

A formal theory for reasoning about parthood, connection, and location

Maureen Donnelly

*Institute for Formal Ontology and Medical Information Science, Saarland University,
D-66041 Saarbrücken, Germany*

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Abstract

In fields such as medicine, geography, and mechanics, spatial reasoning involves reasoning about entities that may coincide without overlapping. Some examples are: cavities and invading particles, passageways and valves, geographic regions and tropical storms. The purpose of this paper is to develop a formal theory of spatial relations for domains that include coincident entities. The core of the theory is a clear distinction between mereotopological relations, such as parthood and connection, and relative location relations, such as coincidence. To guide the development of the formal theory, I construct mathematical models in which nontrivial relative location relations are defined.

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1. Introduction

Two entities *overlap* when they share a common part. Two entities *coincide* when they occupy overlapping regions of space.¹ The Mississippi River and Minnesota overlap—the first ten kilometers of the Mississippi River are part of both the river and the state. The river and the state also coincide—the region occupied by the first ten kilometers of the

E-mail address: maureen.donnelly@ifomis.uni-saarland.de (M. Donnelly).

¹ Note that with this usage coincident objects need only occupy *overlapping* spatial regions. I will use the term “complete coincidence” for the stronger relation that holds between objects that occupy *identical* spatial regions.

Mississippi River is part of both the region occupied by the entire river and the region occupied by the state. Similarly, my hand and my body both overlap and coincide. It is easy to see that any overlapping spatial entities must also coincide. Their locations will overlap at their common parts.

But the relation of coincidence is broader than that of overlap. In other words, there are pairs of coincident objects which do not share parts. The food that is currently being digested in my stomach cavity coincides with, but does not overlap, my stomach cavity. A tropical storm covering Acapulco coincides with, but does not overlap, Mexico. Any object coincides with, but does not overlap, the spatial region at which it is located at a given point of time.

A mereotopology is a formal theory of parthood and connection relations. It has long been recognized that mereotopology forms an essential part of formal ontology. Several different mereotopologies have been proposed in recent literature, including [1,4,11,17]. These theories are ultimately intended for reasoning about relations among a variety of spatial entities including material objects (amoebas, mechanical devices, etc.) and geographical entities (islands, bays, etc.). However, it is assumed in nearly all of this work that the immediate domains of application are restricted to spatial regions. When material objects are introduced, as in [8], mereotopological relations are still restricted to regions. The material objects have only a second-hand mereotopological structure that is inherited from the regions at which they are located. Thus, a distinct coincidence relation is not usually introduced in the mereotopology: on domains of regions, coincidence is just overlap.

Likewise, mathematical models for these theories typically use simple domains consisting just of subsets of a topological space. See, for example, [1,3,9]. On these domains, there is no natural way of giving the coincidence relation a broader interpretation than that of the overlap relation. Here, overlap is generally interpreted as non-empty intersection.² The coincidence relation could have a broader interpretation only if it were artificially extended to pairs of subsets with an empty intersection.

The goal of this paper is to construct a mereotopology for domains that include coincident but non-overlapping entities. I present a primary theory, called Layered Mereotopology, and two variants of it. Domains for Layered Mereotopology may include both material objects and the regions at which they are located, in addition to other types of entities, such as holes or geopolitical entities, which may coincide with material objects. Layered Mereotopology allows spatial relations to apply directly to all entities within the domain, be they regions, material objects, holes, or what have you. It extends mereotopology by adding *relative location relations*—relations such as coincidence that depend only on the objects' locations—and by making explicit the relation between a spatial entity and the region at which it is located. To guide the development of the formal theory, I construct a class of mathematical structures, called Layered Models, in which a coincidence relation distinct from overlap is defined.

Layered Mereotopology borrows much from the theory of location of [7]. It differs in that it divides the domain into different layers, each of which is mereologically independent of the others. Also, no models are provided for the Casati and Varzi theory other than

² The interpretation in [1] is *slightly* different but in the same spirit.

the standard topological models that conflate coincidence and overlap. Additional work on combining either relative location relations or a location function with other spatial relations can be found in [2,5,8,15].

The outline for this paper is as follows. In Section 2, I construct the Layered Models. These structures include both mereotopological relations and relative location relations. Layered Models are the target models of the formal theory, Layered Mereotopology, which is presented in Sections 3–5. The mereological subtheory, Layered Mereology, is developed in Section 3. In Section 4, a function that maps each member of the domain to its region is added and the first group of relative location relations are defined. Topological relations and more relative location relations are added in Section 5. In the last part of the paper, I present two variations of Layered Mereotopology, both of which make weaker assumptions about the composition of the spatial domain. Section 6 proposes a version of Layered Mereotopology with weakened requirements on the formation of sums. Section 7 develops a version of Layered Mereotopology that does not assume that the spatial domain includes a special sub-domain of regions.

Layered Mereotopology and its two variants are presented here as time-independent theories. They can be used in this form either to describe instantaneous time-slices of a three-dimensional domain or to describe space-time relations among changeless four-dimensional entities such as processes. The theory can also be naturally extended to include time-dependent relations which can be used to describe change in spatial domains.

2. Layered Models

In this section, I introduce a class of mathematical structures in which both mereotopological and relative location relations are defined. I call these structures *Layered Models* because their domains are partitioned into non-overlapping *layers*. Members of the same layer coincide only when they overlap. Members of different layers never stand in mereotopological relations, but they may coincide or stand in other relative location relations.

Layered Models are intended to represent the actual spatial world in such a way that spatial entities of distinct types—regions, material objects, holes, geographic objects, etc.—are assigned to distinct layers. In particular, a special layer, covering the entire spatial domain, represents the collection of all regions. For other types of spatial entities (material objects, holes, geographical objects, etc.), I leave open the question of whether all tokens of that type reside on one layer or whether they are distributed to different layers. For example, at one extreme, all material objects may be assigned to the same layer and, at the other extreme, each independent material object (my desk, your car, etc.) may be assigned to its own layer.

Layered Models are defined as follows. Let $T = \langle X, cl \rangle$ be a topological space, where X is the set of points and cl is the closure operator. Let I be any index set that includes 0. The domain, D , of a Layered Model is a nonempty set of ordered pairs, $x_i = \langle x, i \rangle$ where $\emptyset \neq x \subseteq X$ and $i \in I$. (I will generally use the abbreviation x_i for $\langle x, i \rangle$. All variables referring to objects in Layered Models appear in Arial font to distinguish them from the variables of the formal theory.) The first component of each ordered pair determines its location. The

second component determines the layer to which it belongs. All pairs of the form $\langle x, 0 \rangle$ (i.e., x_0) belong to a special layer, called the *region layer*.

I require that the domain, D , of any Layered Model satisfy the following conditions:

1. For any $i \in I$, if $x_i \in D$, then $x_0 \in D$.
2. For any $i \in I$ and any $Y \subseteq \wp(X)$, if $y_i \in D$ for all $y \in Y$, then $(\bigcup_{y \in Y} (y))_i \in D$.
3. For any $i \in I$ and any $x_i, y_i \in D$, if $x \not\subseteq y$, then there is a $z_i \in D$ such that $z \subseteq x$ and $z \cap y = \emptyset$.
4. If $x_i, y_i \in D$ and $x \cap y \neq \emptyset$, then $(x \cap y)_i \in D$.

Note that conditions 1–4 are not very restrictive. Even once the topological space and index set are fixed, domains with very different compositions are possible. As a token example, suppose T is the usual topological space based on the real numbers, \mathfrak{R} , and $I = \{0, 1\}$. Then $\{x_i: \emptyset \neq x \subseteq \mathfrak{R} \text{ and } i = 0, 1\}$ satisfies conditions 1–4. In this case, the domain has uncountably many members. At the other end of the spectrum, $\{[7, 8]_0, [7, 8]_1\}$ is an example of a layered domain with only two members constructed from the same topological space and index set.

Given a domain D of a layered model, it follows from condition 2 that for any $i \in I$, the sum of all members of D of the form x_i is also in D . More precisely, for any $i \in I$, let $Y_i = \{x: x \subseteq X \text{ \& } x_i \in D\}$. Then, by 2, $L_i = (\bigcup_{y \in Y_i} (y))_i \in D$. I will call L_i the Layer i of D . Notice that it follows from condition 1 that for any $i \in I$, if $y_i = \text{Layer } i$ and $z_0 = \text{Layer } 0$, then $y \subseteq z$. The region layer always covers the entire space of possible locations. The other layers may or may not cover the entire background space. Thus, either $y = z$ or $y \subset z$ is possible. For example, both $\{[7, 8]_0, [7, 8]_1\}$ and $\{[6, 7)_0, [6, 8]_0, [7, 8]_0, [7, 8]_1\}$ are possible domains for layered models.

