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Knowledge transformation and fusion in diagnostic systems ☆

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Abstract

Diagnostic systems depend on knowledge bases specifying the causal, structural or functional interactions among components of the diagnosed objects. A diagnostic specification in a diagnostic system is a semantic interpretation of a knowledge base. We introduce the notion of diagnostic specification morphism and some operations of diagnostic specifications that can be used to model knowledge transformation and fusion, respectively. The relation between diagnostic methods in the source system and the target system of a specification morphism is examined. Also, representations of diagnostic methods in a composed system modelled by operations of specifications are given in terms of the corresponding diagnostic methods in its component systems.

Keywords: Diagnostic system; Diagnostic specification; Notion of diagnosis; Knowledge transformation; Knowledge fusion; Specification morphism; Operations of specification

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1. Introduction

To diagnose is to determine the nature of a trouble (for example, a disease) from observations of signs and symptoms, and it is an important human ability, with important applications in medicine, industrial processes and computer software, among others. Due to its importance, diagnostic reasoning has long been an active research area of Artificial Intelligence. Throughout the 1970's, several expert systems aimed in whole or in part at diagnosis were developed (e.g., MYCIN [21]), exploring different knowledge representation and reasoning techniques, but the field lacked unified underlying principles.

One of the first formal theories of diagnosis is Reggia, Nau and Wang's *set-covering model* for diagnostic expert systems [18], where causal knowledge of abnormality is represented by binary relations. Diagnosis then reduces to determining whether actually observed findings can be inferred from observed defects and the causal relations.

In 1987, a logical theory of diagnosis was proposed by Reiter [19], and it is usually called the theory of *consistency-based diagnosis*. This theory was largely extended by de Kleer et al. [10] in 1992. Their main idea is to establish a model of the normal structure and behavior of the diagnosed objects. Diagnosis is then modelled as finding a discrepancy between the normal behavior predicted from the model and the actually observed abnormal behavior. The discrepancy in this approach is formalized as logical inconsistency.

Another logical theory of diagnosis, called *abductive diagnosis*, was developed by Cox and Pietrzykowski [9], Console et al. [3,7,8] and others around 1990. They used logical implications from causes to effects to represent causal knowledge, and a diagnosis is then formalized as reasoning from effects (observed findings) to causes (abnormalities or faults).

Lucas [13] recently introduced a framework allowing these and other formal theories (for example, heuristic classification [6], goal-directed diagnosis [20] and explicit means-end model [12]), to be compared in a unified way. It consists mainly of two parts: (1) diagnostic specification, a mapping from defects to observable findings, specifying the causal relation from defects to findings; and (2) notion of diagnosis, a mapping from observed findings to defects, modelling how to get a diagnostic solution from the observed findings. This is a high-level formalism of diagnosis, and various formal theories of diagnosis can be expressed in it, including consistency-based diagnosis, abductive diagnosis and heuristic classification. In this framework a diagnostic specification need not correspond to a unique notion of diagnosis. Different strategies of diagnosis can be introduced according to varied philosophical considerations or practical purposes. In [13], given a diagnostic specification, Lucas proposed a hierarchy consisting of six notions of diagnosis induced by it, namely, *most general subset diagnosis, most general superset diagnosis, most general intersection diagnosis, most specific subset diagnosis, most specific superset diagnosis* as well as *most specific intersection diagnosis*. The six notions of diagnosis form a flexible spectrum in which one notion may refine another. Thus, they provide the user with an opportunity to choose a diagnosis method suited to his own criterion.

A diagnostic specification in Lucas's formalism is intended to serve as a semantic interpretation of the knowledge base in a diagnostic system. But the cost of gathering and processing knowledge is often very high. Such a situation makes effective reuse of knowledge essential. One of the mechanisms that support reuse of knowledge is knowledge transformation which maps various different knowledge bases to each other, enabling a common interfaces between different domains and application systems. Many different formal representations of knowledge transformation have already been proposed; for example, conditional rules [4], function [5], logical relations [11], and tables and procedures [22]. Another important mechanism for effective use and reuse of knowledge is the fusion and merging of different knowledge resources, often represented in terms of operations on knowledge bases. Examples include Stanford's ontology algebra [24] and Barwise and Seligman's theory of information flow [1].

This leads us to explore the possibility of reusing knowledge in diagnostic systems. In this paper, we consider the following two problems concerning change, evolution of knowledge for diagnosis as well as gathering and combining diagnostic knowledge from multiple sources: (1) if the knowledge base in one diagnostic system is transformed into the knowledge base in another, then to what extent can the diagnostic method adopted in the first system be reused in the second? and (2) if the knowledge bases in a set of diagnostic systems are fused or merged to construct a larger one, how we can produce a suitable diagnostic method for the composed system from the diagnostic methods of its components? In order to solve the first problem, the notion of diagnostic specification morphism is introduced. As a solution to the second problem, some algebraic operations of diagnostic specifications are proposed to model the construction of a complex diagnostic system composed from simpler ones.

This paper is organized as follows: in Section 2, we recall some basic notions from [13]. We also present some new results in this section. First, it is shown that some global properties such as monotonicity and interaction freeness of partial diagnostic specifications, can be extended to the whole specifications generated by them. Second, for the six diagnostic notions in the Lucas refinement diagnosis spectrum, some properties of the Galois connection style are observed. Third, some necessary and sufficient conditions under which the six diagnostic notions respects the given diagnostic specification are found. Fourth, we show that certain relations between diagnostic specifications are preserved and some global properties of diagnostic specifications are inherited by the six diagnostic notions induced from them. These results are useful in the analysis and comparison of various notions of diagnosis. In Section 3, the notion of diagnostic specification morphism is introduced for modelling transformations of knowledge bases in different diagnostic systems. It is shown that the relation that a notion of diagnosis respects a diagnostic specification can be preserved by some morphisms. Also, it is demonstrated that certain global properties of the source diagnostic specification may be transferred by a specification morphism to the target specification. Thus, some diagnosis methods depending heavily on these properties can be safely reused after knowledge transformation. We prove that some partial specification morphism can be smoothly extended to a specification morphism. This gives a convenient technique for constructing specification morphisms because in many applications diagnostic knowledge bases are often specified only partially. The relationship between the diagnostic strategies in the source diagnostic system and the target system of a morphism is thoroughly analyzed. The obtained results provide us with a logical support for knowledge reuse in diagnostic systems. Section 4 is devoted to examining carefully various operations of diagnostic specifications. These operations aim at describing different ways to fuse and merge diagnostic knowledge bases. They include optimistic and pessimistic fusions, optimistic merging and pessimistic merging, sum, and direct product. For each of them, we examine how the global properties of component systems are preserved by the composed system. We also clarify the relationship between the diagnostic methods in the component systems and those in the composed system. These results enable us to know to what an extent the diagnostic strategies used in a diagnostic system can be reused when it is embedded into a larger system.

To conclude this introduction, we would like to comment on applicability of the concepts and results presented in the current paper. Although the work reported in the paper is mainly concerned with the problem of system diagnosis, the formal methods developed for modelling knowledge transformation and fusion may be used in some other areas of Artificial Intelligence and related subjects. The Semantic Web is envisaged as the Web enriched with numerous domain ontologies, which specify formal semantics of data existing on the Web [2]. Recent successful projects in the ontology area have resulted at creation of thousands of ontologies [25]. However, the absence of efficient techniques of knowledge transformation and fusion hampers further development of the Semantic Web. The formalism established in this paper might provide some useful mathematical tools, supporting the development of knowledge transformation and fusion technology in such an rapidly growing area. Another potential application area of this paper is knowledge management [15], where various technologies that support knowledge transformation have been developed, but solid theoretical foundations are still to be found. Knowledge fusion and merging are key issues in the area of multi-agent systems (MAS) [23]. A very interesting problem for further study is to model learning in MAS with the fusion operations introduced here.

2. Lucas formalism of system diagnosis

Our work will be carried out entirely within the Lucas formal framework of diagnosis. So, for convenience of the reader, here we first recall some basic notions from [13]. For detailed explanations and examples illustrating these notions we refer to [13].

Each diagnostic system requires a knowledge base as the basis of implementing diagnostic task. Such a knowledge base usually specifies certain interactions between defects and observable findings. In the Lucas formalism it is interpreted as an *evidence function* which associates a set of observable findings to a set of defects and intends to use these findings to represent the evidence of occurrence of the defects.

The Lucas formalism is established in the set-theoretical setting. Let Δ_P and Φ_P be two nonempty sets. The elements of Δ_P will be used to denote positive defects, and the elements of Φ_P will be positive findings. We write

$$\Delta_N = \{\neg d\colon\ d \in \Delta_P\} \quad \text{and} \quad \Phi_N = \{\neg f\colon\ f \in \Phi_P\}$$

for the sets of negative defects and findings, respectively. Furthermore, let

$$\Delta = \Delta_P \cup \Delta_N \quad \text{and} \quad \Phi = \Phi_P \cup \Phi_N,$$

where it is assumed that $\Delta_P \cap \Delta_N = \emptyset$ and $\Phi_P \cap \Phi_N = \emptyset$, and $\neg\neg x = x$ for every $x \in \Delta_P \cup \Phi_P$. A subset D of Δ will be used to represent a set of defects. Here, we adopt an interpretation of three-valued logic in the following sense: for each $d \in \Delta_P$,

(i) d indicates the presence of defect d;
(ii) $\neg d$ indicates the absence of defect d; and
(iii) if both d and $\neg d$ are not in D, then it is understood that defect d is unknown.

Similarly, a subset E of Φ will be seen as a set of findings, with the same three-valued logical interpretation.

For any set X, we use $\wp(X)$ to express the power set of X, i.e., the set of all subsets of X. After introducing the above notations, we are able to present the first key notion in the Lucas formalism of diagnosis.

Definition 1 (*Diagnostic specification*; [13, Definition 1]). A *diagnostic specification* is a triple $\Sigma = (\Delta, \Phi, e)$, where Δ and Φ are sets of defects and findings, respectively, as explained before, and

$$e : \wp(\Delta) \to \wp(\Phi) \cup \{\perp\}$$

is a mapping, called *evidence function*, such that

(i) for any $D, D' \subseteq \Delta$, if $d, \neg d \in D$ then $e(D) = \perp$; and
(ii) for any $D, D' \subseteq \Delta$, if $e(D) \neq \perp$ and $D' \subseteq D$ then $e(D') \neq \perp$.

In addition, if e satisfies the following condition

(iii) for each $f \in \Phi$ there exists a set $D \subseteq \Delta$ with $f \in e(D)$ or $\neg f \in e(D)$, then Σ is said to be *complete*.

For each $D \subseteq \Delta$, if $e(D) \neq \perp$, then D is called *consistent*.

Intuitively, for each set D of defects, allowing both positive and negative occurrences (i.e., presence or absence) of defects, $e(D)$ expresses the set of findings which are observable when defects in D simultaneously occur.

The above definition is a slightly modified version of Definition 1 in [13]. The difference between them is that a weaker concept of diagnostic specification with only the conditions (1) and (2) is introduced, and the original concept of diagnostic specification is renamed as complete specification.

The nature of a diagnostic specification has a heavy influence on the choice of our diagnostic methods in the diagnostic system with this specification as its knowledge base. Thus, it is worthwhile to carefully analyze various properties of diagnostic specifications. The following two definitions give some common global properties of diagnostic specifications.

Definition 2 (*Monotonicity*; [13, Definition 7]). A diagnostic specification $\Sigma = (\Delta, \Phi, e)$ is called *increasing* (respectively *decreasing*) if for all $D, D' \subseteq \Delta$,

$$D \subseteq D' \text{ implies } e(D) \subseteq e(D') \quad (\text{respectively } e(D' \subseteq e(D))$$

provided D' is consistent.

Monotonicity is very familiar to us and does not need any further explanation. In the above definition, monotonicity is required to hold globally, i.e., to be valid for all sets D and D' of defects. Some localized versions of monotonicity were also introduced in [13].

Definition 3 (*Interaction freeness*; [13, Definition 8]). A diagnostic specification $\Sigma = (\Delta, \Phi, e)$ is said to be *interaction free* if for each consistent set of defects $D \subseteq \Delta$, it holds that

$$e(D) = \bigcup_{d \in D} e\big(\{d\}\big).$$

A slightly different presentation of interaction freeness is that

$$e\left(\bigcup_{i \in I} D_i\right) = \bigcup_{i \in I} e(D_i)$$

for any family $\{D_i\}_{i \in I}$ of consistent subsets of Δ, where I is an arbitrary nonempty index set. The intuitive meaning of interaction freeness is then that the evidence for the union of a family of defect sets is simply the union of their respective evidences, and thus no interaction among defects exist.

One of the most important relations between diagnostic specifications is the subspecification relation. A diagnostic specification is a subspecification of another if the former gives less evidences than the latter for the same defects.

Definition 4 (*Subspecification*). Let $\Sigma = (\Delta, \Phi, e)$ and $\Sigma' = (\Delta, \Phi, e')$ be two specifications with the same sets of defects and findings. If for any $D \subseteq \Delta$, $e(D) \subseteq e'(D)$ whenever $e(D) \neq \bot$ and $e'(D) \neq \bot$, then Σ is called a subspecification of Σ', and we write $\Sigma \preceq \Sigma'$.

It is often very difficult or even impossible to specify the whole knowledge base when the diagnostic system is very large and too many defects have to be considered. A solution to this problem that one may naturally conceive is that we only specify a small part of the knowledge base and the remaining part can be generated automatically in some way from the part specified already. This simple idea motivates the following two definitions.

Definition 5 (*Partial specification*). (1) A partial specification is a quadruple $\Sigma = (\Delta, \Phi, V, e)$, where Δ and Φ are as in Definition 2.1, $V \subseteq \wp(\Delta)$, and $e : V \to \wp(\Phi) \cup \{\bot\}$ is a mapping satisfying conditions (1) and (2) in Definition 1.

(2) A partial specification $\Sigma = (\Delta, \Phi, V, e)$ is said to be up-inductive (respectively down-inductive) if any chain $W \subseteq V$ (i.e., $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$ for all $D_1, D_2 \in W$) has an upper (respectively a lower) bound D (i.e., $D' \subseteq D$ (respectively $D \subseteq D'$) for each $D' \in W$).

It is clear that the unique difference between a diagnostic specification and a partial specification is that the domain V of the evidence function in the latter is allowed to be a proper subset of $\wp(\Delta)$; in other words, the evidences of some defects can be unspecified in a partial specification. If $V = \wp(\Delta)$, then a partial specification $\Sigma = (\Delta, \Phi, V, e)$ is exactly a diagnostic specification. Note that a partial specification $\Sigma = (\Delta, \Phi, V, e)$ is automatically up-inductive and down-inductive when Δ is finite.

Given a partial specification, there will be many different ways to recover a whole specification. Two of the ways that we use most often are presented in the following definition.

Definition 6 (*Bottom-up and top-down partial specifications*; [13, Definitions 12 and 17]). Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification, and let $\Sigma' = (\Delta, \Phi, V, e')$ be a partial specification with the same sets of defects and findings.

(1) If for any $D \in \wp(\Delta)$,

$$e(D) = \bigcup \bigl\{e'(D)\colon\ D' \in m(V, D)\bigr\}$$

$$\left(\text{respectively } e(D) = \bigcap \bigl\{e'(D)\colon\ D' \in m(V, D)\bigr\}\right),$$

where $m(V, D)$ is the set of maximal elements of $\{D' \in V\colon\ D' \subseteq D\}$ with respect to set inclusion $\subseteq$, then Σ' is called an increasing (respectively a decreasing) bottom-up partial specification of Σ.

(2) If for any $D \in \wp(\Delta)$,

$$e(D) = \bigcap \bigl\{e'(D)\colon\ D' \in M(V, D)\bigr\}$$

$$\left(\text{respectively } e(D) = \bigcup \bigl\{e'(D)\colon\ D' \in M(V, D)\bigr\}\right),$$

where $M(V, D)$ is the set of minimal elements of $\{D' \in V\colon\ D \subseteq D'\}$ with respect to set inclusion $\subseteq$, then Σ' is called an increasing (respectively a decreasing) top-down partial specification of Σ.

The philosophy of bottom-up partial specifications is to use the specified evidences to approximate the unspecified evidences from bottom, and top-down partial specifications are defined in a dual fashion. Recall from [16, p. 20] that Zorn's lemma asserts if every chain in a partially ordered set P has an upper bound then P has a maximal element. Thus, Zorn's lemma guarantees the existence of maximal (respectively minimal) elements of $\{D' \in V\colon\ D' \subseteq D\}$ (respectively $\{D' \in V\colon\ D \subseteq D'\}$) in the above definition when Σ' is up-inductive (respectively down-inductive), and $e(D)$ is well-defined for all $D \subseteq \Delta$.

Some global properties of diagnostic specifications, such as monotonicity and interaction freeness, can be easily generalized to partial specifications.

Definition 7 (*Increasing, decreasing and interaction free partial specifications*). Let $\Sigma = (\Delta, \Phi, V, e)$ be a partial specification. Then

(1) Σ is called increasing (respectively decreasing) if for any consistent $D, D' \in V$, $D \subseteq D'$ implies $e(D) \subseteq e(D')$ (respectively $e(D) \supseteq e(D')$).
(2) Σ is called interaction free if
  (i) $\{d\} \in V$ for all $d \in \Delta$; and
  (ii) for all consistent $D \in V$, $e(D) = \bigcup_{d \in D} e(\{d\})$.

The following proposition demonstrates how some properties of partial specifications can be extended to the whole specifications generated by them.

