Heliyon 8 (2022) e09775

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

The statistical properties and applications of the alpha power Topp-Leone-G distribution

Joseph Thomas Eghwerido [a,*], Friday Ikechukwu Agu [b]

[a] Department of Statistics, Federal University of Petroleum Resources, Effurun, Delta State, Nigeria
[b] Department of Statistics, University of Calabar, Calabar, Cross-River State, Nigeria

A R T I C L E I N F O

Keywords:
Alpha power distribution
Beta distribution
Data analysis
Maximum likelihood
Record values
Weighted quantile function

A B S T R A C T

A two-parameter class of generator named the alpha power Topp-Leone-G (APTL-G) distribution for generating and engendering up to the minute family of continuous contemporary distributions is proposed. The propounded model has a bathtub and J-shapes hazard rate characteristics in data science. Thus, it has enhanced data analysis because of its productivity. A weighted function of the parent distribution was used to obtain the quantile function for the different sub-models considered. The record values of the differences of the k^{th} lower and upper values were obtained in a closed-form to enable its applicability. The APTL-G model parameters values were captured and obtained in their simplified cases by the maximum likelihood approach. A simulation was also adopted to examine and investigate the performance, productivity, efficiency and tractability of the APTL-G sub-models. The outcomes of the real-life application and simulation study show that the performance of the goodness-of-fit of the APTL-G model is more flexible and tractable with real-life data sets concerning its goodness-of-fit.

1. Introduction

Despite existing numerous statistical distributions to unravel the distributions of life processes in modelling, there are yet numerous newer tractable and flexible emerging statistical models to harness the true characteristics of observable data set. However, the variability intractability, performance and flexibility of the statistical models may be a result of variability in locations (see [9],[10]). Nevertheless, statistical decision making depends on the probability distribution of the outcome of the underline random processes.

In statistical distribution, most newly developed traditional distributions do not characterise the true characteristics of the data set. This can be seen in their goodness-of-fit with data applications. However, to improve these distributions, families of statistical distributions are being developed to extend, make adequate, and improve existing traditional distributions. Nevertheless, the flexibility, simplicity and tractability of any model is depending on the performance of the goodness-of-fit, parsimoniousness in parameter(s), tractability and simplicity of the true model characteristics in data representation and analysis. Thus, achieving underlined features becomes paramount in data analysis for applicable inference. Hence, in some cases, an additional parameter is added to the classical distributions to obtain a simplified, modified and applicable distribution.

The beta distribution has very important in varieties of scenarios that deal with percentages and proportions that occur randomly (uncertainty). For example, it has been used as a conjugate prior model for geometric, binomial, Bernoulli, and negative binomial distributions. Regardless of how important the beta distribution is, there is a need to improve the tractability and flexibility of the beta distribution for better performance. Thus, the Topp-Leone distribution has served as a better alternative to the beta distribution in reliability theory. This is a result of its J-shaped and boundedness characteristics. Hence, it has found wide importance in the field of reliability and fuzzy theories, risk analysis, economic, actuarial science, spatial statistics, decision sciences, operation research, probabilistic random processes, machine learning, cryptography and uncertainty quality control. Thus, based on the major relevance of the Topp-Leone distribution, there is a need to improve its performance to be able to capture the true character of the data set. Hence, this research introduces a contemporary new family of the generator using the alpha power characteristics.

* Corresponding author.
E-mail address: eghwerido.joseph@fupre.edu.ng (J.T. Eghwerido).

https://doi.org/10.1016/j.heliyon.2022.e09775
Received 28 November 2021; Received in revised form 21 February 2022; Accepted 17 June 2022

Table 1. The skewness (S), kurtosis (K), median (M) and quantile function (Q) for parameter $\hat{\alpha}, \hat{\lambda}, \hat{\beta}$, and $\hat{\mu}$ for the APTL-G model.

Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\mu}$	S	K	Q_1	M	Q_3
W	0.5	1.0	0.5	0.5	0.4766	1.2142	0.0378	0.1890	0.6157
	1.5	1.0	0.5	0.5	0.3741	0.8988	0.0881	0.3607	0.9593
	1.5	1.5	0.5	0.5	0.2029	0.4562	0.2495	0.6385	1.2255
	1.5	1.5	1.0	0.5	0.1194	0.2619	0.6175	1.0851	1.6794
	1.5	1.5	1.0	1.0	0.1194	0.2619	0.3087	0.5425	0.8397
	1.5	1.5	1.5	1.5	0.0970	0.2102	0.2911	0.4511	0.6455
	1.5	1.5	1.5	2.0	0.0970	0.2102	0.2183	0.3383	0.4841
	1.5	1.5	2.0	1.5	0.0876	0.1883	0.3541	0.5140	0.7047
F	0.5	1.0	0.5	0.5	0.5650	2.1312	0.0632	0.1727	0.5670
	1.5	1.0	0.5	0.5	0.5745	2.1396	0.1011	0.3081	1.0740
	1.5	1.5	0.5	0.5	0.5745	2.1396	0.1517	0.4622	1.6110
	1.5	1.5	1.0	0.5	0.5475	1.9790	0.4395	1.2165	3.8744
	1.5	1.5	1.0	1.0	0.3258	0.8807	0.8120	1.3508	2.4107
	1.5	1.5	1.5	1.5	0.2392	0.5927	1.2082	1.6700	2.4225
	1.5	1.5	1.5	2.0	0.1976	0.4702	1.2753	1.6258	2.1489
	1.5	1.5	2.0	1.5	0.2381	0.5898	1.3775	1.8857	2.7116
G	0.5	1.0	0.5	0.5	0.4502	1.0798	0.0188	0.0923	0.2863
	1.5	1.0	0.5	0.5	0.3316	0.7510	0.0436	0.1727	0.4299
	1.5	1.5	0.5	0.5	0.3448	0.7934	0.0291	0.1167	0.2966
	1.5	1.5	1.0	0.5	0.1959	0.4354	0.1112	0.2501	0.4566
	1.5	1.5	1.0	1.0	0.1635	0.3576	0.1083	0.2361	0.4140
	1.5	1.5	1.5	1.5	0.0907	0.1965	0.1690	0.2950	0.4461
	1.5	1.5	1.5	2.0	0.0711	0.1559	0.1628	0.2775	0.4099
	1.5	1.5	2.0	1.5	0.0703	0.1515	0.2181	0.3447	0.4904

Several families have proposed generating new distributions in the statistical literature. For example, the transmuted alpha power generator model was proposed in [12]. [4] proposed the Gompertz-G generator. [2] proposed the family of beta transmuted-H distributions. The Transmuted Odd Log-Logistic generator was proposed in [6]. [11] proposed the alpha power Marshall-Olkin generator for improving distributions. The exponentiated generalized generator Poisson distribution was proposed in [7] for generalizing the Poisson models. [13] proposed the alpha power Teissier distribution. The Topp-Leone odd Lindley generator of distributions was proposed in [19]. The transmuted Weibull generator of distributions was proposed in [5]. The Marshall-Olkin generalized generator of distributions was proposed in [22] for adding a parameter to existing models. The transmuted Topp-Leone generator of distributions was proposed in [23].

Motivated by the tractability and performance of the Topp-Leone generator of models, this research introduces a new up to the minute contemporary generator for continuous distribution using the alpha power model transformation for improving contemporary reliability theories and risk analysis. More so, the record values of the proposed model were examined in a closed form to complement its productiveness.

Let X be a sampled random variable, thus the probability density function (pdf) of the Topp-Leone generator is defined as [3] and [20] as

$$w(x) = 2\lambda g(x)\big(1-G(x)\big)\big(G(x)\big)^{-(1-\lambda)}\big(2-G(x)\big)^{-(1-\lambda)} \quad x \in \Re,\ \lambda > 0, \tag{1}$$

and its cumulative distribution function (cdf) is given as

$$W(x) = \big(G(x)\big)^{\lambda}\big(2-G(x)\big)^{\lambda} \quad x \in \Re,\ \lambda > 0, \tag{2}$$

where $g(x)$ and $G(x)$ are the parent pdf and cdf.

Nevertheless, [16] defined the alpha power transformation pdf and cdf as

$$f(x) = \begin{cases} w(x)\log\alpha(\alpha-1)^{-1}\frac{1}{\alpha^{-W(x)}}, & \text{if } \alpha > 0, \alpha \neq 1 \\ w(x), & \text{otherwise}, \alpha = 1, \end{cases} \tag{3}$$

and

$$F(x) = \begin{cases} \frac{\alpha^{W(x)}-1}{(\alpha-1)}, & \text{if } \alpha > 0, \alpha \neq 1 \\ W(x), & \text{otherwise}, \alpha = 1, \end{cases} \tag{4}$$

2. The APTL-G model

Let X be a random variable sample from the APTL-G distribution. Then, the pdf and cdf of the APTL-G are given as

$$f(x) = \begin{cases} \frac{2\lambda g(x)\log\alpha}{(\alpha-1)}\bar{G}(x)G^{\lambda-1}(x)[2-G(x)]^{\lambda-1} \\ \times\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}, & \text{if } \lambda > 0, \alpha > 0, \alpha \neq 1 \\ 2\alpha g(x)\bar{G}(x)G^{\lambda-1}(x)[2-G(x)]^{\lambda-1}, & \text{otherwise}, \lambda > 0, \alpha = 1, \end{cases} \tag{5}$$

and

$$F(x) = \begin{cases} \frac{\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}-1}{(\alpha-1)}, & \text{if } \lambda > 0, \alpha > 0, \alpha \neq 1 \\ G^{\lambda}(x)[2-G(x)]^{\lambda}, & \text{otherwise}, \lambda > 0, \alpha = 1, \end{cases} \tag{6}$$

where $\bar{G} = 1 - G(x)$; $g(x)$ and $G(x)$ are the parent or baseline pdf and cdf respectively.

