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On the expressivity of inconsistency measures

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ABSTRACT

We survey recent approaches to inconsistency measurement in propositional logic and provide a comparative analysis in terms of their *expressivity*. For that, we introduce four different expressivity characteristics that quantitatively assess the number of different knowledge bases that a measure can distinguish. Our approach aims at complementing ongoing discussions on rationality postulates for inconsistency measures by considering expressivity as a desirable property. We evaluate 16 different measures on the proposed characteristics and conclude that the distance-based measure $\mathcal{I}^{\Sigma}_{\text{dalal}}$ from Grant and Hunter (2013) [8] and the proof-based measure $\mathcal{I}_{P_m}$ from Jabbour and Raddaoui (2013) [16] have maximal expressivity along all considered characteristics. In our study, we discovered several interesting relationships of inconsistency measurement to e.g. set theory and Boolean functions and we also report these findings.

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1. Introduction

Inconsistency measurement is about the quantitative assessment of the severity of inconsistencies in knowledge bases. Consider the following two knowledge bases $\mathcal{K}_1$ and $\mathcal{K}_2$ formalized in propositional logic:

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\} \qquad \mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

Both knowledge bases are classically inconsistent as for $\mathcal{K}_1$ we have $\{a, \neg a \wedge \neg b\} \models \perp$ and for $\mathcal{K}_2$ we have, e.g., $\{a, \neg a\} \models \perp$. These inconsistencies render the whole knowledge bases useless for reasoning if one wants to use classical reasoning techniques. In order to make the knowledge bases useful again, one can either rely on non-monotonic/paraconsistent reasoning techniques [23,27] or one revises the knowledge bases appropriately to make them consistent [9]. Looking at the knowledge bases $\mathcal{K}_1$ and $\mathcal{K}_2$ one can observe that the *severity* of their inconsistency is different. In $\mathcal{K}_1$, only two out of four formulas (a and $\neg a \wedge \neg b$) are "participating" in making $\mathcal{K}_1$ inconsistent while for $\mathcal{K}_2$ all formulas contribute to its inconsistency. Furthermore, for $\mathcal{K}_1$ only two propositions (a and b) are conflicting and using e.g. paraconsistent reasoning one could still infer meaningful statements about c and d. For $\mathcal{K}_2$ no such statement can be made. This leads to the assessment that $\mathcal{K}_2$ should be regarded *more* inconsistent than $\mathcal{K}_1$.

Inconsistency measures can be used to analyze inconsistencies and to provide insights on how to repair them. An inconsistency measure $\mathcal{I}$ is a function on knowledge bases, such that the larger the value $\mathcal{I}(\mathcal{K})$ the more severe the inconsistency in $\mathcal{K}$. A lot of different approaches of inconsistency measures have been proposed, mostly for classical propositional logic [10,12,13,22,25,32,7,8,24,15], but also for classical first-order logic [6], description logics [21,33], default logics [5], and probabilistic and other weighted logics [19,30,26]. Due to this plethora of inconsistency measures it is hard to determine which

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measure to use for an application and which measure is meaningful. Rationality postulates have been proposed that address the issue of assessing the quality of a measure—see e.g. [11,25]—but many of these properties have been criticized to address only a specific point of view, see [2] for a recent discussion on this topic.

In this paper, we take a different perspective on the evaluation of inconsistency measures by considering a *quantitative* analysis of their *expressivity*, that is, we study how many different (inconsistent) knowledge bases can be distinguished by a given inconsistency measure. By the term *expressivity* we here refer to the property of a semantical concept—here, an inconsistency measure—and its capability to distinguish syntactical constructs—here, knowledge bases—, similarly as it has been done for the analysis of expressivity of semantics for other logical languages, see e.g. skepticism relations for formal argumentation [1]. Our analysis is meant to complement the study on rationality postulates and is, of course, not meaningful on its own as the compliance of measures with the basic intuitions behind inconsistency measures can only be assessed by rationality postulates. However, we introduce expressivity of inconsistency measures as an *additional* method to evaluate their quality. In particular, we propose four different *expressivity characteristics* that quantify the relation between the number of different values of an inconsistency measure wrt. different notions of the size of the knowledge base, such as number of formulas or number of propositions. We conduct a thorough comparative analysis of 16 different inconsistency measures from the literature [12,13,7,17,31,8,25,16,32,5] and classify these measures in a hierarchy of expressivity. In our study, we made several interesting observations, such as the relation between the measure $\mathcal{I}_{\mathsf{MI}}$ [7] and Sperner families [28] and of the measure $\mathcal{I}_{\mathsf{MI^C}}$ [7] with profiles of Boolean functions. One of our results is that the distance-based measure $\mathcal{I}^{\Sigma}_{\text{dalal}}$ from Grant and Hunter [8] and the proof-based measure $\mathcal{I}_{P_m}$ from Jabbour and Raddaoui [16] have maximal expressivity along all considered characteristics.

In summary, the contributions of this paper are as follows:

1. We conduct a focused survey of 16 inconsistency measures from the recent literature (Section 3).
2. We propose four different expressivity characteristics, evaluate the considered inconsistency measures wrt. these characteristics, and study our findings (Section 4).
3. We classify the evaluated measures into hierarchies of expressivity and thus provide a means to quantitatively compare different measures (Section 5).

We give necessary preliminaries in Section 2 and provide a summary in Section 6. Appendix A contains proofs of technical results and Appendix B lists all example knowledge bases and families of knowledge bases used in the paper. All inconsistency measures discussed in this paper have been implemented and an online interface to try out these measures is available.[1]

2. Preliminaries

Let At be some fixed propositional signature, i.e., a (possibly infinite) set of propositions, and let $\mathcal{L}(\mathsf{At})$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*and*), $\vee$ (*or*), and $\neg$ (*negation*).

Definition 1. A knowledge base $\mathcal{K}$ is a finite set of formulas $\mathcal{K} \subseteq \mathcal{L}(\mathsf{At})$. Let $\mathbb{K}$ be the set of all knowledge bases.

If X is a formula or a set of formulas we write $\mathsf{At}(X)$ to denote the set of propositions appearing in X. Semantics to a propositional language is given by *interpretations* and an *interpretation* ω on At is a function $\omega : \mathsf{At} \to \{\mathsf{true}, \mathsf{false}\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At. An interpretation ω *satisfies* (or is a *model* of) a proposition $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \mathsf{true}$. The satisfaction relation $\models$ is extended to formulas in the usual way.

As an abbreviation we sometimes identify an interpretation ω with its *complete conjunction*, i.e., if $a_1, \ldots, a_n \in \mathsf{At}$ are those propositions that are assigned true by ω and $a_{n+1}, \ldots, a_m \in \mathsf{At}$ are those propositions that are assigned false by ω we identify ω by $a_1 \ldots a_n \overline{a_{n+1}} \ldots \overline{a_m}$ (or any permutation of this). For example, the interpretation ω_1 on $\{a, b, c\}$ with $\omega(a) = \omega(c) = \mathsf{true}$ and $\omega(b) = \mathsf{false}$ is abbreviated by $a\overline{b}c$.

For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$. Define furthermore the set of models $\mathsf{Mod}(X) = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models X\}$ for every formula or set of formulas X. If $\mathsf{Mod}(X) = \emptyset$ we also write $X \models \perp$ and say that X is *inconsistent*.

3. Inconsistency measures

Let $\mathbb{R}^{\infty}_{\geq 0}$ be the set of non-negative real values including ∞. Inconsistency measures are functions $\mathcal{I} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ that aim at assessing the severity of the inconsistency in a knowledge base $\mathcal{K}$, cf. [7]. The basic idea is that the larger the inconsistency in $\mathcal{K}$ the larger the value $\mathcal{I}(\mathcal{K})$ and $\mathcal{I}(\mathcal{K}) = 0$ if and only if $\mathcal{K}$ is consistent. However, inconsistency is a concept that is not easily quantified and there have been a couple of proposals for inconsistency measures so far, in particular for

[1] http://tweetyproject.org/w/incmes/.

Table 1
Definitions of the considered inconsistency measures.

$$\mathcal{I}_d(\mathcal{K}) = \begin{cases} 1 & \text{if } \mathcal{K} \models \perp \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = |\mathsf{MI}(\mathcal{K})|$$

$$\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}(\mathcal{K}) = \sum_{M \in \mathsf{MI}(\mathcal{K})} \frac{1}{|M|}$$

$$\mathcal{I}_\eta(\mathcal{K}) = 1 - \max\{\xi \mid \exists P \in \mathcal{P}(\mathsf{At}) : \forall \alpha \in \mathcal{K} : P(\alpha) \geq \xi\}$$

$$\mathcal{I}_c(\mathcal{K}) = \min\{|\upsilon^{-1}(B)| \mid \upsilon \models^3 \mathcal{K}\}$$

$$\mathcal{I}_{LP_m}(\mathcal{K}) = \mathcal{I}_c(\mathcal{K}) / |\mathsf{At}(\mathcal{K})|$$

$$\mathcal{I}_{mc}(\mathcal{K}) = |\mathsf{MC}(\mathcal{K})| + |\mathsf{SC}(\mathcal{K})| - 1$$

$$\mathcal{I}_p(\mathcal{K}) = |\bigcup_{M \in \mathsf{MI}(\mathcal{K})} M|$$

$$\mathcal{I}_{hs}(\mathcal{K}) = \min\{|H| \mid H \text{ is a hitting set of } \mathcal{K}\} - 1$$

$$\mathcal{I}^{\Sigma}_{\mathrm{dalal}}(\mathcal{K}) = \min\{\sum_{\alpha \in \mathcal{K}} d_{\mathrm{d}}(\mathsf{Mod}(\alpha), \omega) \mid \omega \in \Omega(\mathsf{At})\}$$

$$\mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K}) = \min\{\max_{\alpha \in \mathcal{K}} d_{\mathrm{d}}(\mathsf{Mod}(\alpha), \omega) \mid \omega \in \Omega(\mathsf{At})\}$$

$$\mathcal{I}^{\mathrm{hit}}_{\mathrm{dalal}}(\mathcal{K}) = \min\{|\{\alpha \in \mathcal{K} \mid d_{\mathrm{d}}(\mathsf{Mod}(\alpha), \omega) > 0\}| \mid \omega \in \Omega(\mathsf{At})\}$$

$$\mathcal{I}_{D_f}(\mathcal{K}) = 1 - \Pi_{i=1}^{|\mathcal{K}|}(1 - R_i(\mathcal{K})/i)$$

$$\mathcal{I}_{P_m}(\mathcal{K}) = \sum_{a \in \mathsf{At}} |P_m^{\mathcal{K}}(a)| \cdot |P_m^{\mathcal{K}}(\neg a)|$$

$$\mathcal{I}_{mv}(\mathcal{K}) = \frac{|\bigcup_{M \in \mathsf{MI}(\mathcal{K})} \mathsf{At}(M)|}{|\mathsf{At}(\mathcal{K})|}$$

$$\mathcal{I}_{nc}(\mathcal{K}) = |\mathcal{K}| - \max\{n \mid \forall \mathcal{K}' \subseteq \mathcal{K} : |\mathcal{K}'| = n \Rightarrow \mathcal{K}' \not\models \perp\}$$

classical propositional logic, see e.g. [2,24,15,14] for some recent works. We selected 16 inconsistency measures from the literature in order to conduct our analysis on expressivity, taken from [12,13,7,17,31,8,25,16,32,5]. We briefly introduce these measures in this section for the sake of completeness, but we refer for a detailed explanation to the corresponding original papers.

To illustrate the different inconsistency measures we will use the knowledge bases $\mathcal{K}_1$ and $\mathcal{K}_2$ from the introduction as running examples.

Example 1. Let $\mathsf{At} = \{a, b, c, d\}$ and define knowledge bases $\mathcal{K}_1$ and $\mathcal{K}_2$ as follows:

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\}$$

$$\mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

A summary of the formal definitions of the considered inconsistency measures can be found in Table 1. We will discuss these measures in more detail below.

3.1. The drastic inconsistency measure

The basic motivation for measuring inconsistency is to provide a *graded* assessment of inconsistency and not only a "consistent"/"inconsistent" assessment. However, in order to evaluate more sophisticated measures and the usefulness of rationality postulates (cf. beginning of Section 4), the *drastic inconsistency measure* $\mathcal{I}_d$—that can only distinguish between consistent and inconsistent knowledge bases—is usually used as a baseline approach, cf. e.g. [12].

Definition 2. The *drastic inconsistency measure* $\mathcal{I}_d : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_d(\mathcal{K}) = \begin{cases} 1 & \text{if } \mathcal{K} \models \perp \\ 0 & \text{otherwise} \end{cases}$$

for $\mathcal{K} \in \mathbb{K}$.

In other words, $\mathcal{I}_d(\mathcal{K}) = 1$ if and only if $\mathcal{K}$ is inconsistent (and 0 otherwise).

Example 2. We continue Example 1 and consider

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\}$$
$$\mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

As both $\mathcal{K}_1$ and $\mathcal{K}_2$ are inconsistent we obtain $\mathcal{I}_d(\mathcal{K}_1) = \mathcal{I}_d(\mathcal{K}_2) = 1$.

3.2. Inconsistency measures based on minimal inconsistencies

One approach to assess the severity of inconsistency in a knowledge base is to focus on its set of *minimal inconsistent subsets*. A set $M \subseteq \mathcal{K}$ is called *minimal inconsistent subset* (MI) of $\mathcal{K}$ if $M \models \perp$ and there is no $M' \subset M$ with $M' \models \perp$. Let $\mathsf{MI}(\mathcal{K})$ be the set of all MIs of $\mathcal{K}$. Informally speaking, minimal inconsistent subsets contain the *essence* of the inconsistency in a knowledge base. Every formula participating in creating an inconsistency is part of at least one minimal inconsistent subset.

A straightforward approach to use minimal inconsistent subsets for measuring inconsistency is to take their number as an indicator [12].

Definition 3. The MI-*inconsistency measure* $\mathcal{I}_{\mathsf{MI}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = |\mathsf{MI}(\mathcal{K})|$$

for $\mathcal{K} \in \mathbb{K}$.

One drawback of $\mathcal{I}_{\mathsf{MI}}$ is that it treats every inconsistent subset of $\mathcal{K}$ equally. A knowledge base with one minimal inconsistent subset of size 2 has the same inconsistency value as another knowledge base with one minimal inconsistent subset of size 10. It is usually acknowledged that a smaller minimal inconsistent subset is more severe than a larger one, cf. [12]. The MI^c *inconsistency measure* takes this into account and is defined as follows.

Definition 4. The MI^c-*inconsistency measure* $\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}(\mathcal{K}) = \sum_{M \in \mathsf{MI}(\mathcal{K})} \frac{1}{|M|}$$

for $\mathcal{K} \in \mathbb{K}$.

In other words, for every minimal inconsistent subset M of $\mathcal{K}$, $1/|M|$ is added up in order to obtain $\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}(\mathcal{K})$. In this way, larger minimal inconsistent subset contribute less to the overall inconsistency value than smaller ones.

Example 3. We continue Example 1 and consider

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\}$$
$$\mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

Here we have

$$\mathsf{MI}(\mathcal{K}_1) = \{\{a, \neg a \wedge \neg b\}\}$$
$$\mathsf{MI}(\mathcal{K}_2) = \{\{a, \neg a\}, \{b, \neg b\}\}$$

Therefore we obtain $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}_1) = 1$ and $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}_2) = 2$ and

$$\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}(\mathcal{K}_1) = \frac{1}{|\{a, \neg a \wedge \neg b\}|} = \frac{1}{2}$$
$$\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}(\mathcal{K}_2) = \frac{1}{|\{a, \neg a\}|} + \frac{1}{|\{b, \neg b\}|} = 1$$

The measure $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ has been further extended in [25] where not only the sizes of the different minimal inconsistent subsets but also their *distribution* in a knowledge base has been considered. For every knowledge base $\mathcal{K}$ and $i = 1, \ldots, |\mathcal{K}|$, define

$$\mathsf{MI}^{(i)}(\mathcal{K}) = \{M \in \mathsf{MI}(\mathcal{K}) \mid |M| = i\}$$
$$\mathsf{CN}^{(i)}(\mathcal{K}) = \{C \subseteq \mathcal{K} \mid |C| = i \wedge C \not\models \perp\}$$

That is, $\mathsf{MI}^{i}(\mathcal{K})$ is the set of minimal inconsistent subsets of $\mathcal{K}$ of size i and $\mathsf{CN}^{(i)}(\mathcal{K})$ is the set of consistent subsets of $\mathcal{K}$ of size i. Furthermore define

$$R_i(\mathcal{K}) = \begin{cases} 0 & \text{if } |\mathsf{MI}^{(i)}(\mathcal{K})| + |\mathsf{CN}^{(i)}(\mathcal{K})| = 0 \\ |\mathsf{MI}^{(i)}(\mathcal{K})| / (|\mathsf{MI}^{(i)}(\mathcal{K})| + |\mathsf{CN}^{(i)}(\mathcal{K})|) & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, |\mathcal{K}|$. The value $R_i(\mathcal{K})$ thus gives the ratio of minimal inconsistent sets of size i to the number of minimal inconsistent and consistent subsets of size i. Obviously, if for two knowledge bases $\mathcal{K}$ and $\mathcal{K}'$, a value $R_i(\mathcal{K})$ is larger than $R_i(\mathcal{K}')$ (for some i and everything else being equal) than $\mathcal{K}$ should be regarded more inconsistent than $\mathcal{K}'$. The idea of the approach of Mu et al. [25] is to weigh these values also wrt. the sizes, i.e., a large $R_i(\mathcal{K})$ has more impact than a large $R_j(\mathcal{K})$, with $j > i$.

Definition 5. The D_f*-inconsistency measure* $\mathcal{I}_{D_f} : \mathbb{K} \to \mathbb{R}^\infty_{\geq 0}$ is defined as

$$\mathcal{I}_{D_f}(\mathcal{K}) = 1 - \Pi_{i=1}^{|\mathcal{K}|}(1 - R_i(\mathcal{K})/i)$$

for $\mathcal{K} \in \mathbb{K}$.

Note that the above definition of $\mathcal{I}_{D_f}$ represents only a single instance of the family introduced in [25]. Other variants can be obtained by other ways of aggregating the values $R_1(\mathcal{K}), \ldots, R_{|\mathcal{K}|}(\mathcal{K})$.

Example 4. We continue Example 3 and recall

$$\mathsf{MI}(\mathcal{K}_1) = \{\{a, \neg a \wedge \neg b\}\}$$
$$\mathsf{MI}(\mathcal{K}_2) = \{\{a, \neg a\}, \{b, \neg b\}\}$$

Then we have

$$\mathsf{MI}^{(2)}(\mathcal{K}_1) = \{\{a, \neg a \wedge \neg b\}\} \qquad \text{and } \mathsf{MI}^{(i)}(\mathcal{K}_1) = \emptyset \quad \text{for } i \neq 2$$
$$\mathsf{MI}^{(2)}(\mathcal{K}_2) = \{\{a, \neg a\}, \{b, \neg b\}\} \qquad \text{and } \mathsf{MI}^{(i)}(\mathcal{K}_2) = \emptyset \quad \text{for } i \neq 2$$

Furthermore, we have

$$\mathsf{CN}^{(1)}(\mathcal{K}_1) = \{\{a\}, \{b \vee c\}, \{\neg a \wedge \neg b\}, \{d\}\}$$
$$\mathsf{CN}^{(2)}(\mathcal{K}_1) = \{\{a, b \vee c\}, \{a, d\}, \{b \vee c, \neg a \wedge \neg b\}, \{b \vee c, d\}, \{\neg a \wedge b, d\}\}$$
$$\mathsf{CN}^{(3)}(\mathcal{K}_1) = \{\{a, b \vee c, d\}, \{b \vee c, \neg a \wedge \neg b, d\}\}$$
$$\mathsf{CN}^{(4)}(\mathcal{K}_1) = \emptyset$$

and

$$\mathsf{CN}^{(1)}(\mathcal{K}_2) = \{\{a\}, \{\neg a\}, \{b\}, \{\neg b\}\}$$
$$\mathsf{CN}^{(2)}(\mathcal{K}_2) = \{\{a, b\}, \{a, \neg b\}, \{\neg a, b\}, \{\neg a, \neg b\}\}$$
$$\mathsf{CN}^{(3)}(\mathcal{K}_2) = \emptyset$$
$$\mathsf{CN}^{(4)}(\mathcal{K}_2) = \emptyset$$

yielding

$$R_1(\mathcal{K}_1) = 0 \qquad R_2(\mathcal{K}_1) = 1/6 \qquad R_3(\mathcal{K}_1) = 0 \qquad R_4(\mathcal{K}_1) = 0$$

and

$$R_1(\mathcal{K}_2) = 0 \qquad R_2(\mathcal{K}_2) = 2/6 \qquad R_3(\mathcal{K}_2) = 0 \qquad R_4(\mathcal{K}_2) = 0$$

Finally, we obtain

$$\begin{aligned}
\mathcal{I}_{D_f}(\mathcal{K}_1) &= 1-(1-R_1(\mathcal{K}_1))(1-R_2(\mathcal{K}_1)/2)(1-R_3(\mathcal{K}_1)/3)(1-R_4(\mathcal{K}_1)/4)\\
&= \frac{1}{6\cdot 2} = \frac{1}{12}\\
\mathcal{I}_{D_f}(\mathcal{K}_2) &= 1-(1-R_1(\mathcal{K}_2))(1-R_2(\mathcal{K}_2)/2)(1-R_3(\mathcal{K}_2)/3)(1-R_4(\mathcal{K}_2)/4)\\
&= \frac{2}{6\cdot 2} = \frac{1}{6}
\end{aligned}$$

Another simple approach for utilizing minimal inconsistent subsets is to use the number of formulas occurring in some minimal inconsistent subsets—that is, the number of *problematic* formulas—as the inconsistency value, cf. e.g. [7].

Definition 6. The *problematic inconsistency measure* $\mathcal{I}_p : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_p(\mathcal{K}) = |\bigcup_{M\in\mathsf{MI}(\mathcal{K})} M|$$

for $\mathcal{K} \in \mathbb{K}$.

Example 5. We continue Example 3 and recall

$$\mathsf{MI}(\mathcal{K}_1) = \{\{a, \neg a \wedge \neg b\}\}$$

$$\mathsf{MI}(\mathcal{K}_2) = \{\{a, \neg a\}, \{b, \neg b\}\}$$

Then $\mathcal{I}_p(\mathcal{K}_1) = 2$ and $\mathcal{I}_p(\mathcal{K}_2) = 4$.

3.3. Inconsistency measures based on maximal consistency

Another family closely related to the family of measures based on minimal inconsistent subsets is the one based on maximal consistent subsets. Let $\mathsf{MC}(\mathcal{K})$ be the set of maximal consistent subsets of $\mathcal{K}$, i.e.

$$\mathsf{MC}(\mathcal{K}) = \{\mathcal{K}' \subseteq \mathcal{K} \mid \mathcal{K}' \not\models \perp \wedge \forall \mathcal{K}'' \supsetneq \mathcal{K}' : \mathcal{K}'' \models \perp\}$$

Note, that a maximal consistent subset can be obtained by removing one formula from each minimal inconsistent subset from the knowledge base. Thus, the number of maximal consistent subsets and the number of minimal inconsistent sets correlate.

Furthermore, let $\mathsf{SC}(\mathcal{K})$ be the set of self-contradictory formulas of $\mathcal{K}$, i.e.

$$\mathsf{SC}(\mathcal{K}) = \{\phi \in \mathcal{K} \mid \phi \models \perp\}$$

An inconsistency measure that takes both maximal consistent subsets and self-contradictory formulas into account can be defined as follows, cf. e.g. [7].

Definition 7. The MC-*inconsistency measure* $\mathcal{I}_{mc} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{mc}(\mathcal{K}) = |\mathsf{MC}(\mathcal{K})| + |\mathsf{SC}(\mathcal{K})| - 1$$

for $\mathcal{K} \in \mathbb{K}$.

Note that the subtraction of 1 in the definition of $\mathcal{I}_{mc}$ is to ensure that a consistent knowledge base has inconsistency value 0 (a consistent knowledge base has one maximal consistent subset, itself, and no self-contradictory formulas).

