., soundness theorem. There are calculi that live well without such a theorem. Still, we need not rehearse the historic examples for the tremendous insight delivered by such soundness theorems. If the calculus looks sensible, if the semantics is intelligible, and if a soundness theorem proves them to be equivalent, then mutual support makes for a nearly unassailable theory.

AGM belief revision theory has these virtues. Originally, it came in a logical disguise; its beginnings reach back to Gärdenfors' [12] epistemic approach to the logic of counterfactuals. Its soundness theorem was that the revision postulates (K$*$1–8) (cf. Alchourrón et al. [1, Sections 2+3], or Gärdenfors [13, Section 3.3]) and the contraction postulates (K$\div$1–8) (cf. Alchourrón et al. [1, Sections 2+3], or Gärdenfors [13, Section 3.4]) were proved to be exactly those justified by an underlying entrenchment relation (cf. Alchourrón et al. [1, Section 4], or Gärdenfors [13, Chapter 4]). Indeed, many of those proposing postulates for iterated belief change also offered a model relative to which the postulates are correct and complete. By contrast, ranking theory did not offer a comparable result, thus giving rise to the impression that ranks are somehow arbitrary.

The aim of this paper is to show that ranking theory, despite its greater strength, can meet these concerns. It will present a rigorous measurement of ranks on a ratio scale in terms of iterated contractions, thus fully satisfying whatever operationalistic requirements one tends to impose. So, when Rott [28] emphasizes in his concluding Section 15 that he has made "it fully clear that no numbers are needed for any of the belief change methods considered", we think this is simply a false opposition; the appropriate method of belief change automatically guarantees the numbers. Moreover, the paper will specify a complete set of laws of iterated contraction, something much desired in its own right and in the present context comparable to a soundness theorem in logic. The connection will, of course, be that the measurement result is in effect the proof of the completeness of the laws proposed.

The basic idea of this paper is quite simple. It is to exploit iterated contractions for getting information about the comparative size of rank differences. If the iterated contractions behave appropriately, these rank difference comparisons will behave appropriately, too, i.e., such that the theory of difference measurement as propounded in Krantz et al. [21, Chapter 4] applies. It requires some skill, though, to find an elaboration of this guiding idea that is intuitively illuminating as well as formally sound.

The plan of the paper is straightforward. In Section 2 we shall briefly introduce ranking theory as far as needed in the rest of the paper. In Section 3 we shall equally briefly introduce the required basics of the theory of difference measurement. Section 4 works up to the announced measurement result. Section 5 then states the complete laws of iterated contraction entailed by this result and gives a comparative discussion of them, in order to explain their content as well as how far they go beyond the present discussion of iterated contraction. Section 6, finally, proves that the measurement theory indeed entails these laws. A conclusion will round up the paper.

A few words about the history of this paper: Its core ideas are already found in Hild [18], a rough first draft that remained unpublished; in particular the present Sections 5 and 6 were already far developed there. The other author independently had the same ideas, less well and less completely realized in Spohn [32], a mere internet publication. Somehow, it took us a long time to start elaborating these ideas in full detail. To our knowledge, the present paper is the first mature presentation of the issue.

2. A brief sketch of ranking theory

Ranking theory assumes propositions to be the objects of belief, and not sentences or sentence-like representations. This is an important and debatable decision right at the beginning of all epistemological theorizing. As things presently stand, it is at the same time a decision between being able and not being able to pursue a substantial way of epistemological theorizing. AGM belief revision theory only apparently proceeds in a different way. It takes sentences and sets of sentences as being in the domain of their belief change operators. At the same time it postulates so-called "extensionality" axioms stating that logically equivalent sentences show exactly the same behavior (are members of the same belief sets, produce the same revision results, etc.). So, it differs only superficially. Then, however, it always seemed to us easier to deal with identical propositions than with logically equivalent sentences (even though the more complicated way has become standard in the literature).

Anyway, let us simply stick to propositions without further discussion. Let W be a set of possibilities, e.g., possible worlds, centered worlds, or small worlds, or what have you, and let $\mathcal{A}$ be any Boolean algebra of subsets of W; the elements of $\mathcal{A}$ are called *propositions*. Only in Section 4 we shall require some further assumptions about the richness of the Boolean algebra considered.

The core notion of ranking theory is this:

Definition 2.1. κ is a *negative ranking function* for $\mathcal{A}$ iff κ is a function from $\mathcal{A}$ into $\mathbf{R}^+ = \mathbf{R} \cup \{\infty\}$ such that for all $A, B \in \mathcal{A}$:

(a) $\kappa(A) \geqslant 0$, $\kappa(W) = 0$, and $\kappa(\emptyset) = \infty$,
(b) $\kappa(A \cup B) = \min\{\kappa(A), \kappa(B)\}$ [*the law of disjunction* (*for negative ranks*)].

Spohn [30,31] originally referred to such functions as ordinal conditional functions. Since Goldszmidt, Pearl [15], they were mostly called ranking functions. We have now added the adjective "negative". The reason is their standard interpretation: negative ranks (that are non-negative numbers) are degrees of *disbelief*. Thus, $\kappa(A) = 0$ says that A is not disbelieved at all according to κ; $\kappa(A) > 0$ says that A *is* disbelieved, and the stronger the larger $\kappa(A)$. Hence, A is *believed* iff $\bar{A}$ is disbelieved to some degree, i.e., iff $\kappa(\bar{A}) > 0$. So, the axioms (2.1a) and (2.1b) say that $\emptyset$ is maximally disbelieved and W thus maximally believed and in any case not disbelieved, and that a disjunction is exactly as disbelieved as its less disbelieved disjunct. Definition 2.1 entails

Corollary 2.2. *Either* $\kappa(A) = 0$ *or* $\kappa(\bar{A}) = 0$ *or both* [the law of negation].

The authors have diverging opinions about the range of ranking functions. Spohn always argued for strengthening axiom (2.1b) to infinite disjunctions (without weakening minimum to infimum). This would force the range to be well-ordered; and then ordinal or natural numbers are a natural choice. In this case, we could also define the set function κ for propositions from a point function for worlds (the rank of a proposition is just the minimum of the ranks of the worlds in that proposition); this was the order of explanation chosen by Spohn [31]. However, Hild prefers not to burden our investigation with this strengthened axiom; there is indeed no point in doing so. Given only the finite axiom (2.1b), the set function need not be definable from the point function; however, the issue is completely cleared up by Huber [19]. Moreover, letting ranking functions range over the real numbers is a more natural choice given this weaker basis, and it facilitates the connection with measurement theory. There, the measurement scales usually consist of real numbers and could artificially be restricted to the natural numbers, whereas to our knowledge no measurement theory exists for scales of ordinal numbers in general.

Let us illustrate our core notion with good ol' Tweetie; the example will pervade the paper.

Example. Tweetie has, or fails to have, each of the three properties: being a bird (B), being a penguin (P), and being able to fly (F). This makes for eight possibilities or atoms of the propositional algebra. Suppose you have no idea who or what Tweetie is, for all you know it might even be a car. Then your negative ranks for the eight atoms (which determine the ranks for all other propositions) may be the following (simply chosen in some plausible way):

κ	$B \cap \bar{P}$	$B \cap P$	$\bar{B} \cap \bar{P}$	$\bar{B} \cap P$
F	0	4	0	11
$\bar{F}$	2	1	0	8

In this case, the strongest proposition you believe is that Tweetie is *either* no penguin and no bird ($\bar{B} \cap \bar{P}$) *or* a flying bird and no penguin ($F \cap B \cap \bar{P}$); all other possibilities are disbelieved. Hence, you neither believe that Tweetie is a bird nor that it is not a bird. You are also neutral concerning its ability to fly. But you believe, for instance: if Tweetie is a bird, it is not a penguin and can fly ($\bar{B} \cup (\bar{P} \cap F)$); and if Tweetie is not a bird, it is not a penguin

$(B \cup \bar{P})$—each if-then taken as material implication. The large ranks in the last column indicate that you strongly disbelieve that penguins are not birds. And so on. This may suffice as an illustration of negative ranking functions.

Negative ranks have positive counterparts:

Definition 2.3. π is a *positive ranking function* for $\mathcal{A}$ iff π is a function from $\mathcal{A}$ into $\mathbf{R}^+$ such that for all $A, B \in \mathcal{A}$:

(a) $\pi(A) \geqslant 0$, $\pi(\emptyset) = 0$, and $\pi(W) = \infty$,
(b) $\pi(A \cap B) = \min\{\pi(A), \pi(B)\}$ [*the law of conjunction* (*for positive ranks*)].

Positive ranks are degrees of *belief*, and if π is defined from a negative ranking function κ by $\pi(A) = \kappa(\bar{A})$, then π obviously is a positive ranking function, and vice versa. This suggests to prefer positive ranking functions and to neglect negative ranking functions for their circumstantial doubly negative description of belief. Or it suggests to introduce

Definition 2.4. τ is a *two-sided ranking function* for $\mathcal{A}$ iff there is a negative ranking function κ for $\mathcal{A}$ such that $\tau(A) = \kappa(\bar{A}) - \kappa(A)$, or a positive ranking function π for $\mathcal{A}$ such that $\tau(A) = \pi(A) - \pi(\bar{A})$, for all $A \in \mathcal{A}$.

A two-sided ranking function thus takes positive as well as negative values. The intended interpretation is very natural: a proposition A is believed if $\tau(A) > 0$, disbelieved if $\tau(A) < 0$, and neutral or unopinionated if $\tau(A) = 0$. The latter notions obviously have their intuitive advantages. We shall make free use of all three interdefinable notions. (Note that reversely $\kappa(A) = 0$ for $\tau(A) \geqslant 0$ and $\kappa(A) = -\tau(A)$ for $\tau(A) < 0$.)

As just explained, the range of unopinionatedness is very small; it consists only of the two-sided rank 0. However, we may be more liberal. We can enlarge the neutral zone to any interval $[z, -z]$, so that belief is expressed by $\tau(A) > z$, disbelief by $\tau(A) < -z$, and neutrality may take different degrees. The two-sided rank 0 is still distinguished then, since it represents *central* neutrality: $\tau(A) = \tau(\bar{A})$ entails that they are both 0. In these terms, the interpretation of ranking theory works just as adequately. The point is that the notion of belief is vague. How strong must belief be in order to be belief? The parameter z might be used to resolve the vagueness. Here, however, we may neglect the parameter z; only rank 0 expresses unopinionatedness.

The structure introduced so far is well known in the literature under varying labels, in its positive as well as its negative version; it may be found in Shackle's [29] functions of potential surprise or Cohen's [6] operators of inductive support as well as in the possibility measures of Dubois, Prade [9]. The distinctive feature of ranking functions is that they were the first to supplement this structure by a reasonable notion of conditional ranks (cf. Spohn [30, Section 5.3] and [31, Section 5]). Dubois, Prade [8] proposed a different form of conditional possibility measures; later on they considered various forms with no or at best technically motivated preferences; cf., e.g., Dubois et al. [10, Section 4.2]. In our view their uncertainty is due to their indeterminate interpretation of possibility measures. One form, though, is equivalent to conditional ranks. (For more extensive comparative remarks see Spohn [33, Section 4]; see also Halpern [16, Chapter 3].)

Definition 2.5. Let κ be a negative and π and τ the corresponding positive and two-sided ranking function for $\mathcal{A}$, and $A \in \mathcal{A}$ with $\kappa(A) = \pi(\bar{A}) < \infty$. Then, for any $B \in \mathcal{A}$ the *conditional negative rank of* B *given* A is defined as $\kappa(B \mid A) = \kappa(A \cap B) - \kappa(A)$, the *conditional positive rank of* B *given* A as $\pi(B \mid A) = \pi(\bar{A} \cup B) - \pi(\bar{A})$, and the *conditional two-sided rank of* B *given* A as $\tau(B \mid A) = \kappa(\bar{B} \mid A) - \kappa(B \mid A) = \pi(B \mid A) - \pi(\bar{B} \mid A)$.

This is tantamount to

Corollary 2.6.

(a) $\kappa(A \cap B) = \kappa(A) + \kappa(B \mid A)$ [the law of conjunction (for negative ranks)],
(b) $\pi(\bar{A} \cup B) = \pi(B \mid A) + \pi(\bar{A})$ [the law of material implication (for positive ranks)].

(b) says that the degree of belief in a material implication is the degree of belief in the consequent given the antecedent plus the degree of belief in the vacuous truth of the material implication, i.e., in $\bar{A}$. This is also intuitively most plausible. (a) is equally plausible, saying that the disbelief in a conjunction amounts to the disbelief in one conjunct plus the disbelief in the other conjunct given the first one. Moreover, (a) will be more handsome.

Example *(continued).* For illustration let us look at the conditional beliefs contained in your ranking function on Tweetie. We can see that precisely the if-then propositions non-vacuously held true correspond to conditional beliefs. According to the κ specified, you believe, e.g., that Tweetie can fly given it is a bird (since $\kappa(\bar{F} \mid B) = 1$) and also given it is a bird, but not a penguin (since $\kappa(\bar{F} \mid B \cap \bar{P}) = 2$), that Tweetie cannot fly given it is a penguin (since $\kappa(F \mid P) = 3$) and even given it is a penguin, but not a bird (since $\kappa(F \mid \bar{B} \cap P) = 3$). You also believe that it is not a penguin given it is a bird (since $\kappa(P \mid B) = 1$) and much more strongly that it is a bird given it is a penguin (since $\kappa(\bar{B} \mid P) = 7$). And so forth.

The notion of conditional ranks helps us to various further notions of deep significance. (Just think of the importance of conditional probabilities.) One such notion is that of a reason, a terminological choice intended to maintain the connection with traditional epistemology. A is a *reason* for B if A supports or speaks for B or if A strengthens the belief in B, that is, if the belief in B given A is firmer (or the disbelief weaker) than given $\bar{A}$, or, in still other words, if A is positively relevant to B. This notion obviously corresponds to probabilistic confirmation in Bayesian epistemology. Of course, positive relevance is accompanied by the derivative notions of negative relevance and irrelevance and their conditional versions. All this is directly expressed in terms of ranking theory:

Definition 2.7. Let κ be a negative, and τ the corresponding two-sided ranking function for $\mathcal{A}$, and $A, B \in \mathcal{A}$. Then A is a *reason for B* or *positively relevant to B* w.r.t. κ iff $\tau(B \mid A) > \tau(B \mid \bar{A})$, i.e., iff $\kappa(\bar{B} \mid A) > \kappa(\bar{B} \mid \bar{A})$ or $\kappa(B \mid A) < \kappa(B \mid \bar{A})$. A is a *reason against B* or *negatively relevant to B* w.r.t. κ iff $\tau(B \mid A) < \tau(B \mid \bar{A})$. Finally, A is *irrelevant to* or *independent of B* w.r.t. κ iff A is a reason neither for nor against B w.r.t. κ. Moreover, A is a *reason for B conditional on* or *given C* w.r.t. κ iff $\tau(B \mid A \cap C) > \tau(B \mid \bar{A} \cap C)$. Likewise for the conditional versions of the other notions of (ir)relevance.

The formal behavior of these notions is quite remarkable. Trivially, the reason or positive relevance relation is reflexive. It is easy to see, moreover, that positive relevance (like the other relevance notions) is symmetric, but not transitive, in sharp contrast to what we are used from deductive reasons. So, reasons rather yield mutual support and not arbitrarily extendible chains of inference. It is obviously an important task to describe and defend the philosophical significance of this notion of a reason, though not a task for this paper. Here, we are content with the fact that we have an excellent intuitive grasp of positive relevance, i.e., of reasons thus explained, a fact heavily exploited by the subsequent method of measuring ranks.