The quantile function of a random variable X for $U \sim$ Uniform $(0,1)$ of the APTL-G distribution can be obtained in a closed form as

$$x_u \begin{cases} G^{-1}\left[1 \pm \left[(1-[(\log\alpha)^{-1}\log[u(\alpha-1)+1]]^{\frac{1}{\lambda}})\right]^{\frac{1}{2}}\right], & \text{if } \lambda > 0, \alpha > 0, \alpha \neq 1 \\ G^{-1}\left[1 \pm (1-u^{\frac{1}{\lambda}})^{\frac{1}{2}}\right], & \text{otherwise}, \lambda > 0, \\ & \alpha = 1. \end{cases} \tag{7}$$

Table 1 illustrates the skewness and the kurtosis of the proposed model. In Table 1, an increase in parameters values decreases the skewness and kurtosis with the Weibull (W), Gompertz (G) and Frechet (F) models. The kurtosis and skewness were positive in all models. The quantiles in all models are increasing.

Theorem 1. *The characteristic demeanor of the APTL-G distribution can be investigated by the attributes of $Q'(x)$, $f(x)$, and $Q''(x)$ with $Q(x)$ given as $Q(x) = \log\ f(x)$ for $\alpha > 0, \alpha \neq 1$.*

Proof. Suppose that $Q(t)$ is given as $Q(t) = \log\ f(x)$, then, we can express

$$Q(x) = 1 - \log(\alpha - 1) + \log(\lambda) + \log g(x) + \log(\bar{G}(x))$$
$$+(\lambda - 1)\log G(x) + (\lambda - 1)\log(2 - G(x)) - 1 + \log\log(\alpha) + \log 2$$
$$+[G(x)(2 - G(x))]^{\lambda}\log(\alpha).$$

Thus,

$$Q'(x) = \frac{g'(x)}{g(x)} + \frac{g(x)}{\bar{G}(x)} + \frac{(\lambda-1)g(x)}{G(x)} - \frac{(\lambda-1)g(x)}{(2-G(x))}$$
$$+\lambda G^{\lambda-1}(x)g(x)(2-G(x))^{\lambda}\log(\alpha) - \lambda g(x)G^{\lambda}(x)(2-G(x))^{\lambda-1}\log(\alpha).$$

However, for all x and $Q' < 0$, $F(x)$ is monotonically non-increasing. The mode of the APTL-G proposed model can be obtained by taking the second partial derivative of Q as Q'' for values of $\eta = 0, \beta = 0$. More so, suppose the differential $f(x)''$ alternates signs from positive to non-positive and to positive and again negative as the values of x increase continuously viz-a-viz, then, the proposed APTL-G distribution will be bimodal. Thus, the J-shape of the hazard rate function of the proposed model is established. □

3. Parameter estimation

In this section, the parameters of the proposed model were estimated using the maximum likelihood and a Monte Carlo simulation study.

3.1. Maximum likelihood

Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a random sample obtained from the APTL-G distribution with unknown parameter vector $\Theta = (\alpha, \lambda, \phi)^T$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a sample values of random sample $\mathbf{X}$. Then, we can obtain the log-likelihood as

$$\ell = n\log 2 + n\log\lambda + n\log(\log\alpha) - n\log(\alpha - 1) + \sum_{d=1}^{n}\log g(x_i,\phi) + p$$
$$+\sum_{d=1}^{n}\log\bar{G}(x_i,\phi) + (\lambda-1)\sum_{d=1}^{n}\log G(x_i,\phi) + (\lambda-1)\sum_{d=1}^{n}\log(2-G(x_i,\phi)), \tag{8}$$

where

$$p = \sum_{d=1}^{n}\log G^{\lambda}(x_i,\phi)(2-G(x_i,\phi))^{\lambda}\log\alpha.$$

Nevertheless, taking the partial derivative with respect to the parameters and equating to zero, we have

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + \sum_{d=1}^{n}\log G(x_i,\phi) + \sum_{d=1}^{n}\log(2-G(x_i,\phi)) + p'_{\lambda} = 0, \tag{9}$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha log\alpha} + \frac{n}{\alpha - 1} + p'_{\alpha} = 0, \tag{10}$$

and

$$\frac{\partial\ell}{\partial\phi} = \sum_{d=1}^{n}\frac{g'(x_i,\phi)}{g(x_i,\phi)} + (\lambda-1)\sum_{d=1}^{n}\frac{g(x_i,\phi)}{G(x_i,\phi)} - \sum_{d=1}^{n}\frac{g(x_i,\phi)}{\bar{G}(x_i,\phi)}$$
$$+p'_{\phi} - (\lambda-1)\sum_{d=1}^{n}\frac{g(x_i,\phi)}{2-G(x_i,\phi)} = 0, \tag{11}$$

where ($\prime$) denote derivative. The estimates can not be obtained in a closed form analytically, thus by solving the nonlinear equations in Equations (9), (10) and (11) by utilizing the Newton Raphson algorithm embedded in R software [18].

The APTL-G density and distribution functions is expanded in power series so that some properties of the model can be easily expressed in linear form. However, recall that for variable k, $\alpha^k = \sum_{q=0}^{\infty}\frac{(\log\alpha)^q k^q}{q!}$ and $(x+m)^w = \sum_{a=0}^{w}\binom{w}{a}x^a m^{w-a}$, for $w > 0$. Thus, the APTL-G density function can be expressed as

$$f(x) = \sum_{q=0}^{\infty}\sum_{a=0}^{\lambda(q+1)-1}2^{\lambda(q+1)-a}\lambda(-1)^a\binom{\lambda(q+1)-1}{a}$$
$$\times\frac{(\log\alpha)^{q+1}}{q!(\alpha-1)}g(x)\bar{G}(x)G^{\lambda(q+1)+a-1}(x). \tag{12}$$

4. Some statistical properties

In this section, we examined some general statistical properties of the APTL-G model.

The r^{th} moment is obtained as

$$\mu'_r = \begin{cases} \sum_{q=0}^{\infty}\sum_{a=0}^{\lambda(q+1)-1}P_{q,a}\int_0^{\infty}x^r g(x)\bar{G}(x) & \\ \times G^{\lambda(q+1)+a-1}(x)dx, & \text{if } \lambda>0, \alpha\in\Re^+ - \{1\} \\ 2\int_0^{\infty}x^r\alpha g(x)\bar{G}(x)G^{\lambda-1}(x)[2-G(x)]^{\lambda-1}dx, & \text{otherwise}, \lambda>0, \alpha=1, \end{cases} \tag{13}$$

where

$$P_{q,a} = 2^{\lambda(q+1)-a}\lambda(-1)^a\binom{\lambda(q+1)-1}{a}\frac{(\log\alpha)^{q+1}}{q!(\alpha-1)}.$$

The first moment is obtained when $r = 1$. The variance is obtained as

$$Var(x) = \mu'_2 - [\mu'_1]^2. \tag{14}$$

The probability and moment generating functions of the proposed model can be expressed as

$$p(t) = \begin{cases} \sum_{q,z=0}^{\infty}\sum_{a=0}^{\lambda(q+1)-1}P_{q,a}\frac{(\log t)^z}{z!}\int_0^{\infty}x^z g(x)\bar{G}(x) \\ \times G^{\lambda(q+1)+a-1}(x)dx,\ \lambda>0, \alpha\in\Re^+ - \{1\} \\ 2\frac{(\log t)^z}{z!}\int_0^{\infty}x^z\alpha g(x)\bar{G}(x)G^{\lambda-1}(x) \\ \times[2-G(x)]^{\lambda-1}dx,\ \text{for } \lambda>0, \alpha=1, \end{cases} \tag{15}$$

and

$$M(t) = \begin{cases} \sum_{q=0}^{\infty}\sum_{a=0}^{\lambda(q+1)-1}P_{q,a}\int_0^{\infty}e^{tx}g(x)\bar{G}(x) \\ \times G^{\lambda(q+1)+a-1}(x)dx,\ \lambda>0, \alpha\in\Re^+ - \{1\} \\ 2\int_0^{\infty}e^{tx}\alpha g(x)\bar{G}(x)G^{\lambda-1}(x) \\ \times[2-G(x)]^{\lambda-1}dx,\ \text{for } \lambda>0, \alpha=1. \end{cases} \tag{16}$$

