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Logical characterizations of regular equivalence in weighted social networks ☆

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ABSTRACT

Social network analysis is a methodology used extensively in social science. Classical social networks can only represent the qualitative relationships between actors, but weighted social networks can describe the degrees of connection between actors. In a classical social network, regular equivalence is used to capture the similarity between actors based on their links to other actors. Specifically, two actors are deemed regularly equivalent if they are equally related to equivalent others. The definition of regular equivalence has been extended to weighted social networks in two ways. The first definition, called *regular similarity*, considers regular equivalence as an equivalence relation that commutes with the underlying graph edges; while the second definition, called *generalized regular equivalence*, is based on the notion of role assignment or coloring. A role assignment (resp. coloring) is a mapping from the set of actors to a set of roles (resp. colors). The mapping is regular if actors assigned to the same role have the same roles in their neighborhoods. Recently, it was shown that social positions based on regular equivalence can be syntactically expressed as well-formed formulas in a kind of modal logic. Thus, actors occupying the same social position based on regular equivalence will satisfy the same set of modal formulas. In this paper, we present analogous results for regular similarity and generalized regular equivalence based on many-valued modal logics.

1. Introduction

Social network analysis (SNA) is a methodology used extensively in social and behavioral sciences, as well as in political science, economics, organization theory, and industrial engineering [58,34,64]. Positional analysis of a social network tries to find similarities between nodes in the network [6,9,25,45,65]. While many traditional clustering methods are based on the attributes of the individual nodes, SNA is more concerned with the structural similarity between the nodes. In SNA, a category, called a *social role* or *social position*, is defined in terms of the similarities of the patterns of relations between the nodes, rather than the attributes of the nodes. For example, one useful way to think about the social role "husband" is to consider it as a set of patterned interactions with a member or members of some other social categories, such as "wife" and "child" (and probably others) [34]. One of the most widely studied notions in the positional analysis of social networks is called *regular equivalence* [6,20,56,57]. According to Borgatti and Everett [6], two actors are regularly equivalent if they are equally related to equivalent others.

☆ This is a significantly extended version of [26].

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Interestingly, Marx and Masush [51] showed that social positions based on regular equivalence can be syntactically expressed as well-formed formulas in a kind of modal logic. Thus, actors that have the same social position based on regular equivalence will satisfy the same set of modal formulas. Traditionally, modal logic has been considered the logic for reasoning about modalities, such as necessity, possibility, time, actions, beliefs, knowledge, and obligations. However, semantically, it is essentially a language for describing relational structures [3]. A relational structure is simply a collection of relations on a given universe; therefore, social networks can be represented by relational structures in mathematics. The logical characterization of social positions implies that modal formulas are semantically invariant with respect to regular equivalence.

In recent years, weighted social networks have also received considerable attention because they can represent both the qualitative relationships and the degrees of connection between nodes [2,27,28,43,54,63]. The notion of regular equivalence is extended to weighted social networks based on two alternative definitions of regular equivalence [28]. While the two definitions are equivalent for ordinary networks, they induce different generalizations for weighted networks. The first generalization, called *regular similarity*, is based on the definition of regular equivalence as an equivalence relation that commutes with the underlying graph edges [9]. By the definition, regular similarity is a fuzzy relation that describes the degree of similarity between actors in the network. The second generalization, called *generalized regular equivalence*, is based on the definition of role assignment or coloring [45]. A role assignment (resp. coloring) is a mapping from the set of actors to a set of roles (resp. colors). The mapping is regular if actors assigned to the same role have the same roles in their neighborhoods. Consequently, generalized regular equivalence is an equivalence relation that can determine the role partition of actors in a weighted social network.

Because of the importance of weighted social networks, we explore the logical characterizations of regular similarity and generalized regular equivalence. In this paper, we use many-valued modal logics to characterize the two kinds of relations. On one hand, we show that the truth values of many-valued modal logic formulas are invariant with respect to generalized regular equivalence. On the other hand, we demonstrate that the maximum regular similarity between any two actors is equal to the minimum equivalence between degrees of the two actors satisfying many-valued modal logic formulas.

The remainder of this paper is organized as follows. In Section 2, we review some basic concepts about social networks, fuzzy relations, and positional analysis. In Sections 3 and 4, we present the logical characterizations of regular similarity and generalized regular equivalence respectively. In Section 5, we discuss issues related to further generalizations and applications of the logical characterizations. Section 6 contains our concluding remarks.

2. Preliminaries

2.1. Social networks

Social networks are defined by actors and relations (or nodes and edges in terms of graph theory) [34]. Generally, a social network is defined as a relational structure $\mathfrak{N} = (U, (R_i)_{i\in I})$, where U is the set of nodes; I is an index set; and for each $i \in I$, $R_i \subseteq U^{k_i}$ is a k_i-ary relation on the domain U, where k_i is a positive integer. If $k_i = 1$, then R_i is called an attribute or a property. In practice, most SNA methods only consider a simplified version of a social network with binary relations. For ease of presentation, we focus on a social network with unary and/or binary relations. Thus, the *social network* considered in this paper is a structure $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$, where the universe U is a *finite set* of actors; $P_i \subseteq U$ for all $i \in I$; and $R_j \subseteq U \times U$ for all $j \in J$. Although practical social networks are always concerned with finite sets of attributes and relations, our results do not rely on the finitary assumptions about attributes and relations. Therefore, we only assume that the set of actors is finite and do not impose additional restrictions on the index sets I and J. In terms of graph theory, $\mathfrak{N}$ is a labeled graph, where U is a set of nodes labeled with subsets of I, and each R_j denotes a set of (labeled) edges. For each $x \in U$, the out-neighborhood and in-neighborhood of x with respect to a binary relation R, denoted respectively by Rx and R^-x, are defined as follows:

$$Rx = \{y \in U \mid (x, y) \in R\}, \tag{1}$$

$$R^-x = \{y \in U \mid (y, x) \in R\}. \tag{2}$$

If E is an equivalence relation on U and x is an actor, the E-equivalence class of x is equal to its neighborhood, i.e., $[x]_E = Ex = E^-x$. Note that the latter equality holds because of the symmetry of E. For any $X \subseteq U$, we use $[X]_E$ to denote the set $\{[x]_E \mid x \in X\}$.

Several equivalence relations have been proposed for exploring the structural similarity between actors. Among them, regular equivalence has been studied extensively [6,9,25,45,65]. Although there are several definitions of regular equivalence, we only consider two of them in this paper. The first, proposed by Boyd and Everett [9], states that an equivalence relation E is a *regular equivalence* with respect to a binary relation R if it commutes with R; i.e.,

$$E \cdot R = R \cdot E, \tag{3}$$

where $E \cdot R = \{(x, y) \mid \exists z \in U, (x, z) \in E \wedge (z, y) \in R\}$ is the composition of E and R. By this definition, if E is a regular equivalence with respect to R and $(x, y) \in E$, then for each $z \in Rx$ (resp. R^-x), there exists $z' \in Ry$ (resp. R^-y) such that $(z, z') \in E$. The property leads naturally to the second definition of regular equivalence, which is based on role assignment [45]. It states that an equivalence relation E is a regular equivalence with respect to a binary relation R if for $x, y \in U$,

$$(x, y) \in E \Rightarrow \left([Rx]_E = [Ry]_E \text{ and } \left[R^- x\right]_E = \left[R^- y\right]_E\right). \tag{4}$$

According to this definition, if x and y are regularly equivalent, they are connected to equivalent neighborhoods.

Obviously, the above definitions are equivalent. Thus, we have the following definition.

Definition 1. Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a social network and let E be an equivalence relation on U. Then, E is a regular equivalence with respect to $\mathfrak{N}$ if

(1) $(x, y) \in E$ implies that $x \in P_i$ iff $y \in P_i$ for all $i \in I$; and
(2) E is a regular equivalence with respect to R_j for all $j \in J$.

By this definition, there may exist more than one regular equivalence for a given network. However, it has been shown that there is always a maximum (i.e., coarsest) regular equivalence for a network [45]. Then, two actors, x and y, are deemed regularly equivalent in a network $\mathfrak{N}$, denoted by $x \equiv_{\mathfrak{N}} y$, if (x, y) is in the coarsest regular equivalence of the network.

2.2. Modal logic and regular equivalence

Modal logics were originally developed as formalizations for reasoning about modalities. The initial formalization is the alethic modal logic (the logic of necessity and possibility). The modalities $\Box$ (for necessity) and $\Diamond$ (for possibility) have now become standard notations in modal logic literature. Several variants of modal logic have been proposed to deal with different classes of modalities; for example, temporal logic for tense operators, deontic logic for obligation and permission, dynamic logic for actions, and epistemic logic for beliefs and knowledge. The development of these logics was motivated by philosophical enquiry as well as by technical applications in computer science, artificial intelligence, and economic game theory.

While modal logics were initially presented in the form of reasoning systems, the relational semantics have influenced the continuing development of the field [41,42]. The semantics, proposed independently by J. Hintikka, S. Kanger, and S. Kripke, and now known as Kripke semantics, show that modal logics are in fact logics for reasoning about relational structures. From this semantic perspective, standard modal logics can be regarded as fragments of first- or second-order predicate logics, where the necessity and possibility modalities correspond to universal and existential quantifiers respectively. Despite this correspondence, quantification in modal logic tends to be bounded in some way to worlds that are "relevant to" or "accessible from" the current one. Consequently, although a number of properties of modal logics follow immediately from those of their classical quantificational counterparts, the modal operators typically have less expressive power than full quantification. This yields many interesting properties not available in classical predicate logic. One of the most striking results is that semantic invariances between models are actually various forms of *bisimulation*, which preserve the local properties of worlds and their transition patterns [3]. Interestingly, it has been shown that bisimulation in Kripke models corresponds exactly to regular equivalence in social networks [51]. The implication of the results is that position-equivalent actors can be characterized by an appropriate set of modal formulas.

As with any other formal logic, the presentation of a modal logic requires the specification of its syntax and semantics. The syntactic aspect includes the language of the logic—its alphabet and formation rules for formulas, as well as a deductive system for logical reasoning. For the purposes of this paper, we only need consider the language part. With regard to the semantic aspect, we have to define the models within which the formulas can be interpreted, as well as the satisfaction of a formula in a model.

We start with propositional modal logic (PML) [14]. The alphabet of PML comprises a set of propositional symbols PV, the logical constant $\bot$, the logical connectives $\wedge$ and $\rightarrow$, and the modal operator $\Diamond$. The set of formulas of PML is the smallest set containing $PV \cup \{\bot\}$ that satisfies the following conditions:

- if φ and ψ are formulas, then $\varphi \wedge \psi$ and $\varphi \rightarrow \psi$ are formulas,
- if φ is a formula, then $\Diamond\varphi$ is a formula.

As usual, we abbreviate $\varphi \rightarrow \bot$ as $\neg\varphi$; $\neg(\neg\varphi \wedge \neg\psi)$ as $\varphi \vee \psi$; $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ as $\varphi \leftrightarrow \psi$; $\neg\Diamond\neg\varphi$ as $\Box\varphi$; and $\neg\bot$ as $\top$. A Kripke model for PML is a triple $\mathfrak{M} = (W, R, V)$, where W is a set of possible worlds, R is a binary relation on W, called an accessibility relation, and $V : W \times PV \rightarrow \{0, 1\}$ is a truth assignment for evaluating the truth value of each propositional symbol in each world. The satisfaction of a formula φ in a world w of the model $\mathfrak{M}$, denoted by $\mathfrak{M}, w \models \varphi$, is defined by the following clauses:

(1) $\mathfrak{M}, w \models p$ if $V(w, p) = 1$ for each $p \in PV$;
(2) $\mathfrak{M}, w \not\models \bot$;

(3) $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$;
(4) $\mathfrak{M}, w \models \varphi \rightarrow \psi$ iff $\mathfrak{M}, w \not\models \varphi$ or $\mathfrak{M}, w \models \psi$;
(5) $\mathfrak{M}, w \models \Diamond\varphi$ iff there exists $(w, u) \in R$ such that $\mathfrak{M}, u \models \varphi$.

Bisimulation, an important notion in the model theory of modal logics, is very useful for studying the expressibility of modal languages. We present the following definition of bisimulation for PML [3].

Definition 2. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two Kripke models. A non-empty binary relation $Z \subseteq W \times W'$ is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$, denoted by $Z : \mathfrak{M} \underline{\leftrightarrow} \mathfrak{M}'$ if the following conditions are satisfied:

(1) If $(w, w') \in Z$, then $V(w, p) = V(w', p)$ for any $p \in PV$ (base).
(2) If $(w, w') \in Z$ and $(w, u) \in R$, then there exists $u' \in W'$ such that $(u, u') \in Z$ and $(w', u') \in R'$ (forth).
(3) If $(w, w') \in Z$ and $(w', u') \in R'$, then there exists $u \in W$ such that $(u, u') \in Z$ and $(w, u) \in R$ (back).

When Z is a bisimulation that links two worlds w in $\mathfrak{M}$ and w' in $\mathfrak{M}'$ (i.e., $(w, w') \in Z$), we say that w and w' are *bisimular*. In the context of this paper, we are especially interested in the bisimulation between a model and itself. Thus, when we discuss bisimular worlds in this paper, we always mean possible worlds in the same model.

Propositional multi-modal logic (PMML) is an extension of PML that allows more than one modality [36]. It is particularly suitable for reasoning about relational structures that contain a number of binary relations, e.g., social networks. The alphabet of PMML comprises a set of propositional symbols PV, a set of relational symbols REL, the logical constant, the logical connectives, the relational converse symbol $^-$, and the modality-forming symbol $\langle\rangle$. The set of formulas of PMML is the smallest set containing $PV \cup \{\bot\}$ that satisfies the following conditions:

- if φ and ψ are formulas, then $\varphi \wedge \psi$ and $\varphi \rightarrow \psi$ are formulas,
- if φ is a formula and α is a relational symbol, then $\langle\alpha\rangle\varphi$ and $\langle\alpha^-\rangle\varphi$ are formulas.