At some point, the authors had to make a decision over another divergence. Spohn traditionally preferred to work with negative ranking functions, because of their far-reaching and most fruitful similarity to probability measures which makes them also formally more perspicuous. Hild prefers the positive way, because of its more direct intuitive access and because of its highly illuminating account of material implication. The subsequent formal developments could be couched in either terms more or less equally well. There would be no sense, though, in presenting them in both terms. So, we simply had to choose—and shall from now on continue to use the negative way as the primary one and thus often suppress the adjective "negative".

This understood, we may move to another important point, namely that conditional ranks allow us to state a general dynamic law for ranking functions. The idea is not that upon receiving information A you move to the ranks conditional on A. Since the rank of $\bar{A}$ would then rise to ∞, this would make sense only if you were absolutely certain of A. The idea developed in Spohn [30, Section 5.3] and [31, Section 5] is rather to copy generalized probabilistic conditionalization as proposed by Jeffrey [20, Chapter 11], that is, to assume that upon directly receiving information only about A you assign to A and $\bar{A}$ new degrees of belief depending on the firmness of information, while your ranks conditional on A and $\bar{A}$ remain the same. This suffices to completely determine the dynamic law:

Definition 2.8. Let κ be a negative ranking function for $\mathcal{A}$ and $A \in \mathcal{A}$ such that $\kappa(A), \kappa(\bar{A}) < \infty$, and $x \in \mathbf{R}^+$. Then the $A \to x$-conditionalization $\kappa_{A \to x}$ of κ is defined by $\kappa_{A \to x}(B) = \min\{\kappa(B \mid A), \kappa(B \mid \bar{A}) + x\}$.

Thus, the effect of the $A \to x$-conditionalization is to shift the possibilities in A (downwards or possibly not at all) so that $\kappa_{A\to x}(A) = 0$ and the possibilities in $\bar{A}$ (upwards or maybe downwards) so that $\kappa_{A\to x}(\bar{A}) = x$. The parameter x characterizes the information process, the way in which the information is received (and its interaction with the prior doxastic state); no fixed value of x is the right one for all cases. The crucial point is that this dynamic law is iterable; obviously, this kind of conditionalization can be arbitrarily repeated as long as the condition of Definition 2.8 is satisfied.

Example *(continued).* For illustration, suppose you first learn and accept with firmness 2 that Tweetie is a bird (say, because you heard Tweetie chirping like a bird; if you would have seen it, you probably would accept the information more firmly). Thus you shift $\bar{B}$ up by 2 and change to the $B \to 2$-conditionalization κ' of the κ above:

κ'	$B \cap \bar{P}$	$B \cap P$	$\bar{B} \cap \bar{P}$	$\bar{B} \cap P$
F	0	4	2	13
$\bar{F}$	2	1	2	10

In κ' you believe that Tweetie is a bird able to fly, but not a penguin. So, in κ' you believe more than in κ, and we might also call κ' an expansion of κ. Next, to your surprise and, suppose, with firmness 1, you tentatively learn and accept that Tweetie is indeed a penguin, thus shifting P down by 1 and $\bar{P}$ up by 1 and moving to the $P \to 1$-conditionalization κ'' of κ':

κ''	$B \cap \bar{P}$	$B \cap P$	$\bar{B} \cap \bar{P}$	$\bar{B} \cap P$
F	1	3	3	12
$\bar{F}$	3	0	3	9

Now, you believe in κ'' that Tweetie is a penguin bird that cannot fly. So, you have changed your mind, and we may also call κ'' a revision of κ'. Finally, suppose you are told that the previous pieces of information are not yet confirmed and that you should treat it as an open question whether or not Tweetie can fly—may be, may be not. That is, in κ'' you disbelieve with rank 1 that Tweetie can fly, and now you should give it rank 0. Thus, you shift F up by 1 and arrive at the $F \to 0$-conditionalization κ''' of κ'':

κ'''	$B \cap \bar{P}$	$B \cap P$	$\bar{B} \cap \bar{P}$	$\bar{B} \cap P$
F	0	2	2	11
$\bar{F}$	3	0	3	9

In κ''' you have given up your belief that Tweetie cannot fly; so, we might call κ''' a contraction of κ'' by $\bar{F}$. In κ''' you believe only that Tweetie is a bird that can fly iff it is not a penguin. At the same time these moves illustrate the iterable applicability of conditionalization.

As the example showed already, $A \to x$-conditionalization comprises expansion, revision, and contraction, the three kinds of belief change studied in AGM belief revision theory. For any $x > 0$, the $A \to x$-conditionalization of κ is an *expansion*, if $\kappa(A) = \kappa(\bar{A}) = 0$, i.e., if A is initially neutral, and a *revision*, if $\kappa(A) > 0$, i.e., if A is initially disbelieved. And the $A \to 0$-conditionalization of κ, after which neither A nor $\bar{A}$ is believed, is a (genuine or vacuous) contraction by A, if $\kappa(A) = 0$, i.e., if A is initially not disbelieved, and a contraction by $\bar{A}$, if $\kappa(\bar{A}) = 0$. Obviously, not all ways of $A \to x$-conditionalization are thereby exhausted; conditionalization is a substantially more general notion.

The conceptions of expansion, revision, and contraction indeed agree also formally (as already noted in Spohn [31, footnote 20], and elaborated in Gärdenfors [13, Section 3.7]). It will be useful to state this precisely:

First, within our propositional framework a (consistent) *belief set* $\mathcal{K}$ is any subset of $\mathcal{A}$ containing W, but not $\emptyset$ and closed under intersection and the superset relation, i.e.: $W \in \mathcal{K}$; $\emptyset \notin \mathcal{K}$, if $A, B \in \mathcal{K}$, then $A \cap B \in \mathcal{K}$; and if $A \in \mathcal{K}$ and $A \subseteq B \in \mathcal{A}$, then $B \in \mathcal{K}$. In other words, $\mathcal{K}$ is a filter in the mathematical sense. Let $\boldsymbol{\mathcal{F}}(\mathcal{A})$ denote the set of filters or belief sets in $\mathcal{A}$. Moreover, we say that the filter or belief set $\mathcal{K}$ is *generated* by $\mathcal{B} \subseteq \mathcal{A}$ iff $\mathcal{K}$ is the smallest filter in $\mathcal{F}(\mathcal{A})$ comprising $\mathcal{B}$. (The corresponding operation on the sentential level is the deductive closure of a given set of sentences.) Often, it would suffice to consider principal filters of the form $\{B \in \mathcal{A} \mid A \subseteq B\}$. In the general infinite case, however, this special case will not do.

The complementary notion is that of an ideal. For any set $\mathcal{B} \subseteq \mathcal{A}$ of propositions, let $\mathcal{B}^c = \{\bar{A} \mid A \in \mathcal{B}\}$. Then $\mathcal{I} \subseteq \mathcal{A}$ is an *ideal* iff $\mathcal{I}^c$ is a filter, i.e., if $\mathcal{I}$ contains $\emptyset$, but not W and is closed under union and the subset relation. Let $\boldsymbol{\mathcal{I}}(\mathcal{A})$ denote the set of ideals in $\mathcal{A}$.

AGM belief revision theory defines their belief change operators for all belief sets at once. This appears to assume one belief change disposition for all possible belief sets. However, as noticed , e.g., by Spohn [31, p. 129] or Darwiche, Pearl [7, Section 3], it is better to have them defined for a single belief set only; this allows to associate different belief change dispositions with different belief sets. With this in mind we may define:

Definition 2.9. Let $\mathcal{N} \in \boldsymbol{\mathcal{I}}(\mathcal{A})$ be an ideal in $\mathcal{A}$. Then $*$ is a *single revision* for $\mathcal{A} - \mathcal{N}$ iff $*$ is a function assigning to each proposition $A \in \mathcal{A} - \mathcal{N}$ a belief set $*(A) \in \boldsymbol{\mathcal{F}}(\mathcal{A})$ such that:

(a) $A \in *(A)$,
(b) if $\bar{B} \notin *(A)$, then $*(A \cap B)$ is the belief set generated by $*(A) \cup \{B\}$.

It is obvious that (a) and (b) are equivalent to the revision postulates (K*1)–(K*8) of Gärdenfors [13, Section 3.3]. (K*1), closure, is entailed by our definition of a belief set. (K*2), success, is the same as condition (a). (K*5), consistency preservation, is replaced by having $*$ defined only on $\mathcal{A} - \mathcal{N}$. (K*6), "extensionality", is incorporated in our propositional framework. (K*7) and (K*8), which generalize (K*3) and (K*4) and are sometimes called superexpansion and subexpansion, are equivalent to condition (b). No wonder that we find the two conditions of Definition 2.10 simpler than Gärdenfors' eight axioms.

Gärdenfors prefers to have revisions defined for all propositions; some revisions then result either in the universal or contradictory belief set ("epistemic hell") or in the belief set containing nothing but the information and its consequences ("epistemic desert"). In the present context it is slightly preferable to deny to the whole of $\mathcal{A}$ the status of a belief set and thus to have revision undefined for the exceptional set $\mathcal{N}$ (of 'null' propositions). This is so because we do not require that ranking functions have to be regular, i.e., that $\kappa(A) = \infty$ only for $A = \emptyset$ and because ranking functions cannot be conditionalized on 'null' propositions A with $\kappa(A) = \infty$.

Likewise, we may define:

Definition 2.10. Let $\mathcal{N} \in \boldsymbol{\mathcal{I}}(\mathcal{A})$ be an ideal in $\mathcal{A}$. Then $\div$ is a *single contraction* for $\mathcal{A} - \mathcal{N}^c$ iff $\div$ is a function assigning to each proposition $A \in \mathcal{A} - \mathcal{N}^c$ a belief set $\div(A) \in \boldsymbol{\mathcal{F}}(\mathcal{A})$ such that:

(a) $A \notin \div(A) \subseteq \div(\emptyset)$,
(b) if $A \notin \div(A \cap B)$, then $\div(A) \cap \div(B) \subseteq \div(A \cap B) \subseteq \div(A)$.

In a similar way as before it may be seen that conditions (a) and (b) are equivalent to the contraction postulates (K$\div$1)–(K$\div$8) of Gärdenfors [31, Section 3.4], given the slight change that contraction remains undefined for W (the belief in W cannot be given up) and some other propositions forming a filter $\mathcal{N}^c$.

Moreover, single revisions and contractions are related by the *Levi* and the *Harper identity*:

Corollary 2.11. *If $\div$ is a single contraction and $*(A)$ is the belief set generated by $\div(\bar{A}) \cup \{A\}$, then $*$ is a single revision; and conversely, if $\div(A)$ is defined as $*(W) \cap *(\bar{A})$.*

All this is directly related to ranking theory. Obviously, a belief set $\mathcal{K}(\kappa) = \{A \in \mathcal{A} \mid \kappa(\bar{A}) > 0\}$ is associated with each negative ranking function κ for $\mathcal{A}$. This allows us to define and observe:

Definition 2.12. Let κ be a negative ranking function for $\mathcal{A}$. Then the *single revision* $*_\kappa$ induced by κ is defined by $*_\kappa(A) = \mathcal{K}(\kappa_{A \to x})$ for all A with $\kappa(A) < \infty$ and some $x > 0$.

This is indeed well defined, since we have

Corollary 2.13. $*_\kappa$ *is a single revision for* $\mathcal{A} - \mathcal{N}$, *where* $\mathcal{N} = \{A \mid \kappa(A) = \infty\}$. *Moreover,* $\mathcal{K}(\kappa_{A\to x}) = \mathcal{K}(\kappa_{A\to y})$ *for all* $x, y > 0$.

Proof. Since $C \in \mathcal{K}(\kappa_{A\to x})$ iff $\kappa(\bar{C} \mid A) > 0$, a condition not depending on x, $\mathcal{K}(\kappa_{A\to x})$ is the same for all $x > 0$. As to condition (a) of Definition 2.9, $\kappa_{A\to x}(\bar{A}) > 0$ entails $A \in *_\kappa(A)$. As to condition (b), suppose that $\kappa_{A\to x}(B) = 0$, i.e. $\kappa(B \mid A) = 0$. Then $C \in *_\kappa(A \cap B)$ iff $\kappa_{A\cap B\to x}(\bar{C}) > 0$ iff $\kappa(\bar{C} \mid A \cap B) > 0$ iff $\kappa(B \cap \bar{C} \mid A) > 0$ iff $\bar{B} \cup C \in *_\kappa(A)$. □

Similarly, we may define and observe:

Definition 2.14. Let κ be a negative ranking function for $\mathcal{A}$ and $A \in \mathcal{A}$ such that $\kappa(\bar{A}) < \infty$. Then the *contraction* $\kappa_{\div A}$ *of* κ *by* A is defined as

$$\kappa_{\div A} = \begin{Bmatrix} \kappa, & \text{if } \kappa(\bar{A}) = 0, \\ \kappa_{A\to 0}, & \text{if } \kappa(\bar{A}) > 0 \end{Bmatrix}.$$

And the *single contraction* $\div_\kappa$ induced by κ is defined as the function assigning to each $A \in \mathcal{A}$ such that $\kappa(\bar{A}) < \infty$ the belief set $\div_\kappa(A) = \mathcal{K}(\kappa_{\div A})$.

Corollary 2.15. $\div_\kappa$ *is a single contraction for* $\mathcal{A} - \mathcal{N}^c$, *where* $\mathcal{N} = \{A \mid \kappa(A) = \infty\}$.

Again, $*_\kappa$ and $\div_\kappa$ are related by the Levi and the Harper identity (2.11).

There is a salient difference between (2.12) and (2.14). In (2.14) it made sense to define contraction on the level of ranking functions, since this contraction is unique on this level; it then induces contraction on the level of belief sets. By contrast, there is no unique revision on the level of ranking functions; for each $x > 0$ $A \to x$-conditionalization gives a different result. As Corollary 2.13 shows, it is only on the level of belief sets it that does not matter on which $x > 0$ we base the revision.

This difference has an important consequence. If revision and contraction are special cases of conditionalization and if the latter is iterable, then, one might think, (2.12) and (2.14) help us to notions of iterated revision and contraction. This is indeed true for contraction. Contraction on the level of ranking functions is clearly iterable; it thus induces a unique behavior of iterated contraction on the level of belief sets. It is this feature that we shall exploit in the rest of the paper for a measurement of ranks and a complete axiomatization of iterated contraction as induced by a ranking function. The same does not work for revision, however, since only the first, but not the subsequent revisions are independent of the conditionalization parameter x. Does iterated contraction induce iterated revision via the Levi identity? No, since expansion, too, is not unique on the level of ranking functions.

