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Preferential reasoning in the perspective of Poole default logic

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Abstract

The sceptical inference relation associated with a Poole system without constraints is known to have a simple semantic representation by means of a smooth order directly defined on the set of interpretations associated with the underlying language. Conversely, we prove in this paper that, on a finite propositional language, any preferential inference relation defined by such a model is induced by a Poole system without constraints. In the particular case of rational relations, the associated set of defaults may be chosen to be minimal; it then consists of a set of formulae, totally ordered through classical implication, with cardinality equal to the height of the given relation. This result can be applied to knowledge representation theory and corresponds, in revision theory, to Grove's family of spheres. In the framework of conditional knowledge bases and default extensions, it implies that any rational inference relation may be considered as the rational closure of a minimal knowledge base. An immediate consequence of this is the possibility of replacing any conditional knowledge base by a minimal one that provides the same amount of information. © 1998 Elsevier Science B.V.

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1. Introduction

In [12], a variation on Reiter's default logic [14] was presented, leading to the following notion: a *Poole system* is a pair (Δ, K) of sets of sentences, called respectively the set of "defaults" and the set of "constraints" of the system. Such a system may be used to determine the nonmonotonic consequences of a premiss α, assuming as true the maximal subsets of Δ that are consistent with α and K. As shown by Makinson

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[11, Section 3.3], Poole's original *liberal* conception of the "extension family function" associated with the pair (Δ, K) can be modified to a *sceptical* approach, providing a preferential inference relation when the Poole system is one "without constraints", i.e. when its set K of constraints is empty. Such a Poole system can be identified with a set of prerequisite-free normal defaults in the sense of Reiter, and the associated preferential inference relation then corresponds to the sceptical Reiter extension of Δ. It was noticed by Makinson [11] and Poole [13] that the preferential inference relation associated with such a Poole system can be represented by a special kind of preferential model, where the set of states is the set of all worlds. Makinson [11] mentioned that the converse of this property was not settled, and conjectured that it may hold.

We shall give an affirmative answer to this conjecture for languages that are logically finite, and prove that *any consistency-preserving preferential relation defined by an injective model is the inference relation associated with a Poole system without constraints*. Thus, preferential reasoning, when determined by a preferential injective model, is (at least in the finite case) essentially the same as default reasoning: the logic of any agent using injective preferential reasoning is fully determined by an implicit set of basic defaults.

This result holds in particular for *rational* reasoning, as it is known that rational inference relations may always be defined by means of ranked injective models [2]. It follows that any consistency-preserving rational inference relation defined on a logically finite language is induced by a Poole system without constraints. This Poole system is not uniquely determined, and two sets of defaults D and D' may induce the same inference relation, but among the different sets of defaults that induce a given rational inference relation, one of them, called *the characteristic set* of the relation, satisfies some interesting properties: it is minimal, simple to describe, and its elements are linearly ordered through classical implication. The characteristic set of a rational inference relation therefore appears to be a most useful tool for the study of this relation: in particular, if we denote by $\delta_1, \ldots, \delta_n$ the elements of this set, with δ_{i+1} (classically) implied by δ_i for $i < n$, the given relation simply reads "$\alpha \mid\sim \beta$ iff there exists an index i such that δ_i is consistent with α but inconsistent with $\alpha \wedge \neg\beta$". This observation explains, in the perspective of Poole systems, some classical results established in revision theory by Grove [7] and Linström and Rabinowicz [10]. Moreover, it leads to some interesting applications in the field of conditional knowledge bases and default extensions: thus we prove that any consistency-preserving rational inference relation may be considered as the rational closure of a minimal conditional base, and we show how to explicitly determine this base.

This paper is self-contained and is organized as follows: in Section 2, we recall the definitions and main properties of preferential inference relations. Poole systems are introduced in Section 3. There, we examine Makinson's conjecture and show that, while not true in general, this conjecture holds in the case of finite languages. The section concludes with some considerations on the dynamics of Poole systems: the set of defaults of a Poole system, analyzed as a belief base, may be revised or updated, and this affects the behaviour of the associated inference relation. Section 4 is the central part of this paper, and is devoted to the case of rational inference relations and their characteristic sets. Section 5 is an application of the results of Section 4 to some aspects of knowledge representation theory concerning conditional bases extension. We conclude in Section 6.

2. Background

We denote by $\mathcal{L}$ a set of well-formed formulae over a set of atomic propositions, closed under the classical propositional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$. When there are only finitely many atomic propositions, the language is said to be logically finite. Semantics is provided by the set W of all assignments of truth values to the propositional variables. Elements of W will be referred to as *worlds* and the satisfaction relation between a world m and a formula α is defined as usual and written $m \models \alpha$. Thus $m \models \alpha \vee \beta$ iff $m \models \alpha$ or $m \models \beta$, and $m \models \neg\alpha$ iff it is not the case that $m \models \alpha$.

For every subset A of $\mathcal{L}$, we write $m \models A$ iff m satisfies all the elements of A. The set of formulae of $\mathcal{L}$ satisfied by a world m will be denoted by $|m|$.

The classical consequence operation attached to $\mathcal{L}$ and W will be denoted by Cn: for any subset A of $\mathcal{L}$, $\mathrm{Cn}(A)$ is the set of all formulae α of $\mathcal{L}$ such that $m \models \alpha$ for all worlds m that satisfy A. Given a subset A of $\mathcal{L}$, we say that A is *consistent* iff $\mathrm{Cn}(A) \neq \mathcal{L}$ or, equivalently, iff there exists a world m such that m satisfies A. The set A is said to be consistent with the set B iff $A \cup B$ is a consistent set. We write $\mathrm{Cn}(A, B)$ for $\mathrm{Cn}(A \cup B)$, $\mathrm{Cn}(\alpha)$ for $\mathrm{Cn}(\{\alpha\})$ and $\alpha \vdash \beta$ for $\beta \in \mathrm{Cn}(\alpha)$.

2.1. Preferential inference relations

Following Kraus, Lehmann and Magidor [8], we call *preferential inference relation* on $\mathcal{L}$ a relation $\mid\!\sim$ that satisfies the following rules:

Reflexivity. $\alpha \mid\!\sim \alpha$.

Left Logical Equivalence. If $\mathrm{Cn}(\alpha) = \mathrm{Cn}(\beta)$ and $\alpha \mid\!\sim \gamma$, then $\beta \mid\!\sim \gamma$.

Right Weakening. If $\beta \in \mathrm{Cn}(\alpha)$ and $\gamma \mid\!\sim \alpha$, then $\gamma \mid\!\sim \beta$.

Cut. If $\alpha \wedge \beta \mid\!\sim \gamma$ and $\alpha \mid\!\sim \beta$, then $\alpha \mid\!\sim \gamma$.

Or. If $\alpha \mid\!\sim \gamma$ and $\beta \mid\!\sim \gamma$, then $\alpha \vee \beta \mid\!\sim \gamma$.

Cautious Monotonicity. If $\alpha \mid\!\sim \beta$ and $\alpha \mid\!\sim \gamma$, then $\alpha \wedge \beta \mid\!\sim \gamma$.

Given such a relation, we shall denote by $C_{\mid\sim}(\alpha)$—or $C(\alpha)$ when there is no ambiguity—the set of all $\mid\!\sim$-consequences of a formula α, that is the set of all β's such that $\alpha \mid\!\sim \beta$. We will indifferently refer to "the inference relation $\mid\!\sim$" or to the "inference relation C". The above rules imply that for any preferential inference relation C, the sets $C(\alpha)$ are closed with respect to Cn, that is $\mathrm{Cn}[C(\alpha)] = C(\alpha)$ for all formulae α. An inference relation C is said to be *consistency-preserving* iff $C(\alpha)$ is a consistent set for any consistent formula α. Thus a preferential inference relation C is consistency-preserving iff $C(\alpha) \neq \mathcal{L}$ whenever $\mathrm{Cn}(\alpha) \neq \mathcal{L}$.

2.2. Preferential models

A *preferential structure* is a triple $M = (S, <, l)$ where $<$ is an irreflexive and transitive relation defined on a set S (the set of "states"), and l (the "label function") is a mapping from S into the set of worlds W. For any state s, we say that s *satisfies* a formula α (written $s \models \alpha$) iff $l(s)$ does, and we denote by α^* the set of all states s satisfying the formula α.

A *preferential model*, as defined in [8], is a preferential structure $(S, <, l)$ that satisfies the following condition of *smoothness*: given any formula α of $\mathcal{L}$ and any state s of α^* that is not minimal in α^*, there exists a state t minimal in α^* such that $t < s$. This condition is always satisfied when the preferential structure is finite (i.e. when its set of states is finite), and in particular when the underlying language is supposed to be logically finite.

A preferential model determines a preferential relation $\mid\!\sim_M$ by:

(def) $\quad \alpha \mid\!\sim_M \beta$ iff all minimal elements of α^* satisfy β.

Conversely, it was shown in [8] that, for any preferential relation $\mid\!\sim$ defined on a language $\mathcal{L}$ (respectively a logically finite language $\mathcal{L}$), there exists a preferential model (respectively a finite preferential model) M that represents $\mid\!\sim$, i.e. is such that $\mid\!\sim = \mid\!\sim_M$.