Another approach utilizing the idea of maximum consistency is the approach of Doder et al. [5].

Definition 8. The *nc-inconsistency measure* $\mathcal{I}_{nc} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{nc}(\mathcal{K}) = |\mathcal{K}| - \max\{n \mid \forall \mathcal{K}' \subseteq \mathcal{K} : |\mathcal{K}'| = n \Rightarrow \mathcal{K}' \not\models \perp\}$$

for $\mathcal{K} \in \mathbb{K}$.

In other words, the inconsistency of $\mathcal{K}$ is assessed by seeking a maximal value $n \in \{1, \ldots, |\mathcal{K}|\}$ such that all subsets of size n of $\mathcal{K}$ are consistent. The larger this value n, the smaller the inconsistency. Note that the above definition of $\mathcal{I}_{nc}$ differs from the original definition in [5] (where only the max-term was considered) in order to ensure that consistent knowledge bases receive a value of zero and the inconsistency value increases with increasing inconsistency.

Example 6. We continue Example 1 and consider

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\}$$
$$\mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

Here we have

$$\mathsf{MC}(\mathcal{K}_1) = \{\{a, b \vee c, d\}, \{b \vee c, \neg a \wedge \neg b, d\}\}$$
$$\mathsf{MC}(\mathcal{K}_2) = \{\{a, b\}, \{a, \neg b\}, \{\neg a, b\}, \{\neg a, \neg b\}\}$$

and $\mathsf{SC}(\mathcal{K}_1) = \mathsf{SC}(\mathcal{K}_2) = \emptyset$. Therefore we obtain

$$\mathcal{I}_{mc}(\mathcal{K}_1) = 1$$
$$\mathcal{I}_{mc}(\mathcal{K}_2) = 3$$

Furthermore, note that for both $\mathcal{K}_1$ and $\mathcal{K}_2$ we can find subsets of size 2 that are inconsistent: $\{a, \neg a \wedge \neg b\}$ for $\mathcal{K}_1$ and $\{a, \neg a\}$ for $\mathcal{K}_2$. Furthermore, all one-element subsets of $\mathcal{K}_1$ and $\mathcal{K}_2$ are consistent, respectively. Therefore, we obtain

$$\mathcal{I}_{nc}(\mathcal{K}_1) = 3$$
$$\mathcal{I}_{nc}(\mathcal{K}_2) = 3$$

3.4. Probabilistic inconsistency measures

One of the first approaches to measuring inconsistency is Knight's measure $\mathcal{I}_\eta$, which is based on probability functions over the underlying propositional language [17]. Recall that $\Omega(\mathsf{At})$ is the set of interpretations of the propositional language $\mathcal{L}(\mathsf{At})$. A *probability function* P on $\mathcal{L}(\mathsf{At})$ is a function $P : \Omega(\mathsf{At}) \to [0, 1]$ with $\sum_{\omega \in \Omega(\mathsf{At})} P(\omega) = 1$. We extend P to assign a probability to any formula $\phi \in \mathcal{L}(\mathsf{At})$ by defining

$$P(\phi) = \sum_{\omega \models \phi} P(\omega)$$

Let $\mathcal{P}(\mathsf{At})$ be the set of all those probability functions. The idea of Knight [17] is to seek a probability function that maximizes the probability of each formula of a knowledge base $\mathcal{K}$. If we can find a probability function that assigns probability 1 to each formula this means that the knowledge base is consistent. If the knowledge base is inconsistent, then the probability mass must be distributed (recall that an inconsistent set of formulas cannot be satisfied by a single interpretation ω; thus the probability $P(\omega)$ can only be associated with a subset of this set). Therefore, the smaller the maximal probability that can be assigned to all formulas the more inconsistent the knowledge base.

Definition 9. The *η-inconsistency measure* $\mathcal{I}_\eta : \mathbb{K} \to \mathbb{R}^\infty_{\geq 0}$ is defined as

$$\mathcal{I}_\eta(\mathcal{K}) = 1 - \max\{\xi \mid \exists P \in \mathcal{P}(\mathsf{At}) : \forall \alpha \in \mathcal{K} : P(\alpha) \geq \xi\}$$

for $\mathcal{K} \in \mathbb{K}$.

Note that we modified the definition of $\mathcal{I}_\eta$ slightly compared to the original definition in order to ensure that consistent knowledge bases receive an inconsistency value of zero.

Instead of seeking a probability function to maximize the probabilities of the formulas in $\mathcal{K}$, one can simplify this idea and seek only a minimal set of interpretations that need to receive a positive probability in order to ensure that every formula has a positive probability [31]. In other words, a subset $H \subseteq \Omega(\mathsf{At})$ is called a *hitting set* of $\mathcal{K}$ if for every $\phi \in \mathcal{K}$ there is $\omega \in H$ with $\omega \models \phi$. Focusing only on minimizing the number of interpretations needed to form a hitting set we can define another measure as follows.

Definition 10. The *hitting-set inconsistency measure* $\mathcal{I}_{hs} : \mathbb{K} \to \mathbb{R}^\infty_{\geq 0}$ is defined as

$$\mathcal{I}_{hs}(\mathcal{K}) = \min\{|H| \mid H \text{ is a hitting set of } \mathcal{K}\} - 1$$

for $\mathcal{K} \in \mathbb{K}$ with $\min \emptyset = \infty$.

Note that $\mathcal{I}_{hs}(\mathcal{K}) = \infty$ if and only if $\mathcal{K}$ contains a self-contradictory formula, i.e., $\alpha \in \mathcal{K}$ with $\alpha \models \perp$. In this case, no hitting set of $\mathcal{K}$ exists.

Table 2
Truth tables for propositional three-valued logic [27].

α	β	$\upsilon(\alpha \wedge \beta)$	$\upsilon(\alpha \vee \beta)$
T	T	T	T
T	B	B	T
T	F	F	T
B	T	B	T
B	B	B	B
B	F	F	B
F	T	F	T
F	B	F	B
F	F	F	F

α	$\upsilon(\neg\alpha)$
T	F
B	B
F	T

Example 7. We continue Example 1 and consider

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\}$$

$$\mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

Consider the probability function $P_1 \in \mathcal{P}(\{a, b, c, d\})$ defined via

$$P_1(abcd) = P_1(\bar{a}\bar{b}cd) = 0.5$$

$$P_1(\omega) = 0 \quad \text{for } \omega \in \Omega(\{a, b, c, d\}) \setminus \{abcd, \bar{a}\bar{b}cd\}$$

Then we obtain

$$P_1(a) = P_1(\neg a \wedge \neg b) = 0.5$$

$$P_1(b \vee c) = P_1(d) = 1$$

and thus $P_1(\phi) \geq 0.5$ for all $\phi \in \mathcal{K}_1$. Furthermore, there can be no other P' that assigns larger probability to all $\phi \in \mathcal{K}_1$. Hence, we have $\mathcal{I}_\eta(\mathcal{K}_1) = 1 - 0.5 = 0.5$. The function P_1 can also be used to determine $\mathcal{I}_\eta(\mathcal{K}_2) = 0.5$.

The set $H_1 = \{abcd, \bar{a}\bar{b}cd\}$ is also a hitting set of both $\mathcal{K}_1$ and $\mathcal{K}_2$ and there is no smaller set that is a hitting set. Therefore we obtain $\mathcal{I}_{hs}(\mathcal{K}_1) = \mathcal{I}_{hs}(\mathcal{K}_2) = 1$.

3.5. Variable-based inconsistency measures

Another approach to assess the severity of inconsistency is to take the number of propositions from At that participate in the inconsistency. The approach of Xiao and Ma [32] is to take the ratio of the propositions appearing in a minimal inconsistent subset wrt. the total number of propositions as the inconsistency value.

Definition 11. The *mv inconsistency measure* $\mathcal{I}_{mv} : \mathbb{K} \to \mathbb{R}^\infty_{\geq 0}$ is defined as

$$\mathcal{I}_{mv}(\mathcal{K}) = \frac{|\bigcup_{M \in \mathsf{MI}(\mathcal{K})} \mathsf{At}(M)|}{|\mathsf{At}(\mathcal{K})|}$$

for $\mathcal{K} \in \mathbb{K}$.

In other words, $\mathcal{I}_{mv}(\mathcal{K})$ is the ratio of the signature involved in minimal inconsistent subsets.

Instead of utilizing minimal inconsistent subsets one can also using paraconsistent semantics to identify the part of the signature involved in inconsistency. In this paper, we will only consider the *contension measure* $\mathcal{I}_c$—cf. e.g. [7]—and its normalized variant $\mathcal{I}_{LP_m}$ from [13] as representatives of this family of measures. Similar approaches relying on the same ideas can be found in e.g. [21,20].

The contension measure $\mathcal{I}_c$ utilizes three-valued interpretations for propositional logic [27]. A three-valued interpretation υ on At is a function $\upsilon : \mathsf{At} \to \{T, F, B\}$ where the values T and F correspond to the classical true and false, respectively. The additional truth value B stands for *both* and is meant to represent a conflicting truth value for a proposition. The function υ is extended to arbitrary formulas as shown in Table 2. Then, an interpretation υ satisfies a formula α, denoted by $\upsilon \models^3 \alpha$ if either $\upsilon(\alpha) = T$ or $\upsilon(\alpha) = B$. Then inconsistency can be measured by seeking an interpretation υ that assigns B to a minimal number of propositions.

Definition 12. The *contension inconsistency measure* $\mathcal{I}_c : \mathbb{K} \to \mathbb{R}^\infty_{\geq 0}$ is defined as

$$\mathcal{I}_c(\mathcal{K}) = \min\{|\upsilon^{-1}(B)| \mid \upsilon \models^3 \mathcal{K}\}$$

for $\mathcal{K} \in \mathbb{K}$.

In [13] a variant $\mathcal{I}_{LP_m}$ of this measure was defined that further normalizes the inconsistency value by the number of propositions appearing in $\mathcal{K}$.

Definition 13. The *normalized contension inconsistency measure* $\mathcal{I}_{LP_m} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{LP_m}(\mathcal{K}) = \frac{\mathcal{I}_c(\mathcal{K})}{|\mathsf{At}(\mathcal{K})|}$$

for $\mathcal{K} \in \mathbb{K}$.

Example 8. We continue Example 1 and consider

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\}$$
$$\mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

and recall

$$\mathsf{MI}(\mathcal{K}_1) = \{\{a, \neg a \wedge \neg b\}\}$$
$$\mathsf{MI}(\mathcal{K}_2) = \{\{a, \neg a\}, \{b, \neg b\}\}$$

Then we have

$$\mathcal{I}_{mv}(\mathcal{K}_1) = \frac{|\{a, b\}|}{|\{a, b, c, d\}|} = 1/2$$
$$\mathcal{I}_{mv}(\mathcal{K}_2) = \frac{|\{a, b\}|}{|\{a, b\}|} = 1$$

Furthermore, consider $\upsilon_1 : \{a, b, c, d\} \to \{T, F, B\}$ defined via

$$\upsilon_1(a) = B \qquad \upsilon_1(b) = F \qquad \upsilon_1(c) = \upsilon_1(d) = T$$

Then $\upsilon_1 \models^3 \phi$ for all $\phi \in \mathcal{K}_1$ and there is no other υ' that assigns B to fewer propositions, yielding $\mathcal{I}_c(\mathcal{K}_1) = 1$ and $\mathcal{I}_{LP_m}(\mathcal{K}_1) = 1/4$. For $\upsilon_2 : \{a, b\} \to \{T, F, B\}$ defined via

$$\upsilon_2(a) = \upsilon_2(b) = B$$

we have $\upsilon_2 \models^3 \phi$ for all $\phi \in \mathcal{K}_2$ and there is no other υ' that assigns B to fewer propositions, yielding $\mathcal{I}_c(\mathcal{K}_2) = 2$ and $\mathcal{I}_{LP_m}(\mathcal{K}_2) = 2/2 = 1$.

3.6. Distance-based inconsistency measures

In [8] three families of inconsistency measures are defined that are based on a notion of distance to consistency. More precisely, an *interpretation distance d* is a function $d : \Omega(\mathsf{At}) \times \Omega(\mathsf{At}) \to [0, \infty)$ that satisfies (let $\omega, \omega', \omega'' \in \Omega(\mathsf{At})$)

1. $d(\omega, \omega') = 0$ if and only if $\omega = \omega'$ (*reflexivity*),
2. $d(\omega, \omega') = d(\omega', \omega)$ (*symmetry*), and
3. $d(\omega, \omega'') \leq d(\omega, \omega') + d(\omega', \omega'')$ (*triangle inequality*).

One prominent example of such a distance is the *Dalal distance* d_d defined via

$$d_\mathrm{d}(\omega, \omega') = |\{a \in \mathsf{At} \mid \omega(a) \neq \omega'(a)\}|$$

for all $\omega, \omega' \in \Omega(\mathsf{At})$. In other words, $d_\mathrm{d}(\omega, \omega')$ is the number of propositions where ω and ω' assign different truth values. If $X \subseteq \Omega(\mathsf{At})$ is a set of interpretations we define $d_\mathrm{d}(X, \omega) = \min_{\omega' \in X} d_\mathrm{d}(\omega', \omega)$ (if $X = \emptyset$ we define $d_\mathrm{d}(X, \omega) = \infty$). While [8] consider arbitrary distances, we will focus here on the Dalal distance for reasons of simplicity.

The basic idea of the approaches in [8] is to measure and aggregate the distances of the models of the formulas in a knowledge base $\mathcal{K}$. For example, if a knowledge base $\mathcal{K}$ has two formulas and their models have a large distance to each other, then $\mathcal{K}$ should be regarded as more inconsistent compared to a knowledge base $\mathcal{K}'$ with two formulas, where this is not the case. More precisely, the first approach from [8] considered here seeks an interpretation ω such that the sum of all distances of the sets of models to ω is minimal.

Definition 14. The Σ-*distance inconsistency measure* $\mathcal{I}^{\Sigma}_{\text{dalal}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}) = \min \left\{ \sum_{\alpha \in \mathcal{K}} d_{\mathrm{d}}(\mathsf{Mod}(\alpha), \omega) \mid \omega \in \Omega(\mathsf{At}) \right\}$$

for $\mathcal{K} \in \mathbb{K}$.

Another approach of [8] is to seek an interpretation ω such that the maximum distance of the sets of models is minimal.

Definition 15. The max-*distance inconsistency measure* $\mathcal{I}^{\max}_{\text{dalal}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}) = \min \left\{ \max_{\alpha \in \mathcal{K}} d_{\mathrm{d}}(\mathsf{Mod}(\alpha), \omega) \mid \omega \in \Omega(\mathsf{At}) \right\}$$

for $\mathcal{K} \in \mathbb{K}$.

The final approach of [8] is to minimize the number of formulas, where their corresponding sets of models have a positive distance.

Definition 16. The *hit-distance inconsistency measure* $\mathcal{I}^{\text{hit}}_{\text{dalal}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}) = \min \{ |\{\alpha \in \mathcal{K} \mid d_{\mathrm{d}}(\mathsf{Mod}(\alpha), \omega) > 0\}| \mid \omega \in \Omega(\mathsf{At}) \}$$

for $\mathcal{K} \in \mathbb{K}$.

Example 9. We continue Example 1 and consider

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\}$$
$$\mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

Observe that for the interpretation $\omega_1 = a\bar{b}cd \in \Omega(\{a, b, c, d\})$ we have

$$\begin{aligned} d_{\mathrm{d}}(\mathsf{Mod}(a), \omega_1) &= 0 \\ d_{\mathrm{d}}(\mathsf{Mod}(b \vee c), \omega_1) &= 0 \\ d_{\mathrm{d}}(\mathsf{Mod}(\neg a \wedge \neg b), \omega_1) &= 1 \\ d_{\mathrm{d}}(\mathsf{Mod}(d), \omega_1) &= 0 \end{aligned}$$

and therefore $\sum_{\alpha \in \mathcal{K}_1} d_{\mathrm{d}}(\mathsf{Mod}(\alpha), \omega_1) = 1$. There is no other interpretation ω' with a smaller total distance, so we have $\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}_1) = 1$. Furthermore, we have $\max_{\alpha \in \mathcal{K}_1} d_{\mathrm{d}}(\mathsf{Mod}(\alpha), \omega_1) = 1$ and there is also no other interpretation ω' with a smaller maximum distance. Hence, we have $\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}_1) = 1$ and similarly $\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}_1) = 1$. For $\mathcal{K}_2$ we obtain

$$\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}_2) = 2$$
$$\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}_2) = 1$$

with a similar argumentation as above. For $\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}_2)$ observe that every interpretation ω must always falsify exactly one formula in $\{a, \neg a\}$ and exactly one formula in $\{b, \neg b\}$. Therefore we obtain $\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}_2) = 2$.

3.7. Proof-based inconsistency measures

The final measure we consider in this paper is the proof-based measure from [16]. The basic idea is to count, for all propositions $x \in \mathsf{At}$, both the number of *minimal proofs* for x and its negation $\neg x$. If there are many proofs for both, this indicates a large inconsistency in the knowledge base. In the context of [16] a *minimal proof* for $\alpha \in \{x, \neg x \mid x \in \mathsf{At}\}$ in $\mathcal{K}$ is a set $\pi \subseteq \mathcal{K}$ such that

1. α appears as a literal in π
2. $\pi \models \alpha$, and
3. π is minimal wrt. set inclusion.

Note that this definition does not require that π is consistent. In particular, the set $\{a \wedge \neg a\}$ is a minimal proof for both a and $\neg a$. Note furthermore, that item 1.) requires that α appears in the exact same form in π, e.g. a appears in $a \wedge b$ but not in $\neg a \wedge b$ (this is a syntactic criterion).

Let $P_m^{\mathcal{K}}(x)$ be the set of all minimal proofs of x in $\mathcal{K}$. The proof-based measure of Jabbour and Raddaoui [16] can then be defined by summing up the products of the number of minimal proofs for complementary literals.

Definition 17. The *proof-based inconsistency measure* $\mathcal{I}_{P_m} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{P_m}(\mathcal{K}) = \sum_{a \in \mathsf{At}} |P_m^{\mathcal{K}}(a)| \cdot |P_m^{\mathcal{K}}(\neg a)|$$

for $\mathcal{K} \in \mathbb{K}$.

Note that the definition of $\mathcal{I}_{P_m}$ is not the original definition but a characterization also provided in [16].

Example 10. We continue Example 1 and consider

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\}$$
$$\mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

Observe that

$$\begin{aligned}
P_m^{\mathcal{K}_1}(a) &= \{\{a\}\} \\
P_m^{\mathcal{K}_1}(\neg a) &= \{\{\neg a \wedge \neg b\}\} \\
P_m^{\mathcal{K}_1}(b) &= \{\{a, b \vee c, \neg a \wedge \neg b\}\} \\
P_m^{\mathcal{K}_1}(\neg b) &= \{\{\neg a \wedge \neg b\}\} \\
P_m^{\mathcal{K}_1}(c) &= \{\{a, b \vee c, \neg a \wedge \neg b\}\} \\
P_m^{\mathcal{K}_1}(\neg c) &= \emptyset \\
P_m^{\mathcal{K}_1}(d) &= \{\{d\}\} \\
P_m^{\mathcal{K}_1}(\neg d) &= \emptyset
\end{aligned}$$

and

$$\begin{aligned}
P_m^{\mathcal{K}_2}(a) &= \{\{a\}\} \\
P_m^{\mathcal{K}_2}(\neg a) &= \{\{\neg a\}\} \\
P_m^{\mathcal{K}_2}(b) &= \{\{b\}\} \\
P_m^{\mathcal{K}_2}(\neg b) &= \{\{\neg b\}\}
\end{aligned}$$

It follows that

$$\mathcal{I}_{P_m}(\mathcal{K}_1) = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 = 2$$
$$\mathcal{I}_{P_m}(\mathcal{K}_2) = 1 \cdot 1 + 1 \cdot 1 = 2$$

4. Expressivity characteristics

In the literature, inconsistency measures are usually analytically evaluated on a set of *rationality postulates*.[2] Some basic example postulates given in [11] are the following (let $\mathcal{I}$ be any inconsistency measure)

Consistency $\mathcal{I}(\mathcal{K}) = 0$ if and only if $\mathcal{K}$ is consistent
Monotony if $\mathcal{K} \subseteq \mathcal{K}'$ then $\mathcal{I}(\mathcal{K}) \leq \mathcal{I}(\mathcal{K}')$
Independence for all $\alpha \in \mathcal{K}$, if $\alpha \notin M$ for every $M \in \mathsf{MI}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$

Satisfaction of the property *consistency* ensures that all consistent knowledge bases receive a minimal inconsistency value and every inconsistent knowledge base receives a positive inconsistency value (we already implicitly required satisfaction of

2 Some few works also consider empirical evaluation on computational performance and accuracy of algorithms approximating existing inconsistency measures, see e.g. [22,24,31].

this postulate in the definition of an inconsistency measure). The postulate *monotony* states that the value of inconsistency can only increase when adding new information. *Independence* states that removing "harmless" formulas from a knowledge base does not change the value of inconsistency. Besides these three postulates a series of other postulates have been proposed in the literature, see e.g. [11,25,2]. However, some of these postulates are disputed as each of them usually covers only a single aspect of inconsistency, such as *independence*, which focuses on the role of minimal inconsistent subsets. An excellent discussion on the rationality of various postulates for inconsistency measures can be found in [2]. Besides Besnard, several other authors have also criticized the rationality of individual postulates—discussions can be found in almost all papers cited before—and so there is some disagreement on which postulates are meaningful and which are not. One the one hand this calls for more work on rationality postulates and, on the other hand, it also suggests to investigate additional means for comparison. In the following, we propose a novel quantitative approach to evaluate and compare inconsistency measures that aims at complementing the existing approach of rationality postulates.

The drastic inconsistency measure $\mathcal{I}_d$ (see Table 1) is usually considered as a very naive baseline approach for inconsistency measurement. Surprisingly, this measure already satisfies many rationality postulates such as *consistency*, *monotony*, and *independence* (the proofs are straightforward). What sets it apart from other more sophisticated inconsistency measures is that it cannot differentiate between different inconsistent knowledge bases. However, this demand is exactly what inconsistency measures are supposed to satisfy. While the *qualitative* behavior of inconsistency measures is being discussed quite deeply using rationality postulates, their *quantitative* properties in terms of *expressivity* have been almost neglected so far.[3] With expressivity of inconsistency measures we here mean the number of different values an inconsistency measure can attain. We investigate the expressivity of inconsistency measures along four different dimensions of subclasses of knowledge bases.

Definition 18. Let ϕ be a formula. The *length* $\mathsf{l}(\phi)$ of ϕ is recursively defined as

$$\mathsf{l}(\phi) = \begin{cases} 1 & \text{if } \phi \in \mathsf{At} \\ 1 + \mathsf{l}(\phi') & \text{if } \phi = \neg\phi' \\ 1 + \mathsf{l}(\phi_1) + \mathsf{l}(\phi_2) & \text{if } \phi = \phi_1 \wedge \phi_2 \\ 1 + \mathsf{l}(\phi_1) + \mathsf{l}(\phi_2) & \text{if } \phi = \phi_1 \vee \phi_2 \end{cases}$$

Definition 19. Define the following subclasses of the set of all knowledge bases $\mathbb{K}$:

$$\mathbb{K}^v(n) = \{\mathcal{K} \in \mathbb{K} \mid |\mathsf{At}(\mathcal{K})| \leq n\}$$
$$\mathbb{K}^f(n) = \{\mathcal{K} \in \mathbb{K} \mid |\mathcal{K}| \leq n\}$$
$$\mathbb{K}^l(n) = \{\mathcal{K} \in \mathbb{K} \mid \forall \phi \in \mathcal{K} : \mathsf{l}(\phi) \leq n\}$$
$$\mathbb{K}^p(n) = \{\mathcal{K} \in \mathbb{K} \mid \forall \phi \in \mathcal{K} : |\mathsf{At}(\phi)| \leq n\}$$

In other words, $\mathbb{K}^v(n)$ is the set of all knowledge bases that mention at most n different propositions, $\mathbb{K}^f(n)$ is the set of all knowledge bases that contain at most n formulas, $\mathbb{K}^l(n)$ is the set of all knowledge bases that contain only formulas with maximal length n, and $\mathbb{K}^p(n)$ is the set of all knowledge bases that contain only formulas that mention at most n different propositions each. The motivation for considering these particular subclasses of knowledge bases is that each of them considers a different aspect of the size of a knowledge base. As a syntactical object, a knowledge base is a set of formulas, and both the number of formulas (considered by the class $\mathbb{K}^f(n)$) and the length of each formula ($\mathbb{K}^l(n)$) are the essential parameters that define its size. From a semantical point of view, the number of propositions appearing in each formula ($\mathbb{K}^p(n)$) and in the complete knowledge base ($\mathbb{K}^v(n)$) define the scope of the knowledge. Larger numbers for both of them also indicate larger scope and thus greater size. Inconsistency measures should adhere to the size of the knowledge base in terms of their expressivity. For example, the number of possible inconsistency values of a particular measure should not decrease when moving from a set $\mathbb{K}^v(n)$ to set $\mathbb{K}^v(n')$ with $n' > n$, as knowledge bases with n' formulas should provide a larger variety in terms of inconsistency as knowledge bases of size n. Indeed, this property is true for all considered measures as $\mathbb{K}^v(n) \subseteq \mathbb{K}^v(n')$ (the same holds for all classes above).

