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A logic-based axiomatic model of bargaining

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ABSTRACT

This paper introduces an axiomatic model for bargaining analysis. We describe a bargaining situation in propositional logic and represent bargainers' preferences in total pre-orders. Based on the concept of minimal simultaneous concessions, we propose a solution to n-person bargaining problems and prove that the solution is uniquely characterized by five logical axioms: *Consistency*, *Comprehensiveness*, *Collective rationality*, *Disagreement*, and *Contraction independence*. This framework provides a naive solution to multi-person, multi-issue bargaining problems in discrete domains. Although the solution is purely qualitative, it can also be applied to continuous bargaining problems through a procedure of discretization, in which case the solution coincides with the Kalai–Smorodinsky solution.

1. Introduction

As one of the most fundamental models in modern economic theory, the Nash bargaining solution [23] has been developed into a highly sophisticated theory with extensive applications in economics, social science, political science and management science [1,13,23,24,26,40,42]. Computer scientists, especially researchers in the area of artificial intelligence (AI), have found it useful in modeling interactions among distributed computer systems and autonomous software agents since the early 90s [12,17,30,38]. Many applications have been developed for the design and evaluation of high-level interaction protocols among autonomous agents for task assignment, resource allocation, conflict resolution, electronic trading and web services [16,27,30,41,48].

Traditionally, a bargaining situation is modeled as a numerical game, using the language of utility. In his seminal paper, Nash [23] defined a bargaining situation as a pair (S, d), where $S \subseteq \Re^2$ represents the set of utility pairs that can be derived from possible agreements and $d \in S$ is the utility pair that follows disagreement.[1] A *solution* is a rule that associates to each bargaining situation (S, d) a feasible utility pair of S. Nash proposed a set of axioms that he thought a solution should satisfy and established the existence of a unique solution satisfying all the axioms [23]. Numerous extensions and alternative solutions have been proposed in the past sixty years after this first axiomatic model of bargaining [42]. The subsequent work has diverged in two different directions: the *cooperative models* and the *non-cooperative models*. The former, following Nash's approach and thus also called *axiomatic models*, provide an axiomatic characterization of bargaining solutions [23,42]. A bargaining problem is modelled as a one-shot game and solutions are characterized by a set of axioms, such as *Pareto optimality*, *Symmetry*, and so on. The non-cooperative models, also called *strategic models*, establish explicit constructions of negotiation procedures and identify the bargaining outcome as an equilibrium [1]. The attempt to establish the relationship between the two models is known as the *Nash program* [1,24].

The Nash bargaining model provides simple and mathematically elegant solutions to bargaining problems and facilitates quantitative analysis of bargaining situations. However, in many real-world bargaining situations, the utility of a bargainer

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[1] $\Re^2$ represents 2-dimensional real Euclidean space.

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cannot be measured using a numeric scale, therefore the solutions that are built on the Nash model become inapplicable.[2] Examples of such bargaining situations can be easily found in political/legal negotiations, household bargaining, labor disputes and so on. For instance, it is difficult to imagine an analysis of the Six-Party Talks on North Korea's Nuclear Program that is based on a numerical measure of each party's utility gains or losses from the negotiations.[3]

An alternative method of bargaining analysis, initially suggested by Shapley and Shubik, is modeling a bargaining situation in terms of bargainers' preference orderings over possible agreements [40, p. 91]. Formally, a bargaining situation can be represented as a tuple $(A, D, \geqslant_1, \geqslant_2)$, where A is a set of possible agreements (described in physical terms), D is the disagreement, and $\geqslant_1$ and $\geqslant_2$ are preference orderings over $A \cup \{D\}$. The interpretation is that $a \geqslant_i b$ if and only if player i either prefers a to b or is indifferent [26, p. 9]. This allows us to assess a bargainer's utility through pairwise comparisons among the possible agreements instead of quantitative measurement. Such a model of bargaining problems is called an *ordinal bargaining model*. A bargaining solution is *ordinal* if it can be built on an ordinal bargaining model. We would like to remark that a bargaining solution built on the Nash bargaining model can also be ordinal as long as it is invariant under any order-preserving transformations of utilities (ordinal invariance) because such a solution can be expressed in an ordinal model [31]. Therefore the judgement of whether a solution is ordinal or cardinal is not by the use of numbers but its structure. In fact, most of the existing work on ordinal bargaining solutions in the literature was built on numerical models [3,25,35,37].

The ordinal bargaining models are of interest because ordinal information is relatively easier to obtain than cardinal utilities [36]. Asked if they prefer coffee or tea, anyone can provide a preference. However, if asked to value their preference in cardinal scale, they would find it difficult [8]. Nevertheless, Shapley observed that there is no non-trivial ordinal bargaining solution to two-player bargaining problems [39].[4] The reason is, as pointed out by many researchers, that the information about bargainers' attitudes towards risk, which in fact determines the negotiation power of a bargainer, is not describable by ordinal preferences [26,31,40].[5] With the Nash bargaining model, bargainers' risk attitudes, combined with the preferences on the possible agreements, are represented by utility scales, i.e., cardinal utility, through the non-linearity or curvature of utility functions (for an intuitive example see [44]). However, such information is lost when the model is converted to an ordinal model through an order-preserving transformation. Therefore the information about bargainers' risk attitudes is lost. This suggests that additional components have to be introduced to the ordinal bargaining models to express bargainers' risk attitudes.

Rubinstein et al. introduced a variation of the ordinal bargaining model in which the preference ordering of each player is extended to the space of lotteries over the possible agreements and the disagreement [33]. A player can express her attitudes towards risk through her preference on the lotteries. However, an ordinal preference on the lotteries is by no means easier to elicit than the utility scales on the possible agreements because the space of lotteries is also a continuum. More recently, O'Neill et al. introduced an ordinal bargaining solution based on the idea of gradual bargaining [25]. Instead of modeling a bargaining problem as a one-shot game, they look at bargaining as a family of bargaining games, parameterized by time. The bargaining outcome can then be viewed as the limit of a step-by-step bargaining in which the agreement of the last negotiation becomes the disagreement point for the next. Players' risk attitudes can then be observed through the variations of their utilities over time. However, there is no explicit representation of players' attitudes towards risk. Zhang and Zhang proposed a purely qualitative model of bargaining based on bargainers' ordinal preferences [44]. Similar to but different from Rubinstein et al.'s framework, the preference ordering of each player is defined on the player's demand items (instead of the lotteries of possible agreements). More precisely, the physical demands of each player are expressed by logical statements. A possible agreement is a logically consistent set of demands from each player. The risk attitudes of a player are represented based on a relative ranking of the player's demands: *a risk-lover tends to insist on conflicting demands more firmly than a risk-averse player and therefore may rank these conflicting demands higher, and vice versa.* Bargaining then is viewed as a procedure of conflict resolution over two sets of ranked demands (for two-player bargaining). A solution concept was proposed based on minimal changes but no axiomatic characterization was provided [44].

This paper will develop an axiomatic model of bargaining based on the ordinal preference structure proposed in [44]. We describe a bargaining situation in propositional logic and represent bargainers' preferences in total pre-orders. Following the tradition of cooperative bargaining theory, we assume that any negotiation is conducted through an impartial arbitrator who has complete information about the negotiation [22, Chapter 8].[6] The agreement of a negotiation is the outcome of a sequence of concessions simultaneously made by all players. We assume that whenever a player has to make a concession, she always tries to make the concession as small as possible provided it is enough to break even. Based on the assumptions, we propose a bargaining solution and establish that it is uniquely characterized by five plausible logical axioms. Our approach

[2] More precisely, most classical bargaining solutions built on the Nash bargaining model assume that individuals' utility scales can be defined up to separate increasing linear transformations. Such a utility measurement is often referred to as *cardinal utility* and a bargaining solution built on cardinal utilities is called a *cardinal solution* [40, p. 98].

[3] As Rubinstein comments "*the language of utility allows the use of geometrical presentations and facilitates analysis*; *in contrast, the numerical presentation results in an unnatural statement of the axioms and the solutions*" [34, p. 83].

[4] More precisely, the result says that a two-player bargaining problem has no single-valued solution satisfying ordinal invariance and strong individual rationality (see also [31, p. 70]).

[5] Although it was found, also by Shapley himself, that Shapley's impossibility result does not apply to three-player bargaining problems, the problem of representing risk attitudes in ordinal structures still exists.

[6] A cooperative model of bargaining can also be used as a tool for off-line analysis of bargaining situations and for predicting bargaining outcomes.

differs from most of the existing research on ordinal bargaining in that it is purely qualitative. However, we will demonstrate that the solution is applicable also to the continuous bargaining problems through a procedure of discretization, in which case the solution converges to the Kalai–Smorodinsky solution.