The following simple result will be used in the next sections:

Lemma 1. *Let $\mid\!\sim$ be a preferential inference relation represented by a preferential model $M = (S, <, l)$. Then one has $\alpha \mid\!\sim \beta$ iff for any state t that satisfies $\alpha \wedge \neg\beta$, there exists a state $s < t$ that satisfies α.*

Proof. If $\alpha \mid\!\sim \beta$ and t satisfies $\alpha \wedge \neg\beta$, t is not minimal in α^*, hence there exists a state $s \in \alpha^*$ such that $s < t$. Conversely, if it is not the case that $\alpha \mid\!\sim \beta$, there exists a state t minimal in α^* that satisfies $\neg\beta$, and the inequality $s < t$ cannot hold for an element $s \in \alpha^*$. $\square$

2.3. Injective models and faithfully representable preferential inference relations

An *injective* preferential model is a model $M = (S, <, l)$ where the label function l is injective [2]. Injective models were initially proposed by Shoham [15] as a framework for preferential logic, but the wider notion of preferential model was adopted by the authors of [8], as they noticed the existence of inference relations that satisfy properties (1)–(6) without admitting a preferential injective model. We will mainly deal with the case where the map l is a bijection. Then, the set of states can be identified with the set W of all worlds, and the model M is of the form $(W, <)$ where $<$ is a strict smooth partial order on W. Such a model will be called a *faithful model*. We shall say that a preferential relation is *faithfully representable* iff it can be represented by a faithful model. Such a relation is clearly consistency-preserving. The following lemma shows that the converse holds in logically finite languages:

Lemma 2. *In a logically finite language, a preferential inference relation is faithfully representable if and only if it admits an injective model and is consistency-preserving.*

Proof. The language being supposed to be finite, let $\mid\!\sim$ be a consistency-preserving preferential inference relation that admits an injective model $(S, <)$, where S is a subset of W. We claim that $S = W$. Indeed, let m be any element of M and χ_m the (finite) conjunction of all formulae satisfied by m. We have to prove that $m \in S$. Since the relation $\mid\!\sim$ preserves consistency, we do not have $\chi_m \mid\!\sim$ *false*, and there exists therefore an element s of S that satisfies χ_m. By the choice of χ_m, we have $s = m$, and it follows that $m \in S$ as desired. $\square$

In a finite language, the model $(W, <)$ that represents a faithfully representable inference relation C is unique, and we will refer to it as the *standard model* of C. The order $<$ on W may be then defined by:

(def) $\quad m < n$ iff $m \models \neg\alpha$ for all formulae α such that $n \models C(\alpha)$.

(See the proof of Lemma 4.12 in [2] for details.)

2.4. *The basic set of defaults associated with a preferential model* $(W, <)$

Given a strict partial order $<$ on the set W of worlds associated with an arbitrary propositional language, there exists a set $\Delta_<$ of formulae that play a prominent role in the study of faithfully representable inference relations. This set will be referred to as *the basic set of defaults associated with the model* $(W, <)$; it consists of all formulae α that satisfy the following condition:

$(*)$ $\quad$ If α is true at a world n, then α is true at all worlds m such that $m < n$.

It is not possible, in the general case, to define the basic set of defaults "associated with a faithfully representable preferential inference relation $\mid\!\sim$", since such an inference relation may be defined by different faithful models. Nevertheless, when the language is logically finite, this set may be defined as the basic set of defaults associated with the *standard model* of $\mid\!\sim$. In this particular case of a logically finite language, we will always suppose fixed a representative of each class under classical equivalence, and choose the elements α of the basic set of defaults among the representatives of these equivalence classes. Thus in a logically finite language, the basic set of defaults associated with a faithfully representable inference relation defined by the model $(W, <)$ is a finite set, that consists of all the representatives of the formulae α satisfying $(*)$.

Note that, in the general case, the basic set of default associated with a preferential model $(W, <)$ is stable under conjunction and disjunction, and contains the tautologies as well as the contradictions of the language $\mathcal{L}$.

3. Poole systems

The definition and the principal properties of the inference relation associated with a Poole system can be found in [5,6,11]. We briefly recall some basic facts.

Let D be any subset of $\mathcal{L}$. We may identify D with a set of prerequisite-free normal Reiter-style defaults, every formula δ of D corresponding then to the default $\frac{:\delta}{\delta}$. Given a formula α of $\mathcal{L}$, it therefore makes sense to build the intersection of all the Reiter extensions of (α, D), as defined in [14]. In this framework, a formula β will be considered as a (nonmonotonic) consequence of α if and only if β lies in this intersection. This leads to the construction of the inference relation $\mid\!\sim_D$ associated with the Poole system $(D, \emptyset)$, which is defined by

$$\text{(def)} \qquad \alpha \mid\!\sim_D \text{ iff } \beta \in \bigcap \mathrm{Cn}(\alpha, D_\alpha),$$

where the intersection is taken over all the subsets D_α of D that are consistent with α and maximal for that property.

A simpler syntactic characterization will be given in Corollary 4. To take a simple example, in the particular case where $D = \{\delta\}$ is a singleton, one sees that the relation $\mid\!\sim_\delta$ boils down to:

- $\alpha \mid\!\sim_\delta \beta$ iff $\beta \in \mathrm{Cn}(\alpha \wedge \delta)$ in the case where α is consistent with δ, and
- $\alpha \mid\!\sim_\delta \beta$ iff $\beta \in \mathrm{Cn}(\alpha)$ when α is inconsistent with δ.

Thus, given the premiss α, the conclusion β is believed iff β classically follows from α together with δ in the case where $\alpha \wedge \delta$ is not a contradiction, and β classically follows from α when α is inconsistent with δ. We emphasize that in the formal theory T obtained by taking δ as an *axiom*, β is T-implied by α iff β is classically implied by $\alpha \wedge \delta$. It follows that a consistent formula α may T-imply a contradiction, whilst this is not possible in the example of $\mid\!\sim_\delta$, which is clearly consistency-preserving.

3.1. The semantics of Poole systems

It is known that the inference relation $\mid\!\sim_D$ associated with a Poole system $(D, \emptyset)$ without constraints is a *preferential inference relation* (see for instance [11]), and it is immediate from its definition that this inference relation preserves consistency. We will refer to it as *the inference relation induced by the set D*. Note that if a set D induces the inference relation $\mid\!\sim$, the sets $D' = D \cup \{\mathit{false}\}$ and $D'' = D \cup \{\mathit{true}\}$ induce the same inference relation. The *semantics* of such a relation turns out to be particularly simple to describe. Indeed, recalling that for any world $|p|$ denotes the set of all formulae satisfied by a world p, one has the following result, observed independently by Makinson [11] and Poole [13]:

Observation 3 (Makinson and Poole). *If $\mid\!\sim_D$ is the preferential inference relation induced by a set D, and $<_D$ the relation defined on the set of worlds W by*

$$\text{(def)} \qquad m <_D n \textit{ iff } |n| \cap D \subsetneq |m| \cap D,$$

the structure $(W, <_D)$ is a preferential model that represents $\mid\!\sim_D$.

Proof. The relation $<_D$ is clearly a strict partial order on W and the condition of *smoothness* readily follows from Zorn's lemma. We have to prove that $(W, <_D)$ represents $\mid\!\sim_D$, that is that $\alpha \mid\!\sim_D \beta$ iff all minimal elements of α^* satisfy β.

Suppose first that we have $\alpha \mid\!\sim_D \beta$, and let m be a minimal element in α^*. We want to show that $m \models \beta$. Note that, since $m \models \alpha$, the set $|m| \cap D$ is a subset of D that is consistent with α. We claim that it is maximal for that property: indeed, suppose that there exists a subset D' of D consistent with α such that $|m| \cap D \subsetneq D'$. Let n be a world that satisfies D' and α. We have then $D' \subseteq |n| \cap D$, so $|m| \cap D \subsetneq |n| \cap D$, and therefore $n <_D m$, contradicting the choice of m. This shows that $|m| \cap D$ is maximal among the subsets of D that are consistent with α. It follows from the definition of $\mid\!\sim_D$ that $\neg\alpha \vee \beta \in \mathrm{Cn}(|m| \cap D)$, so $m \models \neg\alpha \vee \beta$. But m satisfies α, and therefore m must satisfy β.

Suppose now that we do not have $\alpha \mid\!\sim_D \beta$. Then there exists a subset D' of D, maximal consistent with α, such that $\neg\alpha \vee \beta \notin \mathrm{Cn}(D')$. Let m be a world that satisfies D' and does not satisfy $\neg\alpha \vee \beta$. We see that $m \models \alpha \wedge \neg\beta$. Furthermore, m is minimal among the worlds that satisfy α, otherwise there would exist a world n such that $n \models \alpha$ and we would have $D' \subseteq |m| \cap D \subsetneq |n| \cap D$, contradicting the maximality of D'. We have therefore proven that, if one does not have $\alpha \mid\!\sim_\Delta \beta$, there exists a world m that is minimal in α^* and that does not satisfy β. This completes the proof of Observation 3. □

Note that for any set D, one has $D \subseteq \Delta$ where Δ is the basic set of defaults associated with the model $(W, <_D)$. Indeed, let α be any element of D, n a world that satisfies α and m a world such that $m <_D n$. By definition of $<_D$, $|n| \cap D$ is a (strict) subset of $|m| \cap D$, so that m must satisfy α. Condition $(*)$ in Section 2 is therefore satisfied and $\alpha \in \Delta$.