I will now define relations on the domains of Layered Models. Each model-theoretic relation will be the target interpretation of a relation that is introduced formally in Layered Mereotopology (Sections 3–6). I will use bold-faced letters for the model-theoretic relations and plain text for their counterparts in the formal theory.

The mereotopological relations are introduced first.

The parthood relation, **P**, is defined on the domain, D , of a Layered Model as follows:

$$\mathbf{P}(x_i, y_j) =: x \subseteq y \text{ and } i = j.$$

Notice that it follows from this definition that (i) any $x_i \in D$ is part of Layer j if and only if $i = j$ and (ii) only parts of the same layer can stand in the parthood relation.

The overlap relation, **O**, is defined:

$$\mathbf{O}(x_i, y_j) =: x \cap y \neq \emptyset \text{ and } i = j.$$

As with parthood, (i) any $x_i \in D$ overlaps Layer j if and only if $i = j$ and (ii) only parts of the same layer can overlap. Also notice that it follows from condition 4 above that two members of D overlap if and only if they have a common part.

The underlap relation, **U**, is defined:

$$\mathbf{U}(x_i, y_j) =: i = j.$$

It follows that two members of the domain underlap if and only if they are parts of the same layer.

Finally a connection relation, **C**, also restricted to parts of the same layer, is defined:

$$\mathbf{C}(x_i, y_j) =: (\text{cl}(x) \cap y \neq \emptyset \text{ or } x \cap \text{cl}(y) \neq \emptyset) \text{ and } i = j.^3$$

Other mereotopological relations, such as proper parthood, external connection, and tangential parthood, can be defined in the obvious ways, so that they hold only among parts of the same layer.

In contrast to the definitions above, the definitions of the relative location relations refer to only the first coordinates of the ordered pairs. This gives us relations that depend only on location and may hold between parts of different layers.

$$\begin{aligned} \mathbf{Cov}(x_i, y_j) &=: x \subseteq y && \text{(covering)} \\ \mathbf{Coin}(x_i, y_j) &=: x \cap y \neq \emptyset && \text{(coincidence)} \\ \mathbf{CCoin}(x_i, y_j) &=: x = y && \text{(complete coincidence)} \\ \mathbf{M}(x_i, y_j) &=: \text{cl}(x) \cap y \neq \emptyset \text{ or } x \cap \text{cl}(y) \neq \emptyset && \text{(meets)} \\ \mathbf{A}(x_i, y_j) &=: x \cap y = \emptyset \ \& \ (\text{cl}(x) \cap y \neq \emptyset \text{ or } x \cap \text{cl}(y) \neq \emptyset) && \text{(abuts)} \end{aligned}$$

Finally, I add the function, **r**, from *D* to Layer 0 which assigns each member of the domain to its representative on the region layer:

$$\mathbf{r}(x_i) = x_0.$$

When restricted to Layer 0, **r** is the identity function: for any $x_0 \in D$, $\mathbf{r}(x_0) = x_0$. Also notice that two objects stand in a given relative location relation if and only if their regions stand in the corresponding mereotopological relation. For example, two objects coincide if and only if their regions overlap, two objects meet if and only if their regions are connected, and so on.

I will conclude this section with a simple example designed to illustrate the way in which Layered Models can be used to represent spatial relations among regions, material objects, and immaterial entities such as holes. The background topological space for this model is \mathcal{R}^3 with its standard topology. The region layer has as parts the members of the set $\{x_0: \emptyset \neq x \subseteq \mathcal{R}^3\}$.

Suppose that we wish to represent relations holding among a vase, a portion of water in the vase, a flower standing in the vase, and the interior of the vase. The vase, water, and flower can be represented, respectively, by parts, v_1 , w_1 , and f_1 , of Layer 1 where the subsets, *v*, *w*, and *f*, of \mathcal{R}^3 are disjoint, but connected. The interior of the vase, a hole,

³ Nothing important in what follows hinges on this particular interpretation of the connection relation.

$$\mathbf{C}x_iy_j =: \text{cl}(x) \cap \text{cl}(y) \neq \emptyset \text{ and } i = j \quad \text{or}$$

$$\mathbf{C}x_iy_j =: x \cap y \neq \emptyset \text{ and } i = j$$

may be used instead. But these alternative interpretations are better suited for somewhat different domains. In the first case, we probably want to restrict the domains to open regular subsets of the topological space and, in the second case, to closed regular subsets. In the second case, we should also strengthen the definition of **O** to keep it distinct from **C**.

is represented by h_2 on Layer 2 where h and v are disjoint, but connected, w is a proper subset of h ($w \subset h$), and h and f have a nonempty intersection ($h \cap f \neq \emptyset$). According to this representation, the water is not part of the hole, but the fact that the water is contained in the interior of the vase is represented in the model by the covering relation, **Cov**(w_1, h_2), holding between w_1 and h_2 . Similarly, the flower does not overlap the interior of the vase, but the fact that the flower is partially contained in the interior of the vase is represented by the coincidence relation: **Coin**(f_1, h_2). The vase and its interior do not coincide. Nor are they connected. But the fact that the vase touches its interior is represented by the abutment relation: **A**(v_1, h_2). Finally, every object is exactly co-located with its spatial region. This is represented in the model by the complete coincidence of the object and its region. For example, **CCoin**(v_1, v_0).

3. Layered Mereology

A mereology is a formal theory of the binary parthood relation. My aim in this section is to develop a mereology, called *Layered Mereology*, that is satisfied by the parthood relation, **P**, defined on Layered Models. Layered Mereology is extended in Sections 4 and 5 to Layered Mereotopology, a theory that also includes relative location and connection relations.

3.1. Axioms and basic definitions of Layered Mereology

Layered Mereology (like its extension, Layered Mereotopology) is formulated in first-order logic. u, v, w, x, y, z are used as variables and a, b, c, d are used as constants. Relations and functions are represented in the formal theory with plain text letters to distinguish them from the relations and functions defined in Layered Models. Layered Mereology assumes one primitive, the binary relation **P** which, on the intended interpretation, represents parthood.

The following relations are defined in the formal theory in terms of **P**:

(D1) $PPxy =: Pxy \ \& \ \sim Pyx$

(x is a proper part of y)

(D2) $Oxy =: \exists z (Pzx \ \& \ Pzy)$

(x and y overlap)

(D3) $Uxy =: \exists z (Pxz \ \& \ Pyz)$

(x and y underlap)

It is easy to see that, if **P** is interpreted as **P** in Layered Models, the defined relations **O** and **U** will be interpreted as **O** and **U**, respectively.

The axioms of Layered Mereology will be somewhat nonstandard. For example, they cannot require that any pair of objects have a mereological sum. My goal is to axiomatize

P in such a way that, when restricted to a single layer, it satisfies the axioms (and axiom schema) of General Extensional Mereology (GEM):⁴

(P1) Pxx

(P2) $Pxy \ \& \ Pyx \rightarrow x = y$

(P3) $Pxy \ \& \ Pyz \rightarrow Pxz$

(P4) $\sim Pxy \rightarrow \exists z(Pzx \ \& \ \sim Ozy)$

(GEM5) $\exists x \phi[x] \rightarrow \exists z \forall w (Owz \leftrightarrow \exists x (\phi[x] \ \& \ Owx))$

$\phi[x]$ in (GEM5) stands for any first-order formula in GEM (or an extension of GEM) in which only x occurs free. (GEM5) states that if any member of the domain satisfies the formula $\phi[x]$, then there must be a sum of all objects satisfying $\phi[x]$. (GEM5) must be altered for Layered Mereology because Layered Models only include the sums of objects that are parts of the same layer.⁵

I will discuss summation in Layered Mereology in more detail shortly. For now, notice that the relation **P** in Layered Models satisfies each of the first four axioms of GEM. They are therefore used in their original forms as axioms for Layered Mereology.⁶

The first three axioms require that **P** is a partial ordering. (P1) says that **P** is reflexive. (P2) says that **P** is anti-symmetric. (P3) says that **P** is transitive. That (P1)–(P3) are satisfied by **P** in Layered Models follows immediately from the fact that **P** is just the subset relation restricted to the separate layers.

(P4) says that, if x is not a part of y , then there is some part, z , of x that does not overlap y . For instance, suppose x is a table and y is one of its proper parts, say one of its legs. Then, according to (P4), x must have some part (its top or another leg) that does not overlap y .

(P4) is satisfied in Layered Models by virtue of condition 3 of Section 2.

The following theorem is very useful for proving theorems listed later in this paper. It is derived from (P1), (P3), and (P4).

(PT1) $Pxy \leftrightarrow \forall z (Ozx \rightarrow Ozy)$

(x is part of y if and only if for all z , if z overlaps x then z overlaps y)

It follows from (PT1) and (P2) that overlap, **O**, is extensional. This means that any two members of the domain that overlap the same objects are identical.