The rest of the paper is organized as follows. In Section 2, we will introduce a logic-based bargaining model to specify arbitrary n-person bargaining games. Section 3 will introduce the axioms and construct the solution based on the extended model. Section 4 contains our main result, the axiomatic characterization. In Section 5, we will use a typical game-theoretic bargaining problem to show that the proposed solution is also applicable to the continuous bargaining problem even though it is best suited to multi-issue, discrete bargaining problems. In Section 6, we conclude the work with a discussion of the related work. Finally, to make the paper self-contained, we list the basic facts of the two most influential game-theoretic bargaining solutions, the *Nash solution* and the *Kalai–Smorodinsky solution*.

2. The bargaining model

Bargaining is a process through which a set of agents interact to reach an agreement. To model a bargaining situation, we describe the demands of each bargainer in logical statements. The demands of the bargainers may conflict, and therefore collectively, they may be logically inconsistent. Successful bargaining will result in a mutually acceptable agreement that compromises the demands of the bargainers into logically consistent statements.

To facilitate a logical model of bargaining, we assume a propositional language $\mathcal{L}$. The language consists of a finite set of propositional variables and the standard propositional connectives $\{\neg, \vee, \wedge, \rightarrow\}$. We will apply the standard syntax and semantics of propositional logic. Propositional sentences will be denoted by $\varphi, \psi, \ldots$. As usual, the symbol $\vdash$ denotes derivability, and the concept of logical consistency is defined as usual.

Let $\succcurlyeq$ be a binary relation on a non-empty set X. $\succcurlyeq$ is a *total pre-order*, or *complete transitive reflexive order*, on X if it satisfies the following properties:

- *Completeness* or *totality*: For all $\varphi, \psi \in X$, $\varphi \succcurlyeq \psi$ or $\psi \succcurlyeq \varphi$.
- *Reflexivity*: For all $\varphi \in X$, $\varphi \succcurlyeq \varphi$.
- *Transitivity*: For all $\varphi, \psi, \chi \in X$, if $\varphi \succcurlyeq \psi$ and $\psi \succcurlyeq \chi$, then $\varphi \succcurlyeq \chi$.

2.1. Demands of a bargainer

Given a bargaining situation with n participants, assume that the final agreement is represented by a collection of logical statements in the language we specified above. Each participant in the bargaining requests that the final agreement contains a set of statements, which are referred to as the *demands* of the bargainer.[7] For example, a buyer in a price negotiation may have the demands "*the price of the product is no more than \$10*" and "*the warranty is no fewer than three years*". In addition, we assume that each bargainer has a preference on the statements of her demands, indicating how eagerly she wants the statements to be written in the final agreement. Formally, we can represent a bargainer's demands and preferences in the following structure:

Definition 1. A *demand set* in $\mathcal{L}$ is a pair $(X, \succcurlyeq)$, where X is a finite, logically consistent set of sentences in $\mathcal{L}$ and $\succcurlyeq$ is a total pre-order on X.

As mentioned above, a demand set represents the statements an agent wants the agreement of a negotiation to stipulate. The ordering of the demand set represents how firmly the agent insists on her demands: *the higher the firmer*. In some situations, such an ordering can be recorded by observing the sequence of demands the agent gives up during a course of bargaining: *the later it is dropped, the firmer*. Note that the preference ordering of a player does not represent the payoff the player receives from the associated demands. For instance, suppose that two players bargain over the partition of a cake (Section 5). A player receives the highest payoff if she gets the whole cake. However, getting the whole cake should be the least entrenched demand for a player unless she does not want to reach an agreement, because if both players insist on the demand, no agreement could be reached. In addition, the preference ordering of a player is purely private information. A player can get great advantage if she knows other players' preference orderings. However, how to take advantage of other players' information is a research topic of strategic models (or called non-cooperative models) of bargaining [1].

We would like to remark that, in general, the demand set of a player is non-empty, which means that no player would participate in a negotiation for nothing. However, we will use the empty demand set $(\emptyset, \succcurlyeq)$ to indicate the situation in which a player discontinues a negotiation.

Let $(X, \succcurlyeq)$ be a demand set. For any $\varphi, \psi \in X$, we write $\varphi \approx \psi$ to denote $\varphi \succcurlyeq \psi$ and $\psi \succcurlyeq \varphi$. Obviously $\approx$ is an equivalence relation on X. Furthermore, $\varphi \succ \psi$ denotes $\varphi \succcurlyeq \psi$ but $\psi \not\succcurlyeq \varphi$. Occasionally we also use $\preccurlyeq$ and $\prec$ as the reverse order of $\succcurlyeq$ and $\succ$, respectively.

[7] We call these statements "demands" but they may be the player's beliefs, goals, desired constraints or commonsense; whatever the player wants the final agreement to contain.

In the game-theoretical bargaining model, the concept of comprehensiveness plays an important role [32]. Formally, a set $S \subseteq \Re^n_+$ is called to be *comprehensive* if for all $x, y \in \Re^n_+$, $x \in S$ and $y \leqslant x$ implies $y \in S$ [42]. In belief revision, we also have a similar concept, called *cut* [10].

Definition 2. Given a demand set $(X, \succcurlyeq)$, a subset Y of X is *comprehensive* w.r.t. X if for any $\varphi \in Y$, $\psi \in X$ and $\psi \succcurlyeq \varphi$ implies $\psi \in Y$.

In other words, if Y contains an item of X, it contains all the items of X that are at least as firmly kept as this item. In another view, a comprehensive subset is always an upper segment of the original set, i.e., a cut of X.

2.2. Bargaining game

With the representation of a single agent's demands, we are now able to describe a bargaining situation with multiple agents. Consider a finite set $N = \{1, 2, \ldots, n\}$ of agents or players where $n \geqslant 2$. We model an n-agent bargaining problem as follows:

Definition 3. An *n-agent bargaining game* is a tuple $((X_1, \succcurlyeq_1), \ldots, (X_n, \succcurlyeq_n))$, where $(X_i, \succcurlyeq_i)$ is the demand set of agent i in $\mathcal{L}$.

We also write a bargaining game as $((X_i, \succcurlyeq_i))_{i \in N}$. The set of all n-agent bargaining games in language $\mathcal{L}$ is denoted by $\mathcal{G}^{n,\mathcal{L}}$.

There are two special kinds of bargaining situations. The first kind is that one of the agents discontinues an ongoing negotiation by withdrawing all her demands. In such a case, we consider that the negotiation procedure terminates with a disagreement. The second special kind of bargaining situations is that there is no conflict among the demands from all the players in a negotiation, therefore no arbitration or compromises are actually needed. Formally, we introduce the following concepts:

Definition 4. A bargaining game $((X_i, \succcurlyeq_i))_{i \in N}$ is said to represent a *disagreement situation* if $X_k = \emptyset$ for some k. It is said to be *non-conflictive* if it does not represent a disagreement situation and $\bigcup_{i \in N} X_i$ is logically consistent.

For a better understanding of the above bargaining model, let us consider a negotiation scenario.

Example 1. Two political parties in the Parliament bargain over a government rescue plan in response to the 2008 financial crisis. Each party may identify specific benefits from a different approach and therefore propose different rescue plans. Party A wants to put almost all the available funds into the major banks but leave a small portion for creating job opportunities. Party B has a close relationship with the automobile industry and thus insists on investing in car manufacturers. Party B also proposes that some money should be spent on helping people who hold significant mortgage debt.[8] Both parties know that there is no need to give money to both sides of house mortgages: *mortgagees* and *mortgagers*. Also both parties realize that if putting money in both auto industry and financial institutions, the government will incur unprecedented heavy deficit. However, Party A holds a low deficit policy.[9]

Suppose that we let
cars: "rescue automobile industry";
banks: "rescue the major banks";
mortgagers: "help house mortgagers";
jobs: "create job opportunities";
deficit: "heavy government deficit".

Then Party A's demands can be written as

$$X_A = \{banks, jobs, \neg deficit, \neg(banks \wedge mortgagers), (cars \wedge banks) \to deficit\}$$

and Party B's demands are

$$X_B = \{cars, mortgagers, \neg(banks \wedge mortgagers), (cars \wedge banks) \to deficit\}$$

Now suppose that Party A's preference on its demands is:

[8] For instance, there were some house owners whose property values became even lower than their mortgage debts after the financial crisis.
[9] The scenario was refined by taking one of the anonymous reviewers' suggestions.

$$(cars \wedge banks) \rightarrow deficit \approx_A \neg deficit \succ_A \neg(banks \wedge mortgagers) \succ_A banks \succ_A jobs$$

Party B's preference is:

$$\neg(banks \wedge mortgagers) \approx_B (cars \wedge banks) \rightarrow deficit \succ_B cars \succ_B mortgagers$$

Therefore, the bargaining problem can be modeled by the game $((X_A, \succcurlyeq_A), (X_B, \succcurlyeq_B))$. Obviously we cannot satisfy all demands from both parties (i.e., write all the demands into the final agreement) because they are logically inconsistent. The inconsistency can be identified as follows:

- $banks, mortgagers, \neg(banks \wedge mortgagers) \vdash \bot$,
- $cars, banks, \neg deficit, (cars \wedge banks) \rightarrow deficit \vdash \bot$.