Observation 3 together with Lemma 1 provides a rather simple definition of the preferential inference relation associated with a set of defaults D:

Corollary 4. *For any preferential inference relation $\mid\!\sim_D$ induced by a set D of defaults, one has $\alpha \mid\!\sim_D \beta$ iff, for every world n that satisfies $\alpha \wedge \neg\beta$, there exists a world m that satisfies α with $|n| \cap D \subsetneq |m| \cap D$.*

Proof. Clear. □

The order $<_D$ will be referred to as *the order induced by D*. Note that it follows from the above observation that *the inference relation associated with any Poole system $(D, \emptyset)$ is faithfully representable*. Evenmore, it is worth pointing out that the model described in the proposition above is particularly interesting because it is simple to build, and corresponds to the intuitive idea that *a world m is less exceptional, or more normal, than a world n iff it satisfies a greater subset of D than n does.* If D is considered as a knowledge base or as a set of expectations, it is only natural that we consider as most normal the worlds that confirm as much as possible of the information provided by D.

Example 5 (*The Alchourron–Makinson theorem of triviality*). Suppose that D is a *closed consistent* subset of $\mathcal{L}$, that is $D = \mathrm{Cn}(D)$, $D \neq \mathcal{L}$ and let us determine the order $<$ induced by D. Clearly, if a world m satisfies D and a world n does not, one has $m < n$. Conversely, suppose that m and n are two worlds such that $m < n$, and let us show that m satisfies D, while n does not: by definition of the order $<$, there exists a formula $\delta \in D$ such that $m \models \delta$ and $n \models \neg\delta$. The world n therefore does not satisfy D. Let α be any element of D. To prove that $m \models \alpha$, note that $\alpha \vee \neg\delta \in \mathrm{Cn}(D) = D$, and that $n \models \alpha \vee \neg\delta$. It follows that $m \models \alpha \vee \neg\delta$, hence that m satisfies α. This shows that $m \models D$, and we have proven that, *when D is a consistent closed subset of $\mathcal{L}$, one has $m < n$ iff m satisfies D and m does not.* It follows that the model $(W, <)$ is a ranked model with two ranks: at rank 0 we find all the worlds that satisfy D, and at rank 1 all the other ones. One shows then easily that the induced relation C_D is given by:

$$C_D(\alpha) = \begin{cases} \mathrm{Cn}(\alpha, D) & \text{when } \alpha \text{ is consistent with } D, \\ \mathrm{Cn}(\alpha) & \text{otherwise.} \end{cases}$$

When the language is logically finite, the standard model $(W, <)$ of the preferential inference relation induced by a set of defaults D (cf. Section 2) coincides with the model $(W, <_D)$ and is therefore quite easy to write down:

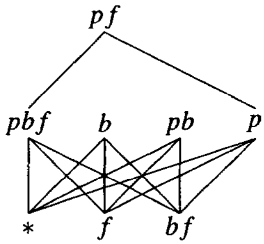
Example 6 (*Penguins*). We consider the information provided by the well-known triangle "penguins are birds, birds fly, penguins don't fly", which can be represented by the set of defaults

$$D = \{p \to b, b \to f, p \to \neg f\} = \{\neg p \vee b, \neg b \vee f, \neg p \vee \neg f\}.$$

Let us denote the worlds associated with the language $\mathcal{L}$ on p, b and f by the sequence of elementary propositions that they satisfy. Thus, the world p is the world that satisfies $p \wedge \neg b \wedge \neg f$, the world * is the world that satisfies $\neg p \wedge \neg b \wedge \neg f$, etc. To determine the preferential model associated with the inference relation induced by D, we first write down the eight worlds attached with $\mathcal{L}$ together with the formulae in D that they satisfy:

m	$p \to b$	$b \to f$	$p \to \neg f$
$*$	1	1	1
p	0	1	1
b	1	0	1
f	1	1	1
pb	1	0	1
pf	0	1	0
bf	1	1	1
pbf	1	1	0

The inference relation $\mid\!\sim$ induced by D is thus given by the preferential model

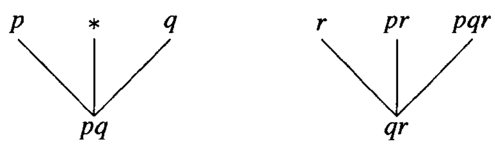


where the lines represent connection between worlds through the order $<_D$.

Note that the default $b \rightarrow f$ is the only element of D that is preserved in the resulting inference relation: one has indeed $b \mid\!\sim f$, but neither $p \mid\!\sim \neg f$, nor $p \mid\!\sim b$. In this sceptical approach of default reasoning, rather than choosing a particular Reiter extension in which two of the three initial defaults would be preserved, the defaults $p \rightarrow b$ and $p \rightarrow \neg f$ are simply replaced by the weaker conditionals $p \wedge f \mid\!\sim b$ and $p \wedge \neg b \mid\!\sim \neg f$.

Example 7. Let $\mathcal{L}$ be the language built on three propositional variables p, q and r, and D the set $D = \{\alpha, \beta, \gamma, \delta\}$, where $\alpha = (p \vee q) \wedge r$, $\beta = \neg p \wedge r$, $\gamma = (\neg p \vee q) \wedge \neg r$, $\delta = p \wedge \neg r$.

We leave it as an exercise for the reader to check that the preferential model



is the standard model associated with the induced relation $\mid\!\sim_D$.

3.2. Faithfully representable inference relations and Poole systems

As results from Observation 3, the preferential inference relation induced by a set of defaults is faithfully representable. The question naturally arises whether the converse is true, that is whether any faithfully representable inference relation might be induced by some set of defaults. The following example shows that this is not in general the case:

Example 8. Let $p_1, p_2, \ldots, p_n, \ldots$ be a countable set of propositional variables and $\mathcal{L}$ the propositional language of the p_i's. For each index i, denote by m_i the world that takes value 1 on p_i and 0 on the p_j's, $j \neq i$. Observe that any element of the intersection $\bigcap_i |m_i|$ can be written as a conjunction of formulae $s_1 \wedge s_2 \wedge \cdots \wedge s_r$ where each s_i is a disjunction of literals that cannot be all positive. It follows that this set is satisfied by the world m^* that takes value 0 on all the p_i's. Let $<$ be the ranked order on W with the following two ranks: at rank 0, put all the worlds m_i, and at rank 1 all the other worlds. The minimal worlds are therefore all the m_i's. *We claim that* $<$ *thus defined cannot be induced by a set* D. Suppose indeed that one has $< = <_D$ for some set D. We can suppose that D contains no contradiction. Let us denote by D_k the maximal consistent subsets of D. Clearly, a world is minimal iff it satisfies one of the D_k's. Note that D cannot be a consistent set: otherwise, D would be a subset of $\bigcap_i |m_i|$ and would therefore be satisfied by m^*, contradicting the fact that m^* is not minimal. Since D is not consistent, there exists in D at least two different maximal consistent subsets D_k and D_l, and it follows readily that there exists a formula α that lies in D_k and not in D_l. Denote by q any world that satisfies D_l. Such a world is minimal. Observe that any world n that satisfies α must also be minimal, since otherwise, by the choice of the order relation $<$, one would have $q <_D n$ and q would satisfy $D_l \cup \{\alpha\}$. Therefore, if a world n satisfies α, n must be one of the m_i's. But if h and j are two index such that p_h and p_j are not elements of $\mathrm{Var}(\alpha)$, one readily sees that there exists a world that satisfies $\alpha \wedge p_h \wedge p_j$ and that this world cannot be any of the m_i's.

We do not know of any necessary and sufficient condition that should satisfy a faithfully representable inference relation in order to be induced by a set of defaults. Nevertheless, as will be shown in Theorem 13, the following sufficient condition turns out to provide such a characterization in the case where the underlying language is logically finite:

Observation 9. *Let* $\mid\!\sim$ *be a preferential inference relation defined by a faithful model* $(W, <)$ *and* Δ *its associated basic set of defaults. Suppose that the following condition* $(**)$ *holds*:

$(**)$ *For any world* p, *there exists a non-empty subset* Δ_p *of* Δ *such that* $m \models \Delta_p$ *iff* $m = p$ *or* $m < p$.

Then $\mid\!\sim$ *is induced by its basic set of defaults.*

Proof. Observe first that it follows readily from the definition of Δ that the sets Δ_p are subsets of Δ. We have to show that $< = <_\Delta$. Suppose first that one has $m < n$. If α is a formula of Δ that is satisfied by n, one has $m \models \alpha$ by definition of Δ, and $|n| \cap \Delta$ is therefore a subset of $|m| \cap \Delta$. Moreover, the existence of the set Δ_m, a subset of Δ that is readily satisfied by m and not by n, implies that the above inclusion is strict, and we have shown that $m <_\Delta n$. Conversely, suppose that we have $m <_\Delta n$. Note that $m \neq n$. Since n satisfies Δ_n, the definition of $<_\Delta$ shows that the same is true for m. It follows then from the definition of Δ_n that one has $m < n$. $\square$

We noticed that when a preferential inference relation $\mid\!\sim$ is induced by a set of defaults D, this set must be embedded in the basic set of defaults Δ associated with the model $(W, <_D)$ of $\mid\!\sim$. When condition $(**)$ of Observation 9 holds, it is possible to determine a strict subset of Δ that induces the same relation:

Observation 10. *Suppose that the condition* $(**)$ *of Observation* 9 *is satisfied, and denote by* Δ^* *the union of all the subsets* Δ_p. *Then the relation* $\mid\!\sim$ *is induced by* Δ^*.