We abbreviate $\neg\langle\alpha\rangle\neg\varphi$ and $\neg\langle\alpha^-\rangle\neg\varphi$ as $[\alpha]\varphi$ and $[\alpha^-]\varphi$ respectively. A Kripke model for PMML is a triple $\mathfrak{M} = (W, (R_\alpha)_{\alpha\in REL}, V)$, where W and V are the same as above, and for each $\alpha \in REL$, R_α is a binary relation on W. The satisfaction of PMML formulas is defined in the same way as that of PML formulas, except that clause (5) is replaced by

(5) $\mathfrak{M}, w \models \langle\alpha\rangle\varphi$ iff there exists $(w, u) \in R_\alpha$ such that $\mathfrak{M}, u \models \varphi$;
(6) $\mathfrak{M}, w \models \langle\alpha^-\rangle\varphi$ iff there exists $(u, w) \in R_\alpha$ such that $\mathfrak{M}, u \models \varphi$,

for any $\alpha \in REL$. Note that we need the converse modalities $\langle\alpha^-\rangle$ because all positional equivalences are defined with respect to their in-neighborhoods and out-neighborhoods. The modality $\langle\alpha\rangle$ allows us to access the out-neighborhood of a world, while $\langle\alpha^-\rangle$ is needed to access the in-neighborhood. It is straightforward to extend the notion of bisimulation to PMML models.

The first logical characterization of positional equivalences is presented for regular equivalences with a particular type of PMML [51]. It is demonstrated by the connection between regular equivalence and bisimulation. Given a social network $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$, we define a PMML language with the following basic symbols:

(1) $PV = \{p_i \mid i \in I\}$;
(2) $REL = \{\alpha_j \mid j \in J\}$.

The social network $\mathfrak{N}$ is transformed into a Kripke model $\mathfrak{M}_\mathfrak{N} = (U, (R_j)_{j\in J}, V)$, where V is defined by $V(x, p_i) = 1$ iff $x \in P_i$ for all $x \in U$ and $i \in I$, and R_j denotes R_{α_j} for $j \in J$. Let $\mathcal{L}_\mathfrak{N}$ be the set of formulas of the PMML. Then, we say that two actors, x and y, are *modally equivalent* with respect to $\mathfrak{N}$ if for all $\varphi \in \mathcal{L}_\mathfrak{N}$, $\mathfrak{M}_\mathfrak{N}, x \models \varphi$ iff $\mathfrak{M}_\mathfrak{N}, y \models \varphi$.

Theorem 1. *(See [51].) Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a social network. Then, for all $x, y \in U$, $x \equiv_\mathfrak{N} y$ iff x and y are modally equivalent with respect to $\mathfrak{N}$.*

Proof. Marx and Masuch [51] showed that regular equivalences on $\mathfrak{N}$ are bisimulations between $\mathfrak{M}_\mathfrak{N}$ and itself, i.e., $x \equiv_\mathfrak{N} y$ iff x and y are bisimular. By the Hennessy–Milner Theorem (see Theorem 2.24 in [3]), x and y are bisimular iff they are modally equivalent with respect to $\mathfrak{N}$. $\square$

2.3. Weighted social networks

Social networks can model the interactions and connections between actors. However, in most real-world networks, the ties in a network do not have the same capacity. In fact, ties are often associated with weights that differentiate them in terms of their strength, intensity, or capacity [2,10,18,19,27,28,43,44,52,54,59,63]. Mathematically, we can use fuzzy sets and relations to model weighted social networks. The elements of fuzzy sets have different degrees of membership [66], which are typically drawn from the unit interval $[0,1]$. Formally, a fuzzy set P on the domain U is a membership function $P: U \to [0,1]$. Alternatively, a fuzzy set can be represented as a class of crisp sets $\{P_c \mid c \in [0,1]\}$, where $P_c = \{u \in U \mid P(u) \geq c\}$ is called the c-cut of P. A *t-norm* operation on $[0,1]$ is normally used to define the intersection of fuzzy sets. A t-norm is a binary operation $\otimes$ on $[0,1]$ that satisfies commutativity and associativity, and is non-decreasing in both arguments; and $1 \otimes c = c$ and $0 \otimes c = 0$ for all $c \in [0,1]$ [31]. The *residuum* of a t-norm $\otimes$ is a binary operation $\Rightarrow$ on $[0,1]$ defined as $a \Rightarrow b = \sup\{c \mid a \otimes c \leq b\}$ for all $a, b \in [0,1]$. Note that the residuum operation satisfies the following property:

$$\min(a \Rightarrow c, b \Rightarrow c) = \max(a, b) \Rightarrow c. \tag{5}$$

Furthermore, the residuum defines its corresponding unary operation of the *precomplement*[1] $-c = c \Rightarrow 0$. In this paper, we mainly use the well-known Gödel t-norm $a \otimes b = \min(a, b)$. Hence, its corresponding residuum is defined by

$$a \Rightarrow b = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{otherwise} \end{cases} \tag{6}$$

and its corresponding precomplement is the Gödel negation defined by

$$-a = \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

In addition, we use $a \Leftrightarrow b$ to denote $\min(a \Rightarrow b, b \Rightarrow a)$. It is easy to see that

$$a \Leftrightarrow b = \begin{cases} 1, & \text{if } a = b, \\ \min(a, b), & \text{otherwise.} \end{cases} \tag{8}$$

In fuzzy set theory, a fuzzy binary relation R on U is a fuzzy set on the domain $U \times U$. Thus, a fuzzy binary relation R can be represented as the membership function $R: U \times U \to [0,1]$. Obviously, a fuzzy binary relation is a generalization of a binary relation, so the upper-case letters R, S, T, etc., are used to denote both fuzzy and crisp relations. Because we only consider fuzzy binary relations in this paper, we call them fuzzy relations hereafter; and the term "binary relation" means crisp relations only. A fuzzy relation R is included in another fuzzy relation S, denoted by $R \subseteq S$, if $R(x, y) \leq S(x, y)$ for all $x, y \in U$. Several basic operations for binary relations can be easily generalized to fuzzy relations.

Definition 3. Given two fuzzy relations R and S on U, the following fuzzy relations can be derived:

(1) the identity relation Id:

$$Id(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise;} \end{cases} \tag{9}$$

(2) the converse of R, R^-:

$$R^-(x, y) = R(y, x); \tag{10}$$

(3) the composition of R and S, $R \cdot S$:

$$R \cdot S(x, y) = \sup_{z \in U} \min\big(R(x, z), S(z, y)\big); \tag{11}$$

(4) the union of R and S, $R \cup S$:

$$R \cup S(x, y) = \max\big(R(x, y), S(x, y)\big); \tag{12}$$

(5) the intersection of R and S, $R \cap S$:

$$R \cap S(x, y) = \min\big(R(x, y), S(x, y)\big). \tag{13}$$

The composition of R with itself k times is denoted by R^k and the transitive closure of R is defined as $R^\infty = \bigcup_{k \geq 1} R^k$. Based on these definitions, an equivalence relation can be generalized to a similarity relation in fuzzy set theory.

[1] The operation corresponds to a logical negation in many-valued logics. We follow the terminology in [31] and call it the precomplement.

Definition 4. A fuzzy relation S is called a similarity relation if it satisfies the following properties:

- reflexivity: $Id \subseteq S$,
- symmetry: $S = S^-$, and
- (sup-min) transitivity: $S^2 \subseteq S$.

Intuitively, if S is a similarity relation, then $S(x, y)$ specifies the degree of similarity between x and y. As with equivalence relations, the set of all similarity relations on a domain U form a lattice. The meet and join of two similarity relations S_1 and S_2 in the lattice are defined as $S_1 \sqcap S_2 = S_1 \cap S_2$ and $S_1 \sqcup S_2 = (S_1 \cup S_2)^\infty$ respectively.

Given the basic notations of fuzzy sets and relations, a *weighted social network* can be defined as a structure $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$, where U is a *finite set* of actors; P_i is a fuzzy subset of U for all $i \in I$; and R_j is a fuzzy relation on U for all $j \in J$. Although the membership degrees of fuzzy sets and relations may be any real numbers from the unit interval, in practice, the weights in a weighted social network are rarely irrational numbers. Thus, we further assume that the membership degrees of P_i's and R_j's are all rational.

We have defined regular equivalence in two ways. Although the definitions coincide for classical social networks, they behave quite differently in weighted social networks. Based on the commutativity between the similarity relation and the underlying fuzzy relations, we can induce a kind of structural similarity between actors. Such a similarity is called a regular similarity. Formally, a similarity relation S is called a *regular similarity* with respect to a fuzzy relation R if it commutes with R, i.e., $S \cdot R = R \cdot S$. Hence, the regular similarity of a weighted social network can be defined as follows.

Definition 5. Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a weighted social network, and let S be a similarity relation on U. Then, S is a regular similarity with respect to $\mathfrak{N}$ if

(1) for all $x, y \in U$, $S(x, y) \le \inf_{i\in I}(P_i(x) \Leftrightarrow P_i(y))$; and
(2) S is a regular similarity with respect to R_j for all $j \in J$.

In [28], Fan et al. show that regular similarities are closed with respect to the usual join of similarity relations. Thus, we can define the maximum (with respect to fuzzy inclusion) regular similarity of a weighted social network.

On the other hand, based on the notion of role assignment, we can derive the concept of *generalized regular equivalence* (GRE). Although regular similarity is a fuzzy relation, GRE yields a crisp partition of the actors in a weighted network. To define GRE, we need to consider the neighborhoods of the nodes in weighted networks. Let R be a fuzzy relation on U. Then, for each $x \in U$, the out-neighborhood and in-neighborhood of x, still denoted by Rx and R^-x respectively, are two fuzzy subsets of U with the following membership functions:

$$Rx(y) = R(x, y), \tag{14}$$

$$R^-x(y) = R(y, x), \tag{15}$$

for any $y \in U$. Let F be a fuzzy subset of U and E be an equivalence relation on U. Then, $[F]_E$ is a fuzzy subset of the quotient set $U/E = \{[x]_E \mid x \in U\}$ with the following membership function:

$$[F]_E(X) = \max_{y\in X} F(y) \tag{16}$$

for any $X \in U/E$. Based on the notations, a GRE with respect to a fuzzy relation R is defined as an equivalence relation E such that

$$[Rx]_E = [Ry]_E \quad \text{and} \quad \left[R^-x\right]_E = \left[R^-y\right]_E \tag{17}$$

for any $(x, y) \in E$. Hence, the GRE of a weighted social network can be defined as follows.

Definition 6. Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a weighted social network, and let E be an equivalence relation on U. Then, E is a GRE with respect to $\mathfrak{N}$ if

(1) $(x, y) \in E$ implies that $P_i(x) = P_i(y)$ for all $i \in I$; and
(2) E is a GRE with respect to R_j for all $j \in J$.

Like regular equivalences, GRE is also closed with respect to the usual join of equivalence relations. Thus, we can define the maximum (i.e., the coarsest) GRE of a weighted social network. In addition, we use $x \equiv^g_{\mathfrak{N}} y$ to denote that (x, y) is in the maximum GRE of the network.

The main difference between regular similarity and GRE is that the former is a fuzzy relation, whereas the latter is a crisp relation. Thus, they can be applied in different situations. For example, an on-line social network user may wish to query the data center about actors that are similar to himself for the purposes of advertising, seeking friends, or simply sharing information. By using GRE to partition the network into equivalence classes, the user can find all actors that are in his equivalence class. On the other hand, the regular similarity can provide a ranking of actors that satisfy the user's requirement in terms of the degree of similarity to himself.

Despite the difference between regular similarity and GRE, GRE is actually a special kind of regular similarity (i.e., the kind of crisp regular similarity), as shown by the following lemma.

Lemma 1. *Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a weighted social network, and let E be an equivalence relation on U. Then, E is a GRE with respect to $\mathfrak{N}$ iff E is a regular similarity with respect to $\mathfrak{N}$.*

Proof.

(1) If E is a GRE, then the first condition of GRE implies that $E(x, y) \le \inf_{i\in I}(P_i(x) \Leftrightarrow P_i(y))$. Thus, we only need to prove that $E \cdot R = R \cdot E$ for any $R \in \{R_j \mid j \in J\}$. Since U is finite, for any $x, z \in U$, there exist x_0 and z_0 such that $(x, x_0) \in E$, $(z, z_0) \in E$,

$$E \cdot R(x, z) = \max_{y\in U} \min\bigl(E(x, y), R(y, z)\bigr) = \max_{y\in[x]_E} R(y, z) = R(x_0, z),$$

and

$$R \cdot E(x, z) = \max_{y\in U} \min\bigl(R(x, y), E(y, z)\bigr) = \max_{y\in[z]_E} R(x, y) = R(x, z_0).$$

Because $(x, x_0) \in E$ and $(z, z_0) \in E$, by the second condition of GRE, we have

$$\max_{y\in[z]_E} R(x, y) = [Rx]_E\bigl([z]_E\bigr) = [Rx_0]_E\bigl([z]_E\bigr) = \max_{y\in[z]_E} R(x_0, y)$$

and

$$\max_{y\in[x]_E} R(y, z) = \bigl[R^- z\bigr]_E\bigl([x]_E\bigr) = \bigl[R^- z_0\bigr]_E\bigl([x]_E\bigr) = \max_{y\in[x]_E} R(y, z_0),$$

which imply

$$R \cdot E(x, z) = \max_{y\in[z]_E} R(x, y) = \max_{y\in[z]_E} R(x_0, y) \ge R(x_0, z) = E \cdot R(x, z)$$

and

$$E \cdot R(x, z) = \max_{y\in[x]_E} R(y, z) = \max_{y\in[x]_E} R(y, z_0) \ge R(x, z_0) = R \cdot E(x, z)$$

respectively.

