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Measuring inconsistency in probabilistic logic: rationality postulates and Dutch book interpretation

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ABSTRACT

Inconsistency measures have been proposed as a way to manage inconsistent knowledge bases in the AI community. To deal with inconsistencies in the context of conditional probabilistic logics, rationality postulates and computational efficiency have driven the formulation of inconsistency measures. Independently, investigations in formal epistemology have used the betting concept of Dutch book to measure an agent's degree of incoherence. In this paper, we show the impossibility of joint satisfiability of the proposed postulates, proposing to replace them by more suitable ones. Thus we reconcile the rationality postulates for inconsistency measures in probabilistic bases and show that several inconsistency measures suggested in the literature and computable with linear programs satisfy the reconciled postulates. Additionally, we give an interpretation for these feasible measures based on the formal epistemology concept of Dutch book, bridging the views of two so far separate communities in AI and Philosophy. In particular, we show that incoherence degrees in formal epistemology may lead to novel approaches to inconsistency measures in the AI view.

1. Introduction

"when you can measure what you are speaking about, you know something about it; but when you cannot [...] your knowledge is of a meagre and unsatisfactory kind;"

— Lord Kelvin [45]

Measuring has been a prominent activity in advancing scientific and technological development. Not all measures are alike and good measures express intuitive notions in a useful way. In the field of deductive logical reasoning, one usually has an intuition expressing that one theory is *more inconsistent* than other, capturing the idea that the "effort" to restore consistency is greater in one case than the other. Also, no effort is required to restore the consistency of a consistent theory.

Based on those intuitions, there are several proposals for measuring inconsistency in knowledge bases over purely logical languages [19]. Some of these proposals involved attaching probabilities to formulas [29], or the combination of inconsistency factors [20]. Some of these measures are discrete or even qualitative, while others are more like distances, but all these measures have to behave like an *information measure* [6]. And to adhere to certain intuitions, a series of postulates

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for inconsistency measures for purely logical knowledge bases were proposed [21,22]; for example, the *consistency postulate* states that the inconsistency measure of a consistent base is 0.

Purely logical bases are known to be expressively limited in representing uncertainty required for real-world applications. In this work, we are interested in measuring the inconsistency of knowledge bases over logical probabilistic languages, which combine the deductive power of logical systems with the well-founded theory of probability. This kind of extension of purely logical systems can be traced back to the work of Boole [2], but has gained attention of AI researchers since the work of Nilsson [33], and has been extended to conditional probabilistic logic [37].

In AI, one of the main uses of measuring inconsistency in a knowledge base is to guide the consolidation of inconsistent pieces of information. Within propositional logic, Grant and Hunter [13] showed how inconsistency measures can be used to direct the stepwise resolution of conflicts via the weakening or the discarding of formulas.

In probabilistic bases, inconsistencies are rather common, specially when knowledge is gathered from different sources. To fix these probabilistic knowledge bases, one can, for instance, delete pieces of information, or change the probabilities' numeric values (or intervals). In this case, an inconsistency measure helps one to detect if a change approximates consistency or not. In other areas, inconsistency measures for probabilistic logic have found applications in merging conflicting opinions, leading to an increased predictive power [47,25], and in quantifying the incoherence of procedures from classical statistical hypothesis testing [41].

Example 1.1. Consider we are devising an expert system to assist medical diagnosis. Suppose a group of experts on a disease D is required to quantify the relationship between D and its symptoms. Suppose three conditional probabilities are presented:

- the probability of a patient exhibiting symptom S_1 given he/she has disease D is 50%;
- the probability of a patient exhibiting symptom S_2 given he/she exhibits symptom S_1 and has disease D is 80%;
- the probability of a patient exhibiting symptom S_2 given he/she has disease D is 30%.

A knowledge engineer, while checking those facts, finds that they are inconsistent: according to the first two items, the probability of symptom S_2, given disease D, should be at least $50\% \times 80\% = 40\%$, instead of 30%. He does not even know where each probability came from, but plans to change the probabilities, since consistency is a requirement. How should he proceed? Which probabilities is the degree of inconsistency most sensitive to? Once chosen which number to change, should it be raised or lowered in order to approximate consistency? These are the kind of questions an inconsistency measure can help to answer.

The issue of measuring inconsistency in probabilistic bases has more recently been tackled by Thimm [44], Muiño [31] and Potyka [34], who developed measures based on distance minimization, tailored to the probabilistic case. Potyka focused on computational aspects, looking for efficiently computable measures [34]. Muiño was driven by the CADIAG-2 knowledge base, presenting its infinitesimal inconsistency degree, however based on a different semantics [31]. Thimm [44] adapted Hunter and Konieczny's [22] desirable properties for inconsistency measures to the probabilistic setting, developing measures that satisfy a set of rationality postulates.

It was Thimm [44] who realized the importance of *continuity* as a Postulate for the probabilistic case, namely the property that a small change in the probability associated to formula (absent in the purely logical case) should lead only to small changes in the inconsistency measure. It was just natural that, (conditional) probabilistic logic being an extension of the classical cases, the continuity postulate was simply added to the postulates defining classical inconsistency measures.

In this work, we argue that continuity cannot hold together with classical postulates such as consistency and independence, and some of these postulates must be abandoned or exchanged for other ones that restore joint satisfiability. So the first contribution of this work is that we *identify* and *fix* the possible problem with the postulates proposed by Thimm [44].

Another contribution lies in showing that these measures of inconsistency have a direct counterpart in formal epistemology research over the coherence of an agent's degrees of belief. It is known that inconsistent probabilistic beliefs correspond to a set of bets with guaranteed loss to the agent, which is called a "Dutch Book" [8,27]. This agent's incoherence has been measured by formalizing the intuition that the greater the inconsistency the greater the corresponding sure loss, and vice versa [40,43]. Thus we interpret these incoherence measures via guaranteed losses as inconsistency measures, showing that existing measures based on distance minimization correspond to guaranteed losses that quantify an agent's incoherence. To the best of our knowledge, no clear link has been shown between these two areas.

Here is a bird's-eye view of how we achieve these goals.

After introducing probabilistic knowledge bases in Section 2, this paper develops three main contributions, in three different sections, dealing closely with three other works. In the following, we overview such contributions, together with the organization of the paper and their relation to the existing literature.

Inconsistency measures for probabilistic knowledge bases were analyzed via rationality postulates by Thimm [44]. In Section 3, we argue for the incompatibility of such desirable properties. Firstly, we introduce the problematic postulates: consistency, independence and continuity. The *independence postulate* claims that a free conditional — a (conditional) probability assignment that does not belong to any minimal inconsistent set — can be rule out without changing the degree of

Table 1
Inconsistency measures, where they are defined and a brief description.

Notation	Section	Explanation
$\mathcal{I}_p$	4.3	Minimum p-norm of the vector composed by the adjustments on the probability bounds to reach consistency.
$\mathcal{I}_p^{\varepsilon}$	5.1	Minimum p-norm of the vector composed by the violations of each restriction corresponding to a probability bound. In the unconditional case, $\mathcal{I}_p^{\varepsilon} = \mathcal{I}_p$.
$\mathcal{I}_{SSK}^{sum}$	5.2	Maximum sure loss in a Dutch book if the sum of the stakes' absolute values is at most 1.
$\mathcal{I}_{SSK}^{max}$	5.2	Maximum sure loss in a Dutch book if the maximum of the stakes' absolute values is at most 1.
$\mathcal{I}_{SSK}^{a,sum}$, $\mathcal{I}_{SSK}^{b,sum}$	5.3	Maximum sure loss in a Dutch book if the sum of the agent's or the bettor's possible losses is at most 1, respectively.
$\mathcal{I}_{SSK}^{a,max}$, $\mathcal{I}_{SSK}^{b,max}$	5.3	Maximum sure loss in a Dutch book if the maximum of the agent's or the bettor's possible losses is at most 1, respectively.

inconsistency. We also present the MIS-separability property, which deals with decomposability (through Minimal Inconsistent Sets) and implies independence. As Thimm's work regards precise probabilities, these four concepts are then introduced in this way. In a second step, Section 3 brings the first contribution of this work: the presented postulates are shown to be incompatible.

In Section 4, we search for a reasonable way to reconcile the incompatible postulates. First of all, we argue that independence is the requirement to be weakened, together with the stronger property of MIS-separability. Afterwards, the concept of free conditional is analyzed, for independence is based on it. We find that free conditional is a notion linked to classical consolidation and contraction (*i.e.*, discarding formulas to reach consistency), and it is not suitable for probabilistic bases. The *innocuous* conditional concept is introduced, by investigating a natural consolidation procedure for probabilities: through widening their intervals, instead of ruling them out. The i-independence postulate is put forward based on innocuous conditionals. In a similar way, we observe that MIS (minimal inconsistent set) is a notion that fails to capture all causes of inconsistency in the probabilistic knowledge bases. We define the alternative concept of *inescapable conflict*, which yields the IC-separability property. We show that innocuous conditionals are to inescapable conflicts as free conditionals are to minimal inconsistent sets. At the end of Section 4, the second main contribution of this paper emerges when i-independence and IC-separability are shown to be compatible with consistency and continuity, besides other desirable properties from the literature that are then presented.

Once a consistent package of postulates is laid out, a myriad of inconsistency measures can still be considered rational. Hence, in Section 5, further criteria to evaluate inconsistency measures are discussed: computational efficiency and meaningful interpretation. On the one hand, two inconsistency measures computable through linear programs are adapted to imprecise probabilities from the work of Potyka [34]. It is shown that both measures satisfy the core postulates, and we present the additional desirable properties each one satisfies. On the other hand, we review two measures from Schervish, Kadane and Seidenfeld that quantify the incoherence of an agent using the betting concept of Dutch book [40], under the operational interpretation that her degrees of belief (probabilities) determine her gambling behavior. The third main contribution of this paper lies in showing the connection between inconsistency measures for probabilistic knowledge bases and incoherence measures for agents from formal epistemology. We prove that the two measures adapted from Potyka that are computable through linear programs are equivalent to two measures from Schervish et al. — that is, these two measures are rather efficient and have a meaningful interpretation. The final part of the section reviews other measures from the work of Schervish et al., showing that they also satisfy the postulates and are computationally feasible.

As several families of inconsistency measures are discussed throughout the paper, Table 1 compiles their notation, the section in which they are defined and a short explanation of each.

2. Preliminaries

A propositional logical language is a set of formulas formed by atomic propositions combined with logical connectives, possibly with punctuation elements (parentheses). We assume a finite set of symbols $X_n = \{x_1, x_2, x_3, \dots, x_n\}$ corresponding to *atomic propositions (atoms)*. Formulas are constructed inductively with connectives $(\neg, \wedge, \vee, \rightarrow)$ and atomic propositions as usual. The set of all these well-formed formulas is the propositional language over X_n, denoted by $\mathcal{L}_{X_n}$. Additionally, $\top$ denotes $x_i \vee \neg x_i$ for some $x_i \in X_n$, and $\bot$ denotes $\neg\top$.

Given a signature X_n, a *possible world* w is a conjunction of $|X_n| = n$ atoms containing either x_i or $\neg x_i$ for each $x_i \in X_n$. We denote by $W_{X_n} = \{w_1, \dots, w_{2^n}\}$ the set of all possible worlds over X_n and say a $w \in W_{X_n}$ *entails* a $x_i \in X_n$ $(w \models x_i)$ iff x_i is not negated in w. This entailment relation can be extended to all $\varphi \in \mathcal{L}_{X_n}$ as usual.

A *probabilistic conditional* (or simply *conditional*) is a statement of the form $(\varphi|\psi)[\underline{q}, \bar{q}]$, with the underlying meaning "the probability that φ is true given that ψ is true lies within the interval $[\underline{q}, \bar{q}]$", where $\varphi, \psi \in \mathcal{L}_{X_n}$ are propositional formulas and $\underline{q}, \bar{q} \in [0, 1]$ are real numbers. Note that we do not assume $\underline{q} \leq \bar{q}$, since we are going to measure inconsistency. If ψ is a tautology, a conditional like $(\varphi|\psi)[\underline{q}, \bar{q}]$ is called an *unconditional probabilistic assessment*, usually denoted by $(\varphi)[\underline{q}, \bar{q}]$. We say a conditional in the format $(.)[q, q]$ is *precise* and denote it by $(.)[q]$.

A *probabilistic interpretation* $\pi : W_{X_n} \to [0,1]$, with $\sum_j \pi(w_j) = 1$, is a probability mass over the set of possible worlds, which induces a probability measure $P_\pi : \mathcal{L}_{X_n} \to [0,1]$ by means of $P_\pi(\varphi) = \sum\{\pi(w_j) | w_j \models \varphi\}$. A conditional $(\varphi|\psi)[\underline{q}, \bar{q}]$ is *satisfied* by π iff $P_\pi(\varphi \wedge \psi) \geq \underline{q} P_\pi(\psi)$ and $P_\pi(\varphi \wedge \psi) \leq \bar{q} P_\pi(\psi)$. Note that when $P_\pi(\psi) > 0$, a probabilistic conditional $(\varphi|\psi)[\underline{q}, \bar{q}]$ is constraining the conditional probability of φ given ψ; but any π with $P_\pi(\psi) = 0$ trivially satisfies the conditional $(\varphi|\psi)[\underline{q}, \bar{q}]$ (this semantics is adopted by Halpern [15], Frisch and Haddawy [11] and Lukasiewicz [30], for instance). A *knowledge base* is a finite set Γ of probabilistic conditionals such that, if $(\varphi|\psi)[\underline{q}, \bar{q}], (\varphi|\psi)[\underline{q}', \bar{q}'] \in \Gamma$, then $[\underline{q}, \bar{q}] = [\underline{q}', \bar{q}']$. That is, for each pair φ, ψ, only one probability interval can be assigned to $(\varphi|\psi)$ in a knowledge base.[3] A knowledge base Γ is *consistent* (or *satisfiable*) if there is a probability mass satisfying all conditionals $(\varphi|\psi)[\underline{q}, \bar{q}] \in \Gamma$. It is precise if all intervals are singletons.

The problem of verifying the consistency of a knowledge base is called *probabilistic satisfiability* (or *PSAT*) [12]. Probabilistic satisfiability has been rediscovered several times, and an analytical and unconditional version was actually proposed by Boole [2]. Hailperin [14], Bruno and Gilio [3], and Nilsson [33] suggested solutions via linear programs. This linear programming approach can be easily extended to handle conditional probabilities under the semantics we are using [16]. Recent advances in algorithms for PSAT solving can be found in [17,9,28].

If any probability mass π satisfying $(\varphi|\psi)[\underline{q}, \bar{q}]$ implied $P_\pi(\psi) > 0$, in an alternative semantics, the latter restriction could be added to the program, although losing the linear program standard format; this is the semantics adopted by Muiño [31], for instance. De Finetti proposed an alternative setting in which the conditional probability is fundamental [8] and the satisfaction of probabilistic conditionals does not trivialize when the conditioning event has null probability. In such scenario, the consistency is called *coherence*, and its checking demands solving a sequence of linear programs [5].

When all interval bounds are rational numbers, PSAT is an NP-complete problem [12]; if there is a solution, there is a solution with only $m + 1$ possible worlds receiving positive probability mass, where m is the knowledge base size. Nevertheless, column generation methods can handle large problems [26,24], and several approaches have recently appeared [28,9,17,7]. Note that this linear programming approach can be applied to other probabilistic logics (see, for instance, [1] and [23]).

3. Proposed postulates for inconsistency measures

Approaches to measuring inconsistency in probabilistic knowledge bases have been put forward by Muiño [31], Thimm [44] and Potyka [34], with different semantics for the conditionals. We follow the one adopted by Thimm and Potyka, in which a conditional is also satisfied by any measure assigning null probability to the conditioning formula. Thimm has done a groundlaying work [44], extending Hunter's postulates for inconsistency measures to the probabilistic case, which is our starting point. Potyka suggests feasible measures [34] we will review in Section 5.1, after investigating carefully the postulates. In this section, we begin with some desirable properties proposed by Thimm and then argue against their joint satisfiability.

3.1. Postulates

Let $\mathbb{K}$ ($\mathbb{K}_{prec}$) be the set of all (precise) knowledge bases. An inconsistency measure for knowledge bases is a function $\mathcal{I} : \mathbb{K} \to [0, \infty)$. Thimm's investigation is restricted to measures $\mathcal{I} : \mathbb{K}_{prec} \to [0, \infty)$ over knowledge bases with precise probabilities, to what we narrow our focus in this section. The author proposes some desirable properties such a function should satisfy, following Hunter and Konieczny's work for classical logic [20]. Although Thimm investigates a total of ten postulates, we describe in this section only four of these properties that we consider problematic. The first one claims that an inconsistency measure must at least discriminate between consistent and inconsistent bases:

Postulate 3.1 *(Consistency). $\mathcal{I}(\Gamma) = 0$ iff Γ is consistent.*

A second desirable property has to do with probabilistic conditionals one can ignore while measuring inconsistency, since they are not involved with the unsatisfiability, in some sense. Some notation is needed to formalize it.

Definition 3.2. A set Γ of probabilistic conditionals is a *minimal inconsistent set (MIS)* if Γ is inconsistent and every set $\Gamma' \subsetneq \Gamma$ is consistent.

Minimal inconsistent sets can be considered the purest form of inconsistency [21], capturing its causes. The focus on MISes is derived from the seminal work of Reiter [36] on the diagnosis problem. Reiter investigated how formulas from a base could be ruled out in order to restore consistency, by choosing at least one element from each MIS, computing thusly a hitting set of their collection.

[3] Note that this requirement is not too restrictive. Since nothing was said about logically equivalent propositions, a knowledge base may contain different probability intervals assigned to φ and $\varphi \wedge \top$, for instance.

Let $MIS(\Gamma)$ denote the collection of all MISes in Γ. Now we can define the central concept of free probabilistic conditional, following Thimm [44]:

Definition 3.3. A *free probabilistic conditional* of Γ is a probabilistic conditional $\alpha \in \Gamma$ such that, for all $\Delta \in MIS(\Gamma)$, $\alpha \notin \Delta$.

Analogously, a free probabilistic conditional of Γ is in all its maximal consistent subsets. The postulate of independence then claims that ruling out a free probabilistic conditional from a knowledge base should not change its inconsistency degree [44].