Proof. Suppose first that we have $m < n$. If α is a formula of Δ^*, we have $\alpha \in \Delta$. Hence, if n satisfies α, the same is true for m. Moreover, $m \models \Delta_m$, and n does not satisfy this set. There exists therefore an element $\delta \in \Delta_m \subseteq \Delta^*$ that is satisfied by m and not by n. It follows that $|n| \cap \Delta^*$ is a strict subset of $|m| \cap \Delta^*$, and we have shown that $m <_{\Delta^*} n$. Conversely, suppose that $m <_{\Delta^*} n$. As n satisfies the set Δ_n, which is a subset of Δ^*, we have also $m \models \Delta_n$. Therefore, we have $m = n$ or $m < n$. But the equality is impossible since we supposed that $m <_{\Delta^*} n$, and the proof of the observation is complete. □

We are now able to prove the main result of this section: *when the underlying language is finite, any consistency-preserving preferential inference relation defined by an injective model is induced by its associated basic set of defaults.* In view of Observation 9, it is enough to prove the following.

Observation 11. *Condition* $(**)$ *holds for any faithfully representable preferential inference relation* $\mid\!\sim$ *defined on a logically finite language. Such a relation is therefore induced by its basic set of defaults.*

Proof. For any world q, denote by χ_q the complete formula associated with q, that is the conjunction of all literals that are true in q. Let $(W, <)$ be the standard model of $\mid\!\sim$ and Δ its associated basic set of defaults. For any world p, let δ_p be the formula equal to χ_p if p is minimal and to $\chi_p \vee (\bigvee_{m<p} \chi_m)$ otherwise. Note that a world m satisfies δ_p iff $m = p$ or $m < p$, so that condition $(**)$ of Observation 9 is readily satisfied for the set $\Delta_p = \{\delta_p\}$. □

As observed in Section 2, the faithful model of a faithfully representable preferential inference relation is unique when the underlying language in finite. Therefore, the set $\Delta^* = \{\delta_p \mid p \in W\}$ only depends on the given inference relation $\mid\!\sim$. We will refer to this set as the *determinant of the inference relation* $\mid\!\sim$. As an immediate consequence of Observations 10 and 11, we have the following corollary:

Corollary 12. *Any faithfully representable preferential inference relation defined on a logically finite language is induced by its determinant.*

Proof. Clear. □

We can now summarize our results:

Theorem 13. *Given any consistency-preserving preferential inference relation* $\mid\!\sim$ *defined on a logically finite language* $\mathcal{L}$ *by an injective model, there exists a subset* D *of* $\mathcal{L}$ *such that* $\mid\!\sim$ *coincides with the inference relation* $\mid\!\sim$ *associated with the Poole system* $(D, \emptyset)$.

Proof. Immediate by Lemma 2 and Observation 11. □

Theorem 13 may be equivalently restated as follows:

Theorem 14. *For any strict partial order $<$ defined on the set W of worlds associated with a logically finite language $\mathcal{L}$, there exists a subset D of $\mathcal{L}$ such that $< \, = \, <_D$.*

Proof. Since the language is finite, the structure $(W, <)$ is smooth, and is therefore a faithful preferential model. One concludes, applying Theorem 13. □

The meaning of the above theorems is that, under some mild conditions, preferential reasoning à la Shoham is the same as reasoning à la Poole. But its interest is also to point out *the existence of a basic set of defaults that comes to conditionalize, implicitly or explicitly, any agent that uses a faithfully representable logic on a finite language. As will be seen in the next section, this applies in particular to any agent that uses consistency-preserving rational logic.* This result is reminiscent of some well-known theories about human behaviour, in as much as they claim that human beings are determined in their judgements and actions, by a set of primary *affects*. In this sense, it is worth pointing out the formal analogy that exists between the result stated above and the most classical psychoanalytic theories.

Applying Corollary 4, one sees that, in a finite language, given any consistency-preserving inference relation C defined by an injective model, there exists a set D such that $\beta \in C(\alpha)$ *iff for any world n that satisfies $\alpha \wedge \neg\beta$, there exists a world m that satisfies α with $|n| \cap D \subsetneq |m| \cap D$.* If the given inference relation is not supposed to be consistency-preserving, this result is no longer true, but it may be applied to the consistency-preserving component C_c of C, defined by

$$C_c(\alpha) = \begin{cases} C(\alpha) & \text{if } C(\alpha) \text{ is consistent,} \\ \mathrm{Cn}(\alpha) & \text{otherwise.} \end{cases}$$

One easily checks indeed that C_c is a faithfully representable preferential inference relation, and it follows that there exists a set D such that for all formulae α, either $C(\alpha)$ is an inconsistent set, or $C(\alpha)$ is given by a condition identical to the one above. The proofs are straightforward and we leave them to the reader.

Although both the basic set of defaults Δ and the determinant Δ^* induce, in a finite language, the same given faithfully representable inference relation, the set Δ^* often turns out to be easier to write down than the set Δ. Note that the cardinality of Δ^* is equal to 2^n, where n is the number of atomic propositions of the language. The link between Δ and Δ^* is fully described by the following:

Observation 15. *Let $<$ be a strict partial order on the set W of worlds associated with a logically finite language, Δ its associated basic set of defaults and Δ^* its determinant. Then, modulo classical equivalence, a formula α is an element of Δ iff it is a disjunction of formulae of Δ^*.*

Proof. We noticed (at the end of Section 2) that Δ is stable by disjunction. Since Δ^* is a subset of Δ, any disjunction of formulae of Δ^* is an element of Δ. To show that the converse is true, let α be any element of Δ. If α is inconsistent, α is a disjunction over an empty set. Suppose that α is consistent, and denote by $\delta(\alpha)$ the disjunction of all the elements δ_m for m maximal among the worlds that satisfy α. We claim that α is classically equivalent to $\delta(\alpha)$. Suppose first that a world q satisfies α. If q is maximal among the formulae that satisfy α, we have $q \models \delta_q$, and therefore $q \models \delta(\alpha)$. If q is not maximal among these formulae, there exists a world m such that $q < m$, m maximal among the worlds that satisfy α. We have then $q \models \delta_m$, hence $q \models \delta(\alpha)$. This shows that $\alpha \models \delta(\alpha)$. Let us check conversely that $\delta(\alpha) \models \alpha$: take a world p that satisfies $\delta(\alpha)$. There exists a world m, maximal among the worlds that satisfy α, such that $p \models \delta_m$. We have then $p = m$, in which case $p \models \alpha$, or $p < m$. But α is an element of Δ, and therefore, since $m \models \alpha$, condition $(*)$ implies that $p \models \alpha$. This shows that $\delta(\alpha) \models \alpha$, and the proof of the observation is complete. □

We now compute the set Δ^* in two examples:

Example 5 *(continued)*. Let δ be any formula of a logically finite language, and $D = \mathrm{Cn}(\delta)$. We saw that the order $<$ of the standard model $(W, <)$ of the inference relation induced by D is defined by: $m < n$ iff $m \models \delta$ and $n \models \neg\delta$. We have

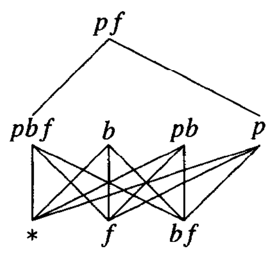
$$\begin{aligned}\Delta^* &= \{\chi_m \mid m \models \delta\} \cup \left\{\chi_p \vee \left(\bigvee \chi_n\right)_{n \models \delta} \,\middle|\, p \models \neg\delta\right\} \\ &= \{\chi_m \mid m \models \delta\} \cup \{\chi_p \vee \delta \mid p \models \neg\delta\}.\end{aligned}$$

It can be shown that the basic set of defaults Δ associated with the model $(W, <)$ consists of all the formulae ϕ that either classically imply δ, or are classically implied by δ. □

Example 6 *(continued)*. The preferential model associated with

$$D = \{p \to b,\ b \to f\, p \to \neg f\}$$

was shown to be



Let us list the elements of Δ^*:

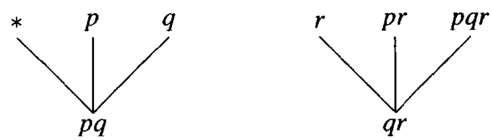
$$\delta_* = \neg p \wedge \neg b \wedge \neg f,$$

$$\delta_f = \neg p \wedge \neg b \wedge f,$$

$$\delta_{bf} = \neg p \wedge b \wedge f,$$
$$\delta_{pbf} = \chi_{pbf} \vee \chi * \vee \chi_f \vee \chi_{bf} = (\neg p \wedge (b \to f)) \vee (f \wedge (p \to b)),$$
$$\delta_b = \neg p,$$
$$\delta_{pb} = (p \wedge b \wedge \neg f) \vee (\neg p \wedge (b \to f)),$$
$$\delta_p = \neg b \wedge (p \to \neg f) \vee (\neg p \wedge (b \to f)),$$
$$\delta_{pf} = b \to f.$$

One sees from this simple example that the determinant is generally not the simplest of the subsets that induce a given inference relation.

Example 7 (*continued*). The faithful model $(W, <)$ of the inference relation defined in Example 6 was shown to be



The set Δ^* then consists of the eight formulae δ_m where $m \in W$, namely

$$\Delta^* = \{p \wedge q \wedge \neg r,\ \neg p \wedge q \wedge r,\ \neg r \wedge (\neg p \wedge \neg q) \vee (p \wedge q),\ p \wedge \neg r,\ q \wedge \neg r;\\ \neg p \wedge r,\ r \wedge (p \wedge \neg q) \wedge (q \wedge \neg p),\ q \wedge r\}.$$

Note that $q \in \Delta - \Delta^*$.