(2) If E violates the first condition of GRE, then there exist $(x, y) \in E$ and $i \in I$ such that $P_i(x) \ne P_i(y)$. Thus, $E(x, y) = 1 > \inf_{i\in I}(P_i(x) \Leftrightarrow P_i(y))$ and E is not a regular similarity. If E violates the second condition of GRE, then there exist $x, y, z \in U$ such that $(x, y) \in E$ but $\max_{z'\in[z]_E} R(x, z') < \max_{z'\in[z]_E} R(y, z') = R(y, z)$. Thus, $R \cdot E(x, z) < R(y, z) \le \max_{y'\in[y]_E} R(y', z) = \max_{y'\in[x]_E} R(y', z) = E \cdot R(x, z)$ and E is not a regular similarity. □

Note that each c-cut ($c \in [0, 1]$) of a similarity relation is an equivalence relation. Hence, it would be interesting to see whether a c-cut of a regular similarity is also a GRE. The following example shows that this is not the case.

Example 1. Let us consider a degenerated weighted social network $\mathfrak{N} = (U, R)$, i.e., J is a singleton and $I = \emptyset$. Assume that $U = \{x_1, x_2, x_3\}$ and the incidence matrix[2] of R is

$$\begin{bmatrix} a & a & c \\ b & b & c \\ c & c & d \end{bmatrix},$$

where $d > c > a > b$. Then, the similarity relation

$$S = \begin{bmatrix} 1 & 1 & c \\ 1 & 1 & c \\ c & c & 1 \end{bmatrix}$$

[2] An incidence matrix of a fuzzy relation R on U is simply the $n \times n$ matrix $[R(x, y)]_{x,y\in U}$.

satisfies

$$R \cdot S = S \cdot R = \begin{bmatrix} c & c & c \\ c & c & c \\ c & c & d \end{bmatrix}.$$

Thus, S is a regular similarity of $\mathfrak{N}$ because the set of attributes is empty. Furthermore, we can show that for any similarity relation

$$S' = \begin{bmatrix} 1 & 1 & e_1 \\ 1 & 1 & e_2 \\ e_1 & e_2 & 1 \end{bmatrix}$$

such that $\min(e_1, e_2) \geq c$ and $\max(e_1, e_2) > c$, we have $S' \cdot R(x_1, x_3) = \min(e_1, d) \neq c = R \cdot S'(x_1, x_3)$ or $S' \cdot R(x_2, x_3) = \min(e_2, d) \neq c = R \cdot S'(x_2, x_3)$. Hence, S is the maximum regular similarity of $\mathfrak{N}$. However, by Lemma 1, the 1-cut of S is not a GRE because $S_1 \cdot R(x_2, x_1) = a \neq b = R \cdot S_1(x_2, x_1)$. $\square$

3. Regular similarity and modal logic

As with ordinary social networks, we want to find a logical language that can characterize regular similarity in weighted social networks. Many-valued modal logic is a candidate language because its formulas typically have a degree of truth in a lattice. It is an extension of modal logic based on many-valued logics. There are various many-valued logics that differ in terms of the choice of syntax and semantics. Hajek [31] introduced a family of [0, 1]-valued logics in which the most important instances are Łukasiewicz, Gödel and product logics. These logic systems are interpreted in algebraic structures called residuated lattices such that continuous t-norms and their corresponding residua in the algebras are taken as the truth functions of the conjunction and the implication respectively. In this paper, we focus on Gödel modal logic $G(\Box\Diamond)$, which we introduce below.

The alphabet of $G(\Box\Diamond)$ is similar to that of PMML, but some subtle differences. First, the necessity and possibility modalities of PMML are inter-definable in a dual way, but such duality does not exist in $G(\Box\Diamond)$. Second, to represent a partial truth, $G(\Box\Diamond)$ is extended with the set of truth constants $\bar{c}$ for each rational $c \in [0, 1]$. Thus, the alphabet of $G(\Box\Diamond)$ comprises a set of propositional symbols PV; the set of truth constant $\{\bar{c} \mid c$ a rational number in $[0, 1]\}$; a set of relational symbols REL; the logical connectives $\wedge$ and $\rightarrow$; the relational converse symbol $^-$; and the modality-forming symbols $[\,]$ and $\langle\rangle$. The set of formulas of $G(\Box\Diamond)$ is the smallest set that contains PV and the set of truth constants that satisfies the following conditions:

- if φ is a formula and α is a relational symbol, then $[\alpha]\varphi$, $[\alpha^-]\varphi$, $\langle\alpha\rangle\varphi$ and $\langle\alpha^-\rangle\varphi$ are formulas;
- if φ and ψ are formulas, then $\varphi \wedge \psi$ and $\varphi \rightarrow \psi$ are formulas.

We abbreviate $\varphi \rightarrow \bar{0}$ as $\neg\varphi$, $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ as $\varphi \leftrightarrow \psi$, and $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$ as $\varphi \vee \psi$. Furthermore, we use $G(\Box)$ and $G(\Diamond)$ to denote the $\Box$-fragment and the $\Diamond$-fragment of the Gödel modal logic respectively.

A Kripke model for $G(\Box\Diamond)$ is $\mathfrak{M} = (W, (R_\alpha)_{\alpha \in REL}, V)$, where W is a set of possible worlds; R_α is a fuzzy relation on W for each $\alpha \in REL$; and $V : W \times PV \rightarrow [0, 1]$ is a truth assignment for evaluating the truth value of each propositional symbol in each world. Let $\mathcal{G}(\Box\Diamond)$ denote the set of all $G(\Box\Diamond)$ formulas. Then, the truth assignment V can be iteratively extended to a function $V : W \times \mathcal{G}(\Box\Diamond) \rightarrow [0, 1]$ as follows:

(1) $V(w, \bar{c}) = c$,
(2) $V(w, \varphi \wedge \psi) = \min(V(w, \varphi), V(w, \psi))$,
(3) $V(w, \varphi \rightarrow \psi) = V(w, \varphi) \Rightarrow V(w, \psi)$,
(4) $V(w, [\alpha]\varphi) = \inf_{u \in W} R_\alpha(w, u) \Rightarrow V(u, \varphi)$,
(5) $V(w, [\alpha^-]\varphi) = \inf_{u \in W} R_\alpha^-(w, u) \Rightarrow V(u, \varphi)$,
(6) $V(w, \langle\alpha\rangle\varphi) = \sup_{u \in W} \min(R_\alpha(w, u), V(u, \varphi))$,
(7) $V(w, \langle\alpha^-\rangle\varphi) = \sup_{u \in W} \min(R_\alpha^-(w, u), V(u, \varphi))$.

Obviously, we can derive the following expressions:

(1) $V(w, \neg\varphi) = -V(w, \varphi)$,
(2) $V(w, \varphi \leftrightarrow \psi) = (V(w, \varphi) \Leftrightarrow V(w, \psi))$,
(3) $V(w, \varphi \vee \psi) = \max(V(w, \varphi), V(w, \psi))$.

A formula φ is true in a model $\mathfrak{M} = (W, (R_\alpha)_{\alpha\in REL}, V)$, denoted by $\mathfrak{M} \models \varphi$, if $V(w,\varphi) = 1$ for any $w \in W$. Let Σ be a set of formulas of $G(\Box\Diamond)$ and let $\mathfrak{M} \models \Sigma$ denote that $\mathfrak{M} \models \varphi$ for all $\varphi \in \Sigma$. Then, φ is a $G(\Box\Diamond)$-consequence[3] of Σ, denoted by $\Sigma \models_{G(\Box\Diamond)} \varphi$, if for any model $\mathfrak{M}$, $\mathfrak{M} \models \Sigma$ implies $\mathfrak{M} \models \varphi$.

As with ordinary social networks, given a weighted social network $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$, we define a $G(\Box\Diamond)$ language with the following basic symbols:

(1) $PV = \{p_i \mid i \in I\}$;
(2) $REL = \{\alpha_j \mid j \in J\}$.

The weighted social network $\mathfrak{N}$ is transformed into a Kripke model for the language $\mathfrak{M}_\mathfrak{N} = (U, (R_j)_{j\in J}, V)$, where V is defined by $V(x, p_i) = P_i(x)$ for $x \in U$ and $i \in I$; and R_j denotes R_{α_j} for $j \in J$. Let $\mathcal{G}(\Box\Diamond)_\mathfrak{N}$ denote the set of formulas of this language, and let $\mathcal{G}(\Diamond)_\mathfrak{N}$ denote the subset of formulas of the $\Diamond$-fragment of the language. Then, the logical characterization of regular similarity can be presented as the following theorem.

Theorem 2. *Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a weighted social network, S be its maximum regular similarity, and $\mathfrak{M}_\mathfrak{N} = (U, (R_j)_{j\in J}, V)$ be its corresponding Kripke model. Then, for any $x, y \in U$, we have*

$$S(x,y) = \inf_{\varphi\in\mathcal{G}(\Box\Diamond)_\mathfrak{N}} \big(V(x,\varphi) \Leftrightarrow V(y,\varphi)\big) \tag{18}$$

$$= \inf_{\varphi\in\mathcal{G}(\Diamond)_\mathfrak{N}} \big(V(x,\varphi) \Leftrightarrow V(y,\varphi)\big). \tag{19}$$

The theorem can be proved by the following two lemmas.

Lemma 2. *Let S be a regular similarity of $\mathfrak{N}$. Then, for any formula $\varphi \in \mathcal{G}(\Box\Diamond)_\mathfrak{N}$ and any $x, y \in U$, we have $S(x,y) \le (V(x,\varphi) \Leftrightarrow V(y,\varphi))$.*

Proof. The lemma can be proved by induction on the complexity of the formulas.

(1) Induction base: If $\varphi = p_i \in PV$, then $V(x,\varphi) = P_i(x)$ by the definition of $\mathfrak{M}_\mathfrak{N}$. Thus, the result follows from the definition of regular similarity. If φ is a truth constant $\bar{c}$, then the result holds trivially because $V(x,\varphi) = V(y,\varphi)$ for any x, y.
(2) Induction step:
 (a) Suppose $\varphi = \psi_1 \wedge \psi_2$ and the result holds for ψ_1 and ψ_2. Let c_1, c_2, d_1, and d_2 denote $V(x,\psi_1), V(x,\psi_2), V(y,\psi_1)$, and $V(y,\psi_2)$ respectively. Then, by the induction hypothesis, we have

$$S(x,y) \le c_1 \Leftrightarrow d_1 \quad \text{and} \quad S(x,y) \le c_2 \Leftrightarrow d_2.$$

 Furthermore, by definition,

$$V(x,\varphi) \Leftrightarrow V(y,\varphi) = \min(c_1,c_2) \Leftrightarrow \min(d_1,d_2).$$

 According to (8), $c_1 \Leftrightarrow d_1 = 1$ or $\min(c_1,d_1)$, and $c_2 \Leftrightarrow d_2 = 1$ or $\min(c_2,d_2)$. By case analysis using the different values of $c_1 \Leftrightarrow d_1$ and $c_2 \Leftrightarrow d_2$, we can prove that

$$\min(c_1 \Leftrightarrow d_1, c_2 \Leftrightarrow d_2) \le \min(c_1,c_2) \Leftrightarrow \min(d_1,d_2).$$

 The result then follows immediately from the three equations.
 (b) The case of $\varphi = \psi_1 \to \psi_2$ can be proved similarly except that we have to show

$$\min(c_1 \Leftrightarrow d_1, c_2 \Leftrightarrow d_2) \le (c_1 \Rightarrow c_2) \Leftrightarrow (d_1 \Rightarrow d_2)$$

 by case analysis.
 (c) Suppose $\varphi = \langle\alpha_j\rangle\psi$ and the result holds for ψ. Let us consider an enumeration of the universe $U = \{x_1, x_2, \cdots, x_n\}$ and denote $V(x_k,\psi)$ by c_k for $1 \le k \le n$. By the induction hypothesis, for any $1 \le k, k' \le n$,

$$S(x_k, x_{k'}) \le (c_k \Leftrightarrow c_{k'}) \le (c_{k'} \Rightarrow c_k),$$

 which implies that

$$\min\big(S(x_k, x_{k'}), c_{k'}\big) \le c_k.$$

 Thus, for any $1 \le k \le n$, we have

[3] Note that the notion corresponds to the global consequence relation in [3].

$$\max_{1\le k'\le n}\min\bigl(S(x_k,x_{k'}),c_{k'}\bigr)\le c_k.$$

This can be conveniently written in a matrix notation as

$$S\cdot\mathbf{c}\le\mathbf{c},$$

where S is regarded as its incidence matrix, $\mathbf{c}$ denotes the $n\times 1$ matrix $[c_1,c_2,\cdots,c_n]^t$, and the $\le$ symbol between the matrices represents the pointwise comparison. Let us further denote $V(x_k,\varphi)$ by d_k for $1\le k\le n$. Then, according to the semantics of $G(\Box\Diamond)$, $\mathbf{d}=R_j\cdot\mathbf{c}$. Hence,

$$\begin{aligned}S\cdot\mathbf{d}&=S\cdot R_j\cdot\mathbf{c}\\&=R_j\cdot S\cdot\mathbf{c}\quad\text{(by the definition of regular similarity)}\\&\le R_j\cdot\mathbf{c}\\&=\mathbf{d}\end{aligned}$$

This means that, for any $x,y\in U$, $\min(S(x,y),V(y,\varphi))\le V(x,\varphi)$; hence, $S(x,y)\le V(y,\varphi)\Rightarrow V(x,\varphi)$ by the definition of residuum. Because x and y are arbitrary and S is symmetric, we can exchange the roles of x and y above and obtain $S(x,y)\le V(x,\varphi)\Leftrightarrow V(y,\varphi)$.