One may notice that the statements '$\neg(banks \wedge mortgagers)$' and '$(cars \wedge banks) \rightarrow deficit$' are more like constraints than demands. Both of the parties request writing these statements into the final agreement because the statements also represent the parties' standpoints. For Party B, if Party A does not agree on $\neg(banks \wedge mortgagers)$, she will not go on. However, dropping this statement can be an option of Party A if she has chance to keep other more crucial demands. Therefore a demand of a player can be a standpoint the player wants the agreement to support.

2.3. Subgames

A bargaining situation changes with the progress of bargaining. The following concepts capture the relationship between the bargaining situations before and after players make concessions.

Definition 5. Given a bargaining game $G = ((X_i, \succcurlyeq_i))_{i \in N}$, a bargaining game $G' = ((X'_i, \succcurlyeq'_i))_{i \in N}$ is a *subgame* of G, denoted by $G' \sqsubseteq G$, if for all $i \in N$,

1. X'_i is a comprehensive subset of X_i,
2. $\preccurlyeq'_i = \preccurlyeq_i \cap (X'_i \times X'_i)$.

Furthermore, G' is a *proper subgame* of G, denoted by $G' \sqsubset G$, if $X'_i \subset X_i$ for all $i \in N$.

Intuitively, a subgame represents a contraction of the original problem: *a few players make concessions by dropping some demands*. The first condition requires each agent to retain its most preferred segment of demands whenever a concession is made. The second condition requires each agent to retain their preference ordering.

We remark that, to be a proper subgame, each player has to make a concession, i.e., $X'_i \subset X$. Therefore, we can view a subgame as the result of a concession made by all players simultaneously, i.e., a *simultaneous concession*. Note that $G' \sqsubseteq G$ and $G' \not\sqsubset G$ do not imply $G' = G$ (due to a bargaining game being a vector).

Definition 6. G' is the *maximal proper subgame* of G, denoted by $G' \sqsubset_{\max} G$, if

1. $G' \sqsubset G$;
2. $G'' \sqsubset G$ implies $G'' \sqsubseteq G'$.

The concept of a maximal proper subgame in fact captures the idea of minimal simultaneous concessions: "*if every agent has to make a concession, what is the smallest step an agent can take?*" The first condition requires every agent to make a concession, that is, the concession is done simultaneously by all agents. The second condition guarantees that there is no other simultaneous concession with fewer losses of demands. Note that the minimal loss or maximal gain we identify here is not simply in quantity, but more importantly in quality. Each agent gives up only their least preferred demands.

3. Bargaining solution

We have presented an n-person bargaining model with logical representation of demands and ordinal representation of preferences. In this section, we propose a solution concept based on this bargaining model.

3.1. Solution concept and the axioms

Generally speaking, the outcome of bargaining is an agreement or contract that is accepted by all players. Therefore the major concern of a player is which and how many demands of the player are included in the final agreement. Formally, given a bargaining game $G = ((X_i, \succcurlyeq_i))_{i \in N}$, a *possible outcome* of the game is a tuple $(O_1, \ldots, O_n)$ such that $O_i \subseteq X_i$ for all i. The agreement with respect to the possible outcome can then be defined as $\bigcup_{i \in N} O_i$.

Definition 7. A *bargaining solution* f is a function that assigns to a bargaining game in $\mathcal{G}^{n,\mathcal{L}}$ a possible outcome of the game. In other words, for any $G = ((X_i, \succcurlyeq_i))_{i\in N}$, $f(G) = (f_1(G), \ldots, f_n(G))$, where $f_i(G) \subseteq X_i$. We will write $f_i(G)$ as the i-th component of $f(G)$. $\bigcup_{i\in N} f_i(G)$ is called the *agreement* of the game, denoted by $A(G)$.

We now consider what properties a solution should have. Following the tradition of cooperative bargaining theory, we call these properties axioms.

We require first that a bargaining solution should resolve all possible conflicts in the demands. In other words, the agreement needs to be logically consistent.

Axiom 1 *(Consistency).* $\bigcup_{i\in N} f_i(G)$ is consistent.

The next axiom assumes that whenever an agent has to make a concession, she should only consider giving up those least preferred demands (see Definition 2).

Axiom 2 *(Comprehensiveness).* $f_i(G)$ is comprehensive for all i.

The following two axioms specify two special situations of bargaining. The first axiom deals with the situation when there is no conflict among the players' demands. In such a case we assume that all players mutually accept each others' demands. The concept of a non-conflict bargaining game is referred to Definition 4.

Axiom 3 *(Collective rationality).* If G is non-conflictive, then $f_i(G) = X_i$ for all i.

The second special case deals with the disagreement situations, that is, there is a k such that $X_k = \emptyset$ (see Definition 4). In such a case, no agreement is reached.

Axiom 4 *(Disagreement).* If G represents a disagreement situation, then $f_i(G) = \emptyset$ for all i.

The final axiom requires that a bargaining solution should be independent of any minimal simultaneous concession of the bargaining game unless no concession is needed. Obviously the axiom is an analog to Nash's Independence of Irrelevant Alternatives (IIA) (see Appendix A).[10]

Axiom 5 *(Contraction independence).* If $G' \sqsubset_{\max} G$, then $f(G) = f(G')$ unless G is non-conflictive.

Recall that $G' \sqsubset_{\max} G$ represents that G' is a maximal proper subgame of G, which means that G' is a minimal simultaneous concession of G (see Definition 6). The axiom actually implies another assumption: "*Whenever a bargaining situation requires a compromise due to conflicting demands, every agent has to make a concession by dropping a number of demands.*" The axiom says, if each agent gives up only a minimal number of the least preferred demands, the bargaining solution should not be affected because all the highly preferred demands of all players remain in the subgame.

3.2. Solution construction

In this subsection we seek a concrete bargaining solution that can be exactly characterized by the axioms we proposed in the previous section. Before we present the formal construction of our bargaining solution, let us describe the intuition behind the construction.

A bargaining solution can be interpreted as an arbitration procedure; i.e., a rule an arbitrator uses to decide what outcome to select [31]. Suppose now that a negotiation is carried out through the following arbitration procedure. At the beginning of the negotiation, all participants submit their demands to the arbitrator who is in charge of the negotiation. If there is no conflict among the demands, the negotiation terminates and the agreement is simply the collection of all the demands; otherwise, the negotiation moves to the next stage, in which each agent has two options: make a concession by withdrawing a number of demands (an agent is not allowed to add new demands), or declare a breakdown, in which case the negotiation terminates with an empty agreement.[11] The procedure continues until the remaining demands are consistent.

The following subsections will formalize the procedure.

[10] This axiom is also referred to as *contraction independence* [42].

[11] We could consider another possibility that the negotiation continues with one less agent. However, no matter whether an agreement is reached in the reduced game or not, the agreement should not be considered as the agreement of the original game because at least one player does not agree.

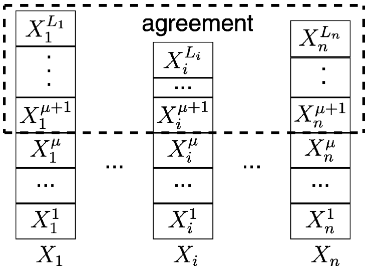

Fig. 1. The simultaneous concession solution.

3.2.1. Demand hierarchies

Given a demand set $(X_i, \preccurlyeq_i)$ of player i where $X_i \neq \emptyset$, we can define an equivalence relation $\approx$ on X_i in terms of the ordering $\preccurlyeq_i$ (see Section 2.1).[12] Let $\{X_i^1, \ldots, X_i^{L_i}\}$, or $\{X_i^l\}_{l=1}^{L_i}$ in short, be the partition of X_i induced by the equivalence relation $\approx$. We assume that this partition satisfies the following conditions:

1. $X_i^l \subseteq X_i$ and $X_i^l \neq \emptyset$ for all l $(1 \leqslant l \leqslant L_i)$;
2. $X_i = \bigcup_{l=1}^{L_i} X_i^l$;
3. $X_i^k \cap X_i^l = \emptyset$ for any $k \neq l$;
4. for any $\varphi \in X_i^k$ and $\psi \in X_i^l$, $\varphi \prec \psi$ if and only if $k < l$.

Furthermore, we extend the partition into an infinite sequence $\{X_i^l\}_{l=1}^{+\infty}$, simply assuming that $X^l = \emptyset$ when $l > L_i$. Note that with the special case that $X_i = \emptyset$, we assume that $L_i = 0$. We call $\{X^l\}_{l=1}^{+\infty}$ the *hierarchy of the demand set* $(X_i, \preccurlyeq)$ and L_i is called the *height of the hierarchy*.