Postulate 3.4 *(Independence). If α is a free probabilistic conditional of Γ, then $\mathcal{I}(\Gamma) = \mathcal{I}(\Gamma \setminus \{\alpha\})$.*

A stronger condition, also introduced by Hunter and Konieczny and adopted by Thimm, deals with a sort of decomposability of the inconsistency measure through its minimal inconsistent sets. We call it a property, saving the name "postulate" to the most basic properties required from every measure. The version we present is tailored from Hunter and Konieczny's work [20]:

Property 3.5 *(MIS-separability). If $\Gamma = \Delta \cup \Psi$, $\Delta \cap \Psi = \varnothing$ and $MIS(\Gamma) = MIS(\Delta) \cup MIS(\Psi)$, then $\mathcal{I}(\Gamma) = \mathcal{I}(\Delta) + \mathcal{I}(\Psi)$.*

The idea behind this property is that the inconsistency of the whole knowledge base should be the sum of the inconsistency of its parts, whenever the partition does not break any minimal inconsistent set. For instance, consider $\Delta = \{(x_1)[0.5], (\neg x_1)[0.6]\}$, $\Psi = \{(x_2)[0.7], (x_2 \wedge x_3)[0.8]\}$ and $\Gamma = \Delta \cup \Psi$. It is clear that Δ and Ψ are the only minimal inconsistent sets in Γ. MIS-separability posits that the measure of inconsistency of Γ is obtained by summing the measures of Δ and Ψ; formally, $\mathcal{I}(\Gamma) = \mathcal{I}(\Delta) + \mathcal{I}(\Psi)$. MIS-separability is stronger than independence [44]:

Proposition 3.6. *If $\mathcal{I}$ satisfies MIS-separability, then $\mathcal{I}$ satisfies independence.*

These properties can be found in Hunter and Konieczny's work [21], in the definition of a "MinInc" separable basic inconsistency measure for knowledge bases over classical propositional logic. The measures they introduce are shown to fit such desiderata. Thimm revises the adaptation of these classical inconsistency measures to the probabilistic case and convincingly argues that they are not suitable to the *quantitative* nature of probabilities, since classical logic is *qualitative*.

To motivate the search for new inconsistency measures for probabilistic knowledge bases, while dispensing with measures from classical logic, Thimm puts forward the postulate of continuity. Intuitively, one expects that small changes in the probabilities of a knowledge base yield small changes in its degree of inconsistency. To formalize the continuity concept in precise knowledge bases, we introduce some notation, following Thimm [44].

That work studies precise knowledge bases of the form $\Gamma = \{(\varphi_i|\psi_i)[q_i] | 1 \leq i \leq m\}$. For each precise knowledge base Γ, there is a *characteristic function* $\Lambda_\Gamma : [0,1]^{|\Gamma|} \to \mathbb{K}_{prec}$ that, roughly speaking, changes the probabilities q_i in the base; *i.e.*, $\Lambda_\Gamma(\langle q'_1, q'_2, \ldots, q'_m \rangle) = \{(\varphi_i|\psi_i)[q'_i] | 1 \leq i \leq m\}$. To handle the (consistent) empty knowledge base, we define $\Lambda_\varnothing : \{\varnothing\} \to \{\varnothing\}$. Thimm imposes some order on the set Γ, building a sequence, for the function Λ_Γ be unique and well-defined. For simplicity, we just suppose there is some order (say, lexicographic) over the probabilistic conditionals used to uniquely specify Λ_Γ.[4] Now the continuity postulate can be enunciated, with $\circ$ denoting function composition:

Postulate 3.7 *(Continuity (for precise probabilities)). For all $\Gamma \in \mathbb{K}_{prec}$, the function $\mathcal{I} \circ \Lambda_\Gamma : [0,1]^{|\Gamma|} \to [0, \infty)$ is continuous.*

To find inconsistency measures holding the desirable properties, including continuity, Thimm introduces a family of measures based on distance minimization, taking into account the numerical value of the probabilities. The basic idea is to quantify the inconsistency through the minimum changes, according to some distance, one has to apply on the probabilities to make the base consistent. The compatibility of consistency, independence and continuity is implicitly stated when it is proved that this whole family of inconsistency measures based on distance minimization satisfies them; and another family is proved to hold MIS-separability as well [44].

3.2. The postulates' incompatibility

The work done by Thimm [44] has carefully analyzed the problem of measuring inconsistency in knowledge bases over probabilistic logic. Desirable properties were borrowed from classical logic [20], and the crucial postulate of continuity was added. To attend these properties, measures based on distance minimization were introduced and some important results were proved. However, under a closer examination, the proposed postulates are incompatible.

[4] Technically, we could use the lexicographic order over the pairs $(\varphi_i|\psi_i)$ to construct a function *Lex* taking each set Γ to the corresponding sequence $\Psi = Lex(\Gamma)$, uniquely specifying a function Λ'_Ψ that changes the probabilities of the sequence Ψ. Then it could be defined $\Lambda_\Gamma(q) = Lex^{-1}(\Lambda'_\Psi(q))$.

Theorem 3.8. *There is no inconsistency measure $\mathcal{I} : \mathbb{K}_{prec} \to [0, \infty)$ that satisfies consistency, independence and continuity.*

Proof. To prove by contradiction, suppose there is a measure $\mathcal{I}$ satisfying consistency, independence and continuity. Consider the following knowledge bases:

$$\Gamma = \{(x_1 \wedge x_2)[0.5 + \varepsilon], (x_1 \wedge \neg x_2)[0.5]\} \text{ for some } 0 < \varepsilon \leq 0.1 \tag{1}$$

$$\Delta = \Gamma \cup \{\alpha\}, \alpha = (x_1)[0.8] \tag{2}$$

We are going to use $\mathcal{I}$ to measure the inconsistency of Δ when $\varepsilon \to 0$. To apply independence, we are going to show that α is free in Δ; we prove that Γ is the only MIS in Δ. Note that $\{(x_1 \wedge x_2)[0.5 + \varepsilon], (x_1)[0.8]\}$ is consistent for any $\varepsilon \in (0, 0.1]$, for such set is satisfied by the probability measure induced by the following probability mass: $\pi_1(x_1 \wedge x_2) = 0.5 + \varepsilon, \pi_1(x_1 \wedge \neg x_2) = 0.3 - \varepsilon, \pi_1(\neg x_1 \wedge x_2) = \pi_1(\neg x_1 \wedge \neg x_2) = 0.1$. To prove that $\{(x_1 \wedge \neg x_2)[0.5], (x_1)[0.8]\}$ is consistent, consider the following probability mass: $\pi_2(x_1 \wedge x_2) = 0.3, \pi_2(x_1 \wedge \neg x_2) = 0.5, \pi_2(\neg x_1 \wedge x_2) = \pi_2(\neg x_1 \wedge \neg x_2) = 0.1$. Hence, all MISes of Δ must contain $\Gamma = \{(x_1 \wedge x_2)[0.5 + \varepsilon], (x_1 \wedge \neg x_2)[0.5]\}$, for other subsets are all consistent. Furthermore, note that Γ is inconsistent and minimal, so it is a MIS. We can conclude that Γ is the only MIS in Δ, for any value of $0 < \varepsilon \leq 0.1$. As α is a free probabilistic conditional of Δ, we can apply independence:

$$\mathcal{I}(\Delta) = \mathcal{I}(\Gamma),$$

for any $0 < \varepsilon \leq 0.1$.

To exploit the continuity of $\mathcal{I}$, we need the characteristic function of Δ, $\Lambda_\Delta : [0,1]^3 \to \mathbb{K}_{prec}$, to be well-defined; so, we need an order over the probabilistic conditionals. Suppose that Γ and Δ are ordered as they were defined in (1) and (2). Let q^* be the vector $\langle 0.5, 0.5, 0.8\rangle$. It follows that $\Lambda_\Delta(q^*)$ differs from Δ only in its first conditional, which becomes $(x_1 \wedge x_2)[0.5]$. Now we prove that $\Lambda_\Delta(q^*)$ is inconsistent. For any probability measure P_π, $P_\pi(x_1 \wedge x_2) = P_\pi(x_1 \wedge \neg x_2) = 0.5$ implies $P_\pi(x_1) = 1$, contradicting $\alpha = \{(x_1)[0.8]\}$. As $\mathcal{I}$ satisfies consistency,

$$\mathcal{I} \circ \Lambda_\Delta(q^*) > 0. \tag{3}$$

By the continuity of $\mathcal{I}$, the function $\mathcal{I} \circ \Lambda_\Delta : [0,1]^3 \to [0, \infty)$ must be continuous, so there must be a limit at the point q^*, and such limit must be unique for any path approaching q^*:

$$\lim_{q \to q^*} \mathcal{I} \circ \Lambda_\Delta(q) = \lim_{\varepsilon \to 0^+} \mathcal{I} \circ \Lambda_\Delta(\langle 0.5 + \varepsilon, 0.5, 0.8\rangle) = \lim_{\varepsilon \to 0^+} \mathcal{I}(\Delta).$$

By independence, we also have:

$$\lim_{\varepsilon \to 0^+} \mathcal{I}(\Delta) = \lim_{\varepsilon \to 0^+} \mathcal{I}(\Gamma).$$

As $\mathcal{I}$ satisfies continuity and $\{(x_1 \wedge x_2)[0.5], (x_1 \wedge \neg x_2)[0.5]\}$ is satisfiable, the consistency of $\mathcal{I}$ implies

$$\lim_{\varepsilon \to 0^+} \mathcal{I}(\Gamma) = \mathcal{I}(\{(x_1 \wedge x_2)[0.5], (x_1 \wedge \neg x_2)[0.5]\}) = 0 = \lim_{q \to q^*} \mathcal{I} \circ \Lambda_\Delta(q). \tag{4}$$

The continuity of $\mathcal{I}$ requires that $\mathcal{I} \circ \Lambda_\Gamma(q^*) = \lim_{q \to q^*} \mathcal{I} \circ \Lambda_\Gamma(q)$, which by (3) and (4) is a contradiction, finishing the proof. □

Corollary 3.9. *There is no inconsistency measure $\mathcal{I} : \mathbb{K}_{prec} \to [0, \infty)$ that satisfies consistency, MIS-separability and continuity.*

Looking at the counterexample given in the proof of Theorem 3.8 may shed some light on what is the cause of such conflict among the desirable properties. The only minimal inconsistent set in Δ is Γ, and so independence forces the degree of inconsistency of Δ to be the same as that of Γ, but this is not generally the case when inconsistency is measured via probability changing. This happens due to the fact that changing the probabilities in Γ to some consistent setting does not in general imply that Δ becomes consistent. Although Γ is the only minimal inconsistent set of Δ, there is another way to prove the contradiction. Note that Γ implies $(x_1)[1 + \varepsilon]$, with $\varepsilon > 0$, which contradicts a probability axiom, but also contradicts $\alpha = (x_1)[0.8]$. While $\varepsilon = 0$ consolidates Γ, consolidating Δ requires a bigger change in probabilities, which is ignored by independence. By demanding $\mathcal{I}(\Delta) = \mathcal{I}(\Gamma)$ for $\varepsilon > 0$, the postulate of consistency forces a discontinuity on $\varepsilon = 0$. When $\varepsilon \to 0$, the inconsistency degree of Γ tends to zero (by continuity), and independence requires the same from Δ. But this contradicts continuity, given consistency, for $\{(x_1 \wedge x_2)[0.5], (x_1 \wedge \neg x_2)[0.5]\}$ would still contradict $(x_1)[0.8]$, and Δ would be inconsistent.

4. Reconciling the postulates

The findings from the previous section suggest that in order to drive the rational choice of an inconsistency measure for knowledge bases, we must abandon at least one postulate among consistency, independence and continuity. We claim that a weakening of the desired properties can restore their compatibility, and in this section we investigate paths to achieve that goal. After reconciling the problematic postulates, we review other proposed properties for inconsistency measures and extend them to the general case of knowledge bases with imprecise probabilities, showing some measures to satisfy them.

The consistency postulate seems to be indisputable, since the least one can expect from an inconsistency measure is that it separates inconsistent from consistent cases, or some inconsistency from none. The answer to the question of which property we should relax to restore compatibility is thus reduced to either independence or continuity. Hunter and Konieczny have already noted problems with independence in knowledge bases over classical logic, proposing to relax it [22]. Intuition shall be inclined towards keeping continuity, for it reflects the particular quantitative nature of probabilistic reasoning. A pragmatic reason to give up independence (and so MIS-separability) is simply to keep continuity, given consistency, to save inconsistency measures based on distance minimization. In the sequel, the withdrawal of independence within probabilistic logic is argued for in a more compelling way.

The notion of free conditional and the postulate of independence are strongly related to the idea that minimal inconsistent sets are the causes of inconsistencies, as suggested by Hunter and Konieczny [20]. Thimm says that free conditionals are "harmless", in some sense, to the consistency of a knowledge base [44]. What is behind these notions is the classical way of handling inconsistency through ruling out formulas, as Reiter proposed in his diagnosis problem [36] and as the standard AGM paradigm of belief revision defines base contraction (see [18] for a general view of the AGM paradigm). Reiter's hitting sets technique views a repair of some inconsistency set of formulas as giving up of at least one element from each minimal inconsistent set. For such repair to be minimal, no free formula should be discarded. In the AGM paradigm, the *consolidation process* of a belief base can be interpreted as the contraction of $\perp$, the contradiction. The inclusion postulate claims that the result of a contraction is a subset of the belief base in question, and the relevance postulate forces the contraction of $\perp$ to contain all free formulas of the base.

When we move from classical to probabilistic logic, there is a natural way to relax formulas without completely losing their information. Note that ruling out a probabilistic conditional $(\varphi|\psi)[\underline{q},\bar{q}]$ is semantically equivalent to changing it to $(\varphi|\psi)[0,1]$, so it is a particular (and extreme) case of widening the probability interval. If we need to give up the belief on $(\varphi|\psi)[\underline{q},\bar{q}]$ to restore consistency, perhaps there are some $\underline{q}' \leq \underline{q}$ and $\bar{q}' \geq \bar{q}$ such that $(\varphi|\psi)[\underline{q}',\bar{q}']$ can still be consistently believed. When inconsistency is measured continuously, through changes in probabilities, it is this more general kind of consolidation process that is being suggested. As it is indicated in the proof of Theorem 3.8, consolidating all minimal inconsistent sets (Γ) through probability changing does not imply consolidating the whole base (Δ). We can conclude that the concepts of free conditional and minimal inconsistent set are not suitable to analyze continuous inconsistency measures based on distance minimization.

Furthermore, it seems that the definition of free conditional, and so independence, can be refined to be suitable for analyzing continuous measures, while continuity is a harder definition to be contrived to be compatible with independence. Hence, we can try to weaken independence, and perhaps MIS-separability, by modifying the notion of free conditional, instead of fully forgetting this postulate.

As both independence and MIS-separability are defined via minimal inconsistent sets, in order to weaken these properties to reach compatibility with consistency and continuity, it seems reasonable to replace MIS by an alternative concept that could reconcile the desirable properties altogether. However, to do it in a principled way, we first analyze the concept of free probabilistic conditional as to the corresponding consolidation procedure and then modify it to save independence. Afterwards, a related notion of conflict that also fixes MIS-separability is introduced.

4.1. Refining the free probabilistic conditional concept

A weaker form of independence has already been suggested in the literature. Thimm [44] defines a *safe* conditional as one whose atomic propositions are disjoint from those in the rest of the base. We also demand that the conditional be satisfiable in order to be safe.[5] The *weak independence* postulate then posits that ruling a (satisfiable) safe conditional out should not change the inconsistency measure of a base. Hunter and Konieczny have suggested the same weakening for independence, in the classical setting, when they acknowledge that independence may be too strong a property to require [22]. Weak independence is compatible with consistency and continuity, since Potyka's measures satisfy them [34]. Although safe conditionals are easily recognizable, we expect that they be rare in practice, due to the natural logical dependencies among propositions within a base. We are looking for a stronger, more useful notion of independence, between the safe-based and the free-based ones, hence we look for a concept between safe and free.

Besides defining free probabilistic conditional through minimal inconsistent sets, one could equivalently do it via the notion of consolidation as giving up conditionals to restore consistency. Let us formalize this concept.

Definition 4.1. Let Γ be a knowledge base in $\mathbb{K}$. An *abrupt repair* of Γ is any set $\Delta \subseteq \Gamma$ such that $\Gamma' = \Gamma \setminus \Delta$ is consistent — we call Γ' an *abrupt consolidation*. If an abrupt repair Δ is such that, for every $\Psi \subsetneq \Delta$, $\Gamma \setminus \Psi$ is inconsistent, Δ is a *minimal abrupt repair* — and $\Gamma' = \Gamma \setminus \Delta$ is a *maximal abrupt consolidation*.

[5] Thimm [44] only considers conditionals $(\varphi|\psi)[\underline{q},\bar{q}]$ such that $\varphi \wedge \psi$ and $\neg\varphi \wedge \psi$ are (classically) satisfiable, so the conditional is also satisfiable.

We can now prove[6] a result that states different ways to define a free probabilistic conditional, as being part of no minimal abrupt repairs (of all maximal consistent sets) or being consistent with any abrupt repair. We say a conditional α is *consistent with* a knowledge base Γ if there is a probability mass π that satisfies α and Γ.

Theorem 4.2. *Consider a knowledge base $\Gamma \in \mathbb{K}$ and a probabilistic conditional $\alpha \in \Gamma$. The following statements are equivalent:*

1. *There is no minimal abrupt repair Δ of Γ such that $\alpha \in \Delta$.*
2. *For all maximal abrupt consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
3. *If $\Gamma' = \Gamma \setminus \Delta$ is an abrupt consolidation of Γ (equivalently, Δ is an abrupt repair of Γ), then α is consistent with Γ'.*
4. *There is no minimal inconsistent set $\Delta \subseteq \Gamma$ such that $\alpha \in \Delta$.*

Note that the fourth statement above is the definition of free probabilistic conditional given in Section 3.1. The first and the second statements are clearly dual to each other, so we have presented two new ways of equivalently defining a free probabilistic conditional without mentioning minimal inconsistent sets, but using abrupt repair and abrupt consolidation. As it is suggested in the previous section, ruling a conditional out is equivalent to widening the corresponding interval to $[0, 1]$ — that is why we call it an *abrupt* repair. However, a probabilistic logic allows for a more general notion of consolidation, formalized below. To save notation, we write $(\varphi|\psi)[\underline{q}, \bar{q}] \subseteq (\varphi|\psi)[\underline{q}', \bar{q}']$ if $\underline{q}' \leq \underline{q}$ and $\bar{q}' \geq \bar{q}$; and $\subsetneq$ is defined from $\subseteq$ as usual.