Proposition 8. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification, and let $\Sigma' = (\Delta, \Phi, V, e')$ be a partial specification with the same sets of defects and findings.*

(1) *If Σ' is an increasing bottom-up or top-down partial specification of Σ, and Σ' is increasing, then Σ is also increasing.*
(2) *If Σ' is an decreasing bottom-up or top-down partial specification of Σ, and Σ' is decreasing, then Σ is also decreasing.*
(3) *If Σ' is an increasing bottom-up partial specification of Σ, and Σ' is interaction free, then Σ is also interaction free.*

Proof. We only consider the case where Σ' is an increasing top-down partial specification of Σ, and Σ' is decreasing. For any consistent $D_1, D_2 \subseteq \Delta$, if $D_1 \subseteq D_2$, then for each $D_2' \in M(V, D_2)$, we have $D_2' \in V$ and $D_2' \supseteq D_2 \supseteq D_1$. Hence, $D_2' \in \{D \in V_2\colon\ D \supseteq D_1\}$. Note that Σ' is down-inductive. We known from Zorn's lemma that there exists $D_1' \in M(V_2, D_1)$ with $D_2' \supseteq D_1'$. Since Σ is decreasing, it holds that $e'(D_2') \subseteq e'(D_1')$. Therefore, we obtain

$$\begin{aligned} e(D_2) &= \bigcup \bigl\{e'(D_2')\colon\ D_2' \in M(V, D_2)\bigr\} \\ &\subseteq \bigcup \bigl\{e'(D_1')\colon\ D_1' \in M(V, D_1)\bigr\} = e(D_1), \end{aligned}$$

and Σ is decreasing. $\square$

We now turn to present the second key component in the Lucas formalism of diagnosis. The evidence function in a diagnostic specification gives the expected evidences for the combined occurrences of defects. Conversely, a notion of diagnosis will seek the defects that may cause the observed findings. In a sense, diagnostic specification and notion of diagnosis are two concepts conjugate to each other.

Definition 9 (*Notion of diagnosis, diagnostic problem and diagnostic solution*; [13, Definition 20]). (1) A notion of diagnosis is a triple $\Pi = (\Delta, \Phi, R)$, where Δ and Φ are respectively sets of defects and findings, $R = \{R_H\colon\ H \subseteq \Delta\}$, and $R_H : \wp(\Phi) \to \wp(\Delta) \cup \{u\}$ is a mapping for each hypothesis $H \subseteq \Delta$, called diagnostic function.

(2) A diagnostic problem is a triple $P = (\Delta, \Phi, E)$ in which Δ and Φ are as in (1), and $E \subseteq \Phi$ is a set of observed findings such that if $f \in E$ then $\neg f \notin E$; i.e., contradictory observed findings are not allowed.

(3) Let $\Pi = (\Delta, \Phi, R)$ be a notion of diagnosis, let $P = (\Delta, \Phi, E)$ be a diagnostic problem with the same sets of defects and findings, and let $H \subseteq \Delta$ be a hypothesis. Then the diagnostic solution of P under Π with respect to H is defined to be $R_H(E)$.

The intuitive meaning of a notion of diagnosis is already clear from its formal definition. The set $H \subseteq \Delta$ in the above definition is a hypothesis. This means that we already know all possible defects must be in H, and thus we only need to conduct the diagnostic task with the scope of H. For any set $E \subseteq \Phi$, denoting the actually observed findings, $R_H(E)$ is viewed as the diagnostic solution to E under the hypothesis H; that is, the defects that possibly cause E. For the case of $R_H(E) = u$, no diagnostic solution exists. For a visual interpretation of the relation among all components of a diagnostic system, including diagnostic specification, notion of diagnosis, and diagnostic problem and solution, we refer to [13, Fig. 9].

We often need to compare strictness of different notions of diagnosis. A suitable mathematical tool for this purpose is given in the following definition.

Definition 10 (*Sub-diagnostic relation*; [13, Definition 29]). Let $\Pi = (\Delta, \Phi, R)$ and $\Pi' = (\Delta, \Phi, R')$ be two notions of diagnosis with the same sets of defects and findings. If for any $E \subseteq \Phi$ and for any $H \subseteq \Delta$, $R_H(E) \subseteq R'_H(E)$ provided $R_H(E) \neq u$ and $R'_H(E) \neq u$, then Π is said to be sub-diagnostic to Π', and we write $\Pi \preceq \Pi'$.

It is not the case that any pair consisting of a diagnostic specification and a notion of diagnosis forms a reasonable diagnostic system. Usually, some conditions must be imposed to a notion of diagnosis so that it gives a suitable diagnostic method with respect to a given diagnostic specification. One of such conditions is presented in the following definition.

Definition 11 (*A notion Π of diagnosis respects a diagnostic specification Σ*; [13, Definition 22]). Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification, and let $\Pi = (\Delta, \Phi, R)$ be a notion of diagnosis with the same sets of defects and findings. It is said that Π respects Σ if

(i) for each set of observed findings $E \subseteq \Phi$, there exists a hypothesis $H \subseteq \Delta$ such that $e(R_H(E)) = E$; and
(ii) for each consistent $D \subseteq \Delta$, there exists a hypothesis $H \subseteq \Delta$ such that $R_H(e(D)) = D$.

If condition (ii) is strengthened as follows:

(ii)′ for each consistent $D \subseteq \Delta$, there exists a hypothesis $H \subseteq \Delta$ such that $R_H(e(D)) = D$ and $R_{H'}(e(D)) = u$ for all $H' \not\supseteq H$,

then we say that Π strictly respects Σ.

A notion Π of diagnosis respects a diagnostic specification Σ actually means that the diagnostic function R in Π is a pseudo-inverse of the evidence function e in Σ. In order to explain further the main idea of the above definition, it is worth comparing it with the notion of Galois connection. Let A and B be two partially ordered sets, and let $G : A \to B$ and $F : B \to A$ be order-preserving functions. Recall from [14, p. 93] that (F, G) is called a Galois connection provided the following equivalence holds: $Fb \leqslant a$ if and only if $b \leqslant Ga$ for all $a \in A$ and $b \in B$. Then it may be noted that the above definition is given in a style similar to the Galois connection.

Except the case considered in the above definition, there are many different requirements for a notion of diagnosis to be suitable with respect to a given diagnostic specification. This flexibility comes from different philosophical considerations that the user takes when choosing his diagnostic method. Thus, a spectrum of different notions of diagnosis could be introduced for a diagnostic system with a given diagnostic specification as its knowledge base. Indeed, six of such notions of diagnosis, namely, *most general subset diagnosis, most general superset diagnosis, most general intersection diagnosis, most specific subset diagnosis, most specific superset diagnosis and most specific intersection diagnosis*, were proposed by Lucas [13], and they give rise to a refinement hierarchy of diagnostic methods.

Definition 12 (*Most general subset diagnosis*; [13, p. 333]). Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification. Then the notion of most general subset diagnosis generated by Σ is defined to be the notion of diagnosis $\Pi_{GS}(\Sigma) = (\Delta, \Phi, GS)$, where for each hypothesis $H \subseteq \Delta$, and for each set $E \subseteq \Phi$ of observed findings,

$$GS_H(E) = \begin{cases} \bigcup\{H' \subseteq H\colon\ e(H') \subseteq E\}, & \text{if } H \text{ is consistent, and} \\ & e(H') \subseteq E \text{ for some } H' \subseteq H, \\ u & \text{otherwise.} \end{cases}$$

The idea behind the notion of most general subset diagnosis is that if a specific diagnosis is not acceptable, then the 'nearest' acceptable sub-hypothesis should be taken instead. We refer to [13] for more detailed explanations for the above definition as well as the other five notions of diagnosis in the Lucas refinement (see Definitions 15, 18, 21, 24 and 27 below).

Some basic properties of most general subset diagnosis are presented in the following proposition.

Proposition 13. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification, and let $\Pi_{GS}(\Sigma) = (\Delta, \Phi, GS)$ be the notion of most general subset diagnosis generated by Σ.*

(1) *If H is consistent, then $GS_H(e(D)) \supseteq D$ for any $D \subseteq H$.*
(2) *If Σ is interaction free, and H is consistent, then $e(GS_H(E)) \subseteq E$ for any $E \subseteq \Phi$.*
(3) *If Σ is interaction free, then for each $H \subseteq \Delta$, and for each $E \subseteq \Phi$,*

$$GS_H(E) = \begin{cases} \{d \in H\colon\ e(\{d\}) \subseteq E\} & \text{if } H \text{ is consistent,} \\ u & \text{otherwise.} \end{cases}$$

(4) *If Σ is decreasing and $GS_H(E) \neq u$, i.e., H is consistent, and $e(H) \subseteq E$, then $e(GS_H(E)) \subseteq E$.*
(5) *If Σ is decreasing, then*

$$GS_H(E) = \begin{cases} H & \text{if } H \text{ is consistent and } e(H) \subseteq E, \\ u & \text{otherwise.} \end{cases}$$

Proof. (1), (2), (4) and (5) are straightforward.

(3) First we note that $e(\emptyset) = \emptyset \subseteq E$ because Σ is interaction free. Thus, $GS_H(E) \neq u$ whenever H is consistent. We now only need to consider the case that H is consistent. Let

$$X = \big\{d \in H\colon\ e\big(\{d\}\big) \subseteq E\big\}.$$

From interaction freeness of Σ it follows that

$$e(X) = \bigcup_{d \in X} e\big(\{d\}\big) \subseteq X.$$

Then

$$X \in \big\{H' \subseteq H\colon e(H') \subseteq E\big\}$$

and

$$X \subseteq \bigcup \big\{H' \subseteq H\colon e(H') \subseteq E\big\}.$$

On the other hand, for any $H' \subseteq H$, if $e(H') \subseteq E$, then for each $d \in H'$, $e(\{d\}) \subseteq e(H') \subseteq E$. This is because e is increasing when Σ is interaction free. Consequently, $d \in X$, and $H' \subseteq X$. This implies further that

$$X = \bigcup \big\{H' \subseteq H\colon e(H') \subseteq E\big\}. \qquad \square$$

The parts (1) and (2) of the above proposition show that the evidence function e and the most general subset diagnosis GS_H form a Galois connection. The part (5) indicates that the notion of most general subset diagnosis is trivial for decreasing diagnostic specifications.

The next proposition gives a sufficient and necessary condition under which the notion of most general subset diagnosis generated by a diagnostic specification respects the specification.

Proposition 14. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification and $\Pi_{GS}(\Sigma) = (\Delta, \Phi, GS)$ be the notion of most general subset diagnosis generated by Σ. Then $\Pi_{GS}(\Sigma)$ respects Σ if and only if e is surjective.*

Proof. Suppose that e is surjective. For any consistent $D \subseteq \Delta$, we have

$$GS_D\big(e(D)\big) = \bigcup \big\{H' \subset D\colon e(H' \subseteq e(D)\big\} = D.$$

For any $E \subseteq \Phi$, since e is surjective, there must be $H_0 \subseteq \Delta$ such that $e(H_0) = E$. Then H_0 is consistent,

$$GS_{H_0}(E) = \bigcup \big\{H' \subseteq H_0\colon e(H') \subseteq E\big\} = H_0,$$

and $e(GS_{H_0}(E)) = e(H_0) = E$. This means that $\Pi_{GS}(\Sigma)$ respects Σ.

Conversely, if $\Pi_{GS}(\Sigma)$ respects Σ, then for any $E \subseteq \Phi$, there exists $H \subseteq \Delta$ such that $e(GS_H(E)) = E$. Then it is clear that e is surjective. $\square$

The second notion of diagnosis that forms the Lucas refinement diagnosis [13] is most general superset diagnosis. It is similar to the notion of most general subset diagnosis, and the unique difference between them is that the 'nearest' acceptable super-hypothesis is used to replace an unacceptable hypothesis when necessary in the most general superset diagnosis, as indicated by its name.

Definition 15 (*Most general superset diagnosis*; [13, p. 335]). Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification. Then the notion of most general superset diagnosis generated by Σ is defined to be $\Pi_{GO}(\Sigma) = (\Delta, \Phi, GO)$, where for each $E \subseteq \Phi$, and for each $H \subseteq \Delta$,

$$GO_H(E) = \begin{cases} \bigcup\{H' \subseteq H\colon\ e(H') \supseteq E\} & \text{if } H \text{ is consistent and} \\ & e(H') \supseteq E \text{ for some } H' \subseteq H, \\ u & \text{otherwise.} \end{cases}$$

It may be observed that most general subset diagnosis and most general superset diagnosis approach the observed findings from opposite directions. The following proposition is similar to Proposition 14, presenting some fundamental properties of most general superset diagnosis in the Galois connection style.

Proposition 16. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification, and let $\Pi_{GO}(\Sigma) = (\Delta, \Phi, GO)$ be the notion of most general superset diagnosis generated by Σ.*

(1) *If H is consistent, then $GO_H(e(D)) \supseteq D$ for any $D \subseteq H$.*
(2) *If Σ is increasing and $GO_H(E) \neq u$, i.e., H is consistent, and $e(H) \supseteq E$, then $e(GO_H(E)) \supseteq E$.*
(3) *If Σ is increasing, then*

$$GO_H(E) = \begin{cases} H & \text{if } H \text{ is consistent and } e(H) \supseteq E, \\ u & \text{otherwise.} \end{cases}$$

Proof. Straightforward. □

The part (3) of the above proposition points out that the notion of most general superset diagnosis generated by an increasing diagnostic specification is trivial.

It is interesting to note that a necessary and sufficient condition under which the notion of most general superset diagnosis respects the diagnostic specification generating it is the same as that for most general subset diagnosis. This fact is exposed by the next proposition.

Proposition 17. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification and $\Pi_{GO}(\Sigma) = (\Delta, \Phi, GO)$ be the notion of most general superset diagnosis generated by Σ. Then $\Pi_{GO}(\Sigma)$ respects Σ if and only if e is surjective.*

Proof. Similar to Proposition 14. □

As pointed out above, if we replace the subset relation in the defining equation of most general subset diagnosis with the superset relation, then we obtain the notion of most general superset diagnosis. For most general subset diagnosis, it is required that all possible evidences must be observed; but for most general superset diagnosis, the condition is instead that no observed findings are not evidences specified by the diagnostic knowledge base. In a sense, we may think that the subset relation and the superset relation are at the two extremes of the conditions that we can impose on a notion of diagnosis. An alternative at the middle is then the relation of nonempty intersection. This motivates the following definition.

Definition 18 (*Most general intersection diagnosis*; [13, p. 336]). Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification. Then the notion of most general intersection diagnosis generated by Σ is defined to be $\Pi_{GI}(\Sigma) = (\Delta, \Phi, GI)$, where for any $E \subseteq \Phi$ and $H \subseteq \Delta$,

$$GI_H(E) = \begin{cases} \bigcup\{H' \subseteq H\colon\ e(H') = \emptyset \text{ or } e(H') \cap E \neq \emptyset\} \\ \qquad \text{if } H \text{ is consistent, } E \neq \emptyset \text{ and } e(H') = \emptyset \text{ or } e(H') \cap E \neq \emptyset \\ \qquad \text{for some } H' \subseteq H, \\ H \quad \text{if } H \text{ is consistent and } E = \emptyset; \\ u \quad \text{otherwise.} \end{cases}$$

The properties of most general intersection diagnosis are much more complicated than those of most general subset or superset diagnosis, and some of them are presented in the following proposition.

Proposition 19. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification, let $\Pi_{GI}(\Sigma) = (\Delta, \Phi, GI)$ be the notion of most general intersection diagnosis generated by Σ, and let $H \subseteq \Delta$ be consistent and $\emptyset \neq E \subseteq \Phi$.*

(1) *$GI_H(e(D)) \supseteq D$ for any $D \subseteq H$.*
(2) *Suppose that Σ is increasing. Then*
  (i) *if $e(\emptyset) \neq \emptyset$ and $e(H) \subseteq \Phi - E$, then $GI_H(E) = u$;*
  (ii) *if $e(H) \cap E \neq \emptyset$, then $GI_H(E) = H$; and*
  (iii) *if $e(H) \subseteq \Phi - E$, then $GI_H(E) = \bigcup\{H' \subseteq H\colon\ e(H) = \emptyset\}$, and $GI_H(E) = \{d \in H\colon\ e(\{d\}) = \emptyset\}$ whenever Σ is interaction free.*
(3) *Suppose that Σ is decreasing. Then*
  (i) *if $e(H) \neq \emptyset$ and $e(\emptyset) \subseteq \Phi - E$, then $GI_H(E) = u$;*
  (ii) *if $e(H) = \emptyset$, then $GI_H(E) = H$; and*
  (iii) *if $e(\emptyset) \cap E \neq \emptyset$, then $GI_H(E) = \bigcup\{H' \subseteq H\colon\ e(H') \cap H \neq \emptyset\}$.*

Proof. Straightforward. □

Part (3) of the above proposition points out that the notion of most general superset diagnosis generated by an increasing diagnostic specification is trivial.

A necessary and sufficient condition under which the notion of most general intersection diagnosis respects the diagnostic specification that generates it is also found to be the same as that for most general subset diagnosis, and it is given by the next proposition.

Proposition 20. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification and $\Pi_{GI}(\Sigma) = (\Delta, \Phi, GI)$ be the notion of most general superset diagnosis generated by Σ. Then $\Pi_{GI}(\Sigma)$ respects Σ if and only if e is surjective.*

Proof. Similar to Proposition 14. □

We may see that in the above definitions we approximate an unacceptable hypothesis with acceptable ones from the bottom. Of course, an alternative is to do the same from top. This observation suggests defining the notions of most specific subset diagnosis, most specific superset diagnosis and most specific intersection diagnosis. What we need to do is to simply replace the union operation by intersection in the defining equation of the corresponding notions. This leads us to the following three definitions.

Definition 21 (*Most specific subset diagnosis*; [13, p. 337]). Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification. Then the notion of most general subset diagnosis generated by Σ is defined to be $\Pi_{SS}(\Sigma) = (\Delta, \Phi, SS)$, where for all $E \subseteq \Phi$, and $H \subseteq \Delta$,

$$SS_H(E) = \begin{cases} \bigcap\{H' \subseteq H\colon e(H') \subseteq E\} & \text{if } H \text{ is consistent and } e(H') \subseteq E \\ & \text{for some } H' \subseteq H; \\ u & \text{otherwise.} \end{cases}$$

In a sense, the notion of diagnosis given in the above definition is dual to that in Definition 12. The following proposition gives some properties of most specific subset diagnosis in the Galois style with respect to the diagnostic specification generating it. Also, it presents a simplified version of most specific subset diagnosis for increasing diagnostic specifications.

Proposition 22. *Suppose that $\Sigma = (\Delta, \Phi, e)$ is a diagnostic specification, and $\Pi_{SS}(\Sigma) = (\Delta, \Phi, SS)$ the notion of most specific subset diagnosis generated by Σ.*

(1) *If $D \subseteq H$ is consistent, then $SS_H(e(D)) \subseteq D$.*
(2) *If Σ is increasing and $SS_H(E) \neq u$, then $e(SS_H(E)) \subseteq E$.*
(3) *If Σ is increasing, then*

$$SS_H(E) = \begin{cases} \emptyset & \textit{if } H \textit{ is consistent and } e(\emptyset) \subseteq E, \\ u & \textit{otherwise.} \end{cases}$$

Proof. Straightforward. □

The last part of this proposition indicates that the notion of most specific subset diagnosis is not reasonable for increasing diagnostic specifications.