(d) Suppose $\varphi=[\alpha_j]\psi$ and the result holds for ψ. If $S(x,y)\le\min(V(x,\varphi),V(y,\varphi))$, then according to (8), we have $S(x,y)\le(V(x,\varphi)\Leftrightarrow V(y,\varphi))$. Hence, we only need to consider the case where $S(x,y)>\min(V(x,\varphi),V(y,\varphi))$. Without loss of generality, we can assume that $S(x,y)>V(x,\varphi)$. Let c denote $S(x,y)$. Then, we have

$$\begin{aligned}V(x,\varphi)&=\min\bigl(c,V(x,\varphi)\bigr)&&(20)\\&=\min\Bigl(c,\inf_{u\in U}R_j(x,u)\Rightarrow V(u,\psi)\Bigr)&&(21)\\&=\inf_{u\in U}\min\bigl(c,R_j(x,u)\Rightarrow V(u,\psi)\bigr)&&(22)\\&=\inf_{u\in U}\min\bigl(c,\min\bigl(c,R_j(x,u)\bigr)\Rightarrow V(u,\psi)\bigr)&&(23)\\&=\min\Bigl(c,\inf_{u\in U}\min\bigl(c,R_j(x,u)\bigr)\Rightarrow V(u,\psi)\Bigr),&&(24)\end{aligned}$$

where (23) follows from a property of the Gödel t-norm and residuum.[4] Because $S\cdot R_j(y,u)=R_j\cdot S(y,u)$ for any $u\in U$, we can find a $u'\in U$ such that $\min(c,R_j(x,u))\le\min(S(u,u'),R_j(y,u'))$ for any $u\in U$. Thus, for any $u\in U$, we can find u' such that

$$\min\bigl(c,R_j(x,u)\bigr)\Rightarrow V(u,\psi)\ge\min\bigl(S\bigl(u,u'\bigr),R_j\bigl(y,u'\bigr)\bigr)\Rightarrow V(u,\psi)$$

since $\Rightarrow$ is monotonically decreasing on its first argument. We can now consider two cases. First, if $V(u,\psi)=V(u',\psi)$, then

$$\begin{aligned}\min\bigl(S\bigl(u,u'\bigr),R_j\bigl(y,u'\bigr)\bigr)\Rightarrow V(u,\psi)&\ge R_j\bigl(y,u'\bigr)\Rightarrow V(u,\psi)\\&=R_j\bigl(y,u'\bigr)\Rightarrow V\bigl(u',\psi\bigr)\end{aligned}$$

by the anti-monotonicity of $\Rightarrow$ on its first argument. Second, if $V(u,\psi)\ne V(u',\psi)$, then by the induction hypothesis, $S(u,u')\le(V(u,\psi)\Leftrightarrow V(u',\psi))\le V(u,\psi)$. Thus, $\min(S(u,u'),R_j(y,u'))\Rightarrow V(u,\psi)=1\ge R_j(y,u')\Rightarrow V(u',\psi)$. In both cases, we have shown that for any $u\in U$, we can find u' such that

$$\min\bigl(c,R_j(x,u)\bigr)\Rightarrow V(u,\psi)\ge R_j\bigl(y,u'\bigr)\Rightarrow V\bigl(u',\psi\bigr).$$

Therefore,

$$\begin{aligned}\inf_{u\in U}\min\bigl(c,R_j(x,u)\bigr)\Rightarrow V(u,\psi)&\ge\inf_{u'\in U}R_j\bigl(y,u'\bigr)\Rightarrow V\bigl(u',\psi\bigr)\\&=V(y,\varphi),\end{aligned}$$

which leads to

$$V(x,\varphi)\ge\min\bigl(S(x,y),V(y,\varphi)\bigr)$$

[4] For the Gödel t-norm and residuum, we can easily verify that $\min(c,a\Rightarrow b)=\min(c,\min(c,a)\Rightarrow b)$ for any $a,b,c\in[0,1]$.

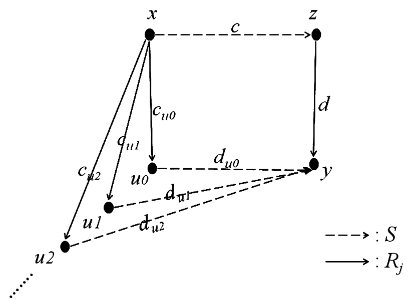

Fig. 1. Visualization of the proof of Lemma 3.

when combined with (24). Because of the assumption that $S(x, y) > V(x, \varphi)$, we have $V(x, \varphi) \geq V(y, \varphi)$ and $S(x, y) > V(y, \varphi)$. Since $S(x, y) > V(y, \varphi)$, we can exploit the whole argument again to prove that $V(y, \varphi) \geq V(x, \varphi)$. Hence, we have $V(y, \varphi) = V(x, \varphi)$, and $S(x, y) \leq (V(x, \varphi) \Leftrightarrow V(y, \varphi)) = 1$ when $S(x, y) > \min(V(x, \varphi), V(y, \varphi))$.

(e) The case where $\varphi = \langle\alpha_j^-\rangle\psi$ $\varphi[\alpha_j^-]\psi$ can be proved analogously except that R_j is replaced by R_j^-. Note that $S \cdot R_j^- = R_j^- \cdot S$ still holds due to the symmetry of S. $\square$

This lemma represents a fuzzified version of the semantic invariance of any formula with respect to the regular similarity. Intuitively, a formula can be seen as the descriptive property of actors. Thus, $V(x, \varphi)$ is the degree that the property φ is true of the actor x. Essentially„ the lemma means that the greater the similarity of two actors, the higher will be the equivalence of their descriptive properties.

Lemma 3. *Let S be a fuzzy relation such that its membership function is defined as*

$$S(x, y) = \inf_{\varphi \in \mathcal{G}(\diamondsuit)_{\mathfrak{N}}} \big(V(x, \varphi) \Leftrightarrow V(y, \varphi)\big). \tag{25}$$

Then, S is a regular similarity of $\mathfrak{N}$.

Proof. We have to prove that S is a similarity relation that satisfies the two conditions of Definition 5. It is straightforward to verify that S is reflexive, symmetric, and transitive. Moreover, the first condition holds trivially because PV is a subset of $\mathcal{G}(\diamondsuit)_{\mathfrak{N}}$. Hence, we only need to show that $S \cdot R_j = R_j \cdot S$ for any $j \in J$. It suffices to show that $S \cdot R_j \leq R_j \cdot S$ and $S \cdot R_j^- \leq R_j^- \cdot S$ because the latter implies that $R_j \cdot S \leq S \cdot R_j$ by the relation's converse and the symmetry of S. First, we show $S \cdot R_j \leq R_j \cdot S$ for any $j \in J$. Assume this is not the case. Then, there exist $j \in J$ and $x, y \in U$ such that $S \cdot R_j(x, y) > R_j \cdot S(x, y)$. According to the definition of relational composition, this means there exists $z \in U$ such that

$$\min\big(S(x, z), R_j(z, y)\big) > \max_{u \in U} \min\big(R_j(x, u), S(u, y)\big).$$

Let c, d, c_u, and d_u $(u \in U)$ denote $S(x, z), R_j(z, y), R_j(x, u)$, and $S(u, y)$ $(u \in U)$ respectively, as visualized in Fig. 1. Then, for any $u \in U$, either $d_u < \min(c, d)$ or $c_u < \min(c, d)$. If $d_u = S(u, y) < \min(c, d)$, then by the definition of S, there exist $\varphi_u \in \mathcal{G}(\diamondsuit)_{\mathfrak{N}}$ such that $(V(u, \varphi_u) \Leftrightarrow V(y, \varphi_u)) < \min(c, d)$. Let e_u denote $V(y, \varphi_u)$ in such a case. Then, we can define ψ_u for any $u \in U$ as follows:

$$\psi_u = \begin{cases} \bar{1}, & \text{if } c_u < \min(c, d), \\ \varphi_u \leftrightarrow \overline{e_u}, & \text{otherwise.} \end{cases} \tag{26}$$

In the case of $c_u \geq \min(c, d)$, we have

$$V(u, \psi_u) = \big(V(u, \varphi_u) \Leftrightarrow e_u\big) = \big(V(u, \varphi_u) \Leftrightarrow V(y, \varphi_u)\big) < \min(c, d)$$

and

$$V(y, \psi_u) = \big(V(y, \varphi_u) \Leftrightarrow e_u\big) = \big(V(y, \varphi_u) \Leftrightarrow V(y, \varphi_u)\big) = 1.$$

Let $\psi = \bigwedge_{u \in U} \psi_u$. Then,

$$\begin{aligned} V\big(x, \langle\alpha_j\rangle\psi\big) &= \max_{u \in U} \min\big(R_j(x, u), V(u, \psi)\big) \\ &= \max_{u \in U} \min\big(c_u, V(u, \psi)\big) \end{aligned}$$

$$\begin{aligned} &\leq \max\Bigl(\max_{u\in U, c_u<\min(c,d)} c_u, \max_{u\in U, c_u\geq \min(c,d)} V(u,\psi_u)\Bigr) \\ &< \min(c,d). \end{aligned}$$

On the other hand,

$$\begin{aligned} V\bigl(z, \langle\alpha_j\rangle\psi\bigr) &= \max_{u\in U} \min\bigl(R_j(z,u), V(u,\psi)\bigr) \\ &\geq \min\bigl(R_j(z,y), V(y,\psi)\bigr) \\ &= d \\ &\geq \min(c,d). \end{aligned}$$

Hence, by the definition of S and $\Leftrightarrow$,

$$c = S(x,z) \leq \bigl(V\bigl(x, \langle\alpha_j\rangle\psi\bigr) \Leftrightarrow V\bigl(z, \langle\alpha_j\rangle\psi\bigr)\bigr) = V\bigl(x, \langle\alpha_j\rangle\psi\bigr) < \min(c,d),$$

which is impossible. Therefore, we can conclude that $S \cdot R_j \leq R_j \cdot S$ for any $j \in J$. The proof that $S \cdot R_j^- \leq R_j^- \cdot S$ for any $j \in J$ can be derived in a similar manner by considering the modality $\langle\alpha_j^-\rangle$. $\square$

Theorem 2 then follows immediately from the above two lemmas and the fact that $\mathcal{G}(\Diamond)_{\mathfrak{N}}$ is a subset of $\mathcal{G}(\Box\Diamond)_{\mathfrak{N}}$. The theorem shows that the regular similarity in weighted social networks can be characterized by the equivalence of formulas in many-valued modal logic. This is also a fuzzified version of the Hennessy–Milner theorem.

It is well-known that the $\Box$-fragment and the $\Diamond$-fragment usually exhibit a kind of asymmetry in deductive systems of many-valued modal logics [8,11–13]. The proof of Lemma 3 requires that the logical language contains possibility modalities. Hence, the proof cannot be applied to the $\Box$-fragment of the same logic. However, as we do not have a counterexample of the lemma for $G(\Box)$, it is still unclear if the $\Box$-fragments of the above-mentioned logics can also characterize regular similarity. Furthermore, the remark about the $\Box$-fragment of a many-valued modal logic applies to all the results in this paper.

While regular similarity is a kind of fuzzy relation, there is a special case whose characterization looks like Theorem 1. The following corollary illustrates the special case. We say that two actors, x and y, are $\mathcal{G}(\Box\Diamond)$-equivalent (resp. $\mathcal{G}(\Diamond)$-equivalent) with respect to $\mathfrak{N}$ if for all $\varphi \in \mathcal{G}(\Box\Diamond)_{\mathfrak{N}}$ (resp. $\varphi \in \mathcal{G}(\Diamond)_{\mathfrak{N}}$), $V(x,\varphi) = V(y,\varphi)$.

Corollary 1. *Let S be the maximum regular similarity of $\mathfrak{N}$. Then, for any actors $x, y \in U$, the following three statements are equivalent:*

(1) $S(x,y) = 1$,
(2) *x and y are $\mathcal{G}(\Box\Diamond)$-equivalent with respect to $\mathfrak{N}$,*
(3) *x and y are $\mathcal{G}(\Diamond)$-equivalent with respect to $\mathfrak{N}$.*

4. Generalized regular equivalence and modal logic

To present the logical characterization of GRE, we have to extend $G(\Box\Diamond)$ with the projection operator Δ. As the operator was first used by Baaz for Gödel logic, it is also called the Baaz Delta [1]. The projection operator $\Delta : [0,1] \to \{0,1\}$ is defined by

$$\Delta a = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

Let $G_\Delta(\Box\Diamond)$ be the extension of $G(\Box\Diamond)$ with the unary projection connective Δ. Then, the formation rules for the formulas of $G_\Delta(\Box\Diamond)$ are the same as those for $G(\Box\Diamond)$ plus the following rule:

- if φ is a formula, then $\Delta\varphi$ is a formula.

