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On the logic of iterated belief revision

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Abstract

We show in this paper that the AGM postulates are too weak to ensure the rational preservation of conditional beliefs during belief revision, thus permitting improper responses to sequences of observations. We remedy this weakness by proposing four additional postulates, which are sound relative to a qualitative version of probabilistic conditioning. Contrary to the AGM framework, the proposed postulates characterize belief revision as a process which may depend on elements of an epistemic state that are not necessarily captured by a belief set. We also show that a simple modification to the AGM framework can allow belief revision to be a function of epistemic states. We establish a model-based representation theorem which characterizes the proposed postulates and constrains, in turn, the way in which entrenchment orderings may be transformed under iterated belief revision.

Keywords: Iterated revision; AGM postulates; Conditional beliefs; Probabilistic conditioning; Epistemic states; Qualitative probability

1. Introduction

The process of belief change has been formalized in several frameworks, most notably nonmonotonic logic, probabilistic reasoning, and belief revision. In nonmonotonic logic (e.g., [21]), belief change is viewed as a byproduct of extending a database containing new facts in accordance with a set of extension-construction rules called "defaults". In probabilistic reasoning (e.g., [10,13,23,24]), belief change is viewed as a byproduct of conditioning a probability function (or some qualitative abstraction thereof) on

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new evidence, in accordance with Bayes' rule. In the belief revision framework, belief changes are characterized by a set of constraints (called "postulates") on an *operator* $\circ$ which modifies the set ψ of currently held beliefs to produce a new belief set $\psi \circ \mu$, implying the new information μ.

While many studies have emphasized features that are common to the three frameworks above (e.g., [8, 9, 14, 22]), serious incompatibilities have also been observed that point to some fundamental limitations and inadequacies of the operator-based approach to belief revision [3–5, 13]. This paper addresses one such limitation, the failure of the standard belief revision framework, as encapsulated in the AGM postulates [1], to properly regulate *iterated* belief revision, that is, the sequential revision of beliefs in response to a string of observations.

We will first demonstrate that the AGM postulates, as they currently stand, are too permissive to enforce plausible iterated revisions, and need to be strengthened by additional constraints. We will then argue that any rational system of belief change should comply with four postulates which are not part of the AGM lexicon, and which are necessary to maintain plausible behavior under iterated belief change. Finally, we will show that one of these postulates stands contrary to the basic tenet of the operator-based framework and, hence, the framework should be broadened to permit operations on *epistemic states*, rather than *belief sets*.

To understand the requirements imposed by iterated revision we should start by recalling the distinction between belief sets and epistemic states. A belief set ψ characterizes the set of propositions to which an agent is committed at any given time. An epistemic state contains, in addition to ψ, the entire information needed for coherent reasoning, including, in particular, the very strategy of belief revision which the agent wishes to employ at that given time. Any such strategy encodes, and is equivalent to, a set of "conditional beliefs", that is, beliefs that one is prepared to adopt conditioned on any hypothetical evidence [2, 3, 15, 16, 19, 20]. To fully specify behavior under successive observations, one must encode, not merely how beliefs are to be revised (this is enough for the first stage only) but also how the revision strategy itself is to be modified by each new evidence. This amounts, in turns, to specifying which conditional beliefs are to be retained and which ones deleted with each piece of evidence.

The hallmark of the AGM postulates is the principle of minimal belief change, that is, the need to preserve as much of earlier beliefs as possible and to add only those beliefs that are absolutely compelled by the revision specified. But despite this emphasis on preserving propositional beliefs, the AGM postulates place almost no constraints on the preservation of *conditional beliefs*. The reason is that the AGM theory is expressed mainly in terms of one-step postulates which tell us what properties the next belief set ought to have, given the current belief set and the current evidence. However, the language of one-step postulates is not rich enough to regulate conditional beliefs because such a language deals only with transformation of belief sets and not with transformation of revision policies as encoded in epistemic states.

In fact, a central result of the AGM theory states that the postulates are equivalent to the existence of a total pre-order on all propositions according to their degree of *epistemic entrenchment* such that belief revisions always retain more entrenched propositions in preference to less entrenched ones. But this ordering, which carries the information necessary for belief revision, cannot be always constrained using the language of one-step postulates, hence, the postulates cannot always regulate how the ordering transforms during belief revision.

Since the relative entrenchment among hypothetical beliefs is crucial for distinguishing future beliefs from future disbeliefs, the preservation of this relative entrenchment in accordance with some minimal-change principle is as important as the preservation of beliefs themselves. Moreover, since the information content of this relative entrenchment is equivalent to that of conditional beliefs, the preservation of the former requires postulates about the latter, namely, two-step postulates about the revision of conditional beliefs.

The over-permissiveness of the AGM postulates relative to changes in conditional beliefs has been noted by several workers [3, 19], including the AGM authors themselves [2, 8], but attempts at applying preservation principles to conditional beliefs have not been very successful. Gärdenfors, for example, has tried the sweeping remedy of including in the belief set not merely propositional beliefs but conditional beliefs as well, and quickly faltered into an inconsistency known as Gärdenfors' triviality result [8, pp. 156–166]. Attempts at circumventing this result now make up voluminous literature which, by and large, seems still reluctant to accept the fact that conditional and propositional beliefs are two different species which require totally different preservation policies. More recently, Boutilier has suggested a promising approach by devising a belief revision operator, called *natural revision*, which still restricts a belief set to propositional beliefs, but provably preserves as many conditional beliefs as the AGM postulates would permit [3]. We show in this paper, however, that this strategy, too, is an excessive remedy to the AGM weakness and leads to counterintuitive results. As it turns out, if one insists on preserving *all* conditional beliefs permitted by AGM, then one is forced to retract some propositional beliefs that ought to be preserved.

The solution we suggest for preserving conditional beliefs is more cautious. Viewing belief revision as an operation on epistemic states, we show that conditional beliefs can be classified succinctly into two distinct categories; those that may compromise propositional beliefs if preserved, and those that may not. We then insist on preserving only the second category of conditional beliefs, and we do this proposing four postulates. [2]

The rest of the paper is structured as follows. In Section 2 we review the AGM proposal and present a number of scenarios that are consistent with the AGM postulates and yet exhibit counterintuitive changes in conditional beliefs. Next, we propose a modification of the AGM postulates in which revisions are applied to epistemic states instead of belief sets and argue that such modification is necessary for a satisfactory treatment of iterated belief revision. We then analyze in Section 4 the minimal-change principle of conditional beliefs. Based on this analysis, we propose four postulates in Section 5 that properly preserve conditional beliefs—hence, regulating iter-

[2] The postulates we propose are inspired by a method for belief change suggested by Spohn [23, 24] and extended by Goldszmidt [11, 12].

ated revisions—and provide a representation theorem for the newly proposed postulates which extends the one provided by Katsuno and Mendelzon for the AGM postulates [17]. We then show in Section 6 that the new postulates are sound with respect to a qualitative version of Jeffrey's rule of probabilistic conditioning. In Section 7, we provide further insights behind the choice of our postulates and conclude in Section 8 by discussing current and future related work. Proofs of theorems are delegated to Appendix B.

2. Belief revision

Belief revision is the process of changing a belief set to accommodate evidence that is possibly inconsistent with existing beliefs. Alchourrón, Gärdenfors, and Makinson have proposed eight postulates to govern the process of belief revision, which are phrased in a very general setting and are known as the AGM postulates [1,8]. Katsuno and Mendelzon (KM) rephrased these postulates assuming a propositional logic setting, yielding postulates (R1)–(R6) that are shown below [17].

According to the KM formulation, a belief set is represented by a sentence ψ in a propositional language $\mathcal{L}$, where any sentence that is entailed by ψ is part of the belief set. Evidence is also represented using a sentence μ in language $\mathcal{L}$. The result of revising ψ with μ is a sentence belonging to $\mathcal{L}$ that is denoted by $\psi \circ \mu$, where $\circ$ is called a belief revision operator. The KM formulation of the AGM postulates follows:

(R1) $\psi \circ \mu$ implies μ.
(R2) If $\psi \wedge \mu$ is satisfiable, then $\psi \circ \mu \equiv \psi \wedge \mu$.
(R3) If μ is satisfiable, then $\psi \circ \mu$ is also satisfiable.
(R4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 \circ \mu_1 \equiv \psi_2 \circ \mu_2$.
(R5) $(\psi \circ \mu) \wedge \phi$ implies $\psi \circ (\mu \wedge \phi)$.
(R6) If $(\psi \circ \mu) \wedge \phi$ is satisfiable, then $\psi \circ (\mu \wedge \phi)$ implies $(\psi \circ \mu) \wedge \phi$.

Katsuno and Mendelzon provided a representation theorem for postulates (R1)–(R6) which shows an equivalence between the postulates and a revision mechanism based on total pre-orders [17]:

Definition 1. Let W be the set of all worlds (interpretations) of a propositional language $\mathcal{L}$. A function that maps each sentence ψ in $\mathcal{L}$ to a total pre-order $\leqslant_\psi$ on worlds W is called a *faithful assignment* if and only if:

(1) $\omega_1, \omega_2 \models \psi$ only if $\omega_1 =_\psi \omega_2$;
(2) $\omega_1 \models \psi$ and $\omega_2 \not\models \psi$ only if $\omega_1 <_\psi \omega_2$; and
(3) $\psi \equiv \phi$ only if $\leqslant_\psi = \leqslant_\phi$.

Here, $\omega_1 <_\psi \omega_2$ is defined as $\omega_1 \leqslant_\psi \omega_2$ and $\omega_2 \not\leqslant_\psi \omega_1$; $\omega_1 =_\psi \omega_2$ is defined as $\omega_1 \leqslant_\psi \omega_2$ and $\omega_2 \leqslant_\psi \omega_1$.

The following representation theorem shows that a revision operator is equivalent to a faithful assignment where the result of a revision $\psi \circ \mu$ is determined by μ and the total pre-order assigned to ψ:

Theorem 2 (Katsuno and Mendelzon [17]). *A revision operator $\circ$ satisfies postulates* (R1)–(R6) *precisely when there exists a faithful assignment that maps each sentence ψ into a total pre-order $\leqslant_\psi$ such that*

$$Mods(\psi \circ \mu) = \min(Mods(\mu), \leqslant_\psi).$$

Here, $Mods(\mu)$ is the set of all worlds satisfying μ; and $\min(Mods(\mu), \leqslant_\psi)$ contains all worlds that are minimal in $Mods(\mu)$ according to the total pre-order $\leqslant_\psi$.

In the remainder of this section, we shall consider a number of revision operators that conform to the AGM postulates but lead to counterintuitive changes in conditional beliefs.

Definition 3. A belief set ψ *accepts* proposition β *given* proposition α precisely when β is entailed by the revision of ψ with α; that is, $\psi \circ \alpha \models \beta$. We also say in this case that $\beta \mid \alpha$ is a conditional belief of ψ. [3]

Our first scenario shows that an agent consistent with the AGM postulates may give up a conditional belief unjustifiably.