The $(s,r)^{th}$ probability weighted moments of the APTL-G density can be expressed as

$$p(s,r) = \begin{cases} \sum_{q=0}^{\infty}\sum_{a=0}^{\lambda(q+1)-1}\sum_{b=0}^{s}J_{q,a,b}\int_0^{\infty}x^r g(x)\bar{G}(x) \\ \times G^{\lambda(2q+1)+2a-1}(x)dx,\ \lambda>0, \alpha\in\Re^+ - \{1\} \\ 2\int_0^{\infty}x^r\alpha g(x)\bar{G}(x)G^{\lambda s+\lambda-1)}(x) \\ \times[2-G(x)]^{\lambda-1}[2-G^s(x)]^{\lambda}dx,\ \text{for } \lambda>0, \alpha=1, \end{cases} \tag{17}$$

where

$$J_{q,a,b} = P_{q,a}\binom{s}{b}(\alpha-1)^{-s}(-1)^{s-b}A^s$$

and

$$A^s = \binom{\lambda q}{a}(-1)^a 2^{\lambda q+a}(\log\alpha)^q\frac{1}{q!}.$$

The Renyi entropy $R(x)$ of the APTL-G density can be obtained as

$$R(x) = \begin{cases} \frac{1}{1-k}\sum_{q=0}^{\infty}\sum_{a=0}^{\lambda(q+1)-1}\log P_{q,a}^k\int_0^{\infty}g^k(x)\bar{G}^k(x)G^{k(\lambda(q+1)+a-1)}(x)dx, \\ \lambda>0, \alpha\in\Re^+ - \{1\}, k>0\ k\neq 0 \\ \frac{2^k}{1-k}\log\int_0^{\infty}(\alpha g(x)\bar{G}(x)G^{\lambda-1}(x) \\ \times[2-G(x)]^{\lambda-1})^k dx,\ \text{for } \lambda>0, \alpha=1. \end{cases} \tag{18}$$

The moment of the residual and reversed residual life can be expressed as

$$b_n(t)=\begin{cases}\frac{1}{1-F(t)}\sum_{i=0}^{n}\sum_{q=0}^{\infty}\sum_{a=0}^{\lambda(q+1)-1}(-1)^{n-i}t^{n-i}P_{q,a}\int_t^{\infty}x^i g(x)\bar{G}(x)\\ \times G^{(\lambda(q+1)+a-1)}(x)dx,\ \lambda>0,\ \alpha\in\Re^+-\{1\},k>0\ k\neq 0\\ \frac{2\alpha}{1-F(t)}\sum_{i=0}^{n}(-1)^{n-i}t^{n-i}\int_t^{\infty}x^i g(x)\bar{G}(x)G^{\lambda-1}(x)\\ \times[2-G(x)]^{\lambda-1}dx,\ \text{for}\ \lambda>0,\ \alpha=1,\end{cases} \tag{19}$$

and

$$B_n(t)=\begin{cases}\frac{1}{F(t)}\sum_{c=0}^{n}\sum_{q=0}^{\infty}\sum_{a=0}^{\lambda(q+1)-1}(-1)^{c}t^{n-c}P_{q,a}\int_0^{t}x^c g(x)\bar{G}(x)\\ \times G^{(\lambda(q+1)+a-1)}(x)dx,\ \lambda>0,\alpha\in\Re^+-\{1\},k>0\ k\neq 0\\ \frac{2\alpha}{F(t)}\sum_{c=0}^{n}(-1)^{c}t^{n-c}\int_0^{t}x^c g(x)\bar{G}(x)G^{\lambda-1}(x)\\ \times[2-G(x)]^{\lambda-1}dx,\ \text{for}\ \lambda>0,\alpha=1.\end{cases} \tag{20}$$

5. Order statistics

In this section, the record values of the APTL-G model will be examined. This will enhance its usefulness in extreme value theory and reliability analysis.

Suppose $X_{(1)},X_{(2)},X_{(3)},\dots,X_{(n)}$ are the order statistics for random variable $X_1,X_2,X_3,\dots,X_n$ sampled from the APTL-G distribution. Then, the APTL-G density k^{th} order statistics for $\alpha\in\Re^+-\{1\}$, and $\lambda>0$ is given as

$$f_k(x)=\frac{n!}{(k-1)!(n-k)!}\frac{2\lambda\log\alpha}{(\alpha-1)}g(x)\bar{G}(x)G^{\lambda-1}(x)[2-G(x)]^{\lambda-1}\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}\times\Big[\frac{\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}-1}{(\alpha-1)}\Big]^{k-1}\Big[\frac{\alpha-\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}}{(\alpha-1)}\Big]^{n-k}. \tag{21}$$

Otherwise for $\alpha=1$ and $\lambda>0$, we have

$$f_k(x)=\frac{2n!}{(k-1)!(n-k)!}\alpha g(x)\bar{G}(x)G^{\lambda-1}(x)[2-G(x)]^{\lambda-1}\times\big[G^{\lambda}(x)[2-G(x)]^{\lambda}\big]^{k-1}\big[1-G^{\lambda}(x)[2-G(x)]^{\lambda}\big]^{n-k}. \tag{22}$$

The maximum and minimum record values are obtained when $k=n$ and when $k=1$ respectively.

The cdf of the k^{th} order statistics for $\alpha\in\Re^+-\{1\}$, and $\lambda>0$ is given as

$$F_k(x)=\sum_{u=0}^{n-k}\binom{k+u-1}{k-1}\Big[\frac{\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}-1}{(\alpha-1)}\Big]^{k}\Big[\frac{\alpha-\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}}{(\alpha-1)}\Big]^{u}. \tag{23}$$

Otherwise for $\alpha=1$, we have

$$F_k(x)=\sum_{u=0}^{n-k}\binom{k+u-1}{k-1}\big[G^{\lambda}(x)[2-G(x)]^{\lambda}\big]^{k}\big[1-G^{\lambda}(x)[2-G(x)]^{\lambda}\big]^{u}. \tag{24}$$

Let $Y_n^k=X_{U_k(n)}$ for $n\geq 1$ be the k^{th} upper record value and $Y_1^k=\min(X_1,X_2,X_3,\cdots,X_k)$. Also, let $Z_n^k=X_{L_k(n)}$ for $n\geq 1$ be the k^{th} lower record value with $L_k(1)=1$ and $Z_1^k=\max(X_1,X_2,X_3,\cdots,X_k)$ and Z_n^1 the lower record value for the sequence $\{X_i\geq 1\}$. Thus, for $1\leq k<s\leq n$), we define the difference of the upper value record as

$$U_n^k=Y_{n+1}^k-Y_n^k=w_{ks}=y-x, \tag{25}$$

and the lower value records as

$$L_n^k=Z_n^k-Z_{n+1}^k. \tag{26}$$

However, the joint APTL-G density for $k<s$ with k^{th} and s^{th} upper record values for $\alpha\in\Re^+-\{1\}$, and $\lambda>0$ is defined as

$$f_{w_{ks}}(w_{ks})=\frac{2^2n!}{(k-1)!(s-k-1)!}\Big(\frac{\lambda\log\alpha}{(\alpha-1)}\Big)^2 w_{ks}^{\bullet} \tag{27}$$

where

$$w_{ks}^{\bullet}=\int_0^{\infty}g(x)\bar{G}(x)G^{\lambda-1}(x)[2-G(x)]^{\lambda-1}\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}\times\Big[\frac{\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}-1}{(\alpha-1)}\Big]^{k-1}\bar{G}(x+w_{ks})G^{\lambda-1}(x+w_{ks})[2-G(x+w_{ks})]^{\lambda-1}\times g(x+w_{ks})\alpha^{G^{\lambda}(x+w_{ks})[2-G(x+w_{ks})]^{\lambda}}\Big[\frac{\alpha-\alpha^{G^{\lambda}(x+w_{ks})[2-G(x+w_{ks})]^{\lambda}}}{(\alpha-1)}\Big]^{n-s}\times\Big[\frac{\alpha^{G^{\lambda}(x+w_{ks})[2-G(x+w_{ks})]^{\lambda}}-1}{(\alpha-1)}-\frac{\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}-1}{(\alpha-1)}\Big]^{s-k-1}dx,$$

and $f_{w_{ks}}(w_{ks})=0$ for $k\geq s$. Thus for $ks=n(n+1)$, we have

$$f_{w_{n(n+1)}}(w_{n(n+1)})=\frac{2^2(n+1)!}{(k-1)!}\Big(\frac{\lambda\log\alpha}{(\alpha-1)}\Big)^2 w_{n(n+1)}^{\bullet}. \tag{28}$$