The definition of Kripke models for $G_\Delta(\Box\Diamond)$ is the same as that for $G(\Box\Diamond)$; however, for a model $\mathfrak{M} = (W, (R_\alpha)_{\alpha\in REL}, V)$, the truth assignment V satisfies the additional condition:

$$V(w, \Delta\varphi) = \Delta\bigl(V(w,\varphi)\bigr).$$

The projection connective can be also defined by an involutive negation. A negation operator is involutive if it satisfies the double negation law. It is easy to see that the negation in $G(\Box\Diamond)$ is not involutive. That is, $\neg\neg\varphi \leftrightarrow \varphi$ is not a 1-tautology in $G(\Box\Diamond)$. The extensions of many-valued logic with an additional involutive negation have been studied extensively in

[15,24]. Let $G_{\sim}(\Box\Diamond)$ be the extension of $G(\Box\Diamond)$ with the involutive negation $\sim$. Then, in addition to the formation rules for formulas of $G(\Box\Diamond)$, we have the following rule:

- if φ is a formula, then $\sim\varphi$ is a formula;

and for a model $\mathfrak{M} = (W, (R_\alpha)_{\alpha\in REL}, V)$, the truth assignment V satisfies the following condition:

$$V(w, \sim\varphi) = 1 - V(w, \varphi).$$

In $G_{\sim}(\Box\Diamond)$, we can define $\Delta\varphi$ as the abbreviation of $\neg\sim\varphi$. Thus, $G_{\sim}(\Box\Diamond)$ is more expressive than $G_{\Delta}(\Box\Diamond)$. As above, the $\Box$-fragment and $\Diamond$-fragment of these two logics are denoted by $G_{\Delta}(\Box)$, $G_{\sim}(\Box)$, $G_{\Delta}(\Diamond)$, and $G_{\sim}(\Diamond)$.

Given a weighted social network $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$, we can define the basic symbols in PV and REL from $\mathfrak{N}$ and the Kripke model $\mathfrak{M}_{\mathfrak{N}}$ in the same way as in the preceding section. Let L denote one of the logics $G_{\Delta}(\Box\Diamond)$, $G_{\sim}(\Box\Diamond)$, $G_{\Delta}(\Diamond)$, or $G_{\sim}(\Diamond)$, and let $\mathcal{L}_{\mathfrak{N}}$ denote the set of formulas of its corresponding language derived from $\mathfrak{N}$. Then, we say that two actors $x, y \in U$ are $\mathcal{L}$-equivalent with respect to $\mathfrak{N}$ iff for any $\varphi \in \mathcal{L}_{\mathfrak{N}}$, $V(x,\varphi) = V(y,\varphi)$. Furthermore, we use $\Delta\mathcal{L}_{\mathfrak{N}}$ to denote the subset of formulas

$$\{\Delta\varphi \mid \varphi \in \mathcal{L}_{\mathfrak{N}} \text{ and does not start with } \Delta\}$$

and define the $\Delta\mathcal{L}$-equivalence relation on U accordingly. This leads to the following theorem.

Theorem 3. *Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a weighted social network, and let L be one of the logics $G_{\Delta}(\Box\Diamond)$, $G_{\sim}(\Box\Diamond)$, $G_{\Delta}(\Diamond)$, or $G_{\sim}(\Diamond)$. Then, for any $x, y \in U$, the following three statements are equivalent:*

(1) $x \equiv^g_{\mathfrak{N}} y$;
(2) *x and y are $\mathcal{L}$-equivalent with respect to $\mathfrak{N}$;*
(3) *x and y are $\Delta\mathcal{L}$-equivalent with respect to $\mathfrak{N}$.*

Proof.

(1) First, we prove that $\equiv^g_{\mathfrak{N}}$ implies $\mathcal{G}_{\sim}(\Box\Diamond)$-equivalence by induction on the complexity of the formulas. The induction base follows trivially from the first condition of Definition 6 and the fact that any truth constant has the same truth value in all possible worlds. The induction steps for the connectives $\wedge, \rightarrow$, and $\sim$ are also straightforward because they are all truth functional. Thus, we only need to prove the result for the modal formulas.
   (a) Suppose $\varphi = \langle\alpha_j\rangle\psi$ for some $j \in J$ and the result holds for ψ. Let us consider an enumeration of the quotient set $U/\equiv^g_{\mathfrak{N}} = \{X_1, X_2, \cdots, X_n\}$. By the induction hypothesis, for a fixed k, $V(u,\psi)$ is the same for any $u \in X_k$. Thus, we can assume that $V(u,\psi) = c_k$ for any $u \in X_k$. Then,

$$\begin{aligned}
V(x,\varphi) &= \max_{u\in U}\min\big(R_j(x,u), V(u,\psi)\big)\\
&= \max_{1\le k\le n}\max_{u\in X_k}\min\big(R_j(x,u), c_k\big)\\
&= \max_{1\le k\le n}\min\Big(c_k, \max_{u\in X_k} R_j(x,u)\Big)\\
&= \max_{1\le k\le n}\min\big(c_k, [R_j x]_{\equiv^g_{\mathfrak{N}}}(X_k)\big) \quad \big(\text{Eq. (16)}\big)\\
&= \max_{1\le k\le n}\min\big(c_k, [R_j y]_{\equiv^g_{\mathfrak{N}}}(X_k)\big) \quad \big(\text{Definition 6 and Eq. (17)}\big)\\
&= \max_{1\le k\le n}\min\Big(c_k, \max_{u\in X_k} R_j(y,u)\Big)\\
&= \max_{1\le k\le n}\max_{u\in X_k}\min\big(R_j(y,u), c_k\big)\\
&= \max_{u\in U}\min\big(R_j(y,u), V(u,\psi)\big)\\
&= V(y,\varphi).
\end{aligned}$$

   (b) Suppose $\varphi = [\alpha_j]\psi$ for some $j \in J$ and the result holds for ψ. Then, by using the above notations, we have

$$\begin{aligned}
V(x,\varphi) &= \min_{u\in U}\big(R_j(x,u) \Rightarrow V(u,\psi)\big)\\
&= \min_{1\le k\le n}\min_{u\in X_k}\big(R_j(x,u) \Rightarrow c_k\big)
\end{aligned}$$

$$
\begin{aligned}
&= \min_{1\le k\le n}\Big(\max_{u\in X_k} R_j(x,u)\Big) \Rightarrow c_k \quad \big(\text{Eq. (5)}\big)\\
&= \min_{1\le k\le n} [R_j x]_{\equiv^g_{\mathfrak{N}}}(X_k) \Rightarrow c_k \quad \big(\text{Eq. (16)}\big)\\
&= \min_{1\le k\le n} [R_j y]_{\equiv^g_{\mathfrak{N}}}(X_k) \Rightarrow c_k \quad \big(\text{Definition 6 and Eq. (17)}\big)\\
&= \min_{1\le k\le n}\Big(\max_{u\in X_k} R_j(y,u)\Big) \Rightarrow c_k\\
&= \min_{1\le k\le n}\min_{u\in X_k}\big(R_j(y,u)\Rightarrow c_k\big)\\
&= \min_{u\in U}\big(R_j(y,u)\Rightarrow V(u,\psi)\big)\\
&= V(y,\varphi).
\end{aligned}
$$

(c) The cases where $\varphi=\langle\alpha_j^-\rangle\psi$ and $\varphi=[\alpha_j^-]\psi$ can be proved in the same way except that R_j is replaced by R_j^-.

(2) It is easy to see that $\mathcal{G}_\sim(\Box\Diamond)$-equivalence implies $\mathcal{G}_\Delta(\Box\Diamond)$-equivalence and $\mathcal{G}_\sim(\Diamond)$-equivalence, both of which imply $\mathcal{G}_\Delta(\Diamond)$-equivalence due to the expressive powers of the respective languages and their fragments.

(3) To prove that $\mathcal{G}_\Delta(\Diamond)$-equivalence implies $\equiv^g_{\mathfrak{N}}$, we let E denote the $\mathcal{G}_\Delta(\Diamond)$-equivalence relation on U and show that E is a GRE. Then, as $\equiv^g_{\mathfrak{N}}$ is the maximum GRE, E must be its subset. Trivially, E is an equivalence relation that satisfies the first condition of Definition 6 because PV is a subset of $\mathcal{G}_\Delta(\Diamond)_{\mathfrak{N}}$. Assume E does not satisfy the second condition of Definition 6. That is, there exist $j\in J$ and $x,y\in U$ such that $(x,y)\in E$, but

$$[R_j x]_E \ne [R_j y]_E \quad \text{or} \quad \big[R_j^- x\big]_E \ne \big[R_j^- y\big]_E.$$

Let us first consider the case where $[R_j x]_E \ne [R_j y]_E$. Without loss of generality, we can assume that an enumeration of the quotient set $U/E=\{X_1,X_2,\cdots,X_n\}$ and $[R_j x]_E(X_1)\ne[R_j y]_E(X_1)$. By slightly abusing the notation, given $1\le k\le n$ and any formula φ, we can use $V(X_k,\varphi)$ to denote $V(u,\varphi)$ for any $u\in X_k$ because X_k is an equivalence class of E. Now, as E is the $\mathcal{G}_\Delta(\Diamond)$-equivalence relation, there exists ψ_k such that $V(X_1,\psi_k)\ne V(X_k,\psi_k)$ for every $2\le k\le n$. We let c_k denote $V(X_1,\psi_k)$ and define $\varphi_k=\Delta(\psi_k\leftrightarrow\overline{c_k})$ for $2\le k\le n$ and $\varphi=\bigwedge_{2\le k\le n}\varphi_k$. Then, we have $V(X_1,\varphi)=1$ and $V(X_k,\varphi)=0$ for $2\le k\le n$. Hence,

$$
\begin{aligned}
V\big(x,\langle\alpha_j\rangle\varphi\big) &= \max_{u\in U}\min\big(R_j(x,u),V(u,\varphi)\big)\\
&= \max_{1\le k\le n}\max_{u\in X_k}\min\big(R_j(x,u),V(X_k,\varphi)\big)\\
&= \max_{u\in X_1} R_j(x,u)\\
&= [R_j x]_E(X_1) \quad \big(\text{Eq. (16)}\big)\\
&\ne [R_j y]_E(X_1) \quad (\text{assumption})\\
&= \max_{u\in X_1} R_j(y,u)\\
&= \max_{1\le k\le n}\max_{u\in X_k}\min\big(R_j(y,u),V(X_k,\varphi)\big)\\
&= \max_{u\in U}\min\big(R_j(y,u),V(u,\psi)\big)\\
&= V\big(y,\langle\alpha_j\rangle\varphi\big).
\end{aligned}
$$

Consequently, the formula $\langle\alpha_j\rangle\varphi$ has different truth values in x and y, which violates the assumption that $(x,y)\in E$ and leads to a contradiction. Hence, it is impossible that $[R_j x]_E\ne[R_j y]_E$. The impossibility of $[R_j^- x]_E\ne[R_j^- y]_E$ can be proved similarly. Therefore, E must satisfy the second condition of Definition 6, so it is a GRE.

(4) To prove that $\Delta\mathcal{L}$-equivalence and $\mathcal{L}$-equivalence are the same, we only need to prove that the former implies the latter. Assume that $V(x,\Delta\varphi)=V(y,\Delta\varphi)$ for any $\varphi\in\mathcal{L}_{\mathfrak{N}}$ that does not start with Δ, and assume that x and y are not $\mathcal{L}$-equivalent. Then, there exists $\psi\in\mathcal{L}_{\mathfrak{N}}$ such that $V(x,\psi)\ne V(y,\psi)$. Without loss of generality, let $a=V(x,\psi)<V(y,\psi)=b$ and let $\varphi=\psi\leftrightarrow\bar{a}$. Then, $\varphi\in\mathcal{L}_{\mathfrak{N}}$ and does not start with Δ. However, $1=V(x,\Delta\varphi)\ne V(y,\Delta\varphi)=0$, which contradicts the assumption that x and y are $\Delta\mathcal{L}$-equivalent. Thus, x and y must be $\mathcal{L}$-equivalent. □

The third item of the theorem shows that the connective Δ is crucial for the logical characterization of GRE. The result cannot be further strengthened to $\Delta\mathcal{G}(\Box,\Diamond)$-equivalence (or $\Delta\mathcal{G}(\Diamond)$-equivalence) because it coincides with $\mathcal{G}(\Box,\Diamond)$-equivalence as shown in the proof of the theorem. However, by Corollary 1, $\mathcal{G}(\Box,\Diamond)$-equivalence is equal to the 1-cut of the maximum regular similarity, which is not necessarily a GRE, as shown in Example 1. Furthermore, Theorem 3 and Example 1 provide an alternative proof of the following well-known result in many-valued (modal) logic.

Corollary 2. *Neither the involutive negation $\sim$ nor the Baaz Delta Δ are definable in $G(\Box\Diamond)$.*

Thus, $G_\Delta(\Box\Diamond)$ and $G_\sim(\Box\Diamond)$ are indeed proper extensions of $G(\Box\Diamond)$.

4.1. Special case: hybrid social networks

In weighted social networks, each actor is associated with fuzzy attributes and connected with other actors by fuzzy relations. However, there is a special kind of weighted social network in which the attributes are crisp although the relations between the actors are still weighted. For example, in a friendship network, the strength of ties determines the degree of friendship between the actors. Hence, the friendship relation is modeled as a fuzzy relation. However, all personal attributes of each actor, such as gender, age, and occupation, may be crisp. To model such networks, we say that a weighted social network $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ is a *hybrid social network* if for each $i \in I$, P_i is a crisp subset of U.

In addition to the characterizations above, we can use quantitative modal logic (QML) [46–49] to derive an alternative characterization of GRE in hybrid social networks. QML is a modal version of possibilistic logic, which is a logic for reasoning about uncertainty based on possibility theory [21,22]. In the theory, a *possibility distribution* on the universe U is a function $\pi : U \to [0, 1]$; and two measures on U, called possibility and necessity measures and denoted by Π and N respectively, can be derived from π. Formally, $\Pi, N : 2^U \to [0, 1]$ are defined as

$$\Pi(X) = \sup_{u\in X} \pi(u), \qquad N(X) = 1 - \Pi(\overline{X}),$$

where $\overline{X}$ is the complement of X with respect to U. In a weighted social network, each actor's out-neighborhood and in-neighborhood with respect to a fuzzy relation can be seen as possibility distributions. In other words, the membership functions in Eqs. (14) and (15) correspond to such possibility distributions. The modalities in QML can represent the lower bounds of the possibility and necessity measures of propositions, each of which is interpreted as a subset of possible worlds (or actors).

