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Enhanced qualitative probabilistic networks for resolving trade-offs ☆

Silja Renooij *, Linda C. van der Gaag

Department of Information and Computing Sciences, Utrecht University, P.O. Box 80.089, 3508 TB Utrecht, The Netherlands

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ABSTRACT

Qualitative probabilistic networks were designed to overcome, to at least some extent, the quantification problem known to probabilistic networks. Qualitative networks abstract from the numerical probabilities of their quantitative counterparts by using signs to summarise the probabilistic influences between their variables. One of the major drawbacks of these qualitative abstractions, however, is the coarse level of representation detail that does not provide for indicating strengths of influences. As a result, the trade-offs modelled in a network remain unresolved upon inference. We present an enhanced formalism of qualitative probabilistic networks to provide for a finer level of representation detail. An enhanced qualitative probabilistic network differs from a basic qualitative network in that it distinguishes between strong and weak influences. Now, if a strong influence is combined, upon inference, with a conflicting weak influence, the sign of the net influence may be readily determined. Enhanced qualitative networks are purely qualitative in nature, as basic qualitative networks are, yet allow for resolving some trade-offs upon inference.

1. Introduction

The formalism of *probabilistic networks* introduced in the 1980s [26], is an intuitively appealing formalism for capturing knowledge of complex problem domains along with the uncertainties involved. Associated with the formalism are powerful algorithms for reasoning with uncertainty in a mathematically correct way. These algorithms for probabilistic inference allow for causal reasoning, diagnostic reasoning as well as case-specific reasoning; probabilistic inference, however, is known to be NP-hard [7]. Applications of probabilistic networks can be found in areas such as (medical) diagnosis and prognosis, planning, monitoring, vision, and information retrieval (see, for example, [1,2,4,5,20,31]).

A probabilistic network basically is a concise representation of a joint probability distribution on a set of statistical variables. It consists of an acyclic directed graph encoding the relevant variables from a domain of application along with their probabilistic interrelationships. Associated with each variable is a set of conditional probability distributions describing the relationship of the variable with its predecessors in the graph. The first task in constructing a probabilistic network is to identify the important domain variables, their values, and their interdependencies. This knowledge is then modelled in a directed graph, referred to as the network's qualitative part. The final task is to obtain the probabilities that constitute the network's quantitative part. As (conditional) probability distributions are to be stated for each variable in the graph, the number of required probabilities can be quite large, even for small applications. While the construction of the qualitative part of a probabilistic network is generally considered feasible, its quantification is a far harder task. Probabilistic information available from literature or data is often insufficient or unusable, and domain experts have to be relied upon to assess the

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* Corresponding author.
E-mail addresses: silja@cs.uu.nl (S. Renooij), linda@cs.uu.nl (L.C. van der Gaag).

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required probabilities [14]. Unfortunately, experts are often uncomfortable with having to provide probabilities. Moreover, the problems of bias encountered when directly eliciting probabilities from experts are widely known [19]. The usually large number of probabilities required for a probabilistic network, as a consequence, tends to pose a major obstacle to their application [14,18].

To mitigate the quantification bottleneck to at least some extent, *qualitative probabilistic networks* have been introduced [34]. Qualitative networks in essence are qualitative abstractions of probabilistic networks. Like a probabilistic network, a qualitative network encodes variables and the probabilistic relationships between them in a directed graph. However, while the relationships between the represented variables are quantified by conditional probabilities in a probabilistic network, these relationships are summarised in its qualitative abstraction by qualitative signs capturing stochastic dominance. The probabilistic information captured by signs is more robust than exact numbers are and is more easily obtained from domain experts [10]. Elicitation methods to this end are being designed [33].

Originally, the benefits of using qualitative probabilistic networks included the complexity of inference: for reasoning with a qualitative probabilistic network, an efficient algorithm is available, based on the idea of propagating and combining these signs [11]. In practice, however, nowadays the complexity of probabilistic inference is less of a problem and interest in qualitative probabilistic networks has shifted more to the construction and validation phase of probabilistic networks for real-life application domains. As the assessment of the various probabilities required is a hard task, it is performed only when the probabilistic network's graph is considered robust. Now, by assessing signs for the influences modelled in the graph, a qualitative network is obtained that can be exploited for studying the projected probabilistic network's reasoning behaviour prior to the assessment of probabilities. Patterns of qualitative influences can also be used to recognise different types of causal interaction, such as the noisy-or, which greatly simplify the quantification effort [24]. In addition, qualitative signs can be used in several ways as constraints on the quantification. For example, by interpreting the signs as continuous subintervals of the probability interval, the constraints they impose on the conditional probability distributions involved can be used for stepwise quantification of a probabilistic network: once a conditional probability table for a certain variable is filled, the interval associated with all direct influences upon that variable can be tightened [28]. These semi-qualitative probabilistic networks can also include assessments based on probabilistic logic and credal sets [6]. More recently, the signs of qualitative probabilistic networks have been used to constrain the probabilities learned from small data sets [3,15,17]. At a somewhat higher level, the constraints imposed by qualitative influences can be used to bound the entire space of possible joint probability distributions over the network's variables [13]. Finally, the qualitative signs can be used for verifying monotonicity properties in a probabilistic network [32], and for explanation of the (qualitative) probabilistic network's reasoning processes [10]. Given the increasing variety of useful applications of qualitative probabilistic networks, it is important to derive as much information as possible from such networks.

Qualitative probabilistic networks, by their nature, have a coarse level of representation detail. Influential relationships between variables can be modelled as positive, negative, zero or ambiguous, but no indication of their strengths can be provided as in a quantified network. One of the major drawbacks of this coarse level of representation detail is the ease with which the ambiguous '?'-sign arises upon inference. Ambiguous signs typically arise from trade-offs. A qualitative network models a trade-off if two nodes in the network's digraph are connected by multiple parallel reasoning chains with conflicting signs. In the absence of a notion of strength of influences, qualitative networks do not provide for resolving such trade-offs. Inference with a qualitative network for a real-life domain of application, as a consequence, often introduces ambiguous signs. Moreover, once an ambiguous sign has been generated, it will spread throughout major parts of the network. Although not incorrect, ambiguous signs provide no information whatsoever about the influence of one variable on another and are therefore not very useful in practice.

Ambiguous results from inference can be averted by enhancing the formalism of qualitative probabilistic networks to provide for a finer level of representation detail. Roughly speaking, the finer the level of detail, the more trade-offs can be resolved during inference. The finer levels of detail, however, typically come at the price of a higher computational complexity of inference. The problem of trade-off resolution for qualitative networks has been addressed by various researchers and we detail the relation between their work and ours in the *Related work* section of this paper. In short, S. Parsons, for example, has introduced the concept of categorical influence, which is either an influence that serves to increase a probability to 1, or an influence that decreases a probability to 0, and thus serves to resolve any trade-off in which it is involved [25]. Parsons has also studied the use of order-of-magnitude reasoning in the context of qualitative probabilistic networks [25]. C.-L. Liu and M.P. Wellman have designed two methods for resolving trade-offs based upon the idea of reverting to numerical probabilities whenever necessary [23]. While only some trade-offs can be resolved by the use of categorical influences, the methods of Liu and Wellman provide for resolving any trade-off, but require the availability of a fully quantified probabilistic network.

To provide for qualitative trade-off resolution without resorting to numerical probabilities, we have designed an intuitively appealing formalism of *enhanced qualitative networks*. An enhanced qualitative probabilistic network differs from a basic qualitative network in that it introduces a notion of relative strength by distinguishing between strong and weak influences. The distinction between strong and weak influences is very intuitive and domain experts should have no problems providing and interpreting the associated signs. Now, if a trade-off is modelled in an enhanced network and the positive influence, for example, is known to be stronger than the conflicting negative one, we may upon inference conclude the net influence to be positive. Trade-off resolution during inference thus builds upon the idea that strong influences dominate over conflicting weak influences. To provide for inference with an enhanced network, we have generalised the sign-propagation algorithm for basic qualitative networks to deal with strong and weak influences. This generalisation is straightforward once we establish that the properties upon which the basic sign-propagation algorithm is based are also provided for in an enhanced network. The new inference algorithm takes into account that the effect of one variable on another diminishes as variables are further apart in the network's graph; it also takes into account that a variable may affect another variable along multiple pathways with differing strengths. To maintain the correct strengths of indirect influences, the algorithm has to do some additional bookkeeping, as a result of which it may become less efficient than the inference algorithm for basic qualitative networks.

The paper is organised as follows. In Section 2, we provide some preliminaries from the fields of probabilistic networks and qualitative networks to introduce our notational conventions. In Section 3, we present our new formalism of enhanced qualitative probabilistic networks. In Section 4, we detail various properties of enhanced networks, on which we build a new sign-propagation algorithm. Section 5 provides an example of inference with an enhanced qualitative probabilistic network and discusses some complexity issues concerning sign-propagation. Related work is reviewed in Section 6. The paper is rounded off with our conclusions and directions for future research in Section 7.

2. Preliminaries

In this section we briefly review probabilistic networks and their qualitative counterparts.

2.1. Probabilistic networks

A probabilistic network models a domain of application basically by representing, in a concise way, the joint probability distribution on the set of statistical variables relevant to the application domain [26]. A probabilistic network $B = (G, \Pr)$ encodes, in an acyclic directed graph $G = (V(G), A(G))$, these relevant variables along with their probabilistic interrelationships. Each node $A \in V(G)$ represents a statistical variable that can take one of a finite set of values. We assume a total order '$>$' on the values of a variable. Variables will be indicated by capital letters. We will restrict ourselves to binary-valued variables, where we write a to denote $A = \mathit{true}$ and $\bar{a}$ to denote $A = \mathit{false}$, with $a > \bar{a}$. As there is a one-to-one correspondence between variables and nodes, we will use the terms 'node' and 'variable' interchangeably.

The probabilistic relationships between the represented variables are captured by the digraph's set of arcs $A(G)$. Informally speaking, we take an arc $A \to B$ in G to represent an influential relationship between the variables A and B, designating B as the effect of cause A. Given an arc $A \to B$, node A is called a (immediate) predecessor of node B and node B is called a successor of node A. We write $\pi(A)$ to denote the set of all predecessors of node A in G, and $\pi^*(A)$ to denote the set of its ancestors; similarly, $\sigma(A)$ is used to denote the set of all successors of node A and $\sigma^*(A)$ to denote its descendants. Two variables A and B are said to be connected by a (simple) trail in G iff they are connected by a (simple) path in the underlying undirected graph of G. Absence of an arc between two variables in the digraph of a probabilistic network means that the variables do not influence each other directly and, hence, are (conditionally) independent. More formally, probabilistic independence can be read from the digraph by means of the d-separation criterion, which builds on the concept of blocking [26]. A trail between two variables is said to be *blocked* by the available evidence if it includes either an observed variable with at least one outgoing arc, or an unobserved variable with two incoming arcs and no observed descendants. Two variables are now said to be *d-separated* if all trails between them are blocked, in which case they are considered conditionally independent given the available evidence. A trail that is not blocked is called *active*. If an active trail connects two Markov-blanket neighbours (i.e. two variables sharing an arc or a common child), then the two variables are said to be *active neighbours*.

Associated with each variable $A \in V(G)$ in the network's digraph G is a set of conditional probability distributions $\Pr(A \mid \pi(A))$ that describe the strengths of the various dependences between A and its (immediate) predecessors. These (conditional) probabilities with each other provide all information necessary for uniquely defining a joint probability distribution on the network's variables: the probabilistic network $B = (G, \Pr)$ defines the distribution $\Pr$ on $V(G)$ with

$$\Pr\bigl(V(G)\bigr) = \prod_{A \in V(G)} \Pr\bigl(A \mid \pi(A)\bigr)$$

that respects the independences portrayed by the digraph G. Since a probabilistic network thus captures a unique joint probability distribution, it provides for computing any prior or posterior probability over its variables. To this end, various algorithms are available [22,26].

We now describe a piece of fictitious medical knowledge that will serve as our example domain of application throughout the paper.

Example 2.1. Our example application domain pertains to the effects of administering antibiotics on a patient and involves five statistical variables together with their interrelationships. Node A represents whether or not a patient has been taking antibiotics. Node T models whether or not the patient is suffering from typhoid fever, node D represents the presence or absence of diarrhoea in the patient, and node H represents whether or not the patient is dehydrated. Node F, to conclude, describes whether or not the composition of the bacterial flora in the patient's intestines has changed. Typhoid fever and a change in bacterial flora are the possible causes of diarrhoea. Diarrhoea, in turn, can cause dehydration. Antibiotics can cure

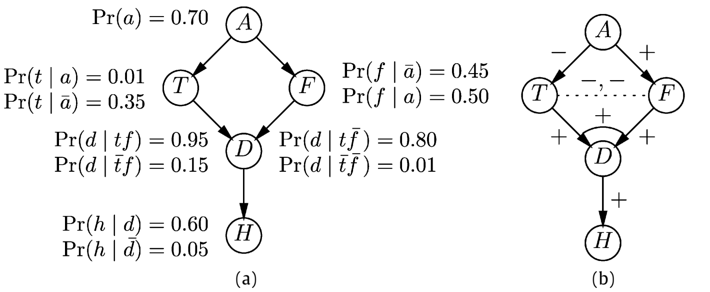

Fig. 1. The *Antibiotics* domain captured by (a) a probabilistic network, and (b) a qualitative probabilistic network.

typhoid fever by killing the bacteria that cause the infection. As a result, the probability of a patient contracting diarrhoea decreases. However, antibiotics can also change the composition of the intestinal bacterial flora, thereby increasing the risk of diarrhoea. Fig. 1(a) depicts the probabilistic network which captures the knowledge from our domain.

