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A causal approach to nonmonotonic reasoning

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Abstract

We introduce logical formalisms of production and causal inference relations based on input/output logics of Makinson and Van der Torre [J. Philos. Logic 29 (2000) 383–408]. These inference relations will be assigned, however, both standard semantics (giving interpretation to their rules), and natural nonmonotonic semantics based on the principle of explanation closure. The resulting nonmonotonic formalisms will be shown to provide a logical representation of abductive reasoning, and a complete characterization of causal nonmonotonic reasoning from McCain and Turner [Proc. AAAI-97, Providence, RI, 1997, pp. 460–465]. The results of the study suggest production and causal inference as general nonmonotonic formalisms providing an alternative representation for a significant part of nonmonotonic reasoning.

Keywords: Nonmonotonic reasoning; Causality; Abduction; Reasoning about action and change

1. Introduction

The field of nonmonotonic reasoning is so abundant with different formalisms, that an attempt to introduce and justify yet another one appears to be doomed from the very beginning. Nevertheless, this is precisely the main aim of this study.[1] Accordingly, we have to explain, first of all, what was the problem such that the new formalism is the suggested solution.

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[1] A preliminary version of this paper has appeared as [7].

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To begin with, studies in nonmonotonic reasoning have given rise to two basically different approaches that could be called, respectively, *preferential* and *explanatory* nonmonotonic reasoning, with little interaction between them.[2] The first approach encompasses nonmonotonic inference relations of [20], as well as a general theory of belief change. A detailed description of this approach can be found in [6]. The second approach includes various default and modal nonmonotonic logics, as well as logic programming. In fact, all the papers in the famous 1980 issue of the Artificial Intelligence on Nonmonotonic Reasoning could be seen as belonging to this latter camp (though McCarthy's circumscription is expressible also by the first approach). The formalism suggested in the present paper will also belong to the explanatory approach.

The above two approaches reflect, respectively, two different senses in which a logical system can be nonmonotonic. First, its rules may not admit addition of new premises, that is, they do not satisfy Strengthening the Antecedent. Second, adding further rules to the system may possibly invalidate earlier conclusions. These two kinds of nonmonotonicity are largely independent. Thus, preferential inference relations are nonmonotonic in the first sense, but monotonic in the second sense, since addition of new conditionals does not invalidate previous derivations. On the other hand, default logic exemplifies monotonicity of the first kind and nonmonotonicity of the second kind. Default rules freely admit strengthening of their pre-requisites and justifications, since this does not change the set of extensions (see [5]). However, adding arbitrary new rules to a default theory may create new extensions, so the nonmonotonic conclusions made earlier will not, in general, be preserved.

We believe that nonmonotonic reasoning should give us a more direct and adequate description of the actual ways we think about the world than, say, the classical logic. In this respect, the preferential nonmonotonic reasoning has a definite advantage over existing explanatory counterparts in that it provides a direct semantic representation for its main nonmonotonic objects, namely default conditionals "If A, then normally B". This semantic representation allows us to assess our default claims and determines, ultimately, the actual choice of default assumptions made in particular circumstances. Default logic and its relatives take a different, less direct, route to assessing what can be inferred from a given set of default rules. Namely, they require from the user to provide an explicit information about when one default can 'block' another default. This information is used as a sole factor in determining acceptable combinations of defaults. This strategy can be remarkably successful in resolving difficult cases of default interaction, which can be seen as the main reason why the explanatory nonmonotonic reasoning so far has had a greater impact on practical applications of nonmonotonic reasoning in AI. Still, the explanatory approach remains largely syntactic in nature, and does not give us a transparent and systematic way of representing empirical data. More precisely, due to the fact that default *rules* do not have a direct semantic interpretation, the task of knowledge representation in these formalisms becomes really an art rather than a systematic methodology.[3]

[2] They have been called, respectively, *classical* and *argumentative* nonmonotonic reasoning in [6].

[3] As was rightly mentioned by the reviewer, this is actually the problem with many other KR formalisms as well.

An additional, more specific, problem with the explanatory approach consists in the epistemic understanding of default rules it presupposes. A distinctive feature of both default and modal nonmonotonic logics is that they are inherently *epistemic* formalisms. Namely, they are essentially based on such notions as belief and knowledge, unlike the extensional classical logic used for a direct representation of facts about the world. Accordingly, the intended semantic models of these formalisms represent (possibly incomplete) sets of beliefs one can have, while their rules allow to make inferences based on absence of belief, or consistency, with respect to candidate belief sets (cf. the introductory sections of [33]). It seems that this modal formulation of many nonmonotonic formalisms is mainly due to historical reasons: at the time these formalisms have emerged, modal logics already reigned in the literature as a standard paradigm of logical representation. The epistemic interpretation strongly influenced also logic programming in that the negation as failure has often been formulated as absence of knowledge (or derivation).

Due to its modal character, default logic has turned out to be a logically weak formalism that does not support many classical inference principles (such as reasoning by cases). It has also other well-known shortcomings, and numerous variants of default logic have been suggested in attempts to overcome them and make it more in accord with our intuitions. On our opinion, however, a relatively modest success of these attempts has shown that it is impossible to radically improve default logic without abandoning its underlying epistemic interpretation.

The shortcomings of default logic became especially vivid in attempts to apply it to one of the primary application fields of nonmonotonic reasoning, a formal representation of actions and change. It was realized quite early that classical logic alone cannot provide an efficient representation for reasoning about change due to the famous *frame problem* [30]—a problem of giving an efficient description for the state of the world after performing an action that would avoid computationally unbearable reproduction of all the facts that remain unaffected. There was also a related *ramification problem* of determining the indirect effects (ramifications) of actions that arise due to the laws of the domain. And it was only natural to expect that nonmonotonic reasoning should help in resolving these problems.

After some less successful attempts to formalize temporal nonmonotonic reasoning in existing nonmonotonic logics, a dominant recent approach to solving these problems has been based on causal reasoning. Given a set of action and causal rules describing the domain, the causal approach employs a distinction between facts that hold in a situation versus facts that are caused (explained) by other facts and the rules. The corresponding *explanation closure assumption* amounts to a requirement that all facts that hold in a situation should be either caused by other occurrent facts, or else preserve their truth-values in time (due to the associated *inertia assumption*). A natural formalization of these principles has been given in the framework of causal theories, introduced in [26]. A causal theory is a set of causal rules that express causal (or explanatory) relations among propositions. The nonmonotonic semantics of such theories is determined by causally explained models, namely the models that both satisfy the causal rules and such that every fact holding in them is explained by some causal rule. The resulting nonmonotonic formalism has been shown to provide a plausible and efficient solution for both the frame and ramification problem (see [17,21,37] for a detailed exposition and applications in representing action domains). Related causal approaches to representing actions and change have been suggested in [23,36, 38], to mention only a few.

From the point of view of the present study, the causal reasoning constitutes an important conceptual shift in the general framework of explanatory nonmonotonic reasoning, since it is based on a direct and transparent description of factual and causal (explanatory) information about the world. In other words, it shows that the epistemic view of nonmonotonic reasoning is not the only possibility. Accordingly, the primary aim of our study will consist in laying down logical foundations for this kind of nonmonotonic reasoning. As we will see, the resulting nonmonotonic formalism will form a most natural and immediate generalization of classical logic that allows for nonmonotonic reasoning. We will try to demonstrate also that the suggested framework constitutes a general-purpose formalism that covers significant parts of general nonmonotonic reasoning. In this respect, it forms a viable non-epistemic alternative to default and modal nonmonotonic logics.

The logical foundations of the suggested formalism will be built in the framework of an inference system for causal rules, called *production inference relations,* that originates in input/output logics of [25]. These logics will be assigned below two semantics. The first is an ordinary logical (monotonic) semantics that will give a semantic interpretation for the causal rules. The second is a natural nonmonotonic semantics, which gives rise to a new kind of nonmonotonic reasoning.

As a first general application, we describe a special kind of *abductive* production inference relations that provide a logical, syntax-independent representation of abductive reasoning. It will be shown, in particular, that in many regular cases such inference relations will be sufficient for describing the nonmonotonic semantics of causal theories.

As a next step, we will describe a particular kind of production inference, called *causal inference relations*. Such inference relations will already admit an objective interpretation as reasoning systems about the world. Accordingly, the nonmonotonic semantics for such inference relations will also be based on worlds. It will be shown that the resulting nonmonotonic formalism provides both an adequate and complete characterization of the reasoning with causal theories of [26]. It will be shown also that any causal theory is reducible with respect to this nonmonotonic semantics to a *determinate* causal theory that contain only Horn causal rules $A \Rightarrow l$, where l is a literal. The importance of determinate theories lies in the fact, established already in [26], that the explained interpretations of such a theory are precisely the interpretations of its classical *completion*. Consequently, the nonmonotonic consequences of such causal theories are obtainable by the use of classical inference tools.

We will assume in this paper that our underlying language is an ordinary classical propositional language with the usual connectives and constants $\{\wedge, \vee, \neg, \rightarrow, \mathbf{t}, \mathbf{f}\}$. In addition, $\vDash$ and Th will stand, respectively, for the classical entailment and the associated logical closure operator. We will reserve also the letters $p, g, r, \ldots$ for denoting propositional atoms, while $A, B, C, \ldots$ will denote arbitrary classical propositions of the language.

For a finite set a of propositions, $\bigwedge a$ and $\bigvee a$ will denote, respectively, the conjunction and disjunction of all propositions from a. As a special case, $\bigwedge \emptyset$ will denote $\mathbf{t}$, while $\bigvee \emptyset$ will denote $\mathbf{f}$.

In what follows, we will use also a few facts about ordinary Tarski consequence relations based on the classical language (see, e.g., [6]). Such a consequence relation $\vdash$ is called *supraclassical*, if it subsumes the classical entailment, that is, $\vDash \subseteq \vdash$. The sets of premises in the rules of a supraclassical consequence relation can be replaced by their conjunctions, that is, $a \vdash A$ is equivalent to $\bigwedge a \vdash A$. This means that such consequence relations can be represented as binary relations on the set of classical propositions, namely as relations satisfying, for instance, the following postulates:

Dominance If $A \vDash B$, then $A \vdash B$.
Transitivity If $A \vdash B$ and $B \vdash C$, then $A \vdash C$.
And If $A \vdash B$ and $A \vdash C$, then $A \vdash B \wedge C$.

A consequence relation will be called *classical*, if it is supraclassical and satisfies one of the following postulates:

Deduction If $A \wedge B \vdash C$, then $A \vdash B \to C$;
Or If $A \vdash C$ and $B \vdash C$, then $A \vee B \vdash C$.

In a classical consequence relation, any rule $A \vdash B$ is reducible to an axiom $\vdash A \to B$, so it can be viewed as a classical entailment augmented with a set of auxiliary axioms.

2. Production inference

We will begin with a general notion of production inference that will be just a slight modification of input–output logic from [25].

Definition 2.1. A *production inference relation* is a binary relation $\Rightarrow$ on the set of classical propositions satisfying the following conditions:

(Strengthening) If $A \vDash B$ and $B \Rightarrow C$, then $A \Rightarrow C$;
(Weakening) If $A \Rightarrow B$ and $B \vDash C$, then $A \Rightarrow C$;
(And) If $A \Rightarrow B$ and $A \Rightarrow C$, then $A \Rightarrow B \wedge C$;
(Truth) $\mathbf{t} \Rightarrow \mathbf{t}$;
(Falsity) $\mathbf{f} \Rightarrow \mathbf{f}$.

Conditionals of the form $A \Rightarrow B$ will be called *production rules*. A characteristic property of production inference is that reflexivity $A \Rightarrow A$ does not necessarily hold.

Production inference relations almost coincide with input-output logics, except for the last postulate, Falsity. The latter allows to restrict the universe of discourse to classically consistent theories. This will give us more smooth semantic characterizations, and some additional important properties. Our description of different kinds of production relations, however, will closely follow the classification of input-output logics given in [25].

We will extend production rules to rules having arbitrary sets of propositions as premises using the familiar compactness recipe: for any set u of propositions, we define $u \Rightarrow A$ as follows:

$$u \Rightarrow A \equiv \bigwedge a \Rightarrow A, \quad \text{for some finite } a \subseteq u$$

$\mathcal{C}(u)$ will denote the set of propositions ‘produced’ by u, that is

$$\mathcal{C}(u) = \{A \mid u \Rightarrow A\}.$$

As could be expected, the production operator $\mathcal{C}$ will play much the same role as the usual derivability operator for consequence relations. Note first that $\mathcal{C}(u)$ is always a deductively closed set:

Lemma 2.1. *For any set u of propositions, $\mathcal{C}(u)$ is a deductively closed set.*

Proof. If $A \in \mathcal{C}(u)$ and $A \vDash B$, then $\bigwedge a \Rightarrow A$, for some $a \subseteq u$, and hence $\bigwedge a \Rightarrow B$ by Weakening, that is, $B \in \mathcal{C}(u)$. In addition, if $A, B \in \mathcal{C}(u)$, then $\bigwedge a \Rightarrow A$ and $\bigwedge b \Rightarrow B$, for some $a, b \subseteq u$. Consequently, $\bigwedge(a \cup b) \Rightarrow A \wedge B$ by Strengthening and And, and therefore $A \wedge B \in \mathcal{C}(u)$. □

In addition, $\mathcal{C}$ satisfies the following basic property:

Monotonicity If $u \subseteq v$, then $\mathcal{C}(u) \subseteq \mathcal{C}(v)$.

Actually, due to compactness, $\mathcal{C}$ is not only monotonic, but also a *continuous* operator: $\mathcal{C}(\bigcup u_i) = \bigcup \mathcal{C}(u_i)$, for any chain $\{u_i \mid i \in I\}$ totally ordered with respect to set inclusion. Still, $\mathcal{C}$ is not inclusive, that is, $u \subseteq \mathcal{C}(u)$ does not always hold. Also, it is not idempotent, that is, $\mathcal{C}(\mathcal{C}(u))$ can be distinct from $\mathcal{C}(u)$.

