

Pouring liquids: A study in commonsense physical reasoning

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ABSTRACT

This paper presents a theory that supports commonsense, qualitative reasoning about the flow of liquid around slowly moving solid objects; specifically, inferring that liquid can be poured from one container to another, given only qualitative information about the shapes and motions of the containers. It shows how the theory and the problem specification can be expressed in a first-order language; and demonstrates that this inference and other similar inferences can be justified as deductive conclusions from the theory and the problem specification.

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1. Introduction

Carrying liquids in containers and pouring or ladling liquids from one container to another are among the most common ways in which people interact with liquids in daily life. People are very familiar with these phenomena and can reason about them easily. In particular, people understand how the physical behavior of the liquids is largely determined by the geometrical characteristics of the liquid, the containers, and the motions involved; they can reason about physical behavior using only partial knowledge of the geometry, without full geometric specifications; and they can use the same knowledge in multiple inferential directions.

For instance, people know that, if a cup has a small hole through the bottom, then liquid in the cup will leak out through the hole, but that a dent in the bottom will not cause the liquid to leak. They can use this knowledge in many ways: prediction—given that there is a hole, predict that the liquid will leak; explanation—given that the liquid is leaking from the bottom, deduce that there is a hole; design—if you want the liquid to drain (e.g. you are designing a colander), put a hole in the bottom; and so on. These various forms of reasoning can be carried out without knowing or positing a precise shape description for the cup or the hole.

It is very desirable that an automated reasoner likewise be able to deal with partial geometric information. Precise geometric information may be unavailable for a number of different reasons. It may not be possible for the agent to perceive or measure the features accurately. The features may be inferred rather than perceived. The object may be in a preliminary state of design, and the precise geometry may not yet have been specified. The features may be a result of a future event which is not yet fully known; for instance, a reasoner may be concerned that an object may spring a leak and worry about the effect on the liquid inside, without knowing where exactly the leak will be or what its shape will be. A reasoner may need to reason generically about classes of similar objects and similar actions rather than about a single manipulation of a single object.

The theory of fluid dynamics, of course, contains a very large body of mathematics, mathematical physics, and scientific software devoted to the question of predicting the flow of fluids; and these computations can now be done with very great

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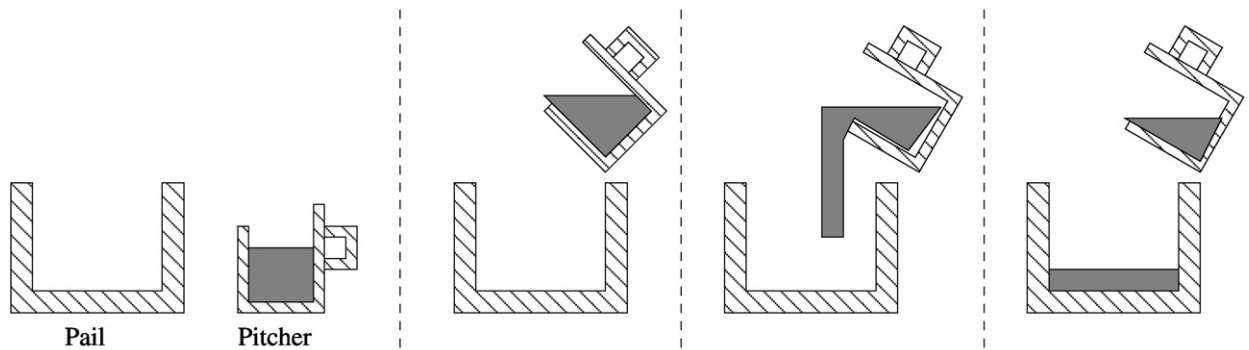


Fig. 1. Pouring from pitcher to pail.

accuracy and speed. However, these techniques all work, either by using a fine-grained division of space and time, and by calculating the force and flow of each small piece of liquid at small step of time; or, if the PDE's are solved or analyzed exactly, by calculating the force and flow at literally every point and instant. The techniques deliver extremely precise predictions of fluid flow, but they require correspondingly precise specification of the boundary conditions (the shapes of the solid objects in contact the liquid).

As the evolution of forces and flows may be extremely variable over a range of circumstances where the overall qualitative behavior is quite stable, there is an inherent mismatch between these techniques and the objectives of qualitative reasoning. In AI applications, precise boundary conditions are generally not known and detailed predictions are not necessary. Different ways of pouring from a pitcher to a pail, or different shapes of the pitcher and the pail, may give rise to flow and force patterns that are completely different; but the commonsensically important inference, that the liquid pours from the pitcher to the pail, remains stable.

For this reason, we are looking for a characterization of the behavior of liquids that does not require calculating of velocity, acceleration, momentum and forces. Not that these concepts lie outside a commonsense understanding of physics—on the contrary these, or something similar, are part of a commonsense understanding—but it must often be possible for a qualitative reasoner to reason about the large scale behavior of liquids without invoking these concepts.

The objective of this paper is to characterize some cases of commonsense reasoning about liquids at the knowledge level [28]; that is, to demonstrate that, for some types of simple qualitative reasoning about liquids, one can develop a representation language and a theory such that the knowledge used in the reasoning and the specifications of particular problems can be (approximately) expressed in the language and the reasoning itself can be carried out as inference from the general theory and specifications. This paper is thus part of the general programme proposed by Hayes and by McCarthy [18, 24,25] of developing automated commonsense reasoners by representing commonsense knowledge in logic-based languages. (We will discuss the goals of the representation in more detail in Section 1.1.)

In this paper we develop a large part, though not all, of a commonsense theory of liquid flowing around slowly moving solid objects. We illustrate the adequacy of the theory by showing that it suffices for correct prediction in a number of scenarios, including carrying a liquid in a closed container or in an open container, pouring a liquid from one container to another, or ladling liquid out of a container using a spoon.

The primary example we will use here is pouring from one container to another. Specifically, we consider the following scenario (Fig. 1): There is a pitcher, partly full of liquid, and an empty pail. Both of these “hold water”. The pail remains in a fixed position throughout the scenario. The pitcher is lifted, keeping it sufficiently upright that the liquid inside does not reach its spout. Once it is in position, with the spout (though not necessarily all of the pitcher) centered over the pail, the pitcher is tilted until the capacity of the part of the inside of the pitcher lower than the spout is less than the volume of the liquid. At this point, the liquid pours out of the pitcher, and falls downward into the pail, where it remains. At the end of the scenario, the liquid is divided into a section that remains in the pitcher and a section that is in the pail. We demonstrate that, given qualitative characterizations of the shapes of the pitcher and the pail and of the motion of the pitcher, our theory allows us to infer the behavior of the liquid.

(Note: all of the figures in this paper are cross-sections in the x - z plane. Throughout this paper, solid objects are indicated with diagonal lines, and liquid is indicated in grey. The fact that the pictures show liquid flowing in polygonal patterns reflects my personal limitations in using the drawing software; it is not at all a requirement of the theory.)

Many aspects of the commonsense understanding of liquids are omitted from our analysis here. Some of the most important of these are:

- Liquids in modes that are not “bulk”, in Hayes' [19] terminology, such as mists, wettings of surfaces, liquids absorbed in sponges, and so on.
- Liquids in *energetic* modes, again in Hayes' terminology, such as fountains or even splashes.
- Mixtures or solutions of any kind.
- Interactions of liquids with the atmosphere or other gasses.

- The effect of liquids on the solids with which they are in contact. We assume that the motion of the solids is given by external constraints. Thus, our theory does not include waterwheels or other mechanisms controlled by hydraulics, solid objects floating on liquids, swimming, and so on. (The theory may be capable in such cases of predicting the liquid flow given the motion of the solid objects, but it certainly cannot predict the motion of the solid objects.)
- Pressure and any consequences of pressure differences. In particular, we assume that all parts of the top surface of a liquid meet the open atmosphere and are therefore at equal height.
- Viscosity, surface tension, cohesion, adhesion, absorption, and so on. We deal only with “dry water”, in von Neumann’s sardonic phrase.
- Any consideration of heat, temperature, and phase transitions, such as evaporation and freezing.
- The feasibility of actions by an agent. The theory developed for physical feasibility of actions on solid objects in [11] can be extended to this domain, but we will not discuss this in this paper.
- The theory yields incorrect predictions for liquids flowing down a channel (Section 6). This is probably the most important gap in the theory.

The paper is organized as follows. Section 1.1 discusses the various goals of this representational work in more detail. Section 2 reviews previous relevant work. Section 3 gives a pre-formal account of the physical theory that we develop here. Section 4 shows how the theory can be formalized; it presents the ontology, the formal language, and the axioms. Section 5 describes a problem specification for a scenario of carrying a liquid in a pitcher and pouring it into a pail; sketches the structure of the formal inference; and demonstrates that the axioms and specifications are consistent by exhibiting a specific model satisfying them. (A complete formal proof of the inference is given in the online appendix <http://www.cs.nyu.edu/faculty/davise/papers/liqAppa.pdf>.) Section 6 discusses the problem of flow in a channel, which is a major gap in this theory. Section 7 presents the conclusion and discusses possible next steps for research.

1.1. Objectives

This paper undertakes a representational project with the following parts:

- We define a microworld of solid objects and liquids, which includes many of the basic large-scale behaviors of liquids carried in solid containers, poured out of and into solid containers, and flowing over solids. We state a number of physical laws that govern the behavior of liquid in the microworld.
- We develop a formal ontology and a first-order representation language in which we can formally state these laws.
- Extending this representation language with some additional geometric predicates, we construct a specification of a scenario of a pitcher pouring liquid into a pail with incomplete geometric specifications of the shapes and motions of the pitcher and the pail. We demonstrate that, from the rules and the specifications, we can infer that some of the liquid pours from the pitcher.

Of course, the physical theory is not intended to apply only to this particular specification; it should apply to many different specifications and different directions of inference within this microworld, and we are quite confident that it does (Section 5.3).

There are many ways in which such a representational project could be carried out, and numerous different desiderata for the theory and for the problem specification:

1. **The theory and specifications should be logically sufficient.** The conclusion must be provable from the specifications and the physical theory.
2. **The specifications should be qualitative.** It would be (comparatively) easy to give geometrically precise descriptions of shape and motion that would suffice to justify the conclusion. But that kind of inference can be done much better—more precisely, robustly, and generally—by conventional simulation methods. The whole point of the theory we have developed is that it supports reasoning from incomplete specifications. Other things being equal, the more general the specifications the better.
3. **The theory should be general.** The theory should not apply only to the problem in question but to a wide range of problems in the microworld. It should be usable not only for prediction but also for other directions of inference (postdiction, planning, design, etc.).
4. **The specifications for a prediction problem should be geometric.** A well-formed prediction problem should consist of specifications of the shape of the objects; the initial state of the liquid; constraints on the motion of the objects; and isolation conditions excluding outside interference (no other object or liquid comes close to the scene of the action). This is essentially the “No function in structure” rule of [15].
5. **Specifications should be logically simple and use a fixed geometric vocabulary.** In a knowledge base that uses this representation, the general theory corresponds to the fixed knowledge base whereas the specification corresponds to the run-time query. Therefore, complex logical form or specialized predicates may be tolerable in the theory, since these can be incorporated off-line into workable algorithms, but it is important that queries in the specification language be

logically simple combinations of known predicates (e.g. conjunction of ground formulas) if there is to be any hope of achieving run-time efficiency.

6. **The specifications should be ecologically valid.** The constraints should correspond to the kinds of partial information that are available in actual applications, or should be easily derivable from that information.
7. **The theory and specifications should be computationally tractable.** It should be possible for a program to draw the desired inferences from the specifications in reasonable time. Until there is a corpus of examples and a proposal on the table for an inference algorithm, there is no way to evaluate this. The theoretical worst-case results are all discouraging; spatial languages much more restricted than this one are intractable or undecidable [2].
8. **The theory should bear some relation to physical reality.** The theory should be approximately correct, or correct in the limit in some sense, or correct as an average in some sense, or in one way or another correspond to the physical reality. A theory with physically very unrealistic assumptions is likely to go very badly wrong sooner or later.
9. **The theory should be plausible as a cognitive model.** To the extent that this can be judged reliably, it should be plausible that the predicates correspond to concepts available to naive human reasoners. It should be noted that almost nothing is known about what concepts are used for qualitative physical reasoning by human reasoners, so it would be a mistake to let one's own guesses on the subject dictate the form of one's theory.
10. **It is desirable that the predicates should be standard in computational geometry.** It would be helpful, though it is not critical, if the geometric predicates used were well-known and well-understood in the literatures of computational geometry, graphics, CAD, physics, scientific computing etc. so that we could draw on methods from those disciplines and interface easily with them.

The theory and problem specifications developed in this paper presented below certainly satisfy conditions 1–4 and, in their current form, certainly do not satisfy condition 7. On the whole we have very much favored achieving the generality advocated in condition 3 over conditions 6–10. In most respects, the theory is approximately physically correct (condition 8); in some aspects, which we will point out, the approximation is quite crude. Condition 5 is hard to evaluate until many different types of problems have been formalized, which we have not yet done. Some of the predicates in our representation are standard (condition 10); many as far as I know as new here and of very limited applicability.

As regards cognitive plausibility (condition 9): At first blush it would seem absurd to claim any degree of cognitive plausibility for a theory with concepts like “continuity relative to the Hausdorff distance” (Section 3.7) or “bubble-free attachment” (Section 5). But the truth is that the commonsensical human reasoner has some conceptualization of the fact that liquids move continuously and of the geometric relation between a container and an opening. The formal definitions here are intended as mathematical definitions that idealize or approximate the intuitive conception, and though the formal definition is not very readable, I would argue that the actual concept is reasonably close. That is, this jargon is just a translation of the intuitive conceptualization of “continuous liquid motion” into “math-talk”, and, it seems to me, a reasonably close translation. The major physical rules that are developed in this paper (Section 3.1) are, I would argue, quite reasonable candidates for a cognitive model, though the details are designed in part to avoid the need for plausible reasoning, which a true cognitive model would, of course, deal with. At minimum, it seems to me that the concepts and rules here constitute a worthwhile first stab at a cognitive theory of the domain. I do not claim that the cognitive *processes* involved at all resemble the formal proofs in the appendix.

The important sticking point here is condition 6. As argued at length in [8], though it is clear that many AI applications require reasoning from partial information, it is not at all easy to know, for a given application, exactly what kinds of partial information are involved; and it is often very hard, for a given domain of interest, to find an application that naturally generates a collection of inferences from partial information to satisfy the requirements of condition 4. It is obvious that people can reason about pouring liquid from a pitcher without knowing the precise shape of the pitcher; it is much harder to say what it is they do know about the pitcher. Moreover, in realistic applications, some of the information that is available may not satisfy either condition 1 or condition 3 above. For instance, one may know “I have often poured liquid from this pitcher without trouble” or “The text refers to this object as a ‘pitcher’; hence, it is presumably possible to pour from it”. But clearly neither of these facts support very strong deductive conclusions, nor are they geometric.

2. Related work

It is unusual, particularly in a fast-moving field like AI, for a survey of previous related work to be dominated by a single paper—Pat Hayes’ “Ontology for Liquids” [19], a companion paper to Hayes’ “Naive Physics Manifesto” [18]—which was written almost thirty years ago,² immediately became very famous, and has barely been touched from then to now.

Besides proposing liquids as a challenging and fruitful domain for research in automated commonsense physical reasoning, and proposing as an initial step the axiomatization of the domain in a first-order language, “Ontology for Liquids” made a number of critical contributions. The most important of these (certainly in terms of its influence on this paper) was to show that a critical category in reasoning about liquids is that of a region of space that evolves over time, and that such an

² “Ontology for Liquids” was written as a working paper in 1979 and circulated in photocopies for six years before it was published in 1985.

evolving region should be considered as an entity in the domain of discourse. (Hayes calls this a “history”, whereas in this paper it is called a “region-valued fluent”.) Other important contributions of Hayes’ paper include:

- The categorization of different modes of liquids, in a commonsense understanding, along four, largely independent, dimensions: bulk vs. divided, still vs. moving and lazy vs. energetic, unsupported vs. supported, in an extended space vs. close to a surface. (Of the 24 logical combinations, Hayes claims that 15 are physically possible.)
- The development of a qualitative language of geometry guided by the demands of a physical theory. This is one of the earliest works in AI on qualitative geometry, and, despite all the subsequent work in this area [2], still one of comparatively few that is designed to fit with a physical theory. (DeKleer’s NEWTON [14] predates it in both respects, but is much more limited in both the geometry and the physics.)

Moreover, the discussion in “Ontology for Liquids” is in many respects broader than this paper. Hayes touches on such subjects as the fifteen modes of liquids mentioned above, liquids adhering to surfaces, solid objects floating on liquids, and others, which we do not consider here.

However, the logic presented in “Ontology for Liquids” has substantial gaps and flaws. The most serious of these is that, in developing his representation, Hayes assumes that “Almost all the histories we consider in this paper ... consist[] of a certain piece of space in which something happens for a certain length of time; they are *rectangular*” ([19], p. 90). Both the representation and the axiomatization depend strongly on this assumption. But the two examples he considers (a bathtub overflowing, a liquid flowing from a tilted container onto a surface) do not at all correspond to this assumption; almost none of the significant histories involved are rectangular. For that reason, it is actually impossible to fill in the details of the logical analysis of these examples given in Section 9 of “Ontology for Liquids”. This gap is very much obscured for the reader (and, probably, for Hayes himself) by the fact that Hayes does not give a logical statement of the specification of the problems being addressed in Section 9.³

Rectangular histories are the norm only when the system is in the steady-state, and over the class of steady-state histories, Hayes’ theory is indeed often adequate. But pouring is not a steady-state system, and rectangular histories do not come close to dealing adequately with it.

Another problem, which as we shall see arises also with theories of very different kinds, has to do with the physics of overflowing. As we shall discuss in more detail in Section 3, when an open container overflows at the top, the liquid necessarily rises somewhat above the top of the container and spills out some horizontal distance beyond the edge of the container. Neither of these is consistent with Hayes’ theory. In particular, as soon as a liquid goes beyond the edge of the container, then it must fall straight down; therefore, it cannot travel outward any finite distance.

2.1. Other AI theories of liquids

A number of other AI theories have dealt with geometrically and physically rich models of liquids. Gardin and Meltzer [17] simulate liquids in terms of particles interacting in a two-dimensional space (one vertical and one horizontal). They also simulate rigid objects, flexible objects, and strings in the same way. Obviously, no such representation can deal adequately with qualitative spatial reasoning. Less obviously, the rules of physical interaction they propose lead to a system whose behavior is determined by the grain-size of the discretization of time and space. (In standard scientific computation, as the grain-size becomes increasingly fine, the predictions hopefully *converge* on the actual behavior.) Specifically, in the “pouring” scenario, once a particle of water has gone past the lip of the pitcher, it is unsupported, and therefore must fall straight down. The result is that the liquid pours out of the pitcher in a column one molecule thick. (This is quite clearly illustrated in [17] p. 679, Fig. 19.10.)

DeCuyper et al. [13] point out, correctly, some of the limitations of Hayes’ analysis, particularly in terms of the spatial language, and propose a hybrid architecture for reasoning about liquids combining three representations: a particle-based representation which tracks the movement of particles of liquid; a fixed grid partition of space which tracks the occupancy of each voxel of space, and a qualitative representation similar to Hayes’. They do not give the details, and, frankly, it seems unlikely that this was ever worked out in detail. It is not difficult to imagine a system that can compute a simulation and abstract a qualitative representation, though this is certainly a non-trivial challenge; but it is very difficult to devise a system in which there is significant information flow in the opposite direction, from the qualitative analysis to the simulation.

Kim [21] developed a system that carried out qualitative predictions of the motions of liquids in response to the motions of pistons. She also included in her model a special case of solids being acted on by liquids, namely the opening and closing of one-way valves. Both the geometric and physical language of this system were quite limited.

Many AI physical reasoning systems have used a geometry-free model of liquids, characterizing the state of liquid purely in terms of the quantity of liquid contained in specified containers, and characterizing flow just as a process that decreases the quantity in the source and increases the quantity in the destination. For instance Forbus [16] uses this as an example

³ This gap was also critical to the debate over Hayes’ argument between McDermott in his “Critique of Pure Reason” [26] and Hayes in his response [20]. McDermott claimed that Hayes’ bathtub examples required non-monotonic reasoning; Hayes claimed that they required only *modus tollens*. The truth is that, in the absence of a specific problem specification, there is no way to know which closed world assumptions are to be considered part of the problem specification, and which are to be considered as deduced using non-monotonic principles.

for qualitative prediction; Shin and Davis [34] use it as an example for planning with quantities that are linear functions of time (the system assume that all flows are constant-rate). Collins and Forbus [3] combine this with a model of liquids in terms of collections of molecules, which supports some more sophisticated inferences.

2.2. Logical theories of commonsense physical domains

The work described in this paper is part of an ongoing project to develop formal theories of commonsense physical domains. This work is mostly presented in three previous papers.

In [4] I presented a formal theory of a very small part of solid object dynamics, capable of supporting the inference that a marble dropped inside a funnel would come out the bottom. The most significant technical innovation here is the concept of a “pseudo-object”, a geometric entity that “moves around” with a rigid object, such as the hole of a doughnut or the center of mass of an object (Section 4.5). This theory has been very much extended in [11], which supports the conclusion that a collection of objects can be loaded into a box and then carried in the box. That paper, unlike this one, includes an explicit theory of manipulation by an agent.

An axiomatization of a kinematic model of one solid object cutting another is given in [7]. Two theories are presented. The “object” theory views the process of a blade cutting a target object as involving a continuous change in the shape of the target until it splits, when it becomes two objects. The “chunk” theory views the same process in terms of the chunks of solid material contained in the target. (Every separate region defines a separate chunk.) A chunk persists until it is penetrated by the blade, at which point it ceases to exist. These chunks are closely analogous to the “liquid chunks” studied in this paper.

All of these theories are mutually compatible, and I hope to present them in an integrated form in a monograph currently under preparation [12].

I have also done other, more minor and less well integrated, work on the formalization of physical reasoning.

Chapter 7 of [5] gives preliminary axiomatizations for a number of physical domains, including liquids. An axiomatization of qualitative process theory is given in [6]. The main issue here is to formulate the closed world assumptions correctly.

The methodological paper [8] advocates structuring domain theories around microworlds rather than clusters, as proposed in Hayes’ “Naive Physics Manifesto” [18].

Other than the above, there has been little work on logical formalizations of physical domains since Hayes’ original papers, despite their fame and popularity. Schmolze [32] presents an axiomatization for a domain that includes actions, events, processes, liquids, solid containers, and faucets. A liquid is modelled as a collection of “granules”.