We write $X_i^{>k} = \bigcup_{l>k} X_i^l$. In particular, $X_i^{>0} = X_i$.

3.2.2. The simultaneous concession solution

Now we present the construction of our bargaining solution.

Definition 8. A bargaining solution F on $\mathcal{G}^{n,\mathcal{L}}$ is the *simultaneous concession solution* (SCS) if for any bargaining game $G = ((X_i, \succcurlyeq_i))_{i \in N}$,

$$F(G) = \begin{cases} (X_1^{>\mu}, \ldots, X_n^{>\mu}), & \text{if } \mu < L \\ (\emptyset, \ldots, \emptyset), & \text{otherwise} \end{cases}$$

where $L = \min_{i \in N} L_i$ and $\mu = \min\{k\colon \bigcup_{i=1}^n X_i^{>k}$ is consistent$\}$. We call L the *height* of G and μ is the *minimal rounds of concessions* of the game.

The construction of the simultaneous solution actually simulates the arbitration procedure described in the beginning of Section 3.2. Given a bargaining game, if the game represents a disagreement situation (there is an agent who has an empty demand), then the height of the game is zero, i.e., $L = 0$. In this case, no agreement is reached (empty agreement). If the game is non-conflictive, then $\mu = 0$. In this case, $F_i(G) = X_i^{>0} = X_i$ for all i. No agent has to make any concession. In the cases when the arbitration procedure has to run a few rounds, each agent makes a minimal concession in each round. Here a minimal concession means whenever a player has to make a concession, she always gives up all the least preferred demands in her demand hierarchy.[13] Based on the assumption, after μ rounds, all the remaining demands are consistent. In the case when one agent has nothing to give up ($\mu \geqslant L$), we assume that the agent will declare a breakdown. In all other cases, the negotiation will end up with all the remaining demands. Fig. 1 illustrates the construction of the solution. The dashed rectangle contains the demands that constitute the solution.

Example 2. Based on the bargaining game described in Example 1, the demand hierarchies of these two parties can be illustrated by the following table (the higher the more preferred):

[12] Note that this equivalence relation has nothing to do with logical equivalence. Two logically equivalent statements can belong to difference equivalent classes.

[13] Thinking in Experimental Economics, this gives us a way to observe a bargainer's demand hierarchy.

Party A	Party B
$(cars \wedge banks) \to deficit$ $\neg deficit$	
$\neg(banks \wedge mortgagers)$	$(cars \wedge banks) \to deficit$ $\neg(banks \wedge mortgagers)$
banks	*cars*
jobs	*mortgagers*

It is easy to see that $L_A = 4$, $L_B = 3$ and $\mu = 2$. The simultaneous concession solution of the game is:

$$F_A(G) = \{(cars \wedge banks) \to deficit, \neg deficit, \neg(banks \wedge mortgagers)\}$$
$$F_B(G) = \{(cars \wedge banks) \to deficit, \neg(banks \wedge mortgagers)\}$$

Then, the outcome of the negotiation (the agreement) is:

$$A(G) = \{(cars \wedge banks) \to deficit, \neg deficit, \neg(banks \wedge mortgagers)\}$$

which means that none of the proposed rescue plans were passed except that Party A's low-deficit policy is confirmed.

As mentioned in Section 1, the most difficulty establishing an ordinal bargaining theory is that the language of ordinal utility is less expressive than the language of cardinal utility. An ordinal bargaining theory has to offer a way to express bargainers' attitudes towards risk. The following example demonstrates how this can be done with our model.

Example 3. Consider the bargaining game in Example 1 again. Suppose that Party A has an additional demand, named *coffee*, meaning requesting a cup of coffee. The demand hierarchy is reordered as follows (Party B's demands and preference remain the same):

Party A	Party B
$(cars \wedge banks) \to deficit$ $\neg deficit$	
$\neg(banks \wedge mortgagers)$	
banks	$(cars \wedge banks) \to deficit$ $\neg(banks \wedge mortgagers)$
jobs	*cars*
coffee	*mortgagers*

It is not hard to calculate that the simultaneous concession solution gives the following outcome:

$$\{\neg deficit, (cars \wedge banks) \to deficit, \neg(banks \wedge mortgagers), banks\}$$

Obviously, Party A gets benefit from the extra demand, which actually helps it delay one compulsory concession. We call such a demand a *dummy demand*.[14] A dummy demand can be placed anywhere in the demand hierarchy to represent a desired delay of concession.[15] It is easy to see that the use of dummy demands mimics the non-linearity of cardinal preferences.

It is worth mentioning that delaying a compulsory concessions is risky. If Party B instead were more aggressive, as shown in the demand hierarchy below (nothing negotiable except *mortgagers*), the outcome of the bargaining would have been totally different.

Party A	Party B
$(cars \wedge banks) \to deficit$ $\neg deficit$	
$\neg(banks \wedge mortgagers)$	
banks	
jobs	$(cars \wedge banks) \to deficit$ $\neg(banks \wedge mortgagers)$ *cars*
coffee	*mortgagers*

[14] More precisely, a dummy demand is a demand that does not conflict with other demands. The use of the dummy demand was firstly proposed by Zhang and Zhang [44].
[15] There are also many other ways to delay a compulsory concession, such as reclaiming an equivalent statement.

The outcome of the above bargaining game is an empty set because $L \geqslant \mu$.

In general, a player can express her attitudes towards risk via her schedule of concessions. A risk-averse player may choose to give up the demands that more likely contradict other players' demands in an earlier stage of a negotiation in order to increase the possibility of reaching an agreement. Thus she may give these demands lower preference. In contrast, a risk-loving player could place less radical demands or a dummy demand at lower levels of her demand hierarchy to protract a negotiation and gain more bargaining power.

4. Characterization

We now establish that the axioms we presented in Section 3.1 exactly characterize the simultaneous concession solution defined in Section 3.2.2.

4.1. The main theorem

Theorem 1. *A bargaining solution is the simultaneous concession solution F if and only if it satisfies the axioms:* Consistency, Comprehensiveness, Collective rationality, Disagreement, *and* Contraction independence.

The proof of the theorem requires the following technical lemma.

Lemma 1. *Let G' be the maximal proper subgame of $G = ((X_i, \succcurlyeq_i))_{i \in N}$. Then $G' = ((X_i^{>1}, \preccurlyeq'_i))_{i \in N}$, where $\preccurlyeq'_i = \preccurlyeq_i \cap (X_i^{>1} \times X_i^{>1})$.*

Proof. Firstly, for all $i \in N$, $X_i \neq \emptyset$ because G has at least one proper subgame G'. Thus $X_i \neq X_i^{>1}$, or $X_i^{>1} \subset X_i$, for all i. Obviously for each $i \in N$, $X_i^{>1}$ is a comprehensive subset of X_i. Therefore we have $((X_i^{>1}, \preccurlyeq'_i))_{i \in N} \sqsubset G$.

Secondly, let $G'' = ((X''_i, \preccurlyeq''_i))_{i \in N}$ be any proper subgame of G, we prove that $G'' \sqsubseteq ((X_i^{>1}, \preccurlyeq'_i))_{i \in N}$. Note that $X_i^{>1} = X_i \setminus X_i^1$. Let $\varphi \in X''_i$. Since G'' is a subgame of G, we know that $\varphi \in X_i$. Now let us assume that $\psi \in X_i$. If $\varphi \in X_i^1$, we have $\psi \succcurlyeq_i \varphi$. By the comprehensiveness of X''_i, we yield that $\psi \in X''_i$. This means that $X_i \subseteq X''_i$. However, G'' is a proper subgame of G, which implies $X''_i \subset X_i$, a contradiction. It turns out that $\varphi \notin X_i^1$. We yield that $\varphi \in X_i^{>1}$. We have proven that $X''_i \subseteq X_i^{>1}$ for all $i \in N$, which implies that $((X''_i, \preccurlyeq''_i))_{i \in N} \sqsubseteq ((X_i^{>1}, \preccurlyeq'_i))_{i \in N}$.

According to Definition 6, the above arguments imply that $((X_i^{>1}, \preccurlyeq'_i))_{i \in N}$ is the maximal proper subgame of G (obviously one game can have at most one maximal proper subgame). □

Proof of Theorem 1. "⇒". We prove that F satisfies all the axioms.

Firstly, it is easy to show that F satisfies *Consistency*. In fact, for any $G = ((X_i, \succcurlyeq_i))_{i \in N}$, if $\mu \geqslant L$, $F(G) = (\emptyset, \ldots, \emptyset)$; otherwise, $F(G) = (X_1^{>\mu}, \ldots, X_n^{>\mu})$. By the definition of μ, $\bigcup_{i=1}^n X_i^{>\mu}$ is consistent. In both cases, $\bigcup_{i \in N} F_i(G)$ is consistent.