Example 4. We see a strange new animal X at a distance, and it appears to be barking like a dog, so we conclude that X is not a bird, and that X does not fly. Still, in the event that X turns out to be a bird, we are prepared to change our mind and conclude that X flies. Observing the animal closely, we realize that it actually can fly. The question now is whether we should retain our willingness to believe that X flies in case X turns out to be a bird after all. We submit that it would be strange to give up this conditional belief merely because we happened to observe that X can fly. Yet, we provide later an AGM-compatible revision operator $\circ$ that permits such behavior: [4]

$$\begin{aligned}
&\psi \equiv \neg bird \wedge \neg flies,\\
&\psi \circ bird \equiv bird \wedge flies,\\
&(\psi \circ flies) \circ bird \equiv bird.
\end{aligned}$$

The example we just considered involves an agent that gives up a conditional belief unjustifiably, while remaining consistent with the AGM postulates. Our next example shows that an agent consistent with the AGM postulates may acquire a conditional belief unjustifiably.

[3] This definition should not be viewed as a position on how to interpret "conditionals". The phrase "conditional belief $\beta \mid \alpha$" is simply a shortcut for the more elaborate statement "β will be accepted after revising our current beliefs by α". All of our discussion below can be made free of the term "conditional belief" if we opt to, except that it will generate sentences that are not easy to parse. We stress this point since traditional problems associated with the treatment of conditionals are mostly irrelevant to our current topic.

[4] We are using the same revision operator to accommodate different pieces of evidence in this and further examples. Some may argue, however, that the AGM theory does not sanction any form of iterated revisions, or, more specifically, that it does not propose using the same revision operator for handling iterated revisions. Our examples, however, are applicable even if one uses different revision operators to accommodate different pieces of evidence.

Example 5. We are introduced to a lady X who sounds smart and looks rich, so we believe that X is smart and X is rich. Moreover, since we profess to no prejudice, we also maintain that X is smart even if found to be poor and, conversely, X is rich even if found to be not smart. Now, we obtain some evidence that X is in fact not smart and we remain of course convinced that X is rich. Still, it would be strange for us to say, "If the evidence turns out false, and X turns out smart after all, we would no longer believe that X is rich". If we currently believe X is smart and rich, then evidence first refuting then supporting that X is smart should not in any way change our opinion about X being rich. Strangely, the AGM postulates do permit such a change of opinion. We will provide later an AGM-compatible revision operator $\circ$ such that

$$\begin{aligned}
&\psi \equiv smart \wedge rich,\\
&\psi \circ \neg rich \equiv smart \wedge \neg rich,\\
&\psi \circ \neg smart \equiv \neg smart \wedge rich,\\
&(\psi \circ \neg smart) \circ smart \equiv smart \wedge \neg rich.
\end{aligned}$$

The common feature permitting us to construct these examples is that while the AGM postulates constrain what revisions are permissible from a given belief set ψ, under different propositions μ, they, in general, do not constrain how revisions must cohere when starting from different belief states. This is seen more clearly from Theorem 2, where the order $\leqslant_\psi$ does not constrain the order $\leqslant_{\psi\circ\mu}$ except trivially.[5]

3. Epistemic states versus belief sets

The examples we presented in the previous section show that the AGM postulates are too weak to regulate changes in conditional beliefs. Our solution to this problem is given in Section 5 where we augment these postulates with four additional ones that regulate such change. The choice of proposed postulates is not arbitrary, however. It is motivated by a careful analysis of such regulation which we conduct in Section 4. In this section, we present a modification of postulates (R1)–(R6), which we argue is necessary for turning the operator $\circ$ into a genuine belief revision operator, worthy of the expectation that such a title evokes. The modified set of postulates will be the basis of our treatment of iterated revisions.

The modification we shall suggest to postulates (R1)–(R6) is a weakening of postulate (R4), which makes belief revision a function of an epistemic state instead of a belief set.[6] Each epistemic state Ψ has an associated belief set, denoted $Bel(\Psi)$, which is a propositional sentence. The belief set of Ψ does not characterize Ψ uniquely; therefore, it is possible to have two different epistemic states with equivalent belief sets.

[5] The orders $\leqslant_\psi$ and $\leqslant_{\psi\circ\mu}$ are constrained by the properties of faithful assignments.

[6] A similar modification has independently been proposed in [7].

The modification of postulates (R1)–(R6) leads to postulates (R*1)–(R*6) which are shown below. To simplify notation, we are adopting the following convention in the rest of the paper: We use Ψ instead of $Bel(\Psi)$ whenever it is embedded in a propositional formula. For example, we will write $\Psi \models \alpha$ to mean $Bel(\Psi) \models \alpha$; $\Psi \wedge \phi$ to mean $Bel(\Psi) \wedge \phi$; $\Psi \equiv \Phi$ to mean $Bel(\Psi) \equiv Bel(\Phi)$; and so on. However, $\Psi \circ \mu$ will stand for the epistemic state, not belief set, that results from the revision by μ. With this notation at hand, the modified AGM postulates are:

(R*1) $\Psi \circ \mu$ implies μ.
(R*2) If $\Psi \wedge \mu$ is satisfiable, then $\Psi \circ \mu \equiv \Psi \wedge \mu$.
(R*3) If μ is satisfiable, then $\Psi \circ \mu$ is also satisfiable.
(R*4) If $\Psi_1 = \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Psi_1 \circ \mu_1 \equiv \Psi_2 \circ \mu_2$.
(R*5) $(\Psi \circ \mu) \wedge \phi$ implies $\Psi \circ (\mu \wedge \phi)$.
(R*6) If $(\Psi \circ \mu) \wedge \phi$ is satisfiable, then $\Psi \circ (\mu \wedge \phi)$ implies $(\Psi \circ \mu) \wedge \phi$.

There are only two differences between these postulates and (R1)–(R6). First, a revision is applied to an epistemic state Ψ instead of a belief set ψ. Second, postulate (R*4) is a weakening of postulate (R4), which, in our notation, reads:

(R4) If $\Psi_1 \equiv \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Psi_1 \circ \mu_1 \equiv \Psi_2 \circ \mu_2$.

Postulate (R4) says that if epistemic states Ψ_1 and Ψ_2 have equivalent belief sets ($\Psi_1 \equiv \Psi_2$), then they must lead to equivalent belief sets when revised using equivalent evidence. Postulate (R*4), in contrast, is more cautious; it requires the epistemic states to be identical ($\Psi_1 = \Psi_2$) for this to be the case.

Having broadened the AGM framework to operate on epistemic states, we also broaden Definition 3 accordingly.

Definition 6. An epistemic state Ψ *accepts* proposition β *given* proposition α precisely when β is entailed by the revision of Ψ with α; that is, $\Psi \circ \alpha \models \beta$. We also say in this case that $\beta | \alpha$ is a conditional belief of Ψ, written $\Psi \models \beta | \alpha$.

In Section 5 we will strengthen this new framework with additional postulates, so as to properly regulate iterated belief revision. But, first, we offer further rationale for weakening (R4) into (R*4).

While several researchers have recognized the need to formulate revision at the epistemic state level [4,7,15,16,19,22], the specific modification of the AGM postulates in the manner proposed above was inspired by recent studies of Freund and Lehmann who have effectively shown that (R1)–(R6) clash with one of the postulates, called (C2), that we propose later [6]. It turns out (Section 6), however, that postulate (R4) alone is the culprit for the clash. Thus the problem arises whether one should retain postulate (R4) and weaken (C2) or the other way around, weaken (R4) to uphold (C2). We argue for the latter approach by demonstrating that postulate (R4) stands contrary to common standards of plausibility, because it encapsulates the overly restrictive requirement that revision should be a function of belief sets instead of epistemic states. We will next illustrate by example the counterintuitive consequences of this restriction.

Example 7 (*Goldszmidt and Pearl* [13]). Two jurors in a murder trial possess different biases; juror-1 believes "A is the murderer, B is a remote but unbelievable possibility while C is definitely innocent". Juror-2 believes "A is the murderer, C is a remote but unbelievable possibility while B is definitely innocent". The two jurors share the same belief set $\psi_1 \equiv \psi_2$ = "A is the only murderer". A surprising evidence now obtains: μ = "A is not the murderer" (A has produced a reliable alibi). Clearly, any rational account of belief revision should allow juror-1 to uphold a different belief set than juror-2. Yet any approach based on a revision operator that satisfies postulate (R4) dictates that $\psi_1 \circ \mu \equiv \psi_2 \circ \mu$, which is an indefensible position.

We conclude this section by providing a representation theorem for postulates (R*1)–(R*6), which parallels Theorem 2:

Definition 8. Let W be the set of all worlds (interpretations) of a propositional language $\mathcal{L}$ and suppose that the belief set of any epistemic state belongs to $\mathcal{L}$. A function that maps each epistemic state Ψ to a total pre-order $\leqslant_\Psi$ on worlds W is said to be a *faithful assignment* if and only if:
(1) $\omega_1, \omega_2 \models \Psi$ only if $\omega_1 =_\Psi \omega_2$;
(2) $\omega_1 \models \Psi$ and $\omega_2 \not\models \Psi$ only if $\omega_1 <_\Psi \omega_2$; and
(3) $\Psi = \Phi$ only if $\leqslant_\Psi = \leqslant_\Phi$.
Here, $\omega_1 <_\Psi \omega_2$ is defined as $\omega_1 \leqslant_\Psi \omega_2$ and $\omega_2 \not\leqslant_\Psi \omega_1$; $\omega_1 =_\Psi \omega_2$ is defined as $\omega_1 \leqslant_\Psi \omega_2$ and $\omega_2 \leqslant_\Psi \omega_1$.

Theorem 9. *A revision operator $\circ$ satisfies postulates* (R*1)–(R*6) *precisely when there exists a faithful assignment that maps each epistemic state Ψ to a total pre-order $\leqslant_\Psi$ such that*

$$Mods(\Psi \circ \mu) = \min(Mods(\mu), \leqslant_\Psi).$$

That is, the representation theorem for postulates (R1)–(R6) continues to hold for the modified set of postulates, with only one difference. The equivalence $Bel(\Psi) \equiv Bel(\Phi)$ is not sufficient to imply $\leqslant_\Psi = \leqslant_\Phi$; the stronger condition $\Psi = \Phi$ is needed instead.

4. Minimizing changes in conditional beliefs

The examples we presented in Section 2 show that the AGM postulates are too weak to regulate changes in conditional beliefs, thus permitting improper responses to sequences of observations. To address this weakness, we shall propose four postulates in Section 5 that properly preserve conditional beliefs and, hence, provide new criteria for testing the coherence of iterated belief revision.

A subtle issue relating to our postulates is identifying those changes in conditional beliefs that *must* be minimized. For example, if we were to opt for a radical strategy of change minimization, then adding postulate (CB) below to the AGM postulates will suffice because it guarantees that conditional beliefs are preserved as much as the AGM postulates permit:
(CB) If $\Psi \circ \mu \models \neg\alpha$, then $(\Psi \circ \mu) \circ \alpha \equiv \Psi \circ \alpha$.

However, such a radical strategy would be excessive. We will first discuss the reason why postulate (CB) minimizes changes in conditional beliefs and then show why it leads to counterintuitive results.