Definition 4.3. Let Γ be a knowledge base in $\mathbb{K}$. $\Gamma' \in \mathbb{K}$ is a *widening* of Γ if there is a bijection $f : \Gamma \to \Gamma'$ such that $\alpha \subseteq f(\alpha)$ for all $\alpha \in \Gamma$; furthermore, if a widening Γ' is consistent, we say it is a *consolidation* of Γ.

In other words, a consolidation of Γ is the result of widening the probability intervals of its conditionals to a consistent setting. Analogously to the maximal abrupt consolidation, related to a minimal abrupt repair, we can define a sort of consolidation with minimal changes, we call dominant.

Definition 4.4. A consolidation Γ' of Γ is a *dominant consolidation* (or simply a *d-consolidation*) of Γ if, for all consolidations Ψ of Γ, if Γ' is a widening of Ψ, then $\Gamma' = \Psi$.

A d-consolidation Γ' of Γ is such that if some probability interval of Γ were less widened, fixing the others, the resulting base would not be consistent. In other words, it is not possible to give up strictly less information than a d-consolidation while restoring consistency; for an interval to be less widened, another must be more enlarged. In these sense, the changes in the probability bounds are minimal, and the consolidation is maximal.

From these concepts, two new definitions for free probabilistic conditional could be derived: a conditional is free if it is in any d-consolidation; or a conditional is free if it is consistent with any consolidation. We can prove these definitions are actually equivalent:

Lemma 4.5. *Consider a knowledge base $\Gamma \in \mathbb{K}$ and a probabilistic conditional $\alpha \in \Gamma$. The following statements are equivalent:*

1. *For all d-consolidations Γ' of Γ, $\alpha \in \Gamma'$.*
2. *If Γ' is a consolidation of Γ, then α is consistent with Γ'.*

A modification of the free probabilistic conditional concept is suggested by the comparison of Lemma 4.5 with Theorem 4.2, which would yield a different postulate of independence. To not overload the concept of free conditional, we say these probabilistic conditionals are *innocuous*, for they are consistent with any consolidation of the knowledge base.

Definition 4.6. An *innocuous probabilistic conditional* of Γ is a probabilistic conditional $\alpha \in \Gamma$ such that, for every dominant consolidation Γ' of Γ, $\alpha \in \Gamma'$.

The difference between free and innocuous conditionals can be seen in the knowledge base from the proof of Theorem 3.8, as the following example shows.

Example 4.7. Consider the following knowledge base:

$$\Delta = \{(x_1 \wedge x_2)[0.6], (x_1 \wedge \neg x_2)[0.5], (x_1)[0.8]\} .$$

[6] Long proofs of technical results are in a separate Appendix A.

As it was claimed in the proof of Theorem 3.8, $\{(x_1 \wedge x_2)[0.6], (x_1 \wedge \neg x_2)[0.5]\}$ is the only minimal inconsistent set of Δ; so $\alpha = (x_1)[0.8]$ is a free probabilistic conditional. Nonetheless, Δ has no innocuous probabilistic conditional. This can be noted through the following dominant consolidation of Δ:

$$\Delta' = \{(x_1 \wedge x_2)[0.55, 0.6], (x_1 \wedge \neg x_2)[0.45, 0.5], (x_1)[0.8, 1]\} .$$

Δ' is consistent and any consolidation $\Psi \neq \Delta'$ has at least one wider probability interval; so Δ' is dominant. But no original conditional of Δ is in Δ', so none is innocuous. Equivalently, any $\beta \in \Delta$ is inconsistent with Δ'. An example of innocuous conditional can be given in the knowledge base $\Psi = \Delta \cup \{(x_2)[0.3, 0.8]\}$, since $(x_2)[0.3, 0.8]$ would be consistent with any consolidation of Ψ.

An innocuous probabilistic conditional of Γ is consistent with any abrupt consolidation of Γ, since it is semantically equivalent to a consolidation with $[0, 1]$ probability intervals; furthermore, a safe conditional of Γ is clearly consistent with any consolidation of Γ:

Proposition 4.8. *Consider a probabilistic conditional $\alpha \in \Gamma$. If α is safe, it is innocuous; if α is innocuous, it is free.*

As to the independence postulate, we modify it in a corresponding way:

Postulate 4.9 *(i-Independence). If α is an innocuous probabilistic conditional of Γ, then $\mathcal{I}(\Gamma) = \mathcal{I}(\Gamma \setminus \{\alpha\})$.*

From Proposition 4.8 follows the relation among weak independence, i-independence and independence:

Corollary 4.10. *If $\mathcal{I}$ satisfies independence, then $\mathcal{I}$ satisfies i-independence. If $\mathcal{I}$ satisfies i-independence, then $\mathcal{I}$ satisfies weak independence.*

4.2. Refining the minimal conflict concept

To redefine MIS-separability, we need a new notion of minimal conflict, related to the consolidation we introduced. Note that the union of minimal inconsistent sets is equal to the union of minimal abrupt repairs of a knowledge base, so that it forms the complement of the set of free probabilistic conditionals. To be consistent, we should provide a definition of conflicting sets such that their union is complementary to the set of innocuous conditionals. A set with all probabilistic conditionals that are not innocuous would be inconsistent when not empty, but would not have the minimality we are looking for. Such a set would be analogous to the union of all minimal inconsistent sets, but we search for a more fundamental notion of conflict, that can be derived by analyzing the consolidation properties of minimal inconsistent sets.

A minimal inconsistent set is minimal regarding set inclusion, and this is related to the abrupt consolidation:

Proposition 4.11. *A knowledge base Γ is a minimal inconsistent set iff Γ is inconsistent and there are no $\Delta_1, \ldots, \Delta_k \subsetneq \Gamma$, with $k \geq 1$, such that:*

1. $\bigcup_{i=1}^{k} \Delta_i = \Gamma$;
2. *For every $\Gamma' \subseteq \Gamma$, if $\Gamma' \cap \Delta_i$ is an abrupt consolidation of Δ_i for all $1 \leq i \leq k$, then Γ' is an abrupt consolidation of Γ.*

Intuitively, a minimal inconsistent set Γ is a conflict that cannot be analyzed in smaller subsets such that abruptly consolidating them implies abruptly consolidating Γ. Starting with a single inconsistent base Γ, we can find smaller subsets Δ_i satisfying both items of 4.11. We can do this recursively on the inconsistent sets Δ_i until we reach unanalyzable conflicts, which happens to be minimal inconsistent sets. So, abruptly consolidating these sets is abruptly consolidating Γ. Substituting consolidation for abrupt consolidation, we have an analogous definition of conflict:

Definition 4.12. A knowledge base Γ is an *inescapable conflict* if Γ is inconsistent and there are no $\Delta_1, \ldots, \Delta_k \subsetneq \Gamma$, with $k \geq 1$, such that:

1. $\bigcup_{i=1}^{k} \Delta_i = \Gamma$;
2. If Δ'_i is a consolidation of Δ_i for all $1 \leq i \leq k$ and $\bigcup_{i=1}^{k} \Delta'_i$ is a widening of Γ, then $\bigcup_{i=1}^{k} \Delta'_i$ is a consolidation of Γ.

The extra condition in the second item of Definition 4.12 forces consolidations of different knowledge bases $\Delta_i, \Delta_j \subsetneq \Gamma$ with some probabilistic conditional in common to agree in that probability interval; otherwise, $\bigcup_{i=1}^{k} \Delta'_i$ would not be a knowledge base. In other words, the second item says that if we widen the probability intervals of Γ making each Δ_i consistent, then Γ becomes consistent. Inescapable conflicts could equivalently be defined in an alternative way:

Lemma 4.13. *A knowledge base Γ is an inescapable conflict iff there is a widening Γ' of Γ such that Γ' is a minimal inconsistent set.*

Lemma 4.13 captures the intuition behind the proof of Theorem 3.8, where there is a widening that consolidates any proper subset of the knowledge base without consolidating the whole base. As it happens with abrupt consolidation and MISes, to consolidate Γ, one only needs to widen its probability intervals in such a way that each inescapable conflict is solved.

Corollary 4.14. *Consider two knowledge bases $\Gamma, \Gamma' \in \mathbb{K}$ such that Γ' is a widening of Γ. If for every inescapable conflict $\Delta \subseteq \Gamma$ its widening $\{\beta \in \Gamma' \mid \alpha \in \Delta$ and $\alpha \subseteq \beta\}$ is consistent, then Γ' is a consolidation of Γ.*

As all abrupt consolidations can be viewed as consolidations and each knowledge base is a widening of itself, an inescapable conflict is something weaker than a minimal inconsistent set:

Corollary 4.15. *If Δ is a minimal inconsistent set, then Δ is an inescapable conflict.*

Example 4.16. Consider again the knowledge base from Example 4.7:

$$\Delta = \{(x_1 \wedge x_2)[0.6], (x_1 \wedge \neg x_2)[0.5], (x_1)[0.8]\} .$$

As it was already shown, $\{(x_1 \wedge x_2)[0.6], (x_1 \wedge \neg x_2)[0.5]\}$ is the only minimal inconsistent set of Δ — and, by Corollary 4.15, it is an inescapable conflict. Nevertheless, it can be proved that the whole Δ is an inescapable conflict as well.

Suppose, by contradiction, there are $\Delta_1, \ldots, \Delta_k \subsetneq \Delta$ such that $\bigcup_{i=1}^{k} \Delta_i = \Delta$ and, if Δ'_i is a consolidation of Δ_i for all $1 \le i \le k$ and $\bigcup_{i=1}^{k} \Delta'_i$ is a widening of Δ, then $\bigcup_{i=1}^{k} \Delta'_i$ is a consolidation of Δ. To build $\bigcup_{i=1}^{k} \Delta'_i$, we pick a consolidation Δ'_i for each $\Delta_i \subsetneq \Delta$. There are two cases: (a) $(x_1 \wedge x_2)[0.6] \in \Delta_i$; and (b) $(x_1 \wedge x_2)[0.6] \notin \Delta_i$. In case (a), we construct Δ'_i by widening the probability interval of the conditional $(x_1 \wedge x_2)[0.6]$ to $(x_1 \wedge x_2)[0.5, 0.6]$; formally, $\Delta'_i = (\Delta_i \setminus \{(x_1 \wedge x_2)[0.6]\}) \cup \{(x_1 \wedge x_2)[0.5, 0.6]\}$. In case (b), we choose the trivial consolidation $\Delta'_i = \Delta_i$. Even though the proof is omitted, we claim that each Δ'_i is consistent. Consider then the following knowledge base:

$$\Delta' = \bigcup_{i=1}^{k} \Delta'_i = \{(x_1 \wedge x_2)[0.5], (x_1 \wedge \neg x_2)[0.5], (x_1)[0.8]\} .$$

By the premises, Δ' is a consolidation of Δ, but it is inconsistent, since $\Delta' \setminus \{(x_1)[0.8]\}$ implies $(x_1)[1]$ (as shown in Section 3.2). Finally, there cannot exist such $\Delta_1, \ldots, \Delta_k \subsetneq \Delta$, and Δ is an inescapable conflict.

We can now change MIS-separability to respect inescapable conflicts (IC) instead of minimal inconsistent sets. Let $IC(\Gamma)$ denote the collection of all inescapable conflicts of Γ.

Property 4.17 *(IC-separability). If $\Gamma = \Delta \cup \Psi$, $\Delta \cap \Psi = \varnothing$ and $IC(\Gamma) = IC(\Delta) \cup IC(\Psi)$, then $\mathcal{I}(\Gamma) = \mathcal{I}(\Delta) + \mathcal{I}(\Psi)$.*

As inescapable conflict is a weaker concept than MIS, MIS-separability is stronger than IC-separability.

Corollary 4.18. *If $\mathcal{I}$ satisfies MIS-separability, then $\mathcal{I}$ satisfies IC-separability.*

Recall that a free probabilistic conditional is defined in the standard way as not belonging to any minimal inconsistent set. We prove the analogous result for innocuous conditionals and inescapable conflicts, linking all concepts introduced in this section.

Theorem 4.19. *The following statements are equivalent:*

1. *For all d-consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
2. *If Γ' is a consolidation of Γ, then α is consistent with Γ'.*
3. *There is no inescapable conflict Δ in Γ such that $\alpha \in \Delta$.*
4. *α is an innocuous probabilistic conditional in Γ.*

A result analogous to Proposition 3.6 follows:

Corollary 4.20. *If $\mathcal{I}$ satisfies IC-separability, then $\mathcal{I}$ satisfies i-independence.*

As already mentioned, inescapable conflicts are to consolidations as minimal inconsistent sets are to abrupt consolidations. If consolidation via conditionals withdrawal, as in Reiter's and AGM approaches, can focus on the collection of minimal inconsistent sets (ignoring free conditionals), consolidation through widening probability intervals can be done by watching only for the inescapable conflicts (ignoring innocuous conditionals). All these relations among free and innocuous probabilistic conditionals, minimal inconsistent sets and inescapable conflicts argue in favor of the new proposed postulates, whose compatibility with consistency and continuity we will prove.

4.3. Compatible postulates for imprecise probabilities

To replace the postulate of independence and the property of MIS-separability, we propose the weaker pair of i-independence and IC-separability towards building a compatible package together with consistency and continuity. Before proving such compatibility, the postulates have to be generalized to imprecise knowledge bases. To generalize consistency, i-independence and IC-separability is straightforward, we just enlarge their intended scope from knowledge bases in $\mathbb{K}_{prec}$ to bases in $\mathbb{K}$, but the continuity postulate demands some notation.

Let $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]|1 \leq i \leq m\}$ be a knowledge base. The characteristic function of Γ can be generalized as a function $\Lambda_\Gamma : [0,1]^{2m} \to \mathbb{K}$ that changes both upper and lower bounds of each probabilistic conditional in Γ; formally, $\Lambda_\Gamma(\langle \underline{q}'_1, \bar{q}'_1, \ldots, \underline{q}'_m, \bar{q}'_m\rangle) = \{(\varphi_i|\psi_i)[\underline{q}'_i, \bar{q}'_i]\,|1 \leq i \leq m\}$. Now the continuity postulate can be generalized, with $\circ$ denoting function composition:

Postulate 4.21 *(Continuity). For all $\Gamma \in \mathbb{K}$, the function $\mathcal{I} \circ \Lambda_\Gamma : [0,1]^{2|\Gamma|} \to [0,\infty)$ is continuous.*

Note that the postulate above implies Postulate 3.7, which defines continuity for precise probabilities. Given a base Γ of size m, Postulate 3.7 considers a function $f : [0,1]^m \to \mathbb{K}_{prec}$ (the characteristic function when probabilities are precise) such that $f(\langle q'_1, q'_2, \ldots, q'_m\rangle) = \Lambda_\Gamma(\langle q'_1, q'_1, q'_2, q'_2, \ldots, q'_m, q'_m\rangle)$ and requires that $\mathcal{I} \circ f$ be continuous. But note that, if $\mathcal{I} \circ \Lambda_\Gamma$ is continuous, so is $\mathcal{I} \circ f$. Therefore, Theorem 3.8 and Corollary 3.9 also hold within the imprecise probability framework.

Hunter and Konieczny proposed another basic postulate for inconsistency measures [20] that was also adopted by Thimm [44].

Postulate 4.22 *(Monotonicity). For any knowledge bases Γ, $(\Gamma \cup \{\alpha\}) \in \mathbb{K}$, $\mathcal{I}(\Gamma \cup \{\alpha\}) \geq \mathcal{I}(\Gamma)$.*

Thimm actually suggests a stronger principle, super-additivity, which implies monotonicity. Since super-additivity is incompatible with normalization [44] — as also is IC-separability —, we state them as properties, and not postulates.

Property 4.23 *(Super-additivity). For any knowledge base $\Gamma \cup \Delta \in \mathbb{K}$, if $\Gamma \cap \Delta = \varnothing$, then $\mathcal{I}(\Gamma \cup \Delta) \geq \mathcal{I}(\Gamma) + \mathcal{I}(\Delta)$.*

Property 4.24 *(Normalization). For any knowledge base $\Gamma \in \mathbb{K}$, $\mathcal{I}(\Gamma) \in [0,1]$.*

To attend the desirable properties, we generalize the inconsistency measures based on distance minimization proposed by Thimm [44] to the case of imprecise probabilities. Muiño introduced similar ideas under a different semantics for conditional probabilities [31]. Firstly, we define a family of p-norms.

Definition 4.25. Consider a (positive) $m \in \mathbb{N}_{>0}$ and a $p \in \mathbb{N}_{>0} \cup \{\infty\}$. Given a vector $q = \langle q_1, q_2, \ldots, q_m\rangle$ over the real numbers, the p-norm of q is $\|q\|_p = \sqrt[p]{\sum_{i=1}^{m} |q_i|^p}$ if p is finite; otherwise it is $\|q\|_\infty = \max_i |q_i|$.

Thimm defines a family $\mathcal{I}_p$ of inconsistency measures based on the p-norms, which we modify to also consider $p = \infty$ and handle the empty base.

Definition 4.26. Consider a $p \in \mathbb{N}_{>0} \cup \{\infty\}$ and a $\Gamma \in \mathbb{K}$. The function $\mathcal{I}_p : \mathbb{K} \to [0,\infty)$ is the *d^p-inconsistency measure*, defined as

$$\mathcal{I}_p(\Gamma) = \min\{\|q - q'\|_p \mid \Lambda_\Gamma(q) = \Gamma \text{ and } \Lambda_\Gamma(q') \text{ is consistent}\},$$

for any non-empty Γ, and $\mathcal{I}_p(\varnothing) = 0$.

Finally, we are in a position to show inconsistency measures satisfying the wanted properties. We extend Thimm's results to prove that all d^p-inconsistency measures satisfy the reconciled postulates and that some of them hold additional properties. Muiño has similar results, though under a different semantics [31].