We are only able to find a sufficient condition for a diagnostic specification to be respected by its most specific subset diagnosis.

Proposition 23. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification. If e is surjective, and it satisfies the condition that $D \subset D'$ implies $e(D) \nsubseteq e(D')$ for all consistent $D, D' \subseteq \Delta$, then $\Pi_{SS}(\Sigma)$ respects Σ.*

Proof. Similar to Proposition 14. □

The relation between the following definition and Definition 21 is similar to that between Definitions 12 and 15; that is, we can derive the defining equation of $SO_H(E)$ in the following definition by replacing directly $\subseteq$ in the defining equation of $SS_H(E)$ with $\supseteq$.

Definition 24 (*Most specific superset diagnosis*; [13, p. 339]). Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification. Then the notion of most general superset diagnosis generated by Σ is defined to be $\Pi_{SO}(\Sigma) = (\Delta, \Phi, SO)$, where for all $E \subseteq \Phi$, and $H \subseteq \Delta$,

$$SO_H(E) = \begin{cases} \bigcap\{H' \subseteq H\colon\ e(H') \supseteq E\} & \text{if } H \text{ is consistent and} \\ & e(H') \supseteq E \text{ for some } H' \subseteq H; \\ u & \text{otherwise.} \end{cases}$$

The following proposition presents some basic properties of most specific superset diagnosis, and it is dual to Proposition 16.

Proposition 25. *Suppose that $\Sigma = (\Delta, \Phi, e)$ is a diagnostic specification, and $\Pi_{SO}(\Sigma) = (\Delta, \Phi, SO)$.*

(1) *If $D \subseteq H$ is consistent, then $SO_H(e(D)) \subseteq D$.*
(2) *If Σ is decreasing and $SO_H(E) \neq u$, then $e(SO_H(E)) \supseteq E$.*
(3) *If Σ is decreasing, then*

$$SO_H(E) = \begin{cases} \emptyset & \textit{if } H \textit{ is consistent and } e(\emptyset) \supseteq E, \\ u & \textit{otherwise.} \end{cases}$$

Proof. Straightforward. □

From the above proposition, we see that the notion of most specific superset diagnosis is not suited to act as a diagnostic method with respect to a decreasing diagnostic specification.

The following proposition gives a sufficient condition under which the notion of most specific superset diagnosis respects its diagnostic specification. It is interesting to compare it with the condition in Proposition 23. The only difference between them is the converse non-inclusion relations of $e(D)$ and $e(D')$.

Proposition 26. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification. If e is surjective and satisfies the condition that $D \subset D'$ implies $e(D) \not\supseteq e(D')$ for all consistent $D, D' \subseteq \Delta$, then $\Pi_{SO}(\Sigma)$ respects Σ.*

Proof. Similar to Proposition 14. □

It is still an open problem to find a necessary and sufficient condition under which $\Pi_{SS}(\Sigma)$ or $\Pi_{SO}(\Sigma)$ respects Σ.

By replacing the union operation in the defining equation of most general intersection diagnosis, we obtain:

Definition 27 (*Most specific intersection diagnosis*; [13, p. 340]). Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification. Then the notion of most specific intersection diagnosis generated by Σ is defined to be $\Pi_{SI}(\Sigma) = (\Delta, \Phi, SI)$, where for any $E \subseteq \Phi$ and for any $H \subseteq \Delta$,

$$SI_H(E) = \begin{cases} \bigcap\{H' \subseteq H\colon\ e(H') = \emptyset \text{ or} & \\ \qquad\qquad e(H') \cap E \neq \emptyset\} & \text{if } H \text{ is consistent,} \\ & E \neq \emptyset \text{ and } e(H') = \emptyset \text{ or} \\ & e(H') \cap E \neq \emptyset \text{ for some } H' \subseteq H, \\ H & \text{if } H \text{ is consistent and } E = \emptyset; \\ u & \text{otherwise.} \end{cases}$$

Some basic properties of most specific intersection diagnosis are given in the following proposition.

Proposition 28. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification, let $\Pi_{SI}(\Sigma) = (\Delta, \Phi, SI)$, and let $H \subseteq \Delta$ be consistent and $\emptyset \neq E \subseteq \Phi$.*

(1) *If $D \subseteq H$, then $SI_H(e(D)) \subseteq D$.*
(2) *Suppose that Σ is increasing. Then*
  (i) *if $e(\emptyset) \neq \emptyset$ and $e(H) \subseteq \Phi - E$, then $SI_H(E) = u$;*
  (ii) *if $e(\emptyset) = \emptyset$, then $SI_H(E) = \emptyset$; and*
  (iii) *if $e(\emptyset) \neq \emptyset$, then $SI_H(E) = \bigcap\{H' \subseteq H\colon\ e(H') \cap E \neq E\}$.*
(3) *Suppose that Σ is decreasing. Then*
  (i) *if $e(H) \neq \emptyset$ and $e(\emptyset) \subseteq \Phi - E$, then $SI_H(E) = u$;*
  (ii) *if $e(\emptyset) \cap E \neq \emptyset$, then $SI_H(E) = \emptyset$; and*
  (iii) *if $e(\emptyset) \cap E = \emptyset$, then $SI_H(E) = \bigcap\{H' \subseteq H\colon\ e(H') = \emptyset\}$.*

Proof. Straightforward. □

A sufficient condition under which the notion of most specific intersection diagnosis respects its diagnostic specification is presented in the following proposition.

Proposition 29. *If $\Sigma = (\Delta, \Phi, e)$ is a diagnostic specification fulfilling the condition that $D' \subset D$ and $e(D) \neq \emptyset$ implies $e(D') \neq \emptyset$ and $e(D') \cap e(D) = \emptyset$ for all consistent $D, D' \subseteq \Delta$, then $\Pi_{SI}(\Sigma)$ respects Σ.*

Proof. Similar to Proposition 14. □

To conclude this section, we examine the influence of the global properties of a diagnostic specification on various notions of diagnosis generated from it and how certain relations between diagnostic specifications are preserved by the notions of diagnosis generated by them. The next proposition shows that most general superset diagnosis and most specific subset diagnosis preserve the sub-relation of the diagnostic specifications generating them, but most general subset diagnosis and most specific superset diagnosis reverse this relation. Unfortunately, the sub-relation of most general or specific intersection diagnoses is not completely determined by the corresponding relation of diagnosis specifications that generate them.

Proposition 30. *Let Σ and Σ' be two diagnostic specifications with the same sets of defects and findings. If $\Sigma \preceq \Sigma'$, then*

(1) $\Pi_{GS}(\Sigma') \preceq \Pi_{GS}(\Sigma)$;
(2) $\Pi_{GO}(\Sigma) \preceq \Pi_{GO}(\Sigma')$;
(3) $\Pi_{SS}(\Sigma) \preceq \Pi_{GS}(\Sigma')$; *and*
(4) $\Pi_{SO}(\Sigma') \preceq \Pi_{SO}(\Sigma)$.

Proof. By a routine argument. □

In order to present the last proposition of this section in a more compact way, we need to introduce a notation expressing some global properties for diagnostic specifications and notions of diagnosis.

Definition 31. Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification. Then for each $A, B \in \{\cup, \cap\}$ and $C \in \{\subseteq, \supseteq\}$, the property (ABC) is defined as follows:

(ABC) $e(A_{i\in I} D_i) C B_{i\in I} e(D_i)$

for any $D_i \subseteq \Delta$ $(i \in I)$ with $e(D_i) \neq \perp$ $(i \in I)$ and $e(A_{i\in I} D_i) \neq \perp$, where I is an arbitrary index set.

Similarly, we can define the corresponding properties for notion of diagnosis.

It is easy to see that interaction freeness is equivalent to $(\cup\cup \subseteq)$ plus $(\cup\cup \supseteq)$.

The properties defined above are very interesting. For example, the property $(\cup\cap \subseteq)$ may be rewritten as

$$R_H(E) \subseteq \bigcap_{f\in E} R_H\bigl(\{f\}\bigr)$$

for each $E \subseteq \Phi$ and $H \subseteq \Delta$. It depicts a method of diagnosis that we often adopt in our daily life. Suppose that a set E of findings are observed, and we want to find a diagnostic solution to E. Usually, we first find all defects $R_H(\{f\})$ that may cause the single finding f for each $f \in E$. Then it allows us to locate the true solution among the common defects for all findings in E. The next proposition indicates that global properties of a diagnosis specification of type (ABC) are inherited by most general (or specific) subset (or superset) diagnosis generated from it. However, both most general and specific intersection diagnoses do not enjoy such an inheritance.

Proposition 32. *Let $\Sigma = (\Delta, \Phi, e)$ be a diagnostic specification, and let $A, B \in \{\cup, \cap\}$.*

(1) *If Σ satisfies* $(\mathrm{AB} \subseteq)$, *then $\Pi_{GS}(\Sigma)$ satisfies* $(\mathrm{BA} \supseteq)$.
(2) *If Σ satisfies* $(\mathrm{AB} \supseteq)$, *then $\Pi_{GO}(\Sigma)$ satisfies* $(\mathrm{BA} \supseteq)$.

(3) *If Σ satisfies* (AB $\subseteq$), *then $\Pi_{SS}(\Sigma)$ satisfies* (BA $\subseteq$).
(4) *If Σ satisfies* (AB $\supseteq$), *then $\Pi_{SO}(\Sigma)$ satisfies* (BA $\subseteq$).

Proof. We prove (1) as an example. From the definition of most general subset diagnosis it follows that

$$GS_H(B_{i\in I} E_i) = \bigcup \bigl\{H' \subseteq H\colon\ e(H') \subseteq B_{i\in I} E_i\bigr\}$$

and

$$\begin{aligned} A_{i\in I} GS_H(E_i) &= A_{i\in I} \cup \bigl\{H_i' \subseteq H\colon\ e(H_i') \subseteq E_i\bigr\} \\ &= \bigcup \bigl\{A_{i\in I} H_i'\colon\ H_i' \subseteq H \text{ and } e(H_i') \subseteq E_i\ (i \in I)\bigr\}. \end{aligned}$$

Now we only need to show that if for each $i \in I$, $H_i' \subseteq H$ and $e({}'_i) \subseteq E_i$, then $A_{i\in I} H_i' \subseteq H$ and $e(A_{i\in I} H_i') \subseteq B_{i\in I} E_i$. The first inclusion is obvious, and the second one is guaranteed by the property (AB$\subseteq$) of Σ. $\square$

3. Diagnostic specification morphisms

This section is devoted to establish a mathematical model of knowledge transformation in diagnostic problem solving, namely, diagnostic specification morphism. We will carefully compare the diagnostic strategies in the source system of a specification morphism and its target system. First, we introduce a formal definition of specification morphism.

Definition 33 (*Specification morphism*). Let $\Sigma_1 = (\Delta_1, \Phi_1, e_1)$ and $\Sigma_2 = (\Delta_2, \Phi_2, e_2)$ be two diagnostic specifications. A *specification morphism* from Σ_1 to Σ_2 is a pair $M = (g, h)$ of mappings $g : \Delta_1 \to \Delta_2$ and $h : \Phi_1 \to \Phi_2$ fulfilling the following two conditions:

(i) for all $d \in \Delta_1$ and $f \in \Phi_1$,

$$g(\neg d) = \neg g(d) \quad \text{and} \quad h(\neg f) = \neg h(f); \text{ and}$$

(ii) for each $D \subseteq \Delta_1$, it holds that

$$\bar{g}\bigl(e_1(D)\bigr) = e_2\bigl(\overline{f}(D)\bigr);$$

in other words, the following diagram commutes:

$$\begin{tikzcd} \wp(\Delta_1) \arrow[r, "\overline{f}"] \arrow[d, "e_1"'] & \wp(\Delta_2) \arrow[d, "e_2"] \\ \wp(\Phi_1) \arrow[r, "\bar{g}"'] & \wp(\Phi_2) \end{tikzcd}$$

where $\overline{f}$ and $\bar{g}$ are respectively the extensions of g and h to $\wp(\Delta_1)$ and $\wp(\Phi_1)$, i.e., for any $D \subseteq \Delta_1$ and $E \subseteq \Phi_1$,

$$\bar{g}(D) = \bigl\{g(d)\colon\ d \in D\bigr\}, \quad \text{and}$$

$$\bar{h}(E) = \bigl\{h(f)\colon\ f \in E\bigr\}, \quad \bar{h}(\bot) = \bot.$$

Table 1
Knowledge in system A

D	$\emptyset$	$\{d_1\}$	$\{d_2\}$	$\{d_3\}$	$\{d_1, d_2\}$	$\{d_1, d_3\}$	$\{d_2, d_3\}$	$\{d_1, d_2, d_3\}$
$e_A(D)$	$\emptyset$	$\{f_1, f_2\}$	$\{f_2\}$	$\{f_3\}$	$\{f_1, f_2\}$	$\{f_1, f_2, f_3\}$	$\{f_2, f_3\}$	$\{f_1, f_2, f_3\}$

Table 2
Knowledge in system B

D	$\emptyset$	$\{v_1\}$	$\ldots$	$\{v_4\}$	$\ldots$	$\{v_1, v_4\}$	$\ldots$
$e_B(D)$	$\emptyset$	$\{w_3\}$	$\ldots$	$\{w_1, w_3\}$	$\ldots$	$\{w_1, w_3\}$	$\ldots$

For simplicity, we will write g and h in place of $\overline{g}$ and $\overline{h}$, respectively.

Obviously, (id_Δ, id_Φ) is a morphism from diagnostic specification $\Sigma = (\Delta, \Phi, e)$ to itself, where id_X stands for the identity function on set X. In addition, it is easy to verify that if both $M_1 = (g_1, h_1) : \Sigma_1 \to \Sigma_2$ and $M_2 = (g_2, h_2) : \Sigma_2 \to \Sigma_3$ are specification morphisms, then $M_2 \circ M_1 = (g_2 \circ g_1, h_2 \circ h_1) : \Sigma_1 \to \Sigma_3$ is a morphism too. Thus, we have a category of diagnostic specifications together with specification morphisms.

To illustrate the above definition, consider the following simple example.

Example 34. A typical application of specification morphism is analyzing the relationship between different medical systems, say, traditional Chinese medicine and the western medicine. Suppose that A and B are two different medical systems. A piece of medical knowledge in system A is represented by the diagnostic specification $\Sigma_A = (\Delta_A, \Phi_A, e_A)$, where $\Delta_A = \{d_1, d_2, d_3\}$, $\Phi_A = \{f_1, f_2, f_3\}$, d_1, d_2, d_3 are the names of three symptoms, f_1, f_2, f_3 are the names of three diseases, and the evidence function e_A depicting the causal knowledge between symptoms and diseases is given by Table 1.

Furthermore, we assume that a piece of medical knowledge in system B is described by the diagnosis specification $\Sigma_B = (\Delta_B, \Phi_B, e_B)$ in which $\Delta_B = \{v_1, v_2, v_3, v_4\}$, $\Phi_B = \{w_1, w_2, w_3\}$, and (a fragment of) the evidence function e_B is given by Table 2.

We now compare the two medical systems. The symptom d_1 in system A is renamed as v_4 in system B, both symptoms d_2 and d_3 are called v_1 in system B, and in system A there is no counterpart of symptoms v_2 and v_3 in system B. This gives a defect mapping $g : \Delta_A \to \Delta_B$. On the other hand, a finding mapping $h : \Phi_A \to \Phi_B$ is defined by $h(f_1) = w_1$ and $h(f_2) = h(f_3) = w_3$. Then it is easy to verify that (g, h) is a specification morphism from Σ_A to Σ_B, and it establishes a reasonable link between the two medical systems A and B.

The following proposition shows that a specification morphism is able to carry some global properties, including monotonicity and interaction freeness, of its source specification forward to its target specification.

Proposition 35. *Suppose that $M = (g, f) : \Sigma_1 = (\Delta_1, \Phi_1, e_1) \to \Sigma_2 = (\Delta_2, \Phi_2, e_2)$ is a specification morphism, and g is surjective. Then*

(1) *if* Σ_1 *is increasing* (*respectively decreasing*), *then* Σ_2 *is also increasing* (*respectively decreasing*);
(2) *if* Σ_1 *is interaction free, so is* Σ_2.

Proof. (1) We only consider the increasing case. For any $D_2, D_2' \subseteq \Delta_2$, we have

$$D_2 = g\big(g^{-1}(D_2)\big) \quad \text{and} \quad D_2' = g\big(g^{-1}(D_2')\big)$$

because g is surjective. If $D_2 \subseteq D_2'$ and D_2' is consistent in Σ_2, then

$$h\big(e_1\big(g^{-1}(D_2')\big)\big) = e_2\big(g\big(g^{-1}(D_2')\big)\big) = e_2(D_2') \neq \bot.$$

This implies that $e_1(g^{-1}(D_2')) \neq \bot$; i.e., $g^{-1}(D_2')$ is also consistent in Σ_1. Otherwise, it follows that $e_2(D_2') = h(\bot) = \bot$, a contradiction. Note that $g^{-1}(D_2) \subseteq g^{-1}(D_2')$. Since Σ_1 is increasing, it holds that

$$\begin{aligned} e_2(D_2) &= e_2\big(g\big(g^{-1}(D_2)\big)\big) = h\big(e_1\big(g^{-1}(D_2)\big)\big) \\ &\subseteq h\big(e_1\big(g^{-1}(D_2')\big)\big) = e_2(D_2'). \end{aligned}$$

(2) For each $D \subseteq \Delta_2$, if D is consistent in Σ_2, then from (1) we know that $g^{-1}(D)$ is consistent in Σ_1. Since e_1 is interaction free, it holds that

$$\begin{aligned} e_2(D) = e_2\big(g\big(g^{-1}(D)\big)\big) = h\big(e_1\big(g^{-1}(D)\big)\big) &= h\left(e_1\left(\bigcup_{d\in D} g^{-1}\big(\{d\}\big)\right)\right) \\ &= h\left(\bigcup_{d\in D} e_1\big(g^{-1}\big(\{d\}\big)\big)\right) = \bigcup_{d\in D} h\big(e_1\big(g^{-1}\big(\{d\}\big)\big)\big) \\ &= \bigcup_{d\in D} e_2\big(g\big(g^{-1}\big(\{d\}\big)\big)\big) = \bigcup_{d\in D} e_2\big(\{d\}\big). \qquad \square \end{aligned}$$

For reason of limited space, we are not going to examine carefully how other global properties, such as those of type (ABC) defined at the end of the previous section, of diagnostic specification is preserved by specification morphism.