The range is obtained when $k=1,\ s=n$. Hence, using the [8] approach, the pdf of U_n^k is given as

$$f_{U_n^k}(u)=\frac{k^{n+1}}{(n-1)!}\Big(\frac{2\lambda\log\alpha}{(\alpha-1)}\Big)^2\psi(u+v) \tag{29}$$

where

$$\psi(u+v)=\int_0^{\infty}\Big[-\log\Big(\frac{\alpha-\alpha^{G^{\lambda}(u+v)[2-G(u+v)]^{\lambda}}}{(\alpha-1)}\Big)\Big]^{n-1}\times g(u+v)\bar{G}(u+v)G^{\lambda-1}(u+v)[2-G(u+v)]^{\lambda-1}\alpha^{G^{\lambda}(u+v)[2-G(u+v)]^{\lambda}}\times g(v)\bar{G}(v)G^{\lambda-1}(v)[2-G(v)]^{\lambda-1}\alpha^{G^{\lambda}(v)[2-G(v)]^{\lambda}}\times\Big[\frac{\alpha-\alpha^{G^{\lambda}(u+v)[2-G(u+v)]^{\lambda}}}{(\alpha-1)}\Big]^{-1}\Big[\frac{\alpha^{G^{\lambda}(v)[2-G(v)]^{\lambda}}-1}{(\alpha-1)}\Big]^{k-1}dv.$$

Suppose the APTL-G pdf of Z_n^k is given as

$$f_{Z_n^k}(x)=\frac{k^n}{(n-1)!}\frac{2\lambda\log\alpha}{(\alpha-1)}\Big[-\log\Big(\frac{\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}-1}{(\alpha-1)}\Big)\Big]^{n-1}\times\Big[\frac{\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}-1}{(\alpha-1)}\Big]^{k}g(x)\bar{G}(x)G^{\lambda-1}(x)[2-G(x)]^{\lambda-1}\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}. \tag{30}$$

Then, using [8] method, the joint pdf of (Z_n^k,Z_{n+1}^k) for $x>y$ and $\alpha\in\Re^+-\{1\}$, and $\lambda>0$ is given as

$$f_{n,n+1}^k(x,y)=\frac{k^{n+1}}{(n-1)!}\Big(\frac{2\lambda\log\alpha}{(\alpha-1)}\Big)^2\Big[\log\Big(\frac{\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}-1}{(\alpha-1)}\Big)\Big]^{n-1}\times\Big(\frac{\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}-1}{(\alpha-1)}\Big)^{-1}\Big[\frac{\alpha^{G^{\lambda}(y)[2-G(y)]^{\lambda}}-1}{(\alpha-1)}\Big]^{k-1}\times g(x)\bar{G}(x)G^{\lambda-1}(x)[2-G(x)]^{\lambda-1}\alpha^{G^{\lambda}(x)[2-G(x)]^{\lambda}}\times g(y)\bar{G}(y)G^{\lambda-1}(y)[2-G(y)]^{\lambda-1}\alpha^{G^{\lambda}(y)[2-G(y)]^{\lambda}}. \tag{31}$$

However, $f_{n,n+1}^k(x,y)=0$, for $x\leq y$. The pdf of the lower record value difference is given as

$$f_{L_n^k}(u)=\frac{k^{n+1}}{(n-1)!}\Big(\frac{2\lambda\log\alpha}{(\alpha-1)}\Big)^2\psi_L \tag{32}$$

where

$$\psi_L=\int_0^{\infty}\Big[-\log\Big(\frac{\alpha^{G^{\lambda}(u+v)[2-G(u+v)]^{\lambda}}-1}{(\alpha-1)}\Big)\Big]^{n-1}\Big[\frac{\alpha^{G^{\lambda}(u+v)[2-G(u+v)]^{\lambda}}-1}{(\alpha-1)}\Big]^{-1}\times g(u+v)\bar{G}(u+v)G^{\lambda-1}(u+v)[2-G(u+v)]^{\lambda-1}\alpha^{G^{\lambda}(u+v)[2-G(u+v)]^{\lambda}}\times\Big[\frac{\alpha^{G^{\lambda}(v)[2-G(v)]^{\lambda}}-1}{(\alpha-1)}\Big]^{k-1}g(v)\bar{G}(v)G^{\lambda-1}(v)[2-G(v)]^{\lambda-1}\alpha^{G^{\lambda}(v)[2-G(v)]^{\lambda}}dv,$$

for $u>0$ and $f_{L_n^k}(u)=0$ for $u\leq 0$ and its cdf is given as

$$F_{L_n^k}(u)=1-\frac{k^n}{(n-1)!}\frac{2\lambda\log\alpha}{(\alpha-1)}\phi_L \tag{33}$$

where

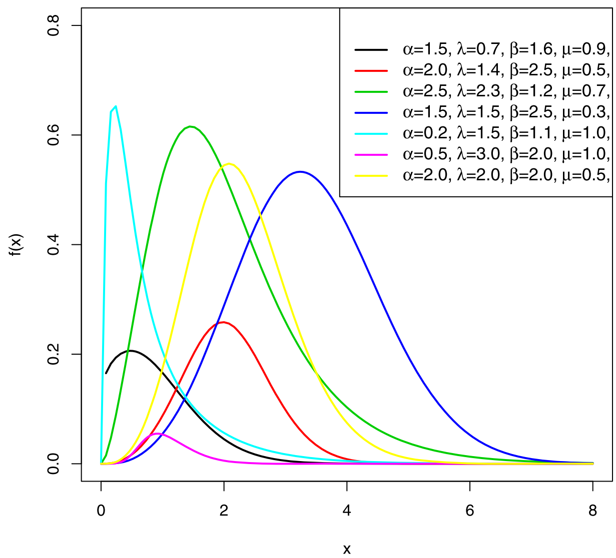

Fig. 1. The plots of the APTL-GW pdf model for some parameter value.

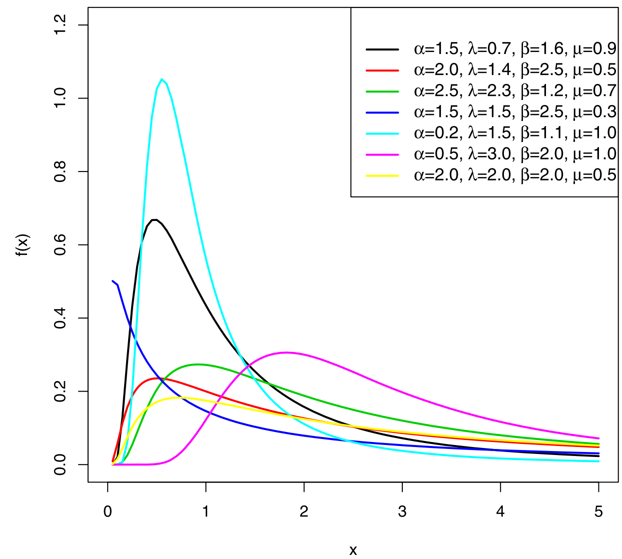

Fig. 2. The plots of the proposed APTL-GF density.

$$\phi_L = \int_0^\infty [-\log(\frac{\alpha^{G^\lambda(s)[2-G(s)]^\lambda}-1}{(\alpha-1)})]^{n-1}[\frac{\alpha^{G^\lambda(s-z)[2-G(s-z)]^\lambda}-1}{(\alpha-1)}]^k$$
$$\times[\frac{\alpha^{G^\lambda(s)[2-G(s)]^\lambda}-1}{(\alpha-1)}]^{-1} g(s)\bar{G}(s)G^{\lambda-1}(s)[2-G(s)]^{\lambda-1}\alpha^{G^\lambda(s)[2-G(s)]^\lambda}ds$$

for $z \geq 0$.

6. The APTL-G sub-models

The performance and tractableness of the APTL-G family of density will be examined using the following models. The models were examined based on their performance in the existing literature. These models include the Frechet (F) and Weibull (W) models.

6.1. The APTL-GW distribution

Suppose the pdf and cdf (for $x \geq 0$) of the Weibull distribution is considered, say $g(x) = \beta\mu^\beta x^{\beta-1}e^{(-(\mu x)^\beta)}$ and $G(x) = 1 - e^{(-(\mu x)^\beta)}$ respectively, (for $\mu > 0, \beta > 0$). Then, we can express for $\alpha \in \Re^+ - \{1\}$ and $\lambda > 0$ the cdf and pdf of the proposed APTL-GW distribution as

$$F(x) = \frac{\alpha^{(1-exp(-(\mu x)^\beta))^\lambda[1+exp(-(\mu x)^\beta)]^\lambda}-1}{(\alpha-1)}, \tag{34}$$

and

$$f(x) = \frac{2\lambda\log\alpha}{(\alpha-1)}\beta\mu^\beta x^{\beta-1}exp(-2(\mu x)^\beta)(1-exp(-(\mu x)^\beta))^{\lambda-1}$$
$$\times[1+exp(-(\mu x)^\beta)]^{\lambda-1}\alpha^{(1-exp(-(\mu x)^\beta))^\lambda[1+exp(-(\mu x)^\beta)]^\lambda}. \tag{35}$$

Otherwise, for $\alpha = 1$ and $\lambda > 0$, we have the pdf and the cdf as

$$f(x) = 2\alpha\beta\mu^\beta x^{\beta-1}exp(-2(\mu x)^\beta)(1-exp(-(\mu x)^\beta))^{\lambda-1}[1+exp(-(\mu x)^\beta)]^{\lambda-1},$$

and

$$F(x) = (1-exp(-(\mu x)^\beta))^\lambda[1+exp(-(\mu x)^\beta)]^\lambda.$$

The density plot for the proposed APTL-GW distribution for selected values cases of parameters α, λ, β and μ are shown in Fig. 1. The plots in Fig. 1 show that the APTL-G Weibull density could be decreasing, increasing, or skewed to the right or left depending on the desirable values of the parameters adopted.

