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Qualitative reasoning with directional relations ☆

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ABSTRACT

Qualitative spatial reasoning (QSR) pursues a symbolic approach to reasoning about a spatial domain. Qualitative calculi are defined to capture domain properties in relation operations, granting a relation algebraic approach to reasoning. QSR has two primary goals: providing a symbolic model for human common-sense level of reasoning and providing efficient means for reasoning. In this paper, we dismantle the hope for efficient reasoning about directional information in infinite spatial domains by showing that it is inherently hard to decide consistency of a set of constraints that represents positions in the plane by specifying directions from reference objects. We assume that these reference objects are not fixed but only constrained through directional relations themselves. Known QSR reasoning methods fail to handle this information.

1. Introduction

Qualitative spatial reasoning (QSR) [1] is the subfield of knowledge representation and symbolic reasoning that represents knowledge about spatial domains by finite sets of named *qualitative relations*. One particular aim of qualitative approaches is to model human common-sense understanding of space. This makes qualitative approaches useful, for instance, in human–machine interaction. Qualitative reasoning is considered to provide efficient means for reasoning about continuous, infinite but structured domains such as space or time.

Qualitative relations state relationships of variables ranging over a spatial domain. Thus, consistency problems in qualitative spatial reasoning are closely related to constraint-based reasoning over mostly infinite domains and so QSR shares much of the terminology of constraint-based reasoning. One central task in QSR is to decide consistency of *qualitative constraint networks*, i.e., constraint networks in which only qualitative relations are used as constraints. In the following we refer to this problem as the *consistency problem*. Deciding consistency of qualitative constraint networks differs from classical constraint satisfaction problems (CSP) in that the infinite domain prevents exhaustive search. QSR techniques rely on the relation algebraic structure of *qualitative calculi* [2] that is captured in converse and composition tables. While reasoning in full qualitative calculi is mostly NP-complete, tractable sub-algebras have been identified for some calculi [3,4].

Directional calculi consist of a set of qualitative directional relations that coarsely specify the direction in which an object is positioned. Positions are considered to be points in the Euclidean plane and directions are given with respect to a frame of reference. Qualitative representations of directional information may involve a single, global frame of reference or they may employ different frames of reference that are determined by reference objects. In this paper we are concerned with directional relations that involve different reference objects, i.e., we are not concerned with cardinal directions that use a single frame of reference and for which reasoning is known to be tractable [5]. Two important examples for reference

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objects are directed lines (establishing directions "left of" or "right of" the line, for instance) or pairs of points to determine triangle orientations (see for example [6]). Directional calculi are important for handling knowledge that makes use of relative or egocentric frames of references. In particular, directional calculi draw their motivation from tasks in high-level agent control [7] or from interpreting natural language for robot instruction [8]. In this article we show that reasoning about directional relations is inherently intractable. By reducing the problem of *matroid realizability* to the consistency problem we show that reasoning with directional relations is NP-hard, NP membership being an open question. Our result has impact on reasoning with any qualitative calculus that is expressive enough to distinguish "left of" from "right of" which includes flip-flop [6,9], double cross [10,11], dipole [12], $\mathcal{OPRA}$ [13], TPCC [14]. For all such calculi, the existing relation algebraic approach is too weak for deciding consistency problems and all reasonable sub-algebras remain NP-hard.

This paper is organized as follows. First we give basis definitions of qualitative reasoning and discuss related work. In Section 3 we explain the principle steps of our proof. After formally introducing oriented matroids (Section 4) we give in Section 5 new intractability results for several directional calculi. In Section 6 we sketch a new approach to deciding consistency in directional calculi. We conclude by discussion and outlook.

2. Qualitative constraint-based reasoning

The basic concept of qualitative spatial reasoning is the *qualitative calculus* [2] which comprises a set of *qualitative relations* and relation algebraic operations that for many calculi meet conditions for a relation algebra in the sense of Tarski. For the context of this paper, only the relations are important.

Definition 1 *(Qualitative relation).* Let D be a non-empty set called *domain* and let $B = \{r_1, r_2, \ldots, r_n\}$ be a set of k-ary relations over D. B is called the set of *base relations* and the set of all unions of base relations $R = \{\bigcup_{r \in b} \mid b \in 2^B\}$ is called the set of *qualitative relations*. Commonly, a qualitative relation $r_i \cup r_j$ is denoted $\{r_i, r_j\}$.

Qualitative relations express the relationship of variables ranging over the domain by base relations or disjunctions thereof.

Definition 2 *(QCSP).* Let $\mathcal{R} = \{r_1, r_2, \ldots, r_n\}$ be a set of k-ary qualitative relations over domain D and let $\mathcal{X}$ be a set of variables ranging over D. A *qualitative constraint* is a formula $X_1 \ldots X_{k-1} r_i X_k$ with variables $X_j \in \mathcal{X}$. For a valuation $\phi : \mathcal{X} \to D$ we say that a qualitative constraint $X_1 \ldots X_{k-1}\, r\, X_k$ is satisfied if $(\phi(X_1), \phi(X_2), \ldots, \phi(X_k)) \in r$ holds.

A *qualitative constraint network* is a set of variables and constraints such that for any k-tuple of variables exactly one constraint is defined. If constraints only involve base relations, it is called a *scenario* for short.

The problem of deciding whether there exists a valuation satisfying all qualitative constraints over a set of qualitative relations $\mathcal{R}$ is called QCSP($\mathcal{R}$).