The aim of our expressivity analysis is to investigate the number of different values that a specific inconsistency measure can attain on different subclasses of knowledge bases. We formalize this idea using *expressivity characteristics* as follows.

Definition 20. Let $\mathcal{I}$ be an inconsistency measure and $n > 0$. Let $\alpha \in \{v, f, l, p\}$. The *$\alpha$-characteristic* $\mathcal{C}^\alpha(\mathcal{I}, n)$ of $\mathcal{I}$ wrt. n is defined as

$$\mathcal{C}^\alpha(\mathcal{I}, n) = |\{\mathcal{I}(\mathcal{K}) \mid \mathcal{K} \in \mathbb{K}^\alpha(n)\}|$$

[3] Some few rationality postulates such as *Attenuation* [25] are addressing this issue only in some very limited form and from a particular point of view.

Table 3
Characteristics of inconsistency measures ($n \geq 1$).

	$\mathcal{C}^v(\mathcal{I},n)$	$\mathcal{C}^f(\mathcal{I},n)$	$\mathcal{C}^l(\mathcal{I},n)$	$\mathcal{C}^p(\mathcal{I},n)$
$\mathcal{I}_d$	2	2	2^*	2
$\mathcal{I}_{\mathsf{MI}}$	∞	$\binom{n}{\lfloor n/2 \rfloor}+1$	∞^*	∞
$\mathcal{I}_{\mathsf{MI^C}}$	∞	$\leq \Psi(n)^{\mathrm{a}}$	∞^*	∞
$\mathcal{I}_\eta$	$\Phi(2^n)^{\mathrm{a}}$	$\leq \Phi(\binom{n}{\lfloor n/2 \rfloor}))^{\mathrm{b}}$	∞^{**}	∞^*
$\mathcal{I}_c$	$n+1$	∞	∞^*	∞
$\mathcal{I}_{LP_m}$	$\Phi(n)$	∞	∞^*	∞
$\mathcal{I}_{mc}$	∞	$\binom{n}{\lfloor n/2 \rfloor}^{**}$	∞^*	∞
$\mathcal{I}_p$	∞	$n+1$	∞^*	∞
$\mathcal{I}_{hs}$	2^n+1	$n+1$	∞^{**}	∞^*
$\mathcal{I}^{\Sigma}_{\mathrm{dalal}}$	∞	∞^*	∞^*	∞
$\mathcal{I}^{\max}_{\mathrm{dalal}}$	$n+2$	∞^*	$\lfloor (n+7)/3 \rfloor^{**}$	$n+2$
$\mathcal{I}^{\mathrm{hit}}_{\mathrm{dalal}}$	∞	$n+1$	∞^*	∞
$\mathcal{I}_{D_f}$	∞	$\leq \Psi(n)^{\mathrm{a}}$	∞^*	∞
$\mathcal{I}_{P_m}$	∞	∞	∞^*	∞
$\mathcal{I}_{mv}$	$n+1$	∞^*	∞^*	∞
$\mathcal{I}_{nc}$	∞	$n+1$	∞^*	∞

[*] Only for $n > 1$.
[**] Only for $n > 3$.
[a] $\Psi(n)$ is the number of profiles of monotone Boolean functions of n variables, see e.g. http://oeis.org/A220880.
[b] $\Phi(x)$ is the number of fractions in the Farey series of order x and can be defined as $\Phi(x) = |\{k/l \mid l = 1,\ldots,x, k = 0,\ldots,l\}|$, see e.g. http://oeis.org/A005728.

In other words, $\mathcal{C}^\alpha(\mathcal{I},n)$ is the number of different inconsistency values $\mathcal{I}$ assigns to knowledge bases from $\mathbb{K}^\alpha(n)$. Note that these characteristics are not always the same as the *maximal* value of an inconsistency measure on a specific set of knowledge bases, even if the codomain of the measure is the natural numbers. Indeed, it can be the case that intermediate values cannot be attained.

Example 11. Consider $\mathcal{I}_\eta$ which has the codomain $[0,1]$ as each value $\mathcal{I}_\eta(\mathcal{K})$ can be associated with a probability value, cf. Table 1. In Knight [17] it has already been shown that $\mathcal{I}_\eta(\mathcal{K})$ is always a rational number, so $\sqrt{2}/2 \notin \mathrm{Im}\,\mathcal{I}_\eta$.[4] Furthermore, the possible values of $\mathcal{I}_\eta$ are further constrained when considering specific subclasses from above. For example, for every arbitrary knowledge base $\mathcal{K}$ which contains at most 2 formulas, the only possible values of $\mathcal{I}_\eta(\mathcal{K})$ are 0, 1/2, 1, so we have $\mathcal{C}^f(\mathcal{I}_\eta,2) = 3$.

We now come to the main contribution of this paper, which is a thorough study of the 16 considered inconsistency measures in terms of our four proposed expressivity characteristics.

Theorem 1. *The α-characteristics $\mathcal{C}^\alpha(\mathcal{I},n)$ ($\alpha \in \{f,v,l,p\}$) for the inconsistency measures $\mathcal{I}_d$, $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{\mathsf{MI^C}}$, $\mathcal{I}_\eta$, $\mathcal{I}_c$, $\mathcal{I}_{LP_m}$, $\mathcal{I}_{mc}$, $\mathcal{I}_p$, $\mathcal{I}_{hs}$, $\mathcal{I}^{\Sigma}_{\mathrm{dalal}}$, $\mathcal{I}^{\max}_{\mathrm{dalal}}$, $\mathcal{I}^{\mathrm{hit}}_{\mathrm{dalal}}$, $\mathcal{I}_{D_f}$, $\mathcal{I}_{P_m}$, $\mathcal{I}_{mv}$, and $\mathcal{I}_{nc}$ are as shown in Table 3.*

The complete proof of the above theorem can be found in Appendix A. However, some of these proofs provide some interesting insights into the behavior of particular inconsistency measures and provide relations to other specific mathematical branches. Therefore, we will discuss these insights in more detail in the following subsections before we continue with the actual discussion on comparing expressivity in Section 5.

4.1. Sperner families and minimal inconsistent sets

Many of the inconsistency measures discussed above use the notion of minimal inconsistent subset as a central tool for assessing the inconsistency of a knowledge base. In its simplest implementation, the inconsistency measure $\mathcal{I}_{\mathsf{MI}}$ is defined to be exactly the number of the minimal inconsistent subsets of a knowledge base. Accordingly, in order to determine the number $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}},n)$ it is necessary to investigate how many different minimal inconsistent subsets a knowledge base with n formulas may possess. This question has already been investigated from a more abstract perspective in set theory under the notion of *Sperner families* (also called *independent systems*).

Definition 21. Let $\mathcal{S} = \{S_1,\ldots,S_n\}$ with $n \geq 1$ be a family of sets over a set $X \neq \emptyset$, i.e., $S_i \subseteq X$ for all $i = 1,\ldots,n$. The family S is called a *Sperner family* over X if for all $S, S' \in \mathcal{S}$ with $S \neq S'$, $S \not\subseteq S'$.

[4] $\mathrm{Im}\,f$ is the image of a function f.

In other words, $\mathcal{S}$ is a Sperner family if none of its elements is contained in another. It can easily be seen that for every *inconsistent* knowledge base $\mathcal{K}$ the set $\mathsf{MI}(\mathcal{K})$ is also a Sperner family over $\mathcal{K}$: for $M, M' \in \mathsf{MI}(\mathcal{K})$ with $M \neq M'$ it cannot hold $M \subseteq M'$, otherwise M' would not be a minimal inconsistent set. Note that a consistent knowledge base yields $\mathsf{MI}(\mathcal{K}) = \emptyset$ which is not covered by the above definition.

As $\mathsf{MI}(\mathcal{K})$ is a Sperner family over $\mathcal{K}$ its maximal cardinality is bounded by the maximal cardinality of any Sperner family over $\mathcal{K}$. For the latter we have the following result.

Theorem 2. *(See [28].) Let X be a set with $n = |X|$.*

1. *There is a Sperner family $\mathcal{S}_{\max}$ over X with $|\mathcal{S}_{\max}| = \binom{n}{\lfloor n/2 \rfloor}$.*[5]
2. *For every Sperner family $\mathcal{S}'$ over X, $|\mathcal{S}'| \leq |\mathcal{S}_{\max}|$.*

A corollary of the above theorem is that $|\mathsf{MI}(\mathcal{K})| \leq \binom{|\mathcal{K}|}{\lfloor |\mathcal{K}|/2 \rfloor}$ for every knowledge base $\mathcal{K}$. Also taking into account that $|\mathsf{MI}(\mathcal{K})|$ is always a non-negative integer we can directly entail $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}}, n) \leq \binom{n}{\lfloor n/2 \rfloor} + 1$ (the addition of 1 is due to the fact that a consistent knowledge base $\mathcal{K}$ yields $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = 0$, which is not covered by the above theorem).

Interestingly, the set $\mathsf{MI}(\mathcal{K})$ is not only a Sperner family, but every Sperner family can be represented as $\mathsf{MI}(\mathcal{K})$ of some knowledge base $\mathcal{K}$. Let $X = \{\alpha_1, \ldots, \alpha_n\}$ be any set and let $\mathcal{S}$ be a Sperner family over X. Consider a propositional signature $\mathsf{At} = \{a_1, \ldots, a_n\}$ where each proposition $a_i \in \mathsf{At}$ corresponds to the element $\alpha_i \in X$. Consider now a knowledge base $\mathcal{K}_n^{\mathcal{S}} = \{\phi_1, \ldots, \phi_n\}$ defined via

$$\phi_i = a_i \wedge \bigwedge_{M \in \mathcal{S}, \alpha_i \in M} \bigvee_{\alpha_j \in M \setminus \{\alpha_i\}} \neg a_j$$

for $i = 1, \ldots, n$. Every ϕ_i states that a_i is accepted and for each set M in $\mathcal{S}$ which contains α_i at least one of the other elements must not be accepted. Thus, every ϕ_i lists the conditions under which any set containing α_i does *not* contain an element of the Sperner family.

Example 12. Let $X = \{\alpha, \beta, \gamma, \delta\}$ be a set and consider the Sperner family $\mathcal{S} = \{S_1, S_2, S_3\}$ over X defined via

$$S_1 = \{\beta, \gamma, \delta\} \qquad S_2 = \{\alpha, \beta\} \qquad S_3 = \{\alpha, \gamma\}$$

Consider now the signature $\mathsf{At} = \{a, b, c, d\}$ where the proposition a corresponds to α, b to β, c to γ, and d to δ. Then $\mathcal{K}_4^{\mathcal{S}} = \{\phi_1, \phi_2, \phi_3, \phi_4\}$ is defined via

$$\phi_1 = a \wedge \neg b \wedge \neg c$$
$$\phi_2 = b \wedge (\neg c \vee \neg d) \wedge \neg a$$
$$\phi_3 = c \wedge (\neg b \vee \neg d) \wedge \neg a$$
$$\phi_4 = d \wedge (\neg b \vee \neg c)$$

For example, ϕ_3 states that if some set S contains γ (c), either not β or not δ $(\neg b \vee \neg d)$, and not α $(\neg a)$, then S does not contain any element of $\mathcal{S}$.

By construction, it follows that $M = \{\alpha_{k_1}, \ldots, \alpha_{k_m}\}$ (for some $k_1, \ldots, k_m \in \{1, \ldots, n\}$) is an element of the Sperner family $\mathcal{S}$ if and only if the set $\{\phi_{k_1}, \ldots, \phi_{k_m}\}$ is a minimal inconsistent subset of $\mathcal{K}_n^{\mathcal{S}} = \{\phi_1, \ldots, \phi_n\}$.

Example 13. We continue Example 12 and consider $S_1 = \{\beta, \gamma, \delta\} \in \mathcal{S}$. The set S_1 corresponds to the set $\{\phi_2, \phi_3, \phi_4\} \subseteq \mathcal{K}_4^{\mathcal{S}}$ with

$$\phi_2 = b \wedge (\neg c \vee \neg d) \wedge \neg a$$
$$\phi_3 = c \wedge (\neg b \vee \neg d) \wedge \neg a$$
$$\phi_4 = d \wedge (\neg b \vee \neg c)$$

As one can see $\{\phi_2, \phi_3, \phi_4\}$ is also a minimal inconsistent subset of $\mathcal{K}_4^{\mathcal{S}}$. Further, the set of minimal inconsistent subsets of $\mathcal{K}_4^{\mathcal{S}}$ is indeed

$$\mathsf{MI}(\mathcal{K}_4^{\mathcal{S}}) = \{\{\phi_2, \phi_3, \phi_4\}, \{\phi_1, \phi_2\}, \{\phi_1, \phi_3\}\}$$

which is in direct correspondence to $\mathcal{S}$.

[5] $\mathcal{S}_{\max}$ can be constructed by taking the set of all subsets $S \subseteq X$ with $|S| = \lfloor n/2 \rfloor$.

From these observations it follows that $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}_n^{\mathcal{S}}) = |\mathsf{MI}(\mathcal{K}_n^{\mathcal{S}})| = |\mathcal{S}|$. Observe that if $\mathcal{S}$ is a Sperner family over X and $S \in \mathcal{S}$ then $\mathcal{S} \setminus \{S\}$ is also a Sperner family over X (provided that it is non-empty). Then for a set X with $|X| = n$ and for every $i = 1, \ldots, \binom{n}{\lfloor n/2 \rfloor}$ we can define a Sperner family $\mathcal{S}_i$ over X with $|\mathcal{S}_i| = i$. This means in our setting, for every $i = 1, \ldots, \binom{n}{\lfloor n/2 \rfloor}$ we can define a knowledge base $\mathcal{K}$ with $|\mathcal{K}| = n$ such that $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = i$. Together with the fact that 0 is also a possible value of $\mathcal{I}_{\mathsf{MI}}$ we obtain $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}}, n) = \binom{n}{\lfloor n/2 \rfloor} + 1$.

Furthermore, also the set of maximal consistent subsets $\mathsf{MC}(\mathcal{K})$ of a knowledge base $\mathcal{K}$ form a Sperner family over $\mathcal{K}$, which is the reason why the value $\mathcal{C}^f(\mathcal{I}_{mc}, n) = \binom{n}{\lfloor n/2 \rfloor}$ is almost the same as $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}}, n)$ (please see Appendix A for the detailed proof of $\mathcal{C}^f(\mathcal{I}_{mc}, n) = \binom{n}{\lfloor n/2 \rfloor}$).

4.2. Monotone Boolean functions and minimal inconsistent sets

Another interesting observation is the relation of the measures $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ and $\mathcal{I}_{D_f}$ with the number $\Psi(n)$ of profiles of monotone Boolean functions of n variables [29]. Let us recall some basics on Boolean functions.

Definition 22. A *Boolean function* f of n variables ($n \in \mathbb{N}_0$) is a function $f : \{0,1\}^n \to \{0,1\}$.

Let $B(n)$ be the set of all Boolean functions of n variables. It can easily be seen that $|B(n)| = 2^{2^n}$.

Definition 23. A Boolean function f of n variables ($n \in \mathbb{N}_0$) is *monotone* if for every $i = 1, \ldots, n$ we have

$$f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \leq f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

Let $MB(n)$ be the set of all monotone Boolean functions of n variables. Obviously $|MB(n)| \leq |B(n)|$. However, no closed form for $m(n) = |MB(n)|$—which is also called *Dedekind number*[6] [3]—is known and only values up to $n = 8$ could be determined yet, cf. [29]. The Dedekind number $m(n)$ has also a meaning in the context of Sperner families (see previous subsection). In fact, $m(n)$ is also the number of *different* Sperner families over a set X with $|X| = n$.

For the issue of analyzing inconsistency measures, the Dedekind number $m(n)$ itself is not directly applicable as it is somewhat *syntax-sensitive*. More precisely, consider the set $X = \{\alpha, \beta, \gamma\}$ and the two Sperner families $\{\{\alpha, \beta\}\}$ and $\{\{\beta, \gamma\}\}$. Each of these families count one in the Dedekind number $m(3)$. From the perspective of inconsistency measurement, the sets of minimal inconsistent subsets $\mathsf{MI}(\mathcal{K}_1) = \{\{\alpha, \beta\}\}$ and $\mathsf{MI}(\mathcal{K}_2) = \{\{\alpha, \gamma\}\}$ for some knowledge bases $\mathcal{K}_1, \mathcal{K}_2 \in \mathbb{K}^f(3)$ are *indistinguishable* for all inconsistency measures solely based on utilizing minimal inconsistent sets. While for $\mathcal{I}_{\mathsf{MI}}$ only the *number* of minimal inconsistent sets is important, even for more elaborate measures such as $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ and $\mathcal{I}_{D_f}$ these sets are equivalent as they coincide in both the number of minimal inconsistent sets and the cardinalities of each of those. Recall that in order to define the measure $\mathcal{I}_{D_f}$ we defined for a knowledge base $\mathcal{K}$ the sets

$$\mathsf{MI}^{(i)}(\mathcal{K}) = \{M \in \mathsf{MI}(\mathcal{K}) \mid |M| = i\}$$

for $i = 1, \ldots, |\mathcal{K}|$. Given these values, we can also redefine the inconsistency measure $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ via

$$\mathcal{I}_{\mathsf{MI}^\mathsf{C}}(\mathcal{K}) = \sum_{i=1}^{|\mathcal{K}|} \frac{|\mathsf{MI}^{(i)}(\mathcal{K})|}{i}$$

Let us call $\mathsf{profile}(\mathcal{K}) = (|\mathsf{MI}^{(1)}(\mathcal{K})|, \ldots, |\mathsf{MI}^{(|\mathcal{K}|)}(\mathcal{K})|) \in \mathbb{N}_0^{|\mathcal{K}|}$ the MI-*profile* of $\mathcal{K}$. One can see that $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ depends only on $\mathsf{profile}(\mathcal{K})$ and not the actual structure of the minimal inconsistent subsets.[7] However, this property of indifference has a corresponding property in the context of Boolean functions.

Definition 24. Two Boolean functions f_1, f_2 of n variables ($n \in \mathbb{N}_0$) are *equivalent*, denoted by $f_1 \sim f_2$, if there is a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that

$$f_1(x_1, \ldots, x_n) = f_2(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for all $x_1, \ldots, x_n \in \{0,1\}$.

Let $MB(n)/_\sim$ be the quotient set of $MB(n)$ wrt. $\sim$, i.e.

$$MB(n)/_\sim = \{[f]_\sim | f \in MB(n)\}$$

[6] See also http://oeis.org/A000372.

[7] The same holds for $\mathcal{I}_{D_f}$ as an MI-profile $(\mathsf{MI}^0(\mathcal{K}), \ldots, \mathsf{MI}^n(\mathcal{K}))$ also uniquely determines the corresponding CN-profile $(\mathsf{CN}^0(\mathcal{K}), \ldots, \mathsf{CN}^n(\mathcal{K}))$.

where $[f]_\sim$ is the *equivalence class* (also called *profile*) of f, i.e., $[f]_\sim = \{f' \in MB(n) \mid f' \sim f\}$. Then $\Psi(n)$ is defined as $\Psi(n) = |MB(n)/{\sim}|$, i.e., $\Psi(n)$ is the number of profiles of monotone Boolean functions of n variables. As for the Dedekind number $m(n)$, no closed form for $\Psi(n)$ is known [29].[8] However, we can observe an intriguing relationship of this number to inconsistency measures via

$$\Psi(n) = |\{\mathsf{profile}(\mathcal{K}) \mid \mathcal{K} \in \mathbb{K}^f(n)\}|$$

In other words, the number of different MI-profiles of knowledge bases of size n is the same as the number of profiles of monotone Boolean functions of n variables. In order to see this, recall that there is a one-to-one correspondence of $\mathsf{MI}(\mathcal{K})$ with Sperner families. It has already been mentioned that there is also a relationship between Sperner families and monotone Boolean functions. More precisely, let $\mathcal{S}$ be any Sperner family over $X = \{a_1, \ldots, a_n\}$ and define a Boolean function $f_\mathcal{S}$ via

$$f_\mathcal{S}(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } \exists S \in \mathcal{S} : S \subseteq \{a_i \mid x_i = 1\} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for all $x_1, \ldots, x_n \in \{0, 1\}$. In other words, the function $f_\mathcal{S}(x_1, \ldots, x_n)$ evaluates to 1 if there is a member $S \in \mathcal{S}$ such that the corresponding variables of the elements of S are 1. Observe that $f_\mathcal{S}$ is monotone for every Sperner family $\mathcal{S}$. Moreover, it can also be seen that for every monotone Boolean function f there is a uniquely determined Sperner family $\mathcal{S}$ such that $f = f_\mathcal{S}$. Consider now two monotone Boolean functions f_1, f_2 with $f_1 \sim f_2$. Then these correspond to two Sperner families $\mathcal{S}_1$, $\mathcal{S}_2$ over X, where $\mathcal{S}_2$ can be obtained from $\mathcal{S}_1$ by only permuting the elements of X. The only invariant between the Sperner families corresponding to the functions in $[f]_\sim$ is the number of sets in each, and the sizes of each set. In the context of minimal inconsistent sets, this means that there is a one-to-one correspondence between any MI-profile and an equivalence class $[f]_\sim$, leading to Equation (1).

For the specific case of $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ it has to be observed that the assignment of an MI-profile to the inconsistency value is not injective, i.e., there may be more than one MI-profile that is mapped to the same inconsistency value.

Example 14. Consider the knowledge bases $\mathcal{K}_3, \mathcal{K}_4 \in \mathbb{K}^f(5)$ defined via

$$\mathcal{K}_3 = \{a, \neg a, b, c, d\}$$
$$\mathcal{K}_4 = \{a, b, c, \neg a \vee \neg b \vee \neg c, \neg(a \wedge b \wedge c)\}$$

Here we have

$$\mathsf{MI}(\mathcal{K}_3) = \{\{a, \neg a\}\}$$
$$\mathsf{MI}(\mathcal{K}_4) = \{\{a, b, c, \neg a \vee \neg b \vee \neg c\}, \{a, b, c, \neg(a \wedge b \wedge c)\}\}$$

and therefore

$$\mathsf{profile}(\mathcal{K}_3) = (0, 1, 0, 0, 0)$$
$$\mathsf{profile}(\mathcal{K}_4) = (0, 0, 0, 2, 0)$$

yielding $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}(\mathcal{K}_3) = \mathcal{I}_{\mathsf{MI}^\mathsf{C}}(\mathcal{K}_4) = 1/2$.