4.1. Absolute minimization

Consider the following lemma:

Lemma 10. *$\Psi \models \beta \mid \alpha$ precisely when there exists a world ω such that $\omega \models \alpha \wedge \beta$ and $\omega <_\Psi \omega'$ for any $\omega' \models \alpha \wedge \neg\beta$.*

Therefore, the pre-order $\leqslant_\Psi$ associated with an epistemic state Ψ encodes the conditional beliefs accepted by Ψ and, similarly, the pre-order $\leqslant_{\Psi \circ \mu}$ encodes the conditional beliefs accepted by $\Psi \circ \mu$. Hence, one can minimize changes in conditional beliefs due to a revision by making the pre-orders $\leqslant_\Psi$ and $\leqslant_{\Psi \circ \mu}$ as similar as possible, which is exactly what postulate (CB) does:

Theorem 11. *Suppose that a revision operator satisfies postulates* (R*1)–(R*6). *The operator satisfies postulate* (CB) *iff the operator and its corresponding faithful assignment satisfy:*

(CBR) *If $\omega_1, \omega_2 \models \neg(\Psi \circ \mu)$, then $\omega_1 \leqslant_\Psi \omega_2$ iff $\omega_1 \leqslant_{\Psi \circ \mu} \omega_2$.*

That is, according to postulate (CB), the order imposed by $\leqslant_{\Psi \circ \mu}$ on two worlds in $Mods(\neg(\Psi \circ \mu))$ should be the same as that imposed on them by $\leqslant_\Psi$. Note also that the order imposed by $\leqslant_{\Psi \circ \mu}$ on other types of worlds is determined by the AGM postulates. Specifically, the faithfulness of $\leqslant_{\Psi \circ \mu}$ ensures that:

(1) If $\omega_1, \omega_2 \models \Psi \circ \mu$, then $\omega_1 =_{\Psi \circ \mu} \omega_2$.

(2) If $\omega_1 \models \Psi \circ \mu$ and $\omega_2 \models \neg(\Psi \circ \mu)$, then $\omega_1 <_{\Psi \circ \mu} \omega_2$.

Therefore, once the total pre-order $\leqslant_\Psi$ is known, postulate (CB) together with the AGM postulates determine the total pre-order $\leqslant_{\Psi \circ \mu}$ completely.

4.2. Is absolute minimization desirable?

Absolute minimization of changes in conditional beliefs is due to Boutilier who suggested minimizing these changes as much as the AGM postulates permit [3]. In fact, condition (CBR) is effectively Boutilier's definition of natural revision, and postulate (CB) is a property that Boutilier has proven about his method of revision [3].

Although postulate (CB) rules out the counterintuitive revision scenarios discussed in Section 2, the postulate is somewhat of an overkill because it does compromise propositional beliefs. In particular, the postulate says that accommodating an evidence α should totally wash out a previous evidence μ whenever μ contradicts α in the light of Ψ. But this does not always constitute enough grounds for evidence α to undermine an earlier evidence μ because the source of contradiction may lie with Ψ not with μ.

Example 12. We encounter a strange new animal and it appears to be a bird, so we believe the animal is a bird. As it comes closer to our hiding place, we see clearly that the animal is red, so we believe that it is a red bird. To remove further doubts about the animal birdness, we call in a bird expert who takes it for examination and concludes that it is not really a bird but some sort of mammal. The question now is whether we should still believe that the animal is red. Postulate (CB) tells us that we should no longer believe that the animal is red. This can be seen by substituting $\Psi \equiv \neg\alpha = bird$ and $\mu = red$ in postulate (CB), instructing us to totally ignore the color observation μ as if it never took place (see Example A.5 in Appendix A for more details).

The reason for this behavior is that retaining the belief in the animal's color means that we are implicitly acquiring a new conditional belief—that the animal is red given that it is not a bird—which we did not have before. That is, if before observing the animal's color someone were to ask us, "Would you say that the animal is red, given that it is not a bird?" our answer would have been, "No, because we are not in possession of any color information". Strangely, according to the minimal-change principle, we should maintain this same color ignorance now that the red animal proved to be a non-bird. The fact that we actually observed the animal's redness prior to calling the expert does not matter, as it only pertains to our belief set during that observation; namely, it renders the animal red, provided the animal is a bird, but says nothing about the animal's color if it turns out to be a non-bird.

This is counterintuitive; once the animal is seen red, it should be presumed red no matter what ornithological classification it obtains. And if this belief preservation introduces new conditional beliefs, so be it.

5. Postulates for iterated revision

We have presented a number of belief revision scenarios that involve counterintuitive changes in conditional beliefs, and yet they are admitted by the AGM postulates for belief revision. This means that the AGM postulates fail to rule out some counterintuitive belief revision operators. We have also shown that although postulate (CB) does preserve conditional beliefs, it also leads to counterintuitive results because it often compromises propositional beliefs while protecting conditional ones.

Our solution to the problem is to divide conditional beliefs into two categories; those that may compromise propositional beliefs if preserved, and those that do not. We then insist that only the second category of conditional beliefs be preserved, and we do this by proposing additional postulates. In fact, for clarity of exposition, we break down the conditional beliefs we want to preserve into four classes and propose one postulate for preserving each class.

We first present these postulates, and then discuss the reason why they do not compromise propositional beliefs as does postulate (CB). That these postulates correspond to four disjoint classes of conditional beliefs will be obvious from the representation theorem of these postulates, which we present later. Conditional beliefs whose protection compromises propositional beliefs are the subject of Section 7.

The proposed postulates are:

(C1) If $\alpha \models \mu$, then $(\Psi \circ \mu) \circ \alpha \equiv \Psi \circ \alpha$.
Explanation: when two pieces of evidence arrive, the second being more specific than the first, the first is redundant; that is, the second evidence alone would yield the same belief set. One can also phrase this postulate as $(\Psi \circ \alpha) \circ (\alpha \wedge \mu) \equiv \Psi \circ (\alpha \wedge \mu)$ with the interpretation that learning full information should wash out any previously learned partial information [18].

(C2) If $\alpha \models \neg\mu$, then $(\Psi \circ \mu) \circ \alpha \equiv \Psi \circ \alpha$.
Explanation: when two contradictory pieces of evidence arrive, the last one prevails; that is, the second evidence alone would yield the same belief set.

(C3) If $\Psi \circ \alpha \models \mu$, then $(\Psi \circ \mu) \circ \alpha \models \mu$.
Explanation: an evidence μ should be retained after accommodating a more recent evidence α that implies μ given current beliefs.

(C4) If $\Psi \circ \alpha \not\models \neg\mu$, then $(\Psi \circ \mu) \circ \alpha \not\models \neg\mu$.
Explanation: no evidence can contribute to its own demise. If μ is not contradicted after seeing α, then it should remain uncontradicted when α is preceded by μ itself.

By examining the postulates carefully, we see that none of them does lead to the unnecessary discredit of evidence. In particular, according to postulate (C1), the later evidence α could never discredit the previous evidence μ because α entails μ. Postulate (C2), on the other hand, permits the later evidence α to discredit the previous evidence μ but justifiably so; α logically contradicts μ. Postulate (C3) clearly insists that the previous evidence μ be retained after accommodating the more recent evidence α. And postulate (C4) concerns a case under which the previous evidence μ should not be contradicted as a result of accommodating the more recent evidence α.

Postulates (C1)–(C4) were phrased in terms of iterated revisions, but following is an equivalent formulation, in terms of conditional beliefs using Definition 6, that highlights the change-minimization role of these postulates:

(C1) If $\alpha \models \mu$, then $\Psi \models \beta \mid \alpha$ iff $\Psi \circ \mu \models \beta \mid \alpha$.
Explanation: accommodating evidence μ should not perturb any conditional beliefs that are conditioned on a premise more specific than μ.

(C2) If $\alpha \models \neg\mu$, then $\Psi \models \beta \mid \alpha$ iff $\Psi \circ \mu \models \beta \mid \alpha$.
Explanation: accommodating evidence μ should not perturb any conditional beliefs that are conditioned on a premise that contradicts μ.

(C3) If $\Psi \models \mu \mid \alpha$, then $\Psi \circ \mu \models \mu \mid \alpha$.
Explanation: the conditional $\mu \mid \alpha$ should not be given up after accommodating evidence μ.

(C4) If $\Psi \not\models \neg\mu \mid \alpha$, then $\Psi \circ \mu \not\models \neg\mu \mid \alpha$.
Explanation: the conditional $\neg\mu \mid \alpha$ should not be acquired after accommodating evidence μ.

Appendix A presents four AGM-compatible revision operators that contradict each of our proposed postulates, thus demonstrating that none of (C1)–(C4) is derivable from the AGM postulates. In the following section, we provide concrete real-life scenarios demonstrating the plausibility of the proposed postulates.

5.1. Examples

Postulate (C1)

I have a circuit containing an adder and a multiplier. I believe both the adder and multiplier are working, hence the circuit as a whole is working. If someone were to tell me that the circuit failed, I would blame the multiplier, not the adder (trick of the trade: multipliers are known to be more troublesome). However, if someone tells me that the adder is bad, I would believe that the multiplier is fine (because failures are presumed independent, so, two simultaneous failures are much less likely that one). Now, they tell me the circuit is faulty, and immediately after, that the adder is bad. Should I be tempted to claim that the multiplier is bad too? A naive argument: "After hearing of the fault in the circuit I blamed the multiplier. Learning that the adder is bad is perfectly consistent with my current belief that the multiplier is bad, therefore, I have no reason to change my mind about the multiplier being bad." Plausible reasoning (and postulate (C1)) on the other hand claim that I *should* change my mind because the only reason I blamed the multiplier was to explain the failing circuit. Otherwise, by my own admission, I would presume the multiplier is fine. Moreover, I also admitted that the two components do not affect each other. Hence, learning that the adder is bad perfectly explains away whatever reasons I had in blaming the multiplier; I should revert to my initial belief that the multiplier is fine. Postulate (C1) enforces this line of reasoning. In particular, by letting

$$\Psi \equiv adder\_ok \wedge multiplier\_ok,$$
$$\mu = \neg(adder\_ok \wedge multiplier\_ok),$$
$$\alpha = \neg adder\_ok,$$
$$\beta = multiplier\_ok,$$

one can conclude that $(\Psi \circ \mu) \circ \alpha \models \beta$ using postulate (C1) and given $\alpha \models \mu$ and $\Psi \circ \alpha \models \beta$. The AGM postulates, however, are too weak to draw such a conclusion, as demonstrated by Example A.1 in Appendix A.

Postulate (C2)

Consider Example 5 in Section 2: I believe that lady X is smart and rich. Moreover, I am disposed to maintain that X is smart even if found to be poor and, conversely, that X is rich even if found to be not smart. Now, I obtain evidence that X is in fact not smart, followed by evidence that X is indeed smart. What should happen to my belief in X being rich after accommodating these pieces of evidence? Postulate (C2) forces one to maintain this belief. Specifically, by letting

$$\Psi \equiv smart \wedge rich,$$
$$\mu = \neg smart,$$
$$\alpha = smart,$$
$$\beta = rich,$$

one can conclude that $(\Psi \circ \mu) \circ \alpha \models \beta$ using postulate (C2) and given that $\alpha \models \neg\mu$ and $\Psi \circ \alpha \models \beta$. Example A.2 in Appendix A, however, demonstrates that the AGM postulates are too weak to reach this conclusion.