Theorem 4.27. *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}_p$ is well-defined and satisfies the postulates of consistency, continuity, i-independence and monotonicity.*

The compatibility of IC-separability and super-additivity with consistency, continuity, monotonicity and i-independence is confirmed by the $\mathcal{I}_1$ measure:

Lemma 4.28. *$\mathcal{I}_p$ satisfies super-additivity and IC-separability iff $p = 1$.*

If normalization is required, we can use the following result due to Muiño [31]:

Lemma 4.29. *$\mathcal{I}_p$ satisfies normalization iff $p = \infty$.*

5. Feasible principled inconsistency measures

Although we have compatible postulates to drive the rational choice of inconsistency measures, these desirable properties are satisfied by a myriad of functions. We may use other arguments to pick some particular inconsistency measures among those obeying the postulates. This section investigates computational aspects of measuring inconsistency through distance minimization, reviewing and generalizing measures proposed by Potyka [34] that can be handled via linear programming. In a second moment, we show how the concrete measures introduced can be justified by means of Dutch books, displaying the maximum guaranteed loss an agent would be exposed to, if stakes are limited. We also show that Dutch books offer other interesting measures.

5.1. Measuring inconsistency with linear programs

In order to better understand the connection between Potyka's inconsistency measures and incoherence measures based on Dutch books, it is worth detailing the construction of the corresponding linear programs. Furthermore, due to such link, every property we prove for Potyka's inconsistency measures shall be inherited by the equivalent Dutch book measures presented in the next section.

To check the consistency of a knowledge base, one can use the well-known formulation of PSAT as a linear program [16]. Consider a knowledge base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \le i \le m\}$. Under the semantics adopted, each assessment $(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]$ is equivalent to the pair $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \ge 0$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \le 0$ of restrictions on P_π. The knowledge base is consistent iff these $2m$ restrictions can be jointly satisfied by a probability measure P_π induced by a probability mass π. Consider two $(m \times 2^n)$-matrices, $A = [a_{ij}]$ and $B = [b_{ij}]$, with $a_{ij} = I_{w_j}(\varphi_i \wedge \psi_i) - \underline{q}_i I_{w_j}(\psi_i)$ and $b_{ij} = I_{w_j}(\varphi_i \wedge \psi_i) - \bar{q}_i I_{w_j}(\psi_i)$, in which $I_{w_j} : \mathcal{L}_{X_n} \to \{0, 1\}$ is the indicator function of the set $\{\varphi \in \mathcal{L}_{X_n} | w_j \models \varphi\}$ — I_{w_j} is the valuation relative to the possible world w_j. The knowledge base Γ is satisfiable iff there is a $(2^n \times 1)$-vector π satisfying the system:

$$A\pi \ge 0 \tag{5}$$

$$B\pi \le 0 \tag{6}$$

$$\sum \pi = 1 \tag{7}$$

$$\pi \ge 0 \,. \tag{8}$$

Restrictions in (5) correspond to $P_\pi(\varphi_i|\psi_i) \ge \underline{q}_i$, and those in (6) codify $P_\pi(\varphi_i|\psi_i) \le \bar{q}_i$; Constraints (7) and (8) force π to be a probability mass over the possible worlds $w_1, w_2, \dots, w_{2^n}$. As all constraints are linear, this system can be solved by linear programming techniques as Simplex. Despite the exponential number of columns, column generation methods can be used to handle them implicitly [26,24], keeping the computation efficient enough to solve large knowledge bases (thousands of probabilities in [17,10]).

To measure inconsistency using distance minimization with $\mathcal{I}_p$, we can add to the system variables $\underline{\varepsilon}_i \ge 0$ ($\bar{\varepsilon}_i \ge 0$) corresponding to decrements (increments) in lower (upper) bounds of each probability interval. Any conditional $(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]$ yields a pair of restrictions $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \ge -\underline{\varepsilon}_i P_\pi(\psi_i)$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \le \bar{\varepsilon}_i P_\pi(\psi_i)$. Computing the $\mathcal{I}_p$ measure is thus reduced to minimizing the p-norm of the vector $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \dots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$.[7] Nonetheless, the constraints contain non-linear terms (from $\underline{\varepsilon}_i P_\pi(\psi_i)$ and $\bar{\varepsilon}_i P_\pi(\psi_i)$), and Potyka points out that these programs have (non-global) local optima [34], so convex optimization techniques cannot be directly applied. Thus, computing $\mathcal{I}_p$ is typically less efficient than deciding PSAT, as empirical results indicate [34].

Potyka emphasizes this impracticability and suggests a new family of inconsistency measures, the *minimal violation measures* [34], which we adapt here to the case of imprecise probabilities. In order to keep constraints linear, "violation"

[7] Note that if we allow $\underline{\varepsilon}_i < 0$ (and $\bar{\varepsilon}_i < 0$), it would represent the tightening of a bound, useless when searching for consistency, and the minimization would avoid it anyway.

variables $\underline{\varepsilon}_i, \bar{\varepsilon}_i \geq 0$ are inserted in the right-hand side of $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \geq 0$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq 0$, yielding $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \geq -\underline{\varepsilon}_i$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq \bar{\varepsilon}_i$. Potyka's minimal violation measures are obtained when the p-norm of $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \dots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$ is minimized with such constraints. We denote by $\mathcal{I}^\varepsilon_p$ the optimal value from the following program, where $\underline{\varepsilon} = [\underline{\varepsilon}_i]$ and $\bar{\varepsilon} = [\bar{\varepsilon}_i]$ are $(m \times 1)$-vectors:

$$\min \|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \dots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_p \text{ subject to:} \tag{9}$$

$$A\pi \geq -\underline{\varepsilon} \tag{10}$$

$$B\pi \leq \bar{\varepsilon} \tag{11}$$

$$\sum \pi = 1 \tag{12}$$

$$\pi, \underline{\varepsilon}, \bar{\varepsilon} \geq 0\,. \tag{13}$$

The restrictions are all linear, and non-linear terms may appear only within the objective function. We can ignore the monotone function $\sqrt[p]{\cdot}$ within the p-norm definition, applying it only after the minimization stops. The degree of each term in the new objective function is p, and for $p = 1$ a linear program is recovered, since $\|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \dots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_1 = \sum_{i=1}^m \underline{\varepsilon}_i + \bar{\varepsilon}_i$. Hence, one can apply the standard Simplex and column generation methods to compute $\mathcal{I}^\varepsilon_1$ with practically the same efficiency as deciding PSAT [34].

For any finite p different from 1, the system (9)–(13) has non-linear terms in its objective function, but this is not the case when we consider $p = \infty$. The ∞-norm is equivalent to take the maximum of the vector $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \dots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$, but this is the same as considering all $\underline{\varepsilon}_i, \bar{\varepsilon}_i$ equal to a single scalar $\varepsilon \geq 0$. The measure $\mathcal{I}^\varepsilon_\infty$ is the solution of the following program [34], in which $\underline{\varepsilon} = \bar{\varepsilon} = [\varepsilon\ \varepsilon\ \dots\ \varepsilon]^T$ are $(m \times 1)$-vectors:

$$\min \varepsilon \text{ subject to:} \tag{14}$$

$$A\pi \geq -\underline{\varepsilon} \tag{15}$$

$$B\pi \leq \bar{\varepsilon} \tag{16}$$

$$\sum \pi = 1 \tag{17}$$

$$\pi, \varepsilon \geq 0\,. \tag{18}$$

The system (14)–(18) is also a linear program, like (9)–(13) when $p = 1$, but has a lesser number of variables. However, Potyka remarks that the variable ε in (14)–(18) is involved in $2m$ restrictions, while each variable $\underline{\varepsilon}_i, \bar{\varepsilon}_i$ appears in only one constraint in (9)–(13); therefore, the computation of $\mathcal{I}^\varepsilon_\infty$ may in practice be slightly less efficient than computing $\mathcal{I}^\varepsilon_1$ [34].

For sets of unconditional probabilistic assessments, when all conditioning events ψ_i are equivalent to $\top$, the inconsistency measures $\mathcal{I}_p$ and $\mathcal{I}^\varepsilon_p$ are extensionally identical for all p. The reason is that the restriction on P_π and $\underline{\varepsilon}_i, \bar{\varepsilon}_i$ corresponding to a conditional is the same when computing both measures. For instance, any constraint $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq 0$ becomes equivalent to $P_\pi(\varphi_i) - \bar{q}_i \leq 0$ when ψ_i is a tautology, and inserting an error to the probability bound, $P_\pi(\varphi_i) - (\bar{q}_i + \bar{\varepsilon}_i) \leq 0$, is the same as placing it in the right-hand side.

Potyka has proved that these measures, $\mathcal{I}^\varepsilon_p$ with $p \in \mathbb{N}_{>0} \cup \{\infty\}$, besides being computable via linear programming, satisfy the postulates of consistency, continuity and monotonicity for the case of precise probabilities [34]:

Proposition 5.1. *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}^\varepsilon_p : \mathbb{K}_{prec} \to [0, \infty)$ is well-defined and satisfies consistency, continuity, weak independence and monotonicity. $\mathcal{I}^\varepsilon_1$ also satisfies super-additivity.*

We can generalize the result above to encompass probability intervals and the new postulates we introduced:

Theorem 5.2. *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}^\varepsilon_p : \mathbb{K} \to [0, \infty)$ is well-defined and satisfies consistency, continuity, i-independence and monotonicity. $\mathcal{I}^\varepsilon_1$ also satisfies super-additivity and IC-separability; and $\mathcal{I}^\varepsilon_\infty$ satisfies normalization.*

Now we have a set of compatible postulates for inconsistency measures and two particular measures satisfying them that can be computed rather efficiently using linear programming techniques. On the one hand, $\mathcal{I}^\varepsilon_1$ also satisfies super-additivity and IC-separability; on the other hand, $\mathcal{I}^\varepsilon_\infty$ additionally satisfies normalization. Nonetheless, one can argue that these measures lack some proper justification, despite satisfying some postulates and being feasible, as Capotorti, Regoli and Vattari did [4]. They claim that distances between conditional probabilities are meaningless, being only geometrical measures. This might be the case, but it would only undermine the $\mathcal{I}_p$ family. For $\mathcal{I}^\varepsilon_p$ measures, distances between probabilities are computed weighting by the probabilities of the conditioning formulas, so to speak, allowing some operational interpretation. In the next section, we provide a rationale for $\mathcal{I}^\varepsilon_1$ and $\mathcal{I}^\varepsilon_\infty$ based on Dutch books.

Table 2
Prizes earned by Alice (in \$) for each bet and each possible world.

Possible world	$x_1 \wedge x_2$	$x_1 \wedge \neg x_2$	$\neg x_1 \wedge x_2$	$\neg x_1 \wedge \neg x_2$
Bet on Hot (x_1)	10	10	0	0
Bet on Sunny (x_2)	10	0	10	0
Bet on Hot and Sunny ($x_1 \wedge x_2$)	−10	0	0	0
Total	10	10	10	0

5.2. Inconsistency measures and Dutch books

In formal epistemology, there is an interest in measuring the incoherence of an agent whose beliefs are given as probabilities or lower previsions over propositions or random variables — a *Bayesian* agent. If we have propositions from classical logic, the formalized problem at hand is exactly the one we are investigating. When the agent's degrees of belief are represented by a knowledge base, to measure the agent's incoherence is to measure the inconsistency of such knowledge base. Schervish, Kadane and Seidenfeld [39,40,38] have proposed ways to measure incoherence of an agent based on Dutch books.

Dutch book arguments are based on the agent's betting behavior induced by her degrees of belief, typically used to show their irrationality. To introduce the concept of Dutch book, we start with the unconditional case. Dutch book arguments rely on an operational interpretation of (imprecise) degrees of belief, in which their lower/upper bounds are defined through bet buying/selling prices. Suppose an agent believes that the probability of proposition φ_i being true lies within $[\underline{q}_i, \bar{q}_i]$, for $1 \leq i \leq m$. Consider a bet ticket on the proposition φ_i that returns a prize (from the ticket seller) of $\lambda_i \geq 0$ if φ_i is the case, and is worthless if φ_i is not the case. Dutch book arguments generally use a *willingness-to-bet assumption* that this agent is willing to buy such a bet ticket on φ_i for $\underline{q}_i\lambda_i \geq 0$ and to sell it for $\bar{q}_i\lambda_i \geq 0$, for any $\lambda_i \geq 0$. Then, if a bettor can buy and/or sell a set of bet tickets from/to the agent that will cause her a sure loss no matter which possible world is the case, we say she is exposed to a Dutch book. This set of bet tickets that causes a guaranteed loss to the agent is called a Dutch book.

Example 5.3. Alice (the agent) and Bob (the bettor) are flying to the beach. To spend the time on the plane, they discuss and gamble on the destination weather, to be checked on arrival — will it be sunny and/or hot (say, at least 20°*C*)? Alice assigns probability intervals for three propositions, formed by the atoms x_1 =“the weather is sunny” and x_2 =“the weather is hot”:

- she believes the probability of the weather being hot is between 70% and 80%; which we represent by the conditional $(x_1)[0.7, 0.8]$;
- she thinks the probability of the weather being sunny is between 50% and 60%; which is represented by $(x_2)[0.5, 0.6]$;
- she also says that the probability of the weather being both hot and sunny is at most 10%; formalized into $(x_1 \wedge x_2)[0, 0.1]$.

Now Bob can choose which bet tickets he wants to buy from Alice, and which ones he wants to sell, under the willingness-to-bet assumption. He can also set the prize λ_i each ticket will return if the corresponding proposition turns out to be the case. So Bob decides to trade the following bet tickets with Alice:

- He sells to Alice a bet ticket that pays back \$10 if the weather is hot (x_1 is true); and 0 otherwise. Alice pays $\$10 \times 0.7 = \$7$ for it, which she considers fair;
- Bob also makes Alice buy for $\$10 \times 0.5 = \$5$ a bet ticket that will return \$10 to her only if the weather is sunny (x_2 is true);
- Finally, Bob buys from Alice a bet ticket whose prize \$10 is paid back only in case the weather is hot and sunny ($x_1 \wedge x_2$ is true); and Alice sells it for $\$10 \times 0.1 = \$1$.

Combining the three bet tickets, Bob received $\$7 + \$5 = \$12$ from Alice and returned \$1 to her, so that before they arrive and prizes are paid Alice is losing \$11. Table 2 shows the prizes Alice can earn from each bet ticket traded for each possible weather; a negative quantity means Alice has to pay it to Bob.

Note that, no matter how is the weather when they arrive and pay the prizes, the total quantity Alice can receive from Bob is at most \$10. Since she was losing \$11 before landing and checking the weather, she will lose at least \$1 in the end. Given this sure loss scenario, this set of three bets is said to be a Dutch book.

Instead of the agent (the bettor) paying for a bet ticket and eventually getting its prize back from the bettor (the agent), we can view this whole operation as a single contract between these two players.

Definition 5.4. A *gamble* on $\varphi \in \mathcal{L}_{X_n}$ is an agreement between the agent and the bettor with two parameters, the *stake* $\lambda \in \mathbb{R}$ and the *relative price* $q \in [0, 1]$, stating that:

- the agent pays $\lambda \times q$ to the bettor if φ is false;
- the bettor pays $\lambda \times (1-q)$ to the agent if φ is true.

A gamble on φ with stake $\lambda \geq 0$ and relative price q is equivalent to the agent buying from the bettor a bet ticket for $\lambda \times q$ that returns λ only if φ is the case; if the stake λ is negative, the gamble is equivalent to the bettor buying the same ticket from the agent. The willingness-to-bet assumption translates to gambles in the following way: if an agent believes that the probability of a proposition φ being true lies within $[\underline{q}, \bar{q}]$, she finds acceptable gambles on φ with any stake $\lambda \geq 0$ and relative price $\underline{q}$ and gambles with any stake $\lambda \leq 0$ and relative price $\bar{q}$. In Example 5.3, the tickets trading is equivalent to a set of three gambles: a gamble on x_1 with stake \$10 and relative price 0.7; a gamble on x_2 with stake \$10 and relative price 0.5; and a gamble on $x_1 \wedge x_2$ with stake \$−10 and relative price 0.1.

A gamble on φ can be generalized to consider a conditioning event ψ. Consider a bet ticket that, when ψ is true, pays a prize of λ if φ is the case and returns 0 if φ is false. In other words, this bet ticket works as a gamble on φ when ψ is the case. However, suppose this bet ticket pays back to the agent the same amount that was spent in its buying if ψ is false — that is, the gamble is canceled. The following generalization of gambles capture these "conditional bets":

Definition 5.5. A *(conditional) gamble* on $\varphi|\psi \in \mathcal{L}_{X_n}|\mathcal{L}_{X_n}$ is an agreement between the agent and the bettor with two parameters, the *stake* $\lambda \in \mathbb{R}$ and the *relative price* $q \in [0,1]$, stating that:

- the agent pays $\lambda \times q$ to the bettor if ψ is true and φ is false;
- the bettor pays $\lambda \times (1-q)$ to the agent if ψ is true and φ is true;
- the gamble is called off, causing neither profit nor loss to the involved parts, if ψ is false.

Accordingly, we generalize the willingness-to-bet assumption: if an agent believes that the probability of a proposition φ being true given that another proposition ψ is true lies within $[\underline{q}, \bar{q}]$, she finds acceptable gambles on $\varphi|\psi$ with stake $\lambda \geq 0$ and relative price $\underline{q}$ and gambles with stake $\lambda \leq 0$ and relative price $\bar{q}$. A *Dutch book* is a set of (conditional) gambles that the agent sees as fair, under the willingness-to-bet assumption, that causes her a guaranteed loss no matter which possible world is the case. We assume Dutch books contain exactly two gambles on $(\varphi_i|\psi_i)$ per each conditional $(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] \in \Gamma$, the base formalizing the agent's beliefs: one with stake $\underline{\lambda}_i \geq 0$ and the other with stake $-\bar{\lambda}_i \leq 0$. This is not restrictive, since gambles on the same $(\varphi_i|\psi_i)$ with the same relative price can be merged by summing the stakes, and the absence of a gamble is equivalent to a stake equal to zero. We can thus denote a Dutch book simply by the absolute value of its stakes $\underline{\lambda}_1, \bar{\lambda}_1, \dots, \underline{\lambda}_m, \bar{\lambda}_m \geq 0$, where $m = |\Gamma|$. Actually, any set of gambles involving an agent whose epistemic state is represented by Γ can be represented by these $2m$ (absolute value of) stakes, since the relative prices are set in Γ.