To show F satisfies *Comprehensiveness*, for all i, if $F_i(G) = \emptyset$, $F_i(G)$ is a comprehensive subset of X_i; otherwise, we assume that $\varphi \in F_i(G)$. Thus $\varphi \in X_i^{>\mu}$. Let $\varphi \in X_i^k$ where $k > \mu$. Given any $\psi \in X_i$, let $\psi \in X_i^l$. If $\psi \succcurlyeq \varphi$, we have $l \geqslant k$. Therefore $\psi \in X_i^{>\mu}$, which implies $\psi \in F_i(G)$.

To see that F satisfies *Collective rationality*, let $G = ((X_1, \succcurlyeq_1), \ldots, (X_n, \succcurlyeq_n))$ be a non-conflictive bargaining game (see Definition 4). In such a situation, it is easy to know that $L \geqslant 1$ and $\mu = 0$. It then follows from the solution construction that for each i, $F_i(G) = X_i^{>0} = X_i$, as desired.

Obviously F satisfies *Disagreement* because if a game represents a disagreement situation (see Definition 4), $L = 0$ but $\mu \geqslant 0$, which implies $\mu \nless L$.

Finally we prove that F satisfies *Contraction independence*. Consider a bargaining game $G = ((X_i, \succcurlyeq_i))_{i \in N}$. In the case when $L = 0$, there exists a k such that $X_k = \emptyset$, which means that G has no proper subgame. Therefore F satisfies the axiom trivially. We then can assume that $L > 0$. In the case that $\mu = 0$, the game is non-conflictive because $\bigcup_{i \in N} X_i$ is consistent and $X_i \neq \emptyset$ for all i. This situation has been excluded by the axiom. We can now assume that $\mu > 0$. Let $G' = ((X'_i, \succcurlyeq'_i))_{i \in N}$ be the maximal proper subgame of G. We write L' and μ' to represent the height and the minimal rounds of concessions of G'. According to Lemma 1, for each i, we have $X'_i = X_i^{>1}$ and $\succcurlyeq'_i = \succcurlyeq_i \cap (X'_i \times X'_i)$. Obviously, $L' = L - 1$ and $\mu' = \mu - 1$. Therefore, $\mu < L$ if and only if $\mu' < L'$. In the case that $\mu \geqslant L$, $F(G) = F(G') = (\emptyset, \ldots, \emptyset)$. In the other case when $\mu < L$, we have $F(G) = (X_1^{>\mu}, \ldots, X_n^{>\mu})$ and $F(G') = ({X'}_1^{>\mu'}, \ldots, {X'}_n^{>\mu'})$. For each $i \in N$, we then have

$$\begin{aligned} X_i^{>\mu} &= \bigcup_{k>\mu} X_i^k = X_i \setminus \bigcup_{k=1}^{\mu} X_i^k = \left(X_i \setminus X_i^1\right) \setminus \bigcup_{k=2}^{\mu} X_i^k \\ &= X'_i \setminus \bigcup_{k=1}^{\mu'} {X'}_i^k = \bigcup_{k>\mu'} {X'}_i^k = {X'}_i^{>\mu'} \end{aligned} \tag{1}$$

Therefore $F(G) = F(G')$.

"$\Leftarrow$". Suppose that a bargaining solution f satisfies the axioms: *Consistency*, *Comprehensiveness*, *Collective rationality*, *Disagreement*, and *Contraction independence*. We show that for any bargaining game $G = ((X_i, \succcurlyeq_i))_{i \in N}$, $f(G) = F(G)$ by induction on μ.

For the base case that $\mu = 0$, we know that $\bigcup_{i \in N} X_i$ is consistent. If there is a k such that $X_k = \emptyset$, then G represents a disagreement situation. By the axiom *Disagreement*, we have $f(G) = (\emptyset, \ldots, \emptyset)$. On the other hand, for any disagreement game, we have $L = 0$. Thus $\mu \not< L$, which implies that $F(G) = (\emptyset, \ldots, \emptyset)$ by the construction of F. Therefore, $f(G) = F(G)$. If $X_i \neq \emptyset$ for all i, G is a non-conflictive game. By *Collective rationality*, we have $f(G) = (X_1, \ldots, X_n)$. Meanwhile, $F(G) = (X_1^{>0}, \ldots, X_n^{>0}) = (X_1, \ldots, X_n)$. Therefore we also have $f(G) = F(G)$.

Now we assume that for any game G' such that $\mu' = k$, $f(G') = F(G')$. We consider a game $G = ((X_i, \succcurlyeq_i))_{i \in N}$ with which $\mu = k + 1$. If G represents a disagreement situation, then the height of the game $L = 0$. By *Disagreement*, we have $f(G) = (\emptyset, \ldots, \emptyset)$. Since $\mu \geqslant L$, by the construction of F, we have $F(G) = (\emptyset, \ldots, \emptyset)$. Therefore $f(G) = F(G)$. Now we consider the non-trivial case in which $L > 0$. Let $G' = ((X_i', \succcurlyeq_i'))_{i \in N}$, where

1. $X_i' = X_i^{>1}$, and
2. $\succcurlyeq_i' = \succcurlyeq_i \cap (X_i' \times X_i')$.

Since $L > 0$ with respect to game G, we have that $X_i' \neq X_i$ for all i. Thus G' is a proper subgame of G. By Lemma 1, G' is actually the maximal proper subgame of G. It is easy to see that $\mu' = k$. By the inductive assumption, we have

$$f\left(G'\right) = F\left(G'\right) = \left({X'}_1^{>\mu'}, \ldots, {X'}_n^{>\mu'}\right)$$

According to *Contraction independence*, we have

$$f(G) = f\left(G'\right) = \left({X'}_1^{>\mu'}, \ldots, {X'}_n^{>\mu'}\right)$$

Similar to Eq. (1), we have ${X'}_i^{>\mu'} = X_i^{>\mu}$ for each i. Therefore

$$f(G) = \left(X_1^{>\mu}, \ldots, X_n^{>\mu}\right) = F(G)$$

That is, $f(G) = F(G)$. $\square$

4.2. Game-theoretic properties of the bargaining solution

Among the axioms that characterize the simultaneous concession solution, Axioms 1, 3 and 4 are purely logical properties. Axioms 2 and 5, although represented in logical form, are rooted in game theory. Axiom 2 is the logical counterpart of comprehensiveness, which is a fundamental assumption in cooperative bargaining theory [42]. Axiom 5 is an analogue of Nash's *Independence of Irrelevant Alternatives* (see Appendix A). One may wonder whether the simultaneous concession solution satisfies other game-theoretic properties, such as *Pareto optimality*, *Individual rationality*, and so on.

It is worth mentioning that there is a subtle difference between the standard game-theoretic ordinal bargaining model and our model. The preference orderings in the classical ordinal model are defined based on the set of possible outcomes, while in our model the preference orderings are based on the demand set of each player. To examine the game-theoretic properties of the bargaining solutions in our model, we need to restate these properties in our language.

Given a bargaining game $G = ((X_1, \succcurlyeq_1), \ldots, (X_n, \succcurlyeq_n))$, we let $\Omega(G) = \{(O_1, \ldots, O_n)\colon\ O_i \subseteq X_i \text{ for all } i \in N\}$, i.e., the set of possible outcomes of G (see Section 3.1). A possible outcome $(O_1, \ldots, O_n)$ is *comprehensive* if for each i, O_i is a comprehensive subset of X_i. It is *consistent* if $\bigcup_{i \in N} O_i$ is consistent.

For any two possible outcomes $O, O' \in \Omega(G)$, we write

- $O \succcurlyeq O'$ iff $O_i \supseteq O_i'$ for all i;
- $O \succcurlyeq O'$ iff $O \succcurlyeq O'$ but $O \neq O'$;
- $O \succ O'$ iff $O_i \supset O_i'$ for all i.

Based on the above concepts, we are now able to discuss the game-theoretic properties of the simultaneous concession solution. Firstly, the solution trivially satisfies the following *Individual rationality*:

$$f(G) \succcurlyeq (\emptyset, \ldots, \emptyset)$$

This is not interesting because the property is satisfied by any bargaining solution (see Definition 7). However, the following strong version of the property is appealing, which says that as long as a negotiation does not end with disagreement, none of the players receives empty gain.

Proposition 1. Disagreement *and* Contract independence *imply the following property*

- (Strong individual rationality) $f(G) \succ (\emptyset, \ldots, \emptyset)$ *unless* $f(G) = (\emptyset, \ldots, \emptyset)$.