Sandewall [31] developed a logical description of a microworld of points objects moving along surfaces. The chief focus of this work was integrating non-monotonic logic with a continuous model of time. Fluents were assumed non-monotonically to be continuous at each point in time; thus a model with a minimal class of discontinuities was preferred.

Three parallel papers by Lifschitz, Morgenstern, and Shanahan [23,27,33] axiomatize various aspects of the process of cracking an egg into a bowl.

Bennett et al. [1] present an axiomatization of solid object kinematics built up from geometrical primitives.

2.3. Fluid dynamics

The elephant in the room here, which I can neither ignore nor adequately deal with, is the immense, deep, and sophisticated mathematical and computational theory of fluid dynamics. It would require years of study to gain a sufficient mastery of this field to be confident that the problems I am studying cannot be solved by existing techniques; or that an expert in the field could not develop a theory of power comparable to the one here, but grounded in standard fluid dynamics. Such a theory would by definition be more physically correct than the one here; it would almost certainly be easier to integrate with standard methods of physical calculation; it might well be more extensible, easily to validate for consistency, perhaps even a more plausible cognitive model. All I can say is that I have not found any evidence that the kind of reasoning studied in this paper has been addressed in the existing theory of fluid dynamics, or indeed is of any interest to researchers in fluid dynamics. As far as I have been able to determine, none of the techniques that have been developed in fluid dynamics can be applied at all directly to the kinds of qualitative information that we are considering here.

3. The physical theory

The physical theory developed in this paper deals with the motion of liquid around slowly moving solid objects. We begin with a high-level sketch of the theory (Section 3.1); we then work through a detailed formulation of our proposed physical rules, (Sections 3.2–3.5). Section 3.6 discusses the ontology of liquids. Section 3.7 discusses the constraint that liquids move continuously.

3.1. The physical theory: High-level view

Our ontology, like Hayes’, has two dual categories that we use to characterize the motion of liquids. The first category is *region-valued fluents* (Hayes calls these “histories”): fluents whose value at each time is a region, such as “the region

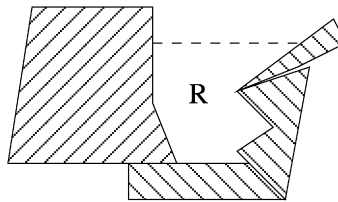


Fig. 2. Region R is cupped.

contained inside pitcher P1". Many aspects of liquid motion can be characterized in terms of the quantity of liquid contained in significant region-valued fluents, but some aspects, such as steady-state flow or cyclic flow, cannot. The second category is *bodies* or *pieces* or *chunks* of liquid (we will use these terms interchangeably). A piece of liquid is a particular collection of matter; a set of molecules, if you like, though our theory is not based on molecules. Thus we can say, "Some of the piece of liquid that was in the pitcher at the start of the scenario is in the pail at the end of the scenario". By contrast, in the language of region-based fluents all that we can say is that at the end of the scenario there is less liquid in the pitcher and more liquid in the pail than at the beginning. To say that the liquid that is in the pail at the end is the *same* as the liquid that was in the pitcher at the start and is no longer in the pitcher requires an identity criterion on liquid over time, and thus requires the category of pieces of liquid. Hayes prefers using histories to using pieces of liquids; we use the two concepts about equally.

Essentially, our theory of liquids consists of the following rules:

Rule 1. Liquids cannot penetrate solid objects.

Rule 2. The volume of a particular body of liquid is constant (incompressibility).

Rule 3. The motion of liquid through space is continuous over time (we will define the sense of "continuity" required here in Section 3.7).

Rule 4. A connected region R is said to be "cupped" at a given time if the outside of the boundary of R consists of solid objects everywhere except at a top, horizontal surface (Fig. 2). If a region (more precisely, a region-valued fluent) R is cupped over a period of time and is never full of liquid, then no liquid flows out of R .

Rule 5. If a body of liquid can flow straight down without encountering a solid object or a cupped liquid, then it does flow straight down.

Rule 6. If a body of liquid is in contact with a solid or cupped liquid but can flow downward around it, then it does flow downward around it.

Given these rules, one can (informally) justify the analysis of the pouring scenario as follows: By Rule 4, the liquid remains in the pitcher as the pitcher is lifted into position. By Rule 2, once the volume inside the pitcher below the spout falls below the volume of the liquid, the liquid can no longer be held in the pitcher; since the pitcher prevents it from flowing out the bottom or out the sides (Rule 1), it flows out the top. At this point, the part of the liquid above the top of the spout cannot flow straight downward because of the cupped liquid underneath it, but it can flow at an angle downward over the spout following Rule 6. Once it has cleared the spout, it can flow straight downward, so it does so by Rule 5. This brings it inside the pail. Once it is inside the pail, it cannot escape by Rule 4.

The objective here is to formulate these rules in a way that they are mutually consistent, strong enough to predict the result of the pouring scenario, and flexible enough to be usable with qualitative spatial information.

The first problem is that the overflow of the pitcher in the pouring scenario necessarily contradicts Rules 5 and 6. If liquid pours out of a spout, then it must necessarily flow up somewhat higher than the level of the spout and flow out horizontally somewhat beyond the spout (Fig. 3). Suppose that the liquid rises to height DH over the top of the spout and flows out to a distance DW beyond the spout. Then at the earlier time when it had only flowed out to distance $DW/2$ beyond the spout it could have flowed straight down, and should have, if Rule 5 applied. Likewise, at the time when the liquid had risen to height $DH/2$ above the spout, it should have started to flow downward over the spout if Rule 6 applied, and not have continued to flow upward.

In a continuous model, such as we will use here, this leads to a Zeno-like paradox. In a molecular theory, it leads to the conclusion that the liquid rises one molecule about the level of the spout and then pours out in a column one molecule thick. As mentioned above, this is exactly the conclusion that Gardin and Meltzer's program does come to [17].

The same problem can arise in any situation where a flow over an object turns into free flow. If a liquid has been flowing down a slanted surface, and then comes to a precipice (Fig. 4) then it must flow some distance beyond the edge before

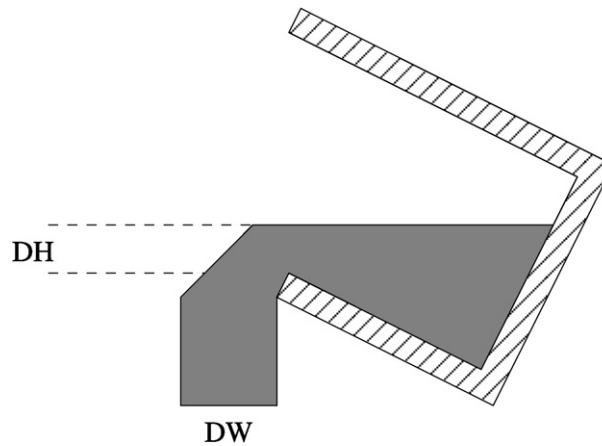


Fig. 3. Flowing over and past of the spout.

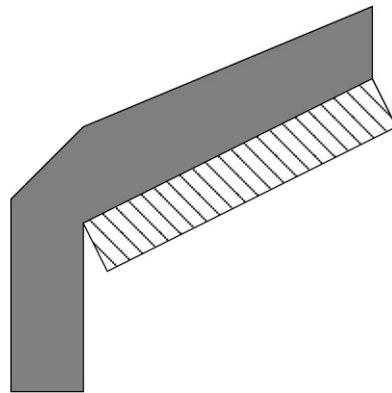


Fig. 4. Overflow past a precipice.

falling straight down, but, again, if it does flow a finite distance beyond the edge, then it is violating Rule 5 by not starting to flow straight down sooner.

Finally, the problem arises, though in a less clear-cut form, when a downward flow terminates in an object. If a vertical column of liquid is flowing downward and hits a surface, then any part of the liquid above the object can flow downward and by Rule 5 must do so; but it is not possible that the horizontal flow of the liquid begins exactly at the surface.

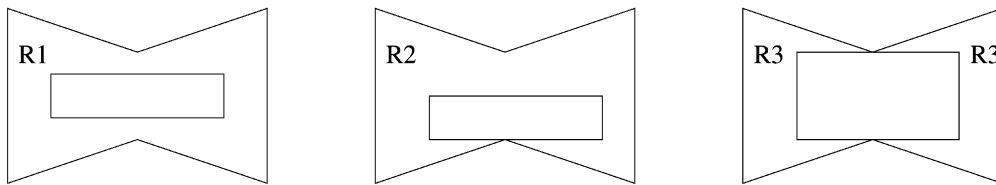
The true explanation here, of course, involves the momentum of the liquid; but as we have discussed, we wish if possible to avoid reasoning paths that require reasoning about momentum. Instead, what we will do here is to posit that these kinds of scenarios generate exceptions to Rules 5 and 6.

This solution in turn raises two further difficulties. The first is that, in a monotonic logic such as we use here, if one has stated that a rule has exceptions, then ever after, in order to apply the rule, one must ascertain that the exception does not hold. As we shall see, this places demands both on the domain axiomatization and on the problem specification.

The second, more important, problem is that some limits must be given on how far, spatially, these exceptions extend: How far can liquid rise above the wall of a cup before it is required to flow down, and how far horizontally can it flow out beyond the rim of a cup before it is required to flow straight down? These are necessary, first because it is obviously the case that liquid in this scenario cannot flow indefinitely far up or out, and more specifically because this conclusion is necessary to carry out the desired inference. If we cannot bound how far the liquid flows beyond the spout of the pitcher, then we cannot predict that it will flow into the pail; it may overshoot the pail.

In this paper, we kludge the problem for the sake of simplicity. We define a characteristic distance `maxOutflow`, and we posit that liquid overflowing from a cup rises at most `maxOutflow` above the top of the cup and flows out at most `maxOutflow` beyond the rim.⁴ This is a very crude approximation, and commonsense understanding is actually substantially richer, but for our purposes here it will suffice. The actual distance followed up and out by liquid overflowing from a cup is a complex function of the shapes and motions involved. My guess is that what a human commonsense reasoner actually knows, and what an automated commonsense reasoner should know consists of three parts:

⁴ There is no physical reason that these should be the same constant; but there is nothing at all to be gained from splitting it into two separate constants.



R1 and R2 are thickly connected. R3 (the union of the two outer brackets) is not.

Fig. 5. Thickly connected regions.

1. A rather sketchy knowledge of the absolute quantities involved. One does know that in pouring liquid at a normal rate from a dining room pitcher, the distance of the overflow and the outflow are somewhere between a substantial fraction of an inch and a couple of inches, not millimeters and not feet. One knows that the outflow pouring liquid out of a 10 gallon pail is larger, perhaps several inches, but still not feet; and so on. This knowledge is presumably gotten by interpolating from observed instances using instance-based learning of the conventional kind. Even so, the choice of the features to use presumably rests on an domain understanding of the kind we are studying here.
2. A more precise knowledge of the signs of the influences involved, in the sense of the classic qualitative calculus of signs [15,16]. In particular, the reasoner knows that the more rapidly a pitcher is tipped, the thicker the stream comes out of the spout.
3. Most importantly, the ability to use the knowledge in (2) in control regimes for pouring liquid to a targeted region without spilling. The reasoner, now an agent, knows that if the liquid is not yet pouring or pouring too slowly, he should increase the angle, and if it is pouring too fast or too far out, he should decrease the angle.

Unfortunately, expressing the control strategy described in (3), which involves feedback from perception to manipulation, requires theories of perception, manipulation, planning, and control which are substantially beyond the scope of this paper. It would not be difficult to extend the theory in this paper to represent knowledge of the kind in (1) and (2), but it would not lead to any very interesting new inferences.

3.2. Rule 4: Cupped liquids

We now proceed to a detailed account of our theory. Rules 1, 2, and 3 are correct as formulated on page 1546. As we have discussed above, Rules 4, 5, and 6 interact in rather intricate ways, and stating them correctly requires some care. In formulating these rules, it will be helpful to go through a series of approximations; that is, we will first state the rule in a form that is nearly correct, explain the cases in which it works, explain the cases it doesn't work, revise the rule, and if necessary iterate.

We begin our formulation of Rule 4 with a few basic definitions,

Definition 3.2.1. A *region* is a topologically regular⁵ set of points in \mathbb{R}^3 , not necessarily either connected or bounded. A *region-valued fluent* is a function from time to regions.

Definition 3.2.2. A region R is *thickly connected* if the interior of R is connected (Fig. 5).

Definition 3.2.3. A region R is *cupped* at time T if:

1. R is thickly connected;
2. No solid object overlaps R ;
3. Any boundary point of R that is lower than the highest point in R is a boundary point of some solid object.

Thus the boundary of a cupped region R consists of (a) a horizontal top surface (possibly disconnected, but of constant height) where R meets the open air; and (b) the rest of the boundary, where R is bounded by solid objects. Note that the cupped region may be formed by a single solid object or by many.

We can now formulate a first approximation to Rule 4:

Rule 4.A. Let Q be a region-valued fluent. Suppose that over time interval I , Q is always cupped and is a continuous function of time. (Again, we defer to Section 3.7 the question of what it means to say that a function of time to regions is “continuous”.) If Q is never full of liquid in I , then there is no outflow from Q during I .

⁵ A set of points is *regular* if it is equal to the closure of its interior. The *regularization* of set S is the closure of the interior of S .

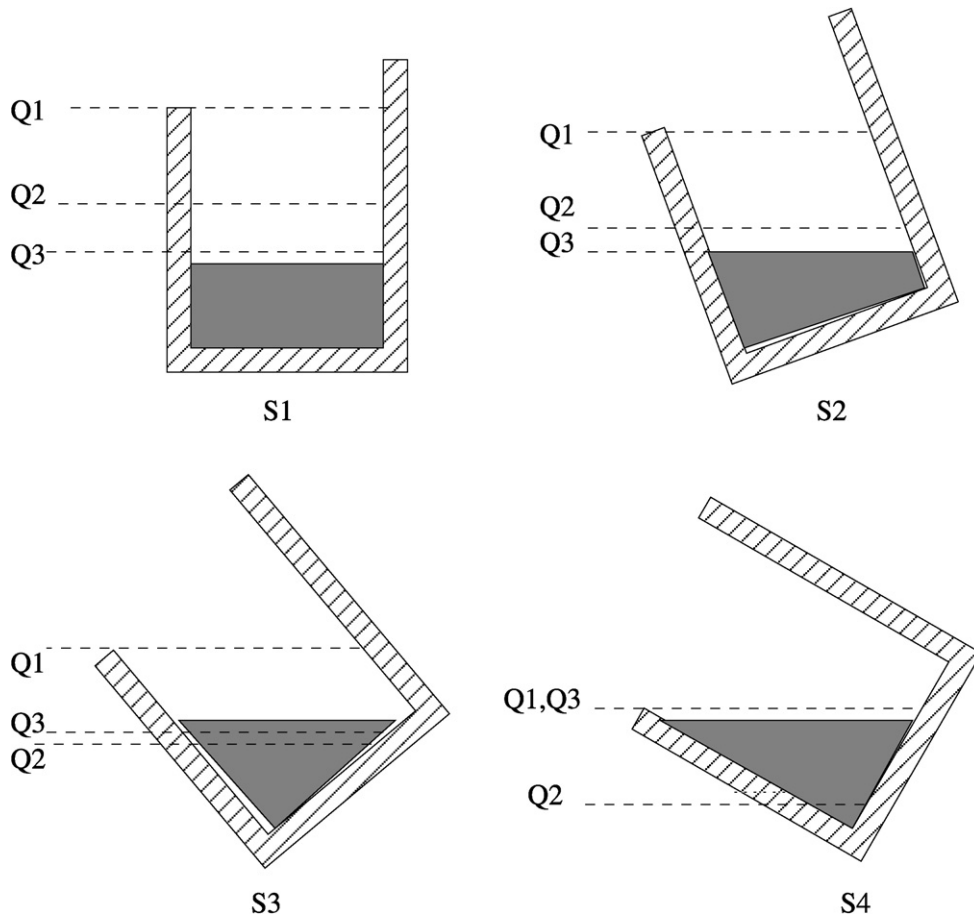


Fig. 6. Fluents onto cupped regions for Rule 4.

Before explaining the problem with this formulation and correcting it, let me first discuss a couple of critical aspects of the proposed rule, and then show that this rule works correctly in some cases where one might suppose there was a problem.

First, when we say that liquid L “flows out of” fluent Q during time interval I , what that means is at some time in I , L is inside Q , and at some later time in I , L is outside Q . This can happen, either because the liquid L moves, or because Q moves, or both.

Second, the chief feature of Rule 4.A is that it applies to *all* region-valued fluents that are continuous and always cupped, not just the largest-possible cup at each time. For instance, in Fig. 6, one might define the following fluents:

- $Q1$ is the largest cupped region formed by the cup.
- $Q2$ is the cupped region whose top is always 8 inches below the top of $Q1$.
- $Q3$ is the cupped region whose top is always 8 inches above the bottom of the cup.

All of these satisfy the conditions of continuity and of always being cupped. There are infinitely many such fluents. The condition that Q is not full of liquid is met by $Q1$ throughout the scenario; it is met by $Q2$ from times $S1$ to $S2$, and ends sometime between $S2$ and $S3$; it is met by $Q3$ from $S1$ to $S2$, becomes false in $S2$, and becomes true again some time between $S3$ and $S4$. During the time that it is met, for a given fluent, the rule states that no liquid flows out of the fluent, which is clearly the case. The point is that, since the fluent Q is always cupped, any flow out of Q can only come out of the top of Q ; and the liquid inside Q only reaches the top of Q if its volume becomes equal to the volume of Q .

Third, the boundaries of the cup can be formed either by a single object, as in the pitcher example, or by several objects; in fact, the collection of objects that form the cup can vary over time, as long as the interior of the cup is a continuous function of time. For example, if you drop pebbles into a vase, then the “cupped region” is the interior of the vase minus the pebbles; the rule then states that as long as the volume of the liquid is less than the volume of the vase minus the volume of the pebbles, the vase will not overflow. Or one might have a cup formed by a cylinder and a piston that can move the bottom of the cupped region up and down (Fig. 7); this will be a useful example for us in our discussion, since it is easier to see what happens to the volume of the cup in this case than in the case of a tilting pitcher. In all these cases,

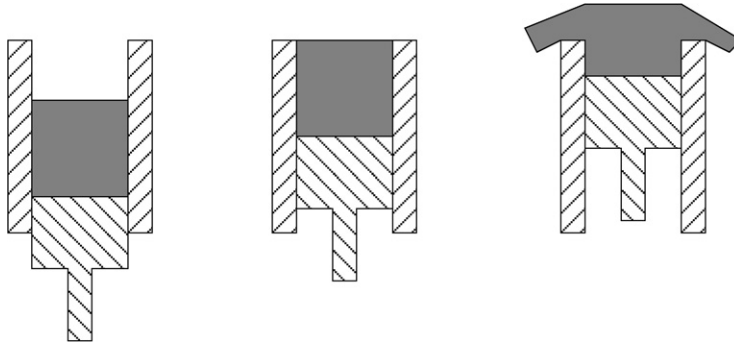


Fig. 7. Cylinder with piston.

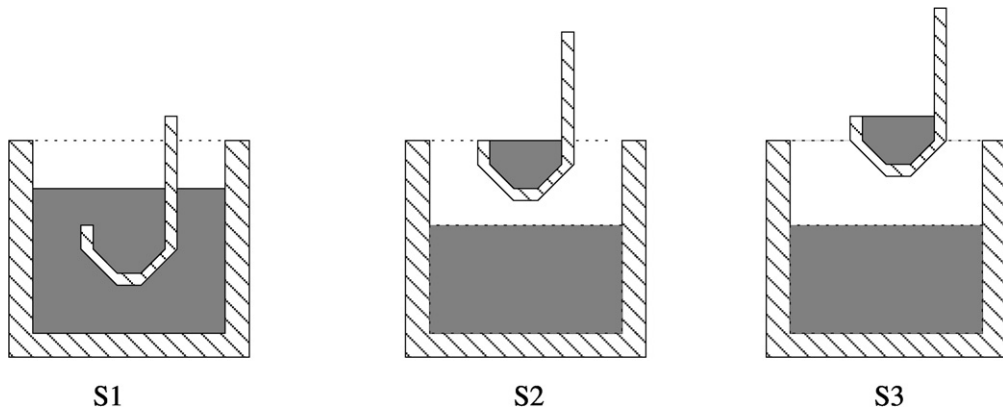


Fig. 8. Ladling from a pail.

we assume that the liquid inside the cup can always reshape itself to the interior of the cup as fast as the objects move around; this is part of the assumption that objects move slowly.

One might suppose that a scenario like Fig. 8, in which liquid is lifted out of a pail in a ladle, would contradict rule 4.A. After all, the interior of the pail is a cupped region and the liquid in the ladle comes out of the pail even though the interior of the pail is not full of liquid. But in fact there is no problem here. The interior of the pail, minus the solid material of the ladle, does indeed constitute a cupped region at time S1, and continues to do so right until time S2, when the rim of the ladle reaches the top of the pail. At that moment, however, the interior of the pail as a whole ceases to be a cupped region, because it ceases to be thickly connected. In S2 there are two separate cupped region: One region that is inside the pail and outside the ladle and one region that is inside the ladle, and there is no cupped region that includes them both. Therefore, the fluent “inside of the pail minus the material of the ladle” does not satisfy the conditions of the lemma, since it is not cupped in S2 and after.

One might also suppose that inflow into a cupped region could cause trouble for the rule; surely if there is liquid flowing into a region there can be liquid flowing out of it? The answer is no; there can be outflow only if the region is full of liquid, in which case the condition of the rule is not satisfied.

The cases where the rule does fail are more recondite. Consider the case shown in Fig. 9: the piston cup, full of liquid, is inside an empty pail and the piston is pushed upward. Suppose that the motion of the piston causes the liquid to rise a distance DH over the top of the cylinder. Now consider the region bounded by the pail whose top is $DH/2$ above the top of the cylinder. This fluent does violate Rule 4.A; it is always cupped, it is a continuous function of time, it is not full of liquid, and yet liquid flows out.

We therefore have to make an exception to Rule 4.A in the case of a cupped region that contains another cupped region that is overflowing. Specifically, we say that a piece of liquid is “driven” if it is overflowing from a cup (the precise definition is given in the next sentence). Driven liquids are exceptions to Rules 4, 5, and 6.

We now can formulate Rule 4 correctly:

Rule 4. Let Q be a region-valued fluent. Suppose that over time interval I , Q is always cupped, is a continuous function of time, and does not contain any driven liquids. If Q is never full of liquid in I , then there is no outflow from Q during I .