Remark 16. It is worth noticing that *the set Δ^* separate the worlds*: if two worlds m and n agree on Δ^*, they must be equal. Indeed, since $m \models \delta_m \in \Delta^*$, we have $n \models \delta_m$, so $n = m$ or $n < m$. Similarly, one sees that $m = n$ or $m < n$, hence the equality.

3.3. Comparison with Gärdenfors–Makinson's work

It is interesting to compare the result stated in Theorem 13 with those proven by Gärdernfors and Makinson in [5]. There, the authors introduced the notion of a *selection function* in the following way: given a set D of defaults, a selection function S_D associates with any formula α a set of subsets of D that are maximal consistent with α. The interest of these functions, as follows from [5, Theorem 2.4], is that for any preferential inference relation C defined on an arbitrary propositional language $\mathcal{L}$, there exists a subset D of $\mathcal{L}$ and a selection function S_D such that, for all formula α, one has $C(\alpha) = \bigcap \mathrm{Cn}(\alpha, D_\alpha)$, where the intersection is taken over all the elements of $S_D(\alpha)$. *Theorem* 13 *shows that, in the particular case where C is a faithfully representable preferential relation defined on a logically finite language, the selection function can be chosen to be the trivial one*, that is the one where, for each formula α, $S_D(\alpha)$ is the set of *all* the subsets of D that are maximal consistent with α. Note, nevertheless, that the set D whose existence is proven in Theorem 13 is not in general closed under logical consequence, whilst in [5] this set is always taken to be $C(true)$, and is therefore closed under logical consequence.

3.4. The dynamics of Poole systems

Since, in finite languages, the preferential logic of an agent is always conditionalized by a set of defaults, it is natural to study the effect, on such a logic, of a perturbation of this set. This perturbation may or may not be analyzed as a revision of the set of defaults, considered as a belief base, but it seems clear that the "classical" postulates for revision are inadequate in this framework: indeed, the *gestalt* that is at the origin of these postulates considers the knowledge base of an agent as a whole, independently of the resulting behaviour that is going to be adopted by this agent after her beliefs have been revised. Thus, revision theory is only concerned with some principles that regulate the evolution of a knowledge base in the presence of new information, and does not take into account the influence of this evolution on the induced logic. In the perspective of Gärdenfors–Makinson, for instance, the process of revision is completely described by means of an inference relation, which will eventually determine the way a belief should change when new information is provided (see [5] and [6]); but this formal link with nonmonotonic logic is of little use in the framework of Poole systems: Gärdenfors and Makinson [6] proposed to analyze the revision of K by a formula α as *an inference operation from the proposition* α, so that the resulting theory $K * \alpha$ is interpreted as the set of consequences of α *modulo* K. In this proposal, the theory K is considered as a static one, and not as a *medium* which only device is to induce an inference relation. Quite different is the framework of Poole systems, where default sets of formulae are considered as equivalent when they induce the same inference relation. Thus, the dynamics of Poole systems may be indifferently studied through a revision of either of these sets. Furthermore, in this perspective, the admissible changes of a set of defaults should obey rules that fundamentally differ from those applicable to classical revision theory. One may require for instance that the consistency of the basic set of defaults is preserved, that the new order $<_{D}$, differs from the old one only for a minimal number of couples of worlds, and that the property of rationality is preserved: the admissible changes of a set D of defaults that induces a rational inference relation would have thus to be chosen among those that will still induce a relation of this type. These questions can be only evoked here and will be more deeply studied in a forthcoming paper. Some examples will be nevertheless discussed in the next section.

4. The rational case

In this section, we consider the particular case of rational inference relations and we shall show that these relations may be advantageously studied through a set that is more tractable than the associated determinant. We first recall some basic definitions and facts.

A *rational* inference relation is a preferential inference relation that satisfies the condition of *rational monotony*:

(RM) If $\alpha \mid\!\sim \beta$, and not $\alpha \mid\!\sim \neg\gamma$, then $\alpha \wedge \gamma \mid\!\sim \beta$.

The semantics of rational inference relations is particularly simple to describe: a preferential inference relation is rational iff it can be represented by an injective preferential model $(S, <)$, where $<$ is a smooth *modular* order defined on a subset S of W. We recall that a order on S is modular iff there exists a totally ordered set $(T, <')$ and a ranking function κ from $(S, <)$ into $(T, <')$ such that $m < n$ iff $\kappa(m) <' \kappa(n)$. Equivalently, an order $<$ on S is modular iff, for all elements m, n, p of S such that $m < n$, either $m < p$, or $p < n$.

In a logically finite language, a rational relation is faithfully representable if and only if it is consistency-preserving. When this is the case, one can always suppose that the ranking function κ is defined on the whole set W, and takes value in a finite set of integers. The *height* h of such a rational relation or, equivalently, of its standard model, is defined by $h = \|\kappa(W)\|$. We will always suppose that the ranking function is normalized, i.e. that $\kappa(W)$ is the set $[0, h-1] = \{0, 1, 2, \ldots, h-1\}$.

Example 8 shows that, in the general case of arbitrary languages, there exists rational inference relations that are not induced by any set of defaults. No characterization of the rational relations that can be determined by such a set is known in the general case. Nevertheless, the following analogue of Observation 9 provides an interesting result:

Observation 17. *Let $\mid\!\sim$ be a faithfully representable rational relation defined by the ranked model $(W, <)$. Suppose that the associated ranking function κ takes its value in the set of integers, and that its satisfies the following condition $(***)$:*

$(***)$ *For any non-zero integer i, there exists a formula δ_i such that $m \models \alpha_i$ iff $\kappa(m) < i$.*

Then exists a set D, totally ordered through classical implication, that induces the relation $\mid\!\sim$.

Conversely, any family D of formulae $D = (\delta_1, \delta_2, \ldots, \delta_k, \ldots)$ with $\delta_i \vdash \delta_{i+1}$ induces a rational inference relation, the ranking of which satisfies condition $(***)$.

Proof. We suppose first that $(***)$ holds and show that $< \; = \; <_D$, where D is the family of all the formulae δ_i. Clearly, one has $\delta_i \vdash \delta_{i+1}$, so that D is totally ordered through classical implication. If $m < n$, and δ_i is an element of D satisfied by n, we have $\kappa(n) < i$, hence $\kappa(m) < i$, and $m \models \delta_i$. Moreover, if $j = \kappa(m)$, we see that the formula δ_{j+1} is satisfied by m but not by n, and it follows that $|n| \cap D$ is a strict subset of $|m| \cap D$, that is $m <_D n$. Conversely, suppose that $m <_D n$. If i is the rank of n, one sees that n satisfies δ_{i+1}, and the same is therefore true for m, whence $j = \kappa(m) \leqslant i$. As there exists an index k such that $m \models \delta_k$ and $n \models \neg\delta_k$, we do not have $n < m$, and therefore we have $j < i$, that is $m < n$, which completes the proof of the first part of the observation.

Suppose now that $D = \{\delta_1, \delta_2, \ldots, \delta_n, \ldots\} \subseteq \mathcal{L}$ is such that, for all i's, $\delta_i \vdash \delta_{i+1}$, and let $\mid\!\sim_D$ be the preferential inference relation induced by D. This relation is represented by the faithful model $(W, <_D)$. We claim that for any worlds m and n, we have $m <_D n$ *iff there exists an element $\delta \in D$ that is satisfied by m and not by n*: indeed, if $m <_D n$, such an element exists by the definition of $<_D$. Conversely, if such an element δ exists, observe that $|m| \cap D \neq |n| \cap D$. Furthermore, for any element $\delta' \in D$ satisfied by n, the hypothesis made on D together with the fact that n does not satisfy δ implies that $\delta \vdash \delta'$, so that m satisfies δ'. It follows that $|n| \cap D \subseteq |m| \cap D$, and we have shown that $m <_D n$.

It is now easy to prove that $<_D$ is a modular order: if m, n and p are three worlds such that $m <_D n$ but not $m <_D p$, there exists an element δ of D that is satisfied by m and not by n, and this element must be satisfied by p. It follows that $p <_D n$, and the modularity is proven. To prove Observation 17, it remains only to show that condition $(***)$ holds when D is of the form $D = \{\delta_1, \delta_2, \ldots, \delta_k, \ldots\}$ with $\delta_j \vdash \delta_{j+1}$. Clearly, the set $D' = D \cup \{\mathit{True}\}$ is totally ordered through classical implication and induces the same rational inference relation as D. Let κ be the ranking function associated with the modular order $<_D = <_{D'}$, i any element of $\kappa(W)$ and n a world of rank i. If $|n| \cap D \neq \emptyset$, let $k(i)$ be the first index such that $n \models \delta_{k(i)}$; set $\delta_{k(i)} = \mathit{True}$ otherwise. We have then for any world m, $m \models \delta_{k(i)}$ iff $\kappa(m) \leqslant \kappa(n)$, that is iff $\kappa(m) < i + 1$, which shows that condition $(***)$ is satisfied. $\square$

It is worth noticing that condition $(***)$ is not in general satisfied by an arbitrary faithfully representable rational inference relation, even though this latter may be induced by a set of defaults: for instance, in Example 5 with D a closed subset of $\mathcal{L}$ such that $D \neq \mathrm{Cn}(\delta)$ for any formula δ, there exists no formula δ such that $m \models \delta$ iff $\kappa(m) \leqslant 1$, since this latter inequality is equivalent to $m \models D$. We shall nevertheless show that condition $(***)$ always holds in logically finite languages.