We now want to observe how a morphism between partial specifications can be extended to a morphism between the total specifications generated from them. To this end, we first introduce the following definition.

Definition 36 (*Partial specification morphism*). Let $\Sigma_1 = (\Delta_1, \Phi_1, V_1, e_1)$ and $\Sigma_2 = (\Delta_2, \Phi_2, V_2, e_2)$ be two partial specifications. Then a specification morphism from Σ_1 to Σ_2 is a pair $M = (g, h)$ of mappings $g : \Delta_1 \to \Delta_2$ and $h : \Phi_1 \to \Phi_2$ such that

(i) $g(V_1) = \{g(D)\colon\ D \in V_1\} \subseteq V_2$; and
(ii) $h(e_1(D)) = e_2(g(D))$ for each $D \in V_1$.

The notion of specification morphism for partial specifications is obviously a generalization of the one for (total) diagnostic specifications. The next proposition shows that a partial specification morphism can also serve as a specification morphism for the case of increasing bottom-up construction. To save space, we omit a detailed discussion of the corresponding problem for the other constructions given in Definition 6.

Proposition 37. *Let $\Sigma_i' = (\Delta_i, \Phi_i, V_i, e_i')$ be an increasing bottom-up partial specification of $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ $(i = 1, 2)$, and $M = (g, h)$ is a specification morphism from Σ_1' to Σ_2'. If*

(1) $g^{-1}(D_2) \in V_1$ *for any* $D_2 \in V_2$;
(2) $g^{-1}(g(D_1)) = D_1$ *for any* $D_1 \in V_1$; *and*
(3) $g(g^{-1}(D_2)) = D_2$ *for any* $D_2 \in V_2$,

then M is also a specification morphism from Σ_1 to Σ_2.

Proof. For each $D_1 \subseteq \Delta_1$, it follows that

$$\begin{aligned} h\big(e_1(D_1)\big) &= h\Big(\bigcup\big\{e_1'(D_1'):\ D_1' \in m(V_1, D_1)\big\}\Big) \\ &= \bigcup\big\{h\big(e_1'(D_1')\big):\ D_1' \in m(V_1, D_1)\big\} \\ &= \bigcup\big\{e_2'\big(g(D_1')\big):\ D_1' \in m(V_1, D_1)\big\}. \end{aligned}$$

If $D_1' \in m(V_1, D_1)$, then $D_1' \in V_1$ and $D_1' \subseteq D_1$. We have $g(D_1') \subseteq g(D_1)$. Moreover, since M is a specification morphism from Σ_1' to Σ_2', $g(D_1') \in V_2$. Then Zorn's lemma warrants that $g(D_1') \subseteq D_2'$ for some $D_2' \in m(V_2, g(D_1))$. Consequently,

$$h\big(e_1(D_1)\big) \subseteq \bigcup\big\{e_2'(D_2'):\ D_2' \in m\big(V_2, g(D_1)\big)\big\} = e_2\big(g(D_1)\big).$$

Conversely, for any $D_2' \in m(V_2, g(D_1))$, it holds that $D_2' \in V_2$ and $D_2' \subseteq g(D)$. From condition (1) we obtain $g^{-1}(D_2') \in V_1$, and from (2) we have $g^{-1}(D_2') \subseteq g^{-1}(g(D_1)) = D_1$. Again, Zorn's lemma tells us that $g^{-1}(D_2') \subseteq D_1'$ for some $D_1' \in m(V_1, D_1)$. Thus, it follows from condition (3) that $D_2' = g(g^{-1}(D_2')) \subseteq g(D_1')$, and

$$\begin{aligned} e_2\big(g(D_1)\big) &= \bigcup\big\{e_2'(D_2'):\ D_2' \in m\big(V_2, g(D_1)\big)\big\} \\ &\subseteq \bigcup\big\{e_2'\big(g(D_1')\big):\ D_1' \in m(V_1, D_1)\big\} = h\big(e_1(D_1)\big). \qquad \square \end{aligned}$$

We now come to present the main results of this section. The following group of propositions will provides us with a close connection between a transformation of knowledge bases in different diagnostic systems and a transformation of their diagnostic strategies. The intuitive idea of transformation between diagnostic strategies is captured by the concept of diagnosis morphism given in the following definition.

Definition 38 (*Diagnosis morphism*). Let $\Pi_1 = (\Delta_1, \Phi_1, R_1)$ and $\Pi_2 = (\Delta_2, \Phi_2, R_2)$ be two notions of diagnosis. A diagnosis morphism from Π_1 to Π_2 is a pair $M = (g, h)$

Table 3
Diagnosis notion in system A

E	$\emptyset$	$\{f_1\}$	$\{f_2\}$	$\{f_3\}$	$\{f_1, f_2\}$	$\{f_2, f_3\}$	$\{f_1, f_3\}$	$\{f_1, f_2, f_3\}$
$R_{A,\emptyset}(E)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{A,\{d_1\}}(E)$	$\emptyset$	u	$\emptyset$	$\emptyset$	$\{d_1\}$	$\emptyset$	$\{d_1\}$	$\{d_1\}$
$R_{A,\{d_2\}}(E)$	$\emptyset$	u	$\{d_2\}$	$\{d_2\}$	$\emptyset$	$\{d_2\}$	$\emptyset$	$\emptyset$
$R_{A,\{d_1,d_2\}}(E)$	$\emptyset$	$\{d_1, d_2\}$	$\{d_1\}$	$\{d_1\}$	$\{d_1\}$	$\{d_1\}$	$\{d_1\}$	$\emptyset$

Table 4
Diagnosis notion in system B

E	$\emptyset$	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_1, w_2\}$	$\{w_2, w_3\}$	$\{w_1, w_3\}$	$\{w_1, w_2, w_3\}$
$R_{B,\emptyset}(E)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{B,\{v_1\}}(E)$	$\emptyset$	u	$\emptyset$	$\{v_1\}$	u	$\{v_1\}$	$\emptyset$	$\{v_2\}$
$R_{B,\{v_2\}}(E)$	$\emptyset$	u	$\emptyset$	$\emptyset$	u	$\emptyset$	$\{v_2\}$	$\emptyset$
$R_{B,\{v_1,v_2\}}(E)$	$\emptyset$	$\{v_1, v_2\}$	$\emptyset$	$\{v_2\}$	$\{v_1, v_2\}$	$\{v_2\}$	$\{v_2\}$	$\emptyset$

of mappings $g : \Delta_1 \to \Delta_2$ and $h : \Phi_1 \to \Phi_2$ satisfying condition (i) in Definition 33 and commutativity of the following diagram:

$$\begin{tikzcd}
\wp(\Delta_1) \arrow[r, "g"] & \wp(\Delta_2) \\
\wp(\Phi_1) \arrow[u, "R_{1,H}"] \arrow[r, "h"'] & \wp(\Phi_2) \arrow[u, "R_{2,g(H)}"']
\end{tikzcd}$$

in other words, for any $E \subseteq \Phi$ and $H \subseteq \Delta$, $g(R_{1,H}(E)) = R_{2,g(H)}(h(E))$, where it is assumed that $g(u) = u$.

The following simple example illustrates the notion of diagnosis morphism very well.

Example 39. Suppose A and B are two expert systems for medical diagnosis. Let $\Delta_A = \{d_1, d_2\}$, $\Delta_B = \{v_1, v_2\}$, $g(d_1) = v_2$ and $g(d_2) = v_1$, and let Φ_A, Φ_B and h be the same as in Example 34. Part of diagnosis method used in system A is represented by Table 3, and part of the diagnosis method used in B is given in Table 4. Then it is easy to check that (g, h) is a diagnosis morphism from R_A to R_B, and we may think that it provides a mechanism for reusing diagnosis method of system A in system B.

We now investigate some fundamental properties of diagnosis morphism and its connection to specification morphism. The next proposition tells us that the information that a notion of diagnosis respects a diagnostic specification can be carried forward and backward by specification morphisms and diagnosis morphisms under a certain condition.

Proposition 40. *Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ be a diagnostic specification and $\Pi_i = (\Delta_i, \Phi_i, R_i)$ a notion of diagnosis with the same sets of defects and findings $(i = 1, 2)$. Suppose that Π_1 respects Σ_1. If there is a pair $M = (g, h)$ of mappings such that M is a specification morphism from Σ_1 to Σ_2, it is also a diagnosis morphism from Π_1 to Π_2, and both g and*

h are surjective, then Π_2 respects Σ_2. Furthermore, if Π_1 strictly respects Σ_1, then Π_2 also strictly respects Σ_2.

Proof. For each $E \subseteq \Phi_2$, there exists $H \subseteq \Delta_1$ such that $e_1(R_{1,H}(h^{-1}(E))) = h^{-1}(E)$ because Π_1 respects Σ_1. Since h is surjective, we have $h(h^{-1}(E)) = E$. This yields that

$$E = h\big(h^{-1}(E)\big) = h\big(e_1\big(R_{1,H}\big(h^{-1}(E)\big)\big)\big) = e_2\big(g\big(R_{1,H}\big(h^{-1}(E)\big)\big)\big)$$
$$= e_2\big(R_{2,g(H)}\big(h\big(h^{-1}(E)\big)\big)\big) = e_2\big(R_{2,g(H)}(E)\big).$$

On the other hand, for each consistent $D \subseteq \Delta_2$, we are able to find some $H \subseteq \Delta_1$ such that $R_{2,g(H)}(e_2(D)) = D$. Therefore, Π_2 respects Σ_2.

For the case that Π_1 strictly respects Σ_1, we can assume that $R_{1,H'}(e_1(g^{-1}(D))) = u$ for all $H' \not\supseteq H$. Our purpose is to show that $R_{2,H''}(e_2(D)) = u$ for each $H'' \not\supseteq g(H)$. If not so; i.e., there is $H'' \subseteq \Delta_2$ with $H'' \not\supseteq g(H)$ and $R_{2,H''}(e_2(D)) \neq u$, then from the fact that g is surjective we know that

$$R_{2,H''}\big(e_2(D)\big) = R_{2,g(g^{-1}(H''))}\big(e_2(D)\big) = g\big(R_{1,g^{-1}(H'')}\big(e_1\big(g^{-1}(D)\big)\big)\big),$$

and $R_{1,g^{-1}(H'')}(e_1(g^{-1}(D))) \neq u$. If $g^{-1}(H'') \supseteq H$, then $g(H) \subseteq g(g^{-1}(H'')) = H''$, and it is impossible. Thus, it holds that $g^{-1}(H'') \not\supseteq H$. This contradicts to the previous assumption. □

The following two propositions clarify the relationship between the six diagnostic strategies of the Lucas refinement diagnosis [13] in the source diagnostic systems of a specification morphism and those in the target system. Suppose we are given a diagnostic problem in the source system. The next proposition carefully compares the following two paths: (i) we first find a diagnostic solution by using the diagnostic method in the source system, and then map it into the target system; and (ii) we map our diagnostic problem and hypothesis into the target system, and then find a diagnostic solution by employing the diagnostic methods in the target system. For example, if we adopt the notion of most general intersection diagnosis in both the source and target systems, then Propositions 41(5) and (6) indicate that path (i) always gives a stricter solution than path (ii), but the two solutions according to paths (i) and (ii) are the same when the defect mapping is bijective and the finding mapping is injective.

Proposition 41. *Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ be a diagnostic specification $(i = 1, 2)$, and let $M = (g, h)$ be a specification morphism from Σ_1 to Σ_2. For any $E \subseteq \Phi_1$ and $H \subseteq \Delta_1$, we have*

(1) $g(GS_{1,H}(E)) \subseteq GS_{2,g(H)}(h(E))$ *if* $GS_{1,H}(E) \neq u$;
(2) $GS_{2,g(H)}(h(E)) \subseteq g(GS_{1,H}(E))$ *if* $GS_{2,g(H)}(h(E)) \neq u$, *g is bijective and h is injective*;
(3) $g(GO_{1,H}(E)) \subseteq GO_{2,g(H)}(h(E))$ *if* $GO_{1,H}(E) \neq u$;
(4) $GO_{2,g(H)}(h(E)) \subseteq g(GO_{1,H}(E))$ *if* $GO_{2,g(H)}(h(E)) \neq u$, *g is bijective and h is injective*;
(5) $g(GI_{1,H}(E)) \subseteq GI_{2,g(H)}(h(E))$ *if* $GI_{1,H}(E) \neq u$;

(6) $GI_{2,g(H)}(h(E)) \subseteq g(GI_{1,H}(E))$ *if* $GI_{2,g(H)}(h(E)) \neq u$, g *is bijective and* h *is injective*;
(7) $g(SS_{1,H}(E)) \subseteq SS_{2,g(H)}(h(E))$ *if* $SS_{2,g(H)}(h(E)) \neq u$, g *is bijective and* h *is injective*;
(8) $SS_{2,g(H)}(h(E)) \subseteq g(SS_{1,H}(E))$ *if* $SS_{1,H}(E) \neq u$ *and* g *is injective*;
(9) $g(SO_{1,H}(E)) \subseteq SO_{2,g(H)}(h(E))$ *if* $SO_{2,g(H)}(h(E)) \neq u$, g *is bijective and* h *is injective*;
(10) $SO_{2,g(H)}(h(E)) \subseteq g(SO_{1,H}(E))$ *if* $SO_{1,H}(E) \neq u$ *and* g *is injective*;
(11) $g(SI_{1,H}(E)) \subseteq SI_{2,g(H)}(h(E))$ *if* $SI_{2,g(H)}(h(E)) \neq u$, g *is bijective and* h *is injective*; *and*
(12) $SI_{2,g(H)}(h(E)) \subseteq g(SI_{1,H}(E))$ *if* $SI_{1,H}(E) \neq u$ *and* g *is injective.*

Proof. We only demonstrate (7) as an instance. We note that injectivity of g implies

$$\begin{aligned} g\big(SS_{1,H}(E)\big) &= g\left(\bigcap\big\{H' \subseteq H\colon\ e_1(H') \subseteq E\big\}\right) \\ &= \bigcap\big\{g(H')\colon\ H' \subseteq H \text{ and } e_1(H') \subseteq E\big\}, \end{aligned}$$

and

$$SS_{2,g(H)}\big(h(E)\big) = \bigcap\big\{K' \subseteq g(H)\colon\ e_2(K') \subseteq h(E)\big\}$$

whenever $SS_{1,H}(E) \neq u$ and $SS_{2,g(H)}(h(E)) \neq u$. Thus, it suffices to prove the following two items:

(i) H is consistent if and only if $g(H)$ is consistent. Indeed, if $e_1(H) \neq \perp$, then $e_2(g(H)) = h(e_1(H)) \neq \perp$, and $g(H)$ is consistent. Conversely, if $e_2(g(H)) \neq \perp$, we must have $e_1(H) \neq \perp$.

(ii) $\{g(H')\colon\ H' \subseteq H \text{ and } e_1(H') \subseteq E\} = \{K' \subseteq g(H)\colon\ e_2(K') \subseteq h(E)\}$. In fact, if $H' \subseteq H$ and $e_1(H') \subseteq E$, then $g(H') \subseteq g(H)$, and $e_2(g(H')) = h(e_1(H')) \subseteq h(E)$. Conversely, if $K' \subseteq g(H)$ and $e_2(K') \subseteq h(E)$, then we set $H' = g^{-1}(K')$, and it holds that $K' = g(H')$ and $H' \subseteq g^{-1}(g(H)) = H$ because g is an bijection. Furthermore,

$$e_1(H') \subseteq h^{-1}\big(h\big(e_1(H')\big)\big) = h^{-1}\big(e_2\big(g(H')\big)\big) = h^{-1}\big(e_2(K)\big) \subseteq h^{-1}\big(h(E)\big) = E.$$

The last equality comes from the fact that h is injective. $\square$

The following simple corollary shows that sometimes a specification morphism can act as a diagnosis morphism too.

Corollary 42. *Let* $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ *be a diagnostic specification and* $\Pi_{GS}(\Sigma_i) = (\Delta_i, \Phi_i, GS_i)$ *be the notion of most general subset diagnosis generated by* Σ_i $(i = 1, 2)$, *and let* $M = (g, h)$ *be a specification morphism from* Σ_1 *to* Σ_2. *If* g *is a bijection and* h *is an injection, then* M *is also a diagnosis morphism from* $\Pi_{GS}(\Sigma_1)$ *to* $\Pi_{GS}(\Sigma_2)$. *The same conclusion also holds for* Π_{GO}, Π_{GI}, Π_{SS}, Π_{SO} *and* Π_{SI}.

Proof. Immediate from Proposition 41. $\square$

What was considered in Proposition 41 is a forward transformation of various diagnostic notions. The next proposition considers a backward transformation of diagnostic methods. Now the diagnostic problem is given in the target system, and the two paths under comparison are: (i) we find a diagnostic solution in the target system, and then map it into the source system via the inverse of defect mapping; and (ii) we map the observed findings and hypothesis into the source system through the inverses of finding mapping and defect mapping respectively, and then find a solution using the diagnostic notion in the source system. The following proposition examines the relation between the diagnostic solutions obtained through these two paths.