(PT2) $x = y \leftrightarrow \forall z (Ozx \leftrightarrow Ozy)$

(x is identical to y if and only if for all z , z overlaps x if and only if z overlaps y)

⁴ Throughout this paper, initial universal quantifiers are suppressed unless they are needed for clarity.

⁵ In some mereologies, a stronger second-order formula, requiring there to be a sum of any nonempty set of individuals, is used instead of the first-order schema (GEM5). See, e.g., [13] and [18]. For reasons of simplicity, I will consider only the first-order version of GEM and develop Layered Mereology as a first-order alternative to GEM. But the same strategy can be used to construct a second-order version of Layered Mereology that parallels the mereologies of [13] or [18].

⁶ All axioms of Layered Mereology are labeled with a “P”. “PT” is used for theorems of Layered Mereology.

(PT2) lets us use the overlap relation to uniquely identify sums when they exist. Because O is extensional, for any formula $\phi[x]$, if we can assign z to a member of the domain that satisfies

$$\forall w (Owz \leftrightarrow \exists x (\phi[x] \& Ow x)) \quad (*)$$

then this object is the *unique* sum of all ϕ -ers. However, there need not be an object satisfying $(*)$ even if some member of the domain satisfies $\phi[x]$. For example, let $\phi[x]$ be $x = x$ and let D be the domain of a Layered Model. Then every member of D satisfies $\phi[x]$. But if there are $x_i, y_j \in D$ with $i \neq j$, then no member of D satisfies $(*)$. This is because such an object would have to overlap every member of D and there can be no member of D that overlaps both x_i and y_j for $i \neq j$.

Thus, we need a restricted version of (GEM5) that requires sums to exist only if all summands are part of the same layer. Such an axiom schema will be satisfied in all Layered Models by virtue of condition 2 of Section 2. Given that two objects in a Layered Model are parts of the same layer if and only if they underlap, the restricted summation axiom schema can read as follows:

$$(P5) (\exists x \phi[x] \& \forall x, y (\phi[x] \& \phi[x/y] \rightarrow Uxy)) \rightarrow \exists z \forall w (Owz \leftrightarrow \exists x (\phi[x] \& Ow x))$$

Here, $\phi[x/y]$ is the formula $\phi[x]$ with all free instances of x replaced by y and where variable substitution is performed as necessary so y is free in $\phi[x/y]$ exactly where x is free in $\phi[x]$. (P5) says that if there is some object that satisfies $\phi[x]$ and any two objects that satisfy $\phi[x]$ underlap, then there is a sum of all objects satisfying $\phi[x]$.

For convenience, I will use the abbreviation $z\Sigma_x(\phi[x])$ for specific substitution instances of the formula $(*)$. For example, $z\Sigma_x(Pxa \vee Pxb)$ is the open formula:

$$\forall w (Owz \leftrightarrow \exists x ((Pxa \vee Pxb) \& Ow x))$$

and $c\Sigma_y(Pya \vee Pyb)$ is the closed formula:

$$\forall w (Ow c \leftrightarrow \exists y ((Pya \vee Pyb) \& Ow y)).$$

We would like to be able to say more things about layers within the mereology. So far, we can only say that two objects are on the same layer. We would like to be able to say that a certain object is a layer or is the layer of a particular object. (D4) defines a relation holding between y and z when z is the sum of all objects that y underlaps. I use this for the definition of the layer relation.

$$(D4) \text{ } L_y z =: z\Sigma_x(Uxy)$$

(z is y 's layer: z is the sum of all objects that y underlaps)

It is easy to see that when P is interpreted as \mathbf{P} in Layered Models, $L_{x_i} y_j$ holds if and only if $y_j = \text{Layer } i$. However, axioms (P1)–(P5) do not allow us to infer that for any object, y , there is some object that is the sum of all objects underlapping y . In other words, our axioms so far do not allow us to infer that every object has a layer. This would follow from (P5) if we knew that any two objects that underlap y must underlap each other. Given that U is already reflexive and symmetric, the antecedent of the relevant substitution instance of (P5) will be satisfied if U is also transitive. Notice that the underlap relation, U , for Layered

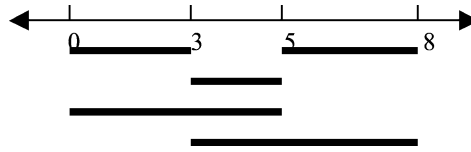


Fig. 1.

Models is in fact an equivalence relation (reflexive, symmetric, and transitive) and that the sets consisting of all objects in a single layer are the equivalence classes determined by the U relation.

But although it follows from (P1)–(P5) that U is reflexive and symmetric, it does not follow that U is transitive. To see this, consider the model of (P1)–(P5) which consists of the intervals, $[0, 5]$, $[0, 3]$, $[3, 5]$, $[5, 8]$, $[3, 8]$, with P interpreted as the subset relation. See Fig. 1.

$[0, 3]$ and $[3, 5]$ underlap (they are both part of $[0, 5]$). $[3, 5]$ and $[5, 8]$ underlap (they are both part of $[3, 8]$). But $[0, 3]$ and $[5, 8]$ do not underlap. Notice also that $[3, 5]$ has no layer. $[3, 5]$ underlaps every member of the domain. Thus, its layer would have to overlap every member of the domain. But there is no object in this model that overlaps every member of the domain.

I, therefore, add a final axiom to Layered Mereology:

(P6) $(Uxy \ \& \ Uyz) \rightarrow Uxz$
(underlap is transitive)

It follows from (P5), (P6), and the reflexivity and symmetry of U that every object has a layer:

(PT3) $\forall y \exists z Lyz$
(every object has a layer)

Thus the relation, L , is a function. I will use the function term $l(x)$ to stand for the layer of x .

The next group of theorems about layers follow from (D4) and the fact that U is an equivalence relation:

(PT4) $Pxl(x)$
(every object is part of its layer)

(PT5) $Uxy \leftrightarrow l(x) = l(y)$
(two objects underlap if and only if they have the same layer)

(PT6) $Uxy \leftrightarrow Pyl(x)$
(x underlaps y if and only if y is part of x 's layer)

It also follows from (P6) that overlap implies underlap.

(PT7) $Oxy \rightarrow Uxy$
(if x and y overlap, then they underlap)

3.2. Layers and maximal individuals

We can introduce the unary predicate, LY, which distinguishes certain members of the domain as layers.

(D5) $LYz =: \exists x Lxz$
(z is a layer)

When P is interpreted as **P** in Layered Models, LY is interpreted as {Layer i: $i \in I$ }.

It follows easily from (PT4), (PT5), and (D5) that:

(PT8) $LYz \leftrightarrow Lzz$
(z is a layer iff z is its own layer)

(PT8) tells us that layers are those members of the domain that are the mereological sums of all objects they underlap.

Unfortunately, (PT8) may not be especially helpful if our goal is to distinguish layers in a practical reasoning context. We do not usually talk about the underlap relation in practical contexts and it is, therefore, somewhat difficult to form an intuitive picture of the collection of all individuals that a given individual underlaps. We can get a better understanding of what layers might be by introducing the following predicate, defined in terms of parthood and overlap, which is equivalent to LY in Layered Mereology.

(D6) $MIz =: \forall x (Oxz \rightarrow Pxz)$
(z is a maximal individual: for all x, if x overlaps z, then x is part of z)

Maximal individuals are individuals that only overlap their own parts. In particular, maximal individuals are never proper parts.

(PT9) $MIz \rightarrow \sim \exists x PPzx$
(if z is a maximal individual, then there is no x such that z is a proper part of x)

It follows from (P1)–(P4) and (P6) that being a maximal individual is equivalent to being a layer.

(PT10) $MIz \leftrightarrow LYz$
(z is a maximal individual if and only if z is a layer)

In fact, given (P1)–(P4), the following two formulae are, taken together, equivalent to (P5) and (P6):

(P5*) $(\exists x \phi[x] \ \& \ \exists z (MIz \ \& \ \forall x (\phi[x] \rightarrow Pxz))) \rightarrow \exists z \forall w (Owz \leftrightarrow \exists x (\phi[x] \ \& \ Owx))$
(if some member of the domain satisfies $\phi[x]$ and there is a maximal individual, z, such that all ϕ -ers are part of z, then there is a sum of all ϕ -ers)

(P6*) $\forall x \exists z (MIz \ \& \ Pxz)$
(every individual is part of some maximal individual)

Thus, Layered Mereology can be equivalently axiomatized by (P1)–(P4), (P5*), and (P6*). In this alternative axiomatization, all formulae referring to the underlap relation are replaced by formulae referring to maximal individuals.