This explains why we focus on contractions instead of revisions. Still, our procedure may raise qualms. Despite the continuous efforts of Isaac Levi to establish expansions and contractions as the basic epistemic movements (see, e.g., Levi [23, Chapter 2]), the prevailing attitude seems to be that contractions have only an auxiliary status, since each belief change must be provoked by some kind of information and hence be an expansion or a revision. We think that both attitudes forget about the frame-relativity of belief change. Each model of belief change considers only a limited set of propositions, a restricted frame, and never *all* propositions we have beliefs about. Hence, it is always possible that we receive information that is not explicitly represented in the frame and induces a contraction within that frame. (Bochman [2] is noticeable for expressly addressing the issue of what kind of belief changes on restricted languages or frames are induced by belief changes on richer languages.) Imagine someone tells you: "Forget everything you have heard about the assassination of JFK!" If the propositional framework only contains possible facts about the assassination of JFK, this message, if accepted, results in a contraction; only in a richer framework considering also that people say this or that about the possible facts, it could be represented as a revision. Hence, revisions and contractions are on an equal footing, and iterated contraction deserves the same interest as iterated revision.

Perhaps, iterated revision is not out of reach of our investigation. One might consider a fixed conditionalization parameter for all revisions. Then iterated revisions of ranking functions and belief sets would be uniquely determined. This should also lead to a measurement procedure for ranking functions and consequently to a complete axiomatization of iterated revisions based on a fixed conditionalization parameter. The results of the present paper should provide all the essentials needed for such theorems. We have not pursued them, however, because the fixing of this parameter seems arbitrary and because it seems arbitrary to assume all revisions to be based on a fixed conditionalization parameter.

3. A brief sketch of difference measurement

Having thus collected all the required basics of ranking theory and its connection to single ABM belief contraction and revision, let us briefly introduce the other pillar on which this paper rests: measurement theory. This is a huge topic, as may be seen from the three volumes of Krantz et al. [21]. We shall refer, however, only to a tiny part, a specific version of difference measurement.

The general problem is to map some empirical realm X into some scale, usually consisting in the set $\mathbf{R}$ of real numbers, and to show how some empirically testable relations and operations on X determine the mapping, the measurement, in a sufficiently unique way.

The realm X may be anything: rods (for measuring length), locations (for measuring temperature), goods (for measuring their subjective utility), or propositions (for measuring their degrees of belief). The starting point is always some order relation $\trianglelefteq$ on X, usually a weak order; i.e., $\trianglelefteq$ is transitive (if $x \trianglelefteq y$ and $y \trianglelefteq z$, then $x \trianglelefteq z$) and complete (either $x \trianglelefteq y$ or $y \trianglelefteq x$ or both for all $x, y \in X$). If the order type of $\trianglelefteq$ is not too rich, X can thus be mapped into $\mathbf{R}$ in an order preserving way. This, however, yields only an ordinal scale; any order preserving transformation of the map would do as well. This is not yet a genuine numerical scale.

There are many ways for strengthening the empirical information and thus arriving at a more determinate scale. One idea is to assume that somehow the differences between two items from X with respect to the dimension to be measured can be compared. Krantz et al. [21, Section 4.1.1] give many examples. Subjects are often able to compare utility differences between goods. In psychophysics the task may be to choose a stimulus y, the brightness (or loudness, etc.) of which is intuitively in the middle between two other stimuli x and z so that the differences between x and y and between y and z are judged equal—a task usually well managed by subjects. And so on.

Thus, the idea is to start with a relation $\preceq$ on $X \times X$ that is again assumed to be a weak order ($\approx$ being the equivalence relation and $\prec$ being the irreflexive order relation induced by $\preceq$). And the task is to find further properties of $\preceq$ sufficient to generate a numerical scale, usually an interval scale that is fully fixed after choosing a zero point and a unit.

Example *(continued).* Let us illustrate how this might work in our example above. There we had specified a ranking function κ for the eight propositional atoms, entailing ranks for all 256 propositions involved. Focusing on the atoms, we are thus dealing with a realm $X = \{x_1, \dots, x_8\}$ and a numerical function f such that

$$f(x_1) = 0, \qquad f(x_2) = 4, \qquad f(x_3) = 0, \qquad f(x_4) = 11,$$
$$f(x_5) = 2, \qquad f(x_6) = 1, \qquad f(x_7) = 0, \qquad f(x_8) = 8.$$

This induces a lot of difference comparisons. Denoting pairs by $(x - y)$ for mnemonic reasons, we have, e.g., $(x_5 - x_6) \preceq (x_2 - x_1)$ or $(x_8 - x_4) \preceq (x_4 - x_8)$. Do these comparisons help to determine f? Yes, the example was so constructed:

First, we have $(x_1 - x_3) \approx (x_3 - x_1) \approx (x_1 - x_7)$. This entails $f(x_1) = f(x_3) = f(x_7)$. Let us choose this as the zero point of our scale; i.e., $f(x_1) = 0$. Next, we have $(x_5 - x_6) \approx (x_6 - x_1)$. If we choose $f(x_6) = 1$ as our unit, this entails $f(x_5) = 2$. Then, we have $(x_2 - x_5) \approx (x_5 - x_1)$, entailing $f(x_2) = 4$, and $(x_8 - x_2) \approx (x_2 - x_1)$, entailing $f(x_8) = 8$. Finally, we have $(x_4 - x_8) \approx (x_2 - x_6)$ so that $f(x_4) = 11$. In this way, the difference comparisons determine f uniquely up to a unit and a zero point.

The case is instructive. However, it worked only because the numbers were luckily chosen. What is the general theory? How must a difference comparison behave so that such a construction is guaranteed to work? Here are the results of Krantz et al. [21, vol. 1, p. 151]:

Definition 3.1. Suppose X is a non-empty set and $\preceq$ a quaternary relation on X, i.e., binary relation on $X \times X$. Then $\preceq$ is an *algebraic-difference relation* for X iff for all $x, y, z, w, x', y', z' \in X$ and all sequences $(x_1, x_2, \dots)$ in X the following five axioms are satisfied:

(a) $\preceq$ is a weak order, i.e., a transitive and complete relation on $X \times X$ [*weak order*],
(b) if $(x - y) \preceq (z - w)$, then $(w - z) \preceq (y - x)$ [*sign reversal*],
(c) if $(x - y) \preceq (x' - y')$ and $(y - z) \preceq (y' - z')$, then $(x - z) \preceq (x' - z')$ [*monotonicity*],
(d) if $(x - x) \preceq (x - y) \preceq (z - w)$, then there exist $w', w'' \in X$ such that $(x - w') \approx (z - w) \approx (w'' - y)$ [*fullness*],
(e) $(x_1, x_2, \ldots)$ is finite, if it is a strictly bounded standard sequence, i.e., $(x_{i+1} - x_i) \approx (x_2 - x_1)$ for all i, not $(x_2 - x_1) \approx (x_1 - x_1)$, and there exist $y', y'' \in X$ such that $(y' - y'') \preceq (x_i - x_1) \preceq (y'' - y')$ for all x_i in the sequence [*Archimedean axiom*].

(a)–(c) are called necessary axioms; if they were violated, the numerical inequalities into which the difference relations translate could not be solved. The Archimedean axiom (e) is also necessary; otherwise, we could not find a numerical representation of X within $\mathbf{R}$. Krantz et al. refer to (d) as a structural axiom. It is not entailed by the numerical representation. Some such structural axiom is usually needed; otherwise, all the numerical inequalities might not have a sufficiently unique solution. Very rarely there are conditions for measurement that are both necessary and sufficient.

Then Krantz et al. [21, vol. 1, pp. 151 + 158] prove the following

Theorem 3.2. *If $\preceq$ is an algebraic-difference relation for X, then there exists a function f from X into $\mathbf{R}$ such that for all $x, y, z, w \in X$ $(x - y) \preceq (z - w)$ iff $f(x) - f(y) \leqslant f(z) - f(w)$. Moreover, f is unique up to a positive linear transformation, i.e., if f' relates to $\preceq$ in the same way as f, then there are real numbers $\alpha > 0$ and β such that $f' = \alpha f + \beta$.*

This is the theorem we shall need for measuring ranking functions. We shall be happy to have found one way of measurement; but, of course, the field is open then for searching for variants possibly more elegant, more intuitive, or more general.

4. Measuring ranks by iterated contractions

Our goal is to utilize Section 3 for measuring ranks via iterated contractions. So, let us start by making explicit our talk about iterated contraction. The ranking theoretic terminology is fixed in

Definition 4.1. Let κ be a negative ranking function for $\mathcal{A}$ and $A_1, \ldots, A_n \in \mathcal{A}$ $(n \geqslant 0)$ such that $\kappa(\bar{A}_i) < \infty$ $(i = 1, \ldots, n)$. Then the *iterated contraction* $\kappa_{\div\langle A_1,\ldots,A_n\rangle}$ of κ by $\langle A_1, \ldots, A_n\rangle$ is defined as $\kappa_{\div\langle A_1,\ldots,A_n\rangle} = (\ldots(\kappa_{\div A_1})\ldots)_{\div A_n}$; this includes the iterated contraction $\kappa_{\div\langle\rangle} = \kappa$ by the empty sequence $\langle\rangle$. The *iterated contraction* $\div_\kappa$ induced by κ is defined as that function which assign to any finite sequence $\langle A_1, \ldots, A_n\rangle$ of propositions with $\kappa(\bar{A}_i) < \infty$ the belief set $\div_\kappa\langle A_1, \ldots, A_n\rangle = \mathcal{K}(\kappa_{\div\langle A_1,\ldots,A_n\rangle})$. Hence, $\div_\kappa\langle\rangle = \mathcal{K}(\kappa)$.

Let us note right away, for later reference, that iterated contraction as induced by a ranking function is not commutative. It is so only under special conditions:

Lemma 4.2. *Let κ be a negative ranking function for $\mathcal{A}$ and $A, B \in \mathcal{A}$. Then $\div_\kappa\langle A, B\rangle \neq \div_\kappa\langle B, A\rangle$ if and only if $A, B \in \mathcal{K}(\kappa)$, $\kappa(B \mid \bar{A}) = 0$ or $\kappa(A \mid \bar{B}) = 0$, and $\kappa(\bar{B} \mid \bar{A}) < \kappa(\bar{B} \mid A)$ (which is equivalent to $\kappa(\bar{A} \mid \bar{B}) < \kappa(\bar{A} \mid B)$).*

This may at first be surprising. However, Hansson [17, p. 648] gives an intuitively compelling example showing that this is exactly what we should expect. It is about a belief set containing A, $\bar{A} \cup B$, and $\bar{A} \cup C$, and their logical consequences, where $\bar{A} \cup B$ is better entrenched than A and A better entrenched than $\bar{A} \cup C$. Thus, if one contracts C, $\bar{A} \cup C$ has to give way, and if one then contracts B, $\bar{A} \cup B$ finally remains. If, however, one first contracts B, A has to give way, and if one then wants to contract C, there is nothing to contract, and both, $\bar{A} \cup B$ and $\bar{A} \cup C$, remain.

Instead of a proof, let us sketch the gist of Lemma 4.2. It is, roughly, that the last condition requires the positive relevance of A to B (and vice versa) and that the first conditions then have the effect either that $A \cap \bar{B}$ is disbelieved (or, "if A, then B" believed) after contracting first by A and then by B, but not after the reverse contraction, or that $\bar{A} \cap B$ is disbelieved (or "if B, then A" believed) after contracting first by B and then by A, but not after the reverse contraction (or that both is the case). This is exactly how non-commutativity of contraction may come about. Indeed, the survival of material implications in iterated contractions will play a crucial role below.

The general format of our discussion is fixed in

Definition 4.3. Let $\mathcal{A}$ be an algebra of propositions over W and $\mathcal{N} \in \mathcal{I}(\mathcal{A})$ an ideal (of 'null' propositions) in $\mathcal{A}$. Let $\boldsymbol{\mathcal{A}_{\mathcal{N}}}$ denote the set of all finite (possibly empty) sequences of propositions from $\mathcal{A} - \mathcal{N}^c$. Then $\div$ is a *potential iterated contraction*, a *potential IC*, for $(\mathcal{A}, \mathcal{N})$ iff $\div$ is a function from the set $\boldsymbol{\mathcal{A}_{\mathcal{N}}}$ of such finite sequences into the set $\mathcal{F}(\mathcal{A})$ of consistent belief sets. A potential IC $\div$ is an *iterated ranking contraction*, an *IRC*, for $(\mathcal{A}, \mathcal{N})$ iff there is a negative ranking function κ such that $\mathcal{N} = \{A \in \mathcal{A} \mid \kappa(A) = \infty\}$ and $\div = \div_\kappa$.

Thus, we use the same framework as has been proposed by Lehmann [22, Section 4] for revision and confirmed by Rott [26, Section 5] after a careful discussion of alternative frameworks.

Given this terminology, our principal aim is to measure ranks with the help of iterated contraction on a ratio scale. This means to reconstruct a ranking function κ from its iterated contraction $\div_\kappa$, indeed uniquely up to a multiplicative constant, via a mediating algebraic-difference relation. This is what we shall do in this section. The further aim, completing the investigation in Sections 5 and 6, is to state which properties a potential IC must have in order to be an IRC, i.e., an IC suitable for measuring ranks. Of course, Definition 4.3 does not count as an answer; we shall be searching for informative properties not referring to ranking functions.

We shall reach our principal aim in four simple steps. The first step is familiar; it consists in the observation already made in AGM belief revision theory that the ordering of negative ranks, i.e., of disbelief, may be inferred from single contractions. In our terms, this means that we have for each negative ranking function κ and all $A, B \in \mathcal{A}$:

Lemma 4.4. *$\kappa(A) \leqslant \kappa(B)$ iff $\div_\kappa$ is not defined for $\langle \bar{B} \rangle$ or $\bar{A} \notin \div_\kappa \langle \bar{A} \cap \bar{B} \rangle$.*

That is, B is at least as disbelieved as A, or $\bar{B}$ is at least as firmly believed as $\bar{A}$, either if $\bar{B}$ is maximally believed or if giving up the belief in $\bar{A} \cap \bar{B}$ entails giving up the belief in $\bar{A}$ (cf. Gärdenfors [13, p. 96]). Let us fix this connection without reference to ranks:

Definition 4.5. Let $\div$ be a potential IC for $(\mathcal{A}, \mathcal{N})$. Then the *potential disbelief comparison $\trianglelefteq_\div$ associated with $\div$* is the binary relation on $\mathcal{A}$ such that for all $A, B \in \mathcal{A}$: $A \trianglelefteq_\div B$ iff $B \in \mathcal{N}$ or $\bar{A} \notin \div \langle \bar{A} \cap \bar{B} \rangle$. The associated *disbelief equivalence* $\stackrel{\Delta}{=}_\div$ and the *strict disbelief comparison* $\triangleleft_\div$ are defined in the usual way. The disbelief comparison associated with the IRC $\div_\kappa$ is denoted by $\trianglelefteq_\kappa$, so that Lemma 4.4 entails that $A \trianglelefteq_\kappa B$ iff $\kappa(A) \leqslant \kappa(B)$.

Of course, such a potential disbelief comparison $\trianglelefteq_\div$ is well-behaved and thus a proper disbelief comparison only if the associated potential IC $\div$ is well-behaved. For instance, $\trianglelefteq_\div$ is a weak order only if the potential IC $\div$ restricted to one-term sequences is a single contraction according to Definition 2.10 (cf. Gärdenfors [13, Section 4.6]). Let us defer, though, the systematic inquiry what good behavior amounts to in the end. At present, the relevant point is that single contractions yield no more than a measurement of ranks on an ordinal scale.