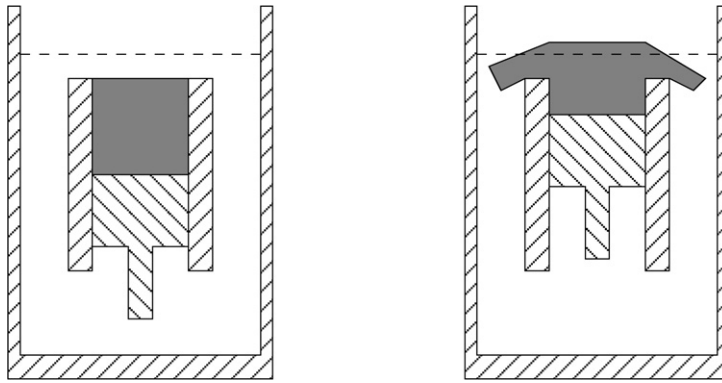


Fig. 9. Exception to Rule 4.A.

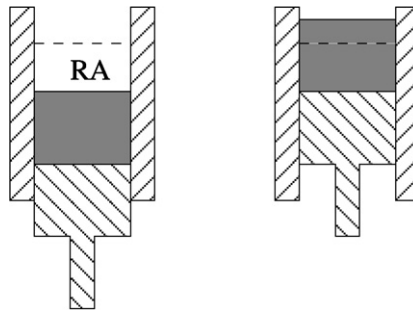


Fig. 10. Not a case of overflow.

3.3. Driven liquids

To complete the formulation of Rule 4, we need next to define precisely what is meant by a “driven” liquid. This definition, as we will see, constitutes our theory of overflowing; overflow occurs in the region above the top of a cup where a driven liquid is allowed to violate rules 4, 5, and 6.

The definition that we will present is obviously rather arbitrary and does not correspond closely to any physical reality. Given the nature of the theory here, this arbitrariness is, I think, both inevitable and unimportant. It is inevitable, because we are trying to characterize what is a very complex, dynamic process in terms of a few simple geometric constraints; necessarily, the fit will not be very good. And it is unimportant because the question of exactly what is happening at the layer where liquid is overflowing is probably not very well understood by a human commonsense reasoner, and does not much affect the large-scale inference that the liquid pours from the pitcher into the pot. What is important is that the rule should be stated in a way that (a) successfully resolves the contradiction between the unqualified version of Rules 4, 5, and 6 and the scenario of an overflowing cup; and (b) justifies the inference that overflow liquid pouring from a spout can only go a fairly small distance before Rule 5 starts to apply and the liquid pours straight downward.

Since driven liquid is caused by an overflowing cupped region, we must first characterize the circumstance of overflowing. Unlike Rule 4, which applies to all cupped regions, it is not the case that overflowing occurs whenever *any* cupped region becomes too small for the liquid it contains. For instance, in Fig. 10, the liquid reaches the top of RA and rises above it, but there is no overflow, in the sense of an exception to Rules 4, 5, and 6; the liquid just continues to occupy the cupped region of the correct volume at the current bottom of the cylinder. True overflow and driven liquid occurs when it is not possible for the liquid to adjust instantaneously to a new bottom of a cup, either because it is not inside any cupped region at all any more, or because the nearest containing cupped region is some finite distance beyond the overflowing cup, and the liquid requires some finite time to flow out to it.

We are thus led to the definition of a *locally maximal cup*:

Definition 3.3.1. Region R is a *locally maximal cup* at time T if:

1. R is a cupped region at T .
2. There exists a distance $D > 0$ such that, if $R1$ is a cupped region in S and $R1$ properly contains R , then some point in $R1$ is at least distance D from R .

Thus, RA in Fig. 10 is not a locally maximal cup, since there are cupped regions that are supersets of RA and arbitrarily close to RA . By contrast, the inside of the ladle in Fig. 8 and the inside of the cylinder in Fig. 9 are locally maximal cups,

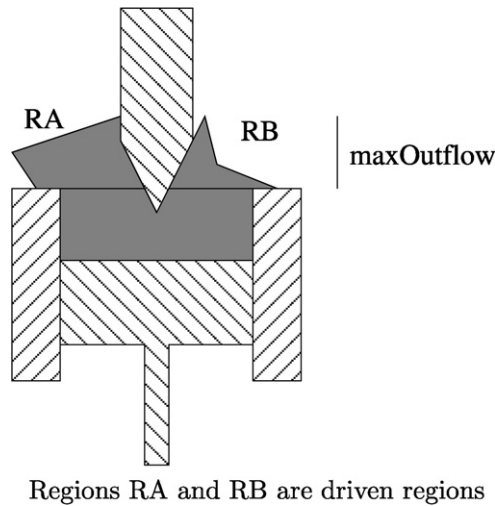


Fig. 11. Driven regions.

because the cupped regions that properly contain them all involve the inside of the pail and thus contain points that are far from the inside of the ladle/cylinder.

Definition 3.3.2. Let Q be a region-valued fluent that, throughout time interval $[TS, TE]$, is continuous and is always a locally-maximal cup. If Q is full at TS , and the volume of Q strictly monotonically decreases throughout $[TS, TE]$, then Q overflows throughout I .

Definition 3.3.3. Let Q be a region-valued fluent that overflows throughout time interval $[TS, TE]$. Let T be a time between TS and TE , and let $Q(T)$ be the value of Q at T . Suppose that a piece of liquid L fills region R at T . L is driven by the overflow from Q at T (Fig. 11) if:

1. $R \cup Q(T)$ is thickly connected.
2. R is entirely within distance maxOutflow of $Q(T)$.
3. For every point P in R there is a point P' in $Q(T)$ such that the line from P to P' is entirely in R and the height of P' is less than or equal to the height of P .

Note that in these definitions, the liquid overflowing from Q is driven only as long as the volume of Q is monotonically decreasing; as soon as Q stops decreasing in volume, the overflow ceases to be driven and starts being governed by Rules 4 and 5.

3.4. Rule 6: Flowing down

Definition 3.4.1. A liquid L flows down during a time interval if the heights of the top and of the bottom of L are monotonically strictly decreasing functions of time.

Rule 6. If a piece of liquid L can flow down at time T , then it does flow down.

The question is, what do we mean by saying that a liquid “can flow down”? Rather than analyze this in terms of a general theory of possible behaviors of liquids, which raises rather difficult technical issues, we will give an explicit geometric characterization.

Definition 3.4.2. At any time T , *nonFlowingSpace* is the union of all regions of space occupied by solid objects, cupped liquids, driven liquids, and “weakly cupped” liquids.

A liquid is weakly cupped if it is cupped by some combination of solid objects, cupped liquid, and driven liquid. Since driven liquid and cupped liquid are, to a degree, resistant to being pushed out of the way, weakly cupped liquid must likewise be, since there is nowhere it can flow freely.

Definition 3.4.3. Point P is a *flow stopping point* associated with region R if P is a local minimum of the top surface of R . (R is the region occupied by an obstacle. A liquid incident on P cannot flow downward at P .)

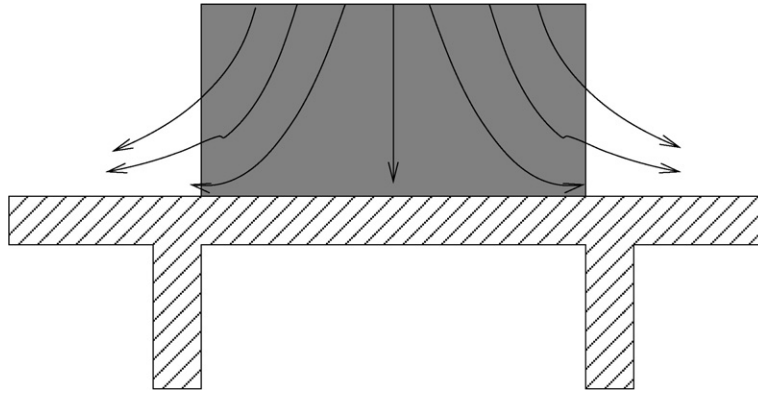


Fig. 12. Column of liquid flowing down.

Definition 3.4.4. A piece of liquid L *cannot flow down* if either L overlaps `nonFlowingSpace` or if a bottom point of L is incident on a flow stopping point of `nonFlowingSpace`. If it is not the case that L cannot flow down, then L *can flow down*.

If a piece of liquid L can flow down, in the sense of Definition 3.4.3 then either there is nothing below it, or it can flow around whatever is below it, or whatever is below it is itself flowing down and thus making room for L .

Suppose, for instance, that in a given coordinate system there is a fixed solid table with a round horizontal top of radius 2 lying in the x - y plane at $z = 0$, and suppose that there is a cylinder of water of radius 1 and height 1 sitting above the center of the table at time $t = 0$. By our definition any piece of this cylinder that is incident on the table cannot flow down; any piece that is not incident on the table can flow down. And indeed, assuming that the cylinder flows outward and downward, every piece not incident on the table will flow down (Fig. 12). Consider, for example, the following flow pattern: a point of liquid whose position at $t = 0$ is $\langle r_0, \theta_0, z_0 \rangle$ in cylindrical coordinates at $t = 0$ flows to position $\langle r_0/\sqrt{1-t}, \theta_0, z_0(1-t) \rangle$ for t between 0 and $3/4$ (at $t = 3/4$ the liquid reaches the edge of the table). It is easily verified that in this flow pattern every piece of liquid has a constant volume, and move continuously, and that every piece of liquid not incident on the table flows downward.

3.5. Rule 5: Flowing straight downward

There are two main questions about Rule 5. First, what is the dividing line between regions where Rule 5 applies and regions where only Rule 6 applies? Second, what is meant by the statement that a liquid “flows straight down”?

As regards the first question: As with our definition of “driven liquid” in Section 3.3, our definition here is rather arbitrary, justified on the grounds that in this kind of theory precision is neither possible nor necessary.

Definition 3.5.1. At time T , R is a *flow disruption region* if the following hold:

1. R is thickly connected.
2. R is filled with liquid.
3. For every point P in R , there is a point P' at the top of `nonFlowingSpace` such that $\text{height}(P) \geq \text{height}(P')$; P is within `maxOutflow` of P' ; and the line from P to P' does not go through any solid objects.

Definition 3.5.2. The region `flowDisruptedSpace` is the union of all flow disruption regions. The region `flowUndisruptedSpace` is the complement of `flowDisruptedSpace` \cup `nonFlowingSpace`.

As regards “flowing straight down”: The obvious interpretation of “flows straight down” would be a simple vertical translation downward. However, it is obviously impossible to have a steady flow of liquid in a column that (a) translates straight downward; (b) preserves volume; (c) does not break into droplets; and (d) accelerates under gravity. Rather, if (b), (c), and (d) hold, then the stream of flowing liquid must get narrower, and indeed this is what happens. But if the stream is growing narrower, then the parts of the liquid at the periphery are not translating straight down, they are moving toward the center. It is not necessary, at our level of precision, to have a theory that mandates that falling liquid accelerates, but it seems overly restrictive to have a theory that prohibits it. Therefore we define “falling straight down” as a characteristic, not of all pieces of liquids, but of a stream as a whole (or, more precisely, of a horizontal slice of a stream).

Definition 3.5.3. A piece of liquid L *falls straight down* if it falls downward and the horizontal projection of L is a monotonically non-increasing function of time. (The complete statement of this definition requires an additional condition to ensure continuity. See Section 4.9.)

Rule 5. At time T , let L be any piece of liquid in `flowUndisruptedSpace`. Then L is contained in a piece of liquid LX with the same vertical span that flows straight downward for some time interval starting in T .

3.6. The ontology of liquids

There are a number of different ontologies that can be used for modelling the dynamics of liquid flow:

I. Discrete models

- I.1 Molecular model. A quantity of liquid on the scale of easy manipulations (between about 1 gram and 100 kilograms) consists of between about 10^{22} and 10^{27} molecules, interacting through van der Waals forces.
- I.2 Small particle model. Liquid consists of a collection of particles that are very small on the human scale but very large as compared to a molecule. They are too small, and therefore too numerous, to allow reasoning that depends on enumerating them individually, but large enough that each one can be meaningfully assigned properties that are in fact statistics over large numbers of molecules, such as temperature [3].⁶
- I.3 Large particle model. The behavior of liquid can be calculated by simulating the interactions of a fairly small number (dozens or at most hundreds) of large-scale particles [17].
- I.4 Spatial cellular model. Space is divided into a grid of cells, and the state of the liquid is characterized in terms of the occupancy of the cells.

II. Continuous models

- II.1 Fixed point model. The behavior of the liquid is characterized in terms of the evolving state of the liquid at fixed points in space. (This is the Eulerian model in fluid dynamics.)
- II.2 Flowing point model. The behavior of the liquid is characterized by tracking “points” of liquid as they move through space over time, and describing the position of each point and the state of the liquid at the point. (This is the Lagrangian model in fluid dynamics.) For instance, the specification of fluid flow in Section 3.4 is a simple instance of a flowing point model.
- II.3 Fixed region models. The behavior of the liquid is characterized in terms of the occupancy of fixed regions.
- II.4 Pieces of liquid. The behavior of the liquid is characterized in terms of the motion of pieces of liquid over time.

Hayes’ theory [19] uses primarily the fixed regions model and to a lesser extent the “pieces of liquid” model. Kim’s program [21] likewise uses primarily the fixed regions model. DeCuyper et al. [13] propose a hybrid architecture combining a large particle model, a spatial cellular model, and a fixed region model, but as mentioned it seems unlikely that the details of this were ever worked out. As our above discussion indicates, our theory here uses primarily the “pieces of liquids” model and to a lesser extent the fixed regions model.

Space here does not permit an extended comparison of these ontologies, but let me discuss briefly the difficulties that arise in using the other ontologies for the kinds of reasoning addressed here.

The true molecular model I.1 is of course the most correct of these; but it is by far the most difficult to use. The relationship between the small scale interactions of the molecules and the large scale dynamics of liquid is mediated by statistical mechanics, and the reasoning needed to get from one to the other is generally more complex than the kinds of reasoning used in this paper or in continuous fluid dynamics.

The small particle discrete model of [3] was developed to enable rather specific lines of inference; for instance, to reason about liquid that heats up while flowing past a heat source. No general dynamic theory has been developed using this ontology.

The large particle model and the cellular model are both inherently limited in their precision by the grain-size of the discretization. Neither is applicable in any straightforward way to reasoning with qualitative spatial information.

It should be noted that the large particle model is no less an abstraction of the true molecular model than any of the other models described here; and it is a mistake to suppose that the superficial resemblance between the large particle model and the molecular model implies that there is any actual advantage to using the large particle model in reasoning. A collection of 10^{23} molecules actually behaves in almost every respect more like the partial differential equations of fluid dynamics than like a collection of 100 particles.

The two point-based models II.1 and II.2 are the basis of fluid dynamics. They have the advantages, first, that an exact specification of flow and other dynamic change can easily be represented in terms of closed-form formulas, as we have done in above in Section 3.4; and that a very precise dynamic theory can be stated in partial differential equations. However, neither of these is suited to qualitative reasoning. The evolution of the flow field and the partial derivatives involved are unstable under the kind of qualitative variation we wish to consider, even in circumstances where the overall behavior is very stable.

Note, also, that even in stating the standard scientific theory, there are facts that are more easily stated in the region-based ontology than in point-based ontologies. In particular, the incompressibility of liquid is trivial to state in the region-

⁶ Gibbs used a somewhat similar model in order to apply statistical mechanic arguments to thermodynamics without needing to resolve the ongoing controversy over the kinetic model of heat.

based model. In the flowing point model, it is substantially more complicated to state, and in the fixed point model, it can only be stated quite indirectly, using the divergence theorem.

Ideally, one would think that there ought to be a language for describing large-scale physical phenomena that is agnostic and indifferent to the small-scale structure. After all, the naive human reasoner knows how liquids behave on the scales in which he can perceive and manipulate it and he neither knows nor cares what is going on at scales much smaller than he can perceive; a commonsense theory should do likewise, certainly if it is making claims to cognitive validity. Unfortunately, despite many attempts, no one has succeeded in formulating such a theory. Space here does not permit an analysis of the difficulties involved; I hope to address this in [12].

3.7. Continuous motion of liquids

An important constraint on liquids is that they move continuously in space. In a “piece of liquid” ontology, it is not obvious how this constraint should be expressed, since there are a number of different possible topologies over the space of regions, and hence a number of different notions of continuity [9].

One condition that is indisputable is that the volume of liquid in any bounded region must be a continuous function of time; e.g. the volume of liquid in a pail cannot instantaneously change from 1 pint to 2 pints. This can be guaranteed as follows:

Definition 3.7.1. Let R_1 and R_2 be two bounded, topologically regular regions. The *symmetric difference* of R_1 and R_2 , denoted $R_1 \ominus R_2$ is defined as the regularization of $(R_1 - R_2) \cup (R_2 - R_1)$. We define the function $d_V(R_1, R_2) = \text{volume}(R_1 \ominus R_2)$.

For example, let \mathbf{A} be the square with vertices $\langle 1, 1 \rangle, \langle -1, 1 \rangle, \langle -1, -1 \rangle, \langle 1, -1 \rangle$ and let \mathbf{B} be the circle centered at the origin of radius 1.2 (Fig. 13). The symmetric difference between \mathbf{A} and \mathbf{B} consists of the four corners of the square that lie outside the circle together with the four “sides” of the circle that lie outside the square. The area of this region is 0.92.

It is easily shown that d_V is a metric over the space of regular, bounded regions.

Definition 3.7.2. Let Q be a region-valued fluent and let I be a time interval. Q is *volume-continuous* during I if it is continuous with respect to the metric d_V ; that is, for any $\epsilon > 0$ there exists $\delta > 0$ such that, for all T_1, T_2 in I , if $|T_1 - T_2| < \delta$ then $d_V(Q(T_1), Q(T_2)) < \epsilon$.

It is certainly the case that the region occupied by any piece of liquid is volume-continuous over time. However, this constraint is too weak; it would permit a liquid to gradually disappear from one container and simultaneously reappear in a container some distance away. Such a behavior would be volume-continuous but is clearly not what we mean by liquid moving continuously.

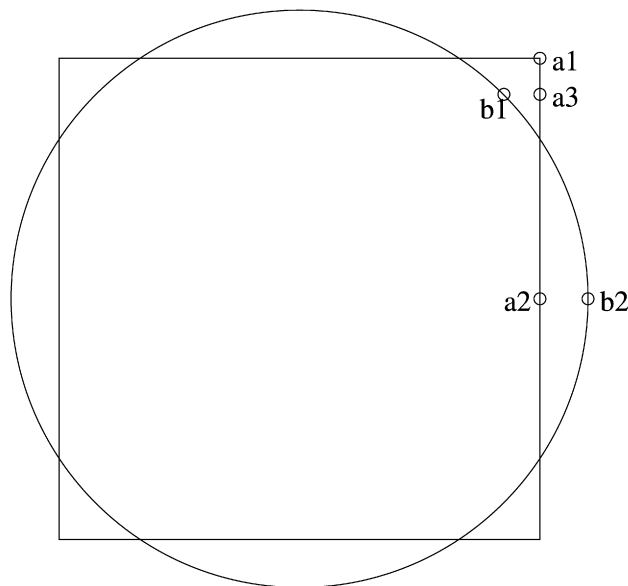


Fig. 13. Measures of distance between regions.

The obvious reason that such a behavior is impossible is that it requires molecules of the liquid to teleport from one container to the other; whereas of course each molecule moves continuously. We can't say that exactly in a continuous, molecule-free model; but one plausible equivalent is to require continuity in the Hausdorff metric, defined as follows:

Definition 3.7.3. Let $R1$ and $R2$ be bounded regular regions. The *one-sided Hausdorff distance* from $R1$ to $R2$, denoted " $d_{H1}(R1, R2)$ " is the maximal distance from any point in $R1$ to the nearest point in $R2$.

$$d_{H1}(R1, R2) = \max_{P1 \in R1} (\min_{P2 \in R2} d(P1, P2)).$$

The *Hausdorff distance*, denoted $d_H(R1, R2)$ from $R1$ to $R2$ is the maximum of the one-sided Hausdorff distance from $R1$ to $R2$ and the one-sided Hausdorff distance from $R2$ to $R1$.

$$d_H(R1, R2) = \max(d_{H1}(R1, R2), d_{H1}(R2, R1)).$$

For example, in Fig. 13, the distance from the point $\mathbf{a1} = \langle 1, 1 \rangle$ in \mathbf{A} to the closest point $\mathbf{b1} = \langle 0.6\sqrt{2}, 0.6\sqrt{2} \rangle$ in \mathbf{B} is $\sqrt{2} - 1.2 \approx 0.214$. Moreover, this is the greatest distance from any point in \mathbf{A} to the closest point in \mathbf{B} . The distance from the point $\mathbf{b2} = \langle 1.2, 0 \rangle$ in \mathbf{B} to the nearest point $\mathbf{a2} = \langle 1, 0 \rangle$ in \mathbf{A} is 0.2. Moreover, this is the greatest distance from any point in \mathbf{B} to the nearest point in \mathbf{A} . Therefore, the Hausdorff distance between \mathbf{A} and \mathbf{B} , $d_H(\mathbf{A}, \mathbf{B}) = \max(0.214, 0.2) = 0.214$.

Definition 3.7.4. Let Q be a region-valued fluent and let I be a time interval. Q is *Hausdorff-continuous* during I if it is continuous with respect to the metric d_H ; that is, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $|T1 - T2| < \delta$ then $d_H(Q(T1), Q(T2)) < \epsilon$.

Clearly, if a liquid consists of the union of molecules each of which is moving continuously, then the region occupied by a piece of liquid is a Hausdorff continuous function of time, since each point occupied by a molecule at one time is close to a point occupied by the same molecule at a sufficiently recent time.

Nonetheless, there is a plausible argument that in certain important cases the constraint that the region occupied by a piece of liquid is Hausdorff continuous is overly restrictive. There are two main cases of this kind.

Case 1. Suppose that solid objects $O1$ and $O2$ have flat surfaces that can meet, and suppose that those surfaces are immersed in liquid and pushed together (Fig. 14). Then the liquid fills the gap between the two surface up until the moment that the two objects meet, at which point the gap disappears. At the moment that the gap disappears, the region occupied by the liquid has a discontinuity in the Hausdorff metric equal to the radius of the largest circle that can be inscribed in the common surface (assuming that the two objects meet in a planar surface. If they meet in a non-planar surface, the analysis is more complex).

Case 2. Let $O1$ be a closed box with a flat inner bottom surface and let $O2$ be an object with a flat bottom sitting inside $O1$ (Fig. 15). Suppose that the entire inner cavity of $O1$ other than $O2$ is filled with liquid. Now, if $O2$ is lifted off the surface of $O1$ the gap created between $O1$ and $O2$ is filled with liquid; since the volume of the liquid is constant, the gap must be filled instantaneously. Again, the region occupied by the liquid is discontinuous with respect to the Hausdorff metric.