Qualitative spatial reasoning exploits the algebraic structure of qualitative relations. The consistency problem is tackled using the algebraic closure algorithm [15], an adaption of Mackworth's AC-3 algorithm [16] for enforcing path-consistency in finite domain CSPs. Algebraic closure exploits the composition operation to rule out local inconsistencies in a constraint network. For some calculi algebraic closure implies path-consistency and can already be a sufficient condition for consistency [17]. In order to apply decision procedures for the consistency problem it is commonly required that algebraic closure is applicable to decide consistency of scenarios [15,18]. For example, this is the case in the RCC calculus [19] or Allen's interval algebra [20]. Given that algebraic closure decides consistency for scenarios, networks involving disjunctions can then be refined to base relations by means of a backtracking search and consistency can be decided [15]. This approach gains efficiency from exploiting *maximal tractable subsets*, i.e., maximal sets of relations for which algebraic closure decides consistency [21].

To put it in a nutshell, qualitative spatial reasoning pursues a relation algebraic approach which relies on the existence of efficient decision algorithms for consistency of scenarios such that reasoning in the full algebra (i.e., including disjunctive relations) can still be tackled in NP.

Previous research investigating the tractability of directional calculi identified intractable sub-algebras that involve disjunctions of base relations [11,14,12]. Particularly ternary point calculi are so expressive that encoding NOT-ALL-EQUAL-3-SAT or BETWEENNESS instances is straightforward (cf. [11,22]) when using disjunctions of base relations. In this paper we significantly refine these results by showing that directional information is inherently intractable, i.e., even deciding consistency of scenarios is intractable.

3. Proof sketch

In the following we describe the general idea of how to show NP-hardness of consistency problems that constrain a point position to be either left of or right of a line. Essentially, we develop a reduction from a realizability problem in combinatorial geometry to a consistency problem of qualitative constraints. This is captured by the central Theorem 8 that directly applies to all calculi that contain relations "left of" and "right of". As our reductions are reversible we are also able

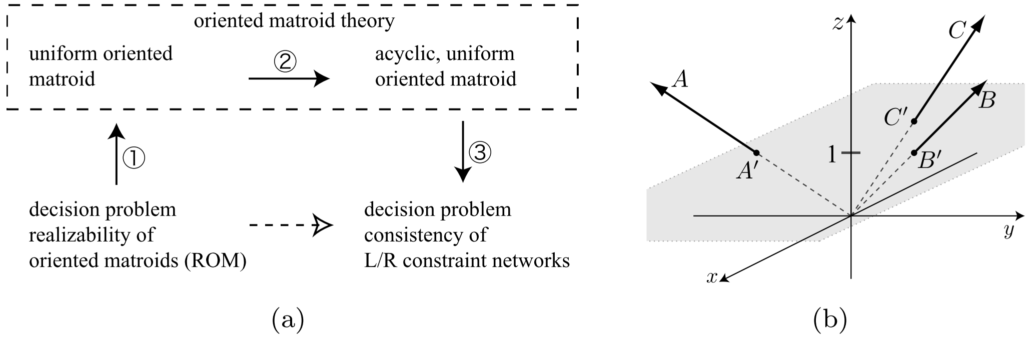

Fig. 1. (a) Steps in the reduction of decision problems about directional information to NP-hard matroid realizability. (b) Projective plane $z = 1$.

to show that if the geometric realizability problem turns out to be in NP, consistency in *left*/*right* networks can also be decided in NP. Hence, both problems are tightly related to one another. Motivated by Theorem 14 we conclude a conjecture that NP-hardness also applies to any calculus which refines *left*/*right* relations. Fig. 1(a) gives an overview of our proof which consists of three steps that all make use of the theory of oriented matroids (see [23]). In short, oriented matroids generalize the notion of geometric arrangements from a combinatorial perspective.

We start with the NP-hard problem of matroid realizability (ROM) which remains NP-hard if we restrict it to so-called uniform matroids (step 1 in the proof diagram). To represent oriented matroids we choose the notion of chirotopes that allows us to connect the combinatorial view of chirotopes to that of orientation of vector sequences.

In the second step of the proof we enforce a certain property (acyclicity) in the oriented matroid that is a necessary condition of geometric realizability in the plane (step 2 in the proof diagram, Lemma 7 and Algorithm 1 in the proof).

Step 3 concludes the proof by exploiting a duality between orientation of vectors in $\mathbb{R}^3$ and *left*/*right* relations between triples of points in the plane. To illustrate this duality, consider the projection on the plane $\{(x, y, z) \in \mathbb{R}^3 \mid z = 1\}$ shown in Fig. 1(b) which identifies $(x, y, z) \in \{(x, y, z) \in \mathbb{R}^3 \mid z \neq 0\}$ with $(\frac{x}{z}, \frac{y}{z}, 1)$. Suppose points A, B, C are above the XY-plane and form a positively oriented basis of $\mathbb{R}^3$ (i.e., interpreting the 3D points as column vectors of a 3×3 matrix, the determinant of this matrix is positive). Then, under the given projection, C' is left of the directed line from A' to B' (cp. Fig. 1(b)). Using a similar projection we construct a *left*/*right* constraint problem which is consistent only if the initial matroid is realizable.

4. Capturing directional information by oriented matroids

Oriented matroids can be considered combinatorial generalizations of spatial arrangements. They provide a broad model to describe information about position and orientation geometrically (with respect to given set of points and lines, Definition 3) as well as purely combinatorially (Definition 5), and have also been proposed as a discrete spatial representation [24,25]. Oriented Matroids allow us to abstract a concrete spatial reasoning problem in $\mathbb{R}^2$ to a problem in combinatorial geometry.