This behavior is the reason that $\Psi(n)$ is only an upper bound for $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}^\mathsf{C}}, n)$ in Theorem 1. As for $m(n)$ and $\Psi(n)$, no closed form for $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}^\mathsf{C}}, n)$ could be found in our investigation. Using a computational brute-force approach we could however determine the first five values for $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}^\mathsf{C}}, n)$ $(n = 1, \ldots, 5)$ which are listed in Table 4 together with their corresponding upper bounds $\Psi(n)$. The measure $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ is quite simplistic in its way to aggregate an MI-profile into a single inconsistency measure. A more elaborated measure is $\mathcal{I}_{D_f}$ (see Definition 5) where this aggregation is more fine-grained. For this measure, we obtain $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}^\mathsf{C}}, n) \le \mathcal{C}^f(\mathcal{I}_{D_f}, n) \le \Psi(n)$, i.e., $\mathcal{I}_{D_f}$ is more expressive than $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ but still bounded by $\Psi(n)$. Note again that Definition 5 represents only a single instance of a more general family of inconsistency measures presented by Mu et al. [25]. Using an injective function h from the set of MI-profiles to real numbers one can instantiate this family with an instance $\mathcal{I}'_{D_f}$ where we could actually have $\mathcal{C}^f(\mathcal{I}'_{D_f}, n) = \Psi(n)$.[9] In any case, $\Psi(n)$ is always an upper bound for every measure following the paradigm of $\mathcal{I}_{D_f}$.

[8] See also http://oeis.org/A220880.

[9] Note that injective functions of the form $h : \mathbb{N}^k \to \mathbb{R}$ do indeed exist for arbitrary (and infinite) k but require complex constructions. However, it is questionable whether there are instances that would lead to meaningful inconsistency measures.

Table 4
Values of $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI^C}}, n)$ and $\Psi(n)$ for $n = 1, \ldots, 5$.

n	$\mathcal{C}^f(\mathcal{I}_{\mathsf{MI^C}}, n)$	$\Psi(n)$
1	2	2
2	4	4
3	7	9
4	15	25
5	43	95

4.3. Knight's inconsistency measure and the Farey series

Let us now turn to another interesting relationship, namely that of the inconsistency measure $\mathcal{I}_\eta$ [17] with the *Farey series*.[10,11] The latter is defined as the series of numbers generated by the function $\Phi : \mathbb{N} \to \mathbb{N}$ defined via

$$\Phi(x) = |\{k/l \mid l = 1, \ldots, x, k = 0, \ldots, l\}|$$

for all $x \in \mathbb{N}$. In other words, $\Phi(x)$ is the number of *different* fractional expressions in $[0, 1]$ with maximal denominator x (where both nominator and denominator are natural numbers). For example, for $x = 3$ we have

$$\left\{\frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}\right\} = \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$$

yielding $\Phi(3) = 5$. Let us now recall the measure $\mathcal{I}_\eta$ which is defined as

$$\mathcal{I}_\eta(\mathcal{K}) = 1 - \max\{\xi \mid \exists P \in \mathcal{P}(\mathsf{At}) : \forall \alpha \in \mathcal{K} : P(\alpha) \geq \xi\}$$

for every $\mathcal{K} \in \mathbb{K}$. So in order to determine $\mathcal{I}_\eta(\mathcal{K})$ we are seeking a probability function $P \in \mathcal{P}(\mathsf{At})$ that maximizes the minimum probability it assigns to formulas in $\mathcal{K}$. For the remainder of this section, let $P_\mathcal{K}$ be any probability function that maximizes the minimum probability of all formulas of $\mathcal{K}$, i.e., we have $\mathcal{I}_\eta(\mathcal{K}) = 1 - \xi_\mathcal{K}$ with $\xi_\mathcal{K} = \min\{P_\mathcal{K}(\alpha) \mid \alpha \in \mathcal{K}\}$.

A first observation one can make about $P_\mathcal{K}$ and $\mathcal{I}_\eta$—which also sets it apart from many other inconsistency measures—is that it does not care for the syntactic representation of formulas. More precisely, for any knowledge base $\mathcal{K}$, formulas ϕ, ϕ' with $\phi \equiv \phi'$ we have $\mathcal{I}_\eta(\mathcal{K} \cup \{\phi\}) = \mathcal{I}_\eta(\mathcal{K} \cup \{\phi'\})$ as $P(\phi) = P(\phi')$ for *every* probability function. Moreover, we also have $\mathcal{I}_\eta(\mathcal{K} \cup \{\phi\}) = \mathcal{I}_\eta(\mathcal{K} \cup \{\phi, \phi'\})$ for the same reason; adding syntactic variations of already present formulas does not change the inconsistency value. It follows that we can identify every formula $\phi \in \mathcal{K}$ with its set of models $\mathsf{Mod}(\phi)$ in all matters related to determining $\mathcal{I}_\eta(\mathcal{K})$. By abusing notation we therefore can rewrite $\mathcal{K}$ as

$$\mathcal{K}' = \{\mathsf{Mod}(\phi) \mid \phi \in \mathcal{K}\} \subseteq \mathfrak{P}(\Omega(\mathsf{At}))$$

where $\mathfrak{P}(X)$ is the power set of a set X and At is the signature of the underlying propositional language. Assume that $\mathsf{At} = \{a_1, \ldots, a_n\}$, then $\Omega(\mathsf{At})$ has 2^n elements and every $\mathsf{Mod}(\phi)$ is a subset of those.

Let us first consider the question of how $\mathcal{K}'$ and $\mathcal{K}$ could look like if we want to have $\xi_{\mathcal{K}'} = k/l$ for $l \in \{1, \ldots, 2^n\}$ and $k \in \{1, \ldots, l\}$. Consider an arbitrary set $\{\omega_1, \ldots, \omega_l\} \subseteq \Omega(\mathsf{At})$ of interpretations (as $l \leq 2^n$ there are enough different interpretations) and define $P_{\mathcal{K}'}(\omega) = 1/l$ for all $\omega \in \{\omega_1, \ldots, \omega_l\}$ and $P_{\mathcal{K}'}(\omega') = 0$ for all remaining ω'. Then $P_{\mathcal{K}'}$ is indeed a probability function that assigns equal probability to all $\{\omega_1, \ldots, \omega_l\}$. Then define a formula ϕ in such a way that $\mathsf{Mod}(\phi)$ is any k-element subset of $\{\omega_1, \ldots, \omega_l\}$. Then we obtain $P_\mathcal{K}(\phi) = k/l$. Populating $\mathcal{K}$ with *all* ϕ that can be defined as such, we obtain a knowledge base $\mathcal{K}$ with $\xi_\mathcal{K} = k/l$.

Example 15. Consider the propositional signature $\mathsf{At}_3 = \{a_1, a_2, a_3\}$, i.e., we have $n = 3$ and $2^n = 8$. Choose $\xi_{\mathcal{K}'} = 5/6$ (note that $6 \in \{1, \ldots, 8\}$ and $5 \in \{1, \ldots, 6\}$) and consider the following 6 interpretations $\omega_1, \ldots, \omega_6 \in \Omega(\mathsf{At}_3)$:

$$\omega_1 = a_1a_2a_3 \qquad \omega_2 = a_1a_2\overline{a_3} \qquad \omega_3 = a_1\overline{a_2}a_3$$
$$\omega_4 = a_1\overline{a_2}\overline{a_3} \qquad \omega_5 = \overline{a_1}a_2a_3 \qquad \omega_6 = \overline{a_1}a_2\overline{a_3}$$

The following sets $M_1, \ldots, M_6$ are all 5-element subsets of $\omega_1, \ldots, \omega_6$:

$$M_1 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} \qquad M_2 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_6\}$$
$$M_3 = \{\omega_1, \omega_2, \omega_3, \omega_5, \omega_6\} \qquad M_4 = \{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6\}$$
$$M_5 = \{\omega_1, \omega_3, \omega_4, \omega_5, \omega_6\} \qquad M_6 = \{\omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$$

[10] See http://oeis.org/A005728 for more information on the Farey series.

[11] The measure $\mathcal{I}_{LP_m}$ also has a relationship with the Farey series via $\mathcal{C}^v(\mathcal{I}_{LP_m}, n) = \Phi(n)$, but it is quite straightforwardly explained. See the proof of Theorem 1 for details.

Consider formulas ϕ_i with $\mathsf{Mod}(\phi_i) = M_i$ for $i = 1, \ldots, 6$. For example, we have $\phi_1 = a_1 \vee (a_2 \wedge a_3)$. For the knowledge base $\mathcal{K} = \{\phi_1, \ldots, \phi_6\}$ consider the probability function $P_\mathcal{K}$ with $P_\mathcal{K}(\omega_1) = \ldots = P_\mathcal{K}(\omega_6) = 1/6$ and $P(\omega) = 0$ for $\omega \in \Omega(\mathsf{At}_3) \setminus \{\omega_1, \ldots \omega_6\}$. By construction we have $P_\mathcal{K}(\phi_1) = \ldots = P_\mathcal{K}(\phi_6) = 5/6$. Note also that there cannot be any other probability function that gives larger probability to all formulas.

So given a signature $\mathsf{At} = \{a_1, \ldots, a_n\}$ and any $l \in \{1, \ldots, 2^n\}$ and $k \in \{1, \ldots, l\}$ we can construct a knowledge base $\mathcal{K}$ such that $\xi_\mathcal{K} = k/l$ (and therefore $\mathcal{I}_\eta(\mathcal{K}) = 1 - k/l$). This gives us $\mathcal{C}^v(\mathcal{I}_\eta, n) \geq \Phi(2^n)$.

The remaining question is whether there are numbers $x \in [0, 1]$ that are not of the form $x = k/l$ with $l \in \{1, \ldots, 2^n\}$ and $k \in \{1, \ldots, l\}$ and for which a knowledge base $\mathcal{K}$ can be found such that $\xi_\mathcal{K} = x$. Knight already showed in Knight [17] that $\xi_\mathcal{K}$ must always be a rational number in the unit interval, so it is clear that $x = p/q$ for some $p, q \in \mathbb{N}$ with $p \leq q$. So what about e.g. $x = 1/(2^n + 1)$? It can be shown (see the complete proof in Appendix A) that due to combinatorial reasons a value such as $1/(2^n + 1)$ cannot be attained for $\xi_\mathcal{K}$ if the underlying signature has n elements. For example, the uniform probability function P with $P(\omega) = 1/2^n$ already yields $P(\phi) \geq 1/2^n$ for every formula $\phi \in \mathcal{K}$ as it has at least one model (note that if $\mathcal{K}$ contains a contradictory formula we always have $\xi_\mathcal{K} = 0$).

Although the expressivity of $\mathcal{I}_\eta$ is characterized by $\Phi(2^n)$ it has to be noted that $\Phi(2^n)$ increases quite rapidly which makes $\mathcal{I}_\eta$ a quite expressive inconsistency measure (see Section 5).

4.4. Normal forms for knowledge bases

Many proofs of statements in Theorem 1 (in particular those showing infinite characteristics) involve the construction of particular families of knowledge bases that exhibit extreme inconsistency values, such as for $\mathcal{C}^f(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = \infty$ (for $n > 1$). Recall that $\mathcal{I}^\Sigma_{\mathrm{dalal}}$ is defined by determining an interpretation $\omega \in \Omega(\mathsf{At})$ such that each formula $\phi \in \mathcal{K}$ has minimal distance to ω (measured in the number of propositions that have to be flipped in ω order to obtain a model of ϕ). Then $\mathcal{I}^\Sigma_{\mathrm{dalal}}$ is the sum of all these distances.

In the proof of $\mathcal{C}^f(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = \infty$ (for $n > 1$) the following family of knowledge bases $\mathcal{K}_i$ is used:

$$\mathcal{K}^1_i = \{a_1 \wedge \ldots \wedge a_i, \neg a_1 \wedge \ldots \wedge \neg a_i\}$$

for $i \in \mathbb{N}$. Note that each $\mathcal{K}^1_i$ consists of only two formulas but the number of mentioned propositions increases with increasing i. It can be seen that for *every* interpretation $\omega \in \Omega(\mathsf{At})$ the sum of its distances to both formulas amounts to exactly i, i.e., $\mathcal{I}^\Sigma_{\mathrm{dalal}}(\mathcal{K}^1_i) = i$ and for $i \to \infty$ we obtain $\mathcal{I}^\Sigma_{\mathrm{dalal}}(\mathcal{K}^1_i) \to \infty$ and thus $\mathcal{C}^f(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = \infty$.

Constructions such as the above can be used to characterize *normal forms* of knowledge bases for inconsistency measures. For example, the above family $\mathcal{K}^1_i$ for $i \in \mathbb{N}$ exhaustively describes the image of $\mathcal{I}^\Sigma_{\mathrm{dalal}}$, i.e.,

$$\mathsf{Im}\, \mathcal{I}^\Sigma_{\mathrm{dalal}} = \{\mathcal{I}^\Sigma_{\mathrm{dalal}}(\mathcal{K}^1_0), \mathcal{I}^\Sigma_{\mathrm{dalal}}(\mathcal{K}^1_1), \mathcal{I}^\Sigma_{\mathrm{dalal}}(\mathcal{K}^1_2), \ldots\}$$

Note that $\mathcal{K}^1_0 = \emptyset$. Every other knowledge base can be transformed into one of these knowledge bases while retaining its inconsistency value. While this transformation, of course, does not maintain semantic equivalence (even in a paraconsistent context), it can be used for illustration purposes. In each $\mathcal{K}^1_i$ the inconsistency is boiled down to its essential core as it is measured by $\mathcal{I}^\Sigma_{\mathrm{dalal}}$. Inspecting this normal form, instead of the original knowledge base where the inconsistency might be obfuscated, can lead to better understanding of the severity of the inconsistency. We leave a deeper investigation of this matter for future work.

4.5. About the distinction between $\{\alpha, \beta\}$ and $\{\alpha \wedge \beta\}$

By studying Table 3 it can be observed that almost all inconsistency measures have trivial characteristic values, i.e., a value of ∞, wrt. $\mathcal{C}^l$ and $\mathcal{C}^p$, and the characteristics $\mathcal{C}^v$ and $\mathcal{C}^f$ seem to be much better suited for assessing the expressivity. The reason for this is that for many inconsistency measures some conjunctions $\alpha \wedge \beta$ can be replaced by two distinct formulas α and β without decreasing the inconsistency value, so large inconsistency values can be attained by either having few long formulas or many short formulas. As $\mathcal{C}^l$ and $\mathcal{C}^p$ only consider the formula-length as fixed (or the number of propositions per formula), arbitrary different inconsistency values can be attained by considering arbitrary large knowledge bases. The only exception, besides the drastic inconsistency measure $\mathcal{I}_d$, is the measure $\mathcal{I}^{\max}_{\mathrm{dalal}}$. Recall that the measure $\mathcal{I}^{\max}_{\mathrm{dalal}}$ is defined as the maximal distance of an optimally chosen $\omega \in \Omega(\mathsf{At})$ to each formula of the knowledge base. If formulas are short, i.e., they each mention only few propositions, this distance is bounded, independently of the number of formulas in the knowledge base.

The important distinction between a set of formulas $\{\alpha, \beta\}$ and the conjunction $\alpha \wedge \beta$ has already been recognized within e.g. the fields of inconsistent-tolerant reasoning and belief revision [18,4]. For example, in the context of contracting from a knowledge base $\mathcal{K}_5 = \{a, b\}$ the inference a, the usually accepted result should be $\mathcal{K}_5 - a = \{b\}$. However, contracting from a knowledge base $\mathcal{K}_6 = \{a \wedge b\}$ the inference a would result in $\mathcal{K}_6 - a = \emptyset$. More generally, a conjunction $\alpha \wedge \beta$ establishes a relationship between the formulas α and β and stipulates that they have to appear together (if one does not appear then the other one should also not appear). For a more detailed discussion see [18,4].

Our study on the characteristics $\mathcal{C}^l$ and $\mathcal{C}^p$ (for details see the proofs in Appendix A) shows that many inconsistency measures do not recognize this difference and, moreover, behave quite incoherently in the general case of adding either separate formulas or a conjunction of the formulas to a knowledge base. Consider the following three properties for inconsistency measures. Let $\mathcal{I}$ be an inconsistency measure, $\mathcal{K} \in \mathbb{K}$, and $\phi, \psi \in \mathcal{L}(\mathsf{At})$ be arbitrary.

$\wedge$-Indifference $\mathcal{I}(\mathcal{K} \cup \{\alpha, \beta\}) = \mathcal{I}(\mathcal{K} \cup \{\alpha \wedge \beta\})$.
$\wedge$-Penalty $\mathcal{I}(\mathcal{K} \cup \{\alpha, \beta\}) \leq \mathcal{I}(\mathcal{K} \cup \{\alpha \wedge \beta\})$.
$\wedge$-Mitigation $\mathcal{I}(\mathcal{K} \cup \{\alpha, \beta\}) \geq \mathcal{I}(\mathcal{K} \cup \{\alpha \wedge \beta\})$.

Note that Besnard [2] proposed $\wedge$-Indifference under the name *Adjunction Invariancy* as a desirable property. However, we do not aim to discuss which (if any) of these properties may be desirable.

But interestingly, only very few of the discussed measures satisfy any of them.

Theorem 3.

1. *The measures $\mathcal{I}_d$, $\mathcal{I}_c$, and $\mathcal{I}_{LP_m}$ satisfy $\wedge$-Indifference, $\wedge$-Penalty, and $\wedge$-Mitigation.*
2. *The measures $\mathcal{I}_\eta$, $\mathcal{I}_{hs}$, and $\mathcal{I}^{\max}_{\text{dalal}}$ satisfy $\wedge$-Penalty, but not $\wedge$-Mitigation.*
3. *The measures $\mathcal{I}^{\text{hit}}_{\text{dalal}}$ and $\mathcal{I}_{P_m}$ satisfy $\wedge$-Mitigation, but not $\wedge$-Penalty.*
4. *None of the measures $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$, $\mathcal{I}_{mc}$, $\mathcal{I}_p$, $\mathcal{I}^{\Sigma}_{\text{dalal}}$, $\mathcal{I}_{D_f}$, $\mathcal{I}_{mv}$, $\mathcal{I}_{nc}$ satisfies any of $\wedge$-Indifference, $\wedge$-Penalty, or $\wedge$-Mitigation.*

As a consequence, the inconsistency values for many measures change quite arbitrarily when a conjunction $\alpha \wedge \beta$ is replaced by it conjuncts α and β.

Example 16. Consider the knowledge base $\mathcal{K}_7$ given via

$$\mathcal{K}_7 = \{a, \neg a\}$$

and

$$\mathsf{MI}(\mathcal{K}_7) = \{\{a, \neg a\}\} \tag{2}$$

$$\mathsf{MI}(\mathcal{K}_7 \cup \{a, b\}) = \{\{a, \neg a\}\} \tag{3}$$

$$\mathsf{MI}(\mathcal{K}_7 \cup \{a \wedge b\}) = \{\{a, \neg a\}, \{a \wedge b, \neg a\}\} \tag{4}$$

$$\mathsf{MI}(\mathcal{K}_7 \cup \{a \wedge \neg a, \neg\neg a\}) = \{\{a, \neg a\}, \{\neg a, \neg\neg a\}, \{a \wedge \neg a\}\} \tag{5}$$

$$\mathsf{MI}(\mathcal{K}_7 \cup \{a \wedge \neg a \wedge \neg\neg a\}) = \{\{a, \neg a\}, \{a \wedge \neg a \wedge \neg\neg a\}\} \tag{6}$$

As one can see, the set $\mathsf{MI}(\mathcal{K})$ may change quite differently when adding separate formulas or conjunctions to $\mathcal{K}$. In Equations (3) and (4) the addition of a conjunction leads to more minimal inconsistent sets than the addition of separate formulas. In Equations (5) and (6) it is exactly the other way around. It follows that for measures $\mathcal{I}$ based on minimal inconsistent subsets—such as $\mathcal{I}_{\mathsf{MI}}$—there is no general relationship such as $\mathcal{I}(\mathcal{K} \cup \{\alpha, \beta\}) \leq \mathcal{I}(\mathcal{K} \cup \{\alpha \wedge \beta\})$ or $\mathcal{I}(\mathcal{K} \cup \{\alpha, \beta\}) \geq \mathcal{I}(\mathcal{K} \cup \{\alpha \wedge \beta\})$ for arbitrary knowledge bases $\mathcal{K}$ and formulas α and β.

5. Expressivity orders

Let us now come back to the original motivation of comparing inconsistency measures wrt. their expressivity. Definition 20 provides the basis for a comparative analysis of inconsistency measures wrt. their expressivity, which we address with the following definition.

Definition 25. An inconsistency measure $\mathcal{I}$ is *at least as expressive* as an inconsistency measure $\mathcal{I}'$ wrt. a characteristic $\mathcal{C}^\alpha$ ($\alpha \in \{f, v, l, p\}$), denoted by $\mathcal{I} \succeq_\alpha \mathcal{I}'$, if there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\mathcal{C}^\alpha(\mathcal{I}, n) \geq \mathcal{C}^\alpha(\mathcal{I}', n)$.

If both $\mathcal{I} \succeq_\alpha \mathcal{I}'$ and $\mathcal{I}' \succeq_\alpha \mathcal{I}$, we say that $\mathcal{I}$ and $\mathcal{I}'$ are *equally expressive* wrt. $\mathcal{C}^\alpha$ and denote this by $\mathcal{I} \sim_\alpha \mathcal{I}'$. If $\mathcal{I} \succeq_\alpha \mathcal{I}'$ but not $\mathcal{I} \sim_\alpha \mathcal{I}'$ we write $\mathcal{I} \succ_\alpha \mathcal{I}'$ ($\mathcal{I}$ is strictly more expressive than $\mathcal{I}'$). Note that the expressivity order $\succeq$ is not to be confused with the refinement order $\sqsubseteq$ sometimes used for pairwise comparisons of inconsistency measures, see e.g. [31]. The refinement order $\sqsubseteq$ is defined as $\mathcal{I}_1 \sqsubseteq \mathcal{I}_2$ iff $\mathcal{I}_2(\mathcal{K}) \geq \mathcal{I}_2(\mathcal{K}')$ implies $\mathcal{I}_1(\mathcal{K}) \geq \mathcal{I}_1(\mathcal{K}')$ for all $\mathcal{K}$, $\mathcal{K}'$. If $\mathcal{I}_1 \sqsubseteq \mathcal{I}_2$ this means that $\mathcal{I}_2$ is a refinement of $\mathcal{I}_1$. Note that $\succeq_\alpha$ compares measures in a quantitative way and also allows comparison of measures that induce totally different orders on knowledge bases. However, it is also easy to see that $\mathcal{I}_1 \sqsubseteq \mathcal{I}_2$ implies $\mathcal{I}_2 \succeq_\alpha \mathcal{I}_1$ (for all $\alpha \in \{f, v, l, p\}$).

By exploiting the results from Theorem 1 we obtain the following simple corollary.

$$\boxed{\mathcal{I}_{\mathsf{MI}}, \mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}, \mathcal{I}_{mc}, \mathcal{I}_{p}, \mathcal{I}^{\Sigma}_{\text{dalal}}, \mathcal{I}^{\text{hit}}_{\text{dalal}}, \mathcal{I}_{D_f}, \mathcal{I}_{P_m}, \mathcal{I}_{nc}} \succ_v \boxed{\mathcal{I}_{\eta}} \succ_v \boxed{\mathcal{I}_{hs}} \succ_v \boxed{\mathcal{I}_{LP_m}} \succ_v \boxed{\mathcal{I}^{\max}_{\text{dalal}}} \succ_v \boxed{\mathcal{I}_{c}, \mathcal{I}_{mv}} \succ_v \boxed{\mathcal{I}_{d}}$$

$$\boxed{\mathcal{I}_{c}, \mathcal{I}_{LP_m}, \mathcal{I}^{\Sigma}_{\text{dalal}}, \mathcal{I}^{\max}_{\text{dalal}}, \mathcal{I}_{P_m}, \mathcal{I}_{mv}} \succ_f \boxed{\mathcal{I}_{\mathsf{MI}}} \succ_f \boxed{\mathcal{I}_{mc}} \succ_f \boxed{\mathcal{I}_{p}, \mathcal{I}_{hs}, \mathcal{I}^{\text{hit}}_{\text{dalal}}, \mathcal{I}_{nc}} \succ_f \boxed{\mathcal{I}_{d}}$$

$$\boxed{\mathcal{I}_{c}, \mathcal{I}_{LP_m}, \mathcal{I}^{\Sigma}_{\text{dalal}}, \mathcal{I}_{P_m}, \mathcal{I}_{mv}, \mathcal{I}_{p}, \mathcal{I}_{hs}, \mathcal{I}^{\text{hit}}_{\text{dalal}}, \mathcal{I}_{nc}, \mathcal{I}_{D_f}, \mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}, \mathcal{I}_{\eta}, \mathcal{I}_{\mathsf{MI}}, \mathcal{I}_{mc}} \succ_l \boxed{\mathcal{I}^{\max}_{\text{dalal}}} \succ_l \boxed{\mathcal{I}_{d}}$$

$$\boxed{\mathcal{I}_{c}, \mathcal{I}_{LP_m}, \mathcal{I}^{\Sigma}_{\text{dalal}}, \mathcal{I}_{P_m}, \mathcal{I}_{mv}, \mathcal{I}_{p}, \mathcal{I}_{hs}, \mathcal{I}^{\text{hit}}_{\text{dalal}}, \mathcal{I}_{nc}, \mathcal{I}_{D_f}, \mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}, \mathcal{I}_{\eta}, \mathcal{I}_{\mathsf{MI}}, \mathcal{I}_{mc}} \succ_p \boxed{\mathcal{I}^{\max}_{\text{dalal}}} \succ_p \boxed{\mathcal{I}_{d}}$$

Fig. 1. Expressivity orders of the investigated inconsistency measures; measures within the same box are equally expressive wrt. the particular expressivity characteristic.