Postulate (C3)

Consider Example 4 in Section 2: I believe that X is not a bird and that X does not fly. Still, in the event that X turns out a bird, I am prepared to change my mind and conclude that X flies. What should happen to this conditional belief upon observing that X can fly? Postulate (C3) forces one to maintain this conditional belief after accommodating the observation. That is, by letting

$$\Psi \equiv \neg bird \wedge \neg flies,$$
$$\mu = flies,$$
$$\alpha = bird,$$

one can conclude that $(\Psi \circ \mu) \circ \alpha \models \mu$ using postulate (C3) and given $\Psi \circ \alpha \models \mu$. Example A.3 in Appendix A, however, demonstrates that the AGM postulates are too weak to draw this conclusion.

Postulate (C4)

A philosopher wakes up in the morning and says: "The sun is shining, great!, I have no reason to believe that it will be a nasty day". His wife tells him: "In fact, just before you woke up they said on the radio that it is going to be a nice day". The philosopher says: "Did they really say that? They are usually right on the radio, I will have to take it back then, it is going to be nasty after all". Readers who feel there is something strange in this dialogue will be pleased to know that postulate (C4) will weed out this sort of logic from conversation. In particular, letting

$$\Psi \equiv \neg shining\_sun,$$
$$\mu = nice\_day,$$
$$\alpha = shining\_sun,$$

one can conclude that $(\Psi \circ \mu) \circ \alpha \not\models \neg\mu$ using postulate (C4) and given that $\Psi \circ \alpha \not\models \neg\mu$. In other words, the philosopher's final statement is inconsistent with postulate (C4). Example A.4 in Appendix A, however, demonstrates that the AGM postulates are too weak to rule out such a statement.

5.2. *A representation theorem*

Theorem 9 shows that a revision operator satisfying the modified AGM postulates is equivalent to a set of total pre-orders $\leqslant_\Psi$, each of which is associated with an epistemic state Ψ and is used to revise this state in the face of further evidence. One observation about this result, however, is that the total pre-orders associated with different epistemic states are not related to one another except by requiring that the pre-orders be faithful. This explains the permissiveness of the AGM postulates regarding some changes in conditional beliefs when evidence is accommodated. Postulates (C1)–(C4), on the other hand, which strongly constrain such changes, should also strongly constrain the relationship between the pre-orders $\leqslant_\Psi$ and $\leqslant_{\Psi\circ\mu}$. This is exactly what the following theorem shows:

Theorem 13. *Suppose that a revision operator satisfies postulates* (R*1)–(R*6). *The operator satisfies postulates* (C1)–(C4) *iff the operator and its corresponding faithful assignment satisfy*:

(CR1) *If* $\omega_1 \models \mu$ *and* $\omega_2 \models \mu$, *then* $\omega_1 \leqslant_\psi \omega_2$ *iff* $\omega_1 \leqslant_{\psi\circ\mu} \omega_2$.
(CR2) *If* $\omega_1 \models \neg\mu$ *and* $\omega_2 \models \neg\mu$, *then* $\omega_1 \leqslant_\psi \omega_2$ *iff* $\omega_1 \leqslant_{\psi\circ\mu} \omega_2$.
(CR3) *If* $\omega_1 \models \mu$ *and* $\omega_2 \models \neg\mu$, *then* $\omega_1 <_\psi \omega_2$ *only if* $\omega_1 <_{\psi\circ\mu} \omega_2$.
(CR4) *If* $\omega_1 \models \mu$ *and* $\omega_2 \models \neg\mu$, *then* $\omega_1 \leqslant_\psi \omega_2$ *only if* $\omega_1 \leqslant_{\psi\circ\mu} \omega_2$.

By examining the above representation theorem, we see how each of postulates (C1)–(C4) concerns itself with preserving some part of the pre-order $\leqslant_\psi$ into the pre-order $\leqslant_{\psi\circ\mu}$. It is also clear from the above theorem that there are two parts of the pre-order $\leqslant_\psi$ that postulates (C1)–(C4) do not preserve into $\leqslant_{\psi\circ\mu}$. Specifically, if $\omega_1 \leqslant_\psi \omega_2$ (or $\omega_1 <_\psi \omega_2$), where $\omega_1 \models \neg\mu$ and $\omega_2 \models \mu$, then the postulates do not insist on $\omega_1 \leqslant_{\psi\circ\mu} \omega_2$ (nor on $\omega_1 <_{\psi\circ\mu} \omega_2$). The rationale behind this will be discussed at length in Section 7.

6. Properties of iterated revision postulates

We provide in this section a concrete revision operator that satisfies postulates (R*1)–(R*6) and postulates (C1)–(C4), thus proving their consistency. The operator is based on a proposal by Spohn for revising *ordinal conditional functions*, which can be viewed as representations of epistemic states [8,22–24]. Spohn's method for belief change, called (μ, m)-conditionalization, can be interpreted as a qualitative version of Jeffrey's rule of probabilistic conditioning [8, 11, 12]. Using a dynamic version of Spohn conditionalization, we will construct a revision operator $\bullet$ that satisfies all our postulates, thus showing that the postulates we propose for characterizing iterated belief revision, in addition to being consistent, are also compatible with a qualitative version of probabilistic conditioning.

An ordinal conditional function (ranking) is a function κ from a given set of worlds into the class of ordinals such that some worlds are assigned the smallest ordinal 0. Intuitively, the ordinals represent degrees of plausibility. The smallest the ordinal, the more plausible a world is. A ranking is extended to propositions by requiring that the rank of a proposition by the smallest rank assigned to a world that satisfies the propositions:

$$\kappa(\mu) = \min_{\omega \models \mu} \kappa(\omega).$$

This also implies that

$$\kappa(\mu \vee \nu) = \min(\kappa(\mu), \kappa(\nu)).$$

A ranking accepts a proposition μ if the negation of the proposition is implausible: $\kappa(\neg\mu) > 0$. One can characterize the set of propositions accepted by a ranking, denoted $Bel(\kappa)$, as follows:

$$Mods(Bel(\kappa)) \stackrel{\text{def}}{=} \{\omega\colon\ \kappa(\omega) = 0\}.$$

Any sentence that has the set of 0-rank worlds as its models is a characterization of these accepted propositions, that is, κ accepts μ precisely when $Bel(\kappa) \models \mu$.

One property of ranking functions is that $Bel(\kappa)$ is guaranteed to be satisfiable since at least one world must be assigned the 0-rank by κ. This does not admit epistemic states with unsatisfiable belief sets, which is a restriction when viewed in light of postulate (R*1). Specifically, if we accept this postulate, we cannot allow revisions with an unsatisfiable μ because this should lead to an unsatisfiable belief set according to (R*1). Therefore, we will relax the assumption that at least one world has the 0-rank, and will permit rankings κ with unsatisfiable belief sets $Bel(\kappa)$.

In addition to proposing rankings as a representation of epistemic states, Spohn proposed a method for changing a ranking in face of new evidence. Specifically, evidence is represented as a pair (μ, m), where μ is a proposition and m is the post-revision degree of plausibility of μ. A rank κ is updated in face of such evidence as follows:

$$\kappa_{(\mu,m)}(\omega) = \begin{cases} \kappa(\omega) - \kappa(\mu), & \text{if } \omega \models \mu; \\ \kappa(\omega) - \kappa(\neg\mu) + m, & \text{if } \omega \models \neg\mu. \end{cases}$$

Spohn called $\kappa_{(\mu,m)}$ the (μ, m)-conditionalization of κ.

One feature of (μ, m)-conditionalization is that μ ends up with a rank of m regardless of its pre-update rank $\kappa(\mu)$. By letting m be a function of $\kappa(\mu)$ a wide variety of belief revision schemes can be synthesized. To construct our belief revision operator • we will choose one such scheme, ensuring that a revision by μ will always strengthen the belief in μ.[7] Specifically, we let m, the post-revision degree of plausibility of μ, be one degree higher than its current value, $\kappa(\neg\mu)$:

$$(\kappa \bullet \mu)(\omega) \stackrel{\text{def}}{=} \kappa_{(\mu,\kappa(\neg\mu)+1)}(\omega) = \begin{cases} \kappa(\omega) - \kappa(\mu), & \text{if } \omega \models \mu; \\ \kappa(\omega) + 1, & \text{if } \omega \models \neg\mu. \end{cases}$$

Note that if μ is unsatisfiable, the belief set of $\kappa \bullet \mu$ will also be unsatisfiable.

The following theorem shows that the proposed postulates are satisfied by Spohn's proposal for belief change (restricted to revision scenarios).

Theorem 14. *The revision operator • satisfies postulates* (R*1)–(R*6) *and* (C1)–(C4).

This theorem also shows that the iterated revision postulates we have proposed are consistent with the modified AGM postulates in which belief revision is a function of an epistemic state instead of a belief set.[8]

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[7] Clearly, other updating schemes will also suit our purpose; for example, leaving $\kappa(\mu)$ unaltered whenever μ is already believed, or incrementing $\kappa(\mu)$ by a number which measures the strength of evidence for μ, in the spirit of L-conditionalization [13].

[8] It is commonly believed that Spohn's conditionalization provides a successful realization of AGM-style revision. Gärdenfors, for example, claims in [8, p. 73]: "... let us define the belief set K *associated with* the ordinal conditional function κ as the set of all propositions that are accepted in κ. If we let K^*_A denote the belief set associated with $\kappa^*(A, a)$, where $a > 0$, then it can be shown that the revision function defined in this way satisfies postulates (K*1)–(K*8)." This is not in fact the case; Gärdenfors construction requires that operator $*$ not be a function since different ordinal functions κ can have the same associated belief sets, thus violating the basic tenet of the original AGM framework.

Lehmann has shown that the AGM postulates together with postulate (C1) are sufficient to imply postulates (C3) and (C4) [18].[9] The following theorem shows that this result is only valid in light of postulate (R4), which requires belief revisions to depend only on the current belief set. If revisions are a function of the current epistemic state (as in (R*1)-(R*6)), then postulates (C3) and (C4) are independent of (C1):

Theorem 15. *There is a revision operator that satisfies postulates* (R*1)-(R*6) *and* (C1), *but does not satisfy postulate* (C3) *or* (C4).

7. Legitimate changes in conditional beliefs

Given Theorem 13, it is not hard to see that postulate (CB) implies, but is not equivalent to, postulates (C1)-(C4). Therefore, postulates (C1)-(C4) do admit some changes in conditional beliefs. What are these changes and why are they legitimate?

To answer these questions, we show that adding the following two postulates to postulates (C1)-(C4) will lead to absolute minimization of changes in conditional beliefs:

(C5) If $\Psi \circ \mu \models \neg\alpha$ and $\Psi \circ \alpha \not\models \mu$, then $(\Psi \circ \mu) \circ \alpha \not\models \mu$.
Explanation: if evidence μ rules out the premise α, then the conditional belief $\mu \mid \alpha$ should not be acquired after observing μ.

(C6) If $\Psi \circ \mu \models \neg\alpha$ and $\Psi \circ \alpha \models \neg\mu$, then $(\Psi \circ \mu) \circ \alpha \models \neg\mu$.
Explanation: if evidence μ rules out the premise α, then the conditional belief $\neg\mu \mid \alpha$ should not be given up after observing μ.

That postulates (C5) and (C6) attain absolute minimal change in conditional beliefs can be seen from the following representation theorem, which, together with Theorem 13, shows that the total pre-order $\leqslant_{\Psi \circ \mu}$ is as similar to the total pre-order $\leqslant_{\Psi}$ as the AGM postulates permit.