In this section we introduce oriented matroids first as a mathematical object from a concrete vector space and then as an abstract combinatorial object. For in-depth coverage refer to [23]. From the different ways of defining oriented matroids, the approach using the notion of *chirotopes* presents itself for characterizing directional information. This leads to the following definition of an oriented matroid with respect to a finite vector sequence V.

Definition 3 *(The oriented matroid of V).* Let $V = (v_1, \ldots, v_n)$ be a finite sequence of vectors in $\mathbb{R}^r$ spanning the space $\mathbb{R}^r$, $\mathrm{sign} : \mathbb{R} \to \{-1, 0, +1\}$ a function that returns the sign of its argument, and $\det(v_{i_1}, v_{i_2}, \ldots, v_{i_r})$ the determinant of a $r \times r$ matrix having $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ as its column vectors. The *oriented matroid of V* is given by the map

$$\begin{aligned} \chi_V &: \{1, 2, \ldots, n\}^r \to \{-1, 0, +1\} \\ &(i_1, i_2, \ldots, i_r) \mapsto \mathrm{sign}\big(\det(v_{i_1}, v_{i_2}, \ldots, v_{i_r})\big) \end{aligned}$$

which is called the *chirotope* of V. For $r = 3$ the map χ_V records for each vector triple whether it consists of linearly dependent vectors, a positively oriented basis of $\mathbb{R}^3$, or a negatively oriented basis of $\mathbb{R}^3$ (0, +1, −1, respectively).

Example 4. The oriented matroid of $V = (v_1, v_2, v_3)$ with $v_1 = (1, 0, 0)^T$, $v_2 = (0, 1, 0)^T$, $v_3 = (0, 0, 1)^T$ is the map $\chi_V : \{1, 2, 3\}^3 \to \{-1, 0, +1\}$ with $\chi_V(1, 2, 3) = \chi_V(2, 3, 1) = \chi_V(3, 1, 2) = +1$ and $\chi_V(2, 1, 3) = \chi_V(1, 3, 2) = \chi_V(3, 2, 1) = -1$. All other triples represent linearly dependent vector triples, and thus map to 0.

In the following we introduce oriented matroids as combinatorial objects. Unlike the previous definition, the following one is defined *without* a vector sequence, i.e., it abstracts from an underlying geometry.

Definition 5 *(Oriented matroid).* An *oriented matroid* of rank r on $E = \{1, 2, \dots, n\}$ is a map given by

$$\chi : E^r \to \{-1, 0, +1\},$$

called a *chirotope*, which satisfies the following three properties:

1. χ is not identically zero.
2. χ is alternating, that is, $\chi(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}) = \mathrm{sign}(\sigma)\chi(i_1, i_2, i_3)$ for all $i_1, i_2, i_3 \in E$ and every permutation σ on $\{1, 2, 3\}$.
3. For every $i_1, i_2, i_3, i_4, i_5 \in E$ the set

   $$\bigl\{\chi(i_1, i_2, i_3) \cdot \chi(i_1, i_4, i_5),\ -\chi(i_1, i_2, i_4) \cdot \chi(i_1, i_3, i_5),\ \chi(i_1, i_2, i_5) \cdot \chi(i_1, i_3, i_4)\bigr\}$$

   either contains $\{-1, +1\}$, or it equals $\{0\}$.

We note that the second condition implies $\chi(i_1, i_2, i_3) = 0$ if two of three arguments coincide. An oriented matroid is said to be *uniform*, if $\chi(i_1, i_2, i_3) \in \{-1, +1\}$ for all pairwise different $i_1, i_2, i_3 \in E$. We also note that an oriented matroid χ_V of a finite vector sequence V as defined in Definition 3 is an oriented matroid on E, where E is the index set of V. In what follows we restrict ourselves to oriented matroids of rank 3, which are relevant for the results of this paper.

Example 6. The map $\chi : \{1, 2, 3, 4\}^3 \to \{-1, 0, +1\}$ defined by $\chi(1, 2, 3) = \chi(1, 3, 4) = -1$, and $\chi(1, 2, 4) = \chi(2, 3, 4) = +1$, where the remaining values for χ are to be derived by permuting the triples and changing the signatures appropriately (e.g., $\chi(1, 3, 2) = +1 = -\chi(1, 2, 3)$), is a uniform oriented matroid of rank 3.

Now that we have the abstract definition of an oriented matroid, a natural question to ask is:

Given an oriented matroid χ on $E = \{1, \dots, n\}$, is there a sequence $V = \{v_1, \dots, v_n\}$ of spanning vectors in $\mathbb{R}^r$, such that χ is the oriented matroid of V, i.e., $\chi = \chi_V$?

To exemplify this question, we take the oriented matroid from Example 6. Then a realization of χ is

$$v_1 = (-1, 0, -1), \qquad v_2 = (0, 1, 1), \qquad v_3 = (0, -1, 1), \qquad v_4 = (1, 0, -1),$$

since $\chi(i, j, k) = \mathrm{sign}(\det(v_i, v_j, v_k)) = \chi_V(i, j, k)$ for all $i, j, k \in \{1, \dots, 4\}$.

The aforementioned problem is the so-called *realizability problem for oriented matroids* (*ROM*) which is NP-hard for oriented matroids of rank 3 and higher [26,27], the tightest complexity bound following from [28] is exponential time with respect to the number of vectors.