The following fact will be used in proving completeness theorems for different kinds of production inference.

Lemma 2.2. *If $A \notin \mathcal{C}(u)$, then there exists a consistent deductively closed set v such that $u \subseteq v$, $A \notin \mathcal{C}(v)$, and $A \in \mathcal{C}(v')$, for any $v' \supset v$.*

Proof. If $\mathcal{U}$ is an inclusion-ordered family of sets that do not produce A, then a usual compactness argument shows that $\bigcup \mathcal{U}$ also does not produce A. Consequently, if $A \notin \mathcal{C}(u)$, then u is included in some maximal set v that does not produce A. Suppose now that v is not deductively closed, that is, $v \vDash B$, for some $B \notin v$. Then $A \in \mathcal{C}(v \cup \{B\})$ due to maximality of v, and hence $V \wedge B \Rightarrow A$, where V is a conjunction of some propositions from v. But v implies $V \wedge B$, and hence $V_0 \vDash V \wedge B$, where V_0 is again a conjunction of some propositions from v. Then $V_0 \Rightarrow A$ holds by Strengthening, and therefore $A \in \mathcal{C}(v)$—a contradiction. Consequently, v is a deductively closed set. Moreover, $\mathbf{f} \notin v$, since $\mathbf{f} \Rightarrow A$ by Falsity. Therefore, v is also consistent. □

The reader should note that the theory v in the formulation of the above lemma need not be a world, that is, a complete deductively closed set.

2.1. Semantics

We will describe now a general semantic framework for production relations. Our basic semantic object will be a pair of deductively closed theories that will be called a bimodel. In accordance with the 'input–output' understanding of productions, a bimodel will represent an initial state (input) and a possible final state (output) of a production derivation based on a given set of production rules. The set of such bimodels will give a semantic description for these production rules.

Definition 2.2. A pair of consistent deductively closed sets will be called a *bimodel*. A set of bimodels will be called a *production semantics*.

Bimodels have been defined above in a syntactic fashion, namely as pairs of theories. This formulation will make subsequent constructions simpler and more transparent. Still, any bimodel (u, v) can be safely identified with a pair of sets of worlds (or propositional interpretations):

$$(u, v) \equiv \bigl(\{\alpha \mid u \text{ is valid in } \alpha\}, \{\beta \mid v \text{ is valid in } \beta\}\bigr).$$

All our subsequent constructions will permit such a purely semantic reformulation. Note that a production semantics can also be viewed as a *binary relation* on the set of deductive theories.

Now we will define the notion of validity of production rules with respect to a production semantics.

Definition 2.3. A production rule $A \Rightarrow B$ will be said to be *valid* in a production semantics $\mathcal{B}$ if, for any bimodel (u, v) from $\mathcal{B}$, $A \in u$ only if $B \in v$.

We will denote by $\Rightarrow_{\mathcal{B}}$ the set of all production rules that are valid in a semantics $\mathcal{B}$. It can be easily verified that this set satisfies all the postulates for production relations, and hence we have

Lemma 2.3. *For any production semantics $\mathcal{B}$, $\Rightarrow_{\mathcal{B}}$ is a production relation.*

In order to prove completeness, for any production relation $\Rightarrow$ we will construct its *canonical semantics* $\mathcal{B}_{\Rightarrow}$ as the set of all bimodels of the form $(w, \mathcal{C}(w))$, where w is an arbitrary consistent and deductively closed set. Then the following result is actually a representation theorem showing that this semantics is *strongly complete* for the source production relation.

Theorem 2.4. *If $\mathcal{B}_{\Rightarrow}$ is the canonical semantics for a production relation $\Rightarrow$, then, for any set of propositions u and any A,*

$$u \Rightarrow A \quad \textit{iff} \quad A \in v, \ \textit{for any bimodel } (w, v) \in \mathcal{B}_{\Rightarrow} \textit{ such that } u \subseteq w.$$

Proof. If $u \Rightarrow A$ and $u \subseteq w$, for some deductively closed theory w, then clearly $A \in \mathcal{C}(w)$. In the other direction, if $u \not\Rightarrow A$, then by Lemma 2.2, u is included in some theory v such that $v \not\Rightarrow A$. Clearly, $(v, \mathcal{C}(v))$ is a bimodel from $\mathcal{B}_{\Rightarrow}$ such that $u \subseteq v$ and $A \notin \mathcal{C}(v)$. This completes the proof. $\square$

As an immediate consequence of the above results, we obtain the basic completeness result:

Corollary 2.5. *A binary relation $\Rightarrow$ on the set of propositions is a production inference relation if and only if it is determined by a production semantics.*

2.2. Causal theories

In what follows, by a *causal theory* we will mean an arbitrary set of production rules. Since all the postulates for production relations are Horn ones (that is, they have non-disjunctive conclusions), intersection of production relations is again a production relation, and hence for any causal theory Δ there exists a least production relation that includes Δ. We will denote it by $\Rightarrow_\Delta$, while $\mathcal{C}_\Delta$ will denote the corresponding production operator. Clearly, $\Rightarrow_\Delta$ is the set of all production rules that can be derived from Δ using the postulates for production relations.

For a set u of propositions and a causal theory Δ, we will denote by $\Delta(u)$ the set of all propositions that are directly produced from u by Δ, that is,

$$\Delta(u) = \{A \mid B \Rightarrow A \in \Delta,\ \text{for some } B \in u\}.$$

In what follows we will use the following explicit description of $\Rightarrow_\Delta$, given, in effect, in [25]:

Proposition 2.6. $\mathcal{C}_\Delta(u) = \mathrm{Th}(\Delta(\mathrm{Th}(u)))$.

3. Regular production inference

A production inference relation will be called *regular* if it satisfies the following well-known rule:

(Cut) If $A \Rightarrow B$ and $A \wedge B \Rightarrow C$, then $A \Rightarrow C$.

Cut is one of the basic rules for ordinary consequence relations. In the context of production inference it plays the same role, namely, allows for a reuse of produced propositions as premises in further productions.[4] It corresponds to the following characteristic condition on the production operator:

$$\mathcal{C}\big(u \cup \mathcal{C}(u)\big) \subseteq \mathcal{C}(u).$$

Regular production relations have a number of additional properties. Thus, any such relation will already be transitive, that is, it will satisfy

(Transitivity) If $A \Rightarrow B$ and $B \Rightarrow C$, then $A \Rightarrow C$.

[4] Such production relations correspond to input–output logics with reusable output in [25].

Note, however, that Transitivity is a weaker postulate than Cut, since it does not imply the latter (see below).

A production rule of the form $A \Rightarrow \mathbf{f}$ will be called a *constraint* in what follows. Such rules can be used for incorporating a purely factual information into causal theories: a rule $A \Rightarrow \mathbf{f}$ says, in a sense, that A is production- or explanatory inconsistent, and hence it should not hold in any intended model.

Now, an important property of regular relations is that any production rule implies the corresponding constraint:[5]

(Constraint) If $A \Rightarrow B$, then $A \wedge \neg B \Rightarrow \mathbf{f}$.

(Indeed, if $A \Rightarrow B$, then $A \wedge \neg B \Rightarrow B$ by Strengthening and $A \wedge B \wedge \neg B \Rightarrow \mathbf{f}$ by Falsity. Hence $A \wedge \neg B \Rightarrow \mathbf{f}$ by Cut.[6])

In particular, we have that $\mathbf{t} \Rightarrow A$ implies $\neg A \Rightarrow \mathbf{f}$. Note, however, that the reverse entailment does not hold even in the latter special case.

As a special case of Constraint, we have also the rule

(Coherence) If $A \Rightarrow \neg A$, then $A \Rightarrow \mathbf{f}$

that says that if a proposition produces propositions that are incompatible with it, then it is explanatory inconsistent. Actually, Coherence turns out to be equivalent to Constraint for all production inference relations (see Lemma 7.1 below).

Remark. Regular production inference sanctions, in effect, an *atemporal* understanding of the notion of production. For example, the rule $p \wedge q \Rightarrow \neg q$ cannot be understood as saying that p and q jointly produce $\neg q$ (afterwards) in a temporal sense; instead, by Coherence it implies $p \wedge q \Rightarrow \mathbf{f}$, which means, in effect, that $p \wedge q$ cannot hold. Speaking generally, production-consistent propositions cannot produce a result incompatible with them. Just as in classical logic, however, a representation of temporal domains in this formalism can be obtained by adding explicit temporal arguments to propositions; this is what has been actually done in [26].

Regular production relations can already be described in terms of a usual notion of a theory.

Definition 3.1. A set u of propositions will be called a *theory* of a production relation, if it is deductively closed, and $\mathcal{C}(u) \subseteq u$.

A theory of a production relation is closed with respect to its rules: if $A \in u$ and $A \Rightarrow B$, then $B \in u$. Accordingly, such theories have much the same properties as ordinary theories of consequence relations. Thus, theories that are worlds have a very simple characterization:

[5] Cf. Corollary 1 in [17].

[6] Notice that the use of Falsity is essential here.

Lemma 3.1. *A world α is a theory of a regular production relation if and only if $\alpha \not\Rightarrow \mathbf{f}$.*

Proof. Clearly, if $\alpha \Rightarrow \mathbf{f}$, then $\mathcal{C}(\alpha)$ is an inconsistent theory, and hence $\mathcal{C}(\alpha) \subseteq \alpha$ cannot hold. Assume now that a world α is not a theory. Then $A, \neg B \in \alpha$, for some propositions A, B such that $A \Rightarrow B$. But then we have also $A \wedge \neg B \Rightarrow \mathbf{f}$ by Constraint, and therefore $\alpha \Rightarrow \mathbf{f}$. □

Theories of a production relation will determine its canonical semantics, described in the proof of the completeness theorem below. Notice, however, that the production relation will be determined not only by what its theories are, but also by what they produce.

The semantic characterization of regular production relations can be obtained by considering only bimodels (u, v) such that $v \subseteq u$. We will call such bimodels (and the corresponding semantics) *inclusive* ones.

Theorem 3.2. $\Rightarrow$ *is a regular production relation if and only if it is generated by an inclusive production semantics.*

Proof. It is easy to check that the production relation generated by an inclusive semantics satisfies Cut. In the other direction, for a regular production relation $\Rightarrow$, we will construct a canonical semantics as the set of bimodels of the form $(w, \mathcal{C}(w))$, where w is a theory of $\Rightarrow$. Clearly, if $u \Rightarrow A$ and $u \subseteq w$, then $A \in \mathcal{C}(w)$. In the other direction, if $u \not\Rightarrow A$, then by Lemma 2.2, u is included in some maximal deductively closed set w such that $w \not\Rightarrow A$. Assume that $w \Rightarrow B$, for some B, though $B \notin w$. Due to maximality of w, we have $w \cup \{B\} \Rightarrow A$. Since w is deductively closed, there are propositions $C, D \in w$ such that $C \Rightarrow B$ and $D \wedge B \Rightarrow A$. But then by Strengthening and Cut we obtain $C \wedge D \Rightarrow A$, contrary to our assumption that $w \not\Rightarrow A$. Hence w is a theory of $\Rightarrow$, and therefore $(w, \mathcal{C}(w))$ is an inclusive bimodel that invalidates $u \Rightarrow A$. This completes the proof. □

We will denote by $\Rightarrow^r_\Delta$ the least regular production relation containing a causal theory Δ, while $\mathcal{C}^r_\Delta$ will denote the corresponding production operator.

A set u of propositions will be called a *Δ-theory*, if it is closed both deductively and with respect to the rules from Δ. As can be anticipated, Δ-theories are precisely the theories of $\Rightarrow^r_\Delta$. The following proposition provides a constructive description of this production relation, obtained in [25].

Proposition 3.3 [25].

$$v \Rightarrow^r_\Delta A \quad \textit{iff} \quad A \in \mathrm{Th}\big(\Delta(u)\big), \ \textit{for any } \Delta\textit{-theory } u \supseteq v.$$

If we denote by $Cl(v)$ the least Δ-theory containing v, we immediately obtain a simpler characterization (that will be used later):

Corollary 3.4. $\mathcal{C}^r_\Delta(v) = \mathrm{Th}(\Delta(Cl(v)))$.

As a final result, we will show that regular production relations allow to define an appropriate notion of equivalence among propositions such that equivalent propositions would be substitutable in any production rule. Namely, let us say that propositions A and B are *production-equivalent* with respect to a production inference relation, if $\mathbf{t} \Rightarrow (A \leftrightarrow B)$ holds. Then we have

Lemma 3.5. *Propositions A and B are production-equivalent with respect to a regular production relation $\Rightarrow$ if and only if any occurrence of A can be replaced by B in the rules of $\Rightarrow$.*

Proof. If A can be replaced by B in any rule of $\Rightarrow$, then it can be replaced also in $\mathbf{t} \Rightarrow (A \leftrightarrow A)$, which holds by Truth. Hence, $\mathbf{t} \Rightarrow (A \leftrightarrow B)$ holds in $\Rightarrow$.

Let X be a propositional formula. We will denote by $X(A/B)$ an arbitrary formula obtained from X by replacing some of the occurrences of A in it by B. Clearly, $A \leftrightarrow B \vDash X \leftrightarrow X(A/B)$. Assume now that A and B are production-equivalent, and $X \Rightarrow Y$. Then $X \Rightarrow (A \leftrightarrow B)$ by Strengthening, and hence $X \Rightarrow (Y \leftrightarrow Y(A/B))$ by Weakening. Consequently, $X \Rightarrow Y(A/B)$ by And and Weakening. Thus, B can replace A in the heads of the rules from $\Rightarrow$. In addition, we have $X(A/B), A \leftrightarrow B \vDash X$, and therefore $X(A/B) \wedge (A \leftrightarrow B) \Rightarrow Y$ by Strengthening. But we have also $X(A/B) \Rightarrow (A \leftrightarrow B)$, so we can apply Cut and obtain $X(A/B) \Rightarrow Y$. This shows that A can be replaced by B also in the bodies of the rules from $\Rightarrow$. $\square$

Due to the above result, production-equivalence can be used, in particular, to describe *definitional extensions* of the underlying language with new propositions (cf. [37]).