Hence, we must take further steps that will culminate in an appropriate algebraic-difference relation. The second step is crucial; it consists in the observation that the reason relations as specified in Definition 2.7, i.e., positive relevance, negative relevance, and irrelevance, can also be expressed in terms of contractions, albeit only iterated ones. This is the content of

Lemma 4.6. *Let κ be a negative ranking function, and let $A \to B = \bar{A} \cup B$ be the set-theoretical version of material implication. Then we have*:

(a) *A is a reason for B, i.e., positively relevant to B, w.r.t. κ iff $A \to B \in \div_\kappa \langle A, \bar{A}, B, \bar{B} \rangle$ or $\bar{A} \to \bar{B} \in \div_\kappa \langle A, \bar{A}, B, \bar{B} \rangle$ or both,*
(b) *A is negatively relevant to B w.r.t. κ iff $\bar{A} \to B \in \div_\kappa \langle A, \bar{A}, B, \bar{B} \rangle$ or $A \to \bar{B} \in \div_\kappa \langle A, \bar{A}, B, \bar{B} \rangle$ or both,*
(c) *A is irrelevant to B w.r.t. κ iff none of $A \to B$, $A \to \bar{B}$, $\bar{A} \to B$, and $\bar{A} \to \bar{B}$ is a member of $\div_\kappa \langle A, \bar{A}, B, \bar{B} \rangle$.*

This lemma will be generalized in, and hence be proved by, Theorem 4.7. However, that theorem is best understood via the lemma. It requires a few commentaries. The most important point, we think, is the tremendous intuitive appeal of these assertions. If A and B are irrelevant to each other, if they have nothing to do with one another, then after eliminating any belief or disbelief about A and about B nothing weaker concerning A and B, no disjunction or material implication should remain; this is what (c) says. Conversely, if A and B are somehow connected to each other, this connection should survive that elimination, and then it is obvious that the survivors in (a) express positive relevance and those in (b) negative relevance. Recall also the law of material implication (2.6b) for positive ranks; its intuitive plausibility has the same source.

Formally, Lemma 4.6 refers to fourfold contraction. But it is obvious that at most two of them can be genuine contractions; if the contraction by A is genuine, that by $\bar{A}$ must be vacuous. However, the theorem requires eliminating any opinion about A; this is why it had to be stated as it is. Likewise for B.

One may suspect an incoherence in Lemma 4.6, since we emphasized after Definition 2.7 that the relevance notions are symmetric and in Lemma 4.2 that iterated contractions do not commute. However, it does not make a difference in Lemma 4.6 whether we refer to $\div_\kappa\langle A, \bar{A}, B, \bar{B}\rangle$ or to $\div_\kappa\langle B, \bar{B}, A, \bar{A}\rangle$. In case (a), for instance, the two iterated contractions may not retain the same of the two specified material implications, but both retain at least one of the two.

We introduced Lemma 4.6 only because it most perspicuously shows how reasons are reflected in contractions. For our measurement purposes the most convenient notion is non-negative relevance, i.e., the complement of case (b). Moreover, we have to more generally refer to conditional relevance. These two points are taken care of in the next theorem, which is immediately intelligible on the basis of Lemma 4.6.

Theorem 4.7. *Let κ be a negative ranking function for $\mathcal{A}$ and $A, B, C \in \mathcal{A}$ such that $\kappa(C) < \infty$. Then A is not a reason against B, or non-negatively relevant to B, given C w.r.t. κ iff $\kappa(A \mid C)$ or $\kappa(\bar{A} \mid C)$ or $\kappa(B \mid C)$ or $\kappa(\bar{B} \mid C)$ is infinite or $\kappa(A \cap B \cap C) - \kappa(A \cap \bar{B} \cap C) \leqslant \kappa(\bar{A} \cap B \cap C) - \kappa(\bar{A} \cap \bar{B} \cap C)$, i.e., iff neither $(C \cap A) \to \bar{B}$ nor $(C \cap \bar{A}) \to B$ is a member of $\div_\kappa\langle C \to A, C \to \bar{A}, C \to B, C \to \bar{B}\rangle$ or the latter is undefined.*

Proof. A is non-negatively relevant to B given C according to Definition 2.7 iff $\tau(B \mid A \cap C) \geqslant \tau(B \mid \bar{A} \cap C)$ iff $\kappa(\bar{B} \mid A \cap C) - \kappa(B \mid A \cap C) \geqslant \kappa(\bar{B} \mid \bar{A} \cap C) - \kappa(B \mid \bar{A} \cap C)$ iff $\kappa(A \mid C)$ or $\kappa(\bar{A} \mid C)$ or $\kappa(B \mid C)$ or $\kappa(\bar{B} \mid C)$ is infinite or $\kappa(A \cap B \cap C) - \kappa(A \cap \bar{B} \cap C) \leqslant \kappa(\bar{A} \cap B \cap C) - \kappa(\bar{A} \cap \bar{B} \cap C)$. This proves the first equivalence.

As to the second equivalence, it is clear that the exceptional clauses are equivalent. So let us suppose that all four contractions are defined, let κ_i denote the ranking function resulting from performing the first i of the 4 contractions considered, and let us abbreviate $\kappa(A \cap B \cap C) = x$, $\kappa(A \cap \bar{B} \cap C) = y$, $\kappa(\bar{A} \cap B \cap C) = z$, and $\kappa(\bar{A} \cap \bar{B} \cap C) = w$. What we have to show then is that $x - y \leqslant z - w$ iff $\kappa_4(A \cap B \cap C) = \kappa_4(\bar{A} \cap \bar{B} \cap C) = 0$. We should observe that for any proposition D, if $\kappa_i(D) = 0$ and $j \geqslant i$, then $\kappa_j(D) = 0$; this is so because belief sets never get larger by contractions. Now we have to distinguish four cases:

First, $x \leqslant y$ and $z \leqslant w$: Then, we have first $\kappa_1(\bar{A} \cap B \cap C) = 0$ and then $\kappa_2(A \cap B \cap C) = 0$, $\kappa_2(A \cap \bar{B} \cap C) = y - x$, and $\kappa_2(\bar{A} \cap \bar{B} \cap C) = w - z$. Hence, $\kappa_3(\bar{A} \cap \bar{B} \cap C) = 0$ iff $w - z \leqslant y - x$. Thus, our equivalence holds in this case.

Second, $x \leqslant y$ and $w < z$: This already entails $x - y \leqslant z - w$. However, we also have first $\kappa_1(\bar{A} \cap \bar{B} \cap C)$ and then $\kappa_2(A \cap B \cap C) = 0$. So, the equivalence holds in this case, too.

Third, $y < x$ and $z \leqslant w$: In this case, we cannot have $x - y \leqslant z - w$. However, we also have first $\kappa_1(\bar{A} \cap B \cap C) = 0$ and then $\kappa_2(A \cap \bar{B} \cap C) = 0 < \kappa_2(A \cap B \cap C)$. Since the third and the fourth contraction run empty in this case, we also have $\kappa_4(A \cap B \cap C) > 0$. Thus, again, the equivalence holds.

Fourth, $y < x$ and $w < z$: Then, we have first $\kappa_1(\bar{A} \cap \bar{B} \cap C) = 0$ and then $\kappa_2(A \cap \bar{B} \cap C) = 0$, $\kappa_2(A \cap B \cap C) = x - y$, and $\kappa_2(\bar{A} \cap B \cap C) = z - w$. The third contraction is vacuous. The fourth is not. Rather, we have $\kappa_4(A \cap B \cap C) = 0$ iff $x - y \leqslant z - w$. Thus, the equivalence also holds in the final case. □

Note that in contrast to Lemma 4.6 the minimal number of vacuous contractions occurring in this proof was only one. Hence, in the general case conditional relevance shows up in up to three genuine contractions.

Note, moreover, that we can express the equivalence of Theorem 4.7 also in the following way: For any four mutually disjoint propositions $A, B, C, D \in \mathcal{A}$ with finite ranks $\kappa(A) - \kappa(B) \leqslant \kappa(C) - \kappa(D)$ iff $A \cup B$ is non-negatively relevant to $A \cup C$ given $A \cup B \cup C \cup D$ w.r.t. κ, i.e., iff neither $\bar{A}$ nor $\bar{D}$ is a member of $\div_\kappa\langle \bar{A} \cap \bar{B}, \bar{C} \cap \bar{D}, \bar{A} \cap \bar{C}, \bar{B} \cap \bar{D}\rangle$.

Thus, we can now make the same transition as we did from (4.4) to (4.5) and adopt the following

Definition 4.8. Let $\div$ be a potential IC for $(\mathcal{A}, \mathcal{N})$. Then the *potential disjoint difference comparison* (*potential DisDC*) $\preceq^d_\div$ *associated with* $\div$ is the four-place relation defined for all quadruples of mutually disjoint propositions in $\mathcal{A} - \mathcal{N}$ such that for all such propositions $A, B, C, D (A - B) \preceq^d_\div (C - D)$ iff $\bar{A}, \bar{D} \notin \div\langle \bar{A} \cap \bar{B}, \bar{C} \cap \bar{D}, \bar{A} \cap \bar{C}, \bar{B} \cap \bar{D}\rangle$—where the ordered pair of A and B is denoted by $(A - B)$ simply for mnemonic reasons. The associated *disjoint difference equivalence* $\approx^d_\div$ and the *strict disjoint difference comparison* $\prec^d_\div$ are defined in the usual way. The potential DisDC associated with the IRC $\div_\kappa$ is denoted by $\preceq^d_\kappa$, so that Theorem 4.7 entails that $(A - B) \preceq^d_\kappa (C - D)$ iff $\kappa(A) - \kappa(B) \leqslant \kappa(C) - \kappa(D)$.

Example *(finished).* Let us once more use our example for illustrating this step. We chose the κ of our example so that

(a) $\kappa(F \cap B \cap P) - \kappa(\bar{F} \cap B \cap P) = \kappa(F \cap \bar{B} \cap P) - \kappa(\bar{F} \cap \bar{B} \cap P)$,

Thus, the associated disjoint difference comparison is such that

(b) $(F \cap B \cap P - \bar{F} \cap B \cap P) \approx^d_\kappa (F \cap \bar{B} \cap P - \bar{F} \cap \bar{B} \cap P)$.

According to Theorem 4.7, this is tantamount to B being irrelevant to F given P. This is something intuitively well assessable, positively in this case. One may not be fully sure whether penguins are really birds, but independently one thinks that penguins cannot fly. So, given Tweetie is a penguin, the further information about its being a bird or not does not influence one's opinion about its ability to fly. If you accept this, your ranks should be as in (a) and your difference comparison as in (b). According to Theorem 4.7 and Definition 4.8, (b) is manifested in the following contraction behavior:

(c) $\bar{F} \cup \bar{B} \cup \bar{P}, F \cup \bar{B} \cup \bar{P}, \bar{F} \cup B \cup \bar{P}, F \cup B \cup \bar{P} \notin \div_\kappa\langle \bar{B} \cup \bar{P}, B \cup \bar{P}, \bar{F} \cup \bar{P}, F \cup \bar{P}\rangle$.

Now, if one runs these four contractions on the initial κ, one sees how the first three contractions successively reduce all the numbers 4, 1, 11, and 8 to 0, that they would not have done so if (a) would not hold, that the fourth contraction runs empty, and hence that the resulting ranking function, κ^*, say, looks thus:

κ^*	$B \cap \bar{P}$	$B \cap P$	$\bar{B} \cap \bar{P}$	$\bar{B} \cap P$
F	0	0	0	0
$\bar{F}$	2	0	0	0

κ^* holds only the disbelief in $\bar{F} \cap B \cap \bar{P}$, i.e., only the belief in $F \cup \bar{B} \cup P$, and thus indeed none of the four beliefs in question. This is how we can verify comparisons of rank differences in terms of iterated contractions.

It should be clear now what we are heading for. On the one hand, we have shown how to derive such a difference comparison from iterated contractions. On the other hand, we know that if such a difference comparison is indeed an algebraic-difference relation, we can use it for a difference measurement of ranking functions.

However, we are not yet fully prepared for this final step. If we want to apply the theory of difference measurement as presented in Section 3 to the present case, the difference comparison must hold for any four propositions, not only for any four mutually disjoint propositions. The required extension is the third step of our measurement procedure.

There are various options at this point. What is in any case required is a certain richness of the set of propositions. So, we are about to state a structural condition with the help of which our measurement procedure will succeed, but not a necessary condition entailed by ranking theory and its definition of iterated contraction. Therefore, one might strive for as parsimonious a structural condition as possible. However, parsimony has technical costs. Therefore, we prefer to choose an extremely simple structural condition that is moderately demanding and intuitively highly intelligible.

The idea is to straightforwardly require that for each proposition with a finite rank there are at least n mutually disjoint equally ranked propositions for some $n \geqslant 4$. With this assumption we can extend any potential DisDC to all quadruples of propositions. This is the content of the next two definitions:

Definition 4.9. A potential IC $\div$ for $(\mathcal{A}, \mathcal{N})$ is called *n-rich* iff for each $A \in \mathcal{A} - \mathcal{N}$ there are n mutually disjoint propositions $E_1, \ldots, E_n$ such that $A \stackrel{\Delta}{=}_{\div} E_i$ for $i = 1, \ldots, n$. A ranking function κ is called *n-rich* iff the associated IRC $\div_\kappa$ is n-rich.

Definition 4.10. Let $\div$ be a 4-rich potential IC for $(\mathcal{A}, \mathcal{N})$. Then the *potential doxastic difference comparison* (*potential DoxDC*) $\preceq_{\div}$ *associated with* $\div$ is the quarternary relation defined for all propositions in $\mathcal{A} - \mathcal{N}$ such that for all $A, B, C, D \in \mathcal{A} - \mathcal{N} (A - B) \preceq_{\div} (C - D)$ iff there are four mutually disjoint propositions $A', B', C', D' \in \mathcal{A} - \mathcal{N}$ such that $A \stackrel{\Delta}{=}_{\div} A'$, $B \stackrel{\Delta}{=}_{\div} B'$, $C \stackrel{\Delta}{=}_{\div} C'$, $D \stackrel{\Delta}{=}_{\div} D'$, and $(A' - B') \preceq^d_{\div} (C' - D')$, i.e., $\bar{A}', \bar{D}' \notin \div\langle \bar{A}' \cap \bar{B}', \bar{C}' \cap \bar{D}', \bar{A}' \cap \bar{C}', \bar{B}' \cap \bar{D}' \rangle$. Again, the associated *doxastic difference equivalence* $\approx_{\div}$ and the *strict doxastic difference comparison* $\prec_{\div}$ are defined in the usual way.

In fact, it takes very little to be rich. Suppose the ranking function κ for $\mathcal{A}$ is not rich. How can we extend it to a rich one? Very easily: Just take k new propositions $N_1, \ldots, N_k$ that are entirely neutral, i.e., we have no beliefs about them so that $\tau(N_i) = 0$ for $i = 1, \ldots, k$ and $N_1, \ldots, N_k$ are irrelevant to each other and to all the old propositions in $\mathcal{A}$. Let $\mathcal{A}^*$ be the algebra generated by $\mathcal{A} \cup \{N_1, \ldots, N_k\}$. The assumptions determine how to extend κ to $\mathcal{A}^*$; in particular, we have for any $A \in \mathcal{A}\ \kappa(A) = \kappa(A \cap N'_1 \cap \cdots \cap N'_k)$ for all $N'_i \in \{N_i, \bar{N}_i\}$ $(i = 1, \ldots, k)$. Thus, the extended κ *is* 2^k-rich. And surely, it is easy to find k propositions that are neutral in this sense; take, for instance, k fair coins not occurring in the propositions of $\mathcal{A}$ and put $N_i =$ "coin i shows head in the next throw". Richness thus appears to be a very modest structural condition. Observe also that our little ranking story about the k neutral propositions $N_1, \ldots, N_k$ could as well be expressed in terms of iterated contractions. We shall see that our proofs go through most smoothly with the assumption of 6-richness.