4.1. The characteristic set of a rational relation defined on a finite language

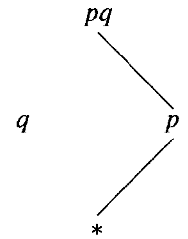
Supposing we work in a finite language, our first result is that condition $(***)$ of Observation 17 holds.

Observation 18. *Condition $(***)$ holds in finite languages.*

Proof. Let $\mid\!\sim$ be a rational inference relation defined on a logically finite language and $(W, <)$ its associated standard model. Denote by h the height of the associated ranking function κ. For any integer i, $0 < i \leqslant h - 1$, let ψ_i be a representative of the disjunction over all worlds n of rank less than i of the complete formulae associated with n, that is $\psi_i = \bigvee_{\kappa(n)<i} \chi_n$. Set $\psi_h = \mathit{True}$. One has then clearly $m \models \psi_i$ iff $\kappa(m) < i$. $\square$

The set $\Psi = \{\psi_i\}$ will be referred to as the *characteristic set* of the relation (or, equivalently, as the characteristic set of the model $(W, <)$. Note that it has cardinality equal to h, and that its satisfies the property of *separating the ranks*: two worlds agree on Ψ iff they have same rank. Its main feature is that, given two worlds m and n, one has $m < n$ *iff there exists an index i such that ψ_i is satisfied by m and not by n*, or, equivalently, *iff m satisfies strictly more elements in Ψ than n*. The elements of the characteristic set Ψ form a *chain* with last term equal to *true*, in the sense that they are totally ordered through classical implication. We will call such an h-tuple a *logical chain*.

A notion similar to that of a characteristic set could have been defined in an analogous way in the case where the preferential model $(W,<)$ is not supposed to be ranked, defining, for any non-minimal world m, the formula ψ_m as the disjunction over all worlds $n < m$ of the complete formulae associated with n. Thus the "generalized characteristic set" Ψ^* of the model



could have been defined as the set $\{\chi_*, \chi_p \vee \chi_*, true\}$. But one notices that the order $<_{\Psi^*}$ induced on W by the set Ψ^* is the modular order where the ranks are given by the following diagram:

$$\begin{array}{cc} pq \quad q & 2 \\ p & 1 \\ * & 0 \end{array}$$

Therefore, one does not have $< \, = \, <_{\Psi^*}$. For this reason, the interest of defining the set Ψ in the general case of preferential relations is not obvious. Nevertheless, we shall see at the end of this section that this set may play an important role in the framework of the dynamics of a Poole system. For the moment we note that in the case where the order $<$ defined on W is supposed to be modular, the equality $< \, = \, <_\Psi$ does hold:

Theorem 19. *In a finite language, any rational inference relation is induced by its characteristic set.*

Proof. Immediate, using Observations 17 and 18. □

Note that the above result leaves open the question of characterizing *all* the sets D such that $\mid\!\sim_D$ is rational. Theorem 19 only shows that, given such a set D, there exists a set Ψ totally ordered through classical implication such that $\mid\!\sim_D \, = \, \mid\!\sim_\Psi$.

A consequence of Theorem 19 is the existence of a simple syntactic characterization of all rational relations defined in a finite language:

Theorem 20. *For any consistency-preserving rational inference relation C defined on a logically finite language, there exists a logical chain $(\psi_1, \psi_2, \ldots, \psi_h)$ with $\psi_h = true$ and $\psi_i \vdash \psi_{i+1}$ for all $i < h$, such that, for any consistent formula α, one has $C(\alpha) = \mathrm{Cn}(\alpha, \psi_i)$ where i is the first index such that α is consistent with ψ_i.*

Proof. Let Ψ be the characteristic set of $\mid\!\sim$. By Corollary 4, one has $\beta \in C(\alpha)$ iff for any world n that satisfies $\alpha \wedge \neg\beta$, there exists a world m that satisfies α, with $m <_{\psi} n$. If i is the first index such that α is consistent with ψ_i, the formula $\alpha \wedge \psi_i$ is satisfied by all worlds of rank strictly less than i, and only by these worlds. Therefore one has $\alpha \mid\!\sim \beta$ iff the formula $\alpha \wedge \neg\beta$ is inconsistent with ψ_i, whence the result. □

A consequence of Theorem 19 is that a rational inference relation is fully determined by its characteristic set. Conversely, one easily shows that *there exists only one logical chain that induces a given rational inference relation.* This remark is of interest in the framework of theory change: if the admissible perturbations of a given rational inference relation are to be chosen among those that preserve rationality, it is natural to require that the revision of a logical chain yields again a logical chain. This implies that there is no stability under revision, as the initial chain and the revised chain, if different, will necessarily give rise to different inference relations.

Before looking at some examples, we note that the set Ψ is of minimal cardinality among the subsets of $\mathcal{L}$ that induce a rational inference relation:

Observation 21. *Let $(W,<)$ be a ranked model defined on a logically finite language, Ψ its characteristic set and D a subset of $\mathcal{L}$ such that $< \, = \, <_D$. Then the number of elements of Ψ is at most equal to the number of elements of D.*

Proof. Let h be the number of elements of Ψ. Then h is the height of the model $(W,<)$. Note that if the order $<_A$ induced on W by a subset A of $\mathcal{L}$ is modular, the height of this order is the length of a strictly decreasing sequence of subsets of A, so that this length is at most equal to the number of elements of A. Since we supposed that $< \, = \, <_D$, it follows that h is at most equal to the number of elements of D. □

Example 8 (*continued*). Consider a consistent formula α, and set $D = \mathrm{Cn}(\alpha)$. Recall that the order $< \, = \, <_D$ on W is a ranked order, defined by $m < n$ iff α is satisfied by m and not by n. The height of this model is thus equal to 2, and its characteristic set is $\Psi = \{\alpha, \mathit{true}\}$.

Example 22. One checks easily that the characteristic set of a consistency-preserving rational inference relation is the singleton *true* iff this relation is equal to the classical consequence operation Cn.

4.2. Comparison with Grove's work

In [7], Grove proposed a model for belief revision in terms of a family K_i of "spheres" around an agent theory K. These spheres are subtheories of K, that are obtained by removing from K all sentences that are not "sufficiently entrenched" (see also [10] for a presentation of this work). More precisely, Grove showed that, given any revision $*$ of K satisfying the extended set of AGM postulates, there exists a total ordered family of subtheories K_i such that, for any formula α, the revision $K * \alpha$ of K by α is equal to the expansion $\mathrm{Cn}(K_i, \alpha)$ of one of the spheres by α. This sphere K_i

is the greatest one that is consistent with α. Conversely, given a totally ordered family of subtheories K_i of K, the operation $*$ on K that associates with any formula α the expansion by α of the greatest K_i consistent with α is a revision operation that satisfies the extended set of AGM postulates.

In the finite case, this result may be seen as a direct consequence of Theorem 20: indeed, any revision $*$ of K that satisfies the extended set of AGM postulates yields a consistency-preserving rational inference relation $\mid\!\sim_*$ defined by: $\alpha \mid\!\sim_* \beta$ iff $\beta \in K * \alpha$. If $(\psi_1, \psi_2, \ldots, \psi_h)$ is the associated characteristic set and if, for all i, K_i is the set $K_i = \mathrm{Cn}(\psi_i)$, one has $\alpha \mid\!\sim_* \beta$ iff $\neg\alpha \vee \beta \in K_i$ where i is the first index such that α is consistent with K_i. If follows that $K * \alpha = \mathrm{Cn}(K_i, \alpha)$ is equal to the expansion of K_i by α.

4.3. Rationalizing an irrational behaviour

Suppose that the behaviour of an agent is conditionalized by a preferential logic, and that it appears desirable or necessary that this agent acts *rationally*. The initial behaviour, represented by a preferential inference relation $\mid\!\sim_P$ is therefore to be changed into a *rational* inference relation, $|_R$. One way to proceed is to take for $\mid\!\sim_R$ the *rational closure of* $\mid\!\sim_P$ (see [9] for a discussion of the existence of the rational closure). The main feature of this "closure" is that it is an extension of the given relation $\mid\!\sim_P$, so that the conditional $\alpha \mid\!\sim_R \beta$ holds whenever $\alpha \mid\!\sim_P$ holds. Rationalizing an agent's behaviour through rational closure will therefore consist in adding some missing connections, without suppressing any of the existing ones. The procedure of taking the rational closure of a preferential relation is nevertheless far from being simple. Another way, that seems much simpler, is to first "revise" the determinant Δ^* or the "generalized characteristic set" Ψ^* of $\mid\!\sim_P$ so as to get a logical chain Ψ, which will induce the desired rational relation. There exists of course several ways of transforming Δ^* into a logical chain $(\psi_1, \psi_2, \ldots, True)$, and we will only mention two of them.