Theorem 16. *Suppose that a revision operator satisfies postulates* (R*1)-(R*6). *The operator satisfies postulates* (C5) *and* (C6) *iff the operator and its corresponding faithful assignment satisfy*:

(CR5) *If* $\omega_1, \omega_3 \models \mu$ *and* $\omega_2 \models \neg\mu$, *then* $\omega_3 <_{\Psi} \omega_1$ *and* $\omega_2 \leqslant_{\Psi} \omega_1$ *only if* $\omega_2 \leqslant_{\Psi \circ \mu} \omega_1$.

(CR6) *If* $\omega_1, \omega_3 \models \mu$ *and* $\omega_2 \models \neg\mu$, *then* $\omega_3 <_{\Psi} \omega_1$ *and* $\omega_2 <_{\Psi} \omega_1$ *only if* $\omega_2 <_{\Psi \circ \mu} \omega_1$.

The remaining changes in conditional beliefs that are not eliminated by postulates (C1)-(C4) are those identified by postulates (C5)-(C6). The first of these changes is acquiring a conditional belief $\mu \mid \alpha$ only because evidence μ was acquired. And the second of these changes is giving up a conditional belief $\neg\mu \mid \alpha$ only because evidence μ was acquired. Postulates (C5)-(C6), and also postulate (CB), eliminate these changes, but the following analysis shows that such elimination is premature.

[9] That is, when (C1), (C3) and (C4) are phrased using belief sets instead of epistemic states.

To show that postulate (C5) can prohibit some legitimate changes, consider Example 12, which was presented as counterexample to postulate (CB). This example is a clear cut contradiction with postulate (C5) because it shows that the revision suggested by postulate (C5) is wrong: All we believe initially is that X is a bird. We then observe that X is red, followed by an observation that X is not a bird. Postulate (C5) tells us that we should dismiss the observation of X's color in this case. That is, since the conditional $red \mid \neg bird$ was not believed by the belief set *bird*, it should neither be believed by the new belief set $bird \circ red$. But this falsely means that when $\neg bird$ is observed, *red* must be retracted, which is a counterintuitive behavior.

To show that postulate (C6) prohibits some legitimate changes in conditional beliefs, consider the following example.

Example 17. We face a murder trial with two main suspects, John and Mary. Initially, it appears that the murder was committed by one person, hence, we believe that

$$\Psi \equiv (John \wedge \neg Mary) \vee (\neg John \wedge Mary).$$

Given the AGM postulates, we also believe in the two conditionals $\neg Mary \mid John$ and $\neg John \mid Mary$. As the trial unfolds, however, we receive a very reliable testimony incriminating John, followed by another reliable testimony incriminating Mary. At this point, it is only reasonable to believe that both suspects took part in the murder, thus dismissing the one-person theory together with the two conditional beliefs $\neg Mary \mid John$ and $\neg John \mid Mary$. Postulate (C6), on the other hand, will force us to maintain the two conditionals and dismiss the testimony against John, no matter how compelling. That is, by substituting $\alpha = Mary$ and $\mu = John$, postulate (C6) forces the conclusion $(\Psi \circ John) \circ Mary \models \neg John$ given that $\Psi \circ John \models \neg Mary$ and $\Psi \circ Mary \models \neg John$.

This is counterintuitive; whether we should dismiss the testimony against John should depend on how strongly we believe in it compared with how strongly we believe in the one-person theory. Postulate (C6), however, does not take these factors into consideration and always prefers the conditional belief over the propositional one.

8. Future work

The counterexamples to postulates (C5) and (C6) that we discussed in Section 7 show that the outcome of belief change depends on the strength of evidence triggering the change. The language of AGM, however, is too weak to represent evidence strength and is therefore inappropriate for phrasing some plausible properties of belief change, such as qualified versions of postulates (C5) and (C6).

To remedy this inexpressiveness, we have been investigating the refinement of revision operators so that one can express the strength of evidence with which one is revising beliefs. In particular, instead of one revision operator $\circ$, we are investigating a sequence of revision operators $\circ_0, \circ_1, \circ_2, \ldots$, where $\Psi \circ_m \mu$ denotes the revision of Ψ with evidence μ having strength m.

The notion of evidence strength leads us to another important notion: degree of acceptance. Specifically, we will say that proposition μ is accepted by Ψ to degree m if it takes an evidence $\neg\mu$ with strength m to retract μ from Ψ. Formally, we have the following definition.

Definition 18. Proposition μ is accepted by an epistemic state Ψ to degree m (written $\Psi \models_m \mu$) precisely when

(1) $\Psi \not\models \neg\mu$;
(2) $\Psi \circ_m \neg\mu \not\models \neg\mu$; and
(3) $\Psi \circ_m \neg\mu \not\models \mu$.

This refinement to the AGM language is intended to allow expressing qualified versions of postulate (C5) and (C6) by taking into account the degrees to which conditional beliefs are accepted and the strength of competing evidence. Moreover, the refined language allows one to express stronger versions of postulates (C1)–(C4) that insist on the selective preservation of not only conditional beliefs, but also their degrees of acceptance.

9. Conclusions

We have demonstrated that adequate preservation of conditional beliefs is a necessary component in any account of rational belief revision, and that such preservation must be applied at the epistemic state, rather than belief set level. The AGM postulates are inadequate for regulating iterated belief revision because they apply to belief sets and, even when broadened to accommodate epistemic state revision, they remain too weak—two-step postulates are necessary. We have also shown that full, indiscriminate preservation of conditional beliefs leads to counterintuitive results because it comes at the expense of compromising propositional beliefs.

Accordingly, we have proposed an epistemic state version of the AGM framework, together with four additional postulates that preserve the proper mix of conditional and propositional beliefs. The resulting system provides a new criterion for testing the coherence of iterated belief revision. Finally, we extended the Katsuno and Mendelzon representation theorem of the AGM postulates to cover the newly proposed postulates.

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Table A.1
An AGM-compatible operator contradicting postulate (C1)

world	*adder_ok*	*multiplier_ok*	$\leqslant_{\Psi}$	$\leqslant_{\Psi \circ \mu}$
ω_1	T	T	0	1
ω_2	T	F	1	0
ω_3	F	T	2	2
ω_4	F	F	3	1

Table A.2
An AGM-compatible operator contradicting postulate (C2)

world	*smart*	*rich*	$\leqslant_{\Psi}$	$\leqslant_{\Psi \circ \mu}$
ω_1	T	T	0	2
ω_2	T	F	1	1
ω_3	F	T	1	0
ω_4	F	F	2	1

Appendix A. Concrete examples

We will represent a total pre-order $\leqslant_{\Psi}$ by a mapping κ from worlds to positive integers, where $\omega_1 \leqslant_{\Psi} \omega_2$ precisely when $\kappa(\omega_1) \leqslant \kappa(\omega_2)$.

Example A.1 (*Postulate* (C1)). Consider the AGM revision operator given partially in Table A.1. Let

$$\Psi \equiv \mathit{adder\_ok} \wedge \mathit{multiplier\_ok},$$
$$\mu = \neg(\mathit{adder\_ok} \wedge \mathit{multiplier\_ok}),$$
$$\alpha = \neg \mathit{adder\_ok}.$$

Although $\alpha \models \mu$, we have

$$\Psi \circ \alpha \equiv \neg \mathit{adder\_ok} \wedge \mathit{multiplier\_ok},$$
$$(\Psi \circ \mu) \circ \alpha \equiv \neg \mathit{adder\_ok} \wedge \neg \mathit{multiplier\_ok},$$

thus violating postulate (C1), which requires that $\Psi \circ \alpha \equiv (\Psi \circ \mu) \circ \alpha$.

Moreover, $\omega_3, \omega_4 \models \mu$, $\omega_3 \leqslant_{\Psi} \omega_4$, yet $\omega_3 \not\leqslant_{\Psi \circ \mu} \omega_4$, thus violating condition (CR1).

Example A.2 (*Postulate* (C2)). Consider the AGM revision operator given partially in Table A.2. Let

$$\Psi \equiv \mathit{smart} \wedge \mathit{rich},$$
$$\mu = \neg \mathit{smart},$$
$$\alpha = \mathit{smart}.$$

Table A.3
An AGM-compatible operator contradicting postulate (C3)

world	*bird*	*flies*	$\leqslant_{\Psi}$	$\leqslant_{\Psi \circ \mu}$
ω_1	T	T	2	1
ω_2	T	F	3	1
ω_3	F	T	1	0
ω_4	F	F	0	1

Table A.4
An AGM-compatible operator contradicting postulate (C4)

world	*shining_sun*	*nice_day*	$\leqslant_{\Psi}$	$\leqslant_{\Psi \circ \mu}$
ω_1	T	T	1	2
ω_2	T	F	1	1
ω_3	F	T	0	0
ω_4	F	F	0	1

Although $\alpha \models \neg\mu$, we have

$$\Psi \circ \alpha \equiv smart \wedge rich,$$
$$(\Psi \circ \mu) \circ \alpha \equiv smart \wedge \neg rich,$$

thus violating postulate (C2), which requires that $\Psi \circ \alpha \equiv (\Psi \circ \mu) \circ \alpha$.

Moreover, $\omega_1, \omega_2 \models \neg\mu$, $\omega_1 \leqslant_{\Psi} \omega_2$, yet $\omega_1 \not\leqslant_{\Psi \circ \mu} \omega_2$, thus violating condition (CR2).

Example A.3 (*Postulate* (C3)). Consider the AGM revision operator given partially in Table A.3. Let

$$\Psi \equiv \neg bird \wedge \neg flies,$$
$$\mu = flies,$$
$$\alpha = bird.$$

We have,

$$\Psi \circ \alpha \equiv bird \wedge flies,$$
$$(\Psi \circ \mu) \circ \alpha \equiv bird.$$

That is, although $\Psi \circ \alpha \models \mu$, we have $(\Psi \circ \mu) \circ \alpha \not\models \mu$, thus violating postulate (C3).

Moreover, $\omega_1 \models \mu$, $\omega_2 \models \neg\mu$, $\omega_1 <_{\Psi} \omega_2$, yet $\omega_1 \not<_{\Psi \circ \mu} \omega_2$, thus violating condition (CR3).

Example A.4 (*Postulate* (C4)). Consider the AGM revision operator given partially in Table A.4. Let

$\Psi \equiv \neg shining\_sun,$
$\mu = nice\_day,$
$\alpha = shining\_sun.$

We have,

$$\Psi \circ \alpha \equiv shining\_sun,$$
$$(\Psi \circ \mu) \circ \alpha \equiv shining\_sun \wedge \neg nice\_day.$$

That is, although $\Psi \circ \alpha \not\models \neg\mu$, we have $(\Psi \circ \mu) \circ \alpha \models \neg\mu$, thus violating postulate (C4).

Moreover, $\omega_1 \models \mu$, $\omega_2 \models \neg\mu$, $\omega_1 \leqslant_\Psi \omega_2$, yet $\omega_1 \not\leqslant_{\Psi \circ \mu} \omega_2$, thus violating condition (CR4).

Example A.5 (*Postulate* (CB)). Let

$\Psi \equiv bird,$
$\mu = red,$
$\alpha = \neg bird,$

and assume that $\Psi \circ \neg bird \equiv \neg bird$. Substituting in postulate (CB), we get

$$\text{if } \Psi \circ red \models bird, \text{ then } (\Psi \circ red) \circ \neg bird \equiv \Psi \circ \neg bird.$$

Given the AGM postulates, this implies

$$(\Psi \circ red) \circ \neg bird \equiv \Psi \circ \neg bird.$$

Given our assumption, this reduces to

$$(\Psi \circ red) \circ \neg bird \equiv \neg bird,$$

which is a counterintuitive conclusion.