After this auxiliary move, we can take the fourth and final step and complete our measurement procedure: Potential DoxDC's have to be algebraic-difference relations. So, let us simply copy Definition 3.1 and slightly adapt it for our purposes:

Definition 4.11. $\preceq$ is a *doxastic difference comparison* (*DoxDC*) for $(\mathcal{A}, \mathcal{N})$ (with $\approx$ being the associated equivalence and $\prec$ the associated strict comparison) iff $\preceq$ is a quarternary relation on $\mathcal{A} - \mathcal{N}$ such that for all $A, B, C, D, E, F \in \mathcal{A}$:

(a) $\preceq$ is a weak order on $(\mathcal{A} - \mathcal{N}) \times (\mathcal{A} - \mathcal{N})$ [*weak order*],
(b) if $(A - B) \preceq (C - D)$, then $(D - C) \preceq (B - A)$ [*sign reversal*],
(c) if $(A - B) \preceq (D - E)$ and $(B - C) \preceq (E - F)$, then $(A - C) \preceq (D - F)$ [*monotonicity*],
(d) if $(A - W) \preceq (B - W)$, then $(A - W) \approx (A \cup B - W)$ [*law of disjunction*].
The DoxDC $\preceq$ is *Archimedean* iff, moreover, for any sequence $A_1, A_2, \ldots$ in $\mathcal{A} - \mathcal{N}$:
(e) if $A_1, A_2, \ldots$ is a strictly bounded standard sequence, i.e., if for all $i (A_1 - A_1) \prec (A_2 - A_1) \approx (A_{i+1} - A_i)$ and if there is a $D \in \mathcal{A} - \mathcal{N}$ such that for all $i (A_i - A_1) \prec (D - W)$, then the sequence $A_1, A_2, \ldots$ is finite.
Finally, the DoxDC $\preceq$ is *full* iff for all $A, B, C, D \in \mathcal{A} - \mathcal{N}$:
(f) if $(A - A) \preceq (A - B) \preceq (C - D)$, then there exist $C', D' \in \mathcal{A}$ such that $(A - B) \approx (C' - D) \approx (C - D')$.

As before, (a)–(c) are necessary axioms. The Archimedean axiom (e) is necessary as well. We have already explained the point of the structural axiom (f). (d), finally, is an additional necessary axiom that distinguishes ranking theory from other possible representations of degrees of belief.

We are not claiming that DoxDC's are intuitively well accessible. Indeed, they are not, we find. Disbelief comparisons or their positive counterparts, entrenchment relations, are highly accessible. By contrast, we have no good intuitive assessment of doxastic differences between four arbitrary propositions, even if they are mutually disjoint. Therefore we did not start this section with Definition 4.11 that plays only a mediating role, but rather explained how difference judgments reduce to well accessible relevance judgments and how all these assessments reduce to even better accessible iterated contractions.

Now, we can finally apply Theorem 3.2 and get

Theorem 4.12. *Let $\preceq$ be a full Archimedean DoxDC for $(\mathcal{A}, \mathcal{N})$. Then there is a negative ranking function κ for $\mathcal{A}$ such that for all $A \in \mathcal{A}\ \kappa(A) = \infty$ iff $A \in \mathcal{N}$ and for all $A, B, C, D \in \mathcal{A} - \mathcal{N} (A - B) \preceq (C - D)$ iff $\kappa(A) - \kappa(B) \leqslant \kappa(C) - \kappa(D)$. If κ' is another negative ranking function with these properties, then there is an $\alpha > 0$ such that $\kappa' = \alpha\kappa$.*

Proof. Since $\preceq$ satisfies (a)–(f) of Definition 4.11, Theorem 3.2 tells that there is a real-valued function f on $\mathcal{A} - \mathcal{N}$ such that $(A - B) \preceq (C - D)$ iff $f(A) - f(B) \leqslant f(C) - f(D)$, and that f is unique up to a positive linear transformation. Because of condition (4.11d) this function must satisfy the law of disjunction for negative ranking functions. We can extend f to $\mathcal{N}$ by putting $f(A) = \infty$ for $A \in \mathcal{N}$; because $\mathcal{N}$ is an ideal, f still satisfies the law of disjunction. Since $W \notin \mathcal{N}$, we have $f(W) < \infty$. Thus, we can choose $f(W) = 0$, and then f *is* a negative ranking function. As such it is unique up to a positive multiplicative constant. □

5. The laws of iterated contraction

We have thus shown that the right kind of potential IC's measure ranks and that IC's inducing DoxDC's as defined in (4.11) and in particular IRC's are the right kind. However, we still miss a general characterization of what the right kind is that avoids reference to ranking functions. The required information is, of course, implicit in Definitions 4.11 and 4.10 and the auxiliary Definitions 4.5, 4.8, and 4.9. Yet, the implicit information needs to be explicitly elaborated in a perspicuous way, so that content and import of our measurement result becomes intelligible.

Thereby we can also close a gap in the current literature. Despite 15 years of efforts, it was not possible to find and agree upon a stronger or even complete set of laws of iterated contraction (cf. the subsequent comparative discussion). In our view, this is so because there was no accepted semantics or no model to guide the search for such laws. As explained, ranking theory provides such a model, and hence these laws will fall right into our lap as a consequence of our measurement theory. That is, Theorem 4.12 shows that the laws we shall find are complete given the structural conditions of richness and fullness.

In this section we shall introduce the required characterization and explain and comparatively discuss its axioms. Only in the next section we shall show that this characterization is indeed the required one.

Definition 5.1. Let $\mathcal{A}$ be an algebra of propositions over W and $\mathcal{N} \in \boldsymbol{\mathcal{I}}(\mathcal{A})$ an ideal in $\mathcal{A}$. Let again be $\mathcal{N}^c = \{\bar{A} \mid A \in \mathcal{N}\}$ and $\boldsymbol{\mathcal{A}_{\mathcal{N}}}$ the set of all finite sequences of propositions from $\mathcal{A} - \mathcal{N}^c$. We shall use S as a variable for elements of $\boldsymbol{\mathcal{A}_{\mathcal{N}}}$. Then $\div$ is an *iterated contraction* (*IC*) for $(\mathcal{A}, \mathcal{N})$ iff $\div$ is a potential IC for $(\mathcal{A}, \mathcal{N})$ such that for all $A, B, C \in \mathcal{A} - \mathcal{N}^c$, and $S \in \boldsymbol{\mathcal{A}_{\mathcal{N}}}$:

(IC1) the function $A \mapsto \div\langle A\rangle$ is a single contraction (as specified in Definition 2.10) [*single contraction*],
(IC2) if $A \notin \div\langle\rangle$, then $\div\langle A, S\rangle = \div\langle S\rangle$ [*strong vacuity*],
(IC3) if $\bar{A} \cap \bar{B} = \emptyset$, then $\div\langle A, B, S\rangle = \div\langle B, A, S\rangle$ [*restricted commutativity*],
(IC4) if $A \subseteq B$ and $A \cup \bar{B} \notin \div\langle A\rangle$, then $\div\langle A \cup \bar{B}, B, S\rangle = \div\langle A, B, S\rangle$ [*path independence*],
(IC5) if $A \subseteq \bar{C}$ or $A, B \subseteq C$ and $A \trianglelefteq_{\div} B$, then $A \trianglelefteq_{\div\langle C\rangle} B$, and if the inequality in the antecedent is strict, that of the consequent is strict, too [*order preservation*],
(IC6) $\div_{\langle S\rangle}$ is an IC [*iterability*].

$\div_{\langle S\rangle}$ in (IC6) denotes the function assigning the value $\div_{\langle S,S'\rangle}$ to each sequence S' in $\boldsymbol{\mathcal{A}_{\mathcal{N}}}$. Thus, $\div_{\langle S\rangle}$ is a potential IC in turn and indeed an IC according to (IC6). (IC6) goes without saying; it lies at the heart of iteration that it can be carried out without limit. Of course, (IC6) does not make Definition 5.1 circular; it only states succinctly what we could have attained by stating all the other axioms more clumsily for all $\div_{\langle S\rangle}$.

(IC5) is not couched in the basic terms of iterated contraction, but could of course be with the help of Definition 4.5. We find that it would look less perspicuous then. As will be explained soon, (IC5) is equivalent to the Darwiche/Pearl postulates. Moreover, a notational slip occurs in (IC5). Disbelief comparisons, difference comparisons, etc., are explained relative to potential IC's; thus, strictly taken, the notation "$\trianglelefteq_{\div\langle C\rangle}$" is nonsense. However, for typographic reasons we shall always write "$\trianglelefteq_{\div\langle S\rangle}$" instead of the more correct "$\trianglelefteq_{\div_{\langle S\rangle}}$"; there is no danger of confusion. This understood, let us immediately add a most useful consequence of (IC5):

Corollary 5.2. *For any IC $\div$ we have*:

(IC7) *if $A, B \subseteq C$ or $A, B \subseteq \bar{C}$, then $A \trianglelefteq_{\div} B$ iff $A \trianglelefteq_{\div\langle C\rangle} B$* [order equivalence].

Proof. If $A \trianglelefteq_{\div} B$, then $A \trianglelefteq_{\div\langle C\rangle} B$ due to (IC5). If $B \triangleleft_{\div} A$, then $B \triangleleft_{\div\langle C\rangle} A$, again due to (IC5). Because both, $\trianglelefteq_{\div}$ and $\trianglelefteq_{\div\langle C\rangle}$, are weak orders, as we shall show in Lemma 6.1, the latter says that if not $A \trianglelefteq_{\div} B$, then not $A \trianglelefteq_{\div\langle C\rangle} B$. This proves the reverse direction. □

Before proceeding to the formal business in the next section, we should look at the intuitive and formal content of these axioms. At the same time, it is interesting to examine the extent to which they go beyond the existing efforts to come to grips with iterated contraction.

Iterated contractions must, of course, behave like single contractions at each single step; therefore (IC1). Strong vacuity (IC2) goes beyond vacuity for single contractions (expressed by condition (b) of Definition 2.10 for $B = \emptyset$), since it says that a vacuous contraction does not only leave the beliefs unchanged, but indeed the entire doxastic state as reflected in possible further contractions. This is certainly how vacuous contractions were intended, even though it was not expressible in terms of single contractions.

Obviously, hence, (IC3)–(IC5) are the proper laws of iterated contraction. We find them intuitively convincing, though, of course, our intuitions are already shaped by ranking theory. Let us briefly discuss them.

The first point we want to make is that (IC5) is equivalent to the postulates proposed by Darwiche, Pearl [7] and henceforth widely accepted. This requires a little bit of explanation:

The Darwiche/Pearl postulates were stated as conditions for iterated revision. They started from the well-known observation that single revisions (or contractions) are tantamount to an entrenchment order (in positive terms of belief) or its negative counterpart (in terms of disbelief). And they stated conditions on what this entrenchment order should look like after the revision by some proposition C. That posterior entrenchment order governs the subsequent belief change; so, the more we know about it, the more we know about iterated revision.

In our present terms, their postulates may be expressed in the following way: Let $\trianglelefteq$ be the prior negative version of the entrenchment order as defined in Definition 4.5, and let $\trianglelefteq_{*\langle C\rangle}$ be the posterior order after revision by C. Then Darwiche, Pearl [7] proposed four conditions partially characterizing $\trianglelefteq_{*\langle C\rangle}$:

Postulates 5.3.

(DP1) if $A, B \subseteq C$, then $A \trianglelefteq_{*\langle C\rangle} B$ iff $A \trianglelefteq B$,
(DP2) if $A, B \subseteq \bar{C}$, then $A \trianglelefteq_{*\langle C\rangle} B$ iff $A \trianglelefteq B$,
(DP3) if $A \subseteq C$, $B \subseteq \bar{C}$, and $A \triangleleft B$, then $A \triangleleft_{*\langle C\rangle} B$, and
(DP4) if $A \subseteq C$, $B \subseteq \bar{C}$, and $A \trianglelefteq B$, then $A \trianglelefteq_{*\langle C\rangle} B$.

(DP1) and (DP2) say that revision by C does not change the entrenchment order within C and within $\bar{C}$. This corresponds to the idea of Jeffrey [20, Chapter 11] that probabilities conditional on the evidence are not changed by the evidence, it was carried over to the belief revision context in Spohn [31, Section 3] (in the form of $A \to x$-conditionalization, Definition 2.8), and it seems generally accepted. (DP3) and (DP4) say that subsets of C do not worsen their position relative to subsets of $\bar{C}$ through revision by C. There is indeed a slight improvement of the latter postulates. If the revision by C is a genuine one (because C is initially not believed), then $A \subseteq C$ should actually improve its position relative to $B \subseteq \bar{C}$. However, we need not deepen that point.

These ideas carry over to iterated contraction. If $\trianglelefteq$ is again the prior negative version of the entrenchment order as specified in Definition 4.5 and if $\trianglelefteq_{\div\langle C\rangle}$ is the posterior order after contraction by C, then (DP1–DP4) directly translate into (DP1′–DP4′); one must only observe that in contraction by C it is $\bar{C}$ that does not worsen its position relative to C:

Postulates 5.4.

(DP1′) if $A, B \subseteq C$, then $A \trianglelefteq_{\div\langle C\rangle} B$ iff $A \trianglelefteq B$,
(DP2′) if $A, B \subseteq \bar{C}$, then $A \trianglelefteq_{\div\langle C\rangle} B$ iff $A \trianglelefteq B$,
(DP3′) if $A \subseteq \bar{C}$, $B \subseteq C$, and $A \triangleleft B$, then $A \triangleleft_{\div\langle C\rangle} B$, and
(DP4′) if $A \subseteq \bar{C}$, $B \subseteq C$, and $A \trianglelefteq B$, then $A \trianglelefteq_{\div\langle C\rangle} B$.

In the form (5.4) the Darwiche/Pearl postulates are easily compared with (IC5). (DP1′) and (DP2′) are obviously tantamount to *order equivalence* (IC7) that is implied by *order preservation* (IC5). (IC5) goes beyond (IC7) exactly in its asymmetric part, i.e., by maintaining order preservation not only under the supposition $A, B \subseteq C$, but also under the weaker supposition $A \subseteq \bar{C}$ and thus also captures (DP3′) and (DP4′).

Hence, it is exactly (IC3) and (IC4) by which our axiomatization of iterated contraction goes beyond the present state of the art. As to (IC3), *restricted commutativity*, we had mentioned that iterated contractions cannot be expected to always commute, and Lemma 4.2 described the conditions under which they do not do so. One condition was that the two contracted propositions A and B are positively relevant to each other (and the point then was that, though at least one of the two material implications expressing the positive relevance must survive the two contractions, it may be the one after $\div\langle A, B\rangle$ and the other after $\div\langle B, A\rangle$). However, in (IC3) we assumed an extreme negative relevance, i.e., that "if A, then *not* B" is logically true. This deductive relation holds across all doxastic states whatsoever. So, Lemma 4.2 can never apply to such A and B, whatever the doxastic state, and we may accept *restricted commutativity* as an axiom. In other words, if two disbeliefs are logically incompatible, there can be no interaction between giving up these disbeliefs, and hence it seems also intuitively convincing that the order in which they are given up should not matter at all.

