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The well-designed logical robot: Learning and experience from observations to the Situation Calculus

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ABSTRACT

The well-designed logical robot paradigmatically represents, in the words of McCarthy, the abilities that a robot-child should have to reveal the structure of reality within a “language of thought”. In this paper we partially support McCarthy's hypothesis by showing that early perception can trigger an inference process leading to the “language of thought”. We show this by defining a systematic transformation of structures of different formal languages sharing the same signature kernel for actions and states. Starting from early vision, visual features are encoded by descriptors mapping the space of features into the space of actions. The densities estimated in this space form the observation layer of a hidden states model labelling the identified actions as observations and the states as action preconditions and effects. The learned parameters are used to specify the probability space of a first-order probability model. Finally we show how to transform the probability model into a model of the Situation Calculus in which the learning phase has been reified into axioms for preconditions and effects of actions and, of course, these axioms are expressed in the language of thought. This shows, albeit partially, that there is an underlying structure of perception that can be brought into a logical language.

To John and his surprising innate abilities

1. Introduction

1.1. Foreword

In this paper I discuss McCarthy's exploration of the relation between appearance and reality in the “well-designed child” [77,79]. McCarthy's arguments, in my opinion, are twofold. The first addresses the need to axiomatise the appearance–reality relation within a *language of thought*, while the second, addressing the *archetype* of inner abilities, contests the ability of current learning methods to explain the appearance–reality relation. My thesis here is to show that early learning, via perception, is necessary to tune knowledge formation. I argue that the perception of reality, although local and particular, can be shaped by learning the parameters of the appearance of events, and that these parameters provide a structure for a symbolic representation supporting the inference from appearance to reality.

Other than the cited papers and many works that are on McCarthy's web site, some of his arguments about learning are taken from my personal conversations with him during these last 18 years of friendship.

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1.2. The well-designed child

In his notes on *The well-designed child* [77,79], John McCarthy argues that the Lockean view that knowledge is built from sensations is not adequate for what he calls the *designer stance*, a theme taken from Dennett [31], who paradigmatically used it to understand and predict the structure and behaviour of biological and artificial systems.

"In so far as we have an idea what innate knowledge of the world would be useful, AI can work on putting it into robots, and cognitive science and philosophy can look for evidence of how much of it evolved in humans. This is the designer stance."

Evidence of evolution efficiency [77,79] supersedes the point of view, shared by behaviourism in psychology and positivist philosophy, that knowledge proceeds from sensations. Evolution had gradually shaped optimal predispositions in humans, in so determining a correspondence between an innate mental structure and facts about the world. The criticism of the Lockean child, nevertheless, is not addressed as a mind/body problem; instead, stepping aside it, McCarthy offers the bold view of the robot-child experiment: an AI (or possibly a psychological) system replicating the intelligent behaviour of a child. So what is innate and what can be learned? McCarthy recognises learning as central to the experiment design, but grounded in meanings independent of sensations and facts, "what a child learns about the world is based on its innate mental structure" [77,79]. Specifically, the innate mental structure is based on three interdependent aspects:

1. "world characteristics", such as *appearance and reality*, *continuity of motion*, *gravity*, and *relations*,
2. "mental characteristics" such as *central decision making*, *incompleteness of appearance*, *senses*,
3. "innate abilities" such as *introspection*, *noise rejection*, *principle of mediocrity*.

These innate mental structures evolved for survival and control purposes, thus are necessitated by adaptation "A baby innately equipped to deal with them will outperform a Lockean baby". They have been investigated also in neurophysiological studies; for example, it is known that the vestibular system has specialised sensors, such as canals and otoliths, to detect gravity direction and disambiguate spatial orientation and localisation. Likewise there is a specialised motion sensitive area in the temporal cortex for both control of locomotion and perception of spatial relations. Most important, in connection with what McCarthy calls the *principle of mediocrity*, both imitation and the innate ability to *put oneself in the place of others* have been explained by the mirror neurons discovered by Rizzolatti and colleagues [103,63,39].

Psychological, neurological and neurophysiological findings on the function and structure of the brain justify the above-mentioned innate abilities. Innate structures take a step beyond the view (see e.g. Russell in [105]) that sensations alone determine knowledge. The question, from the designer stance, is whether it is possible, with a *language of thought*, in McCarthy words, to represent the innate child's brain structure by means of logical sentences?

1.3. Appearance and Reality: the language of thought

McCarthy discusses the implementation of four innate structures, namely *the relation of appearance and reality, persistent objects, the spatial and temporal continuity of perception* and *the language of thought*. In his notes on *Appearance and Reality: a challenge to machine learning* [78], that complements those on *Appearance and Reality* included in the *Well-designed child*, McCarthy says that classifying "experience is inadequate as a model of human learning and also inadequate for robotic applications". He argues that mere classification does not "dis-cover" (in the sense of "un-cover" and "reveal") the hidden reality behind appearance. To unlock reality behind appearance McCarthy suggests "*brute force logical attitude* toward making their relations (appearance–reality) explicit", and proposes to axiomatise their relation in a formal language, such as the Situation Calculus, as follows:

$$holds\big(appears(appearance, object), s\big) \tag{1}$$

So, for each single object one would have an accurate description through the term appearance. Instances of the term *appearance* would need some recursive definition or some intended interpretation. These descriptions of the term "appearance" are not qualia,[1] in the sense of attributes of pure perceptual experience [70,44], but designate quantifiable attributes of physical objects, e.g., as projected on the retina, however not necessarily evoking the sensory experience.

A well-known argument about the relation between innate knowledge (abilities) and learned knowledge is that of *Mary's Room* [56,57,32] and it is interesting to compare it with McCarthy's arguments. We recall that Jackson's knowledge argument was the following. Mary is a perfect scientist who cannot see colours but knows all the cause-effect laws that regulate the human perception of colours and the utterance of colour nouns of any object whatsoever [56,57]. When she eventually is made capable to see colours would she learn anything? Would she have an experience inducing knowledge different from her current knowledge which, by hypothesis is complete? Daniel Dennett in [32] rejects the argument that Mary derives incremental knowledge when she is allowed to see colours, based on the fact that her knowledge is complete. Furthermore, having reformulated the argument to make Mary a perfect logician [33,55] he argues that when Mary finally can see the colours and she is offered a blue banana, then she is able to discover that she has been deceived as she knows

[1] Note that also in [76] McCarthy proposes a *basic consciousness* made of propositions and images of scenes and objects.

what bananas look like (see also [32]). Dennett's statement [33,55] is to clarify the abstract precondition of the knowledge argument, describing the best circumstances under which Mary's (RoboMary's) knowledge is to be complete, so that she can infer everything, just applying useful lemmas (actually her knowledge should be equivalent to its deductive closure). "What matters is whether Mary (or RoboMary) can deduce what it's like to see red from her complete physical knowledge, not whether one could use one's physical knowledge in some way or other to acquire knowledge of what it's like to see in colour" [33]. Obviously if RoboMary has complete knowledge [33] her perceptual experience (or any form of knowledge acquisition whatsoever) is unnecessary and irrelevant. In fact, by definition, nothing can be added to a complete theory, without making it inconsistent.

However, as far as visual perception is concerned, it should be pointed out, that there is no complete axiomatisation even for the partial structure concerning colour perception, as it would at least include arithmetic, so there would be true appearances that could not be derived.

Clearly McCarthy is not proposing a complete axiomatisation of the appearance–reality relation. Nevertheless a concrete question about the "brute force logical attitude" is whether any such term designating the appearance of a physical object can make the relation "appear" a logical truth, accounting for the innate brain structure, such as colour-selective perceptive fields, chromatic adaptation, and the Purkinje effect, and with correlated phenomena, such as light change or background changes.

1.4. From the language of thought to learning

An immediate and concrete problem is how the robot-child uses its sensors. A robot-child is endowed with sound and visual sensors, and proprioperception sensors for moving. The term designating "appearance", as posed in (1), cannot match the continuous stimuli obtained from the sensor instruments. It might be possible that the appearance term is either not accurate enough or too accurate, or that there is a variability in the object phenomenology or a variability in the instruments that has not been taken care of by the amanuensis instructor who "if any, should have to know the subject matter and very little about how the program or hardware works" [77]. So a typical situation would be a mismatch between the term designating appearance and the measurements, since for the robot-child there is no way (in the language of thought) to sample data from its instrument and obtain an adequate model of the current stimulus. Even if the innate abilities could be hardwired in the sensors, the pathway from sensors to reasoning must be explained. Perceptual stimuli, driven from appearance, though informing the robot-child's reasoning, undergo a complex and hardly predictable processing that cannot be described by logical sentences. In our view the variability of the *appearance* induces a stimulus gradient which, through the learning experience, solicits the "innate abilities" to determine (possibly new) correlations between events and objects. An innate ability is, for example, finding hidden parameters for any partially perceivable phenomenon, and eliciting from these a prediction of future appearance.

A single statement in which "both appearances of objects and physical objects will be represented as logical objects, i.e. as the values of variables and terms" [77,79] seems to reduce the power of scientific findings to the Kantian synthetic truth [60]. Whereas, as Sloman would note [111,112], there is a tradeoff between deterministic and probabilistic causation (Humean versus Kantian view).

Furthermore, appearance categorisation in the language of thought requires the language to name all the phenomena denoted in the ontology. An enumeration of objects and phenomena would be against a parsimony principle that entia non sunt multiplicanda praeter necessitatem. The parsimony principle, instead, is applied when learning from experience. When we look at a scene, we can distinguish more entities than we can designate, yet the detailed acknowledgement of several "un-named" phenomena allows us to abstract concepts, find correlations, indicate causal power, and infer the deep and essential structure of the scene.

Designation of appearance, in the language, is the result of some episodic inference producing knowledge of causation through perceptual evidence of correlation.

It still remains open how to determine what could be an appropriate explanation of the appearance–reality relation. Recall that we started with "the design of a robot-child that has some chance of learning from experience and education". In their developmental studies, Gopnik, Meltzoff and colleagues argue that children's theories converge to accurate descriptions, by learning mechanisms that are analogous to "scientific-theory formation" [47], see also [46,45,93,80]. On this basis we can view the problem of modelling the relation appearance reality as the relation between science and experimental data. In this way the appearance–reality problem is transformed into a problem of causation and prediction. Sloman, in [111, 112], discussing the Gopnik and colleagues' view of a child as a tiny scientist, proposes a cleavage between Humean and Kantian attitude towards causation. The first involving the direct experience of co-occurrence and correlations that can be compared to what Gopnik and Shulz [47] consider the children's learning mechanisms. That is, the Humean view specifies correlations among physical objects in terms of probabilities and causation. On the other hand, the Kantian view concerns the inner structure of causation that affects universal and deterministic laws. He argues that a child starts with a Humean approach to correlation and ends up with a Kantian causal structure of the environment. Sloman foresees a progression from one attitude towards the other "Typically in science causation starts off being Humean until we acquire a deep (often mathematical) theory of what is going on: then we use a Kantian concept of causation".

These issues are close to the reification steps explained by Quine in [97], initiated with *observation sentences* and accomplished with the whole sketch of epistemological settings. *Observation* sentences, also *occasion* sentences, vehicle perceptual stimuli to the formation of a scientific theory. The empirical significance of the observation sentences is in the relation between a scientific theory and its sensory evidence [95]. For Quine it is folly to separate the synthetic sentences "which hold contingently on experience" and the analytic ones "which hold come what may". He speaks of a varying distance from a sensory periphery, where statements are closer to experience, stimuli and measurements, to the centre which is held by "highly theoretical statements of physics or logic or ontology" [94]. Highly theoretical statements constitute the backlog of the scientific theory. To cope with prediction Quine introduces *observation categorical* whose doctrine is to explain "the bearing of sensory stimulation" (see [96,97]).

Although our discussion is quite limited, it seems that different notions of causation are involved in learning mechanisms that take place, initially, with perception, experience and observations.

1.5. Commonsense and learning from experience

At this point we can say that modelling the appearance–reality relation can be reduced to the problem of the scientific theory formation through its natural steps: from the perceptual stimulus to the elicitation of the hypotheses and their confirmation. These steps lead to a description of the appearance and its underlying correlation with objects, possibly counting also refutation.

As it is, at least in its generality, the problem has been addressed since the early approaches to commonsense reasoning, from the foundational *Programs with commonsense* [75] to the recent reviews of Davies and Morgenstern in [35]. In fact, with the well-designed robot-child McCarthy intends explicitly an AI experiment in the commonsense world.

However, in the huge literature on commonsense AI, few approaches have effectively faced the appearance–reality relation. Here we distinguish between attempts to formalise perception and learning. Approaches seeking to formalise perception from the stimulus, here with the specific meaning of the observation and its variability, have been at earliest taken by Levesque in [67] and by Bacchus, Halpern and Levesque in [8] (see also [9]). These are the most thorough attempts to epistemologically formalise sensory data. In particular, [8] has been the very first meaningful attempt to explicate the appearance–reality relation in a simulated system. Similarly the works of [68] and [87,102].

While there is a prodigious literature on learning from perception in the computer vision and statistical learning communities, this is not the case for the knowledge representation and commonsense communities. Often, learning from perception is confused with learning from the *representation of perception*. If the representation of perception is already provided (through specific terms or predicates or any symbolic structure not dealing with measure, scale, dimension, and other quantities) nothing is left to learning, nothing is left to understanding the real problem: what is learned from appearance and how, what is mapped from appearance to reality, and how.

Nevertheless we have to bear in mind that McCarthy starting point, for the robot-child experiment, is the innate mental structures of the child, which is paramount to both the designer stance and the notion of the child as a "tiny scientist".

McCarthy's designer stance could be implemented taking care of a crucial passage. That is, providing a plausible representation for those patterns of sensory stimuli, first classified by computational models of pattern recognition.

An empirical method of learning from experience should formally take care of the inductive attitude (in the sense mentioned by Polya in [89]) towards abstraction, and the determination of "the spatio-temporal world behind appearance" can be obtained by shifting these forms of empirical reasoning into the "language of thought". This coincides with Quine's argument [94] of shifting from the sensory periphery towards the centre of the system, using the categorical observations [97].

1.6. The designer stance

Dennett's *design stance* is concerned with predicting both purpose and function of an entity from its design while McCarthy's *designer stance* is concerned with what features have to be *put* in an intelligent system to implement a behaviour. Thus the designer stance *predicts a behaviour* by defining the rules governing a robot-child's function.

The design of a database of actions begins by specifying a signature, spelling out the words to denote actions, objects, properties and fluents, to formalise a set of tasks. This is the *representation context*. In so doing the designer has already in mind an abstraction layer which is a schematisation of an agent activity, whose execution laws are formalised by a set of axioms. The designer, jumping beyond the learning steps, separates distinctly what is essential from what is inessential, according to her judgement, expressed in form of axioms. These judgements form the instructions of a robot-system tasks execution. This schema is induced not from data but from what she, the designer, theorises is an execution process. The structure of data is not even considered.

On the other hand, here we try to go upstream and conjecture a more general setting for the designer stance. The design of a learning process that starts with perception and ends with an ontology of actions is exemplified in Fig. 1. Here it is supposed that the designer shows a task, such as, for example, opening a door, to someone that does not speak the language and has never seen the task. So she would begin by designating the action names such as *unlock* or *push the door*. But the language would not be self-explaining if at the same time she did not point out the uttered action names while the learner was looking at the sequence. On the other hand, this pointing would not be effective if, at the same time, the learner would not be able to infer, from an enormous amount of data gathered from its stimuli, that these and those events are from a single specific action, the very one indicated by the designer while uttering the action name. So we have added two further

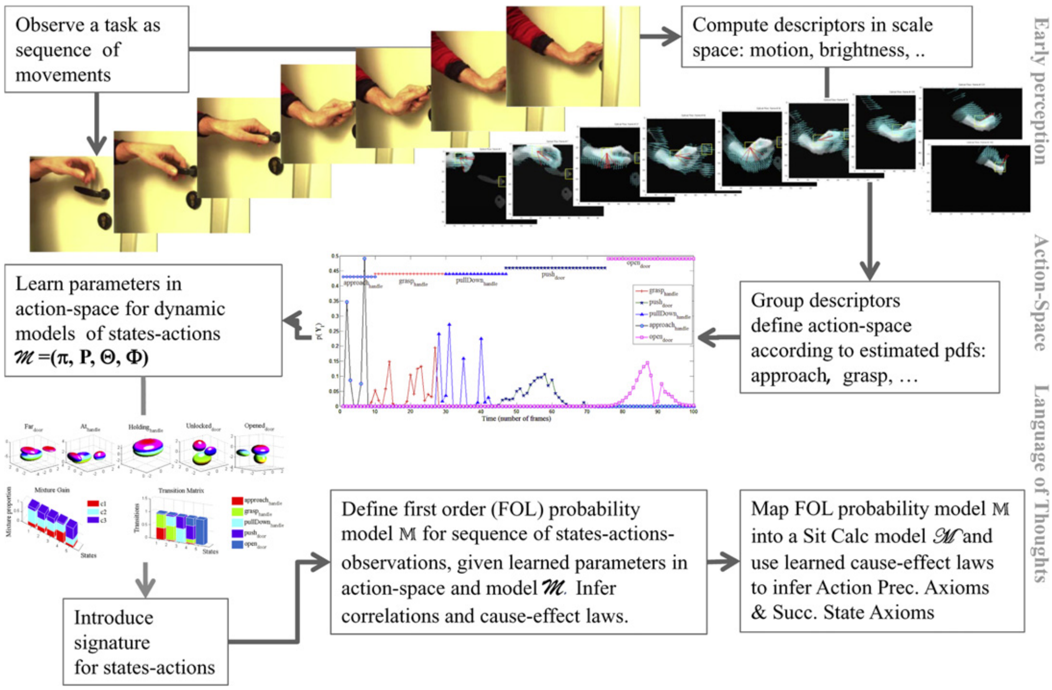


Fig. 1. Structure of the learning model, from observations to axioms, for the task *open the door*.

inference processes: inferring actions from movements and inferring a correlation among actions and between terms and actions.

Showing and pointing is often successful, and not only with children; even dogs and cats learn how to open a door just by looking at adult humans doing so. A dog understands two essential concepts: an aperture needs to be created, somehow, to quickly pass through. The actions are that the door needs to be unlocked by turning or lowering the handle and that it has to be pushed in some direction. How does the dog infer these two concepts, not having hands, and often using the mouth, not even knowing the name of actions and the name of the afforded objects, like the handle or the door?

To implement our interpretation of the designer stance, as hinted above, we formalise a continuous learning process from the perceptual stimulus to the language of thought. Indeed, we show how a simple task such as *opening a door* is learned in three steps of the pathway from perceptual stimulus to categorisation.

The first step is concerned with early visual perception, eliciting a number of actions from a sequence of movements. This step leads to learning to categorise an action from a large number of simpler movements and to correlate actions on the basis of speed of movements, positions in space, distances, centres of mass and their brightness. The second step is concerned with understanding the local causal relations among the actions. This step, which we identify with the induction hypothesis, extends the basic terms to a signature and to a language for the quantities learned in the first step, mapping the specific parameters into a probability space. The third step, the inductive generalisation, is concerned with establishing the rules of what has been learned, for example, that at the end of the task the door is open.

Starting from a number of videos of different people opening a door (actually the same door), we show that a sequence of events, such as the sequence of frames illustrating a hand opening a door, can be used to assess the basic laws of cause and effect that specify the task.

Although this is a simple example it gives a substantial insight into a crucial issue, that is, which are the critical steps of the process that from observations leads to the formation of abstract concepts like the cause and effect laws involved in a simple task. The most important aspect of the transformation is to move from quantitative information (the observations) to qualitative judgements (the laws). The information collected from observations lies in a high-dimensional space (colours, intensity, velocity, positions, etc.) and the simplest way to reduce the dimension and move towards a qualitative assessment of the underpinning properties of the observed events, is to use probability functions. Stochastic variables, in a true Bayesian conception, are by far the only entities that can convey the beliefs about observations, expressed as measures in terms of priors and posterior probabilities, into epistemic states that can further be named by words or specified by properties. For example, the probability of a certain motion direction, given the parameters of a class, becomes the belief that a certain action can be executed after another one has produced its effects.

Our idea, that we illustrate below with the aid of the *opening the door* task, is that the data underlying the observations can be explained with stochastic variables denoting values of actions and states. These values initially describe the probability of events, for example, a sequence of movements is specified by descriptors assessed via visual perception. Further, the induced densities generate the action space by gathering the movements into main actions that can be named. For

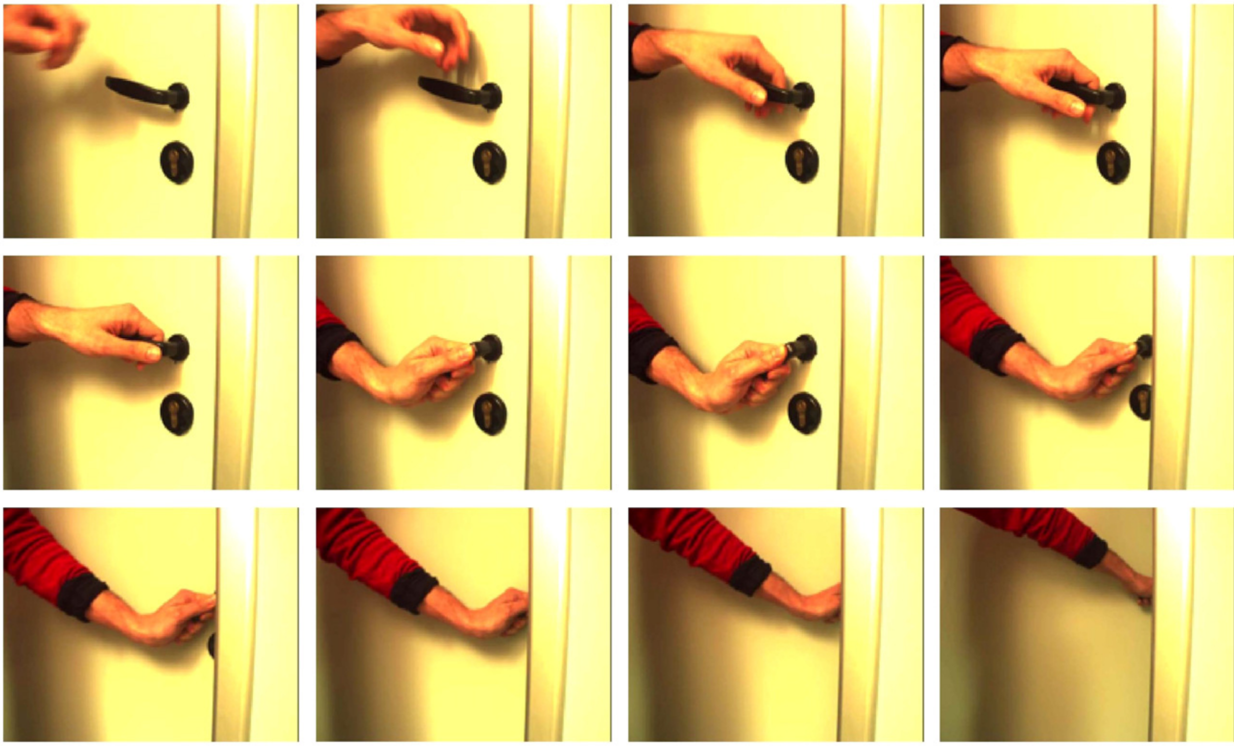


Fig. 2. A set of frames showing a sequence of moves of a hand opening a door for the task *open the door*.

example, in a sequence of 50 frames showing a hand lowering the handle of a door, there are 50 movements that can be grouped together in the unique action *lowering the handle*, thus a signature naming the chosen actions can be provided. When actions leads to states the signature can be expanded with more terms and functions, which can be given a more abstract representation via predicates and formulae of a logical language. The role of the probabilistic logic of processes, that we introduce, is to provide a structure in which the observations can obtain their representation in a formal language that conjugate measures and truth values. In the final transformation measures can be eliminated and the ontology can be fully expressed in the "language of thought". That is, the relationships between actions and states can be expressed in the formal language because they are no more contingent to the observations, nor to the specific measures gathered from the example, and thus they can be taken to be general. And at this point we are where the designer stance begun: designing the causal and effect laws governing a specific task, like opening the door; but now prediction of behaviours is based on the results of learning the inner structure of actions and their preconditions and effects, from the specific perceptual experience. The designer stance is to allow the full pathway for prediction, not limited by the particular, although postulated as general, categories of the designer.