What, then, are the layers, or maximal individuals, of a spatial domain? They are the individuals that share parts only with their own parts. Just what, more specifically, these individuals are will depend on one's assumptions about the structure of the world. Layered Mereology is designed to accommodate different kinds of ontologies. If one holds that a common-sense material object, such as a table, a television, or a human body, is not part of any larger object and does not share parts with any object that extends beyond itself, then each of these objects is its own layer. If, on the other hand, one holds that there is a sum of any collection of material objects, but that no entity includes both material objects and other types of entities (regions, holes, etc.) as parts, then the sum of all material objects is a layer. At the furthest extreme, one may even hold that there is a sum of any collection of individuals. In this case, the sum of all individuals is the only layer.

Similar considerations apply to other types of entities. For example, if one holds that the interior of my coffee cup (a hole) overlaps only its own parts, then the interior of my coffee cup is a layer. If, on the other hand, one holds that there is a sum of any collection of holes, then all holes belong to the same layer.

In the full theory, Layered Mereotopology, only one restriction is put on the way in which different types of entities are divided into layers. I add in Section 4 axioms requiring that there is a layer consisting of all regions. It follows from these axioms that if there is only one layer, then every individual is a region.

3.3. Additional mereological relations

We can use the formula schema $z\Sigma_x(\phi[x])$ to introduce relational counterparts of familiar Boolean operators. (D7)–(D9) are the standard mereological definitions of the sum, product, and difference relations. (D10) defines a *relative* complement relation.

(D7) $+(v, y, z) =: z\Sigma_x(Pxv \vee Pxy)$
(z is the binary sum of v and y)

(D8) $\times(v, y, z) =: z\Sigma_x(Pxv \& Pxy)$
(z is the binary product of v and y)

(D9) $-(v, y, z) =: z\Sigma_x(Pxy \& \sim Oxv)$
(z is the difference of v in y)

(D10) $-(v, z) =: z\Sigma_x(Uxv \& \sim Oxv)$
(z is the relative complement of v)

The following existence theorems can be derived:

(PT11) $\exists z(+(x, y, z)) \leftrightarrow Uxy$
(x and y have a sum if and only if x and y underlap)

(PT12) $\exists z(\times(x, y, z)) \leftrightarrow Oxy$
(x and y have a product if and only if x and y overlap)

(PT13) $\exists z(-(x, y, z)) \leftrightarrow \sim Pyx$
(there is a difference of x in y if and only if y is not part of x)

(PT14) $\exists z(-\langle x, z \rangle) \leftrightarrow \sim LYx$

(x has a relative complement if and only if x is not a layer)

We can also derive the following correspondence between the relative complement relation and the difference relation.

(PT15) $-\langle x, z \rangle \leftrightarrow -\langle x, l(x), z \rangle$

(z is x's relative complement if and only if z is the difference of x in x's layer)

Using (PT5) it is easy to prove that, when they exist, sums, products, and relative complements belong to the same layer as the original object(s). If it exists, the difference of x in y belongs to the same layer as y. This need not be also x's layer. In fact, if x does not belong to the same layer as y, then y itself is the difference of x in y.

3.4. Layered Mereology and General Extensional Mereology (GEM)

Meta-Theorem 1. *Layered Mereology is a subtheory of GEM.*

Proof. It is obvious that (P1)–(P4) can be derived from the axioms of GEM—they are included in the axiomatization of GEM given in Section 3.1. (P5) is an obvious consequence of (GEM5). To see that (P6) can also be derived, we need only note that

$$\forall x \forall y Uxy$$

is a theorem of GEM. (Taking $x = x$ for $\phi[x]$ in (GEM5), we can prove that there is an individual of which every member of the domain is part.) It follows immediately that U is transitive (P6). \square

Thus, any model of GEM is a one-layer model of Layered Mereology.

Conversely, any one-layer model of Layered Mereology is a model of GEM. We can show, more generally, that any layer of a model of Layered Mereology is a model of GEM.

Meta-Theorem 2. *Let M be any model of Layered Mereology with domain, D. Note that M need not belong to the class of Layered Models defined in Section 2. Let \mathbf{P} , \mathbf{O} , \mathbf{U} , \mathbf{I} be, respectively, the interpretations of P, O, U, I in M. Let $c \in D$ and let $D_c = \{y: y \in D \ \& \ I(y) = I(c)\}$. Let M_c be the structure whose domain is D_c with P interpreted as $\mathbf{P}|_{D_c}$ (i.e., the restriction of \mathbf{P} to D_c). Then M_c satisfies axioms (P1)–(P4) and axiom schema (GEM5).*

Proof. Note that, because M is a model of Layered Mereology, whenever $x \in D_c$ and either $\langle x, z \rangle$ or $\langle z, x \rangle$ is a member of \mathbf{P} , \mathbf{O} , or \mathbf{U} , then $I(z) = I(x) = I(c)$ and $z \in D_c$. Also, note that when P is interpreted as $\mathbf{P}|_{D_c}$ in D_c , each defined relation of Layered Mereology is interpreted as the restriction of its M-interpretation to D_c . For example, O and U are interpreted in M_c as, respectively, $\mathbf{O}|_{D_c}$ and $\mathbf{U}|_{D_c}$. Notice further that $\mathbf{U}|_{D_c}$ is $D_c \times D_c$.

It follows trivially that, since \mathbf{P} is a partial ordering, so is $\mathbf{P}|_{D_c}$. Thus, (P1)–(P3) are satisfied in M_c .

To see that (P4) is satisfied in M_c , suppose that $x, y \in D_c$ and $\langle x, y \rangle \notin \mathbf{P}_{|D_c}$. Then $\langle x, y \rangle \notin \mathbf{P}$. Since (P4) is satisfied in M , there is some $z \in D$ such that $\langle z, x \rangle \in \mathbf{P}$ & $\langle z, y \rangle \notin \mathbf{O}$. Since $\langle z, x \rangle \in \mathbf{P}$, $z \in D_c$. Thus, $\langle z, x \rangle \in \mathbf{P}_{|D_c}$. Since $\langle z, y \rangle \notin \mathbf{O}$, $\langle z, y \rangle \notin \mathbf{O}_{|D_c}$.

For (GEM5): Let $\phi[x]$ be a first-order formula (with only x free) in an extension, T , of Layered Mereology where the interpretation of each primitive of T on D_c is the restriction of its interpretation on D . Suppose there is some $b \in D_c$ such that b satisfies $\phi[x]$ in M_c . Let $\phi^*[x]$ be a formula obtained from $\phi[x]$ by: (1) constructing a formula $\phi'[x]$ equivalent to $\phi[x]$ in which only primitives of T occur and then (2) restricting the range of all quantification in $\phi'[x]$ to D_c .

Lemma 1. *If $x \in D_c$, then x satisfies $\phi^*[x]$ in M if and only if x satisfies $\phi[x]$ in M_c .*

Lemma 1 is proved by induction on the number of connectives in $\phi'[x]$. The proof is tedious, but straightforward and will be omitted.

Let b be a constant denoting b . Then:

Lemma 2. *$(\phi^*[x] \& Uxb)$ is satisfied in M by $x \in D$ if and only if x satisfies $\phi[x]$ in M_c .*

Lemma 2 follows immediately from Lemma 1, and the fact that x satisfies $(\phi^*[x] \& Uxb)$ in M if and only if x satisfies $\phi^*[x]$ in M and $x \in D_c$.

By Lemma 2, since b satisfies $\phi[x]$ in M_c , b satisfies $(\phi^*[x] \& Uxb)$ in M . Thus,

$$M \models \exists x (\phi^*[x] \& Uxb).$$

Also,

$$M \models \forall x, y (((\phi^*[x] \& Uxb) \& (\phi^*[x/y] \& Uyb)) \rightarrow Uxy).$$

(To see this, note that by the transitivity and symmetry of U , $M \models \forall x, y ((Uxb \& Uyb) \rightarrow Uxy)$.) Thus, since (P5) is satisfied in M ,

$$M \models \exists z \forall w (Owz \leftrightarrow \exists x ((\phi^*[x] \& Uxb) \& Owx)).$$

Let s be the unique member of D that satisfies

$$\forall w (Owz \leftrightarrow \exists x ((\phi^*[x] \& Uxb) \& Owx))$$

in M . Since $\langle s, b \rangle \in \mathbf{O}$, $s \in D_c$. I will show that s satisfies $\forall w (Owz \leftrightarrow \exists x (\phi[x] \& Owx))$ in M_c :

(i) Suppose $\langle w, s \rangle \in \mathbf{O}_{|D_c}$. Then $\langle w, s \rangle \in \mathbf{O}$ and there is $x \in D$ such that x satisfies $(\phi^*[x] \& Uxb)$ in M and $\langle w, x \rangle \in \mathbf{O}$. By Lemma 2, x satisfies $\phi[x]$ in M_c . Thus, w satisfies $\exists x (\phi[x] \& Owx)$ in M_c .