Corollary 1. *The expressivity orders wrt. α-characteristics $\mathcal{C}^{\alpha}(\mathcal{I}, n)$ ($\alpha \in \{f, v, l, p\}$) for the inconsistency measures $\mathcal{I}_d$, $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}$, $\mathcal{I}_{\eta}$, $\mathcal{I}_c$, $\mathcal{I}_{LP_m}$, $\mathcal{I}_{mc}$, $\mathcal{I}_p$, $\mathcal{I}_{hs}$, $\mathcal{I}^{\Sigma}_{\text{dalal}}$, $\mathcal{I}^{\max}_{\text{dalal}}$, $\mathcal{I}^{\text{hit}}_{\text{dalal}}$, $\mathcal{I}_{D_f}$, $\mathcal{I}_{P_m}$, $\mathcal{I}_{mv}$, and $\mathcal{I}_{nc}$ are as shown in Fig. 1.*

The proof of the above corollary is omitted as the results follow directly from Theorem 1.

In Fig. 1 the order $\succ_f$ does not show the placement of the measures $\mathcal{I}_{D_f}$, $\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}$, and $\mathcal{I}_{\eta}$ as we only provided upper bounds for the corresponding characteristics in Theorem 1. However, we can give the following partial classification.

Corollary 2. *For $\mathcal{I} \in \{\mathcal{I}_c, \mathcal{I}_{LP_m}, \mathcal{I}^{\Sigma}_{\text{dalal}}, \mathcal{I}^{\max}_{\text{dalal}}, \mathcal{I}_{P_m}, \mathcal{I}_{mv}\}$, $\mathcal{I} \succ_f \mathcal{I}_{D_f}$, $\mathcal{I} \succ_f \mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}$, $\mathcal{I} \succ_f \mathcal{I}_{\eta}$.*

The proof of the above corollary is straightforward as, e.g., we provided a finite bound for $\mathcal{C}^f(\mathcal{I}_{\eta}, n)$ (for every n) while $\mathcal{C}^f(\mathcal{I}_c, n)$ is unbounded. Empirical evidence suggests also the following relationships, but a formal proof has yet to be found.

Conjecture 1. $\mathcal{I}_{D_f} \succ_f \mathcal{I}_{\mathsf{MI}^{\mathsf{C}}} \succ_f \mathcal{I}_{\eta} \succ_f \mathcal{I}_{\mathsf{MI}}$.

Fig. 1 shows that the measures $\mathcal{I}^{\Sigma}_{\text{dalal}}$ and $\mathcal{I}_{P_m}$ are the only measures that have maximal expressivity wrt. all four expressivity characteristics (among the considered inconsistency measures) and, as expected, the drastic inconsistency measure $\mathcal{I}_d$ is the least expressive one. One can also observe that for many measures their positioning in the orders $\succ_v$ and $\succ_p$ is complementary, i.e., if a measure has high expressivity wrt. $\mathcal{C}^f$ it has low expressivity wrt. $\mathcal{C}^v$ (consider e.g. $\mathcal{I}_c$ and $\mathcal{I}_p$). This is due to the fact that many measures measure only a specific aspect of inconsistency and usually belong either to the MI-based family of inconsistency measures—which focus on using minimal inconsistent subsets for measuring—or the variable-based family—which focus on conflicting propositions—, cf. [12]. Therefore, they are constrained in their expressivity if one of these dimensions is limited. For example, if the number of formulas in a knowledge base is restricted, so is the number of minimal inconsistent subsets.

Again, it should be noted that expressivity characteristics are meant to complement the investigation of rationality postulates, not to replace them. Rationality postulates are important to analyze the meaningfulness of the values of inconsistency measures, while our characteristics provide a quantitative assessment of their expressivity. However, we believe that the concept of expressivity characteristics and the results reported in this work will nurture general comparative analyses of inconsistency measures.

The expressivity characteristics considered in this paper each tackle one specific aspect of *size* of a knowledge base. Of course, one can also combine these characteristics to obtain hybrid versions via

$$\mathcal{C}^{\alpha,\alpha'}(\mathcal{I}, n, m) = |\{\mathcal{I}(\mathcal{K}) \mid \mathcal{K} \in \mathbb{K}^{\alpha}(n) \cap \mathbb{K}^{\alpha'}(m)\}|$$

with $\alpha, \alpha' \in \{v, f, l, p\}$, $\alpha \neq \alpha'$, and $n, m > 0$. For example, $\mathcal{C}^{v,f}(\mathcal{I}, n, m)$ is the number of different inconsistency values on knowledge bases which mention at most n propositions and consist of at most m formulas. A simple observation on these new characteristics is the following one.

Proposition 1. *Let $\alpha, \alpha' \in \{v, f, l, p\}$, $\alpha \neq \alpha'$, and $n, m > 0$. Then*

$$\mathcal{C}^{\alpha,\alpha'}(\mathcal{I}, n, m) \leq \min\{\mathcal{C}^{\alpha}(\mathcal{I}, n), \mathcal{C}^{\alpha'}(\mathcal{I}, m)\}$$

The proof of the above proposition is straightforward. An investigation of these hybrid and other characteristics—and the resulting expressivity orders—is left for future work.

Table A.5
Characteristics of inconsistency measures ($n \geq 1$).

	$\mathcal{C}^v(\mathcal{I},n)$	$\mathcal{C}^f(\mathcal{I},n)$	$\mathcal{C}^l(\mathcal{I},n)$	$\mathcal{C}^p(\mathcal{I},n)$
$\mathcal{I}_d$	2	2	2^*	2
$\mathcal{I}_{\mathsf{MI}}$	∞	$\binom{n}{\lfloor n/2 \rfloor}+1$	∞^*	∞
$\mathcal{I}_{\mathsf{MI^C}}$	∞	$\leq \Psi(n)^{\mathrm{a}}$	∞^*	∞
$\mathcal{I}_\eta$	$\Phi(2^n)^{\mathrm{b}}$	$\leq \Phi(\binom{n}{\lfloor n/2 \rfloor})^{\mathrm{b}}$	∞^{**}	∞^*
$\mathcal{I}_c$	$n+1$	∞	∞^*	∞
$\mathcal{I}_{LP_m}$	$\Phi(n)$	∞	∞^*	∞
$\mathcal{I}_{mc}$	∞	$\binom{n}{\lfloor n/2 \rfloor}^{**}$	∞^*	∞
$\mathcal{I}_p$	∞	$n+1$	∞^*	∞
$\mathcal{I}_{hs}$	2^n+1	$n+1$	∞^{**}	∞^*
$\mathcal{I}^{\Sigma}_{\mathrm{dalal}}$	∞	∞^*	∞^*	∞
$\mathcal{I}^{\max}_{\mathrm{dalal}}$	$n+2$	∞^*	$\lfloor (n+7)/3 \rfloor^{**}$	$n+2$
$\mathcal{I}^{\mathrm{hit}}_{\mathrm{dalal}}$	∞	$n+1$	∞^*	∞
$\mathcal{I}_{D_f}$	∞	$\leq \Psi(n)^{\mathrm{a}}$	∞^*	∞
$\mathcal{I}_{P_m}$	∞	∞	∞^*	∞
$\mathcal{I}_{mv}$	$n+1$	∞^*	∞^*	∞
$\mathcal{I}_{nc}$	∞	$n+1$	∞^*	∞

[*] Only for $n > 1$.
[**] Only for $n > 3$.
[a] $\Psi(n)$ is the number of profiles of monotone Boolean functions of n variables, see e.g. http://oeis.org/A220880.
[b] $\Phi(x)$ is the number of fractions in the Farey series of order x and can be defined as $\Phi(x) = |\{k/l \mid l = 1, \ldots, x, k = 0, \ldots, l\}|$, see e.g. http://oeis.org/A005728.

6. Summary and conclusion

We conducted a focused but extensive comparative analysis of 16 inconsistency measures from the recent literature in terms of their expressivity. For that, we introduced 4 different expressivity characteristics and conducted an analytical evaluation of the considered measures wrt. these expressivity characteristics. Our findings revealed some interesting relationships of inconsistency measures to, e.g., set theory and monotone Boolean functions. Finally, the measures $\mathcal{I}^{\Sigma}_{\mathrm{dalal}}$ [8] and $\mathcal{I}_{P_m}$ [16] have been proven to be maximally expressive wrt. all our characteristics.

Expressivity characteristics provide a novel evaluation method for assessing the quality of inconsistency measures. It has to be noted again, however, that high expressivity alone is not a sufficient criterion for doing this. It is straightforward to construct measures that exhibit maximal expressivity along all discussed dimensions, but fail to comply with the intuitions one expects from inconsistency measures. The use of rationality postulates—such as the ones presented and discussed in Hunter and Konieczny [11]; Mu et al. [25]; Besnard, [2]—must still serve as first-level evaluation criterion. If measures satisfy the same (or a similar set of) rationality postulates, expressivity can be used to make further quality assessments.

To the best of our knowledge, our work is the most extensive comparative analysis of inconsistency measures so far. All inconsistency measures discussed in this paper have been implemented and an online interface to try out these measures is available.[12]

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Appendix A. Proofs of technical results

Theorem 1. *The α-characteristics $\mathcal{C}^\alpha(\mathcal{I},n)$ ($\alpha \in \{f, v, l, p\}$) for the inconsistency measures $\mathcal{I}_d$, $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{\mathsf{MI^C}}$, $\mathcal{I}_\eta$, $\mathcal{I}_c$, $\mathcal{I}_{LP_m}$, $\mathcal{I}_{mc}$, $\mathcal{I}_p$, $\mathcal{I}_{hs}$, $\mathcal{I}^{\Sigma}_{\mathrm{dalal}}$, $\mathcal{I}^{\max}_{\mathrm{dalal}}$, $\mathcal{I}^{\mathrm{hit}}_{\mathrm{dalal}}$, $\mathcal{I}_{D_f}$, $\mathcal{I}_{P_m}$, $\mathcal{I}_{mv}$, and $\mathcal{I}_{nc}$ are as shown in Table A.5.*

Proof. Let $n > 0$ except in proofs regarding $\mathcal{C}^l$ where $n > 1$ is assumed (note that $\mathcal{C}^l(\mathcal{I}, 1) = 1$ for every measure $\mathcal{I}$ as every $\mathcal{K} \in \mathbb{K}^l(1)$ does not contain a negation and is therefore always consistent).

1. $\mathcal{C}^v(\mathcal{I}_d, n) = 2$
   By definition, $\mathcal{I}_d$ has co-domain $\{0, 1\}$ and therefore $\mathcal{C}^v(\mathcal{I}_d, n) \leq 2$. For the knowledge bases $\mathcal{K}_8 = \{a\}$ and $\mathcal{K}_9 = \{a \wedge \neg a\}$ we get $\mathcal{I}_d(\mathcal{K}_8) = 0$ and $\mathcal{I}_d(\mathcal{K}_9) = 1$ and therefore $\mathcal{C}^v(\mathcal{I}_d, n) \geq 2$. As $\mathcal{K}_8$ and $\mathcal{K}_9$ use only one proposition the statement is true for all $n > 0$.

[12] http://tweetyproject.org/w/incmes/.

2. $\mathcal{C}^f(\mathcal{I}_d, n) = 2$
   Analogous to 1.
3. $\mathcal{C}^l(\mathcal{I}_d, n) = 2$
   Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_d, 1) = 1$. Analogous to 1 but consider $\mathcal{K}_8 = \{a\}$ and $\mathcal{K}_7 = \{a, \neg a\}$.
4. $\mathcal{C}^p(\mathcal{I}_d, n) = 2$
   Analogous to 1.
5. $\mathcal{C}^v(\mathcal{I}_{\mathsf{MI}}, n) = \infty$
   Consider for $i \in \mathbb{N}$ the knowledge bases $\mathcal{K}_i^2 = \{\neg a, a, a \wedge a, a \wedge a \wedge a, \ldots, \bigwedge_{j=1}^{i} a\}$. Then $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}_i^2) = i$ and $\lim_{i\to\infty} \mathcal{I}_{\mathsf{MI}}(\mathcal{K}_i^2) = \infty$. As each $\mathcal{K}_i^2$ only uses one proposition the statement is true for every $n > 0$.
6. $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}}, n) = \binom{n}{\lfloor n/2 \rfloor} + 1$
   Note that for every inconsistent knowledge base $\mathcal{K}$ the set $\mathsf{MI}(\mathcal{K})$ is a *Sperner family* of $\mathcal{K}$, i.e. a set S of subsets from a set T for which $X \subseteq Y$ for no two $X, Y \in S$. According to Sperner's theorem the maximal cardinality (which is also attained) of any Sperner family of a set T with $|T| = n$ is $\binom{n}{\lfloor n/2 \rfloor}$ [28]. If $\mathcal{K}$ is consistent we have $\mathsf{MI}(\mathcal{K}) = \emptyset$ and $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = 0$, yielding $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}}, n) \leq \binom{n}{\lfloor n/2 \rfloor} + 1$. To show $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}}, n) \geq \binom{n}{\lfloor n/2 \rfloor} + 1$ we show that every Sperner family can be represented through $\mathsf{MI}(\mathcal{K})$ of a knowledge base $\mathcal{K}$. Let $T = \{\alpha_1, \ldots, \alpha_n\}$ be a set and define a propositional signature $\mathsf{At} = \{a_1, \ldots, a_n\}$. Let S be any Sperner family of T with cardinality $\binom{n}{\lfloor n/2 \rfloor}$. Define a knowledge base $\mathcal{K}_n^S = \{\phi_1, \ldots, \phi_n\}$ via

   $$\phi_i = a_i \wedge \bigwedge_{M \in S, \alpha_i \in M} \bigvee_{\alpha_j \in M \setminus \{\alpha_i\}} \neg a_j$$

   for $i = 1, \ldots, n$. Informally, every ϕ_i states that a_i is accepted and for each set M in S which contains α_i at least one of the other elements must not be accepted. If follows that $M = \{\alpha_{k_1}, \ldots, \alpha_{k_m}\}$ (for some $k_1, \ldots, k_m \in \{1, \ldots, n\}$) is an element of S if and only if the set $\{\phi_{k_1}, \ldots, \phi_{k_m}\}$ is a minimal inconsistent set. It follows that $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}_n^S) = |\mathsf{MI}(\mathcal{K}_n^S)| = |S| = \binom{n}{\lfloor n/2 \rfloor}$. As removing any element from a Sperner family still yields a Sperner family, every value between 1 and $\binom{n}{\lfloor n/2 \rfloor}$ can be attained. Together with the fact that 0 is also a possible value of $\mathcal{I}_{\mathsf{MI}}$ we obtain $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}}, n) \geq \binom{n}{\lfloor n/2 \rfloor} + 1$ and thus $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}}, n) = \binom{n}{\lfloor n/2 \rfloor} + 1$.
7. $\mathcal{C}^l(\mathcal{I}_{\mathsf{MI}}, n) = \infty$
   Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_{\mathsf{MI}}, 1) = 1$. Consider the family of knowledge bases $\mathcal{K}_i^3 = \{a_1, \ldots, a_i, \neg a_1, \ldots, \neg a_i\}$ for $i \in \mathbb{N}$. Then $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}_i^3) = i$ and $\mathcal{C}^l(\mathcal{I}_{\mathsf{MI}}, n) = \infty$ as only formulas of maximum length two have been used.
8. $\mathcal{C}^p(\mathcal{I}_{\mathsf{MI}}, n) = \infty$
   Analogous to 7 (note that every formula in $\mathcal{K}_i^3$ mentions only one proposition).
9. $\mathcal{C}^v(\mathcal{I}_{\mathsf{MI^C}}, n) = \infty$
   Consider the family of knowledge bases $\mathcal{K}_i^2$ from 5. Observe that $\mathcal{I}_{\mathsf{MI^C}}(\mathcal{K}_i^2) = i/2$ and therefore $\mathcal{C}^v(\mathcal{I}_{\mathsf{MI^C}}, n) = \infty$.
10. $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI^C}}, n) \leq \Psi(n)$
    Consider the vector $\mathsf{profile}(\mathcal{K}) = (\mathsf{MI}^0(\mathcal{K}), \ldots, \mathsf{MI}^n(\mathcal{K}))$, called MI-profile of $\mathcal{K}$ in the following, where $\mathsf{MI}^i(\mathcal{K})$ is the set of i-size minimal inconsistent subsets of $\mathcal{K}$. Note that every MI-profile induces the inconsistency value wrt. $\mathcal{I}_{\mathsf{MI^C}}$ of its corresponding knowledge base by $\mathcal{I}_{\mathsf{MI^C}}(\mathcal{K}) = \sum_{i=1}^{n} |\mathsf{MI}^i(\mathcal{K})| \cdot 1/i$. Furthermore, note that two distinct MI-profiles may yield the same inconsistency value, e.g. $(1, 0, 0)$ and $(0, 2, 0)$ yield the same inconsistency value 1. It follows that

    $$\begin{aligned}\mathcal{C}^f(\mathcal{I}_{\mathsf{MI^C}}, n) \leq |\{(|\mathsf{MI}^0(\mathcal{K})|, \ldots, |\mathsf{MI}^n(\mathcal{K})|) \mid \\ (\mathsf{MI}^0(\mathcal{K}), \ldots, \mathsf{MI}^n(\mathcal{K})) \text{ is an} \\ \text{MI-profile for some } \mathcal{K} \in \mathbb{K}^f(n)\}|\end{aligned}$$

    As discussed in 6, for every knowledge base $\mathcal{K}$ the set $\mathsf{MI}(\mathcal{K})$ is a Sperner family. It is well-known that there is an equivalence between Sperner families and monotone boolean functions, cf. [29]. In [29] the number of inequivalent monotone boolean functions has been investigated, see also http://oeis.org/A220880. These numbers are the same of inequivalent Sperner families as well. Here, two Sperner families $\mathsf{MI}(\mathcal{K})$ and $\mathsf{MI}(\mathcal{K}')$ are equivalent if they yield the same MI-profiles. The number of different MI-profiles is also the number on the right-hand side of the above equation, thus showing the claim $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI^C}}, n) \leq \Psi(n)$ where $\Psi(n)$ is the number of inequivalent monotone Boolean functions on n variables.
11. $\mathcal{C}^l(\mathcal{I}_{\mathsf{MI^C}}, n) = \infty$
    Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_{\mathsf{MI^C}}, 1) = 1$. Consider then the family of knowledge bases $\mathcal{K}_i^3$ from 7 and observe that $\mathcal{I}_{\mathsf{MI^C}}(\mathcal{K}_i^3) = i/2$ and therefore $\mathcal{C}^l(\mathcal{I}_{\mathsf{MI^C}}, n) = \infty$.
12. $\mathcal{C}^p(\mathcal{I}_{\mathsf{MI^C}}, n) = \infty$
    Analogous to 11 (note that every formula in $\mathcal{K}_i^3$ mentions only one proposition).
13. $\mathcal{C}^v(\mathcal{I}_\eta, n) = \Phi(2^n)$
    We first show $\mathcal{C}^v(\mathcal{I}_\eta, n) \leq \Phi(2^n)$. In [17] it has already been shown that $\mathcal{I}_\eta(\mathcal{K}) \in [0, 1] \cap \mathbb{Q}$ for every $\mathcal{K}$ (Definition 2.7 and Theorem 2.28). Hence, assume $\eta = k/l$ and $\mathcal{I}_\eta(\mathcal{K}) = 1 - \eta$ for $k, l \in \mathbb{N}$ and $k \leq l$. We also assume for now that $\mathcal{K}$

contains no contradictory formula. Furthermore, we assume that $\mathcal{K}$ contains no free formulas (as $\mathcal{I}_\eta$ satisfies independence they have no influence on the inconsistency value, cf. [30]). Let P be a probability function such that $P(\phi) \geq k/l$ for all $\phi \in \mathcal{K}$. It can be assumed that there is no $\omega \in \Omega(\mathsf{At})$ such that $P(\omega) > 0$ but $\omega \not\models \phi$ for every $\phi \in \mathcal{K}$ (otherwise one could set $P(\omega) = 0$ and distribute the "probability mass" $P(\omega)$ on the remaining interpretations which have already a positive probability; this cannot change the fact that $P(\phi) \geq k/l$ for all $\phi \in \mathcal{K}$). So for all $\omega \in \Omega(\mathsf{At})$, if $P(\omega) > 0$ then $\omega \models \phi$ for some $\phi \in \mathcal{K}$. Define $F_\mathcal{K}(\omega) = \{\phi \in \mathcal{K} \mid \omega \models \phi\}$ for all $\omega \in \Omega(\mathsf{At})$, i.e., $F_\mathcal{K}(\omega)$ is the set of formulas in $\mathcal{K}$ that are satisfied by ω. We can furthermore assume that for all $\omega, \omega' \in \Omega(\mathsf{At})$ with $P(\omega) > 0$ and $P(\omega') > 0$ we have $F_\mathcal{K}(\omega) \nsubseteq F_\mathcal{K}(\omega')$ (otherwise we could set $P(\omega) = 0$ and add the probability mass $P(\omega)$ to $P(\omega')$, without decreasing the probabilities of the formulas). Assume furthermore, that among all probability functions that satisfy the above constraints, P is one such that $|\{\omega | P(\omega) > 0\}|$ is minimal.

Now consider the case $|\{\omega \mid P(\omega) > 0\}| = 2$, i.e., there are two interpretations ω_1, ω_2 that receive positive probability. For every formula $\phi \in \mathcal{K}$, either $\phi \in F_\mathcal{K}(\omega_1)$, or $\phi \in F_\mathcal{K}(\omega_2)$, or $\phi \in F_\mathcal{K}(\omega_1) \cap F_\mathcal{K}(\omega_2)$. Note that the latter case cannot be possible for all $\phi \in \mathcal{K}$ as otherwise $F_\mathcal{K}(\omega_1) = F_\mathcal{K}(\omega_2)$. Furthermore, there is one formula $\phi' \in \mathcal{K}$ with $P(\phi') = P(\omega_1)$ and one formula $\phi'' \in \mathcal{K}$ with $P(\phi'') = P(\omega_2)$, otherwise we would have $F_\mathcal{K}(\omega_1) \subseteq F_\mathcal{K}(\omega_2)$ or $F_\mathcal{K}(\omega_2) \subseteq F_\mathcal{K}(\omega_1)$. As $P(\phi)$ has to be maximal for all $\phi \in \mathcal{K}$ we can conclude $P(\omega_1) = P(\omega_2)$ and therefore $\eta = 1/2$.