1.7. Paper organisation

In the following sections we discuss the process of learning the task *opening the door* in its full generality. The task is illustrated to the observer (ideally the robot-child) via a number of video sequences similar to Fig. 2. Thus the next section, Section 2, describes early perception and how this analysis is devoted to the definition of an action space, shaping descriptors mainly from motion analysis [72,53,17,16]. Section 3 shows the preliminary formalisation of the observations and prove effectiveness of the action space, estimating their density and grouping with mixtures of principal component analysers [104,113,114]. These mixtures form the observation model of an HMM, described in Section 4, focusing also on how to move from the action space to the dynamic of actions.

Further, in Section 5 we show how to map the probability space of the learned distribution parameters to the domain of a probability model of these processes, in which states and actions are formalised in a first-order language. To this end we show that we can transform a hidden Markov model $\mathcal{M}$ into a first-order model $\mathbb{M}$ of processes (see Section 5). Differently from other approaches to statistical logic with probability on the domain [7,49,6], our focus is on setting the parametric distribution on the domain and hence on the interpretation of the language. In fact, we introduce a method for fitting the learned parameters to the formulae of the language. A typical problem with probabilistic logics is that it is not known where probabilities come from, these have to be specified a priori, or obtained by uniform distributions on the structures (counting measures). In contrast to these approaches, our transformations deal with stochastic processes but also determine the learned distribution, indeed learned from the observations, and map it to the domain of the first-order interpretation. Our approach is also rather different from the work of, for example, [24,25,83,99] and [107] because we can directly map the computational learning model to the first-order domain. In this sense our formalisation of a first-order probability model is closer to the concept of BLOG [81], inasmuch as the domain inherits a distribution just by suitably fixing the interpretation of the atoms, according to a transformation of the terms of the statistical model. This can be clearly extended to other graphical models and sampling methods. Finally (see Section 6) we show how the probability first-order model $\mathbb{M}$ can be transformed into a model $\mathscr{M}$ of a basic theory of actions in the language of the Situation Calculus. Here, at last, the generalisation is accomplished and thus, for learning purposes, the probabilities can be discarded.

2. Features descriptors and action space

The study of human actions via motion analysis has received a good deal of attention in recent years. We refer the reader to the recent reviews of [54,82,92,65], to get an idea of the progress made on the application of motion analysis to the understanding of human actions and behaviour. Beside being focused on a specific example, the contribution of our approach, with respect to this early phase of the designer stance, is mainly in the specification of a mapping from a feature space to an eigen action space. The eigen action space is obtained by grouping the motion features into actions, by both combining the smoothed functions of the velocity, and estimating the features descriptors density, with mixtures of probabilistic principal component analysers.

The first step of the design is to obtain a set of descriptors of the motion of a hand opening a door. A video sequence of the task *open the door* is composed of 200 to 280 frames at 35–40 fps, where each image has size 640×420 as in Fig. 2. Thus, a sequence is a volume or 3D matrix $640 \times 420 \times T$ of frames $I(t)$, $t = 1, \dots, T$. For each sampled video sequence three new volumes are generated. The first one is the result of *early* segmentation obtained by combining attention-based motion detection (see [13]) with an intensity-based mean-shift tracking (see [21,22]). The result of segmentation is a sequence of images in which the non-interesting pixels, with respect to both motion and saliency, are blurred and uniformly set to null. This segmented volume is denoted $\mathbb{G}^T$, and its binarisation, used for further analysis, is defined as:

$$\delta_{\mathcal{G}}(x, y, t) = \begin{cases} 1 & \text{if } \mathbb{G}(x, y, t) > 0,\ \forall x, y, \text{ and } t = 1, \dots, T \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

That is $\delta_{\mathcal{G}}^T$ is like $\mathbb{G}^T$ but the pixel values different from null are set to 1.

Two more volumes are generated from $\mathbb{G}^T$, for the optical flow vector components. The optical flow problem amounts to computing the displacement field between two images, from the image intensity variation. Here it is assumed that the 2D velocity is the projection, on the image plane, of the space–time path of a 3D point. The optical flow vector $w = (u, v, 1)$ is computed between two successive images of an image sequence using a natural constraint based on the principle of brightness constancy.

The optical flow for each frame in the video sequence, computed according to Eq. (36) (see A.3), returns the flow vector $w = (u, v, t)$ for each pixel in the image (Brox [16]). Namely, the optical flow vectors $w = (u, v, t)$ are represented at each time step t by two images $V(t)$ and $U(t)$. Fig. 3 illustrates the direction of the flow vector w at each pixel of the segmented images where the size of the arrows at each pixel (x, y, t) is given by the magnitude. The flow is the main feature for the descriptors of each image (in the volume). Descriptors are a compact representation of the relevant features of the sub-behaviours, specified by motion direction and magnitude. Thus, descriptors are events inducing a grouping of these observed sub-behaviours into actions. For example when the motion is concentrated on the fingers (see the second and third frame of the first row in Fig. 4) the sequence of frames, holding this property, induces a perceptual grouping of the *motion-lets* into a single action, that is, in fact, the action of *grasping*.

The principal directions of the velocity vectors at each time t, $t = 1, \dots, T$, along with a smoothed velocity magnitude are obtained as follows:

$$\begin{aligned} &1.\quad dir(x, y, t) = \frac{\pi \delta_{\mathcal{G}}(x, y, t)}{2k}\left(\left\lceil k \arctan\left(\frac{V(x, y, t)}{U(x, y, t)}\right)\frac{1}{\pi}\right\rceil + \left\lfloor k \arctan\left(\frac{V(x, y, t)}{U(x, y, t)}\right)\frac{1}{\pi}\right\rfloor\right) \\ &\qquad \left(\theta_j = \pm\frac{(2j-1)\pi}{2k}, j = 1, \dots, 2k\right) \\ &2.\quad M(x, y, t) = \left(\left(V(x, y, t)^2 + U(x, y, t)^2\right)^{1/2}\right)\left(\delta_{\mathcal{G}}(x, y, t)\right) \end{aligned} \tag{3}$$

Here $\lceil z \rceil$ and $\lfloor z \rfloor$ are, respectively, the ceiling and floor operators, $dir(x, y, t)$ is the (x, y) element, at time t, of the image of principal directions of the velocity for the segmented binarised image $\delta_{\mathcal{G}}(x, y, t)$, and $M(x, y, t)$ is the (x, y) element, at time t, of the magnitude of the velocity applied to the binarised segmented image. Furthermore, here $V(x, y, t)$ is the (x, y) element of the image of the v component of the optical flow at time t, $t = 1, \dots, T$, and $U(x, y, t)$ is the (x, y) element of the image of the u component of the optical flow at time t, $\delta_{\mathcal{G}}(x, y, t)$ is the (x, y) element of the binarised image at time t. Here k is half the number of the principal directions. So for 12 principal directions $k = 6$, therefore θ_j are the specific angles (w.r.t. the x-axis).

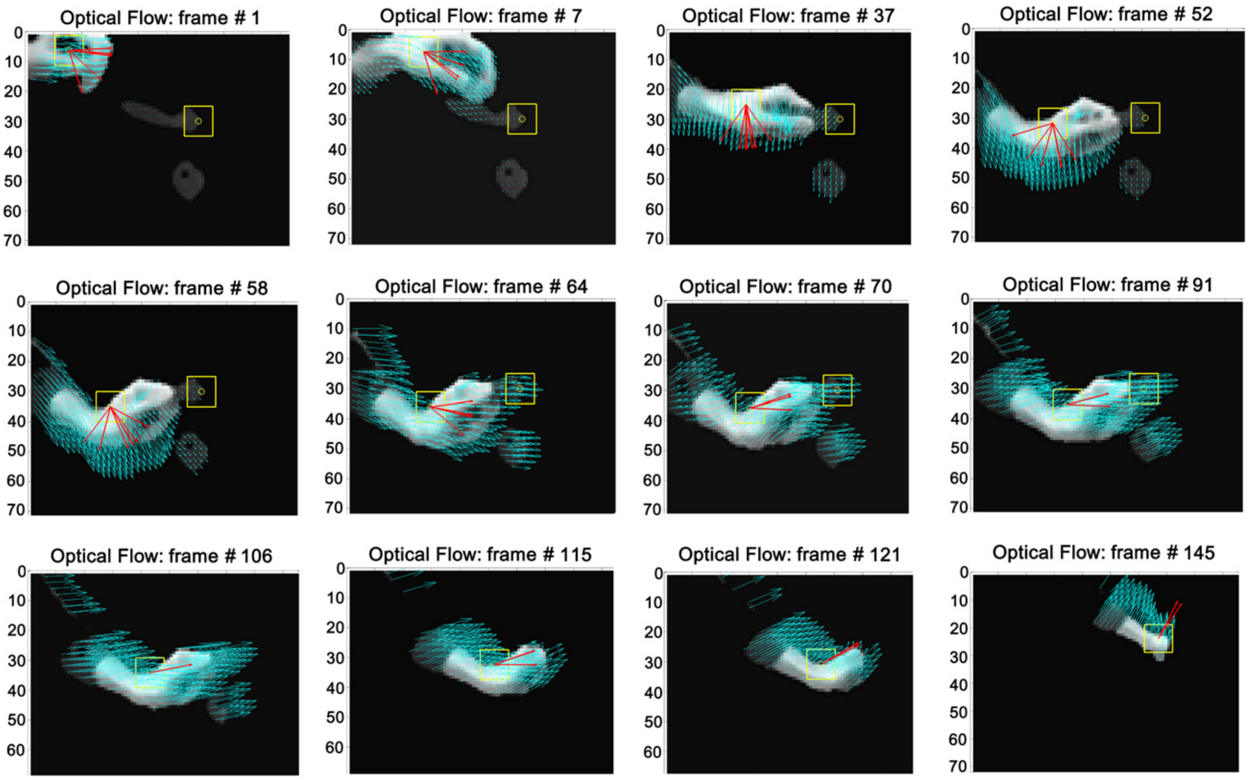


Fig. 3. Frames, corresponding to those shown in Fig. 2, illustrating part of the observations descriptors, see Section 2. Here the optical flow is shown, highlighted by the arrows indicating motion direction.

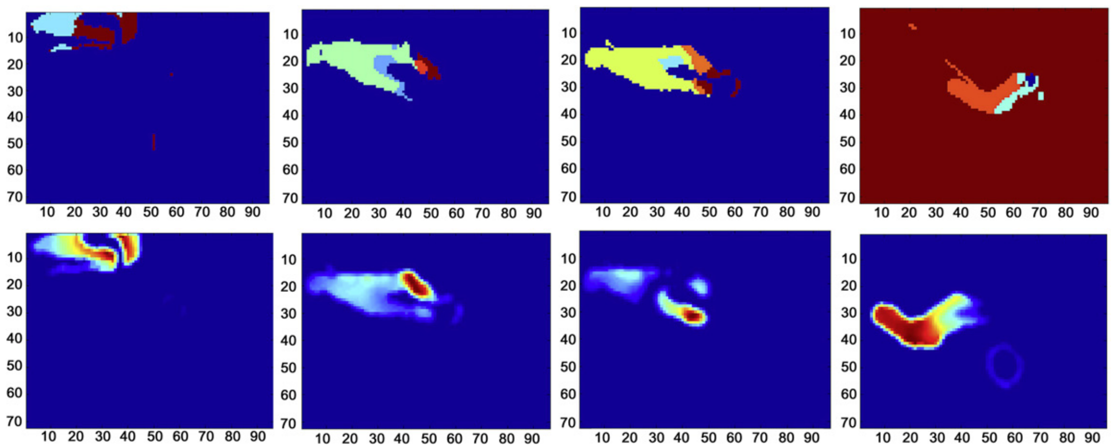


Fig. 4. The upper row shows the directions of the optical flow grouped according to the first equation in (3), into 12 directions. The lower row illustrates the magnitude of the velocity as obtained from the optical flow. The images are from a sequence of 272 frames.

Finally the smoothed magnitude is $Mg(t) = M(t) \star g(\sigma)$ where $M(t)$ is the magnitude image given above (3) at time t and $g(\sigma)$ is a Gaussian kernel of size 3×3 and with variance $\sigma = 0.6$, with $\star$ is the convolution operation.

Fig. 4 illustrates, in the upper row, the main directions of 4 frames taken from a sequence and in the lower row the magnitude $Mg(t)$ for the same frames.

The centre of mass of the extrema of the velocity magnitude in the segmented and smoothed image, at each time step t, $t = 1, \dots, T$, are located at:

$$\mathbf{c}_M(t) = \frac{1}{\alpha} \sum \bigl\{(x, y) \bigm| \forall z_1, \forall z_2\colon Mg(x, y, t) \geqslant Mg(z_1, z_2, t)\bigr\} \tag{4}$$

Here α is a normalisation factor that depends on the size of $\arg\max_{(x,y)} Mg(x, y, t)$.

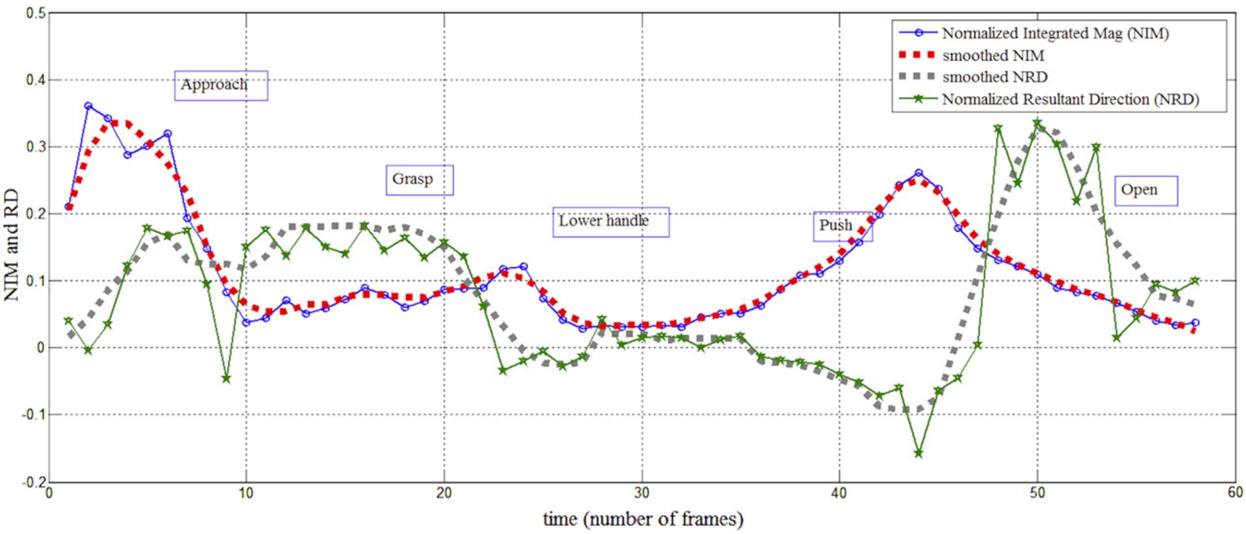


Fig. 5. The graph shows the flow of motion directions (expressed in radians) along a set of 58 sampled frames, from the 272 frames of the video sequence.

Another crucial aspect in the action description is the concept of object-related actions (see [66]), which is here identified with the number of regions of interest, in the segmented image. This is simply devised by labelling the 8-connect regions of the binary image $\delta_{\mathcal{G}}(t)$. These are the active regions with values not equal to 0. For example the hand and the handle are distinct regions up to the point in which the hand grasps the handle and the key hole is a distinct region up to the point in which the door is opened. To better capture the hand action, in the motion tracking, the extreme edge of the region in which the highest magnitude values fall are identified at each time t. These, indeed, coincide with the hand, at each time step.

Therefore, in the described feature space a feature descriptor, for each time step t (for each frame) is formed by: 10 values for motion directions from $dir(t)$, (Eq. (3).1), 1 value for time, 2 values for the centre of mass of the location of the extrema for the velocity magnitude (1×2) from $\mathbf{c}_M$ (Eq. (4)), 1 value for the integration of the velocity magnitude over the whole frame obtained by integrating the whole image $Mg(t)$ (Eq. (3).2), 2 values for the centre of mass of the region where $\mathbf{c}_M$ lies, which is different from the location of $\mathbf{c}_M$ (1×2), and 1 value for the number (a scalar) of active regions, i.e. the number of non-overlapping objects in each image, from the viewer vantage point.

This data set forms, for each frame of each video sequence, the set of descriptors of an instantaneous actions, and it is a 17-dimensional vector $\mathbf{Y}$.

3. Defining and testing the action space with mixtures of PPCA

The descriptors introduced above are used to define the action space. We do so using again the perceived motion as the initial estimate. Indeed, the perceived motion direction is along the resultant of motion direction at time t. To capture a meaningful difference in the motion speed and direction, which would tell us when an action ends and the next action begins, it is necessary to smooth the velocity field. Both the resultants along time and the integrated magnitude of the velocity are smoothed. Smoothing is obtained with both Laplace and moving-average filters. Let $n_j(\theta_j, t) = \#\{(x, y) \mid dir(x, y, t) = \theta_j\}$. We indicate the velocity components, according to $dir(x, y, t)$ (Eq. (3)), of the principal directions, as $dir(\theta_j(u), t) = \cos(\theta_j) Mg(x, y, t)$ and $dir(\theta_j(v), t) = \sin(\theta_j) Mg(x, y, t)$:

$$NRD(t) = \arctan\left[\frac{\sum_j dir(\theta_j(v), t) n_j(\theta_j, t)}{\sum_j dir(\theta_j(u), t) n_j(\theta_j, t)}\right], \qquad NIM(t) = \frac{\sum_x \sum_y Mg(x, y, t)}{\| \sum_x \sum_y Mg(x, y, t)\|} \tag{5}$$

Here $\|\mathbf{z}\| = \sqrt{\mathbf{z}^\top \mathbf{z}}$. Then the smoothed functions $\Delta NDR(t)$ and $\Delta NIM(t)$ are obtained, respectively, by convolving $NRD(t)$ with a moving average kernel of size 3 and $NIM(t)$ with a moving average kernel of size 5 and both with the negative Laplace filter. Fig. 5 illustrates the plot of both $NRD(t)$ and $NIM(t)$, and both the smoothed functions $\Delta NDR(t)$ and $\Delta NIM(t)$ for 58 uniformly sampled frames, from a video sequence of 272 frames, the smoothed functions are represented by dashed lines. The extrema of the gradients of the two smoothed functions show that the time evolution of motion can be divided into 5 main regions, hence the action space can be formed by 5 principal actions.

Two video sequences, A with 245 frames and B with 272 frames have been used: A for testing and B for training. Each frame of both sequences is transformed into a descriptor, namely a 17-dimensional vector $\mathbf{Y}_t$, according to the specification given in the previous section, thus for a sequence of T frames a data matrix of size $T \times 17$ is obtained.

Let us denote the time intervals during which the appearance of motion is ascribed to the j-th action, $j = 1, \ldots, N$, with α_j. We expect that if the action labelled a_j is described by any $\mathbf{Y}$ specifying a frame taken in the time interval α_j, then

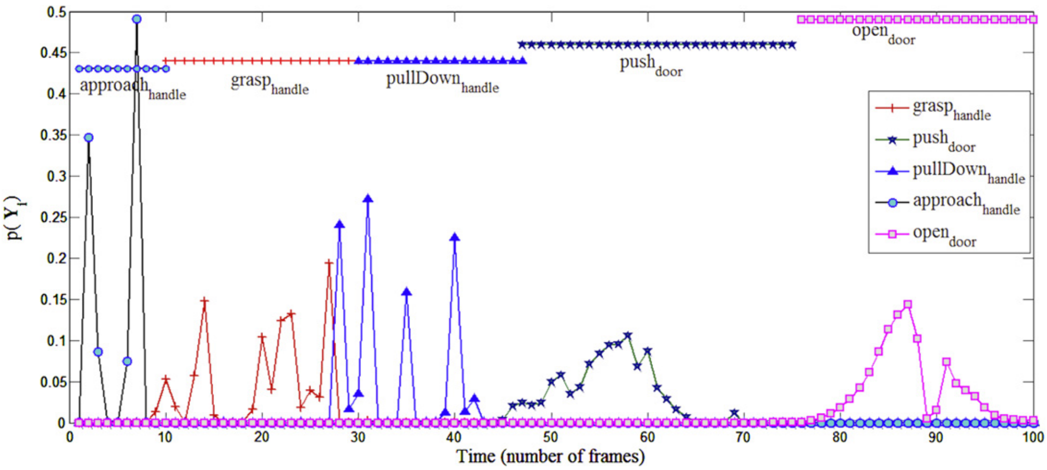


Fig. 6. The graph shows the sampled probabilities of each of the 17-dimensional vectors coding 100 samples taken from a video of 272 frames, from the PPCA mixture. The lines on the top of the graph indicate the x-axis ground truth, that is, the action observed at the specific frame, $1, \dots, 100$. Note that the 1-D sampled probabilities are approximately zero everywhere, but on the specific action observed, in so proving that the coding is effective.

$P(\mathbf{Y}|\alpha_j) > P(\mathbf{Y}'|\alpha_j)$, with $\mathbf{Y}'$ a descriptor from another time interval. Thus we use the data from the training sequence B, first for clustering and then for establishing a correspondence between action labelling and time intervals. First the data of 100 frames randomly selected from B are partitioned into $N = 5$ initial clusters: one cluster for each of the actions. For each i-th cluster, $i = 1, \dots, N$, the sample covariance Σ_i and the sample mean μ_i are estimated from the data, together with the proportion c_i of each cluster. Let $\Lambda_i = \mathrm{diag}(\lambda_{i1}, \dots, \lambda_{i17})$, with $\lambda_{i1} \geqslant \lambda_{i2} \cdots \geqslant \lambda_{i17}$ be the eigenvalues of Σ_i; these values, together with the corresponding eigenvectors, define the eigen action space, for each j-th action. The observed data can be reassigned to the clusters that maximise the likelihood of the observation in the eigen action space. Using PPCA clustering (see [114]) the principal sub-space of the data is made possible via maximum-likelihood estimation of the parameters in a Gaussian latent variable model [113]. This model will be further used for building the dynamic action model (see next section).

Thus $\mathbf{Y}$ can be expressed as a linear transformation of the unobserved ρ-dimensional variable $\mathbf{z}$ as $\mathbf{Y} = \mathcal{A}\mathbf{z} + \mu + \epsilon$. Here $\mathcal{A}$ is a $17 \times \rho$ linear transformation matrix, μ is a 17×1-dimensional vector locating the observations and ϵ the residual error of the combination. The space of observations has been set as follows:

$$\{\mathbf{Y}_n\}_{n=1}^{T} \subset \mathbb{R}^D, \qquad \{z_n\}_{n=1}^{T} \subset \mathbb{R}^\rho \tag{6}$$

Here D is the dimension of $\mathbf{Y}$, T is the number of frames (272) and ρ is the dimension of the latent space. For more details on the estimation see Appendix A.4. A correspondence between time intervals and mixture parameters is established as follows. The time-labels α_i have been grouped by the ground truth, according to the region defined using smoothed direction and magnitude of the optical flow, as defined above. That is, frames $t = 1:30$, are labelled with $a_1 = \mathit{approach}_{\mathit{handle}}$, frames $t = 31:85$ with $a_2 = \mathit{grasp}_{\mathit{handle}}$, frames $t = 86:133$ with $a_3 = \mathit{pullDown}_{\mathit{handle}}$, frames $t = 134:210$ with $a_4 = \mathit{push}_{\mathit{door}}$, and frames $t = 211:272$ with $a_5 = \mathit{open}_{\mathit{door}}$. Note that the action names are manually forced into the system as we give no method to generate, together with the action space, the action denotations.