Proposition 43. *Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ be a diagnostic specification $(i = 1, 2)$, and let $M = (g, h)$ be a specification morphism from Σ_1 to Σ_2. For any $E \subseteq \Phi_2$ and $H \subseteq \Delta_2$, we have*

(1) $g^{-1}(GS_{2,H}(E)) \subseteq GS_{1,g^{-1}(H)}(h^{-1}(E))$ *if* $GS_{2,H}(E) \neq u$;
(2) $GS_{1,g^{-1}(H)}(h^{-1}(E)) \subseteq g^{-1}(GS_{2,H}(E))$ *if* $GS_{1,g^{-1}(H)}(h^{-1}(E)) \neq u$ *and H is consistent, and in particular* $GS_{1,g^{-1}(H)}(h^{-1}(E)) = g^{-1}(GS_{2,H}(E))$ *if* $GS_{2,H}(E) \neq u$ *and* $GS_{1,g^{-1}(H)}(h^{-1}(E)) \neq u$;
(3) $g^{-1}(GO_{2,H}(E)) \subseteq GO_{1,g^{-1}(H)}(h^{-1}(E))$ *if* $GO_{2,H}(E) \neq u$, *g is surjective and h is injective*;
(4) $GO_{1,g^{-1}(H)}(h^{-1}(E)) \subseteq g^{-1}(GO_{2,H}(E))$ *if* $GO_{1,g^{-1}(H)}(h^{-1}(E)) \neq u$, *$H$ is consistent and h is surjective*;
(5) $g^{-1}(GI_{2,H}(E)) \subseteq GI_{1,g^{-1}(H)}(h^{-1}(E))$ *if* $GO_{2,H}(E) \neq u$, *g is surjective and h is bijective*;
(6) $GI_{1,g^{-1}(H)}(h^{-1}(E)) \subseteq g^{-1}(GI_{2,H}(E))$ *if* $GI_{1,g^{-1}(H)}(h^{-1}(E)) \neq u$, $h^{-1}(E) \neq \emptyset$ *and H is consistent*;
(7) $SS_{1,g^{-1}(H)}(h^{-1}(E)) \subseteq g^{-1}(SS_{2,H}(E))$ *if* $SS_{2,H}(E) \neq u$;
(8) $g^{-1}(SS_{2,H}(E)) \subseteq SS_{1,g^{-1}(H)}(h^{-1}(E))$ *if* $SS_{1,g^{-1}(H)}(h^{-1}(E)) \neq u$, *$H$ is consistent and g is injective*;
(9) $SO_{1,g^{-1}(H)}(h^{-1}(E)) \subseteq g^{-1}(SO_{2,H}(E))$ *if* $SO_{2,H}(E) \neq u$, *g is surjective and h is injective*;
(10) $g^{-1}(SO_{2,H}(E)) \subseteq SO_{1,g^{-1}(H)}(h^{-1}(E))$ *if* $SO_{1,g^{-1}(H)}(h^{-1}(E)) \neq u$, *$H$ is consistent, g is injective and h is surjective*;
(11) $SI_{1,g^{-1}(H)}(h^{-1}(E)) \subseteq g^{-1}(SI_{2,H}(E))$ *if* $SI_{2,H}(E) \neq u$, *g is surjective and h is bijective;*
(12) $g^{-1}(SI_{2,H}(E)) \subseteq SI_{1,g^{-1}(H)}(h^{-1}(E))$ *if* $SI_{1,g^{-1}(H)}(h^{-1}(E)) \neq u$, *$H$ is consistent and g is injective.*

Proof. As examples, we demonstrate (1), (2) and (6).

(1) If $GS_{2,H}(E) \neq u$, then H is consistent in Σ_2, i.e., $e_2(H) \neq \bot$, and there is $H' \subseteq H$ such that $e_2(H') \subseteq E$. First, we show that $g^{-1}(H)$ is consistent in Σ_1. If not so, then $e_1(g^{-1}(H)) = \bot$, and

$$e_2\big(g\big(g^{-1}(H)\big)\big) = h\big(e_1\big(g^{-1}(H)\big)\big) = \bot.$$

Note that $g(g^{-1}(H)) \subseteq H$. From condition (ii) in Definition 2.1 it follows that $e_2(H) = \perp$, a contradiction.

Second, it holds that

$$g^{-1}\big(GS_{2,H}(E)\big) = g^{-1}\left(\bigcup\big\{H' \subseteq H\colon\ e_2(H') \subseteq E\big\}\right)$$
$$= \bigcup\big\{g^{-1}(H')\colon\ H' \subseteq H \text{ and } e_2(H') \subseteq E\big\},$$

and

$$GS_{1,g^{-1}(H)}\big(h^{-1}(E)\big) = \bigcup\big\{K' \subseteq g^{-1}(H)\colon\ e_1(K' \subseteq h^{-1}(E)\big\}.$$

For any $H' \subseteq H$ with $e_2(H') \subseteq E$, we have $g^{-1}(H') \subseteq g'(H)$, and

$$h\big(e_1\big(g^{-1}(H')\big)\big) = e_2\big(g\big(g^{-1}(H')\big)\big) \subseteq e_2(H) \subseteq E.$$

Then

$$e_1\big(g^{-1}(H')\big) \subseteq h^{-1}\big(h\big(e_1\big(g^{-1}(H')\big)\big)\big)h^{-1}(E).$$

This shows that

$$\big\{g^{-1}(H')\colon\ H' \subseteq H \text{ and } e_2(H' \subseteq E\big\} \subseteq \big\{K' \subseteq g^{-1}(H)\colon\ e_1(K' \subseteq h^{-1}(E)\big\},$$

and $g^{-1}(GS_{2,H}(E)) \subseteq GS_{1,g^{-1}(H)}(h^{-1}(E))$ follows.

(2) For any $K' \subseteq g^{-1}(H)$ with $e_1(K') \subseteq h^{-1}(E)$, we need to find a set $H' \subseteq H$ such that $e_2(H') \subseteq E$ and $K' \subseteq g^{-1}(H')$. It is easy to see that we can take $H' = g(K')$.

(6) If $GI_{1,g^{-1}(H)}(h^{-1}(E)) \neq u$, $h^{-1}(E) \neq \emptyset$ and H is consistent, then $E \neq \emptyset$, and we obtain

$$GI_{1,g^{-1}(H)}\big(h^{-1}(E)\big) = \bigcup\big\{K' \subseteq g^{-1}(H)\colon\ e_1(K') = \emptyset \text{ or } e_1(K') \cap h^{-1}(E) \neq \emptyset\big\},$$

and

$$g^{-1}\big(GI_{2,H}(E)\big) = \bigcup\big\{g^{-1}(H')\colon\ H' \subseteq H,\ \text{and } e_2(H') = \emptyset \text{ or } e_2(H') \cap E \neq \emptyset\big\}.$$

Now it suffices to show that for any $K' \subseteq g^{-1}(H)$ with $e_1(K') = \emptyset$ or $e_1(K') \cap h^{-1}(E) \neq \emptyset$, there exists $H' \subseteq H$ with $e_2(H') = \emptyset$ or $e_2(H') \cap E \neq \emptyset$, and $K' \subseteq g^{-1}(H')$. Indeed, we take $H' = g(K')$. Then

$$K' \subseteq g^{-1}\big(g(K)\big) = g^{-1}(H'),$$

and

$$H' = g(K') \subseteq g\big(g^{-1}(H)\big) \subseteq H.$$

If $e_1(K') = \emptyset$, then

$$e_2(H') = e_2\big(g(K')\big) = h\big(e_1(K')\big) = h(\emptyset) = \emptyset.$$

If $e_1(K') \cap h^{-1}(E) \neq \emptyset$, then

$$\begin{aligned} e_2(H') \cap E = e_2\big(g(K')\big) \cap E &= h\big(e_1(K')\big) \cap E \\ &\supseteq h\big(e_1(K')\big) \cap h\big(h^{-1}(E)\big) \\ &\supseteq h\big(e_1(K') \cap h^{-1}(E)\big) \neq \emptyset. \qquad \square \end{aligned}$$

4. Operations of diagnostic specifications

The aim of this section is to introduce several operations of diagnostic specifications which can model knowledge gathering, fusion, merging and combination in the processes of diagnostic problem solving. These operations include optimistic and pessimistic fusions, sum, direct product as well as optimistic and pessimistic mergings. The diagnostic strategies in a composite system modelled by a certain operation of diagnostic specifications will be analyzed in terms of the corresponding diagnostic strategies in its component systems.

Definition 44 (*Optimistic and pessimistic fusions of specifications*). Let $\Sigma_i = (\Delta, \Phi, e_i)$ $(i \in I)$ be a family of diagnostic specifications with the same sets of defects and findings. Then their optimistic and pessimistic fusions are defined to be $\bigcup_{i\in I} \Sigma_i = (\Delta, \Phi, \bigcup_{i\in I} e_i)$ and $\bigcap_{i\in I} \Sigma_i = (\Delta, \Phi, \bigcap_{i\in I} e_i)$, respectively, where for each $D \subseteq \Delta$,

$$\left(\bigcup_{i\in I} e_i\right)(D) = \begin{cases} \bigcup_{i\in I} e_i(D) & \text{if } e_i(D) \neq \perp \text{ for all } i \in I, \\ \perp & \text{otherwise,} \end{cases}$$

$$\left(\bigcap_{i\in I} e_i\right)(D) = \begin{cases} \bigcap_{i\in I} e_i(D) & \text{if } e_i(D) \neq \perp \text{ for all } i \in I, \\ \perp & \text{otherwise.} \end{cases}$$

It is easy to see that the notions of optimistic and pessimistic fusions are well-defined. Moreover, if all Σ_i $(i \in I)$ are complete, then $\bigcup_{i\in I} \Sigma_i$ is also complete, but $\bigcap_{i\in I} \Sigma_i$ is not necessary to be complete.

Some other interesting fusion operators for diagnostic specifications may be introduced. A typical example is contradiction-finding operator. Remember that an evidence function allows contradictory values f and $\neg f$. So, a fusion operator can be defined by modifying slightly the above definition to indicate conflicting opinions about a topic and to resolve this conflict. We are not going to examine these extra fusion operators in detail.

Example 45. We imagine a medical expert system which aggregates medical knowledge from different doctors. Certainly, there will be many different ways for such an aggregation. It is reasonable to say that optimistic and pessimistic fusions are at the two extremes of the whole spectrum formed by these aggregation ways. Let $\Delta = \{d_1, d_2, d_3\}$ and $\Phi = \{f_1, f_2, f_3\}$, where d_1, d_2 and d_3 stand for three symptoms and f_1, f_2 and f_3 three diseases. Suppose that a piece of medical knowledge of doctor A and a piece of medical knowledge of doctor B (both concerning the causal relation between symptoms d_1, d_2, d_3 and f_1, f_2, f_3) are represented by evidence e_A and e_B respectively (see Table 5). Then a simple calculation gives their optimistic fusion $e_A \cup e_B$ and pessimistic fusion $e_A \cap e_B$ as shown in Table 6.

Table 5
Knowledge of doctors A and B

D	$\emptyset$	$\{d_1\}$	$\{d_2\}$	$\{d_3\}$	$\{d_1, d_2\}$	$\{d_1, d_3\}$	$\{d_2, d_3\}$	$\{d_1, d_2, d_3\}$
e_A	$\emptyset$	$\{f_1\}$	$\emptyset$	$\{f_2\}$	$\{f_1, f_3\}$	$\{f_1, f_2\}$	$\{f_2\}$	$\{f_1, f_2, f_3\}$
e_B	$\emptyset$	$\emptyset$	$\{f_3\}$	$\{f_2\}$	$\{f_3\}$	$\{f_1, f_2\}$	$\{f_2, f_3\}$	$\{f_1, f_2, f_3\}$

Table 6
Optimistic and pessimistic fusions

D	$\emptyset$	$\{d_1\}$	$\{d_2\}$	$\{d_3\}$	$\{d_1, d_2\}$	$\{d_1, d_3\}$	$\{d_2, d_3\}$	$\{d_1, d_2, d_3\}$
$e_A \cup e_B$	$\emptyset$	$\{f_1\}$	$\{f_3\}$	$\{f_2\}$	$\{f_1, f_3\}$	$\{f_1, f_2\}$	$\{f_2, f_3\}$	$\{f_1, f_2, f_3\}$
$e_A \cap e_B$	$\emptyset$	$\emptyset$	$\emptyset$	$\{f_2\}$	$\{f_3\}$	$\{f_1, f_2\}$	$\{f_2\}$	$\{f_1, f_2, f_3\}$

The next proposition indicates that some global properties of diagnostic specifications, such as monotonicity and interaction freeness, are preserved by the fusion operations. The optimistic fusion of a family of increasing (respectively decreasing, interaction free) diagnostic specifications is also increasing (respectively decreasing, interaction free), and the pessimistic fusion of a family of decreasing diagnostic specifications is decreasing.

Proposition 46. *Let $\Sigma_i = (\Delta, \Phi, e_i)$ $(i \in I)$ be a family of diagnostic specifications with the same sets of defects and findings.*

(1) *If all Σ_i $(i \in I)$ are increasing, then both $\bigcup_{i \in I} \Sigma_i$ and $\bigcap_{i \in I} \Sigma_i$ are increasing.*
(2) *If all Σ_i $(i \in I)$ are decreasing, so is $\bigcup_{i \in I} \Sigma_i$.*
(3) *If all Σ_i $(i \in I)$ are interaction free, so is $\bigcup_{i \in I} \Sigma_i$.*

Proof. (1) We only consider pessimistic fusion. Suppose that $D \subseteq D' \subseteq \Delta$ and D' is consistent in $\bigcap_{i \in I} \Sigma_i$. From the condition (2) in Definition 1 we know that for each $i \in I$, if $e_i(D') \neq \perp$ then $e_i(D) \neq \perp$. This implies that

$$\begin{aligned}
\Big(\bigcap_{i \in I} e_i\Big)(D) &= \bigcap\big\{e_i(D)\colon\ i \in I \text{ and } e_i(D) \neq \perp\big\} \\
&\subseteq \bigcap\big\{e_i(D)\colon\ i \in I \text{ and } e_i(D') \neq \perp\big\} \\
&\subseteq \bigcap\big\{e_i(D')\colon\ i \in I \text{ and } e_i(D') \neq \perp\big\} \\
&= \Big(\bigcap_{i \in I} e_i\Big)(D')
\end{aligned}$$

because $e_i(D) \subseteq e_i(D')$ for all $i \in I$.

(2) Similar to (1).

(3) If $(\bigcup_{i \in I} e_i)(D) \neq \perp$, then $e_i(D) \neq \perp$ for each $i \in I$, and interaction freeness of Σ_i $(i \in I)$ leads to

$$\begin{aligned}
\Big(\bigcup_{i \in I} e_i\Big)(D) &= \bigcup_{i \in I} e_i(D) = \bigcup_{i \in I}\Big(\bigcup_{d \in D} e_i\big(\{d\}\big)\Big) \\
&= \bigcup_{d \in D}\Big(\bigcup_{i \in I} e_i\big(\{d\}\big)\Big) = \bigcup_{d \in D}\Big(\Big(\bigcup_{i \in I} e_i\Big)\big(\{d\}\big)\Big). \qquad \square
\end{aligned}$$

The relationship between the diagnostic methods used in a fused diagnostic system and the diagnostic methods in its component systems are established by the following two propositions. For example, if we adopt the notion of most general subset diagnosis, then Proposition 47(1) shows that the diagnostic solution in an optimistic fusion is always included in the intersection of the solutions in its component systems, and they are the same when these component diagnostic specifications are all increasing. On the other hand, if we adopt the notion of most specific subset diagnosis, then Proposition 47(3) indicates that the diagnostic solution in an optimistic fusion includes the union of the solutions in its component systems, and they are equal provided all component diagnostic specifications are decreasing.

Proposition 47. *Let $\Sigma_i = (\Delta, \Phi, e_i)$ $(i \in I)$ be a family of diagnostic specifications with the same sets of defects and findings, and let $\bigcup_{i\in I} \Sigma_i = (\Delta, \Phi, \bigcup_{i\in I} e_i)$ be their optimistic fusion.*

(1) *If $\Pi_{GS}(\Sigma_i) = (\Delta, \Phi, R_i)$ is the notion of most general subset diagnosis generated by Σ_i $(i \in I)$ and $\Pi_{GS}(\bigcup_{i\in I} \Sigma_i) = (\Delta, \Phi, R)$ the notion of most general subset diagnosis generated by $\bigcup_{i\in I} \Sigma_i$, then for any $E \subseteq \Phi$ and $H \subseteq \Delta$,*
   (1.1) *$R_H(E) \subseteq \bigcap_{i\in I} R_{i,H}(E)$ when $R_H(E) \neq u$; and*
   (1.2) *$R_H(E) = \bigcap_{i\in I} R_{i,H}(E)$ if all Σ_i $(i \in I)$ are increasing.*

(2) *If $\Pi_{GO}(\Sigma_i) = (\Delta, \Phi, R_i)$ is the notion of most general superset diagnosis generated by Σ_i $(i \in I)$ and $\Pi_{GO}(\bigcup_{i\in I} \Sigma_i) = (\Delta, \Phi, R)$ the notion of most general superset diagnosis generated by $\bigcup_{i\in I} \Sigma_i$, then for all $E \subseteq \Phi$ and $H \subseteq \Delta$,*
   (2.1) *$\bigcup_{i\in I} R_{i,H}(E) \subseteq R_H(E)$ whenever H is consistent in each Σ_i $(i \in I)$, and $R_{i_0,H}(E) \neq u$ for some $i_0 \in I$; and*
   (2.2) *$R_H(E) = \bigcup_{i\in I} R_{i,H}(E)$ if E is finite, and $\{\Sigma_i\}_{i\in I}$ is directed with respect to the sub-specification relation $\preceq$, i.e., for any $i_1, i_2 \in I$, there is an $i_0 \in I$ such that $\Sigma_{i_1} \preceq \Sigma_{i_0}$ and $\Sigma_{i_2} \preceq \Sigma_{i_0}$.*

(3) *If $\Pi_{SS}(\Sigma_i) = (\Delta, \Phi, R_i)$ is the notion of most specific subset diagnosis generated by Σ_i $(i \in I)$ and $\Pi_{SS}(\bigcup_{i\in I} \Sigma_i) = (\Delta, \Phi, R)$ the notion of most specific subset diagnosis generated by $\bigcup_{i\in I} \Sigma_i$, then for any $E \subseteq \Phi$ and for any $H \subseteq \Delta$,*
   (3.1) *$\bigcup_{i\in I} R_{i,H}(E) \subseteq R_H(E)$ provided $R_H(E) \neq u$; and*
   (3.2) *$R_H(E) = \bigcup_{i\in I} R_{i,H}(E)$ if all Σ_i $(i \in I)$ are decreasing.*

(4) *If $\Pi_{SO}(\Sigma_i) = (\Delta, \Phi, R_i)$ is the notion of most specific superset diagnosis generated by Σ_i $(i \in I)$ and $\Pi_{SO}(\bigcup_{i\in I} \Sigma_i) = (\Delta, \Phi, R)$ the notion of most specific superset diagnosis generated by $\bigcup_{i\in I} \Sigma_i$, then for all $E \subseteq \Phi$ and $H \subseteq \Delta$,*
   (4.1) *$R_H(E) \subseteq \bigcap_{i\in I} R_{i,H}(E)$ provided $R_{i,H}(E) \neq u$ for each $i \in I$; and*
   (4.2) *$R_H(E) = \bigcap_{i\in I} R_{i,H}(E)$ when E is finite, and $\{\Sigma_i\}_{i\in I}$ is directed with respect to the sub-specification relation $\preceq$.*

Proof. We only prove (1.1), (1.2) and (4.2); the others are similar and so omitted.