Proof. We apply induction on the height of a game. For any $G \in \mathcal{G}^{n,\mathcal{L}}$ whose height $L = 0$, the game represents a disagreement situation. Thus *Disagreement* implies $f(G) = (\emptyset, \ldots, \emptyset)$. Assume that the proposition holds for any game in which the height equals k. Now consider a game G whose height $L = k + 1$. Let G' be the maximal proper subgame G. By *Contract independence*, $f(G) = f(G')$. According to Lemma 1, the height of G' is k. By induction assumption, $f(G') \succ (\emptyset, \ldots, \emptyset)$ unless $f(G') = (\emptyset, \ldots, \emptyset)$. So does $f(G)$ because $f(G) = f(G')$. □

Next we consider Pareto optimality. Obviously it is not reasonable to require a bargaining solution to be maximal in terms of set inclusion without requiring the outcome to be comprehensive and consistent. Even though we impose comprehensiveness and consistency on any possible outcome, strong Pareto optimality is still not attainable because the domain of problems to which our model applies is discrete (so is non-convex, see [5]). However, we have the following variation of weak Pareto optimality.

Proposition 2. *The simultaneous concession solution satisfies the following property*

- (Weak Pareto optimality) *If $f(G) \neq (\emptyset, \ldots, \emptyset)$, there is no comprehensive consistent outcome $O \in \Omega(G)$ such that $O \succ f(G)$.*

Proof. Given a bargaining game $G = ((X_i, \succcurlyeq_i))_{i \in N}$, by the construction of the simultaneous concession solution, if $F(G) \neq (\emptyset, \ldots, \emptyset)$, then $F(G) = (X_1^{>\mu}, \ldots, X_n^{>\mu})$, where μ is the minimal rounds of concessions of the game (see Definition 8). Suppose that there were a comprehensive consistent outcome $O = (O_1, \ldots, O_n) \in \Omega(G)$ such that $O \succ F(G)$. For each $i \in N$, we have $F_i(G) \subset O_i \subseteq X_i$, which implies that $X_i^{>\mu-1} \subseteq O_i$ (obviously $\mu > 0$). Since $\bigcup_{i \in N} O_i$ is consistent, so does $\bigcup_{i \in N} X_i^{>\mu-1}$, which contradicts the definition of μ. □

5. Continuous domain

We have introduced a purely qualitative solution to the bargaining problems in discrete domains. In this section, we demonstrate that our solution is also applicable to continuous domains via a procedure of discretization. More importantly, the process of discretization reveals the connection between the logic-based bargaining solution and the numerical bargaining solutions.

To make the process easy to understand, let us consider a well-known bargaining scenario: *bargaining over the partition of a cake* (see, for instance, [21]). Two players, A and B, bargain over the partition of a cake of size π ($\pi > 0$). The set of possible agreements is $\Phi = \{(x_A, x_B)\colon\ 0 \leqslant x_A \leqslant \pi \text{ and } x_B = \pi - x_A\}$. For each $x_i \in [0, \pi]$, $U_i(x_i)$ is player i's utility from obtaining a share x_i of the cake, where player i's utility function $U_i : [0, \pi] \to \Re$ is strictly increasing and continuous. Without loss of generality, we assume that the disagreement point is $d = (U_A(0), U_B(0))$. Let

$$\Omega = \{(u_A, u_B)\colon\ U_A(x_A) = u_A \text{ and } U_B(x_B) = u_B \text{ for some } (x_A, x_B) \in \Phi\}$$

Then (Ω, d) is the numerical game that represents the bargaining problem.

To represent the bargaining situation in the logical model, we first have to discretize the problem. Let L_A and L_B be two non-zero natural numbers, the granularity in which each player wants to discretize the problem. Let $L = \min\{L_A, L_B\}$. For each $i \in \{A, B\}$ and each l ($1 \leqslant l \leqslant L_i$), let $P_i(l)$ represent the following proposition:

$$x_i \geqslant U_i^{-1}\left(\left(1 - \frac{l}{L_i}\right)U_i(0) + \frac{l}{L_i}U_i(\pi)\right) \tag{2}$$

Informally, we can assume that the utility of a player represents her degree of satisfaction on the share of the cake she gains. Each player i ranks her satisfaction in L_i levels from $U_i(0)$ to $U_i(\pi)$. Note that L_A is not necessarily equal to L_B. $P_i(l)$ then represents the demand of player i that she requests a share, which can bring her a satisfaction at least to level l. For instance, $P_i(L_i)$ represents the demand that player i wants the whole cake, which will bring her to the highest level of satisfaction.

We assume that each player knows the rule of the game: *the total shares the two players can request should not be more than* π. This rule can be expressed by a set, R, of logical statements. The elements of R are in the form $\neg(P_A(k) \wedge P_B(l)) \in R$. $\neg(P_A(k) \wedge P_B(l)) \in R$ if and only if

$$U_A^{-1}\left(\left(1 - \frac{k}{L_A}\right)U_A(0) + \frac{k}{L_A}U_A(\pi)\right) + U_B^{-1}\left(\left(1 - \frac{l}{L_B}\right)U_B(0) + \frac{l}{L_B}U_B(\pi)\right) > \pi \tag{3}$$

Let T represent the following commonsense: *for each player, as long as a demand for a bigger share is satisfied, any demand for a smaller share is also satisfied.* Formally, T contains the statements in the form $P_i(k) \to P_i(l)$, where $P_i(k) \to P_i(l) \in T$ if and only if $k \geqslant l$ for all $1 \leqslant k, l \leqslant L_i$ and $i \in \{A, B\}$. We write $C = R \cup T$.

Let $G = ((X_A, \succcurlyeq_A), (X_B, \succcurlyeq_B))$ be the bargaining game, where

$$X_A = C \cup \{P_A(1), P_A(2), \ldots, P_A(L_A)\}$$
$$X_B = C \cup \{P_B(1), P_B(2), \ldots, P_B(L_B)\}$$

Table 1
Demand hierarchies of numerical bargaining.

Player A	Player B
$C \cup \{P_A(1)\}$	$C \cup \{P_B(1)\}$
$P_A(2)$	$P_B(2)$
...	...
$P_A(L_A)$	$P_B(L_B)$

The orderings $\succcurlyeq_A$ and $\succcurlyeq_B$ are denoted by Table 1 (the higher the more firmly kept).[16]

To find the solution of the game, let us calculate the minimal rounds of concessions of the game first:

$$\begin{aligned}
\mu &= \min\left\{k\colon \bigcup_{i=1}^{n} X_i^{>k} \text{ is consistent}\right\} \\
&= \min\bigl\{k\colon C \cup \bigl\{P_A(1), \ldots, P_A(L_A-k)\bigr\} \cup \bigl\{P_B(1), \ldots, P_B(L_B-k)\bigr\} \text{ is consistent}\bigr\} \\
&= \min\bigl\{k\colon C \cup \bigl\{P_A(L_A-k)\bigr\} \cup \bigl\{P_B(L_B-k)\bigr\} \text{ is consistent}\bigr\}^{17} \\
&= \min\left\{k\colon U_A^{-1}\left(\frac{k}{L_A}U_A(0) + \left(1-\frac{k}{L_A}\right)U_A(\pi)\right)\right. \\
&\qquad \left. + U_B^{-1}\left(\frac{k}{L_B}U_B(0) + \left(1-\frac{k}{L_B}\right)U_B(\pi)\right) \leqslant \pi\right\}
\end{aligned} \tag{4}$$

Therefore the simultaneous concession solution of the game gives the following outcome:

1. if $\mu \geqslant L$, $F(G) = (\emptyset, \emptyset)$; otherwise,
2. $F(G) = (C \cup \{P_A(L_A - \mu)\}, C \cup \{P_B(L_B - \mu)\})$.

The condition $\mu \geqslant L$ means that the bargaining goes to the situation in which one of the players is forced to give up everything (the game knowledge and a share of the cake). Thus the game will end with disagreement. If this is not the case, an agreement will be reached, which should contain, among others, the statements $P_A(L_A - \mu)$ and $P_B(L_B - \mu)$. In other words, the partition of the cake, (x_A, x_B), satisfies the following conditions:

$$x_A \geqslant U_A^{-1}\left(\frac{\mu}{L_A}U_A(0) + \left(1-\frac{\mu}{L_A}\right)U_A(\pi)\right)$$

$$x_B \geqslant U_B^{-1}\left(\frac{\mu}{L_B}U_B(0) + \left(1-\frac{\mu}{L_B}\right)U_B(\pi)\right)$$

Combining with the condition $x_A + x_B \leqslant \pi$, we have:

$$x_A \in \left[U_A^{-1}\left(\frac{\mu}{L_A}U_A(0) + \left(1-\frac{\mu}{L_A}\right)U_A(\pi)\right), \pi - U_B^{-1}\left(\frac{\mu}{L_B}U_B(0) + \left(1-\frac{\mu}{L_B}\right)U_B(\pi)\right)\right] \tag{5}$$

$$x_B \in \left[U_B^{-1}\left(\frac{\mu}{L_B}U_B(0) + \left(1-\frac{\mu}{L_B}\right)U_B(\pi)\right), \pi - U_A^{-1}\left(\frac{\mu}{L_A}U_A(0) + \left(1-\frac{\mu}{L_A}\right)U_A(\pi)\right)\right] \tag{6}$$

Note that instead of giving the actual value of the bargaining outcome, the solution specifies an interval as its estimation of the bargaining outcome. This is due to the process of discretization. Proposition 3 will show that with the refinement of discretization, the estimation converges to a single point.