Now, consider a mixture component g_j with parameters (c_j, μ_j, C_j), where $C_j = \mathcal{A}_j\mathcal{A}_j^\top + \sigma_j^2\mathcal{I}$, see Appendix A.4. Then g_j is labelled by the action label a_j if, given the training sequence B, and the time interval α_j:

$$\frac{\sum_{t\in\alpha_j}(Y_t - \mu_j)^\top C_j^{-1}(Y_t - \mu_j)}{\sum_{t\in\alpha_j}(Y_t - \mu_k)^\top C_k^{-1}(Y_t - \mu_k)} < 1 \quad (\forall k \neq j)$$

In other words, action space labelling is achieved using the time intervals specified within the training sequence. Given the above labelling then the test is to predict the correct label a_j for each descriptor $\mathbf{Y}_t$, $t = 1, \dots, 245$, from the test sequence A. This is verified using the responsibility of the mixture component g_j in the computation of $f(\mathbf{Y}_t)$, namely $p(\mathbf{Y}_t|g_j)$. The sampled probabilities from the mixture of PPCA for each action cluster are shown in Fig. 6. The figure shows that the probability of the sampled descriptors at each observed frame (time t) are greater than zero only on those frames labelled by the specific action. This proves that the frame coding, with the 17-dimensional descriptors vector $\mathbf{Y}$, is effective.

4. From feature space to the space of actions and states

In the previous two sections we have shown how to obtain a good set of descriptors to model the observed actions, providing an action space. In particular, we proved the effectiveness of the descriptors with PPCA, noting that the eigen action space is formed by the ρ-dimensional eigen-space spanned by the latent variable $\mathbf{z}$. We have, thus, seen that the pdf of each descriptor is a Gaussian with variance maximised in the eigen-space, and that its likelihood is thus maximal when the descriptor comes from the space of the action it represents.

However, the PPCA does not capture the dynamic of actions or their time–space relation. Given an observation sequence $\mathbf{Y}_1, \ldots, \mathbf{Y}_T$, where now the observation is the *description of the visual process*, we want to find out if each observation $\mathbf{Y}_t$ can be explained by a condition and effect on the action it predicts. These conditions and effects are the unobservable states. We made the hypothesis that the appearance of an action starting and ending depends on scale, motion evolution, light change and space location of afforded objects (the definition of $\mathbf{Y}_t$). Since the descriptors do not capture the interaction between an action ending and another action beginning, these are in fact unobserved, in terms of the visual features of each frame. Thus a state is unobserved and records the executability of an action in the following sense: the transition from a state to another state specifies under what conditions an action starts and what conditions are rated at the end of the action. Thus, according to the results of the previous section, there are 5 states and 5 actions.

A continuous observation HMM is a suitable dynamic model for actions and states, requiring estimation of a transition matrix $\mathbf{P}$ between states, a distribution π on the initial states, and the mixture parameters Ψ modelling the local evolution of actions (that is, with respect to the observed sequences) and their interactions. HMM are useful when a chain cannot be observed directly but only through another process; that is, of the two processes $\{X_i, Y_i\}_{i\geqslant 0}$, only $\{Y_i\}_{i\geqslant 0}$ is observed. More precisely:

Definition 1. Consider the bivariate discrete time process, with continuous observations, $\{\mathbf{Y}_i, X_i\}_{i\geqslant 0}$, where $\{X_i\}_{i\geqslant 0}$, is a chain and $\{\mathbf{Y}_i\}_{i\geqslant 0}$, is a set of continuous random variables. Let $\{X_i\}_{i\geqslant 0}$ satisfy the independence properties of a Markov chain (see Appendix A.4) and let $\mathbf{Y}_i$ be conditionally independent of all other $\mathbf{Y}_j$, $i \neq j$, and any other X_k, $k \neq j$, given a specified assignment to the variable X_k, it depends on. The model $\mathcal{M}$ for the bivariate process is identified by the parameters vector $(\pi, \mathbf{P}, \Psi, \gamma)$ where:

1. $(\pi, \mathbf{P})$ is a Markov chain.
2. Ψ is the family of parameters of the mixtures of normal densities, specifying the state emissions. Given M mixture components, N states, the probability of the observation $\mathbf{Y}$ at state j is, according to a mixture PPCA-HMM:

$$b_j(\mathbf{Y}) = \sum_{k=1}^{M} c_{jk}\mathcal{N}\big(\mathbf{Y}|\mu_{jk}, \mathcal{A}_{jk}\mathcal{A}_{jk}^{\top} + \sigma_{jk}^2\mathcal{I}\big), \quad j = 1, \ldots, N \tag{7}$$

   Here c_{jk} is the probability of the k-th mixture at state j, μ_{jk} is the mean of the k-th normal density of the mixture at state j, and $\mathcal{A}_{jk}$, σ_{jk}^2 are the variances of observed and hidden parameters of the k-th normal density at state j, these are specified for the mixture of PPCA in Eqs. (44), (45) and (46) in Appendix A.4.1.
3. γ is the family of parameters of the mixtures of PPCA densities, used to determine and test the action space (see Section 3).

The re-estimation procedure of the model parameters $(\pi, \Psi, \mathbf{P})$ for HMMs with Gaussian observation densities is described in [98,71,59], based on Expectation Maximisation (EM), this does not concern the action space. The adaptation of the re-estimation procedure for mixture of PPCA to the HMM-PPCA is described in Appendix A.4.

An example of the HMM for the open-the-door problem described in the paper, with the 17-dimensional observations, with 5 states, and with 5 mixtures of PPCA, each formed by three components, is illustrated in Fig. 7: above, the mixtures represented as 3D ellipsoids and, below, the figures illustrate the mixture matrix on the left, which in this case is of dimension 5×3, each element of which is the mixture gain c_{jk}, indicating the probability that in state j the mixture component k is chosen, and on the right the transition matrix $\mathbf{P}$. The states of the HMM are labelled with terms specifying the preconditions/effect of each action.

We recall useful facts about estimation of sequences for HMMs with continuous observation densities, given a model $\mathcal{M}$, with states S:

1. The probability of a state sequence $\omega_T = s_{j_1}, \ldots, s_{j_T}$ is:

$$P(\omega_T) = \sum_{s_j \in S} \pi(s_{j_1}) \prod_{k=1}^{T-1} P(s_{j_{k+1}}|s_{j_k}) \tag{8}$$

2. The probability of an observation sequence $\mathbf{Y}_1, \ldots, \mathbf{Y}_T$, given a specific state sequence $\omega_T = s_1, \ldots, s_T$ is, given independence of observations:

$$p(\mathbf{Y}_1, \ldots, \mathbf{Y}_T|\omega_T) = \prod_{j=1}^{T} p(\mathbf{Y}_j|s_j) = \prod_{j=1}^{T} b_j(\mathbf{Y}_j) \tag{9}$$

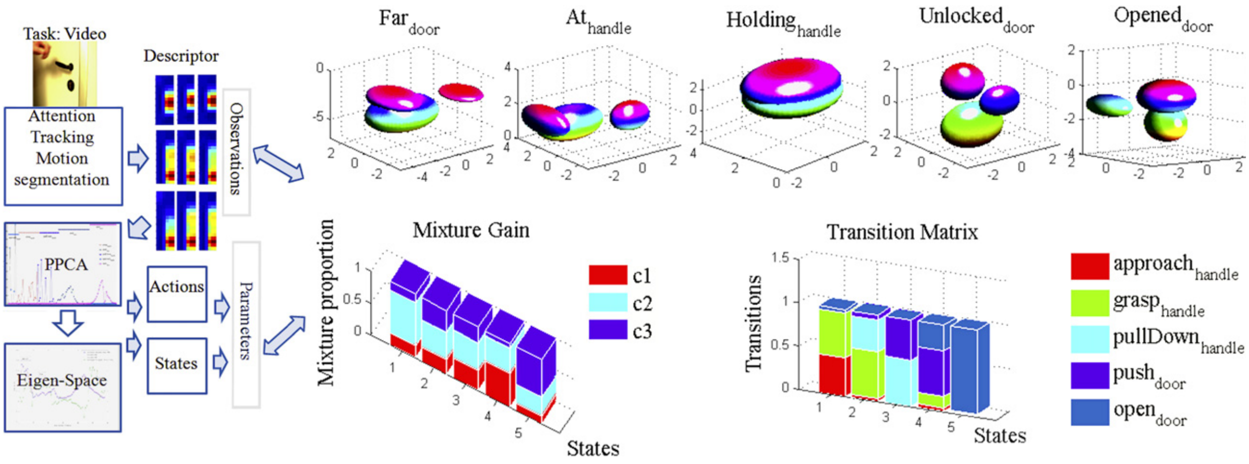


Fig. 7. On the left the steps from motion perception to the definition of the descriptors leading to the eigen action space. On the right the estimated HMM with mixtures of probabilistic PCA. The HMM is composed of 5 states, as estimated from descriptors. The emission is a 17-dimensional Gaussian mixture with 3 components. The HMM-graph, in the upper part, illustrates the 3-dimensional mixture components obtained from the posterior distribution $p(\mathbf{z}|\mathbf{Y})$ of the latent variables. The HMM-graph, below, illustrates on the left the mixture matrix (also in 3d), that is, the probability of each mixture component given a state. On the right the graph shows the transition matrix, that is the probability to transit from a state, represented on the x-axis, to a the state specified by the colour.

Table 1
Transition matrix for a sequence of images sampled from the video describing the task *open the door*.

Transition matrix					
	Far	At_{handle}	$Holding_{handle}$	*Unlocked*	*DoorOpened*
Far	8.67E−01	1.33E−01	3.82E−107	7.15E−237	0.00E+00
At_{handle}	1.30E−12	8.68E−01	1.32E−01	5.23E−106	0.00E+00
$Holding_{handle}$	2.66E−203	1.10E−31	8.58E−01	1.42E−01	5.56E−194
Unlocked	0.00E+00	6.51E−123	1.57E−25	8.62E−01	1.38E−01
DoorOpened	0.00E+00	0.00E+00	0.00E+00	2.76E−35	1.00E+00

3. The joint probability of an observation sequence $\mathbf{Y}_1, \ldots, \mathbf{Y}_T$ and a specific state sequence $\omega_T = s_1, \ldots, s_T$ is:

$$P(\mathbf{Y}_1, \ldots, \mathbf{Y}_T, \omega_T) = \prod_{j=1}^{T} p(\mathbf{Y}_j|s_j)P(\omega_T) = \prod_{j=1}^{T} b_j(\mathbf{Y}_j)P(\omega_T) \tag{10}$$

4. The probability of an observation sequence, given any state sequence with fixed length T is:

$$\begin{aligned} p(\mathbf{Y}_1, \ldots, \mathbf{Y}_T) &= p(\mathbf{Y}_1, \ldots, \mathbf{Y}_T|\omega_T, \mathcal{M})P(\omega_T) \\ &= \sum_{s_j \in S} \pi(s_{j_1})b_{j_1}(\mathbf{Y}_{j_1}) \prod_{k=1}^{T-1} P(s_{j_{k+1}}|s_{j_k})b_{j_{k+1}}(\mathbf{Y}_{j_{k+1}}) \end{aligned} \tag{11}$$

An HMM $\mathcal{M}$ can tell several aspects of the combination of actions and states. The probability of observing an action at a specific state amounts to the estimation of the expected number of times such an action has been observed to be executed in such a state. Likewise the probability of a transition between state s_i, and state s_j depends on how many times, in the temporal sequences of frames presented to the system, the action that most probably can be seen to be executed, at state s_i, has been effectively seen. Thus learning in terms of basic and local cases is effective: by looking at a sequence of motions induced by actions, which for us is the video sequence, it is possible to learn for each sequence the precondition of an action, and the probability that the action leads to a specific state. Here we give some examples.

Let $\mathcal{M} = \langle \pi, \mathbf{P}, \Psi, \gamma \rangle$ be an HMM learned according to the estimation process described above, whose transition matrix $\mathbf{P}$ is illustrated in Table 1, and whose mixture gain matrix is illustrated in Table 2.

Given a sequence of descriptors $\mathbf{Y}_i$, specifying the observations, some outcomes of the learned process are in order:

1. The probability of each action being generated by the state that rationally makes the action possible: e.g., the precondition of *grasp the handle* is that the hand has to be *at the handle*, so the likelihood for *grasping*, according to the Ψ parameters, should be maximised in the state labelled by At_{handle}.

Table 2
The mixtures proportion matrix, each element c_{ij} indicates the proportion of the j-th mixture component activated in state i.

Mixture components matrix			
	c_{1i}	c_{2i}	c_{3i}
At_{handle}	3.653E−01	2.693E−01	3.654E−01
$Holding_{handle}$	2.763E−01	4.408E−01	2.829E−01
Unlocked	3.118E−01	2.425E−01	4.458E−01
DoorOpened	2.440E−01	3.005E−01	4.555E−01
Far	5.536E−01	2.677E−01	1.787E−01

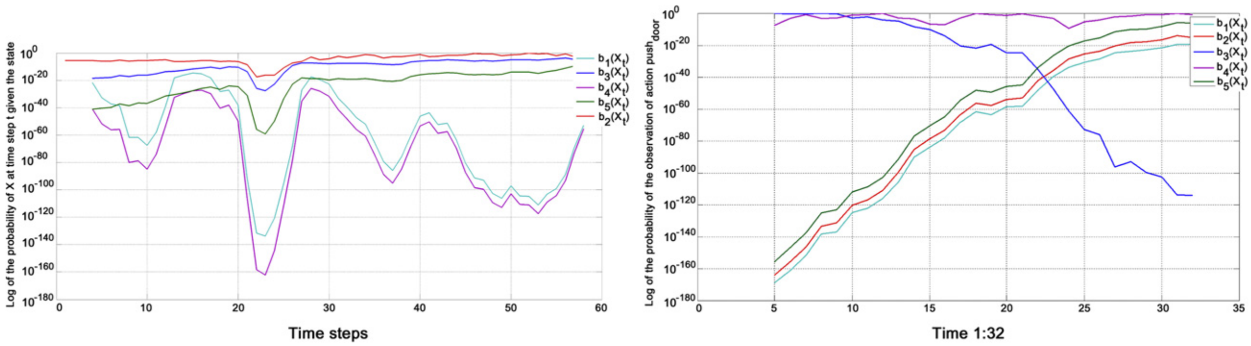


Fig. 8. The two graphs illustrate the sampled probabilities of a sequence of observations showing $grasp_{handle}$ on the left and of a sequence observations showing $push_{door}$ on the right. The graph shows that the probability is maximal in state 2 for $grasp_{handle}$ and in state 4 for $push_{door}$, as expected.

2. The likelihood of a sequence of states $s_1, \ldots, s_T$, if at each state the action with maximal probability is executed.
3. The probability of observing the, presumably correct, sequence of actions that is, the likelihood of ($approach_{handle}$, $grasp_{handle}$, $pullDown_{handle}$, $push_{door}$, $open_{door}$).
4. The probability that, given the correct sequence of actions, the correct sequence of states is obtained, and the probability that at the end of the sequence the door is opened, that is, the likelihood that the last state is the one we have denoted *DoorOpened*.

These and many other questions can be easily answered by a model $\mathcal{M}$ according to the estimation process of the HMM parameters that we have described.

First note that item (1) above is a consequence of the maximum-likelihood estimation of the parameters of each mixture, with covariance obtained by PPCA. On the other hand, item (2) can be checked by the Viterbi algorithm.

Example 1. To show that the labelling is correct, we illustrate 1) the sampled values from the pdfs, from a sequence of 58 frames of the action $grasp_{handle}$ (that is, the 58 frames illustrate the movements concurring in the formation of the action $grasp_{handle}$); 2) the sampled values from a sequence of 32 frames of the action $push_{door}$, given each of the 5 states. It can be seen from the two graphs in Fig. 8 that the likelihood is maximal in the second (At_{handle}) and fourth ($Unlocked_{door}$) states, respectively, as expected.

When a sequence of motions of the same repeated action $grasp_{handle}$ is observed, the learned model $\mathcal{M}$ generates a sequence of states starting in the first state *Far*, further transiting to state At_{handle} and remaining in this state. This is due to the fact that the estimated prior (actually for all $\mathcal{M}$) is $\arg\max_x \pi(x) = s_1$ and the transition probability from *Far* to At_{handle} is relevant (see Table 1). The maximum likelihood of a sequence of actions of only $grasp_{handle}$ is close to 0. In the case of a sequence of actions $push_{door}$, the learned model $\mathcal{M}$ generates a sequence of states starting in the first state *Far* but remaining in such a state, as there is no way to make a transition to the further states, and the maximum likelihood is close to 0 too. Thus, unwanted sequences are not even created.

Finally, consider the case of a sequence of frames (motions) illustrating the correct sequence of actions. The obtained sequence of states is, indeed, the expected one, in all the 10 experiments performed with test data, with the log likelihood computed with the Viterbi algorithm [116]. That is, the list of best states computed with Viterbi is the one maximising at each step t, $P(\mathbf{Y}_1, \ldots, \mathbf{Y}_t, X_1, \ldots, X_t = x_j|\mathcal{M})p_{ij}b_j(\mathbf{Y}_t)$, and assigning to a list of best states, the argument of the maximum. Thus, given the list of best states $x_{15}, \ldots, x_0$ which is $s_1^2, s_2^2, s_3^4, s_4^3, s_5^4$, where the superscript indicates states repetition, $P(DoorOpened|\mathbf{X})$ is $\pi(x_0)\prod_{t=1}^{T-1} P(x_{t+1}|x_t) = 2.6516\text{E}{-004}$. Indeed, all sequences ended in *DoorOpened*. The stationary distribution of the illustrated transition is obtained at $T = 87$.

The stationary distribution is described in Appendix A.2, and it ensures the finite property:

Definition 2. Let $\mathcal{M}$ be an HMM such that $\mathbf{P}$ is either irreducible or it satisfies the conditions of Lemma 5 (see Appendix A.3), then we say that $\mathcal{M}$ has the finite property.

At last, given our example of learning the task *opening the door*, we have seen that a specific HMM model $\mathcal{M}$ is obtained from each sequence and each starting parameters set. Each model, as long as it is trained according to the correct observed sequence, starting with the hand approaching the handle, and ending with the door opened, will lead to the expected results, with slight variations. Thus we have a whole class of models that are tuned to the task, i.e. to the specific observations sequence. For the parameters estimation see Appendices A.3 and A.4.

5. First-order parametric models

In this section we start to take care of the induction step. We have settled the basic step by learning the parameters of the probability space of actions and the hidden values, specifying the states. Hidden state transitions have been learned, likewise the likelihood of observations given the states. These amount to a set of parameters $(\pi, \mathbf{P}, \Psi, \gamma)$ for each initial data set, that is, for each finite set of sequences, via their induced transformations and their initial parameters, used to train the model, as described in previous sections. There is, thus, an infinite number of possible models, one for each set of initial parameters, though learning a task requires few observations. Continuing with the *open the door* example we look for a generalisation step transforming the parameter space, learned from a small number of sequences, into general rules of behaviours. In the next subsections we show that the learned parameters can be used to extend the simple signature of action terms to a new signature encoding predicates, functions and terms. We show that a language can be obtained, so that there is a correspondence between both formulae and field terms, in a first-order probability structure, and the random variables in the HMM.

5.1. Probability structures

Usually we assume that real world events are described by random variables behaving according to some unknown distribution that has to be estimated from the observed cases and, possibly, some prior belief about the distribution. In other words we see the random variable *in practice* and we want to determine its distribution. In most of the studies on probabilistic logic this problem is not faced, and the axiomatisation is pushed forward to assess properties of probabilistic inference. For example, Keisler in [61] shows that in the first-order probabilistic logic with quantifiers it is possible to assert that two random variables X_1 and X_2 are independent. Indeed McCarthy in [74] argues that "The information necessary to assign numerical probabilities is not ordinarily available. Therefore a formalism that required numerical probabilities would be epistemologically inadequate",[2] exactly because of the lack of methods on how to feed parameter learning into formalisms for reasoning.

Since the early studies on first-order probability by Gaifman [40], Krauss and Scott [110], Keisler [62,61] and Hoover [52], a wealth of research has been dedicated to the integration of logic and probability. Nowadays the seminal contributions for computational purposes have been the works of Abadi, Bacchus, Fagin and Halpern in [36,7,6,1,49,38,48], introducing a first-order probabilistic logic with both first-order quantifiers and real valued formulae. More recently several streams of research on logic and probability integration have appeared concerning probabilistic logic programs, Poole [90], likewise [24,99,83] connected learning with stochastic logic programs. Koller and colleagues [64] and [42] introduced probability relational models, while Domingo and colleagues introduced Markov logics [4] (see also a general overview in [34]). Milch and Russell [81] introduced first-order models with unknown objects. These are examples of a wealth of approaches that we cannot enumerate here. The clear direction has become that of adapting first-order probability languages to the statistical needs of computational learning or, the other way round, to lift statistical learning to first-order logic. Here our aim is different. Here we aim at embedding the distribution learned into a first-order probability model. More precisely, we are not interested to learn within the probability logic (as, for example, in PRISM [106]), but to use the parameters to feed the logic as an inductive step towards the *language of thought*. The task is to build a first-order probability model that accepts formulae enunciating all the events an HMM describes with random variables, such as the probability of an observed action, given a sequence of states, or the probability of a state, given a sequence of actions, or comparing probabilities between events. This seems to be a natural approach that, on one side preserves the declarative-relational structure of logic and, on the other side, prepares the ground for embedding learning, viewed as an early computational process, into reasoning.

For our example we are thus concerned only with the model construction and not with the proof theory and axiomatisation. For this aspects we mainly refer to the seminal works of Gaifman, Halpern and Bacchus [41,7,48]. In fact, the main results of Bacchus and Halpern work is the specification of a proof theory, which is complete only under alternative restrictive conditions. That is, the logic is complete if either

(a) The domain is assumed to be countable and the measure functions are field-valued (non-Archimedean) and the generated algebra of events is finitely additive [7,6].

[2] Bacchus in [6] also refers to this sentence.

(b) The collection of axioms includes the proper axiomatisation for reals, the language excludes random variables and the domain is bounded in size [49].
(c) The language includes the random variables, the domain is bounded in size, but the Archimedean property is given up [7,6].

In fact, while Halpern chooses to axiomatise the reals, using the theory of the real closed field, Bacchus admits measure functions ranging over non-Archimedean ordered fields, in the sense that his logic admits infinitesimals.[3,4]

Let us consider the first-order language *Lp* with probabilities on the domain as presented in [7,6]. The language *Lp* is formed by all formulae and sentences of first-order logic plus the field terms $[\alpha(\vec{x})]_{\vec{x}}$, where $\alpha(\vec{x})$ is a first-order formula with free variables $\vec{x}$ and $[\cdot]$ is the probability term constructor. A statistical probability structure for *Lp* is $\mathcal{M} = \langle \mathbb{A}, \mathbb{F}, (\Pi_n, \mu_n)_{n<\omega}\rangle$, where $\mathbb{A} = (D, I)$ is a classical first-order structure, $\mathbb{F}$ is the totally ordered field of numbers and, for each n, Π_n is a field of subsets of D^n, and μ_n is a probability function on Π_n whose range is $\mathbb{F}$. The language $\mathcal{L}_1(\Phi)$ presented in [49] is like *Lp*, but the field terms range over the reals, hence $\mathbb{F}$ is the real closed field, having the same first-order properties as the field of real numbers, and field terms are denoted by $w_{\vec{x}}(\alpha)$. A structure for $\mathcal{L}_1(\Phi)$ is $\mathcal{M} = (D, \pi, \mathcal{X}, \mu)$ where D is the domain, π is an interpretation, $\mathcal{X}$ is a σ-algebra over D and μ is a probability function on $\mathcal{X}$, which is essentially a counting measure. In this section we substantially refer to the work of [7,49].

Let us start with the learned HMM $\mathcal{M} = \langle S, H, (\pi, \mathbf{P}, \Psi, \gamma)\rangle$ with S the finite set of states, H the set of observations, of fixed dimension, with respect to the model. For example for the task *open the door* the observations are 17-dimensional vectors and the parameters Ψ have been tuned for this 17-dimensional space, as shown in Section 4. However here we assume that variables are 1-dimensional to simplify the notation. Finally, the whole set of parameters is formed by a transition matrix $\mathbf{P}$, the mixture parameters Ψ, the initial distribution parameter π, selecting an initial state from S, and the action space parameters γ. We define the canonical (i.e. parametric, because the parameters $\pi, \mathbf{P}$ and Ψ are given) probability structure $\mathbb{M}$ for $\mathcal{M}$, as follows.

5.1.1. The signature

The signature $\Im_{\mathbb{M}}$ of the language includes terms of sort state to denote the set S of states, terms of sort observations to denote the set H of observations. Note that states and observations induce random variables (see Appendix A.1), and while the domain of states is finite and of dimension n, the domain of observations is $\mathbb{R}$. To these sorts we add the terms of sort sequence of states $w : S^{\mathbb{N}} \to \mathcal{W}$, endowed with the constructor $\circ$ and $\mathcal{W}$ the domain of countable sequences of states. Finally, the signature includes terms of sort natural numbers, for indexing.