A slightly modified version of ROM is *the realizability problem for uniform oriented matroids* (*RUOM*), where only oriented matroids are considered that do not contain zero in the range. RUOM is also NP-hard in the number of vectors for matroids of rank 3 and higher [27].

Now we establish a connection between a point configuration in a plane (the domain of many qualitative spatial calculi) and a uniform oriented matroid. Assume there exists a linear map $l : \mathbb{R}^3 \to \mathbb{R}$ (i.e., a *linear form*), such that the vector sequence V consists of vectors $v_1, \dots, v_n$ with $l(v_i) > 0$ for all i, i.e., the vectors as points in $\mathbb{R}^3$ are entirely contained in one of the open half-spaces determined by the (hyper-)plane $\{x \in \mathbb{R}^3 \mid l(x) = 0\}$. Then we can project the vectors v_i to points in an affine plane $\mathbb{A}^2$ defined by

$$\mathbb{A}^2 := \bigl\{x \in \mathbb{R}^3 \bigm| l(x) = 1\bigr\},$$

where we associate each vector v_i with point $\frac{1}{l(v_i)} v_i \in \mathbb{A}^2$ for all i. An oriented matroid of V with this property is called *acyclic* (see Fig. 2). The following lemma states that determinants of vector triples give us a necessary and sufficient condition for deciding whether an oriented matroid is acyclic.

Lemma 7. *Given a vector sequence $V = (v_1, \dots, v_n)$ in $\mathbb{R}^3$ with $\det(v_i, v_j, v_k) \neq 0$ for all pairwise different $1 \leqslant i, j, k \leqslant n$, then there exists a linear form l with $l(v_i) > 0$ for all i, if and only if there is a pair of two distinguished vectors in V, say v_1 and v_2, such that either $\det(v_1, v_2, v_i) > 0$ for all $i > 2$, or $\det(v_1, v_2, v_i) < 0$ for all $i > 2$.*

Before proving the lemma let us consider the vector sequences in Fig. 2 as an example: regarding the vector sequence on the left-hand side, there is no vector pair $(v_i, v_j), i \neq j$, such that the determinant of the 3×3 matrix (v_i, v_j, v_k) is positive for all $k, k \neq i, k \neq j$ or negative for all $k, k \neq i, k \neq j$. The lemma states that therefore there exists no (hyper-)plane such that all vectors are contained in one of the two open half-spaces determined by that plane; the oriented matroid of this vector sequence is not acyclic. Considering the vector sequence on the right-hand side of the figure, vectors v_1 and v_2 give rise to positive determinants $\det(v_1, v_2, v_3)$ and $\det(v_1, v_2, v_4)$. According to the lemma a half-space containing all vectors

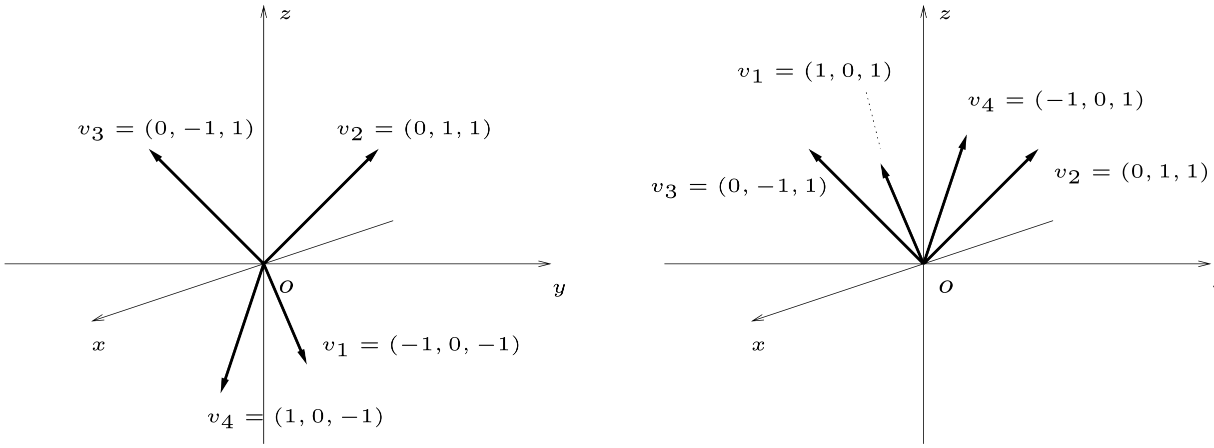

Fig. 2. Examples of non-acyclic and acyclic vector sequences in $\mathbb{R}^3$. The vector sequence on the left represents an oriented matroid that is not acyclic, i.e., there is no open half-space containing all the vectors $v_1, \ldots, v_4$. By contrast, the vector sequence on the right is contained in the open half-space $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ and accordingly, it is acyclic. Its affine representation is equivalent to the one in Fig. 4.

must exist and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ is one example. We note that by negating the vectors v_1 and v_4 in the first vector sequence we obtain the second one. This is an essential step in the proof of Theorem 8 for enforcing acyclicity of non-acyclic oriented matroids.

Proof. Assume that there exists such a linear form l. Then there exists a cone covering the convex hull of V which has a supporting hyperplane H in $\mathbb{R}^3$ given by linear combinations of two vectors of V, say v_1, v_2. Since the remaining vectors of V are entirely contained in one of the open half-spaces determined by H, it must be either $\det(v_1, v_2, v_i) > 0$ for all $i > 2$, or $\det(v_1, v_2, v_i) < 0$ for all $i > 2$.