What are we to do about these two cases? One natural answer might be as follows. In Case 1, the idealization of the liquid as continuous breaks down once the gap between $O1$ and $O2$ becomes a single molecule wide, and the molecular nature of the liquid becomes critical. In Case 2, the idealization of liquid as incompressible breaks down; clearly what happens in fact is that the liquid is slightly compressed until it can flow to fill up the gap. Since the idealizations are not sustainable in these cases, these cases naturally lie outside the scope of the theory I am trying to develop. (Another explanation is that the idealization of the solid objects as perfectly rigid breaks down; if the surfaces can bend, then they can meet or separate gradually.)

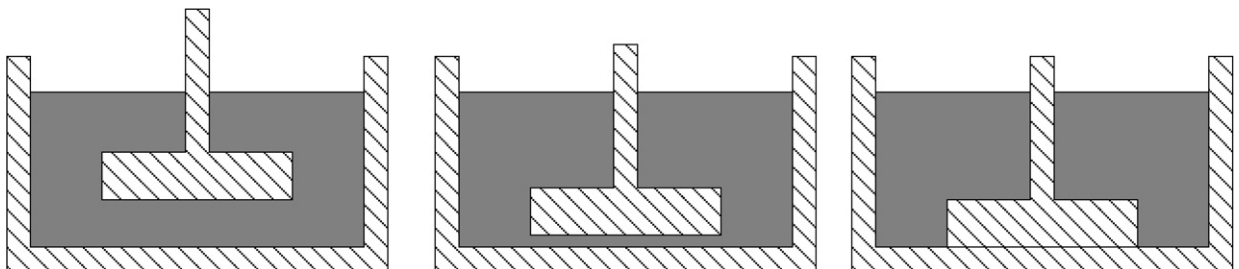


Fig. 14. Hausdorff discontinuous flow: Case 1.

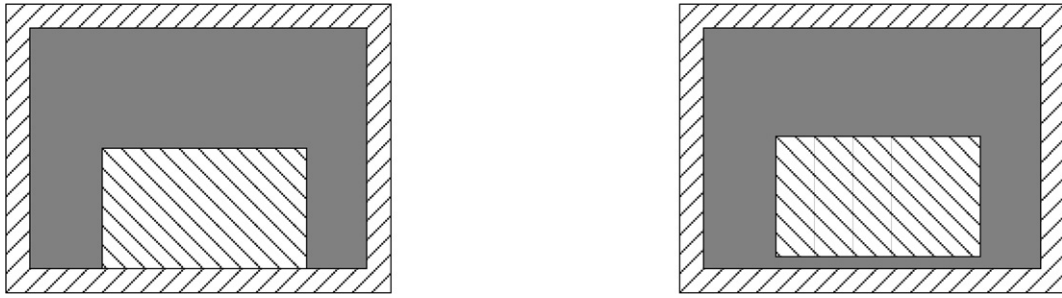


Fig. 15. Hausdorff discontinuous flow: Case 2.

It is possible that this is, in fact, the best answer to the question, that we should simply exclude these cases and posit that liquids move continuously in the Hausdorff distance. (One aspect of knowledge representation that is in general very poorly understood is what to do about cases where an idealization breaks down.) But I find this solution problematic. In particular, note that case 1 appears in exactly the same form if O_1 and O_2 are immersed in a gas (air). I could believe that when you push two objects together in liquid, there is always a residual film of liquid between them, but saying that any two solid objects that appear to be in contact actually have a separating film of air does not seem at all plausible. Moreover, it will hugely complicate our theory of the dynamics of solid objects if no two solid objects are ever actually in contact, and all forces between solid objects are actually mediated by a layer of fluid. And a theory that excludes the case of one solid object from being placed against another would seem to be unacceptably limited.

Instead, in order to maintain at least the option of being able to place one solid against another, I have weakened the constraint that the region occupied by a piece of liquid is Hausdorff continuous to read that it is *almost everywhere* Hausdorff continuous; that is, any piece of liquid contains a subset that is Hausdorff continuous. This allows discontinuities like Case 1 and Case 2 to take place, but only with protrusions that are initially “infinitesimally thin” and connected to a previously existing body of liquid.

Similarly, though it is critical in a theory of liquids is that a piece of liquid can break into disconnected parts and that a piece of liquid can deform itself into a shape of arbitrarily large diameter, we want to prohibit flows patterns in which every piece of liquid breaks into parts, or in which every piece of liquid becomes very long and thin. Therefore, we impose the similar condition that, over any finite time interval, every piece of liquid contains a subpiece that is thickly connected and has a small diameter throughout the time interval.

Definition 3.7.5. A piece of liquid L has a *simple flow of maximal diameter D during time interval $[TS, TE]$* if throughout $[TS, TE]$, L is Hausdorff continuous, is thickly connected, and has a diameter at most D .

Rule 3. For any piece of liquid L and time interval I ,

- The region occupied by L is volume continuous;
- For any distance $D > 0$ there is a subset L_1 of L that has simple flow of maximal diameter D during I .

It is easily shown that Rule 3 as stated above is flexible enough to allow Cases 1 and 2 above but strong enough to prohibit a liquid from disappearing from one place and reappearing at a separated place. To show the latter we use a proof by contradiction. Suppose that liquid L gradually disappears from R_1 and reappears in R_2 , where R_1 and R_2 are separated. Then there must exist a subset L_1 of L that has simple flow. But L_1 must likewise move from R_1 to R_2 , which contradicts both the condition that L_1 is Hausdorff continuous and the condition that it is always thickly connected.

We also wish to impose a rule that discontinuities like Cases 1 and 2 can only happen under special circumstances. For our purposes in this paper, it suffices to say that they cannot happen in `flowUndisruptedSpace`. The exact statement is given in axiom `DOWN.4`, and will be discussed in Section 4.9.

(One of the reviewers suggested an alternative solution to this problem, which would be to add an axiom stating, in effect,

for each piece of liquid L and times TS, TE
 there exists a finite set of time points $T_1 \dots T_k$ between TS and TE
 such that for every subset L_1 of L ,
 L_1 is Hausdorff continuous throughout TS, TE except at $T_1 \dots T_k$.

With a small extension to the temporal vocabulary, this can be recast in valid first-order form. I don't know whether this alternative axiom is adequate or how it relates logically to Rule 3 above.)

4. The formal representation and theory

This section describes how the physical theory developed in Section 3 can be represented in formal logic. This is actually reasonably straightforward; we have covered most of the tricky issues already. In this section we first describe the foundational theories: notational conventions and system of sorts (Section 4.1), theories of real arithmetic and of Boolean operations on finite sets of objects (4.2) and the language and theory of time (4.3), and space (4.4). We then proceed to the parts of the paper specific to the theory of liquids. Section 4.5 describes the ontology of solids and liquids. Section 4.6 formalizes the kinematic theory, which includes Rules 1, 2, and 3. Section 4.7 formalizes our theory of overflowing, particularly the definition of driven liquid (Definition 3.3.3 above). Section 4.8 formalizes the theory of cupped liquids, particularly Rule 4. Section 4.9 formalizes the theory of downward flow, particularly Rules 5 and 6.

It may be of methodological interest to note that, though the exposition in this paper follows a linear sequence of first working through the pre-formal theory, next formalizing the theory, then verifying the proof, the actual progression of the research project did not proceed in anything like so systematic a fashion. Rather, at the time when I first thought I had completed the representational work and could now easily work out the formal proof of the example (in early April 2007), both the physical theory and the problem specifications were quite different from when I finally completed the proof (in mid July); inadequate in many respects, overly detailed or overly constrained or even inconsistent in others. Many large holes in the physical theory and the specifications were found through the impossibility of closing surprisingly minor details of desired lemmas. I am not advocating this as a method of research—in fact, I suspect I could have saved time overall if I had spent more time earlier on the pre-formal theory before entering on to the work of formalization—just saying that this is, in fact, how this project developed.

We assume the standard mathematical theories of integer and real arithmetic, Boolean operators on finite sets, Euclidean geometry, and continuous functions. Therefore, in our axiomatization, we will enumerate the sorts and symbols we need, but we will not list the axioms. In our formal proof, we cite theorems from these theories as needed without axiomatic proofs (though when the theorems are not obvious, we give proofs in the usual sense).

4.1. Notational conventions

All this is largely standard and self-explanatory, but it is as well to lay it out systematically.

Our axioms are stated in a sorted first-order logic. We use symbols in Roman font beginning with a lower-case letter, such as “openBox” for constant, function, and predicate symbols. We use upper case symbols in italics for variables. Standard mathematical functions and predicates are used in the standard way; e.g. $X1 + X2$ is an infix function; the curly brackets of $\{E1, E2\}$ is an outfix function (mapping $E1$ and $E2$ to the set $\{E1, E2\}$). To aid readability, the sort of a variable is generally indicated by the first letter; however, all axioms are stated so that this convention is not necessary. Quantified variables are subscripted to their quantifier. For brevity, we use two forms of restricted quantification in the subscript: a variable may be restricted by sort or by membership in a set. Logical operators follow the following precedence, from highest to lowest: negation \neg , conjunction \wedge , disjunction \vee , implication \Rightarrow , equivalence \Leftrightarrow , definitional equivalence \equiv , and quantifiers \forall and \exists . Thus, the scope of a quantifier is to the end of the formula or to a right bracket that contains it. Free variables are taken to be universally quantified, where the scope of the implicit quantifier is the entire formula. Greek letters are infrequently used as meta-variables.

We have a hierarchical system of sorts, which is interpreted as syntactic sugar for a standard first-order theory with a null element \perp . (The null element is never used explicitly in our formulas, except in the axioms of sorts in Table 1.) In the translation to the base theory, an individual sort such as “object” or “time” is considered an entity. There are two predicates over sorts: “sortOf(X, S)”, meaning that X is an entity of sort S , and “subsort($S1, S2$)”, meaning that $S1$ is a subsort of $S2$. We will abuse notation in our formulas by writing “ $S(X)$ ” where S is a sort instead of “sortOf(X, S)”; e.g. we will write “object(O)” instead of “sortOf(X, object)”. The sorts of the arguments to predicate and function symbols and to the values of function and constant symbols will be declared in a self-explanatory notation when these symbols are defined. Table 1 shows the translation of sort declarations into the base language.

As axiom SORT.2 indicates, all functions are required to be total over the sort on which they are defined. Hence, when we have a mapping which is single-valued but not total, we will use a relation symbol for it and add an axiom stating that it is single valued (when necessary). The “value” of the mapping will conventionally be the last argument. For this reason, in cases where we do use a function symbol, even if that is defined and not primitive, we do not have to add an axiom stating that the function is total; such an axiom is implicit from the use of the function symbol.

Axiom SORT.4 combined with axioms SORT.1 and SORT.2 implies that any ground term with argument \perp evaluates to \perp and that any ground atomic formula with argument \perp is false, unless the predicate is the equals sign.

4.2. Real arithmetic; Set theory

As stated above, we use real arithmetic and Boolean set theory over sets of elements. (As we shall see below, the sets that we need in this paper are sets of objects and sets of geometric points; these two sorts are therefore defined as subsorts of “element”.) Table 2 enumerates the sorts and the symbols used.

Table 1

Axioms of sorts

SORT.1	The declaration of predicate symbol α as taking arguments of sorts $\sigma_1 \dots \sigma_k$ corresponds to the axiom $\forall X_1 \dots X_k. \alpha(X_1 \dots X_k) \Rightarrow \text{sortOf}(X_1, \sigma_1) \wedge \dots \wedge \text{sortOf}(X_k, \sigma_k)$. (Note: The equality symbol $X = Y$ has no sort declaration, so this axiom does not apply.)
SORT.2	The declaration of function symbol β as taking arguments of sorts $\sigma_1 \dots \sigma_k$ and returning a value of sort σ_0 corresponds to the two axioms $\forall X_0, X_1 \dots X_k. X_0 = \beta(X_1 \dots X_k) \neq \perp \Rightarrow$ $\text{sortOf}(X_1, \sigma_1) \wedge \dots \wedge \text{sortOf}(X_k, \sigma_k) \wedge \text{sortOf}(X_0, \sigma_0)$ and $\forall X_0, X_1 \dots X_k. \text{sortOf}(X_1, \sigma_1) \wedge \dots \wedge \text{sortOf}(X_k, \sigma_k) \Rightarrow \beta(X_1 \dots X_k) \neq \perp$
SORT.3	The declaration of constant symbol γ as having sort σ corresponds to the axiom $\text{sortOf}(\gamma, \sigma)$
SORT.4	$\forall S. \neg \text{sortOf}(\perp, S)$. (The null element has no sort.)
SORT.5	$\text{subsort}(S1, S2) \equiv \forall X. \text{sortOf}(X, S1) \Rightarrow \text{sortOf}(X, S2)$. (Definition of subsort.)

Table 2

Sorts and symbols for real arithmetic and set theory

Sorts:

Real numbers (X).
 Integers (I). Subsort of reals.
 Elements. (E)
 Sets of elements (U).

Symbols:

$X1 + X2, X1 - X2, X1 \cdot X2, X1/X2, |X|, \min(X1, X2), X1 < X2, X1 \leq X2, X1 > X2, X1 \geq X2$.
 $\emptyset, \{E1, E2 \dots Ek\}, E \in U, U1 \cup U2, U1 \subset U2, U1 - U2$.

4.3. Time

Since this paper does not deal with the feasibility of actions, and since all the actions carried out consist of the exogenous motion of solid objects, a very simple temporal ontology will suffice; there is a single time line, isomorphic to the real numbers, and there are fluents (functions of time) that take on different values at different times. Times and fluents are indeed the only temporal sorts that we will need. For brevity, we will treat times as real numbers, though of course a more rigorous treatment would take “time” to be a separate sort which has a real-valued measure in a given temporal coordinate system.

A *fluent* is an entity that takes on a particular value at a particular time [30]. A *Boolean fluent* such as “fullOfLiquid(R)” (region R is full of liquid) is true or false at a time; a *non-Boolean fluent* such as “place(O)” (the region occupied by object O) takes on values of some other sort. If a fluent Q takes on values of sort σ , we denote the sort of Q as “fluent[σ]”; e.g. grasping(A, O) has sort fluent[Bool] and place(O) has sort fluent[region]. In translating the sort language to the base language, the symbol “fluent” here is a function mapping a sort like “region” to the sort fluent[region].

The temporal language is likewise simple and requiring only two primitives holds(T, Q) and value(T, Q) in addition to the primitives over the reals. However, it is useful to have some additional defined predicates as convenient abbreviations.

We assert two axioms of time in Table 3 (in addition to the implicit axioms of the real numbers). These are comprehension axioms for fluents; they state that any definable property of time can be reified as a fluent.

It will also be convenient to define some additional syntactic conventions for constructing functions over fluents from functions and predicates over atemporal entities. First, if X is an atemporal entity, then we define value(T, X) = X for all times T ; that is, we conflate a fluent that is constant with its value.

Let $\Phi(X_1 : \sigma_1 \dots X_k : \sigma_k)$ be a predicate (or equality sign) over atemporal sorts $\sigma_1 \dots \sigma_k$. For $i = 1 \dots k$ let Q_i be a fluent of sort fluent[σ_i]. Then we define $\Phi^\#(Q_1 \dots Q_k)$ to be the Boolean fluent satisfying

$$\forall T:\text{time} \text{ holds}(T, \Phi^\#(Q_1 \dots Q_k)) \Leftrightarrow \Phi(\text{value}(T, Q_1) \dots \text{value}(T, Q_k)).$$

Similarly let $\Psi(X_1 \dots X_k) : \sigma$ be a function with arguments of atemporal sorts $\sigma_1 \dots \sigma_k$ and value of sort σ . Then $\Psi^\#(Q_1 \dots Q_k)$ is the fluent of sort fluent[σ] satisfying

$$\forall T:\text{time} \text{ value}(T, \Psi^\#(Q_1 \dots Q_k)) = \Psi(\text{value}(T, Q_1) \dots \text{value}(T, Q_k)).$$

Finally, if Δ is a Boolean operator then $\Delta^\#$ is the corresponding function over Boolean fluents (used with the same syntax as the operator). That is,

$$\forall T:\text{time} \text{ holds}(T, Q1 \Delta^\# Q2) \Leftrightarrow [\text{holds}(T, Q1) \Delta \text{ holds}(T, Q2)].$$

For example, if $Q1$ and $Q2$ are fluents whose value at each time is a set, then “ $Q1 =^\# \emptyset$ ” is the Boolean fluent that holds at those times where $Q1$ is empty. “ $Q1 \subset^\# Q2$ ” is the Boolean fluent that holds when $Q1$ is a subset of $Q2$. “ $Q1 \cup^\# Q2$ ”

Table 3

Temporal theory: Proper axioms

Sorts:Time (T) = real.Fluent (Q).**Symbols:** $\text{holds}(T: \text{time}, Q: \text{fluent}[\text{Bool}])$. Boolean fluent Q holds at time T . $\text{value}(T: \text{time}, Q: \text{fluent}[\gamma]) \rightarrow \gamma$.The value of non-Boolean fluent Q at time T . γ is a meta-variable ranging over sorts. $\text{throughout}(T1, T2: \text{time}, Q: \text{fluent}[\text{Bool}])$. Boolean fluent Q holds from $T1$ to $T2$ inclusive. $\text{throughoutxE}(T1, T2: \text{time}, Q: \text{fluent})$. Boolean fluent Q holds from $T1$ until $T2$ but possibly not at $T2$. $\text{foreverAfter}(T: \text{time}, Q: \text{fluent})$. Fluent Q holds from time T onward. $\text{eventuallyForever}(Q: \text{fluent})$. Q eventually becomes true and remains true forever.**Definitions:**TD.1 $\text{throughout}(T1, T2, Q) \equiv \forall T \ T1 \leq T \leq T2 \Rightarrow \text{holds}(T, Q)$.TD.2 $\text{throughoutxE}(T1, T2, Q) \equiv \forall T \ T1 \leq T < T2 \Rightarrow \text{holds}(T, Q)$.TD.3 $\text{foreverAfter}(T, Q) \equiv \forall T1 \ T \leq T1 \Rightarrow \text{holds}(T1, Q)$.TD.4 $\text{eventuallyForever}(Q) \equiv \exists T1 \forall T2 \ T1 < T2 \Rightarrow \text{holds}(T2, Q)$.**Axioms:**T.1 Let $\Phi(T)$ be an open formula where T is a variable of sort “time” such that the variable Q does not appear free in Φ . Then the following is an axiom:

$$\exists Q \ \forall T: \text{time} \ \text{holds}(T, Q) \Leftrightarrow \Phi(T).$$

T.2 Let $\Phi(X, T)$ be an open formula where T is a variable of sort “time”, and X is an atemporal variable (i.e. not of sort “time” or “fluent”) such that the variable Q does not appear free in Φ . Then the following is an axiom:

$$[\forall T: \text{time} \ \exists X \ \Phi(X, T)] \Rightarrow \exists Q \ \forall T: \text{time} \ \Phi(\text{value}(T, Q), T).$$

is the fluent whose value at each time S is the union of the values of $Q1$ and $Q2$. “ $Q1 \subset^\# Q2 \vee^\# Q2 \subset^\# Q1$ ” is the fluent that holds at all times in which either $Q1$ is a subset of $Q2$ or $Q2$ is a subset of $Q1$.

4.4. Space

The ontology we use for space is Euclidean geometry (\mathfrak{N}^3). Our discussion here is entirely *ad hoc* for the example of pouring liquids from one container to another. That is, Table 4 enumerates the sorts and the predicates that we use in the physical axioms of Sections 4.5–4.9; it does not attempt any systematic discussion of geometric reasoning. In our formal proof, we will cite standard or easily proven geometric theorems as needed.

We use the following geometric sorts: *distance*, *height* (abbreviated “ht”), *volume*, *point*., *point set*, *vector*, *region*, *bregion*, and *rigid mapping*. The general category “geometric entity” includes points, point sets, vectors, regions, bregions, and rigid mappings.

It is useful to posit a standard unit length and a standard coordinate system, with a vertical z -axis. Relative to these, “distance”, “height” and “volume” can be considered subsorts of the real numbers.

A *point set* is any non-empty set of points. The exclusion of empty point sets together with our rule excluding partial functions means that many useful functions such as intersection have to be defined as predicates. (Including empty point sets would necessitate adding the condition “ PS is non-empty” to all kinds of logical statements; my feeling is that overall it is about an even trade-off.)

A *region* is a topologically regular point set. A region need not be either bounded or connected. A *bregion* is a bounded (but not necessarily connected) region.

A *rigid mapping* is a positive, orthonormal mapping of three-dimensional space to itself; that is, the composition of a rotation and a translation. (Reflections are not allowed.)

The other geometric sorts are self-explanatory.

The large number of geometric sorts that we use here may surprise readers who are used to more ontologically pure theories, in which the only geometric sort allowed is the sort of well-behaved, fully dimensional regions. However, it turns out that, strictly speaking, the greater ontological profligacy here is illusory. Specifically, if you have a first-order language that allows quantification over regions, then even if you restrict the language to the single predicate “ $\text{closer}(X, Y, Z)$ ”, meaning “region X is closer to Y than to Z ” and you restrict the universe of entities to include only simple polygons, nonetheless all of these ontological categories and the whole range of standard geometric concepts can be defined as first-order constructions in this language [10].

Because of the large number of geometric sorts, we are less systematic about the use of initial letters to indicate sort of variables. In most cases, variables of sort “region” start with R ; points start with start with P ; pointSets start with PS ; rigid mappings start with M ; distances start with D ; other geometric variables start with G .

Table 4
Spatial sorts and symbols

Sorts:

distance. $\text{subsort}(\text{distance}, \text{real})$.
 ht (height). $\text{subsort}(\text{ht}, \text{real})$.
 geomEntity (G). Any geometric entity.
 point (P). $\text{subsort}(\text{point}, \text{geomEntity})$. $\text{subsort}(\text{point}, \text{element})$.
 pointSet (PS). $\text{subsort}(\text{pointSet}, \text{set})$. $\text{subsort}(\text{pointSet}, \text{geomEntity})$.
 vector. $\text{subsort}(\text{vector}, \text{geomEntity})$.
 region (R). $\text{subsort}(\text{region}, \text{pointSet})$.
 bregion(R). $\text{subsort}(\text{bregion}, \text{region})$.
 rigidMapping (M). $\text{subsort}(\text{rigidMapping}, \text{geomEntity})$.

Symbols: (These are mostly explained in the text.)