2.2. Qualitative probabilistic networks

Qualitative probabilistic networks bear a strong resemblance to their quantitative counterparts [34]. Instead of representing the joint probability distribution on the set of statistical variables relevant to the application domain, however, it only represents qualitative constraints on this distribution. A qualitative probabilistic network $Q = (G, \Delta)$ also comprises an acyclic digraph $G = (V(G), A(G))$ modelling variables and the probabilistic relationships between them. Moreover, the set of arcs $A(G)$ of this digraph again models probabilistic independence. Instead of conditional probability distributions, however, a qualitative probabilistic network associates with its digraph a set Δ of qualitative influences and qualitative synergies.

A *qualitative influence* between two variables expresses how the values of one variable influence the probabilities of the values of the other variable; the direction of the shift in distribution (i.e. do higher values become more likely or less likely) is indicated by the *sign* of the influence. A *positive qualitative influence* of a variable A on a variable B, for example, expresses that observing a higher value for A makes a higher value for B more likely, *regardless* of any other influences on B [34].

Definition 2.2. Let $G = (V(G), A(G))$ be an acyclic digraph and let Pr be a joint probability distribution on $V(G)$ that respects the independences in G. Let A, B be variables in G with $A \to B \in A(G)$. Then, variable A *positively influences* variable B along arc $A \to B$, written $S^+(A, B)$, iff

$$\Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \geqslant 0$$

for any combination of values x for the set $\pi(B) \setminus \{A\}$ of predecessors of B other than A.

A *negative qualitative influence*, denoted by S^-, and a *zero qualitative influence*, denoted by S^0, are defined analogously, replacing $\geqslant$ in the above formula by $\leqslant$ and $=$, respectively. If the influence of variable A on variable B is not monotonic or if it is unknown, we say that it is *ambiguous*, denoted $S^?(A, B)$.

With each arc in the digraph of a qualitative probabilistic network, a qualitative influence is associated. Variables, however, not only influence each other directly along arcs, they can also exert indirect influences on one another. The definition of qualitative influence trivially extends to indirect influences, that is, influences along active trails. We denote an indirect influence of sign δ along an active trail t from variable A to variable B by $\hat{S}^\delta(A, B, t)$. From here on, the term *trail* will be used to refer to either a simple trail, basically consisting of a *concatenation* of arcs, or to a subgraph containing all simple trails between two variables. The latter type of trail is said to consist of a *composition* of simple trails. The set of all variables on a trail t will be denoted $V(t)$.

The set of influences of a qualitative probabilistic network exhibits various convenient properties that constitute the basis for an efficient algorithm for qualitative probabilistic inference [34]. The property of *symmetry* guarantees that, if a network includes the influence $S^\delta(A, B)$, then it also includes $S^\delta(B, A)$ with the same sign $\delta \in \{+, -, 0, ?\}$. The property of *transitivity* asserts that qualitative influences along an active trail without head-to-head nodes, that is, without nodes with two incoming arcs on the trail, combine into an indirect influence whose sign is determined by the $\otimes$-operator from Table 1. The property of *composition* asserts that multiple qualitative influences between two variables along parallel active trails combine into a composite influence whose sign is determined by the $\oplus$-operator. From Table 1, we observe that combining non-ambiguous qualitative influences with the $\oplus$-operator can yield an ambiguous result. Such an ambiguity, in fact, results whenever parallel influences with opposite signs are combined. We say that the *trade-off* that is reflected by the conflicting influences cannot be *resolved*. Note that, in contrast with the $\oplus$-operator, the $\otimes$-operator cannot introduce ambiguities upon combining signs. The operators in Table 1 adhere to the standard algebraic properties of commutativity, associativity, and distributivity of $\otimes$ over $\oplus$.

In addition to influences, a qualitative probabilistic network includes synergies that model the interactions between triples of variables. An *additive synergy*, for example, captures the joint influence of two variables on a common succes-

Table 1
The ⊗- and ⊕-operators

$\otimes$	$+$	$-$	0	?		$\oplus$	$+$	$-$	0	?
$+$	$+$	$-$	0	?		$+$	$+$	?	$+$	?
$-$	$-$	$+$	0	?		$-$	?	$-$	$-$	?
0	0	0	0	0		0	$+$	$-$	0	?
?	?	?	0	?		?	?	?	?	?

sor [34]. A positive additive synergy of two variables A and B on a variable C, more specifically, expresses that the joint influence of A and B on C is greater than their separate influences.

Definition 2.3. Let $G = (V(G), A(G))$ be an acyclic digraph and let Pr be a joint probability distribution on $V(G)$ that respects the independences in G. Let A, B, C be variables in G with $A \to C, B \to C \in A(G)$. Then, variables A and B exhibit a *positive additive synergy* on C iff

$$\Pr(c \mid abx) + \Pr(c \mid \bar{a}\bar{b}x) - \Pr(c \mid a\bar{b}x) - \Pr(c \mid \bar{a}bx) \geqslant 0$$

for any combination of values x for the set $\pi(C) \setminus \{A, B\}$ of predecessors of C other than A and B.

Negative, *zero*, and *ambiguous additive synergies* are defined analogously.

If two variables A and B have a common successor C, then observation of a value for variable C serves to activate the trail $A \to C \leftarrow B$. The observation thus induces a dependence between A and B. This dependence can be represented by a qualitative influence of A on B, or vice versa. Such an induced influence is commonly known as an *intercausal influence*. The sign of the intercausal influence is captured by the sign of the *product synergy* associated with the variables involved and the observation. A product synergy thus expresses how the value of one variable influences the probabilities of the values of another variable in view of a given value for a third variable [12]. A negative product synergy of A and B on C with value c, for example, expresses that, given c, a high value for A renders a high value for B less likely; this reasoning pattern is known as *explaining away* [26].

Definition 2.4. Let $G = (V(G), A(G))$ be an acyclic digraph and let Pr be a joint probability distribution on $V(G)$ that respects the independences in G. Let A, B, C be variables in G with $A \to C, B \to C \in A(G)$. Then, variables A and B exhibit a *negative product synergy* on variable C with value c, denoted $X^-(\{A, B\}, c)$, iff

$$\Pr(c \mid abx)\cdot\Pr(c \mid \bar{a}\bar{b}x) - \Pr(c \mid a\bar{b}x)\cdot\Pr(c \mid \bar{a}bx) \leqslant 0$$

for any combination of values x for the set $\pi(C) \setminus \{A, B\}$ of predecessors of C other than A and B.

Positive, *zero*, and *ambiguous product synergies* again are defined analogously.

With each triple of variables A, B, C in $V(G)$ such that $A \to C$, $B \to C \in A(G)$, an additive synergy and two product synergies are associated. Note that a product synergy is defined for every possible value of C. These qualitative synergies are again trivially extended to trails and also exhibit symmetry, transitivity and composition properties. For details, we refer to [27,34].

Example 2.5. We consider the qualitative probabilistic network representation of the *Antibiotics* domain in Fig. 1(b). The figure displays the signs of the qualitative influences along the arcs, of the additive synergy over the curve over node D, and of the product synergies over the dotted edge.

The qualitative influence $S^-(A, T)$ specifies that the difference in conditional probabilities $\Pr(t \mid a) - \Pr(t \mid \bar{a})$ should be zero or less. Indeed, from the conditional probabilities specified for the variable T in the probabilistic network representation of the domain in Fig. 1(a), we have that

$$\Pr(t \mid a) - \Pr(t \mid \bar{a}) = 0.01 - 0.35 = -0.34 \leqslant 0$$

Similar observations hold for $S^+(A, F)$ and $S^+(D, H)$. From the conditional probabilities specified for the variable D, we have that

$$\Pr(d \mid tf) - \Pr(d \mid \bar{t}f) = 0.95 - 0.15 = 0.80 \geqslant 0, \quad \text{and}$$

$$\Pr(d \mid t\bar{f}) - \Pr(d \mid \bar{t}\bar{f}) = 0.80 - 0.01 = 0.79 \geqslant 0$$

which indeed obey the constraints posed by $S^+(T, D)$. We similarly find that the qualitative influence $S^+(F, D)$ is preserved in the quantified network. In both networks, the variables T and F exert a positive additive synergy $Y^+(\{T, F\}, D)$ on variable D:

$$\Pr(d \mid tf) + \Pr(d \mid \bar{t}\bar{f}) - \Pr(d \mid \bar{t}f) - \Pr(d \mid t\bar{f}) = 0.01 \geqslant 0$$

```
procedure PropagateObservation(Q, O, sign, Observed):
    for each V_i ∈ V(G)
    do sign[V_i] ← '0';
    PropagateSign(∅, O, sign).

procedure PropagateSign(trail, to, messagesign):
    sign[to] ← sign[to] ⊕ messagesign;
    trail ← trail ∪ {to};
    for each active neighbour V_i of to given {O} ∪ Observed
    do linksign ← sign of (induced) influence between to and V_i;
        messagesign ← sign[to] ⊗ linksign;
        if V_i ∉ trail and sign[V_i] ≠ sign[V_i] ⊕ messagesign
        then PropagateSign(trail, V_i, messagesign).
```

Fig. 2. The sign-propagation algorithm for qualitative probabilistic inference.

Either value for D, in addition, indeed induces a negative intercausal influence between the variables T and F in the probabilistic network representation. For example, we have that

$$\Pr(\bar{d} \mid tf) \cdot \Pr(\bar{d} \mid \bar{t}\bar{f}) - \Pr(\bar{d} \mid \bar{t}f) \cdot \Pr(\bar{d} \mid t\bar{f}) = -0.12 \leqslant 0$$

For reasoning with a qualitative probabilistic network, an efficient algorithm is available from M.J. Druzdzel and M. Henrion [11]; this algorithm, termed the *sign-propagation algorithm*, is summarised in pseudocode in Fig. 2. The basic idea of the algorithm is to trace the effect of observing a variable's value on the probabilities of the values of all other variables in the network by message-passing between neighbouring nodes. In essence, the algorithm computes the sign of the net influence along all active trails between the newly observed variable and the other variables in the network, building upon the properties of symmetry, transitivity and composition of influences. For each variable, it summarises the net influence in a *node-sign* that indicates the direction of the shift in the variable's probability distribution that is occasioned by the new observation.

The sign-propagation algorithm takes for its input a qualitative probabilistic network Q, a set *Observed* of previously observed variables, the variable O for which an observation has become available, and the sign *sign* of the new observation, that is, either a '$+$' for the value *true* or a '$-$' for the value *false*. Prior to the propagation of the new observation, for all variables V_i the node-sign sign[V_i] is set to '0'. For the newly observed variable O the appropriate sign is now entered into the network. The observed variable updates its node-sign to the sign-sum of its original sign and the entered sign. It thereupon reports its change of sign to all its *active* neighbours, that is to all variables in its Markov-blanket that can be reached through a trail not blocked by the set *Observed*. This notification is done by passing to each of them a message containing an appropriate sign, which is the sign-product of the variable's current node-sign and the sign *linksign* of the influence associated with the arc or induced intercausal link it traverses. Each message further records its origin in the variable *trail*; this information is used to prevent messages being passed on to nodes that have already been visited on the same trail. Upon receiving a message, a variable *to* updates its node-sign to the sign-sum of its current node-sign sign[*to*] and the sign *messagesign* from the message it has just received. The variable then sends a copy of the message to all *its* neighbours that need to reconsider their node-signs. In doing so, the variable changes the sign in each copy to the appropriate sign and adds itself to *trail* as the origin of the copy. Note that as this process is repeated throughout the network, the trails along which messages have been passed are recorded. Also note that as messages travel simple trails only, it is sufficient to just record the nodes on these trails.

During sign-propagation, variables are only visited if they need a change of node-sign. A node-sign can change at most twice, once from '0' to '$+$', '$-$' or '?', and then only from '$+$' or '$-$' to '?'. From this observation we have that no variable is ever visited more than twice upon inference. The algorithm is therefore guaranteed to halt. For a proof of the algorithm's correctness we refer the reader to [11].

We illustrate the sign-propagation algorithm by means of our running example.

Example 2.6. We consider once again the qualitative *Antibiotics* network from Fig. 1(b). Suppose that a patient is taking antibiotics. This observation is entered into the network by updating the node-sign of the variable A to '$+$'. Variable A thereupon propagates a message, with sign $+ \otimes - = -$, towards T. Variable T updates its node-sign to '$-$' and sends a message with sign $- \otimes + = -$ to D. Variable D updates its sign to '$-$' and sends a message with sign $- \otimes + = -$ to H. Variable H updates its node-sign to '$-$'; it sends no messages as it has no neighbours that need to update their sign. Variable D does not pass on a sign to F, since the trail from T via D to F is not active.

Variable A also sends a message, with sign $+ \otimes + = +$, to F. Variable F updates its node-sign accordingly and passes a message with sign $+ \otimes + = +$ to D. Variable D thus receives the additional sign '$+$'. This sign is combined with the previously updated node-sign '$-$', which results in the ambiguous node-sign $- \oplus + = ?$ for D. Note that the ambiguous sign arises from the trade-off represented for variable D. D now sends a message with sign $? \otimes + = ?$ to H, which updates its sign to $? \oplus - = ?$. Note that, had the network contained additional variables beyond the variables D and/or H, then these variables would have all ended up with the node-sign '?' after inference.

3. The enhanced formalism

Qualitative probabilistic networks capture the knowledge from a problem domain at a coarse level of representation detail. Qualitative influences between variables, for example, are captured by simple signs without any indication of their strengths. As a consequence, any trade-off encountered during inference will remain unresolved. In this section, we present a new formalism for qualitative probabilistic networks that allows for a finer level of representation detail, which will enable the resolving of trade-offs to at least some extent. In this new formalism, we enhance qualitative probabilistic networks by associating an indication of relative strength with their influences. Now, if, for example, upon encountering a trade-off during inference, the positive influence is known to be stronger than the conflicting negative one, then we may conclude the combined influence to be positive, thereby effectively resolving the trade-off.