Now assume $\det(v_1, v_2, v_i) > 0$ for all $i > 2$. Then $(v_1 \times v_2)^T \cdot v_i = \det(v_1, v_2, v_i) > 0$ for all $i \geqslant 3$, where $v_1 \times v_2$ denotes the vector product of v_1 and v_2. We then define a linear form l by

$$l(x) = \left(v_1 \times v_2 + \epsilon\left(\frac{1}{\|v_1\|}v_1 + \frac{1}{\|v_2\|}v_2\right)\right)^T x,$$

where $\epsilon > 0$ is small enough, such that $l(v_i) > 0$ for all $i > 2$. The fact $l(v_1) > 0$ follows from $l(v_1) = 0 + \epsilon(\|v_1\| + \frac{v_1^T v_2}{\|v_2\|})$ and the Cauchy–Schwartz inequality

$$\|v_1\|\|v_2\| \geqslant \left|v_1^T v_2\right|,$$

$l(v_2) > 0$ can be shown analogously. Altogether, $l(v_i) > 0$ for all $1 \leqslant i \leqslant n$. We get the proof for the other case by switching the signs. □

5. Hardness of directional calculi

In this section we show NP-hardness for individual directional calculi by encoding RUOM into QCSP for individual calculi.

5.1. $\mathcal{LR}$ calculus

The $\mathcal{LR}$ calculus [6,9] defines 9 ternary base relations for points positioned in the Euclidean plane $\mathbb{R}^2$. Fig. 3(a) shows the 7 base relations for pairwise disjoint points, namely **l**eft, **r**ight, **b**ack, **s**tart, **i**nbetween, **e**nd, and **f**ront. Additionally, two relations of point superposition are considered, namely $dou = \{(a, a, c) \mid a, c \in \mathbb{R}^2,\ a \neq c\}$ and $tri = \{(a, a, a) \mid a \in \mathbb{R}^2\}$.

Theorem 8. QCSP(*left*, *right*) *is NP-hard.*

Proof. Since RUOM is NP-hard, it suffices to show that the encoding of the RUOM into QCSP(*left*, *right*) can be done in polynomial time in the number of vectors. Let a uniform oriented matroid $\chi : \{1, \ldots, n\}^3 \mapsto \{-1, +1\}$ of rank 3 be given. Since the $\mathcal{LR}$ calculus represents information about the plane but the realization of a rank 3 matroid can cover the full 3D space, we generate a new uniform oriented matroid χ' which is equivalent in realizability and acyclic, i.e., the realization of χ' can be identified with a point configuration in an affine space. This is accomplished by Algorithm 1, in which we make use of Lemma 7. Since there are three loops ranging over n, the complexity of the algorithm is $\mathcal{O}(n^3)$.

Algorithm 1 Algorithm for converting oriented matroid into an acyclic one used by Theorem 8.

1: **function** `flipChi`(χ)
2: $\quad\quad \triangleright$ The elements 1 and 2 corresponds to v_1 and v_2 in Lemma 7.
3: $\quad \chi' \leftarrow \chi$
4: $\quad$ **for** $i \in \{3, 4, \dots, n\}$ **do**
5: $\quad\quad \triangleright$ We enforce $\chi'(1, 2, i) = 1$ for all $i = 3, \dots, n$ to apply Lemma 7.
6: $\quad\quad$ **if** $\chi'(1, 2, i) = -1$ **then**
7: $\quad\quad\quad \chi'(1, 2, i) \leftarrow 1$
8: $\quad\quad\quad \chi'(i, 1, 2) \leftarrow 1$
9: $\quad\quad\quad \chi'(2, i, 1) \leftarrow 1$
10: $\quad\quad\quad \chi'(1, i, 2) \leftarrow -1$
11: $\quad\quad\quad \chi'(i, 2, 1) \leftarrow -1$
12: $\quad\quad\quad \chi'(2, 1, i) \leftarrow -1$
13: $\quad\quad\quad\quad \triangleright$ Switch other signs of χ' that involve i accordingly.
14: $\quad\quad\quad$ **for** $j \in \{3, 4, \dots, n\}, k \in \{3, 4, \dots, n\}, i \neq j, i \neq k, j \neq k$ **do**
15: $\quad\quad\quad\quad \chi'(i, j, k) \leftarrow -\chi'(i, j, k)$
16: $\quad\quad\quad\quad \chi'(j, i, k) \leftarrow -\chi'(j, i, k)$
17: $\quad\quad\quad\quad \chi'(j, k, i) \leftarrow -\chi'(j, k, i)$
18: $\quad\quad\quad\quad \chi'(i, k, j) \leftarrow -\chi'(i, k, j)$
19: $\quad\quad\quad\quad \chi'(k, j, i) \leftarrow -\chi'(k, j, i)$
20: $\quad\quad\quad\quad \chi'(k, i, j) \leftarrow -\chi'(k, i, j)$
21: $\quad\quad\quad$ **end for**
22: $\quad\quad$ **end if**
23: $\quad$ **end for**
24: $\quad$ **return** χ'
25: **end function**