4. General nonmonotonic semantics

In the preceding sections, we have given a formalization and standard (monotonic) semantics for a logical system of production inference. It turns out, however, that production inference relations determine also a natural nonmonotonic semantics, and provide thereby a logical basis for a particular form of nonmonotonic reasoning. Namely, the fact that the production operator $\mathcal{C}$ is not reflexive creates an important distinction among theories of a production relation.

Definition 4.1.

- A theory u of a production inference relation will be called *exact*, if it is consistent, and $u = \mathcal{C}(u)$.
- A set u of propositions is an *exact theory of a causal theory Δ*, if it is an exact theory of $\Rightarrow_\Delta$.

An exact theory describes an informational state in which every proposition is produced, or *explained*, by other propositions accepted in this state. Accordingly, restricting our universe of discourse to exact theories amounts to imposing a kind of an *explanation closure assumption* on a production relation. Namely, it amounts to requiring that any accepted proposition should also have reason, or explanation, for its acceptance. This suggests the following notion of a nonmonotonic semantics:

Definition 4.2. A *general nonmonotonic semantics* of a production inference relation (or a causal theory) is the set of all its exact theories.

The general nonmonotonic semantics, as defined above, is a certain set of propositional theories. This abstract definition still leaves us much freedom in determining what can be seen as *nonmonotonic consequences* of a causal theory. The most obvious choice consists in taking propositions that belong to all exact theories. As we will see, however, this means that we accept only propositions that belong to the least exact theory. But we can also choose propositions that belongs to all *maximal* exact theories. Similarly, we can consider all propositions that do not belong to any such theory as nonmonotonically *rejected*. All these options determine common variants of a *cautious* (sceptical) nonmonotonic semantics. A *credulous* nonmonotonic semantics can be defined by considering propositions that belong (or not belong) to at least one exact theory. As studies in nonmonotonic reasoning show, each of these options could be appropriate for particular reasoning tasks. Speaking generally, all the information that can be discerned from the general nonmonotonic semantics of a causal theory, or a production inference relation, can be seen as nonmonotonically implied by the latter.

The general nonmonotonic semantics for causal theories is indeed nonmonotonic in the sense that adding new rules to the production relation may lead to a nonmonotonic change of the associated semantics, and thereby to a nonmonotonic change in the derived information (a number of examples will be presented below). This happens even though production rules themselves are monotonic, since they satisfy Strengthening (the Antecedent).

Exact theories are precisely fixed points of the production operator $\mathcal{C}$. Since the latter operator is monotonic and continuous, exact theories (and hence the nonmonotonic semantics) always exist. Moreover, there always exists a least exact theory. For regular inference relations, the least exact theory coincides with $\mathcal{C}(\mathbf{t})$, that is, with the set of propositions that are produced by tautologies. In addition, the union of any chain of exact theories (with respect to set inclusion) is an exact theory, so any exact theory is included in a maximal such theory.

A useful property of exact theories for regular production relations is described in the next lemma.

Lemma 4.1. *If $\Rightarrow$ is a regular production relation, and $u \subseteq \mathcal{C}(u)$, then $\mathcal{C}(u)$ is an exact theory of $\Rightarrow$.*

Proof. If $u \subseteq \mathcal{C}(u)$, then $\mathcal{C}(u) \subseteq \mathcal{C}(\mathcal{C}(u))$ by monotonicity of $\mathcal{C}$. In addition, the characteristic property of regular productions $\mathcal{C}(u \cup \mathcal{C}(u)) \subseteq \mathcal{C}(u)$ implies in our case $\mathcal{C}(\mathcal{C}(u)) \subseteq \mathcal{C}(u)$, and therefore $\mathcal{C}(u) = \mathcal{C}(\mathcal{C}(u))$. □

Unfortunately, the following example shows that exact theories are not closed with respect to arbitrary intersections.

Example. Let us consider a causal theory $\Delta = \{p_i \Rightarrow p_i, p_i \Rightarrow q \mid i \geqslant 0\}$. Then, for any natural n, the set $u_n = \mathrm{Th}(q, p_n)$ is an exact theory of $\Rightarrow_\Delta$. However, $\bigcap u_n = \mathrm{Th}(q)$ is not an exact theory of $\Rightarrow_\Delta$.

As a result, a least exact theory containing a given set of propositions does not always exist.

The following simple lemma gives a constructive description of the general nonmonotonic semantics of a causal theory. The proof is immediate by Proposition 2.6.

Lemma 4.2. *u is an exact theory of a causal theory Δ iff $u = \mathrm{Th}(\Delta(u))$.*

We are going to show now that regular production inference provides an adequate and maximal logical framework for reasoning with exact theories.

Definition 4.3. Two causal theories will be called *nonmonotonically equivalent* if they have the same general nonmonotonic semantics.

Note that production inference relations can also be considered as (rather big) causal theories. As before, let $\Rightarrow^r_\Delta$ denote the least regular production relation that contains a causal theory Δ. Then we have

Lemma 4.3. *Any causal theory Δ is nonmonotonically equivalent to $\Rightarrow^r_\Delta$.*

Proof. If v is a Δ-theory, then $v = Cl(v)$, and hence $\mathrm{Th}(\Delta(v)) = \mathrm{Th}(\Delta(Cl(v)))$. Consequently, $v = \mathrm{Th}(\Delta(v))$ iff $v = \mathrm{Th}(\Delta(Cl(v)))$. By Corollary 3.4, this implies that v is an exact theory of Δ if and only if it is an exact theory of $\Rightarrow^r_\Delta$. $\square$

The above lemma implies that the postulates of regular inference are adequate for reasoning with exact theories, since they preserve the latter. Moreover, we will show now that regular inference relations constitute a maximal logic suitable for the general nonmonotonic semantics.

Let us say that causal theories Δ and Γ are *regularly equivalent*, if each can be obtained from the other using the postulates of regular production inference. Or, equivalently, when $\Rightarrow^r_\Delta = \Rightarrow^r_\Gamma$. Now, as an immediate consequence of Theorem 4.3, we obtain

Corollary 4.4. *Regularly equivalent theories are nonmonotonically equivalent.*

The reverse implication in the above corollary does not hold, and a deep reason for this is that the regular equivalence is a monotonic (logical) notion, and hence, unlike the nonmonotonic equivalence, it is preserved under addition of new causal rules. What we need, therefore, is a stronger, monotonic counterpart of the notion of nonmonotonic equivalence that would be preserved under addition of new causal rules. This immediately suggests the following definition.

Definition 4.4. Two causal theories Δ and Γ will be said to be *strongly equivalent* if, for any set Φ of causal rules, $\Delta \cup \Phi$ is nonmonotonically equivalent to $\Gamma \cup \Phi$.

Strongly equivalent theories are 'equivalent forever', that is, they are interchangeable in any larger causal theory without changing the general nonmonotonic semantics. Consequently, strong equivalence can be seen as an equivalence with respect to the background monotonic logic of causal theories. And the next result shows that this logic is precisely the logic of regular production relations.

Theorem 4.5. *Two causal theories are strongly equivalent if and only if they are regularly equivalent.*

Proof. To simplify notation, $\Rightarrow_\Psi$ below will denote the least regular production relation containing a causal theory Ψ, while $\mathcal{C}_\Psi$ will denote the associated production operator.

The direction from right to left follows from Corollary 4.4 and the fact that if Δ and Γ are regularly equivalent, then, for any Φ, $\Delta \cup \Phi$ and $\Gamma \cup \Phi$ are also regularly equivalent.

Assume now that Δ is not regularly equivalent to Γ. Then we may assume for certainty that there are propositions A, B such that $A \Rightarrow_\Delta B$ and $A \not\Rightarrow_\Gamma B$. The latter fact means that there is a theory u of Γ such that $A \in u$ and $B \notin \mathcal{C}_\Gamma(u)$. Let us consider two cases.

Suppose first that u is not a theory of Δ. Then we choose $\Phi = \{C \Rightarrow C \mid C \in u\}$. Clearly, u is an exact theory for $\Gamma \cup \Phi$, though not for $\Delta \cup \Phi$.

Suppose now that u is a theory of Δ. Since $B \in \mathcal{C}_\Delta(u)$, we have $B \in u$. Let β be a world that includes both $\mathcal{C}_\Gamma(u)$ and $\neg B$ (since $B \notin \mathcal{C}_\Gamma(u)$, such a world should exist). Then we define Φ as $\{C \Rightarrow C \mid C \in u \cap \beta\}$. Note first that $\mathcal{C}_{\Gamma\cup\Phi}(u) = u \cap \beta$, and hence u is not an exact theory for $\Gamma \cup \Phi$. But u is the least theory that contains both $\mathcal{C}_\Delta(u)$ and $u \cap \beta$, and hence it is an exact theory for $\Delta \cup \Phi$. Indeed, if $C \in u$, then $B \to C \in u \cap \beta$, and therefore $B \to C \in \mathcal{C}_{\Delta\cup\Phi}(u)$. In addition, $B \in \mathcal{C}_{\Delta\cup\Phi}(u)$ (since $B \in \mathcal{C}_\Delta(u)$). Consequently, $C \in \mathcal{C}_{\Delta\cup\Phi}(u)$, and therefore $u \subseteq \mathcal{C}_{\Delta\cup\Phi}(u)$. $\square$

The above result implies, in effect, that regular production relations are maximal inference relations that are adequate for reasoning with causal theories: any postulate that is not valid for regular production relations can be 'falsified' by finding a suitable extension of two causal theories that would determine different nonmonotonic semantics, and hence would produce different nonmonotonic conclusions.

Note that 'discriminating' sets of causal rules Φ were restricted in the proof to rules of the form $C \Rightarrow C$. Such rules will play below an important role in representing abductive reasoning in the framework of production inference.

As a last result in this section, we will show that, under some special conditions, the nonmonotonic semantics of causal theories grows monotonically with the growth of these causal theories.

Definition 4.5. Causal theories Γ and Δ will be called *supraclassically equivalent*, if $\Rightarrow_\Delta$ and $\Rightarrow_\Gamma$ have the same theories.

Recall that theories of $\Rightarrow_\Delta$ are precisely deductively closed sets that are closed also with respect to the rules of Δ. Accordingly, supraclassical equivalence amounts, in effect, to the equivalence of Δ and Γ viewed as ordinary rules of a consequence relation. More exactly,

Δ and Γ are supraclassically equivalent if and only if the rules of Γ and Δ are inter-derivable using the postulates of supraclassical consequence relations (see Introduction).

Now we can show the following

Lemma 4.6. *If causal theories Γ and Δ are supraclassically equivalent, and $\Gamma \subseteq \Delta$, then any exact theory of Γ is an exact theory of Δ.*

Proof. If $u = \mathcal{C}_\Gamma(u)$, then clearly $u \subseteq \mathcal{C}_\Delta(u)$. But u is a theory of Γ, and hence it is a theory of Δ, that is, $\mathcal{C}_\Delta(u) \subseteq u$. Therefore $u = \mathcal{C}_\Delta(u)$, and consequently u is an exact theory of Δ. $\square$

The above result clarifies, in effect, the reasons why the semantics of exact theories is nonmonotonic. Indeed, if the production rules added to a causal theory do not change the set of theories of the latter, then they extend, in general, the set of its exact theories, and therefore not all the conclusions made earlier will be preserved.

4.1. *On informational content of causal theories*

The existence of two kinds of semantics for production inference relations, monotonic and nonmonotonic ones, poses a nontrivial question about the meaning, or *informational content*, of causal theories. Moreover, this conceptual question becomes a practical problem when we need to compare such theories from informational point of view, or update a causal theory with new information in such a way that would produce a minimal change in its informational content.

The above question is clearly not specific to production inference, and it can be posed about practically any nonmonotonic formalism. And unfortunately, a common (though often tacit) answer to this question consists in a direct identification of the meaning of a nonmonotonic theory with its nonmonotonic semantics. Furthermore, in the case of default logic, the nonmonotonic semantics (of extensions) is even presented as the only semantics that we have for default theories.

The above solution to the problem of informational content of nonmonotonic theories turns out to be problematic, however. To begin with, quite diverse theories can have the same ‘meaning’ according to this understanding, witness all default theories that do not have extensions. Moreover, adding new rules to a theory does not necessarily mean that the extended theory contains more information. Speaking generally, the meaning associated with a theory by this solution is *non-modular*, and hence cannot be viewed as composed from the meanings of its components. As a result, meaning of this kind turns out to be useless for most purposes we could possibly have in invoking this notion, be it knowledge representation or informational updates.

Intuitively, an informational content of a causal theory should contain all the information needed to determine the properties and behavior of this theory. Thus, it is natural to suppose that when two causal theories have the same informational content, they should be interchangeable in any larger causal theory without changing its properties. Moreover, it seems natural to expect also that adding new rules to a theory should normally increase its informational content (unless the added rules are derivable somehow from the source theory). All these requirements will not be met if we will identify the content of the causal theory with its nonmonotonic semantics. Indeed, it is nonmonotonic precisely in the sense that adding new production rules to a theory may result in removal of some of the previously derivable information. Consequently, the nonmonotonic semantics is patently insufficient for capturing the full content of a causal theory.