Notation. Indices are denoted by natural numbers n, m or t. When we refer to states as elements of the domain we denote them by s, when they are referred to by variables are denoted by x, possibly indexed. Sequences of states are denoted by w, we use w both to denote a variable of sort sequence of states and the term $(x_n \circ \cdots \circ x_0)$. When we refer to observations in the domain we denote them by h, when they are referred to by variables of sort observations are denoted by y.

The language includes monadic predicate symbols R_i, one for each state $s_i \in S$, taking as argument a sequence in $\mathcal{W}$, binary predicates A_j, one for each observation (of an action), taking as argument an observation $h \in H$ and a state $s_i \in S$, and binary predicate symbols O_{ji}, that can be defined as the conjunction of R_i and A_j and take as arguments an observation and a sequence of states. Finally the language includes the relation $\leqslant$, abbreviating $< \vee =$, between sequences.

To the above sorts we add the real line $\mathbb{R}$, which we assume represented as in Matlab, and the Borel σ-field on the real line $\mathcal{B}(\mathbb{R})$. Sorts taking values in the reals include, beside observations, field terms, the mixture gain matrix $\mathbf{c} : S \times [0,1] \to [0,1]$, the normal distribution $\mathcal{N}(\mu_{ik}, \mathcal{A}_{ik}\mathcal{A}_{ik}^{\top} + \sigma_{ik}^2\mathcal{I})$, with μ_{ik}, $\mathcal{A}_{ik}$ and σ_{ik}, elements of Ψ, similarly for γ, and random variables X_i, Y_i, $i = 1, 2, \ldots$. Sorts mapping to the reals include the initial distribution π on states, with $\pi : S \to [0,1]$, the transition matrix $\mathbf{P} : S \times S \to [0,1]$.

5.1.2. Formulae of the language

First-order formulae are defined as usual. Field terms and formulae including field terms are defined as follows, where classical laws for connectives and quantifiers are implicitly assumed.

Definition 3. A field-base α, for a field term, is defined inductively as follows:

1. If y, x and w are, respectively, terms of sort observation, of sort state and sequence of states, then $A_j(y, x)$, $R_i(w)$ and $O_i(y, w)$ are all field-bases.
2. If w, w' are terms of sort $\mathcal{W}$ then $w = w$, $w < w'$ and $w > w'$ are field-bases.
3. If α_1 and α_2 are field-bases, then $\neg\alpha_i$, $\alpha_1 \wedge \alpha_2$ are field-bases.

---

[3] The Archimedean property says that any set of reals have a positive upper bound, that is, if $x \in \mathbb{R}$ and $x > 0$ there is an $n \in \mathbb{N}$ such that $xn > 1$.
[4] See Hammond [50] for a discussion on the advantages of non-Archimedean ordered field in game theory.

4. If $\alpha(x)$ is a field-base, with x a free variable (possibly among others) varying on sort state, observation and sequence, then $\forall x.\alpha(x)$ is a field-base.

Definition 4. A field term constructor is $[\cdot] : \alpha \mapsto [0,1]$ where α is a field-base formula. A field term is defined as follows:

1. If x_i, x_j are terms of sort states, $\mathbf{P}(x_i, x_j)$ is a field term, likewise $\pi(x_i)$; $\mu_{ij}, \sigma_{ij}, i = 1, \ldots, n, j = 1, \ldots, k$, are field terms. Also $X_i(x)$, $i = 1, 2, \ldots, T$, are field terms. If y is a term of sort observation then $\mathcal{N}(y|\mu_{ij}, \sigma_{ij})$, $Y_j(y)$, $i, j \geqslant 1$, are field terms. In particular these are the *atomic* field terms.
2. If α_1 and α_2 are field-bases with free variables $var(\alpha_i) = V$, $i = 1, 2$, then: if V is of sort state or sequence of states then $[\alpha_i(V)]$ is a discrete probability term, if V ranges over observations then $[\alpha_i(V)]$ is a continuous probability term, both are defined field terms.
3. $[\alpha_1(V)] \cdot [\alpha_2(V')]$, $[\alpha_1(V)] + [\alpha_2(V')]$, $[\alpha_1(V)]/[\alpha_2(V')]$, $1 - [\alpha_i(V)]$ are field terms.
4. $[\alpha_i(V)|\alpha_j(V')]$ is a field term.

Definition 5. If $f(z)$ is a field term, with z a variable of sort either state, observation, or sequence of states, and $p \in [0,1]$ then formulae can be formed using field terms as follows:

1. $f(z) = p$, $f(z) < p$ and $f(z) > p$ are formulae, we shall abbreviate $< \vee =$ with $\leqslant$ and $> \vee =$ with $\geqslant$.
2. If $f(z)$ and $g(z')$ are field terms, $f(z) \leqslant g(z')$ and $f(z) \geqslant g(z')$ are formulae.
3. If $\phi(z)$ and $\psi(z')$ are formulae including field terms, then $\phi(z) \wedge \psi(z')$ and $\neg\phi(z)$ are formulae.
4. If $\varphi(z)$ is a formula, in which z is a variable occurring free (possibly among others) and such that z can be of sort either state or observation $\exists z.\varphi(z)$ is a formula.

To these terms the measure terms η_X, with X an index to be specified, and their products are added, as detailed in the next section.

5.1.3. Domain and probability space

The domain D is partitioned into the following sets: a finite space S of states, the space H of observations taking value in $\mathbb{R}$, and the space $W = S \times S \times \cdots = S^T$ of sequences of states denoted by ω, $T \leqslant \mathbb{N}$. Let Δ_0 be the σ-field formed by all subsets of S and $\Delta = \Delta_0 \times \Delta_0 \times \cdots = \Delta_0^T$. Given the distribution π, the initial distribution on S, and the transition matrix $\mathbf{P}$, as specified by the HMM parameters, we consider discrete random variables $X_1, X_2, \ldots$, with Markov property, all defined on the same probability space, mapping S^T into S, such that $X_t(\omega) = s_{j_t}$. A probability measure $\eta_s : \Delta \mapsto \mathbb{R}$, satisfying the Markov property, is

$$\eta_s\bigl(\{X_1(\omega) = s_{j_1}, \ldots, X_t(\omega) = s_{j_t}\}_{j=1,\ldots,|S|}\bigr) = \sum_{s_{j_1},\ldots,s_{j_t}} \pi(s_{j_1}) \prod_{i=1}^{t-1} \mathbf{P}(s_{j_{i+1}}|s_{j_i}) \tag{12}$$

η_s is a countably additive measure (see [15]). Since the sum of the initial density π is 1 and $\mathbf{P}$ is a stochastic matrix, then $(\mathcal{W}, \Delta, \eta_s)$ is a probability space.

H is the set of observations with domain $\mathbb{R}$. Let Θ be the smallest σ-field generated by the Borel sets on $\mathbb{R}$. The elements of Θ are Borel sets denoted B. Let (H, Θ) be a measure space with measure η. On the space (H, Θ, η) we consider together with the elements of H random variables Y_j, as identities. The probability measure on this measure space induced by the random variables is defined as

$$\eta(B) = \int_B f(y)\,dy \quad (B \in \Theta)$$
$$f(y) = \sum_{k=1}^{M} c_k \mathcal{N}\bigl(y|\mu_k, \mathcal{A}_k\mathcal{A}_k^\top + \sigma_k^2\mathcal{I}\bigr) \quad \bigl(\mu_k, \mathcal{A}_k, \sigma_k^2 \in \gamma\bigr) \tag{13}$$

Here $f(y)$ is the density specified by the mixture of PPCA, as defined in Section 3, see also Appendix A.4, Eq. (40), and $\mathcal{N}$ is the normal density. When the random events are specified with respect to a fixed state $s \in S$ then η is extended to η_I on the product space $H \times \mathcal{W}$ with σ-field $\Theta \times \Delta$ formed by the measurable rectangles $B \times Q$, $B \in \Theta$ and $Q \in \Delta$. That is, if $E \in \Theta \times \Delta$ then $Q = \{\omega \mid (h, \omega) \in E\}$ lies in Δ and $B = \{h \mid (h, \omega) \in E\}$ lies in Θ. On the measure space $(H \times W, \Theta \times \Delta)$ the density of the random variables is b_i, taking two arguments y and, implicitly, a state s_i.

$$\eta_I(B) = \int_B b_i(y)\,dy \quad (B \in \Theta)$$
$$b_i(y) = \sum_{k=1}^{M} c_{ik} \mathcal{N}\bigl(y|\mu_{ik}, \mathcal{A}_{ik}\mathcal{A}_{ik}^\top, \sigma_{ik}^2\mathcal{I}\bigr) \quad \bigl(\mu_{ik}, \mathcal{A}_{ik}, \sigma_{ik}^2 \in \Psi, i = 1, \ldots, N\bigr) \tag{14}$$

Here N is the number of states, M the number of mixture components. When B is $\mathbb{R}$ then the b_i integrates to 1 and when $B=\emptyset$ then $\eta_I(\emptyset)=0$. Indeed, $(H\times\mathcal{W},\Theta\times\Delta,\eta_I)$ is a probability space. We can thus introduce the structure $\mathbb{M}$ as follows:

Definition 6. A probability structure of first-order with parameters fixed by an HMM model $\mathcal{M}=(S,H,(\pi,\mathbf{P},\Psi,\gamma))$ is $\mathbb{M}=(D,\Psi,\Phi,\mathscr{I})$. Here D is the domain, $D=(S,\mathcal{W},H)$, Ψ is the set of parameters defined by the HMM $\mathcal{M}=(S,H,(\pi,\mathbf{P},\Psi,\gamma))$, Φ is the probability space defined as

$$\Phi=\big((\mathcal{W},\Delta,\eta_s),(H,\Theta),(H\times S,\Theta\times\Delta,\eta_I)\big). \tag{15}$$

Here $\Delta,\Theta\times\Delta$ are the *sigma*-fields defined above and η_s,η_I are the associated probability measures.

Finally $\mathscr{I}$ is the interpretation for the signature, defined as follows.

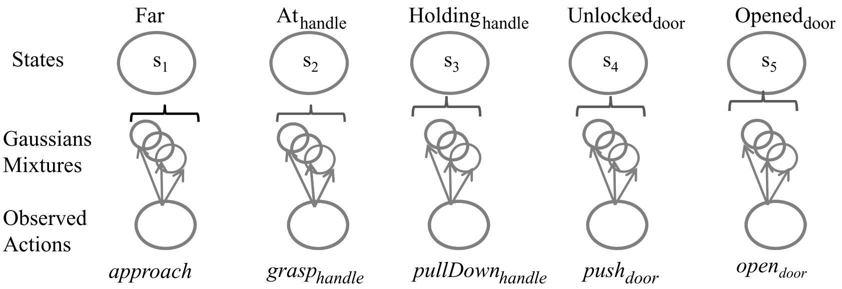
1. Interpretation of predicates R_i, A_j, O_{ji}, with $i,j=1,\dots,N$, N the number of states and the number of observation actions:
   Let $X_i: S^T\mapsto S$ be a discrete random variable
   $$R_i(\cdot)^{\mathscr{I}}=\bigcup_t\big\{\omega \;\big|\; X_1(\omega)=\arg\max_s \pi(s),\, X_t(\omega)=s_i,\, t\geqslant i\big\}\quad (i=1,\dots,N) \tag{16}$$
   Let N be both the number of observation actions (hence the mixture components for observations independently of states) and the number of states, and M the number of mixture components for the observations at each state:
   Let $z\in\mathbb{R}$
   $$A^\star_j(\cdot)^{\mathscr{I}}=\left\{h \;\middle|\; \frac{(h-\mu_j)^2}{\sigma_j^2}\leqslant z\right\}\quad \big(j=1,\dots,N,\ \mu_j,\sigma_j \text{ as in (13)}\big) \tag{17}$$
   Let $z\in\mathbb{R}$
   $$A_j(\cdot,s_i)^{\mathscr{I}}=\left\{h \;\middle|\; \frac{(h-\mu_{ik})^2}{\sigma_{ik}^2}\leqslant z, h\in A^\star_j(\cdot)^{\mathscr{I}}, k=1,\dots,M\right\}\quad \big(i,j=1,\dots,N,\ \mu_{jk},\sigma_{jk} \text{ as in (14)}\big)$$
   $$O_{ji}(\cdot,\cdot)^{\mathscr{I}}=\big\{\langle h, s_{i_t}\circ w_{t-1}\rangle \;\big|\; h\in A_j(\cdot,s_{i_t})^{\mathscr{I}}\wedge s_{i_t}\circ w_{t-1}\in R_i(\cdot)^{\mathscr{I}}\big\}\quad (i,j=1,\dots,N) \tag{18}$$
2. Interpretation of field terms. Let v be any assignment to the free variables:
   $$\begin{aligned}\big[R_i(w_t)\big]^{(\mathscr{I},v)}&=\eta_s\big(\big\{v\big(w_t/\big(X_1(\omega),\dots,X_t(\omega)\big)\big) \;\big|\; \omega\in R^{\mathscr{I}}, t\geqslant i\big\}\big)\\ &=\sum_{v(w_t/(X_1(\omega),\dots,X_t(\omega)))}\pi\big(X_1(\omega)=s_{j_1}\big)\prod_{k=1}^{t-1}\mathbf{P}\big(X_{k+1}(\omega)=s_{j_{k+1}} \;\big|\; X_k(\omega)=s_{j_k}\big)\quad (s_{j_t}=s_i)\end{aligned} \tag{19}$$
   Let $Y_j: H\mapsto H$, be the identity:
   $$\begin{aligned}\big[A_j(y,s_i)\big]^{(\mathscr{I},v)}&=\eta_I\big(\big\{h \;\big|\; v\big(y/Y_j(h)\big)\in A_j(\cdot,s_i)^{\mathscr{I}},\ i=1,\dots,N\big\}\big)\\ &=\int_B b_i(h)\,dh\quad (B\in\Theta)\end{aligned} \tag{20}$$
   Finally:
   $$\big[O_{ji}(\cdot,\cdot)\big]^{(\mathscr{I},v)}=\big[A_j(\cdot,\cdot),R_i(\cdot)\big]^{(\mathscr{I},v)} \tag{21}$$
3. The field terms $\pi,\mathbf{P}$ and $\mathcal{N}$ are interpreted as themselves.

In particular $\mathscr{I}$ ensures that the distribution on the domain agree with the model $\mathcal{M}$. This is established in the following:

Lemma 1. *Given an HMM $\mathcal{M}=(S,H,(\mathbf{P},\pi,\Psi,\gamma))$, with finite property (see Definition 2) there exists a probability structure $\mathbb{M}=(D,\Psi,\Phi,\mathscr{I})$, with domain $D=(S,H,\mathcal{W},\mathbb{R})$, probability space Φ generated from D, such that the atoms and terms are interpreted according to the $\mathcal{M}$-parameters $\mathbf{P},\pi$ and Ψ. Furthermore, according to the given interpretation $\mathscr{I}$, for each field-based atom φ there is a corresponding measurable set such that the field term for φ has the intended distribution.*

Proof. See Appendix A.5, proof of Lemma 1. □

We have, thus, shown that the interpretation of atoms fully determines the distribution of field terms and the structure $\mathbb{M}$ for the HMM $\mathcal{M}$. Before showing how to extend this to formulae we illustrate how the above lemma is implemented, with an example.



Terms of sort state	$s_1, s_2, s_3, s_4, s_5,..$	
Terms of sort observations	$Y_1, Y_2, Y_{3,..}$	
Predicates for states	$Far(x)$, $At_{handle}(x)$, $Holding_{handle}(x)$, $Unlocked_{door}(x)$, $Opened_{door}(x)$.	
Predicates for observations	$approach(Y)$, $grasp_{handle}(Y)$, $pullDown_{handle}(Y)$, $push_{door}(Y)$, $open_{door}(Y)$.	
Predicates for observations and states	$Poss(Y,w)$	
Field terms		
Transition matrix	$P(s_i, s_j)$	transition from state s_i to state s_j
Prior	π	probability of initial state
Mixture gain matrix	c_{jk}	probability of mixture component j, in state s_k
PPCA Mixture	$b_j(Y_t)=\sum_{k=1}^{3} c_{jk} G_j\left(Y_t \mid \mu_{jk}, A_{jk}A_{jk}^T+\sigma^2 I\right)$	probability of the action coded by observation Y_t, at time t, in state s_j
Field based atoms $A_j(y,x), R_i(x), O_{ji}(y,x)$	$[A_j(y,x)]_{\partial}, [R_i(x)], [O_{ji}(y,x)]$	probability measures on the domain.

Fig. 9. Structures $\mathcal{M}$ and $\mathbb{M}$ for the example *opening a door*, interpretation and signature.

Example 2. In the example we illustrate how we have implemented the interpretation of the atoms, for the considered task, the assignments to the free variables,[5] and the evaluation of the atomic field terms. For more details we refer the reader to the proof of Lemma 1 in Appendix A.5.

Let $\mathcal{M} = (\pi, \mathbf{P}, \Psi, \gamma)$ be the learned HMM structure and $\mathbb{M}$ the first-order parametric structure specified in Definition 6. The signature for the language $\mathcal{L}_{\mathbb{M}}$ is resumed in Fig. 9. The implementation we have considered is based on both Matlab and Prolog and an interface between them, suitably implemented. The atomic field terms are evaluated in Matlab, appealing to the functions implemented for the HMM $\mathcal{M}$, whose data and results are reported in Sections 3 and 4.

Each predicate R_i, $i = 1, \ldots, N$, with $R \in \{Far, At_{handle}, Holding_{handle}, Unlocked_{door}, Opened_{door}\}$ takes as argument the sort sequence of states. Its interpretation is defined as the set of all sequences of states in $\mathcal{W}$ such that the n-th element of the first n-outcomes is s_i, with n up to length 87. In fact, $\mathbf{P}^{87}$ is the stationary distribution for the model $\mathcal{M}$. Sequences start in state s_1, assuming $\pi(s_1) = 1$. The interpretation for each R_i is specified as in Eq. (16). Now, consider $Far(w_t)$, given an assignment v to the free variable with $v(w_t) \in \{X_1(\omega) = s_1, X_2(\omega) = s_{j_2}, X_3(\omega) = s_1\}_{j=1,\ldots,5}$, it follows that $Far(w_t)$ is true in $\mathbb{M}$. If $v(w_t) \in \{X_1(\omega) = s_1, X_2(\omega) = s_{j_2}, X_3(\omega) = s_3\}_{j=1,\ldots,5}$ then $Far(w_t)$ is not satisfied in $\mathbb{M}$. Thus, if w_t indicates a sequence of length t, we are considering an event in which the experiment is repeated t times and $X_i(\omega)$ indicates the outcome at time i.

The probability measure of the field term $[Far(w_t)]^{(\mathbb{M},v)}$ is as specified in (19):

$$\begin{aligned}\left[Far(w_t)\right]^{(\mathbb{M},v)} &= \pi(X_1 = s_1)\big(\mathbf{P}(X_2 = s_1|X_1 = s_1)\mathbf{P}(X_3 = s_1|X_2 = s_1)\big) + \cdots + \\ &\big(\mathbf{P}(X_2 = s_2|X_1 = s_1)\mathbf{P}(X_3 = s_1|X_2 = s_2) + \mathbf{P}(X_2 = s_3|X_1 = s_1)\mathbf{P}(X_3 = s_1|X_2 = s_3)\big)\end{aligned} \tag{22}$$

Consider, now, the predicates A_j. Each predicate A_j, $j = 1, \ldots, N$ (the number of states equals the number of observed actions) is specified by the following action names:

$$A_j \in \{approach, grasp_{handle}, pullDown_{handle}, push_{door}, open_{door}\}$$

An observation action predicate A_j takes as argument an element of sort observation $h \in H$ denoted by a variable y and an element of sort state $s \in S$, denoted by a variable x. Its interpretation, Eq. (17), is defined by the events specified by all

[5] We indicate the assignment to a free variable z of a domain element d either with $v(z/d)$ or with $v(z) = d$.

those descriptors, real valued variables, having Mahalanobis distance from the mean, with covariance σ^2, less than z, see (17) in Definition 6. The domain of each A_j is generated as follows. Let $U \in \mathbb{R}$ be a set of values generated from normal distributions with parameters fixed by the HMM $\mathcal{M}$. Take from U the interval B_z delimited by those values $y \in U$ having Mahalanobis distance, according to the parameters $\mu_{iu}, \sigma_{iu} \in \Psi$, $u = 1, \ldots, M$, less than a specified value z and assign it to $A_j^{\mathscr{I}}$, for each state s_i. To ensure that the values are centred on the mixture generated by $A_j^{\mathscr{I}}$ we first check the Mahalanobis distance for the mixture specified in Eq. (13). In general we assume that z is decided according to the distances between means of different mixtures. For example, if μ_{ju} and μ_{ku} are the means of the component u of two mixtures at states s_k and s_j, we require that $(\mu_{ju} - \mu_{ku})^2 > \sigma_{ju}^2 \sigma_{ku}^2 z$. This would ensure that the interpretation of two observations does not overlap at the modes. The probability measure η_I of the field term is the distribution function of the random variable Y_j over the domain of $A_j(\cdot, s_i)$. Given an assignment $v(y/h) \in A_j^{\mathscr{I}}$, the field term for $A_j(h, s_i)$ is evaluated by the density $b_i(h)$, specified in ((14), see also (1)), that is, the derivative of the distribution function.

Finally the denotation of the predicate O_{ji} is intended as $Poss_{ji}(action, state)$, where the j indicates the observation action and the i the state. The action is meant as the observation of the action via the descriptor. The interpretation of O_{ji} is the conjunction of A_j and R_i as defined in (18). Its field term is as defined in (21).

We can, now, extend the interpretation to formulae, as stated in the following theorem.

Theorem 1. *Let $\mathbb{M}$ be a probability structure of first-order, with probability space Φ. Let Φ be extended to the probability space $\Phi^\star$ with product measures η_s^n, η_I^n, $(\eta_s \times \eta_I)^n$, $n \geqslant 1$, and their space product. For each field-base formula φ there exists a measurable set, whose distribution agree with the HMM.*

Proof. See Appendix A.5, proof of Theorem 1. □

We can now extend the relation $\models$ to field terms and formulae mentioning field terms as follows, let $P \in \{\eta_s, \eta_I, \eta_s \times \eta_I\}$ and let Γ_φ be the measurable set of φ:

$$\begin{array}{lll} 1. & \mathbb{M}, v \models \big[\varphi(\vec{x})\big] \geqslant p & \text{iff } P(\Gamma_\varphi) \geqslant p \\ 2. & \mathbb{M}, v \models \forall y.\big[\varphi(\vec{x})\big] \geqslant p & \text{iff } \mathbb{M}, v(y/d) \models \big[\varphi(\vec{x})\big] \geqslant p \text{ for all } d \in D \text{ of the right sort} \\ 4. & \mathbb{M}, v \models \big[\psi(\vec{y}) | \varphi(\vec{x})\big] \geqslant p & \text{iff } \mathbb{M}, v \models \big[\psi(\vec{y}) \wedge \varphi(\vec{x})\big] \geqslant p \cdot \big[\varphi(x)\big] \end{array} \tag{23}$$

Proposition 1. *Let v and v' agree on all the assignments to the free variables then $\mathbb{M}, v \models \varphi$ iff $\mathbb{M}, v' \models \varphi$.*

Proof. See Appendix A.5, proof of Lemma 1. □

Theorem 2. *Let $\mathcal{M} = (S, H, (\pi, \mathbf{P}, \Psi, \gamma))$ then there exists a probability model $\mathbb{M}$ which is a structure for $\mathcal{M}$, that is $\mathcal{M}$ and $\mathbb{M}$ agree on the distribution on the domain.*

Proof. See Appendix A.5, proof of Theorem 2. □

Example 3. Let us continue with Example 2 as illustrated in Fig. 9. Let us assume that the interpretation for each atom has been specified, as described in Example 2, according to Lemma 1. Let the binary predicate O_{ji} be $Poss_{ji}$, indicating that action A_j is observable (hence executable) at the final state s_{n_i} of a n-length sequence. Let y and y' be variables of sort observation. Consider two sequences, one ending at the state labelled by $Unlocked_{door}$ and the other at At_{handle}. We want to verify the likelihood of observing $Push_{door}$ at the state $Unlocked_{door}$ and the likelihood of observing $Push_{door}$ at the state At_{handle}. We would express this fact as follows. Let w_{n-1} and w_{m-1} be two variables of sort sequence of states, referring to sequences of length $m-1$ and $n-1$, $m = 3$ and $n = 4$:

$$\begin{aligned} &\big[Poss_{jk}(y, x_n \circ w_{n-1}) \wedge Push_{door}(y, x_n) \wedge Unlocked_{door}(x_n \circ w_{n-1})\big] \\ &\quad > \big[Poss_{jk}\big(y', x_m \circ w_{m-1}\big) \wedge Push_{door}\big(y', x_m\big) \wedge At_{handle}(x_m \circ w_{m-1})\big] \end{aligned} \tag{24}$$

Let us assume that the assignment to the free variables of sort sequence of states is as follows $v(w_3) \in \{X_1(\omega) = s_1, X_2(\omega) = s_2, X_3(\omega) = s_3\}$, $v(w_2) \in \{X_1(\omega') = s_1, X_2(\omega') = s_2\}$, the terms of sort state are $\{v(x_3/s_3), v(x_4/s_4)\}$. Furthermore assume that $\pi(s_1) = 1$ and s_1 is the first state, labelled by *Far*. Let $h \in H$ with $Y_j(h) < z$ so that $\mathbb{M} \models Push_{door}(v(y/h), s_2)$ and $\mathbb{M} \models Push_{door}(v(y'/h), s_4)$. We recall from the previous example that $Push_{door}$ is a predicate verified in the parametric model $\mathbb{M}$ according to a Mahalanobis distance, and its atomic field term is evaluated by Eq. (7).