6.2. The APTL-GF distribution

Assume, we consider the Frechet pdf as $g(x) = \mu\beta^\mu x^{-\mu-1}exp(-(\frac{\beta}{x})^\mu)$ and its cdf as $G(x) = exp(-(\frac{\beta}{x})^\mu)$ for $\beta > 0$ and $\mu > 0$. Then, we can express the pdf and cdf of the proposed APTL-GF model for $\alpha \in \Re^+ - \{1\}$ and $\lambda > 0$ as

$$f(x) = \frac{2\lambda\log\alpha}{(\alpha-1)}\mu\beta^\mu x^{-\mu-1}e^{-(\frac{\beta}{x})^\mu}(1-e^{-(\frac{\beta}{x})^\mu})(e^{-(\frac{\beta}{x})^\mu})^{\lambda-1}[2-e^{-(\frac{\beta}{x})^\mu}]^{\lambda-1}$$
$$\times\alpha^{(e^{-(\frac{\beta}{x})^\mu})^\lambda(x)[2-e^{-(\frac{\beta}{x})^\mu}]^\lambda}, \tag{36}$$

and

$$F(x) = \frac{\alpha^{(e^{-(\frac{\beta}{x})^\mu})^\lambda[2-e^{-(\frac{\beta}{x})^\mu}]^\lambda}-1}{(\alpha-1)}. \tag{37}$$

Otherwise, for $\alpha = 1$ and $\lambda > 0$, the pdf and the cdf become

$$f(x) = 2x^{-\mu-1}e^{-(\frac{\beta}{x})^\mu}(1-e^{-(\frac{\beta}{x})^\mu})(e^{-(\frac{\beta}{x})^\mu})^{\lambda-1}(x)[2-e^{-(\frac{\beta}{x})^\mu}]^{\lambda-1}\alpha\mu\beta^\mu$$

and

$$F(x) = (e^{-(\frac{\beta}{x})^\mu})^\lambda[2-e^{-(\frac{\beta}{x})^\mu}]^\lambda.$$

The density plot of the proposed APTL-GF distribution for some selected values cases of parameters α, λ, β and μ are shown in Fig. 2. The plot in Fig. 2 indicates that the proposed APTL-G model could be left-skewed, right-skewed, decreasing, and increasing depending on the values of the parameters.

6.3. Simulation study

A Monte Carlo simulation study is performed to illustrate the performance of the proposed model using the quantile function. The Table 2 shows simulation results. The Weibull model was simulated with $\hat{\alpha} = 3.0, \hat{\lambda} = 3.0, \hat{\beta} = 0.7$, and $\hat{\mu} = 1.5$. The Gompertz model was simulated with $\hat{\alpha} = 1.5, \hat{\lambda} = 1.7, \hat{\beta} = 1.5$, and $\hat{\mu} = 0.9$. Also, the Frechet model was simulated with $\hat{\alpha} = 1.5, \hat{\lambda} = 1.7, \hat{\beta} = 1.5$, and $\hat{\mu} = 0.9$. In Table 2, the performance of the proposed model was examined. The mean estimated value tends to the true values in all the model considered. More so, the mean squared errors decrease as the sample size increases.

7. Data applications

The empirical illustrations of the APTL-G distribution were examined using the goodness-of-fit of real-life data sets. The APTL-G sub-models were compared with the Weibull Gompertz (WGz),

Table 2. The simulation results for mean estimates (ME), biases and mean squared errors (MSE) for the APTL-G model.

Distribution	n	ME	Bias	MSE
Frechet	05	1.4256, 1.8125, 1.1594, 1.2262	0.0744, 0.1125, 0.1594, 0.3262	0.7168, 0.3427, 0.2906, 0.3791
	10	1.5294, 1.7754, 1.3026, 1.0554	0.0694, 0.0714, 0.1026, 0.1554	0.8153, 0.3131, 0.1794, 0.1258
	50	1.5628, 1.7605, 1.4104, 0.9272	0.0628, 0.0605, 0.0244, 0.0272	0.5208, 0.2193, 0.0744, 0.0141
	100	1.5434, 1.7709, 1.4273, 0.9109	0.0434, 0.0600, 0.0073, 0.0109	0.4105, 0.1846, 0.0476, 0.0069
	150	1.5103, 1.7310, 1.4407, 0.9063	0.0403, 0.0510, 0.0007, 0.0063	0.3487, 0.1425, 0.0339, 0.0049
	200	1.5060, 1.7222, 1.4907, 0.9049	0.0310, 0.0502, -0.0093, 0.0049	0.2736, 0.1184, 0.0266, 0.0036
	250	1.5054, 1.7172, 1.4915, 0.9032	0.0234, 0.0472, -0.0095, 0.0032	0.2719, 0.1085, 0.0236, 0.0030
	300	1.5048, 1.7072, 1.4823, 0.9011	0.0168, 0.0472, -0.0107, 0.0011	0.2963,0.1007, 0.0215, 0.0027
	350	1.5617, 1.7039, 1.4843, 0.9009	0.0117, 0.0429, -0.0127, 0.0000	0.2612, 0.0868, 0.0179, 0.0022
	400	1.5029, 1.7034, 1.4868, 0.9002	0.0109, 0.0404, -0.0132, 0.0002	0.2424, 0.0841, 0.0181, 0.0021
	450	1.5015, 1.7022, 1.4894, 0.9000	0.0105, 0.0322, -0.0146, 0.0000	0.2246, 0.0781, 0.0153, 0.0019
	500	1.5006, 1.7000, 1.4897, 0.9000	0.0101, 0.0290, -0.0153, -0.0006	0.2010, 0.0816, 0.0153, 0.0017
Weibull	05	1.6360, 3.8475, 0.7412, 1.3215	1.5063, 2.1475, -0.7588, 0.4715	0.4481, 0.6849, 0.5934, 0.3703
	10	1.8874, 3.6887, 0.7211, 1.3551	1.5009, 1.9887, -0.7589, 0.4551	0.0458, 0.5611, 0.0379, 0.3362
	50	2.3295, 3.3408, 0.7119, 1.3654	1.5971, 0.6408, -0.7681, 0.4354	0.0302, 0.4199, 0.0292, 0.0275
	100	2.5020, 3.2663, 0.7087, 1.3837	1.4194, 0.5663, -0.7713, 0.3837	0.0275, 0.2487, 0.0231, 0.0189
	150	2.6371, 3.2270, 0.6993, 1.4054	1.4514, 0.5270, -0.7907, 0.2964	0.0222, 0.0564, 0.0184, 0.0176
	200	2.7503, 3.0889, 0.7024, 1.4060	1.3158, 0.3889, -0.7806, 0.2060	0.0212, 0.0238, 0.0161, 0.0161
	250	2.8158, 3.0398, 0.6927, 1.4434	1.2503, 0.3398, -0.7973, 0.1434	0.0150, 0.0226, 0.0152, 0.0134
	300	2.9514, 3.0215, 0.6926, 1.4457	1.1371, 0.3215, -0.8044, 0.0457	0.0131, 0.0153, 0.0105, 0.0101
	350	2.9194, 3.0208, 0.6951, 1.4465	1.0020, 0.3108, -0.8079, 0.0415	0.0129, 0.0095, 0.0097, 0.0096
	400	3.0971, 3.0119, 0.6868, 1.4717	0.8295, 0.3019, -0.9132, 0.0410	0.0125, 0.0084, 0.0081, 0.0070
	450	3.0009, 3.0036, 0.6893, 1.4849	0.3874, 0.3006, -0.9107, 0.0349	0.0113, 0.0065, 0.0040, 0.0066
	500	3.0063, 3.0003, 0.6907, 1.4912	0.1360, 0.2763, -0.9593, 0.0112	0.0069, 0.0047, 0.0019, 0.0013
Gompert	05	1.3223, 2.4789, 1.4168, 2.0016	0.1777, 0.7789, 0.0832, 1.1016	0.9587, 2.7461, 0.8061, 3.4446
	10	1.3749, 2.2110, 1.4473, 1.5700	0.1251, 0.5110, 0.0527, 0.6700	0.7331, 1.7958, 0.5585, 1.9589
	50	1.4326, 1.8307, 1.4658, 1.0939	0.0674, 0.1307, 0.0342, 0.1939	0.4040, 0.4035, 0.1852, 0.5110
	100	1.4954, 1.7746, 1.4975, 0.9776	0.0546, 0.0746, 0.0325, 0.0776	0.2814, 0.1663, 0.0977, 0.2321
	150	1.5416, 1.7438, 1.4935, 0.9515	0.0416, 0.0438, 0.0265, 0.0515	0.2358, 0.1159, 0.0678, 0.1406
	200	1.5312, 1.7285, 1.4980, 0.9425	0.0292, 0.0285, 0.0220, 0.0425	0.1971, 0.0761, 0.0554, 0.1137
	250	1.5294, 1.7212, 1.4995, 0.9337	0.0194, 0.0212, 0.0205, 0.0337	0.1801, 0.0657, 0.0472, 0.0946
	300	1.5276, 1.7135, 1.4935, 0.9284	0.0176, 0.0195, 0.0165, 0.0284	0.1849, 0.0493, 0.0377, 0.0745
	350	1.5241, 1.7082, 1.4986, 0.9227	0.0141, 0.0182, 0.0114, 0.0227	0.1602, 0.0406, 0.0364, 0.0660
	400	1.5159, 1.7050, 1.4961, 0.9126	0.0059, 0.0150, 0.0009, 0.0226	0.1164, 0.0362, 0.0286, 0.0547
	450	1.5139, 1.7041, 1.5006, 0.9058	0.0039, 0.0081, 0.0006, 0.0158	0.1434, 0.0342, 0.0266, 0.0504
	500	1.5006, 1.7006, 1.5001, 0.9007	0.0016, 0.0006, 0.0002, 0.0097	0.1169, 0.0323, 0.0227, 0.0430