The intuitive content of (IC4), *path independence*, may be described as follows: Suppose you believe A. Then you also believe the logical consequences of A. Let B be one of them; $B \to A$ is another. Now you contract by A. This entails that you have to give up at least one of B and $B \to A$. Suppose you keep B and give up $B \to A$. What *path independence* claims is that it does not make a difference then whether you give up $A (= (B \to A) \cap B)$ and then B or whether you give up $B \to A$ right away and then B. The description is still simpler in terms of disbelief: Suppose you disbelieve two logically incompatible propositions, and you have to contract both of them. Then you can either contract one after the other. Or you can first contract their disjunction, and if you still disbelieve one of them, you then contract it as well. (IC4) says that both ways result in the same doxastic state. This seems entirely right to us.

This may suffice as an explanation of the intuitive appeal of our iterated contraction axioms. A further point that we find remarkable is that these axioms make assertions only about one-step and two-step contractions. (IC1) and (IC2) characterize single contractions, (IC3) and (IC4) both say that two different two-step contractions come to the same, and (IC5) compares the results of a one-step and a two-step contraction (since claims about "$\trianglelefteq_{\div}$" are claims about single contractions). What is remarkable about this is that we do not need any independent assumptions about the interaction or relation between three or more steps of contraction, even though we have been referring to such longer contraction chains all the time; recall that conditional relevance showed up only in threefold contractions.

We just said that it is exactly (IC3) and (IC4) by which our axiomatization goes beyond the present state of the art. Is that to say that no further axioms have been discussed in the literature? To the contrary: there are almost confusingly many proposals. It only seems to us that none of them enjoys general assent. This is a situation we hope to change. Let us explain.

One confusing aspect is that single belief change is not at all a settled affair and that the issues there radiate to iterated belief change. For instance, one fundamental issue is whether belief change is discussed in terms of belief bases or (deductively closed) belief sets. Iterated belief change in terms of belief bases seems to be an even more difficult topic (see, e.g., the remarks in Rott [27, Section 5.2.3]); Williams [35] copes with it by applying ranking theoretic means in her theory of transmutations. However, Rott [28] carries out his comparative discussion by starting from belief bases and thus bridges the difference. We abstain from deepening the point; it is obvious that our investigation is entirely on the side of belief sets.

Even then, though, single belief change is contested. The full AGM postulates of belief revision and contraction have been vigorously criticized, mainly as too strong; see the impressive list of alternative postulates in Rott [27, Sections 4.2 and 4.3]. Of course, this debate heavily affects iterated belief change, although there seem to be little efforts to work out the consequences. A noticeable exception is Bochman [2] who provides a general framework for representing iterated contraction not presupposing the full AGM postulates; however, the emphasis is on the representation of weaker postulates for single contractions and not on iterations.

Again, we are silent on this issue. We have simply accepted the full AGM postulates, since, as explained in the definitions and Corollaries 2.12–2.15, ranking theory embraces them (though it may be interesting to study the extent to which ranking theory provides resources for understanding variants to these postulates). Thereby, we agree with the main discussion of iterated belief change that proceeds on the basis of belief sets and the full AGM postulates. It is here where we find general agreement on the Darwiche/Pearl postulates and little agreement beyond.

The general problem here is this, to put it in a summarizing non-formal way. As is known from Alchourrón et al. [1], an entrenchment order on the sentences or propositions considered is equivalent to single belief changes obeying the full AGM postulates. However, they govern only single changes of belief sets; they do not say how to get from a prior to a posterior entrenchment order. Thus, characterizing iterated belief change is the same task as characterizing the posterior entrenchment order. How might we tackle the latter task?

If we consider a belief change by the sentence or proposition A (expansion, revision, or contraction), the Darwiche/Pearl postulates tell us to preserve the entrenchment order given A and given $\bar{A}$. If we accept this, as is generally done, the only remaining problem is to integrate the two partially defined entrenchment orders given A and given $\bar{A}$ into one embracive posterior entrenchment order. At this point proposals proliferate. Conservative revision (Boutilier [3,4]) proposes to turn the best entrenched propositions given A into the best entrenched propositions after revision by A and to leave the prior entrenchment order unchanged otherwise. Moderate or lexicographic revision (Nayak [24]) proposes to take the entrenchment order given A followed by the entrenchment order given $\bar{A}$ as the entrenchment order after revision by A. (Both proposals have been criticized already by Spohn [31, Section 3].) Revision by comparison or raising (Cantwell [5], Fermé, Rott [11]) introduces a second proposition B as standard of comparison. Roughly, all the propositions better entrenched given A than B is in the prior entrenchment order are so also according to posterior entrenchment order, whereas other propositions get equally or less well entrenched than B. And there are more proposals. Similarly for contraction. There is no point in going through all these proposals in a detailed manner; we refer here to Rott [28] where on the whole 27 different methods of iterated expansion, revision, and contraction are presented and discussed. To a large extent, these proposals for rearranging the posterior entrenchment order are accompanied by characteristic postulates of iterated revision or contraction.

In our view, though, all these proposals are hampered by an ordinalist prejudice. They all stand under the restriction that it is the entrenchment orders given A and given $\bar{A}$ that have to be integrated into the posterior entrenchment order. Then we face this ingenious, but also discouraging variety of choices. We take this variety as confirming the early opinion of Spohn [31, Section 3] that no generally acceptable posterior entrenchment order can be found under this restriction. The ranking-theoretic approach then proposed to preserve not only the entrenchment orders given A and given $\bar{A}$, but also the *ranks* given A and given $\bar{A}$, as realized in $A \to x$-conditionalization (Definition 2.8). Given the conditionalization parameter x (that is 0 in contraction), the posterior entrenchment order and indeed the posterior ranking function is thereby uniquely determined. Of course, this meant preserving cardinal features of epistemic states. The point of the present paper thus is to show precisely how these cardinal features are fully determined by the appropriate laws of iterated contraction. There is no ground whatsoever for sticking to the ordinalist restriction.

This entails, of course, that our axioms (IC1)–(IC6) partially agree and partially disagree with the postulates found in the literature. It would be a tedious exercise to go through all proposals. Let us note, though, a partial agreement. Nayak et al. [25] propose a principle of iterated contraction that they call “Principled Factored Insertion”. In our notation it says:

(PRI) If $A \cup B \in \div\langle A, B\rangle$, then $\div\langle A, B\rangle = \div\langle A\rangle \cap \div\langle \bar{A} \cup B\rangle$;
if $\bar{A} \cup B \in \div\langle A, B\rangle$, then $\div\langle A, B\rangle = \div\langle A\rangle \cap \div\langle A \cup B\rangle$;
if $A \cup B, \bar{A} \cup B \notin \div\langle A, B\rangle$, then $\div\langle A, B\rangle = \div\langle A\rangle \cap \div\langle A \cup B\rangle \cap \div\langle \bar{A} \cup B\rangle$.

It is easily verified that if $\div$ is an IRC, then $\div$ satisfies (PRI). So, (PRI) must be entailed by (IC1)–(IC6). Contrary to appearances, though, (PRI) is not a reduction of iterated contraction to single contraction, since the single contractions $\div\langle A\rangle$, $\div\langle A \cup B\rangle$, and $\div\langle \bar{A} \cup B\rangle$ alone do not tell which of the three cases of (PRI) applies. Of course, disagreement would start again, if (PRI) is considered half way of completely characterizing lexicographic contraction, as Nayak et al. [25] do, since lexicographic contraction is not compatible with ranking contraction.

6. From the laws of iterated contraction to doxastic difference comparisons

Let us return to the formal business. What we have to show is that the potential DoxDC $\preceq_\div$ generated by an IC $\div$ is indeed a DoxDC. In order to reach this peak, we have to climb some antecedent hills.

First, we may observe that the disbelief comparison induced by an IC has the expected properties:

Lemma 6.1. *For any IC $\div$ $\trianglelefteq_\div$ is a weak order on $\mathcal{A}$* [weak order].

Lemma 6.2. *For any IC $\div$ if $A \trianglelefteq_\div B$, then $A \triangleq_\div A \cup B$* [law of disjunction].

These are the familiar consequences of (IC1), i.e., of the properties of single contractions.

Next, we should note that for IC's we can express the induced potential DisDC in terms of the induced disbelief comparison:

Lemma 6.3. *For any IC $\div$ and any four mutually disjoint $A, B, C, D \in \mathcal{A} - \mathcal{N}$, $(A - B) \preceq^d_\div (C - D)$ iff $A \trianglelefteq_{\div\langle \bar{A}\cap\bar{B}, \bar{C}\cap\bar{D}\rangle} C$ and $D \trianglelefteq_{\div\langle \bar{A}\cap\bar{B}, \bar{C}\cap\bar{D}\rangle} B$; and the strict inequality holds on the left-hand side iff at least one of the inequalities of the right-hand side is strict.*

Proof. Let us abbreviate the quadruple $\langle \bar{A}\cap\bar{B}, \bar{C}\cap\bar{D}, \bar{A}\cap\bar{C}, \bar{B}\cap\bar{D}\rangle$ as S_4, its initial triple as S_3, and its initial pair as S_2. Then, we have $A \trianglelefteq_{\div\langle S_2\rangle} C$ iff $\bar{A} \notin \div\langle S_3\rangle$ by definition. If $\bar{B}\cap\bar{D} \notin \div\langle S_3\rangle$, then $\div\langle S_3\rangle = \div\langle S_4\rangle$ due to (IC2). If $\bar{B}\cap\bar{D} \in \div\langle S_3\rangle$, then (IC1) entails that $A \notin \div\langle S_3\rangle$ iff $\bar{B}\cap\bar{D} \to A \notin \div\langle S_4\rangle$. Due to mutual disjointness, we have $\bar{B}\cap\bar{D} \to A = A$. Hence, in any case $A \notin \div\langle S_3\rangle$ iff $A \notin \div\langle S_4\rangle$. By a similar argument, we can show that $D \trianglelefteq_{\div\langle S_2\rangle} B$ iff $D \notin \div\langle S_2, \bar{B}\cap\bar{D}, \bar{A}\cap\bar{C}\rangle$ iff $D \notin \div\langle S_4\rangle$ (with IC3). By Definition 4.8, though, $A, D \notin \div\langle S_4\rangle$ is equivalent to $(A - B) \preceq^d_\div (C - D)$. The strict version follows by the same argument. □

Indeed, the binary disbelief comparison and the quaternary disjoint difference comparison induced by an IC cohere in the expected way. This is the point of a series of lemmata we show next:

Lemma 6.4. *For any IC $\div$ and any mutually disjoint $A, B, C, D \in \mathcal{A} - \mathcal{N}$, if $A \triangleq_\div B$, then $(A - B) \preceq^d_\div (C - D)$ iff $D \trianglelefteq_\div C$.*

Proof. Let us abbreviate $\langle \bar{A}\cap\bar{B}\rangle$ as S_1 and $\langle \bar{A}\cap\bar{B}, \bar{C}\cap\bar{D}\rangle$ as S_2. Then two applications of (IC7) show that $A \triangleq_\div B$ iff $A \triangleq_{\div\langle S_1\rangle} B$ iff $A \triangleq_{\div\langle S_2\rangle} B$. (IC1) tells us that moreover $W \triangleq_{\div\langle S_2\rangle} A$ (indeed already $W \triangleq_{\div\langle S_1\rangle} A$). Likewise, we can see that $D \trianglelefteq_\div C$ iff $D \trianglelefteq_{\div\langle S_2\rangle} C$ and that $W \triangleq_{\div\langle S_2\rangle} D$. In Lemma 6.3 we have shown that $(A - B) \preceq^d_\div (C - D)$ iff $A \trianglelefteq_{\div\langle S_2\rangle} C$ and $D \trianglelefteq_{\div\langle S_2\rangle} B$. Now, since $\trianglelefteq_{\div\langle S_2\rangle}$ is a weak order due to Lemma 6.1, Lemma 6.4 may be immediately seen to follow. □

In the same way we may prove:

Lemma 6.5. *For any IC $\div$ and any mutually disjoint $A, B, C, D \in \mathcal{A} - \mathcal{N}$, if $C \triangleq_\div D$, then $(A - B) \preceq^d_\div (C - D)$ if $A \trianglelefteq_\div B$.*

Further expected connections are:

Lemma 6.6. *For any IC $\div$ and mutually disjoint $A, B, C, D \in \mathcal{A} - \mathcal{N}$, if $(A - B) \preceq^d_\div (C - D)$ and $B \trianglelefteq_\div A$, then $D \trianglelefteq_\div C$; and if at least one of the inequalities in the antecedent is strict, so is that of the consequent.*

Proof. Let us again abbreviate $\langle \bar{A}\cap\bar{B}, \bar{C}\cap\bar{D}\rangle$ as S_2. Then $(A - B) \preceq^d_\div (C - D)$ is equivalent to $A \trianglelefteq_{\div\langle S_2\rangle} C$ and $D \trianglelefteq_{\div\langle S_2\rangle} B$ according to Lemma 6.3, and two applications of (IC7) show that $B \trianglelefteq_\div A$ iff $B \trianglelefteq_{\div\langle S_2\rangle} A$ and $D \trianglelefteq_\div C$ iff $D \trianglelefteq_{\div\langle S_2\rangle} C$. From all this Lemma 6.6 follows by the transitivity of $\trianglelefteq_{\div\langle S_2\rangle}$ and of $\triangleleft_{\div\langle S_2\rangle}$ due to (IC1). □

Lemma 6.7. *For any IC $\div$ and mutually disjoint $A, B, C, D \in \mathcal{A} - \mathcal{N}$, if $A \trianglelefteq_\div C$ and $D \trianglelefteq_\div B$, then $(A - B) \preceq^d_\div (C - D)$; and if at least one of the inequalities in the antecedent is strict, that in the consequent is strict, too.*

Proof. We again abbreviate $\langle \bar{A}\cap\bar{B}\rangle$ as S_1 and $\langle \bar{A}\cap\bar{B}, \bar{C}\cap\bar{D}\rangle$ as S_2. First assume $A \triangleleft_\div B$. In this case (IC1) tells us that $A \triangleq_{\div\langle S_1\rangle} W$ and thus that $A \triangleq_{\div\langle S_2\rangle} W$. Hence, by (IC1) $A \trianglelefteq_{\div\langle S_2\rangle} C$. Now assume $B \trianglelefteq_\div A$. In that case, we have with the premises $D \trianglelefteq_\div B \trianglelefteq_\div A \trianglelefteq_\div C$. (IC7) yields that $A \trianglelefteq_\div C$ iff $A \trianglelefteq_{\div\langle \bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D}\rangle} C$, and the latter obtains iff $A \trianglelefteq_{\div\langle \bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D}, \bar{C}\cap\bar{D}\rangle} C$ due to (IC2), since, D being the least disbelieved, the second revision is vacuous after the first. (IC4) (*path independence*) entails that $\div\langle\bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D},\bar{C}\cap\bar{D}\rangle = \div\langle\bar{A}\cap\bar{B},\bar{C}\cap\bar{D}\rangle = \div\langle S_2\rangle$. Hence, in that case $A \trianglelefteq_{\div\langle S_2\rangle} C$, too. So, in any case, our premises entail $A \trianglelefteq_{\div\langle S_2\rangle} C$. By the same argument, we arrive first at $D \trianglelefteq_{\div\langle\bar{C}\cap\bar{D},\bar{A}\cap\bar{B}\rangle} B$ and hence with (IC3) at $D \trianglelefteq_{\div\langle S_2\rangle} B$. According to Lemma 6.3, these two conclusions are equivalent with $(A-B) \preceq^d_\div (C-D)$.