$\text{rccC}(R1, R2: \text{region})$.
 $\text{rccEC}(R1, R2: \text{region})$.
 $\text{rccDS}(R1, R2: \text{region})$.
 $\text{rccO}(R1, R2: \text{region})$.
 $\text{rccNTPP}(R1, R2: \text{region})$.
 $\text{bounded}(PS: \text{pointSet})$.
 $\text{mappingImage}(M: \text{rigidMapping}, G: \text{geomEntity}) \rightarrow \text{geomEntity}$. The image of G under M .
 $\text{interior}(PS: \text{region}) \rightarrow \text{pointSet}$. The topological interior of PS .
 $\text{boundary}(R: \text{region}) \rightarrow \text{pointSet}$. The topological boundary surface of R ($R\text{-interior}(R)$).
 $\text{diameter}(R: \text{bregion}) \rightarrow \text{distance}$.
 \hat{z} : vector. The absolute upward direction.
 $\text{pointAbove}(P1, P2: \text{point})$. $P1$ is vertically above $P2$.
 $\text{height}(P: \text{point}) \rightarrow \text{ht}$.
 $\text{bottom}(R: \text{bregion}) \rightarrow \text{ht}$. The minimum value of $\text{height}(P)$ for $P \in R$.
 $\text{top}(R: \text{bregion}) \rightarrow \text{ht}$. The maximum value of $\text{height}(P)$ for $P \in R$.
 $\text{openBox}(RB: \text{region}, RI: \text{bregion})$.
 $\text{thicklyConnected}(R: \text{region})$.
 $\text{volumeOf}(R: \text{bregion}) \rightarrow \text{volume}$.
 $\text{topPoint}(P: \text{point}, R: \text{region})$.
 $\text{bottomPoint}(P: \text{point}, R: \text{region})$.
 $\text{weakTopPoint}(P: \text{point}, R: \text{region})$.
 $\text{dist}(P1, P2: \text{point}) \rightarrow \text{distance}$.
 $\text{dist}(PS1, PS2: \text{pointSet}) \rightarrow \text{distance}$.
 $\text{xyProj}(PS: \text{pointSet}) \rightarrow \text{pointSet}$.
 $\text{regDif}(R1, R2, RD: \text{region})$.
 $\text{regCompl}(R1, R2: \text{region})$.
 $\text{regInt}(R1, R2, RI: \text{region})$.
 $\text{hausdorff1}(PS1, PS2: \text{pointSet}, D: \text{distance})$.
 $\text{hausdorff}(PS1, PS2: \text{pointSet}, D: \text{distance})$.
 $\text{expand}(PS: \text{pointSet}, D: \text{distance}) \rightarrow \text{pointSet}$.
 $\text{lineSeg}(PS: \text{pointSet}, PA, PB: \text{point})$.
 $\text{upExpand}(PS: \text{pointSet}, D: \text{distance}, R, RE: \text{region})$.

We use some of the RCC [29] topological relations between regions. However, since our vocabulary of symbols is so large, we preface the name with “rcc”. Thus, the predicate “ $\text{rccC}(R1, R2)$ ” is the relation usually designated “ $C(R1, R2)$ ” in the qualitative spatial reasoning literature; namely, regions $R1$ and $R2$ are connected. Similarly “ $\text{rccEC}(R1, R2)$ ” is the QSR relation $EC(R1, R2)$, $R1$ and $R2$ are externally connected; “ $\text{rccDS}(R1, R2)$ ” is the QSR relation $DS(R1, R2)$, $R1$ and $R2$ are disjoint (either disconnected or externally connected); “ $\text{rccO}(R1, R2)$ ” is the QSR relation $O(R1, R2)$, $R1$ and $R2$ overlap; and “ $\text{rccNTPP}(R1, R2)$ ” is the QSR relation $NTPP(R1, R2)$, $R1$ is a non-tangential proper part of $R2$. The RCC relations are only applied to regions, not to other point sets.

The function “ $\text{mappingImage}(M, G)$ ” denotes the image of G under rigid mapping M . The sort of $\text{mappingImage}(M, G)$ is the same as the sort of G (in other words, all our geometric sorts are closed under rigid mappings).

The predicates “ $\text{regDif}(R1, R2, RD)$ ”, “ $\text{regCompl}(R1, RC)$ ”, and “ $\text{regInt}(R1, R2, RI)$ ” assert that RD is the regularized difference of $R1$ and $R2$, RC is the regularized complement of $R1$, and RI is the regularized intersection of $R1$ and $R2$.

The predicate “ $\text{openBox}(RB, RI)$ ” is the geometric equivalent of the definition of a “cupped” region in Definition 3.2.3. It holds for regions RB, RI if RI is thickly connected; RB and RI are disjoint; and every boundary point of RI either has height equal to $\text{top}(RI)$ or is a boundary point of RB .

The predicate “ $\text{diameter}(PS, X)$ ” means that X is the diameter of point set PS ; that is, the least upper bound on $\text{distance}(P1, P2)$ where $P1, P2$ are points in PS . If PS is unbounded, then this does not hold for any X .

The predicate “ $\text{topPoint}(P, R)$ ” holds if point P is a boundary point of R and there is a vertical line segment upward from P that lies outside R . The predicate “ $\text{bottomPoint}(P, R)$ ” is defined analogously. The predicate “ $\text{weakTopPoint}(P, R)$ ” holds if P is a boundary point of R and there is a vertical line segment upward from P that does not enter the interior of R (but may lie in the boundary of R).

Table 5

Some spatial definitions

Definitions:

- SD.1 $\text{bregion}(R) \equiv \text{region}(R) \wedge \text{bounded}(R)$.
- SD.2 $\text{localMin}(P, PS) \equiv$
 $\exists_D D > 0 \wedge \forall_{P1 \in PS} \text{dist}(P, P1) < D \Rightarrow \text{height}(P) \leq \text{height}(P1)$.
- SD.3 $\text{bottom}(R) = H \equiv [\exists_{P \in R} \text{height}(P) = H] \wedge [\forall_{P \in R} \text{height}(P) \geq H]$.
- SD.4 $\text{top}(R) = H \equiv [\exists_{P \in R} \text{height}(P) = H] \wedge [\forall_{P \in R} \text{height}(P) \leq H]$.
- SD.5 $\text{topPoint}(P, R) \equiv$
 $P \in \text{boundary}(R) \wedge$
 $\exists_{D1: \text{distance}} 0 < D1 \wedge \forall_{D2: \text{distance}, P \in PS} 0 < D2 < D1 \Rightarrow P + D2 \cdot \hat{z} \notin R$.
- SD.6 $\text{bottomPoint}(P, R) \equiv$
 $P \in \text{boundary}(R) \wedge$
 $\exists_{D1: \text{distance}} 0 < D1 \wedge \forall_{D2: \text{distance}, P \in PS} 0 < D2 < D1 \Rightarrow P - D2 \cdot \hat{z} \notin R$.
- SD.7 $\text{dist}(PS1, PS2) = D \equiv$
 $[\forall_{P1 \in PS1, P2 \in PS2} \text{dist}(P1, P2) \geq D] \wedge$
 $[\forall_{D1} D1 > D \Rightarrow \exists_{P1 \in PS1, P2 \in PS2} \text{dist}(P1, P2) < D1]$.
- SD.8 $\text{regDif}(R1, R2, R3) \equiv$
 $\forall_{R: \text{region}} R \subset R3 \Leftrightarrow R \subset R1 \wedge \text{rccDS}(R, R2)$.

Table 6

More spatial definitions

- SD.9 $\text{regComp}(R1, R2) \equiv \forall_{R: \text{region}} R \subset R2 \Leftrightarrow \text{rccDS}(R, R1)$.
- SD.10 $\text{regInt}(R1, R2, R3) \equiv \forall_{R: \text{region}} R \subset R3 \Leftrightarrow R \subset R1 \wedge R \subset R2$.
- SD.11 $\text{openBox}(RB, RI) \equiv$
 $\text{thicklyConnected}(RI) \wedge$
 $[\forall_P P \in \text{boundary}(RI) \Rightarrow \text{top}(RI) = \text{height}(P)] \vee P \in \text{boundary}(RB)]$
- SD.12 $\text{pointAbove}(PA, PB) \equiv$
 $\exists_D 0 \leq D \wedge PA = PB + D \cdot \hat{z}$.
- SD.13 $\text{hausdorff1}(PS1, PS2, D) \equiv$
 $[\forall_{P1 \in PS1} \text{dist}(P1, PS2) \leq D] \wedge$
 $[\forall_{D1} D1 < D \Rightarrow \exists_{P1 \in PS1} \text{dist}(P1, PS2) > D1]$.
- SD.14 $\text{hausdorff}(PS1, PS2, D) \equiv$
 $\exists_{D1, D2} \text{hausdorff1}(PS1, PS2, D1) \wedge \text{hausdorff1}(PS2, PS1, D2) \wedge D = \max(D1, D2)$.
- SD.15 $\text{weakTopPoint}(P, R) \equiv$
 $P \in \text{boundary}(R) \wedge$
 $\exists_{D1: \text{distance}} 0 < D1 \wedge \forall_{D2: \text{distance}, P \in PS} 0 < D2 < D1 \Rightarrow P + D2 \cdot \hat{z} \notin \text{interior}(R)$.
- SD.16 $P \in \text{expand}(PS, D) \Leftrightarrow \text{dist}(P, PS) \leq D$.
- SD.17 $\text{upExpand}(PS, D, R, RE) \equiv$
 $\forall_P P \in RE \Leftrightarrow$
 $\exists_{PA, PL} PA \in PS \wedge \text{lineSeg}(PL, PA, P) \wedge PL \in R \wedge$
 $\text{height}(P) \geq \text{height}(PA) \wedge \text{dist}(P, PA) \leq D$.

The predicates “ $\text{hausdorff1}(PS1, PS2, D)$ ” and “ $\text{hausdorff}(PS1, PS2, D)$ ” assert that D is the one-sided Hausdorff distance/the true Hausdorff distance between point sets $PS1$ and $PS2$. They are defined as predicates, because they may be undefined (i.e. infinite) for unbounded point sets.

The function “ $\text{xyProj}(PS)$ ” is the projection of point set PS in the standard coordinate system.

The function “ $\text{expand}(PS, D)$ ” is the set of all points within distance D of some point in PS . The predicate “ $\text{upExpand}(PS, D, R, RE)$ ” states that RE is the *upward expansion* of point set PS by distance D through region R ; that is, RE is the set of all points P such that there is a point PA in PS within distance D of P such that the line segment from P to PA goes through R and P is at least as high as PA .

The meanings of the remaining symbols in Table 4 are immediate. Functions and predicates defined over point sets are overloaded to apply to individual points by coercing the point P to the point set $\{P\}$. Tables 5 and 6 give formal definitions of non-standard symbols.

4.5. Ontology

There are two basic kinds of physical entities in our theory. An *object* is a rigid solid object.

A *liquid chunk* is a piece of liquid. Think of it as an abstraction of a set of particular molecules. At any time, any bounded region, connected or disconnected, which is filled with liquid has a corresponding liquid chunk. Each liquid chunk is eternal. The volume of liquid chunk L , denoted “ $\text{liqVolume}(L)$ ” is constant over time. The set theoretic relation between two liquid chunks is constant over time; in particular, the atemporal relation “ $\text{subchunk}(L1, L2)$ ” asserts that $L1$ is a subset of $L2$.

In physical reasoning about solid objects it is often useful to be able to reify *pseudo-objects* [4]; geometric entities that “move around” with an object, such as the center of mass of an object, the hole of a donut, the apex of a cone, and so on. The *source* of pseudo-object B is the object to which B is “attached”.

Table 7

Ontology: Sorts and symbols

Sorts:

movableEntity(G).
 gObject (G). Subsort of movableEntity.
 object (O). Subsort of gObject.
 pseudo (B). Subsort of gObject.
 liquidChunk (L). Subsort of movableEntity.
 objectSet(U). Subsort of set.

Symbols:

place(G : movableEntity) \rightarrow fluent[geomEntity].
 source(G : gObject) \rightarrow object.
 shape(G : gObject) \rightarrow geomEntity.
 placement(G : gObject) \rightarrow fluent[rigidMapping].
 motionless(O : object) \rightarrow fluent[Bool]
 liquidSpace \rightarrow fluent[region].
 solidSpace \rightarrow fluent[region].
 solidFreeSpace \rightarrow fluent[region].
 subChunk($L1, L$: liquidChunk).
 liqVolume(L : liquidChunk) \rightarrow volume.
 fullOfLiquid(R : region) \rightarrow fluent[Bool].
 empty(R : region) \rightarrow fluent[Bool].
 emptyLiquid(R : region) \rightarrow fluent[Bool].

Table 8

Ontology: Definitions and axioms

Definitions:

ONTD.1 $\text{fullOfLiquid}(R) = R \subset^{\#} \text{liquidSpace}$.
 ONTD.2 $\text{emptyLiquid}(R) = \text{rccDS}^{\#}(R, \text{liquidSpace})$
 ONTD.3 $\text{holds}(T, P \in^{\#} \text{solidSpace}) \equiv \exists_{O:\text{object}} \text{holds}(T, P \in^{\#} \uparrow O)$.
 ONTD.4 $\text{solidFreeSpace} = \text{regCompl}^{\#}(\text{solidSpace})$.
 ONTD.5 $\text{empty}(R) = \text{emptyLiquid}(R) \wedge^{\#} R \subset^{\#} \text{solidFreeSpace}$.
 ONTD.6 $\text{objectSet}(U) \wedge O \in U \Rightarrow \text{object}(O)$.

Axioms:

ONT.1 $\text{object}(O) \Rightarrow \text{source}(O) = O$.
 ONT.2 $\text{gObject}(G) \Rightarrow \text{placement}(G) = \text{placement}(\text{source}(G))$.
 ONT.3 $\text{object}(O) \Rightarrow \text{bregion}(\text{shape}(O))$.
 ONT.4 $\text{gObject}(G) \Rightarrow \text{value}(T, \text{place}(G)) = \text{mappingImage}(\text{value}(T, \text{placement}(G)), \text{shape}(G))$.
 ONT.5 $\forall_{T:\text{time}, L1, L2:\text{liquidChunk}} \text{subChunk}(L1, L2) \Leftrightarrow \text{holds}(T, \uparrow L1 \subset^{\#} \uparrow L2)$
 ONT.6 $\text{bregion}(R) \wedge \text{holds}(T, \text{fullOfLiquid}(R)) \Rightarrow \exists_{L:\text{liquidChunk}}^1 \text{value}(T, \uparrow L) = R$.
 ONT.7 $\forall_{T:\text{time}, L:\text{liquidChunk}} \text{holds}(T, \text{bregion}^{\#}(\uparrow L) \wedge^{\#} \uparrow L \subset^{\#} \text{liquidSpace})$
 ONT.8 $\text{throughout}(T1, T2, \text{motionless}(O)) \Leftrightarrow \exists_P \text{throughout}(T1, T2, P =^{\#} \text{placement}(O))$.

Since objects, pseudo-objects, and liquid chunks have important properties in common, it will be convenient to group these into supersorts and define these properties over the supersorts. A *generalized object* (gObject) is either an object or a pseudo-object, and a *movable entity* is either an object, a pseudo-object or a liquid chunk.

We characterize an object O and its associated pseudo-objects in terms of an arbitrary standard position. The *shape* of O is the region that it occupies in its standard position. The shape of pseudo-object Q is the geometric entity that instantiates Q when O is in its standard position.

The position of an object O at a given time T is specified by a rigid mapping from its standard position to its position at T , denoted $\text{value}(T, \text{placement}(O))$. The region occupied by O at time T is thus the image of $\text{shape}(O)$ under the mapping $\text{value}(T, \text{placement}(O))$; this is denoted $\text{value}(T, \text{place}(O))$. The same holds for pseudo-objects. The constraint that a pseudo-object must move around with its source is expressed by axiom ONT.2, which states that, for any pseudo-object B , $\text{placement}(B) = \text{placement}(\text{source}(B))$.

Similarly, if L is a liquid chunk, then the fluent “ $\text{place}(L)$ ” is the region occupied by L at each time T .

Since the function $\text{place}(E)$ is used so frequently in our theory, we abbreviate it using the symbol $\uparrow E$. For example, the formula, “ $\text{holds}(T, \text{rccEC}^{\#}(\uparrow O, \uparrow L))$ ” is an abbreviation for “ $\text{holds}(T, \text{rccEC}^{\#}(\text{place}(O), \text{place}(L)))$ ”, meaning that O and L are externally connected at time T .

The constants “liquidSpace” and “solidSpace” denote the fluents corresponding to all of the space in the world occupied by liquid/solids at a given time. The constant “solidFreeSpace” denotes the fluent corresponding to all the space not in the interior of a solid at a given time. The function “fullOfLiquid(R)” denotes the Boolean fluent that holds at time T if R is full of liquid. The function “empty(R)” holds when the interior of R is empty of both liquids and solids. The fluent “emptyLiquid(R)” holds when the interior of R is empty of liquid but not necessarily of solids.

Table 9
Continuity

Symbols:	
continuous(Q :fluent[real], TS, TE :time)	
continuousPoint(Q :fluent[point], TS, TE :time)	
continuousPlacement(Q :fluent[rigidMapping], TS, TE :time)	
volumeDiff($R1, R2$:bregion) \rightarrow volume	
volumeSymDiff($R1, R2$:bregion) \rightarrow volume	
continuousVolume(Q :fluent[bregion], TS, TE :time)	
continuousHausdorff(Q :fluent[pointSet], TS, TE :time)	
Definitions:	
CONTD.1	$\text{continuous}(Q, TS, TE) \equiv$ $\forall_E E > 0 \Rightarrow$ $\exists_D D > 0 \wedge$ $\forall_{T1, T2} TS \leq T1 < T2 \leq TE \wedge T2 - T1 < D \Rightarrow$ $ \text{value}(T1, Q) - \text{value}(T2, Q) < E.$
CONTD.2	$\text{continuousPoint}(Q, TS, TE) \equiv$ $\forall_E E > 0 \Rightarrow$ $\exists_D D > 0 \wedge$ $\forall_{T1, T2} TS \leq T1 < T2 \leq TE \wedge T2 - T1 < D \Rightarrow$ $\text{dist}(\text{value}(T1, Q), \text{value}(T2, Q)) < E$
CONTD.3	$\text{continuousPlacement}(Q, TS, TE) \equiv$ $\forall_P \text{continuousPoint}(\text{mappingImage}^\#(Q, P), TS, TE).$
CONTD.4	$\text{volumeDiff}(R1, R2) = V \Leftrightarrow$ $[R1 \subset R2 \wedge V = 0] \vee [\exists_{R3} \text{regDif}(R1, R2, R3) \wedge V = \text{volumeOf}(R3)].$
CONTD.5	$\text{volumeSymDiff}(R1, R2) = \text{volumeDif}(R1, R2) + \text{volumeDif}(R2, R1).$
CONTD.6	$\text{continuousVolume}(Q, TS, TE) \equiv$ $\forall_E E > 0 \Rightarrow$ $\exists_D D > 0 \wedge$ $\forall_{T1, T2} TS \leq T1 < T2 \leq TE \wedge T2 - T1 < D \Rightarrow$ $\text{volumeSymDiff}(\text{value}(T1, Q), \text{value}(T2, Q)) < E$
CONTD.7	$\text{continuousHausdorff}(Q, TS, TE) \equiv$ $\forall_E E > 0 \Rightarrow$ $\exists_D D > 0 \wedge$ $\forall_{T1, T2} TS \leq T1 < T2 \leq TE \wedge T2 - T1 < D \Rightarrow$ $\exists_{E1} \text{hausdorff}(\text{value}(T1, Q), \text{value}(T2, Q), E1) \wedge E1 < E.$

Tables 7 and 8 present ontological definitions and axioms. ONT.4 asserts that the place occupied by any object or pseudo-object G is the image of $\text{shape}(G)$ under the rigid mapping $\text{placement}(G)$. Since $\text{shape}(G)$ is constant over time rather than a fluent, this implies that objects are rigid. ONT.2 asserts that the placement of a pseudo-object is equal to the placement of a source, guaranteeing that the pseudo-object moves around with the corresponding object. ONT.6 asserts that at a given time the region occupied by liquid chunk $L1$ is a subset of the region occupied by liquid chunk $L2$ just if $L1$ is a subchunk of $L2$; since subchunk is not time-dependent, this means that the set theoretical relations among chunks are constant over time. ONT.7 and ONT.8 assert that every bounded region filled with water corresponds to a liquid chunk, and conversely.

4.6. Kinematics

We begin with the representation of Rules 1, 2, and 3, called the *kinematic* constraints. In the domain of solid object mechanisms, reasoning purely with kinematic constraints is an important category of reasoning; there are many mechanisms whose workings can be explained largely or entirely on kinematic principles [22]. In the domain of liquids, purely kinematic reasoning is much less important because the class of kinematic behaviors is much less constrained. Nonetheless, these are very important constraints, because they always apply, as long as the idealizations that objects are rigid and liquids are incompressible are valid.

Table 9 defines the concepts of continuity that we will need, as in Section 3.7. Solid objects have a continuous placement function; liquid chunks are volume-continuous and almost everywhere Hausdorff-continuous.

Table 10 contains the kinematic axioms. Definition KIND.2 defines a liquid chunk L having “simple flow” of diameter D between times TS and TE as in Definition 3.7.5. Axiom KIN.1 asserts that no two solid objects overlap. Axiom KIN.2 asserts that no liquid overlaps a solid object. Axiom KIN.3 asserts that a liquid chunk always occupies a volume equal to its inherent volume; thus liquids are incompressible. Axiom KIN.4 asserts that the place of every liquid chunk is volume continuous. Axiom KIN.5 asserts that every liquid chunk has a subchunk with simple flow of arbitrarily small diameter.

4.7. Overflow

Table 11 gives the formal statement of Definition 3.3.3 of “driven” liquids, since that category is needed as an exception to Rules 4, 5, and 6,

Table 10
Kinematics

Symbols: $\text{simpleFlow}(L:\text{liquidChunk}, D:\text{distance}, TS, TE:\text{time})$	
Definitions:	
KIND.1	$\text{simpleFlow}(L, D, TS, TE) \equiv$ $\text{continuousHausdorff}(\uparrow L, TS, TE) \wedge$ $\text{throughout}(TS, TE, \text{thicklyConnected}^\#(\uparrow L) \wedge^\# \text{diameter}^\#(\uparrow L) <^\# D)$
Axioms:	
KIN.1	$\forall O1, O2:\text{object}, T:\text{time} \quad O1 \neq O2 \Rightarrow \text{holds}(T, \text{rccDS}^\#(\uparrow O1, \uparrow O2))$ (Two objects $O1$ and $O2$ do not overlap at any time T .)
KIN.2	$\forall O:\text{object}, L:\text{liquidChunk}, T:\text{time} \quad \text{holds}(T, \text{rccDS}^\#(\uparrow O, \uparrow L))$ (An object O does not overlap a liquid chunk L at time T .)
KIN.3	$\forall L:\text{liquidChunk}, T:\text{time} \quad \text{volumeOf}(\text{value}(T, \uparrow L)) = \text{liqVolume}(L)$ (The volume of a liquid chunk L is constant.)
KIN.4	$\forall L:\text{liquidChunk}, TS, TE:\text{time} \quad \text{continuousVolume}(\uparrow L, TS, TE)$
KIN.5	$\forall L:\text{liquidChunk}, TS, TE:\text{time}, D:\text{distance}$ $D > 0 \Rightarrow$ $\exists L1:\text{liquidChunk} \quad \text{subchunk}(L1, L) \wedge \text{simpleFlow}(L1, D, TS, TE).$ (Liquids flow continuously, as described in the text.)