Appendix B. Proofs

Proof of Theorem 9

This proof is symmetric to the one provided by Katsuno and Mendelzon for Theorem 3.3 in [17]. We also use the notation $form(\omega_1, \omega_2, \ldots)$ to denote a sentence α that has $\omega_1, \omega_2, \ldots$ as its models: $Mods(\alpha) = \{\omega_1, \omega_2, \ldots\}$.

($\Rightarrow$) Suppose that a revision operator $\circ$ satisfies postulates (R*1)–(R*6). For each epistemic state Ψ, define its corresponding total pre-order $\leqslant_\Psi$ as follows:

$$\omega \leqslant_\Psi \omega' \stackrel{\text{def}}{=} \omega \models \Psi \text{ or } \omega \models \Psi \circ form(\omega, \omega').$$

The binary relation $\leqslant_\Psi$ is a total pre-order:

(1) *Total*: By (R*3), $Mods(\Psi \circ form(\omega, \omega'))$ is a non-empty set. By (R*1), $Mods(\Psi \circ form(\omega, \omega'))$ is a subset of $\{\omega, \omega'\}$. Therefore, for any ω and ω', either $\omega \models \Psi \circ form(\omega, \omega')$ or $\omega' \models \Psi \circ form(\omega, \omega')$. Therefore, $\leqslant_\Psi$ is total.

(2) *Reflexive*: By (R*1) and (R*3), $Mods(\Psi \circ form(\omega)) = \{\omega\}$. Therefore, $\omega \leqslant_\Psi \omega$ and $\leqslant_\Psi$ is reflexive.

(3) *Transitive*: Suppose that $\omega_1 \leqslant_\Psi \omega_2$ and $\omega_2 \leqslant_\Psi \omega_3$. We need to show that $\omega_1 \leqslant_\Psi \omega_3$. We consider three cases:

   (1) $\omega_1 \models \Psi$.
   $\omega_1 \leqslant_\Psi \omega_3$ follows from the definition of $\leqslant_\Psi$.

   (2) $\omega_1 \not\models \Psi$ and $\omega_2 \models \Psi$.
   Since $Mods(\Psi \wedge form(\omega_1, \omega_2)) = \{\omega_2\}$, then $Mods(\Psi \circ form(\omega_1, \omega_2)) = \{\omega_2\}$ by (R*2). Hence, $\omega_1 \not\leqslant_\Psi \omega_2$ follows given that $\omega_1 \not\models \Psi$. This is a contradiction, which means the case is impossible.

   (3) $\omega_1 \not\models \Psi$ and $\omega_2 \not\models \Psi$.
   We have two subcases:

      (1) $Mods(\Psi \circ form(\omega_1, \omega_2, \omega_3)) = \{\omega_3\}$.
      By setting μ to $form(\omega_1, \omega_2, \omega_3)$ and ϕ to $form(\omega_2, \omega_3)$ in (R*5) and (R*6), we obtain

      $$\begin{aligned} &Mods(\Psi \circ form(\omega_1, \omega_2, \omega_3)) \cap \{\omega_2, \omega_3\} \\ &\quad = Mods(\Psi \circ form(\omega_2, \omega_3)). \end{aligned}$$

      Hence, $Mods(\Psi \circ form(\omega_2, \omega_3)) = \{\omega_3\}$ and $\omega_2 \not\leqslant_\Psi \omega_3$ since $\omega_2 \not\models \Psi$. A contradiction, which means the case is impossible.

      (2) $Mods(\Psi \circ form(\omega_1, \omega_2, \omega_3)) \neq \{\omega_3\}$.
      Since $\omega_1 \leqslant_\Psi \omega_2$ and $\omega_1 \not\models \Psi$, we have $\omega_1 \models \Psi \circ form(\omega_1, \omega_2)$. By setting μ to $form(\omega_1, \omega_2, \omega_3)$ and ϕ to $form(\omega_1, \omega_2)$ in (R*5) and (R*6), we obtain

      $$\begin{aligned} &Mods(\Psi \circ form(\omega_1, \omega_2, \omega_3)) \cap \{\omega_1, \omega_2\} \\ &\quad = Mods(\Psi \circ form(\omega_1, \omega_2)). \end{aligned}$$

      Hence, $\omega_1 \models \Psi \circ form(\omega_1, \omega_2, \omega_3)$. By setting μ to $form(\omega_1, \omega_2, \omega_3)$ and ϕ to $form(\omega_1, \omega_3)$ in (R*5) and (R*6), we also obtain

      $$\begin{aligned} &Mods(\Psi \circ form(\omega_1, \omega_2, \omega_3)) \cap \{\omega_1, \omega_3\} \\ &\quad = Mods(\Psi \circ form(\omega_1, \omega_3)). \end{aligned}$$

      Hence, $\omega_1 \models \Psi \circ form(\omega_1, \omega_3)$. Therefore, $\omega_1 \leqslant_\Psi \omega_3$.

The assignment mapping Ψ to $\leqslant_\Psi$ is faithful:

(1) $\omega_1, \omega_2 \models \Psi$ only if $\omega_1 =_\Psi \omega_2$.
Follows immediately from the definition of $\leqslant_\Psi$.

(2) $\omega_1 \models \Psi$ and $\omega_2 \not\models \Psi$ only if $\omega_1 <_\Psi \omega_2$.
Suppose that $\omega_1 \models \Psi$ and $\omega_2 \not\models \Psi$. Then $Mods(\Psi \circ form(\omega_1, \omega_2)) = \{\omega_1\}$ follows from (R*2). Therefore, $\omega_1 \leqslant_\Psi \omega_2$ and $\omega_2 \not\leqslant_\Psi \omega_1$.

(3) $\Psi = \Phi$ only if $\leqslant_\Psi = \leqslant_\Phi$.
Follows immediately from the definitions of $\leqslant_\Psi$ and $\leqslant_\Phi$ and from (R*4).

The equality $Mods(\Psi \circ \mu) = \min(Mods(\mu), \leqslant_\Psi)$ *holds*:

Follows immediately when μ is not satisfiable. Suppose that μ is satisfiable.

- $Mods(\Psi \circ \mu) \subseteq \min(Mods(\mu), \leqslant_\Psi)$.

  Suppose that $\omega \models \Psi \circ \mu$ and $\omega \notin \min(Mods(\mu), \leqslant_\Psi)$. We will prove a contradiction. Given the supposition, we must have $\omega' \models \mu$ where $\omega' <_\Psi \omega$.

  (1) $\omega' \models \Psi$.

  $\Psi \wedge \mu$ is satisfiable and, by (R*2), $\Psi \circ \mu = \Psi \wedge \mu$. Therefore, $\omega \models \Psi$ since $\omega \models \Psi \circ \mu$. This leads to $\omega \leqslant_\Psi \omega'$ which is a contradiction.

  (2) $\omega' \models \Psi \circ form(\omega, \omega')$ and $\omega \not\models \Psi \circ form(\omega, \omega')$.

  By (R*5) and $\mu \wedge form(\omega, \omega') \equiv form(\omega, \omega')$, we have

  $$Mods(\Psi \circ \mu) \cap \{\omega, \omega'\} \subseteq Mods(\Psi \circ form(\omega, \omega')).$$

  Since $\omega \not\models \Psi \circ form(\omega, \omega')$, we conclude $\omega \not\models \Psi \circ \mu$, which is a contradiction.

- $\min(Mods(\mu), \leqslant_\Psi) \subseteq Mods(\Psi \circ \mu)$.

  Suppose that $\omega \in \min(Mods(\mu), \leqslant_\Psi)$ and $\omega \not\models \Psi \circ \mu$. We will prove a contradiction. Since μ is satisfiable, there must exist ω' such that $\omega' \models \Psi \circ \mu$ by (R*3). By (R*5) and (R*6) and since $\mu \wedge form(\omega, \omega') = form(\omega, w')$, we have

  $$Mods(\Psi \circ \mu) \cap form(\omega, \omega') = Mods(\Psi \circ form(\omega, \omega')).$$

  Since $\omega' \models \Psi \circ \mu$ and $\omega \not\models \Psi \circ \mu$, we have $Mods(\Psi \circ form(\omega, \omega')) = \{\omega'\}$. Since $\omega \models \min(Mods(\mu), \leqslant_\Psi)$, we also have $\omega \leqslant_\Psi \omega'$. Given that $\omega \not\models \Psi \circ form(\omega, \omega')$, $\omega \models \Psi$. Therefore, $\omega \models \Psi \circ \mu$ follows from (R*2), which is a contradiction.

($\Leftarrow$) Suppose that a faithful assignment exists which maps each epistemic state Ψ to a total pre-order $\leqslant_\Psi$ such that

$$Mods(\Psi \circ \mu) = \min(Mods(\mu), \leqslant_\Psi).$$

(R*1) $\Psi \circ \mu$ implies μ.

Follows immediately from the definition of $\circ$.

(R*2) If $\Psi \wedge \mu$ is satisfiable, then $\Psi \circ \mu \equiv \Psi \wedge \mu$.

It suffices to show that if $Mods(\Psi \wedge \mu)$ is not empty, then $Mods(\Psi \circ \mu) = Mods(\Psi \wedge \mu)$. Suppose that $Mods(\Psi \wedge \mu)$ is not empty.

- $Mods(\Psi \circ \mu) \subseteq Mods(\Psi \wedge \mu)$.

  Suppose that $\omega \models \Psi \circ \mu$ and $\omega \not\models \Psi \wedge \mu$. Then $\omega \models \mu$ and $\omega \not\models \Psi$. Moreover, there must exist $\omega' \models \Psi \wedge \mu$ and $\omega' <_\Psi \omega$ by properties of faithful assignments. Therefore, ω cannot be minimal in $Mods(\mu)$ under $\leqslant_\Psi$, which is a contradiction with $\omega \models \Psi \circ \mu$.

- $Mods(\Psi \wedge \mu) \subseteq Mods(\Psi \circ \mu)$.

  Suppose that $\omega \models \Psi \wedge \mu$. Then ω must be minimal in $Mods(\mu)$ under $\leqslant_\Psi$ by properties of faithful assignments. Hence, $\omega \in \min(Mods(\mu), \leqslant_\Psi) = Mods(\Psi \circ \mu)$.

(R*3) If μ is satisfiable, then $\Psi \circ \mu$ is also satisfiable.

Follows immediately from the definition of $\circ$.

(R*4) If $\Psi_1 = \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Psi_1 \circ \mu_1 \equiv \Psi_2 \circ \mu_2$.

Follows immediately from the definition of $\circ$ and properties of faithful assignments.

(R*5) $(\Psi \circ \mu) \wedge \phi$ implies $\Psi \circ (\mu \wedge \phi)$.
Suppose that $\omega \models (\Psi \circ \mu) \wedge \phi$ and $\omega \not\models \Psi \circ (\mu \wedge \phi)$. Then $\omega \models \mu \wedge \phi$ and there must exist $\omega' \models \mu \wedge \phi$ where $\omega' <_\Psi \omega$. Therefore, ω cannot be minimal in $Mods(\mu)$ under $\leqslant_\Psi$, which is a contradiction.