As to the strict version of Lemma 6.7: If $A \lhd_\div C$, then in the case $B \trianglelefteq_\div A$ the same argument as above runs through to the conclusion $A \lhd_{\div\langle S_1\rangle} C$. Likewise when $D \lhd_\div B$ and $C \trianglelefteq_\div D$. So, the only case left is $A \lhd_\div B$ and $D \lhd_\div C$. In this case, it is the strict version of the argument that holds. □

After these exercises, the next fore-summit we want to reach is to establish that the disjoint difference comparison $\preceq^d_\div$ induced by an IC $\div$ already has the pertinent properties of a DoxDC. So, let us work through all the axioms of Definition 4.11.

Lemma 6.8. *For any IC $\div$ and four mutually disjoint $A, B, C, D \in \mathcal{A}-\mathcal{N}$, either $(A-B) \preceq^d_\div (C-D)$ or $(C-D) \preceq^d_\div (A-B)$ or both* [completeness].

Proof. If $S_4 = \langle\bar{A}\cap\bar{B}, \bar{C}\cap\bar{D}, \bar{A}\cap\bar{C}, \bar{B}\cap\bar{D}\rangle$ we have to show according to Definition 4.8 that $\bar{A}, \bar{D} \notin \div\langle S_4\rangle$ or $\bar{B}, \bar{C} \notin \div\langle S_4\rangle$. However, if $\bar{A} \in \div\langle S_4\rangle$, (IC1) trivially entails that $\bar{B}, \bar{C} \notin \div\langle S_4\rangle$; and the same holds for $\bar{D} \in \div\langle S_4\rangle$. □

Lemma 6.9. *For any IC $\div$ and six mutually disjoint $A, B, C, D, E, F \in \mathcal{A}-\mathcal{N}$, if $(A-B) \preceq^d_\div (C-D)$ and $(C-D) \preceq^d_\div (E-F)$, then $(A-B) \preceq^d_\div (E-F)$* [transitivity].

Proof. According to Lemma 6.3, the first premise is equivalent to $A \trianglelefteq_{\div\langle\bar{A}\cap\bar{B},\bar{C}\cap\bar{D}\rangle} C$ and $D \trianglelefteq_{\div\langle\bar{A}\cap\bar{B},\bar{C}\cap\bar{D}\rangle} B$; the second to $C \trianglelefteq_{\div\langle\bar{C}\cap\bar{D},\bar{E}\cap\bar{F}\rangle} E$ and $F \trianglelefteq_{\div\langle\bar{C}\cap\bar{D},\bar{E}\cap\bar{F}\rangle} D$; and the conclusion to $A \trianglelefteq_{\div\langle\bar{A}\cap\bar{B},\bar{E}\cap\bar{F}\rangle} E$ and $F \trianglelefteq_{\div\langle\bar{A}\cap\bar{B},\bar{E}\cap\bar{F}\rangle} B$. (IC7) guarantees that all these inequalities are equivalent to the extension of their contractions to the contraction of $\langle\bar{A}\cap\bar{B}, \bar{C}\cap\bar{D}, \bar{E}\cap\bar{F}\rangle$ (where some rearrangement of the contractions with the help of (IC3) may be needed). Then Lemma 6.9 follows with (IC7). □

Lemma 6.10. *For any IC $\div$ and four mutually disjoint $A, B, C, D \in \mathcal{A}-\mathcal{N}$, if $(A-B) \preceq^d_\div (C-D)$, then $(D-C) \preceq^d_\div (B-A)$* [sign reversal].

Proof. This follows immediately from Lemma 6.3 and (IC3). □

As may have been expected, condition (c) of Definition 4.11, monotonicity, is the hardest part:

Lemma 6.11. *For any IC $\div$ and six mutually disjoint $A, B, C, D, E, F \in \mathcal{A}-\mathcal{N}$, if $(A-B) \preceq^d_\div (D-E)$ and $(B-C) \preceq^d_\div (E-F)$, then $(A-C) \preceq^d_\div (D-F)$* [monotonicity].

Proof. Let $S_1 = \langle\bar{A}\cap\bar{B}\cap\bar{C}\rangle$, $S_2 = \langle\bar{D}\cap\bar{E}\cap\bar{F}\rangle$, and $S = \langle S_1, S_2\rangle$. In a first step, we want to show that each of the difference comparisons in Lemma 6.11 holds iff it holds under contraction by S. Let us look at $(A-B) \preceq^d_\div (D-E)$. By Lemma 6.3 this holds iff $A \trianglelefteq_{\div\langle\bar{A}\cap\bar{B},\bar{D}\cap\bar{E}\rangle} D$ and $E \trianglelefteq_{\div\langle\bar{A}\cap\bar{B},\bar{D}\cap\bar{E}\rangle} B$. Now assume that $C \trianglelefteq_\div A, B$, but not $F \trianglelefteq_\div D, E$; this is the most instructive case. In this case the inequalities $A \trianglelefteq D$ and $E \trianglelefteq B$ hold under $\div\langle\bar{A}\cap\bar{B}, \bar{D}\cap\bar{E}\rangle$
iff they hold under $\div\langle\bar{C}, \bar{A}\cap\bar{B}, \bar{D}\cap\bar{E}, \bar{F}\rangle$ (twice (IC7) and (IC3))
iff they hold under $\div\langle S_1, \bar{A}\cap\bar{B}, \bar{D}\cap\bar{E}, \bar{F}\rangle$ (by (IC4) and $C \trianglelefteq_\div A, B$)
iff they hold under $\div\langle S_1, \bar{A}\cap\bar{B}, S_2, \bar{F}\rangle$ (by (IC4) and not $F \trianglelefteq_\div D, E$)
iff they hold under $\div\langle S_1, \bar{A}\cap\bar{B}, S_2, \bar{D}\cap\bar{E}, \bar{F}\rangle$ by (IC2)
iff they hold under $\div\langle S, \bar{A}\cap\bar{B}, \bar{D}\cap\bar{E}\rangle$ (by (IC3) and (IC7)).
If $F \trianglelefteq_\div D, E$ we can reason about F as we just did about C; and if not $C \trianglelefteq_\div A, B$, we can reason about C as we just did about F. So, in any case $(A-B) \preceq^d_\div (D-E)$ iff $(A-B) \preceq^d_{\div\langle S\rangle} (D-E)$. The same holds for the other difference comparisons in Lemma 6.11.

So, our premises now are $(A-B) \preceq^d_{\div\langle S\rangle} (D-E)$ and $(B-C) \preceq^d_{\div\langle S\rangle} (E-F)$, or, what comes to the same by Lemma 6.3:

(a) $A \trianglelefteq_{\div\langle S, \bar{A}\cap\bar{B}, \bar{D}\cap\bar{E}\rangle} D$, and
(b) $E \trianglelefteq_{\div\langle S, \bar{A}\cap\bar{B}, \bar{D}\cap\bar{E}\rangle} B$, and
(c) $B \trianglelefteq_{\div\langle S, \bar{B}\cap\bar{C}, \bar{E}\cap\bar{F}\rangle} E$, and
(d) $F \trianglelefteq_{\div\langle S, \bar{B}\cap\bar{C}, \bar{E}\cap\bar{F}\rangle} C$.

From this we want to deduce

(e) $A \trianglelefteq_{\div\langle S, \bar{A}\cap\bar{C}, \bar{D}\cap\bar{F}\rangle} D$, and
(f) $F \trianglelefteq_{\div\langle S, \bar{A}\cap\bar{C}, \bar{D}\cap\bar{F}\rangle} C$.

For this purpose we have to distinguish six cases; however, the further cases are only variations of the first.

In the first case we assume $C \trianglelefteq_{\div\langle S\rangle} B \trianglelefteq_{\div\langle S\rangle} A$. This entails by Lemma 6.6 that also $F \trianglelefteq_{\div\langle S\rangle} E \trianglelefteq_{\div\langle S\rangle} D$. From this and the premises conclusion (f) follows trivially. As to (e), we may reason as follows: First, premise (c) entails $B \trianglelefteq_{\div\langle S\rangle} E$, because the last two contractions of (c) are vacuous. So, indeed, $B \trianglelefteq_{\div\langle S\rangle} A, E, D$. (IC4) thus yields $\div\langle S, \bar{A}\cap\bar{B}, \bar{D}\cap\bar{E}\rangle = \div\langle S, \bar{A}\cap\bar{B}\cap\bar{D}\cap\bar{E}, \bar{D}\cap\bar{E}\rangle$. Hence, we have $A \trianglelefteq D$ not only under the contraction of premise (a), but also under $\div\langle S, \bar{A}\cap\bar{B}\cap\bar{D}\cap\bar{E}, \bar{D}\cap\bar{E}\rangle$, thus with the strict part of (IC5) under $\div\langle S, \bar{A}\cap\bar{B}\cap\bar{D}\cap\bar{E}\rangle$ and with (IC7) under $\div\langle S\rangle$, and so finally under $\div\langle S, \bar{A}\cap\bar{C}, \bar{D}\cap\bar{F}\rangle$ thanks to (IC2) (*strong vacuity*). This establishes (e).

In the second case we assume $C \trianglelefteq_{\div\langle S\rangle} A \lhd_{\div\langle S\rangle} B$. Lemma 6.6 guarantees $F \lhd_{\div\langle S\rangle} E$. Again, (c) entails $B \trianglelefteq_{\div\langle S\rangle} E$ (even if $D \lhd_{\div\langle S\rangle} F$). With the strict part of Lemma 6.7 this entails $A \trianglelefteq_{\div\langle S\rangle} D$. Hence also $F \trianglelefteq_{\div\langle S\rangle} D$. So, again (f) follows trivially from our assumptions. We might have $E \trianglelefteq_{\div\langle S\rangle} D$, in which case (e) is trivial. Therefore, the interesting case is $F \trianglelefteq_{\div\langle S\rangle} D \lhd_{\div\langle S\rangle} E$. Now, however, we can reason as in the first case.

The third case is characterized by $B \lhd_{\div\langle S\rangle} C \trianglelefteq_{\div\langle S\rangle} A$. So $E \lhd_{\div\langle S\rangle} D$ by Lemma 6.6. Now, (b) entails $E \trianglelefteq_{\div\langle S\rangle} B$ (even if $F \trianglelefteq_{\div\langle S\rangle} E$). Hence, $A \trianglelefteq_{\div\langle S\rangle} D$ by (a), and also $F \trianglelefteq_{\div\langle S\rangle} D$ (since $E \trianglelefteq_{\div\langle S\rangle} B \trianglelefteq_{\div\langle S\rangle} C \trianglelefteq_{\div\langle S\rangle} D \lhd_{\div\langle S\rangle} F$ would contradict (c) and (d) because of the strict part of Lemma 6.7). So, again, (f) follows trivially. We might have $F \trianglelefteq_{\div\langle S\rangle} E$; then (e) is trivial. Therefore, the interesting case is $E \lhd_{\div\langle S\rangle} F \trianglelefteq_{\div\langle S\rangle} D$. In this case, premise (d) and (IC4) yield $\div\langle S, \bar{A}\cap\bar{C}, \bar{D}\cap\bar{F}\rangle = \div\langle S, \bar{A}\cap\bar{C}\cap\bar{D}\cap\bar{F}, \bar{A}\cap\bar{C}\rangle$, from where we can infer (e) in a way analogous to the first case.

The fourth case is given by $B \lhd_{\div\langle S\rangle} A \lhd_{\div\langle S\rangle} C$. Lemma 6.6 again guarantees $E \lhd_{\div\langle S\rangle} D$. If $F \trianglelefteq_{\div\langle S\rangle} D$, the conclusions follow trivially. So, we may focus on the case $E \lhd_{\div\langle S\rangle} D \lhd_{\div\langle S\rangle} F$. Now, conclusion (e) is obvious. Premise (a) yields with (IC4) that $\div\langle S, \bar{A}\cap\bar{C}, \bar{D}\cap\bar{F}\rangle = \div\langle S, \bar{A}\cap\bar{C}\cap\bar{D}\cap\bar{F}, \bar{D}\cap\bar{F}\rangle$, and then the familiar reasoning carries us from premise (d) to conclusion (f).

In the fifth case we have $A \lhd_{\div\langle S\rangle} C \trianglelefteq_{\div\langle S\rangle} B$. Lemma 6.6 now entails $F \trianglelefteq_{\div\langle S\rangle} E$. Again, if $F \trianglelefteq_{\div\langle S\rangle} D$, the conclusions follow trivially, and we may turn to the case $D \lhd_{\div\langle S\rangle} F \trianglelefteq_{\div\langle S\rangle} E$. (e) follows obviously. (IC4) and premise (d) yield $\div\langle S, \bar{B}\cap\bar{C}, \bar{E}\cap\bar{F}\rangle = \div\langle S, \bar{B}\cap\bar{C}\cap\bar{E}\cap\bar{F}, \bar{B}\cap\bar{C}\rangle$, and (f) follows in the same way as before.