(1.1) First, we note that H is consistent in $\bigcup_{i\in I} \Sigma_i$ if and only if it is consistent in each Σ_i $(i \in I)$. Second, we have

$$R_H(E) = \bigcup \left\{ H' \subseteq H \colon \bigcup_{i\in I} e_i(H') \subseteq E \right\} = \bigcup_{H' \in \bigcap_{i\in I} \{H' \subseteq H \colon e_i(H') \subseteq E\}} H'$$

$$\subseteq \bigcap_{i\in I}\Big(\bigcup\big\{H' \subseteq H:\ e_i(H') \subseteq E\big\}\Big) = \bigcap_{i\in I} R_{i,H}(E).$$

(1.2) If $d \in \bigcap_{i\in I} R_{i,H}(E)$, then for any $i \in I$, $d \in R_{i,H}(E)$ and there exists $H'_i \subseteq H$ such that $e_i(H'_i) \subseteq E$ and $d \in H'_i$. Let $H' = \bigcap_{i\in I} H'_i$. Then $H' \subseteq H$, and for any $i \in I$, $e_i(H') \subseteq e_i(H'_i) \subseteq E$ because e_i is increasing. This means that

$$H' \in \bigcap_{i\in I}\big\{H' \subseteq H:\ e_i(H'_i) \subseteq E\big\}.$$

Consequently, we have $d \in H' \subseteq R_H(E)$, and

$$\bigcap_{i\in I} R_{i,H}(E) \subseteq R_H(E).$$

(4.2) If $d \notin R_H(E)$, then there exists $H' \subseteq H$ such that $\bigcup_{i\in I} e_i(H') \supseteq E$ and $d \notin H'$. Assume that $E = \{f_1, f_2, \ldots, f_m\}$. Then for any $k \leqslant m$, we have some $i_k \in I$ with $f_k \in e_{i_k}(H')$. Since $\{\Sigma_i\}_{i\in I}$ is directed with respect to $\preceq$, there must be $i_0 \in I$ such that $e_{i_k}(H') \subseteq e_{i_0}(H')$ for all $k \leqslant m$. Now, it follows that $E \subseteq e_{i_0}(H')$, $d \notin R_{i_0,H}(E)$, and $d \notin \bigcap_{i\in I} R_{i,H}(E)$. Therefore, it follows that

$$\bigcap_{i\in I} R_{i,H}(E) \subseteq R_H(E). \qquad \square$$

The next proposition deals with the case of pessimistic fusion. It is shown that the diagnostic solution in a pessimistic fusion is always looser than the union of the solutions in its component systems if we apply the notion of most general subset diagnosis or most specific superset diagnosis; whereas the diagnosis in a pessimistic fusion is stricter than the intersection of the solutions in its component systems if the notion of most general superset diagnosis or most specific subset diagnosis is employed.

Proposition 48. *Let $\Sigma_i = (\Delta, \Phi, e_i)$ $(i \in I)$ be a family of diagnostic specifications with the same sets of defects and findings, and let $\bigcap_{i\in I} \Sigma_i = (\Delta, \Phi, \bigcap_{i\in I} e_i)$ be their pessimistic fusion.*

(1) *If $\Pi_{GS}(\Sigma_i) = (\Delta, \Phi, R_i)$ $(i \in I)$ and $\Pi_{GS}(\bigcap_{i\in I} \Sigma_i)) = (\Delta, \Phi, R)$, then for any $E \subseteq \Phi$ and $H \subseteq \Delta$, it holds that*

$$\bigcup_{i\in I} R_{i,H}(E) \subseteq R_H(E).$$

(2) *If $\Pi_{GO}(\Sigma_i) = (\Delta, \Phi, R_i)$ $(i \in I)$ and $\Pi_{GO}(\bigcap_{i\in I} \Sigma_i)) = (\Delta, \Phi, R)$, then for any $E \subseteq \Phi$ and $H \subseteq \Delta$, we have*

$$R_H(E) \subseteq \bigcap_{i\in I} R_{i,H}(E),$$

and the equality holds whenever all Σ_i $(i \in I)$ are decreasing.

(3) *If $\Pi_{SS}(\Sigma_i) = (\Delta, \Phi, R_i)$ $(i \in I)$ and $\Pi_{SS}(\bigcap_{i\in I} \Sigma_i)) = (\Delta, \Phi, R)$, then for any $E \subseteq \Phi$ and $H \subseteq \Delta$, it holds that*

$$R_H(E) \subseteq \bigcap_{i\in I} R_{i,H}(E).$$

(4) *If* $\Pi_{SO}(\Sigma_i) = (\Delta, \Phi, R_i)$ $(i \in I)$ *and* $\Pi_{SO}(\bigcap_{i\in I}(\Sigma_i)) = (\Delta, \Phi, R)$*, then for any* $E \subseteq \Phi$ *and* $H \subseteq \Delta$*, it holds that*

$$\bigcup_{i\in I} R_{i,H}(E) \subseteq R_H(E)$$

with the equality when all Σ_i $(i \in I)$ *are increasing.*

Proof. Similar to Proposition 47. □

Definition 49 (*Sum of specifications*). Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ $(i \in I)$ be a family of diagnostic specifications. Then the sum of Σ_i $(i \in I)$ is defined to be $\bigoplus_{i\in I} \Sigma_i = (\bigcup_{i\in I} \Delta_i, \bigcup_{i\in I} \Phi_i, e)$, where for each $D \subseteq \bigcup_{i\in I} \Delta_i$,

$$e(D) = \begin{cases} \bigcup_{i\in I} e_i(D \cap \Delta_i) & \text{if } e_i(D \cap \Delta_i) \neq \bot \text{ for all } i \in I; \\ \bot & \text{otherwise.} \end{cases}$$

It is easy to see that the notion of sum is well-defined; that is, e satisfies the conditions (1) and (2) in Definition 1. Furthermore, $\bigoplus_{i\in I} \Sigma_i$ is complete whenever all Σ_i $(i \in I)$ are complete.

A simple idea behind sum of diagnostic specifications is that we can divide a big system into some independent smaller systems and then examine these subsystems one by one. Thus, the notion of specification sum provides us with a mathematical model of modularization technique in diagnostic problem solving.

Example 50. Suppose we have an industrial system consisting of two subsystems A and B. These two systems are assumed to be independent in the sense that the function of one subsystem cannot be affected by the defects in the other subsystem, for instance, the water and gas systems in a plant. Let $\Sigma_A = (\Delta_A, \Phi_A, e_A)$ and $\Sigma_B = (\Delta_A, \Phi_B, e_B)$ specify respectively the causal interactions among the components of the two subsystems, where $\Delta_A = \{d_1, d_2, d_3\}$, $\Phi_A = \{f_1, f_2\}$, $\Delta_B = \{d_4, d_5\}$, $\Phi_B = \{f_3, f_4\}$, and e_A and e_B are given by Tables 7 and 8. Then

$$\Sigma_A \oplus \Sigma_B = \big(\{d_1, d_2, d_3, d_4, d_5\}, \{f_1, f_2, f_3, f_4\}, e\big),$$

and it may be seen as a diagnostic knowledge base of the whole system. For example,

$$e\big(\{d_3, d_4, d_5\}\big) = e_A\big(\{d_3\}\big) \cup e_B\big(\{d_4, d_5\}\big) = \{f_1, f_3, f_4\};$$

Table 7
Subsystem A

D	$\emptyset$	$\{d_1\}$	$\{d_2\}$	$\{d_3\}$	$\{d_1, d_2\}$	$\{d_1, d_3\}$	$\{d_2, d_3\}$	$\{d_1, d_2, d_3\}$
e_A	$\emptyset$	$\emptyset$	$\emptyset$	$\{f_1\}$	$\emptyset$	$\{f_1\}$	$\{f_1\}$	$\{f_1, f_2, f_3\}$

Table 8
Subsystem B

D	$\emptyset$	$\{d_4\}$	$\{d_5\}$	$\{d_4, d_5\}$
e_B	$\emptyset$	$\{f_4\}$	$\{f_3\}$	$\{f_3, f_4\}$

that is, the observable findings for the simultaneous occurrences of defects d_3, d_4 and d_5 are f_1, f_3 and f_4.

Both monotonicity and interaction freeness are preserved by the operation of diagnostic specification sum.

Proposition 51. *Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ $(i \in I)$ be a family of diagnostic specifications. If all Σ_i $(i \in I)$ are increasing (respectively decreasing, interaction free), so is $\bigoplus_{i\in I} \Sigma_i$.*

Proof. We consider the interaction-free case as an example. For any consistent $D \subseteq \bigcup_{i\in I} \Delta_i$, it holds that

$$\begin{aligned} e(D) &= \bigcup_{i\in I} e_i(D\cap\Delta_i) = \bigcup_{i\in I}\bigcup_{d\in D\cap\Delta_i} e_i\big(\{d\}\big) = \bigcup_{i\in I}\bigcup_{d\in D\cap\Delta_i} e\big(\{d\}\big) \\ &= \bigcup_{d\in\bigcup_{i\in I}(D\cap\Delta_i)} e\big(\{d\}\big) = \bigcup_{d\in D} e\big(\{d\}\big). \qquad \square \end{aligned}$$

One may naturally expects that all component diagnostic systems can be embedded into the sum of them via the inclusion morphisms, and specification morphisms of the component systems can be glued to a morphism of their sum. This is indeed guaranteed by the following proposition.

Proposition 52. (1) *Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ $(i \in I)$ be a family of diagnostic specifications, and for each $k \in I$, let*

$$in_{\Delta_k} : \Delta_k \hookrightarrow \bigcup_{i\in I} \Delta_i \quad \text{and} \quad in_{\Phi_k} : \Phi \hookrightarrow \bigcup_{i\in I} \Phi_i$$

be the inclusion mappings from Δ_k into $\bigcup_{i\in I} \Delta_i$ and from Φ_k into $\bigcup_{i\in I} \Phi_i$, respectively, i.e., for any $d \in \Delta_k$ and $f \in \Phi_k$,

$$in_{\Delta_k}(d) = d \quad \text{and} \quad in_{\Phi_k}(f) = f.$$

If $\Delta_i \cap \Delta_j = \emptyset$ for all $i, j \in I$ with $i \neq j$, and $e_i(\emptyset) = \emptyset$ for each $i \in I$, then $in_k = (in_{\Delta_k}, in_{\Phi_k})$ is a specification morphism from Σ_k to $\bigoplus_{i\in I} \Sigma_i$.

(2) *Suppose that $M_i = (g_i, h_i)$: $\Sigma_i = (\Delta_i, \Phi_i, e_i) \to \Sigma_i' = (\Delta_i', \Phi_i', e_i')$ $(i \in I)$ be a family of specification morphisms, and $\Delta_i \cap \Delta_j = \emptyset$ and $\Phi_i \cap \Phi_j = \emptyset$ provided $i, j \in I$ and $i \neq j$. Let $\bigoplus_{i\in I} M_i = (\bigoplus_{i\in I} g_i, \bigoplus_{i\in I} h_i)$, where*

$$\bigoplus_{i\in I} g_i : \bigcup_{i\in I} \Delta_i \to \bigcup_{i\in I} \Delta_i', \qquad \bigoplus_{i\in I} h_i : \bigcup_{i\in I} \Phi_i \to \bigcup_{i\in I} \Phi_i'$$

and for all $d \in \bigcup_{i\in I} \Delta_i$ and $f \in \bigcup_{i\in I} \Phi_i$,

$$\left(\bigoplus_{i\in I} g_i\right)(d) = g_j(d) \quad \text{when } d \in \Delta_j \quad \text{and} \quad \left(\bigoplus_{i\in I} h_i\right)(f) = h_j(f) \quad \text{when } f \in \Phi_j.$$

If $g_i(\Delta_i) \cap g_j(\Delta_j) = \emptyset$ for all $i, j \in I$ with $i \neq j$, then $\bigoplus_{i\in I} M_i$ is a specification morphism from $\bigoplus_{i\in I} \Sigma_i$ to $\bigoplus_{i\in I} \Sigma_i'$.

Proof. (1) is easy. We only prove (2). The condition that $\{\Delta_i\}$ and $\{\Phi_i\}$ $(i \in I)$ are pairwise disjoint warrants that both $\bigoplus_{i\in I} g_i$ and $\bigoplus_{i\in I} h_i$ are well-defined. For each $D \subseteq \bigcup_{i\in I} \Delta_i$, we have

$$\Big(\bigoplus_{i\in I} h_i\Big)\big(e(D)\big) = \Big(\bigoplus_{i\in I} h_i\Big)\Big(\bigcup_{i\in I} e_i(D\cap\Delta_i)\Big) = \bigcup_{i\in I}\Big(\bigoplus_{i\in I} h_i\Big)\big(e_i(D\cap\Delta_i)\big)$$
$$= \bigcup_{i\in I} h_i\big(e_i(D\cap\Delta_i)\big) = \bigcup_{i\in I} e'_i\big(g_i(D\cap\Delta_i)\big).$$

Since $g_i(\Delta_i)\cap g_j(\Delta_j)=\emptyset$ for all $i, j \in I$ with $i \neq j$, it holds that

$$\Big(\bigcup_{i\in I} g_i(D\cap\Delta_i)\Big)\cap\Delta'_j = g_j(D\cap\Delta_j).$$

Consequently, it follows that

$$\Big(\bigoplus_{i\in I} h_i\Big)\big(e(D)\big) = \bigcup_{i\in I} e'_i\Big(\Big(\bigcup_{i\in I} g_i(D\cap\Delta_i)\Big)\cap\Delta'_j\Big)$$
$$= e'\Big(\bigcup_{i\in I} g_i(D\cap\Delta_i)\Big) = e'\Big(\Big(\bigoplus_{i\in I} g_i\Big)(D)\Big). \qquad \square$$

We now are going to observe the relationship between the notions of diagnosis in a sum system and those in its component systems. To this end, we need to introduce the concept of sum of diagnostic notions.

Definition 53 (*Sum of diagnostic notions*). Let $\Pi_i = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$ be a family of notions of diagnosis. Then the sum of Π_i $(i \in I)$ is defined to be $\bigoplus_{i\in I}\Pi_i = (\bigcup_{i\in I}\Delta_i, \bigcup_{i\in I}\Phi_i, R)$, where for any $E \subseteq \bigcup_{i\in I}\Phi_i$ and $H \subseteq \bigcup_{i\in I}\Delta_i$,

$$R_H(E) = \begin{cases} \bigcup_{i\in I} R_{i,H\cap\Delta_i}(E\cap\Phi_i) & \text{if } R_{i,H\cap\Delta_i}(E\cap\Phi_i) \neq u \text{ for all } i \in I,\\ u & \text{otherwise.}\end{cases}$$

If we have a large diagnostic system consisting of some subsystems, and each subsystem has a notion of diagnosis respected by its diagnostic specification, then the following proposition guarantees that in the whole large system the diagnostic specification respects the sum of the notions of diagnosis in all subsystems.

Example 54. We consider the two notions of diagnosis in Example 39, $\Pi_A = (\Delta_A, \Phi_A, R_A)$ and $\Pi_B = (\Delta_B, \Phi_B, R_B)$. They were originally linked via a diagnosis morphism in Example 39, but here they are treated as two independent subsystems of a larger diagnosis system Π. Thus, Π can be thought of as the sum of Π_A and Π_B; that is, $\Pi = (\{d_1, d_2, v_1, v_2\}, \{f_1, f_2, f_3, w_1, w_2, w_3\}, R)$ and $R = \{R_H\colon\ H \subseteq \{d_1, d_2, v_1, v_2\}\}$. Let $H = \{d_1, v_1, v_2\}$ be a hypothesis and $E = \{f_2, f_3, w_1, w_2\}$ a diagnostic problem in Π. Then the solution to E under hypothesis H is

$$
\begin{aligned}
R_H(E) &= R_{A,H\cap\Delta_A}(E\cap\Phi_A)\cup R_{B,H\cap\Delta_B}(E\cap\Phi_B)\\
&= R_{A,\{d_1\}}\big(\{f_2,f_3\}\big)\cup R_{B,\{v_1,v_2\}}\big(\{w_1,w_2\}\big)\\
&= \emptyset\cup\{v_1,v_2\}=\{v_1,v_2\}.
\end{aligned}
$$

Proposition 55. *Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ $(i \in I)$ be a family of diagnostic specifications and $\Pi_i = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$ a family of notions of diagnosis such that Σ_i and Π_i have the same sets of defects and findings for each $i \in I$. If for all $i \in I$, Π_i (strictly) respects Σ_i, then $\bigoplus_{i\in I}\Pi_i$ (strictly) respects $\bigoplus_{i\in I}\Sigma_i$ too.*

Proof. For each consistent $D \subseteq \bigcup_{i\in I}\Delta_i$, and for any $i \in I$, since Π_i strictly respects Σ_i, there is $H_i \subseteq \Delta_i$ such that $R_{i,H_i}(e_i(D\cap\Delta_i)) = D\cap\Delta_i$, and $R_{i,H_i'}(e_i(D\cap\Delta_i)) = u$ for all $H_i' \subseteq \Delta_i$ with $H_i' \not\supseteq H_i$. Let $H = \bigcup_{i\in I} H_i$. Then

$$
\begin{aligned}
R_H\big(e(D)\big) &= \bigcup_{i\in I} R_{i,H\cap\Delta_i}\big(e_i(D\cap\Delta_i)\big)\\
&= \bigcup_{i\in I} R_{i,H_i}\big(e_i(D\cap\Delta_i)\big) = \bigcup_{i\in I}(D\cap\Delta_i) = D.
\end{aligned}
$$

Furthermore, if $H' \not\supseteq H$, then there must be $i_0 \in I$ such that $H'\cap\Delta_{i_0} \not\supseteq H_{i_0}$. Now, we have $R_{H'\cap\Delta_{i_0}}(e_{i_0}(D\cap\Delta_{i_0})) = u$ and $R_{H'}(e(D)) = u$.