For a better understanding of the discretization procedure, let us consider an instance of the problem (see also [44]). Assume that player A has a linear utility scale of its share, $U_A(x_A) = x_A$, and player B has a utility scale that is proportional to the square of his share, $U_B(x_B) = x_B^2$. Let us assume that $\pi = 100$ and $L_A = L_B = 10$. Then for each l ($1 \leqslant l \leqslant 10$), $P_A(l)$ represents the statement "$x_A \geqslant 10l$" (i.e., $U_A(x_A) \geqslant (1 - \frac{l}{L_A})U_A(0) + \frac{l}{L_A}U_A(\pi)$) and $P_B(l)$ represents "$x_B \geqslant 10\sqrt{10l}$" (i.e., $U_B(x_B) \geqslant (1 - \frac{l}{L_B})U_B(0) + \frac{l}{L_B}U_B(\pi)$). The demand hierarchy of each player represents the player's demands, the schedule of concessions at each round, and the player's knowledge of the game rules (see Table 1). It is not hard to know that $\mu = 7$. Therefore the simultaneous concession solution gives an estimation of the bargaining outcome as: $x_A \in [30, 45.2]$ and $x_B \in [54.8, 70]$ (Table 2 demonstrates the bargaining procedure).

[16] Note that each player ranks the game rule and commonsense highest. The ranking of the other demands follows exactly the natural order of concessions, i.e., the demands that have to be dropped earlier are ranked lower.

[17] Note that $P_i(k) \to P_i(l) \in C$ if $k \geqslant l$.

Table 2
Bargaining over the partition of a cake.

Round	Player A's demand	Player B's demand	Agreement
0	$x_A \geqslant 100$	$x_B \geqslant 100.0$	×
1	$x_A \geqslant 90$	$x_B \geqslant 94.9$	×
2	$x_A \geqslant 80$	$x_B \geqslant 89.4$	×
3	$x_A \geqslant 70$	$x_B \geqslant 83.7$	×
4	$x_A \geqslant 60$	$x_B \geqslant 77.5$	×
5	$x_A \geqslant 50$	$x_B \geqslant 70.7$	×
6	$x_A \geqslant 40$	$x_B \geqslant 63.2$	×
7	$x_A \geqslant 30$	$x_B \geqslant 54.8$	√

Notice that both the Nash solution (33.3, 66.7) and the Kalai–Smorodinsky solution (38.2, 61.8) belong to the range (see Appendix A for the definitions of these solutions). Surprisingly, with the refinement of the discretization, the prediction is no longer in favor of the Nash solution but is approaching the Kalai–Smorodinsky solution. For instance, when $L_A = L_B = 20$, the solution gives the range $x_A \in [35, 40.8]$ and $x_B \in [59.2, 65]$. When $L_A = L_B = 100$, the solution ranges become $x_A \in [38, 38.4]$ and $x_B \in [61.6, 62]$. The following proposition shows that the simultaneous concession solution is actually an approximation of the Kalai–Smorodinsky solution.

Proposition 3. *Let (x_A^*, x_B^*) be the Kalai–Smorodinsky solution of the cake partition game. If $L_A = L_B$, then (x_A^*, x_B^*) satisfies Eqs.* (5) *and* (6)*. Moreover,*

$$\lim_{L\to\infty} U_A^{-1}\left(\frac{\mu}{L}U_A(0) + \left(1 - \frac{\mu}{L}\right)U_A(\pi)\right) = x_A^* \tag{7}$$

$$\lim_{L\to\infty} U_B^{-1}\left(\frac{\mu}{L}U_B(0) + \left(1 - \frac{\mu}{L}\right)U_B(\pi)\right) = x_B^* \tag{8}$$

where $L = L_A = L_B$.

Proof. First we calculate the Kalai–Smorodinsky solution. Since both players' utility functions are strictly increasing, the ideal point of the bargaining game is $(U_A(\pi), U_B(\pi))$. Any point in the segment of the line connecting the disagreement point $(U_A(0), U_B(0))$ and the ideal point can be represented as $(tU_A(\pi) + (1-t)U_A(0), tU_B(\pi) + (1-t)U_B(0))$, where $t \in [0, 1]$. The Kalai–Smorodinsky solution is then:

$$\left(x_A^*, x_B^*\right) = \arg\max_{(x_A,x_B)\in X}\big\{t \in [0,1]:\ U_A(x_A) = tU_A(\pi) + (1-t)U_A(0) \text{ and}$$
$$U_B(x_A) = tU_B(\pi) + (1-t)U_B(0)\big\}$$

Let $t_0 \in [0, 1]$ such that $U_A(x_A^*) = t_0U_A(\pi) + (1-t_0)U_A(0)$ and $U_B(x_B^*) = t_0U_B(\pi) + (1-t_0)U_B(0)$.

Next we show that (x_A^*, x_B^*) satisfies Eqs. (5) and (6). To this end, let $\underline{x}_A = U_A^{-1}(\frac{\mu}{L}U_A(0) + (1-\frac{\mu}{L})U_A(\pi))$ and $\underline{x}_B = U_B^{-1}(\frac{\mu}{L}U_B(0) + (1-\frac{\mu}{L})U_B(\pi))$. It follows that $U_A(\underline{x}_A) = \frac{\mu}{L}U_A(0) + (1-\frac{\mu}{L})U_A(\pi)$ and $U_B(\underline{x}_B) = \frac{\mu}{L}U_B(0) + (1-\frac{\mu}{L})U_B(\pi)$. By Eq. (4), we have $\underline{x}_A + \underline{x}_B \leqslant \pi$. Since $x_A^* + x_B^* = \pi$, we entail $t_0 \geqslant 1 - \frac{\mu}{L}$ (note that U_A and U_B are strictly increasing). We have established that $\underline{x}_A \leqslant x_A^*$ and $\underline{x}_B \leqslant x_B^*$.

On the other hand, $x_A^* = \pi - x_B^* \leqslant \pi - \underline{x}_B$. Similarly, we have $x_B^* = \pi - x_A^* \leqslant \pi - \underline{x}_A$. Therefore we have proven that $x_A^* \in [\underline{x}_A, \pi - \underline{x}_B]$ and $x_B^* \in [\underline{x}_B, \pi - \underline{x}_A]$.

Finally we prove Eqs. (7) and (8). Let $\bar{x}_A = U_A^{-1}(\frac{\mu-1}{L}U_A(0) + (1-\frac{\mu-1}{L})U_A(\pi))$ and $\bar{x}_B = U_B^{-1}(\frac{\mu-1}{L}U_B(0) + (1-\frac{\mu-1}{L})U_B(\pi))$. Note that $\mu > 0$ because $\pi > 0$. By Eq. (4), we entail that $\bar{x}_A + \bar{x}_B > \pi$. Notice that $(U_A(\bar{x}_A), U_B(\bar{x}_B))$ is on the segment of the line connecting the disagreement point and the ideal point, we know that $x_A^* \leqslant \bar{x}_A$ and $x_B^* \leqslant \bar{x}_B$ because U_A and U_B are strictly increasing. Therefore, to show Eqs. (7) and (8), we only have to prove that $\lim_{L\to\infty}(\bar{x}_A - \underline{x}_A) = 0$ and $\lim_{L\to\infty}(\bar{x}_B - \underline{x}_B) = 0$. Obviously $\lim_{L\to\infty}(U_A(\bar{x}_A) - U_A(\underline{x}_A)) = 0$. Since U_A is strictly increasing and continuous, U_A^{-1} exists and is continuous. Therefore $\lim_{L\to\infty}(\bar{x}_A - \underline{x}_A) = 0$. Similarly $\lim_{L\to\infty}(\bar{x}_B - \underline{x}_B) = 0$. □

Although the procedure we described above is based on the cake partition problem, it is easy to see that the approach is applicable to any single-issue continuous bargaining with strictly increasing and continuous utility functions.

6. Conclusion and related work

In this paper we introduced a logic-based axiomatic model of bargaining for qualitative bargaining analysis. With the model a bargaining situation is described by a set of propositional statements and bargainers' preferences are represented by total pre-orders. We proposed a solution to n-person bargaining problems based on the assumption of minimal simultaneous concessions. We proved that the solution can be characterized by five natural and intuitive logical axioms. We demonstrated that the solution is applicable not only to bargaining problems in discrete domains but also problems in continuous domains. In the latter case the solution gives an approximation of the Kalai–Smorodinsky solution.