Now consider the case $|\{\omega \mid P(\omega) > 0\}| = 3$, i.e., there are three interpretations ω_1, ω_2, ω_3 that receive positive probability. For each $\phi \in \mathcal{K}$ let $\Delta_P(\phi) = \{\omega \in \Omega(\mathsf{At}) \mid P(\omega) > 0, \omega \models \phi\}$, i.e., $\Delta_P(\phi)$ is such that $P(\phi) = \sum_{\omega \in \Delta_P(\phi)} P(\omega)$. Note that it cannot be the case that $|\Delta_P(\phi)| = 3$ for any $\phi \in \mathcal{K}$ (otherwise ϕ would be free in $\mathcal{K}$) or that $|\Delta_P(\phi)| = 0$ (then ϕ would be self-contradictory). Consider the following sub-cases:

(a) for all $\phi \in \mathcal{K}$ we have $|\Delta_P(\phi)| = 1$:
Then for all $\phi \in \mathcal{K}$ we have $P(\phi) = P(\omega)$ for some $\omega \in \{\omega_1, \omega_2, \omega_3\}$ and as there are no subset relations between any $F_\mathcal{K}(\omega_1)$, $F_\mathcal{K}(\omega_2)$, and $F_\mathcal{K}(\omega_3)$, it follows that $P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$ maximizes each probability and we have $\eta = 1/3$.

(b) for all $\phi \in \mathcal{K}$ we have $|\Delta_P(\phi)| = 2$:
Then for all $\phi \in \mathcal{K}$ we have $P(\phi) = P(\omega) + P(\omega')$ for some $\omega, \omega' \in \{\omega_1, \omega_2, \omega_3\}$ with $\omega \neq \omega'$ and as there are no subset relations between any $F_\mathcal{K}(\omega_1)$, $F_\mathcal{K}(\omega_2)$, and $F_\mathcal{K}(\omega_3)$, it follows that $P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$ maximizes each probability and we have $\eta = 2/3$.

(c) otherwise:
Let $\phi_1 \in \mathcal{K}$ with $|\Delta_P(\phi_1)| = 1$. Without loss of generality assume $\Delta_P(\phi_1) = \{\omega_1\}$. As $F_\mathcal{K}(\omega_2) \nsubseteq F_\mathcal{K}(\omega_1)$ there is $\phi_2 \in F_\mathcal{K}(\omega_2)$ with $\phi_2 \notin F_\mathcal{K}(\omega_1)$. Consider the case that for all $\phi \in F_\mathcal{K}(\omega_3)$ either $\phi \in F_\mathcal{K}(\omega_1)$ or $\phi \in F_\mathcal{K}(\omega_2)$. Then P' defined via $P'(\omega_1) = 0.5$, $P'(\omega_2) = 0.5$, and $P'(\omega) = 0$ for all other ω yields $P'(\phi) \geq 0.5$ for all $\phi \in \mathcal{K}$. Assuming P obtains a larger probability for all formulas implies that $P(\omega) > 0.5$ (in order to have $P(\phi_1) > 0.5$), but then $P(\phi_2) < 0.5$. So we have a contradiction since P is supposed to be minimal wrt. $|\{\omega | P(\omega) > 0\}|$. It follows that there is $\phi_3 \in F_\mathcal{K}(\omega_3)$ with $\phi_3 \notin F_\mathcal{K}(\omega_1)$ and $\phi_3 \notin F_\mathcal{K}(\omega_2)$, so $P(\phi_3) = P(\omega_3)$. Similarly, it can be assumed that $\phi_2 \notin F_\mathcal{K}(\omega_3)$ as well. As $P(\phi_1) = P(\omega_1)$, $P(\phi_2) = P(\omega_2)$, and $P(\phi_3) = P(\omega_3)$ it follows that $P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$ maximizes each probability and we have $\eta = 1/3$.

So for $|\{\omega \mid P(\omega) > 0\}| = 3$ we have that $\eta \in \{1/3, 2/3\}$. Inductively it follows that for $|\{\omega \mid P(\omega) > 0\}| = h$ we have $\eta \in \{1/h, \ldots, (h-1)/h\}$. As a signature with n propositions has 2^n different interpretations, and together with the cases of a consistent knowledge base (inconsistency value 0) and one that contains a contradictory formula (inconsistency value 1) we obtain $\mathcal{C}^v(\mathcal{I}_\eta, n) \leq |\{k/l \mid l = 1, \ldots, 2^n, k = 0, \ldots, l\}| = \Phi(2^n)$.

We now show that $\mathcal{C}^v(\mathcal{I}_\eta, n) \geq \Phi(2^n)$. For that let $\eta = k/l$ for $l \in \{1, \ldots, 2^n\}$ and $k \in \{1, \ldots, l\}$. Let $X = \{\omega_1, \ldots, \omega_l\} \subseteq \Omega(\mathsf{At})$ be any set of l different interpretations. Define P via $P(\omega) = 1/l$ if $\omega \in X$ and $P(\omega) = 0$ otherwise. Define $\hat{\mathcal{K}}$ via $\phi \in \hat{\mathcal{K}}$ if and only if

$$\phi = \bigvee_{\omega \in X_k} \phi_\omega$$

where X_k is a k-element subset of X and ϕ_ω is the complete conjunction that has $\omega \in \Omega(\mathsf{At})$ as its only model (note that $\hat{\mathcal{K}}$ contains $\binom{l}{k}$ formulas, one for each k-element subset of X). Observe that for all $\phi \in \hat{\mathcal{K}}$ we have

$$P(\phi) = P(\bigvee_{\omega \in X_k} \phi_\omega) = \sum_{\omega \in X_k} P(\phi_\omega) = k/l$$

and that this is obviously the maximal possible value for $\hat{\mathcal{K}}$. It follows that $\mathcal{I}_\eta(\hat{\mathcal{K}}) = 1 - k/l$ and therefore $\mathcal{C}^v(\mathcal{I}_\eta, n) \geq \Phi(2^n)$.

14. $\mathcal{C}^f(\mathcal{I}_\eta, n) \leq \Phi(\binom{n}{\lfloor n/2 \rfloor})$
Analogous to 13. However, note that the maximal number of interpretations that may receive a positive probability is bounded by the number of different $F_\mathcal{K}(\omega)$ for $\omega \in \Omega(\mathsf{At})$. As the set of $F_\mathcal{K}(\omega)$ with $P(\omega) > 0$ form a Sperner family (no two elements have a subset relation) the maximal cardinality of this set is $\binom{n}{\lfloor n/2 \rfloor}$, cf. [28].

15. $\mathcal{C}^l(\mathcal{I}_\eta, n) = \infty$

Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_\eta, 1) = 1$. For $n = 2$ observe that either $\mathcal{I}_\eta(\mathcal{K}) = 0$ (for consistent $\mathcal{K}$) or $\mathcal{I}_\eta(\mathcal{K}) = 0.5$. For the latter, note that $\mathcal{K}$ can only be inconsistent if and only if there is at least one (possibly more) $a \in \mathsf{At}$ such that $a, \neg a \in \mathcal{K}$ (or semantically equivalent formulas). Then any probability function P with $P(\phi)$ maximal for all $\phi \in \mathcal{K}$ has to satisfy $P(a) = P(\neg a) = 0.5$. Therefore we have $\mathcal{C}^l(\mathcal{I}_\eta, 2) = 2$. For $n = 3$ we additionally have the case that a three-element minimal inconsistent subset $\{\neg a_1, \neg a_2, a_1 \vee a_2\}$ may occur with corresponding inconsistency value 1/3, thus $\mathcal{C}^l(\mathcal{I}_\eta, n) = 3$. For $n > 3$ consider the family of knowledge bases $\mathcal{K}_i^4 = \{\neg a_1 \vee a_2, \neg a_2 \vee a_3, \dots, \neg a_{i-1} \vee a_i, \neg a_i \wedge a_1\}$. Note that $\mathcal{K}_i^4$ is a minimal inconsistent set. By Theorem 2.12 of [17] $\mathcal{I}_\eta(\mathcal{K}_i^4) = 1/|\mathcal{K}_i^4| = 1/i$ and therefore $\mathcal{C}^l(\mathcal{I}_\eta, n) = \infty$.

16. $\mathcal{C}^p(\mathcal{I}_\eta, n) = \infty$
First, for $n = 1$ observe that either $\mathcal{I}_\eta(\mathcal{K}) = 0$ (for consistent $\mathcal{K}$), $\mathcal{I}_\eta(\mathcal{K}) = 1$ (for $\mathcal{K}$ containing a contradictory formula), or $\mathcal{I}_\eta(\mathcal{K}) = 0.5$. For the latter, note that $\mathcal{K}$ can only be inconsistent without containing a contradictory formula if and only if there is at least one (possibly more) $a \in \mathsf{At}$ such that $a, \neg a \in \mathcal{K}$ (or semantically equivalent formulas). Then any probability function P with $P(\phi)$ maximal for all $\phi \in \mathcal{K}$ has to satisfy $P(a) = P(\neg a) = 0.5$. For $n > 1$ consider the family of knowledge bases $\mathcal{K}_i^4$ from 15. Note that $\mathcal{K}_i^4$ is a minimal inconsistent set. By Theorem 2.12 of [17] $\mathcal{I}_\eta(\mathcal{K}_i^4) = 1/|\mathcal{K}_i^4| = 1/i$ and therefore $\mathcal{C}^p(\mathcal{I}_\eta, n) = \infty$.
17. $\mathcal{C}^v(\mathcal{I}_c, n) = n + 1$
Consider the propositional signature $\mathsf{At} = \{a_1, \dots, a_n\}$ and for each $i = 0, \dots, n$ consider the knowledge base $\mathcal{K}_i^5 = \{a_1 \wedge \neg a_1, \dots, a_i \wedge \neg a_i\}$ (with $\mathcal{K}_0^5 = \emptyset$). Then $\mathcal{I}_c(\mathcal{K}_i^5) = i$ as every $a_1, \dots, a_i$ has to be set to B in every model of $\mathcal{K}_i^5$. Together with the fact that every model can assign the value B to at most $|\mathsf{At}| = n$ different propositions we have $\mathcal{C}^v(\mathcal{I}_c, n) = n + 1$.
18. $\mathcal{C}^f(\mathcal{I}_c, n) = \infty$
Consider the family of knowledge bases $\mathcal{K}_i^6 = \{a_1 \wedge \ldots \wedge a_i \wedge \neg a_1 \wedge \ldots \wedge \neg a_i\}$. Then $\mathcal{I}_c(\mathcal{K}_i^6) = i$ for $i > 0$ and $\lim_{i\to\infty} \mathcal{I}_c(\mathcal{K}_i^6) = \infty$. As each $\mathcal{K}_i^6$ has only one formula the statement is true for every $n > 0$.
19. $\mathcal{C}^l(\mathcal{I}_c, n) = \infty$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_c, 1) = 1$. Consider the family of knowledge bases $\mathcal{K}_i^3 = \{a_1, \dots, a_i, \neg a_1, \dots, \neg a_i\}$ for $i \in \mathbb{N}$. Then $\mathcal{I}_c(\mathcal{K}_i^3) = i$ and $\mathcal{C}^l(\mathcal{I}_c, n) = \infty$ as only formulas of maximum length two have been used.
20. $\mathcal{C}^p(\mathcal{I}_c, n) = \infty$
Analogous to 19 (note that every formula in $\mathcal{K}_i^3$ mentions only one proposition).
21. $\mathcal{C}^f(\mathcal{I}_{LP_m}, n) = \Phi(n)$
Recall from item that $\mathcal{I}_c(\mathcal{K}) \in \{0, \dots, n\}$ for $\mathcal{K} \in \mathbb{K}^v(n)$ and note $|\mathsf{At}(\mathcal{K})| \in \{1, \dots, n\}$ and $\mathcal{I}_c(\mathcal{K}) \le |\mathsf{At}(\mathcal{K})|$. Together we obtain $\mathcal{I}_{LP_m}(\mathcal{K}) = \mathcal{I}_c(\mathcal{K})/|\mathsf{At}(\mathcal{K})| \in \{k/l \mid l = 1, \dots, |\mathsf{At}(\mathcal{K})|, k = 0, \dots, l\}$ and therefore $\mathcal{C}^f(\mathcal{I}_{LP_m}, n) \le \Phi(n) = |\{k/l \mid l = 1, \dots, n, k = 0, \dots, l\}|$. To see $\mathcal{C}^f(\mathcal{I}_{LP_m}, n) \ge \Phi(n)$ consider the family of knowledge bases $\mathcal{K}_{i,j}^{16} = \{a_1, \dots, a_i, \neg a_1, \dots, \neg a_i\}$ for $i, j \in \mathbb{N}$, $i \le j$ and observe that $\mathcal{I}_{LP_m}(\mathcal{K}_{i,j}^{16}) = i/j$.
22. $\mathcal{C}^f(\mathcal{I}_{LP_m}, n) = \infty$
Consider the family of knowledge bases $\mathcal{K}_i^{15} = \{\neg a_1 \wedge a_1 \wedge a_2 \wedge \ldots \wedge a_i\}$. Then $\mathcal{I}_{LP_m}(\mathcal{K}_i^{15}) = 1/i$ for $i > 0$. As each $\mathcal{K}_i^{15}$ has only one formula the statement is true for every $n > 0$.
23. $\mathcal{C}^l(\mathcal{I}_{LP_m}, n) = \infty$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_{LP_m}, 1) = 1$. Consider the family of knowledge bases $\mathcal{K}_i^{12} = \{\neg a_1, a_1, a_2, \dots, a_i\}$ for $i \in \mathbb{N}$. Then $\mathcal{I}_{LP_m}(\mathcal{K}_i^{12}) = 1/i$ and $\mathcal{C}^l(\mathcal{I}_{LP_m}, n) = \infty$ as only formulas of maximum length two have been used.
24. $\mathcal{C}^p(\mathcal{I}_{LP_m}, n) = \infty$
Analogous to 23 (note that every formula in $\mathcal{K}_i^{12}$ mentions only one proposition).
25. $\mathcal{C}^v(\mathcal{I}_{mc}, n) = \infty$
Consider the family of knowledge bases $\mathcal{K}_i^7 = \{a \wedge \neg a, a \wedge a \wedge \neg a \wedge \neg a, \dots \bigwedge_{j=1}^{i} a \wedge \neg a\}$ and observe that $\mathcal{I}_{mc}(\mathcal{K}_i^7) = |\mathsf{SC}(\mathcal{K}_i^7)| = i + 1$ (all formulas in $\mathcal{K}_i^7$ are self-contradicting and only the empty subset is a maximal consistent subset). It follows that $\mathcal{C}^v(\mathcal{I}_{mc}, n) = \infty$.
26. $\mathcal{C}^f(\mathcal{I}_{mc}, n) = \binom{n}{\lfloor n/2 \rfloor}$
Note first, that if $\mathcal{K}$ contains only self-contradictory formulas we have $\mathsf{MC}(\mathcal{K}) = \emptyset$. Otherwise, analogously to 6, observe that for every other consistent or inconsistent knowledge base $\mathcal{K}$ the set $\mathsf{MC}(\mathcal{K})$ is a *Sperner family* of $\mathcal{K}$, i.e. a set S of subsets from a set T for which $X \subseteq Y$ for no two $X, Y \in S$. According to Sperner's theorem the maximal cardinality (which is also attained) of any Sperner family of a set T with $|T| = n$ is $\binom{n}{\lfloor n/2 \rfloor}$ [28]. Hence we have $0 \le |\mathsf{MC}(\mathcal{K})| \le \binom{n}{\lfloor n/2 \rfloor}$. Also analogously to 6 observe that every value can be attained by some knowledge base. For that let S be any Sperner family of cardinality $\lfloor n/2 \rfloor$ of a set $T = \{\alpha_1, \dots, \alpha_n\}$. Let $\mathsf{At} = \{a_1, \dots, a_n\}$ and define a knowledge base $\hat{\mathcal{K}}_n^S = \{\phi_1, \dots, \phi_n\}$ with

$$\phi_i = \bigvee_{\alpha_i \in M \in S} \Big(\bigwedge_{\alpha_j \in M} a_j \wedge \bigwedge_{\alpha_j \notin M} \neg a_j \Big)$$

for $i = 1, \dots, n$. Informally, every ϕ_i lists all sets $M \in S$ that include α_i. Then a set $M \subseteq \hat{\mathcal{K}}_n^S$ is a maximal consistent subset if and only if it corresponds to an element of S. As removing any element from a Sperner family still yields a Sperner family, we have $\{|\mathsf{MC}(\mathcal{K})| \mid \mathcal{K} \in \mathbb{K}^f(n)\} = \{i \in \mathbb{N} \mid 0 \le i \le \binom{n}{\lfloor n/2 \rfloor}\}$.

Note that for $|\mathcal{K}| = n$, $0 \leq |\mathsf{SC}(\mathcal{K})| \leq n$ (and every value can be attained). However, we cannot simply obtain the value $\mathcal{C}^f(\mathcal{I}_{mc}, n)$ by adding the upper bounds of $|\mathsf{MC}(\mathcal{K})|$ and $|\mathsf{SC}(\mathcal{K})|$ as these two values are dependent. Observe that if $\mathcal{K}$ with $|\mathcal{K}| = n$ contains a self-contradictory formula ϕ then ϕ cannot be part of any maximal consistent subset of $\mathcal{K}$, i.e., we have $\mathsf{MC}(\mathcal{K}) = \mathsf{MC}(\mathcal{K} \setminus \{\phi\})$. In general, we have that if $\mathcal{K}$ contains k contradictory formulas then $|\mathsf{MC}(\mathcal{K})| \leq \binom{(n-k)}{\lfloor (n-k)/2 \rfloor}$. Define $c_{mc}^{n,k} = \binom{(n-k)}{\lfloor (n-k)/2 \rfloor}$ and then we obtain the following characterization of $\mathcal{C}^f(\mathcal{I}_{mc}, n)$:

$$\mathcal{C}^f(\mathcal{I}_{mc}, n) = \max\{c_{mc}^{n,0}, c_{mc}^{n,1} + 1, c_{mc}^{n,2} + 2, \ldots, c_{mc}^{n,n} + n\}$$

That is, the value $\mathcal{C}^f(\mathcal{I}_{mc}, n)$ is either $c_{mc}^{n,0}$ (considering no self-contradictory formulas) or $c_{mc}^{n,1} + 1$ (considering one self-contradictory formula), etc. Observe that the first element of the above maximum is dominant for $n > 3$. For $n = 1$ we obtain $\mathcal{C}^f(\mathcal{I}_{mc}, n) = 2$ (a knowledge base with one formula is either consistent, i.e., $\mathsf{MC}(\mathcal{K}) = \{\mathcal{K}\}$, $\mathsf{SC}(\mathcal{K}) = \emptyset$, and thus $\mathcal{I}_{mc}(\mathcal{K}) = 0$; or it is inconsistent with $\mathsf{MC}(\mathcal{K}) = \{\emptyset\}$, $\mathsf{SC}(\mathcal{K}) = \mathcal{K}$, and thus $\mathcal{I}_{mc}(\mathcal{K}) = 1$). Note that either the empty set or the whole set are the only possible maximal consistent subsets. For $n = 2$ we obtain $\mathcal{C}^f(\mathcal{I}_{mc}, n) = 3$: either $\mathcal{K}$ is consistent ($\mathcal{I}_{mc}(\mathcal{K}) = 0$), or it contains one self-contradictory formula ($\mathcal{I}_{mc}(\mathcal{K}) = 1$), or it contains two contradictory formulas ($\mathcal{I}_{mc}(\mathcal{K}) = 2$). Note that the maximal number of consistent subsets of $\mathcal{K}$ is 2 (for the case that $\mathcal{K}$ is a two-element minimal inconsistent set), but then there cannot be self-contradictory formulas and we have $\mathcal{I}_{mc}(\mathcal{K}) = 1$. For $n = 3$ we obtain $\mathcal{C}^f(\mathcal{I}_{mc}, n) = 4$ (for a consistent knowledge base and knowledge bases with 1 to 3 self-contradictory formulas and one maximal consistent subset). Note that the maximal number of consistent subsets of $\mathcal{K}$ is 3, e.g. all two-element subsets, but then $\mathcal{K}$ cannot contain any self-contradictory formula and we have $\mathcal{I}_{mc}(\mathcal{K}) = 2$ which we can also obtain by having two self-contradictory formulas. For $n > 3$ and $k = 1, \ldots, n$ we have

$$\begin{aligned}
c_{mc}^{n,0} = \binom{n}{\lfloor n/2 \rfloor} &= \binom{n-1}{\lfloor n/2 \rfloor} + \underbrace{\binom{n-1}{\lfloor n/2 \rfloor - 1}}_{\geq n \text{ as } n>3} \\
&\geq \binom{n-1}{\lfloor n/2 \rfloor} + n \\
&= \binom{n-1}{\lfloor (n-1)/2 \rfloor} + n \\
&\geq \binom{n-k}{\lfloor (n-k)/2 \rfloor} + n \\
&\geq \binom{n-k}{\lfloor (n-k)/2 \rfloor} + k = c_{mc}^{n,k} + k
\end{aligned}$$

Note that $\binom{n-1}{\lfloor n/2 \rfloor} = \binom{n-1}{\lfloor (n-1)/2 \rfloor}$ as for odd n we have $\binom{n-1}{\lfloor n/2 \rfloor} = \binom{n-1}{(n-1)/2} = \binom{n-1}{\lfloor (n-1)/2 \rfloor}$ and for even n we have $\binom{n-1}{\lfloor n/2 \rfloor} = \binom{n-1}{n/2}$ which (as $n-1$ is odd) is the same as $\binom{n-1}{n/2-1} = \binom{n-1}{\lfloor (n-1)/2 \rfloor}$. Hence, for $n > 3$ we obtain $\mathcal{C}^f(\mathcal{I}_{mc}, n) = \binom{n}{\lfloor n/2 \rfloor}$.

27. $\mathcal{C}^l(\mathcal{I}_{mc}, n) = \infty$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_{mc}, 1) = 1$. Note furthermore that for $n = 2$ only literals are allowed as formulas in $\mathcal{K}$. Consider the family of knowledge bases $\mathcal{K}_i^3 = \{a_1, \neg a_1, \ldots, a_i, \neg a_i\}$ and observe that $|\mathcal{K}_i^3| = 2i$ and $\mathcal{I}_{mc}(\mathcal{K}_i^3) = 2^i$ (every interpretation ω corresponds to a maximal consistent subset of $\mathcal{K}_i^3$, i.e., the union of all a_i with $\omega(a_i) = \mathsf{true}$ and $\neg a_i$ with $\omega(a_i) = \mathsf{false}$; adding any other formula from $\mathcal{K}_i^3$ makes this set inconsistent). As only formulas of maximum length 2 are used in $\mathcal{K}_i^3$ it follows that $\mathcal{C}^l(\mathcal{I}_{mc}, n) = \infty$.
28. $\mathcal{C}^p(\mathcal{I}_{mc}, n) = \infty$
Consider the family knowledge bases $\mathcal{K}_i^7$ from 25 and observe that every formula mentions only one proposition. It follows that $\mathcal{C}^p(\mathcal{I}_{mc}, n) = \infty$.
29. $\mathcal{C}^v(\mathcal{I}_p, n) = \infty$
Consider the family of knowledge bases $\mathcal{K}_i^2$ from 5. Then $\mathcal{I}_p(\mathcal{K}_i^2) = i$ and $\lim_{i\to\infty} \mathcal{I}_p(\mathcal{K}_i^2) = \infty$. As each $\mathcal{K}_i^2$ only uses one proposition the statement is true for every $n > 0$.
30. $\mathcal{C}^f(\mathcal{I}_p, n) = n + 1$
For $k < n$ let $M_k = \{a_1, \ldots, a_{k-1}, \neg a_1 \vee \ldots \neg a_{k-1}\}$. Note that M_k is a minimal inconsistent set. Consider $\mathcal{K}_{n,k} = M_k \cup \{a_{k+1}, \ldots, a_n\}$. Then $\mathcal{K}_{n,k}$ has exactly one minimal inconsistent subset (M_k) and $\mathcal{I}_p(\mathcal{K}_{n,k}) = k$. Hence, for $k = 1, \ldots, n$ every value in $\{1, \ldots, n\}$ is attained for $\mathcal{I}_p(\mathcal{K}_{n,k})$. Together with $\mathcal{I}_p(\mathcal{K}) = 0$ for any consistent $\mathcal{K}$ of size n we have $\mathcal{C}^f(\mathcal{I}_p, n) = n + 1$.
31. $\mathcal{C}^l(\mathcal{I}_p, n) = \infty$
Analogous to 19.
32. $\mathcal{C}^p(\mathcal{I}_p, n) = \infty$
Analogous to 20.