Table 11
Overflow

Symbols: $\text{localMaxBox}(RB:\text{region}, RI:\text{bregion}) \rightarrow \text{fluent}[\text{Bool}].$ $\text{localMaxCup}(R:\text{bregion}) \rightarrow \text{fluent}[\text{Bool}].$ $\text{monotonic}(Q:\text{fluent}[\text{real}], TS, TE:\text{time}, X:\text{real}).$ $\text{monotonicPrefix}(Q:\text{fluent}[\text{real}], TS:\text{time}).$ $\text{simpleOverflows}(L:\text{liquidChunk}, Q:\text{fluent}[\text{bregion}], TS, TE:\text{history})$ $\text{drivenReg}(R:\text{bregion}) \rightarrow \text{fluent}[\text{Bool}].$ $\text{driven}(L:\text{liquidChunk}) \rightarrow \text{fluent}[\text{Bool}].$	
SPILLD.1	$\text{localMaxBox}(RB, RI) \equiv$ $\text{openBox}(RB, RI) \wedge$ $\exists D \quad D > 0 \wedge$ $\forall R1 \quad \text{openBox}(RB, R1) \wedge RI \subset R1 \wedge R \neq R1 \Rightarrow$ $\exists P \in R1 \quad \text{dist}(P, RI) > D.$
SPILLD.2	$\text{localMaxCup}(R) = \text{localMaxBox}^\#(\text{solidSpace}, R).$
SPILLD.3	$\text{monotonic}(Q, TS, TE, X) \equiv$ $TS < TE \wedge$ $\forall T1, T2 \quad TS \leq T1 < T2 \leq TE \Rightarrow$ $[[X = 0 \wedge \text{value}(T2, Q) = \text{value}(T1, Q)] \vee$ $X \cdot (\text{value}(T2, Q) - \text{value}(T1, Q)) > 0].$
SPILLD.4	$\text{monotonicPrefix}(Q, TS) \equiv \exists TE, X \quad \text{monotonic}(Q, TS, TE, X).$
SPILLD.5	$\text{simpleOverflows}(L, Q, TS, TE) \equiv$ $\text{holds}(TS, \uparrow L =^\# Q) \wedge$ $\text{throughout}(TS, TE, \text{localMaxCup}^\#(Q)) \wedge \text{continuousVolume}(Q, TS, TE) \wedge$ $\text{monotonic}(\text{volumeOf}^\#(Q), TS, TE, -1)$
SPILLD.6	$\text{holds}(T, \text{drivenReg}(R)) \Leftrightarrow$ $\exists LO, QO, TS, TE, RE \quad TS \leq T \leq TE \wedge$ $\text{simpleOverflow}(LO, QO, TS, TE) \wedge R \subset RE \wedge$ $\text{holds}(T, \text{thicklyConnected}^\#(R \cup \uparrow LO) \wedge^\#$ $\text{upExpand}^\#(\text{topSurface}^\#(QO), \text{maxOutflow}, \text{solidFreeSpace}, RE))$
SPILLD.7	$\text{holds}(T, \text{driven}(L)) \Leftrightarrow$ $\exists R \quad \text{holds}(T, \text{drivenReg}(R) \wedge^\# \uparrow L \subset^\# R)$
Axioms:	
SPILL.1	$\text{throughout}(TS, TE, \text{localMaxCup}^\#(Q)) \wedge \text{continuousVolume}(Q, TS, TE) \Rightarrow$ $\text{monotonicPrefix}(\text{volumeOf}(Q), TS).$

Definitions SPILLD.1 and SPILLD.2 define “locally maximal cup” as in Definition 3.3.1. Definition SPILLD.3 defines the relation “ $\text{monotonic}(Q, TS, TE, X)$ ”, meaning that fluent Q is monotonic over interval $[TS, TE]$ with sign X . If X is positive, Q is strictly monotonically increasing; if X is negative, Q is strictly monotonically decreasing; if X is 0, Q is constant.

Definition SPILLD.5 defines the event of a cupped region overflowing. Let Q be a locally maximal cup throughout interval $[TS, TE]$ that is volume continuous. If liquid chunk L fills Q at TS and the volume of Q monotonically decreases during $[TS, TE]$, then Q overflows during $[TS, TE]$. The condition “ $\text{continuousVolume}(Q, H)$ ” is required, first to ensure that Q

Table 12
Flow

Symbols:	
$\text{flowsIn}(L:\text{liquidChunk}, Q:\text{fluent}[\text{bregion}], TS, TE: \text{time})$	
$\text{flowsOut}(L:\text{liquidChunk}, Q:\text{fluent}[\text{bregion}], TS, TE: \text{time})$	
$\text{noInflow}(Q:\text{fluent}[\text{bregion}], TS, TE: \text{time})$	
$\text{noInflowDuring}(Q:\text{fluent}[\text{bregion}], TS, TE: \text{time})$	
$\text{noOutflow}(Q:\text{fluent}[\text{bregion}], TS, TE: \text{time})$	
Definitions:	
FLOWD.1	$\text{flowsIn}(L, Q, TS, TE) \equiv \text{holds}(TS, \text{rccDS}^\#(\uparrow L, Q)) \wedge \text{holds}(TE, \uparrow L \subset^\# Q).$
FLOWD.2	$\text{flowsOut}(L, Q, TS, TE) \equiv \text{holds}(TS, \uparrow L \subset^\# Q) \wedge \text{holds}(TE, \text{rccDS}^\#(\uparrow L, Q)).$
FLOWD.3	$\text{noInflow}(Q, TS, TE) \equiv \neg \exists_L \text{flowsIn}(L, Q, TS, TE).$
FLOWD.4	$\text{noOutflow}(Q, TS, TE) \equiv \neg \exists_L \text{flowsOut}(L, Q, TS, TE).$
FLOWD.5	$\text{noInflowDuring}(Q, TS, TE) \equiv \forall_{T1, T2} TS \leq T1 < T2 \leq TE \Rightarrow \text{noInflow}(Q, T1, T2).$

Table 13
Cups

Symbols:	
$\text{cuppedRegion}(R:\text{bregion}) \rightarrow \text{fluent}[\text{Bool}]$	
$\text{motionlessLiquidIn}(R:\text{bregion}) \rightarrow \text{fluent}[\text{Bool}]$	
$\text{slowObjectMotion}(O:\text{object}) \rightarrow \text{fluent}[\text{Bool}]$	
$\text{slowObjectsInContact}(R:\text{region}) \rightarrow \text{fluent}[\text{Bool}]$	
$\text{motionlessObjectsInContact}(R:\text{region}) \rightarrow \text{fluent}[\text{Bool}]$	
$\text{driven}(L) \rightarrow \text{fluent}[\text{Bool}]$	
$\text{noDrivenLiqIn}(R:\text{region}) \rightarrow \text{fluent}[\text{Bool}]$	
Definitions:	
CUPD.1	$\text{cuppedRegion}(R) = \text{openBox}^\#(\text{solidSpace}, R)$
CUPD.2	$\text{holds}(T, \text{slowObjectsInContact}(R)) \equiv \forall_{O:\text{object}} \text{holds}(T, \text{rccC}^\#(\uparrow O, R)) \Rightarrow \text{holds}(T, \text{slowObjectMotion}(O))$
CUPD.3	$\text{holds}(T, \text{motionlessObjectsInContact}(R)) \equiv \forall_{O:\text{object}} \text{holds}(T, \text{rccC}^\#(\uparrow O, Q)) \Rightarrow \text{holds}(T, \text{motionless}(O))$
CUPD.4	$\text{holds}(T, \text{noDrivenLiqIn}(R)) \equiv \neg \exists_L \text{holds}(T, \text{driven}(L) \wedge^\# \uparrow L \subset^\# R).$
Axioms:	
CUP.1	$\text{throughout}(TS, TE, \text{motionlessLiquidIn}(Q)) \equiv TS < TE \wedge \forall_{L:\text{liquidChunk}, T1, T2} TS \leq T1 \leq TE \wedge TS \leq T2 \leq TE \wedge \text{holds}(T1, \uparrow L \subset^\# Q) \Rightarrow \text{value}(T2, \uparrow L) = \text{value}(T1, \uparrow L).$
CUP.2	$\text{continuousVolume}(Q, TS, TE) \wedge \text{throughout}(TS, TE, \text{slowObjectsInContact}(Q) \wedge^\# \text{cuppedRegion}(Q) \wedge^\# \neg^\# \text{fullOfLiquid}(Q) \wedge^\# \text{noDrivenLiqIn}(Q)) \Rightarrow \text{noOutflow}(Q, TS, TE).$
CUP.3	$\forall_{TS:\text{time}, R:\text{bregion}} [\forall_{T1} TS \leq T1 \Rightarrow \text{holds}(T1, \text{cuppedRegion}(R) \wedge^\# \text{motionlessObjectsInContact}(R)) \wedge \text{noInflowDuring}(R, TS, T1)] \Rightarrow \text{eventuallyForever}(\text{motionlessLiquidIn}(R)).$
CUP.4	$\text{holds}(T, \text{motionless}(O)) \Rightarrow \text{holds}(T, \text{slowObjectMotion}(O)).$

does not simply “jump” from one cup to an entirely separate cup, and second to exclude the case where the extent of a cupped region changes discontinuously, e.g. a plug is pulled from the side of a cup.

Definitions SPILLD.6 and SPILLD.7 defines a “driven region” and a “driven liquid” as in Definition 3.3.3.

Axiom SPILL.1 is required to exclude certain pathological scenarios. SPILLD.6 and SPILLD.7 assert that a liquid is driven only while the corresponding cup is overflowing, and SPILLD.5 states that a cup overflows only over a history where its volume decreases monotonically. However, there exist continuous real-valued functions—variants of the Cantor function—which decrease between the start and end of an interval but do not monotonically decrease over any subinterval. If the volume of a cupped region were to follow one of these, then SPILLD.5 would say that it never “overflows” even though the volume of the cup has fallen below the volume of the liquid it contains. To rule this out, we assert the rather specialized axiom SPILL.1 that if Q is a fluent satisfying the conditions of SPILLD.5 (other than being filled with liquid) at time $T1$, then $\text{volumeOf}(Q)$ has the “monotonic prefix property” defined in SPILLD.4; there exists $T2 > T1$ such that $\text{volumeOf}(Q)$

Table 14
Downward flow: Symbols

Symbols:

$\text{maxOutflow} \rightarrow \text{distance}.$
 $\text{obstacleSpace} \rightarrow \text{fluent}[\text{region}].$
 $\text{weaklyCupped}(L:\text{liquidChunk}) \rightarrow \text{fluent}[\text{Bool}].$
 $\text{nonFlowingSpace} \rightarrow \text{fluent}[\text{region}].$
 $\text{flowDisruptionReg}(R:\text{region}) \rightarrow \text{fluent}[\text{Bool}].$
 $\text{flowDisruptedSpace} \rightarrow \text{fluent}[\text{region}].$
 $\text{flowUndisruptedSpace} \rightarrow \text{fluent}[\text{region}].$
 $\text{flowStoppingPoint}(P:\text{point}, R:\text{region}) \rightarrow \text{fluent}[\text{Bool}].$
 $\text{cantFlowDown}(R:\text{region}) \rightarrow \text{fluent}[\text{Bool}].$
 $\text{canFlowDown}(R:\text{region}) \rightarrow \text{fluent}[\text{Bool}].$
 $\text{flowsDown}(Q:\text{fluent}[\text{region}], TS, TE:\text{time}).$
 $\text{flowsStraightDown}(L:\text{liquidChunk}, TS, TE:\text{time}).$
 $\text{connHorizOutslice}(LA, LB:\text{liquidChunk}) \rightarrow \text{fluent}[\text{Bool}].$

is monotonic over $[T1, T2]$. Essentially, this amounts to a well-behavedness condition on the shape and the motion of the objects forming the cup. It is a quite weak condition; if the shapes and motions are semi-algebraic or probably even if they are semi-analytic, then the condition is guaranteed.

4.8. Cupped liquid

Axiom CUP.1 is the formal statement of rule 4, the first of our substantive dynamic rules: If, throughout an interval $[TS, TE]$ fluent Q is always a cupped region, is continuous, and is never full of liquid, and if the objects in contact with Q move slowly throughout $[TS, TE]$, then there is no outflow from Q during $[TS, TE]$. The definition of “no outflow” and other useful predicates associated with flow are given in Table 12.

Axiom CUP.2 states a dynamic rule which we actually do not need for our pouring example here and did not discuss in Section 3: If a fixed region R is a cupped region forever after a starting time TS and the objects in contact with R are motionless throughout forever, then eventually the liquid in R will likewise become motionless.

Axioms CUP.3–CUP.5 assert some basic properties of “slow object motion”: it is inherited by subhistories and by splicing, and an object that is motionless is (so to speak) moving slowly.

4.9. Downward flow

Finally, Tables 14–16 give the formal statement of our theory of downward flow and rules 5 and 6 from Sections 3.4 and 3.5. Definitions DOWND.1–DOWND.5 define the concepts of weakly cupped liquids, nonFlowingSpace , $\text{flowDisruptedSpace}$, and $\text{flowUndisruptedSpace}$, corresponding to Definitions 3.4.2, 3.5.1, and 3.5.2. Definitions DOWND.7–DOWND.9 define regions that can and can’t flow down, corresponding to Definitions 3.4.3 and 3.4.4. Definitions DOWND.10–DOWND.12 define the event of a liquid flowing down and flowing straight down, corresponding to definitions 3.4.1 and 3.5.3. Definition DOWND.13 defines the concept of a liquid slice with the same vertical span, used in Rule 5.

The definition of “flowing straight down” given in Section 3.5 (Definition 3.5.3) was not quite adequate, and is corrected here. The precise definition is given in DOWND.11 and DOWND.12. We say that a region-value fluent Q *flows straight down* in history $[TS, TE]$ if Q is continuous and flows down and is always thickly connected and its projection on the xy plane is constant. A piece of liquid L flows straight down in $[TS, TE]$ if L flows down TS, TE , and at TE , the place of every connected piece $L2$ of L is equal to the value in TE of some region-valued fluent Q that has flowed straight down in H and has always been part of L in $[TS, TE]$. Note that Q does not have to correspond to $L2$ throughout $[TS, TE]$ —that would require the whole liquid to flow vertically downward—but at each time T between TS and TE , Q corresponds to some part of L . This condition excludes histories in which pieces of L could “sneak around” obstacles, getting underneath other pieces of L that they are not actually connected to, which would satisfy Definition 3.5.3 as originally stated.

Axiom DOWN.2 states Rule 5: a liquid that can flow down throughout an interval does flow down. Axiom DOWN.3 states Rule 6: a liquid chunk in $\text{flowUndisruptedSpace}$ is part of a horizontal slice of the same vertical extent that flows straight down for some finite length of time. The horizontal slice may flow into $\text{flowDisruptedSpace}$ —indeed, it may overlap $\text{flowDisruptedSpace}$ even at the start—but it certainly does not stop flowing straight down before it has come into contact with $\text{flowDisruptedSpace}$.

Axiom DOWN.4 prohibits the phenomenon of liquid flow that is not Hausdorff continuous, discussed in Section 3.7, from occurring in $\text{flowUndisruptedSpace}$. Specifically it states that for any sufficiently small interval $[TS, TE]$, any liquid chunk that is contained within $\text{flowUndisruptedSpace}$ at any time between TS and TE moves only a small Hausdorff distance between TS and TE .

Axiom DOWN.5 states that if a region R never contains any cups, and there is no flow into R , then all the liquid will eventually flow out of R .

Axiom DOWN.6 states that the downward velocity of any downward flowing liquid L is faster than the downward velocity of the top of any region fluent Q which is a maximal cup of any slowly moving object. This is not literally true, but we use

Table 15

Downward flow: Definitions

Definitions:	
DOWND.1	$\text{holds}(T, P \in^{\#} \text{obstacleSpace}) \Leftrightarrow$ $\exists_G \text{ holds}(T, P \in^{\#} \uparrow G) \wedge$ $[\text{object}(G) \vee$ $[\text{liquidChunk}(G) \wedge$ $[\text{holds}(T, \text{cuppedRegion}(\uparrow G) \vee^{\#} \text{driven}(G))]]]$
DOWND.2	$\text{weaklyCupped}(L) = \text{openBox}^{\#}(\text{obstacleSpace}, \uparrow L)$
DOWND.3	$\text{holds}(T, P \in^{\#} \text{nonFlowingSpace}) \Leftrightarrow$
DOWND.4	$\text{holds}(T, P \in^{\#} \text{obstacleSpace}) \vee \exists_L \text{ holds}(T, \text{weaklyCupped}(L) \wedge^{\#} P \in^{\#} L).$ $\text{holds}(T, \text{flowDisruptionReg}(R)) \equiv$ $\text{thicklyConnected}(R) \wedge \text{holds}(T, \text{fullOfLiquid}(R)) \wedge$ $\forall_{P \in R} \exists_{P1 \in R} \text{ holds}(T, \text{weakTopPoint}^{\#}(P1, \text{nonFlowingSpace}) \wedge$ $P \in^{\#} \text{upExpand}^{\#}(P1, \text{maxOutflow}, \text{solidFreeSpace})).$
DOWND.5	$\text{holds}(T, P \in^{\#} \text{flowDisruptedSpace}) \equiv$ $\exists_R P \in R \wedge \text{holds}(T, \text{flowDisruptionReg}(R)).$
DOWND.6	$\text{flowUndisruptedSpace} = \text{regComp}^{\#}(\text{flowDisruptedSpace} \cup^{\#} \text{nonFlowingSpace})$
DOWND.7	$\text{flowStoppingPoint}(P, R) \equiv$ $\neg \text{bottomPoint}(P, R) \wedge \text{localMin}(P, \text{boundary}(R)).$
DOWND.8	$\text{holds}(T, \text{cantFlowDown}(R)) \Leftrightarrow$ $[\text{holds}(T, \text{rccO}^{\#}(R, \text{nonFlowingSpace})) \vee$ $[\exists_P \text{ bottomPoint}(P, R) \wedge \text{holds}(T, \text{flowStoppingPoint}^{\#}(P, \text{nonFlowingSpace}))]]$
DOWND.9	$\text{canFlowDown}(R) = \neg^{\#} \text{cantFlowDown}(R).$
DOWND.10	$\text{flowsDown}(Q, TS, TE) \equiv$ $\forall_{T1, T2} TS \leq T1 < T2 \leq TE \Rightarrow$ $\text{value}(T2, \text{top}^{\#}(Q)) < \text{value}(T1, \text{top}^{\#}(Q)) \wedge$ $\text{value}(T2, \text{bottom}^{\#}(Q)) < \text{value}(T1, \text{bottom}^{\#}(Q))$
DOWND.11	$\text{qFlowsStraightDown}(Q, TS, TE) \equiv$ $\text{continuousVolume}(Q, TS, TE) \wedge \text{flowsDown}(Q, TS, TE) \wedge$ $\text{throughout}(TS, TE, \text{thicklyConnected}^{\#}(Q)) \wedge$ $\exists_{PS1: \text{pointSet}} \text{throughout}(TS, TE, PS1 =^{\#} \text{xyProj}^{\#}(Q))$
DOWND.12	$\text{flowsStraightDown}(L, TS, TE) \equiv$ $\text{flowsDown}(L, TS, TE) \wedge$ $\forall_{T2, L2} TS \leq T2 \leq TE \wedge \text{subchunk}(L2, L) \wedge \text{holds}(T2, \text{thicklyConnected}^{\#}(\uparrow L2)) \Rightarrow$ $\exists_{Q2} \text{ qFlowsStraightDown}(Q2, TS, T2) \wedge$ $\text{holds}(T2, Q2 =^{\#} \uparrow L2) \wedge \text{throughout}(TS, T2, Q2 \subset^{\#} \uparrow L).$
DOWND.13	$\text{holds}(T, \text{connHorizOutslice}(LA, LB)) \Leftrightarrow$ $\text{subchunk}(LA, LB) \wedge$ $\text{holds}(T, \text{thicklyConnected}^{\#}(\uparrow LB) \wedge^{\#}$ $\text{top}^{\#}(\uparrow LA) =^{\#} \text{top}^{\#}(\uparrow LB) \wedge^{\#} \text{bottom}^{\#}(\uparrow LA) =^{\#} \text{bottom}^{\#}(\uparrow LB))$

it to exclude a scenario in which one pours liquid of a cup and catches it again in the same cup.⁷ Note that “slow moving” must actually exclude downward acceleration that is faster than gravitational acceleration, or axiom CUP.2 would be false. Axiom DOWN.6 approximates this constraint on acceleration with a similar constraint on velocity, since we do not deal with acceleration in this paper.

5. Problem specification

In this section, we translate the commonsense inference, “If a partially full pitcher is lifted over a pail and tilted far enough, some of the liquid will pour from the pitcher to the pail”, into a formal specification and a conclusion that is a logical consequence of the specification and the theory presented above. In particular, we must specify the necessary geometric constraints on the pitcher, the pail, and the motion of the pitcher.

This is a more problematic undertaking than one might suppose. It seems plausible to assume that the specifications of individual inference problems will consist entirely or largely of a set of sentences of simple logical form over a fixed and finite language. Determining what concepts should be included in that language is a large part of the research enterprise. That is to say, we need to determine what concepts are important in specifying individual inference problems; this is a different and probably larger set than the concepts needed to state the general theory. The concepts themselves are defined in terms of the theoretical temporal, spatial, and physical primitives. These definitions may be of arbitrary logical complexity, but hopefully they will eventually be a fixed and finite collection. We have followed this structure in our specifications, giving the definitions in Tables 17 and 18, and the scenario description in Table 19; note that the scenario description is effectively a conjunction of ground atomic formulas.

⁷ One of the reviewers of this paper, who is more agile than the author, remarks that he/she has actually succeeded in doing this.