In an *enhanced qualitative probabilistic network*, we distinguish between strong and weak influences. Intuitively, a strong influence of a variable A on a variable B is an influence that is stronger than any weak influence in the network, that is, the property

$$\left|\Pr(b \mid ax) - \Pr(b \mid \bar{a}x)\right| \geqslant \left|\Pr(d \mid cy) - \Pr(d \mid \bar{c}y)\right|$$

holds for all variables C and D with a weak influence between them, for any combination of values x and y for the sets X and Y of relevant predecessors. The basic idea now is to partition the set of all direct influences in a network into disjoint sets in such a way that any influence from the one subset is stronger than any influence from the other subset. To this end, we introduce a *cut-off value* α that serves to partition the set of direct qualitative influences into a set of influences that capture an absolute difference in probabilities of at least α and a set of influences that model an absolute difference of at most α. An influence from the former subset will be termed a *strong* influence; an influence from the latter subset will be termed a *weak* influence.

Definition 3.1. Let $G = (V(G), A(G))$ be an acyclic digraph and let Pr be a joint probability distribution on $V(G)$ that respects the independences in G. Let A, B be variables in G with $A \to B \in A(G)$. Let $\alpha \in [0, 1]$ be a cut-off value. The influence of variable A on variable B along arc $A \to B$ is *strongly positive*, denoted $S^{++}(A, B)$, iff

$$\Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \geqslant \alpha$$

for any combination of values x for the set $\pi(B) \setminus \{A\}$ of predecessors of B other than A. The influence of variable A on variable B along the arc is *weakly positive*, denoted $S^{+}(A, B)$, iff

$$0 \leqslant \Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \leqslant \alpha$$

for any combination of values x.

Strongly negative qualitative influences, denoted S^{--}, and *weakly negative qualitative influences*, denoted S^{-}, are defined analogously; *zero qualitative influences* and *ambiguous qualitative influences* are defined as for basic qualitative probabilistic networks. In the special case where, for an influence of a variable A on a variable B, the difference between the probabilities $\Pr(b \mid ax)$ and $\Pr(b \mid \bar{a}x)$ equals α for *all* x, we take the influence to be strong.

A product synergy is defined to be *strongly negative* if it induces a strongly negative intercausal influence. *Weakly negative*, *strongly positive*, and *weakly positive product synergies* are defined analogously; *zero product synergies* and *ambiguous product synergies* again are defined as for basic qualitative networks. For additive synergies, the distinction between weak and strong is slightly more complicated. Since additive synergies are not used during sign-propagation and therefore do not contribute to the resolution of trade-offs, we will not consider them any further in this paper.

Upon abstracting a quantified probabilistic network to an enhanced qualitative probabilistic network, the cut-off value α would need to be chosen explicitly. This cut-off value will typically vary from application to application, but it is always possible to choose such a cut-off value, since the values $\alpha = 0$ or $\alpha = 1$ yield a trivial partitioning of the set of influences. In real-life applications of enhanced qualitative probabilistic networks, however, the cut-off value need *not* be established explicitly. The partitioning into strong and weak influences is then elicited directly from the domain experts involved in the construction of the network.

Example 3.2. We consider once again Fig. 1 showing the qualitative and quantitative probabilistic network representations of the *Antibiotics* domain from Example 2.1. An enhanced qualitative probabilistic network representation of the domain is given in Fig. 3, showing just the qualitative influences involved. In addition to these qualitative influences, we have a strongly negative product synergy of variables T and F on $D = d$, and a weakly negative product synergy for $D = \bar{d}$.

We note that the basic signs of the qualitative influences in the enhanced representation are consistent with the signs of the qualitative influences in the basic qualitative network representation in Fig. 1(b). We will now show that the enhanced representation is also consistent with the completely quantified representation of Fig. 1(a), if we choose for our cut-off value $\alpha = 0.30$.

From $S^{--}(A, T)$ we conclude that $\Pr(t \mid a) - \Pr(t \mid \bar{a})$ should be negative with an absolute value of at least $\alpha = 0.30$; indeed, we have that

$$\Pr(t \mid a) - \Pr(t \mid \bar{a}) \leqslant 0, \quad \text{and} \quad \left|\Pr(t \mid a) - \Pr(t \mid \bar{a})\right| = 0.34 \geqslant \alpha$$

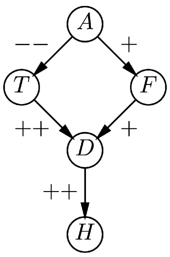

Fig. 3. The enhanced qualitative *Antibiotics* network.

We similarly find that the conditional probabilities associated with variables F and D are consistent with $S^{+}(A, F)$ and $S^{++}(D, H)$, respectively. For the influence of variable T on variable D, we observe that $\Pr(d \mid tF) - \Pr(d \mid \bar{t}F) \geqslant 0$, regardless of the value of F, as well as

$$\Pr(d \mid tf) - \Pr(d \mid \bar{t}f) = 0.80, \quad \text{and} \quad \Pr(d \mid t\bar{f}) - \Pr(d \mid \bar{t}\bar{f}) = 0.79$$

which both exceed the level of the cut-off value α. We therefore indeed have that $S^{++}(T, D)$; we similarly find that $S^{+}(F, D)$. The signs of the product synergies exhibited by the variables T and F on variable D, in the presence of a value for D, equal the signs of the corresponding intercausal influences. The intercausal influences are defined in terms of differences between $\Pr(f \mid tx)$ and $\Pr(f \mid \bar{t}x)$, where x represents different combinations of values for the variables D and A. These probabilities can be found from the network in Example 2.1 by applying Bayes' rule; we list them here for ease of reference:

	$\Pr(f \mid tx)$	$\Pr(f \mid \bar{t}x)$
$x = da$	0.54	0.94
$x = d\bar{a}$	0.52	0.92
$x = \bar{d}a$	0.20	0.49
$x = \bar{d}\bar{a}$	0.17	0.41

For the sign of the intercausal influence of variable T on variable F given the value d for D, we now have that

$$\Pr(f \mid tda) - \Pr(f \mid \bar{t}da) = -0.40 \leqslant -\alpha, \quad \text{and}$$

$$\Pr(f \mid td\bar{a}) - \Pr(f \mid \bar{t}d\bar{a}) = -0.40 \leqslant -\alpha$$

We conclude that the intercausal influence, and therefore its corresponding product synergy, is indeed strongly negative: $X^{--}(\{T, F\}, d)$; we similarly confirm that $X^{-}(\{T, F\}, \bar{d})$.

In our enhanced formalism, the semantics of the sign of an influence has slightly changed: while in a basic qualitative probabilistic network, the sign of an influence represents the sign of differences in probability only, in an enhanced qualitative network a sign in addition captures the relative magnitude of the differences. These relative magnitudes should be correctly preserved upon inference, when the indirect influences between variables are considered. In fact, we will demonstrate in Sections 4.2 and 4.3 that whereas the strength of a direct influence is defined relative only to cut-off value α, the strengths of indirect influences can be described in terms of a *polynomial* expression in α. To capture such a polynomial, we introduce a *multiplication-index list* and augment the signs of indirect influences with such a list. Before defining our multiplication-index list and the augmented signs, we observe that in general an n-th order polynomial in α can be written as

$$\sum_{i=0}^{n} c_i \cdot \alpha^i = \sum_{i=0,\ldots,n:\ c_i>0} c_i \cdot \alpha^i - \sum_{i=0,\ldots,n:\ c_i<0} -c_i \cdot \alpha^i$$

For our purposes, it suffices to consider polynomials in α with coefficients $c_i \in \mathbb{Z}$ and $c_0 \geqslant 0$. Since all exponents i are non-negative, we can represent any such polynomial by listing each exponent i $|c_i|$ times together with an indication of whether the associated term should be added or subtracted. This list of, possibly negated, exponents constitutes our multiplication-index list.

Definition 3.3. A *multiplication-index list* I is a multiset $\{i_1, \ldots, i_n\}$, $n \geqslant 1$, where each *index* $i_j \in I$, $j = 1, \ldots, n$, is an integer. The multiplication-index list I captures the polynomial in α

$$\sum_{i_j \geqslant 0} \alpha^{i_j} - \sum_{i_j < 0} \alpha^{-i_j}$$

A polynomial in α captured by multiplication-index list I will in short be denoted by $[\alpha]^I$.

As an example, consider the polynomial $[\alpha]^I = -2 \cdot \alpha^3 + 1$, which can be written as $-\alpha^3 - \alpha^3 + \alpha^0$. The multiset $\{-3, -3, 0\}$ defines a multiplication-index list I that captures this polynomial.

A sign augmented with a multiplication-index list is now displayed by attaching the multiplication-index list to it as a superscript, omitting the curly braces for the sake of readability. For example, a weakly positive sign with multiplication-index list $I = \{i_1, \ldots, i_n\}$ is written as $+^{i_1,\ldots,i_n}$, or $+^I$ for short. The following definition formally describes the meaning of an indirect influence with such an augmented sign.

Definition 3.4. Let $G = (V(G), A(G))$ be an acyclic digraph in which the variables A and B are connected by an active trail t. Let Pr be a joint probability distribution on $V(G)$ that respects the independences in G. Let $\alpha \in [0, 1]$ be a cut-off value. The influence of variable A on variable B along trail t is *strongly positive with multiplication-index list I*, denoted $\hat{S}^{++^I}(A, B, t)$, iff

$$\Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \geqslant [\alpha]^I \geqslant 0$$

for every combination of values x for the subset $X = (\bigcup_{C \in V(t) \setminus \{A\}} \pi(C) \setminus V(t))$ of relevant ancestors of B. The influence of A on B along t is *weakly positive with multiplication-index list I*, denoted $\hat{S}^{+^I}(A, B, t)$, iff

$$0 \leqslant \Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \leqslant [\alpha]^I$$

for every combination of values x for X.

Strongly and weakly negative influences with a multiplication-index list are again defined analogously. Zero and ambiguous influences are once more defined as in basic qualitative probabilistic networks and are not augmented with multiplication indices.

We would like to remark that the multiplication-index list we introduce is used to augment signs in an enhanced network during inference only. The list is used solely for the purpose of computation and, although possible, we do not intend to output signs augmented with these indices to the user.

4. Enabling inference in an enhanced network

For inference with a basic qualitative probabilistic network, an efficient algorithm is available. We recall from Section 2 that this algorithm builds on the idea of propagating signs throughout a network and combining them with the $\otimes$- and $\oplus$-operators. We further recall that the algorithm thereby exploits the properties of symmetry, transitivity, and parallel composition of influences. In this section we generalise the idea of sign-propagation to inference with an enhanced qualitative probabilistic network by taking into account the strength of influences. Upon initiating inference, the signs of the influences associated with the arcs of the digraph of an enhanced network are now interpreted as having a single multiplication index equal to 1. In Section 4.1, we address the property of symmetry, followed by a discussion and enhancement of the $\otimes$- and $\oplus$-operators to provide for the properties of transitivity and parallel composition of strong and weak influences in Sections 4.2 and 4.3, respectively.

4.1. The property of symmetry

In a basic qualitative probabilistic network, the property of symmetry guarantees that, if a variable A exerts an influence on a variable B, then variable B exerts an influence *of the same sign* on variable A. As a result, signs can be propagated during inference over an arc in both directions. In an enhanced qualitative network, as in a basic qualitative network, an influence and its reverse are both positive, both negative, both zero, or both ambiguous. The symmetry property, however, does not hold with regard to the strength of an influence: the reverse of a strongly positive qualitative influence, for example, may be a weakly positive influence, and vice versa. There are two ways of ensuring, in an enhanced network, that during inference signs can be propagated in both directions of an arc:

- elicit the signs of all influences against the direction of an arc explicitly;
- alternatively, use positive and negative signs of ambiguous strength, that is, signs whose strength is unknown and may be anywhere between 0 and 1.

Both alternatives have their benefits and drawbacks. The latter option is quite straightforward, since the symmetric counterpart of any positive influence, for example, would be an ambiguously positive influence, which we represent by $+^0$. However, upon using such signs of unknown strength, much useful information is lost and we therefore opt for explicitly specifying the signs of influences against the arc directions. Upon explicitly specifying these signs, however, care has to be taken that the two signs specified for an arc are consistent. For example, without going into details, we can derive the following from the definition of qualitative influence and its property of symmetry: for an arc $A \to B$, predecessors X of A and predecessors Y of B other than A, we have that if $\Pr(a \mid xy)$ lies inbetween $\Pr(b \mid xy)$ and $\Pr(\bar{b} \mid xy)$, then the qualitative influence of B on A is necessarily stronger than the qualitative influence of A on B, otherwise it is weaker. In the former case, $S^{+}(B, A)$, for example, would be inconsistent with $S^{++}(A, B)$.

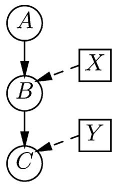

Fig. 4. A fragment of a network.

With respect to intercausal influences we note that since they can be regarded as a qualitative influence, the above observations also hold with respect to the signs of such influences.

4.2. The property of transitivity

For propagating qualitative signs along active trails in an enhanced qualitative probabilistic network, we have to enhance the $\otimes$-operator that is defined for this purpose for basic qualitative networks, to apply to strong and weak influences. We recall that the $\otimes$-operator provides for multiplying signs of influences. In a basic qualitative probabilistic network, an influence in essence captures a difference between two probabilities. Combining two influences with the property of transitivity then amounts to determining the sign of the product of two such differences. In our formalism of enhanced qualitative probabilistic networks, however, we have associated an explicit notion of strength with influences. It will be evident that these strengths need to be taken into consideration when multiplying signs with the $\otimes$-operator.