(ii) Let $w, x \in D_c$. Suppose $(\phi[x] \& Owx)$ is satisfied in M_c when w is interpreted as w and x is interpreted as x . By Lemma 2, $(\phi^*[x] \& Uxb)$ is satisfied by x in M . Since $\langle w, x \rangle \in \mathbf{O}$, $((\phi^*[x] \& Uxb) \& Owx)$ is satisfied in M when w is interpreted as w and x is interpreted as x . Thus, $\langle w, s \rangle \in \mathbf{O}$. So $\langle w, s \rangle \in \mathbf{O}_{|D_c}$.

It follows from (i) and (ii) that the assumption

$$M_c \models \exists x \phi[x]$$

implies

$$M_c \models \exists z \forall w (Owz \leftrightarrow \exists x (\phi[x] \& Ow x)).$$

Thus, any instance of the axiom schema (GEM5) is satisfied in M_c .

4. The region function

In Layered Mereology, we have no way of stating that two objects coincide. Layered Mereology lets us describe the parthood relations between objects. It does not let us describe the relative locations of objects. To do this, I extend Layered Mereology by adding the unary function r which, on the intended interpretation, assigns each object, x , to the region, $r(x)$, at which x is exactly located. In Layered Models, r is interpreted as the function \mathbf{r} .

Using r , we can define a one-place predicate, R , which distinguishes the sub-domain of regions.

$$(D11) \quad Ry =: \exists x (r(x) = y)$$

(y is a region)

When r is interpreted as \mathbf{r} in Layered Models, the interpretation of R is $\{x_i \in D: i = 0\}$.

The axioms for r are added to axioms (P1)–(P6).⁷ It is easy to check that they are satisfied in Layered Models.

$$(R1) \quad Ry \& Rz \rightarrow Uyz$$

(all regions are located in the same layer)

$$(R2) \quad Ry \& Uyz \rightarrow r(z) = z$$

(every member of the region layer is its own region)

The theorems below can now be derived.

$$(RT1) \quad Ry \rightarrow r(y) = y$$

(every region is located at itself)

$$(RT2) \quad r(r(x)) = r(x)$$

(the region function is idempotent)

$$(RT3) \quad Ry \& Uyz \rightarrow Rz$$

(every member of a region's layer is a region)

$$(RT4) \quad \forall x (\phi[x] \rightarrow Rx) \& z \Sigma_x (\phi[x]) \rightarrow Rz$$

(every sum of regions is a region)

Additional axioms relate the region function to parthood.

$$(R3) \quad Pxy \rightarrow \text{Pr}(x)r(y)$$

(if x is part of y , then x 's region is part of y 's region)

⁷ All axioms specific to the r function are marked with “R”. Theorems specific to the r function are marked with “RT”.

The following theorem is derived from (R3) and the mereological axioms.

(RT5) $Oxy \rightarrow Or(x)r(y)$

(if x overlaps y , then x 's region overlaps y 's region)

Notice that the converse of (R3) is not generally satisfied in Layered Models. $r(x_i)$ may be part of $r(y_j)$ even though x_i is not part of y_j . This will be the case whenever $x \subseteq y$, but $i \neq j$. In fact, given the antisymmetry of P (P2) and the idempotency of r (RT2), the converse of (R3) would imply that every member of the domain is identical to its region. Thus, the converse of (R3) would reduce the domain to a single layer of regions.

On the other hand, in Layered Models, if $P(r(x_i), r(y_j))$ and $i = j$, then $P(x_i, y_j)$ must also hold. More generally, if $O(r(x_i), r(y_j))$ and $i = j$, then $O(x_i, y_j)$ must also hold. I therefore add the following axiom.

(R4) $Uxy \ \& \ Or(x)r(y) \rightarrow Oxy$

(if x and y are on the same layer and x 's region overlaps y 's region, then x overlaps y)⁸

From (R4), it follows that whenever the regions of two objects in the same layer stand in a given mereological relation, then the objects themselves also stand in that relation. In particular, we can derive the following theorem.

(RT6) $Uxy \ \& \ Pr(x)r(y) \rightarrow Pxy$

(if x and y are in the same layer and x 's region is part of y 's region, then x is part of y)

Notice that, given (P1)–(P6) and (R1)–(R3), (R4) could not have been derived from (RT6). To see this, consider the model of (P1)–(P6), (R1)–(R3), and (RT6) whose domain consists of

Layer 0: $[0, 1]_0, [1, 1]_0, [1, 2]_0, [0, 2]_0$

Layer 1: $[0, 1]_1, [1, 2]_1, [0, 2]_1$

with both P and r interpreted as in Layered Models. Here, the region, $[0, 1]_0$, of $[0, 1]_1$ overlaps the region, $[1, 2]_0$, of $[1, 2]_1$ ($[1, 1]_0$ is a part of both), but $[0, 1]_1$ and $[1, 2]_1$ do not overlap. Thus, (R4) is not satisfied in this model.

The structure described in the previous paragraph is not a Layered Model because it does not satisfy condition 4 of Section 2, requiring that whenever $x_i, y_i \in D$ and $x \cap y \neq \emptyset$, then $(x \cap y)_i \in D$. We can weaken this condition somewhat and still retain (R4),⁹ but (R4) at least requires that the overlap relations among objects on any given layer mirror those of their images in the region layer. If we wanted, for example, to allow regions to have point-like products, but material objects to have only three-dimensional products, then we would have to give up (R4). We could allow such “mixed granularity” models if, for example, we replaced (R4) with the weaker (RT6).

⁸ To see that (R4) does not follow from the preceding axioms, consider the model of (P1)–(P6) and (R1)–(R3) whose domain consists of Layer 0: $[0, 1]_0$; Layer 1: $[0, 1]_1, [1, 2]_1, [0, 2]_1$; where P is interpreted as in Layered Models, but r is interpreted so that $[0, 1]_0$ is the region of all members of the domain. Here, the region of $[0, 1]_1$ overlaps the region of $[1, 2]_1$, but $[0, 1]_1$ and $[1, 2]_1$ do not overlap.

⁹ For example, if we wanted to restrict the first components of the ordered pairs to non-empty closed regular subsets of the topological space, we could replace 4 with: if $x_i, y_i \in D$ and $\text{int}(x \cap y) \neq \emptyset$, then $\text{cl}(\text{int}(x \cap y))_i \in D$.

The relative location relations are defined formally in terms of the region function, r .

$$(D12) \text{ CCoin}(x, y) =: r(x) = r(y)$$

(x and y completely coincide)

$$(D13) \text{ Cov}(x, y) =: \text{Pr}(x)r(y)$$

(x is covered by y)

$$(D14) \text{ Coin}(x, y) =: \text{Or}(x)r(y)$$

(x and y coincide)

It is easy to check that when P is interpreted as \mathbf{P} and r as \mathbf{r} in Layered Models, CCoin , Cov , and Coin are interpreted as, respectively, **CCoin**, **Cov**, and **Coin**.

The following theorems concerning these relations can be derived.

$$(RT7) \text{ Coin}(x, y) \leftrightarrow \exists z (\text{Cov}(z, x) \ \& \ \text{Cov}(z, y))$$

(x and y coincide if and only if there is some z that is covered by both x and y)

$$(RT8) \text{ CCoin}(x, y) \leftrightarrow \text{Cov}(x, y) \ \& \ \text{Cov}(y, x)$$

(x and y completely coincide if and only if y covers x and x covers y)

$$(RT9) \text{ CCoin}(x, y) \ \& \ \text{CCoin}(x, z) \ \& \ \text{Uyz} \rightarrow y = z$$

(any object can completely coincide with at most one object in any layer)

$$(RT10) \text{ CCoin}(x, y) \ \& \ \text{Uxy} \rightarrow x = y$$

(if x and y completely coincide and are on the same layer, then $x = y$)

$$(RT11) \text{ Cov}(x, y) \ \& \ \text{Uxy} \rightarrow \text{Pxy}$$

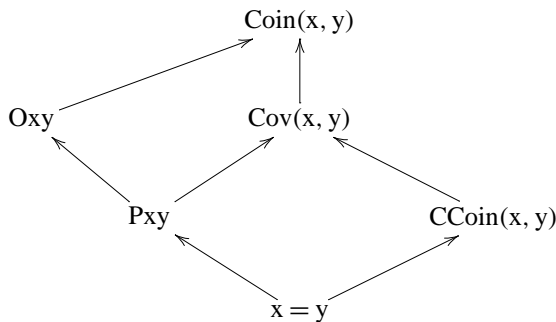
(if y covers x and x and y are on the same layer, then x is part of y)

$$(RT12) \text{ Coin}(x, y) \ \& \ \text{Uxy} \rightarrow \text{Oxy}$$

(if x and y coincide and are on the same layer, then x and y overlap)

(RT10)–(RT12) tell us that when two objects in the same layer stand in one of the relative location relations above, then they must also stand in the corresponding mereological relation. This implies, for example, that two parts of the same individual can coincide only when they overlap.