33. $\mathcal{C}^v(\mathcal{I}_{hs}, n) = 2^n + 1$
Any hitting set can be of maximal size 2^n as there are that many interpretations in a language with n propositions, and as a hitting set may not be defined in case of a contradictory formula we get $\mathcal{C}^v(\mathcal{I}_{hs}, n) \leq 2^n + 1$. Let now $\mathsf{At}_i = \{a_1, \ldots, a_i$ be a propositional signature i propositions and consider the knowledge base $\mathcal{K}_i^{11} = \{\phi_\omega \mid \omega \in \Omega(\mathsf{At}_i)\}$ where ϕ_ω is any formula with $\mathsf{Mod}(\phi_\omega) = \{\omega\}$. Then $\mathcal{K}_i^{11}$ contains 2^i formulas and each of them is satisfied by only one interpretation. We get $\mathcal{I}_{hs}(\mathcal{K}_i^{11}) = 2^i - 1$ and removing any formula from $\mathcal{K}_i^{11}$ reduces the value by one, so all values $0, \ldots, 2^i - 1$ are attained. Taking the case of a knowledge base with a contradictory formula into account, we obtain $\mathcal{C}^v(\mathcal{I}_{hs}, n) \geq 2^n + 1$ and thus $\mathcal{C}^v(\mathcal{I}_{hs}, n) = 2^n + 1$.
34. $\mathcal{C}^f(\mathcal{I}_{hs}, n) = n + 1$
For $\mathcal{K}$ with $|\mathcal{K}| = n$ any hitting set can be of maximal size n, as only formulas in $\mathcal{K}$ need to be hit. Considering also the case of a knowledge base with a contradictory formula we obtain $\mathcal{C}^f(\mathcal{I}_{hs}, n) \leq n + 1$. For $\mathcal{C}^f(\mathcal{I}_{hs}, n) \geq n + 1$ consider a knowledge base with n pairwise inconsistent formulas, such as in 33 (note that the signature can be arbitrarily large). Therefore we get $\mathcal{C}^f(\mathcal{I}_{hs}, n) = n + 1$.
35. $\mathcal{C}^l(\mathcal{I}_{hs}, n) = \infty$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_{hs}, 1) = 1$. For $n = 2$ or $n = 3$ consider the interpretations ω_1, ω_2 with $\omega_1(a) = \mathsf{true}$ and $\omega_2(a) = \mathsf{false}$ for all $a \in \mathsf{At}$. As formulas of maximal length 2 are either a, or $\neg a$, and formulas of length 3 are either $a \wedge b$ or $a \vee b$ for $a, b \in \mathsf{At}$, either ω_1 or ω_2 is a model of each formula. Therefore, $\mathcal{I}_{hs}(\mathcal{K}) = 1$ or $\mathcal{I}_{hs}(\mathcal{K}) = 0$ and $\mathcal{C}^l(\mathcal{I}_{hs}, 2) = \mathcal{C}^l(\mathcal{I}_{hs}, 3) = 2$. For $n > 3$ consider the signature $\mathsf{At}_m = \{a_1, \ldots, a_m\}$ and the knowledge base $\mathcal{K}_m^9 = \{a \wedge b, \neg a \wedge b, a \wedge \neg b \mid a, b \in \mathsf{At}_m, a \neq b\}$. Observe that for $m \to \infty$ we have $\mathcal{I}_{hs}(\mathcal{K}_m^9) \to \infty$ and therefore $\mathcal{C}^l(\mathcal{I}_{hs}, n) = \infty$.
36. $\mathcal{C}^p(\mathcal{I}_{hs}, n) = \infty$
First, for $n = 1$ observe that either $\mathcal{I}_{hs}(\mathcal{K}) = 0$ (for consistent $\mathcal{K}$), $\mathcal{I}_{hs}(\mathcal{K}) = \infty$ (for $\mathcal{K}$ containing a contradictory formula), or $\mathcal{I}_{hs}(\mathcal{K}) = 1$. For the latter, note that given a signature $\mathsf{At}_m = \{a_1, \ldots, a_m\}$ the two interpretations ω_1, ω_2 with $\omega_1(a) = \mathsf{true}$ and $\omega_2(a) = \mathsf{false}$, for all $a \in \mathsf{At}_m$, form a hitting set for every knowledge base where the formulas mention at most one proposition. It follows that $\mathcal{C}^p(\mathcal{I}_{hs}, 1) = 3$. For $n > 1$ consider the signature $\mathsf{At}_m = \{a_1, \ldots, a_m\}$ and the knowledge base $\mathcal{K}_m^9 = \{a \wedge b, \neg a \wedge b, a \wedge \neg b, \neg a \wedge \neg b \mid a, b \in \mathsf{At}_m, a \neq b\}$. Observe that for $m \to \infty$ we have $\mathcal{I}_{hs}(\mathcal{K}_m^9) \to \infty$ and therefore $\mathcal{C}^p(\mathcal{I}_{hs}, n) = \infty$.
37. $\mathcal{C}^v(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = \infty$
Consider the family of knowledge bases

$$\mathcal{K}_i^{10} = \{a, \neg a, a \wedge a, \neg a \wedge \neg a, \ldots, \bigwedge_{j=1}^{i} a, \bigwedge_{j=1}^{i} \neg a\}$$

for $i \in \mathbb{N}$. Then $\mathcal{I}^\Sigma_{\mathrm{dalal}}(\mathcal{K}_i^{10}) = i$ as there are only two interpretations ω_1 and ω_2 with $\omega_1(a) = \mathsf{true}$ and $\omega_2(a) = \mathsf{false}$ and for both interpretations the sets of models of half of the formulas in $\mathcal{K}_i^{10}$ have a distance of one to each of them (note that $|\mathcal{K}_i^{10}| = 2i$). Therefore we have $\mathcal{C}^v(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = \infty$.
38. $\mathcal{C}^f(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = \infty$
For $\mathcal{K}$ with $|\mathcal{K}| = 1$ we have that either $\mathcal{I}^\Sigma_{\mathrm{dalal}}(\mathcal{K}) = 0$ or $\mathcal{I}^\Sigma_{\mathrm{dalal}}(\mathcal{K}) = \infty$ and therefore $\mathcal{C}^f(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = 2$. For $n > 1$ consider the family of knowledge bases $\mathcal{K}_i^1 = \{a_1 \wedge \ldots \wedge a_i, \neg a_1 \wedge \ldots \wedge \neg a_i\}$ for $i \in \mathbb{N}$ and observe that $\mathcal{I}^\Sigma_{\mathrm{dalal}}(\mathcal{K}_i^1) = i$. Therefore we have $\mathcal{C}^f(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = \infty$.
39. $\mathcal{C}^l(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = \infty$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}^\Sigma_{\mathrm{dalal}}, 1) = 1$. Then for $i \in \mathbb{N}$ consider the family of knowledge bases $\mathcal{K}_i^3 = \{a_1, \ldots, a_i, \neg a_1, \ldots, \neg a_i\}$ and observe that $\mathcal{I}^\Sigma_{\mathrm{dalal}}(\mathcal{K}_i^3) = i$. Hence, we obtain $\mathcal{C}^l(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = \infty$.
40. $\mathcal{C}^p(\mathcal{I}^\Sigma_{\mathrm{dalal}}, n) = \infty$
Analogous to 37 (note that in every formula of $\mathcal{K}_i^0$ only one proposition is used).
41. $\mathcal{C}^v(\mathcal{I}^{\max}_{\mathrm{dalal}}, n) = n + 2$
For every consistent knowledge base $\mathcal{K}$ we have $\mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K}) = 0$ and for the knowledge base $\mathcal{K}^9 = \{a \wedge \neg a\}$ we have $\mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K}^9) = \infty$. Furthermore, for the signature $\mathsf{At}_i = \{a_1, \ldots, a_i\}$ and $i = 1, \ldots, n$ consider the family of knowledge bases $\mathcal{K}_i^{11} = \{\phi_\omega \mid \omega \in \Omega(\mathsf{At}_i)\}$ where ϕ_ω is any formula with $\mathsf{Mod}(\phi_\omega) = \{\omega\}$. Observe that for every $\omega \in \Omega(\mathsf{At}_i)$ there is one formula $\phi \in \mathcal{K}_i^{11}$ with $d_{\mathrm{d}}(\mathsf{Mod}(\phi), \omega) = i$ and therefore $\mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K}_i^{11}) = i$. Note also that $d_{\mathrm{d}}(\omega, \omega') \leq i$ for every pair $\omega, \omega' \in \Omega(\mathsf{At}_i)$ as ω and ω' can differ in at most i propositions. Hence, we have $\mathcal{C}^v(\mathcal{I}^{\max}_{\mathrm{dalal}}, n) = n + 2$.
42. $\mathcal{C}^f(\mathcal{I}^{\max}_{\mathrm{dalal}}, n) = \infty$
First, for $\mathcal{K}$ with $|\mathcal{K}| = 1$ observe that either $\mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K}) = 0$ or $\mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K}) = \infty$ ($\mathcal{K}$ can only be inconsistent if it contains a contradictory formula and then the Dalal distance between the set of models of this formula (which is the empty set) to any interpretation is ∞). Therefore we have $\mathcal{C}^f(\mathcal{I}^{\max}_{\mathrm{dalal}}, 1) = 2$. For $n > 1$ consider the family of knowledge bases $\mathcal{K}_i^1 = \{a_1 \wedge \ldots \wedge a_i, \neg a_1 \wedge \ldots \wedge \neg a_i\}$ with $i \in \mathbb{N}$. Then we have $\mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K}_i^1) = \lceil i/2 \rceil$ and therefore $\mathcal{C}^f(\mathcal{I}^{\max}_{\mathrm{dalal}}, n) = \infty$.

43. $\mathcal{C}^l(\mathcal{I}^{\max}_{\text{dalal}}, n) = \lfloor (n+7)/3 \rfloor$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}^{\max}_{\text{dalal}}, 1) = 1$. For $n = 2$ observe that $\mathcal{K}$ only contains propositions or negations of propositions and therefore, the set of models of every formula of $\mathcal{K}$ has at most distance 1 to any interpretation. Therefore, $\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}) = 0$ or $\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}) = 1$ and we have $\mathcal{C}^l(\mathcal{I}^{\max}_{\text{dalal}}, 2) = 2$. For $n = 3$ observe that formulas may only have the form a, $\neg a$, $a \wedge b$, or $a \vee b$ for $a, b \in \mathsf{At}$. Note that only the models of a formula $a \wedge b$ may have distance 2 to some interpretation. However, as a conjunction cannot contain a negation, the minimal maximal distance of any formula from $\mathcal{K}$ is 1 as the models of e.g. $\neg a$ has only distance 1 to any models of $a \wedge b$. As there can also be no contradictory formula (for that a length of 4 of a formula is required) we get $\mathcal{C}^l(\mathcal{I}^{\max}_{\text{dalal}}, 3) = 2$. For $n = 4$ consider the interpretation ω with $\omega(a) = \mathsf{true}$ for every $a \in \Omega(\mathsf{At})$. As every conjunction can contain at most one negation, the distance of the models of every formula to ω is also maximally 1. Additionally, we have self-contradictory formulas which yield in total $\mathcal{C}^l(\mathcal{I}^{\max}_{\text{dalal}}, 4) = 3$. Assume $n > 4$ with $n = 2 + 3k$ with $k > 0$ and let ϕ_ω be the complete conjunction that has $\omega \in \Omega(\mathsf{At})$ as its only model. Then observe that $\mathcal{K}^{11}_{k+1} = \{\phi_\omega \mid \omega \in \Omega(\mathsf{At}_{k+1})\}$ has only formulas of maximal length n and $\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}^{11}_k) = k$ (and that smaller inconsistency values can be attained by removing some formula in $\mathcal{K}^{11}_i$). Observe further that for $n \in \{3 + 3k, 4 + 3k\}$ the maximal distance cannot be larger than for $n = 2 + 3k$. Together with consistent knowledge bases and knowledge bases containing self-contradictory formulas we obtain $\mathcal{C}^l(\mathcal{I}^{\max}_{\text{dalal}}, n) = \lfloor (n-2)/3 \rfloor + 3 = \lfloor (n+7)/3 \rfloor$ for $n > 3$.

44. $\mathcal{C}^p(\mathcal{I}^{\max}_{\text{dalal}}, n) = n + 2$
Consider the signature $\mathsf{At}_i = \{a_1, \ldots, a_i\}$ for $i = 1, \ldots, n$ and the family of knowledge bases $\mathcal{K}^{11}_i = \{\phi_\omega \mid \omega \in \Omega(\mathsf{At}_i)\}$. Note that every $\phi \in \mathcal{K}^{11}_i$ mentions exactly i propositions and, with the same argumentation as in 41, we have $\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}^{11}_i) = i$. Furthermore, for a consistent knowledge base $\mathcal{K}$ we have $\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}) = 0$ and for the knowledge base $\mathcal{K}^9 = \{a \wedge \neg a\}$ we have $\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}^9) = \infty$ and therefore $\mathcal{C}^p(\mathcal{I}^{\max}_{\text{dalal}}, n) = n + 2$.

45. $\mathcal{C}^v(\mathcal{I}^{\text{hit}}_{\text{dalal}}, n) = \infty$
For every $i \in \mathbb{N}$ consider the knowledge base $\mathcal{K}^{10}_i = \{a, \neg a, a \wedge a, \neg a \wedge \neg a, \ldots, \bigwedge_{j=1}^{i} a, \bigwedge_{j=1}^{i} \neg a\}$. Then $\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}^{10}_i) = i$ (consider e.g. the interpretation ω with $\omega(a) = \mathsf{true}$, then the models of half of the formulas of $\mathcal{K}^{10}_i$ have distance one to ω and $|\mathcal{K}^{10}_i| = 2i$). As only one proposition is necessary to create any inconsistency value we have $\mathcal{C}^v(\mathcal{I}^{\text{hit}}_{\text{dalal}}, n) = \infty$.

46. $\mathcal{C}^f(\mathcal{I}^{\text{hit}}_{\text{dalal}}, n) = n + 1$
For every consistent knowledge base $\mathcal{K}$ we have $\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}) = 0$ and for the family of knowledge base $\mathcal{K}^5_i = \{a_1 \wedge \neg a_1, \ldots, a_i \wedge \neg a_i\}$ (for $i = 1, \ldots, n$) we have $\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}^5_i) = i$. Note that n is also the maximal value of $\mathcal{I}^{\text{hit}}_{\text{dalal}}$ as this is the maximal number of formulas in any $\mathcal{K}$. Hence, we have $\mathcal{C}^f(\mathcal{I}^{\text{hit}}_{\text{dalal}}, n) = n + 1$.

47. $\mathcal{C}^l(\mathcal{I}^{\text{hit}}_{\text{dalal}}, n) = \infty$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}^{\text{hit}}_{\text{dalal}}, 1) = 1$. Consider the family of knowledge bases $\mathcal{K}^3_i = \{a_1, \neg a_1, \ldots, a_i, \neg a_i\}$. Then $\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}^3_i) = i$ and therefore $\mathcal{C}^l(\mathcal{I}^{\text{hit}}_{\text{dalal}}, n) = \infty$ as only formula of length 2 are necessary to produce any inconsistency value.

48. $\mathcal{C}^p(\mathcal{I}^{\text{hit}}_{\text{dalal}}, n) = \infty$
Analogous to 45.

49. $\mathcal{C}^v(\mathcal{I}_{D_f}, n) = \infty$
Consider for $i \in \mathbb{N}$ the knowledge bases $\mathcal{K}^2_i = \{\neg a, a, a \wedge a, a \wedge a \wedge a, \ldots, \bigwedge_{j=1}^{i} a\}$. Then $\mathsf{MI}^{(1)}(\mathcal{K}^2_i) = \mathsf{MI}^{(3)}(\mathcal{K}^2_i) = \mathsf{MI}^{(4)}(\mathcal{K}^2_i) = \ldots = \mathsf{MI}^{(|\mathcal{K}^2_i|)}(\mathcal{K}^2_i) = \emptyset$ and $\mathsf{MI}^{(2)}(\mathcal{K}^2_i) = \{\{\neg a, a\}, \{\neg a, a \wedge a\}, \{\neg a, a \wedge a \wedge a\}, \ldots, \{\neg a, \bigwedge_{j=1}^{i} a\}\}$. Therefore $|\mathsf{MI}^{(1)}(\mathcal{K}^2_i)| = |\mathsf{MI}^{(3)}(\mathcal{K}^2_i)| = |\mathsf{MI}^{(4)}(\mathcal{K}^2_i)| = \ldots = |\mathsf{MI}^{(|\mathcal{K}^2_i|)}(\mathcal{K}^2_i)| = 0$ and $|\mathsf{MI}^{(2)}(\mathcal{K}^2_i)| = i$. Furthermore, note that $\mathsf{CN}^{(2)}$ is comprised of every two-element subset of $\mathcal{K}^2_i \setminus \{\neg a\}$ and therefore $|\mathsf{CN}^{(2)}| = \binom{i}{2}$. It follows that $R_1(\mathcal{K}^2_i) = R_3(\mathcal{K}^2_i) = R_4(\mathcal{K}^2_i) = \ldots = R_{|\mathcal{K}^2_i|-1}(\mathcal{K}^2_i) = 0$ and $R_2(\mathcal{K}^2_i) = i/(i + \binom{i}{2})$. We obtain $\mathcal{I}_{D_f}(\mathcal{K}^2_i) = 1 - R_2(\mathcal{K}^2_i)/2 = 1 - i/(2i + 2\binom{i}{2}) = 1 - i/(2i + i(i-1)) = 1 - 1/(i+1)$. As $i \in \mathbb{N}$ we obtain $\mathcal{C}^v(\mathcal{I}_{D_f}, n) = \infty$.

50. $\mathcal{C}^f(\mathcal{I}_{D_f}, n) \leq \Psi(n)$
Analogous to 10. Observe that each MI-profile $(\mathsf{MI}^0(\mathcal{K}), \ldots, \mathsf{MI}^n(\mathcal{K}))$ also uniquely determines the corresponding CN-profile $(\mathsf{CN}^0(\mathcal{K}), \ldots, \mathsf{CN}^n(\mathcal{K}))$. That is, there are no two knowledge bases $\mathcal{K}$ and $\mathcal{K}'$ that have the same MI-profile but different CN-profiles. Then there cannot be more R-profiles $(R_1(\mathcal{K}), \ldots, R_n(\mathcal{K}))$ than there are MI-profiles and, thus, the number of inequivalent monotone Boolean functions on n variables is also an upper bound for $\mathcal{C}^f(\mathcal{I}_{D_f}, n)$.

51. $\mathcal{C}^l(\mathcal{I}_{D_f}, n) = \infty$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_{D_f}, 1) = 1$. Consider then the family of knowledge bases $\mathcal{K}^3_i$ from 7. Observe that $\mathsf{MI}^2(\mathcal{K}^3_i) = i$ and $\mathsf{MI}^j(\mathcal{K}^3_i) = 0$ for $j \neq 2$. Furthermore, it is $\mathsf{CN}^2 = \binom{2i}{2} - i$ and therefore $R_2(\mathcal{K}^3_i) = i/\binom{2i}{2}$. Then

$$\mathcal{I}_{D_f}(\mathcal{K}^3_i) = 1 - \frac{i}{2\binom{2i}{2}}$$

Hence, we have $\mathcal{C}^l(\mathcal{I}_{D_f}, n) = \infty$.

52. $\mathcal{C}^p(\mathcal{I}_{D_f}, n) = \infty$
Analogous to 49.
53. $\mathcal{C}^v(\mathcal{I}_{P_m}, n) = \infty$
Consider the family of knowledge bases $\mathcal{K}_i^2 = \{\neg a, a, a \wedge a, \ldots, \bigwedge_{j=1}^{i} a\}$ for $i \in \mathbb{N}$. Then $\mathcal{K}_i^2$ contains one minimal proof for $\neg a$ and i minimal proofs for a (each formula other than $\neg a$ is a minimal proof for a). Then we have $\mathcal{I}_{D_f}(\mathcal{K}_i^2) = i$ and therefore $\mathcal{C}^v(\mathcal{I}_{D_f}, n) = \infty$.
54. $\mathcal{C}^f(\mathcal{I}_{P_m}, n) = \infty$
Consider the family of knowledge bases $\mathcal{K}_i^6 = \{a_1 \wedge \ldots \wedge a_i \wedge \neg a_1 \wedge \ldots \wedge \neg a_i\}$ for $i \in \mathbb{N}$. Then $\mathcal{K}_i^6$ contains one minimal proof for each a_j and one minimal proof for each $\neg a_j$ for $j = 1, \ldots, i$. Then we have $\mathcal{I}_{D_f}(\mathcal{K}_i^6) = i$ and therefore $\mathcal{C}^f(\mathcal{I}_{D_f}, n) = \infty$.
55. $\mathcal{C}^l(\mathcal{I}_{P_m}, n) = \infty$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_{D_f}, 1) = 1$. Consider the family of knowledge bases $\mathcal{K}_i^3 = \{a_1, \ldots, a_i, \neg a_1, \ldots, \neg a_i\}$ for $i \in \mathbb{N}$. Then $\mathcal{K}_i^3$ contains one minimal proof for each a_j and one minimal proof for each $\neg a_j$ for $j = 1, \ldots, i$. Then we have $\mathcal{I}_{D_f}(\mathcal{K}_i^3) = i$ and therefore $\mathcal{C}^l(\mathcal{I}_{D_f}, n) = \infty$ as only formulas of maximal size 2 have been used.
56. $\mathcal{C}^p(\mathcal{I}_{P_m}, n) = \infty$
Analogous to 55 (note that $\mathcal{K}_i^3$ mentions only one proposition in each formula).
57. $\mathcal{C}^v(\mathcal{I}_{mv}, n) = n + 1$
Consider the propositional signature $\mathsf{At} = \{a_1, \ldots, a_n\}$ and for each $i = 0, \ldots, n$ consider the knowledge base $\mathcal{K}_i^{13} = \{a_1 \wedge \neg a_1, \ldots, a_i \wedge \neg a_i, a_{i+1}, \ldots, a_n\}$ (with $\mathcal{K}_0^{13} = \emptyset$). Then $\mathcal{I}_{mv}(\mathcal{K}_i^{13}) = i/n$ and we obtain $\mathcal{C}^v(\mathcal{I}_{mv}, n) = n + 1$.
58. $\mathcal{C}^f(\mathcal{I}_{mv}, n) = \infty$
If $|\mathcal{K}| = 1$ then observe that either $\mathcal{I}_{mv}(\mathcal{K}) = 0$ or $\mathcal{I}_{mv}(\mathcal{K}) = 1$ (if the knowledge base is inconsistent then all propositions appearing in $\mathcal{K}$ also appear in the only formula of $\mathcal{K}$ and are thus part of a minimal inconsistent subset), so we have $\mathcal{C}^f(\mathcal{I}_{mv}, 1) = 2$. For $n = |\mathcal{K}| > 1$ consider for $i \in \mathbb{N}$ the family of knowledge bases $\mathcal{K}_i^{14} = \{a_1 \wedge \ldots \wedge a_{i-1}, a_i \wedge \neg a_i\}$ and observe that $\mathcal{I}_{mv}(\mathcal{K}_i^{14}) = 1/i$. Hence, we have $\mathcal{C}^f(\mathcal{I}_{mv}, n) = \infty$ for $n > 1$.
59. $\mathcal{C}^l(\mathcal{I}_{mv}, n) = \infty$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_{mv}, 1) = 1$. For $n > 1$ consider $\mathcal{K}_i^{12} = \{\neg a_1, a_1, a_2, \ldots, a_i\}$ with $i > 1$ and note that $\{\neg a_1, a_1\}$ is the only minimal inconsistent subset of $\mathcal{K}_i^{12}$. Then $\mathcal{I}_{mv}(\mathcal{K}_i^{12}) = 1/i$ and therefore $\mathcal{C}^l(\mathcal{I}_{mv}, n) = \infty$.
60. $\mathcal{C}^p(\mathcal{I}_{mv}, n) = \infty$
Consider the family of knowledge bases $\mathcal{K}_i^8 = \{a_1 \wedge \neg a_1, a_2, \ldots, a_i\}$ for $i \in \mathbb{N}$. Then $\mathcal{I}_{mv}(\mathcal{K}_i^8) = 1/i$ and therefore $\mathcal{C}^p(\mathcal{I}_{mv}, n) = \infty$.
61. $\mathcal{C}^v(\mathcal{I}_{nc}, n) = \infty$
Consider the family of knowledge bases $\mathcal{K}_i^2$ from 5. Then $\mathcal{I}_{nc}(\mathcal{K}_i^2) = i$ and $\lim_{i \to \infty} \mathcal{I}_{nc}(\mathcal{K}_i^2) = \infty$. As each $\mathcal{K}_i^2$ only uses one proposition the statement is true for every $n > 0$.
62. $\mathcal{C}^f(\mathcal{I}_{nc}, n) = n + 1$
Obviously, $\mathcal{I}_{nc}(\mathcal{K}) \leq |\mathcal{K}| \in \{0, \ldots, n\}$ for every knowledge base. Consider the family of knowledge bases $\mathcal{K}_{n,k}$ from 30. Then $\mathcal{I}_p(\mathcal{K}_{n,k}) = k - 1$ for $k = 1, \ldots, n$. Furthermore, a knowledge base $\mathcal{K}$ containing at least one contradictory formula has $\mathcal{I}_{nc}(\mathcal{K}) = |\mathcal{K}| = n$. Hence, we have $\mathcal{C}^f(\mathcal{I}_{nc}, n) = n + 1$.
63. $\mathcal{C}^l(\mathcal{I}_{nc}, n) = \infty$
Note that $n > 1$ is assumed as trivially $\mathcal{C}^l(\mathcal{I}_{nc}, 1) = 1$. For $n > 1$ consider $\mathcal{K}_i^{12} = \{\neg a_1, a_1, a_2, \ldots, a_i\}$ with $i > 1$ and note that $\{\neg a_1, a_1\}$ is the only minimal inconsistent subset of $\mathcal{K}_i^{12}$. Then $\mathcal{I}_{nc}(\mathcal{K}_i^{12}) = |\mathcal{K}_i| - 1 = i$ and therefore $\mathcal{C}^l(\mathcal{I}_{nc}, n) = \infty$.
64. $\mathcal{C}^p(\mathcal{I}_{nc}, n) = \infty$
Analogous to 61. □

Theorem 3.