To address the sign-product of two signs in an enhanced qualitative probabilistic network, we consider the network fragment shown in Fig. 4. The fragment includes an (active) trail that is composed of the variables A, B, C, and two qualitative influences between them. In addition, X denotes the set of all predecessors of B other than A, and Y is the set of all predecessors of C other than B. The following lemma now indicates that the strength of the indirect influence of A on C along the given trail equals the product of the strengths of the influences of A on B and of B on C.

Lemma 4.1. *Let $G = (V(G), A(G))$ be an acyclic digraph where $A, B, C \in V(G)$ and $A \to B, B \to C$ is the only active trail between the variables A and C. Let $\Pr$ be a joint probability distribution on $V(G)$ that respects the independences in G. Then,*

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) = \big(\Pr(c \mid by) - \Pr(c \mid \bar{b}y)\big) \cdot \big(\Pr(b \mid ax) - \Pr(b \mid \bar{a}x)\big)$$

for any combination of values x for the set of variables $X = \pi(B) \setminus \{A\}$ and any combination of values y for the set $Y = \pi(C) \setminus \{B\}$.

Proof. We observe that, in G, variable C is independent of the variables A and X, given B and Y; in addition, variable B is independent of variable Y, given A and X. By conditioning on B we now find

$$\begin{aligned}
\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) &= \Pr(c \mid abxy) \cdot \Pr(b \mid axy) + \Pr(c \mid a\bar{b}xy) \cdot \Pr(\bar{b} \mid axy) \\
&\quad - \Pr(c \mid \bar{a}bxy) \cdot \Pr(b \mid \bar{a}xy) - \Pr(c \mid \bar{a}\bar{b}xy) \cdot \Pr(\bar{b} \mid \bar{a}xy) \\
&= \big(\Pr(c \mid by) - \Pr(c \mid \bar{b}y)\big) \cdot \Pr(b \mid ax) + \Pr(c \mid \bar{b}y) \\
&\quad - \big(\Pr(c \mid by) - \Pr(c \mid \bar{b}y)\big) \cdot \Pr(b \mid \bar{a}x) - \Pr(c \mid \bar{b}y) \\
&= \big(\Pr(c \mid by) - \Pr(c \mid \bar{b}y)\big) \cdot \big(\Pr(b \mid ax) - \Pr(b \mid \bar{a}x)\big) \qquad \square
\end{aligned}$$

Similar lemmas hold for the strengths of the influences along any other possible active trail between the variables A and C that can be obtained by reversing one or both arcs in Fig. 4 without introducing a head-to-head node on the trail. The lemma can further be easily extended to apply to the situation where A and B, and B and C, respectively, are connected by indirect active trails rather than direct arcs. We would like to note that the existence of additional (parallel) active trails between the variables A and C is handled by the $\oplus$-operator, and is therefore disregarded here.

The differences $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy)$ for the various combinations of values xy serve to indicate the strength of the indirect influence of variable A on variable C. We informally investigate these differences using the property stated in Lemma 4.1. Suppose that the qualitative influences of A on B and of B on C both are strongly positive, that is, we have $S^{++}(A, B)$ and $S^{++}(B, C)$. Let α be the cut-off value used for distinguishing between strong and weak influences. From the expression stated in the lemma, we now find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geqslant \alpha \cdot \alpha = \alpha^2$$

for any combination of values xy for the set of variables $X \cup Y$. Since $\alpha \leqslant 1$, we have that $\alpha^2 \leqslant \alpha$. Upon multiplying the signs of two strong direct influences, therefore, a sign results that indicates an indirect influence that is not necessarily stronger than a weak direct influence. Similar observations apply to strongly negative influences. Now suppose that both qualitative influences in the network fragment from Fig. 4 are weakly positive, that is, we have $S^+(A, B)$ and $S^+(B, C)$. For the indirect influence of variable A on variable C, we then find that

$$0 \leqslant \Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \leqslant \alpha \cdot \alpha = \alpha^2$$

for any combination of values xy. Similar observations apply to weakly negative influences. While the indirect influence resulting from the product of two strong influences cannot be compared to a weak direct influence, we have from the above observation that this indirect influence is always at least as strong as an indirect influence that results from the product of two weak influences. Finally, suppose that one qualitative influence in the network fragment from Fig. 4 is weakly positive and that the other is strongly positive, for example, $S^+(A, B)$ and $S^{++}(B, C)$. We then find for the indirect influence of variable A on variable C that

$$0 = 0 \cdot \alpha \leqslant \Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \leqslant \alpha \cdot 1 = \alpha$$

for any combination of values xy. Similar observations apply to other combinations of weak and strong influences. We thus have that the strength of an indirect influence resulting from the product of a strong and a weak influence *is* comparable to the strength of a weak direct influence.

From the previous observations, we conclude that to provide for comparing indirect qualitative influences along different trails with respect to their strengths, as required for trade-off resolution, we have to preserve information concerning the number of times signs have been multiplied.

4.2.1. Enhancing the $\otimes$-operator

We employ the *multiplication-index list*, defined in Section 3, to retain information about the strengths of signs which have been multiplied. To be able to combine information from different multiplication-index lists, we now define a sum-operation on these lists.

Definition 4.2. Let I and J be two multiplication-index lists. Then the multiplication-index list $I + J$ is the multiset

$$\left\{\left(|i| + |j|\right) \cdot sgn(i) \cdot \mathrm{sgn}(j) \mid i \in I, j \in J\right\}$$

where $\mathrm{sgn} : \mathbb{Z} \to \{-1, 1\}$ is defined as

$$\mathrm{sgn}(z) = \begin{cases} 1 & \text{if } z \geqslant 0 \\ -1 & \text{otherwise} \end{cases}$$

As an example, consider the polynomials $[\alpha]^I$ with $I = \{1, 2\}$ and $[\alpha]^J$ with $J = \{1, -3\}$, then the multiplication-index list $I + J = \{2, 3, -4, -5\}$ actually captures the polynomial $[\alpha]^{I+J} = [\alpha]^I \cdot [\alpha]^J$:

$$[\alpha]^{I+J} = [\alpha]^I \cdot [\alpha]^J = \left(\alpha^1 + \alpha^2\right) \cdot \left(\alpha^1 - \alpha^3\right) = \alpha^{1+1} + \alpha^{2+1} - \alpha^{1+3} - \alpha^{2+3}$$

Table 2 now defines the enhanced $\otimes$-operator, which shapes the transitivity property for qualitative influences in an enhanced network. From the table, it is readily seen that the '+', '−', '0', and '?' signs in essence combine just as in a basic qualitative probabilistic network; the only difference is in the handling of the multiplication indices. The following proposition shows that the operator correctly captures the sign of the transitive combination of two weakly positive influences.

Proposition 4.3. *Let $Q = (G, \Delta)$ be an enhanced qualitative probabilistic network. Let A, B, and C be variables in G for which there exist an active trail t_1 from A to B and an active trail t_2 from B to C such that their concatenation $t_1 \circ t_2$ is an active trail from A to C. Let I and J be multiplication-index lists. Then,*

$$\hat{S}^{+^I}(A, B, t_1) \wedge \hat{S}^{+^J}(B, C, t_2) \quad \Longrightarrow \quad \hat{S}^{+^{I+J}}(A, C, t_1 \circ t_2)$$

Proof. Let Pr be a joint probability distribution on $V(G)$ that respects the independences in G, and let $\alpha \in [0, 1]$ be the cut-off value used for distinguishing between strong and weak influences. We will start by assuming that the multiplication-index lists I and J each consist of a single index i and j, respectively. Then, the weakly positive influence $\hat{S}^{+^I}(A, B, t_1)$ of variable A on variable B expresses that

$$0 \leqslant \Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \leqslant \alpha^i$$

Table 2
The enhanced $\otimes$-operator

$\otimes$	$++^J$	$+^J$	0	$-^J$	$--^J$	?
$++^I$	$++^{I+J}$	$+^J$	0	$-^J$	$--^{I+J}$	?
$+^I$	$+^I$	$+^{I+J}$	0	$-^{I+J}$	$-^I$	?
0	0	0	0	0	0	0
$-^I$	$-^I$	$-^{I+J}$	0	$+^{I+J}$	$+^I$	?
$--^I$	$--^{I+J}$	$-^J$	0	$+^J$	$++^{I+J}$	?
?	?	?	0	?	?	?

for every combination of values x for the set $X = \bigcup_{D \in V(t_1) \setminus \{A\}} \pi(D) \setminus V(t_1)$ of relevant ancestors of B. Similarly, the weakly positive qualitative influence $\hat{S}^{+^J}(B, C, t_2)$ of variable B on variable C expresses that

$$0 \leqslant \Pr(c \mid by) - \Pr(c \mid \bar{b}y) \leqslant \alpha^j$$

for every combination of values y for the set Y of relevant ancestors of C. For the indirect influence of variable A on variable C, we thus find from Lemma 4.1 that

$$0 \leqslant \Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \leqslant \alpha^i \cdot \alpha^j = \alpha^{i+j}$$

for every combination of values xy for the set $X \cup Y$. More in general, we observe that the strength of the resulting influence lies between 0 and the product of the polynomial expressions in α captured by the multiplication-index lists I and J, respectively. We therefore conclude that $\hat{S}^{+^{I+J}}(A, C, t_1 \circ t_2)$. □

From the above proposition and the appropriate entry in Table 2, we conclude that for two weakly positive influences the enhanced ⊗-operator indeed correctly captures the sign of their transitive combination. Similar observations hold for the transitive combination of any two weak influences or any two strong influences, be they positive or negative.

The following proposition shows that the operator in Table 2 correctly captures the sign of the transitive combination of a weakly positive and a strongly positive influence.

Proposition 4.4. *Let Q, A, B, C, t_1, t_2, $t_1 \circ t_2$, I and J be as in the previous proposition. Then,*

$$\hat{S}^{+^I}(A, B, t_1) \wedge \hat{S}^{++^J}(B, C, t_2) \quad \Longrightarrow \quad \hat{S}^{+^I}(A, C, t_1 \circ t_2)$$

Proof. Let Pr and α be as in the previous proof. We again start by assuming that the multiplication-index lists I and J each consist of a single index i and j, respectively. Then, the weakly positive influence $\hat{S}^{+^I}(A, B, t_1)$ of variable A on variable B expresses that

$$0 \leqslant \Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \leqslant \alpha^i$$

for every combination of values x for the set $X = \bigcup_{D \in V(t_1) \setminus \{A\}} \pi(D) \setminus V(t_1)$ of relevant ancestors of B. The strongly positive qualitative influence $\hat{S}^{++^J}(B, C, t_2)$ of variable B on variable C further expresses that

$$\alpha^j \leqslant \Pr(c \mid by) - \Pr(c \mid \bar{b}y) \leqslant 1$$

for every combination of values y for the set Y of relevant ancestors of C. For the indirect influence of variable A on variable C, we thus find from Lemma 4.1 that

$$0 \leqslant \Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \leqslant \alpha^i \cdot 1$$

for every combination of values xy for the set $X \cup Y$. More in general, we observe that the strength of the resulting influence lies between 0 and 1 times the polynomial expression in α captured by multiplication-index list I. We therefore conclude that $\hat{S}^{+^I}(A, C, t_1 \circ t_2)$. □

From the above proposition and the appropriate entry in Table 2, we conclude that for a weakly positive and a strongly positive influence the enhanced ⊗-operator indeed correctly captures the sign of their transitive combination. Similar observations hold for the transitive combination of any weak influence with any strong influence, be they positive or negative. The proofs for the signs of all other transitive combinations of influences stated in Table 2, are analogous to the proofs of Propositions 4.3 and 4.4.

4.3. The property of parallel composition

For combining multiple qualitative influences between two variables along parallel active trails in an enhanced qualitative probabilistic network, we have to enhance the ⊕-operator that is defined for this purpose for basic qualitative networks, to apply to strong and weak influences. We recall that the ⊕-operator provides for summing signs of influences. We further recall that, upon adding the signs of two conflicting influences during inference with a basic qualitative network, the represented trade-off cannot be resolved and an ambiguous influence results. In our formalism of enhanced qualitative probabilistic networks, we have associated an explicit notion of strength with influences. These strengths can now be taken into consideration when summing the signs of influences and can be used to resolve trade-offs. For example, if a trade-off is encountered during inference, and the negative influence is known to be stronger than the conflicting positive one, then we may conclude that the combined influence is negative, thereby forestalling ambiguous results.

Upon addressing the property of transitivity in the previous section, we have argued that the product of two influences may yield an indirect influence that is weaker than the influences it is built from. We will now see that the sum of two influences, in contrast, may result in a stronger influence. To address the sign-sum of two signs in an enhanced qualitative

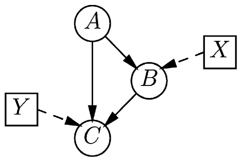

Fig. 5. Another network fragment.

probabilistic network, we consider the network fragment shown in Fig. 5. The fragment includes two active trails between the variables A and C, one of which captures a direct influence of A on C and the other one an indirect influence through B. In addition, the set X denotes the set of all predecessors of B other than A, and Y is the set of predecessors of C other than A and B. The following lemma now relates the strength of the net influence of variable A on variable C to the strengths of the influences it is built from.

Lemma 4.5. *Let $G = (V(G), A(G))$ be an acyclic digraph where $A, B, C \in V(G)$ and $A \to B$, $B \to C$ and $A \to C$ are the only active trails between the variables A and C. Let* Pr *be a joint probability distribution on $V(G)$ that respects the independences in G. Then,*

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) = \big(\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)\big) \cdot \Pr(b \mid ax) + \Pr(c \mid a\bar{b}y) - \big(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)\big) \cdot \Pr(b \mid \bar{a}x) - \Pr(c \mid \bar{a}\bar{b}y)$$

for any combination of values x for the set X of all predecessors of B other than A and any combination of values y for the set Y of all predecessors of C other than A and B.