In addition to the theorems listed above, we can prove that CCoin is an equivalence relation, that Cov is transitive and reflexive, and that Coin is symmetric and reflexive. Also, the implications illustrated in the diagram below can be derived. The arrow indicates that the atomic formula at the start of the arrow implies the atomic formula at the end of the arrow.



5. Layered Mereotopology

The base theory can be extended to Layered Mereotopology in a straightforward way by adding a connection relation, C , where Cxy means “ x is connected to y ”. I assume that only parts of the same maximal individual may be connected. For example, my hand is connected to my forearm—they are parts of my body that touch one another. By contrast, the vase and its interior, as described in Section 2, are not connected because they do not belong to the same maximal individual. Instead, the vase and its interior stand in the relative location relation, *abuts*.

C is interpreted as \mathbf{C} in Layered Models. The axioms for C are as follows.¹⁰

(C1) Cxx

(connection is reflexive)

(C2) $Cxy \rightarrow Cyx$

(connection is symmetric)

(C3) $Pxy \rightarrow \forall z(Czx \rightarrow Czy)$

(if x is part of y , everything connected to x is connected to y)

(C4) $Cxy \rightarrow Uxy$

(if x and y are connected, then they are parts of the same layer)

(C5) $Cxy \rightarrow Cr(x)r(y)$

(if x and y are connected, their regions are also connected)

(C6) $Uxy \ \& \ Cr(x)r(y) \rightarrow Cxy$

(if x and y are members of the same layer and their regions are connected, then x and y are connected)

The external connection relation is defined in terms of C and O in the usual way.

(D15) $ECxy =: Cxy \ \& \ \sim Oxy$

(x and y are externally connected: x and y are connected but do not overlap)

It follows immediately that external connection, like connection, can only hold among parts of the same layer.

(CT1) $ECxy \rightarrow Uxy$

The relative location relations *meets* (M) and *abuts* (A) are defined in terms of the region function:

(D16) $Mxy =: Cr(x)r(y)$

(x and y meet: their regions are connected)

(D17) $Axy =: ECr(x)r(y)$

(x and y abut: their regions are externally connected)

¹⁰ Axioms specific to Layered Mereotopology are marked with a “C”. Theorems are marked with “CT”.

For example, the vase and its interior both meet and abut. The flower standing in the vase meets, but does not abut, the interior of the vase.

It is easy to check that if C is interpreted as **C** in Layered Models, then M is interpreted as **M** and A is interpreted as **A**.

We can prove that M is reflexive and symmetric and that both EC and A are irreflexive and symmetric. We can also derive the following theorems:

(CT2) $Cxy \rightarrow Mxy$

(if x and y are connected, then x and y meet)

(CT3) $ECxy \rightarrow Axy$

(if x and y are externally connected, then x and y abut)

(CT4) $(Uxy \ \& \ Mxy) \rightarrow Cxy$

(if x and y are on the same layer and they meet, then they are connected)

(CT5) $(Uxy \ \& \ Axy) \rightarrow ECxy$

(if x and y are on the same layer and they abut, then they are externally connected)

(CT6) $Axy \leftrightarrow (Mxy \ \& \ \sim \text{Coin}(x, y))$

(x and y abut if and only if they meet and do not coincide)

The tangential part relation, TP , is usually defined in terms of external connection as follows (see, for example, [1,9]):

$$TPxy =: Pxy \ \& \ \exists z (ECzx \ \& \ ECzy) \quad (**)$$

This definition is not appropriate for Layered Mereotopology. To see why, consider the Layered Model whose underlying topological space is \mathfrak{R} with its standard topology and whose layers have the members of the following sets as parts:

Layer 0: $\{x_0: \emptyset \neq x \subseteq \mathfrak{R}\}$

Layer 1: $\{x_1: \emptyset \neq x \subseteq [0, 1]\}$

It would follow from the standard definition of TP that $[0, 1]_1$ has no tangential parts, since it is not externally connected to any member of the domain. For example, it would follow that $[0, 0]_1$ and $[1, 1]_1$ are not tangential parts of $[0, 1]_1$.

In the context of Layered Mereotopology, $(**)$ would tell us that no maximal individual has any tangential parts. Thus, for example, if a human body were considered a maximal object, $(**)$ would not enable us to distinguish between external body parts, like fingers and skin, and internal body parts, like the heart.

I will therefore use instead the following definition of tangential part:

(D18) $TPxy =: Pxy \ \& \ \exists z (Azx \ \& \ Azy)$

(x is a tangential part of y : x is part of y and there is some z such that z abuts both x and y)

Applying this definition to the previous model, it turns out that any part of $[0, 1]_1$ that contains either $[0, 0]_1$ or $[1, 1]_1$ is a tangential part of $[0, 1]_1$. More generally, it follows from (D18) that, for objects x and y in the same layer, x is a tangential part of y if and only if x 's region is a tangential part of y 's region:

(CT7) $U_{xy} \rightarrow (TP_{xy} \leftrightarrow TPr(x)r(y))$.

Interior parthood is then defined as usual:

(D19) $IP_{xy} =: P_{xy} \ \& \ \sim TP_{xy}$

(x is an interior part of y)

It may be useful to add cross-layer counterparts of the tangential part and interior part relations:

(D20) $TCov(x, y) =: TPr(x)r(y)$

(x is t-covered by y: x's region is a tangential part of y's region)

(D21) $ICov(x, y) =: IPr(x)r(y)$

(x is i-covered by y: x's region is an interior part of y's region)

With these two additional relations we can, for example, distinguish between the way in which a table is covered by a room's interior (t-covered) and the way in which a computer on the table is covered by the room's interior (i-covered).

If desired, relational counterparts of topological operators can be added in the usual way. For instance, an object's interior can be defined as the sum of its interior parts:

(D22) $INT(y, z) =: z \Sigma_x (IP_{xy})$

(z is the interior of y)

Since each layer of any model of Layered Mereology is a model of GEM (Section 2, Meta-theorem 2), it is trivial to show that each layer of any model of Layered Mereotopology is a model of the standard mereotopology which uses (**) as the definition of tangential parthood and includes axioms (C1)–(C3) in addition to those of GEM. (See [7] for a discussion of this mereotopology.) However, notice that the interpretations of TP and IP on the reduced single-layer models may not match their interpretations on the larger multi-layer model. In the standard mereotopological model whose domain is limited to Layer 1 of the previous example (with P and C interpreted as in the Layered Model),

$$\{x_1: \emptyset \neq x \subseteq [0, 1]\}$$

$[0, 1]_1$ does not have any tangential parts.

Though the separate layers within a model of Layered Mereotopology function independently as mereological structures, they are not entirely independent as mereotopological structures. It is an obvious consequence of definition (D18) that we may need to look at the spatial structure beyond an object's layer to determine whether it has any tangential parts.

It is easy to see, however, that the region layer, at least, must always have a self-contained mereotopological structure. When the domain of any model of Layered Mereotopology is restricted to the region layer and P and C are interpreted as in the larger model, then the interpretations of EC, TP, and IP (with TP defined via (**)) will always be the restrictions to the region layer of their interpretations in the larger model.

The overall strategy of Layered Mereotopology can be summarized as follows: the spatial domain is partitioned into layers in such a way that mereotopological relations hold

only among parts of the same layer. This allows us, when the context warrants, to down-size to a more standard single-layer mereotopological model, as long as care is taken with the interpretations of TP and IP. The weaker cross-layer relative location relations are distinct from the mereotopological relations in multiple-layered models, but collapse into their mereotopological counterparts when the domain consists of a single layer.

My goal in developing Layered Mereotopology has been to formulate a theory for mixed spatial domains in which the relative location relations are distinct from the mereotopological relations. I have tried to make the theory flexible on other ontological issues so that it will be compatible with a variety of positions. For example, Layered Mereotopology allows models in which each member of the domain is the sum of atoms as well as models in which every member of the domain has a proper part. Also, as discussed in Section 3, the layers may be interpreted so as to include either all or only some entities of a given type. Thus the theory is compatible with the assumption that there is a sum of any collection of material objects. But it is also compatible with the assumption that common-sense objects (tables, chairs, etc.) are maximal individuals.

Some slight changes in the underlying mereotopology can be easily made to accommodate different assumptions about the relation between parthood and connection. For instance, we could use a stronger version of (C3), in which the implication is replaced by a biconditional. In this case, each layer would be a model of the stronger mereotopology that uses the biconditional instead of (C3).