An alternative, and seemingly more plausible, solution to this problem is based on a clear separation between logical (monotonic) and nonmonotonic aspects of reasoning with nonmonotonic theories. Once such a separation is made, the meaning, or an informational content, of a theory could be identified with its logical meaning naturally determined by the underlying monotonic logic and its semantics. The nonmonotonic semantics will play no direct role in this description; the logical meaning will usually determine also the associated nonmonotonic semantics, though not vice versa. For default theories, such an underlying monotonic logic has been described in [5] as a set of inference rules for default rules that preserve the extension-based nonmonotonic semantics. In the present study, we suggest a similar solution for causal theories.

As has been established, regular production relations constitute a maximal logic suitable for the general nonmonotonic semantics. Accordingly, the informational content of a causal theory Δ with respect to the latter can be safely identified with the set of causal rules that are derivable from Δ by the postulates of regular production inference, that is, with $\Rightarrow^r_\Delta$. Clearly, such a definition of an informational content will satisfy all the desired properties, mentioned above. Moreover, since the general nonmonotonic semantics of a production relation is uniquely determined by its canonical production semantics, the nonmonotonic semantics of a causal theory constitutes, in a sense, part of its informational content.

A more detailed description of informational content will be given below for a special case of causal nonmonotonic semantics.

5. Abductive production inference

In this section we will describe a special kind of regular production inference relations that provides a formal representation for general abductive reasoning. In addition to specific results, this will give us also a broader perspective on the representation capabilities of production inference relations as a general-purpose formalism for nonmonotonic reasoning.

A general *abductive framework* can be defined as a pair $\mathbb{A} = (\mathrm{Cn}, \mathcal{A})$, where Cn is a consequence relation, and $\mathcal{A}$ is a distinguished set of propositions called *abducibles*. A proposition A is *explainable* in an abductive framework $\mathbb{A}$ if there exists a consistent set of abducibles $a \subseteq \mathcal{A}$ such that $A \in \mathrm{Cn}(a)$.

A general correspondence between abductive frameworks and production inference has been established in [10]. The correspondence has been based on the identification between the above notion of explainability and production derivability. By this correspondence, abducibles are representable by 'reflexive' propositions satisfying the rule $A \Rightarrow A$, and the abductive frameworks themselves correspond to production inference relations satisfying an additional postulate of Abduction, given below.

Definition 5.1.

- A proposition A will be called an *abducible* in a production relation $\Rightarrow$, if $A \Rightarrow A$;
- A production relation will be called *abductive* if it is regular and satisfies

(Abduction) If $A \Rightarrow B$, then $A \Rightarrow C \Rightarrow B$, for some abducible C.

As can be seen, production inference in abductive production relations is always mediated by abducible (self-explanatory) propositions. Consequently, for such production relations, propositions caused by worlds are caused, in effect, by the abducibles that hold in these worlds:

$$\mathcal{C}(\alpha) = \mathcal{C}(\alpha \cap \mathcal{A}),$$

where $\mathcal{A}$ is the set of abducibles of $\Rightarrow$.

By the correspondence established in [10], the general nonmonotonic semantics of abductive production relations describes, in effect, the *explanatory closure*, or *completion*, in associated abductive frameworks (see [12,18]). These results have shown, in effect, that production inference allows to give a syntax-independent representation of abductive reasoning, a representation in which abducibles are defined not as a syntactically distinguished set of propositions, but logically as propositions that satisfy certain property (namely reflexivity $A \Rightarrow A$).

Example. The following causal theory Δ represents a variant of the well-known example from [31].

$$\mathit{Rained} \Rightarrow \mathit{Grasswet} \quad \mathit{Sprinkler} \Rightarrow \mathit{Grasswet} \quad \mathit{Rained} \Rightarrow \mathit{Streetwet}$$
$$\mathit{Rained} \Rightarrow \mathit{Rained} \quad \neg\mathit{Rained} \Rightarrow \neg\mathit{Rained}$$
$$\mathit{Sprinkler} \Rightarrow \mathit{Sprinkler} \quad \neg\mathit{Sprinkler} \Rightarrow \neg\mathit{Sprinkler}$$
$$\neg\mathit{Grasswet} \Rightarrow \neg\mathit{Grasswet} \quad \neg\mathit{Streetwet} \Rightarrow \neg\mathit{Streetwet}$$

The first three rules give a causal description of the domain, while the rest describes the associated abducibles. It can be verified that the regular production relation determined by Δ is abductive.

By stipulating that both *Rained* and $\neg$*Rained* are abducibles, we make *Rained* an independent (exogenous) parameter. However, since $\neg$*Grassswet* is an abducible, but *Grasswet* is not, non-wet grass does not require explanation, but wet grass does. Thus, any exact theory of Δ that contains *Grasswet* should contain either *Rained*, or *Sprinkler*. Similarly, the nonmonotonic semantics of Δ sanctions in this sense that *Streetwet* implies both *Rained* and *Grasswet*.

5.1. The abductive subrelation

Here we will show that any production relation includes an important abductive subrelation; as we will see, in many regular situations the latter subrelation will determine the same nonmonotonic semantics.

Given a regular production relation $\Rightarrow$, we will define the following production relation:

$$A \Rightarrow^a B \;\equiv\; (\exists C)(A \Rightarrow C \Rightarrow C \Rightarrow B).$$

Theorem 5.1. *If $\Rightarrow$ is a regular production relation, then $\Rightarrow^a$ is the greatest abductive production relation included in $\Rightarrow$.*

Proof. We need to check first that $\Rightarrow^a$ satisfies the postulates of production inference. Strengthening, Weakening, Truth and Falsity are obvious.

And. Assume that $A \Rightarrow^a B$ and $A \Rightarrow^a D$. Then there are C_1 and C_2 such that $A \Rightarrow C_1 \Rightarrow C_1 \Rightarrow B$ and $A \Rightarrow C_2 \Rightarrow C_2 \Rightarrow D$. Note first that $C_1 \wedge C_2 \Rightarrow C_1 \wedge C_2$ by Strengthening and And, as well as $A \Rightarrow C_1 \wedge C_2$ and $C_1 \wedge C_2 \Rightarrow B \wedge D$. Hence $A \Rightarrow^a B \wedge D$. Thus, And holds.

Cut. Assume that $A \Rightarrow^a B$ and $A \wedge B \Rightarrow^a D$. Then there are C_1 and C_2 such that $A \Rightarrow C_1 \Rightarrow C_1 \Rightarrow B$ and $A \wedge B \Rightarrow C_2 \Rightarrow C_2 \Rightarrow D$. As before, we have $C_1 \wedge C_2 \Rightarrow C_1 \wedge C_2$. In addition, since $A \Rightarrow B$ (by Transitivity) and $A \wedge B \Rightarrow C_2$, we have $A \Rightarrow C_2$ by Cut, and therefore $A \Rightarrow C_1 \wedge C_2$. Also, $C_2 \Rightarrow D$ implies $C_1 \wedge C_2 \Rightarrow D$. Therefore, $A \Rightarrow^a D$, which shows that Cut holds for $\Rightarrow^a$.

Thus, $\Rightarrow^a$ is an abductive production relation. Moreover, if $A \Rightarrow^a B$, then $A \Rightarrow B$ by Transitivity, so $\Rightarrow^a$ is included in $\Rightarrow$.

Let $\Rightarrow_1$ be some abductive production relation that is included in $\Rightarrow$, and $A \Rightarrow_1 B$. By Abduction, there is C such that $A \Rightarrow_1 C \Rightarrow_1 C \Rightarrow_1 B$. But $\Rightarrow_1$ is included in $\Rightarrow$, and therefore $A \Rightarrow C \Rightarrow C \Rightarrow B$. The latter implies $A \Rightarrow^a B$, which shows that $\Rightarrow_1$ is included in $\Rightarrow^a$. Thus, $\Rightarrow^a$ is the greatest abductive production relation included in $\Rightarrow$. □

It turns out that the above abductive subrelation of a production relation preserves many properties of the latter. For example, both have the same constraints, that is, $A \Rightarrow \mathbf{f}$ holds if and only if $A \Rightarrow^a \mathbf{f}$. Similarly, both have the same 'axioms': $\mathbf{t} \Rightarrow A$ holds iff $\mathbf{t} \Rightarrow^a A$. Last but not least, both have the same abducibles.

5.2. Quasi-abductive production inference

The general nonmonotonic semantics is based on the explanation closure assumption, according to which any accepted proposition should be produced by other accepted propositions. But the latter should also be produced by accepted propositions, and so on. It should be clear that if our 'production resources' are limited (for example, if our language is finite), such a production process should stop somewhere. More exactly, it should reach abducible (self-explanatory) propositions. This indicates that in many regular cases the nonmonotonic semantics of a production relation should coincide with the nonmonotonic semantics of its abductive subrelation. Below we will make this claim precise.

Definition 5.2. A production relation will be called *quasi-abductive* if it is nonmonotonically equivalent to its abductive subrelation.

The next definition will give us an important sufficient condition for quasi-abductivity.

Definition 5.3. A regular production relation $\Rightarrow$ will be called *well-founded* if any infinite sequence $\{A_0, A_1, A_2, \ldots\}$ of propositions such that $A_{n+1} \Rightarrow A_n$, for every $n \geqslant 0$, contains an abducible.

The above definition describes a variant of a standard notion of well-foundedness with respect to the partial order determined by a regular production relation. Namely, it says that any infinite descending sequence of productions should always contain reflexive elements.

It should be clear that any production relation in a finite propositional language should be well-founded. Moreover, let us say that a regular production relation is *finitary*, if it is a least regular production relation containing some finite causal theory. Then the following result shows that any such production relation is well-founded.

Theorem 5.2. *Any finitary regular production relation is well-founded.*

Proof. Let $\Rightarrow$ be a regular production relation determined by a finite causal theory Δ, and assume that $\{A_i \mid i \geqslant 0\}$ is an infinite sequence of propositions such that $A_{n+1} \Rightarrow A_n$, for any $n \geqslant 0$.

Let us say that a proposition C is *explicit* with respect to $\Rightarrow$ if it is a conjunction of heads of some rules from Δ. Since Δ is finite, the set of explicit propositions will also be finite (up to logical equivalence). Moreover, Proposition 3.3 implies that $A \Rightarrow B$ holds only if $A \Rightarrow C$, for some explicit C such that $C \vDash B$. This means, in particular, that in the above infinite production sequence, any production is 'mediated' by an explicit proposition: $A_{i+1} \Rightarrow C_i \vDash A_i$.

Since the set of explicit propositions is finite, there must exist at least two propositions A_j, A_k in the sequence such that $j > k$, $A_{j+1} \Rightarrow C \vDash A_j$ and $A_{k+1} \Rightarrow C \vDash A_k$, for some explicit C. Then we have $C \vDash A_j$ and $A_j \Rightarrow C$ (by Transitivity), and hence $A_j \Rightarrow A_j$ by Weakening. Hence, any sequence of productions in $\Rightarrow$ should be well-founded. □

Finally, the following basic result shows that a well-founded production relation determines precisely the same general nonmonotonic semantics as its abductive subrelation.

Theorem 5.3. *Any well-founded regular production relation is quasi-abductive.*

Proof. We will denote by $\mathcal{C}^a$ the production operator corresponding to $\Rightarrow^a$.

Assume first that u is an exact theory of $\Rightarrow$, that is, $u = \mathcal{C}(u)$. Then clearly $\mathcal{C}^a(u) \subseteq u$. Let $A_0 \in u$. Then $A_0 \in \mathcal{C}(u)$, and hence there must exist $A_1 \in u$ such that $A_1 \Rightarrow A_0$. Repeating this argument, we obtain an infinite chain $\{A_0, A_1, A_2, \ldots\}$ of propositions such that all A_i belong to u and $A_{i+1} \Rightarrow A_i$, for any $i \geqslant 0$. But our production relation is well-founded, so there must exist some abducible A_n in the chain. Clearly, $A_n \in u$ and $A_n \Rightarrow A_n \Rightarrow A_0$ (by Transitivity). Hence $A_n \Rightarrow^a A_0$, and therefore $A_0 \in \mathcal{C}^a(u)$. This shows that $u \subseteq \mathcal{C}^a(u)$, and hence u is an exact theory of $\Rightarrow^a$.

Assume now that u is an exact theory of $\Rightarrow^a$. Then clearly $u \subseteq \mathcal{C}(u)$. So we have to show only that u is a theory in $\Rightarrow$. Suppose that $A \in u$ and $A \Rightarrow B$. Then $A \in \mathcal{C}^a(u)$, and hence there exists $D \in u$ such that $D \Rightarrow^a A$. Consequently $D \Rightarrow C \Rightarrow C \Rightarrow A$, for some C.

But then $C \Rightarrow B$, and therefore $D \Rightarrow^a B$. The latter implies that $B \in \mathcal{C}^a(u)$, and therefore $B \in u$. Thus, u is a theory in $\Rightarrow$, and consequently it is an exact theory in $\Rightarrow$. □

Recall now that abductive production relations correspond to abductive systems. Accordingly, the above result shows that in the well-founded case the general nonmonotonic semantics of a regular production relation is describable by some abductive framework, and vice versa. As a general conclusion, however, we can say that production inference constitutes a proper generalization of abductive reasoning, a generalization that goes beyond well-foundedness.

6. Basic production inference

Following [25], a production inference relation will be called *basic* if it satisfies

(Or) If $A \Rightarrow C$ and $B \Rightarrow C$, then $A \vee B \Rightarrow C$.

The above inference rule allows for reasoning by cases, and hence basic production relations can already be seen as systems of *objective* production inference, namely as systems of reasoning about complete worlds.

The postulate Or corresponds to the following characteristic property of the production operator: for any deductively closed sets u, v,

$$\mathcal{C}(u \cap v) = \mathcal{C}(u) \cap \mathcal{C}(v).$$

As a consequence of this condition, the set of propositions produced by a theory u coincides with the set of propositions that are produced by every world containing u:

$$\mathcal{C}(u) = \bigcap\{\mathcal{C}(\alpha) \mid u \subseteq \alpha\}.$$

Another important fact about basic production inference is that a production rule $A \vee B \Rightarrow C$ is equivalent to the pair of rules $A \Rightarrow C$ and $B \Rightarrow C$. As a result, any production rule is reducible to a set of *clausal* production rules of the form $\bigwedge l_i \Rightarrow \bigvee l_j$, where l_i, l_j are classical literals.