Proof. The proof of the property stated in the lemma is similar to that of Lemma 4.1. □

Similar lemmas hold for the strengths of the net influences of A on C along other combinations of multiple parallel trails that can be obtained by reversing one or more arcs in Fig. 5, as long as both trails remain active. A similar lemma can also be formulated for situations where one or more of the arcs in Fig. 5 are replaced by active trails. We would like to note that the existence of additional parallel trails between the variables A and C is handled by repeated application of the composition property, and is therefore disregarded here.

The differences $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy)$ for the various combinations of values xy serve to indicate the sum of the strengths of the direct influence and the indirect influence of the variable A on the variable C. If all the arcs in the network fragment from Fig. 5 are associated with a weakly positive influence, for example, we find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \leqslant \alpha + \alpha^2$$

Building upon Lemma 4.5, we will prove this property shortly. From the inequality, we observe that the parallel composition of two weak influences of the same sign may result in a net influence that is stronger than a weak direct influence. Its relation to a strong influence is unknown, however. So, although the basic sign of the resulting influence is known unambiguously, its strength is not readily expressible as a simple power of α. Alternatively, if all the arcs in the network fragment from Fig. 5 are associated with a strongly positive influence, we find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geqslant \alpha + \alpha^2$$

and we observe that the parallel composition of two strong influences of the same sign results in a net influence that is slightly stronger than a strong direct influence.

From these observations we have that the parallel composition of two or more influences may result in an influence for which the strength cannot be expressed by a single power of the cut-off value α without losing information; rather, an entire polynomial in α is required. In the remainder of this section, we present two ways of capturing the strengths of parallel influences, by defining two different $\oplus$-operators for summing signs. The first *enhanced $\oplus$-operator* is discussed in Section 4.3.1 and keeps track of entire polynomials in α by means of the multiplication-index list. The second operator works on signs with single multiplication indices only. In the latter case, we minimise the amount of bookkeeping necessary during inference by discarding the higher-order terms of the polynomial expression in α, leaving a single α-term whose power can be taken as a single multiplication index for the resulting sign; if these higher-order terms cannot be discarded without introducing a possible error, a sign of unknown strength is yielded. This second operator is called the *simple enhanced operator $\oplus_s$* and is briefly described in Section 4.3.2. Obviously, application of the $\oplus_s$-operator can result in loss of available information when adding two signs upon inference.

4.3.1. The enhanced operator $\oplus$

We employ the multiplication-index list defined in Section 3 to retain information about the strengths of possibly conflicting signs which have been summed. To be able to compare the strengths of signs, as captured by their multiplication-index lists, we define three additional list operations.

Table 3
The enhanced $\oplus$-operator for signs with multiplication-index lists

$\oplus$	$++^J$	$+^J$	0	$-^J$	$--^J$	$?$
$++^I$	$++^{I\cup J}$	$++^I$	$++^I$	$a)$	$?$	$?$
$+^I$	$++^J$	$+^{I\cup J}$	$+^I$	$?$	$d)$	$?$
0	$++^J$	$+^J$	0	$-^J$	$--^J$	$?$
$-^I$	$b)$	$?$	$-^I$	$-^{I\cup J}$	$--^J$	$?$
$--^I$	$?$	$c)$	$--^I$	$--^I$	$--^{I\cup J}$	$?$
$?$	$?$	$?$	$?$	$?$	$?$	$?$

where
a) $++^{I\cup -J}$, if $I \leqslant J$; else ?,
b) $++^{-I\cup J}$, if $J \leqslant I$; else ?,
c) $--^{I\cup -J}$, if $I \leqslant J$; else ?,
d) $--^{-I\cup J}$, if $J \leqslant I$; else ?.

Definition 4.6. Let I and J be two multiplication-index lists and let $[\alpha]^I$ and $[\alpha]^J$ be the polynomials in α captured by lists I and J, respectively. Then,

- the multiplication-index list $-I$ is the multiset $\{-i \mid i \in I\}$;
- the multiplication-index list $I \cup J$ is the multiset $\{i \mid i \in I \text{ or } i \in J\}$;
- $I \leqslant J$ iff $[\alpha]^I - [\alpha]^J \geqslant 0$.

The above definition defines a negation operator '$-$' which negates each index in the list to which it is applied, and a union operator '$\cup$' which combines all elements of two multisets into a single multiset; the comparison operator $\leqslant$ captures the idea that in general lower order polynomials in $\alpha \in [0,1]$ correspond to stronger signs. As an example, consider the polynomials $[\alpha]^I$ with $I = \{1, -3\}$ and $[\alpha]^J$ with $J = \{2, -3\}$, then $-I = \{-1, 3\}$ captures the polynomial $[\alpha]^{-I} = -[\alpha]^I$, $I \cup J = \{1, 2, -3, -3\}$, and $I \leqslant J$, since $[\alpha]^I - [\alpha]^J = \alpha - \alpha^2 \geqslant 0$.

Table 3 now defines the enhanced $\oplus$-operator, which shapes the composition property for influences in an enhanced qualitative network. From the table, it is readily seen that the '+', '$-$', '0', and '?' signs combine as in a basic qualitative probabilistic network; the only difference is in the handling of the multiplication indices. The following four propositions show, for four different situations, that the operator correctly captures the sign of a combination of two parallel influences; the proofs for the other combinations of influences are quite similar. The first proposition pertains to the situation where two weakly positive influences along parallel trails are combined.

Proposition 4.7. *Let $Q = (G, \Delta)$ be an enhanced qualitative probabilistic network. Let A, C be variables in G and let t_1 and t_2 be parallel active trails in G from A to C, where $t_1 \parallel t_2$ is their trail composition. Let I and J be multiplication-index lists. Then,*

$$\hat{S}^{+^I}(A, C, t_1) \wedge \hat{S}^{+^J}(A, C, t_2) \quad \Longrightarrow \quad \hat{S}^{+^{I\cup J}}(A, C, t_1 \parallel t_2)$$

Proof. Let Pr be a joint probability distribution on $V(G)$ that respects the independences in G. Let $\alpha \in [0,1]$ be the cut-off value used for distinguishing between strong and weak influences. For ease of exposition, we assume that the trail t_1 consists of a single arc and that the trail t_2 consists of the arcs $A \to B$, $B \to C$ for some variable B, as in the network fragment of Fig. 5. Additional trails between A and C can be handled by repeated application of the composition property, and are therefore disregarded here. We recall that with each arc is associated an influence with multiplication index 1, so we have $I = \{1\}$. We further recall that Lemma 4.5 gives the net influence of variable A on variable C along the trail composition $t_1 \parallel t_2$. We now write the equation from Lemma 4.5 as the difference between two functions f and h:

$$\begin{aligned}
\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) &= \big[\big(\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)\big) \cdot \Pr(b \mid ax) + \Pr(c \mid a\bar{b}y)\big] \\
&\quad - \big[\big(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)\big) \cdot \Pr(b \mid \bar{a}x) + \Pr(c \mid \bar{a}\bar{b}y)\big] \\
&= f\big(\Pr(b \mid ax)\big) - h\big(\Pr(b \mid \bar{a}x)\big)
\end{aligned}$$

for all value combinations x and y for the set X of predecessors of B other than A and the set Y of predecessors of C other than A and B, respectively. We note that the functions f and h are both linear in their respective parameter.

We now assume that the positive influence along trail t_2 is composed of two separate positive influences. From the influence of variable B on variable C being positive, we have that the functions f and h are both linearly increasing, as depicted in Fig. 6; the fact that in the figure the gradient of the function f is larger than the gradient of the function h is an arbitrary choice. From the positive direct influence of variable A on variable C we further have that $f(0) \geqslant h(0)$ and $f(1) \geqslant h(1)$. We therefore have that the functions f and h do not intersect. If the two influences along trail t_2 are both negative, then the functions f and h are decreasing and similar observations apply.

To determine the sign of the composite influence of variable A on variable C, we have to consider the sign of the difference between the functions f and h. We observe that, although the functions f and h are expressed in terms of

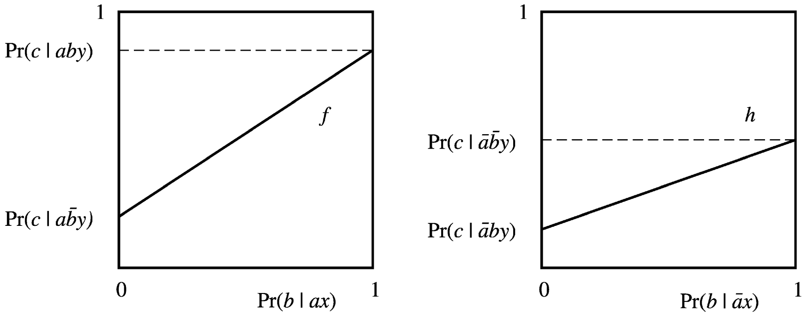

Fig. 6. Possible functions $f(\Pr(b \mid ax))$ and $h(\Pr(b \mid \bar{a}x))$.

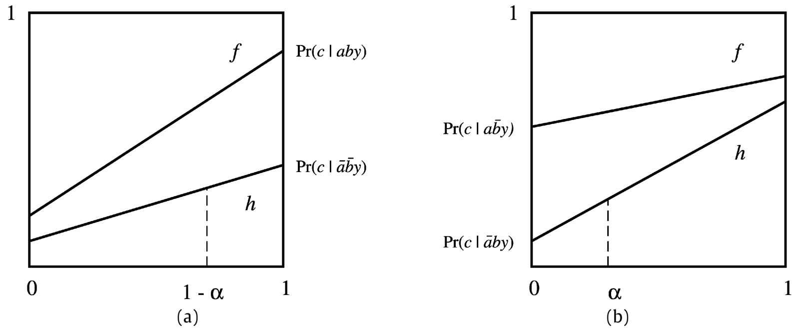

Fig. 7. The functions $f(\Pr(b \mid ax))$ and $h(\Pr(b \mid \bar{a}x))$ depicted in a single graph, with (a) *gradient*$(f) >$ *gradient*(h), and (b) *gradient*$(h) >$ *gradient*(f).

different parameters, these parameters cannot be varied independently as their difference is restricted by the sign of the qualitative influence of variable A on variable B. Under this constraint, we are allowed to compare the function values of f and h for different parameters. For ease of comparison, we have depicted for this purpose the two functions f and h in a single graph, in Fig. 7.

Since the positive indirect influence along trail t_2 is composed of two positive influences, we have three possible situations:

(1) $S^+(A, B)$ and $S^{++}(B, C)$, or
(2) $S^{++}(A, B)$ and $S^+(B, C)$, or
(3) $S^+(A, B)$ and $S^+(B, C)$.

In the first two situations we have $\hat{S}^{+^J}(A, C, t_2)$ with $J = \{1\}$; in situation (3) we find $\hat{S}^{+^J}(A, C, t_2)$ with $J = \{2\}$. Here, we only consider the latter situation; the proofs for the other two situations are quite similar. As the direct influence of the variable A on the variable B is weakly positive, we have that $0 \leqslant \Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \leqslant \alpha$. Therefore, when investigating the difference between the two functions f and h, we have to satisfy the following constraints:

- the parameter $\Pr(b \mid ax)$ for the function f should be greater than or equal to the parameter $\Pr(b \mid \bar{a}x)$ for the function h;
- the difference between the two parameters may not be greater than α.

We now show that under these constraints the difference $f(\Pr(b \mid ax)) - h(\Pr(b \mid \bar{a}x))$ is greater than or equal to zero. To this end, we consider the graph from Fig. 7(a); similar observations hold for the graph from Fig. 7(b). Under the given constraints, we have that the minimal difference between $f(\Pr(b \mid ax))$ and $h(\Pr(b \mid \bar{a}x))$ is attained for $f(0)$ and $h(0)$. We find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geqslant f(0) - h(0) = \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y)$$

The minimal difference is positive as a result of the direct influence of A on C being positive. The sign of the composite influence of variable A on variable C is therefore positive. The maximal difference between $f(\Pr(b \mid ax))$ and $h(\Pr(b \mid \bar{a}x))$ is attained for $f(1)$ and $h(1 - \alpha)$. Once again exploiting the information that the signs of the direct influences are all weakly positive, this difference equals:

$$\begin{aligned}\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) &\leqslant f(1) - h(1-\alpha) = \Pr(c \mid aby) - \Pr(c \mid \bar{a}\bar{b}y) - \big(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)\big) \cdot (1-\alpha) \\ &= \Pr(c \mid aby) - \Pr(c \mid \bar{a}by) + \alpha \cdot \big(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)\big) \leqslant \alpha + \alpha \cdot \alpha = \alpha + \alpha^2\end{aligned}$$

We conclude that the composite influence of variable A on variable C is weakly positive with multiplication-index list $\{1, 2\}$.

More in general, we find that the strength of the composite influence lies between zero and the sum of the two polynomials in α, captured by the multiplication-index lists I and J, respectively, that is, we conclude that the composite influence equals $\hat{S}^{+^{I \cup J}}(A, C, t_1 \parallel t_2)$. □

From the above proposition and the appropriate entry in Table 3, we conclude that for two weakly positive influences the enhanced $\oplus$-operator correctly captures the sign of their composition. Similar observations hold for the composition of two weakly negative signs.

The next proposition addresses the situation where two strongly positive influences along parallel trails are combined into a composite influence.