(R*6) If $(\Psi \circ \mu) \wedge \phi$ is satisfiable, then $\Psi \circ (\mu \wedge \phi)$ implies $(\Psi \circ \mu) \wedge \phi$.
Suppose that $(\Psi \circ \mu) \wedge \phi$ is satisfiable, $\omega \models \Psi \circ (\mu \wedge \phi)$, and $\omega \not\models (\Psi \circ \mu) \wedge \phi$. Since $\omega \models \phi$, we have $\omega \not\models \Psi \circ \mu$. Given that $(\Psi \circ \mu) \wedge \phi$ is satisfiable, there must exist some $\omega' \models (\Psi \circ \mu) \wedge \phi$ and $\omega' \models \mu \wedge \phi$. This implies $\omega \leqslant_\Psi \omega'$ since $\omega \models \Psi \circ (\mu \wedge \phi)$. Given that $\omega' \models \Psi \circ \mu$, we have $\omega' \in \min(Mods(\mu), \leqslant_\Psi)$. Therefore, $\omega \in \min(Mods(\mu), \leqslant_\Psi)$ and $\omega \models \Psi \circ \mu$. A contradiction.

Proof of Theorem 11

($\Rightarrow$) Suppose that (CB) holds. Assume $\omega_1, \omega_2 \models \neg(\Psi \circ \mu)$. We want to show $\omega_1 \leqslant_\Psi \omega_2$ iff $\omega_1 \leqslant_{\Psi \circ \mu} \omega_2$. Let α be such that $Mods(\alpha) = \{\omega_1, \omega_2\}$. Then $\alpha \models \neg(\Psi \circ \mu)$, $\Psi \circ \mu \models \neg\alpha$ and $(\Psi \circ \mu) \circ \alpha \equiv \Psi \circ \alpha$ by postulate (CB). Hence, $\min(\{\omega_1, \omega_2\}, \leqslant_{\Psi \circ \mu}) = \min(\{\omega_1, \omega_2\}, \leqslant_\Psi)$ and $\omega_1 \leqslant_\Psi \omega_2$ iff $\omega_1 \leqslant_{\Psi \circ \mu} \omega_2$.

($\Leftarrow$) Suppose that (CBR) holds. Assume $\Psi \circ \mu \models \neg\alpha$. We want to show $\Psi \circ \alpha \equiv (\Psi \circ \mu) \circ \alpha$. We have $\alpha \models \neg(\Psi \circ \mu)$. Moreover, $\leqslant_\Psi$ and $\leqslant_{\Psi \circ \mu}$ are identical on their subdomains $Mods(\neg(\Psi \circ \mu)) \times Mods(\neg(\Psi \circ \mu))$. Therefore, $\min(Mods(\alpha), \leqslant_\Psi) = \min(Mods(\alpha), \leqslant_{\Psi \circ \mu})$ and $\Psi \circ \alpha \equiv (\Psi \circ \mu) \circ \alpha$.

Proof of Theorem 13

(1) Postulate (C1) is equivalent to (CR1).
($\Rightarrow$) Suppose that (CR1) holds. Assume $\alpha \models \mu$. We want to show that $\Psi \circ \alpha \equiv (\Psi \circ \mu) \circ \alpha$. Condition (CR1) implies that $\leqslant_\Psi$ and $\leqslant_{\Psi \circ \mu}$ are equivalent on their subdomain $Mods(\alpha) \times Mods(\alpha)$ since $\alpha \models \mu$. Hence,

$$\begin{aligned} Mods(\Psi \circ \alpha) &\equiv \min(\alpha, \leqslant_\Psi) \\ &\equiv \min(\alpha, \leqslant_{\Psi \circ \mu}) \\ &\equiv Mods((\Psi \circ \mu) \circ \alpha), \end{aligned}$$

and $\Psi \circ \alpha \equiv (\Psi \circ \mu) \circ \alpha$.
($\Leftarrow$) Suppose that (C1) holds. Assume $\omega_1, \omega_2 \models \mu$. We want to show $\omega_1 \leqslant_\Psi \omega_2$ iff $\omega_1 \leqslant_{\Psi \circ \mu} \omega_2$. Let α be such that $Mods(\alpha) = \{\omega_1, \omega_2\}$. Then $\alpha \models \mu$ and $\Psi \circ \alpha \equiv (\Psi \circ \mu) \circ \alpha$ by postulate (C1). Hence, $\min(\{\omega_1, \omega_2\}, \leqslant_\Psi) = \min(\{\omega_1, \omega_2\}, \leqslant_{\Psi \circ \mu})$ and $\omega_1 \leqslant_\Psi \omega_2$ iff $\omega_1 \leqslant_{\Psi \circ \mu} \omega_2$.

(2) Postulate (C2) is equivalent to (CR2).
Proof is symmetric with the one above.

(3) Postulate (C3) is equivalent to (CR3).
($\Rightarrow$) Suppose that (CR3) holds. Assume $\Psi \circ \alpha \models \mu$. We want to show that $(\Psi \circ \mu) \circ \alpha \models \mu$. By Lemma 10, there exists $\omega \models \alpha \wedge \mu$ such that $\omega <_\Psi \omega'$ for any $\omega' \models \alpha \wedge \neg\mu$. Therefore, by condition (CR3), there exists $\omega \models \alpha \wedge \mu$ such that $\omega <_{\Psi \circ \mu} \omega'$ for any $\omega' \models \alpha \wedge \neg\mu$. Hence, by Lemma 10, $(\Psi \circ \mu) \circ \alpha \models \mu$.

($\Leftarrow$) Suppose that (C3) holds. Assume $\omega_1 \models \mu$, $\omega_2 \models \neg\mu$ and $\omega_1 <_\Psi \omega_2$. We want to show $\omega_1 <_{\Psi\circ\mu} \omega_2$. Let α be such that $Mods(\alpha) = \{\omega_1, \omega_2\}$. Then $\Psi \circ \alpha \models \mu$ by Lemma 10 since $\omega_1 \models \alpha \wedge \mu$, $\omega_1 <_\Psi \omega_2$ and $Mods(\alpha \wedge \neg\mu) = \{\omega_2\}$. Then $(\Psi \circ \mu) \circ \alpha \models \mu$ by postulate (C3). Moreover, $\omega_1 <_{\Psi\circ\mu} \omega_2$ by Lemma 10 since $Mods(\alpha \wedge \mu) = \{\omega_1\}$ and $Mods(\alpha \wedge \neg\mu) = \{\omega_2\}$.

(4) Postulate (C4) is equivalent to (CR4).

($\Rightarrow$) Suppose that (CR4) holds. Assume $(\Psi\circ\mu)\circ\alpha \models \neg\mu$. We want to show $\Psi \circ \alpha \models \neg\mu$.[10] By Lemma 10, there exists $\omega \models \alpha \wedge \neg\mu$ such that $\omega <_{\Psi\circ\mu} \omega'$ for all $\omega' \models \alpha \wedge \mu$. Moreover, by the contrapositive of condition (CR4), there exists $\omega \models \alpha\wedge\neg\mu$ such that $\omega <_\Psi \omega'$ for all $\omega' \models \alpha\wedge\mu$. Hence, by Lemma 10, we have $\Psi \circ \alpha \models \neg\mu$.

($\Leftarrow$) Suppose that (C4) holds. Assume $\omega_1 \models \mu$, $\omega_2 \models \neg\mu$ and $\omega_2 <_{\Psi\circ\mu} \omega_1$. We want to show $\omega_2 <_\Psi \omega_1$.[11] Let α be such that $Mods(\alpha) = \{\omega_1, \omega_2\}$. Then $(\Psi \circ \mu) \circ \alpha \models \neg\mu$ by Lemma 10 since $\omega_2 \models \alpha \wedge \neg\mu$, $\omega_2 <_{\Psi\circ\mu} \omega_1$ and $Mods(\alpha\wedge\mu) = \{\omega_1\}$. Then $\Psi\circ\alpha \models \neg\mu$ by postulate (C4). Moreover, $\omega_2 <_\Psi \omega_1$ by Lemma 10 since $Mods(\alpha \wedge \neg\mu) = \{\omega_2\}$ and $Mods(\alpha \wedge \mu) = \{\omega_1\}$.

Lemma B.1. *Let the total pre-order of a ranking κ be defined as follows:*

$$\omega_1 \leqslant_\kappa \omega_2 \stackrel{\text{def}}{=} \kappa(\omega_1) \leqslant \kappa(\omega_2).$$

We then have

$$Mods(Bel(\kappa \bullet \mu)) = \min(Mods(\mu), \leqslant_\kappa),$$

and

(1) $\omega_1, \omega_2 \models Bel(\kappa)$ *only if* $\omega_1 =_\kappa \omega_2$.

(2) $\omega_1 \models Bel(\kappa)$ *and* $\omega_2 \models \neg Bel(\kappa)$ *only if* $\omega_1 \leqslant_{\kappa\bullet\mu} \omega_2$.

(3) $\kappa^1 = \kappa^2$ *only if* $\leqslant_{\kappa^1} = \leqslant_{\kappa^2}$.

Here, $\omega_1 <_\kappa \omega_2$ *is defined as* $\omega_1 \leqslant_\kappa \omega_2$ *and* $\omega_2 \not\leqslant_\kappa \omega_1$; *and* $\omega_1 =_\kappa \omega_2$ *is defined as* $\omega_1 \leqslant_\kappa \omega_2$ *and* $\omega_2 \leqslant_\kappa \omega_1$.

Proof. To show that $Mods(Bel(\kappa \bullet \mu)) = \min(Mods(\mu), \leqslant_\kappa)$, we show the following:

- If $\omega \models Bel(\kappa \bullet \mu)$ then $\omega \in \min(Mods(\mu), \leqslant_\kappa)$.
  Suppose that $\omega \models Bel(\kappa \bullet \mu)$. Then $(\kappa \bullet \mu)(\omega) = 0$ by definition of $Bel(\kappa \bullet \mu)$, and $\omega \models \mu$ by definition of $\kappa \bullet \mu$. Moreover, $(\kappa \bullet \mu)(\omega) = \kappa(\omega) - \kappa(\mu) = 0$ and $\kappa(\omega) = \kappa(\mu) = \min_{\omega' \models \mu} \kappa(\omega')$. This implies that $\omega \leqslant_\kappa \omega'$ for all $\omega' \models \mu$ and $\omega \in \min(Mods(\mu), \leqslant_\kappa)$.
- If $\omega \in \min(Mods(\mu), \leqslant_\kappa)$ then $\omega \models Bel(\kappa \bullet \mu)$.
  Suppose that $\omega \in \min(Mods(\mu), \leqslant_\kappa)$. Then $\omega \models \mu$ and $\omega \leqslant_\kappa \omega'$ for all $\omega' \models \mu$. Moreover, $\kappa(\omega) \leqslant \kappa(\omega')$ for all $\omega' \models \mu$ and, hence, $\kappa(\omega) = \kappa(\mu)$. This implies that $(\kappa \bullet \mu)(\omega) = \kappa(\omega) - \kappa(\mu) = 0$ and, hence, $\omega \models Bel(\kappa \bullet \mu)$.

[10] We are proving the contrapositive of postulate (C4).