If the set of probabilistic conditionals that represents an agent's epistemic state turns out to be inconsistent, then she is exposed to a Dutch book, and vice-versa [32]. In other words, an agent sees as fair a set of gambles that causes her a guaranteed loss if, and only if, the knowledge base codifying her (conditional) degrees of belief is inconsistent. We can check this connection in Example 5.3: $(x_1)[0.7, 0.8]$ and $(x_2)[0.5, 0.6]$ imply a probability of at least 0.2 for $x_1 \wedge x_2$, which Alice violates. Consequently, she is exposed to a Dutch book, for her three probability interval assignments are not jointly satisfiable. In this way, Dutch book arguments were introduced to show that degrees of belief must obey the axioms of probability and are a standard proof of incoherence (introductions to Dutch books and their relation to incoherence can be found in [42] and [8]). Hence, a natural approach to measuring an agent's degree of incoherence is through the magnitude of the sure loss she is vulnerable to. The intuition says that, the more incoherent an agent is, the greater the guaranteed loss that can be imposed on her through a Dutch book. Nevertheless, with no bounds on the stakes, such loss would also be unlimited for incoherent agents. For instance, in Example 5.3, if stakes were \$100, \$100 and \$−100, Alice would have a net loss of at least \$10, instead of \$1, regardless of the weather on arrival. To better understand the loss a Dutch book causes to an agent, we formalize it in the following.

Consider the knowledge base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$ representing an agent's epistemic state. Let $\underline{\lambda}_i, \bar{\lambda}_i \geq 0$ denote gambles on $(\varphi_i|\psi_i)$, the first with relative price $\underline{q}_i$ and stake $\underline{\lambda}_i \geq 0$, the second with relative price $\bar{q}_i$ and stake $-\bar{\lambda}_i \leq 0$, for $1 \leq i \leq m$. A set of gambles can then be represented by the vector $\langle \underline{\lambda}_1, \bar{\lambda}_1, \dots, \underline{\lambda}_m, \bar{\lambda}_m \rangle$. If a possible world w_j is the case, the net profit for the agent regarding a bet on $\varphi|\psi$ with stake λ and relative price q can be computed via

$$\lambda(I_{w_j}(\varphi \wedge \psi) - qI_{w_j}(\psi)),$$

in which $I_{w_j}: \mathcal{L}_{X_n} \to \{0,1\}$ is the indicator function of the set $\{\varphi \in \mathcal{L}_{X_n} | w_j \models \varphi\}$ — a valuation. For a gamble on $(\varphi_i|\psi_i)$ with stake $\underline{\lambda}_i$ (or $-\bar{\lambda}_i$), the agent's net profit in a possible word w_j is $\underline{\lambda}_i(I_{w_j}(\varphi_i \wedge \psi_i) - \underline{q}_i I_{w_j}(\psi_i))$ (or $-\bar{\lambda}_i(I_{w_j}(\varphi_i \wedge \psi_i) - \bar{q}_i I_{w_j}(\psi_i))$). Recall (from (5)–(8)) that $a_{ij} = I_{w_j}(\varphi_i \wedge \psi_i) - \underline{q}_i I_{w_j}(\psi_i)$ and $b_{ij} = I_{w_j}(\varphi_i \wedge \psi_i) - \bar{q}_i I_{w_j}(\psi_i)$. If a given possible world w_j is the case, the set of gambles $\langle \underline{\lambda}_1, \bar{\lambda}_1, \dots, \underline{\lambda}_m, \bar{\lambda}_m \rangle$ gives the agent a profit of $\sum_{i=1}^m a_{ij}\underline{\lambda}_i + \sum_{i=1}^m -b_{ij}\bar{\lambda}_i$. Let ℓ be the sure loss ($-\ell$ is profit) a set of gambles yields to the agent; *i.e.*, no matter which possible world is the case, the agent loses at least ℓ. Thus, ℓ is such that $\sum_{i=1}^m a_{ij}\underline{\lambda}_i + \sum_{i=1}^m -b_{ij}\bar{\lambda}_i \leq -\ell$ for all possible worlds w_j. When there is no restriction on the stakes, to find the set of gambles $\langle \underline{\lambda}_1, \bar{\lambda}_1, \dots, \underline{\lambda}_m, \bar{\lambda}_m \rangle$ that maximizes the sure loss is to solve the following linear program:

$$\max \ell \text{ subject to:} \tag{19}$$

$$\begin{bmatrix} 1 & a_{11} & \dots & a_{m1} & -b_{11} & \dots & -b_{m1} \\ 1 & a_{12} & \dots & a_{m2} & -b_{12} & \dots & -b_{m2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{12^n} & \dots & a_{m2^n} & -b_{12^n} & \dots & -b_{m2^n} \end{bmatrix} \begin{bmatrix} \ell \\ \underline{\lambda}_1 \\ \vdots \\ \underline{\lambda}_m \\ \bar{\lambda}_1 \\ \vdots \\ \bar{\lambda}_m \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{20}$$

$$\underline{\lambda}_1, \bar{\lambda}_1, \dots, \underline{\lambda}_m, \bar{\lambda}_m \geq 0 \,. \tag{21}$$

The linear program above can be viewed as the dual of that in lines (5)–(8), which checks the consistency of Γ, if we consider that 0 is the function being minimized in the latter, since we are interested only in its feasibility (for duality theory in linear programing, see, for instance, [46]). Note that, in (5)–(8), $B\pi \leq 0$ is equivalent to $-B\pi \geq 0$ and $\sum \pi = 1$ can be inserted into A as a line of 1's. By duality theory, as the program above is feasible, it is unbounded iff (5)–(8) is infeasible. That is, if Γ is inconsistent, sure loss via Dutch book is unlimited.

Different strategies to circumvent this in order to measure incoherence as a finite loss are found in the formal epistemology literature. Schervish et al. propose a flexible formal approach to limiting these stakes generating a family of incoherence measures for upper and lower previsions on bounded random variables [40]. In this section, we are interested in two of them, which we simplify to our case.

Their whole family of incoherence measures is based on the maximum guaranteed loss an agent is exposed to via a Dutch book, varying only on how stakes are limited. The first incoherence measure Schervish et al. introduce that concerns us is when the sum of the absolute values of the stakes is lesser than or equal to one, $\sum_i \underline{\lambda}_i + \bar{\lambda}_i \leq 1$. The second incoherence measure we investigate is defined as the maximum guaranteed loss when each stake have absolute value no greater than one, or $\underline{\lambda}_i, \bar{\lambda}_i \leq 1$.[8] We define the inconsistency measures $\mathcal{I}^{sum}_{SSK} : \mathbb{K} \to [0, \infty)$ and $\mathcal{I}^{max}_{SSK} : \mathbb{K} \to [0, \infty)$ on knowledge bases as these two incoherence measures on the corresponding agents represented by these knowledge bases. That is, we equate $\mathcal{I}^{sum}_{SSK}(\Gamma)$ (and $\mathcal{I}^{max}_{SSK}(\Gamma)$), for any $\Gamma \in \mathbb{K}$, to the maximum sure loss an agent whose epistemic state is represented by Γ is exposed to through a Dutch book when the sum (maximum) of the stakes' absolute values is at most one.

Example 5.6. Recall Example 5.3, in which there are three gambles, with stakes \$10, \$10 and \$−10. These gambles guarantee a loss of at least \$1 to Alice. But now suppose that Bob, while choosing the gambles, must do it so that the absolute values of the stakes sum up to one. He could so arrange the same gambles but changing the stakes to 1/3, 1/3 and −1/3. In this new scenario, Alice would have a sure loss of 1/30. Similarly, if the absolute value of each stake is limited to the interval $[0, 1]$, stakes could be 1, 1 and −1, yielding a guaranteed loss of 1/10 to Alice. In fact, it can be checked (by solving the linear programs) that 1/30 and 1/10 are the greatest amount one can take for sure from Alice via Dutch book if stakes have absolute values summing up to one or are all in $[0, 1]$, respectively. Formalizing, with $\Gamma = \{(x_1)[0.7, 0.8], (x_2)[0.5, 0.6], (x_1 \wedge x_2)[0, 0.1]\}$ codifying Alice's epistemic state, we have $\mathcal{I}^{sum}_{SSK}(\Gamma) = 1/30$ and $\mathcal{I}^{max}_{SSK}(\Gamma) = 1/10$.

Even though incoherence measures based on Dutch books from the formal epistemology community and inconsistency measures based on distance minimization from Artificial Intelligence researchers may seem unrelated at first, they are actually two sides of the same coin. The programs that compute the maximum guaranteed loss an agent is exposed to are technically dual to those that minimize distances to measure inconsistency. Nau has already investigated this matter, mentioning results similar to the following [32]:

Theorem 5.7. *For any* $\Gamma \in \mathbb{K}$, $\mathcal{I}^{sum}_{SSK}(\Gamma) = \mathcal{I}^{\varepsilon}_{\infty}(\Gamma)$.

Proof. Just add the constraint $\underline{\lambda}_1 + \bar{\lambda}_1 + \cdots + \underline{\lambda}_m + \bar{\lambda}_m \leq 1$ to the linear program (19)–(21). The dual of this new program would become the program (14)–(18), which computes $\mathcal{I}^{\varepsilon}_{\infty}(\Gamma)$. So, by the strong duality theorem, $\mathcal{I}^{sum}_{SSK}(\Gamma) = \mathcal{I}^{\varepsilon}_{\infty}(\Gamma)$, for both programs are always feasible. □

Recall that $\mathcal{I}^{\varepsilon}_{\infty}$ is exactly one of the two feasible measures proposed by Potyka [34]. Far from meaningless, such measure quantifies the maximum sure loss an agent is exposed to when the sum of the stakes is no greater than one — or, equivalently, fixed at one.

As to Potyka's other feasible proposal, $\mathcal{I}^{\varepsilon}_{1}$, duality in linear programming provides a correspondence with the second incoherence measure we presented from Schervish et al.:

[8] Schervish et al. [40] actually measure the incoherence as maximum rates between the guaranteed loss and the sum (the maximum) of the stakes' absolute values. Clearly, this is equivalent to maximizing the guaranteed loss when the sum of the stakes' absolute values is no greater than 1 (or these absolute values are in $[0, 1]$).

Theorem 5.8. *For any* $\Gamma \in \mathbb{K}$, $\mathcal{I}^{max}_{SSK}(\Gamma) = \mathcal{I}^{\varepsilon}_1(\Gamma)$.

Proof. Similarly to the proof of Theorem 5.7, insert the constraints $\underline{\lambda}_i, \bar{\lambda}_i \leq 1$, for $1 \leq i \leq m$, into the linear program (19)–(21). The dual of this new program would become the program (9)–(13), with $p = 1$, which computes $\mathcal{I}^{\varepsilon}_1(\Gamma)$. Again, by the strong duality theorem, $\mathcal{I}^{max}_{SSK}(\Gamma) = \mathcal{I}^{\varepsilon}_1(\Gamma)$, since both programs are always feasible. $\square$

Theorem 5.8 states the extensional identity between $\mathcal{I}^{\varepsilon}_1$ and $\mathcal{I}^{max}_{SSK}$. Within the unconditional probabilities scenario, this means that the Manhattan distance between the agent's probabilities and the closest consistent probabilities is equal to the maximum sure loss she is exposed to when stakes' absolute values are not higher than one.

Theorem 5.7 and Theorem 5.8 give an operational interpretation for the inconsistency measures $\mathcal{I}^{\varepsilon}_\infty$ and $\mathcal{I}^{\varepsilon}_1$ based on betting behavior. It was remarked in Section 5.1 that $\mathcal{I}^{\varepsilon}_p$ and $\mathcal{I}_p$ give the same inconsistency degrees to unconditional knowledge bases. Thus, Dutch books with limited stakes ($\underline{\lambda}_i, \bar{\lambda}_i \leq 1$ or $\sum_i \underline{\lambda}_i + \bar{\lambda}_i \leq 1$) can be used to rationalize also $\mathcal{I}_1$ and $\mathcal{I}_\infty$ in the unconditional setting. However, when we take into account conditional probabilities, only $\mathcal{I}^{\varepsilon}_1$ and $\mathcal{I}^{\varepsilon}_\infty$ measure the maximum guaranteed loss an agent would be exposed to, when stakes are limited via $\underline{\lambda}_i, \bar{\lambda}_i \leq 1$ or $\sum_i \underline{\lambda}_i + \bar{\lambda}_i \leq 1$, respectively.

Different strategies for bounding stakes can lead to different inconsistency measures, but our motivation in this section was not to use Dutch books to determine which measures should be adopted — that is the reason of the postulates. The point here is that these two measures ($\mathcal{I}^{\varepsilon}_1$ and $\mathcal{I}^{\varepsilon}_\infty$), besides satisfying the postulates and being computable through linear programs, have a meaningful interpretation. In the next section, we show that other measures based on Dutch books have these qualities as well.

5.3. Other feasible principled measures

In order to measure incoherence as the greatest guaranteed loss in a Dutch book, Schervish et al. have firstly proposed two different ways of normalizing such loss: by limiting either the agent's or the bettor's resources [39]. The authors introduce the concept of *escrow* as the amount committed into a gamble by the agent (or the bettor). For instance, consider a gamble on $\varphi_i|\psi_i$ with stake $\underline{\lambda}_i \geq 0$ and relative price $\underline{q}_i$. The agent might lose $\underline{q}_i\underline{\lambda}_i$ with this gamble, while the bettor is exposed to a loss of $(1 - \underline{q}_i)\underline{\lambda}_i$. Now consider a gamble on the same conditional with stake $-\bar{\lambda}_i \leq 0$ and relative price $\bar{q}_i$. The agent might have to pay $(1 - \bar{q}_i)\bar{\lambda}_i$ to the bettor, whilst the bettor might lose $\bar{q}_i\bar{\lambda}_i \geq 0$ to the agent. Schervish et al. call these quantities the agent's and the bettor's *escrows*. Equivalently, the agent's (or bettor's) escrow for a gamble is how much she (he) has to commit from her (his) resources to cover an eventual loss.

Instead of bounding the sum of the stakes, an agent's degree of incoherence can be measured, as the maximum guaranteed loss in a Dutch book, by limiting the agent's (or the bettor's) total escrow to one.[9] In other words, we are limiting how much the agent (or the bettor) could lose in case that every gamble resolves unfavorably, inflicting a loss to her (him). Schervish et al. give market situations that justifies these choices [39]. We denote by $\mathcal{I}^{a,sum}_{SSK} : \mathbb{K} \to [0, \infty)$ and $\mathcal{I}^{b,sum}_{SSK}\mathbb{K} \to [0, \infty) \cup \{\infty\}$[10] the inconsistency measures corresponding to these two incoherence measures, when the agent's or the bettor's total escrow is at most one, respectively.

Formally, starting with the linear program of lines (19)–(21), $\mathcal{I}^{a,sum}_{SSK}(\Gamma)$ and $\mathcal{I}^{b,sum}_{SSK}(\Gamma)$ are obtained via the maximization of ℓ by adding further constraints. Let $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]|1 \leq i \leq m\}$ be a knowledge base. To compute $\mathcal{I}^{a,sum}_{SSK}(\Gamma)$, one need to insert the restriction $\sum_{i=1}^{m} \underline{q}_i\underline{\lambda}_i + (1 - \bar{q}_i)\bar{\lambda}_i \leq 1$ into (19)–(21). Similarly, $\mathcal{I}^{g,sum}_{SSK}(\Gamma)$ is the solution (on ℓ) of the program (19)–(21) incremented with the constraint $\sum_{i=1}^{m}(1 - \underline{q}_i)\underline{\lambda}_i + \bar{q}_i\bar{\lambda}_i \leq 1$.

The fact that $\mathcal{I}^{g,sum}_{SSK}$ may be unbounded is acknowledged by Schervish et al. [39]. For instance, consider an agent whose belief state is given by $\Gamma = \{(\varphi)[1], (\neg\varphi)[1]\}$. The agent finds acceptable pairs of gambles on (φ and $\neg\varphi$) in which the bettor has escrows equal to zero ($\underline{\lambda}_i(1 - \underline{q}_i) = 0$, for $\underline{q}_i = 1$), and sure loss can be scaled arbitrarily up. In such cases, we define $\mathcal{I}^{b,sum}_{SSK}(\Gamma) = \infty$.

Example 5.9. Recall Example 5.3, its three gambles, with stakes \$10, \$10 and \$−10, and the implied loss of at least \$1 to Alice. But now suppose that Bob has to choose gambles in such a way that his (or Alice's) total escrow sum up to 1. Note that, with stakes \$10, \$10 and \$−10, his total escrow is $\$10 \times (1 - 0.7) + \$10 \times (1 - 0.5) + \$10 \times 0.1 = \$9$ (Alice's is $\$10 \times 0.7 + \$10 \times 0.5 + \$10 \times (1 - 0.1) = \$21$). He could then arrange the same gambles but changing the stakes to 10/9, 10/9 and −10/9 (or 10/21, 10/21 and −10/21) in order to his (Alice's) total escrow be equal to one. In this new scenario, Alice would have a sure loss of 1/9 (or 1/21). Once again, one could verify (by solving the linear programs) that 1/9 and 1/21 are the greatest amount one can take for sure from Alice via Dutch book if Bob's or Alice's total escrow is no greater than 1, respectively. Formalizing, with $\Gamma = \{(x_1)[0.7, 0.8], (x_2)[0.5, 0.6], (x_1 \wedge x_2)[0, 0.1]\}$ codifying Alice's epistemic state, we have $\mathcal{I}^{b,sum}_{SSK}(\Gamma) = 1/9$ and $\mathcal{I}^{a,sum}_{SSK}(\Gamma) = 1/21$.

[9] Again, this is equivalent to measuring incoherence as the maximum ratio between the sure loss and the agent's (the bettor's) total escrow.

[10] For reasons that will be clear soon, we relax in this section the definition of inconsistency measures, allowing their range to include ∞.

Schervish et al. contemplate in detail a whole spectrum of ways to bound the agent's escrows, the bettor's, or their sum in order to measure the maximum sure loss [40]. For each of these three quantities, the author note that the two extreme functions in their framework used to normalize the guaranteed loss are the maximum and the sum, from which six different inconsistency measures arise [38]. Note that the sum of the agent's and the bettor's escrows for a single gamble is equal to the absolute value of its stake, so $\mathcal{I}^{sum}_{SSK}$ and $\mathcal{I}^{max}_{SSK}$ are two inconsistency measures from this same framework. To build the remaining two measures, escrows could be bounded via their maximum, instead of their total. Intuitively, this corresponds to limiting the quantity the agent (or the bettor) accepts to eventually lose in each individual gamble.