Proposition 4.8. *Let Q, A, C, t_1, t_2, $t_1 \parallel t_2$, I and J be as in the previous proposition. Then,*

$$\hat{S}^{++^{I}}(A, C, t_1) \wedge \hat{S}^{++^{J}}(A, C, t_2) \implies \hat{S}^{++^{I \cup J}}(A, C, t_1 \parallel t_2)$$

Proof. The proof proceeds in a similar fashion as the proof of Proposition 4.7; more details are provided in Appendix A. □

From the above proposition and the appropriate entry in Table 3, we conclude that for two strongly positive influences the enhanced $\oplus$-operator correctly captures the sign of their composition. Similar observations hold for the composition of two strongly negative signs.

The next proposition addresses the combination of a strongly positive and a weakly positive influence; its proof can again be found in Appendix A.

Proposition 4.9. *Let Q, A, C, t_1, t_2, $t_1 \parallel t_2$, I and J be as in the previous proposition. Then,*

$$\hat{S}^{++^{I}}(A, C, t_1) \wedge \hat{S}^{+^{J}}(A, C, t_2) \quad \implies \quad \hat{S}^{++^{I}}(A, C, t_1 \parallel t_2)$$

From the above proposition and the appropriate entry in Table 3, we deduce that for a weakly and a strongly positive influence the enhanced $\oplus$-operator correctly captures the sign of their composition. Similar observations hold for the composition of a strongly negative and a weakly negative influence.

The main reason for enhancing qualitative probabilistic networks with a notion of strength has been to provide for a finer level of representation detail that allows for resolving trade-offs upon inference. Trade-off resolution in essence amounts to associating an unambiguous basic sign with the composite influence that is built from two or more conflicting influences along parallel active trails. The next proposition provides for the combination of conflicting influences and describes the type of trade-off that can now typically be resolved.

Proposition 4.10. *Let Q, A, C, t_1, t_2, $t_1 \parallel t_2$, I and J be as in the previous proposition. Then, if $I \leqslant J$,*

$$\hat{S}^{++^{I}}(A, C, t_1) \wedge \hat{S}^{-^{J}}(A, C, t_2) \quad \Rightarrow \quad \hat{S}^{++^{I \cup -J}}(A, C, t_1 \parallel t_2)$$

Proof. Let Pr and α be as before. We again use the functions f and h as defined the proof of Proposition 4.7. Depending on the sign of the influence of variable B on variable C, we have that the functions f and h are either both linearly increasing, or linearly decreasing functions. We assume that the two functions are increasing, which implies that the influence of variable B on variable C is positive. We further assume that the gradient of the function f is larger than the gradient of the function h, as depicted in the graph from Fig. 7(a). Similar observations apply to the graph from Fig. 7(b), and to decreasing functions.

We now distinguish between the two cases (I) and (II) from the proof of Proposition 4.9:

(I) the trail t_1 consists of a single arc and the trail t_2 consists of the arcs $A \to B$, $B \to C$ for some variable B;
(II) the trail t_1 consists of the arcs $A \to B$, $B \to C$ and the trail t_2 consists of the single arc.

First we address case (I), with a strongly positive direct influence of variable A on variable C. From our assumptions we have that the indirect negative influence along trail t_2 is composed of a negative influence of A on B and a positive influence of B on C. More specifically, we have one of the following three situations:

(1) $S^{-}(A, B)$ and $S^{+}(B, C)$, or
(2) $S^{--}(A, B)$ and $S^{+}(B, C)$, or
(3) $S^{-}(A, B)$ and $S^{++}(B, C)$.

The indirect influence of variable A on variable C along trail t_2 has associated the sign $-^J$ with $J = \{2\}$ in situation (1) and with $J = \{1\}$ in the situations (2) and (3).

To establish the sign of the composite influence of A on C, we first establish the minimal difference between the functions f and h. We begin by considering the situations (1) and (3) described above. Since the influence of variable A on variable B is weakly negative, we have that the parameters $\Pr(b \mid ax)$ and $\Pr(b \mid \bar{a}x)$ for the functions f and h, respectively, have to satisfy the following constraints:

- the parameter $\Pr(b \mid ax)$ for function f is smaller than or equal to the parameter $\Pr(b \mid \bar{a}x)$ for function h;
- the difference between the two parameters is at most α.

From Fig. 7(a), we observe that under these constraints the minimal difference between f and h is attained for $f(0)$ and $h(\alpha)$. The minimal difference thus is

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geqslant f(0) - h(\alpha) = \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y) - \big(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)\big) \cdot \alpha$$

The difference between the first two terms is α or more, due to the strongly positive direct influence of A on C. The difference between the last two terms is captured by the influence of B on C, which is weakly positive in situation (1) and strongly positive in situation (3). In situation (1) we now have that $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geqslant \alpha - \alpha \cdot \alpha$; for situation (3) we find that $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geqslant \alpha - 1 \cdot \alpha = 0$.

We now consider the situation (2) described above. The strongly negative influence of variable A on variable B imposes the following constraints on the parameters for f and h:

- the parameter $\Pr(b \mid ax)$ for function f is smaller than the parameter $\Pr(b \mid \bar{a}x)$ for function h;
- the difference between the two parameters is at least α.

From Fig. 7(a), we observe that under these constraints the minimal difference between f and h is attained for $f(0)$ and $h(1)$:

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \;\geqslant\; f(0) - h(1) = \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y) - \big(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)\big)$$

We therefore have that $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geqslant \alpha - \alpha = 0$.

For all three situations (1), (2), and (3), the maximum difference between the functions f and h is attained for $f(1-\alpha)$ and $h(1)$. The maximum difference thus is

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \;\leqslant\; f(1-\alpha) - h(1) = \Pr(c \mid aby) - \Pr(c \mid \bar{a}by) - \big(\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)\big) \cdot \alpha$$

We find that the maximum difference is at most 1 in situations (1) and (2), and $1-\alpha$ in situation (3). We conclude that in case (I), the composite influence of variable A on variable C is positive and at least $\alpha^I - \alpha^J$, that is, $\hat{S}^{++^{I \cup -J}}(A, C, t_1 \parallel t_2)$.

We now address case (II). Since the indirect influence along trail t_1 is strongly positive, it must be composed of two strong direct influences. Recall that we assume that the influence of variable B on variable C is positive, hence both the strong influences are positive, that is,

$$S^{++}(A, B) \quad \text{and} \quad S^{++}(B, C),$$

resulting in the indirect influence $S^{++^2}(A, C)$. As the proposition addresses only situations where $I \leqslant J$, we now assume that the weakly negative direct influence of variable A on variable C has a multiplication index of (at least) 2. The above observations result in the following constraints:

- function f lies below function h, that is, $f(0) \leqslant h(0)$ and $f(1) \leqslant h(1)$;
- the parameter $\Pr(b \mid ax)$ for function f is greater than the parameter $\Pr(b \mid \bar{a}x)$ for function h, with a difference of at least α;
- the functions f and h are both linearly increasing functions.

We again assume the gradient of f to be larger than that of h, with similar observations holding for the opposite case.

To establish the sign of the composite influence of variable A on variable B, we once again investigate the minimal and maximal differences between the functions f and h. Under the constraints above, we find that the minimal difference between f and h is attained for $f(1)$ and $h(0)$, and thus equals

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \;\geqslant\; f(1) - h(0) = \Pr(c \mid aby) - \Pr(c \mid a\bar{b}y) + \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y)$$

From the strongly positive influence of variable B on variable C, we have that $\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y) \geqslant \alpha$; from the weakly negative influence of variable A on variable C we have that $0 \geqslant \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y) \geqslant -\alpha^2$. The minimal difference therefore equals $\alpha - \alpha^2$. Similarly, the maximal difference between the functions f and h is attained for $f(\alpha)$ and $h(0)$, and equals α^2. We conclude that for case (II), the composite influence of variable A on variable C is positive and at least $\alpha^I - \alpha^J$, that is, $\hat{S}^{++^{I \cup -J}}(A, C, t_1 \parallel t_2)$.

To summarise, in all possible situations where the multiplication-index list I of the strong sign is less than or equal to the multiplication-index list J of the weak sign, the composite influence of variable A on variable C is positive and equals $\hat{S}^{++^{I\cup -J}}(A, C, t_1 \parallel t_2)$. Note that if $I > J$, then we cannot guarantee that the composite influence is at all positive. □

From the above proposition and the appropriate entry in Table 3, we observe that for a strongly positive influence with multiplication-index list I and a weakly negative influence with multiplication-index list J, $I \leqslant J$, the enhanced $\oplus$-operator correctly captures the sign of their composition. Similar observations apply to other combinations of strong and weak conflicting influences. We conclude that, under certain conditions, the composition of conflicting strong and weak influences using the enhanced $\oplus$-operator leads to an unambiguous result at the level of the basic sign of the composite influence. The enhanced $\oplus$-operator thus indeed serves to resolve trade-offs upon inference.

From Table 3 we observe that multiplication-index lists tend to grow in size upon combining signs. These lists, however, can often be simplified to a large extent. For example, the list $I = \{1, 2, -1, -3\}$ captures the polynomial $\alpha + \alpha^2 - \alpha - \alpha^3 = \alpha^2 - \alpha^3$, and can be simplified to $I = \{2, -3\}$. That is, two complementing indices can be removed as long as this does not result in an empty multiplication-index list. For example, the list $I = \{1, -1\}$ represents the constant 0 and not $\alpha^0 = 1$. A list of the form $I = \{n, -n\}$ should therefore be represented in the given form; the actual value of n, however, is irrelevant and any non-zero integer value could be used without changing the list's meaning. In the case where a *strong* sign is augmented with a multiplication-index list of the form $\{n, -n\}$, an equivalent representation is given by a weak sign with the single multiplication index 0, since both represent an influence with a strength anywhere between zero and one. So although we cannot simplify a multiplication-index list of the form $\{n, -n\}$, we can, for example, replace the sign $++^{n,-n}$ by the sign $+^0$ without changing its meaning. We finally note that a multiplication-index list is a true multiset from which duplicates cannot be removed, since for example $I = \{1, 1\}$ represents the polynomial $\alpha + \alpha$ which equals $2 \cdot \alpha$ and not simply α.

Although simplifying all multiplication-index lists during inference can save a large amount of bookkeeping, we can be spared even more bookkeeping by approximating the polynomial expressions in α by a single term. The next section briefly describes how we can safely manage the loss of information incurred by this approximation.

4.3.2. The simple enhanced $\oplus_s$-operator

In this section we introduce a simplified version of the enhanced $\oplus$-operator, which assumes that all multiplication-index lists consist of a single positive index only. This single index is the result of essentially discarding the higher-order terms of the polynomial expression in α that captures the strength of a sign. If higher-order terms cannot be discarded without introducing a possible error, a sign of *unknown strength*, denoted $+^0$ or $-^0$, is yielded, which is equivalent to a positive or negative sign, respectively, in a basic qualitative probabilistic network.

The *simple enhanced* $\oplus_s$-operator is defined in Table 4, where the multiplication-index lists of the signs are reduced to single indices. When comparing this table to Table 3 for the enhanced $\oplus$-operator in the previous section, we note the following differences:

(1) upon combining two strong signs having the same basic sign, the multiplication index of the resulting sign is the *minimum* of the multiplication indices of the combined signs, instead of a concatenation;
(2) upon combining two weak signs having the same basic sign, we no longer preserve enough information to conclude whether the resulting sign is strong or weak, so a sign of *unknown* strength results;
(3) upon combining a strong and a weak sign with conflicting basic signs, we can unambiguously conclude the basic sign, if the multiplication index of the strong sign is smaller than that of the weak sign, *but* we do not know its strength.

The fact that the simple enhanced $\oplus_s$-operator correctly captures the sign of a combination of two parallel influences follows directly from the proofs of the propositions in the previous section. The proof of Proposition 4.8, for example, demonstrates that the strength of the sign which results from combining two non-conflicting strong signs with multiplication indices i and j, respectively, is at least $\alpha^i + \alpha^j \geqslant \alpha^{\min(i,j)}$. Similarly, we have from the proof of Proposition 4.7 that

Table 4
The simple enhanced $\oplus_s$-operator for signs with single multiplication indices

$\oplus_s$	$++^j$	$+^j$	0	$-^j$	$--^j$	?
$++^i$	$++^m$	$++^i$	$++^i$	*a*)	?	?
$+^i$	$++^j$	$+^0$	$+^i$	?	*d*)	?
0	$++^j$	$+^j$	0	$-^j$	$--^j$	?
$-^i$	*b*)	?	$-^i$	$-^0$	$--^j$	?
$--^i$	?	*c*)	$--^i$	$--^i$	$--^m$	?
?	?	?	?	?	?	?

where $m = min(i, j)$,
a) $+^0$, if $i \leqslant j$; else ?,
b) $+^0$, if $j \leqslant i$; else ?,
c) $-^0$, if $i \leqslant j$; else ?,
d) $-^0$, if $j \leqslant i$; else ?.

the strength of the sign which results from combining two non-conflicting weak signs with multiplication indices i and j, respectively, is at most $\alpha^i + \alpha^j$ and therefore considered unknown. Finally, the proof of Proposition 4.10 shows that the strength of the sign which results from combining a strong sign with multiplication index i and a conflicting weak sign with multiplication index j is at least $\alpha^i - \alpha^j$ and therefore considered unknown.

We stress that contrary to purely ambiguous signs, signs of unknown strength are valuable since they do not necessarily spread throughout a network once they occur upon inference. We conclude that application of the simple enhanced $\oplus_s$-operator rather than the enhanced $\oplus$-operator results in less computational overhead upon qualitative inference. Due to loss of information at the level of the strengths of signs, however, application of the simple $\oplus_s$-operator may result in less trade-offs being resolved.

