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Statics and dynamics of induced systems

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Abstract

A collection of formulae, regarded as a set of prerequisite-free normal defaults, generates a nonmonotonic inference relation through its Reiter skeptical extension. The structure of the initial set totally determines the behavior of the associated inference relation, and the aim of this paper is to investigate in two directions the link that exists between a set of defaults and its induced inference relation. First, we determine the structural conditions corresponding to the important property of *rationality*. For this purpose, we introduce the notion of *stratification* for a set of defaults, and prove that stratified sets are exactly those that induce a rational inference relation. This result is shown to have interesting consequences in belief revision theory, as it can be used to define a nontrivial *full meet revision* operator for belief bases. Then, we adopt a dynamic point of view and study the effects, on the induced inference relation, of a change in the set of defaults. In this perspective, the set of defaults, considered as a knowledge base, together with its induced inference relation is treated as an *expert system*. We show how to modify the original set of defaults in order to obtain as output a rational relation. We propose a revision procedure that enables the user to incorporate a new data in the knowledge base, and we finally show what changes can be performed on the original set of defaults in order to take into account a particular conditional that has to retracted from or added to the primitive induced inference relation.

Keywords: Nonmonotonic reasoning; Preferential relations; Rationality; Stratification; Conditional revision; Rationalization; Full meet base revision

Introduction

Any set of formulae D of a propositional language may be seen as a set of Reiter normal prerequisite-free defaults [16], or, equivalently, as a Poole system without constraints [14]. As such, it generates an inference relation that corresponds to the Reiter extension of D.

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It was noticed in [13,15] that this induced inference relation is a preferential inference relation, in the sense of [11], and that it can be represented by a special kind of preferential model, where the set of states coincides with the set of all worlds attached to the language. Conversely, we proved in [4] that any consistency-preserving preferential inference relation defined on a finite language by an injective model is always induced by a set of defaults. Thus, preferential reasoning, when determined by a preferential injective model, is, in the finite case, essentially the same as default reasoning: the logic of any agent using injective preferential reasoning is fully determined by a set of normal prerequisite-free defaults. This applies in particular to *rational* reasoning, as it is known that rational inference relations may always be defined by means of a ranked injective model [3,11].

In this paper, we investigate the link that exists between a set of defaults D and its induced inference relation $\mid\!\sim_D$. First, on a *statics* point of view, we characterize the sets of defaults for which the induced inference relation satisfies the property of *rationality* (postulate $K*8$ in belief revision theory). We show that a set satisfies this condition if and only if it has a *stratified* structure, analogous to a logical chain of sets. This study turns out to have interesting applications in the framework of *belief revision*: as an example, we show that our results may be used to construct a nontrivial base revision operator that satisfies the extended set of AGM postulates as well as the categorial matching principle. This solves a problem posed in [9]. After this statics study, we evoke in the second part of the paper the *dynamics* of induced systems, and examine some problems connected with changes occurring in a given set of defaults: in this perspective, the set D is understood as representing all the information available to an agent, and the inference relation $\mid\!\sim_D$ that it induces corresponds to the inferences drawn by the agent. Thus the couple $(D, \mid\!\sim_D)$ is considered as an *expert system*. In this perspective, it might be necessary to operate changes in the knowledge base D for different reasons: first, in order to improve the system, one may require that the resulting inference process is of a *rational* type, so that one has to replace D by a stratified set. Then, it might appear necessary to incorporate in D some new information. The problem is then to revise D by this information without altering too much the resulting inference process. A last problem finally is the one that occurs when one wants to modify the set of defaults D because either it appears necessary to obtain a particular conditional that was not originally entailed by D, or, on the contrary, one wishes to remove a conditional that was part of the primitive induced relation. For all these problems, we shall see that several solutions exist, and we shall compare their merits.