For any causal theory Δ, a least basic production relation containing Δ will be denoted by $\Rightarrow^b_\Delta$. The following characterization of this relation has been obtained in [25]:

Proposition 6.1. *$u \Rightarrow^b_\Delta A$ iff $A \in \mathrm{Th}(\Delta(\alpha))$, for every world $\alpha \supseteq u$.*

6.1. Possible worlds semantics

The semantic characterization of basic production relations can be obtained from the general production semantics by restricting the set of bimodels to world-based ones, namely to bimodels of the form (α, β), where α, β are worlds. Clearly, the corresponding production semantics can be identified with a *relational possible worlds model* $\mathbb{W} = (W, \mathcal{B})$, where W is a set of worlds, and $\mathcal{B}$ is an accessibility relation on W. The following definition provides the corresponding notion of validity for productions:

Definition 6.1. A production rule $A \Rightarrow B$ will be said to be *valid* in a possible worlds model $(W, \mathcal{B})$ if, for any pair of worlds $\alpha, \beta \in W$ such that $\alpha\mathcal{B}\beta$, if A holds in α, then B holds in β.

We will denote by $\Rightarrow_{\mathbb{W}}$ the set of all productions that are valid in a possible worlds model $\mathbb{W}$. It can be easily verified that this set is closed with respect to the rule Or, and hence we have

Lemma 6.2. $\Rightarrow_{\mathbb{W}}$ *is a basic production relation.*

In order to prove completeness, for any basic production relation $\Rightarrow$ we will construct its *canonical possible worlds model* $\mathbb{W}_{\Rightarrow}$ as a pair $(W, \mathcal{B})$ such that W is a set of all worlds (maximal classically consistent subsets) of the language, and $\mathcal{B}$ is a relation on W defined as follows:

$$\alpha\mathcal{B}\beta \ \equiv\ \mathcal{C}(\alpha) \subseteq \beta.$$

Then the following result shows that this semantics is *strongly complete* for the source production relation.

Lemma 6.3. *If $\mathbb{W}_{\Rightarrow}$ is the canonical semantics of a basic production relation $\Rightarrow$, then, for any set of propositions u and any A,*

$$u \Rightarrow A \quad \textit{iff} \quad A \in \beta,\ \textit{for any}\ \alpha, \beta \in W\ \textit{such that}\ \alpha\mathcal{B}\beta\ \textit{and}\ u \subseteq \alpha.$$

Proof. If $u \Rightarrow A$ and $u \subseteq \alpha$, for some world α, then clearly $A \in \mathcal{C}(\alpha)$, and therefore $A \in \beta$, for any world β that includes $\mathcal{C}(\alpha)$. Thus, $u \Rightarrow A$ is valid in the canonical semantics. In the other direction, if $u \not\Rightarrow A$, then by Lemma 2.2, u is included in a maximal deductively closed set α that does not produce A. Now, by the rule Or, $B \vee C \in \alpha$ only if either $B \in \alpha$ or $C \in \alpha$. Hence α should be a world. Now, since $A \notin \mathcal{C}(\alpha)$, there must exist a world β containing $\mathcal{C}(\alpha)$ and such that $A \notin \beta$. Clearly, (α, β) is a bimodel from $\mathcal{B}_{\Rightarrow}$ that refutes $u \Rightarrow A$. $\square$

As a general conclusion from the above results, we obtain

Corollary 6.4. $\Rightarrow$ *is a basic production relation if and only if it is determined by a possible worlds semantics.*

The above semantics immediately suggests a modal translation of production rules described in [25] (see also [37,38]). Namely, let $\Box$ be the usual modal operator definable in a possible worlds model: $\Box A$ holds in α iff A holds in all β such that $\alpha R\beta$. Then the validity of $A \Rightarrow B$ in a possible worlds model is equivalent to validity of the formula $A \to \Box B$. Consequently, causal rules are representable by modal formulas of the latter form.

6.2. Four-valued semantics

In this section we are going to show that basic production inference relations possess also a natural four-valued semantics. Among other advantages, this semantics has turned out to be useful in establishing the correspondence between production inference relations, on the one hand, and logic programming and related nonmonotonic formalisms, on the other.

According to the well-known *Belnap's interpretation* (see [2]), a four-valued interpretation can be viewed as a function assigning each propositional atom a *subset* of the set $\{t, f\}$ of classical truth-values. More exactly, the four truth-values $\top$, **t**, **f**, $\bot$ are identified, respectively, with $\{t, f\}$, $\{t\}$, $\{f\}$ and $\emptyset$. Accordingly, $\top$ means that a proposition is both true and false (i.e., contradictory), **t** means that it is 'classically' true (that is, true without being false), **f** means that it is classically false (without being true), and $\bot$ means that it is neither true nor false (undetermined).

The above interpretation allows us to see any 4-assignment as a pair of ordinary classical valuations, corresponding, respectively, to independent assignments of truth and falsity to propositions. To be more exact, for any 4-assignment ν (under the above interpretation) and any proposition A we can define the following two valuations saying, respectively, "A is true in ν" and "A is false in ν":

$$\nu \models A \quad \text{iff} \quad t \in \nu(A)$$
$$\nu \mathrel{=\!\!|} A \quad \text{iff} \quad f \in \nu(A).$$

In this setting, any four-valued connective is definable via a pair of definitions stating, respectively, when the formula is true and when it is false (see [3] for details). Two such four-valued connectives will be of special interest for our present study. The first is a well-known conjunction connective:

$$\nu \models A \wedge B \quad \text{iff} \quad \nu \models A \text{ and } \nu \models B$$
$$\nu \mathrel{=\!\!|} A \wedge B \quad \text{iff} \quad \nu \mathrel{=\!\!|} A \text{ or } \nu \mathrel{=\!\!|} B.$$

The second is the following negation connective that can be seen as a most natural extension of the classical negation to the four-valued setting:

$$\nu \models \neg A \quad \text{iff} \quad \nu \not\models A$$
$$\nu \mathrel{=\!\!|} \neg A \quad \text{iff} \quad \nu \mathrel{\not\!\!=\!\!|} A.$$

A common feature of these two connectives is that they have the same definitions for truth as for non-falsity, and that the truth (respectively, falsity) of a compound formula is determined only in terms of truth (respectively, falsity) of its arguments. Four-valued functions of this kind will be called *locally classical*, since they behave as ordinary classical truth-valued functions with respect to each of the two contexts. Moreover, since a classical conjunction and negation form a functionally complete set for classical (two-valued) functions, it should be clear that the above two four-valued connectives are functionally complete for the set of all locally classical connectives.

Recall now that the semantics of a basic production inference is a set of pairs of worlds. By the above correspondence, such pairs can be identified with four-valued interpretations, whereas the classical connectives will correspond then to the relevant locally classical four-valued connectives. Given this understanding, the notion of validity of production rules with respect to a possible worlds semantics is transformed into the following definition of validity with respect to a set of four-valued interpretations:

Definition 6.2. A production rule $A \Rightarrow B$ will be said to be *valid* with respect to a set of four-valued interpretations $\mathbf{I}$ when

$$\nu \dashv A \text{ or } \nu \models B, \quad \text{for any } \nu \in \mathbf{I}.$$

Let us denote by $\Rightarrow_{\mathbf{I}}$ the set of all production rules that are valid in a four-valued semantics $\mathbf{I}$. Then, as an immediate consequence of the above correspondence, we obtain

Theorem 6.5. *A set of production rules $\Rightarrow$ is a basic production inference relation if and only if $\Rightarrow = \Rightarrow_{\mathbf{I}}$, for some set of four-valued interpretations* $\mathbf{I}$.

The above result shows that the four-valued semantics gives a complete semantic characterization of basic production inference relations.

As a by-product of the above semantic representation, we obtain also that production inference relations are closely related to *biconsequence relations* from [3]. Biconsequence relations are sets of 'double' sequents (bisequents) of the form

$$a : b \Vdash c : d,$$

where a, b, c, d are sets of propositions. Such bisequents have the following four-valued interpretation:

> If all propositions from a are true, and all propositions from b are false, then one of the propositions from c is true, or one of the propositions from d is false.

It has been shown in [3] that biconsequence relations provide a structural description of four-valued reasoning. Namely, any four-valued connective is definable in this setting by using appropriate introduction and elimination rules, just as in the usual classical sequent calculus. Biconsequence relations have been used in [4] as a logical basis of logic programming.

Now, it follows from the above four-valued interpretation of bisequents that a bisequent $a{:}b \Vdash c{:}d$ has the same interpretation as the production rule

$$\bigwedge(\neg b \cup d) \Rightarrow \bigvee(\neg a \cup c)$$

while any production rule $A \Rightarrow B$ has the same four-valued interpretation as a bisequent $\Vdash B{:}A$. This shows, in effect, that basic production inference relations constitute an exact logical counterpart of biconsequence relations in the language of local classical connectives. A more detailed description of this correspondence will be given elsewhere.

7. Causal inference

Now we will consider production relations that are both basic and regular.

Definition 7.1. A production inference relation will be called *causal* if it satisfies Or and Cut.

Causal inference relations enjoy the properties of both basic and regular production relations described in the preceding sections. In particular, since they are basic, causal inference can also be considered as a particular (namely, regular) kind of objective production inference.

An interesting syntactic fact about causal inference relations is that the Cut rule turns out to be equivalent to Coherence (see Section 3).

Lemma 7.1. *For basic production relations, Cut is equivalent to Coherence.*

Proof. It has been shown earlier that Cut implies Constraint (and hence Coherence). Assume now that Coherence holds, and we have $A \Rightarrow B$ and $A \wedge B \Rightarrow C$. Then the former implies $A \wedge \neg B \Rightarrow \neg(A \wedge \neg B)$ by Strengthening and Weakening, and therefore $A \wedge \neg B \Rightarrow \mathbf{f}$ by Coherence.[7] Consequently, $A \wedge \neg B \Rightarrow C$ by Weakening, and therefore $A \Rightarrow C$ by Or. Thus, Cut holds. □

Another important fact about causal relations is the following decomposition of causal rules:

Lemma 7.2. *Any causal rule $A \Rightarrow B$ is equivalent to a pair of rules $A \wedge \neg B \Rightarrow \mathbf{f}$ and $A \wedge B \Rightarrow B$.*

Proof. The direction from left to right is immediate. In the other direction, if $A \wedge \neg B \Rightarrow \mathbf{f}$ and $A \wedge B \Rightarrow B$, then $A \wedge \neg B \Rightarrow B$ by Weakening, and hence $A \Rightarrow B$ by Or. □

Causal rules of the form $A \wedge B \Rightarrow B$ are logically trivial, but they play an important explanatory role in causal reasoning. Namely, they say that, in any causally explained interpretation in which A holds, we can freely accept B, since it is self-explanatory in this context. Accordingly, such rules will be called *explanatory rules* in what follows.

Since causal inference is a special kind of an objective (world-based) inference, a constraint $A \wedge \neg B \Rightarrow \mathbf{f}$ says, in effect, that the classical implication $A \to B$ should hold in any causally consistent world.

Now the above lemma says that any causal rule can be decomposed into a constraint and an explanatory rule. This decomposition neatly delineates two kinds of information conveyed by causal rules. One is a factual information that constrains the set of admissible models, while the other is an explanatory information describing what propositions are caused (explainable) in such models.[8]

Now we will turn to a semantic description of causal inference. It has been shown earlier that possible worlds models with arbitrary accessibility relations provide a semantics for basic production relations. It turns out that causal inference relations are determined by possible worlds models in which the accessibility relation is *quasi-reflexive*, that is, satisfies the following condition for any two worlds:

[7] This shows, in effect, that Constraint and Coherence are equivalent rules.

[8] Cf. a similar decomposition of causal rules in [36].

(Quasi-Reflexivity) If $\alpha R\beta$, then $\alpha R\alpha$.

The next lemma shows that quasi-reflexive models determine causal inference relations.

Lemma 7.3. *A production inference relation determined by a quasi-reflexive possible worlds model is causal.*

Proof. Suppose that (W, R) is a quasi-reflexive model in which $A \Rightarrow B$ and $A \wedge B \Rightarrow C$ are valid, while $A \Rightarrow C$ is not valid. Then there are worlds α, β such that $\alpha R\beta$, $A \in \alpha$ and $C \notin \beta$. Consequently, $\alpha R\alpha$ by quasi-reflexivity, and hence $A \Rightarrow B$ implies that $B \in \alpha$. Thus, $A \wedge B \in \alpha$, and therefore $A \wedge B \Rightarrow C$ gives us that C should hold in β, contrary to our assumptions. Hence, production relations generated by quasi-reflexive models are causal. □

The following theorem shows that causal relations are actually complete for such models.

Theorem 7.4. *A production inference relation is causal if and only if it is determined by a quasi-reflexive possible worlds model.*

Proof. Given a causal relation $\Rightarrow$, we construct the corresponding canonical model (W, R_c) by defining R_c as follows: $\alpha R_c\beta \;\equiv\; \mathcal{C}(\alpha) \subseteq \alpha \cap \beta$. Notice that this definition directly implies quasi-reflexivity of R_c.

As in the proof of Lemma 6.3, it can be shown that if $A \Rightarrow B$ holds, then it is valid in (W, R_c). Now if $A \not\Rightarrow B$, the same proof shows that there are worlds α, β such that $A \in \alpha$, $B \notin \beta$ and $\mathcal{C}(\alpha) \subseteq \beta$. Moreover, we will show that $\mathcal{C}(\alpha) \subseteq \alpha$.