5. Probabilistic inference revisited

In Section 3 we introduced the formalism of enhanced qualitative probabilistic networks. In Section 4, we enhanced the standard $\otimes$- and $\oplus$-operators for combining signs of influences upon inference and have addressed propagation of signs against the direction of arcs. Building upon the new, enhanced operators, the basic sign-propagation algorithm for probabilistic inference with a qualitative network is generalised straightforwardly to apply to enhanced networks: instead of the standard $\otimes$- and $\oplus$-operators, the enhanced operators are used for propagating and combining signs. In this section we illustrate the application of the resulting algorithm, for both versions of the enhanced $\oplus$-operator, by means of our running example; the qualitative networks associated with the example are reproduced in Fig. 8. In addition, we discuss the algebraic properties of the enhanced operators which may affect inference results. Finally, we briefly discuss some complexity issues concerning the different versions of the sign-propagation algorithm.

5.1. Inference using the enhanced operators

The idea behind the sign-propagation algorithm is basically to establish the net influence between an observed variable and all other variables in a qualitative probabilistic network, and multiply the sign of this net influence with the sign of the observation to return the effect of the observation on all variables. For ease of implementation, the algorithm starts by sending the sign of observation, a '$+$' or a '$-$', to the observed variable, thereby already incorporating the effect of the observation in all messages that are subsequently sent. Due to the algebraic properties of the basic $\otimes$- and $\oplus$-operators, the actual implementation does not affect the results.

In the next section we will demonstrate that in an enhanced qualitative network, the enhanced operators do not adhere to the algebraic properties that ensure that the order in which signs are combined does not affect the result of their combination. As a consequence, multiplying the sign of a net influence with the sign of the observation may lead to a different result than that obtained by directly incorporating the sign of observation in the messages sent by the observed variable. To disturb the computation of the signs of net influences as little as possible, we propose entering an observation using an "identity" sign with respect to strength. More specifically, we require a sign s such that for arbitrary sign t, the result of $s \otimes t$ has the strength of t. We note from Table 2 that the signs $++^0$ and $--^0$ are suitable for this purpose, since they can be taken to represent the constants 1 and -1, respectively. We now present an example that illustrates sign-propagation with the enhanced $\otimes$- and $\oplus$-operators.

Example 5.1. We consider once again the qualitative *Antibiotics* network, which is reproduced in Fig. 8(a). Recall that entering the sign '$+$' for variable A results upon inference with the basic sign-propagation algorithm in the ambiguous sign $- \oplus + =$ '?' for variable D, which in turn causes an ambiguous sign for variable H. Now, consider the enhanced *Antibiotics* network reproduced in Fig. 8(b); the signs specified are taken to hold in the direction of the corresponding arcs. We recall that initially all influences associated with the arcs in the network's digraph have signs with a multiplication-index of 1. We once again apply the sign-propagation algorithm, now using our enhanced operators. We enter the sign $++^0$ for variable A, reflecting a positive observation for A. Variable A propagates a message with sign $++^0 \otimes --^1 = --^1$ towards variable T. Variable T updates its node-sign to $--^1$ and sends a message with sign $--^1 \otimes ++^1 = --^2$ to variable D. Variable D

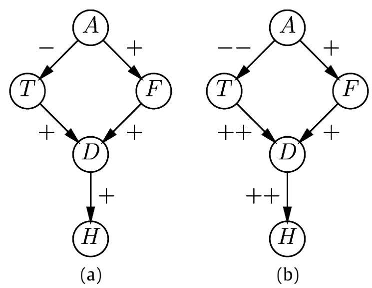

Fig. 8. The (a) qualitative *Antibiotics* network and (b) its enhanced version.

updates its node-sign to $--^2$ and only sends a message with sign $--^2 \otimes ++^1 = --^3$ to variable H. Variable H updates its sign accordingly and sends no messages.

Variable A also sends a message, with sign $++^0 \otimes +^1 = +^1$, to variable F. Variable F updates its sign and passes a message with sign $+^1 \otimes +^1 = +^2$ to variable D. Variable D receives the additional sign $+^2$. Variable D will now combine the signs it has received from the two parallel trails originating in A. The result of this combination depends on the enhanced operator used. More specifically, if the sign-propagation algorithm employs the fully enhanced $\oplus$-operator then variable D updates its sign to $--^2 \oplus +^2 = --^{2,-2}$, and then computes for variable H a message with sign $--^{2,-2} \otimes ++^1 = --^{3,-3}$. On the other hand, if the simple enhanced $\oplus_s$-operator is applied, then variable D updates its sign to $--^2 \oplus_s +^2 = -^0$, and computes a message with sign $-^0 \otimes ++^1 = -^0$ for variable H. Variable H, however, does not need a sign update as its current sign is already correct, regardless of the operator used ($--^3 \oplus --^{3,-3} = --^3$ and $--^3 \oplus_s -^0 = --^3$). The variables D and H therefore send no further messages and the algorithm halts.

From the above example, it seems at first glance that the results from sign-propagation with the fully enhanced $\oplus$-operator are similar to the results from using the simple enhanced $\oplus_s$-operator, with only the node-sign for node D differing. This illusion, however, is caused by the specific example network used. With the $\oplus$-operator, the node-sign of variable D is of the form $--^{i,-i}$ due to the fact that the two trails with conflicting influences have the same length; recall that this node-sign captures the same information as the negative sign $-^0$ of unknown strength returned by the $\oplus_s$-operator, and therefore the results of inference in essence do not differ for the two operators. If the conflicting trails have different lengths, however, then the difference between the two $\oplus$-operators becomes more important: for those situations in which the algorithm using the simple enhanced $\oplus_s$-operator leads to a node-sign $-^0$, the algorithm using the fully enhanced $\oplus$-operator will result in a node-sign $--^{I \cup -J}$, with $I \neq J$; this latter sign captures more information than the ambiguous negative sign and may aid in resolving even more trade-offs. We again stress that contrary to purely ambiguous signs, signs of unknown strength do not necessarily spread throughout a network once they occur. In addition, these signs do convey useful information about the basic sign of influence between two variables.

We conclude that, while in the basic framework of qualitative networks trade-offs cannot be resolved upon inference and result in an ambiguous net influence, enhanced qualitative probabilistic networks allow for resolving at least some trade-offs.

5.2. Algebraic properties

As a result of the algebraic properties of the basic $\otimes$- and $\oplus$-operators used for qualitative inference in a basic qualitative probabilistic network, the node-signs computed for the variables in such a network do not depend on the order in which the variables receive the messages from their neighbours. Unfortunately, this observation no longer holds in an enhanced qualitative network and the node-signs computed can in fact be different depending on the order in which messages are received. Given that the enhanced operators correctly capture the transitivity and parallel composition properties of qualitative influences, the computed node-signs are always correct but may be less informative than they could have been.

The following proposition states which algebraic properties the enhanced operators still adhere to; proofs are given in Appendix A.

Proposition 5.2. *Consider the enhanced operators defined in Tables* 2, 3 *and* 4 *for combining signs in an enhanced qualitative network. Then*

- *the enhanced $\otimes$-operator is commutative*;
- *the enhanced $\otimes$-operator is associative*;
- *the enhanced $\oplus$- and $\oplus_s$-operators are commutative.*

The enhanced $\oplus$- and $\oplus_s$-operators for combining signs of parallel influences are no longer associative, which can result in loss of information upon combining several trails having strong and weak conflicting influences. This is illustrated by the following example:

$$\left(++^i \oplus +^i\right) \oplus -^i = ++^i \oplus -^i = ++^{i,-i} \quad \left(\text{or } +^0,\ \text{if } \oplus_s \text{ is used}\right)$$
$$++^i \oplus \left(+^i \oplus -^i\right) = ++^i \oplus ? = ?$$

We stress that both combinations in this example lead to correct results, regardless of the operator ($\oplus$ or $\oplus_s$) used, the first is just more informative than the second. Heuristics, such as, for example, separately adding all positive and all negative signs before combining them, can be designed to prevent unnecessary ambiguous results due to order of combination. Such heuristics, however, could increase the complexity of inference.

Finally, we observe that the enhanced $\otimes$-operator distributes over neither the $\oplus$-operator nor the $\oplus_s$-operator. Compare, for example, the following:

$$\left(++^i \oplus +^i\right) \otimes -^i = ++^i \otimes -^i = -^i$$
$$\left(++^i \otimes -^i\right) \oplus \left(+^i \otimes -^i\right) = -^i \oplus -^{2i} = -^{i,2i} \quad \left(\text{or} -^0, \text{ if } \oplus_s \text{ is used}\right)$$

Again, we stress that all these results are correct, but they differ in level of informativeness with respect to the strength of the resulting sign.

5.3. Complexity of probabilistic inference

For quantitative probabilistic networks, in general, exact computation of probabilities is NP-hard [7]. The algorithms for probabilistic inference in a probabilistic network, however, are known to behave polynomially under certain restrictions concerning the topology of the network's digraph. In general, the sparser the digraph, the better most algorithms perform.

The basic sign-propagation algorithm for inference in a basic qualitative network has a worst-case runtime complexity that is polynomial in the number of nodes of the network's digraph, regardless of the digraph's topology. In a singly connected digraph, each pair of nodes is connected by a single simple trail. Upon sign-propagation, therefore, each variable A is visited at most once to receive the single sign which is the sign-product of the sign of observation and the signs associated with the arcs on the trail between A and the observed variable. In a multiply connected graph, two nodes can be connected by more than one simple trail. As a result, a variable should be visited as many times as the number of active simple trails between that variable and the observed variable to receive the sign of influence along each of those trails. To limit this possibly exponential number of visits to a variable, the basic propagation algorithm exploits the fact that node signs can only change twice: once from '0' to '+', '−' or '?' and then only to '?'. As variables therefore need to be visited at most twice, and each visited variable inspects and constructs a message for at most all other variables, we have that the basic propagation algorithm halts after a number of operations that is polynomial in the number of nodes in the network's digraph.

The basic formalism of qualitative probabilistic networks does not allow for resolving trade-offs, as combining two conflicting influences with the basic $\oplus$-operator immediately results in an ambiguous node-sign. From the example in the previous section, we have that the enhanced $\oplus$- and $\oplus_s$-operators do provide for resolving some trade-offs, using the additional information carried by the enhanced and augmented signs. The possibility of resolving trade-offs, however, comes at the expense of efficiency of sign-propagation. This is not surprising, since qualitative trade-off resolution is also known to be NP-hard [23]. The main difference between sign-propagation in a basic qualitative probabilistic network and sign-propagation in an enhanced network is that, in multiply connected digraphs, the limit of two visits to each variable no longer applies. Although a variable's basic enhanced node-sign can change at most three times—from zero to strong, to weak and then to ambiguous—the multiplication-index lists associated with the sign may require updating each time the variable is inspected. The difference between using the fully enhanced $\oplus$-operator and the simple enhanced $\oplus_s$-operator is that in the latter case the multiplication-index list of a sign is restricted to a single index, which may require an update less often. Using the simple enhanced $\oplus_s$-operator may therefore turn out to be more efficient in practice. Further research is still required, however, to determine the exact complexity class to which inference in an enhanced qualitative network belongs and to determine whether the enhanced $\oplus$- and $\oplus_s$-operators really differ in complexity. Recent results indicate that for networks in which the enhanced qualitative signs are translated into intervals, interval-propagation is NP-hard [21].

We conclude that there exists a trade-off between the amount of information present in inference results after sign-propagation and the complexity of the propagation algorithm. Inference using the basic sign-propagation algorithm has a runtime complexity that is polynomial in the number of nodes of a qualitative network's digraph, but always leads to ambiguous results when the network models a trade-off. Inference using the enhanced operators may perhaps become exponential, but does enable the resolving of trade-offs without resorting to numerical information.

6. Related work

The problem of trade-off resolution within the framework of qualitative networks has been addressed before by different researchers. In this section we briefly review this related work.

S. Parsons introduced the concept of categorical influence [25]. A categorical influence is a qualitative influence that serves either to increase a probability to 1 or to decrease a probability to 0, disregarding all other influences. For example, a positive categorical influence $S^{[++]}(A, B)$ of a variable A on a variable B is defined as $\Pr(b \mid ax) = 1$ for all relevant variables X. A categorical influence thus serves to resolve any trade-off in which it is involved, but can only capture deterministic relationships between nodes; in real-life applications few to none of such relationships will exist. Parsons also studied the use of both relative and absolute order-of-magnitude reasoning in the context of qualitative probabilistic networks [25]. Using the relative order-of-magnitude system ROM[K] [8], Parsons relates different qualitative influences to each other by specifying one qualitative influence as being, respectively, *negligible* with respect to, *distant* from, *comparable* to, or *close* to another influence. The use of relative orders of magnitude thus serves to relate the strengths of different influences, but it requires the specification of a relation between all pairs of influences, instead of a notion of strength per influence. In addition, due to the vague interpretations of the above terms, the relations used seem to be ill-defined, which makes reasoning with them anything but intuitive. For absolute order-of-magnitude reasoning, Parsons proposes a method that revolves around the propagation of abstract intervals between −1 and 1, that correspond to labels like 'Strongly Positive', 'Weakly Positive', etc. Two different sets of labels are required: one for modelling influences that are associated with the arcs in the network's digraph, and one for modelling changes that occur at the nodes in the graph (comparable to 'node-signs'). The intervals corresponding to a set of labels do not overlap and together span the interval $[-1, 1]$. The boundaries of the intervals, however, are not actually quantified, but set to be α, β, etc.; this approach is therefore comparable to our treatment of the cut-off value. Probabilistic inference is based on propagating and combining the abstract intervals; the interval comparisons required to this end are done using $\geqslant_{int}$, where $[\alpha, \beta] \geqslant_{int} [\gamma, \delta]$ iff $\alpha \geqslant \gamma$ and $\beta \geqslant \delta$. Note that if one interval is considered larger than another with this operator, then they may in fact overlap. To prevent considerable loss of information, assumptions about the actual values of the interval boundaries have to be made.