1. *The measures $\mathcal{I}_d$, $\mathcal{I}_c$, and $\mathcal{I}_{LP_m}$ satisfy $\wedge$-Indifference, $\wedge$-Penalty, and $\wedge$-Mitigation.*
2. *The measures $\mathcal{I}_\eta$, $\mathcal{I}_{hs}$, and $\mathcal{I}_{\mathrm{dalal}}^{\max}$ satisfy $\wedge$-Penalty, but not $\wedge$-Mitigation.*
3. *The measures $\mathcal{I}_{\mathrm{dalal}}^{\mathrm{hit}}$ and $\mathcal{I}_{P_m}$ satisfy $\wedge$-Mitigation, but not $\wedge$-Penalty.*
4. *None of the measures $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$, $\mathcal{I}_{mc}$, $\mathcal{I}_p$, $\mathcal{I}_{\mathrm{dalal}}^{\Sigma}$, $\mathcal{I}_{D_f}$, $\mathcal{I}_{mv}$, $\mathcal{I}_{nc}$ satisfies any of $\wedge$-Indifference, $\wedge$-Penalty, or $\wedge$-Mitigation.*

Proof. Let $\mathcal{K}$ be some arbitrary knowledge base and $\alpha, \beta \in \mathcal{L}(\mathsf{At})$ formulas.

1. Note first that if an inconsistency measure $\mathcal{I}$ satisfies $\wedge$-Indifference it also satisfies $\wedge$-Penalty and $\wedge$-Mitigation. Therefore, we only show that $\mathcal{I}_d$ and $\mathcal{I}_c$ satisfy $\wedge$-Indifference.
   (a) $\mathcal{I}_d$: the statement follows directly from that fact that every knowledge base $\mathcal{K} \cup \{\alpha, \beta\}$ is inconsistent if and only if $\mathcal{K} \cup \{\alpha \wedge \beta\}$ is inconsistent.

(b) $\mathcal{I}_c$: let υ be a three-valued interpretation $\upsilon : \mathsf{At} \to \{T, F, B\}$ with $\upsilon \models^3 \mathcal{K} \cup \{\alpha, \beta\}$. Then $\upsilon(\alpha), \upsilon(\beta) \in \{T, B\}$. From Table 2 it follows that $\upsilon(\alpha \wedge \beta) \in \{T, B\}$ and therefore $\upsilon \models^3 \mathcal{K} \cup \{\alpha \wedge \beta\}$ as well. Note that the converse holds as well. Then we have

$$\begin{aligned}\mathcal{I}_c(\mathcal{K} \cup \{\alpha, \beta\}) &= \min\{|\upsilon^{-1}(B)| \mid \upsilon \models^3 \mathcal{K} \cup \{\alpha, \beta\}\} \\ &= \min\{|\upsilon^{-1}(B)| \mid \upsilon \models^3 \mathcal{K} \cup \{\alpha \wedge \beta\}\} \\ &= \mathcal{I}_c(\mathcal{K} \cup \{\alpha \wedge \beta\})\end{aligned}$$

(c) $\mathcal{I}_{LP_m}$: this follows directly from $\mathcal{I}_c$ satisfying $\wedge$-Indifference and $\mathsf{At}(\mathcal{K} \cup \{\alpha, \beta\}) = \mathsf{At}(\mathcal{K} \cup \{\alpha \wedge \beta\})$.

2. (a) $\mathcal{I}_\eta$: let P be a probability distribution $P : \Omega(\mathsf{At}) \to [0, 1]$ with $P(\phi) \geq \hat{\xi}$ for all $\phi \in \mathcal{K} \cup \{\alpha \wedge \beta\}$ with $\hat{\xi}$ being maximal. Then $P(\alpha) \geq \hat{\xi}$ and $P(\beta) \geq \hat{\xi}$ as well. It follows that

$$\begin{aligned}\hat{\xi} &= \max\{\xi \mid \exists P \in \mathcal{P}(\mathsf{At}) : \forall \phi \in \mathcal{K} \cup \{\alpha \wedge \beta\} : P(\phi) \geq \xi\} \\ &\leq \max\{\xi \mid \exists P \in \mathcal{P}(\mathsf{At}) : \forall \phi \in \mathcal{K} \cup \{\alpha, \beta\} : P(\phi) \geq \xi\}\end{aligned}$$

and therefore $\mathcal{I}_\eta(\mathcal{K} \cup \{\alpha \wedge \beta\}) \geq \mathcal{I}_\eta(\mathcal{K} \cup \{\alpha, \beta\})$. To see that $\mathcal{I}_\eta$ does not satisfy $\wedge$-Mitigation consider $\mathcal{K}_{10} = \{\neg a \wedge b, a \wedge \neg b\}$ and

$$\begin{aligned}\mathcal{I}_\eta(\mathcal{K}_{10} \cup \{a \wedge b\}) &= 2/3 \\ \mathcal{I}_\eta(\mathcal{K}_{10} \cup \{a, b\}) &= 1/2\end{aligned}$$

(b) $\mathcal{I}_{hs}$: let H be a hitting set of $\mathcal{K} \cup \{\alpha \wedge \beta\}$ and let $\omega \in H$ be such that $\omega \models \alpha \wedge \beta$. Then $\omega \models \alpha$ and $\omega \models \beta$ as well and H is a hitting set of $\mathcal{K} \cup \{\alpha, \beta\}$. So we have that every hitting set of $\mathcal{K} \cup \{\alpha \wedge \beta\}$ is also a hitting set of $\mathcal{K} \cup \{\alpha, \beta\}$ and therefore

$$\begin{aligned}&\{H \mid H \text{ is a hitting set of } \mathcal{K} \cup \{\alpha \wedge \beta\}\} \\ \subseteq\ &\{H \mid H \text{ is a hitting set of } \mathcal{K} \cup \{\alpha, \beta\}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{hs}(\mathcal{K} \cup \{\alpha \wedge \beta\}) &= \min\{|H| \mid H \text{ is a hitting set of } \mathcal{K} \cup \{\alpha \wedge \beta\}\} \\ &\geq \min\{|H| \mid H \text{ is a hitting set of } \mathcal{K} \cup \{\alpha, \beta\}\} \\ &= \mathcal{I}_{hs}(\mathcal{K} \cup \{\alpha, \beta\})\end{aligned}$$

To see that $\mathcal{I}_{hs}$ does not satisfy $\wedge$-Mitigation consider $\mathcal{K}_{10} = \{\neg a \wedge b, a \wedge \neg b\}$ and

$$\begin{aligned}\mathcal{I}_{hs}(\mathcal{K}_{10} \cup \{a \wedge b\}) &= 2 \\ \mathcal{I}_{hs}(\mathcal{K}_{10} \cup \{a, b\}) &= 1\end{aligned}$$

(c) $\mathcal{I}^{\max}_{\mathrm{dalal}}$: For every $\omega \in \Omega(\mathsf{At})$, as $\mathsf{Mod}(\alpha \wedge \beta) \subseteq \mathsf{Mod}(\alpha)$ and $\mathsf{Mod}(\alpha \wedge \beta) \subseteq \mathsf{Mod}(\beta)$ we have

$$\begin{aligned}d_{\mathrm{d}}(\mathsf{Mod}(\alpha \wedge \beta), \omega) &\geq d_{\mathrm{d}}(\mathsf{Mod}(\alpha), \omega) \\ d_{\mathrm{d}}(\mathsf{Mod}(\alpha \wedge \beta), \omega) &\geq d_{\mathrm{d}}(\mathsf{Mod}(\beta), \omega)\end{aligned}$$

It follows that

$$\max_{\phi \in \mathcal{K} \cup \{\alpha \wedge \beta\}} d_{\mathrm{d}}(\mathsf{Mod}(\phi), \omega) \geq \max_{\phi \in \mathcal{K} \cup \{\alpha, \beta\}} d_{\mathrm{d}}(\mathsf{Mod}(\phi), \omega)$$

and

$$\begin{aligned}\mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K} \cup \{\alpha \wedge \beta\}) &= \min\{\max_{\phi \in \mathcal{K} \cup \{\alpha \wedge \beta\}} d_{\mathrm{d}}(\mathsf{Mod}(\phi), \omega) \mid \omega \in \Omega(\mathsf{At})\} \\ &\geq \min\{\max_{\phi \in \mathcal{K} \cup \{\alpha, \beta\}} d_{\mathrm{d}}(\mathsf{Mod}(\phi), \omega) \mid \omega \in \Omega(\mathsf{At})\} \\ &= \mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K} \cup \{\alpha, \beta\})\end{aligned}$$

To see that $\mathcal{I}^{\max}_{\mathrm{dalal}}$ does not satisfy $\wedge$-Mitigation consider $\mathcal{K}_{11} = \{\neg a \wedge b, a \wedge \neg b, \neg a \wedge \neg b\}$ and

$$\begin{aligned}\mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K}_{11} \cup \{a \wedge b\}) &= 2 \\ \mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K}_{11} \cup \{a, b\}) &= 1\end{aligned}$$

3. (a) $\mathcal{I}^{\text{hit}}_{\text{dalal}}$: Let $\omega \in \Omega(\text{At})$ be arbitrary. Note that $\omega \not\models \alpha \wedge \beta$ if and only if either $\omega \not\models \alpha$, $\omega \not\models \beta$, or both. Therefore, $d_{\text{d}}(\text{Mod}(\alpha \wedge \beta), \omega) > 0$ if and only if either $d_{\text{d}}(\text{Mod}(\alpha), \omega) > 0$, $d_{\text{d}}(\text{Mod}(\beta), \omega) > 0$, or both. Then

$$\begin{aligned} &|\{\phi \in \mathcal{K} \cup \{\alpha \wedge \beta\} \mid d_{\text{d}}(\text{Mod}(\phi), \omega) > 0\}| \\ \leq\ &|\{\phi \in \mathcal{K} \cup \{\alpha, \beta\} \mid d_{\text{d}}(\text{Mod}(\phi), \omega) > 0\}| \end{aligned}$$

and $\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K} \cup \{\alpha \wedge \beta\}) \leq \mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K} \cup \{\alpha, \beta\})$. To see that $\mathcal{I}^{\text{hit}}_{\text{dalal}}$ does not satisfy $\wedge$-Penalty consider $\mathcal{K}_{12} = \{\neg a, \neg b\}$ and

$$\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}_{12} \cup \{a \wedge b\}) = 1$$
$$\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}_{12} \cup \{a, b\}) = 2$$

(b) $\mathcal{I}_{P_m}$: observe that if π is a minimal proof for some γ in $\mathcal{K} \cup \{\alpha \wedge \beta\}$ and $\alpha \wedge \beta \in \pi$ then either
   i. $\pi \setminus \{\alpha \wedge \beta\} \cup \{\alpha\}$, or
   ii. $\pi \setminus \{\alpha \wedge \beta\} \cup \{\beta\}$,
   iii. both of the above, or
   iv. $\pi \setminus \{\alpha \wedge \beta\} \cup \{\alpha, \beta\}$

is a minimal proof (are minimal proofs) for γ in $\mathcal{K} \cup \{\alpha, \beta\}$. In any case, $|P_m^{\mathcal{K} \cup \{\alpha \wedge \beta\}}(\gamma)| \leq |P_m^{\mathcal{K} \cup \{\alpha, \beta\}}(\gamma)|$ for every γ and therefore $\mathcal{I}_{P_m}(\mathcal{K} \cup \{\alpha \wedge \beta\}) \leq \mathcal{I}_{P_m}(\mathcal{K} \cup \{\alpha, \beta\})$. To see that $\mathcal{I}_{P_m}$ does not satisfy $\wedge$-Penalty consider $\mathcal{K}_{13} = \{\neg a\}$ and

$$\mathcal{I}_{P_m}(\mathcal{K}_{13} \cup \{a \wedge \neg\neg a\}) = 1$$
$$\mathcal{I}_{P_m}(\mathcal{K}_{13} \cup \{a, \neg\neg a\}) = 2$$

4. (a) $\mathcal{I}_{\text{MI}}$: consider Example 16 with $\mathcal{K}_7 = \{a, \neg a\}$. Here we have

$$\mathcal{I}_{\text{MI}}(\mathcal{K}_7 \cup \{a, b\}) = 1 < 2 = \mathcal{I}_{\text{MI}}(\mathcal{K}_7 \cup \{a \wedge b\})$$

and

$$\mathcal{I}_{\text{MI}}(\mathcal{K}_7 \cup \{a \wedge \neg a, \neg\neg a\}) = 3 > 2 = \mathcal{I}_{\text{MI}}(\mathcal{K}_7 \cup \{a \wedge \neg a \wedge \neg\neg a\})$$

(b) $\mathcal{I}_{\text{MI}^{\text{C}}}$: consider Example 16 with $\mathcal{K}_7 = \{a, \neg a\}$. Here we have

$$\mathcal{I}_{\text{MI}^{\text{C}}}(\mathcal{K}_7 \cup \{a, b\}) = 1/2 < 1 = \mathcal{I}_{\text{MI}^{\text{C}}}(\mathcal{K}_7 \cup \{a \wedge b\})$$

and

$$\mathcal{I}_{\text{MI}^{\text{C}}}(\mathcal{K}_7 \cup \{a \wedge \neg a, \neg\neg a\}) = 2 > 3/2 = \mathcal{I}_{\text{MI}^{\text{C}}}(\mathcal{K}_7 \cup \{a \wedge \neg a \wedge \neg\neg a\})$$

(c) $\mathcal{I}_{mc}$: consider Example 16 with $\mathcal{K}_7 = \{a, \neg a\}$. Here we have

$$\mathcal{I}_{mc}(\mathcal{K}_7 \cup \{a, \neg a\}) = 1 < 2 = \mathcal{I}_{mc}(\mathcal{K}_7 \cup \{a \wedge \neg a\})$$

and for $\mathcal{K}_9 = \{a, b\}$ we have

$$\mathcal{I}_{mc}(\mathcal{K}_9 \cup \{\neg a, \neg b\}) = 2 > 1 = \mathcal{I}_{mc}(\mathcal{K}_9 \cup \{\neg a \wedge \neg b\})$$

(d) $\mathcal{I}_p$: consider $\mathcal{K}_{10} = \{\neg a, a, \neg b, b\}$. Here we have

$$\mathcal{I}_p(\mathcal{K}_{10} \cup \{a, b\}) = 4 < 5 = \mathcal{I}_p(\mathcal{K}_{10} \cup \{a \wedge b\})$$

and for $\mathcal{K}_5 = \{a, b\}$ we have

$$\mathcal{I}_p(\mathcal{K}_5 \cup \{\neg a, \neg b\}) = 4 > 3 = \mathcal{I}_p(\mathcal{K}_5 \cup \{\neg a \wedge \neg b\})$$

(e) $\mathcal{I}^{\Sigma}_{\text{dalal}}$: consider $\mathcal{K}_5 = \{a, b\}$. Here we have

$$\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}_5 \cup \{\neg a, a\}) = 1 < \infty = \mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}_5 \cup \{\neg a \wedge a\})$$

and for $\mathcal{K}_{14} = \{a, b, a \wedge b, \neg a \wedge \neg b\}$ we have

$$\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}_{14} \cup \{\neg a, \neg b\}) = 4 > 2 = \mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}_{14} \cup \{\neg a \wedge \neg b\})$$

(f) $\mathcal{I}_{D_f}$: consider Example 16 with $\mathcal{K}_7 = \{a, \neg a\}$. Here we have

$$\mathcal{I}_{D_f}(\mathcal{K}_7 \cup \{a, b\}) = 1/6 < 1/3 = \mathcal{I}_{D_f}(\mathcal{K}_7 \cup \{a \wedge b\})$$

and for $\mathcal{K}_{15} = \{a \wedge b, a \wedge c, a \wedge d\}$ we have

$$\mathcal{I}_{D_f}(\mathcal{K}_{15} \cup \{\neg a, \neg a \wedge e\}) = 3/10 > 1/4 = \mathcal{I}_{D_f}(\mathcal{K}_{15} \cup \{\neg a \wedge \neg a \wedge e\})$$

(g) $\mathcal{I}_{mv}$: consider $\mathcal{K}_8 = \{a\}$. Here we have

$$\mathcal{I}_{D_f}(\mathcal{K}_8 \cup \{\neg a, b\}) = 1 < 2 = \mathcal{I}_{D_f}(\mathcal{K}_8 \cup \{\neg a \wedge b\})$$

and for $\mathcal{K}_6 = \{a \wedge b\}$ we have

$$\mathcal{I}_{D_f}(\mathcal{K}_6 \cup \{\neg a, a\}) = 2 > 1 = \mathcal{I}_{D_f}(\mathcal{K}_6 \cup \{\neg a \wedge a\})$$

(h) $\mathcal{I}_{nc}$: consider $\mathcal{K}_8 = \{a\}$. Here we have

$$\mathcal{I}_{nc}(\mathcal{K}_8 \cup \{\neg a, a\}) = 1 < 2 = \mathcal{I}_{nc}(\mathcal{K}_8 \cup \{\neg a \wedge a\})$$

and for $\mathcal{K}_7 = \{a, \neg a\}$ from Example 16 we have

$$\mathcal{I}_{nc}(\mathcal{K}_7 \cup \{b, c\}) = 3 > 2 = \mathcal{I}_{nc}(\mathcal{K}_7 \cup \{b \wedge c\}) \qquad \square$$

Appendix B. List of knowledge bases

$$\begin{aligned}
\mathcal{K}_1 &= \{a, b \vee c, \neg a \wedge \neg b, d\} \\
\mathcal{K}_2 &= \{a, \neg a, b, \neg b\} \\
\mathcal{K}_3 &= \{a, \neg a, b, c, d\} \\
\mathcal{K}_4 &= \{a, b, c, \neg a \vee \neg b \vee \neg c, \neg(a \wedge b \wedge c)\} \\
\mathcal{K}_5 &= \{a, b\} \\
\mathcal{K}_6 &= \{a \wedge b\} \\
\mathcal{K}_7 &= \{a, \neg a\} \\
\mathcal{K}_8 &= \{a\} \\
\mathcal{K}_9 &= \{a \wedge \neg a\} \\
\mathcal{K}_{10} &= \{\neg a \wedge b, a \wedge \neg b\} \\
\mathcal{K}_{11} &= \{\neg a \wedge b, a \wedge \neg b, \neg a \wedge \neg b\} \\
\mathcal{K}_{12} &= \{\neg a, \neg b\} \\
\mathcal{K}_{13} &= \{\neg a\} \\
\mathcal{K}_{14} &= \{a, b, a \wedge b, \neg a \wedge \neg b\} \\
\mathcal{K}_{15} &= \{a \wedge b, a \wedge c, a \wedge d\} \\
\mathcal{K}_i^1 &= \{a_1 \wedge \ldots \wedge a_i, \neg a_1 \wedge \ldots \wedge \neg a_i\} && i \in \mathbb{N} \\
\mathcal{K}_i^2 &= \{\neg a, a, a \wedge a, a \wedge a \wedge a, \ldots, \bigwedge_{j=1}^{i} a\} && i \in \mathbb{N} \\
\mathcal{K}_i^3 &= \{a_1, \ldots, a_i, \neg a_1, \ldots, \neg a_i\} && i \in \mathbb{N} \\
\mathcal{K}_i^4 &= \{\neg a_1 \vee a_2, \neg a_2 \vee a_3, \ldots, \neg a_{i-1} \vee a_i, a_i \wedge \neg a_1\} && i \in \mathbb{N} \\
\mathcal{K}_i^5 &= \{a_1 \wedge \neg a_1, \ldots, a_i \wedge \neg a_i\} && i \in \mathbb{N} \\
\mathcal{K}_i^6 &= \{a_1 \wedge \ldots \wedge a_i \wedge \neg a_1 \wedge \ldots \wedge \neg a_i\} && i \in \mathbb{N} \\
\mathcal{K}_i^7 &= \{a \wedge \neg a, a \wedge a \wedge \neg a \wedge \neg a, \ldots \bigwedge_{j=1}^{i} a \wedge \neg a\} && i \in \mathbb{N}
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_i^8 &= \{a_1 \wedge \neg a_1, a_2, \ldots, a_i\} && i \in \mathbb{N}\\
\mathcal{K}_i^9 &= \{a \wedge b, \neg a \wedge b, a \wedge \neg b \mid a, b \in \mathsf{At}_i, a \neq b\} && i \in \mathbb{N}\\
\mathcal{K}_i^{10} &= \{a, \neg a, a \wedge a, \neg a \wedge \neg a, \ldots, \bigwedge_{j=1}^{i} a, \bigwedge_{j=1}^{i} \neg a\} && i \in \mathbb{N}\\
\mathcal{K}_i^{11} &= \{\phi_\omega \mid \omega \in \Omega(\mathsf{At}_i)\} && i \in \mathbb{N}\\
\mathcal{K}_i^{12} &= \{\neg a_1, a_1, a_2, \ldots, a_i\} && i \in \mathbb{N}\\
\mathcal{K}_i^{13} &= \{a_1 \wedge \neg a_1, \ldots, a_i \wedge \neg a_i, a_{i+1}, \ldots, a_n\} && i \in \mathbb{N}\\
\mathcal{K}_i^{14} &= \{a_1 \wedge \ldots \wedge a_{i-1}, a_i \wedge \neg a_i\} && i \in \mathbb{N}\\
\mathcal{K}_i^{15} &= \{\neg a_1 \wedge a_1 \wedge a_2 \wedge \ldots \wedge a_i\} && i \in \mathbb{N}\\
\mathcal{K}_{i,j}^{16} &= \{a_1, \ldots, a_i, \neg a_1, \ldots, \neg a_i\} && i, j \in \mathbb{N}
\end{aligned}$$

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