The purpose of the final part of this paper is to discuss two more substantial possible changes in Layered Mereotopology. In Section 6, I present an alternate version of Layered Mereotopology in which the summation requirements are weakened. In Section 7, I present a version of Layered Mereotopology that does not assume its domains contain a distinct layer of regions.

6. Layered Mereotopology with Weakened Summation requirements

At one extreme, the spatial domain can be represented in terms of Layered Mereotopology as a single-layer model. (P5) would then imply that there is a sum of any nonempty collection of spatial entities satisfying a given formula $\phi[x]$. (R1) and (R2) would imply that every member of the spatial domain is a region.

At the other extreme, we can treat each common-sense individual (my table, the interior of your coffee cup) as a maximal individual. On this interpretation, there could be no sum of all material objects, no sum of all holes, and so on. But, (P5) would still require that there is a sum of any nonempty collection of parts of a given maximal individual satisfying a formula $\phi[x]$. Thus, for example, given that all four legs of my table are parts of the table, (P5) requires that the table has as additional parts the sums of any two of its legs (e.g., the sum of the back left leg and the right front leg). In particular, (P5) requires here that there are scattered material objects, where a *scattered object* is an object that is the sum of disconnected parts (like the table legs).

These sums, though perhaps unintuitive, are not necessarily problematic. After all, we can, when it is convenient, talk comfortably about scattered sums such as my pair of gloves. Nevertheless, if desired, it is easy to construct a weaker version of Layered Mereotopology

which does not commit us to arbitrary sums of the parts of a given maximal individual. I will call the weaker version of the full theory Weak Summation Layered Mereotopology (WSLM).

WSLM can be axiomatized by deleting (P5*) (which asserts that if some member of the domain satisfies $\phi[x]$ and all ϕ -ers are part of the same maximal individual, then there is a sum of all ϕ -ers) from the alternative axiomatization for Layered Mereology given in Section 3.2. The region function axioms and topological axioms are left unchanged. The axioms for WSLM are, then, (P1)–(P4), (P6*), (R1)–(R4), and (C1)–(C6).

Surprisingly, we lose very few of the theorems listed in Sections 3–5 in WSLM. Of these, only (PT11)–(PT14) (the theorems giving the conditions for the existence of binary sums, binary products, differences, and relative complements) are not also theorems of WSLM. In particular, when the layer relation, L , and the layer predicate, LY , are defined as in Section 3, we can still prove that every individual has a layer and that being a layer is equivalent to being a maximal individual:¹¹

(WSLMT1) $\forall y \exists z L y z$

(WSLMT2) $LYz \leftrightarrow M I z$.

In addition, (P6) of the original axiomatization of Layered Mereology is a theorem of WSLM:

(WSLMT3) $Uxy \ \& \ Uyz \rightarrow Uxz$

(underlap is transitive)

In WSLM, as opposed to Layered Mereotopology, we are free to hold that some material objects consist only of nonoverlapping, self-connected proper parts. We may, if we like, hold that only its top and its four separate legs are proper parts of my table. Alternatively, we may hold that there are sums of the collection of all instances of a given universal (the sum of all tables, the sum of all human bodies), but not sums of random sub-collections of these (the sum of my table and your table).

Since WSLM is a subtheory of Layered Mereotopology, all models of Layered Mereotopology, including the Layered Models of Section 2, are models of WSLM. But we obtain a broader class of models for WSLM (call them WS Layered Models) by interpreting P , r , and C , as in Layered Models but replacing condition 2 on the domains of the models with:

2*. Let $Y_i = \{x: x \subseteq X \ \& \ x_i \in D\}$. Then, $L_i = (\bigcup_{y \in Y_i} (y))_i \in D$.

Conditions 1, 3, and 4 are retained in their original forms. Condition 2* requires only that the sum of all pairs with the same second coordinate (i.e., belonging to the same layer) belong to the domain. It replaces the much stronger condition 2 which required that for each $i \in I$, $Y_i = \{x: x \subseteq X \ \& \ x_i \in D\}$ is closed under finite and infinite unions. 2* requires only that $\bigcup_{y \in Y_i} (y)$ be in Y_i .

¹¹ Theorems of WSLM are marked with “WSLMT”.

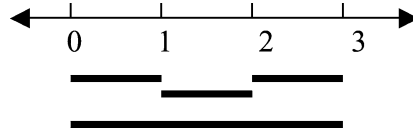


Fig. 2.

An example of a WS Layered Model that is not also a Layered Model is the single-layered model with the following domain:

Layer 0: $[0, 1]_0, (1, 2)_0, [2, 3]_0, [0, 3]_0$

Here, there is a sum, $[0, 3]_0$, of all members of Layer 0, but no sum of $[0, 1]_0$ and $(1, 2)_0$, of $(1, 2)_0$ and $[2, 3]_0$, or of $[0, 1]_0$ and $[2, 3]_0$ (see Fig. 2).

Intermediate theories between WSLM and Layered Mereotopology can be obtained by adding axioms to WSLM. For example, either of the following axioms, when added to WSLM, yields a somewhat stronger theory.

- (i) $Cxy \rightarrow \exists z + (x, y, z)$
(if x and y are connected, there is a sum of x and y)
- (ii) $Uxy \rightarrow \exists z + (x, y, z)$
(if x and y underlap, there is a sum of x and y)

(i) and (ii) are both theorems of Layered Mereotopology. Neither is satisfied by the four-element WS Layered Model above. (i) would be satisfied if the domain were extended as follows:

Layer 0: $[0, 1]_0, (1, 2)_0, [2, 3]_0, [0, 2]_0, (1, 3]_0, [0, 3]_0$

(ii) requires that, in addition, the model contains a sum of $[0, 1]_0$ and $[2, 3]_0$. It would be satisfied if the domain were further extended to include $([0, 1] \cup [2, 3])_0$. With this last addition, the model qualifies as a Layered Model. But even if (ii) is added to WSLM, the resulting theory is still weaker than Layered Mereotopology. For example, the WS Layered Model with the domain

Layer 0: $\{x_0: x \text{ is a non-empty closed subset of } \mathfrak{R}\}$

satisfies (ii) and the axioms of WSLM, but is not a model of Layered Mereotopology.¹²

¹² To see that this structure is not a model of Layered Mereotopology, notice that, for example, there is no difference of $[0, 1]_0$ in $[0, 2]_0$ in this model.

7. Region-Free Layered Mereotopology

Layered Mereotopology assumes that the spatial domain includes a special collection of entities, the regions, which cover the entire space in the sense that every member of the domain completely coincides with some region. We can think of the region layer as the fixed background structure of all possible locations in the universe. In this sense, the regions act like parts of a Newtonian absolute space.

In Layered Mereotopology, all relative location relations (Coin, Cov, CCoin, M, A) are defined in terms of regions. We might wonder whether we can introduce these relations without regions. There are at least two reasons for investigating a region-free variant of Layered Mereotopology. The first is that there are some good philosophical and scientific reasons for at least being cautious about, if not for rejecting outright, the assumption that there is some immaterial structure in which all spatial entities are located. See, for example, Leibniz's criticism of the Newtonian view in his correspondence with Clarke [12] and a more contemporary discussion in [16]. Secondly, even if we do accept some kind of background spatial structure theoretically, in many practical contexts we seem to reason about the relative locations of spatial entities without reference to their regions. For example, to represent a table as contained in (in my terminology: covered by) the interior of a room, we do not seem to need to refer to the regions at which the table and the interior of the room are located.

Regions are likely to be useful for reasoning about motion in a time-inclusive version of Layered Mereotopology. They may also be useful for reasoning about more sophisticated spatial relations, such as orientation and distance relations. These seem to assume some kind of underlying structure similar to a Euclidean space. But for static reasoning about the fairly simple relations introduced in this paper, we may want to do without the assumption that there is a separate region layer.

The purpose of this section is to propose a variant of Layered Mereotopology which I shall call Region-Free Layered Mereotopology (RFLM). The mereological subtheory of RFLM is just Layered Mereology. There is no region function in RFLM. Instead, the binary relation Cov is treated as a primitive.

On the intended interpretation

$$\text{Cov}(x, y)$$

means:

x is contained in y .

Here “contained in” is to be understood in the sense of “the water is contained in the interior of the vase” (i.e., the water occupies part of the interior of the vase) *not* in the sense of “the water is contained in the vase” (i.e., the water is partially surrounded and held in place by the vase). This interpretation of the Cov relation is consistent with the intended interpretation of Cov in Layered Mereotopology.

Coincidence and complete coincidence are defined in terms of Cov as follows:¹³

¹³ Axioms specific to RFLM are marked with “RF”, definitions with “RFD”, and theorems with “RFT”.