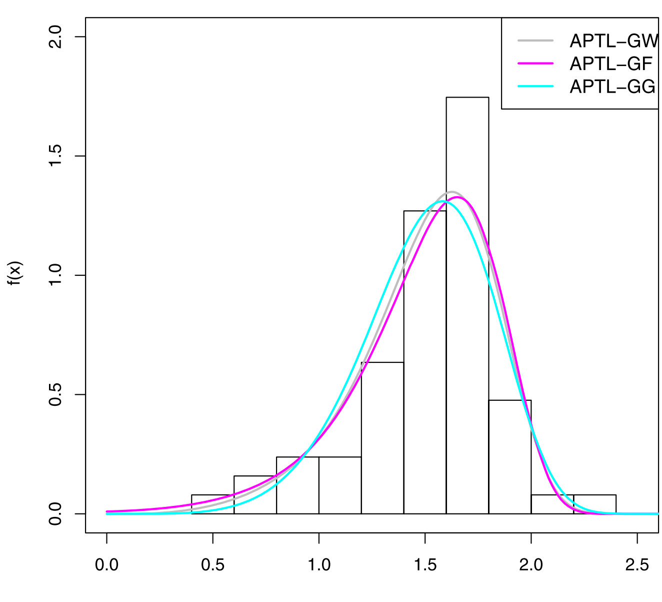

Fig. 3. The Empirical densities with the Glass fiber data set.

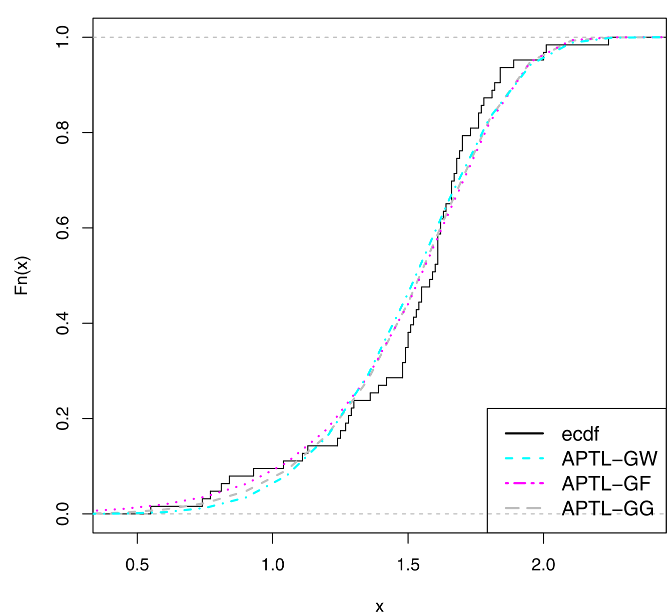

Fig. 4. The Empirical cdfs with the Glass fiber data set.

Kumaraswamy Gompertz (KGz), transmuted Weibull (TW), Topp-Leone Gompertz (TLGz), Kumaraswamy Weibull (KW), alpha power Weibull (APW), Kumaraswamy Frechet (KFr), transmuted Marshall–Olkin Frechet (TMFr), exponentiated Frechet (EFr), Marshall–Olkin Frechet (MFr), Gompertz Frechet (GFr), Lindley Poisson (LP), transmuted Pareto (TP), alpha power inverted Weibull (APIW), exponential generalized Frechet (EGF), Gompertz Weibull (GW), Weibull Marshall-Olkin (WMO-W), Mc-Donald Weibull (MC-W), generalized Kumaraswamy Weibull (GKW-W) distributions. The Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Akaike

Table 3. The goodness-of-fit rating with glass fiber data (standard errors in parentheses.

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-val.
	$\hat{\alpha} = 55.9(48.7)$							
APTL-GF	$\hat{\beta} = 1.6(0.3)$	**32.9**	**33.6**	**41.5**	**36.3**	**0.2**	**1.1**	**0.8**
	$\hat{\lambda} = 0.2(0.2)$							
	$\hat{\mu} = 3.2(0.4)$							
	$\hat{\alpha} = 13.2(21.4)$							
APTL-GG	$\hat{\beta} = 2.2(0.6)$	**33.1**	**33.8**	**41.7**	**36.5**	**0.1**	**0.7**	**0.7**
	$\hat{\lambda} = 1.7(1.0)$							
	$\hat{\mu} = 0.1(0.1)$							
	$\hat{\alpha} = 0.4(0.1)$							
APTL-GW	$\hat{\beta} = 21.4(1.4)$	**34.0**	**34.7**	**42.5**	**37.3**	**0.1**	**0.7**	**0.7**
	$\hat{\lambda} = 0.6(0.3)$							
	$\hat{\mu} = 4.6(0.0)$							
	$\hat{\beta} = 0.0(0.0)$							
TLGz	$\hat{\gamma} = 1.7(0.6)$	34.3	34.7	40.7	36.8	0.2	0.9	0.2
	$\hat{\lambda} = 2.8(0.5)$							
	$\hat{\alpha} = -0.8(0.3)$							
TGz	$\hat{\beta} = 0.1(0.0)$	34.9	35.1	39.2	36.6	0.1	0.8	0.3
	$\hat{\lambda} = 2.9(0.5)$							
	$\hat{a} = 0.1(0.1)$							
WGz	$\hat{b} = 3.2(1.3)$	36.8	37.5	45.4	40.2	0.2	1.0	0.2
	$\hat{\alpha} = 0.8(0.6)$							
	$\hat{\beta} = 0.0(0.4)$							
	$\hat{a} = 1.6(0.4)$							
KGz	$\hat{b} = 0.2(0.0)$	37.5	38.2	46.1	40.9	0.2	1.1	0.1
	$\hat{\alpha} = 0.1(0.0)$							
	$\hat{\beta} = 3.1(0.0)$							
	$\hat{\alpha} = 2.1(4.6)$							
KFr	$\hat{\beta} = 0.7(0.1)$	47.6	48.3	56.2	52.8	0.3	0.57	0.1
	$\hat{a} = 5.5(8.0)$							
	$\hat{b} = 857.4(153.9)$							
	$\hat{\alpha} = 7.8(3.0)$							
EFr	$\hat{\beta} = 1.0(0.1)$	50.5	50.7	56.7	52.8	0.3	0.6	0.1
	$\hat{\mu} = 132.8(116.6)$							
	$\hat{\alpha} = 0.7(0.1)$							
	$\hat{\beta} = 0.2(0.3)$	56.5	57.1	65.1	59.8	0.2	1.3	0.0
TMFr	$\hat{a} = 6.9(0.6)$							
	$\hat{b} = 376.3(246.8)$							
	$\hat{\beta} = 0.2(0.0)$							
MFr	$\hat{\gamma} = 6.5(0.6)$	57.1	57.5	63.5	59.6	0.2	2.8	0.0
	$\hat{\mu} = 161.6(91.5)$							
	$\hat{\alpha} = 0.6(0.0)$							
	$\hat{\beta} = 0.2(0.0)$	35.4	36.1	44.0	38.8	0.2	0.9	0.06
KW	$\hat{a} = 0.7(0.0)$							
	$\hat{b} = 7.1(0.0)$							
	$\hat{\alpha} = -0.5(0.3)$							
TW	$\hat{\beta} = 0.7(0.0)$	36.7	37.4	45.3	40.1	0.2	1.1	0.06
	$\hat{\lambda} = 5.2(0.7)$							
	$\hat{\alpha} = 6.6(8.0)$							
APW	$\hat{\beta} = 0.2(0.1)$	38.2	38.6	44.6	40.7	0.2	1.0	0.03
	$\hat{\lambda} = 4.7(0.8)$							
	$\hat{\alpha} = 0.2(0.8)$							
GW	$\hat{\beta} = 0.0(0.1)$	38.4	39.1	47.0	41.8	0.2	1.3	0.01
	$\hat{a} = 0.8(0.5)$							
	$\hat{b} = 5.6(0.5)$							

Information Criteria (AIC), Hannan and Quinn Information Criteria (HQIC), Anderson Darling (A), and Cramér-von Mises (W) test statistics were adopted to obtain the goodness-of-fit.