The pdfs $b_4(h) = b_{Unlocked_{door}}(h)$ and $b_2(h) = b_{At_{handle}}(h)$ return the likelihood of the field terms $[Push_{door}(h, s_2)]_\delta$ and $[Push_{door}(h, s_4)]_\delta$.

In our implemented model, an observation is a 17-dimensional vector projected in 3D space (in this section we have, actually, further reduced the space to 1D stochastic variables). For this example we consider a real 17-D observation $\hat{h} = [0.05, 0.48, 0.47, 0.47, 0.55, \ldots]^\top$. Hence, $\hat{h}$ is a 17-dimensional descriptor. The results for the projected h' are:

$$
\begin{aligned}
&\big[Push_{door}\big(h', s_4\big)\big]^{\mathbb{M}} = \big[A_4\big(h', s_4\big)\big]^{\mathbb{M}} = b_4\big(h'\big) \sim 0.274\\
&b_1\big(h'\big) = b_2\big(h'\big) = b_3\big(h'\big) \sim 0.001, b_5\big(h'\big) \sim 0.002\\
&\big[Unlocked_{door}(x_4 \circ w_3)\big]^{(\mathbb{M},v)} = \pi(s_1)\prod_{i=1}^{3}\mathbf{P}(s_i, s_{i+1}) \sim 0.006\\
&\big[At_{handle}(x_3 \circ w_2)\big]^{(\mathbb{M},v)} = \pi(s_1)\prod_{i=1}^{2}\mathbf{P}(s_i, s_{i+1}) \sim 0.192\\
&\big[Poss_{44}(y, x_4 \circ w_3)\big]^{\mathbb{M}} = b_4\big(h'\big)\pi(s_1)\prod_{i=1}^{3}\mathbf{P}(s_i, s_{i+1}) \sim 0.006 \cdot 0.274 = 0.0016
\end{aligned}
\tag{25}
$$

On the other hand, according to the assignment v:

$$
\big[Poss_{42}(y, x_3 \circ w_2)\big]^{\mathbb{M}} = b_4(h)\pi(s_1)\prod_{i=1}^{2}\mathbf{P}(s_i, s_{i+1}) \sim 0.001 \cdot 0.192 = 0.0002 \tag{26}
$$

Now, using the definition of O_{ji}, the product rule and Bayes rule, we obtain that the right-hand side of (24) reduces to (26), hence to 0.0002.

Hence we gather that it is more likely that the action $Push_{door}$ is executed at state $Unlocked_{door}$, hence this is the most likely precondition for the observed action.

We can finally note that in the absence of a proof theory we can use Bayes' rule only at the semantic level. Indeed, being our models parametric, hence canonical, a proof theory should be defined modulo parameters. In fact, given a class $\mathbb{A}$ of HMM models $\mathcal{M}_1, \mathcal{M}_2, \ldots$ of some observed event, there exist a class $\mathbb{B}$ of parametric probability models $\mathbb{M}_1, \mathbb{M}_2$, corresponding to $\mathbb{A}$, according to Theorem 2. Now, given that both classes of structures model the same group of events, there must be formulae which are true of all these structures, independently of the variation of the parameters in each structure. Thus there might be a proof theory by lifting from the individual parametric models to a class of models. The analysis of this issue might be a next step of research.

6. The final induction step, concluding with the Situation Calculus

At this step our purpose is to complete the induction obtaining a simple theory of action. This last step is attained by mapping formulae true of the parametric probability model into formulae of the Situation Calculus. Learning has concluded a cycle and reasoning is the next step.

Recently several research efforts have been directed towards the problem of formalising learning steps in a theory of actions, also for planning purposes.

For example, Amir and Chang in [3] have introduced a dynamic model of actions for solving the problem of simultaneous learning and filtering from partial observations. The model constructs a system aware of its real states, with respect to the expected ones. The model, though has aims similar to the one we investigate here with the induction of structures, does not rely on a logic of first-order nor on probabilistic logic. On the other hand Pasula and colleagues, in [86] propose a framework for learning how to suitably represent new concepts and, thus, new rules for handling action effects in planning domains. So, in particular, the objective of [86] is to learn a transition model. This is done in a language with Skolem constants, in which first-order rules for actions are combined with probabilistic rules. As in our approach, a deterministic model of action is learned. However because in [86] learning does not construct from perception most of the effort is done at the representation level, by adding noise. Therefore it is not clear how the combination of rules would allows for reasoning on the learned structure. The differences with our approach are both with respect to the methodology and the formalism. For example, we do not use operators and we model learning as an induction process based on different phases. On the other hand [86] assess the process via a set of rules that are embedded in a specific logical formalisms. Furthermore, in our approach learning is precisely from experience, the action model can be actually constructed online, as shown in the examples, while in [86] approach online learning cannot really be addressed. In fact, learning rules are initialised by the designer and not completely learned as in our method. Under a view similar to [86], indeed, logical learning has been studied in connection with logic in several works. For example in connection with logic programming in the language SLP (Stochastic Logic Programs) by Muggleton [118,83] and Cussens [24] and in PRISM (Programming in statistical logic) by Sato and Kameya [107]. On the other hand BLOG [81] offers a representation paradigm, in which also HMM can be computed, but it cannot really embed parameters into first-order formulae although it offers interesting methods for sampling interpretations. Learning the axioms of an action ontology should be regarded also as a discovery problem.

The Situation Calculus (SC) [74,100,101] is optimal for the purpose illustrated in the example, because it provides a language for a dynamic theory of action. Despite being deterministic, the SC models the first-order Markov property (actually it can model also the n-th order Markov property), it models time-homogeneity, and sequences of situations as histories of actions on an infinite tree of choices [101].

We recall that the Situation Calculus, has been earlier formalised by McCarthy and Hayes in [74] and further extended by Reiter, in order to give a *simple* solution to the frame problem, exploiting the definitional strength of first-order logic without appealing to the meta-language. The version of the Situation Calculus provided initially by Reiter in [100], that McCarthy calls the *Toronto Situation Calculus*, is a multi-sorted language powered by a second order induction axiom. In particular, the language is sorted on three distinguished disjoint domains (Act, Obj, Sit), i.e. *Actions*, *Objects* and *Situations* and it comes with an axiomatisation for situations denoted by Σ, inducing the above mentioned tree, rooted in a special situation called S_0. Situations are bounded to be countable by the second order axiom, specified in Σ, forcing the structures of the language to be the intended ones.

Reiter, as a solution to the frame problem, has also introduced axiom schemata that are classified into three sets: $\mathcal{D}_{una}$, axioms specifying unique names for actions, $\mathcal{D}_{ap}$, axioms specifying preconditions for actions and $\mathcal{D}_{ss}$ which are the successor state axioms. These sort of axiom schemata can be suitably elaborated to obtain a full dynamic theory of actions which is made meaningful by specifying the initial conditions of the state of affairs, that is, those axioms telling what is true in S_0: the state of the world at the very instant in which actions clock in.

This initial description is called $\mathcal{D}_{S_0}$. In [88], it is shown that if this initial theory, called, $\mathcal{D}_{S_0}$ is consistent then the additions of all the others axioms turns out to be a conservative extensions of $\mathcal{D}_{S_0} \cup \mathcal{D}_{una}$, given by induction.

A structure for $\mathcal{L}_{SC}$ is defined as follows.

Definition 7. A structure $\mathscr{M}$ for the language $\mathcal{L}_{SC}$ is defined by $(D, \mathscr{I}_{SC})$, where D is the domain partitioned in three sets, object X, actions A and situations $\mathscr{S}$ and $\mathscr{I}_{SC}$ is the interpretation. The signature for the language $\mathcal{L}_{SC}$ is formed by a countable set of variables for each sort, the constant S_0 of sort situation, the function $do : A \times \mathscr{S} \mapsto \mathscr{S}$, a countable set of distinguished predicates called fluents, taking as last argument a term of sort situation, and a special fluent symbol *Poss*. To this set any other predicate and function symbol can be added. The language $\mathcal{L}_{SC}$ is formed by terms and formulae over the signature, with usual connectives and quantifiers and it is extended to include first-order predicate variables of sort fluent.

The discovery problem here is to generate a very simple theory of actions for the task *open the door*, from the parametric model $\mathbb{M}$. To this end we will build a structure $\mathscr{M}$, of the specified language, in which $\mathcal{D}_{S_0}$, and the set of axioms $\mathcal{D}_{ss}$ and $\mathcal{D}_{una}$, are obtained from sentences with maximal probability values in $\mathbb{M}$.

Let us specify the signature for $\mathcal{L}_{SC}$ on the basis of the signature of $\mathcal{L}_{\mathbb{M}}$ as follows:

Definition 8 *(Signature of $\mathcal{L}_{SC}$).* The signature of $\mathcal{L}_{SC}$ is defined via the signature of $\mathcal{L}_{\mathbb{M}}$ as follows:

1. Variables of sort object, action and situation are freely introduced.
2. A new constant symbol S_0 of sort situation is introduced.
3. Constants of sort object are freely introduced as names for each element devised in the image sequence.[6]
4. For each predicate symbol A_j in $\mathcal{L}_{\mathbb{M}}$ a zero-ary or unary function symbol of sort action is introduced; and for each predicate symbol R_i in $\mathcal{L}_{\mathbb{M}}$ an analogous unary or binary fluent symbol is introduced.
5. For the predicate symbols O_{ji} in $\mathcal{L}_{\mathbb{M}}$, with $j = 1, \ldots, m$, m the number of actions and $i = 1, \ldots, N$, N the number of states, the binary predicate symbol *Poss* is introduced.

Given a structure $\mathscr{M}$ for $\mathcal{L}_{SC}$, by a suitable construction (see [88]) it is possible to ensure that $\mathscr{M}$ satisfies the foundational axioms Σ (see Appendix A.6, Definition 66). Let $\mathscr{M}$ be any structure for $\mathcal{L}_{SC}$ satisfying the foundational axioms Σ, for each introduced symbol A_j, R_i and O_{ij}, with v and v' assignments to the free variables, we define:

$$\begin{aligned}
&\mathscr{M}, v \models A_j(z) \neq A_i(z) && (\forall i\ j, i \neq j)\\
&\mathscr{M}, v \models R_i(z, S_0) && \text{iff } \mathbb{M}, v' \models R_i(x) \text{ and } v'(x) = \arg\max_{s \in S} \pi(s)\\
&\mathscr{M}, v \models Poss\big(A_i(z), S_0\big) && \text{iff } \mathbb{M}, v' \models A_i(y, x) \text{ and } v'(y/h) \in \arg\max_{h \in B} \big[A_i(y, x)\big]_{\delta}^{(\mathbb{M}, v')}\\
&&& \quad \text{and } v'(x) = \arg\max_{s \in S} \pi(s)
\end{aligned} \tag{27}$$

Here $v'(y)$ and $v'(x)$ are, respectively, elements of the domain observation and states in $\mathbb{M}$, π assigns a probability to each initial state in both the structures HMM and $\mathbb{M}$ and $[A_i(y, x)]_{\delta}^{(\mathbb{M}, v')}$, with the subscript δ, indicates the density $b_s(y)$ if

[6] In this paper, these constants denoting objects would not be derived from learning since we have only identified the number of interesting elements in each image, which are recorded in the descriptors, but not an interpretation for each of them. To adequately develop this issue it is either necessary to extend the visual step with tagging or to add a suitable sound analysis, also with tagging.

$v(x/s)$, see Appendix A, proof of Lemma 1. We have thus defined both $\mathcal{D}_{una}$ and $\mathcal{D}_{S_0}$, and ensured that $\mathscr{M}$ is a model of $\Sigma, \mathcal{D}_{una}$ and $\mathcal{D}_{S_0}$.

Now given a structure $\mathscr{M}$ for $\Sigma \cup \mathcal{D}_{una} \cup \mathcal{D}_{S_0}$ in $\mathcal{L}_{SC}$ we can extend it using the first-order probability model $\mathbb{M}$. We can do that because if we obtain, from $\mathbb{M}$, a set of successor state axioms (ssa) and a set of action precondition axioms (apa), then these axioms can be added consistently to the above defined initial axiomatisation, as a consequence of the relative consistency theorem of [88]. The relative consistency theorem, in fact, says that starting with a structure which is a model for the set $\Sigma \cup \mathcal{D}_{una} \cup \mathcal{D}_{S_0}$, this structure will also model the set $\mathcal{D}_{ap} \cup \mathcal{D}_{ss}$ defined in the language (see [88]).

Generating these axioms from $\mathbb{M}$ cannot be done just considering the first-order part of $\mathbb{M}$. In fact, it is clear that relying just on the first-order information recorded in the structure $\mathbb{M}$ would be insufficient because the information in a structure is complete. On the other hand the probabilistic information is more conspicuous. The learned likelihood of the observed (executed) actions of a sequence and the likelihood of states, can tell us how to connect actions, transitions and states. It turns out that only preconditions can be clearly established and the effect of actions can be reasoned about only by referring or projecting to the next state. This is what is presented below.

Going back to Eq. (27), $\arg\max_{v'(y)\in B}[A_i(y,s_k)]_\delta^{(\mathbb{M},v')}$ indicates the value in B, a Borel set on $\mathbb{R}$, at which the density of A_i is maximal, according to the state s_k and the assignment v'. More precisely:

Definition 9. $[A_j(y,s_k)]_\delta^{(\mathbb{M},v)}$ is maximal in $\mathbb{M}$, precisely at s_k iff for $v(y/h) \in B$:

1. $[A_j(y,s_k)]_\delta^{(\mathbb{M},v)} > [A_j(y',s_k)]_\delta^{(\mathbb{M},v)}$, for all $v(y'/h') \notin B$
2. $[A_j(y,s_k)]_\delta^{(\mathbb{M},v)} \geqslant [A_j(y,s_u)]_\delta^{(\mathbb{M},v)}$, for any s_u, $s_u \neq s_k$

$[R_i(x \circ w_t)]$ is maximal (most likely) in $\mathbb{M}$, at $v(x/s_i)$ if:

3. $[R_i(s_i \circ w_t)] \geqslant [R_j(s_j \circ w_t)]$, $s_i \neq s_j$.

The observation values for which $[A_j(y,s_k)]_\delta^{(\mathbb{M},v)}$ is maximal in $\mathbb{M}$, under the assignment v, are defined by $v(y/h) \in \arg\max_{h\in B}([A_j(y,s_k)]_\delta)$.

The above definition can be extended also to terms including $\mathbf{P}_{ij}$.

According to the learned structure $\mathbb{M}$, at each state $s_i \in S$ there exists an action field term which is maximal, independently from the process or sequence that leads to s_i, see also the example in Fig. 6. This is made precise in the following lemma:

Lemma 2. *For each t, there exists a most likely state $R_i(x \circ w_t)$, with $\mathbb{M}, v \models R_i(x \circ w_t)$, and there exists an A_j such that $[A_j(y,x)]_\delta^{(\mathbb{M},v)}$ is maximal. Furthermore $[A_j(y,x)]_\delta^{(\mathbb{M},v)}$ is maximal precisely at $v(x/s)$, for some s, in the interpretation of R_i, but independently of any assignment v to w_t.*

Proof. See Appendix A.6, proof of Lemma 2. □

In a similar way we can show maximality for field terms concerning state predicates and transitions.

Lemma 3. *For each most likely state $R_u(x \circ w_t)$, with $\mathbb{M}, v \models R_u(x \circ w_t)$, there exists an observation action $A_j(y,x)$ and a transition $\mathbf{P}(X_{t+2} = x' | X_{t+1} = x)$ that extends $R_u(x \circ w_t)$ to $R_i(x' \circ x \circ w_t)$, maximally, for some i.*

That is, $[A_j(y,x)]_\delta^{(\mathbb{M},v)}\mathbf{P}(X_{t+2} = x' | X_{t+1} = x)$ is maximal at $v(x/s_u)$.

Proof. See Appendix A.6, proof of Lemma 3. □

An immediate corollary of the above property is:

Corollary 1. *For each state predicate $R_i(x \circ x' \circ w_t)$, with $\mathbb{M}, v \models R_i(x \circ x' \circ w_t)$, there exist two observation actions A_m and A_j and a unique transition $\mathbf{P}(X_{t+2} = x | X_{t+1} = x')$ that extends $R_u(x' \circ w_t)$ to $R_i(x \circ x' \circ w_t)$, maximally. That is, $(\mathbf{P}(X_{t+2} = x | X_{t+1} = x')[A_j(y',x')]_\delta [A_m(y,x)]_\delta)^{(\mathbb{M},v)}$ is maximal at $v(x/s_i)$.*

Proof. See Appendix A.6, proof of Corollary 1. □

We can now make the following remarks concerning observations. Given a state $R_u(x \circ w_t)$, with w_t any sequence, $t = 1, \dots, T$, three cases might occur for any observed action $A_j(y,x')$ relatively to this state:

1. Remaining. $[A_j(y,x')]_\delta^{(\mathbb{M},v)}$ is maximal at x, with $v(x)=v(x')$ and $\mathbf{P}(X_{t+2}=x|X_{t+1}=x)$ is maximal, hence the process remains in the same state.
2. Entering. $[A_j(y,x')]_\delta^{(\mathbb{M},v)}$ is maximal at x' and $\mathbf{P}(X_{t+2}=x|X_{t+1}=x')$ is maximal, $v(x')\neq v(x)$. In this case the process enters in state x' incoming from another state.
3. Exiting. $[A_j(y,x)]_\delta^{(\mathbb{M},v)}$ is maximal at x, and $\mathbf{P}(X_{t+2}=x'|X_{t+1}=x)$ is maximal with $v(x')\neq v(x)$. Hence the process abandons x to go to another state, via the *execution* of A_j.

Note the importance of $[A_j(y,x)]_\delta$ being maximal at x, because this is the action observed, according to the parameters estimation and the distributions in the learned models. Thus the observed action is the one executed at the state, which specifies the action preconditions. However the effects can be only specified (as with Lemma 3) by the joint of the observed action and the transition.

Since the successor state axiom captures the persistence of properties, we need to use the above properties to ensure a somewhat analogous persistence property as follows:

Definition 10 *(Persistence).* Let w_t be a fixed sequence, with $v(w_t)\in\{X_1(\omega)=s_{j_1},\dots,X_t(\omega)=s_{j_t}\}$ and $[R_{j_{t+2}}(x\circ x'\circ w_t)]^{(\mathbb{M},v)}=[R_{j_{t+1}}(x'\circ w_t)]\mathbf{P}(X_{t+2}=x|X_{t+1}=x')$. A state predicate (property) R_j is said to happen at w_t, at t, if $\mathbb{M},v\models R_j(w_t)$ and, for any state predicate at w_t, at $t-k$, $[R_{i_{t-k}}(x\circ w_{t-k})]\mathbf{P}(X_{t-k+1}=x'|X_{t-k}=x)\cdots\mathbf{P}(X_t=x^*|X_{t-1}=x')=[R_j(w_t)]$. When $[R_{j_t}(x\circ x'\circ w_{t-k})]$ happens at t and it does not happen at $t-1$ then it is said to enter w_t; and when it happens at t and it does not happen at $t+1$ then it is said to exit from w_t (see also the definition above).

A predicate R_j that happens for a fixed sequence w_t at $t+k$, $k\geqslant 0$, is *persistent* if $\mathbb{M},v\models R_j(x\circ w_t)$, $[R_i(x\circ w_t)]^{(\mathbb{M},v)}$ is maximal at t and:

1. When it enters w_t, at $t+2$, in the form $R_i(x\circ x'\circ w_t)$, there exist two actions $A_j(y',x')$ and $A_m(y,x)$, with $\mathbb{M},v\models A_j(y',x')\wedge A_m(y,x)$. Moreover the terms $[A_j(y',x')]_\delta,[A_m(y,x)]_\delta$ are maximal at x' and x, with $v(x/s_u)$, for some u, such that the predicate of state $R_u(x'\circ w_t)$ happens at w_t, at $t+1$, and it is persistent.
2. When it exits from w_t (maybe for future reappearance) there exist two actions $A_q(y',x')$ and $A_k(y,x)$, with terms maximal at x' and x, respectively. Furthermore:

   $$\mathbb{M},v\models A_q(y',x')\wedge A_k(y,x)\wedge R_i(x'\circ w_t)\wedge\neg R_i(x\circ x'\circ w_t)$$

   That is, $v(x'/s_i)$, $v(x/s_u)$ for some $u\neq i$. However, it is always possible to delay the exit at w_t, from t to $t+k$, that is (for $k=1$):

   $$\mathbb{M},v\models\neg A_k(y,x')\wedge R_i(x'\circ w_t)\wedge R_i(x'\circ x'\circ w_t)$$

These conditions can be expressed by the maximality of the term

$$\big[R_i(x\circ x'\circ w_t)|A_j(y,x)\big]+\big[R_i(x\circ x\circ w_t)|\neg A_k(y',x)\big] \tag{28}$$

in $\mathbb{M}$, for some v.

The persistence of a predicate of state is stated below:

Theorem 3. *Given $\mathbb{M}$ and an assignment v, with $v(w_t)=\hat{w}_t$, $\hat{w}_t\in\{X_1(\omega)=s_{j_1},\dots,X_t(\omega)=s_{j_t}\}$, $t\geqslant 0$, and for all R_i there exists a sequence $(x\circ x'\circ\hat{w}_t)$ such that $R_i(x\circ x'\circ\hat{w}_t)^{(\mathbb{M},v)}$ is persistent.*

Proof. See Appendix A.6, proof of Theorem 3. □

Given the above results we can finally specify how to obtain the action precondition axioms and the successor state axioms for the learned theory, note that the variables z,z' and z'' are free variables denoting the domain object of $\mathscr{M}$, not specified in $\mathbb{M}$:

1. $\mathscr{M},v'\models\forall s.Poss(A_j(z),s)\equiv R_i(z',s)$

   iff

   $\mathbb{M},v\models\exists x.A_j(y,x)\wedge R_i(x\circ w_t)$, and $[A_j(y,s_i)]_\delta^{\mathbb{M},v}$ is maximal precisely at the s_i, chosen for x
2. $\mathscr{M},v'\models\forall a\,s.R_i(z,do(a,s))\equiv A_j(z')=a\vee\forall z''A_m(z'')\neq a\wedge R_i(z,s)$

   iff

   $\mathbb{M},v\models\big[R_i(x\circ x'\circ w_t)|A_j(y,x')\big]+\big[R_i(x\circ x\circ w_t|\neg A_m(y',x))\big]>0$

   and $R_i(x\circ x'\circ w_t)$ is persistent for some $v(w_t)\in\{X_1(\omega)=s_{j_1},\dots,X_t(\omega)=s_{j_t}\}$, $t\leqslant T_{\mathbf{P}}$ (29)

So it is interesting to note that the universal quantifiers for the sort situation in $\mathcal{L}_{SC}$ are introduced both by action maximality and by persistence, in the domain $\mathcal{W}$ of the structure $\mathbb{M}$. On the other hand the quantifiers on the actions are fixed by the equality in the successor state axioms, and this fact is justified by the choice of the argument $v(y)$ with respect to maximality.