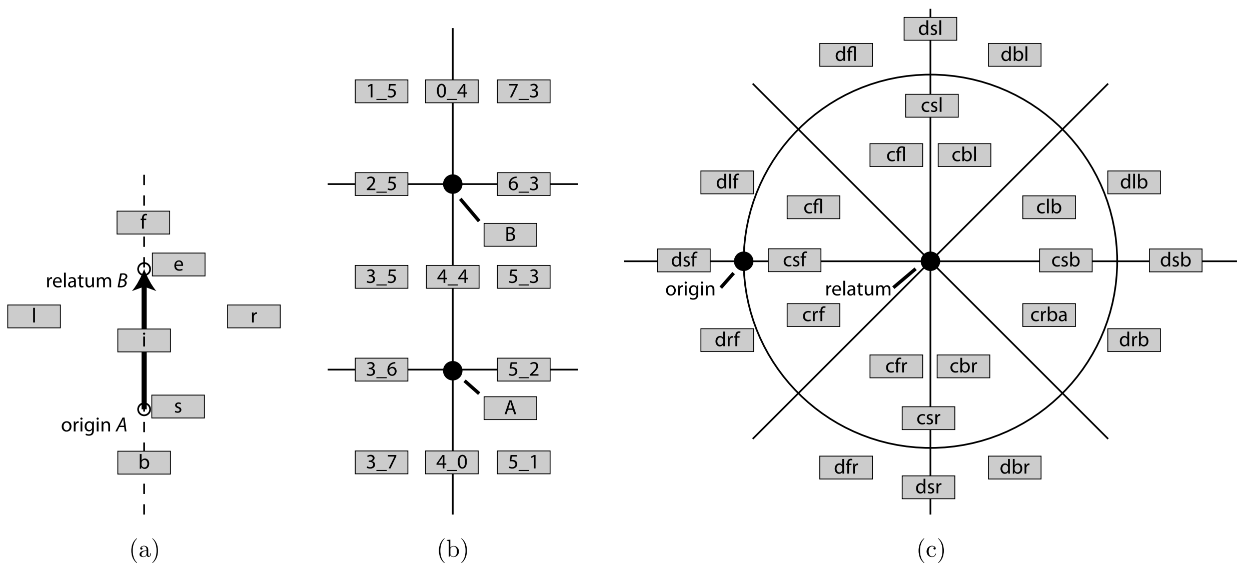

Fig. 3. Qualitative relations defined by ternary directional calculi. (a) $\mathcal{LR}$ calculus point-to-line relations. (b) Double cross calculus [10]. (c) TPCC [14].

Furthermore, if χ is realizable, i.e., χ is the (uniform) oriented matroid of a vector sequence $V = (v_1, \dots, v_n)$, then $\chi(i_1, i_2, i_3) = \operatorname{sign}(\det(v_{i_1}, v_{i_2}, v_{i_3}))$ for all (i_1, i_2, i_3). As the determinant function is alternating, negating a vector v_k, $k \in \{1, \dots, n\}$ changes the signs of $\chi(i_1, i_2, i_3)$, if i_1, i_2 and i_3 are pairwise different and $i_j = k$ for a $j \in \{1, 2, 3\}$. This is reflected in line 8–21 of Algorithm 1, as a consequence of enforcing $\chi(1, 2, i) = \operatorname{sign}(\det(v_1, v_2, v_i)) = 1$ for all $i > 2$ to meet the condition in Lemma 7. With regards to equivalence of realizability of χ and χ' we note that the negations performed by the Algorithm 1 simply correspond to flipping vectors from the "negative" side of the hyperplane given by $v_1 \times v_2$ to the "positive" side (see proof of Lemma 7). Thus, χ' is realizable if χ realizable. Analogously, χ is realizable if χ' realizable.

Finally, we encode χ' into QCSP(*left*, *right*): The domain $\{1, 2, \dots, n\}$ of χ' is represented by variables $\{v_1, v_2, \dots, v_n\}$. For each triple $(i, j, k), i, j, k \in D$ we have $v_i v_j$ *right* v_k, if $\chi'(i, j, k) = -1$, whereas $v_i v_j$ *left* v_k, if $\chi'(i, j, k) = +1$ (see Fig. 4). As χ is uniform, $\chi(i, j, k) = 0$ occurs if and only if two of its three arguments coincide, giving no information about the general point configuration. Thus, the case $\chi(i, j, k) = 0$ does not need to be considered as a constraint for the triple (v_i, v_j, v_k).

According to the translation above, the oriented matroid χ' is realizable if and only if the corresponding qualitative constraint network is satisfiable. $\square$

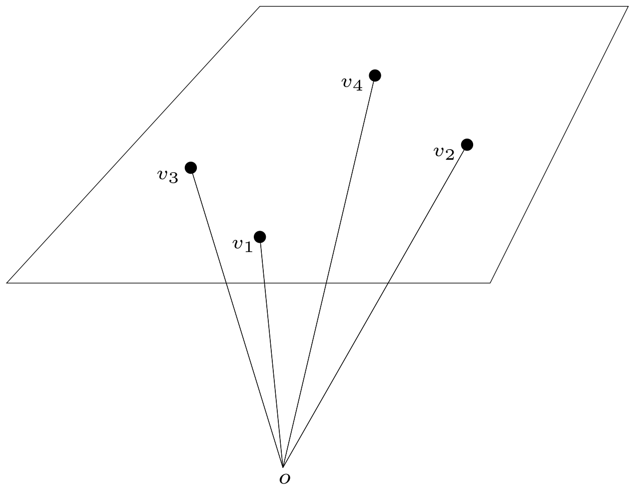

Fig. 4. A realization of a uniform oriented matroid $\chi : \{1,2,3,4\}^3 \to \{-1,+1\}$ with $\chi(1,2,3) = +1$, $\chi(1,2,4) = +1$, $\chi(1,3,4) = -1$ and $\chi(2,3,4) = -1$. Equivalently, we have $v_1 v_2$ *left* v_3; $v_1 v_2$ *left* v_4; $v_1 v_3$ *right* v_4 and $v_2 v_3$ *right* v_4. Note that v_3 and v_4 are entirely lying on one of the half-spaces determined by the hyperplane through v_1 and v_2 as generated by Algorithm 1.