[11] We are proving the contrapositive of condition (CR4).

The rest of the lemma is shown as follows:

(1) $\omega_1, \omega_2 \models Bel(\kappa)$ only if $\omega_1 =_\kappa \omega_2$.
Suppose that $\omega_1, \omega_2 \models Bel(\kappa)$. Then $\kappa(\omega_1) = \kappa(\omega_2) = 0$ by definition of $Bel(\kappa)$. Therefore, $\omega_1 =_\kappa \omega_2$ by definition of $=_\kappa$.

(2) $\omega_1 \models Bel(\kappa)$ and $\omega_2 \models \neg Bel(\kappa)$ only if $\omega_1 \leqslant_{\kappa \bullet \mu} \omega_2$.
Suppose that $\omega_1 \models Bel(\kappa)$ and $\omega_2 \models \neg Bel(\kappa)$. Then $\kappa(\omega_1) = 0$ and $\kappa(\omega_2) > 0$ by definition of $Bel(\kappa)$. Therefore, $\omega_1 \leqslant_{\kappa \bullet \mu} \omega_2$ by definition of $\leqslant_{\kappa \bullet \mu}$.

(3) $\kappa^1 = \kappa^2$ only if $\leqslant_{\kappa^1} = \leqslant_{\kappa^2}$.
Follows immediately from the definitions of $\leqslant_{\kappa^1}$ and $\leqslant_{\kappa^2}$. □

Lemma B.2. *Let $\leqslant_\kappa$ and $\leqslant_{\kappa \bullet \mu}$ be total pre-orders induced by rankings κ and $\kappa \bullet \mu$. We then have*:

(1) *If $\omega_1 \models \mu$ and $\omega_2 \models \mu$, then $\omega_1 \leqslant_\kappa \omega_2$ iff $\omega_1 \leqslant_{\kappa \bullet \mu} \omega_2$.*
(2) *If $\omega_1 \models \neg\mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 \leqslant_\kappa \omega_2$ iff $\omega_1 \leqslant_{\kappa \bullet \mu} \omega_2$.*
(3) *If $\omega_1 \models \mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 <_\kappa \omega_2$ only if $\omega_1 <_{\kappa \bullet \mu} \omega_2$.*
(4) *If $\omega_1 \models \mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 \leqslant_\kappa \omega_2$ only if $\omega_1 \leqslant_{\kappa \bullet \mu} \omega_2$.*

Proof.

$$(\kappa \bullet \mu)(\omega) = \kappa_{(\mu, \kappa(\neg\mu)+1)}(\omega) = \begin{cases} \kappa(\omega) - \kappa(\mu), & \text{if } \omega \models \mu; \\ \kappa(\omega) + 1, & \text{if } \omega \models \neg\mu. \end{cases}$$

Therefore, conditioning is a shifting process in which the ranks of worlds inside $Mods(\mu)$ are all reduced by $\kappa(\mu)$ and the ranks of worlds inside $Mods(\neg\mu)$ are all increased by 1. This implies the following:

(1) The relative order of worlds inside $Mods(\mu)$ does not change.
(2) The relative order of worlds inside $Mods(\neg\mu)$ does not change.
(3) If a world in $Mods(\mu)$ had a lower rank than a world in $Mods(\neg\mu)$ before the shifting, this will continue to be the case after the shifting.
(4) It is impossible for a world in $Mods(\mu)$ to have a higher rank than a world in $Mods(\neg\mu)$ after the shifting if it did not before the shifting.

The four properties (1)–(4) then hold. □

Proof of Theorem 14

The fact that operator $\bullet$ satisfies postulates (R*1)–(R*6) follows immediately from Lemma B.1 and Theorem 9. That it satisfies postulates (C1)–(C4) follows immediately from Lemma B.2 and Theorem 13.

Proof of Theorem 15

The revision operator $\diamond$ is defined as follows:

$$(\kappa \diamond \mu)(\omega) = \begin{cases} \kappa(\omega) - \kappa(\mu), & \text{if } \omega \models \mu; \\ \kappa(\omega) + 1, & \text{if } \omega \models \neg\mu,\ \kappa(\omega) < 2; \\ \kappa(\omega) - 1, & \text{if } \omega \models \neg\mu,\ \kappa(\omega) \geqslant 2; \end{cases}$$

Table B.1
A scenario that violates (C3)

world	κ	$\kappa \diamond \alpha$	$\kappa \diamond \mu$	$(\kappa \diamond \mu) \diamond \alpha$
ω_1	0	1	0	1
ω_2	3	0	3	0
ω_3	4	1	3	0
ω_4	0	1	1	2

Table B.2
A scenario that violates (C4)

world	κ	$\kappa \diamond \alpha$	$\kappa \diamond \mu$	$(\kappa \diamond \mu) \diamond \alpha$
ω_1	0	1	0	1
ω_2	3	0	3	1
ω_3	3	0	2	0
ω_4	0	1	1	2

This operator is exactly like $\bullet$ except for one thing: it decrements the rank of every world inside $Mods(\neg\mu)$ if the world's rank is no less than 2. Therefore, the relative order of worlds inside $Mods(\mu)$ is preserved, but other ordering relations (especially between worlds in $Mods(\mu)$ and $Mods(\neg\mu)$) are purturbed. This causes the operator to violate Properties (2), (3) and (4) of Lemma B.2, while continuing to satisfy (1).

Since operator $\diamond$ is equivalent to $\bullet$ for worlds inside $Mods(\mu)$, Lemma B.1 holds for $\diamond$ as well. Therefore, Theorem 14 also holds for $\diamond$.

Since operator $\diamond$ satisfies (1) of Lemma B.2, it also satisfies postulate (C1) according to Theorem 14.

To show that operator $\diamond$ does not satisfy postulate (C3), consider Table B.1 where $Mods(\alpha) = \{\omega_2, \omega_3\}$ and $Mods(\mu) = \{\omega_1, \omega_2\}$. We have $Bel(\kappa \diamond \alpha) \models \mu$, but $Bel((\kappa \diamond \mu) \diamond \alpha) \not\models \mu$, which violates postulate (C3).

To show that operator $\diamond$ does not satisfy postulate (C4), consider Table B.2 where $Mods(\alpha) = \{\omega_2, \omega_3\}$ and $Mods(\mu) = \{\omega_1, \omega_2\}$. We have $Bel(\kappa \diamond \alpha) \not\models \neg\mu$, but $Bel((\kappa \diamond \mu) \diamond \alpha) \models \neg\mu$, which violates postulate (C4).

Proof of Theorem 16

(1) Postulate (C5) is equivalent to (CR5).

($\Rightarrow$) Suppose that (CR5) holds. Assume $\Psi \circ \mu \models \neg\alpha$ and $(\Psi \circ \mu) \circ \alpha \models \mu$. We want to show $\Psi \circ \alpha \models \mu$.[12] By Lemma 10, there exists $\omega_3 \models \mu \wedge \neg\alpha$ such that $\omega_3 <_\Psi \omega$ for all $\omega \models \mu \wedge \alpha$. Also by Lemma 10, there exists $\omega_1 \models \alpha \wedge \mu$ such that $\omega_1 <_{\Psi \circ \mu} \omega_2$ for all $\omega_2 \models \alpha \wedge \neg\mu$. Therefore, by condition (CR5) and since $\omega_3 <_\Psi \omega_1$, there exists $\omega_1 \models \alpha \wedge \mu$ such that $\omega_1 <_\Psi \omega_2$ for all $\omega_2 \models \alpha \wedge \neg\mu$. Hence, by Lemma 10, we have $\Psi \circ \alpha \models \mu$.

[12] We are proving the contrapositive of postulate (C5).

($\Leftarrow$) Suppose that (C5) holds. Assume $\omega_1, \omega_3 \models \mu$, $\omega_2 \models \neg\mu$, $\omega_3 <_\Psi \omega_1$ and $\omega_1 <_{\Psi\circ\mu} \omega_2$. We want to show $\omega_1 <_\Psi \omega_2$.[13] Let α be such that $Mods(\alpha) = \{\omega_1, \omega_2\}$. Then $\Psi \circ \mu \models \neg\alpha$ by Lemma 10 since $\omega_3 \models \mu \wedge \neg\alpha$, $\omega_3 <_\Psi \omega_1$ and $Mods(\mu \wedge \alpha) = \{\omega_1\}$. Also, $(\Psi \circ \mu) \circ \alpha \models \mu$ by Lemma 10 since $\omega_1 \models \alpha \wedge \mu$, $\omega_1 <_{\Psi\circ\mu} \omega_2$ and $Mods(\alpha \wedge \neg\mu) = \{\omega_2\}$. Then $\Psi \circ \alpha \models \mu$ by the contrapositive of postulate (C5). Moreover, $\omega_1 <_\Psi \omega_2$ by Lemma 10 since $Mods(\mu \wedge \alpha) = \{\omega_1\}$ and $Mods(\alpha \wedge \neg\mu) = \{\omega_2\}$.

(2) Postulate (C6) is equivalent to (CR6).

($\Rightarrow$) Suppose that (CR6) holds. Assume $\Psi \circ \mu \models \neg\alpha$ and $\Psi \circ \alpha \models \neg\mu$. We want to show $(\Psi \circ \mu) \circ \alpha \models \neg\mu$. By Lemma 10, there exists $\omega_3 \models \mu \wedge \neg\alpha$ such that $\omega_3 <_\Psi \omega$ for all $\omega \models \mu \wedge \alpha$. Also by Lemma 10, there exists $\omega_2 \models \alpha \wedge \neg\mu$ such that $\omega_2 <_\Psi \omega_1$ for all $\omega_1 \models \alpha \wedge \mu$. Therefore, by condition (CR6) and since $\omega_3 <_\Psi \omega_1$, there exists $\omega_2 \models \alpha \wedge \neg\mu$ such that $\omega_2 <_{\Psi\circ\mu} \omega_1$ for all $\omega_1 \models \alpha \wedge \mu$. Hence, by Lemma 10, we have $(\Psi \circ \mu) \circ \alpha \models \neg\mu$.

($\Leftarrow$) Suppose that (C6) holds. Assume $\omega_1, \omega_3 \models \mu$, $\omega_2 \models \neg\mu$, $\omega_3 <_\Psi \omega_1$ and $\omega_2 <_\Psi \omega_1$. We want to show $\omega_2 <_{\Psi\circ\mu} \omega_1$. Let α be such that $Mods(\alpha) = \{\omega_1, \omega_2\}$. Then $\Psi \circ \mu \models \neg\alpha$ by Lemma 10 since $\omega_3 \models \mu \wedge \neg\alpha$, $\omega_3 <_\Psi \omega_1$ and $Mods(\mu \wedge \alpha) = \{\omega_1\}$. Moreover, $\Psi \circ \alpha \models \neg\mu$ by Lemma 10 since $\omega_2 \models \alpha \wedge \neg\mu$, $\omega_2 <_\Psi \omega_1$ and $Mods(\alpha \wedge \mu) = \{\omega_1\}$. Then $(\Psi \circ \mu) \circ \alpha \models \neg\mu$ by postulate (C6). Moreover, $\omega_2 <_{\Psi\circ\mu} \omega_1$ since $Mods(\alpha \wedge \neg\mu) = \{\omega_2\}$ and $Mods(\alpha \wedge \mu) = \{\omega_1\}$.

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