The alphabet of QML is the same as that of PMML, but the formation rule for modal formulas is modified as follows:

- if φ is a formula, α is a relational symbol, and c is a rational number in $[0, 1]$, then $\langle\alpha_{\geq c}\rangle\varphi$, $\langle\alpha_{>c}\rangle\varphi$, $\langle\alpha^-_{\geq c}\rangle\varphi$, and $\langle\alpha^-_{>c}\rangle\varphi$ are all formulas.

Because QML is a two-valued logic, the necessity modalities can be defined from the corresponding possibility modalities. For example, $[\alpha_{\geq c}]\varphi$ is an abbreviation of $\neg\langle\alpha_{>1-c}\rangle\neg\varphi$.

For the semantics, a Kripke model of QML is $\mathfrak{M} = (W, (R_\alpha)_{\alpha\in REL}, V)$, where W and V are the same as those of PMML, and for each $\alpha \in REL$, R_α is a fuzzy relation on W. Then, the satisfaction of each modal formula is defined as follows:

(1) $\mathfrak{M}, w \models \langle\alpha_{\geq c}\rangle\varphi$ iff $\Pi_w(|\varphi|) \geq c$,
(2) $\mathfrak{M}, w \models \langle\alpha_{>c}\rangle\varphi$ iff $\Pi_w(|\varphi|) > c$,
(3) $\mathfrak{M}, w \models \langle\alpha^-_{\geq c}\rangle\varphi$ iff $\Pi^-_w(|\varphi|) \geq c$,
(4) $\mathfrak{M}, w \models \langle\alpha^-_{>c}\rangle\varphi$ iff $\Pi^-_w(|\varphi|) > c$,

where $|\varphi| = \{x \in W \mid \mathfrak{M}, x \models \varphi\}$ is the truth set of φ; and Π_w and Π^-_w represent the possibility measures derived from the out-neighborhood and in-neighborhood of w respectively.

The main feature of QML is that the numerical possibility measures are internalized by using modal operators. Thus, unlike many-valued modal logic, its interpretation is two-valued. However, QML can be faithfully interpreted by a subclass of Kripke models for $G_\Delta(\Box\Diamond)$ (and $G_\sim(\Box\Diamond)$), namely, the models in which $V(p) \in \{0, 1\}$ for any $p \in PV$. Let us define a standard translation τ from QML formulas to $G_\Delta(\Box\Diamond)$ formulas as follows:

(1) $\tau(p) = p$ for any $p \in PV$,
(2) $\tau(\neg\varphi) = \neg\tau(\varphi)$,
(3) $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$,
(4) $\tau(\varphi \vee \psi) = \tau(\varphi) \vee \tau(\psi)$,
(5) $\tau(\varphi \to \psi) = \tau(\varphi) \to \tau(\psi)$,
(6) $\tau(\langle\alpha_{\geq c}\rangle\varphi) = \Delta(\bar{c} \to \langle\alpha\rangle\tau(\varphi))$,
(7) $\tau(\langle\alpha_{>c}\rangle\varphi) = \neg\Delta(\langle\alpha\rangle\tau(\varphi) \to \bar{c})$,
(8) $\tau(\langle\alpha^-_{\geq c}\rangle\varphi) = \Delta(\bar{c} \to \langle\alpha^-\rangle\tau(\varphi))$,
(9) $\tau(\langle\alpha^-_{>c}\rangle\varphi) = \neg\Delta(\langle\alpha^-\rangle\tau(\varphi) \to \bar{c})$.

Note that a QML model is also a $G_\Delta(\Box\Diamond)$ model (albeit a special one in which propositional symbols are only $\{0, 1\}$-valued). Thus, we can evaluate both QML and $G_\Delta(\Box\Diamond)$ formulas in the same Kripke models and obtain the following lemma.

Lemma 4. *Let $\mathfrak{M} = (W, (R_\alpha)_{\alpha \in REL}, V)$ be a Kripke model for QML. Then, for any $w \in W$ and QML formula φ, we have*

- $\mathfrak{M}, w \models \varphi$ *iff* $V(w, \tau(\varphi)) = 1$, *and*
- $\mathfrak{M}, w \not\models \varphi$ *iff* $V(w, \tau(\varphi)) = 0$.

Proof. The proof is straightforward by simultaneous induction on the complexity of φ. $\square$

As in the case of $G(\Box\Diamond)$ logic, we can define the consequence relation with respect to QML and $G_\Delta(\Box\Diamond)$. Let $\Sigma \cup \{\varphi\}$ be a set of QML formulas. Then, by Lemma 4, we have $\Sigma \models_{QML} \varphi$ iff $\tau(\Sigma) \cup \Sigma_0 \models_{G_\Delta(\Box\Diamond)} \tau(\varphi)$, where $\tau(\Sigma) = \{\tau(\psi) \mid \psi \in \Sigma\}$ and $\Sigma_0 = \{p \vee \neg p \mid p \in PV\}$.

Given a hybrid social network $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$, we use the following basic symbols to define a QML language:

(1) $PV = \{p_i \mid i \in I\}$;
(2) $REL = \{\alpha_j \mid j \in J\}$.

The social network $\mathfrak{N}$ is transformed into a QML model $\mathfrak{M}_\mathfrak{N} = (U, (R_j)_{j\in J}, V)$, where V is defined by $V(x, p_i) = 1$ iff $x \in P_i$ for $x \in U$ and $i \in I$; and R_j denotes R_{α_j} for $j \in J$. Then, we say that two actors, x and y, are QML-equivalent with respect to $\mathfrak{N}$ if for all φ in the given QML language ($\mathfrak{M}_\mathfrak{N}, x \models \varphi$ iff $\mathfrak{M}_\mathfrak{N}, y \models \varphi$).

Theorem 4. *Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a hybrid social network. Then, for any $x, y \in U$, $x \equiv^g_\mathfrak{N} y$ iff x and y are QML-equivalent with respect to $\mathfrak{N}$.*

Proof. The forward implication is a corollary of Theorem 3 and Lemma 4. For the converse implication, we assume that E is the QML-equivalence relation on U. Then, E is an equivalence relation that satisfies the first condition of Definition 6 because PV is a subset of QML formulas. Assume that E does not satisfy the second condition of Definition 6. That is, there exist $j \in J$ and $x, y \in U$ such that $(x, y) \in E$, but

$$[R_j x]_E \neq [R_j y]_E \quad \text{or} \quad \left[R_j^- x\right]_E \neq \left[R_j^- y\right]_E.$$

First, we consider the case where $[R_j x]_E \neq [R_j y]_E$. Without loss of generality, we can assume that an enumeration of the quotient set $U/E = \{X_1, X_2, \cdots, X_n\}$ and

$$c_x = [R_j x]_E(X_1) > [R_j y]_E(X_1) = c_y.$$

By slightly abusing the notation, given $1 \le k \le n$ and any QML formula φ, we can use $\mathfrak{M}_\mathfrak{N}, X_k \models \varphi$ to denote $\mathfrak{M}_\mathfrak{N}, u \models \varphi$ for any $u \in X_k$ because X_k is an equivalence class of E. As E is the QML-equivalence relation, there exists φ_k such that $\mathfrak{M}_\mathfrak{N}, X_1 \models \varphi_k$, but $\mathfrak{M}_\mathfrak{N}, X_k \not\models \varphi_k$ for every $2 \le k \le n$. Let $\varphi = \bigwedge_{2\le k\le n} \varphi_k$. Then, the truth set of φ is X_1. Thus, $\sup_{u\in|\varphi|} R_\alpha(x, u) = c_x$ and $\sup_{u\in|\varphi|} R_\alpha(y, u) = c_y$ since $|\varphi| = X_1$. Let us choose a rational c arbitrarily such that $c_x > c > c_y$. Then, we have

$$\mathfrak{M}_\mathfrak{N}, x \models \langle\alpha_{>c}\rangle\varphi;$$

however,

$$\mathfrak{M}_\mathfrak{N}, y \not\models \langle\alpha_{>c}\rangle\varphi,$$

which violates the assumption that $(x, y) \in E$. The assumption that $[R_j^- x]_E \neq [R_j^- y]_E$ leads to a contradiction in the same way. Therefore, E must satisfy the second condition of Definition 6 and is therefore a GRE. $\square$

5. Discussion

In this section, we consider some issues related to the generalizations and applications of the presented results.

5.1. Cut-based definition of regular similarity

Example 1 shows that the 1-cut of the maximum regular similarity is not necessarily a GRE. Hence, although a regular similarity can be decomposed into a family of equivalence relations (i.e., its c-cuts), we need logics with different expressive powers to characterize regular similarity and GRE respectively. However, by using the notion of *weak bisimulation* in [23], we can provide an alternative definition of regular similarity in terms of its c-cuts. Weak bisimulation is defined as a family of binary relations between two Heyting-valued possible world models. In the context of weighted social networks, the definition can be simplified as follows.

Definition 7. Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a weighted social network. Then, a weak bisimulation on $\mathfrak{N}$ is a function $Z : (0, 1] \to 2^{U\times U}$ such that

(1) $Z(c_2) \subseteq Z(c_1)$ for any $c_1 \le c_2$;
(2) for any $c \in (0, 1]$ and $(x, y) \in Z(c)$, the following conditions are satisfied for any $P \in \{P_i \mid i \in I\}$ and $R \in \{R_j, R_j^- \mid j \in J\}$:
  (a) (base) $\min(c, P(x)) = \min(c, P(y))$;
  (b) (forth) for any $u \in U$ such that $R(x, u) > 0$, there exists a $u' \in U$ such that $\min(c, R(x, u)) \le R(y, u')$ and $(u, u') \in Z(\min(c, R(x, u)))$;
  (c) (back) for any $u' \in U$ such that $R(y, u') > 0$, there exists a $u \in U$ such that $\min(c, R(y, u')) \le R(x, u)$ and $(u, u') \in Z(\min(c, R(y, u')))$.

For any $c \in (0, 1]$, two actors $x, y \in U$ are called *weakly c-bisimular* in $\mathfrak{N}$ (notation $x \equiv^c_{\mathfrak{N}} y$) if there exists a weak bisimulation Z on $\mathfrak{N}$ such that $(x, y) \in Z(c)$. The truth invariance of Heyting-valued modal logic with respect to weak bisimulation is stated as follows [23, Theorems 3.11 and 3.14].

Theorem 5. *Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a weighted social network and let $\mathfrak{M}_{\mathfrak{N}} = (U, (R_j)_{j\in J}, V)$ be its corresponding Kripke model. Then, for any $x, y \in U$ and $c \in (0, 1]$, $x \equiv^c_{\mathfrak{N}} y$ iff for any formula $\varphi \in \mathcal{G}(\Box\Diamond)_{\mathfrak{N}}$, $\min(c, V(x, \varphi)) = \min(c, V(y, \varphi))$.*

Although it is possible to prove the equivalence between regular similarity and weak bisimulation by using the definitions directly, the equivalence can also be derived from Theorems 2 and 5 as a corollary.

Corollary 3. *Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a weighted social network, and let S be its maximum regular similarity. Then, for any $x, y \in U$ and $c \in (0, 1]$, $x \equiv^c_{\mathfrak{N}} y$ iff $(x, y) \in S_c$.*

Proof. It is easy to verify that for any $x, y \in U$, $c \in (0, 1]$, and $\varphi \in \mathcal{G}(\Box\Diamond)_{\mathfrak{N}}$, $\min(c, V(x, \varphi)) = \min(c, V(y, \varphi))$ iff $(V(x, \varphi) \Leftrightarrow V(y, \varphi)) \ge c$. Then, the corollary follows immediately from Theorems 2 and 5. $\Box$

5.2. Regular similarity in general structures

Following [27,28], we use sup-min composition and transitivity (Definitions 3 and 4) to define regular similarity. However, it has been suggested that (1) the definitions can be generalized by using any t-norm to replace the min operation; and (2) the maximum regular similarity can be found by solving certain systems of fuzzy relation equations and inequalities [16,17,39,40]. Although the generalization extends the application scope of weighted social networks, the logical characterization theorem does not hold for any t-norm and its corresponding residuum. For instance, let us consider another two well-studied t-norms in fuzzy logic, i.e., the Łukasiewicz t-norm and the product t-norm [31]. These two t-norms and their residua are defined as follows:

(1) Łukasiewicz t-norm: $a \otimes_Ł b = \max(a + b - 1, 0)$ and its residuum: $a \Rightarrow_Ł b = \min(1 - a + b, 1)$;
(2) product t-norm: $a \otimes_P b = a \cdot b$ and its residuum: $a \Rightarrow_P b = \min(\frac{b}{a}, 1)$.

For simplicity, we omit the subscripts Ł and P and simply write $\otimes$ and $\Rightarrow$ when the meanings of the notations are clear from the context. With regard to the Łukasiewicz t-norm, we can define $a \Leftrightarrow b = (a \Rightarrow b) \otimes (b \Rightarrow a) = \min(a \Rightarrow b, b \Rightarrow a) = 1 - |a - b|$, and for the product t-norm $a \Leftrightarrow b = (a \Rightarrow b) \otimes (b \Rightarrow a) = \min(a \Rightarrow b, b \Rightarrow a) = \min(\frac{b}{a}, \frac{a}{b})$.