Note that the variables of sort objects are freely (and arbitrarily) added, according to the comment in item (3) above, concerning the signature of $\mathcal{L}_{SC}$.

We have thus obtained a basic theory of actions on the basis of what has been learned from the initial perception.

Example 4 *(A basic theory of actions for the task* open the door*).* According to the above definitions, the action theory turns out to be the following, here we have added (arbitrarily) variables and constants of sort object. The signature is the one specified in Example 2 and illustrated in Fig. 9, with subscripts removed. All the actions and situation variables are universally quantified as gathered above.

Initial database $\mathcal{D}_{S_0}$:

$$Far(z, S_0) \wedge z = door$$

this coincide with the first state in the models $\mathcal{M}$ and $\mathbb{M}$, indeed $\arg\max_x \pi = s_1$ whose label is R_1, whose denotation is *Far*.

Action Precondition axioms:

$$\begin{aligned}
&\forall s.Poss\big(approach(z), s\big) \equiv Far(z, s)\\
&\forall s.Poss\big(grasp(z), s\big) \equiv At(z, s)\\
&\forall s.Poss\big(pullDown(z), s\big) \equiv Holding(z, s)\\
&\forall s.Poss\big(push(z), s\big) \equiv Unlocked(z, s)
\end{aligned} \tag{30}$$

Successor state axiom:

$$\begin{aligned}
&\forall a\, s.Far\big(z, do(a, s)\big) \equiv a \neq approach(z) \wedge Far(z, s)\\
&\forall a\, s.At\big(z, do(a, s)\big) \equiv a = approach(z) \vee At(z, s)\\
&\forall a\, s.Holding\big(z, do(a, s)\big) \equiv a = grasp(z) \vee Holding(z, s)\\
&\forall a\, s.Unlocked\big(z, do(a, s)\big) \equiv a = push(z) \vee Unlocked(z, s)\\
&\forall a\, s.Opened\big(z, do(a, s)\big) \equiv a = open(z) \vee Opened(z, s)
\end{aligned} \tag{31}$$

It remains to prove that the conditions on $\mathbb{M}$ in Eqs. (29) are well defined. That is, we need to ensure that the successor state axioms can be obtained uniquely from a parametric probability model of first-order. In other words, there must be no ambiguities in the choice of the action precondition axioms and in the successor state axioms.

This is stated in the following theorem.

Theorem 4. *Let $\mathbb{M}$ be a first-order parametric probability structure, as defined in Definition* 6 *and v any assignment to the free variables. Let $v(w_t) \in \{X_1(\omega) = s_{j_1}, \ldots, X_t(\omega) = s_{j_t}\}$ and let $R_i(x \circ w_t)$ be any predicate of state, with $\mathbb{M}, v \models R_i(x \circ w_t)$:*

1. *There is at least an action $A_j(y, x)$ that is maximal at $v(x/s)$, for s in the interpretation of R_i, $i = 1, \ldots, N$. Hence the action precondition for each A_j is uniquely determined by R_i.*
2. *There exists at least one action A_j, and possibly A_m, ensuring both persistence and satisfiability of R_i, $i = 1, \ldots, N$. Hence the successor state axiom for R_i is uniquely determined by the maximal values of $[R_i(x \circ x' \circ w_t)|(A_j(y, x)] + [R_i(x \circ x \circ w_t)|\neg A_m(y', x))]^{(\mathbb{M}, v)}$.*

Proof. See Appendix A.6, proof of Theorem 4. □

The method provided ensures that, given a first-order probability structure $\mathbb{M}$ it is possible to construct uniquely a theory of actions and to derive the initial database, the action precondition axioms and the successor state axioms. It is also interesting to note that in the probability model $\mathbb{M}$ it is always possible to establish several properties of states that are stronger than in a classical HMM because states and observations are specified by first-order properties. This has allowed us to obtain the axioms for a basic theory of actions in the Situation Calculus. We do not need, in fact, in $\mathbb{M}$ to deal with observations as single variables.

There are several limits though. First of all the example is still quite poor. The main reason is the fact that we have not been able to learn the signature, and we have just used the labels of the HMM. Furthermore no method has been established to assign elements of sort object to the domain, it could even be possible that there is only one element in the domain of sort object. The limited example and the limited vocabulary make also impossible to define axioms for unobserved fluents. For example as closing the door was not observed, we do not have axioms mentioning the action *close*:

$$Opened_{door} \equiv a = Open_{door} \vee a \neq Close_{door} \wedge Opened_{door}(s)$$

Thus the axioms are quite naive.

Another important limitation is that, in order to construct the theory, maximality on the densities associated with field terms for observation actions is required. These requirements so far are restricted to very simple cases, because of the lack of a clear condition on effect of actions. Indeed what is observed is only the precondition of an action, thus effects are stated only under some maximality requirement. However determining maximality can be a difficult and expensive task.

7. Comparisons and conclusions

We have proposed a completely new method that from perception discovers a genuine theory of actions in the Situation Calculus. We wanted to show that the design of a logical robot-child requires different steps of modelling, starting from a pure visual path. The transformation requires building the concept of observation as action. Then, on top of this the induction process can build a computational model of observations actions and states and finally an action theory for reasoning.

The transformations we have shown start from trials, concerning observed events (phenomena), and move to the HMM model $\mathcal{M}$ of the observed events. Then, from this, to the parametric probability model $\mathbb{M}$, and finally from the probability model to the first-order model $\mathscr{M}$. These are the inductive steps underlying the process of learning *reality behind appearance* [78]. At the very beginning, the robot-child can observe a sequence of phenomena, from which it learns measures of the real-world features, forming its perception of actions. It learns to observe, constructing from motion a space of actions, this is modelled by a computational structure such as, for example, the HMM model $\mathcal{M}$. The terms in the computational structure are, however, volatile so they have no strength to explain the events and the observed phenomena. Thus the next step is to state properties and relations among these terms, according to the given signature. These can be done within a structure embedding the parameters into relations and assessing connections between relations, which is the probability model $\mathbb{M}$. These properties enable the robot-child hypotheses to be formed about what it has seen. At this point it should be able to induce regularities regarding the hidden reality behind appearance.

The robot-child has observed events affected by uncertainty, events that are subject to randomness, and the canonical model proves only a specific sample of observations (independently of how many trials have been performed). Therefore it is quite hazardous to induce general laws, from these hypotheses. This, in fact, is achieved in the last step of the induction obtaining a theory of actions. Still the crucial problem of *effectively* inducing general laws remains open, we have not provided a proof theory for the parametric probability model.

Actually the induction process needs a vocabulary too, i.e. the signature, denoting the observations, and how to obtain this it is not faced here. It would be interesting to find out how to combine visual and audio learning to connote each observation with a word and then, following a suitable transformation, obtain a theory of actions in which also the signature is learned. Another limitation is that no object is learned.

Nevertheless we have not even completely explicated in details all the steps and passages on how to generate a full theory of actions from visual observations. We have just hinted some crucial passage that have to be further developed. However we have shown, with respect to McCarthy argument, that learning reality behind appearance is possible, directly via perception.

The novelty of our approach resides in several aspects. First of all the whole transformation methodology has never been faced, mainly because the communities of vision and knowledge representation and learning are not yet well integrated. Further, although the interpretation and recognition of gestures is an active research area, our approach based on building a multidimensional descriptor and then training a continuous observations PPCA-HMM, to build the action space, is quite new. Finally the construction of a probability first-order model for visual processes, embedding parameters, is, to our knowledge, new.

Despite the novelty, this is not the first research attempt to generate knowledge by learning from observations, however it is certainly the first that directly from perception builds a formal theory of actions.

The concept of *observation* is vague enough to be used in several contexts. In many approaches an observation is not even connected with perception and it is a structured representation of code, as for example in [43,117] or similar approaches, where learning is not burden by the problem of building the data.

On the other hand, robot learning, by sensing and vision, is a wide research area, in which robot actions are learned for both assessing and improving robot performances. A particular prolific research area connected with robot learning is learning by imitation, an example are the works of [73,29,58,27,18,19,12]. Other approaches have otherwise favoured the discovery and planning aspects to inform actions, without stressing a biological inspiration. These are, for example, the approaches of Cohen and colleagues [20,108,109]. Although these approaches introduce a clear connection among discovery, learning and the effect of actions, there is no attempt to generate a theory that would allow a robot to reason on what has been learned, which is, instead the purpose of our approach. A survey on robot learning facing the current state of the art has been recently provided by [5].

A quick comparison with other research approaches to action learning from perception has been done in Section 2, while the methodology behind the approach has been discussed in Appendix A. Apart from the works on probabilistic logic cited in Section 5 several works have studied stochastic processes and the problem of verifying whether a sentence about the process is satisfied in some logical language. These approaches were mainly based on temporal propositional logic, but also with some monadic first-order logic, beginning with the works of Vardi [115]. From [115] a huge literature germinated on probabilistic model checking and automatic verification of processes, starting with CTL and then logics such as DTMC, MDPs, PCTL etc. Model checking algorithms have been developed for both discrete time Markov chains, such as in the work of Alur [2] and Hansson and Jonsson [51] and for Markov decision processes, as in the works of [14,23,28]. These are only few examples of a vast literature on probabilistic model checking essentially based on propositional logic. Further Beauquier, Rabinovich and Slissenko have studied model checking for a specific class of first-order formulae representing liveness properties of Markov chains [11], their work is indeed inspired by the work of Halpern on probabilistic logic [36, 49,37], and the possible worlds approach.

Hidden Markov models have been studied in connection with logic programming for example in the language SLP (Stochastic Logic Programs) by Muggleton [118,83] and Cussens [24] and in PRISM (Programming in statistical logic) of Sato and Kameya [107]. These approaches, however seem to be more attracted by the statistical learning computation, of which the HMM are a particular case, than by the representation problem concerned with probabilistic logic. On the other hand BLOG [81] offers a representation paradigm, in which also HMM can be computed, but it cannot really embed parameters into first-order formulae, although it offers interesting methods for sampling interpretations.

Indeed, as Poole notes in [91] “probability specifies a semantic construction and not a representation of knowledge”. Poole briefly discusses a possible representation for HMM in the language ICL, the Independent Choice Logic [90], as a special case of Dynamic Belief Networks. A discussion on the close relationship existing between the above three frameworks combining probabilities and logic is presented in Cussens in [26].

On the other hand a number of formalisms have been devised for combining probabilistic reasoning, logical representation and machine learning. This integration is nowadays a strong research area, in [99] De Raedt and Kersting show a Venn diagram intersecting probability, learning and logic whose kernel is probabilistic logic learning, they also provide an overview of the different formalisms developed in recent years. The analysis is mostly concerned with stochastic logic programs.

Our approach differs in many respects with the above cited ones. First of all because, even though we have shown that all the computations valid in a specific HMM $\mathcal{M}$ are equivalent to those obtained in the probability model $\mathbb{M}$, we firmly believe that all the computational steps (for the parameters) have to be carried in the computational model (see Example 2). Our view is not to supersede with the logic what can be quite easily achieved with the computational learning models, but to build an induction process.

A limitation of our approach is that we have considered only the construction of a canonical model for the HMM. Furthermore we assume the finite property for the HMM models. More understanding is needed to prove several properties concerning parameters embedding and lifting to first-order logic.

The system has been implemented up to the transformation into the theory of actions. The whole visual process has been implemented using an original script for segmentation, motion detection and optical flow, all has been realised in Matlab. This part of the implementation creates a matrix of 17-dimensional vectors each one standing for the processed frame. The implementation of PPCA-HMM is also done in Matlab, and extends other well-known implementations to the mixture of probabilistic PCA, where the re-estimation parameters of the PPCA-HMM are those discussed in Appendix A.4. To this end we have widely used, as basic work, the implementation provided by Kevin Murphy [84] and the primitives for mixtures of Gaussian provided by Netlab [85]. The probability model of first-order has been implemented using an embedding of Prolog into Matlab, as explained in Example 2. To conclude, the mechanisation of the process of learning reality behind appearance, still lack an automatic introduction of the signature, this will require the integration with either voice or image tagging.

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Appendix A

In Appendix A.1 we introduce some preliminaries on the probability space and measure space. In Appendix A.2 we introduce the basic concepts of Markov chains. Further, in Appendix A.3 we introduce some details concerning the optical flow problem which is used in Section 2 for the features descriptors. In Appendix A.4 we introduce the mixture of principal component analysers [113,114] used to model the action space via the descriptors, and the extension of the re-estimation procedure for mixtures PPCA to Hidden Markov Models with continuous observation densities. In Appendix A.5 and in Appendix A.6 we report the proofs of Sections 5 and 6.

A.1. Probability space

A measurable space is a pair $(\Omega, \mathscr{F})$ with Ω a set and $\mathscr{F}$ a σ-field of subsets of Ω. The collection $\mathscr{F}$ of subsets of Ω is a σ-field, if the empty set is in $\mathscr{F}$, if $A_1, A_2, \ldots$ are elements of $\mathscr{F}$ then also their union is in $\mathscr{F}$ ($\mathscr{F}$ is closed under the operation of taking countable unions), and if $A \in \mathscr{F}$ also the complement $\overline{A}$ of A is in $\mathscr{F}$.

A probability space is a triple $(\Omega, \mathscr{F}, P)$ s.t. Ω is the sample space, and an *event* is any field set of $\mathscr{F}$, with Ω the certain event. If P is a measure, the domain of P is the σ-field of all sets $G \in \mathscr{F}$ for which $P(G)$ is defined. Hence P is a probability measure on $(\Omega, \mathscr{F})$ if $P : \mathscr{F} \mapsto [0,1]$ and it satisfies the conditions: $P(\emptyset) = 0$, $P(\Omega) = 1$ and

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

whenever for all $i \neq j$, $A_i \cap A_j = \emptyset$.

Given a probability space $(\Omega, \mathscr{F}, P)$ and a measurable space $(\Theta, \mathcal{B})$ a random variable X on $(\Omega, \mathscr{F})$ is a measurable $\mathscr{F}$ function $X : \Omega \to \Theta$, such that:

$$\text{if } F \in \mathcal{B}, \quad \text{then } X^{-1}(F) = \{\omega \in \Omega \mid X(\omega) \in F\} \in \mathscr{F}$$

That is, the inverse image of X-measured events are events in the original σ-algebra.

The probability measure induced by X is:

$$P_X(\{\omega \mid X(\omega) \in B\}), \quad B \in \Theta$$

the distribution function F_X is:

$$F_X(z) = \{x \mid X(x) \leqslant z\}$$

A.2. Markov chains

We recall that a chain (discrete time stochastic process), is defined by a family of random variables $\{X_t\}_{t \geqslant 0}$ indexed with discrete time, with T the discrete time-index set. That is, a chain is a function taking as arguments the time $t \in T$ and the states in a sample space S. $X(j)$ or X_j indicates the random variable varying on both states and time and the values of X_j are states, e.g. $(X_j = s_i)$ indicates that the random variable at time j takes as value the state $s_i \in S$.

Given a probability space a process $\{X_i\}_{i \geqslant 0}$ is a discrete time Markov chain if:

$$P(X_{k+1} = s \mid X_0 = s_{j_0}, \ldots, X_k = s_{j_k}) = P(X_{k+1} = s \mid X_k = s_{j_k}) \tag{32}$$

Property (32) is the Markov property. A Markov chain is *homogeneous* if the transition does not depend on the time step. That is,

$$P(X_{i+1} = s_m \mid X_i = s_k) = P(X_1 = s_m \mid X_0 = s_k) \tag{33}$$

We shall only consider discrete time homogeneous Markov chains.

A model $\mathcal{M}$ for a Markov chain $\{X_i\}_{i \geqslant 0}$ is identified by two parameters: $(\pi, \mathbf{P})$.

1. π is a stochastic vector (i.e. it sums to 1, and each entry is nonnegative) defining the *initial distribution of the states*;
2. $\mathbf{P}$, the *state transition* matrix, is a stochastic matrix (i.e. each row is a stochastic vector) defining the probability to move from states to states.

All the entries of $\mathbf{P}$ will be of the kind $P(X_k = s_j \mid X_{k-1} = s_i)$, $s_i, s_j \in S$, abbreviated by p_{ij}. Since we assume that the set of states is finite then $\mathbf{P}$ will have a dimension $n \times n$ if $|S| = n$. At time 1 the probability of each state is determined by the initial distribution π. The matrix $\mathbf{P}^n$ is the n-step transition matrix. The distribution of the Markov chain $(\pi, \mathbf{P})$, at time n is given by the stochastic vector

$$\mathbf{p}_n = \mathbf{p}_{n-1}\mathbf{P} = \pi \mathbf{P}^n$$

For our purpose, we shall deal with transition matrices which are either *irreducible* (i.e. all states communicate) or partitioned into the set of absorbing states, which are *persistent* (i.e. $P(X_n = s_i \mid X_0 = s_i) = 1$), and the set of non-persistent states (i.e. state q is non-persistent or *transient* if starting from q the probability of returning to it is less than 1), but such that there is a path from each non-persistent state to a persistent one. A vector $\mathbf{p}$ is called a stationary distribution of the chain if $\mathbf{p}$ is a stochastic vector and $\mathbf{pP} = \mathbf{p}$. The distribution is said to be stationary because $\mathbf{pP}^n = \mathbf{pP}^{n-1} = \mathbf{p}$. When $\mathbf{P}$ is irreducible and all states are non-null persistent then the stationary distribution exists and it is unique. Let $\mathbf{P}$ be not irreducible, *Tr* be the transient states and *As* the absorbing states.

Lemma 4. *Let $\mathbf{P}$ be finite, and such that some of the states in As are accessible from all the non-absorbing states, then absorption in one or another of these states is certain.*

Lemma 5. *Let* $\mathbf{P}$ *be a finite transition matrix, partitioned in transient states Tr and absorbing states As, such that all absorbing states are accessible and for each transient state there exists a path to an absorbing state. Then there exists an n such that* $\mathbf{P}^n = \mathbf{P}^{n+k}$, $\forall k > 0$.

Proof. Let $\mathbf{P}^m$ be the m-step matrix such that absorption in all absorbing states holds, by the previous lemma. Then $\mathbf{P}^m$ can be expressed as $\mathcal{Q} = \left[\begin{smallmatrix} I & O \\ R & O \end{smallmatrix}\right]$. Here I is the identity matrix of all the absorbing states, s.t. $q_{ii} = 1$, the elements of R are the one step probabilities from non-absorbing to absorbing states, and O is a matrix of zeroes. On the other hands $\mathbf{P}$ can be expressed as $\mathbf{P} = \left[\begin{smallmatrix} I & O \\ R & Q \end{smallmatrix}\right]$, where the elements of Q are the one step probabilities between non-absorbing states. It is easy to see that $\mathcal{Q}\mathbf{P} = \mathcal{Q}$. $\square$

Lemma 6. *Let* $\mathbf{P}$ *be a stochastic matrix,* $m \times m$, *and let* $\mathbf{P}^{n+k} = \mathbf{P}^n$, $k = 0, 1, \ldots,$ *each row* r_i *of* $\mathbf{P}^{n+k}$ *is a stochastic vector and it is a stationary distribution for* $\mathbf{P}^n$, *hence for* $\mathbf{P}$.

Proof. Let $\mathbf{P}^n = Q$, clearly $QQ = Q$ because $\mathbf{P}^{n+n} = \mathbf{P}^n$, let r_i be the i-th row of Q^2,

$$r_i = [q_{i1}, \ldots, q_{im}] = \left[\sum_{k=1}^{m} q_{ik}q_{k1}, \ldots, \sum_{k=1}^{m} q_{ik}q_{km}\right] = r_i Q$$

Since Q is stochastic, r_i is stochastic and it is a stationary distribution of $\mathbf{P}^n$, since $r_i = r_i\mathbf{P}^{n+k} = \mathbf{p}_{n+k}\mathbf{P}$, $\forall k$, then r_i is also a stationary distribution of the chain. $\square$

A.3. Optical flow

The optical flow problem amounts to compute the displacement field between two images, from the image intensity variation. Here is assumed that the 2D velocity is the projection, on the image plane, of the space–time path of a 3D point. The optical flow vector $w = (u, v, 1)$ is computed between two successive images of an image sequence using a natural constraint based on the principle of brightness constancy. That is, a point (pixel) on a moving object does not change intensity between two frames:

$$I(x, y, t) = I(x + u, y + v, t + 1) \quad \Rightarrow \quad \text{optical flow constraint: } I_x u + I_y v + I_t = 0 \tag{34}$$

Here I_z is the image partial derivative with respect to z. We thus have a single constraint for the two unknown (u, v) of the flow vector w. Usually local methods (e.g. [72]) introduce a least square estimation on a small window of nearby pixels, to overcome the problem. However when many brightness patterns move independently (such as the motion of a deformable object, like a hand), it is quite hard to recover the velocities. Global methods combine the optical flow constraint with a global smoothness term. These methods are based on the minimisation of an energy functional $E(u, v)$ (see [53]):

$$E(u, v) = \int_R \left((I_x u + I_y v + I_t)^2 + \alpha\left(|\nabla u|^2 + |\nabla v|^2\right)\right) dx\,dy \quad \text{with smoothness term: } \alpha\left(|\nabla u|^2 + |\nabla v|^2\right) \tag{35}$$

α a regularisation parameter. This functional can be minimised solving the corresponding Euler–Lagrange (PDE) equations with reflecting boundary conditions, $0 = \Delta u - (1/\alpha)(I_x^2 u + I_x I_y v + I_x I_t)$ and $0 = \Delta v - (1/\alpha)(I_y^2 u + I_x I_y v + I_y I_t)$ (see [17]). Recently Brox [16][7] has extended the above mentioned global approach applying a multi-resolution strategy to the image sequence obtaining a better approximation of the optimum of the energy functional $E(u, v) = E_{data} + E_{smooth}$, where

$$E_{Data}(u, v) = \int_R \Psi\left(\left|I(\mathbf{x} + \mathbf{w}) - I(\mathbf{x})\right|^2 + \gamma\left|\nabla I(\mathbf{x} + \mathbf{w}) - \nabla I(\mathbf{x})\right|^2\right) d\mathbf{x} \tag{36}$$

Here $\mathbf{x} = (x, y, 1)^\top$ and $\mathbf{w} = (u, v, 1)^\top$, Ψ is a concave function and γ a weight, and $E_{smooth}(u, v)$ is derived. $E_{smooth}(u, v)$ integrates over the sum of a function taking care of outliers, and the spatio-temporal image gradient. $E(u, v)$ is then minimised with using numerical approximation.

A.4. Probabilistic PCA for the action space and the HMM

Probabilistic PCA [113,114] is here used to build the mixtures of principal component analysers both for clustering the descriptors $\mathbf{Y}_t$ in the action space, and to account for the mixtures of observations at each state in the HMM. PPCA defines a probability model relating two sets of variables: a D-dimensional vector of observations and a ρ-dimensional vector of unobserved variables.

[7] See also http://perception.inrialpes.fr/~chari/myweb/Software/flow_documentation.pdf.