7.1. First data

The first data as used in [1] is made up of 63 workmen's observations of the strength of 1.5 cm glass fibres at the UK National Physical Laboratory in [21]. The results of the contemporary performance test statistics are displayed in Table 3.

Figs. 3, 4 and 5 show the empirical densities, CDFs and QQ-plots with the first data set for some models.

7.2. Second data

A stress-rupture life data of kevlar 49/epoxy strands were subjected to constant sustained pressure at 90 per cent stress level until all strands had failed and the failure times of the complete data were obtained. The data were studied by [15], [14], and [17]. The data were given as:

0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.2, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.4, 0.42, 0.43, 0.52, 0.54, 0.56, 0.6, 0.6, 0.63, 0.65, 0.67,0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07,0.08, 0.09, 0.09, 0.1, 0.1, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.8, 0.8, 1.8, 1.8, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34,

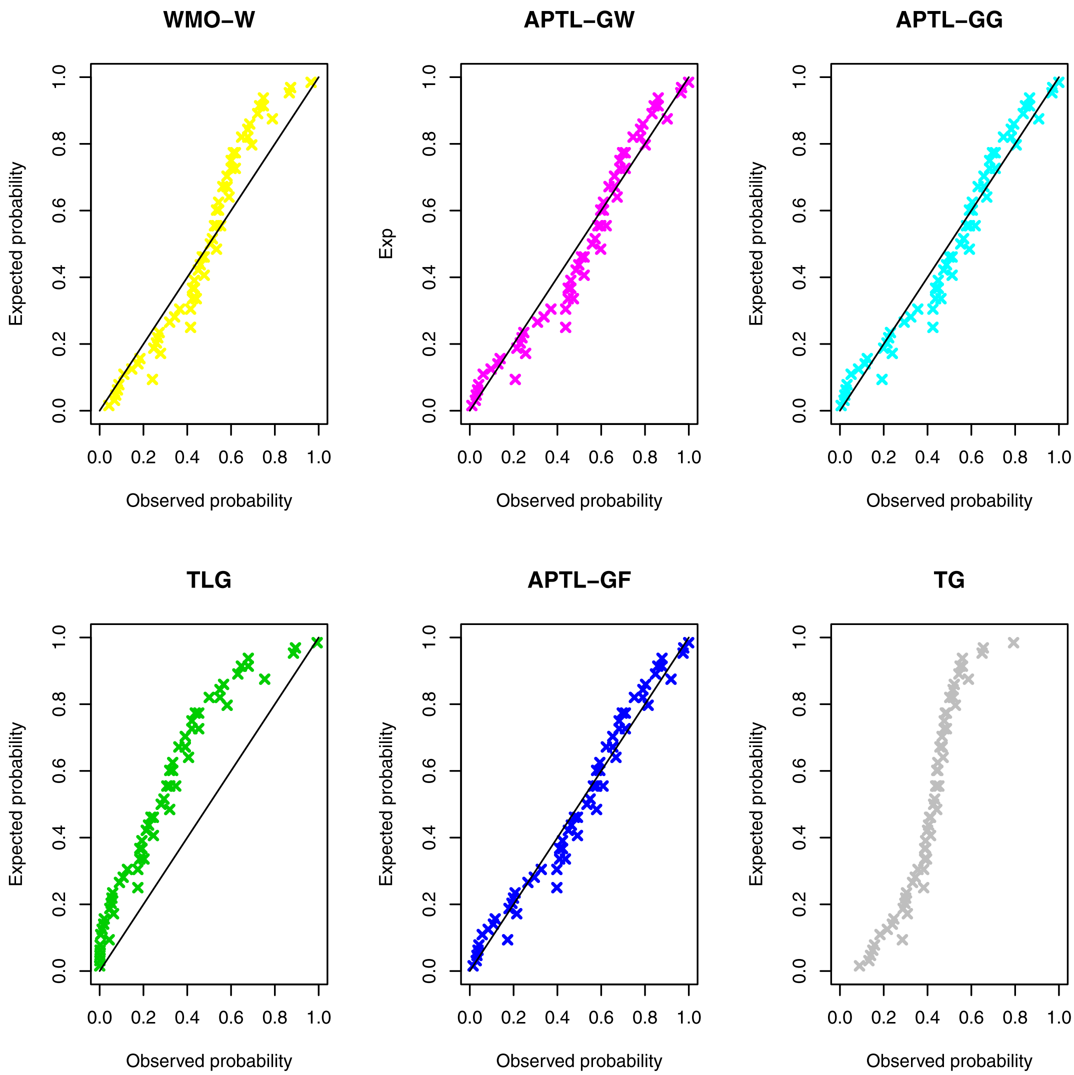

Fig. 5. The QQ-plots of the Glass fiber data set with models.

4.2, 4.69, 7.89 0.83, 0.85, 0.9, 0.92, 0.95, 0.99, 1, 1.01, 1.02, 1.03, 1.05, 1.1, 1.1, 1.11, 1.15, 1.18, 1.2, 1.29, 1.31, 1.33, 1.34, 1.4, 1.43, 1.45, 1.5, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64.

The results of the contemporary performance test statistics are displayed in Table 4.

Figs. 6 and 7 show the empirical densities, CDFs and QQ-plots with the Stress-rupture life data set for some models.

8. Conclusion

A two-parameter alpha power Topp-Leone generator has been introduced and examined for tractability, efficiency, performance and flexibility. The new model quantile function was derived and expressed as a weighted baseline CDF. The APTL-G model has been extended to extreme value and reliability theory. However, the quotients and differences k^{th} lower and upper record values of the APTL-G model were obtained in a closed-form. The parameters of the formulated APTL-G model were obtained in a closed form by the maximum likelihood method of estimation. A Monte Carlo simulation study and two real-life contemporary data were administered to the APTL-G model to validate its productivity of the APTL-G model. The outcomes of the up-to-the-minute Monte Carlo simulation study and a real-life application show that the performance of the goodness-of-fit of the APTL-G model is flexible and tractable with real-life data applications.

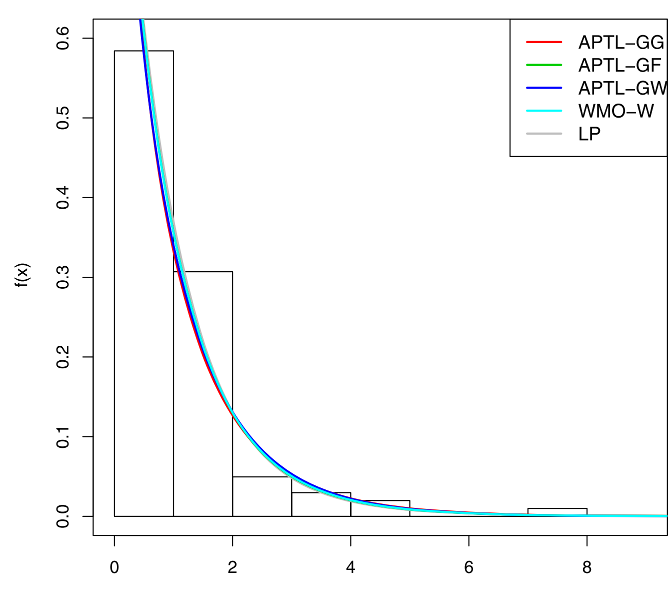

Fig. 6. The Empirical densities with the Stress-rupture life data set.

Table 4. The goodness-of-fit rating with Stress-rupture life data (standard errors in parentheses).