As with Gödel logic, we can define the Łukasiewicz and product modal logics Ł$(\Box\Diamond)$ and $P(\Box\Diamond)$ as follows. The alphabet of Ł$(\Box\Diamond)$ and $P(\Box\Diamond)$ is an extension of that of $G(\Box\Diamond)$ with an additional conjunction symbol &. In $G(\Box\Diamond)$, & is reduced to the ordinary conjunction $\wedge$. Thus, we do not need & in the $G(\Box\Diamond)$ language. However, in Ł$(\Box\Diamond)$ and $P(\Box\Diamond)$, the t-norms are not the min operation, so & and $\wedge$ are interpreted differently. The Kripke model for Ł$(\Box\Diamond)$ and $P(\Box\Diamond)$ is still the same as that for $G(\Box\Diamond)$, but the truth assignment must be modified accordingly. For a model $\mathfrak{M} = (W, (R_\alpha)_{\alpha\in REL}, V)$, we add the following condition to the truth function of $G(\Box\Diamond)$:

$$V(w, \varphi \& \psi) = V(w, \varphi) \otimes V(w, \psi);$$

and the evaluation of modal formulas is modified as follows:

$$V\big(w, \langle\alpha\rangle\varphi\big) = \sup_{u\in W}\big(R_\alpha(w, u) \otimes V(u, \varphi)\big),$$
$$V\big(w, \langle\alpha^-\rangle\varphi\big) = \sup_{u\in W}\big(R^-_\alpha(w, u) \otimes V(u, \varphi)\big),$$
$$V\big(w, [\alpha]\varphi\big) = \inf_{u\in W}\big(R_\alpha(w, u) \Rightarrow V(u, \varphi)\big),$$
$$V\big(w, [\alpha^-]\varphi\big) = \inf_{u\in W}\big(R^-_\alpha(w, u) \Rightarrow V(u, \varphi)\big),$$

where $\Rightarrow$ is interpreted as the corresponding residuum of the t-norm. Furthermore, the evaluation of the implication is $V(w, \varphi \to \psi) = V(w, \varphi) \Rightarrow V(w, \psi)$. We use the following example to show that Theorem 2 fails for both Ł(□◇) and P(□◇).

Example 2. Let us consider a degenerated weighted social network $\mathfrak{N} = (U, P_1, P_2)$ and assume that there exist at least two distinct actors $x, y \in U$ such that $P_1(x) = 1$, $P_1(y) = 0.9$, $P_2(x) = 0.9$, $P_2(y) = 0.8$. Then, for Ł(□◇), the membership function of the maximum regular similarity S satisfies $S(x, y) = \min(1 - |P_1(x) - P_1(y)|, 1 - |P_2(x) - P_2(y)|) = 0.9$. However, by the semantics, $V(x, p_1\&p_2) = 1 + 0.9 - 1 = 0.9$ and $V(y, p_1\&p_2) = 0.9 + 0.8 - 1 = 0.7$. Thus, $(V(x, p_1\&p_2) \Leftrightarrow V(y, p_1\&p_2)) = 1 - |0.9 - 0.7| = 0.8$, which is less than $S(x, y)$, and Theorem 2 fails for Ł(□◇). Analogously, for P(□◇), the membership function of S satisfies $S(x, y) = \min(\frac{P_1(y)}{P_1(x)}, \frac{P_2(y)}{P_2(x)}) = \frac{8}{9}$. However, $V(x, p_1\&p_2) = 1 \cdot 0.9 = 0.9$ and $V(y, p_1\&p_2) = 0.9 \cdot 0.8 = 0.72$, so $(V(x, p_1\&p_2) \Leftrightarrow V(y, p_1\&p_2)) = \frac{0.72}{0.9} = 0.8$, which is less than $S(x, y)$. Consequently, Theorem 2 also fails for P(□◇). It may be argued that the correct formulation of Theorem 2 should replace the inf operation on the right-hand side with the corresponding t-norm for Ł(□◇) and P(□◇), since the symbol $\Leftrightarrow$ has been interpreted by using the respective residua. However, because $a \otimes b \le \min(a, b)$ holds for any t-norm and $a, b \in [0, 1]$, the reformulated theorem still fails in our example. Note that, for the degenerated weighted social network, the logics Ł(□◇) and P(□◇) are reduced to propositional Łukasiewicz logic and propositional product logic respectively, because the set of modalities is empty. Thus, the degenerated weighted social network is simply a set of fuzzy interpretations for these two propositional logics. In general, the definition of the regular similarity of S with respect to a fuzzy relation R should be modified appropriately for Łukasiewicz and product structures. That is, the min operation in (11) should be replaced by Łukasiewicz or product t-norms. However, because the second condition of regular similarity is vacuously true for the degenerated case, Theorem 2 fails no matter how the relational composition is defined. Furthermore, it is unlikely that Theorem 2 holds for the non-degenerated case, because adding modal formulas to the right-hand sides of (18) and (19) simply make them smaller no matter which t-norm is used. □

5.3. GRE in general structures

We have shown several negative results for the generalizations of Theorem 2, but the situation is more encouraging in the case of GRE. As above, we can extend P(□◇) to P_Δ(□◇) and $P_\sim$(□◇) with the Baaz Delta operator or the involutive negation. However, in Ł(□◇), the negation defined by $\neg\varphi = \varphi \to \bar{0}$ is semantically equivalent to the involutive negation $\sim$, so the Gödel negation is missing in Ł(□◇). Thus, we only need to extend Ł(□◇) to $Ł_\Delta$(□◇) with the Baaz Delta operator. Then, the Gödel negation can be defined as $\Delta\neg\varphi$ in $Ł_\Delta$(□◇). Let us define $\mathcal{L}$-equivalence and $\Delta\mathcal{L}$-equivalence on the universe of actors in the same way as above for any $L = P_\sim$(□◇), P_Δ(□◇), $Ł_\Delta$(□◇), $P_\sim$(◇), P_Δ(◇), $Ł_\Delta$(◇). Then, we can show that Theorem 3 also holds for these extended logical systems. Formally, we have

Theorem 6. *Let $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ be a weighted social network and let L be one of the logics $P_\sim$(□◇), P_Δ(□◇), $Ł_\Delta$(□◇), $P_\sim$(◇), P_Δ(◇), $Ł_\Delta$(◇). Then, for any $x, y \in U$, the following statements are equivalent:*

(1) $x \equiv^g_{\mathfrak{N}} y$;
(2) *x and y are $\mathcal{L}$-equivalent with respect to $\mathfrak{N}$;*
(3) *x and y are $\Delta\mathcal{L}$-equivalent with respect to $\mathfrak{N}$.*

Proof. By checking the proof of Theorem 3, it is easy to see that replacing the Gödel t-norm with the Łukasiewicz t-norm or product t-norm, the proof is still valid. The key facts are as follows:

- any t-norm is distributive with respect to the max operation, i.e., $a \otimes \max_{1\le i\le k} b_i = \max_{1\le i\le k}(a \otimes b_i)$ for any $a, b_i \in [0, 1]$;
- Eq. (5) holds for any residuum operation; and
- $a \Leftrightarrow b = 1$ if $a = b$ and $a \Leftrightarrow b < 1$ if $a \ne b$ for both the Łukasiewicz and the product residua. □

It may be thought that the max in Eq. (16) and the sup in the evaluation of modal formulas should be replaced by a t-conorm if we use other t-norms instead of the min operation. In general, a t-conorm $\oplus$ that corresponds to a t-norm $\otimes$ is defined as $a \oplus b = 1 - (1 - a) \otimes (1 - b)$ for any $a, b \in [0, 1]$. However, Theorem 6 would no longer hold if we were to make such modifications. The main reason is that, in general, a t-norm is not distributive with respect to its corresponding t-conorm.

Example 3. Let us consider the weighted social network $\mathfrak{N} = (U, P, R)$ shown in Fig. 2(a). The network has four actors x, y, z, u, one attribute P and a fuzzy relation R. The numbers on the nodes represent the membership values of an actor to P and the numbers on the edges represent the membership values of R between two actors. We omit the zero membership values of R, so only non-zero values are shown in the figure. If we define GRE by using the Łukasiewicz t-conorm (i.e.,

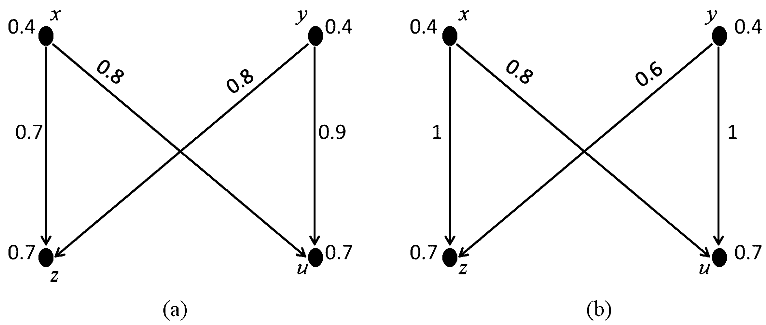

Fig. 2. Two weighted social networks.

$a \oplus b = \min(a+b, 1))$ in Eq. (16), then the maximum GRE E partitions the network into two equivalence classes $X_1 = \{x, y\}$ and $X_2 = \{z, u\}$. This can be verified because we have

$$\begin{aligned}
&[Rx]_E(X_1) = 0 = [Ry]_E(X_1),\\
&[Rx]_E(X_2) = \min(0.7 + 0.8, 1) = 1 = \min(0.8 + 0.9, 1) = [Ry]_E(X_2),\\
&\big[R^- x\big]_E(X) = 0 = \big[R^- y\big]_E(X), \quad \text{for } X = X_1, X_2,\\
&\big[R^- z\big]_E(X_2) = 0 = \big[R^- u\big]_E(X_2),\\
&\big[R^- z\big]_E(X_1) = \min(0.7 + 0.8, 1) = 1 = \min(0.8 + 0.9, 1) = \big[R^- u\big]_E(X_1),\\
&[Rz]_E(X) = 0 = [Ru]_E(X), \quad \text{for } X = X_1, X_2,
\end{aligned}$$

$P(x) = P(y) = 0.4$, and $P(z) = P(u) = 0.7$. However, $V(x, \langle\alpha\rangle p) = (0.7 \otimes 0.7) \oplus (0.7 \otimes 0.8) = (0.7 + 0.7 - 1) + (0.7 + 0.8 - 1) = 0.9$ and $V(y, \langle\alpha\rangle p) = (0.9 \otimes 0.7) \oplus (0.7 \otimes 0.8) = \min((0.9 + 0.7 - 1) + (0.7 + 0.8 - 1), 1) = 1$. Thus, x and y are not $Ł_\Delta(\Box\Diamond)$-equivalent although $x \equiv^g_{\mathfrak{N}} y$ if we use the Łukasiewicz t-conorm in the definition of GRE and the semantics of the modal formulas. Analogously, the network in Fig. 2(b) would be a counterexample of Theorem 6 if we use the product t-conorm (i.e., $a \oplus b = a + b - a \cdot b$) in the definition of GRE and the semantics of the modal formulas. □

5.4. Application to data privacy

In cloud computing, a user can retrieve information from a data center. Thus, the data stored on cloud servers are valuable information sources. However, privacy concerns usually prevent the sharing of information. How to share information while simultaneously preserving data privacy has generated a great deal of interest recently. The information contained in social network data can be released in at least two ways: (1) by publishing the structural information of the anonymized network; and (2) by answering users' queries regarding non-confidential information about the network.

Consider the following realistic scenario. Assume that a weighted social network $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ is stored in a data center and for some subset $I_s \subseteq I$, P_i is a confidential attribute (e.g. a HIV test result) for any $i \in I_s$. To publish the structural information about the network, it is first de-identified so that an isomorphic network $\mathfrak{N}_0 = (W, (P_i)_{i\in I}, (R_j)_{j\in J})$ is released, where W denotes the set of pseudonyms for the actors in U. In other words, each actor is renamed with a unique pseudonym in W so that there exists a 1–1 and onto mapping $de: U \to W$ such that (1) $P_i(x) = P_i(de(x))$ for any $i \in I$ and $x \in U$; and (2) $R_j(x, y) = R_j(de(x), de(y))$ for any $j \in J$ and $x, y \in U$. This allows the structural information in $\mathfrak{N}$ to be published accurately without revealing the identifiers of the actors.

However, an attacker may have access to the non-confidential information about the original network from other data sources. In other words, the attacker may have some background knowledge about the actors in the network. In the logical approach, the background knowledge is represented by the formulas of a logical language. Hence, the attacker can obtain such background knowledge by querying the data sources with the formulas of the language; and the answers are the actors that satisfy the queries (possibly with degrees of satisfaction in the case of the weighted network). Thus, a query language determines whether the queries to the data center are allowable or legal. A more expressive language would allow more detailed information to be released. To restrict queries to the non-confidential part of the network, we assume that the alphabet of the language only contains symbols that denote P_i for $i \in I - I_s$ and R_j for $j \in J$.

The publication of the anonymized network alone would not reveal any confidential information about individual actors. This is because the user can not identify the actor that a pseudonym denotes, i.e., the user does not know what the de-identification mapping *de* refers to. Moreover, the query-answering process alone does not jeopardize personal privacy because the query language does not contain any symbols that denote the confidential attributes. However, there may be a risk to privacy if the two processes are combined because the query-answering process may help the attacker determine the pseudonym of an actor if he possesses all data in the anonymized network $\mathfrak{N}_0$. Let $\mathcal{L}$ denote the query language. Then, for any query φ (i.e., a formula of $\mathcal{L}$) with respect to the network $\mathfrak{N}$, the answer set is $ans(\varphi, \mathfrak{N}) = \{(x, deg(x, \varphi)) \mid x \in U,$

$deg(x,\varphi) \neq 0\}$, where $deg(x,\varphi)$ is the degree of satisfaction of φ for an actor x. Note that we usually omit the actors that do not satisfy φ at all. Now, the query φ can also be evaluated in the anonymized network $\mathfrak{N}_0$. Thus, with the answer set, the user can exclude the possibility of $de(x) = w$ if $deg(x,\varphi) \neq deg(w,\varphi)$ for any $x \in U$ and $w \in W$. In this way, the user can successively eliminate the possible candidates for the real pseudonym of x. However, no matter how many queries are answered, the user can not eliminate any $w \in W$ that is $\mathcal{L}$-equivalent to $de(x)$. Hence, the final candidate set for the real pseudonym of an actor x is the $\mathcal{L}$-equivalence class of W that contains $de(x)$.