Suppose that $\mathcal{C}(\alpha) \nsubseteq \alpha$. This means that there are propositions C, D such that $D \in \alpha$ and $D \Rightarrow C$, but $C \notin \alpha$. Now, α has been defined in the proof of Lemma 6.3 as a maximal set that does not produce B. Consequently, $B \in \mathcal{C}(\alpha, C)$, and hence there must exist $E \in \alpha$ such that $C \wedge E \Rightarrow B$. The latter implies $D \wedge C \wedge E \Rightarrow B$. In addition, $D \Rightarrow C$ implies $D \wedge E \Rightarrow C$. Hence $D \wedge E \Rightarrow B$ by Cut. But $D \wedge E \in \alpha$, and therefore $B \in \mathcal{C}(\alpha)$, contrary to the definition of α. Thus, $\mathcal{C}(\alpha) \subseteq \alpha$.

By the definition of R_c, we have now $\alpha R_c\beta$, and therefore $A \Rightarrow B$ is not valid in (W, R_c). This completes the proof. □

It is interesting to note that the above semantic interpretation gives clear semantic reasons why Transitivity is a weaker property than Cut. Indeed, it is easy to verify that Transitivity holds for all production relations determined by possible world models with a *dense* accessibility relation, that is, a relation satisfying the condition that if $\alpha R\beta$, then there exists γ such that $\alpha R\gamma$ and $\gamma R\beta$. Clearly, quasi-reflexivity is a stronger property than density.

In what follows, $\Rightarrow^c_\Delta$ will denote the least causal inference relation containing a causal theory Δ. In addition, $\vec{\Delta}$ will denote the set of material implications corresponding to the production rules from Δ, namely

$$\vec{\Delta} = \{A \to B \mid A \Rightarrow B \in \Delta\}.$$

As before, [25] provides a constructive description of $\Rightarrow_{\Delta}^{c}$. Such a description can be obtained from the corresponding description for regular production relations, given in Proposition 3.3, simply by restricting the set of Δ-theories to worlds. Notice, however, that a world α is a Δ-theory if and only if $\vec{\Delta} \subseteq \alpha$. Consequently, we obtain

Proposition 7.5. *$u \Rightarrow_{\Delta}^{c} A$ iff $A \in \mathrm{Th}(\Delta(\alpha))$, for any world $\alpha \supseteq u \cup \vec{\Delta}$.*

Thus, derivability in causal inference relations is reducible, in effect, to derivability in basic production relations with an additional set of assumptions $\vec{\Delta}$. This fact implies, in particular, that the material implications corresponding to the production rules can be used as auxiliary assumptions in making derivations. In other words, causal inference relations make valid the following rule:

If $A \Rightarrow B$ and $C \wedge (A \to B) \Rightarrow D$, then $C \Rightarrow D$.

8. The causal nonmonotonic semantics

If we concentrate on an objective understanding of production rules as rules acting in world-based contexts, it is only natural to consider also the corresponding restriction of the general nonmonotonic semantics to exact theories that are worlds. In this case, the principle of explanation closure can be justifiably called the *principle of universal causation* (see [37]).

Definition 8.1. A *causal nonmonotonic semantics* of a production inference relation or a causal theory is the set of all its exact worlds.

Since the causal nonmonotonic semantics forms a subset of the general nonmonotonic semantics, it produces, in general, a larger set of nonmonotonic consequences. Moreover, the causal semantics is just a set of worlds, so its logical content is exhausted by the classical propositional theory that is uniquely associated with this set of worlds. Note, however, that, unlike the general nonmonotonic semantics, the causal nonmonotonic semantics is not guaranteed to exist for any causal theory.

The following lemma gives a useful alternative description of exact worlds.

Lemma 8.1. *A world α is an exact world of a production inference relation if and only if, for any propositional atom p,*

$$p \in \alpha \quad \textit{iff} \quad \alpha \Rightarrow p$$
$$p \notin \alpha \quad \textit{iff} \quad \alpha \Rightarrow \neg p.$$

Proof. Due to the fact that α is a world, the above conditions are sufficient for the equality $\alpha = \mathcal{C}(\alpha)$. □

By Lemma 4.2, the causal nonmonotonic semantics of a causal theory Δ is the set of worlds α such that $\alpha = \mathrm{Th}(\Delta(\alpha))$. Such worlds coincide with *causally explained inter-*

pretations of [26], which determine the nonmonotonic semantics of their causal theories. Consequently the causal nonmonotonic semantics provides an adequate representation for this nonmonotonic system.

An alternative description of exact worlds for causal theories is given below.

Corollary 8.2. *A world α is an exact world of a causal theory Δ if and only if, for any propositional atom p,*

$$p \in \alpha \quad \textit{iff} \quad \Delta(\alpha) \vDash p$$
$$p \notin \alpha \quad \textit{iff} \quad \Delta(\alpha) \vDash \neg p.$$

Proof. Follows from the fact that the above conditions are equivalent to $\alpha = \mathrm{Th}(\Delta(\alpha))$. $\square$

Now we will show that causal inference relations provide an adequate framework of reasoning with respect to the causal nonmonotonic semantics. As before, we will introduce first the following definitions:

Definition 8.2. Causal theories Γ and Δ will be called

- *objectively equivalent* if they have the same causal nonmonotonic semantics;
- *strongly objectively equivalent* if, for any set Φ of production rules, $\Delta \cup \Phi$ is objectively equivalent to $\Gamma \cup \Phi$;
- *causally equivalent* if $\Rightarrow^c_\Delta = \Rightarrow^c_\Gamma$.

Two causal theories are causally equivalent if each theory can be obtained from the other using the inference postulates of causal relations.

Lemma 8.3. *Any causal theory Δ is objectively equivalent to $\Rightarrow^c_\Delta$.*

Proof. Let $\mathcal{C}^c_\Delta$ denote the production operator corresponding to $\Rightarrow^c_\Delta$. Then Proposition 7.5 implies that, for any causally consistent world α, $\mathcal{C}^c_\Delta(\alpha) = \mathrm{Th}(\Delta(\alpha))$. Consequently, $\alpha = \mathcal{C}^c_\Delta(\alpha)$ if and only if $\alpha = \mathrm{Th}(\Delta(\alpha))$. $\square$

The above lemma says that the postulates of causal inference relations are adequate for reasoning with exact worlds, since they preserve the latter. Moreover, the next theorem shows that causal inference relations constitute a maximal logic suitable for the causal nonmonotonic semantics.

Theorem 8.4. *Two causal theories are strongly objectively equivalent if and only if they are causally equivalent.*

Proof. Coincides with the proof of Theorem 4.5, once we notice that, for our present case, the theory u in this proof is a world. $\square$

Finally, just as for the general nonmonotonic semantics, under some conditions, the causal nonmonotonic semantics grows monotonically with the growth of these causal theories.

Definition 8.3. Causal theories Γ and Δ will be called *classically equivalent*, if $\overrightarrow{\Delta}$ is logically equivalent to $\overrightarrow{\Gamma}$.

Causal theories are classically equivalent if they are logically equivalent when viewed as sets of classical material implications. Recall now that causally consistent worlds of a causal theory Δ are the worlds satisfying $\overrightarrow{\Delta}$. Consequently, classically equivalent causal theories have the same causally consistent worlds, and hence as a consequence of Lemma 4.6, we obtain

Corollary 8.5. *If causal theories Γ and Δ are classically equivalent, and $\Gamma \subseteq \Delta$, then any exact world of Γ is an exact world of Δ.*

As a particular application of the above result, suppose that we add a new inference rule for production relations that corresponds to a valid rule of classical logic. Such a rule will not change the causally consistent worlds of a production relation, and consequently the new causal nonmonotonic semantics will include the original one.

8.1. Factual and explanatory content of causal theories

Since causal inference relations form a maximal logic with respect to the causal nonmonotonic semantics, the informational content of a causal theory Δ with respect to the latter can be identified with $\Rightarrow^c_\Delta$, that is, with the set of causal rules that are derivable from Δ by the rules of causal inference. Again, such a definition of an informational content will satisfy all the desired properties, mentioned earlier. For the case of causal inference, however, a more fine-grained analysis of informational content is possible.

Recall that causal rules serve in causal inference simultaneously two informational roles. One consists in determining causally consistent worlds, while the other in establishing explanatory relations between propositions. Fortunately, these two roles can be neatly separated by decomposing any causal rule into a constraint and an explanation (see Lemma 7.2). This suggests the following definition.

Definition 8.4.

- The set of constraints $A \Rightarrow \mathbf{f}$ belonging to a causal relation will be called its *factual content*.
- The *explanatory content* of a causal relation is the set of its *explanatory rules*, namely the rules $A \Rightarrow B$ such that $A \vDash B$.

Constraints restrict the set of worlds that are admissible (causally consistent) with respect to a causal theory. In this sense they play the role of ordinary classical formulas, namely they just express facts about the world. However, they do not explain anything, and hence they can be seen as devoid of explanatory content. The later is expressed, however, by explanatory causal rules. Such rules are 'factually trivial', since they do not impose restrictions on admissible worlds; their only role consists in determining what explains what in admissible worlds. Consequently, the factual and explanatory contents are not only disjoint, but are actually independent of each other. Moreover, the informational content of causal theories can be safely represented as a union of a factual and explanatory contents.

The interplay of the factual and explanatory contents determines, eventually, the nonmonotonic semantics, and it is responsible, in particular, for the nonmonotonic properties of the latter. Namely, the nonmonotonicity arises from the fact that the two kinds of content have opposite impacts on derivability. Thus, addition of constraints leads, as expected, to reduction of the set of admissible worlds (and hence to increase of factual information). However, the addition of explanatory rules leads, in general, to *increase* of exact worlds, and hence to decrease of nonmonotonically derived information (cf. Corollary 8.5). But in all cases, one of the effects of growth in informational content is a monotonic *reduction of the set of non-exact* (*unexplained*) *worlds*.

8.2. Causal nonmonotonic reasoning: between default logic and logic programming

In this section we will briefly address the question how the causal nonmonotonic reasoning is related to default logic, on the one hand, and logic programming, on the other.

As has been shown already in [27], causal theories can be translated into default logic, but the correspondence is quite restricted. Namely, production rules $A \Rightarrow B$ are translatable as prerequisite-free default rules $: A/B$, and then the causal nonmonotonic semantics corresponds to the set of extensions of the resulting default theory that are worlds. The limited character of this correspondence is due, of course, to the essential difference in underlying semantic interpretations, namely an epistemic interpretation of default rules versus the objective interpretation of causal rules. Still, the existence of such a correspondence allows to explain why default and causal representations often produce similar results. Nevertheless, the unidirectional character of this correspondence indicates that default logic is a strictly more general formalism, which means also that causal inference is a more specific formalism with a stronger underlying logic, witness reasoning by cases that is valid for causal inference (by the Or postulate), but not for default logic.

Though distinct from default logic, causal inference relations have turned out to be intimately connected with logic programming. Thus, it has been shown in [9] that the causal logic, coupled with the Closed World Assumption (CWA), can be used as an underlying logic for logic programming involving negation as failure. Namely, a disjunctive program rule $c \leftarrow a, \mathbf{not}\ b$ (where a, b, c are sets of propositional atoms, and **not** a negation as failure) can be interpreted as a causal rule $\neg b \Rightarrow \wedge a \to \vee c$. In addition, CWA is naturally interpretable in causal logic as an additional postulate that all negated atoms are abducibles:

Default Negation $\neg p \Rightarrow \neg p$, where p is a propositional atom.

Then the causal nonmonotonic semantics of the resulting causal theories will correspond precisely to the stable semantics of logic programs [15]. Moreover, unlike known embedding of logic programs into other nonmonotonic formalisms, namely default and autoepistemic logics, the causal interpretation of logic programs turns out to be bi-directional in the sense that any causal theory is reducible to a general logic program. This shows that the causal logic provides a more faithful representation of the declarative meaning of logic programs.

Actually, the very possibility of an alternative, causal interpretation of logic programs allows to explain the well-known differences between logic programming and default logic. Thus, normal logic programs (under the stable semantics) can be translated into either default logic, or modal nonmonotonic logics, but the reverse translation, or reduction, is impossible. Moreover, this translation is not extendable directly to disjunctive programs, and requires a corresponding generalization of default logic to disjunctive default logic (see [16]). The differences have even led to an often expressed view that logic programming constitutes a separate formalism for knowledge representation and nonmonotonic reasoning in its own right (see, e.g., [1]). From the point of view of the present study, however, these differences show in effect, that logic programming constitutes a more specific formalism, and hence a formalism with a reacher underlying logic. This logic can be expressed, however, in the framework of causal inference.

9. Determinate theories and completion

Now we are going to show that any causal theory is objectively equivalent to a causal theory of a special kind, called determinate one.

A production rule will be called a *Horn* one, if it has the form $A \Rightarrow l$, where l is either a literal or a falsity constant $\mathbf{f}$.[9] A causal theory will be called *determinate*, if it contains only Horn rules. Finally, causal theories Δ and Γ will be called *Horn-equivalent*, if $\Rightarrow_\Delta$ and $\Rightarrow_\Gamma$ have the same Horn rules.

Causal theories are Horn-equivalent, if they derive the same Horn rules. Now, Lemma 8.1 shows that the exact worlds of a production relation are uniquely determined by the Horn rules that belong to it. Consequently,

Lemma 9.1. *Horn-equivalent causal theories are objectively equivalent.*

The above result implies that the causal nonmonotonic semantics of a causal theory is determined ultimately by the Horn rules that are derivable from the theory. This immediately suggests that any causal theory can be transformed into a determinate theory that is objectively equivalent to it. Such a transformation is provided below.[10]

For a causal theory Δ, let us denote by Δ_d the set of all Horn rules of the form $\bigwedge A_i \Rightarrow l$, for which Δ contains a (minimal) set of rules $\{A_i \Rightarrow B_i\}$ such that $\bigwedge B_i \vDash l$. Then the following result shows that Δ_d embodies the 'determinate content' of Δ.