Given a Gaussian noise model $\mathcal{N}(0,\sigma^2\mathcal{I})$ for ϵ and a Gaussian prior model $\mathcal{N}(\mathbf{z}|\mathbf{0},\mathcal{I}_\rho)$ for the latent variable $\mathbf{z}$, the marginal distribution of $\mathbf{Y}$ is Gaussian. More precisely [113,104], let $\mathcal{B}=\mathcal{A}^\top\mathcal{A}+\sigma^2\mathcal{I}_\rho$:

$$\begin{aligned}
&P(\mathbf{z}|\mathbf{Y})=\mathcal{N}\big(\mathbf{z}|\mathcal{B}^{-1}\mathcal{A}^\top(Y-\mu),\sigma^2\mathcal{B}^{-1}\big)\\
&P(\mathbf{Y}|\mathbf{z})=\mathcal{N}\big(\mathbf{Y}|\mathcal{A}\mathbf{z}+\mu,\sigma^2\mathcal{I}_D\big)\\
&p(\mathbf{Y})=\mathcal{N}\big(\mathbf{Y}|\mu,\mathcal{A}\mathcal{A}^\top+\sigma^2\mathcal{I}_D\big)
\end{aligned}\tag{37}$$

Here $\mathcal{I}_\rho$ is the identity $\rho\times\rho$ matrix. The maximum-likelihood estimation of the $D\times\rho$ matrix $\mathcal{A}$ relating latent variables and observations is [113]:

$$\mathcal{A}_{ML}=\mathbf{U}_\rho\big(\Lambda_\rho-\sigma^2\mathcal{I}_\rho\big)^{1/2}\mathcal{I}_\rho \tag{38}$$

Here $\mathbf{U}_\rho$ is the $D\times\rho$ vector formed by the eigenvectors corresponding to the greatest eigenvalue $\lambda_1,\dots,\lambda_\rho$, of the sample variance Σ of the whole data, Λ_ρ is the $\rho\times\rho$ diagonal matrix of the eigenvalues of Σ and σ is the average variance per discarded dimension. The ML estimation of the residual variance is [113]:

$$\sigma^2_{ML}=\frac{1}{D-q}\sum_{j=q+1}^{D}\lambda_j \tag{39}$$

The mixture of PPCA is thus defined for the model $(\mathcal{A}_k,c_k,\mu_k,\sigma_k^2)$, $k=1,\dots,N$, and N the number of clusters, and the distribution of the observation $\mathbf{Y}_t$ has the form:

$$f(\mathbf{Y}_t)=\sum_{k=1}^{M}c_k\mathcal{N}\big(\mathbf{Y}_t|\mu_k,\mathcal{A}_k\mathcal{A}_k^\top+\sigma_k^2\mathcal{I}\big),\quad t=1,\dots,T \tag{40}$$

Here c_k is the mixture proportion. The parameters of the mixture of PPCA can be approximated by the re-estimation procedure given in [104,113,114], in which the EM approach is taken to maximise the log-likelihood of the complete-data $\mathcal{L}_C=\sum_{t=1}^{T}\sum_{k=1}^{M}w_{tk}\ln\{c_kp(\mathbf{Y}_t,\mathbf{z}_{tk})\}$ and the re-estimation steps are as follows, see [114]. Let $\mathbf{Y}_1,\dots,\mathbf{Y}_t$ be an observation sequence and $\mathbf{z}_t$ be the latent variable $\mathbf{z}$ at time t (or its t-th row), let $\mathcal{B}_k=\mathcal{A}_k^\top\mathcal{A}_k+\sigma_k^2\mathcal{I}_\rho$. Let N be fixed (here $N=5$). The estimation step accounts for the computation of the expectation $E[\mathcal{L}_C]$, using the expectation of the latent variables:

$$\begin{aligned}
&\mathbb{E}[\mathbf{z}_{tk}]=\mathcal{B}_k^{-1}\mathcal{A}_k^\top(\mathbf{Y}_t-\mu_k)\\
&\mathbb{E}\big[\mathbf{z}_{tk}\mathbf{z}_{tk}^\top\big]=\sigma_k^2\mathcal{B}_k^{-1}+\mathbb{E}[\mathbf{z}_{tk}]\mathbb{E}[\mathbf{z}_{tk}]^\top
\end{aligned}\tag{41}$$

In the maximisation step $E[\mathcal{L}_C]$ is maximised with respect to c_k, μ_k, $\mathcal{A}_k$ and σ_k^2, using the posterior *responsibility* of the k-th mixture to generate $\mathbf{Y}_t$. Here $k=1,\dots,N$ according to the initial clustering of the descriptors defined in Section 3:

$$\gamma_t(k)=p(k|Y_t) \tag{42}$$

This is obtained initially just by the clustering via the smoothed functions, and via the sample mean and variance of each obtained cluster. Then $\mathcal{A}_k$ and σ_k^2 are estimated as follows [114]:

$$\begin{aligned}
&\hat{\mathcal{A}}_k=\frac{\sum_{t=1}^{T}\gamma_t(k)(\mathbf{Y}_t-\hat{\mu}_k)\mathbb{E}[\mathbf{z}_{tk}]^\top}{\sum_{t=1}^{T}\gamma_t(k)\mathbb{E}[\mathbf{z}_{tk}\mathbf{z}_{tk}^\top]}\\
&\hat{\sigma}_k^2=\frac{1}{D\sum_{t=1}^{T}\gamma_t(k)}\left\{\sum_{t=1}^{T}\gamma_t(k)\|\mathbf{Y}_t-\hat{\mu}_k\|^2-2\sum_{t=1}^{T}\gamma_t(k)\mathbb{E}[\mathbf{z}_{tk}]^\top\hat{\mathcal{A}}_k^\top(\mathbf{Y}_t-\hat{\mu}_k)+\sum_{t=1}^{T}\gamma_t(k)tr\big(\mathbb{E}\big[\mathbf{z}_{tk}\mathbf{z}_{tk}^\top\big]\hat{\mathcal{A}}_k^\top\hat{\mathcal{A}}_k\big)\right\}
\end{aligned}\tag{43}$$

Here $\hat{X}$ is the new estimated variable, D is the space dimension. While both $\hat{\mu}_k$ and $\hat{c}_k$ can be obtained as in classical EM [10,30]. For visualization, the projection of the data $\mathbf{Y}_t$ on the mean of the posterior distribution of the latent variables is used.

A.4.1. PPCA-HMM

An HMM with continuous observation densities is a suitable dynamic model for actions and states requiring to estimate a transition matrix $\mathbf{P}$ between states, a distribution π on the initial states, and the mixture parameters Ψ modelling the local evolution of actions (that is, with respect to the observed sequences) and their interactions. In the HMM the observations are estimated with respect to the state they are observed, therefore there are N mixtures, given N states. So we have the observation mixtures f, as described in the previous section, and the observation mixture at each state, as we shall specify below. Now, for the PPCC-HMM we have (differently from above) M mixture components, and a sequence of length T of observations $\mathbf{Y}_t$. Then the probability of $\mathbf{Y}_t$ is defined according to the above discussed PPCA model. Here, however, there is a mixture for each state $j = 1, \dots, N$ $(\mu_{jk}, c_{jk}, \mathcal{A}_{jk}, \sigma^2_{jk})$:

$$b_j(\mathbf{Y}_t) = \sum_{k=1}^{M} c_{jk}\mathcal{N}\big(\mathbf{Y}|\mu_{jk}, \mathcal{A}_{jk}\mathcal{A}_{jk}^{\top} + \sigma^2_{jk}\mathcal{I}\big), \quad t = 1, \dots, T;\ j = 1, \dots, N \tag{44}$$

Here $\mathcal{A}_{jk}$, μ_{jk}, and σ^2_{jk} are specified above (see in particular Eqs. (39) and (38)). The re-estimation procedure for the model parameters $(\pi, \Psi, \mathbf{P})$ with HMM with Gaussian observation densities is provided in [98,71,59], based on EM. To adapt the re-estimation procedure for mixture of PPCA to the HMM-PPCA only the states have to be added. So we begin with the *posterior responsibility* $\gamma_t(j,k)$. This is defined, for a sequence of observations $\mathbf{Y}_1, \dots, \mathbf{Y}_t$, $p(k|j, \mathbf{Y}_1, \dots, \mathbf{Y}_T)$ using the forward and backward variables $\alpha_t(j), \beta_t(j)$ [69], as follows:

$$\gamma_t(j,k) = \left(\frac{\alpha_t(j)\beta_t(j)}{\sum_{j=1}^{N}\alpha_t(j)\beta_t(j)}\right)\left(\frac{c_{jk}\mathcal{N}(\mathbf{Y}_t|\mu_{jk}, \mathcal{A}_{jk}\mathcal{A}_{jk}^{\top} + \sigma^2_{jk}\mathcal{I})}{\sum_{k=1}^{M} c_{jk}\mathcal{N}(\mathbf{Y}_t|\mu_{jk}, \mathcal{A}_{jk}\mathcal{A}_{jk}^{\top} + \sigma^2_{jk}\mathcal{I})}\right) \tag{45}$$

Analogously, the definitions for $\mathcal{A}_{jk}, \sigma^2_{jk}$ are as given in (43), and likewise μ_{jk}, c_{jk} have to be restated using $\gamma_t(j,k)$. Finally the chain parameters are defined as follows:

$$\hat{\pi}_j = \sum_{k=1}^{M}\gamma_1(j,k), \qquad \hat{\mathbf{P}}_{ij} = \frac{\sum_{t=1}^{T-1}(\alpha_t(j)\mathbf{P}_{ij}b_j(\mathbf{Y}_{t+1})\beta_{t+1}(j))}{(\sum_{t=1}^{T-1}\sum_{j=1}^{N}\alpha_t(j)\mathbf{P}_{ij}b_j(\mathbf{Y}_{t+1})\beta_{t+1}(j))^{-1}} \tag{46}$$

A.5. Proofs of Section 5: First-order parametric models

Lemma 1. *Given an HMM $\mathcal{M} = (S, H, (\mathbf{P}, \pi, \Psi, \gamma))$, with finite property (see Definition 2) there exists a probability structure $\mathbb{M}$ of first-order with domain $D = (S, H, \mathcal{W}, \mathbb{R})$, probability space Φ generated from D, such that the atoms and terms are interpreted according to the $\mathcal{M}$-parameters $\mathbf{P}, \pi$ and Ψ. Furthermore, according to the given interpretation $\mathscr{I}$, for each field-based atom ϕ there is a corresponding measurable set such that the field term for ϕ has the intended distribution of the HMM.*

Proof. Let $\mathbb{M}$ be a structure with domain formed by the sorts $H, S, \mathcal{W}$ and probability space as in Definition 6. Where the probability space mentions the σ-fields Δ, Θ and $\Theta \times \Delta$ and the probability measures η_s and η_I.

An assignment v of a domain element d of the appropriate sort to the free variables is indicated either as $v(x) = d$ or as $v(x/d)$, and it maps free variables to their respective domains in D, so $v(x/s)$ maps the variable x to the domain S of states, $v(w_t/(X_1(\omega), \dots, X_t(\omega)))$ maps a variable denoting a sequence of states to a sequence $(s_{j_1}, \dots, s_{j_t})$, chosen by the random variables over ω, with $\omega \in S^T$, and $v(y/Y(h))$ maps a variable denoting an observation to the domain of H which is intended to be the reals $\mathbb{R}$.

By the HMM model $\mathcal{M}$, the following sets of parameters are available to $\mathbb{M}$:

$$\begin{array}{lll}
1.\ \text{The parameters for the states:} & (\pi, \mathbf{P}) & \\
2.\ \text{The parameters linking states and observations:} & c_{ik}\ (i = 1, \dots, N,\ j = 1, \dots M) & \\
3.\ \text{The parameters for the observations at each state:} & \Psi = \{\{\mu_{ij}\}, \{\sigma_{ij}\}, \{\mathcal{A}_{ij}\}\}_{i=1,\dots,N,j=1,\dots,M} & \\
4.\ \text{The parameters for the observations independently of states:} & \gamma = \{\{\mu_j\}, \{\sigma_j\}, \{\mathcal{A}_j\}\}_{j=1,\dots,N} & (47)
\end{array}$$

These are interpreted as field terms in $\mathbb{R}$. To simplify we shall only consider the parameters σ_{ik} for the covariances of the normal pdfs.

Let us first consider the interpretation of the atoms $R_i(\cdot)$ and of their field terms. We have to show that they are measurable sets and that their probability is the same as the one specified by the HMM $\mathcal{M}$. Let $T_{\mathbf{P}}$ be the stationary distribution of $\mathbf{P}$, the transition matrix of $\mathcal{M}$, and let $T > T_{\mathbf{P}}$. By the construction, $\Delta^T = \Delta_0 \times \Delta_0 \times \cdots$, where Δ_0^t consists of all subsets of S^t, and each (S, Δ_0, μ_0) a measure space, with Δ_0 the σ-field of subsets of S. Let $Q^t \in \Delta_0^t$ and $X_1, \dots, X_t$ a sequence of discrete random variables with Markov property:

$$Q_t^i = \big\{\omega \in S^T \bigm| \big(X_1(\omega) = \arg\max_s \pi(s), X_2(\omega) = s_{j_2}, \dots, X_{t-1}(\omega) = s_{j_{t-1}}, X_t(\omega) = s_i\big) \in Q^t\big\} \tag{48}$$

Then Q_t^i, the cylinder with base Q^t, is measurable as Q^t is in Δ_0^t. Hence, by the definition of $R_i(\cdot)^{\mathscr{I}}$, namely:

$$R_i(\cdot)^{\mathscr{I}} = \bigcup_t \big\{\omega \bigm| X_1(\omega) = \arg\max_s \pi(s), X_t(\omega) = s_i, t \geqslant i\big\} \quad (i = 1, \dots, N) \tag{49}$$

it follows that $R_i(\cdot)^{\mathscr{I}} = \bigcup_t Q_t^i \in \Delta$ and $R_i(\cdot)^{\mathscr{I}} \subset \mathcal{W}$, hence it is a measurable set. Now let as assume that $\arg\max_s \pi(s) = 1$, with $s = s_1$, the first state. Let $(s_{j_1}, s_{j_2}, \dots, s_{j_{t-1}}, s_i)$ be a sequence in $\mathcal{M}$ then, by (8), $P(s_{j_1}, \dots, s_{j_t}) = \sum_{s_j \in S}\pi(s_{j_1}) \times \prod_{k=1}^{T-1} P(s_{j_{k+1}}|s_{j_k}) = p$, and $s_1 = s_{j_1}$. Thus, let $\omega \in S^T$ then, by the interpretation of R_i and above (48):

$$\omega \in R_i^{\mathscr{I}} \quad \text{iff} \quad \omega \in Q_t^i \quad \text{iff} \quad \big(X_1(\omega) = s_1, X_2(\omega) = s_{j_2}, \dots, X_{t-1}(\omega) = s_{j_{t-1}}, X_t(\omega) = s_i\big) \in Q^t \quad (t \geqslant i) \tag{50}$$

Hence, if v is an assignment, w_t a free variable of sort sequence, $t \geqslant i$ we have:

$$\begin{aligned}
\left[R_i(w_t)\right]^{(\mathscr{I},v)} = {} & \eta_s\big(\big\{\big(X_1(\omega)=s_1, X_2(\omega)=s_{j_2},\dots,X_{t-1}(\omega)=s_{j_{t-1}}, X_t(\omega)=s_i\big)\big\}\big) \quad \text{iff} \\
& \omega \in R_i^{\mathscr{I}} \quad \text{iff} \\
& v\big(w_t/(s_1, s_{j_2}\dots,s_{j_{t-1}},s_i)\big) \in Q^t \quad \text{iff} \\
& \eta_s\big(\big\{\big(X_1(\omega)=s_1,\dots,X_{t-1}(\omega)=s_{j_{t-1}}, X_t(\omega)=s_i\big)\big\}\big) \\
& \quad = \sum_{s_j \in S} \pi(s_{j_1}) \prod_{k=1}^{T-1} P(s_{j_{k+1}}|s_{j_k}) = p
\end{aligned} \tag{51}$$

Which follows from the fact that the X_i are defined on the same probability space, have the Markov property and η_s is a product measure. Therefore η_s is a probability measure, and the distribution on the discrete random variables, with Markov property, is the same as for the HMM $\mathcal{M}$, for the same domain space S and the parameters $(\pi, \mathbf{P})$.

Consider, now, the atom $A_j(\cdot, s_i)$, for some fixed state s_i the domain H and the σ-field Θ. The elements of H are reals and beside them we consider the random variables $Y_j : H \mapsto H$ as identity.

For each state $s_i \in S$ there is a number of predicates A_j, the same for each state, labelling the observed actions, taking as argument a real number in H, where the real number (actually a vector) is the value of the descriptor of the action.

Now, let $\mathbf{Y}_1,\dots,\mathbf{Y}_t$ be a sequence of observations, with the $\mathbf{Y}_j$ arrays of real variables. We might assume here that instead of the vectors we have single real variables h, then in the HMM $\mathcal{M}$ the likelihood of the sequence $h_1,\dots,h_t$ given a fixed sequence of states $s_{j_1},\dots,s_{j_t}$ is, according to Eq. (9):

$$P(h_1,\dots,h_t|s_{j_1},\dots,s_{j_t}) = \prod_{i=1}^{t} b_{j_i}(h_i) \tag{52}$$

In particular, if we consider a single observation given the sequence $s_{j_1},\dots,s_{j_t}$ then the likelihood of the observation given the sequence of states becomes:

$$P(h_t|s_{j_1},\dots,s_{j_t}) = P(h_t|s_{j_t}) = b_{j_i}(h_t) \tag{53}$$

because of the independence of the observation from the other states, here $b_{j_i}(h_i)$, for the model $\mathcal{M}$, is defined in (7).

On the other hand, by the definition of interpretation of the A_j, for $z \in \mathbb{R}$:

$$A_j(\cdot, s_i)^{\mathscr{I}} = \left\{ h \;\middle|\; \frac{(h-\mu_{ik})^2}{\sigma_{ik}^2} \leqslant z,\ h \in A_j^{\star},\ k=1,\dots,M \right\} \quad (i,j=1,\dots,N) \tag{54}$$

Where $h \in A_j^{\star}$ depends on the initial mixture, whose parameters are specified in γ. Clearly $h \in A_j(\cdot, s_i)^{\mathscr{I}}$ depends on the choice of $z \geqslant 0$. Then $A_j(\cdot, s_i) \subset H$ and $A_j(\cdot, s_i) \in \Theta$. Therefore, let $h \in B_j \subseteq A_j(\cdot, s_i)^{\mathscr{I}}$. Given that $Y_j(h) = h$, we have, by Eqs. (14) and (20):

$$\begin{aligned}
\left[A_j(y, s_i)\right]^{(\mathscr{I},v)} &= \eta_I\big(\big\{h \mid v\big(y/Y_j(h)\big) \in A_j(\cdot, s_i)^{\mathscr{I}},\ i=1,\dots,N\big\}\big) \\
&= \int_{B_j} b_i(h)\,dh
\end{aligned} \tag{55}$$

If $B_j = (h, h+\delta h)$ then $b_i(h)\delta h$, for $\delta h \to 0$ is the probability density as specified in (53). Then it follows that the derivative of η_I at h, which we denote $[A_j(h, s_i)]_{\delta}^{\mathscr{I}}$, is the required density $b_i(h)$ both in $\mathbb{M}$ and in $\mathcal{M}$. In particular, when both the same interval and the same stochastic variable h are chosen then it follows that the value of the density is exactly the same in both models.

It follows that each measurable set $A_i^{\mathscr{I}}$ has a distribution induced by the random variables that agree with the parameters Ψ of the HMM.

The interpretation of the binary predicates O_{ij} is given as follows.

$$\begin{aligned}
&\mathbb{M}, v \models O_{ij}(y, x \circ w_{t-1}) \quad \text{iff} \quad v(y/h) \in A_i^{\mathscr{I}}\big(\cdot, v(x/s_i)\big) \quad \text{and} \\
&\quad v\big(x \circ w_t/\big(s_i, X_1(\omega),\dots,X_t(\omega)\big)\big), \quad (s_i,\omega) \in R_i^{\mathscr{I}}
\end{aligned} \tag{56}$$

Hence $O_{ij} \in \Theta \times \Delta$ and it is a measurable set.

Finally, given an assignment v to the free variables of the different sorts $\{v(y/h), v(x/s_i), v(w/\omega)\}$, the distribution on O_{ji} is defined as follows

$$\begin{aligned}\big[O_{ji}(y, x\circ w_t)\big]^{(\mathbb{M},v)} &= \big[A_j(y,x), R_i(x\circ w_t)\big]^{(\mathbb{M},v)}\\ &= \big[A_j(y,x)\big]^{(\mathbb{M},v)}\big[R_i(x\circ w_t)\big]^{(\mathbb{M},v)}\\ &= b_i(h)\big[R_i(s_i\circ w_t)\big]^{(\mathbb{M},v)}\\ &= b_i(h)\pi(X_1=s_{q_1})\sum_i\left(\prod_{k=1}^{t-1}\mathbf{P}(X_{k+1}=s_{q_{k+1}}|X_k=s_{q_k})\right)\mathbf{P}(s_i|X_{t-1}=s_{q_{t-1}}) \qquad\square \end{aligned}\tag{57}$$

Theorem 1. *Let $\mathbb{M}$ be a probability structure of first-order, with probability space Φ. Let Φ be extended to the probability space $\Phi^\star$ with product measures η_s^n, η_I^n, $(\eta_s\times\eta_I)^n$, $n\geqslant 1$, and their space product. For each field-base formula φ there exists a measurable set, whose distribution agree with the HMM.*

Proof. First note that η_s^n is like η_s as η_s is already a product measure. Thus we are only concerned with η_I^n, therefore the conditions for the Fubini theorem are satisfied because η_I is a probability measure. Thus the integral of the products can be considered.

Let $\Delta_\alpha()$, $\Theta_\alpha()$ and $\Gamma_\alpha()$ be the measurable sets of the atom α, according to Lemma 1.

We prove the statement by induction on the structure of the field-base formulae and on the dimension of the product space. For the basic case.

If $\varphi(w)=R_i(w)$ or $\varphi(y,x)=A_j(y,s_i)$, or $\varphi(y,x\circ w)=O_{ji}(y,x\circ w)$ then for $\Delta_\varphi(\omega)$, $\Theta_\varphi(h,s)$ and $\Gamma_\varphi(h,s\circ\omega)$ the statement is verified by Lemma 1.

If $\varphi(w)=\neg\alpha(w)$ then the statement follows since Δ is a σ-field hence $\Delta_\alpha^c(\omega)\in\Delta$ hence η_s is the same.

Analogously, if $\varphi(y)=\neg\alpha(y)$ then the statement follows from the fact that Θ is a σ-field.

Let $n=2$:

1. (Case a). Let $\varphi(w,w')=\alpha(w)\wedge\beta(w')$. Let $v(w/X_1(\omega),\dots,X_n(\omega))$, $v(w'/X_1(\omega'),\dots,X_m(\omega'))$. Then either $X_1(\omega),\dots,X_n(\omega)$ is a subsequence of $X_1(\omega'),\dots,X_m(\omega')$ or they are independent.
   By induction hypothesis there are the measurable sets $\Delta_\alpha(\omega)$ and $\Delta_\beta(\omega')$, hence:
   (i) If $\Delta_\alpha(\omega)\subseteq\Delta_\beta(\omega')$, then $\Delta_\beta=\Delta_\varphi\in\Delta$ hence they both belong to Δ.
   (ii) Otherwise $\Delta_\alpha(\omega)\cap\Delta_\beta(\omega')=\Delta_\varphi\in\Delta$, hence they both belong to Δ.
   Hence η_s is the same.
2. (Case b). Let $\varphi(y,y')=\alpha(y)\wedge\beta(y')$. Let $v(y/h)$ and $v(y'/h')$. Now $E\in\Theta^2$, if $B=\{h|(h,h')\}\in E$ is in Θ and $B'=\{h'|(h,h')\}\in E$ is in Θ. By induction hypothesis $\Theta_\alpha(h)\in\Theta$ and $\Theta_\beta(h)\in\Theta$ hence $E\in\Theta^2$, and $\Theta_\varphi(h,h')\in\Theta^2$ and $\eta_I^2=\eta_I\times\eta_I$, in particular $\eta_I^2(B,B')=\int_{B\cup B'}b(y|y')b(y')\,dy\,dy'$.
3. (Case c). If $\varphi(\vec{z})=\alpha(y,w)\wedge\beta(y',w')$, then according to v there are $\Gamma_\alpha(h,\omega)$ and $\Gamma_\beta(h',\omega')$ with $\Gamma_\alpha(h,\omega)\in\Theta\times\Delta$ and $\Gamma_\beta(h',\omega')\in\Theta\times\Delta$, then using the same argument as case b we obtain that $\Gamma_\varphi(\vec{z})\in\Theta^2\times\Delta$ and the measures are $\eta_I^2\times\eta_s$, accordingly.
4. (Case d). If $\varphi=\forall w.\alpha$, then by induction hypothesis there is a set Γ_α measurable in some product space. Since α has no free variables the probability space is either the whole space, or the empty space, and hence for any product probability measure its value is 1 or 0.