κ-calculus [30] can be considered another absolute order-of-magnitude system and was proposed as a qualitative version of probabilistic reasoning by M. Goldszmidt and J. Pearl [16]. Using a probabilistic interpretation of the κ-calculus, probabilities can be abstracted to κ-values, where a κ-value of n indicates that the associated probability has the same order of magnitude as ϵ^n for some infinitesimal number ϵ. This interpretation was subsequently applied in the context of probabilistic networks, by replacing all (conditional) probabilities with κ-values and computing posterior κ-values using κ-calculus [9]. More recently, we used the interpretation in another approach to enhance the expressiveness of qualitative probabilistic networks [29]. With this approach, an interval of κ-values is associated with the sign of an influence to capture its possible strengths. These κ-intervals are propagated along with the qualitative network's signs. As a result of the way κ-values are defined, however, propagation results are only guaranteed to be correct for infinitesimal probabilities. Another drawback of the use of κ-values is that their definition is not very intuitive and such values are therefore hard for domain experts to specify and interpret.

Categorical influences, order-of-magnitude reasoning and κ-calculus are of a purely qualitative nature, yet serve for resolving some trade-offs. C.-L. Liu and M.P. Wellman designed methods for resolving trade-offs based upon the idea of reverting to numerical probabilities whenever necessary [23]. They propose to reason with a probabilistic network in a qualitative way, thereby exploiting the efficiency of sign-propagation, and only reverting to the full quantification whenever a trade-off leads to an ambiguous result. Two methods are described for resolving the trade-off. The first method provides for incrementally applying numeric inference to the point where qualitative reasoning can produce a decisive result. That is, a trade-off between two variables is resolved numerically and then abstracted into a net qualitative influence between the two variables. The second method amounts to estimating bounds on the net influence along the trails that give rise to a trade-off. These bounds are then again used to compute the qualitative sign of the net influence. The methods presented by Liu and Wellman resolve any trade-off present in the network, but require the availability of an already fully specified, numerical probabilistic network. As such, their methods are less interesting for use in the construction phase of a probabilistic network.

We would like to mention that several other approaches to dealing with uncertainty in a qualitative way have been proposed in the literature. As these approaches are not tailored specifically to qualitative probabilistic reasoning or for use within the framework of qualitative probabilistic networks, we do not review them here.

7. Conclusions and further research

Qualitative probabilistic networks can be used to overcome, to at least some extent, the quantification problem known to probabilistic networks. Qualitative networks in essence are qualitative abstractions of their quantitative counterparts: while in a probabilistic network relationships between variables are quantified by probabilities, these relationships are expressed by qualitative signs in qualitative probabilistic networks. As a result of their coarse level of representation detail, qualitative networks lack the expressive power that allows for resolving trade-offs the way probabilistic networks do.

Since qualitative probabilistic networks are more and more recognised as useful tools in different stages of the construction and verification of quantitative probabilistic networks for real-life application domains, we feel that it is important that the qualitative formalism is as expressive as possible in order to derive as much information as possible from a qualitative network. The formalism of enhanced qualitative networks provides for a step into making qualitative networks more applicable, by providing for trade-off resolution in a qualitative way. To this end, we have distinguished between strong and weak influences. We have further enhanced the multiplication and addition operators to guarantee the transitivity and parallel-composition properties of influences. Unfortunately, the additional expressiveness of our enhancement comes at the expense of the property of symmetry of influences, where the strength of the influence is concerned. To handle the asymmetry of an influence's strength we have proposed specifying two influences for each arc. With these enhancements we have generalised the basic sign-propagation algorithm to apply to enhanced qualitative networks. We have shown that our formalism provides for resolving at least some trade-offs in a qualitative way, that is, without having to fall back on numerical information.

To distinguish between weak and strong influences, we have introduced additional signs and augmented all signs with multiplication-index lists. Since it may be difficult for domain experts to interpret the meaning of such lists of indices, it is not our intention to output the augmented signs. The multiplication indices are merely used internally for trade-off resolution; the output of inference, as in a basic qualitative network, is a basic sign for each variable that indicates whether the net influence of an observation on that variable is positive, negative, zero or ambiguous. If desirable, an additional level of strength can be added by introducing, for example, '$+++$' and '$---$' signs using an additional cut-off value and redefining the $\otimes$- and $\oplus$-operators. This would, however, render these operators far more complex. Alternatively, signs with a multiplication index other than 1 could be allowed on the arcs of the enhanced network's digraph; this option can be implemented directly in our current enhanced framework. Both options, however, would require domain experts to be able to distinguish between more than two levels of strength.

When the sign-propagation algorithm is used with the enhanced $\oplus$-operators, it becomes less efficient than the basic sign propagation algorithm. In fact, inference may then in theory become infeasible. Further research will be necessary to determine the actual complexity of sign-propagation with the enhanced operators in real-life qualitative networks. Two approaches can, however, be used to bound the complexity of inference. The first approach amounts to posing a limit on the number of sign-additions performed for a single variable. If this limit is reached, the node-sign of the variable is changed into a basic sign ('$+$', '$-$', '0', or '?') and the basic sign-propagation algorithm is used for further propagation. Note that this approach may lead to weaker, but correct, results. The other approach is to use enhanced signs only in small parts of the network, that is, in those parts where trade-offs reside. In constructing the enhanced network, we then focus on the multiply connected parts of the network's digraph and ask the domain experts whether the possible parallel trails between variables consist of conflicting influences. If so, enhanced signs are elicited for the influences on these trails. During inference, the trade-off can be locally resolved using the enhanced sign-propagation algorithm, and the basic sign for the net influence is then used for further propagation with the basic sign-propagation algorithm. Another advantage of such local computation with enhanced signs is that it requires only local specification of such signs. As a consequence, during the elicitation of signs, domain experts then only have to compare differences in strengths for small sets of influences. Since correctly specifying strengths will be harder for experts than correctly specifying the basic sign for an influence, local specification of enhanced signs will make the resulting signs less prone to error. Local specification also allows for different interpretations of strong and weak influences for different parts of the network, that is, it allows for different cut-off values to be (implicitly) used in different parts of the network. This may in addition simplify the elicitation of signs from domain experts.

We conclude that including a notion of strength is a logical extension to the original formalism of qualitative probabilistic networks. We have formalised such a notion of strength and shown how to cope with it upon qualitative probabilistic inference in a mathematically correct way. As such, we have enhanced the expressiveness of qualitative probabilistic networks, albeit at the expense of convenient properties such as symmetry of influences, some algebraic properties of the operators for combining signs, and complexity of inference. Although further research into these latter issues is required, our enhancement has broadened the range of possible applications of qualitative probabilistic networks.

Appendix A. Additional proofs of propositions

Proof of Proposition 4.8 ($++^I \oplus ++^J \Rightarrow ++^{I\cup J}$). The proof proceeds in a similar fashion as the proof of Proposition 4.7. We again assume that the positive influence along trail t_2 is composed of two separate positive influences, with similar observations applying when both influences are negative. Since the indirect influence of variable A on variable C along trail t_2 is strongly positive, it must be composed of two strongly positive direct influences. We thus have that

$$S^{++}(A,B) \quad \text{and} \quad S^{++}(B,C)$$

and therefore that $S^{++^J}(A,C,t_2)$ with $J=\{2\}$. We now investigate the difference between the two functions f and h defined in the proof of Proposition 4.7. Since the influence of the variable A on the variable B is strongly positive, we have that the difference between the two parameters for f and h should be at least α. To establish the minimum difference between $f(\Pr(b \mid ax))$ and $h(\Pr(b \mid \bar{a}x))$, we once again consider the graph from Fig. 7(a); similar observations again hold for the graph from Fig. 7(b). Under the constraint mentioned above, it is readily seen that the minimal difference between $f(\Pr(b \mid ax))$ and $h(\Pr(b \mid \bar{a}x))$ is attained for $f(\alpha)$ and $h(0)$. We find that

$$\begin{aligned}\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) &\geqslant f(\alpha) - h(0)\\ &= \big(\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)\big)\cdot\alpha + \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y)\\ &\geqslant \alpha^2 + \alpha\end{aligned}$$

We conclude that the composite influence of variable A on variable C is strongly positive with multiplication-index list $\{1,2\}$.

More in general, we find that the strength of the composite influence is at least the sum of the two polynomials in α, captured by the multiplication-index lists I and J, respectively, that is, we conclude that the composite influence equals $\hat{S}^{++^{I\cup J}}(A,C,t_1 \parallel t_2)$. $\square$

Proof of Proposition 4.9 ($++^I \oplus +^J \Rightarrow ++^I$). We distinguish between two different cases:

(I) the trail t_1 consists of a single arc and the trail t_2 consists of the arcs $A \to B$, $B \to C$ for some variable B;
(II) the trail t_1 consists of the arcs $A \to B$, $B \to C$ and the trail t_2 consists of the single arc.

For each of these cases, the proof proceeds in a similar fashion as the proof of Proposition 4.7. First we address case (I). As before, we assume that the indirect weakly positive influence of variable A on variable C along trail t_2 is composed of two separate weakly positive influences; the proofs for the other possible situations again are analogous. To establish the minimal difference between the functions f and h defined in the proof of Proposition 4.7, we once again consider the graph from Fig. 7(a). Since the influence of variable A on variable B is weakly positive, the difference between the two parameters for f and h should be at most α. Under this constraint, the minimal difference between $f(\Pr(b \mid ax))$ and $h(\Pr(b \mid \bar{a}x))$ is attained for $f(0)$ and $h(0)$. We thus find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geqslant f(0) - h(0) = \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y)$$

Since the direct influence of variable A on variable C is strongly positive, we have that $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geqslant \alpha$. We conclude that the composite influence of variable A on variable C is strongly positive with multiplication-index list $I = \{1\}$, that is, we conclude that the composite influence equals $\hat{S}^{++^I}(A, C, t_1 \parallel t_2)$ in case (I).

We now consider the minimal difference between the two functions f and h in case (II). We again assume that the indirect positive influence of A on C along trail t_1 is composed of two separate positive influences, with similar observations applying when both influences are negative. Since the indirect influence of A on C now is strongly positive, we have from Table 2 that the two separate influences from A to B and from B to C must both be strongly positive. We thus have that

$$S^{++}(A, B) \quad \text{and} \quad S^{++}(B, C)$$

and, therefore, that $\hat{S}^{++^I}(A, C, t_1)$, with $I = \{2\}$. Since the influence of variable B on variable C is positive, we have that the two functions f and h are both linearly increasing. Since the influence of A on B is strongly positive, we further have that parameter $\Pr(b \mid ax)$ for the function f should be greater than the parameter $\Pr(b \mid \bar{a}x)$ for the function h, with a difference of at least α. To establish the minimum difference between $f(\Pr(b \mid ax))$ and $h(\Pr(b \mid \bar{a}x))$, we again consider the graph from Fig. 7(a), with similar observations applying for the graph from Fig. 7(b). Under the constraints mentioned above, we observe that the minimal difference between $f(\Pr(b \mid ax))$ and $h(\Pr(b \mid \bar{a}x))$ is attained for $f(\alpha)$ and $h(0)$. We thus find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \;\geqslant\; f(\alpha) - h(0) = \big(\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)\big) \cdot \alpha + \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y)$$

Since the direct influence of A on C is weakly positive, we have that $0 \leqslant \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y) \leqslant \alpha$. We conclude that the composite influence of variable A on variable C is strongly positive with multiplication-index list $I = \{2\}$, that is, we conclude that the composite influence equals $\hat{S}^{++^I}(A, C, t_1 \parallel t_2)$ in case (II). □

Proof of Proposition 5.2 (algebraic properties). The fact that the enhanced $\otimes$-, $\oplus$-, and $\oplus_s$-operators are commutative follows directly from the symmetry in their respective tables. To prove that the enhanced $\otimes$-operator is associative, we distinguish between a number of cases.

We first observe that the property trivially holds if one of the signs used with the enhanced $\otimes$-operator is either a '0' or a '?'. Now consider combining with this operator three signs, be they positive or negative, that are either all weak or all strong. Since combining two strong signs with their respective multiplication-index lists results in a strong sign augmented with the sum of those multiplication-index lists, the order of combination of three such signs will not affect the resulting sign and the property of associativity holds. The same argument applies to combining all weak signs.

Now consider the case where the enhanced $\otimes$-operator is used to combine a single weak sign δ^I with two strong signs $\delta\delta^J$ and $\delta\delta^K$. If all signs are positive, then

$$\big(\delta^I \otimes \delta\delta^J\big) \otimes \delta\delta^K = \delta^I \otimes \delta\delta^K = \delta^I \quad \text{and}$$

$$\delta^I \otimes \big(\delta\delta^J \otimes \delta\delta^K\big) = \delta^I \otimes \delta\delta^{J+K} = \delta^I$$

Similar results hold regardless of whether the signs involved are positive or negative.

Finally we consider the case where the enhanced $\otimes$-operator is used to combine two weak signs δ^I and δ^J with a single strong sign $\delta\delta^K$. If all signs are positive, then

$$\big(\delta^I \otimes \delta^J\big) \otimes \delta\delta^K = \delta^{I+J} \otimes \delta\delta^K = \delta^{I+J} \quad \text{and}$$

$$\delta^I \otimes \big(\delta^J \otimes \delta\delta^K\big) = \delta^I \otimes \delta^J = \delta^{I+J}$$

with similar results holding regardless of whether the signs involved are positive or negative.

We conclude based upon the above observations and commutativity of the $\otimes$-operator that the operator is associative. □

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