Lemma 9.2. *Δ is Horn-equivalent to Δ_d.*

[9] Note that any constraint is a Horn rule by this definition.
[10] A similar algorithm has been suggested earlier by Norman McCain (V. Lifschitz, personal communication).

Proof. Note first that any production rule from Δ_d is derivable from Δ by Strengthening, And and Weakening. Assume now that $A \Rightarrow_\Delta l$, for some literal l. Then $\Delta(\mathrm{Th}(A)) \vDash l$ by Proposition 2.6. This can hold only if Δ contains a set of rules $A_i \Rightarrow B_i$ such that $A_i \in \mathrm{Th}(A)$, for any i, and $\bigwedge B_i \vDash l$. But then $\bigwedge A_i \Rightarrow l$ belongs to Δ_d, and therefore $A \Rightarrow_{\Delta_d} l$ by Strengthening. This shows that $\Rightarrow_{\Delta_d}$ includes all Horn production rules from $\Rightarrow_\Delta$, and consequently Δ and Δ_d are Horn-equivalent. □

As a consequence, we obtain that Δ_d is a determinate causal theory that is nonmonotonically equivalent to Δ.

Unfortunately, the above algorithm is neither modular, nor polynomial. Moreover, the complexity considerations confirm that this is as it should be, due to the difference in complexity between arbitrary and determinate causal theories, established in [17]. Still, in many cases the algorithm gives a convenient recipe for transforming an arbitrary causal theory into an objectively equivalent determinate theory. It should be kept in mind, however, that we do not have strong equivalence here. The following example illustrates this.

Example (*Two gears*). This is a simplified (atemporal) version of the example due to Marc Denecker that was described in [29]. Two gears are powered by separate motors that can turn them in opposite (i.e., compatible) directions. In addition, when the gears are connected, each can also turn the other. Let us assume that this domain is given the following causal representation Δ:

$$\mathit{Connected} \Rightarrow \mathit{Turning}1 \leftrightarrow \mathit{Turning}2$$

$$\mathit{Motor}1 \Rightarrow \mathit{Turning}1 \qquad \mathit{Motor}2 \Rightarrow \mathit{Turning}2,$$

plus the assumptions that $(\neg)\mathit{Motor}1$, $(\neg)\mathit{Motor}2$ and $(\neg)\mathit{Connected}$ are abducibles.

The causal theory Δ is not determinate (due to the first rule), but by the above algorithm, it can be reduced to a determinate theory Δ_d obtained from Δ by replacing the first rule with the following two Horn rules:

$$\mathit{Connected} \wedge \mathit{Motor}1 \Rightarrow \mathit{Turning}2 \qquad \mathit{Connected} \wedge \mathit{Motor}2 \Rightarrow \mathit{Turning}1.$$

Δ and Δ_d have, of course, the same causal nonmonotonic semantics, namely the worlds where both gears are turning, and either both motors are working, or else one of them is working, and the gears are connected. Let us add, however, a causal rule $\neg\mathit{Turning}2 \Rightarrow \neg\mathit{Turning}2$ to each of these causal theories. Since the added rule is purely explanatory, it can only extend the set of exact worlds. And indeed, being added to Δ, it creates a new exact world α in which the gears are connected, the two motors are not working, and both gears are not turning. This is because $\neg\mathit{Turning}2$ is now self-explanatory, while $\neg\mathit{Turning}1$ is explained by the rule $\mathit{Connected} \wedge \neg\mathit{Turning}2 \Rightarrow \neg\mathit{Turning}1$ that is derivable from the first rule of Δ and $\neg\mathit{Turning}2 \Rightarrow \neg\mathit{Turning}2$. However, if the latter rule is added to Δ_d, the world α will not be an exact world of the resulting theory. As a result, we can still nonmonotonically infer from the second theory that both gears are turning.

As has been established in [26], for a special kind of determinate theories, the causal nonmonotonic semantics coincides with the classical semantics of their completions. Below we will reproduce this result in our framework.

Definition 9.1. A causal theory will be called

- *locally finite* if any propositional atom appears in heads of no more than a finite number of its causal rules;
- *definite* if it is determinate and locally finite.

Any finite causal theory will be locally finite, though not vice versa. Similarly, any finite determinate theory will be definite.

Given a definite causal theory Δ, we will define its *completion*, *comp*(Δ), as the set of all classical formulas of the forms

$$p \leftrightarrow \bigvee\{A \mid A \Rightarrow p \in \Delta\}$$

$$\neg p \leftrightarrow \bigvee\{A \mid A \Rightarrow \neg p \in \Delta\},$$

for any propositional atom p, plus the set $\{\neg A \mid A \Rightarrow \mathbf{f} \in \Delta\}$.

Note that if Δ does not have rules with the head p, then *comp*(Δ) contains the formula $p \leftrightarrow \bigvee \emptyset$, which amounts to $\neg p$ (and similarly for $\neg p$).

The following result shows that the classical models of *comp*(Δ) precisely correspond to exact worlds of Δ. The proof follows readily from Corollary 8.2 (see also Proposition 6 in [17]).

Proposition 9.3. *The causal nonmonotonic semantics of a definite causal theory coincides with the classical semantics of its completion.*

Example. (*Reiter's simple solution*). As an illustration, we will use the framework of [26] in order to give a toy causal representation of the well-known Reiter's simple solution to the frame problem (see [34,35]). Despite its simplicity, the representation contains the main ingredients of causal reasoning in temporal domains.

We consider a small causal theory for a single propositional fluent F. The temporal behavior of F is described using two propositional atoms F_0 and F_1 saying that F holds now and, respectively, in the next moment.

$$C^+ \Rightarrow F_1 \qquad C^- \Rightarrow \neg F_1$$

$$F_0 \wedge F_1 \Rightarrow F_1 \qquad \neg F_0 \wedge \neg F_1 \Rightarrow \neg F_1$$

$$F_0 \Rightarrow F_0 \qquad \neg F_0 \Rightarrow \neg F_0.$$

The first pair of causal rules describes the factors (actions or natural causes) that can cause F and, respectively, $\neg F$ (C^+ and C^- can be arbitrary formulas, but they normally describe the present situation). Such rules correspond to Reiter's *effect axioms for fluent* F, though in our description they are not relativized to particular actions. Second, instead of Reiter's explanation closure axioms, we have a pair of *inertia axioms*. The latter are instances of explanatory production rules stating that if F holds (does not hold) now, then it is self-explanatory that it will hold (respectively, not hold) in the next moment. The last pair of *initial axioms* states, in effect, that the truth-value of F_0 is an independent (exogenous) parameter.

The above causal theory is clearly definite, and its completion (with respect to F) is as follows:

$$F_1 \leftrightarrow C^+ \vee (F_0 \wedge F_1)$$
$$\neg F_1 \leftrightarrow C^- \vee (\neg F_0 \wedge \neg F_1).$$

Now, it can be easily verified that the latter formulas are logically equivalent to the following two:

$$\neg(C^+ \wedge C^-)$$
$$F_1 \leftrightarrow C^+ \vee (F_0 \wedge \neg C^-).$$

The above formulas provide, in effect, an abstract description of Reiter's simple solution: the first formula corresponds to his *consistency condition*, while the second one—to the *successor state axiom* for F, since it describes the conditions for F to hold in the next state. Note, however, that in our representation the above formulas are basically consequences of the effect axioms and the general principles of the nonmonotonic semantics, while in Reiter's system we need each time to add appropriate explanation closure axioms.[11]

10. Beyond causal inference

Though causal inference relations have turned out to be maximal production relations that are adequate for the causal nonmonotonic semantics, some stronger production relations are also useful for the general study of production and causal inference.

10.1. Quasi-classicality

We consider first production relations that have an especially simple semantic interpretation.

Definition 10.1. A causal inference relation will be called *quasi-classical*, if it satisfies the rule

(Weak Contraposition) If $\neg A \Rightarrow \mathbf{f}$, then $\mathbf{t} \Rightarrow A$.

Weak Contraposition is equivalent to the following special case of the Deduction rule:

(Weak Deduction) If $A \Rightarrow B$, then $\mathbf{t} \Rightarrow (A \to B)$.

The latter condition says, in effect, that material implications corresponding to production rules are universally valid propositions in causal inference.

[11] John McCarthy once called this procedure 'doing nonmonotonic reasoning by hand'.

A semantic interpretation of quasi-classical production relations can be obtained by requiring full reflexivity of the accessibility relation.

Theorem 10.1. *A production relation is quasi-classical if and only if it is determined by a reflexive possible worlds model.*

Proof. It is easily verified that the causal relation determined by a reflexive possible worlds model satisfies Weak Contraposition. In the other direction, it is sufficient to show that the canonical semantics for causal productions, described in the proof of Theorem 7.4 is fully reflexive in our case. Clearly, in evaluating productions, we can restrict the set of worlds to worlds that occur either in the domain or in the range of the accessibility relation. Then, since the canonical relation R_c constructed in Theorem 7.4 was quasi-reflexive, we have to show only that if $\alpha R_c \beta$, then $\beta R_c \beta$. So, let us assume that $\mathcal{C}(\alpha) \subseteq \alpha, \beta$ and that $A \Rightarrow B$ holds. Then $A \wedge \neg B \Rightarrow \mathbf{f}$, and hence $\mathbf{t} \Rightarrow A \to B$ by Weak Contraposition. Consequently, $A \to B \in \mathcal{C}(\alpha)$, and therefore $A \to B \in \beta$. The latter means that β is closed with respect to the production rules, that is $\mathcal{C}(\beta) \subseteq \beta$. Consequently $\beta R_c \beta$, as required. □

A constructive description of the least quasi-classical production relation containing a given causal theory is given in the next lemma.

Lemma 10.2. *If $\Rightarrow^q_\Delta$ is the least quasi-classical production relation containing a causal theory Δ, then*

$$u \Rightarrow^q_\Delta A \quad \textit{iff} \quad A \in \mathrm{Th}(\overrightarrow{\Delta} \cup \Delta(\alpha)), \ \textit{for any world } \alpha \supseteq u \cup \overrightarrow{\Delta}.$$

Proof. It can be directly verified that the description on the right side of the above equivalence determines a quasi-classical causal relation containing Δ. Since $\Rightarrow^q_\Delta$ is a least such relation, this gives as the direction from left to right. Assume now that $u \not\Rightarrow^q_\Delta A$. Then there exists a world α that is a theory of $\Rightarrow^q_\Delta$ and such that $u \subseteq \alpha$ and $A \notin \mathcal{C}^q_\Delta(\alpha)$. Suppose that $A \in \mathrm{Th}(\overrightarrow{\Delta} \cup \Delta(\alpha))$. Then there exists $a \subseteq \overrightarrow{\Delta}$ such that $\bigwedge a \to A$ belongs to $\mathrm{Th}(\Delta(\alpha))$. The latter means that $\alpha \Rightarrow^c_\Delta (\bigwedge a \to A)$ (see Proposition 7.5), and consequently $\alpha \Rightarrow^q_\Delta (\bigwedge a \to A)$. But we have also $\alpha \Rightarrow^q_\Delta \bigwedge a$ due to quasi-classicality, and consequently $\alpha \Rightarrow^q_\Delta A$—a contradiction. Consequently, $A \notin \mathrm{Th}(\overrightarrow{\Delta} \cup \Delta(\alpha))$, which gives us the direction from right to left. □

The above description shows that derivability in quasi-classical production relations amounts to causal derivability which is based, however, on a classical consequence relation that includes $\overrightarrow{\Delta}$ as the set of axioms. As a consequence, we obtain

Corollary 10.3. *A world α is exact for $\Rightarrow^q_\Delta$ iff $\alpha = \mathrm{Th}(\overrightarrow{\Delta} \cup \Delta(\alpha))$.*

Recall that causally consistent worlds are precisely the worlds that satisfy $\overrightarrow{\Delta}$. Accordingly, the above description says that α is an exact world of $\Rightarrow^q_\Delta$ if and only if it is the only *causally consistent* world containing $\Delta(\alpha)$.

Despite apparent plausibility, the following example shows, however, that quasi-classical inference produce seemingly unexpected results.

Example. A causal theory $\Delta = \{p \Rightarrow q,\ \neg q \Rightarrow \neg q\}$ has an empty causal semantics. However, $p \Rightarrow q$ implies $\mathbf{t} \Rightarrow (p \to q)$ by Weak Deduction, and then $\neg q \Rightarrow (p \to q)$ by Strengthening. Coupled with $\neg q \Rightarrow \neg q$, this gives us $\neg q \Rightarrow \neg p$, namely the contraposition of $p \Rightarrow q$. As a result, $\Rightarrow^q_\Delta$ already has an exact world, namely $\{\neg p, \neg q\}$.

The above example shows, of course, that quasi-classical inference can change the causal nonmonotonic semantics. It still does not show, however, that such an inference can produce downright wrong results. An additional argument in favor of quasi-classical inference can be found in the fact, shown in [10], that such production inference corresponds precisely to abductive reasoning based on *classical* consequence relations (see below). Accordingly, the role of quasi-classical inference seems worth further study.

Finally, note that a natural further strengthening of Weak Contraposition is full Contraposition:

(Contraposition) If $A \Rightarrow B$, then $\neg B \Rightarrow \neg A$.

As can be verified, Contraposition is valid for all symmetric accessibility relations. Note, however, that even this property still does not transform production relations into (reflexive) consequence relations, described in the next section.

10.2. Reflexivity and consequence relations

A distinctive feature of production relations as compared with ordinary consequence relations is the absence of the Reflexivity postulate:

Reflexivity $A \Rightarrow A$.