For $n>2$ the proof is as above, going through the cases a–d, also considering that the product measures are not affected by the permutations of the formulae. $\square$

Proposition 1. *Let v and v' agree on all the assignments to the free variables then $\mathbb{M},v\models\varphi$ iff $\mathbb{M},v'\models\varphi$.*

Proof. By induction on the structure of φ, considering all the cases in which φ mentions field terms. $\square$

Theorem 2. *Let $\mathcal{M}=(S,H,(\pi,\mathbf{P},\Psi,\gamma))$ then there exists a probability model $\mathbb{M}$ with probability space $\Phi^\star$ extending Φ such that $\mathcal{M}$ and $\mathbb{M}$ agree on the distribution on the domain.*

Proof. By Lemma 1, given an HMM $\mathcal{M}$ there is a probability model $\mathbb{M}$ of first-order with probability space Φ, such that the atomic field terms have the same distribution as the terms of the HMM. If the probability space is extended to $\Phi^\star$ then the obtained product measures must be unique, because they are defined as products of probability measures.

We shall thus consider the cases of joint terms in HMM,

1. Let $\mathcal{O}=y_1,\dots,y_m$ be a sequence of observations. The probability of $\mathcal{O}$ in $\mathcal{M}$ is given by

$$\sum_{\forall s_{i_1},\dots,s_{i_m}\in S}\pi(s_{i_1})b_{i_1}(y_1)\mathbf{P}(s_{i_2}|s_{i_1})\cdots b_{i_m}(y_m)\mathbf{P}(s_{i_m}|s_{i_{m-1}})\tag{58}$$

Thus the term requires to represent all sequences of states of length m. To represent these sequences we need to define a suitable set of variables. First let $Q_i^m = \{\omega^m | X_m(\omega) = s_i\}$, that is, Q_i^m is formed by all ω whose subsequence of length m have last state s_i. Let $U_i^m = \{w_m^i | v(w_m^i/(X_1(\omega), \dots, X_m(\omega)))$, and $(X_1(\omega), \dots, X_m(\omega)) \in Q_i^m\}$, that is U_i^m provides a variable name, of sort sequence, for each element in Q_i^m, then

$$\varphi = \bigvee_{\substack{i=1,\dots,N \\ w_m^i \in U_i^m}} \bigwedge_{j=1}^{N} O_{ji}\big(y_j, w_m^i\big) \tag{59}$$

Let $\theta^\star = \{y_1, \dots, y_m\}$, $\omega^\star = \bigcup_i Q_i^m$

$$\begin{aligned}[\varphi] &= \big(\eta_I^m \times \eta_s\big)\Gamma_\varphi\big(\theta^\star, \omega^\star\big) \\ &= \sum_{s_{j_1},\dots,s_{j_m} \in \omega^\star} \pi(s_{j_1}) b_{j_1}(y_1) \mathbf{P}(s_{j_2}|s_{j_1}) \cdots b_{j_m}(y_m) \mathbf{P}(s_{j_m}|s_{j_{m-1}})\end{aligned}$$

2. Let $\mathbf{p}$ be a stationary distribution of the Markov chain in $\mathcal{M}$ i.e. $\mathbf{p}\mathbf{P}^t = \mathbf{p}\mathbf{P} = \mathbf{p}$ and such that $\mathbf{p}(t) \neq 0$. Consider, now, any variable w_t^i, such that $w_t^i \in U_i^t$ (as defined in the previous item), then for any sequence w_{t+k}^i, denoting a sequence $(X_1(\omega), \dots, X_{t+1}(\omega))$ ending in state s_i,

$$\big[R_i\big(w_t^i\big)\big] = \big[R_i\big(w_{t+k}^i\big)\big] = \mathbf{p}(t) = \pi(s_{j_1})\mathbf{P}^t \tag{60}$$

By the interpretation of R_i (see (49)).

3. Let $P(h_1, \dots, h_m, s_1, \dots, s_m)$ in $\mathcal{M}$ be the joint probability of a sequence of observations and a sequence of states (see (11)). Consider a formula of the form:

$$\psi(y_1, \dots, y_m, x_1, \dots, x_m \circ w_{m-1}) \wedge \varphi(w_1, \dots, w_m)$$

Since the only predicate mentioning both observations and states is O_{ji} and $O_{ji}(y, x \circ w) \equiv R_i(x \circ w) \wedge A_j(y, x)$ then the above formula can be reduced in normal form:

$$\bigwedge_{i=1}^{m} A_j(y_i, x_i) \wedge \bigwedge_{i=1}^{m} R_i(x_i \circ w_{m-1})$$

Then the joint probability of the observations, given the states is

$$\big[\psi(y_1, \dots, y_m, x_1, \dots, x_m) \wedge \varphi(w_1, \dots, w_m)\big] = \big(\eta_I^m \times \eta_s\big)(\Gamma_{\psi \wedge \varphi})(\Gamma_\varphi) \tag{61}$$

by Theorem 1 and (23).

Furthermore:

$$\begin{aligned}&\mathbb{M}, v \models \neg\big(R_i(w) \wedge R_j(w)\big), \quad \forall i, j,\ i \neq j \\ &\mathbb{M}, v \models \neg\big(O_i(y, w) \wedge O_j(y, w)\big), \quad \forall i, j,\ i \neq j\end{aligned} \tag{62}$$

Simply by the definition of interpretation given in Lemma 1.

Hence $\mathbb{M}$ is the required model. □

A.6. Proofs of Section 6: Final induction step, concluding with the Situation Calculus

We recall that a basic theory of actions is formed by the set of sentences:

$$\Sigma \cup \mathcal{D}_{S_0} \cup \mathcal{D}_{una} \cup \mathcal{D}_{ap} \cup \mathcal{D}_{ss} \tag{63}$$

Where Σ is the set of foundational axioms described below. $\mathcal{D}_{S_0}$ is the set of formulae which either do not mention situation terms or they mention only the situation term S_0. $\mathcal{D}_{una}$ are the set of unique name of action axioms, that is, for each action name A_i and A_j we have $A_i \neq A_j$. $\mathcal{D}_{ap}$ are the axioms specifying the preconditions to execute an action, such as:

$$Poss(push_{door}, s) \equiv Unlocked_{door} \tag{64}$$

Finally $\mathcal{D}_{ss}$ are the axioms specifying the effect of an action: what comes true after the execution of an action, such as, for example:

$$Opened_{door}\big(do(a, s)\big) \equiv a = push_{door} \vee a \neq close_{door} \wedge Opened_{door}(s) \tag{65}$$

The set of foundational axioms Σ is:

$$
\begin{aligned}
&\neg(s \sqsubset S_0)\\
&s \sqsubset do(a, s') \equiv s \sqsubseteq s'\\
&do(a, s) = do(a', s') \equiv a = a' \wedge s = s'\\
&\forall P.P(S_0) \wedge \forall as.P(s) \to P\big(do(a, s)\big) \to \forall s P(s)
\end{aligned}
\tag{66}
$$

In the following $\mathbb{M}$ refers to a probability structure while $\mathscr{M}$ refers to a structure of the Situation Calculus. Thus, when the context is $\mathbb{M}$, A_j is a predicate taking values in the domain of observations and in the domain of states, while in the language of the Situation Calculus A_j refers to an action and the signature is as defined in Definition 8. We shall assume σ^2_{ik} to be the variance of the k-th component of the i-th univariate mixture, and that all the mixtures have as many modes as components. Furthermore we shall assume, for each mixture b_j, $j = 1, \ldots, N$, that no component will have variance $\sigma^2_{jk} < \tau$, with τ a suitable threshold, e.g. 0.01.

Lemma 2. *For each t there exists a most likely state $R_i(x \circ w_t)$, with $\mathbb{M}, v \models R_i(x \circ w_t)$, and there exists an A_j such that $[A_j(y, x)]^{(\mathbb{M},v)}_\delta$ is maximal. Furthermore $[A_j(y, x)]^{(\mathbb{M},v)}_\delta$ is maximal precisely at $v(x/s)$, for some s, in the interpretation of R_i, but independently of any assignment v to w_t.*

Proof. By the re-estimation formulae for the HMM $\mathcal{M}$, the most likely state at time t is specified by $s_i = \arg\max_{1 \leqslant i \leqslant N}[\sum_{k=1}^M \gamma_t(i, k)]$. For $t = 1$, $R_i(s_i)$ is the most likely state if $[R_i(s_i)] \geqslant [R_j(s_j)]$, for all $j \neq i$, and clearly, $\mathbb{M} \models R_i(s_i)$. By Lemma 1 and the re-estimation formula for π_i, $\sum_{k=1}^M \gamma_1(i, k) = \alpha_1(i) = \hat{\pi}_i b_i(y_1)$. Hence $[R_i(s_i)]$ is the most likely state if $b_i(y_1)$ is maximal. Choose from $b_i = \sum_k c_{ik} \mathcal{N}(\mu_{ik}, \sigma^2_{ik})$ the component and the argument maximising b_i, whence for some j, $\mu_{ij} = \arg\max_{v(y_1/\mu_{ij}) \in B}[b_i(y_1)]$. Therefore, for $v(y_1/\mu_{ij})$, $[A_j(y_1, s_i)]^{(\mathbb{M},v)}_\delta$ is maximal at s_i and $[R_i(s_i)]$ is the most likely state.

Let, now, $t > 1$. By the re-estimation formula for $\gamma_t(i, k)$, $i = 1, \ldots, N$, we have that:

$$
\gamma_t(i, k) = \frac{1}{\lambda} \alpha_t(i) \beta_t(i) \frac{1}{b_i(y_t)} c_{ik} \mathcal{N}\big(y_t, \mu_{ik}, \sigma^2_{ik}\big) \tag{67}
$$

Here λ is the normalisation factor. We can fix $\beta_t(i)$ to be 1 as it is the last state, and

$$
\alpha_t(i) = \left(\sum_{q=1}^N \alpha_{t-1}(q) \mathbf{P}_{qi}\right) b_i(y_t) \tag{68}
$$

Replacing $\alpha_t(i)$ with its definition, simplifying and summing over k we obtain that

$$
\sum_{k=1}^M \gamma_t(i, k) = \frac{1}{\lambda}\left(\sum_{q=1}^N \alpha_{t-1}(q) \mathbf{P}_{qi}\right) \sum_{k=1}^M c_{ik} \mathcal{N}\big(y_t, \mu_{ik}, \sigma^2_{ik}\big) = \frac{1}{\lambda}\left(\sum_{q=1}^N \alpha_{t-1} \mathbf{P}_{qi}\right) b_i(y_t) \tag{69}
$$

By the re-estimation formula for the transition matrix coefficients (see (46)), $\mathbf{P}_{qi}$ depends on $b_i(y_t)$, hence the term $\frac{1}{\lambda}(\sum_{q=1}^N \alpha_{t-1} \mathbf{P}_{qi})$ is maximal if $b_i(y_t)$ is maximal, $i = 1, \ldots, N$. Then to obtain the most likely action choose from $b_i = \sum_k c_{ik} \mathcal{N}(y_t, \mu_{ik}, \sigma^2_{ik})$ the component and the argument maximising b_i, whence for some j, $\mu_{ij} = \arg\max_{v(y_t/\mu_{ij}) \in B}[b_i(y_t)]$. Therefore for $v(y_t/\mu_{ij})$, both $[A_j(y_t, s_i)]^{(\mathbb{M},v)}_\delta$ is maximal at s_i and $s_i = \arg\max_{1 \leqslant i \leqslant N}[\sum_{k=1}^M \gamma_t(i, k)]$. Hence $R_i(s_i \circ w_t)$ is the most likely state and $A_j(y_t, s_i)$ the most likely action. Clearly, maximality of $[A_j(y_t, s_i)]^{(\mathbb{M},v)}_\delta$ does not depend on w_t and $\mathbb{M} \models R_i(s_i \circ w_t)$. □

Lemma 3. *For each most likely state $R_u(x \circ w_t)$, with $\mathbb{M}, v \models R_u(x \circ w_t)$, there exists an observation action $A_j(y, x)$ and unique transition $\mathbf{P}(X_{t+2} = x' | X_{t+1} = x)$ that extends $R_u(x \circ w_t)$ to $R_i(x' \circ x \circ w_t)$, maximally, for some i.*

That is, $[A_j(y, x)]^{(\mathbb{M},v)}_\delta \mathbf{P}(X_{t+2}(\omega) = x' | X_{t+1}(\omega) = x)$ is maximal at $v(x/s_u)$.

Proof. If $\mathbb{M}, v \models R_u(x \circ w_t)$ then $v(x/s_u)$ and by Lemma 2, given the s_u there exists A_j maximal at s_u. Now, be the property of the transition matrix, there exists at least a s_k such that $\mathbf{P}(X_{t+2} = s_k | X_{t+1} = s_u) > 0$, choose the maximal transition and the statement holds for $k = i$. □

Corollary 1. *For each state predicate $R_i(x \circ x' \circ w_t)$, with $\mathbb{M}, v \models R_i(x \circ x' \circ w_t)$, there exist two observation actions A_m and A_j and a unique transition $\mathbf{P}(X_{t+2} = x | X_{t+1} = x')$ that extends $R_u(x' \circ w_t)$ to $R_i(x \circ x' \circ w_t)$, maximally. That is, $(\mathbf{P}(X_{t+2} = x | X_{t+1} = x')[A_j(y', x')]_\delta [A_m(y, x)]_\delta)^{(\mathbb{M},v)}$ is maximal at $v(x/s_i)$.*

Proof. Let $\mathbb{M}, v \models R_i(x \circ x' \circ w_t)$, then v must map $X_{t+2} = s_i$. Now, by the same argument of Lemma 3 there exists a transition $\mathbf{P}(X_{t+2} = s_i | X_{t+1} = s_u)$ for some s_u thus $\mathbb{M}, v \models R_u(x' \circ w_t)$ with v mapping $X_{t+1} = s_u$. Choose an observation action A_m whose pdf is maximal at s_i, and an observation action A_j whose pdf is maximal at s_u for some $v(y/h)$, $v(y'/h')$, by Lemma 2. Hence, by Lemma 3, $([A_j(y,s_u)]_\delta \mathbf{P}(X_{t+1}=s_i|X_{t+1}=s_u))^{(\mathbb{M},v)}$ extends $R_u(x'\circ w_t)$ maximally to $R_i(x\circ x'\circ w_t)$ and since the pdf of A_m is maximal at s_i the statement follows. □

Theorem 3. *Given $\mathbb{M}$ and an assignment v, with $v(w_t)=\hat{w}_t$, $\hat{w}_t\in\{X_1(\omega)=s_{j_1},\dots,X_t(\omega)=x_{s_{j_t}}\}$, $t\geqslant 0$, and for all R_i, there exists a sequence $(x\circ x'\circ\hat{w}_t)$ such that $R_i(x\circ x'\circ\hat{w}_t)^{(\mathbb{M},v)}$ is persistent.*

Proof. Note that, by construction, either the transition matrix is irreducible or all transient states are connected with an absorbing state, therefore the probability that a state happens at w_t is greater than zero for all states. By Lemma 2 and Definition 10 for each state $R_i(x\circ w_t)$ that happens at $\hat{w}_t$ at $t+k$, $\mathbb{M},v\models R_j(x\circ w_t)$ and $[R_i(x\circ w_t)]^{(\mathbb{M},v)}$ is maximal at $v(x/s_i)$. Furthermore, by Lemma 2 again, there is an action $A_j(y,x)$ such that $[A_j(y,x)]_\delta$ is maximal at $v(x/s_i)$. In fact, in the proof of Lemma 2 we have shown that, if R_i is maximal, this is due, indeed, to some action A_j maximal at $s_i\in R_i^{\mathscr{I}}$, via the re-estimation formulae.

Let us assume that $[R_i(x\circ x'\circ w_t)]^{(\mathbb{M},v)}$, $t\geqslant 0$, is maximal and $\mathbb{M},v\models R_i(x\circ x'\circ w_t)$. We define the following construction:

$$
\begin{aligned}
&U_{t+2}=s_i \quad \text{(given the choice of } R_i)\\
&T_{t+2}=\arg\max_{v(y/h_j\in A_j^{\mathscr{I}})}\big[A_j(y,s_i)\big]_\delta\\
&M_{t+2}=\big\{R_i\big(s_i\circ x'\circ w_t\big),A_j(T_{t+2},s_i)\big\}\\
&U_{t+1}=\arg\max_{1\leqslant u\leqslant N,v(x'/s_u)}\{\big[R_u\big(x'\circ\hat{w}_t\big)\big]\mathbf{P}(X_{t+2}=s_i|X_{t+1}=U_{t+1})\\
&T_{t+1}=\arg\max_{v(y'/h_m\in A_m^{\mathscr{I}})}\big[A_m\big(y',U_{t+1}\big)\big)\big]_\delta\\
&M_{t+1}=M_{t+2}\cup\big\{R_u\big(U_{t+1}\circ x''\circ w_{t-1}\big),A_m(T_{t+1},U_{t+1}),v\big(x'/U_{t+1}\big)\big\}\\
&\vdots\\
&U_1=\arg\max_{1\leqslant q\leqslant N,v(x^*/s_q)\in S}\big(\big[R_q\big(x^\star\big)\big]\mathbf{P}\big(X_2=U_2|X_1=x^\star\big)\big)\\
&T_1=\arg\max_{v(y^*/h_k\in A_k^{\mathscr{I}})}\big[A_k\big(y^*,U_1\big)\big)\big]_\delta\\
&M_1=M_2\cup\big\{R_q(U_1),A_m(T_1,U_1),v\big(x^\star/U_1\big)\big\}
\end{aligned}
\tag{70}
$$

Let v be as defined in M_1, then it is easy to see that by the construction and by Lemma 2, $\mathbb{M},v\models\bigwedge M^\star$, where $M^\star$ is the set of atoms in M_1. Furthermore for each n, $n=1,\dots,t+2$, if $v(y_n/h_j)\in T_n$ and $v(x_n/s_u)\in U_n$, then $[A_j(h_j,s_u)]$ is maximal for $R_u(s_u\circ w_{n-1})$. Now let us consider:

$$
\begin{aligned}
&\big(A_{j_1}(h_{j_1},s_{i_1}),\dots,A_{j_{t+2}}(h_{j_{t+2}},s_{i_{t+2}})\big)\\
&\big(R_{i_1}(s_{i_1}),\dots,R_{i_{t+2}}(s_{i_{t+2}})\big)
\end{aligned}
$$

By the above construction, each of the field term of the atoms in the two sequences is maximal.

We show by induction on t, on the given construction, that R_i, $i=1,\dots n$, is persistent according to Definition 10. We have to show item (1) and (2) and that the term introduced in Eq. (28) is maximal.

For the basic case item (1) and (2) are straightforward because R_i is entering w_t at $t=1$. Thus (28) reduces to the maximality of $[R_i(U_1)]$.

Let $t>1$. Again (1) is obviously verified by the construction and the inductive hypothesis. Hence $[R_i(U_{t+2}\circ U_{t+1}\circ w_t)\wedge A_j(T_{t+2},U_{t+2})\wedge A_m(T_{t+1},U_{t+1})\wedge R_u(U_{t+1}\circ w_t)]$ is maximal and, thus, $[R_i(U_{t+2}\circ U_{t+1}\circ w_t)|A_j(T_{t+2},U_{t+2})]$ is maximal.

Consider (2). Suppose that at $t+2$ there is a change. In M_1 there are two actions $A_q(T_{t+1},U_{t+1})$ and $A_k(T_{t+2},U_{t+2})$, with terms maximal at U_{t+1} and U_{t+2}, respectively. Because they are in M_1 they are both satisfied in $\mathbb{M}$. Since by hypothesis there is a change, then:

$$\mathbb{M},v\models A_q(T_{t+1},U_{t+1})\wedge A_k(T_{t+2},U_{t+2})\wedge R_i(U_{t+1}\circ w_t)\wedge\neg R_i(U_{t+2}\circ U_{t+1}\circ w_t)$$

We need only to show that a delay is possible. If U_{t+1} is an absorbing state then change cannot occur, thus, it is transient but not null, by definition of the transition matrix of the system, that is, $\mathbf{P}_{ii}>0$. We can suppose that a delay does not affect the observation, hence $[A_q(T_{t+1},U_{t+1})]_\delta$ maximal at $R_i(U_{t+1}\circ w_t)$ is still maximal at $R_i(U_{t+1}\circ U_{t+1}\circ w_t)$. From this fact and the fact that $[A_k(T_{t+2},U_{t+2})]_\delta$ is maximal at state $U_{t+2}\neq U_{t+1}$, it follows that $1-[A_k(T_1,U_1)]_\delta>0$. We have also to note that, by Corollary 1, $[A_q(T_{t+1},U_{t+1})]_\delta\mathbf{P}(X_{t+2}=U_{t+1}|X_{t+1}=U_{t+1})[A_q(T_{t+1},U_{t+1})]_\delta$ extends maximally $[R_i(U_{t+1}\circ w_t)]$ to $[R_i(U_{t+1}\circ U_{t+1}\circ w_t)]$, hence it is possible to delay the change with

$$\mathbb{M},v\models\neg A_k(T_{t+2},U_{t+2})\wedge R_i(U_{t+1}\circ w_t)\wedge R_i(U_{t+1}\circ U_{t+1}\circ w_t)$$

The delay is like having added, between $t+1$ and $t+k$ (here $k=2$) $U_{t+1}, T_{t+1}, M_{t+1}$, k-times, hence both $R_i(U_{t+1} \circ w_t)$ and $R_i(U_{t+1} \circ U_{t+1} \circ w_t)$ are in M_1, thus they are both maximal. But A_q, by the right choice of y, must be the A_j of item (1) hence, renaming the states, $[R_i(s_i \circ U_{t+1} \circ w_t)|A_j(T_{t+2}, s_i)] = [R_i(s_i \circ s_i \circ w_t)|A_q(T_{t+2}, s_i)]$ therefore, without considering normalisation factors:

$$\big[R_i(s_i \circ s_i \circ w_t)|A_j(T_{t+2}, s_i)\big] + \big[R_i(s_i \circ U_{t+1} \circ w_t)|\neg A_k(T_{t+2}, s_i)\big]$$

must be maximal. That is, choosing $\neg A_k$ is like having chosen any maximal action that let R_i remaining at w_t. □

Theorem 4. *Let $\mathbb{M}$ be a first-order parametric probability structure, as defined in Definition* 6 *and v any assignment to the free variables. Let $v(w_t) \in \{X_1(\omega), \ldots, X_t(\omega) = x^*\}$, $\omega \in R^{\mathscr{I}}$, and let $R_i(x \circ w_t)$ be any predicate of state, with $\mathbb{M}, v \models R_i(x \circ w_t)$:*

1. *There is at least an action $A_j(y, x)$ that is maximal at $v(x/s)$, for s in the interpretation of R_i. Hence the action precondition for A_j is uniquely determined by R_i.*
2. *There exists at least one action A_j, and possibly A_m, ensuring both persistence and satisfiability of R_i. Hence the successor state axiom for R_i is uniquely determined by the maximal values of $[R_i(x \circ x' \circ w_t)|A_j(y, x)] + [R_i(x \circ x \circ w_t)|\neg A_m(y', x)]$.*

Proof. 1) follows from Lemma 2 and Eq. (29) and 2) follows from Theorem 3 and Eq. (29). More precisely, by Theorem 3, given a fixed w_t, $t \leqslant \mathbf{Tp}$, each $[R_i(x \circ w_t)]^{(\mathbb{M}, v)}$ is a most likely state, at $t+1$. Then:

1. By Lemma 2 for each $t \geqslant 0$ choose the most likely state $R_i(x \circ w_t)$ and the action $A_j(y, x)$ maximal at the state and add to the action precondition axioms, possibly with free variables z and z' of sort object:

   $$\forall s.Poss\big(A_j(z), s\big) \equiv R_i\big(z', s\big)$$

2. By Theorem 3 and the above item for each time $t \geqslant 0$, choose the most likely state and at least an action $A_j(y, x)$ maximal at the state according to the construction specified in Theorem 3. Further, if R_i does not correspond to an absorbing state, then there is an action A_m that, according to Theorem 3, induces a change, such that R_i exits w_t. Then $[R_i(x \circ x' \circ w_t)|A_j(y, x)] + [R_i(x \circ x \circ w_t)|\neg A_m(y', x)]$ is maximal. Then A_m is the action hence add to the successor state axioms, possibly with free variables z, z' and z'' of sort object:

   $$\forall a\, s.R_i\big(z, do(a, s)\big) \equiv A_j\big(z'\big) = a \vee \forall z'' A_m\big(z''\big) \neq a \wedge R_i(z, s) \qquad \square$$

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