The key to our approach is the way in which the bargainers' attitudes towards risk are expressed. We describe bargainers' physical demands in logical statements together with the ordinal representation of preferences. A player's attitude towards risk is reflected by the player's preferences over conflicting demands. A risk-averse player tends to give lower preferences to conflicting demands in order to reduce the risk of breakdown, whereas a risk-loving player would rather give lower preference to the conflicting demands to protract a negotiation in order to gain more benefit. We have observed that logical relations among bargainers' demands play an important role in the bargaining model. It is the combination of ordinal preferences and logical consistency that provides a medium for the expression of bargainers' risk postures. Such a representation is much more natural than the representation through non-linearity of utility functions.

Our bargaining solution is built on ordinal preferences; therefore it is an ordinal solution. Most existing work on ordinal bargaining solutions comes with conditions. Shapley, Kibris, Safra and Samet showed the existence of ordinal solutions to the problems with no less than three players [14,35,40]. Calvo and Peters showed that ordinal solutions exist if at least one player is cardinal [3]. As mentioned in Section 1, Rubinstein et al.'s solution requires each player to provide a preference ordering on the space of lotteries over the set of possible agreements [33]. O'Neill et al.'s solution gives an outcome which is a limit of a family of bargaining solutions in continuous domains [25]. In contrast, the solution proposed in this paper is applicable to any n-player bargaining problem with no constraints on the number or the type of players.

We would like to remark that our framework provides a naive solution to multi-issue bargaining. The analysis of multi-issue bargaining can be extremely complicated. Multi-issue bargaining problems are mostly procedure-dependent, thus axiomatic analysis becomes much harder. Almost all existing research on multi-issue bargaining is based on non-cooperative models, such as [6,7,11,43]. Axiomatic analysis requires aggregation of preferences on issues. The simplest and most commonly used assumption is additivity of utility over issues [28]. With additivity, a multi-issue bargaining problem can be reduced to a single-issue problem, which is less interesting [29]. Unlike the existing work on multi-issue bargaining, we describe negotiation terms in logical statements. The relations between bargaining issues can be specified by logical connectives. With the benefit of the expressiveness of logical language, our solution has clear advantages in dealing with bargaining situations with complicated relations among bargaining issues and bargaining problems in which quantitative analysis are hard to apply. Such bargaining situations are common in legal bargaining, labor bargaining, political bargaining and family bargaining.

As one of the frameworks for conflict resolution, this work has a close relationship with the axiomatic models of belief revision, belief merging and belief arbitration. In fact, the axioms *Consistency* and *Collective rationality* are fundamental assumptions for all these operations [4,9,15,18]. The idea of simultaneous concessions is also applicable to belief merging.[18] Nevertheless, the differences between bargaining and belief merging are obvious. For merging, the information sources are passive therefore we can optimize the outcome of merging. However, in a bargaining, bargainers take initiative. The outcome of bargaining purely relies on bargainers' strategies. For example, assume a bargaining situation $((X_1, \succcurlyeq_1), (X_2, \succcurlyeq_2))$ as follows:

- $X_1 = \{p, q\}$ and $X_2 = \{p, \neg q\}$;
- $p \approx_1 q$ and $p \approx_2 \neg q$.

Then the bargaining ends with disagreement because no player is prepared to compromise. However, from the belief merging point of view, it is reasonable to assume that the merging of these two information sources contains p (for instance, Axiom A7 in [18]). In addition, a bargaining model can also partially encode players' strategic reasoning. Suppose that player 1 is risk-averse who is willing to give up q in order to keep p, i.e., $p \succ_1 q$; while player 2 is a risk-lover who creates a dummy demand $p \vee \neg p$ and adds it to her demand set in order to circumvent a compulsory concession, i.e., $p \approx_2 \neg q \succ_2 p \vee \neg p$. The outcome of bargaining will then be $(\{p\}, \{p, \neg q\})$. Note that player 2 does not lose anything, which shows another difference between bargaining and belief merging (see footnote 18).

There have been a few other frameworks for negotiation or bargaining built upon, or in the spirit of, belief revision theory. Booth proposed a negotiation model based on multi-agent belief contraction (or social contraction) [2]. Zhang et al. introduced the idea of modeling negotiation as a process of mutual belief revision [45]. Meyer et al. discussed the logical properties of a negotiation model based on AGM theory [19,20]. However, none of these works provided an axiomatic model of negotiation with characterization of players' preference orderings.

This work is part of an ongoing project on logical analysis of bargaining reasoning. In [46], we proposed a bargaining model in which negotiation items were represented in propositional logic but players' preferences are represented numerically. Based on such a semi-quantitative model, we demonstrated that the Nash solution is still applicable if we apply lotteries (randomization) to the set of possible agreements. In [47], we removed the requirement of randomization and constructed a bargaining solution so that it can be characterized by a combination of logical axioms and game-theoretic

[18] For instance, if we impose the rule – "*a merging operation must not give preference to any sources*" – on a belief merging operator, then we can have an axiom saying that "*whenever two belief bases are logically inconsistent, none of them can be fully retained in the merging outcome*" (Axiom A4 in [15]).

axioms. In [44], we introduced a bargaining model in which bargaining items are described by logical statements and players' preferences represented in pre-orders. Based on such a purely qualitative model, a logic-based bargaining solution was constructed. In this paper, we developed an axiomatic model for the solution, thereby laying down a foundation for the logic-based cooperative theory of bargaining. Meanwhile, the logic-based axiomatic model sheds light on the study of logical analysis on strategic bargaining reasoning.

There are a number of challenging issues on bargaining reasoning that need to be addressed in strategic models (or non-cooperative models) of bargaining.[19] Firstly, the axiomatic model abstracts all the items that a player wants the final agreement to contain as the *demands* of the player. However, as we have seen in Example 1, some of these demands may represent the player's beliefs, goals or desired constraints. Identifying the roles of these specific demands in the player's strategic reasoning requires a model of the player and a model of the bargaining procedure. Secondly, a cooperative theory of bargaining assumes that the information about bargainers' preferences is available for bargaining analysis. However, from each player's point of view, such information is private and disclosure of the information would lose the players their bargaining power. The investigations on the possibility of one player manipulating a bargaining game using the knowledge of other players' private information also rely on a strategic model of bargaining. Finally, an interesting research direction for the future is to study the ways in which coalitions of players influence bargaining outcomes, especially for knowledge-based coalitions.

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Appendix A. Two classical game-theoretic bargaining solutions

To make the paper self-contained, we briefly review the basic facts of the two classical game-theoretic bargaining solutions: *the Nash solution* [23] and the *Kalai–Smorodinsky solution* [13]. For more details on these bargaining solutions, the reader is referred to [42].

In game theory, an n-player bargaining game is defined as a pair (S,d), where $S \subseteq \Re^n$ represents the feasible set that can be derived from possible agreements and $d \in S$ stands for the disagreement point. It is assumed that S is convex and compact. A *bargaining solution* f is a function that assigns to each bargaining game (S,d) a unique point of S, i.e., $f(S,d) \in S$. A bargaining solution N is the *Nash solution* if for any bargaining game (S,d), $N(S,d)$ is the maximizer of the product $\prod_{1 \leqslant i \leqslant n}(x_i - d_i)$ over S.[20]

Nash showed that a bargaining solution $f = N$ if and only if it satisfies the following axioms [23]:

- *Pareto optimality*: There is no $y \in S$ such that $y \geqslant f(S,d)$.
- *Symmetry*: If (S,d) is a symmetric game, then $f_i(S,d) = f_j(S,d)$ for all i, j.
- *Scale Invariance*: For any positive affine transformation $\tau = (\tau_1, \ldots, \tau_n)$, $\tau(f(S,d)) = f(\tau(S), \tau(d))$.
- *Independence of Irrelevant Alternatives*: If $S' \subseteq S$ and $f(S,d) \in S'$, then $f(S',d) = f(S,d)$.

A bargaining solution KS is the *Kalai–Smorodinsky solution* (KS-solution) if for any bargaining game (S,d), $KS(S,d)$ is the maximal point of S on the segment connecting d to $a(S,d)$, where $a_i(S,d) = \max\{x_i\colon\ x \in S \ \&\ x \geqq d\}$ for all i.

Kalai and Smorodinsky showed that a solution $f = KS$ for 2-person bargaining games if and only if it satisfies *Pareto optimality*, *Symmetry*, *Scale Invariance* as well as the following *Restricted Monotonicity* [13]:

- *Restricted Monotonicity*: If $S' \subseteq S$ and $a(S',d) = a(S,d)$, then $f(S,d) \geqq f(S',d)$.

In spite of the large number of other solutions that have been proposed in the literature, these two solutions are most influential in game theory (see [42] for a comprehensive survey).

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[20] By default, for any vector $x \in \Re^n$, x_i indicates the ith component of x.

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