The final case is $A \lhd_{\div\langle S\rangle} B \lhd_{\div\langle S\rangle} C$. Once more, the conclusions are trivial given $F \trianglelefteq_{\div\langle S\rangle} D$. So assume $D \lhd_{\div\langle S\rangle} F$. So, again (e) follows immediately. We cannot infer the position of E relative to D and F. We might have $F \trianglelefteq_{\div\langle S\rangle} E$, in which case (e) follows trivially. Or we might have $E \lhd_{\div\langle S\rangle} F$. Then, (c) yields with (IC4) that $\div\langle S, \bar{B}\cap\bar{C}, \bar{E}\cap\bar{F}\rangle = \div\langle S, \bar{B}\cap\bar{C}\cap\bar{E}\cap\bar{F}, \bar{E}\cap\bar{F}\rangle$; and (f) follows from (d) in the usual way. □

After this steep ascent, the rest is easy walking. It remains to show that all the properties established for disjoint difference comparisons carry over to unrestricted difference comparisons. Here, we hardly get along without richness assumptions. We had already seen that Definition 4.10 would not have made sense without the assumption of 4-richness. In translating our findings we should take the liberty to even assume 6-richness; then the generalization will be straightforward. The only point we have to observe is that the translations all agree. This will be guaranteed by the next three lemmata:

Lemma 6.12. *For any IC $\div$ and eight mutually disjoint $A, A', B, B', C, C', D, D' \in \mathcal{A} - \mathcal{N}$, if $(A - B) \preceq^d_{\div} (C - D)$ and $A' \trianglelefteq_{\div} A$, $B \trianglelefteq_{\div} B'$, $C \trianglelefteq_{\div} C'$, and $D' \trianglelefteq_{\div} D$, then $(A' - B') \preceq^d_{\div} (C' - D')$.*

Proof. The disbelief comparisons in Lemma 6.12 entail with Lemma 6.7 that $(A' - B') \preceq^d_{\div} (A - B)$ and $(C - D) \preceq^d_{\div} (C' - D')$. Then the conclusion follows with Lemma 6.9, the transitivity of $\preceq^d_{\div}$. □

Lemma 6.13. *For any IC $\div$ and eight mutually disjoint $A, A', B, B', C, C', D, D' \in \mathcal{A} - \mathcal{N}$, if $A \trianglelefteq_\div A'$, $B \trianglelefteq_\div B'$, $C \trianglelefteq_\div C'$, and $D \trianglelefteq_\div D'$, then $(A - B) \preceq^d_\div (C - D)$ iff $(A \cup A' - B \cup B') \preceq^d_\div (C \cup C' - D \cup D')$.*

Proof. Let $S = \langle \bar{A} \cap \bar{A}' \cap \bar{B} \cap \bar{B}', \bar{C} \cap \bar{C}' \cap \bar{D} \cap \bar{D}' \rangle$. By (IC5) the disbelief comparisons in the premises hold also under the contraction $\div\langle S \rangle$. We know from Lemma 6.3 that $(A - B) \preceq^d_\div (C - D)$ iff $A \trianglelefteq_{\div\langle \bar{A} \cap \bar{B}, \bar{C} \cap \bar{D} \rangle} C$ and $D \trianglelefteq_{\div\langle \bar{A} \cap \bar{B}, \bar{C} \cap \bar{D} \rangle} B$. Now we can reason as follows: The comparison $A \trianglelefteq C$ holds under $\div\langle \bar{A} \cap \bar{B}, \bar{C} \cap \bar{D} \rangle$
iff it holds under $\div\langle \bar{A} \cap \bar{B}, \bar{A}' \cap \bar{B}', \bar{C} \cap \bar{D}, \bar{C}' \cap \bar{D}' \rangle$ (because of (IC5))
iff it holds under $\div\langle \bar{A} \cap \bar{B} \cap \bar{A}' \cap \bar{B}', \bar{A}' \cap \bar{B}', \bar{C} \cap \bar{D} \cap \bar{C}' \cap \bar{D}', \bar{C}' \cap \bar{D}' \rangle$ (IC4) iff it holds under $\div\langle S \rangle$ (again because of (IC4))
iff $A \cup A' \trianglelefteq_{\div\langle S \rangle} C \cup C'$. The same holds for the comparison $B \trianglelefteq D$. □

Lemma 6.14. *For any IC $\div$ and any $A, A', B, B', C, C', D, D' \in \mathcal{A} - \mathcal{N}$ such that A, B, C, and D are mutually disjoint and A', B', C', D' are mutually disjoint, we have: if $A \triangleq_\div A'$, $B \triangleq_\div B'$, $C \triangleq_\div C'$, and $D \triangleq_\div D'$, then $(A - B) \preceq^d_\div (C - D)$ iff $(A' - B') \preceq^d_\div (C' - D')$.*

Proof. Let $A^* = B \cup B' \cup C \cup C' \cup D \cup D'$ and define B^*, C^*, and D^* correspondingly. Since $A, A' \trianglelefteq_\div \bar{A} \cap A' - A^*, A \cap \bar{A}' - A^*$ and likewise for B, B' and C, C' and D, D', the assertions to be proved equivalent are, according to Lemma 6.13, both equivalent to $(A \cup A' - A^* - B \cup B' - B^*) \preceq^d_\div (C \cup C' - C^* - D \cup D' - D^*)$, which is indeed about eight mutually disjoint propositions. □

Now we are in a position to confirm all the expected properties of the difference comparison $\preceq_\div$ induced by the IC $\div$.

Lemma 6.15. *For any 4-rich IC $\div$ and $A, B, C, D \in \mathcal{A} - \mathcal{N}$, either $(A - B) \preceq_\div (C - D)$ or $(C - D) \preceq_\div (A - B)$ or both* [completeness].

Proof. Let us do it once carefully. By definition, the assertion to be proven is: either there exist mutually disjoint A', B', C', D' with $A \triangleq_\div A'$, etc., and $(A' - B') \preceq^d_\div (C' - D')$ or there exist mutually disjoint A'', B'', C'', D'' with $A \triangleq_\div A''$, etc., and $(C'' - D'') \preceq^d_\div (A'' - B'')$; that is, there exist mutually disjoint A', B', C', D' and mutually disjoint A'', B'', C'', D'' such that $A \triangleq_\div A' \triangleq_\div A''$, etc., and $(A' - B') \preceq^d_\div (C' - D')$ or $(C'' - D'') \preceq^d_\div (A'' - B'')$. With Lemma 6.14, however, the latter assertion reduces to Lemma 6.8. □

Lemma 6.16. *For any 6-rich IC $\div$ and $A, B, C, D, E, F \in \mathcal{A} - \mathcal{N}$, if $(A - B) \preceq_\div (C - D)$ and $(C - D) \preceq_\div (E - F)$, then $(A - B) \preceq_\div (E - F)$* [transitivity].

With Lemma 6.14, Lemma 6.16 reduces to Lemma 6.9 in the same way as before. With the assumption of 6-richness this reduction runs smoothly for sure. We did not take the trouble to check whether less richness would suffice as well. In a similar vein we may show:

Lemma 6.17. *For any 4-rich IC $\div$ and $A, B, C, D, \in \mathcal{A} - \mathcal{N}$, if $(A - B) \preceq_\div (C - D)$, then $(C - D) \preceq_\div (B - A)$* [sign reversal];

and

Lemma 6.18. *For any 6-rich IC $\div$ and $A, B, C, D, E, F \in \mathcal{A} - \mathcal{N}$, if $(A - B) \preceq_\div (D - E)$ and $(B - C) \preceq_\div (E - F)$, then $(A - C) \preceq_\div (D - F)$* [monotonicity].

We have not yet attended to the law of disjunction for difference comparison. Of course, it does not pose a difficulty:

Lemma 6.19. *For any 5-rich IC $\div$ and any $A, B \in \mathcal{A} - \mathcal{N}$, if $(A - W) \preceq_\div (B - W)$, then $(A - W) \approx_\div (A \cup B - W)$* [law of disjunction].

Proof. Suppose there are five mutually disjoint A', B', C', D', and E' such that $A' \triangleq_{\div} A$, $B' \triangleq_{\div} B$, $C' \triangleq_{\div} A \cup B$, and $D \triangleq_{\div} E \triangleq_{\div} W$ so that $(A' - D') \preceq^{d}_{\div} (B' - E')$. According to Lemma 6.3, this is equivalent to $A' \trianglelefteq_{\div} B'$ (since the contractions required by Lemma 6.3 are vacuous and the other half of Lemma 6.3 is trivial). Hence, by Lemma 6.2 $A' \triangleq_{\div} C'$, and by the reverse reasoning $(A' - D') \approx^{d}_{\div} (C' - E')$. $\square$

Concerning the remaining properties of IC's we have no ambitions. Let us simply accept

Definition 6.20. Let $\div$ be an IC for $(\mathcal{A}, \mathcal{N})$. Then $\div$ is called *Archimedean* iff the DoxDC $\preceq_{\div}$ induced by $\div$ is Archimedean. And $\div$ is called *full* iff the DoxDC $\preceq_{\div}$ induced by $\div$ is full.

We did not attempt to express the Archimedean property purely in terms of iterated contraction; this appears to be an unilluminating exercise. Likewise, though fullness is easily translated into contractions, its original explanation in terms of difference comparisons is the most perspicuous.

All this can finally be summed up in the following theorem of which we have proved every part:

Theorem 6.21. *For any IC $\div$ for $(\mathcal{A}, \mathcal{N})$, the potential DoxDC $\preceq_{\div}$ induced by $\div$ is a DoxDC for $(\mathcal{A}, \mathcal{N})$. And any 6-rich full Archimedean IC $\div$ for $(\mathcal{A}, \mathcal{N})$ is an IRC for $(\mathcal{A}, \mathcal{N})$, i.e., there is a negative ranking function κ for $\mathcal{A}$ with $\div = \div_{\kappa}$. Moreover, for each ranking function κ' with $\div = \div_{\kappa'}$ there is an $\alpha > 0$ such that $\kappa' = \alpha\kappa$.*

7. Conclusion

What did we achieve? We have seen how potential IC's induce potential DisDC's, how rich potential IC's induce potential DoxDC's, that the right kind of potential IC's, namely those satisfying (IC1)–(IC6) or, what comes to the same, IRC's, indeed induce DoxDC's, and that DoxDC's, if they are full and Archimedean, determine ranking functions uniquely up to a multiplicative constant. In particular, this means that we may start from a rich ranking function κ, then consider only the rich IRC $\div_{\kappa}$ associated with κ, and finally reconstruct the whole of κ from $\div_{\kappa}$ in the unique way indicated.

This appears to be a most satisfying representation result. The only data we need are the beliefs under various iterated contractions. These data reflect not only the comparative strength of beliefs, they also reflect the comparative nature of reasons and relevance. And these inferred comparisons suffice to fix the cardinal structure of ranking functions. Moreover, it shows that (IC1)–(IC6) cannot be essentially improved, at least if iterated contraction is conceived as proposed by ranking theory. This proviso can be dropped, if our comparative discussion of other models of iterated contraction is correct.

In any case, our basic results should open the way for variants, generalizations, and more elegant formulations, and maybe for a characterization of iterated revision along the lines mentioned at the end of Section 2. We hope to have thus forged new means and methods for the topic of iterated revision and contraction.

References

[1] C.E. Alchourrón, P. Gärdenfors, D. Makinson, On the logic of theory change: Partial meet functions for contraction and revision, Journal of Symbolic Logic 50 (1985) 510–530.

[2] A. Bochman, Contraction of epistemic states: A general theory, in: M.-A. Williams, H. Rott (Eds.), Frontiers in Belief Revision, Kluwer, Dordrecht, 1999, pp. 195–220.

[3] C. Boutilier, Revision sequences and nested conditionals, in: R. Bajcsy (Ed.), IJCAI-93—Proceedings of the 13th International Joint Conference on Artificial Intelligence, 1993, pp. 519–525.

[4] C. Boutilier, Iterated revision and minimal change of conditional beliefs, Journal of Philosophical Logic 25 (1996) 263–305.

[5] J. Cantwell, On the logic of small changes in hypertheories, Theoria 63 (1997) 54–89.

[6] L.J. Cohen, The Implications of Induction, Methuen, London, 1970.

[7] A. Darwiche, J. Pearl, On the logic of iterated belief revision, Artificial Intelligence 89 (1997) 1–29.

[8] D. Dubois, H. Prade, Possibilistic inference under matrix form, in: H. Prade, C.V. Negoita (Eds.), Fuzzy Logic in Knowledge Engineering, Verlag TÜV Rheinland, Köln, 1986, pp. 112–126.

[9] D. Dubois, H. Prade, Possibility Theory: An Approach to Computerized Processing of Uncertainty, Plenum Press, New York, 1988.

[10] D. Dubois, S. Moral, H. Prade, Belief change rules in ordinal and numerical uncertainty theories, in: D.M. Gabbay, P. Smets (Eds.), Handbook of Defeasible Reasoning and Uncertainty Management Systems, vol. 3, Kluwer, Dordrecht, 1998, pp. 311–392.

[11] E. Fermé, H. Rott, Revision by comparison, Artificial Intelligence 157 (2004) 5–47.
[12] P. Gärdenfors, Conditionals and changes of belief, in: I. Niiniluoto, R. Tuomela (Eds.), The Logic and Epistemology of Scientific Change (Acta Philosophica Fennica 30), North-Holland, Amsterdam, 1978, pp. 381–404.
[13] P. Gärdenfors, Knowledge in Flux, MIT Press, Cambridge, MA, 1988.
[14] P. Gärdenfors, H. Rott, Belief revision, in: D.M. Gabbay, C.J. Hogger, J.A. Robinson (Eds.), Handbook of Logic in Artificial Intelligence and Logic Programming, vol. 4, Epistemic and Temporal Reasoning, Oxford University Press, Oxford, 1995, pp. 35–132.
[15] M. Goldszmidt, J. Pearl, Qualitative probabilities for default reasoning, belief revision, and causal modeling, Artificial Intelligence 84 (1996) 57–112.
[16] J.Y. Halpern, Reasoning about Uncertainty, MIT Press, Cambridge, MA, 2003.
[17] S.O. Hansson, Reversing the Levi identity, Journal of Philosophical Logic 22 (1993) 637–669.
[18] M. Hild, A representation theorem for iterated contraction, unpublished manuscript, 1997.
[19] F. Huber, Ranking functions and rankings on languages, Artificial Intelligence 170 (2006) 462–471.
[20] R.C. Jeffrey, The Logic of Decision, University of Chicago Press, Chicago, 1965, second ed. 1983.
[21] D.H. Krantz, R.D. Luce, P. Suppes, A. Tversky, Foundations of Measurement, vol. 1, Academic Press, New York, 1971, vol. 2, 1989, vol. 3, 1990.
[22] D. Lehmann, Belief revision, revised, in: C.S. Mellish (Ed.), IJCAI '95—Proceedings of the 14th International Joint Conference on Artificial Intelligence, vol. 2, Morgan Kaufmann, San Mateo, 1995, pp. 1534–1540.
[23] I. Levi, Mild Contraction: Evaluating Loss of Information Due to Loss of Belief, Oxford University Press, Oxford, 2004.
[24] A.C. Nayak, Iterated belief change based on epistemic entrenchment, Erkenntnis 41 (1994) 353–390.
[25] A.C. Nayak, R. Goebel, M.A. Orgun, Iterated belief change from first principles, in: IJCAI-07. Proceedings of the 20th International Joint Conference of Artificial Intelligence, 2007, pp. 2568-2573.
[26] H. Rott, Coherence and conservatism in the dynamics of belief. Part I: Finding the right framework, Erkenntnis 50 (1999) 387–412.
[27] H. Rott, Change, Choice, and Inference. A Study of Belief Revision and Nonmonotonic Reasoning, Oxford University Press, Oxford, 2001.
[28] H. Rott, Shifting priorities: Simple representations for twenty seven iterated theory change operators, in: D. Makinson, J. Malinowski, H. Wansing (Eds.), Towards Mathematical Philosophy, Springer, Dordrecht, 2008.
[29] G.L.S. Shackle, Decision, Order, and Time in Human Affairs, second ed., Cambridge University Press, Cambridge, 1961, 1969.
[30] W. Spohn, Eine Theorie der Kausalität, unpublished Habilitationsschrift, Universität München, 1983, available at: http://www.uni-konstanz.de/FuF/Philo/Philosophie/Spohn/spohn_files/Habilitation.pdf.
[31] W. Spohn, Ordinal conditional functions. A dynamic theory of epistemic states, in: W.L. Harper, B. Skyrms (Eds.), Causation in Decision, Belief Change, and Statistics, vol. II, Kluwer, Dordrecht, 1988, pp. 105–134.
[32] W. Spohn, Ranking functions, AGM style, in: B. Hansson, S. Halldén, N.-E. Sahlin, W. Rabinowicz (Eds.), Internet-Festschrift for Peter Gärdenfors, 1999, http://www.lucs.lu.se/spinning/.
[33] W. Spohn, A survey of ranking theory, in: F. Huber, C. Schmidt-Petri (Eds.), Degrees of Belief. An Anthology, Springer, Dordrecht, 2008.
[34] J. von Neumann, O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, Princeton, 1944, second ed. 1947.
[35] M.-A. Williams, Iterated theory base change: A computational model, in: C.S. Mellish (Ed.), IJCAI '95—Proceedings of the 14th International Joint Conference on Artificial Intelligence, vol. 2, Morgan Kaufmann, San Mateo, 1995, pp. 1541–1547.