We define the inconsistency measure $\mathcal{I}^{a,max}_{SSK} : \mathbb{K} \to [0,\infty)$ (and $\mathcal{I}^{b,max}_{SSK}\mathbb{K} \to [0,\infty) \cup \{\infty\}$) on knowledge bases as the degree of incoherence of the corresponding agents measured via the maximum sure loss she is exposed through a Dutch book if the agent's (the bettor's) escrow for each gamble in no greater than one. To compute $\mathcal{I}^{a,max}_{SSK}(\Gamma)$, for $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \le i \le m\}$, we may again use the linear program (19)–(21) and compute the maximum value of ℓ with extra constraints $\underline{q}_i \underline{\lambda}_i \le 1$ and $(1 - \bar{q}_i)\bar{\lambda}_i \le 1$, for $1 \le i \le m$. Similarly, $\mathcal{I}^{b,max}_{SSK}(\Gamma)$ is the solution (on ℓ) to the program formed by inserting the restrictions $(1 - \underline{q}_i)\underline{\lambda}_i \le 1$ and $\bar{q}_i \bar{\lambda}_i \le 1$, for $1 \le i \le m$, into (19)–(21). As with $\mathcal{I}^{b,sum}_{SSK}$, we define $\mathcal{I}^{b,max}_{SSK}(\Gamma) = \infty$ when such program is unbounded.

Example 5.10. Remember the scenario from Example 5.3, in which three gambles are considered, with stakes \$10, \$10 and \$−10, and Alice has a guaranteed loss of at least \$1. Now suppose that Bob can only choose gambles in which his (Alice's) eventual loss — the escrow — is lesser than or equal to one. In other words, his (her) maximum escrow is no greater than 1. With these constraints, Bob can choose the same three gambles, but with stakes 2, 2 and −2: his escrows are $2 \times (1 - 0.7) = 0.6$, $2 \times (1 - 0.5) = 1$ and $2 \times 0.1 = 0.2$ (with stakes 10/9, 10/9 and −10/9, Alice's escrow are $(10/9) \times 0.7 = 7/9$, $10/9 \times 0.5 = 5/9$ and $10/9 \times (1 - 0.1) = 1$). Note that Bob (Alice) can eventually lose at most 1 in a single gamble. In this new setting, Alice would have a guaranteed loss of 1/5 (or 1/9). By solving the corresponding linear programs, we would find that 1/5 and 1/9 are the greatest amount one can take for sure from Alice via Dutch book if Bob's or Alice's maximum escrow is no greater than 1, respectively. Formalizing, with $\Gamma = \{(x_1)[0.7, 0.8], (x_2)[0.5, 0.6], (x_1 \wedge x_2)[0, 0.1]\}$ codifying Alice's epistemic state, we have $\mathcal{I}^{b,max}_{SSK}(\Gamma) = 1/5$ and $\mathcal{I}^{a,max}_{SSK}(\Gamma) = 1/9$.

These four inconsistency measures ($\mathcal{I}^{a,sum}_{SSK}$, $\mathcal{I}^{b,sum}_{SSK}$, $\mathcal{I}^{a,max}_{SSK}$ and $\mathcal{I}^{b,max}_{SSK}$) based on limiting the escrows have most of the desirable properties we presented.

Theorem 5.11. *$\mathcal{I}^{a,sum}_{SSK}$, $\mathcal{I}^{b,sum}_{SSK}$, $\mathcal{I}^{a,max}_{SSK}$ and $\mathcal{I}^{b,max}_{SSK}$ are well-defined and satisfy consistency, i-independence and monotonicity. $\mathcal{I}^{a,max}_{SSK}$ and $\mathcal{I}^{b,max}_{SSK}$ also satisfy super-additivity and IC-separability.*

Lemma 5.12. *$\mathcal{I}^{a,sum}_{SSK}$, $\mathcal{I}^{a,max}_{SSK}$, $\mathcal{I}^{b,sum}_{SSK}$ and $\mathcal{I}^{b,max}_{SSK}$ are continuous for probabilities within $(0,1)$.*

Lemma 5.13. *$\mathcal{I}^{a,sum}_{SSK}$ satisfies normalization.*

Altogether, $\mathcal{I}^{a,sum}_{SSK}$, $\mathcal{I}^{b,sum}_{SSK}$, $\mathcal{I}^{a,max}_{SSK}$ and $\mathcal{I}^{b,max}_{SSK}$ are all computable through linear programs, have the core desirable properties and can be given an operational interpretation. $\mathcal{I}^{a,max}_{SSK}$ and $\mathcal{I}^{b,max}_{SSK}$ also satisfy super-additivity and IC-separability, while $\mathcal{I}^{a,sum}_{SSK}$ is normalized. These measures can be good alternatives, depending on the context, as the market scenarios described by Schervish et al. [39].

6. Conclusions and future work

Handling inconsistency has been receiving increased attention in the AI community since most inference methods rely on the consistency of the premises; and such requirement is commonly violated in large bases of probabilistic knowledge. A reasonable start point to deal with the inconsistency in probabilistic bases is to know how severe it is, and how this severity changes with the probabilities. In this work, we studied different ways of measuring inconsistency in probabilistic knowledge bases. Three aspects were discussed: postulates the measures should satisfy, the efficiency of the methods used to compute the measures, and possible meaningful interpretations for them. As it was argued for, the independence postulate shall be abandoned in favor of continuity. The causes of such incompatibility were analyzed, and a modification of independence was proposed to restore compatibility. Inconsistency measures that can be computed using linear programs were reviewed and proved to satisfy the postulates, and we gave them a rational by means of Dutch books. Finally, we showed that other measures in the literature based on Dutch books, and computable through linear programming, also satisfy the postulates.

By restoring the compatibility of the postulates for measuring inconsistency in probabilistic knowledge bases, we put forward a new pair of properties one can use to formulate or evaluate measures: i-independence and IC-separability. These desirable properties are based on two new concepts: innocuous conditional and inescapable conflicts. Besides measuring inconsistency, these concepts may be useful for formalizing inference from inconsistent bases or performing probabilistic belief revision/update, for instance. Both concepts are derived from a specific consolidation procedure — the widening of probability intervals. If other consolidation methods are considered, one can define analogous concepts, even in a different logical formalism.

In AI, inconsistency measures for probabilistic knowledge bases have been based on distance minimization, while in Formal Epistemology incoherence measures for Bayesian agents were focused on Dutch books vulnerability. The connections here established can help both communities to investigate their corresponding problems under a different angle.

The introduced concepts of innocuous conditional and inescapable conflict might have practical use in measuring inconsistency only if their instances are recognizable in a reasonable time. Nothing was said here about the complexity of the computational task of finding innocuous conditionals and inescapable conflicts within a knowledge base, but they are clearly very hard problems. Thus, future work includes investigating these problems aiming at devising algorithms to solve them. It would also be interesting to propose concrete procedures to consolidate knowledge bases, as done in [35] for instance. To achieve that, one could rely on the same triplet: rationality postulates, efficiency of computation and meaningful interpretation. Another intended continuation of this work is to study principled ways of inferring probabilistic conclusions from inconsistent bases, using the ideas here presented. For instance, this could be done by defining the set of models of an inconsistent base as the set containing all models of each closest consistent base, construed as the consolidations corresponding to some inconsistency measure here studied.

Appendix A. Proofs of technical results

Proposition 3.6. *If $\mathcal{I}$ satisfies MIS-separability, then $\mathcal{I}$ satisfies independence.*

Proof. Let Γ be a knowledge base and $\alpha \in \Gamma$ a free conditional. By MIS-separability, as α is free and all MISes of Γ are in $\Gamma \setminus \{\alpha\}$, we have $\mathcal{I}(\Gamma) = \mathcal{I}(\Gamma \setminus \{\alpha\}) + \mathcal{I}(\alpha)$. □

Corollary 3.9. *There is no inconsistency measure $\mathcal{I} : \mathbb{K}_{prec} \to [0, \infty)$ that satisfies consistency, MIS-separability and continuity.*

Proof. It follows directly from Theorem 3.8 and Proposition 3.6. □

Theorem 4.2. *Consider a knowledge base $\Gamma \in \mathbb{K}$ and a probabilistic conditional $\alpha \in \Gamma$. The following statements are equivalent:*

1. *There is no minimal abrupt repair Δ of Γ such that $\alpha \in \Delta$.*
2. *For all maximal abrupt consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
3. *If $\Gamma' = \Gamma \setminus \Delta$ is an abrupt consolidation of Γ (equivalently, Δ is an abrupt repair of Γ), then α is consistent with Γ'.*
4. *There is no minimal inconsistent set $\Delta \subseteq \Gamma$ such that $\alpha \in \Delta$.*

Proof. The first two items are clearly dual, and the fourth one is the definition of free conditional. Suppose α is free in Γ. Note that all abrupt consolidations Γ' of Γ are consistent with α. As Γ' is consistent, it has no MIS, and adding α cannot create a MIS, for it is free. Thus, if an abrupt consolidation does not contain α, it is not maximal. Now suppose there is a maximal abrupt consolidation Γ' such that $\alpha \notin \Gamma'$. For Γ' is maximal, α cannot be consistent with it. As Γ' is consistent, it has no MIS, and adding α creates a MIS (that contains α), which also is a MIS of Γ — hence, α cannot be free. □

Lemma 4.5. *Consider a knowledge base $\Gamma \in \mathcal{K}$ and a probabilistic conditional $\alpha \in \Gamma$. The following statements are equivalent:*

1. *For all d-consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
2. *If Γ' is a consolidation, then α is consistent with Γ'.*

Proof. Suppose all d-consolidations of Γ contain α. For any consolidation Ψ, there is a d-consolidation Ψ' such that, for each $\beta' \in \Psi'$, there is a $\beta \in \Psi$ such that $\beta' \subseteq \beta$. Therefore, any probability mass π satisfying Ψ' must also satisfies Ψ, and $\alpha \in \Psi'$ implies π satisfies α as well. Now suppose there is a d-consolidation Ψ that does not contain α. As $\alpha \in \Gamma$, there is a $\beta \in \Psi$ such that $\alpha \subsetneq \beta$. For Ψ is dominant, $(\Psi \setminus \{\beta\}) \cup \{\alpha\}$ cannot be a consolidation and thus is inconsistent. Finally, α is not consistent with Ψ. □

Proposition 4.8. *Consider a probabilistic conditional $\alpha \in \Gamma$. If α is safe, it is innocuous; if α is innocuous, it is free.*

Proof. If $\Gamma = \{\alpha\}$, then α is safe, innocuous and free iff it is satisfiable, thus we focus on $\Gamma \neq \{\alpha\}$. Let Γ be built over the set of atoms $X_n = \{x_1, \ldots, x_n\}$. Suppose α is safe and, without loss of generality, the set of atoms appearing in α is $X_\alpha = \{x_1, \ldots, x_m\}$, for some $m < n$. As α is satisfiable, there is a probability mass $\pi_\alpha : W_{X_\alpha} \to [0, 1]$ satisfying it, where W_{X_α} is the set containing the 2^m possible worlds with atoms from X_α. The base $\Gamma' = \Gamma \setminus \{\alpha\}$ is built over the set of atoms $X_{\Gamma'} = X_n \setminus X_\alpha$. Any consolidation Ψ of Γ' must also be formed by atoms in $X_{\Gamma'}$. If Δ is a consolidation of Γ, there is a consolidation Ψ of Γ' such that $\Delta = \Psi \cup \{\beta\}$, for some β such that $\alpha \subseteq \beta$. Let $\pi_\Psi : W_{X_{\Gamma'}} \to [0,1]$ be the probability mass satisfying Ψ, where $W_{X_{\Gamma'}}$ is the set containing the 2^{n-m} possible worlds with atoms from $X_{\Gamma'}$. Consider the probability mass $\pi : W_{X_n} \to [0,1]$ such that $\pi(w_i \wedge w_j) = \pi_\alpha(w_i) \times \pi_\Psi(w_j)$ for any pair $(w_i, w_j) \in W_{X_\alpha} \times W_{X_{\Gamma'}}$. Note that π satisfies Ψ and α, thus π satisfies Ψ and β. Therefore, α is consistent with any consolidation $\Delta = \Psi \cup \{\beta\}$ of Γ and is innocuous by Lemma 4.5.

Now suppose α is innocuous. Any abrupt consolidation $\Delta \subseteq \Gamma$ is equivalent (and equisatisfiable) to a consolidation $\Delta' \in \Gamma$ such that $\Delta' = \Delta \cup \{(\varphi|\psi)[0,1] | (\varphi|\psi)[q,\bar{q}] \in \Gamma \setminus \Delta\}$. As α is innocuous, it is consistent with any consolidation Δ' and, consequently, any abrupt consolidation Δ. Finally, by Theorem 4.2, α is free. $\square$

Corollary 4.10. *If $\mathcal{I}$ satisfies independence, then $\mathcal{I}$ satisfies i-independence. If $\mathcal{I}$ satisfies i-independence, then $\mathcal{I}$ satisfies weak independence.*

Proof. It follows directly from the definitions and Proposition 4.8. $\square$

Proposition 4.11. *A knowledge base Γ is a minimal inconsistent set iff Γ is inconsistent and there are no $\Delta_1, \dots, \Delta_k \subsetneq \Gamma$, with $k \geq 1$, such that:*

1. $\bigcup_{i=1}^{k} \Delta_i = \Gamma$;
2. *For every $\Gamma' \subseteq \Gamma$ if $\Gamma' \cap \Delta_i$ is an abrupt consolidation of Δ_i for all $1 \leq i \leq k$, then Γ' is an abrupt consolidation of Γ.*

Proof. ($\rightarrow$) Suppose Γ is a MIS and there are $\Delta_1, \dots, \Delta_k \subsetneq \Gamma$ satisfying both items. For any $1 \leq i \leq k$, as $\Delta_i \subsetneq \Gamma$ is consistent, $\Gamma \cap \Delta_i$ is an abrupt consolidation of Δ_i. Thus, by the second item, Γ is an abrupt consolidation of itself, which contradicts the fact that Γ is inconsistent.

($\leftarrow$) Now suppose Γ is inconsistent but not a MIS. Let $MIS(\Gamma) = \{\Delta_1, \dots, \Delta_m\}$ be the set of MISes in Γ, for some $m \geq 1$. Let Δ_{m+1} denote the set of free formulas in Γ. Clearly, $\bigcup_{i=1}^{m+1} \Delta_i = \Gamma$. Now consider a set $\Gamma' \subseteq \Gamma$ such that $\Gamma' \cap \Delta_i$ is consistent for any $1 \leq i \leq m+1$. If Γ' was inconsistent, it would contain a MIS $\Delta_i \in MIS(\Gamma)$ and $\Gamma' \cap \Delta_i$ would be inconsistent — a contradiction. Thus Γ' is an abrupt consolidation of Γ. $\square$

Lemma 4.13. *A knowledge base Γ is an inescapable conflict iff there is a widening Γ' of Γ such that Γ' is a minimal inconsistent set.*

Proof. ($\leftarrow$) Consider a minimal inconsistent set Γ' that is a widening of Γ. To prove by contradiction, suppose Γ is not an inescapable conflict. As its widening Γ' is inconsistent, Γ also is, for each conditional in Γ' is implied by a conditional in Γ. Hence, as Γ is not an inescapable conflict, there must be $\Delta_1, \dots, \Delta_k \subsetneq \Gamma$ such that $\bigcup_{i=1}^{k} \Delta_i = \Gamma$ and, if Δ_i' is a consolidation of Δ_i, for all $1 \leq i \leq k$, and $\bigcup_{i=1}^{k} \Delta_i'$ is a widening of Γ, then $\bigcup_{i=1}^{k} \Delta_i'$ is a consolidation of Γ. Consider such collection $\Delta_1, \dots, \Delta_k \subsetneq \Gamma$. Note that, for each Δ_i, there is a widening $\Psi_i \subsetneq \Gamma'$, defined via $\Psi_i = \{\beta \in \Gamma' | \alpha \in \Delta_i \text{ and } \alpha \subseteq \beta\}$, for $1 \leq i \leq k$. Furthermore any $\Psi_i \subsetneq \Gamma'$ is consistent, for Γ' is a minimal inconsistent set. As $\bigcup_{i=1}^{k} \Psi_i$ is equal to Γ', it is a widening of Γ. As each Ψ_i is consistent, Γ' must be a consolidation of Γ and, thus, Γ' is consistent. This is a contradiction, which proves that Γ is an inescapable conflict.

($\rightarrow$) Let $\Gamma = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ be an inescapable conflict and define $\Delta_i = \Gamma \setminus \{\alpha_i\}$, for $1 \leq i \leq m$. Note that $\Delta_1, \dots, \Delta_m \subsetneq \Gamma$ such that $\bigcup_{i=1}^{m} \Delta_i = \Gamma$. Let Δ_i' denote an arbitrary consolidation of Δ_i, for $1 \leq i \leq m$. If every set $\{\Delta_1', \Delta_2', \dots, \Delta_m' | \bigcup_{i=1}^{m} \Delta_i'$ is a widening of $|\Gamma\}$ is such that $\bigcup_{i=1}^{m} \Delta_i'$ is a consolidation of Γ, Γ would not be an inescapable conflict. So, there are consolidations Δ_i' for each Δ_i $(' \leq i \leq m)$ such that $\bigcup_{i=1}^{m} \Delta_i' = \Gamma'$ is widening of Γ but is not consistent. As Γ' is a widening of Γ, $\Gamma' = \{\alpha_1', \alpha_2', \dots, \alpha_m'\}$ for some $\alpha_i \subseteq \alpha_i'$ for $1 \leq i \leq m$. As $\Delta_i = \Gamma \setminus \{\alpha_i\}$, $\Delta_i' = \Gamma' \setminus \{\alpha_i'\}$, for $1 \leq i \leq m$. Hence, $\Delta_1', \dots, \Delta_m'$ are the maximal proper subsets of Γ, and every proper subset of Γ' is consistent. Thus, Γ' is a minimal inconsistent set. $\square$

Corollary 4.14. *Consider two knowledge bases $\Gamma, \Gamma' \in \mathbb{K}$ such that Γ' is a widening of Γ. If for every inescapable conflict $\Delta \subseteq \Gamma$ its widening $\{\beta \in \Gamma' \,|\, \alpha \in \Delta$ and $\alpha \subseteq \beta\}$ is consistent, then Γ' is a consolidation of Γ.*

Proof. We will prove via the contrapositive: given Γ and its widening Γ', if Γ' is not a consolidation of Γ, then there is an inescapable conflict $\Delta \subseteq \Gamma$ such that the set $\{\beta \in \Gamma' \,|\, \alpha \in \Delta \text{ and } \alpha \subseteq \beta\}$ is inconsistent.