Formally, we can define a *configuration of a weighted social network publication* (CWSNP) model as a tuple $(\mathfrak{N}, \mathfrak{N}_0, de, \mathcal{L})$, where

(1) $\mathfrak{N} = (U, (P_i)_{i\in I}, (R_j)_{j\in J})$ is a weighted social network;
(2) $\mathfrak{N}_0 = (W, (P_i)_{i\in I}, (R_j)_{j\in J})$ is the anonymized weighted social network;
(3) de is an isomorphism from $\mathfrak{N}$ to $\mathfrak{N}_0$; and
(4) $\mathcal{L}$ is a many-valued modal language built from propositional symbols corresponding to P_i for $i \in I - I_s$ and relational symbols corresponding to R_j for $j \in J$, where $\{P_i \mid i \in I_s\}$ is the subset of confidential attributes of the network.

Let $\equiv_{\mathcal{L}}^{\mathfrak{N}}$ denote the $\mathcal{L}$-equivalence relation on the domain of the network $\mathfrak{N}$. Then, for a given CWSNP $(\mathfrak{N}, \mathfrak{N}_0, de, \mathcal{L})$, X is an equivalence class of $\equiv_{\mathcal{L}}^{\mathfrak{N}}$ iff $de(X)$ is an equivalence class of $\equiv_{\mathcal{L}}^{\mathfrak{N}_0}$ for any $X \subseteq U$. Hence, if the quotient set $U/\equiv_{\mathcal{L}}^{\mathfrak{N}}$ is $\{X_1, X_2, \cdots, X_k\}$, then $W/\equiv_{\mathcal{L}}^{\mathfrak{N}_0}$ is $\{de(X_1), de(X_2), \cdots, de(X_k)\}$. Because de is an isomorphism, we have $V(X_i,\varphi) = V(de(X_i),\varphi)$ for $1 \le i \le k$ and $\varphi \in \mathcal{L}$, where the truth value of a formula in an equivalence class is defined in the proof of Theorem 3. Furthermore, for $1 \le i \neq j \le k$, there exists a formula $\varphi \in \mathcal{L}$ such that $V(X_i,\varphi) \neq V(X_j,\varphi)$. This means that an attacker can identify each equivalence class X_i on U with its corresponding equivalence class $de(X_i)$ on W, even though he can not identify $de(x)$ for any individual $x \in U$. Formally, for a given CWSNP $(\mathfrak{N}, \mathfrak{N}_0, de, \mathcal{L})$, the total information (potentially) available to an attacker comprises

(1) the anonymized network $\mathfrak{N}_0$;
(2) the partition of the actors of $\mathfrak{N}$ into $X_1, X_2, \cdots, X_k$;
(3) the partition of the pseudonym of $\mathfrak{N}_0$ into $Y_1, Y_2, \cdots, Y_k$;
(4) the mapping $(X_i \mapsto Y_i)_{1\le i\le k}$;
(5) for any $\varphi \in \mathcal{L}$ and $1 \le i \le k$, the truth value $V(X_i,\varphi) = V(Y_i,\varphi)$.

Note that we do not make any assumption about an attacker's computational power. Hence, an attacker may be not able to derive all the information that is potentially available to him. However, the data center must ensure that privacy is not invaded even though the attacker has unlimited computational power. Thus, to assess the privacy risks of a CWSNP $(\mathfrak{N}, \mathfrak{N}_0, de, \mathcal{L})$, the data center has to assess if any confidential information can be derived from the above-mentioned information. For example, if there exists an equivalence class $X \subseteq U$ such that $|X| = 1$, it means that there exists an actor $x \in U$ such that it is possible to uniquely locate $de(x)$ in $\mathfrak{N}_0$ by querying the data center several times. In this case, we say that the anonymity of x is violated and his/her privacy is invaded. Another case of privacy invasion occurs when there exists an equivalence class $X \subseteq U$ such that for all $u \in X$, $P_i(u)$ is the same for some $i \in I_s$. In this case, the attacker knows $\mathfrak{N}_0$ and X's corresponding equivalence class $Y = de(X) \subseteq W$; therefore, he can evaluate the truth value of the confidential attribute P_i with respect to each element of Y and derive that $P_i(u) = P_i(w)$ for any $w \in Y$ and $u \in X$ because de is an isomorphism. Consequently, the attacker can determine the membership value of u in the confidential attribute P_i for any $u \in X$. In general, the assessment of privacy risk depends on the choice of privacy criteria. Formal definitions of privacy criteria based on the $\mathcal{L}$-equivalence relation can be found in [37,38].

From the above analysis, it is clear that the data center must compute the $\mathcal{L}$-equivalence relation on $\mathfrak{N}$ for the privacy assessment. However, because the number of formulas of a logical language is infinite in general, the direct computation of the $\mathcal{L}$-equivalence relation is not effective. In other words, to directly determine if two actors x and y are $\mathcal{L}$-equivalent, we have to evaluate and compare $V(x,\varphi)$ and $V(y,\varphi)$ for each formula φ in $\mathcal{L}$. However, because the number of formulas is infinite, it is impossible to complete the process in a finite number of steps. However, by using the characterization results provided in this paper, the computation of the $\mathcal{L}$-equivalence relation can be reduced to the computation of the corresponding GRE or regular similarity of $\mathcal{L}$. In [27,28], an effective procedure is presented for the computation of GRE and regular similarity over Gödel structure. More general algorithms that compute GRE and regular similarity over any complete residuated lattices by solving systems of fuzzy relation equations are provided in [16,17,39,40]. Therefore, the $\mathcal{L}$-equivalence relation can be computed effectively. Although the analysis of the complexity of these algorithms and their computational properties are beyond the scope of the paper, we remark that the algorithms are based on an iterative refinement procedure proposed in [7] for computing the coarsest regular equivalence of an ordinary social network, which has $O(n^3)$-time complexity. A more sophisticated algorithm with $O(m \log_2 n)$-time complexity exists for an ordinary social network, where m and n are, respectively, the number of links and the number of nodes in the network [45]. Therefore, it would be worthwhile exploring the generalization of this more efficient algorithm to the computation of regular similarity and GRE.

As an example, let the query language $\mathcal{L}$ be a $G_{\Delta}(\Box\Diamond)$ language with the alphabet $REL = \{\alpha_j \mid j \in J\}$ and $PV = \{p_i \mid i \in I - I_s\}$. Then, $deg(x, \varphi)$ is interpreted as $V(x, \varphi)$ for the truth valuation V in the Kripke model. In this case, we can simply compute the GRE of the subnetwork $\mathfrak{N}_1 = (U, (P_i)_{i \in I - I_s}, (R_j)_{j \in J})$ over a Gödel structure and partition W accordingly. Then, we can check the equivalence classes of the partition and assess the safety of the CWSNP $(\mathfrak{N}, \mathfrak{N}_0, de, \mathcal{G}_{\Delta}(\Box\Diamond))$. If privacy violations are detected, we can try to use a less expressive query language or sanitize the network structure before it is released; however, concrete sanitization techniques are beyond the scope of this paper. In general, our results show that the computation of $\mathcal{L}$-equivalence can be reduced to the computation of GRE if the query language $\mathcal{L}$ is one of the languages mentioned in Theorems 3 and 4; and it can be reduced to the computation of regular similarity if the query language is $\mathcal{G}(\Box\Diamond)$ or $\mathcal{G}(\Diamond)$.

5.5. Related work

Many-valued modal logics have been motivated by different application contexts in the literature. One of the earliest systems was the modal logic valued on finite Heyting algebra introduced by Fitting [29,30], who also provided a multi-expert interpretation of the logic. More recently, the minimum modal logic over a finite residuated lattice (RL) was systematically studied in [8], where $\Box$-fragments of both finite RL-valued modal logics with truth constants and finite MV chain-valued modal logics without truth constants were axiomatized.

While the many-valued modal logics considered in [8,29,30] are quite general, they tend to concentrate on the finite-valued case. For the infinite-valued case, the formalization of the $[0,1]$-valued modal systems S5 and KD45 is investigated in [31,33]. However, these systems impose special constraints on the fuzzy accessibility relations of the Kripke models. For example, the fuzzy S5 system requires that the fuzzy accessibility relation is the universal relation. In recent years, the general framework for Gödel modal logic has also been studied extensively. The $\Box$-fragment and $\Diamond$-fragment of different Gödel modal systems, including K, D, T, S4, and S5, are axiomatized in [12]; and the full Gödel modal logics K, T, S4, and S5 are axiomatized in [13]. Moreover, analytic proof methods for the $\Box$-fragment and $\Diamond$-fragment of the fuzzy K system, including sequent-of-relations and hypersequent calculi, are introduced in [53]. In addition to complete axiomatizations, the finite model property and the decidability of Gödel modal logics are investigated in [11–13,53]. Syntactically, our modal language $G(\Box\Diamond)$ is simply a multi-modal extension of the Gödel modal logics presented in [11–13,53]. However, the different syntax is inessential because each α-modality is interpreted in the same way. In fact, there also exist works on fuzzy description logics [4,5,32], that are inter-translatable with our $G(\Box\Diamond)$ language. However, it seems that the modal logics extended with Baaz Delta operator and involutive negation have not been explored.

Irrespective of whether the logics we consider in this paper have been presented in the literature, the focus of our work is quite different from existing studies, which are mainly concerned with the development of deductive systems for the logics. Hence, there is a dearth of works on the investigation of bisimulation and the resultant modal invariance results. The only exception is the notion of weak bisimulation for Heyting-valued modal logics introduced in [23]. In Section 5.1, we showed that weak bisimulation on a weighted social network (regarded as a model of many-valued modal logics) is equivalent to the family of c-cuts of the maximum regular similarity. However, to the best of our knowledge, the notion of GRE and its corresponding modal invariance results were not investigated before.

Since the pioneering work of Sweeney [60–62], the application of privacy-preserving data publication (PPDP) to data privacy has rapidly evolved into an active research area in computer science. Initially, the objective of PPDP was to prevent an attacker from inferring private information by using a published data table. While most early works on PPDP focused on the publication of tabulated data, privacy-preserving publication of social networks has received a great deal of attention in recent years [35,50,55,68,67].

Generally, the cited studies assume that an attacker could use some background knowledge to compromise an individual's privacy by analyzing the released anonymized network. With different types of background knowledge, an attacker can launch different privacy attacks. In [37,38], Hsu et al. propose a logical framework that unifies existing attack models by considering each type of attack as a subset of admissible queries in a logical language. The main computational task of the framework is to find the indiscernibility relation on the set of actors with respect to admissible queries. It is shown that the modal invariance results of regular equivalence and exact equivalence facilitate efficient computation with respect to query languages based on description logic. In this paper, we show that the framework can be extended to weighted social network. We also demonstrate that the effective computation of indiscernibility relations with respect to several query languages based on many-valued modal logics can be achieved by using existing algorithms to compute regular similarity or GRE.

In addition to anonymizing individual identities, link weight anonymization has been proposed as an effective technique for privacy protection of weighted social networks [18,19]. The technique modifies the weights on links of a network while preserving some properties of the network. The properties to be preserved depend on the type of application. Although our results are mainly applied to privacy risk assessment, link weight anonymization can be regarded as a kind of network sanitization technique if our approach detects privacy violation. However, it seems that assessing the privacy risk of a network after link weight anonymization does not fit into the framework proposed in [37,38]. Thus, our results can not be applied straightforwardly and further investigation is needed in order to incorporate the technique into the logical framework.

6. Concluding remarks

The notion of regular equivalence has been studied extensively in social network analysis, and it has found many applications in block modeling, network clustering, role or position identification, and so on. To represent the intensity of ties and interactions between actors, traditional social networks have been generalized to weighted social networks. There exist different, but equivalent, definitions of regular equivalence in the literature. However, when generalized to weighted social networks, the definitions may result in different notions of similarity. Two kinds of generalizations based on the Gödel t-norm, called regular similarity and generalized regular equivalence (GRE), were proposed in [28]. Regular similarity is generalized based on the commutativity between the similarity relation and the underlying fuzzy relation; while GRE is generalized according to the equivalence of the neighborhoods of equivalent nodes. The definition of regular similarity was further generalized to arbitrary residuated lattices in [16].

In this paper, we show that many-valued modal logic can characterize regular similarity or GRE in weighted social networks. By regarding a weighted social network as a model of many-valued modal logic, similar or equivalent actors satisfy the set of modal logic formulas to the same degree. Specifically, the many-valued modal logic $G(\Box\Diamond)$ based on the Gödel t-norm characterizes regular similarity in the sense that the degree of similarity between two actors is equal to the fuzzy equivalence of the actors' truth degrees for any formulas of the logic. In addition, the extensions of $G(\Box\Diamond)$ with the involutive negation or projection operators characterize GRE in the sense that two actors are equivalent iff they satisfy any formulas of the logics to the same degree. For a special kind of weighted social network, called a hybrid social network, in which all the actors' attributes are crisp, although their ties may be weighted, we also show its logical characterization by a modal version of possibilistic logic, called QML. Furthermore, we show that the characterizations of GRE can be extended to many-valued modal logics based on the Łukasiewicz and product t-norms, while the extensions for regular similarity have negative results. Finally, we demonstrate a potential application of the logical characterizations to privacy-preserving data publication for weighted social networks.

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