As can be verified, any supraclassical consequence relation satisfies already all the postulates of regular production relations. Moreover, Reflexivity is then obviously the only postulate that need to be added to the postulates of a regular production relation in order to obtain a supraclassical consequence relation:

Lemma 10.4. *A production relation is a supraclassical consequence relation if and only if it is regular and satisfies Reflexivity.*

Actually, there is a number of classical inference rules that imply Reflexivity in the context of production relations, such as

Deduction If $A \wedge B \Rightarrow C$, then $A \Rightarrow B \to C$;
Import If $A \Rightarrow B \to C$, then $A \wedge B \Rightarrow C$;
Antecedence If $A \Rightarrow B$, then $A \Rightarrow A \wedge B$.

Indeed, $\mathbf{t} \Rightarrow A \vee \neg A$ implies $A \Rightarrow A$ by Import, $A \Rightarrow \mathbf{t}$ implies $A \Rightarrow A$ by Antecedence, and $A \wedge \neg A \Rightarrow \mathbf{f}$ implies $A \Rightarrow A$ by Deduction.

Clearly, any classical consequence relation will already be a kind of a causal inference relation. In addition, the above observations imply

Corollary 10.5. *A production relation is a classical consequence relation if and only if it is regular and satisfies Deduction.*

A simple and uniform way of transforming production relations into consequence relations has been suggested in [25]. Namely, for any production relation, we can consider the following relation:

$$A \Rightarrow^i B \equiv A \Rightarrow (A \to B).$$

Then we have

Theorem 10.6. *If $\Rightarrow$ is a production relation, then $\Rightarrow^i$ is a least reflexive production relation containing $\Rightarrow$. Moreover,*

- *If $\Rightarrow$ is regular, then $\Rightarrow^i$ is a supraclassical consequence relation*;
- *If $\Rightarrow$ is quasi-classical, then $\Rightarrow^i$ is a classical consequence relation.*

Proof. It is easy to verify that $\Rightarrow^i$ is a reflexive production relation. Let $\Rightarrow_1$ be any reflexive production relation that includes $\Rightarrow$, and $A \Rightarrow^i B$. Then $A \Rightarrow A \to B$, and hence $A \Rightarrow_1 (A \to B)$. But $A \Rightarrow_1 A$, and hence $A \Rightarrow_1 B$ by And and Weakening. Thus, $\Rightarrow^i$ is included in $\Rightarrow_1$, and therefore it is a least reflexive production relation containing $\Rightarrow$.

Assume that $\Rightarrow$ satisfies Cut, and we have $A \Rightarrow^i B$ and $A \wedge B \Rightarrow^i C$. Then $A \Rightarrow (A \to B)$ and $A \Rightarrow (A \wedge B) \to C$ by Cut, and hence $A \Rightarrow (A \to C)$ by And and Weakening. Hence, $\Rightarrow^i$ will also satisfy Cut. As a result, if $\Rightarrow$ is regular, then $\Rightarrow^i$ will satisfy already all the rules for supraclassical consequence relations.

Finally, assume that $\Rightarrow$ is quasi-classical, and we have $A \wedge B \Rightarrow^i C$. Then $A \wedge B \Rightarrow (A \wedge B) \to C$. Consequently, $\mathbf{t} \Rightarrow (A \wedge B) \to C$ by Weak Deduction, and therefore $A \Rightarrow (A \wedge B) \to C$, that is, $A \Rightarrow^i (B \to C)$. This shows that Deduction holds for $\Rightarrow^i$, and therefore the latter is a classical consequence relation. $\square$

Note that the above 'reflexivization' construction does not preserve the Or rule. In other words, even for causal inference relations, $\Rightarrow^i$ may be only supra-classical. That is why we needed a stronger condition of quasi-classicality in order to secure that the resulting consequence relation will be fully classical. And indeed, it has been shown in [10] that abductive reasoning with respect to classical consequence relations corresponds already to quasi-classical production relations.

11. Conclusions and perspectives

Summing up the main results of the paper, we can argue that production and causal inference constitute a natural and powerful formalism for nonmonotonic reasoning. In particular, the general nonmonotonic semantics of production inference relations allows to give a syntax-independent representation of abductive reasoning, while the causal nonmonotonic semantics and causal inference provide an exact description of nonmonotonic causal theories from [26]. If we add to this also the correspondence between causal reasoning and logic programming, described in [9], we can safely conclude that, already at this stage, production inference covers a significant part of general nonmonotonic reasoning.

Turning to applications, the causal nonmonotonic reasoning has already proved its usefulness in representing action domains (see [17]) and planning theory [28]. In addition, it is natural to expect that production inference can be used, for example, in formalizing the numerous application areas of abductive reasoning, such as assumption-based truth maintenance (ATMS), update theory [11], or a theory of diagnosis. In fact, a causal approach to diagnosis has already been suggested in the literature—see, e.g., [13,19,32]. As was rightly observed in [32], however, a proper theoretical development of this approach requires an adequate underlying theory of causal reasoning, and we expect that the causal inference constitutes precisely such a theory.

In addition to the above applications, there is still a number of internal problems that need to be resolved in order to fully realize the opportunities created by the causal approach to nonmonotonic reasoning.

As has been shown in [22], causal reasoning copes fairly well McCarthy's principle of *elaboration tolerance*. Still, the nonmonotonic semantics for causal theories is based on the principle of explanation closure, or universal causation, which is obviously very strong. The principle implies, for example, that if we have no causal rules for a certain proposition, it should be false in all intended (explainable) models. This makes causal theories extremely sensitive to the underlying language in which they are formulated, and creates thereby an apparent conflict with elaboration tolerance. Of course, we can always exempt particular propositions from the burden of explanation (by making them abducibles). Moreover, we can even syntactically restrict the principle of universal causation to a particular subset of *explainable* propositions (as has been suggested in [21]). However, this solution is purely syntactical and hence retains language dependence. A proper and systematic trade-off between the principles of explanation closure and elaboration tolerance still has to be found.

More subtle, yet perceptible difficulties arise also in representing indeterminate causation in causal theories (see [8]). Thus, we have seen that any causal theory is reducible to a determinate theory with respect to the causal nonmonotonic semantics. Consequently, non-Horn causal rules are informative nonmonotonically only to the extent that they imply some Horn rules. This does not mean, however, that we cannot represent indeterminate information in causal theories. Actually, one of the main contributions of [26] consisted in showing how we can do this in common cases (see also [24]). Still, there is yet no systematic understanding whether all kinds of indeterminate information are representable in this way by Horn causal rules.

A more general problem concerns the role and place of causal reasoning in general nonmonotonic reasoning. Though causal reasoning covers many areas of nonmonotonic reasoning, it does not cover them all. Thus, it does not seem suitable for solving the *qualification problem* in representing actions, due to the fact that causal rules are monotonic and freely admit strengthening of their antecedents. Speaking generally, causal nonmonotonic reasoning still belongs to the explanatory approach to nonmonotonic reasoning, and hence it does not cover the kind of nonmonotonicity described by the preferential approach (see Introduction). Nevertheless, the causal reasoning creates a new opportunity here. Namely, both preferential and causal reasoning amount, in effect, to different reasoning systems about *conditionals*. This naturally suggests that the two kinds of nonmonotonic reasoning with conditionals could be combined into a single formalism, a grand uniform theory of nonmonotonic reasoning. Actually this idea is not new; it has been suggested and studied more than ten years ago in [14]. It remains to be seen whether the suggested theory of causal inference can contribute to viability of such a general theory.

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References

[1] C. Baral, M. Gelfond, Logic programming and knowledge representation, J. Logic Programming 19, 20 (1994) 73–148.
[2] N.D. Belnap Jr., A useful four-valued logic, in: M. Dunn, G. Epstein (Eds.), Modern Uses of Multiple-Valued Logic, D. Reidel, Dordrecht, 1977, pp. 8–41.
[3] A. Bochman, Biconsequence relations: a four-valued formalism of reasoning with inconsistency and incompleteness, Notre Dame J. Formal Logic 39 (1) (1998) 47–73.
[4] A. Bochman, A logical foundation for logic programming, I and II, J. Logic Programming 35 (1998) 151–194.
[5] A. Bochman, On the relation between default and modal nonmonotonic reasoning, Artificial Intelligence 101 (1998) 1–34.
[6] A. Bochman, A Logical Theory of Nonomonotonic Inference and Belief Change, Springer, Berlin, 2001.
[7] A. Bochman, A logic for causal reasoning, in: G. Gottlob, T. Walsh (Eds.), Proceedings Internat. Joint Conference on Artificial Intelligence, IJCAI’03, Acapulco, Mexico, Morgan Kaufmann, San Mateo, CA, 2003, pp. 141–146.
[8] A. Bochman, On disjunctive causal inference and indeterminism, in: G. Brewka, P. Peppas (Eds.), Proc. Workshop on Nonmonotonic Reasoning, Action and Change (NRAC’03), 2003, pp. 45–50.
[9] A. Bochman, A causal logic of logic programming, in: D. Dubois, C. Welty, M.-A. Williams (Eds.), Proc. Ninth Internat. Conf. on Principles of Knowledge Representation and Reasoning, KR2004, Whistler, BC, AAAI Press, 2004, pp. 427–437.
[10] A. Bochman, Production inference, nonmonotonicity, and abduction, in: Proc. Eight Internat. Symp. on Artificial Intelligence and Mathematics, Fort Lauderdale, FL, 2004.
[11] C. Boutilier, Abduction to plausible causes: an event-based model of belief update, Artificial Intelligence 83 (1996) 143–166.
[12] L. Console, D. Theseider Dupre, P. Torasso, On the relationship between abduction and deduction, J. Logic Comput. 1 (1991) 661–690.
[13] A. Darwiche, Model-based diagnosis using causal networks, in: Proceedings Internat. Joint Conf. on Artificial Intelligence, IJCAI’95, Montreal, Quebec, 1995, pp. 211–217.
[14] H. Geffner, Default Reasoning. Causal and Conditional Theories, MIT Press, Cambridge, MA, 1992.
[15] M. Gelfond, V. Lifschitz, The stable model semantics for logic programming, in: R. Kowalski, K. Bowen (Eds.), Proc. 5th International Conf./Symp. on Logic Programming, MIT Press, Cambridge, MA, 1988, pp. 1070–1080.

[16] M. Gelfond, V. Lifschitz, H. Przymusińska, M. Truszczyński, Disjunctive defaults, in: Proc. Second Internat. Conf. on Principles of Knowledge Representation and Reasoning, KR'91, Cambridge, MA, 1991, pp. 230–237.

[17] E. Giunchiglia, J. Lee, V. Lifschitz, N. McCain, H. Turner, Nonmonotonic causal theories, Artificial Intelligence 153 (2004) 49–104.

[18] K. Konolige, Abduction versus closure in causal theories, Artificial Intelligence 53 (1992) 255–272.

[19] K. Konolige, Using default and causal reasoning in diagnosis, Ann. Math. Artificial Intelligence 11 (1994) 97–135.

[20] S. Kraus, D. Lehmann, M. Magidor, Nonmonotonic reasoning, preferential models and cumulative logics, Artificial Intelligence 44 (1990) 167–207.

[21] V. Lifschitz, On the logic of causal explanation, Artificial Intelligence 96 (1997) 451–465.

[22] V. Lifschitz, Missionaries and cannibals in the causal calculator, in: Proc. 7th Internat. Conference on Principles of Knowledge Representation and Reasoning, KR'00, Breckenridge, CO, Morgan Kaufmann, San Mateo, CA, 2000, pp. 85–96.

[23] F. Lin, Embracing causality in specifying the indirect effects of actions, in: Proceedings IJCAI-95, Montreal, Quebec, Morgan Kaufmann, San Mateo, CA, 1995, pp. 1985–1991.

[24] F. Lin, Embracing causality in specifying the indeterminate effects of actions, in: Proceedings AAAI-96, Portland, OR, 1996, pp. 670–676.

[25] D. Makinson, L. Van der Torre, Input/Output logics, J. Philos. Logic 29 (2000) 383–408.

[26] N. McCain, H. Turner, Causal theories of action and change, in: Proceedings AAAI-97, Providence, RI, 1997, pp. 460–465.

[27] N. McCain, H. Turner, On relating causal theories to other formalisms, Unpublished manuscript, 1997.

[28] N. McCain, H. Turner, Satisfiability planning with causal theories, in: Proc. Internat. Conf. on Principles of Knowledge Representation and Reasoning, KR-98, Trento, Italy, 1998, pp. 212–223.

[29] N. McCain, Causality in commonsense reasoning about actions, PhD thesis, The University of Texas at Austin, 1997.

[30] J. McCarthy, P. Hayes, Some philosophical problems from the standpoint of artificial intelligence, in: B. Meltzer, D. Michie (Eds.), Machine Intelligence, Edinburg University Press, Edinburg, 1969, pp. 463–502.

[31] J. Pearl, Embracing causality in formal reasoning, Artificial Intelligence 35 (1988) 259–271.

[32] D. Poole, Representing diagnosis knowledge, Ann. Math. Artificial Intelligence 11 (1994) 33–50.

[33] R. Reiter, A logic for default reasoning, Artificial Intelligence 13 (1980) 81–132.

[34] R. Reiter, The frame problem in the situation calculus: a simple solution (sometimes) and a completeness result for goal regression, in: V. Lifschitz (Ed.), Artificial Intelligence and Mathematical Theory of Computation: Papers in Honor of John McCarthy, Academic Press, San Diego, CA, 1991, pp. 359–380.

[35] R. Reiter, Knowledge in Action: Logical Foundations for Specifying and Implementing Dynamical Systems, MIT Press, Cambridge, MA, 2001.

[36] M. Thielscher, Ramification and causality, Artificial Intelligence 89 (1997) 317–364.

[37] H. Turner, A logic of universal causation, Artificial Intelligence 113 (1999) 87–123.

[38] D. Zhang, N. Foo, EPDL: A logic for causal reasoning, in: Proceedings IJCAI'01, Seattle, WA, Morgan Kaufmann, San Mateo, CA, 2001, pp. 131–136.