If Γ' is not a consolidation of Γ, Γ' is inconsistent and must contain at least one minimal inconsistent set, that we denote by Δ'. Let Δ be the set $\{\alpha \in \Gamma | \beta \in \Delta' \text{ and } \alpha \subseteq \beta\}$ — that is, $\Delta' \subseteq \Gamma'$ is a widening of $\Delta \subseteq \Gamma$. By Lemma 4.13, Δ is an inescapable conflict. $\square$

Corollary 4.15. *If Δ is a minimal inconsistent set, then Δ is an inescapable conflict.*

Proof. Just note that Δ is a widening of itself. By Lemma 4.13, Δ is an inescapable conflict. $\square$

Corollary 4.18. *If $\mathcal{I}$ satisfies MIS-separability, then $\mathcal{I}$ satisfies IC-separability.*

Proof. If follows directly from the definitions and Corollary 4.15. $\square$

Theorem 4.19. *The following statements are equivalent:*

1. *For all d-consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
2. *If Γ' is a consolidation of Γ, then α is consistent with Γ'.*
3. *There is no inescapable conflict Δ in Γ such that $\alpha \in \Delta$.*
4. *α is an innocuous probabilistic conditional in Γ.*

Proof. By the definition of innocuous conditionals and Lemma 4.5, the first, the second and the fourth statements are equivalent. It remains to prove that α is innocuous iff there is no inescapable conflict Δ in Γ such that $\alpha \in \Delta$.

($\rightarrow$) Let α be innocuous in Γ. Suppose there is an inescapable conflict $\Delta \subseteq \Gamma$ such that $\alpha \in \Delta$. Consider the base $\Psi = \Delta \setminus \{\alpha\}$. Let Ψ' be a consolidation of Ψ. Thus, $\Gamma' = \Psi' \cup \{(\varphi|\psi)[0,1] | (\varphi|\psi)[\underline{q}, \bar{q}] \in \Gamma \setminus \Psi\}$ is consistent and it is a consolidation of Γ. Due to the fact that α is innocuous, α is consistent with Γ' (by Lemma 4.5) and, therefore, with Ψ'. Consequently, $\Psi' \cup \{\alpha\}$ is a consolidation of Δ for any consolidation Ψ' of Ψ. Furthermore, if $\{\beta\}$ is a consolidation of $\{\alpha\}$ (*i.e.*, $\alpha \subseteq \beta$), $\Psi' \cup \{\beta\}$ is a consolidation of Δ. As $\Psi, \{\alpha\} \subsetneq \Delta$ are such that $\Psi \cup \{\alpha\} = \Delta$, and any consolidations Ψ' and $\{\beta\}$ of theirs are such that $\Psi' \cup \{\beta\}$ is a consolidation of Δ, Δ is not an inescapable conflict, which is a contradiction.

($\leftarrow$) Suppose there is no inescapable conflict Δ in Γ such that $\alpha \in \Delta$. Consider the base $\Psi = \Gamma \setminus \{\alpha\}$. Every consolidation Γ' of Γ can be written as $\Gamma' = \Psi' \cup \{\beta\}$, where Ψ' is a consolidation of Ψ and $\alpha \subseteq \beta$. As all inescapable conflicts of Γ are in Ψ, by Corollary 4.14, $\Psi' \cup \{\alpha\}$ is consistent. Hence, α is consistent with any consolidation $\Gamma' = \Psi' \cup \{\beta\}$ and α is innocuous by Lemma 4.5. $\square$

Corollary 4.20. *If $\mathcal{I}$ satisfies IC-separability, then $\mathcal{I}$ satisfies i-independence.*

Proof. Let Γ be a knowledge base and $\alpha \in \Gamma$ an innocuous conditional. As α is innocuous, all inescapable conflicts of Γ are in $\Gamma \setminus \{\alpha\}$ by Lemma 4.19. By IC-separability, we have $\mathcal{I}(\Gamma) = \mathcal{I}(\Gamma \setminus \{\alpha\}) + \mathcal{I}(\alpha)$. $\square$

Theorem 4.27. *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}_p$ is well-defined and satisfies the postulates of consistency, continuity, i-independence and monotonicity.*

Proof. To show that $\mathcal{I}_p$ is well-defined, we use results from the proof of Theorem 1 in [44]. For any $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$, Thimm shows that the set $Q_\Gamma = \{\langle q_1, \dots, q_m \rangle \in \mathbb{R}^m | \Lambda_\Gamma(\langle q_1, q_1, \dots, q_m, q_m \rangle)$ is consistent$\}$ is compact and closed, where $\Lambda_\Gamma : [0,1]^{2m} \to \mathbb{K}$ is the characteristic function of Γ. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $h(a,b) = \max(0, a-b)$ for any $a, b \in \mathbb{R}$. The measure $\mathcal{I}_p$ is the minimum of $\| f_{\underline{q},\bar{q}}(q) \|_p$ with $q \in Q_\Gamma$, where $f_{\underline{q},\bar{q}} : \mathbb{R}^m \to \mathbb{R}^{2m}$ is a function such that $f_{\underline{q},\bar{q}}(\langle q_1, \dots, q_m \rangle) = \langle h(\underline{q}_1 - q_1), h(q_1 - \bar{q}_1), \dots, h(\underline{q}_m - q_m), h(q_m - \bar{q}_m) \rangle$. Intuitively, $f_{\underline{q},\bar{q}}(q)$ measures, for each point q_i, how much the lower and the upper bounds have to change for we have $q_i \in [\underline{q}_i, \bar{q}_i]$. Finally, $\mathcal{I}_p$ is well defined, for Q_Γ is closed and compact [44].

Consistency: By definition, a p-norm is never negative, thus $\mathcal{I}_p(\Gamma) \geq 0$. Suppose $\Gamma = \Lambda_\Gamma(q)$ is consistent. A vector $q' = q$ is such that $\|q' - q\|_p = 0$ for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, thus $\mathcal{I}_p(\Gamma) = 0$. Now suppose $\Gamma = \Lambda_\Gamma(q)$ is inconsistent. For every $q' \in Q_\Gamma$, $q' \neq q$, then $\|q' - q\|_p > 0$ and $\mathcal{I}_p(\Gamma) > 0$ for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$.

Continuity: Given a base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$, its characteristic function $\Lambda_\Gamma : [0,1]^{2m} \to \mathbb{K}$ and a fixed $q \in Q_\Gamma$, define the function $g_q : \mathbb{R}^{2m} \to \mathbb{R}$ such that $g_q(\langle \underline{q}_1, \bar{q}_1, \dots, \underline{q}_m, \bar{q}_m \rangle) = \| f_{\underline{q},\bar{q}}(q) \|_p$. Note that $\mathcal{I}_p \circ \Lambda_\Gamma(\langle \underline{q}_1, \bar{q}_1, \dots, \underline{q}_m, \bar{q}_m \rangle)$ is computed as the minimum of $\{g_q(\langle \underline{q}_1, \bar{q}_1, \dots, \underline{q}_m, \bar{q}_m \rangle) | q \in Q_\Gamma\}$. Each g_q is continuous, and the minimum of continuous functions is continuous, hence $\mathcal{I}_p \circ \Lambda_\Gamma$ is continuous.

Monotonicity: Let $\Lambda_\Gamma(q')$ be a consolidation of $\Gamma = \Lambda_\Gamma(q)$ such that $\|q' - q\|_p$ is minimized, for a $p \in \mathbb{N}_{>0} \cup \{\infty\}$, and $\mathcal{I}_p(\Gamma) = \|q' - q\|_p$. To prove by contradiction, suppose $\mathcal{I}(\Gamma \cup \{\alpha\}) < \mathcal{I}_p(\Gamma)$, for some $\Psi = \Gamma \cup \{\alpha\} \in \mathbb{K}$. Hence, there is a consolidation $\Psi' = \Lambda_\Psi(r')$ of $\Psi = \Lambda_\Psi(r)$ such that $\|r' - r\|_p < \|q' - q\|_p$. Consider the base $\Gamma' = \Psi' \setminus \{\beta\}$, such that $\alpha \subseteq \beta$. As Ψ' is consistent, $\Gamma' = \Lambda_\Gamma(q'')$ also is, and it is a consolidation of Γ. Since q and q'' are projections (subsets, in sense) of r and r', $q'' - q$ is a projection of $r' - r$ and $\|q'' - q\|_p \leq \|r' - r\|_p < \|q' - q\|_p$. Finally, it would follow that $\mathcal{I}_p(\Gamma) \leq \|q'' - q\|_p < \|q' - q\|_p = \mathcal{I}_p(\Gamma)$, which is a contradiction.

***i*-Independence:** Consider the bases $\Gamma = \Lambda_\Gamma(r)$ and $\Psi = \Gamma \setminus \{\alpha\}$ in $\mathbb{K}$, where $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$ is innocuous in Γ. We are going to prove that $\mathcal{I}_p(\Gamma) \leq \mathcal{I}_p(\Psi)$, and the desired result follows from monotonicity. Let $\Psi' = \Lambda_\Psi(q')$ be a consolidation of $\Psi = \Lambda_\Psi(q)$ such that $\|q' - q\|_p$ is minimized, for a $p \in \mathbb{N}_{>0} \cup \{\infty\}$, and $\mathcal{I}_p(\Psi) = \|q' - q\|_p$. Note that $\Gamma' = \Psi' \cup \{(\varphi|\psi)[0,1]\}$ is a consolidation of Γ. As $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$ is innocuous, α is consistent with Γ' and Ψ'. Hence, $\Psi' \cup \{\alpha\} = \Lambda_\Gamma(r')$ is a consolidation of Γ. Note that $r' - r$ is $q' - q$ with two extra 0's (from alpha). Finally, $\mathcal{I}_p(\Gamma) \leq \|r' - r\|_p = \|q' - q\|_p = \mathcal{I}_p(\Psi)$. $\square$

Lemma 4.28. *$\mathcal{I}_p$ satisfies super-additivity and IC-separability iff $p = 1$.*

Proof. ($\rightarrow$) To note that super-additivity and IC-separability do not hold if $p > 1$, consider the bases $\Psi = \{(\top)[0.9]\}$, $\Delta = \{(\bot)[0.1]\}$, $\Gamma = \Psi \cup \Delta$. By the definition of d-consolidation, if $\mathcal{I}_p(\Gamma) = d$, then there is d-consolidation $\Lambda_\Gamma(q')$ of $\Gamma = \Lambda_\Gamma(q)$ such that $\|q' - q\|_p = d$. The only d-consolidations of Ψ, Δ, Γ are $\Psi' = \{(\top)[0.9, 1]\}$, $\Delta' = \{(\bot)[0, 0.1]\}$, $\Gamma' = \Psi' \cup \Delta'$, for changing the lower bound in Ψ and the upper bound in Δ is useless to reach consistency. For any finite p, $\mathcal{I}_p(\Psi) = \mathcal{I}_p(\Delta) = \sqrt[p]{0.1^p} = 0.1$, and $\mathcal{I}_p(\Gamma) = \sqrt[p]{0.1^p + 0.1^p} = 0.1\sqrt[p]{2}$. For $p = \infty$, $\mathcal{I}_p(\Psi) = \mathcal{I}_p(\Delta) = \max\langle 0.1\rangle = 0.1$ and $\mathcal{I}_p(\Gamma) = \max\langle 0.1, 0.1\rangle = 0.1$. Therefore, for any $p > 1 \in \mathbb{N} \cup \{\infty\}$, $\mathcal{I}_p(\Gamma) < 0.2 = \mathcal{I}_p(\Psi) + \mathcal{I}_p(\Delta)$, and both super-additivity and IC-separability fail.

($\leftarrow$) Now fix $p = 1$. To prove that super-additivity holds, suppose there are bases $\Psi, \Delta, \Gamma = \Psi \cup \Delta$ in $\mathbb{K}$ such that $\Psi \cap \Delta = \varnothing$. Let $\Psi' = \Lambda_\Psi(q')$, $\Delta' = \Lambda_\Delta(r')$, $\Gamma' = \Lambda_\Gamma(s')$ be d-consolidations of $\Psi = \Lambda_\Psi(q)$, $\Delta = \Lambda_\Delta(r)$, $\Gamma = \Lambda_\Gamma(s)$ that minimize $\|q' - q\|_1, \|r' - r\|_1, \|s' - s\|_1$, corresponding to $\mathcal{I}_1(\Psi), \mathcal{I}_1(\Delta), \mathcal{I}_1(\Gamma)$. Clearly, Γ' can be partitioned into $\Psi'' \cup \Delta''$ in such a way that $\Psi'' = \Lambda_\Psi(s'_\Psi)$, $\Delta'' = \Lambda_\Delta(s'_\Delta)$ are consolidations of Ψ, Δ. By the construction of s'_Ψ and s'_Δ, $\|s' - s\|_1 = \|s'_\Psi - q\|_1 + \|s'_\Delta - r\|_1$. Hence, for $\mathcal{I}_1(\Psi) \le \|s'_\Psi - q\|_1$ and $\mathcal{I}_1(\Delta) \le \|s'_\Delta - r\|_1$, it follows that $\mathcal{I}(\Gamma) = \|s' - s\|_1 \ge \mathcal{I}_1(\Psi) + \mathcal{I}_1(\Delta)$.

To prove that IC-separability holds, suppose there are bases $\Psi, \Delta, \Gamma = \Psi \cup \Delta$ in $\mathbb{K}$ such that $\Psi \cap \Delta = \varnothing$, $IC(\Gamma) = IC(\Psi) \cup IC(\Delta)$. Let $\Psi' = \Lambda_\Psi(q')$, $\Delta' = \Lambda_\Delta(r')$, be consolidations of $\Psi = \Lambda_\Psi(q)$, $\Delta = \Lambda_\Delta(r)$ that minimize $\|q' - q\|_1, \|r' - r\|_1$, corresponding to $\mathcal{I}_1(\Psi), \mathcal{I}_1(\Delta)$. As $\Gamma' = \Psi' \cup \Delta' = \Lambda_\Gamma(s')$ is a widening of $\Gamma = \Lambda_\Gamma(s)$ such that, for each $\Phi \in IC(\Gamma) = IC(\Psi) \cup IC(\Delta)$, the base $\{\beta \in \Gamma' \,|\, \alpha \in \Phi \text{ and } \alpha \subseteq \beta\}$ is consistent (all inescapable conflicts are solved), Γ' is a consolidation of Γ by Corollary 4.14. As $\|s' - s\|_1 = \|q' - q\|_1 + \|r' - r\|_1 = \mathcal{I}_1(\Psi) + \mathcal{I}_1(\Delta)$, it follows that $\mathcal{I}_1(\Gamma) \le \mathcal{I}_1(\Psi) + \mathcal{I}_1(\Delta)$. By super-additivity, $\mathcal{I}_1(\Gamma) \ge \mathcal{I}_1(\Psi) + \mathcal{I}_1(\Delta)$, thus $\mathcal{I}_1(\Gamma) = \mathcal{I}_1(\Psi) + \mathcal{I}_1(\Delta)$. □

Lemma 4.29. *$\mathcal{I}_p$ satisfies normalization iff $p = \infty$.*

Proof. ($\rightarrow$) To note that normalization does not hold if p is finite, consider the base $\Gamma = \{(\top)[0], (\bot)[1]\}$. The only d-consolidation of Γ is $\Gamma' = \{(\top)[0, 1], (\bot)[0, 1]\}$, for changing the lower bound in $(\top)[0]$ and the upper bound in $(\bot)[1]$ is useless to reach consistency. For any finite p, $\mathcal{I}_p(\Gamma) = \sqrt[p]{1^p + 1^p} = \sqrt[p]{2} > 1$, and normalization fails.

($\leftarrow$) By definition, $\mathcal{I}_\infty(\Gamma)$ is the minimum of $\|q' - q\|_\infty$ subject to $\Gamma = \Lambda_\Gamma(q)$ and $\Lambda_\Gamma(q')$ being consistent. As the vectors q, q' are in $[0, 1]^{2|\Gamma|}$, $\|q' - q\|_\infty \in [0, 1]$, since $|q'_i - q_i| \in [0, 1]$ for all elements q_i, q'_i of q, q'. □

Proposition 5.1. *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}^\varepsilon_p : \mathbb{K} \to [0, \infty)$ is well-defined and satisfies consistency, continuity, weak independence and monotonicity. $\mathcal{I}^\varepsilon_1$ also satisfies super-additivity.*

Proof. See Section 4 in [34]. □

Theorem 5.2. *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}^\varepsilon_p : \mathbb{K} \to [0, \infty)$ is well-defined and satisfies consistency, continuity, i-independence and monotonicity. $\mathcal{I}^\varepsilon_1$ also satisfies super-additivity and IC-separability; and $\mathcal{I}^\varepsilon_\infty$ satisfies normalization.*

Proof. For well-definedness, consistency, i-independence, monotonicity, super-additivity and IC-separability, see the proof of Theorem 5.11. For continuity, see Lemma 5.12.

For normalization, we note that $\mathcal{I}^\varepsilon_\infty(\Gamma) = \mathcal{I}^{sum}_{SSK}(\Gamma)$ for any $\Gamma \in \mathbb{K}$, by Theorem 5.7. When we are computing $\mathcal{I}^{sum}_{SSK}$, we limit the sum of the absolute values of the stakes to one. As the agent cannot lose more than the absolute value of the stake in each gamble in a Dutch book, and they sum up to one, $\mathcal{I}^{sum}_{SSK} = \mathcal{I}^\varepsilon_\infty$ satisfy normalization. □

Theorem 5.11. *$\mathcal{I}^{a,sum}_{SSK}$, $\mathcal{I}^{b,sum}_{SSK}$, $\mathcal{I}^{a,max}_{SSK}$ and $\mathcal{I}^{b,max}_{SSK}$ are well-defined and satisfy consistency, i-independence and monotonicity. $\mathcal{I}^{a,max}_{SSK}$ and $\mathcal{I}^{b,max}_{SSK}$ also satisfy super-additivity and IC-separability.*

Proof. Let $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \le i \le m\}$ be an arbitrary knowledge base in $\mathbb{K}$ and $\underline{\gamma}_i, \bar{\gamma}_i$ be non-negative real parameters, for $1 \le i \le m$. Consider the following program, with $p \in \mathbb{N}_{>0} \cup \{\infty\}$, where $\underline{\varepsilon}_\gamma$ ($\bar{\varepsilon}_\gamma$) is a $(m \times 1)$-vector whose elements are $\underline{\varepsilon}_i \underline{\gamma}_i$ ($\bar{\varepsilon}_i \bar{\gamma}_i$), for $1 \le i \le m$:

$$\min \|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_p \text{ subject to:} \tag{A.1}$$

$$A\pi \ge -\underline{\varepsilon}_\gamma \tag{A.2}$$

$$B\pi \le \bar{\varepsilon}_\gamma \tag{A.3}$$

$$\sum \pi = 1 \tag{A.4}$$

$$\pi, \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \ge 0\,. \tag{A.5}$$

Define the inconsistency measure $\mathcal{I}^\gamma_p : \mathbb{K} \to [0, \infty) \cup \infty$ in such a way that $\mathcal{I}^\gamma_p(\Gamma)$ is the minimum of the objective function of the program above; or ∞ if it is infeasible. If $\underline{\gamma}_i = \bar{\gamma}_i = 1$ for all $1 \le i \le m$, $\mathcal{I}^\gamma_p(\Gamma) = \mathcal{I}^\varepsilon_p(\Gamma)$, for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$.