Since our reduction can reversed, we are able to state the following theorems that tighten the connection of RUOM and deciding consistency of *left*/*right* constraints.

Theorem 9. QCSP(*left*, *right*) *for scenarios is reducible to RUOM.*

Proof. Let S be a *left*/*right* scenario and n be the number of variables in S. We define the induced oriented matroid χ by assigning $\chi(i,j,k) = +1$ to each constraint $v_i v_j$ *left* v_k in S, and $\chi(i,j,k) = -1$ to each constraint $v_i v_j$ *right* v_k in S, where $i,j,k \in \{1,\dots n\}$ and pairwise different. There are altogether $\mathcal{O}(n^3)$ such assignments.

We then check whether χ is acyclic, which is the case if there exists a pair (i,j), $i,j \in \{1,\dots,n\}$ with $\chi(i,j,k) > 0$ for all $k \in \{1,\dots,n\}$, $k \neq i$, $k \neq j$. Determining the existence of such a pair can also be done in $\mathcal{O}(n^3)$ time by trying out all $\mathcal{O}(n^2)$ candidates.

Since S is consistent if and only if χ is acyclic and realizable, it takes $\mathcal{O}(n^3)$ time to reduce QCSP(*left*, *right*) for scenarios to RUOM. □

Corollary 10. *If RUOM is in NP, than* QCSP(*left*, *right*) *for scenarios is in NP too.*

5.2. Dipole calculus

The dipole calculus [12] has been introduced as qualitative calculus about path segments which are oriented line segments defined by start and end point — see Fig. 5(a) for illustration. The calculus assumes all points to be in general position, i.e., no three different points are positioned on the same line. The 24 dipole relations $\mathcal{DRA}_{24}$ represent all possible relative orientations of two dipoles $A = (s_A, e_A)$, $B = (s_B, e_B)$. Dipole relations can be rewritten as sets $\mathcal{LR}$ relations considering all 3-tuples of points: $(s_A, e_A, s_B), (s_A, e_A, e_B), (s_B, e_B, s_A)$, (s_B, e_B, e_A). This makes rewriting QCSP(*left*, *right*) as QCSP($\mathcal{DRA}_{24}$) straightforward.

Corollary 11. QCSP($\mathcal{DRA}_{24}$) *is NP-hard.*

The original paper on the dipole calculus also considers a refined dipole calculus which also handles multiple points on a line, but this does not affect the orientation relations and their intractability.

5.3. $\mathcal{OPRA}$ calculus

The $\mathcal{OPRA}_n$ family of calculi defines a set of directional relations for oriented points (see Fig. 5(b)–(d)) with adjustable granularity parameter [13], $\mathcal{OPRA}_{2n}$ being a refinement of $\mathcal{OPRA}_n$. The granularity parameter n stands for the number of dividing lines used to construct the relations (see Fig. 5). Refining $\mathcal{OPRA}_1$ to $\mathcal{OPRA}_2$ we have a new line distinguishing *before* (sectors 0,1,7) and *behind* (sectors 3,4,5). Like the $\mathcal{LR}$ calculus, $\mathcal{OPRA}_n$ makes left/right distinctions.

Corollary 12. QCSP($\mathcal{OPRA}_1$) *is NP-hard.*

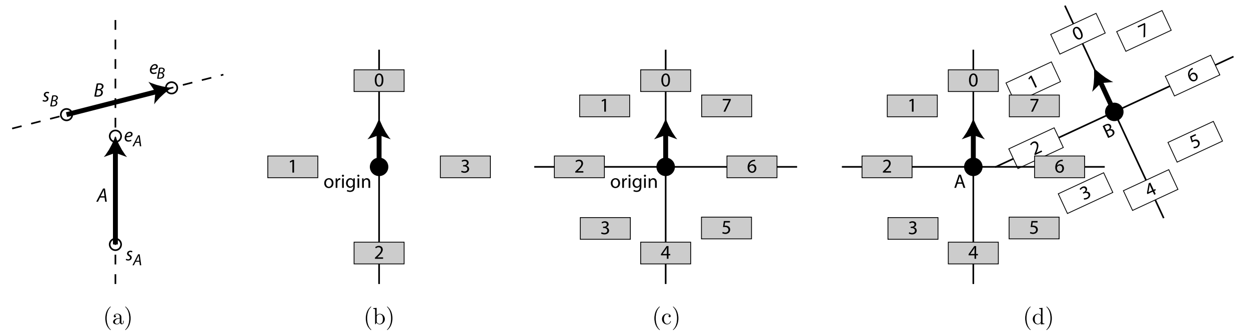

Fig. 5. Binary directional calculi. (a) Example of a dipole base relation $A\,lrrr\,B$ (s_B **L**eft of A, e_B **R**ight of A, s_A **R**ight of B, e_A **R**ight of B). (b) Relation sectors defined by $\mathcal{OPRA}_1$. (c) Relation sectors defined by $\mathcal{OPRA}_2$. (d) Example of a $\mathcal{OPRA}_2$ relation $A\ {}_2\angle_7^1\ B$.