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-val.
	$\hat{\alpha}=1.9(0.2)$							
APTL-GG	$\hat{\beta}=1.4(0.5)$	**185.3**	**186.2**	**192.9**	**188.2**	**1.2**	**2.9**	**1.0**
	$\hat{\lambda}=0.8(0.2)$							
	$\hat{\mu}=0.1(0.0)$							
	$\hat{\alpha}=1.5(0.1)$							
APTL-GW	$\hat{\beta}=1.2(0.5)$	**189.7**	**190.6**	**197.3**	**192.6**	**1.1**	**2.2**	**1.0**
	$\hat{\lambda}=0.1(0.0)$							
	$\hat{\mu}=0.31(0.1)$							
	$\hat{\alpha}=1.8(0.8)$							
APTL-GF	$\hat{\beta}=1.0(0.6)$	**200.4**	**201.3**	**207.9**	**203.3**	**1.1**	**3.1**	**0.9**
	$\hat{\lambda}=0.8(0.2)$							
	$\hat{\mu}=1.6(0.1)$							
	$\hat{\gamma}=6.5(2.5)$							
	$\hat{\alpha}=0.5(0.0)$							
WMO-W	$\hat{\beta}=2.3(0.0)$	202.8	203.2	213.1	206.9	0.2	1.28	0.9
	$\hat{\lambda}=1.3(0.0)$							
	$\hat{\gamma}=0.8(0.4)$							
	$\hat{\alpha}=0.8(0.7)$							
BMO-W	$\hat{\beta}=4.9(5.3)$	209.3	209.9	222.2	214.5	0.20	1.2	0.7
	$\hat{\lambda}=2.3(1.1)$							
	$\hat{\sigma}=0.8(0.3)$							
	$\hat{\gamma}=0.3(0.3)$							
	$\hat{\alpha}=4.3(5.4)$							
MC-W	$\hat{\beta}=4.4(5.7)$	211.0	211.6	223.9	216.2	0.20	1.2	0.5
	$\hat{\lambda}=0.7(0.7)$							
	$\hat{\sigma}=0.7(0.4)$							
	$\hat{\gamma}=3.0(4.2)$							
	$\hat{\alpha}=2.1(1.8)$							
GKW-W	$\hat{\beta}=0.9(0.1)$	209.0	209.7	222.0	214.3	0.15	0.9	0.77
	$\hat{\lambda}=0.4(1.8)$							
	$\hat{\sigma}=0.4(0.2)$							
	$\hat{\alpha}=0.3(0.8)$							
GFr	$\hat{\beta}=3.4(3.9)$	212.0	212.5	222.5	216.3	0.18	1.0	0.4
	$\hat{a}=0.4(0.2)$							
	$\hat{b}=0.3(1.3)$							
	$\hat{\alpha}=1.3(60.8)$							
GW	$\hat{\beta}=0.0(0.5)$	214.0	214.4	224.4	218.2	0.2	1.1	0.4
	$\hat{a}=0.8(41.2)$							
	$\hat{b}=0.9(0.1)$							
	$\hat{\alpha}=0.5(2.5)$							
TP	$\hat{\beta}=48.2(0.0)$	310.5	311.1	316.2	312.7	0.11	0.7	0.5
	$\hat{\lambda}=0.9(1.5)$							
LP	$\hat{\beta}=0.2(0.0)$	319.9	320.2	323.7	321.4	0.06	0.3	0.2
	$\hat{\lambda}=2.4(1.1)$							
	$\hat{\gamma}=32.2(0.0)$							
	$\hat{\alpha}=2.9(0.0)$							
EGF	$\hat{\beta}=1049(44.0)$	317.4	318.3	325,0	320.3	0.22	1.2	0.1
	$\hat{\lambda}=0.3(0.0)$							

Declarations

Author contribution statement

Joseph Thomas Eghwerido: Conceived and designed the analysis; Wrote the paper. Friday Ikechukwu Agu: Performed the experiments; Analyzed and interpreted the data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability statement

Data included in article/supplementary material/referenced in article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

Acknowledgements

The authors wish to acknowledge the Chief Editor and the Editorial management for their effort to improve this study.

References

[1] A.Z. Afify, H.M. Yousof, G.M. Cordeiro, E.M.M. Ortega, Z.M. Nofal, The Weibull Frechet distribution and its applications, J. Appl. Stat. 43 (14) (2016) 2608–2626.

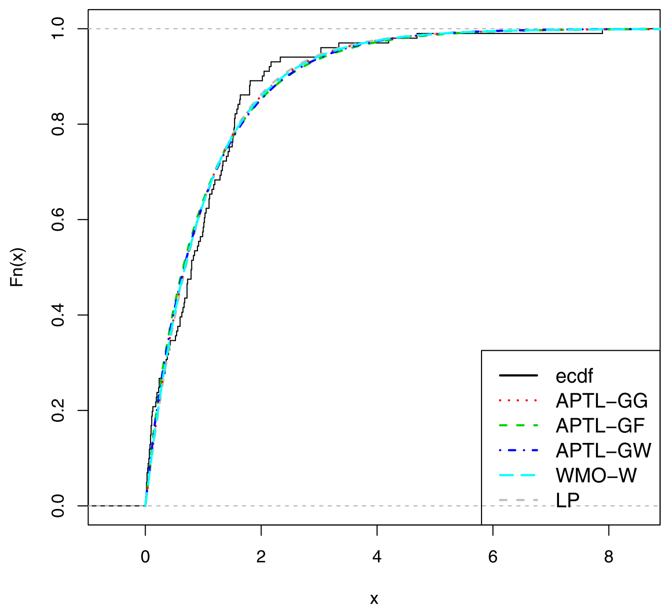

Fig. 7. The Empirical cdfs with the Stress-rupture life data set.

[2] A.Z. Afify, H.M. Yousof, S. Nadarajah, The beta transmuted-H family of distributions: properties and applications, Stat. Interface 10 (2017) 505–520.
[3] A. Al-Shomrani, O. Arif, A. Shawky, S. Hanif, M.Q. Shahbaz, Topp-Leone family of distributions: some properties and applications, Pak. J. Stat. Oper. Res. 12 (3) (2016) 443–451.
[4] M. Alizadeh, G.M. Cordeiro, L.G.B. Pinho, I. Ghosh, The Gompertz-G family of distributions, J. Stat. Theory Pract. 11 (1) (2017) 179–207.
[5] M. Alizadeh, M. Rasekhi, H.M. Yousof, G.G. Hamedani, The transmuted Weibull-G family of distributions, Hacet. J. Math. Stat. 47 (6) (2018) 1–20.
[6] M. Alizadeh, H.M. Yousof, S.M.A. Jahanshahiz, S.M. Najibi, G.G. Hamedani, The transmuted odd log-logistic-G family of distributions, J. Stat. Manag. Syst. (2020).
[7] G.R. Aryal, H.M. Yousof, The exponentiated generalized-G Poisson family of distributions, Econ. Qual. Control 32 (1) (2017) 1–17.
[8] M. Bieniek, D. Szynal, Limiting distributions of differences and quotients of successive k-th upper and lower record values, Probab. Math. Stat. 20 (1) (2000) 189–202.
[9] J.T. Eghwerido, J.I. Mbegbu, Adaptive models for nonstationary spatial covariance structures, Turkiye Klinikleri J. Biostat. 12 (1) (2020) 1–15.
[10] J.T. Eghwerido, J.I. Mbegbu, An adaptive parametric model for nonstationary spatial covariance, Malaysian J. Sci. 39 (2) (2020) 51–70.
[11] J.T. Eghwerido, P.E. Oguntunde, F.I. Agu, The alpha power Marshall-Olkin-G distribution: properties, and applications, Sankhya, Ser. A (2021).
[12] J.T. Eghwerido, S.C. Zelibe, E. Efe-Eyefia, The transmuted alpha power-G family of distributions, J. Stat. Manag. Syst. (2020).
[13] J.T. Eghwerido, The alpha power Teissier distribution and its applications, Afr. Stat. 16 (2) (2021) 2731–2745.
[14] M.A. Haq, S.N. Butt, R.M. Usman, A.A. Fattah, Transmuted power function distribution, Gazi Univ. J. Sci. 29 (1) (2016) 177–185.
[15] M.C. Korkmaz, G.M. Cordeiro, H.M. Yousof, R.R. Pescim, A.Z. Afify, S. Nadarajah, The Weibull Marshall–Olkin family: regression model and application to censored data, Commun. Stat., Theory Methods (2018).
[16] A. Mahdavi, D. Kundu, A new method for generating distributions with an application to the exponential distribution, Commun. Stat., Theory Methods 46 (13) (2017) 6543–6557.
[17] P.F. Paranaiba, E.M.M. Ortega, G.M. Cordeiro, M.A.R. de Pascoa, The Kumaraswamy Burr XII distribution: theory and practice, J. Stat. Comput. Simul. 83 (11) (2013) 2117–2143.
[18] R Core Team, A language and environment for statistical computing, R Foundation for Statistical Computing, Vienna, Austria, https://www.R-project.org/, 2020.
[19] H. Reyad, M. Alizadeh, F. Jamal, S. Othman, The Topp Leone odd Lindley-G family of distributions: properties and applications, J. Stat. Manag. Syst. 21 (7) (2018) 1273–1297.
[20] Y. Sangsanit, W. Bodhisuwan, The Topp-Leone generator of distributions: properties and inferences, Songklanakarin J. Sci. Technol. 38 (5) (2016) 537–548.
[21] R.L. Smith, J.C. Naylor, A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, Appl. Stat. 36 (1987) 258–369.
[22] H.M. Yousof, A.Z. Afify, M. Alizadeh, S. Nadarajah, G.R. Aryal, G.G. Hamedani, The Marshall-Olkin generalized-G family of distributions with applications, Statistica 78 (3) (2018) 273–295.
[23] H.M. You, M. Alizadeh, S.M.A. Jahanshahiand, T.G. Ramires, I. Ghosh, G.G. Hamedani, The transmuted Topp-Leone-G family of distributions: theory, characterizations and applications, J. Data Sci. 15 (2017) 6723–6740.