Table 16
Downward flow: Axioms

Axioms:	
DOWN.1	$\text{maxOutflow} > 0.$
DOWN.2	$\text{liquidChunk}(L) \wedge$ $\text{throughout}(TS, TE, \text{slowObjectsInContact}(\uparrow L) \wedge^{\#} \text{canFlowDown}(\uparrow L)) \Rightarrow$ $\text{flowsDown}(\uparrow L, TS, TE)$
DOWN.3	$\text{liquidChunk}(L) \wedge \text{throughout}(TS, TE, \text{slowObjectsInContact}(\uparrow L)) \wedge$ $\text{holds}(TS, \uparrow L \subset^{\#} \text{flowUndisruptedSpace}) \Rightarrow$ $\exists_{TX, LX} TS < TX \wedge$ $\text{holds}(TS, \text{connHorizOutslice}(L, LX) \wedge^{\#} \text{rccDS}^{\#}(\uparrow LX, \text{nonFlowingSpace})) \wedge$ $\text{flowsStraightDown}(LX, TS, TX) \wedge \text{holds}(TX, \text{rccC}^{\#}(\uparrow LX, \text{flowDisruptedSpace})).$
DOWN.4	$\forall_{TS, TE: \text{time}} TS < TE \wedge \text{bregion}(R) \wedge \text{distance}(D) \wedge D > 0 \Rightarrow$ $\exists_{T1} TS < T1 \leq TE \wedge$ $\forall_{L: \text{liquidChunk}, T}$ $TS \leq T \leq T1 \wedge \text{holds}(T, \uparrow L \subset^{\#} \text{flowUndisruptedSpace} \wedge^{\#} \uparrow L \subset^{\#} R) \Rightarrow$ $\exists_{D1} \text{hausdorff}(\text{value}(T, \uparrow L), \text{value}(TS, \uparrow L), D1) \wedge D1 < D.$
DOWN.5	$\forall_{TS: \text{time}, R: \text{bregion}}$ $[\forall_{TE, RC} TS \leq TE \wedge RC \subset R \Rightarrow$ $\text{noInflowDuring}(R, TS, TE) \wedge \neg \text{holds}(TE, \text{cuppedRegion}(RC))]$ \Rightarrow $\text{eventuallyForever}(\text{emptyLiquid}(R)).$
DOWN.6	$\text{liquidChunk}(L) \wedge \text{flowsDown}(\uparrow L, TS, TE) \wedge \text{continuousVolume}(QIN, TS, TE) \wedge$ $\text{throughout}(TS, TE, \text{slowObjectMotion}(OB) \wedge^{\#} \text{localMaxBox}^{\#}(\uparrow OB, QIN) \wedge^{\#}$ $\text{rccDS}^{\#}(\uparrow L, \text{nonFlowingSpace})) \Rightarrow$ $\text{value}(TS, \text{top}^{\#}(QIN)) - \text{value}(TE, \text{top}^{\#}(QIN)) \leq$ $\text{value}(TS, \text{top}^{\#}(L)) - \text{value}(TE, \text{top}^{\#}(L)).$

Table 17
Problem specification: Symbols

Symbols:

oPitcher, oPail \rightarrow object.
 blnsidePitcher, bSpout, bTopPitcher \rightarrow pseudo.
 l0 \rightarrow liquidChunk.
 t0, t1, t2 \rightarrow time.
 topSurface(PS :pointSet, R :region)
 partialBox($RB, RI1, RI2, ROPEN$: region, D :distance)
 spout1($RB, RI1, RI2, RSPOUT, ROPEN$:region, D :distance)
 spout($RSPOUT, RB$:region, D :distance).
 pitcherShape(ROB, RIN :region, $PTOP$:pointSet, $RSPOUT$:region).
 pitcher(OP :object, BI, BT, BS :pseudo) \rightarrow fluent[Bool].
 volumeBelow(R :region, Z :ht) \rightarrow volume
 psBottom(PS :pointSet, Z :ht).
 simpleSafeTilt($BIN, BTOP$: pseudo, V : volume)
 alwaysSimpleBox(RB : region)
 passOverBox($PASS$:pointSet, $RBOX, RIN$:region)
 upColumn(PS :pointSet, Z :ht, PSC :pointSet).
 pouringRegion \rightarrow fluent[region].
 simpleBox(R :region).
 convexHull(PS :pointSet) \rightarrow pointSet.
 isolated(R :region, U :objectSet, L :liquidChunk) \rightarrow fluent[Bool].
 maxBox(RB, RI :region).
 horizExpand(PS :pointSet, D :distance) \rightarrow pointSet.

Turning now to the details of the specifications: There are two objects oPitcher and oPail, and a liquid chunk l0. There are three pseudo-objects associated with oPitcher: blnsidePitcher, the inside of the pitcher at the starting time, bTopPitcher, the top surface of blnsidePitcher, and bSpout, the spout (PS.3, PS.4). Initially all the liquid l0 is inside blnsidePitcher (PS.5).

There are three distinguished time points: t0, t1, and t2 (PS.1). At the starting time t0, the pitcher is an open box (PS.3) and contains the liquid l0 (PS.4).

From t0 to t1, the pitcher is moved slowly (PS.7) to a position where its spout is over oPail, being kept reasonably upright throughout (PS.8).

From t2 to t3, the pitcher is tipped to a final position where the volume it can contain is less than the volume of l0. After t3, it remains motionless forever in its pouring position (PS.17). Since the pitcher remains motionless after t3, any geometric property that holds in t3 continues to hold after t3.

PS.6 specifies that the bottom of bTopPitcher is always above the bottom of blnsidePitcher. This guarantees that there is always some cupped region inside the pitcher, and thus that some of l0 remains inside the pitcher.

Table 18

Problem specification: Definitions

Definitions:	
PD.1	$\text{bubbleFreeAttachment}(RA, RB) \equiv$ $\text{rccEC}(RA, RB) \wedge$ $\neg \exists_R \text{ rccEC}(R, RA) \wedge \text{rccDS}(R, RB) \wedge$ $\text{boundary}(R) \subset \text{boundary}(RA) \cup \text{boundary}(RB) \wedge$ $\neg [\text{boundary}(R) \subset \text{boundary}(RB)].$
PD.2	$\text{topSurface}(PS, R) \equiv \forall_P P \in PS \Leftrightarrow \text{topPoint}(P, R).$
PD.3	$\text{partialBox}(RB, RI1, RI2, \text{ROpen}, D) \equiv$ $\text{localMaxBox}(RB, RI1) \wedge RI1 \subset RI2 \wedge \text{top}(RI2) = \text{top}(RI1) + D \wedge$ $\text{rccEC}(RB, \text{ROpen}) \wedge \text{openBox}(RB \cap \text{ROpen}, RI2, D)$
PD.4	$\text{spout1}(RB, RI1, RI2, \text{RSpout}, \text{ROpen}, D) \equiv$ $\text{partialBox}(RB, RI1, RI2, \text{ROpen}, D) \wedge$ $\text{bubbleFreeAttachment}(\text{RSpout}, RB) \wedge \text{ROpen} \subset \text{RSpout}.$
PD.5	$\text{spout}(\text{RSpout}, RB, D) \equiv$ $\exists_{RI1, RI2, \text{ROpen}} \text{spout1}(RB, RI1, RI2, \text{RSpout}, \text{ROpen}, D).$
PD.6	$\text{regionBelow}(R, Z, RB) \equiv$ $\forall_{R1: \text{region}} R1 \subset RB \Leftrightarrow R1 \subset R \wedge \text{top}(RB) \leq Z.$
PD.7	$\text{psBottom}(PS, Z) \equiv$ $[\forall_{P \in PS} \text{height}(PS) \geq Z] \wedge$ $[\forall_{Z1} Z1 > Z \Rightarrow \exists_{P \in PS} \text{height}(P) < Z1].$
PD.8	$\text{holds}(T, \text{simpleSafeTilt}(BIN, \text{BTOP}, V)) \equiv$ $\exists_{Z, RB} \text{holds}(T, \text{psBottom}^\#(\uparrow \text{BTOP}, Z) \wedge^\# \text{regionBelow}^\#(\uparrow BIN, Z, RB)) \wedge$ $V \leq \text{volumeOf}(RB)$
PD.9	$\text{simpleBox}(R) \equiv$ $\exists_{R1} \text{localMaxBox}(R, R1) \wedge$ $[\forall_{R2} \text{localMaxBox}(R, R2) \Rightarrow R1 = R2] \wedge$ $[\forall_P \text{flowStoppingPoint}(P, R) \Rightarrow P \in R1].$
PD.10	$\text{holds}(T, \text{isolated}(R, U, L)) \Leftrightarrow$ $\exists_{D: \text{distance}} D > 0 \wedge$ $[\forall_{O: \text{object}} \text{holds}(T, \text{dist}^\#(\uparrow O, R) <^\# D) \Rightarrow O \in U] \wedge$ $[\forall_{L1: \text{liquidChunk}} \text{holds}(T, \uparrow L1 \subset^\# \text{expand}^\#(R, D)) \Rightarrow \text{subchunk}(L1, L)].$
PD.11	$\text{maxBox}(RB, RI) \equiv$ $\text{openBox}(RB, RI) \wedge [\forall_{RC} \text{openBox}(RB, RC) \wedge RI \subset RC \Rightarrow RI = RC].$
PD.12	$\text{columnWithin}(RC, RI) \equiv$ $\exists_{RB, \text{ZTOP}} RB \subset RI \wedge \text{thicklyConnected}(RB) \wedge \text{upColumn}(RB, \text{ZTOP}, RC)$
PD.13	$P \in \text{horizExpand}(PS, D) \Leftrightarrow$ $\exists_{PA \in PS} \text{dist}(P, PS) \leq D \wedge \text{height}(PA) = \text{height}(P).$
PD.14	$\text{onlyChannelsInReg}(RB, RE, RX, \text{ZTOP}, D) \equiv$ $\forall_P \text{weakTopPoint}(P, RB) \wedge \text{height}(P) \leq \text{ZTOP} \wedge P \in RE \Rightarrow$ $[P \in RX \vee \text{horizExpand}(P, D) \subset RE \vee [\exists_{RI} \text{openBox}(RB, RI) \wedge P \in RI]]$

PS.7 asserts that oPitcher is a “simple box” throughout the scenario. This means that (a) throughout j0, oPitcher bounds at most one locally maximal boxed region; and (b) that the only flow stopping points associated with oPitcher are in the inside of the pitcher. Thus, no flow of liquid outside the pitcher can have its flow blocked by a piece of the pitcher.

PS.8 together with PD.8, PD.7, PD.6 asserts that from t0 to t1, the pitcher stays sufficiently upright that the volume of the inside of the pitcher below the bottom of bTopPitcher is always greater than the volume of the liquid l0. This guarantees that bInsidePitcher does not overflow between t0 and t1. (This is a sufficient condition, not a necessary one; the cupped region may actually extend higher than the lowest point of bTopPitcher.)

PS.9 states that bSpout is a “spout” for oPitcher from time t1 onward. This is a geometric constraint whose purpose is to guarantee that any overflow out of the pitcher to the outside must pass through bSpout. The geometric definition is a little complicated. We say (PD.3) that *RB* is a *partial box* with true inside *RI1*, overflow region *RI2*, and opening *ROpen* relative to the overflow distance *D* if the following hold (Fig. 16):

- *RB* is a locally maximal box with inside *RI1*.
- The union of *RB* with *ROpen* forms an open box with inside equal to *RI2*. *RI2* contains *RI1* and has a top *D* higher than the top of *RI1*.

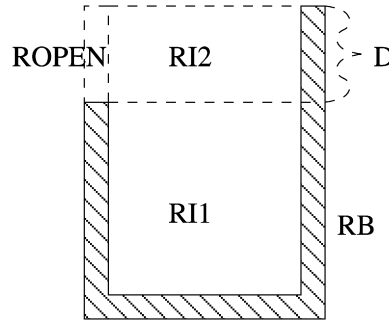
RSpout is a *spout* for *RB* if

- *RB* is a partial box with opening *ROpen* which is a subset of *RSpout*.
- *RSpout* \cup *RB* has no internal cavities other than the ones that are already in *RB* (*RSpout* is a *bubble free attachment* to *RB* (PD.1)).

Table 19

Problem specification: Axioms

Axioms:	
PS.1	$t0 < t1 < t2$
PS.2	$oPitcher = source(bInsidePitcher) = source(bTopPitcher) = source(bSpout).$
PS.3	$holds(t0, openBox^\#(\uparrow oPitcher, \uparrow bInsidePitcher) \wedge^\#$ $\uparrow bTopPitcher =^\# topSurface^\#(\uparrow bInsidePitcher))$
PS.4	$holds(t0, l0 \subset^\# bInsidePitcher)$
PS.5	$foreverAfter(t0, slowObjectMotion(oPitcher)).$
PS.6	$\forall_T t0 \leq T \Rightarrow$ $\exists_H holds(T, psBottom^\#(\uparrow bTopPitcher, H) \wedge^\#$ $H >^\# bottom^\#(\uparrow bInsidePitcher))$
PS.7	$foreverAfter(t0, simpleBox(oPitcher))$
PS.8	$throughout(t0, t1, simpleSafeTilt(bInsidePitcher, bTopPitcher, liqVolume(l0))).$
PS.9	$foreverAfter(t1, spout^\#(\uparrow bSpout, \uparrow oPitcher, maxOutflow)).$
PS.10	$\exists_{RI} holds(t2, maxBox^\#(\uparrow oPitcher, RI)) \wedge volumeOf(RI) < liqVolume(l0).$
PS.11	$foreverAfter(t2, motionless(oPitcher)).$
PS.12	$source(bInsidePail) = oPail.$
PS.13	$\exists_R holds(t0, regionBelow^\#(\uparrow bInsidePail, top^\#(\uparrow bInsidePail) -^\# maxOutflow, R)) \wedge$ $volumeOf(R) > liqVolume(l0).$
PS.14	$holds(t0, maxBox^\#(\uparrow oPail, \uparrow bInsidePail) \wedge^\# simpleBox(oPail))$
PS.15	$foreverAfter(t0, motionless(oPail)).$
PS.16	$foreverAfter(t1, top^\#(\uparrow bInsidePail) <^\# bottom^\#(\uparrow bSpout)).$
PS.17	$holds(t0, columnWithin^\#(pouringRegion, \uparrow bInsidePail, topPouringRegion) \wedge^\#$ $dist^\#(pouringRegion, \uparrow oPail) >^\# maxOutflow)$
PS.18	$foreverAfter(t1, rccNTPP^\#(expand^\#(\uparrow bSpout, 2 \cdot maxOutflow), pouringRegion))$
PS.19	$foreverAfter(t2, onlyChannelsInReg^\#(\uparrow oPitcher, pouringRegion, \uparrow bSpout, top^\#(\uparrow bSpout), maxOutflow))$
PS.20	$foreverAfter(t0, isolated(expand^\#(convexHull^\#(\uparrow oPitcher), 2 \cdot maxOutflow), \{ oPitcher \}, l0))$
PS.21	$foreverAfter(t0, isolated(horizExpand^\#(pouringRegion \cup^\# \uparrow bInsidePail), maxOutflow)$ $\{ oPail, oPitcher \}, l0)).$
Infer:	
$\exists_{L1, L2: liquidChunk}$	$eventuallyForever(\uparrow l0 =^\# \uparrow L1 \cup^\# \uparrow L2 \wedge^\#$ $liqInContainer(L1, oPitcher) \wedge^\# liqInContainer(L2, oPail)).$

**Fig. 16.** Partial box.

PS.10 asserts that at $t2$, the maximal volume of any region cupped by $oPitcher$ is less than the volume of $l0$, so $oPitcher$ must have overflowed before $t2$.

PS.15 states that $oPail$ is motionless throughout the scenario. Therefore any geometric property of the pail that holds at $t0$ holds throughout the scenario.

PS.14 states that $bInsidePail$ is the maximal open box formed by $oPail$. PS.13 states that the part of $bInsidePail$ that is at least $maxOutflow$ below its top has a volume greater than $l0$. Thus, even if $bInsidePail$ were filled with $l0$, the top of $bInsidePail$ is still in $freeFlowingSpace$.

PS.16 states that from $t1$ onward the bottom of the spout is higher than the top of the pail.

PS.17 defines the fixed region “pouringRegion” as a vertical column rising from the inside of $bInsidePail$ and at least $maxOutflow$ from the material of the pail (Fig. 17). The proof will work by showing that, once the liquid has been poured from the spout, it remains inside pouringRegion

PS.18 asserts that, from time $t1$ onward, the spout $bSpout$ is always inside pouringRegion, more than $2 \cdot maxOutflow$ from the boundaries of pouringRegion.

PS.19 enforces the constraint that there are no channels on the outside of $oPitcher$ that would allow liquid to flow from the inside of pouringRegion to within $maxOutflow$ of the boundary of pouringRegion. Otherwise, one could construct a

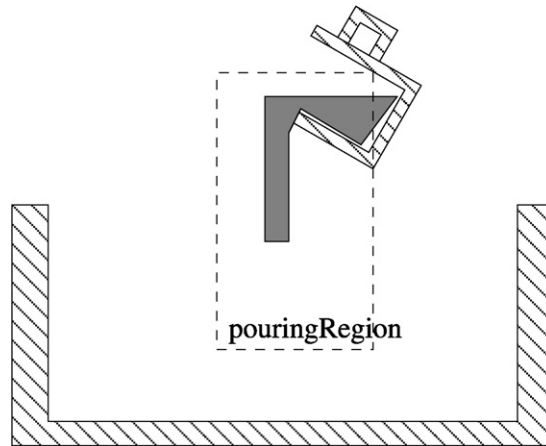


Fig. 17. Pouring region.

scenario in which the liquid pours out of the spout, but is then caught by a channel on the exterior of the pitcher and guided to spill out at a point that is not over the pail.

PS.20 and PS.21 enforce the “isolation” conditions that no objects other than the pitcher and the pail and no liquids other than l_0 interacts with the pouring of the liquid. PS.20 states that no object other than $oPitcher$ and no liquid other than l_0 comes within $2 \cdot \text{maxOutflow}$ of the convex hull of $oPitcher$. PS.21 states that no objects other than $oPitcher$ and $oPail$ and no liquids other than l_0 come within maxOutflow , horizontally, of the pouring region or of the inside of the pail. In particular, the pail is empty at t_0 and no liquid other than l_0 pours into the pail.

5.1. Correctness proof

The proof of the desired inference from the axioms and the problem specification is very much as sketched at the beginning of Section 3. The detailed proof is rather long but not deep; it is given in the online appendix at <http://www.cs.nyu.edu/faculty/davise/papers/liqAppa.pdf>.

The main steps are as follows (we enumerate only the most important axioms as justifications here): By CUP.1, PS.10 the liquid remains in the pitcher as the pitcher is lifted into position. By PS.12 at t_2 the cupped volume in the pitcher is less than the volume of l_0 ; hence using SPILL.1 and SPILLD.5 the pitcher overflows at some point between t_1 and t_2 . It follows from DOWN.2, DOWN.6 using a proof by contradiction that the liquid cannot rise more than maxOutflow above the region cupped by the pitcher. Hence, by PS.11, any liquid that flows out of the pitcher or out of the overflow region must go through the spout. By PS.12, PS.13, after t_2 the cupped region in the pitcher is too small to hold the liquid; hence it must flow out of the pitcher. By PS.8 there is always some cupped region inside the pitcher; hence by CUP.1 some of the liquid remains in the pitcher. By DOWN.2 the liquid in the overflow region must flow downward through the spout. When it flows out of the spout, by PS.20 it is inside *pouringRegion*. Using DOWN.2, DOWN.3, DOWN.4, PS.9, PS.21, PS.15 once the liquid is past the spout it remains inside *pouringRegion* until it is inside *blInsidePail* and at least maxOutflow below the top of the pail. (This is by far the hardest part of the proof; the proof is given in Lemma 64 of the appendix). Once the liquid is inside the pail, then by CUP.1, PS.10 it remains inside the pail. By DOWN.5 the part of *pouringRegion* above the pail and the overflow region above the pitcher are eventually empty. Hence eventually some of the liquid is in the pail, some of it is in the pitcher, and none of it is anywhere else.

5.2. Consistency proof

Of course, the above inference is only significant if the problem specifications are consistent with the domain axioms; if they are not, then one can prove anything at all. One can demonstrate that the specifications are consistent with the axioms by constructing a specific model of the shapes and motions of the objects and the flow of the liquid that satisfies the specifications and axioms.

We sketch one such model below (Fig. 18). Assume a coordinate system where the z -axis is the vertical and where the unit length is one inch.

Atemporal specifications:

Assume that $\text{maxOutflow} = 2$.

The inside of the pail is a $18 \times 18 \times 19$ rectangular solid. The pail is a cubical box around the inside, open on top and with one inch walls on the sides and the bottom.

The pitcher is a $10 \times 10 \times 9$ rectangular box, with the $8 \times 8 \times 8$ block cut out from the center top, and with a notch 2 inches deep cut out of the lower x -face. In other words: the bottom of the pitcher is $10 \times 10 \times 1$. The lower x -side is $1 \times$

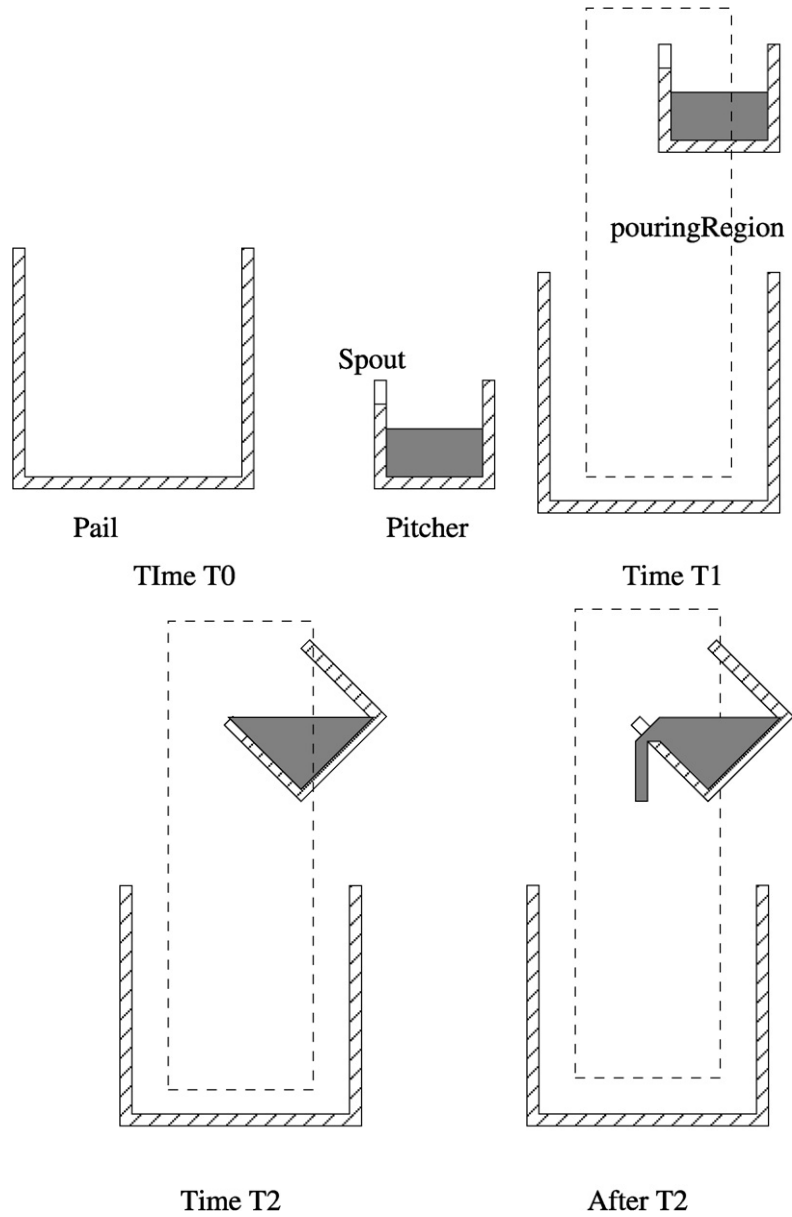


Fig. 18. Model scenario.