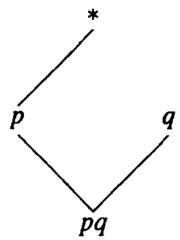
Concatenation of chains

Let Ψ^* be the *generalized characteristic set of* $\mid\!\sim_P$, that is *the set of all formulae* ψ_n, *where* ψ_n *is the disjunction over all worlds* $m < n$ *of the complete formulae associated with* m, *for* n *non-minimal, and* $\psi_n = true$ *for* n *minimal*. The set Ψ^* may then be seen as an union of logical chains. Observe that if $\alpha_0 \vdash \alpha_1 \vdash \cdots \vdash \alpha_n$ and $\beta_0 \vdash \beta_1 \vdash \cdots \vdash \beta_k$ are two chains with, say, $k \leqslant n$, then

$$\alpha_0 \vee \beta_0 \vdash \alpha_1 \vee \beta_1 \vdash \alpha_2 \vee \beta_2 \vdash \cdots \vdash \alpha_k \vee \beta_k \vdash \alpha_{k+1}\beta_k \vdash \cdots \vdash \alpha_n \vee \beta_k$$

is again a chain. Taking for (α_i) a chain of maximal length in Ψ, one can therefore proceed by concatenation to obtain a set Ψ^*, the elements of which form a logical chain, that will induce the desired *rationalization* of $\mid\!\sim$.

Example 23. Let $\mid\!\sim_P$ be induced by the set $\Delta^* = \{\neg p \vee q, p, p \wedge q, q\}$ and represented by the preferential model $(W, <)$

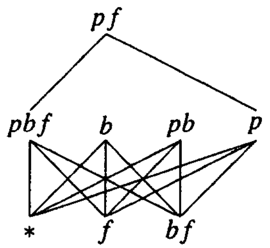


with order $<$ defined by $pq < q$, $pq < p < *$. To write down the elements of the generalized characteristic set, it is simpler to associate to any formula of Ψ^* the set of all worlds that satisfy this formula, remembering that, for n non-minimal, a world m satisfies ψ_n iff $m < n$. We have thus $\psi_* = \{p, pq\} = p$, $\psi_p = \psi_q = \{pq\} = p \wedge q$. The set Ψ^* is therefore equal to $\{p \wedge q, p, true\}$, and consists on a single logical chain. The rationalization of $\mid\!\sim_P$ by the above process yields the ranked model $(W, <_R)$

$$\begin{array}{cc} q \quad * & 2 \\ p & 1 \\ pq & 0 \end{array}$$

Note that the order $<_R$ is an extension of $<$, so that, by Lemma 1, the rationalized relation $\mid\!\sim_R$ extends the relation $\mid\!\sim_P$. $\square$

Example 6 (*continued*). We take for $\mid\!\sim_P$ the inference relation induced by the set $D = \{p \to b, b \to f, p \to \neg f\}$. As noticed above, the associated standard model $(W, <)$ is given by



The elements of the associated generalized characteristic set Ψ^* are

$$\psi_{pbf} = \psi_b = \psi_{pb} = \psi_p = \neg p \wedge (b \to f),$$
$$\psi_{pf} = (p \wedge b \wedge f) \vee (p \wedge \neg b \wedge \neg f) \vee \psi_{pbf} = (b \wedge f) \vee (\neg b \wedge (p \to \neg f).$$

We have $\Psi^* = \{\neg p \wedge (b \to f), (b \wedge f) \vee \neg b \wedge (p \to \neg f), true\}$, and the resulting rationalized relation is determined by the ranked model

$$\begin{array}{cc} b \quad pf \quad pb & 2 \\ pbf \quad p & 1 \\ * \quad f \quad bf & 0 \end{array}$$

Note that $\mid\!\sim_R$ is again an extension of $\mid\!\sim_P$.

The characteristic ranking

The process of concatenation described above is not standard, in the sense that it depends of the chain of maximal length that is chosen. It is possible to rationalize the preferential inference relation induced by its determinant Δ^* by ordering the set W through *the number of elements of the generalized characteristic set Ψ^* that a world may satisfy*. The *characteristic rank* of a world is then simply defined as the number of elements of Ψ^* that are not satisfied by this world. Recalling that a world m satisfies the element ψ_n of Ψ^* iff $m < n$, *we see that the characteristic rank of a world m can be directly computed through the number of worlds n such that $m < n$*. To compute this rank, it is therefore not necessary to display the generalized characteristic set. In the penguin example, for instance, the number of nontrivial elements of Ψ^* satisfied by a world are given by:

m	Number of elements of Ψ satisfied by m = number of n's with $m < n$
$*$	5
f	5
bf	5
pbf	1
p	1
b	0
pb	0
pf	0

We see that the resulting rationalized inference relation $\mid\!\sim_R$ is again determined by the model

$$\begin{array}{cl} b \quad pb \quad pf & 2 \\ pbf \quad p & 1 \\ * \quad f \quad bf & 0 \end{array}$$

The process of rationalization *via* the characteristic ranking is clearly much simpler than that of chains concatenation. Its properties will be studied in detail in a forthcoming paper, but it is worth noticing for the present that the resulting relation $\mid\!\sim_R$ *always extends the original one $\mid\!\sim_P$, and agrees with it iff $\mid\!\sim_P$ is already rational.* The proof of this observation is straightforward, noticing that if D is a logical chain, then the order $<_D$ induced by D satisfies $m <_D n$ iff m satisfies more elements in D than n.

4.4. Revising a rational behaviour

Even when an agent's behaviour is rational, it may be desirable to revise it in order to take into account some new information. The problem of revising a rational inference relation amounts then to revise the (unique) logical chain $\Psi = \{\psi_1, \psi_2, \ldots, \psi_{h-1}, \mathit{true}\}$ that induces a given rational inference relation, in such a way that the resulting set is again a logical chain. If, moreover, one requires that the revision of Ψ by *true* should yield again the set Ψ, one sees that the simplest admissible revision is the one defined, for any formula α, by $\Psi \Diamond \alpha = \{\psi_k \wedge \alpha, \psi_{k+1} \wedge \alpha, \ldots, \psi_{h-1} \wedge \alpha, \alpha, true\}$, where k is the first index such that ψ_k is consistent with α. Thus, in the above example, revising the characteristic set $\Psi = \{\neg p \wedge (b \to f), (b \wedge f) \vee \neg b \wedge (p \to \neg f), true\}$ by the formula p yields the logical chain

$$\Psi \Diamond p = \{p \wedge ((b \wedge f) \vee (\neg b \wedge \neg f)), p, true\},$$

with associated ranked model

$*$	b	bf	f	2
	pb	pf		1
	pbf	p		0

5. Default extensions

We shall now apply to some specific rational inference relations the results proven in the preceding section. From now on, we suppose that we work on a logically finite propositional language.

5.1. Default ranking through rational closure

We suppose given a conditional knowledge base K, that consists of a finite number of conditional assertions $\alpha_i \Rightarrow \beta_i$, where α_i and β_i are elements of a logically finite language A conditional $\alpha \Rightarrow \beta$ represents a piece of information that can be interpreted as "if α, then normally β". Note that such a conditional is not an element of the language, so that the set K is not to be taken as a subset of $\mathcal{L}$. The *material counterpart* of the conditional $\alpha \Rightarrow \beta$ is the formula $\alpha \to \beta$, equivalent to $\neg\alpha \vee \beta$. The material counterpart K^* of a knowledge base K is the conjunction of the material counterparts of its conditionals.

We wish to build from K a *rational* inference relation $\mathrel{|}\!\sim_K$ (or simply $\mathrel{|}\!\sim$ when there is no ambiguity) that will extend K, in the sense that $\alpha \mathrel{|}\!\sim \beta$ should hold for all conditionals $\alpha \Rightarrow \beta$ that are in K. A first solution to this extension problem was given by Lehmann and Magidor [9] and consists of the so-called *rational closure* of K, which we briefly describe.

Following [9], we say that a formula α is *exceptional* for a set A of conditional assertions if and only if α is inconsistent with the material counterpart A^* of A. This means that any world that satisfies α must falsify at least one assertion of A. A conditional assertion $\alpha \Rightarrow \beta$ is said to be exceptional for a set A iff its antecedent α is.

Given the conditional base K, one defines inductively the subsets K_i $(1 \leqslant i \leqslant h)$ in the following way: K_1 is taken to be equal to K and, for each index i such that K_i is not empty, K_{i+1} is the set of all conditionals $\alpha \Rightarrow \beta$ of K_i whose antecedent α is exceptional for K_i. The conditional base K is assumed to be *consistent*: This implies that we will always suppose that *the subsets K_i defined as above form a strictly decreasing* *sequence, with last term equal to the empty set.* We will refer to this sequence as the LM-sequence.

The set W of worlds is ranked through the *LM-ranking* function κ_{LM} that assigns *to every world m the value $i-1$ where i is the least integer such that m satisfies K_i*. The corresponding rational relation $\mid\!\sim_{\text{LM}(K)}$, called the *rational closure of K*, satisfies then the desired property of extending K. Note that the height of this relation is equal to the integer h such that K_h is empty.

Consider the conditional base $K = \{b \Rightarrow f,\ p \Rightarrow b,\ p \Rightarrow \neg f\}$. Its rational closure is represented by the model

$$\begin{array}{lll} p\quad pf\quad pbf & & 2 \\ \quad b\quad bp & & 1 \\ \ast\quad bf\quad f & & 0 \end{array}$$

Let us determine its characteristic set: one has

$$\begin{aligned} \Psi &= \{\neg p \wedge (\neg b \vee f), (\neg p \vee b) \wedge (\neg p \vee \neg f), \mathit{true}\} \\ &= \{\neg p \wedge (b \to f), (p \to b) \wedge (p \to \neg f), \mathit{true}\}, \end{aligned}$$

that is, recalling that the material counterpart of a set A is denoted by A^*, $\Psi = \{K_1^*, K_2^*, \mathit{true}\}$. This result holds in the general case:

Theorem 24. *Let K be a conditional base and K_i, $1 \leqslant i \leqslant h$ its LM-sequence with $K_h = \emptyset$. Then the characteristic set of its rational closure is the set of all K_i^*, $1 \leqslant i \leqslant h$.*

Proof. From the definition of the rank of a world, we see that a world m satisfies K_i^* iff $\kappa_{\text{LM}}(m) < i$. By Observation 17, it follows that $K_i^* \in \Psi$ for all $i \leqslant h$. One concludes, since Ψ has exactly h elements. $\square$

The above result has interesting theoretical and practical consequences. It shows that the rational closure of a conditional knowledge base K is induced by a set of n elements, each of them equal to the material counterpart of a different term of the LM-sequence. Thus, it appears that the complexity of any procedure grounded on rational closure can be evaluated with respect to the height of this closure, independently of the size of the given knowledge base.