Likewise, we can prove that for any $E \subseteq \bigcup_{i\in I}\Phi_i$, there exists $H \subseteq \bigcup_{i\in i}\Delta_i$ with $e(R_H(E)) = E$. This implies that $\bigoplus_{i\in I}\Pi_i$ strictly respects $\bigoplus_{i\in I}\Sigma_i$. □

The following proposition relates the diagnostic method in a sum system to the diagnostic strategies in its component systems. Consider a diagnostic problem with the observed findings E and hypothesis H in a large system consisting of some subsystems. Of course, the best way to solve this problem is to deal with it directly in the whole system. Often, however, this is very difficult. An alternative way is to find a diagnostic solution in each subsystem with the piece of information in this subsystem provided by E and H. We then combine these solutions in all subsystems to form a diagnostic solution for the whole system. Now a natural question is: how far is this alternative solution from our expected solution? The following proposition answers this question: if we employ the notion of most general subset diagnosis or most specific superset diagnosis, then the alternative solution is stricter than the expected one, but for the notion of most general superset diagnosis or most specific subset diagnosis, the alternative solution is looser than the expected one; and a similar result conditionally holds for the notion of most general or specific intersection diagnosis.

Proposition 56. *Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ $(i \in I)$ be a family of diagnostic specifications.*

(1) *If $\Pi_{GS}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$ and $\Pi_{GS}(\bigoplus_{i\in I}\Sigma_i) = (\bigcup_{i\in I}\Delta_i, \bigcup_{i\in I}\Phi_i, R)$, then for any $E \subseteq \bigcup_{i\in I}\Phi_i$, and for any $H \subseteq \bigcup_{i\in I}\Delta_i$,*
   (1.1) *$R_H(E) \subseteq \bigcup_{i\in I} R_{i,H\cap\Delta_i}(E\cap\Phi_i)$; and*
   (1.2) *$R_H(E) = \bigcup_{i\in I} R_{i,H\cap\Delta_i}(E\cap\Phi_i)$ if for any $i, j \in I$, $\Delta_i\cap\Delta_j = \emptyset$ whenever $i \neq j$.*

(2) *If* $\Pi_{GO}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$ *and* $\Pi_{GO}(\bigoplus_{i\in I} \Sigma_i) = (\bigcup_{i\in I} \Delta_i, \bigcup_{i\in I} \Phi_i, R)$, *then for all* $E \subseteq \bigcup_{i\in I} \Phi_i$ *and* $H \subseteq \bigcup_{i\in I} \Delta_i$,
 (2.1) $\bigcup_{i\in I} R_{i,H\cap\Delta_i}(E \cap \Phi_i) \subseteq R_H(E)$; *and*
 (2.2) $R_H(E) = \bigcup_{i\in I} R_{i,H\cap\Delta_i}(E \cap \Phi_i)$ *if for any* $i, j \in I$, $\Phi_i \cap \Phi_j = \emptyset$ *whenever* $i \neq j$.

(3) *If* $\Pi_{GI}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$ *and* $\Pi_{GI}(\bigoplus_{i\in I} \Sigma_i) = (\bigcup_{i\in I} \Delta_i, \bigcup_{i\in I} \Phi_i, R)$, *then for all* $E \subseteq \bigcup_{i\in I} \Phi_i$ *and* $H \subseteq \bigcup_{i\in I} \Delta_i$,

$$\bigcup_{i\in I} R_{i,H\cap\Delta_i}(E \cap \Phi_i) \subseteq R_H(E)$$

provided $\Delta_i \cap \Delta_j = \emptyset$ *for all* $i, j \in I$ *with* $i \neq j$.

(4) *If* $\Pi_{SS}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$ *and* $\Pi_{SS}(\bigoplus_{i\in I} \Sigma_i) = (\bigcup_{i\in I} \Delta_i, \bigcup_{i\in I} \Phi_i, R)$, *then for all* $E \subseteq \bigcup_{i\in I} \Phi_i$ *and* $H \subseteq \bigcup_{i\in I} \Delta_i$,
 (4.1) $\bigcup_{i\in I} R_{i,H\cap\Delta_i}(E \cap \Phi_i) \subseteq R_H(E)$; *and*
 (4.2) $R_H(E) = \bigcup_{i\in I} R_{i,H\cap\Delta_i}(E \cap \Phi_i)$ *if* $\Delta_i \cap \Delta_j = \emptyset$ *for all* $i, j \in I$ *with* $i \neq j$.

(5) *If* $\Pi_{SO}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$ *and* $\Pi_{SO}(\bigoplus_{i\in I} \Sigma_i) = (\bigcup_{i\in I} \Delta_i, \bigcup_{i\in I} \Phi_i, R)$, *then for all* $E \subseteq \bigcup_{i\in I} \Phi_i$, *and for all* $H \subseteq \bigcup_{i\in I} \Delta_i$,
 (5.1) $R_H(E) \subseteq \bigcup_{i\in I} R_{i,H\cap\Delta_i}(E \cap \Phi_i)$; *and*
 (5.2) $R_H(E) = \bigcup_{i\in I} R_{i,H\cap\Delta_i}(E \cap \Phi_i)$ *if* $\Phi_i \cap \Phi_j = \emptyset$ *for all* $i, j \in I$ *with* $i \neq j$.

(6) *If* $\Pi_{SI}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$ *and* $\Pi_{SI}(\bigoplus_{i\in I} \Sigma_i) = (\bigcup_{i\in I} \Delta_i, \bigcup_{i\in I} \Phi_i, R)$, *then for all* $E \subseteq \bigcup_{i\in I} \Phi_i$ *and* $H \subseteq \bigcup_{i\in I} \Delta_i$,

$$R_H(E) \subseteq \bigcup_{i\in I} R_{i,H\cap\Delta_i}(E \cap \Phi_i)$$

whenever $\Delta_i \cap \Delta_j = \emptyset$ *for all* $i, j \in I$ *with* $i \neq j$.

Proof. We only prove (4). From Definition 21 it follows that

$$R_H(E) = \bigcap \left\{ H' \subseteq H\colon\ e(H') = \bigcup_{i\in I} e_i(H' \cap \Delta_i) \subseteq E \right\},$$

and

$$\begin{aligned}\bigcup_{i\in I} R_{i,H\cap\Delta_i}(E \cap \Phi_i) &= \bigcup_{i\in I} \cap \left\{ H_i' \subseteq H \cap \Delta_i\colon\ e_i(H_i') \subseteq E \cap \Phi_i \right\} \\ &= \bigcap \left\{ \bigcup_{i\in I} H_i'\colon\ H_i' \subseteq H \cap \Delta_i \text{ and } e_i(H_i') \subseteq E \cap \Phi_i \text{ for all } i \in I \right\}.\end{aligned}$$

Note that complete distributivity of set union over intersection is applied in the last equality.

Now the conclusion comes immediately from the following two items:

(a) If $H' \subseteq H$ and $e(H') \subseteq E$, then we set $H_i' = H \cap \Delta_i$ for each $i \in I$. It holds that $H' = \bigcup_{i\in I} H_i'$, $H_i' \subseteq H \cap \Delta_i$, and

$$e_i(H_i') = e_i(H_i') \cap \Phi_i \subseteq \bigcup_{j\in I} \left[e_j(H_j') \cap \Phi_i \right] = \left[\bigcup_{j\in I} e_j(H_j') \right] \cap \Phi_i \subseteq E \cap \Phi_i.$$

(b) Conversely, if for each $i \in I$, $H'_i \subseteq H \cap \Delta_i$ and $e_i(H'_i) \subseteq E \cap \Phi_i$, then we set $H' = \bigcup_{i\in I} H'_i$. It is clear that $H' \subseteq H$. Since $\Delta_i \cap \Delta_j = \emptyset$ whenever $i \neq j$, we have $H' \cap \Delta_i = H'_i$ and

$$e(H') = \bigcup_{i\in I} e_i(H' \cap \Delta_i) \subseteq \bigcup_{i\in I} (E_i \cap \Phi_i) = E. \qquad \square$$

As we saw before, the concept of diagnostic specification sum models the modularization technique of dividing a large system into a number of subsystems. Here, we are going to introduce the concept of direct product of diagnostic specifications. It can be used to describe the way that we observe various profiles of a system. The two operations of specification sum and product are orthogonal in a sense, and they complement each other.

Definition 57 (*Direct product of specifications*). Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ $(i \in I)$ be a family of diagnostic specifications. Then their direct product is defined to be $\prod_{i\in I} \Sigma_i = (\prod_{i\in I} \Delta_i, \prod_{i\in I} \Phi_i, e)$, where for each $D \subseteq \prod_{i\in I} \Delta_i$,

$$e(D) = \begin{cases} \prod_{i\in I} e_i(\mathit{proj}_i(D)) & \text{if } e_i(\mathit{proj}_i(D)) \neq \bot \text{ for all } i \in I, \\ \bot & \text{otherwise,} \end{cases}$$

and $\mathit{proj}_i : \prod_{j\in I} \Delta_j \to : \Delta_i$ is the projection on Δ_i; i.e., for any $d = (d_j)_{j\in I} \in \prod_{j\in I} \Delta_j$, $\mathit{proj}_i(d) = d_i$. Moreover, we define $\neg x = (\neg x_i)_{i\in I}$ for all $x = (x_i)_{i\in I} \in \prod_{i\in I} \Delta_i \cup \prod_{i\in I} \Pi_i$.

The notion of direct product is well-defined; i.e., $\prod_{i\in I} \Sigma_i$ is indeed a diagnostic specification. However, $\prod_{i\in I} \Sigma_i$ is not necessary to be complete when all Σ_i $(i \in I)$ are complete.

Example 58. The direct product of diagnostic specifications models the process that one extracts his knowledge bases from different profiles of an object and then aggregate them together into a single knowledge base. Suppose we have a system whose function is determined by two factors A and B. The causal interactions between defects and findings related to factors A and B are described by the diagnostic specifications $\Sigma_A = (\Delta_A, \Phi_A, e_A)$ and $\Sigma_B = (\Delta_B, \Phi_B, e_B)$, respectively, where $\Delta_A = \{d_1, d_2, d_3\}$, $\Phi_A = \{f_1, f_2\}$, $\Delta_B = \{c_1, c_2\}$, $\Phi_B = \{g_1, g_2, g_3\}$, and e_A and e_B are given by Tables 9 and 10. If the two factors A and B are assumed to be interaction free, then the direct product

$$\begin{aligned} \Sigma_A \times \Sigma_B = \big(&\big\{(d_1, c_1), (d_1, c_2), (d_2, c_1), (d_2, c_2), (d_3, c_1), (d_3, c_2)\big\}, \\ &\big\{(f_1, g_1), (f_1, g_2), (f_1, g_3), (f_2, g_1), (f_2, g_2), (f_2, g_3)\big\}, e\big) \end{aligned}$$

provides a diagnostic knowledge base for the whole system. For example,

$$\begin{aligned} e\big(\big\{(d_1, c_2), (d_2, c_1), (d_2, c_2)\big\}\big) &= e_A\big(\{d_1, d_2\}\big) \times e_B\big(\{c_1, c_2\}\big) \\ &= \big\{(f_1, g_1), (f_1, g_2), (f_1, g_3)\big\}, \end{aligned}$$

and the possible findings are (f_1, g_1), (f_1, g_2), (f_1, g_3) when (d_1, c_2), (d_2, c_1) and (d_2, c_2) occur simultaneously. The following proposition shows that the operation of direct product of diagnostic specifications preserves monotonicity.

Table 9
Factor system A

D	$\emptyset$	$\{d_1\}$	$\{d_2\}$	$\{d_3\}$	$\{d_1, d_2\}$	$\{d_1, d_3\}$	$\{d_2, d_3\}$	$\{d_1, d_2, d_3\}$
e_A	$\emptyset$	$\emptyset$	$\{f_1\}$	$\emptyset$	$\{f_1\}$	$\emptyset$	$\{f_1\}$	$\{f_1, f_2\}$

Table 10
Factor system B

D	$\emptyset$	$\{c_1\}$	$\{c_2\}$	$\{c_1, c_2\}$
e_B	$\emptyset$	$\{g_3\}$	$\emptyset$	$\{g_1, g_2, g_3\}$

Proposition 59. *Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ $(i \in I)$ be a family of diagnostic specifications. If Σ_i $(I \in i)$ are all increasing* (*respectively decreasing*), *then $\prod_{i\in I} \Sigma_i$ is also increasing* (*respectively decreasing*).

Proof. Immediate. $\square$

The next proposition demonstrates that the direct product of a family of specification morphisms is a specification morphism from the direct product of their domains to the direct product of their co-domains.

Proposition 60. *Let $M_i = (g_i, h_i) : \Sigma_i = (\Delta_i, \Phi_i, e_i) \to \Sigma'_i = (\Delta'_i, \Phi'_i, e'_i)$ $(i \in I)$ be a family of specification morphisms, and let $\prod_{i\in I} g_i : \prod_{i\in I} \Delta_i \to \prod_{i\in I} \Delta'_i$ and $\prod_{i\in I} h_i : \prod_{i\in I} \Phi_i \to \prod_{i\in I} \Phi'_i$ be defined as follows*:

$$\Big(\prod_{i\in I} g_i\Big)(d) = (g_i(d_i))_{i\in I} \quad \text{for any } d = (d_i)_{i\in I} \in \prod_{i\in I} \Delta_i, \quad \text{and}$$

$$\Big(\prod_{i\in I} h_i\Big)(f) = (h_i(f_i))_{i\in I} \quad \text{for any } f = (f_i)_{i\in I} \in \prod_{i\in I} \Phi_i.$$

Then $\prod_{i\in I} M_i = (\prod_{i\in I} g_i, \prod_{i\in I} h_i)$ is a specification morphism from $\prod_{i\in I} \Sigma_i$ to $\prod_{i\in I} \Sigma'_i$.

Proof. For any $D \subseteq \prod_{i\in I} \Delta_i$,

$$\begin{aligned}
\Big(\prod_{i\in I} h_i\Big)\big(e(D)\big) &= \Big(\prod_{i\in I} h_i\Big)\Big(\prod_{i\in I} e_i\big(proj_i(D)\big)\Big) = \prod_{i\in I} h_i\big(e_i\big(proj_i(D)\big)\big) \\
&= \prod_{i\in I} e'_i\big(g_i\big(proj_i(D)\big)\big) = \prod_{i\in I} e'_i\Big(proj_i\Big(\Big(\prod_{i\in I} g_i\Big)(D)\Big)\Big) \\
&= e'\Big(\Big(\prod_{i\in I} g_i\Big)(D)\Big). \qquad \square
\end{aligned}$$

We now turn to examine diagnostic problem solving in a direct product system. Given a diagnostic problem with the observed findings E and hypothesis H. We may observe them from different profiles $i \in I$. The pieces of information that E and H shine on the profile i are then represented by the projections $proj_i(E)$ and $proj_i(H)$, respectively. Now in the factor system Σ_i, we employ a notion R_i of diagnosis and find a diagnostic solution $R_{i,proj_i(H)}(proj_i(E))$ with information $proj_i(E)$ and $proj_i(H)$. Thus, we are able to present a diagnostic solution to the original problem by taking the direct product of these factor solutions. The following proposition clarify the relation between such a solution and the solution obtained directly by using a corresponding notion of diagnosis in the whole system (when possible). For example, Proposition 61(1) indicates that if we adopt the notion of most general subset diagnosis then the former is stricter than the latter, and they are the same whenever all factor systems are interaction free and E and H appear as cubes. Note that some conditions introduced in Definition 31 are needed here. We use condition ($\cap \subseteq \cup$) in the case of most specific subset diagnosis and ($\cap \supseteq \cap$) in the case of most specific superset diagnosis.

Proposition 61. *Let $\Sigma_i = (\Delta_i, \Phi_i, e_i)$ $(i \in I)$ be a family of diagnostic specifications.*

(1) *If $\Pi_{GS}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ for all $i \in I$, and $\Pi_{GS}(\prod_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \prod_{i\in I} \Phi_i, R)$, then for any $E \subseteq \prod_{i\in I} \Phi_i$ and for any $H \subseteq \prod_{i\in I} \Delta_i$,*
   (1.1) *$R_H(E) \subseteq \prod_{i\in I} R_{i,proj_i(H)}(proj_i(E))$ if for each $i \in I$, $e_i(D) \neq \emptyset$ whenever $D \neq \emptyset$;*
   (1.2) *$R_H(E) = \prod_{i\in I} R_{i,proj_i(H)}(proj_i(E))$ if all Σ_i $(i \in I)$ are interaction free, and $H = \prod_{i\in I} H_i$ and $E = \prod_{i\in I} E_i$ are cubes ($E_i \subseteq \Phi_i$ and $H_i \subseteq \Delta_i$ for each $i \in I$).*
(2) *If $\Pi_{GO}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ for all $i \in I$ and $\Pi_{GO}(\prod_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \prod_{i\in I} \Phi_i, R)$, then for any $E \subseteq \prod_{i\in I} \Phi_i$, and for any $H \subseteq \prod_{i\in I} \Delta_i$,*
   (2.1) *$R_H(E) \subseteq \prod_{i\in I} R_{i,proj_i(H)}(proj_i(E))$; and*
   (2.2) *$R_H(E) = \prod_{i\in I} R_{i,proj_i(H)}(proj_i(E))$ if all Σ_i $(i \in I)$ are interaction free, and $H = \prod_{i\in I} H_i$ is a cube ($H_i \subseteq \Delta_i$ for each $i \in I$).*
(3) *If $\Pi_{GI}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ for all $i \in I$, and $\Pi_{GI}(\prod_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \prod_{i\in I} \Phi_i, R)$, then for any cube $E = \prod_{i\in I} E_i \subseteq \prod_{i\in I} \Phi_i$ and for any cube $H = \prod_{i\in I} H_i \subseteq \prod_{i\in I} \Delta_i$,*

$$\prod_{i\in I} R_{i,H_i}(E_i) \subseteq R_H(E)$$

   provided all Σ_i $(i \in I)$ are interaction free.
(4) *Suppose that $\Pi_{SS}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$, and $\Pi_{SS}(\prod_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \prod_{i\in I} \Phi_i, R)$. If each Σ_i $(i \in I)$ satisfies the following condition:*

$$e_i\left(\bigcap_{t\in T} D_t\right) \subseteq \bigcup_{t\in T} e_i(D_t),$$

   where all $D_t \subseteq \Delta_i$ $(t \in T)$ are consistent, then for all cubes $E = \prod_{i\in I} E_i \subseteq \prod_{i\in I} \Phi_i$ and $H = \prod_{i\in I} H_i \subseteq \prod \Delta_i$, we have

$$R_H(E) \subseteq \prod_{i\in I} R_{i,H_i}(E_i).$$

(5) *Suppose that* $\Pi_{SO}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$ *and* $\Pi_{SO}(\prod_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \prod_{i\in I} \Phi_i, R)$. *If each* Σ_i $(i \in I)$ *satisfies the following condition*:

$$\bigcap_{t\in T} e_i(D_t) \subseteq e_i\left(\bigcap_{t\in T} D_t\right)$$

where all $D_t \subseteq \Delta_i$ $(t \in T)$ *are consistent, then for any* $E \subseteq \prod_{i\in I} \Phi_i$ *and for any cube* $H = \prod_{i\in I} H_i \subseteq \prod \Delta_i$, *it holds that*

$$R_H(E) \subseteq \prod_{i\in I} R_{i,H_i}(proj_i\, E).$$

(6) *If* $\Pi_{SI}(\Sigma_i) = (\Delta_i, \Phi_i, R_i)$ $(i \in I)$, *and* $\Pi_{SI}(\prod_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \prod_{i\in I} \Phi_i, R)$, *then for any cube* $E = \prod_{i\in I} E_i \subseteq \prod_{i\in I} \Phi_i$ *and for any cube* $H = \prod_{i\in I} H_i \subseteq \prod_{i\in I} \Delta_i$,

$$R_H(E) \subseteq \prod_{i\in I} R_{i,H_i}(E_i)$$

provided all Σ_i $(i \in I)$ *are interaction free.*

Proof. We only prove (1); others are similar.