(RFD1) $\text{Coin}(x, y) =: \exists z (\text{Cov}(z, x) \ \& \ \text{Cov}(z, y))$

(x and y coincide: there is some z that is covered by both x and y)

(RFD2) $\text{CCoin}(x, y) =: \text{Cov}(x, y) \ \& \ \text{Cov}(y, x)$

(x and y completely coincide: x is covered by y and y is covered by x)

Axioms for the Cov relation are:

(RF1) $\text{Cov}(x, y) \ \& \ \text{Cov}(y, z) \rightarrow \text{Cov}(x, z)$

(Cov is transitive)

(RF2) $\text{Pxy} \rightarrow \text{Cov}(x, y)$

(if x is part of y, then x is covered by y)

(RF3) $\text{Uxy} \ \& \ \text{Coin}(x, y) \rightarrow \text{Oxy}$

(if x and y are parts of the same layer and x coincides with y, then x overlaps y)

Notice that counterparts of (RF1)–(RF3) are theorems of Layered Mereotopology.

It is easy to see that, in RFLM as in Layered Mereotopology, Cov is reflexive, CCoin is an equivalence relation, and Coin is reflexive and symmetric. In addition, the following theorems can be derived from (P1)–(P6) and (RF1)–(RF3). All of their counterparts using the relative location relations of Layered Mereotopology are theorems of Layered Mereotopology.

(RFT1) $\text{Oxy} \rightarrow \text{Coin}(x, y)$

(if x overlaps y, then x coincides with y)

(RFT2) $\text{Uxy} \ \& \ \text{Cov}(x, y) \rightarrow \text{Pxy}$

(if x and y are in the same layer and x is covered by y, then x is part of y)

(RFT3) $\text{CCoin}(x, y) \ \& \ \text{CCoin}(x, z) \ \& \ \text{Uyz} \rightarrow y = z$

(any object can completely coincide with at most one object in any layer)

(RFT4) $\text{CCoin}(x, y) \ \& \ \text{Uxy} \rightarrow x = y$

(if x and y completely coincide and are on the same layer, then $x = y$)

To transform Layered Models into region-free models, we need to eliminate the **r** function and delete the first condition on the domains (stating that $x_0 \in D$, whenever $x_i \in D$). When P is interpreted as **P** and Cov as **Cov**, axioms (P1)–(P6) and (RF1)–(RF3) are satisfied. But notice that now Coin need not correspond to the relation **Coin**. For example, consider the model with the following two-element domain:

Layer 0: $[0, 2]_0$

Layer 1: $[1, 3]_1$

Here $\{[0, 2]_0, [1, 3]_1\} \in \mathbf{Coin}$, since $[0, 2] \cap [1, 3] \neq \emptyset$. But nothing in this domain is covered by both $[0, 2]_0$ and $[1, 3]_1$. Thus, $[0, 2]_0$ and $[1, 3]_1$ do not stand in the Coin relation defined in (RFD1).

Analogous unintended interpretations would have arisen if either Coin or CCoin were treated as the primitive in terms of which the remaining two relations were defined.

A solution is to strengthen the conditions on the domains of the models in order to make up for the loss of the region layer. As the background collection of all possible locations, the region layer had served as the finest possible grid in terms of which the spatial structure of all members of the domain could be analyzed. Layered Mereotopology allows that the parthood structure of an arbitrary pair of entities may be too coarse for determining their relative location relations (as it is above in the case of $[0, 2]_0$ and $[1, 3]_1$). Since relative location is determined in Layered Mereotopology by relations among regions, we need only assume that the regions have a clear spatial structure. When the region layer is eliminated, we need to strengthen our assumptions about the spatial structure of arbitrary members of the domain. One minimal way of doing this is by replacing condition 4 on the domains of the models with the following stronger condition:

4*. If $x_i, y_j \in D$ and $x \cap y \neq \emptyset$, then $(x \cap y)_i \in D$ or $(x \cap y)_j \in D$.

4* requires that whenever two entities coincide, at least one of them has a part that corresponds to their common location. For example, if a table is only partly in a room, then there is a part of the table or a part of the interior of the room (or a part of each) that represents their common location.

I will call the structures whose domains conform to conditions 2, 3, and 4*, with all of the relations of Section 2 except **r**, RF Layered Models. It is easy to check that when **P** is interpreted as **P** and **Cov** as **Cov** in RF Layered Models, axioms (P1)–(P6) and (RF1)–(RF3) are satisfied, **CCoin** is interpreted as **CCoin**, and **Coin** is interpreted as **Coin**.

For the purposes of adding topological relations to RFLM, the cross-layer relation, **M**, rather than the intra-layer relation, **C**, should be treated as primitive. Again, the loss of the region layer makes definitions of relative location relations problematic. We cannot assume that the meeting of a pair of entities from different layers is reflected by the connecting of a corresponding pair in a single layer. The interior of my glass meets the walls of the glass. But, unless we assume a background structure of regions, there seems to be no completely coincident pair that stands in the connection relation. As another example, consider the following domain of a RF Layered Model.

Layer 0: $[0, 1]_0$

Layer 1: $(1, 2]_1$

$\langle [0, 1]_0, (1, 2]_1 \rangle \in \mathbf{M}$, but **C** is just the identity relation in this model. There is no pair in **C** whose coordinates completely coincide with those of $\langle [0, 1]_0, (1, 2]_1 \rangle$.

Axioms for the meets relation (**M**) can be given as follows.

(RF4) Mxx

(**M** is reflexive)

(RF5) $Mxy \rightarrow Myx$

(**M** is symmetric)

(RF6) $Pxy \rightarrow \forall z(Mzx \rightarrow Mzy)$

(if x is part of y , then everything that meets x also meets y)

The counterparts of (RF4)–(RF6) using the meets relation defined in Section 5 are all theorems of Layered Mereotopology.

Connection (C) and abutment (A) are defined in RFLM as

(RFD3) $C_{xy} =: U_{xy} \& M_{xy}$

(x and y are connected: x and y are on the same layer and x and y meet)

(RFD4) $A_{xy} =: M_{xy} \& \sim \text{Coin}(x, y)$

(x and y abut: x and y meet but do not coincide)

External Connection (EC), Tangential Parthood (TP), and Interior Parthood (IP) are defined as in Layered Mereotopology.

It is easy to see that counterparts of axioms (C1)–(C6) are theorems of RFLM. The counterparts of (C5) and (C6) (the two axioms that use the region function) are:

(RFT5) $C_{xy} \rightarrow M_{xy}$

(if x and y are connected, then they meet)

(RFT6) $U_{xy} \& M_{xy} \rightarrow C_{xy}$

(if x and y are in the same layer and they meet, then they are connected)

Counterparts of all of the other theorems listed in Section 5 are also theorems of RFLM.

When P is interpreted as **P**, Cov as **Cov**, and M as **M** in RF Layered Models, all of the axioms of RFLM are satisfied and all of the defined relations are interpreted as intended.

8. Conclusions, applications, and further work

The goal of this paper was to construct a mereotopology for domains that include coincident but non-overlapping entities. The result is an extension of mereotopology that includes relative location relations and a region function in addition to mereotopological relations.

The models given for Layered Mereotopology and its variants make explicit the intended interpretations of the relations introduced in the theories and show that each theory is sound. One important project for further work is to determine whether each theory is complete with respect to its target class of models and, if not, to further strengthen its axioms.

Layered Mereotopology and its variants are particularly appropriate for applications that involve reasoning about objects that are located in holes. Reasoning about holes is crucial in a wide variety of domains, including medicine (body cavities and orifices, see [14]) and mechanics (valves, pathways formed by piping). For more examples and a further discussion of holes, see [6].

The distinction between mereotopological and relative location relations is also critical for geospatial sciences such as epidemiology and meteorology. Here, we locate entities such as collections of bacteria or storm systems relative to geopolitical entities such cities or countries, but we do not treat the epidemiological or meteorological phenomena as parts of the geopolitical entities in which they are located. For further discussion of the application of Layered Mereotopology to geospatial sciences see [10].

Layered Mereotopology and its variants allow the same mereotopological relations to apply directly to all spatial entities, including regions, material objects, and holes. This approach is an alternative to that of [8] in which mereotopological relations apply to material objects only indirectly, via the spatial regions at which they are located. One advantage of allowing direct descriptions of the mereotopological properties of material objects is that this leaves open the possibility of attributing different structures to material objects and the regions at which they are located. For example, we may wish to represent material objects as having only closed, regular, divisible parts, but represent spatial regions as sums of points. Slight changes in the conditions on the domains of Layered Models (specifically, condition 4) and in the axioms of Layered Mereotopology (specifically, (R4)) would allow models in which the parts of different layers are restricted to different granularities.

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