In fact, Theorem 24 has an important consequence, concerning the representation of rational inference relations by means of conditional knowledge bases. Let indeed $\mid\!\sim$ be a consistency-preserving rational inference relation defined on a logically finite language. Clearly, such an inference relation may be analyzed as the rational closure of the set K of conditional assertions defined by $K = \{\alpha \to \beta \mid \alpha \mid\!\sim \beta\}$. This trivial observation leads nevertheless to an interesting problem, which is the problem of *rational behaviour decoding*: given such a rational inference relation $\mid\!\sim$, is it possible to explicitly find a conditional knowledge base K of *minimal order* such that $\mid\!\sim\ =\ \mid\!\sim_{\text{LM}(K)}$? In other words, is it possible to determine a set of basic rules that conditionalize the behaviour of a rational agent? The answer to this question is a surprisingly simple consequence of Theorem 24:

Theorem 25. *Let $\mid\!\sim$ be a consistency-preserving rational inference relation defined on a logically finite language, Ψ its characteristic set and h its height. Then there exists a conditional knowledge base containing exactly h elements, whose rational closure is the inference relation $\mid\!\sim$. More precisely, if the elements of Ψ are denoted by $\psi_1, \psi_2, \ldots, \psi_h$, with $\psi_i \models \psi_{i+1}$ for all $i < h$ and $\psi_h = true$, the conditional base may be taken equal to the set*

$$K = \{True \Rightarrow \psi_1, \neg\psi_1 \Rightarrow \psi_2, \neg\psi_2 \Rightarrow \psi_3, \ldots, \neg\psi_{h-1} \Rightarrow true\}.$$

Proof. Let us denote $\mid\!\sim_{\mathrm{LM}(K)}$ the rational closure of the conditional base $K = \{True \Rightarrow \psi_1, \neg\psi_1 \Rightarrow \alpha_2, \neg\psi_2 \Rightarrow \psi_3, \ldots, \neg\psi_{h-1} \Rightarrow \psi_h\}$.

To show that $\mid\!\sim\ =\ \mid\!\sim_{\mathrm{LM}(K)}$, we only have to check that the characteristic sets of these relations are equal. Computing the LM-sequence of K, we find, using the fact that $\psi_i \vdash \psi_{i+1}$ for all $i < n$,

$$\begin{array}{ll}
K_1 = K, & K_1^* = \{\psi_1\}, \\
K_2 = \{\neg\psi_1 \Rightarrow \psi_2, \neg\psi_2 \Rightarrow \psi_3, \ldots, \neg\psi_{h-1} \Rightarrow \psi_h\}, & K_2^* = \{\psi_2\}, \\
K_3 = \{\neg\psi_2 \Rightarrow \psi_3, \ldots, \neg\psi_{h-1} \Rightarrow \psi_h\}, & K_3^* = \{\psi_3\}, \\
\vdots & \vdots \\
K_{h-1} = \{\neg\psi_{h-2} \Rightarrow \psi_{h-1}\}, & K_{h-1}^* = \{\psi_{h-11}\}, \\
K_h = \emptyset. &
\end{array}$$

We conclude by Theorem 24. □

When a conditional knowledge base K of cardinality n extends, through rational closure, into a rational inference relation of height h, one has clearly $h \leqslant n$, and the above theorem shows that the base K may be replaced by a base K' whose cardinality is h. Therefore it is in general possible to replace a base K by a base K' with less elements. It should be emphasized, though, that the base K' is not in general a sub-base of K, and is not always of a simpler structure.

It is possible to determine the characteristic set of any of the systems that were proposed to extend a given conditional knowledge base. As an example, we will examine the case of a system that offers much similarity with the rational closure.

5.2. Default ranking through $\mathbb{Z}_\oplus$

In a recent paper [3], we showed that when two knowledge bases K and K' satisfy a certain condition of *independence*, it is possible to extend their union $K \cup K'$ through the ranking function $\kappa_\oplus$ defined by $\kappa_\oplus(m) = \kappa(m) + \kappa'(m)$, where κ and κ' are the $\mathbb{Z}$-rankings associated with the rational closures of K and K'. The motivations and the detailed presentation of this new ranking system can be found in [3], but they are not necessary to understand what follows. We begin with a simple example:

Let $K = \{b \Rightarrow f, p \Rightarrow b, p \Rightarrow \neg f\}$ and $K' = \{p \wedge b \Rightarrow \neg f\}$. These bases are independent, and the ranking $\kappa_\oplus$ is as follows:

$$\begin{array}{lll} & pbf & 3 \\ p & pf & 2 \\ b & pb & 1 \\ * \ f & bf & 0 \end{array}$$

After simplification, the characteristic set of $\kappa_\oplus$ is equal, modulo classical equivalence, to the set $\{\neg p \vee \neg b \vee \neg f, (\neg p \vee b) \wedge (\neg p \vee \neg f), \neg p \wedge (\neg b \vee f), true\}$.

The LM-sequence of K is $K_1 = \neg p \wedge (\neg b \vee f)$, $K_2 = (\neg p \vee b) \wedge (\neg p \vee \neg f)$, $K_3 = \emptyset$; the LM-sequence of K' is $K'_1 = \neg p \vee \neg b \vee \neg f$, $K'_2 = \emptyset$. If we set $K_i = \emptyset$ for $i \geqslant 3$ and $K'_j = \emptyset$ for $j \geqslant 2$, we see that Ψ consists of the representatives of the set

$$\{K_1^* \wedge K_1'^*, (K_1^* \wedge K_2'^*) \vee (K_2^* \wedge K_1'^*), (K_1^* \wedge K_3'^*) \vee (K_2^* \wedge K_2'^*) \vee (K_3^* \wedge K_1'^*)\}.$$

In the general case, the characteristic set of $\kappa_\oplus$ is given by the following:

Theorem 26. *Let K and K' be two knowledge bases with associated LM-sequences K_i and K'_j. Denote by κ and κ' the rankings of their rational closures and by h and h' their respective heights. Let $\kappa_\oplus$ be the ranking defined, for every world m, by $\kappa_\oplus(m) = \kappa(m) + \kappa(m')$. Then the characteristic set Ψ of $\kappa_\oplus$ is equal, modulo classical equivalence, to the set of all formulae $\bigvee_{i+j=l}(K_i^* \wedge K_j'^*)$, where l is any integer $1 \leqslant l \leqslant h + h'$ and the sets K_i^*, $i > h$, and $K_j'^*$, $j > h'$, are empty.*

Proof. It is clear that the height of $\kappa_\oplus$, hence the number of elements of Ψ, is equal to $h+h'$. Therefore, it is enough to prove that any formula of the form $\bigvee_{i+j=l}(K_i^* \wedge K_j'^*)$ is an element of Ψ modulo classical equivalence. Let α be such a formula, and m a world that satisfies α. Then there exists two integers i and j, $i + j = l$, such that m satisfies K_i^* and K_j^*. We have therefore $\kappa(m) < i$ and $\kappa'(m) < j$. It follows that $\kappa_\oplus(m) < l$. Conversely, let m be a world such that $\kappa_\oplus(m) < l$. If we denote by i the κ-rank of m and by j its κ'-rank, we see that m satisfies the formula $K_i^* \wedge K'_j$, and one has $i+j < l$. We have therefore proven that a world m satisfies the formula $\bigvee_{i+j=l}(K_i^* \wedge K_j'^*)$ if and only if $\kappa_\oplus(m) < l$. It follows that, modulo classical equivalence, the formula $\bigvee_{i+j=l}(K_i^* \wedge K_j'^*)$ is an element of Ψ, and the proof of the theorem is complete. □

6. Conclusion

The main result of this paper is the link that exists between faithfully representable preferential inference relations and inference relations associated with Poole systems without constraints. These notions turn out to be equivalent in the case of logically finite languages, but this result is not anymore valid for arbitrary propositional languages, and we do not know how to characterize, in the general case, those relations that may be associated with such a Poole system. Even in the finite case, there still exists some fundamental open questions concerning the link between a set of defaults and its induced inference relation: we do not know, for instance, what properties satisfies a set D whose induce relation $\mid\!\sim_D$ is rational. We know that if D is linearly ordered through classical implication, its induced inference relation is rational, but this strong condition is clearly not a necessary one, and, for the present, we have no way of directly deciding whether the order $<_D$ induced by a subset D of $\mathcal{L}$ is or not modular.

The results that we have established concerning the determinant associated with a preferential inference relations, or the characteristic set associated with a rational relation, nevertheless provide an interesting tool for the study of these relations. They show that arbitrary orderings on W may be replaced by some more familiar orderings, closely related to the inclusion order. It is doubtless that these notions of determinant and characteristic set will play a prominent role in any future work concerning the general study of faithfully representable inference relations or the dynamics of Poole systems.

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