(1.1) We write

$$M = \left\{H' \subseteq H\colon\ e(H') = \prod_{i\in I} e_i\big(proj_i(H')\big) \subseteq E\right\}$$

and

$$N = \left\{\prod_{i\in I} H_i'\colon\ H_i' \subseteq proj_i(H) \text{ and } e_i(H_i') \subseteq proj_i(E) \text{ for all } i \in I\right\}.$$

Then it is clear that $R_H(E) = \bigcup M$, and

$$\prod_{i\in I} R_{i,proj_i(H)}\big(proj_i(E)\big) = \prod_{i\in I} \cup\big\{H_i' \subseteq proj_i(H)\colon\ e_i(H_i') \subseteq proj_i(E)\big\} \supseteq \bigcup N.$$

Now it suffices to show that for any $H' \in M$, there exists $K' \in N$ with $H' \subseteq K'$. If $H' = \emptyset$, it is obvious. We now assume that $H' \neq \emptyset$. It holds that $proj_i(H') \subseteq proj_i(H)$. In addition, $e_i(proj_i(H')) \subseteq proj_i(E)$ follows from that $e(H') = \prod_{i\in I} e_i(proj_i(H')) \subseteq E$ and $e_j(proj_j(H')) \neq \emptyset$ for each $j \in I - \{i\}$. Then $H' \subseteq \prod_{i\in I} proj_i(H') \in N$.

(1.2) Let $E = \prod_{i\in I} E_i$ and $H = \prod_{i\in I} H_i$ be cubes. Then $proj_i(E) = E_i$ and $proj_i(H) = H_i$. We put

$$N_i = \big\{H' \subseteq H_i\colon\ e_i(H_i') \subseteq E_i\big\}.$$

Then $e_i(\bigcup N_i) \subseteq \bigcup_{H_i'\in N_i} e_i(H_i') \subseteq E_i$, and $\bigcup N_i \in N_i$. This yields $\prod_{i\in I} R_{iH_i}(E_i) = \bigcup N$. Furthermore, it is easy to see that $N \subseteq M$ for cubes E and H. This completes the proof. □

Note that in a direct product diagnostic system both the defects and findings are examined from different profiles. However, sometimes we only need to analyze the defects from different angles, leaving the findings unchanged. This motivates the following definition.

Definition 62 (*Optimistic and pessimistic merging of specifications*). Let $\Sigma_i = (\Delta_i, \Phi, e_i)$ $(i \in I)$ be a family of diagnostic specifications with the same set of findings. Then their optimistic and pessimistic merging are defined to be $\bigotimes_{i\in I} \Sigma_i = (\prod_{i\in I} \Delta_i, \Phi, e^+)$ and $\bigodot_{i\in I} \Sigma_i = (\prod_{i\in I} \Delta_i, \Phi, e^-)$, respectively, where for each $D \subseteq \prod_{i\in I} \Delta_i$,

$$e^+(D) = \begin{cases} \bigcup_{i\in I} e_i(\mathit{proj}_i(D)) & \text{if } e_i(\mathit{proj}_i(D)) \neq \bot \text{ for all } i \in I, \\ \bot & \text{otherwise,} \end{cases}$$

$$e^-(D) = \begin{cases} \bigcap_{i\in I} e_i(\mathit{proj}_i(D)) & \text{if } e_i(\mathit{proj}_i(D)) \neq \bot \text{ for all } i \in I, \\ \bot & \text{otherwise.} \end{cases}$$

Comparing the above definition with Definitions 44 and 57, we will find that (optimistic and pessimistic) mergings are mixtures of direct product and (optimistic and pessimistic) fusions. In the operation of merging, the defects take a structure of direct product. This reflects the fact that we will examine defects from different profiles. For each profile, the causal relation between defects and findings is specified separately. Now we have many different way to aggregate these specifications coming from various profiles. The ways used in optimistic and pessimistic mergings are respectively union and intersection which are just at the two extremes of all the possible ways. Obviously, this idea follows directly Definition 44.

Example 63. The two operations of merging express two extreme ways of aggregating knowledge about causal interactions between findings and different profiles of defects. Suppose that $\Sigma_A = (\Delta_A, \Phi, e_A)$ and $\Sigma_B = (\Delta_B, \Phi, e_B)$, where $\Delta_A = \{d_1, d_2, d_3\}$, $\Delta_B = \{c_1, c_2\}$, $\Phi = \{f_1, f_2, f_3\}$ and e_A and e_B are given as shown in Tables 11 and 12. And, we assume that Σ_A and Σ_B depict respectively the causal relation between findings and the profile A of defects and the profile B of defects. Then $\Sigma_A \otimes \Sigma_B = (\Delta_A \times \Delta_B, \Phi, e^+)$ and $\Sigma_A \odot \Sigma_B = (\Delta_A \times \Delta_B, \Phi, e^-)$ represent two different aggregation of Σ_A and Σ_B, and they are able to serve as a diagnostic knowledge base of the whole system, where

$$\Delta_A \times \Delta_B = \big\{(d_1, c_1), (d_1, c_2), (d_2, c_1), (d_2, c_2), (d_3, c_1), (d_3, c_2)\big\}.$$

For example, we have

$$e^+\big(\big\{(d_1, c_1), (d_1, c_2), (d_3, c_2)\big\}\big) = e_A\big(\{d_1, d_3\}\big) \cup e_B\big(\{c_1, c_2\}\big) = \{f_1, f_2\},$$

Table 11
Profile *A*

D	$\emptyset$	$\{d_1\}$	$\{d_2\}$	$\{d_3\}$	$\{d_1, d_2\}$	$\{d_1, d_3\}$	$\{d_2, d_3\}$	$\{d_1, d_2, d_3\}$
e_A	$\emptyset$	$\{f_1\}$	$\{f_1\}$	$\emptyset$	$\{f_1\}$	$\{f_1, f_2\}$	$\{f_1\}$	$\{f_1, f_2, f_3\}$

Table 12
Profile *B*

D	$\emptyset$	$\{c_1\}$	$\{c_2\}$	$\{c_1, c_2\}$
e_B	$\emptyset$	$\{f_2\}$	$\{f_1\}$	$\emptyset$

and

$$e^{-}\big(\big\{(d_1, c_1), (d_1, c_2), (d_3, c_2)\big\}\big) = e_A\big(\{d_1, d_3\}\big) \cap e_B\big(\{c_1, c_2\}\big) = \emptyset.$$

Monotonicity of diagnostic specifications is preserved by both optimistic or pessimistic mergings, but only optimistic merging carries interaction freeness.

Proposition 64. *Let $\Sigma_i = (\Delta_i, \Phi, e_i)$ $(i \in I)$ be a family of diagnostic specifications with the same set of findings.*

(1) *If all Σ_i $(i \in I)$ are increasing (respectively decreasing), then $\bigotimes_{i\in I} \Sigma_i$ and $\bigodot_{i\in I} \Sigma_i$ are also increasing (respectively decreasing).*
(2) *If all Σ_i $(i \in I)$ are all interaction free, so is $\bigotimes_{i\in I} \Sigma_i$.*

Proof. Immediate. □

We consider the problem of constructing a specification morphism between two optimistic (pessimistic) mergings from the morphisms between their component systems. Note that the operands in an optimistic merging are a family of diagnostic specifications with the same set of findings. Thus, the finding mappings between the mergings and between the corresponding components should be same. The following proposition shows that the direct product of defect mapping between the components together with the fixed finding mapping forms a specification morphism between the two optimistic mergings under consideration, and it is a morphism between the two pessimistic mergings whenever the fixed finding mapping is injective.

Proposition 65. *Let $M_i = (g_i, h): \Sigma_i = (\Delta_i, \Phi, e_i) \to \Sigma_i' = (\Delta_i', \Phi', e_i')$ $(i \in I)$ be a family of specification morphisms. Then*

(1) *$\prod_{i\in I} M_i = (\prod_{i\in I} g_i, h)$ is a specification morphism from $\bigotimes_{i\in I} \Sigma_i$ to $\bigotimes_{i\in I} \Sigma_i'$; and*
(2) *$\prod_{i\in I} M_i = (\prod_{i\in I} g_i, h)$ is a specification morphism from $\bigodot_{i\in I} \Sigma_i$ to $\bigodot_{i\in I} \Sigma_i'$ provided h is injective.*

Proof. Straightforward. □

We conclude this section with two propositions concerning the notions of diagnosis that form the Lucas refinement diagnosis in an optimistic or pessimistic merging system. Given a diagnostic problem in an optimistic merging with observed findings E and hypothesis H, since the finding set is fixed, and the defect set possesses a structure of direct product, we can analyze the hypothesis H from its different profiles. These profiles of H are then represented by $proj_i(H)$, where i is the index in the merging construction. For each component system i, a diagnostic problem with the original observed findings E is still present, but the new hypothesis $proj_i(H)$ is given, and we are able to find a solution to it by employing a notion of diagnosis in this subsystem. Now what interests us is the relation between the diagnostic solution that we hope to find directly in the whole system and the family of diagnostic solutions in these subsystems. For most general subset diagnosis, it is shown that the former is included in the direct product of the latter. The same result holds for most specific subset diagnosis, most specific superset diagnosis and most specific intersection diagnosis if certain global properties of specifications are imposed. On the other hand, the direct product of solutions to the subproblems is included in the solution of the original problem for most general superset diagnosis and most general intersection diagnosis with some global properties of specifications.

Proposition 66. *Let $\Sigma_i = (\Delta_i, \Phi, e_i)$ $(i \in I)$ be a family of diagnostic specifications with the same set of findings.*

(1) *If $\Pi_{GS}(\Sigma_i) = (\Delta_i, \Phi, R_i)$ $(i \in I)$, and $\Pi_{GS}(\bigotimes_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \Phi, R)$, then for any $E \subseteq \Phi$ and for any $H \subseteq \prod_{i\in I} \Delta_i$,*
   (1.1) *$R_H(E) \subseteq \prod_{i\in I} R_{i,proj_i(H)}(E)$; and*
   (1.2) *$R_H(E) = \prod_{i\in I} R_{i,proj_i(H)}(E)$ if H is a cube and all Σ_i $(i \in I)$ are interaction free.*

(2) *If $\Pi_{GO}(\Sigma_i) = (\Delta_i, \Phi, R_i)$ $(i \in I)$, $\Pi_{GO}(\bigotimes_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \Phi, R)$, and all Σ_i $(i \in I)$ fulfil the following condition*:

$$e_i\left(\bigcup_{t\in T} D_t\right) \supseteq \bigcap_{t\in T} e_i(D_t)$$

for all $D_t \subseteq \Delta_t$ $(t \in T)$, then for each $E \subseteq \Phi$ and for each cube $H \subseteq \prod_{i\in I} \Delta_i$,

$$\prod_{i\in I} R_{i,proj_i(H)}(E) \subseteq R_H(E).$$

(3) *If $\Pi_{GI}(\Sigma_i) = (\Delta_i, \Phi, R_i)$ $(i \in I)$, $\Pi_{GI}(\bigotimes_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \Phi, R)$, and all Σ_i $(i \in I)$ are interaction free, then for any $E \subseteq \Phi$ and for any cube $H \subseteq \prod_{i\in I} \Delta_i$,*

$$\prod_{i\in I} R_{i,proj_i(H)}(E) \subseteq R_H(E).$$

(4) *If $\Pi_{SS}(\Sigma_i) = (\Delta_i, \Phi, R_i)$ $(i \in I)$, $\Pi_{SS}(\bigotimes_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \Phi, R)$, and all Σ_i $(i \in I)$ satisfy the condition in Proposition* 61(4), *then for all $E \subseteq \Phi$ and cubes $H \subseteq \prod_{i\in I} \Delta_i$,*

$$R_H(E) \subseteq \prod_{i\in I} R_{i,proj_i(H)}(E).$$

(5) *If $\Pi_{SO}(\Sigma_i) = (\Delta_i, \Phi, R_i)$ $(i \in I)$, $\Pi_{SO}(\bigotimes_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \Phi, R)$, and all Σ_i $(i \in I)$ fulfil the condition in Proposition* 61(5), *then for any $E \subseteq \Phi$ and cube $H \subseteq \prod_{i\in I} \Delta_i$,*

$$R_H(E) \subseteq \prod_{i\in I} R_{i,proj_i(H)}(E).$$

(6) *If $\Pi_{SI}(\Sigma_i) = (\Delta_i, \Phi, R_i)$ $(i \in I)$, $\Pi_{SI}(\bigotimes_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \Phi, R)$, and all Σ_i $(i \in I)$ are interaction free, then for any $E \subseteq \Phi$ and cube $H \subseteq \prod_{i\in I} \Delta_i$,*

$$R_H(E) \subseteq \prod_{i\in I} R_{i,proj_i(H)}(E).$$

Proof. Similar to Proposition 61. □

For diagnosis in a pessimistic merging system, we have:

Proposition 67. *Let $\Sigma_i = (\Delta_i, \Phi, e_i)$ $(i \in I)$ be a family of diagnostic specifications with the same set of findings.*

(1) *If $\Pi_{GS}(\Sigma_i) = (\Delta_i, \Phi, R_i)$ $(i \in I)$, and $\Pi_{GS}(\bigodot_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \Phi, R)$, then for any $E \subseteq \Phi$ and for any $H \subseteq \prod_{i\in I} \Delta_i$,*

$$\prod_{i\in I} R_{i,proj_i(H)}(E) \subseteq R_H(E).$$

(2) *If $\Pi_{GO}(\Sigma_i) = (\Delta_i, \Phi, R_i)$ $(i \in I)$, and $\Pi_{GO}(\bigodot_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \Phi, R)$, then for any $E \subseteq \Phi$ and $H \subseteq \prod_{i\in I} \Delta_i$,*

$$R_H(E) \subseteq \prod_{i\in I} R_{i,proj_i(H)}(E),$$

with the equality when H is a cube.

(3) *If $\Pi_{SS}(\Sigma_i) = (\Delta_i, \Phi, R_i)$ $(i \in I)$, and $\Pi_{SS}(\bigodot_{i\in I} \Sigma_i) = (\prod_{i\in I} \Delta_i, \Phi, R)$, then for any $E \subseteq \Phi$ and for any cube $H \subseteq \prod_{i\in I} \Delta_i$,*

$$R_H(E) \subseteq \prod_{i\in I} R_{i,proj_i(H)}(E).$$

The same conclusion holds for most specific superset diagnosis.

Proof. Similar to Proposition 61. □

5. Conclusion

Lucas [13] proposed a set-theoretic framework for diagnostic problem solving in which the knowledge base in a diagnostic system is represented by a diagnostic specification. A diagnostic specification is defined to be a mapping from defects to observable findings, and it establishes a causal relation between defects and findings. The solution to a diagnostic problem is then given by a notion of diagnosis which maps observed findings to defects. A refinement diagnosis consisting of six notions of diagnosis were introduced and carefully analyzed by Lucas in [13].

This paper is a continuation of Lucas [13], and its main aim is to examine the influence of diagnostic specification transformation on diagnostic strategies and to provide some useful mathematical tools supporting knowledge reuse in diagnostic systems. The concept of diagnostic specification morphism is introduced in order to describe diagnostic specification transformation. The diagnostic strategies, including the six in the Lucas refinement diagnosis, in the source and target systems of a specification morphism are compared. At the same time, we propose several operations of diagnostic specifications that can serve as mathematical models of knowledge base fusion and merging in diagnostic systems. Some representations of diagnostic methods in composite diagnostic systems constructed by using our proposed operations are presented in terms of the corresponding diagnostic methods in their subsystems.

It is obvious that the diagnostic systems dealt with here and in [13] are not time-varying, and the dimension of time is ignored. In many application domains, however, the systems to be diagnosed are dynamic, and the assumption that the relation between defects and observations does not depend on time factor is not realistic. In fact, much effort has been made to accommodate the dimension of time into diagnostic systems have been made in the previous researches. Brusoni et al. [2] defined a spectrum of notions of temporal model-based diagnosis. As a problem for further studies, we hope to generalize the Lucas framework of diagnosis so that certain temporal phenomena can be taken into account. Furthermore, we need to explore the possibility of adding the time factor into the results obtained in this paper in order to support knowledge reuse in time-varying diagnostic systems.

Uncertainty management is another important problem in diagnostic systems. This is especially clear in the field of medical diagnosis due to inherent vagueness of human doctor thinking. Indeed, uncertainty has been considered in many early medical diagnostic expert systems such as MYCIN [21]. An important and much more recent work incorporating uncertainty into diagnosis systems was done by Poole [17] using probabilistic Horn clauses to represent diagnosis knowledge with uncertainty; notions of diagnosis in his setting were defined by assuming different probabilistic constraints. The Lucas theory of diagnosis allows modelling particular qualitative approaches to diagnosis, such as those expressed by strong causality or weak causality, but it does not provide us with an explicit mechanism for expressing uncertainty. Nowadays, a dominant method of representing uncertainty in Artificial Intelligence is given by Bayesian networks, or, more generally, by probability theory. Thus, an interesting problem for further studies is how to introduce a suitable mechanism for coping with uncertainty and vagueness into the Lucas framework of diagnosis and how to model knowledge transformation and reuse in diagnostic problem solving when uncertainty involved. In particular, what highly concerns us is how to accommodate probabilistic information in the formal development presented in this paper.

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