Lemma 13. QCSP(*behind*, *before*) *is NP-hard.*

Proof. Relations *behind*, *before* can be bijectively mapped to *left*, *right*. □

Theorem 14. *Let L and R be finite refinements of the $\mathcal{LR}$ relations left and right, i.e., $L = \{l_1, l_2, \ldots, l_n\}$, with left $= l_1 \cup l_2 \cup \cdots \cup l_n$ and analogously for R. If deciding* QCSP($L \cup R$) *is in NP, then Matroid realizability of rank* 3 *oriented matroid is in NP too.*

Proof. Let the rank 3 oriented matroid be given by a chirotope so we can read off relations *left* ($\chi(x, y, z) = -1$) and *right* ($\chi(x, y, z) = +1$). Relations *left*, *right* can be written as disjunctions of $l_1, l_2, \ldots, l_n$ or $r_1, r_2, \ldots, r_m$, respectively. Given a *left*/*right* decision problem one can non-deterministically select one relation from each disjunction and decide the refined problem. □

Corollary 15. *If base relations of $\mathcal{OPRA}_2$ can be decided in NP, then Matroid realizability of rank* 3 *matroid is in NP.*

Since matroid realizability is extensively studied and NP membership of this problem could not been shown yet, we conclude from Theorem 14 the following conjecture.

Conjecture 16. *There is no directional calculus capable of expressing left and right* (*by disjunction of base relations*) *such that consistency of constraint networks over its base relations can be decided in polynomial time.*

5.4. Double cross calculus

The double cross calculus [10] is a ternary point configuration calculus which defines 15 relations between pairwise disjoint points; see Fig. 3(b). As can be seen in the figure, the double cross calculus is a refinement of the $\mathcal{LR}$ calculus.

5.5. Ternary point configuration calculus

In [14] a ternary point configuration calculus TPCC for robot localization and navigation tasks is proposed. From the base relations defined by the calculus (see Fig. 3) it is easy to see that TPCC is a refinement of the $\mathcal{OPRA}_2$ calculus, i.e., $\mathcal{LR}$ relations can be written as disjunctions of TPCC relations.[1]

6. On deciding consistency of directional relations

NP-membership of RUOM is still an open problem and so the computational complexity of qualitative reasoning about directional information remains open too: *left*/*right* consistency can be decided in NP if and only if RUOM can be decided in NP. We note that deciding consistency of directional constraints is equivalent to the existential theory of the reals [23,29]. This theory deals with solvability of systems of polynomial equations and inequalities; only exponential time algorithms are known so far. Therefore, computationally cheap approaches that can decide a significant subset of directional information constraint problems are important. However, the common approach of QSR, decision by algebraic closure on scenarios is not effective for directional relations. Considering the $\mathcal{LR}$ calculus, it is easy to construct algebraically closed, but inconsistent scenarios involving as few as 5 variables [22]: with respect to the classically used binary composition all scenarios only containing the relations *left* and *right* are algebraically closed anyway, but even for the more natural ternary composition

[1] Since TPCC does not define half-line relations for all 45° angles as $\mathcal{OPRA}_4$ does, TPCC is not a refinement of $\mathcal{OPRA}_4$.

(cf. [30]) this counterexample holds. Considering the conditions for oriented matroids we are able to give a much better approximation of QCSP(*left*, *right*) than obtained by algebraic closure. Our approach is based on the following realizability theorem.

Theorem 17 *(Matroid realizability [23]). All oriented matroids of rank* 3 *with* $|E| \leqslant 8$ *are realizable.*

Hence, only testing the conditions of oriented matroids according to Definition 5 we obtain a decision method for small instances that is more effective than algebraic closure. Since we are considering qualitative relations *left* and *right* only, condition 1 of Definition 5 is always met. Condition 2 requires us to check all permutations of triples (which can be done in $\mathcal{O}(n^3)$ time) and it is easy to see that condition 3, also known as Grassmann–Plücker conditions, can be checked $\mathcal{O}(n^5)$ time. Theorem 17 gives us that testing matroid conditions decides consistency for up to size 8 (sub-)networks.

We note that this procedure has little higher complexity than checking algebraic closure with respect to ternary composition for ternary calculi which is $\mathcal{O}(n^4)$, but it is more effective. Considering all 1024 constraint networks with 5 variables and relations *left* and *right*, 53 of these meet the Grassmann–Plücker conditions (and are thus realizable), whereas 544 are algebraically closed with respect to ternary composition, i.e., testing realizability by algebraic closure yields 491 false positives.

7. Conclusion

In this article we have shown that directional calculi are inherently intractable. Nevertheless, dealing with directional information is relevant to applications involving, e.g., robot instruction or natural language semantics [8]. We believe this is a motivation for future QSR research: identifying *new* reasoning methods to handle directional information. So far, QSR has focused on one single core method: deciding consistency of constraint networks only involving (disjoint) base relations by the path-consistency method. As this polynomial-time method does not decide consistency for any of the calculi discussed here, future work in QSR must investigate alternative reasoning methods. Furthermore, it needs to be researched whether there exist tractable refinements of directional constraints. This question is potentially hard to answer, since existence of a polynomially tractable finite refinement of the relations *left*, *right* implies NP-membership of RUOM (cf. Theorem 14) — an open question. For practical applications it is also interesting to learn how good polynomial-time methods can approximate consistency.

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