10×7 (this overlaps with the bottom). The upper x-side is $1 \times 10 \times 10$, and the two y-faces are $10 \times 1 \times 10$. The spout is the $1 \times 8 \times 2$ notch cut out from the top of the lower x-face.

The volume of the liquid is 256 cubic inches.

$$\text{shape}(\text{bInsidePail}) = [1, 19] \times [1, 19] \times [1, 20]$$

$$\text{shape}(\text{oPail}) = ([0, 20] \times [0, 20] \times [0, 20]) - \text{bInsidePail}.$$

$$\text{shape}(\text{bInsidePitcher}) = [31, 39] \times [6, 14] \times [1, 8].$$

$$\text{shape}(\text{oPitcher}) = ([30, 40] \times [5, 15] \times [0, 9]) - (\text{bInsidePitcher} \cup ([30, 39] \times [6, 14] \times [8, 9])).$$

$$\text{shape}(\text{bSpout}) = [30, 31] \times [6, 14] \times [8, 9]$$

$$\text{liqVolume}(l) = 256.$$

$$\text{pouringRegion} = [4, 15] \times [5, 15] \times [4, 43]$$

Time points: $t_0=0$. $t_1=1$. $t_2=2$.

Initial State: At time t_0 , the pitcher and pail are at the positions specified by their shape. The liquid occupies the bottom half of the pitcher.

$$\text{value}(t_0, \text{placement}(\text{oPitcher})) = \text{value}(s_0, \text{placement}(\text{oPail})) = \text{identity}.$$

$$\text{value}(t_0, \text{place}(\text{l}_0)) = [31, 39] \times [6, 14] \times [1, 5].$$

Motion of the rigid objects: The pail is motionless throughout. From $t = 0$ to $t = 1/2$, the pitcher translates vertically 30 inches. From $t = 1/2$ to $t = 1$, the pitcher translates horizontally to the left 20 inches. From t_1 to t_2 the pitcher rotates at a constant rate around the line $x = 15, z = 35$; this line is parallel to the y -axis, sits at the top of the liquid, in the center of the x -extent of the top surface of the liquid. After t_2 the pitcher is motionless.

$$\text{motionless}(\text{oPail}, j_0)$$

$$0 \leq T \leq 1/2 \Rightarrow \text{value}(T, \text{placement}(\text{oPitcher})) = \text{translate}(0, 0, 60T)$$

$$1/2 \leq T \leq 1 \Rightarrow \text{value}(T, \text{placement}(\text{oPitcher})) = \text{translate}(-40(T - 1/2), 0, 30)$$

$$1 \leq T \leq 2 \Rightarrow \text{value}(T, \text{placement}(\text{oPitcher})) = \text{rotateAroundAxis}(\{(15, Y, 35) | \text{real}(Y)\}, \pi(T - 1)/4)$$

(In the above formulas, the function “ $\text{translate}(X, Y, Z)$ ” maps coordinates to a translation and the function “ $\text{rotateAroundAxis}(PSL, A)$ ” maps a line PSL and an angle A to a rotation in the obvious way.)

$$\text{foreverAfter}(t_2, \text{motionless}(\text{oPitcher})).$$

Liquid flow: There is no flow in the y -dimension. Hence, we will describe the flow purely in terms of its x - z component,

From t_0 to t_1 the liquid simply translates along with the pitcher.

The liquid flow from t_1 to t_2 is described as follows: Let Q_2 be the region-valued fluent whose value is always the boxed region bounded by $\text{oPail} \cup \text{bSpout}$. From t_1 to t_2 , the liquid fills the section of that region which is below the plane $z = 35$. That is, we are supposing that the liquid flows in exactly the same way it would if the notch were filled in. Note that the volume of this section is constant; viewed in a coordinate system attached to the pitcher, this flow involves vacating a triangular wedge on the right and flowing into a congruent wedge on the left. Note that at t_1 , the liquid is simply observing rule 4 and in the region cupped by the pitcher. At some time t_m before t_2 , it reaches the edge of the pitcher and then starts to overflow. Between t_m and t_2 the liquid is filling the maximal cupped region in the pitcher, and the volume of that region is monotonically decreasing; hence by SPILLD.5 the liquid is overflowing out of the pitcher. Since it continues to overflow until t_2 , and since the specified region of overflow is all within the driven area, this pattern of flow is allowed by our rules. Within the region occupied by the liquid, the flow consists of a volume-preserving “rotation” around the axis of rotation. That is, as the pitcher rotates, every wedge of liquid centered at the axis replaces the wedge of equal volume to its left.

At t_2 , then, the liquid fills the cupped region bounded by oPitcher , plus a trapezoid over that (Fig. 19.A). After t_2 the pitcher is motionless; hence, the volume of the maximal cupped region is constant; hence, by SPILLD.5 the liquid is no longer considered to be overflowing, and is no longer driven. The liquid in the cupped region remains where it is; the liquid in the overflow trapezoid must all flow down. The liquid in the overflow trapezoid flows downward and to the left into the triangle to the left of the overflow trapezoid, following the flow lines shown in Fig. 19.B. To see intuitively that this can be done while maintaining constant volume, imagine cutting each flow tube into segments of equal volume, in a way that is consistent across the flow tubes. Then in each small time interval ΔT , each segment flows into the place vacated by the next segment in the tube.

Once the liquid has flowed out the bottom of the left hand triangle, it flows vertically in a column at a constant rate down toward the pail. Until the column hits the pail, it maintains its columnar shape. Once it has hit the bottom, there will be a rising pool occupying a cupped region at the bottom of the pail. The part of the falling liquid more than maxOutflow above the top of this pool is in $\text{flowUndisruptedSpace}$ and therefore continues to fall down in a column. The part of the liquid that is above the top of the pool but less than maxOutflow above it is in $\text{flowDisruptedSpace}$; it flows downward and outward, fast enough to make room for the column falling down after it.

It is an easy, but long and very uninteresting, exercise to specify this flow in closed formulas.

Verifying that the specified flow actually satisfies the axioms is not trivial, since some of the axioms involve quantifying over all regions, or all region-valued fluents, but neither is it very difficult.

Of course, it would be undesirable to have to construct an explicit model like this for each new problem specification, to make sure that the theory meaningfully applies to it. Much better would be to have a general meta-theorem stating that all problem specifications of a specified form are consistent with the axioms.

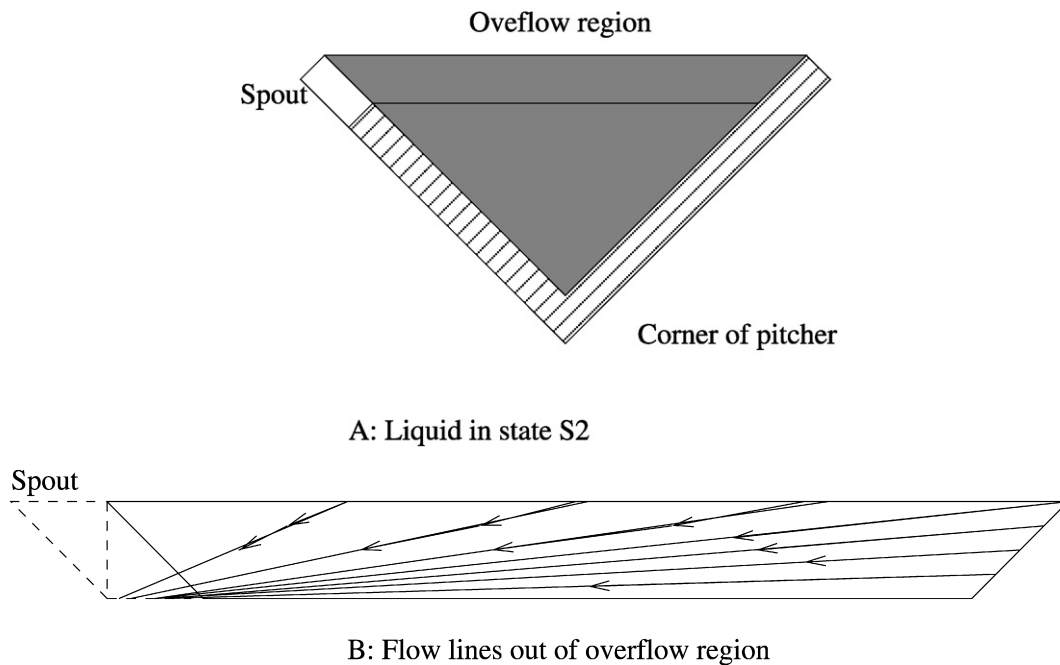


Fig. 19. Down flow from overflow region.

5.3. Other inferences

The above inference is the only one that I have worked through in full detail. However, I am quite confident that the following inferences are also justified in the theory. Moreover, their proof can use many of the same general lemmas already worked through for our example.

- It is possible to ladle liquid out of a bowl with a spoon, if the spoon is immersed in the liquid and then lifted out, maintaining its cupped shape. The spoon may be immersed, either by plunging it in, so that the liquid spills over its side, or by sliding it in.
- If pebbles are gradually dropped into a vase filled with liquid, the level of the liquid will gradually rise. Once the volume of the pebbles plus the volume of the liquid exceeds the capacity of the vase, the vase will overflow.
- If an object with no cups is immersed in a bowl of liquid and then lifted out, the removal of the object does not bring any liquid with it.
- If a large object is lowered gradually into a bowl of liquid, the liquid will reach the rim of the bowl just when the volume of the part of the object below the rim plus the volume of the liquid is equal to the capacity of the bowl. If part of the object is still above the liquid, then lowering the object any further will cause the bowl to overflow.
- (Archimedes). Consider a scenario in which (1) object A is placed inside a bowl; (2) the bowl is filled to the brim with liquid; (3) object A is removed; (4) object B is placed inside the bowl. Assume that object A has no cups in which to carry out liquid. The bowl will overflow during the execution of (4) if and only if the volume of B is greater than the volume of A.
- If liquid is poured over a solid object suspended in the air, and the object does not form any cups then the liquid will flow down over the object, spreading a short distance beyond the bounds of the object. If the object does form cups then all of the liquid that is not caught in those cups will pour down over the object in the same way.
- Suppose there is a body of liquid inside a simple cup with a plug at the bottom and the plug is removed. If there is nothing below the cup, then all the liquid will flow out the bottom. If the cup is itself inside a bowl then the liquid will flow out the bottom of the cup into the bowl until the level in the outer bowl is equal to the level in the cup.
- Suppose you have a W-shaped pail, and you pour from a pitcher into one of the internal cups (Fig. 20). The level of liquid will rise until it is full. Once it is full, the liquid will start to pour off into the other internal cup. Once both internal cups are full, the two pools of liquid will be joined on top, and the level will gradually rise across the entire pail.

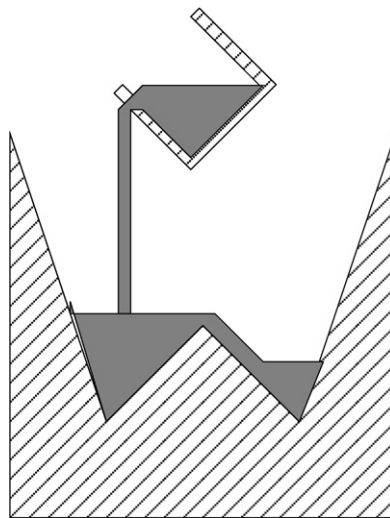
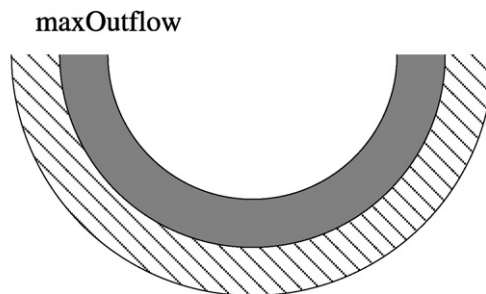


Fig. 20. Pouring into W-shaped pail.



Cross section of channel:
Steady state flow is only possible in the grey annulus.
The central core is all flowUndisruptedSpace.

Fig. 21. Anomalous flow in channel.

6. The problem with channels

The most immediate problem with our theory is that it completely mispredicts the way that water flows in a channel. Not only does it fail to predict the correct behavior, it actually excludes the correct behavior. Suppose that there is a channel whose cross-section has a radius greater than `maxOutflow` filled with flowing water. The liquid chunks that are more than `maxOutflow` from the boundaries of the channel, in the center-top of the channel, are in `flowUndisruptedSpace`, and therefore by Rule 5 flow straight down rather than parallel to the channel. The only steady-state flow pattern allowed by rule 5 is one in which the water flows down in the annulus which is `maxOutflow` from the boundary of the channel (Fig. 21). This is all `flowDisruptedSpace`, and therefore the liquid there is allowed to flow parallel to the channel.

Thus, fixing this problem has two parts. The first part is to fix Rule 5 so that it does not prohibit the true flow down the channel; the second part is to add rules that require the correct flow. It is certainly possible that the first part can be accomplished through some fairly minor change to Rule 5 that will eliminate this problem, while still preserving the ability to predict that liquid poured from the pitcher lands in the pail. I doubt it, however; I suspect that even the first part of the solution will involve explicitly characterizing channels and marking them, like overflow, as an exception to Rule 5.

In any case, for the second part of the solution we need rules that will allow us to predict the flow of liquids in a channel. I intend to address this in future work. In the meantime, the problem provides an interesting context for discussing the issues involved in extending a theory like this. Absent any great simplifying insight, adding channel flow to this theory will probably involve the following:

- Defining a channel geometrically. This is considerably more challenging than defining a cup. A cup has a single distinguished direction—upward—which is fixed. A channel has two – upward and forward—and the latter is defined relative to the channel and may change over the length of the channel. Also channels interact with each other and with other

topographical features in more complicated ways than cups. Channels may split and merge; they may contain pits, bumps, and ridges.

- Defining the dynamics of “normal flow” in a channel. Again this is more complicated than the dynamics of cupped liquid; at its simplest, it is a steady-state flow rather than a static condition. For example, an open channel overflows its banks when the flow rate into the channel exceeds the maximum flow rate that the channel can sustain; expressing this requires a substantially richer theory than the simple rule for an overflowing cup. As with cups, we would wish to at least partially categorize flow through a channel even in the case where the solid object or objects forming the channel are in motion.
- Define the liquid in a channel as an exception to Rule 5.
- Make sure that the rules accommodate the case of non-steady-state flow in a channel; for example, a chunk of liquid falls onto the side of a channel. The prediction in a case like this need not be precise, but it should not be absurd, and it should smoothly interface with the normal flow through a channel. For instance, if a chunk of liquid falls onto the side of a channel which has a channeled flow in it, the chunk should somehow merge with the channeled flow.
- Make sure that the examples that have been worked out still work. For instance the spout of a pitcher may well be a channel, or it may not; make sure that the pitcher example works in either case.
- As far as possible, check that the theory is still consistent.

Another problem is leaky cups. A cup with a small hole in fact works like a cup except for a slow leak, and can be used to carry liquid. But in our theory, a cup with a small hole is not a cup at all; the liquid inside can flow down (Definition 3.4.4) and therefore must flow down (Rule 6). In particular, if the cup is lifted, our theory states that the liquid must flow out the hole fast enough that the absolute height of the top of the liquid always decreases. This problem, however, unlike the problem of channels is easy to fix; it is merely necessary to add a new rule similar to rule 4 stating that leaky cups leak slowly, and to add the liquid cupped in a leaky cup as a part of nonFlowingSpace in Definition 3.4.2.

7. Conclusions and future work

We have constructed and formalized a theory that supports qualitative reasoning about many scenarios involving liquid flowing around slowly moving objects. There are a few scenarios that it handles badly, but it seems likely that these can be corrected and that this theory can be an important part of a larger theory of commonsense physical reasoning.

The most important next steps in this research are:

- To develop a theory of liquid flow in channels.
- To develop a theory of liquid flow around rapidly moving objects.
- To integrate a theory of pressure, including atmospheric pressure.
- To integrate a theory of agents, with feedback from perception to manipulation, so that one can describe the plan of carefully pouring liquid from a pitcher to a pail and reason about its feasibility.
- To define a reasonable subclass of inferences, particularly prediction, that can be effectively implemented. The complexities of the proof in the appendix should not be discouraging, here; this is a complete proof based on very general axioms. If the language of the problem specification and the closed-world assumptions can be standardized, and powerful general lemmas can be subsumed into procedures, then it may well be possible to develop an inference engine whose complexity is not much greater than the English description of the reasoning involved on page 1546.

In the longer term, the critical problems are to identify significant classes of inference in this domain, to develop effective algorithms for these, and to integrate these into knowledge-based systems for applications such as design, text interpretation, and automated tutoring.

References

- [1] B. Bennett, A.G. Cohn, P. Torrini, S.M. Hazarika, Describing rigid body motions in a qualitative theory of spatial regions, in: AAAI-00, pp. 503–509.
- [2] A.G. Cohn, J. Renz, Spatial reasoning, in: F. van Harmelen, V. Lifschitz, B. Porter (Eds.), *The Handbook of Knowledge Representation*, Elsevier, Oxford, 2007.
- [3] J. Collins, K. Forbus, Reasoning about fluids via molecular collections, in: AAAI-87, pp. 590–595.
- [4] E. Davis, A logical framework for commonsense predictions of solid object behavior, *International Journal of AI in Engineering* 3 (3) (1988) 125–140.
- [5] E. Davis, *Representations of Commonsense Reasoning*, Morgan Kaufmann, 1990.
- [6] E. Davis, Axiomatizing qualitative process theory, in: KR-92, pp. 177–188.
- [7] E. Davis, The kinematics of cutting solid objects, *Annals of Mathematics and Artificial Intelligence* 9 (1993) 253–305.
- [8] E. Davis, The naive physics perplex, *AI Magazine* 19 (4) (1998) 51–79.
- [9] E. Davis, Continuous shape transformations and metrics on regions, *Fundamenta Informaticae* 46 (1–2) (2001) 31–54.
- [10] E. Davis, The expressivity of quantifying over regions, *Journal of Logic and Computation* 16 (2006) 891–916.
- [11] E. Davis, How does a box work? A study in the qualitative dynamics of solid objects, *Artificial Intelligence*, submitted for publication.
- [12] E. Davis, *The Logic of Commonsense Physical Reasoning*, in preparation.
- [13] J. DeCuyper, D. Keymeulen, L. Steels, A hybrid architecture for modelling liquid behavior, in: J. Glasgow, B. Chandrasekaran, N.H. Narayanan (Eds.), *Diagrammatic Reasoning*, MIT Press, 1995, pp. 731–751.
- [14] J. de Kleer, Multiple representations of knowledge in a mechanics problem solver, in: *IJCAI-77*, pp. 299–304.

- [15] J. de Kleer, J.S. Brown, A qualitative physics based on confluences, in: D. Bobrow (Ed.), *Qualitative Reasoning about Physical Systems*, MIT Press, Cambridge MA, 1985, pp. 7–83.
- [16] K. Forbus, Qualitative process theory, in: D. Bobrow (Ed.), *Qualitative Reasoning about Physical Systems*, MIT Press, Cambridge MA, 1985, pp. 85–186.
- [17] F. Gardin, B. Meltzer, Analogical representations of naive physics, in: J. Glasgow, B. Chandrasekaran, N.H. Narayanan (Eds.), *Diagrammatic Reasoning*, MIT Press, 1995, pp. 669–689.
- [18] P. Hayes, The naive physics manifesto, in: D. Michie (Ed.), *Expert Systems in the Microelectronic Age*, Edinburgh University Press, Edinburgh, 1979.
- [19] P. Hayes, Ontology for liquids, in: J. Hobbs, R. Moore (Eds.), *Formal Theories of the Commonsense World*, Ablex Pub., Norwood NJ, 1985.
- [20] P. Hayes, A critique of pure reason, *Computational Intelligence* 3 (1987) 179–185.
- [21] H. Kim, Qualitative reasoning about fluids and mechanics, Ph.D. thesis, Institute of Learning Sciences, Northwestern University, 1993.
- [22] L. Joskowicz, E. Sacks, Computational kinematics, *Artificial Intelligence* 51 (1991) 381–416.
- [23] V. Lifschitz, Cracking an egg: An exercise in formalizing commonsense reasoning, in: *Fourth Symposium on Logical Formalizations of Commonsense Reasoning*, 1998.
- [24] J. McCarthy, Programs with common sense, in: M. Minsky (Ed.), *Semantic Information Processing*, MIT Press, Cambridge, MA, 1968, pp. 403–418.
- [25] J. McCarthy, P. Hayes, Some philosophical problems from the standpoint of artificial intelligence, in: B. Meltzer, D. Michie (Eds.), *Machine Intelligence* 4, Edinburgh University Press, 1969.
- [26] D. McDermott, A critique of pure reason, *Computational Intelligence* 3 (1987) 151–160.
- [27] L. Morgenstern, Mid-sized axiomatizations of commonsense problems: A case study in egg cracking, *Studia Logica* 67 (2001) 333–384.
- [28] A. Newell, The knowledge level, *AI Magazine* 2 (2) (1981).
- [29] D.A. Randell, Z. Cui, A.G. Cohn, A spatial logic based on regions and connection, in: *Third International Conference on Principles of Knowledge Representation and Reasoning*, 1992, pp. 165–176.
- [30] R. Reiter, *Knowledge in Action: Logical Foundations for Specifying and Implementing Dynamical Systems*, MIT Press, Cambridge, MA, 2001.
- [31] E. Sandewall, Combining logic and differential equations for describing real-world systems, in: *KR-89*, pp. 412–420.
- [32] J. Schmolze, Physics for robots, in: *AAAI-86*, pp. 44–50.
- [33] M. Shanahan, An attempt to formalise a non-trivial problem in common sense reasoning, *Artificial Intelligence* 153 (2004) 141–165.
- [34] J. Shin, E. Davis, Processes and continuous change in a SAT-based planner, *Artificial Intelligence* 166 (1–2) (2005) 194–253.