When $p=1$, if $\underline{\gamma}_i=\underline{q}_i$ and $\bar{\gamma}_i=1-\bar{q}_i$ for $1\le i\le m$, the program above is the dual of that formed by adding the constraints $\underline{q}_i\underline{\lambda}_i\le 1$ and $(1-\bar{q}_i)\bar{\lambda}_i\le 1$, for $1\le i\le m$, into the program (19)–(21). That is, $\mathcal{I}_1^{\gamma}(\Gamma)=\mathcal{I}_{SSK}^{a,max}(\Gamma)$. Analogously, if $\underline{\gamma}_i=1-\underline{q}_i$ and $\bar{\gamma}_i=\bar{q}_i$ for $1\le i\le m$, $\mathcal{I}_1^{\gamma}(\Gamma)=\mathcal{I}_{SSK}^{g,max}(\Gamma)$.

When $p=\infty$, in all restrictions we can replace $\underline{\varepsilon}_i,\bar{\varepsilon}_i$ by a single scalar ε, as it was done in (14)–(18), creating an equivalent new linear program $\mathcal{P}$ that minimizes ε, computing $\mathcal{I}_{\infty}^{\gamma}(\Gamma)$. If $\underline{\gamma}_i=\underline{q}_i$ and $\bar{\gamma}_i=1-\bar{q}_i$ for $1\le i\le m$, the program $\mathcal{P}$ is the dual of that formed by adding the constraints $\sum_{i=1}^{m}\underline{q}_i\underline{\lambda}_i+(1-\bar{q}_i)\bar{\lambda}_i\le 1$ into (19)–(21). That is, P computes $\mathcal{I}_{\infty}^{\gamma}(\Gamma)=\mathcal{I}_{SSK}^{a,sum}(\Gamma)$. Analogously, if $\underline{\gamma}_i=1-\underline{q}_i$ and $\bar{\delta}_i=\bar{q}_i$ for $1\le i\le m$, P computes $\mathcal{I}_{\infty}^{\gamma}(\Gamma)=\mathcal{I}_{SSK}^{g,sum}(\Gamma)$.

Note that the linear restrictions in the program (A.1)–(A.5), when it is feasible, define a convex, closed region of feasible points (a simplex). The p-norm is a continuous function, so the minimum of the objective function in (A.1) is well-defined for any $p\in\mathbb{N}_{>0}\cup\{\infty\}$. If the program (A.1)–(A.5) is infeasible for some $\Gamma\in\mathbb{K}$, $\mathcal{I}_p^{\gamma}(\Gamma)$ is (well-)defined as ∞.

Consistency: Note that a p-norm is never negative. The base Γ is consistent iff the program (5)–(8) is feasible; and such program is feasible iff the program (A.1)–(A.5) has a feasible solution with $\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle=\langle 0,0,\dots,0\rangle$; which is the case iff $\|\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle\|_p=0$ is the minimum of the objective function in (A.1).

Monotonicity: Consider the program $\mathcal{P}$ from lines (A.1)–(A.5), corresponding to the computation of $\mathcal{I}_p^{\gamma}(\Gamma)$, for some $\Gamma\in\mathbb{K}$. Let $\Psi=\Gamma\cup\{\alpha\}$ be a knowledge base. For any $p\in\mathbb{N}_{>0}\cup\{\infty\}$ and parameters $\underline{\gamma}_1,\bar{\gamma}_1,\dots,\underline{\gamma}_m,\bar{\gamma}_m\ge 0$, the program (A.1)–(A.5) whose solution gives $\mathcal{I}_p^{\gamma}(\Psi)$ has two extra constraints in comparison with $\mathcal{P}$. Thus, the program that computes $\mathcal{I}_p^{\gamma}(\Psi)$ cannot reach a smaller value for $\|\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle\|_p$, the objective function being minimized by $\mathcal{P}$. Furthermore, $\|\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_{m+1},\bar{\varepsilon}_{m+1}\rangle\|_p\ge\|\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle\|_p$, for any $p\in\mathbb{N}_{>0}\cup\{\infty\}$ and parameters $\underline{\gamma}_1,\bar{\gamma}_1,\dots,\underline{\gamma}_m,\bar{\gamma}_m\ge 0$. Hence, $\mathcal{I}_p^{\gamma}(\Gamma\cup\{\alpha\})\ge\mathcal{I}_p^{\gamma}(\Gamma)$, for any $p\in\mathbb{N}_{>0}\cup\{\infty\}$.

***i*-independence:** Let $\Gamma=\{(\varphi_i|\psi_i)[\underline{q}_i,\bar{q}_i]|1\le i\le m\}$ be a knowledge base in $\mathbb{K}$ and $\alpha=(\varphi_m|\psi_m)[\underline{q}_m,\bar{q}_m]$ be an innocuous conditional in Γ, and define $\Psi=\Gamma\setminus\{\alpha\}$. Suppose $\mathcal{I}_p^{\gamma}(\Psi)$ is finite. The solution on $\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_{m-1},\bar{\varepsilon}_{m-1}\rangle$ to the program (A.1)–(A.5) that computes $\mathcal{I}_p^{\gamma}(\Psi)$ corresponds to a consolidation of Ψ given by $\Psi'=\{(\varphi_i|\psi_i)[\underline{q}_i-\underline{\gamma}_i\underline{\varepsilon}_i,\bar{q}_i+\bar{\gamma}_i\bar{\varepsilon}_i]|1\le i\le m-1\}$. For α is innocuous in Γ, it is consistent with $\Psi'\cup(\varphi_m|\psi_m)[0,1]$ (a consolidation of Γ) and $\Psi'\cup\{\alpha\}$ is a consolidation of Γ. Hence, $\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_{m-1},\bar{\varepsilon}_{m-1},0,0\rangle$ corresponds to a feasible solution to the program (A.1)–(A.5) computing $\mathcal{I}_p^{\gamma}(\Gamma)$. As $\|\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_{m-1},\bar{\varepsilon}_{m-1}\rangle\|_p$ is equal to $\|\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_{m-1},\bar{\varepsilon}_{m-1},0,0\rangle\|_p$ for any $p\in\mathbb{N}_{>0}\cup\{\infty\}$, $\mathcal{I}_p^{\gamma}(\Gamma)\le\mathcal{I}_p^{\gamma}(\Psi)$. By monotonicity, $\mathcal{I}_p^{\gamma}(\Gamma)=\mathcal{I}_p^{\gamma}(\Psi)$.

Now suppose $\mathcal{I}_p^{\gamma}(\Psi)$ is infinite. Thus, the program (A.1)–(A.5) that computes $\mathcal{I}_p^{\gamma}(\Psi)$ is infeasible. Constraints in such program are inherited by the program that computes $\mathcal{I}_p^{\gamma}(\Gamma)=\mathcal{I}_p^{\gamma}(\Psi\cup\{\alpha\})$ together with the infeasibility, hence $\mathcal{I}_p^{\gamma}(\Gamma)=\infty$ by definition.

Super-additivity: Suppose there are bases $\Psi,\Delta,\Gamma=\Psi\cup\Delta$ in $\mathbb{K}$ such that $\Psi\cap\Delta=\varnothing$. Without loss of generality, let $\Psi=\{(\varphi_i|\psi_i)[\underline{q}_i,\bar{q}_i]|1\le i\le k\}$, $\Delta=\{(\varphi_i|\psi_i)[\underline{q}_i,\bar{q}_i]|k+1\le i\le m\}$ and $\Gamma=\{(\varphi_i|\psi_i)[\underline{q}_i,\bar{q}_i]|1\le i\le m\}$. If $\mathcal{I}_1^{\gamma}(\Gamma)=\infty$, super-additivity trivially holds, then consider $\mathcal{I}_1^{\gamma}(\Gamma)$ is finite. Let $\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle$ be part of a solution (that includes π) to the program (A.1)–(A.5) that computes $\mathcal{I}_1^{\gamma}(\Gamma)$, minimizing the objective function. As $\Gamma'=\{(\varphi_i|\psi_i)[\underline{q}_i-\underline{\varepsilon}_i\underline{\gamma}_i,\bar{q}_i+\bar{\varepsilon}_i\bar{\gamma}_i]|1\le i\le m\}$ is consistent, so are $\Psi'=\{(\varphi_i|\psi_i)[\underline{q}_i-\underline{\varepsilon}_i\underline{\gamma}_i,\bar{q}_i+\bar{\varepsilon}_i\bar{\gamma}_i]|1\le i\le k\}$ and $\Delta'=\{(\varphi_i|\psi_i)[\underline{q}_i-\underline{\varepsilon}_i\underline{\gamma}_i,\bar{q}_i+\bar{\varepsilon}_i\bar{\gamma}_i]|k+1\le i\le m\}$, which are consolidations of Ψ and Δ. Thus, $\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_k,\bar{\varepsilon}_k\rangle$ and $\langle\underline{\varepsilon}_{k+1},\bar{\varepsilon}_{k+1},\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle$ correspond to feasible solutions to the programs that compute $\mathcal{I}_1^{\gamma}(\Psi)$ and $\mathcal{I}_1^{\gamma}(\Delta)$, respectively. It follows that $\mathcal{I}_1^{\gamma}(\Psi)\le\|\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_k,\bar{\varepsilon}_k\rangle\|_1$ and $\mathcal{I}_1^{\gamma}(\Delta)\le\|\langle\underline{\varepsilon}_{k+1},\bar{\varepsilon}_{k+1},\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle\|_1$. Finally, $\mathcal{I}_1^{\gamma}(\Delta)+\mathcal{I}_1^{\gamma}(\Psi)\le(\sum_{i=1}^{k}\underline{\varepsilon}_i+\bar{\varepsilon}_i)+(\sum_{i=k+1}^{m}\underline{\varepsilon}_i+\bar{\varepsilon}_i)=\sum_{i=1}^{m}\underline{\varepsilon}_i+\bar{\varepsilon}_i=\mathcal{I}_1^{\gamma}(\Gamma)$.

IC-separability: To prove that IC-separability holds, suppose there are bases $\Psi,\Delta,\Gamma=\Psi\cup\Delta$ in $\mathbb{K}$ such that $\Psi\cap\Delta=\varnothing$, $IC(\Gamma)=IC(\Psi)\cup IC(\Delta)$. Without loss of generality, let $\Psi=\{(\varphi_i|\psi_i)[\underline{q}_i,\bar{q}_i]|1\le i\le k\}$, $\Delta=\{(\varphi_i|\psi_i)[\underline{q}_i,\bar{q}_i]|k+1\le i\le m\}$ and $\Gamma=\{(\varphi_i|\psi_i)[\underline{q}_i,\bar{q}_i]|1\le i\le m\}$. If $\mathcal{I}_1^{\gamma}(\Psi)=\infty$ or $\mathcal{I}_1^{\gamma}(\Delta)=\infty$, then $\mathcal{I}_1^{\gamma}(\Gamma)=\infty$ by monotonicity, and IC-separability holds, considering that ∞ plus any non-negative number yields ∞; thus, we assume $\mathcal{I}_1^{\gamma}(\Psi),\mathcal{I}_1^{\gamma}(\Delta)<\infty$. Let $\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_k,\bar{\varepsilon}_k\rangle$ and $\langle\underline{\varepsilon}_{k+1},\bar{\varepsilon}_{k+1},\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle$ be solutions (on $\underline{\varepsilon},\bar{\varepsilon}$) to the programs in the form (A.1)–(A.5) that compute $\mathcal{I}_1^{\gamma}(\Psi)$ and $\mathcal{I}_1^{\gamma}(\Delta)$, respectively, minimizing their objective functions. As all inescapable conflicts of Γ are either in Ψ or in Δ, the union of consolidations of Ψ and Δ is a consolidation of Γ, by Corollary 4.14. Hence, $\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle$ correspond to a feasible solution to the program in the form (A.1)–(A.5) that computes $\mathcal{I}_1^{\gamma}(\Gamma)$ and $\mathcal{I}_1^{\gamma}(\Gamma)\le\|\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle\|_1=(\sum_{i=1}^{k}\underline{\varepsilon}_i+\bar{\varepsilon}_i)+(\sum_{i=k+1}^{m}\underline{\varepsilon}_i+\bar{\varepsilon}_i)=\mathcal{I}_1^{\gamma}(\Psi)+\mathcal{I}_1^{\gamma}(\Delta)$. By super-additivity, $\mathcal{I}_1^{\gamma}(\Gamma)=\mathcal{I}_1^{\gamma}(\Psi)+\mathcal{I}_1^{\gamma}(\Delta)$. □

Lemma 5.12. *$\mathcal{I}_{SSK}^{a,sum}$, $\mathcal{I}_{SSK}^{a,max}$, $\mathcal{I}_{SSK}^{b,sum}$ and $\mathcal{I}_{SSK}^{b,max}$ are continuous for probabilities within* $(0,1)$.

Proof. Consider the inconsistency measure $\mathcal{I}_p^{\gamma}$ defined in the proof of Theorem 5.11, the knowledge base $\Gamma=\{(\varphi_i|\psi_i)[\underline{q}'_i,\bar{q}'_i]|1\le i\le m\}$ and the vector $q=\langle\underline{q}_1,\bar{q}_1,\dots,\underline{q}_m,\bar{q}_m\rangle$. Note that, for any measure $\mathcal{I}_p^{\varepsilon}$, the parameters $\underline{\gamma}_1,\bar{\gamma}_1,\dots,\underline{\gamma}_m,\bar{\gamma}_m$ are positive (for $\mathcal{I}_{SSK}^{a,sum}$, $\mathcal{I}_{SSK}^{a,max}$, $\mathcal{I}_{SSK}^{b,sum}$ and $\mathcal{I}_{SSK}^{b,max}$, they are positive if $q\in(0,1)^{2m}$). When these parameters are positive, every probability mass $\pi:W_{X_n}\to[0,1]$ defines a vector $\varepsilon_{\pi}(q)=\langle\underline{\varepsilon}_1,\bar{\varepsilon}_1,\dots,\underline{\varepsilon}_m,\bar{\varepsilon}_m\rangle$ for each q in the following way: $\underline{\varepsilon}_i=-\min\{0,(1/\underline{\gamma}_i)(P_{\pi}(\varphi_i\wedge\psi_i)-\underline{q}_iP_{\pi}(\psi_i))\}$ and $\bar{\varepsilon}_i=\max\{0,(1/\bar{\gamma}_i)(P_{\pi}(\varphi_i\wedge\psi_i)-\underline{q}_iP_{\pi}(\psi_i))\}$

for every $1 \leq i \leq m$ and $q \in [0,1]^{2m}$ (or $q \in (0,1)^{2m}$). Note that the pair $\pi, \varepsilon_\pi(q)$ is a feasible solution to the program (A.1)–(A.5) that computes $\mathcal{I}_p^\gamma(\Lambda_\Gamma(q))$ for any $q \in [0,1]^{2m}$ (or $q \in (0,1)^{2m}$), since $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \geq -\underline{\varepsilon}_i \underline{\gamma}_i$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq \bar{\varepsilon}_i \bar{\gamma}_i$ for all $1 \leq i \leq m$. Thus, every π yields a value for the objective function $h_\pi(q) = \|\varepsilon_\pi(q)\|$ of the program (A.1)–(A.5), for any $q \in [0,1]^{2m}$ (or $q \in (0,1)^{2m}$). As $\underline{\gamma}_1 = \bar{\gamma}_1 = \cdots = \underline{\gamma}_m = \bar{\gamma}_m = 1$ for $\mathcal{I}_p^\varepsilon$ and any $q \in [0,1]^{2m}$, $\varepsilon_\pi(q)$ is continuous on $q \in [0,1]^{2m}$, and as any p-norm is a continuous function, $h_\pi : [0,1]^{2m} \to [0,\infty)$ also is for any π. (For $\mathcal{I}_{SSK}^{a,sum}$, $\mathcal{I}_{SSK}^{a,max}$, $\mathcal{I}_{SSK}^{b,sum}$ and $\mathcal{I}_{SSK}^{b,max}$, $q \in (0,1)^{2m}$ implies positive parameters $\underline{\gamma}_i, \bar{\gamma}_i$; furthermore, such parameters change continuously — linearly — with $q \in (0,1)^{2m}$. Thus, $h_\pi : (0,1)^{2m} \to [0,\infty)$ is continuous on $q \in (0,1)^{2m}$). To compute $\mathcal{I}_p^\gamma(\Lambda_\Gamma(q))$ for a particular q, one needs to take the minimum in π of $\{h_\pi(q) | \pi : W_{X_n} \to [0,1]$ is a probability mass$\}$. As the minimum of continuous functions is continuous, $\mathcal{I}_p^\gamma \circ \Lambda_\Gamma : [0,1]^{2m} \to [0,\infty) \cup \{\infty\}$ (or $\mathcal{I}_p^\gamma \circ \Lambda_\Gamma : (0,1)^{2m} \to [0,\infty) \cup \{\infty\}$) is continuous for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$. □

Lemma 5.13. *$\mathcal{I}_{SSK}^{a,sum}$ satisfy normalization.*

Proof. When we are computing $\mathcal{I}_{SSK}^{a,sum}$, the maximum sure loss when limiting the agent's total escrows to one. As the agent cannot lose more her total escrow in a Dutch book, $\mathcal{I}_{SSK}^{a,sum}$ is trivially normalized. □

References

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