In order to make the paper self-contained, we have recalled in Section 1 the basic definitions and properties of preferential inference relations. The notion of induced inference relation is introduced in Section 2, where we give a simple proof of the representation theorem of faithfully representable inference relations *via* their associated basic set of defaults. In Section 3, we present a characterization of the default sets that induce *rational* inference relations and propose an application in the framework of base revision theory. Section 4 is an introduction to the general study of the dynamics of induced systems. In Section 5, we discuss the problem of rationalizing an inference relation by a suitable modification of its set of defaults. In Section 6, we treat the problem of revising a set of defaults by a new information and the effect of this revision on the induced inference. Section 7 is concerned with the problem of conditional revision, which occurs when a given conditional has to be forced in the agent's beliefs. We conclude in Section 8. Proofs of the main results are given in Appendix A.

1. Background

We denote by $\mathcal{L}$ set of well-formed formulae over a set of atomic propositions, closed under the classical propositional connectives $\neg$, $\vee$, $\wedge$, $\to$ and $\leftrightarrow$. When there are only finitely many atomic propositions, the language is said to be logically finite. Semantics is provided by the set W of all assignments of truth values to the propositional variables. Elements of W will be referred to as *worlds*, and the satisfaction relation between a world m and a formula α is defined as usual and written $m \models \alpha$. Thus $m \models \alpha \vee \beta$ iff $m \models \alpha$ or $m \models \beta$, and $m \models \neg\alpha$ iff it is not the case that $m \models \alpha$.

For any world m and any subset A of $\mathcal{L}$, we denote by $\mathrm{form}_A(m)$ the set of formulae of A satisfied by m. The set of worlds that satisfy A will be denoted by $\mathrm{Mod}(A)$. We write $m \models A$ iff m satisfies all the elements of A, that is iff $A = \mathrm{form}_A(m)$.

The classical consequence operation attached to $\mathcal{L}$ and W will be denoted by Cn: for any subset A of $\mathcal{L}$, $\mathrm{Cn}(A)$ is the set of all formulae α of $\mathcal{L}$ such that $m \models \alpha$ for all worlds m that satisfy A. Given a subset A of $\mathcal{L}$, we say that A is *consistent* iff $\mathrm{Cn}(A) \neq \mathcal{L}$ or, equivalently, iff there exists a world m satisfying A. The set A is said to be consistent with the set B iff $A \cup B$ is a consistent set. We write $\mathrm{Cn}(A, B)$ for $\mathrm{Cn}(A \cup B)$, and $\mathrm{Cn}(\alpha)$ for $\mathrm{Cn}(\{\alpha\})$. We use the notation $\alpha \vdash \beta$ as an abbreviation for $\beta \in \mathrm{Cn}(\alpha)$.

1.1. Preferential inference relations

Following Kraus et al. [11], we call *preferential inference relation* on $\mathcal{L}$ a relation $\mid\!\sim$ that satisfies the following rules:

Reflexivity. $\alpha \mid\!\sim \alpha$.

Left Logical Equivalence. If $\mathrm{Cn}(\alpha) = \mathrm{Cn}(\beta)$ and $\alpha \mid\!\sim \gamma$, then $\beta \mid\!\sim \gamma$.

Right Weakening. If $\beta \in \mathrm{Cn}(\alpha)$ and $\gamma \mid\!\sim \alpha$, then $\gamma \mid\!\sim \beta$.

Cut. If $\alpha \wedge \beta \mid\!\sim \gamma$ and $\alpha \mid\!\sim \beta$, then $\alpha \mid\!\sim \gamma$.

Or. If $\alpha \mid\!\sim \gamma$ and $\beta \mid\!\sim \gamma$, then $\alpha \vee \beta \mid\!\sim \gamma$.

Cautious Monotonicity. If $\alpha \mid\!\sim \beta$ and $\alpha \mid\!\sim \gamma$, then $\alpha \wedge \beta \mid\!\sim \gamma$.

Given such a relation, we shall denote by $C_{\mid\sim}(\alpha)$—or $C(\alpha)$ when there is no ambiguity—the set of all consequences of a formula α, that is the set of all β's such that $\alpha \mid\!\sim \beta$. We will indifferently refer to "the inference relation $\mid\!\sim$" or to "the inference relation C". The above rules imply that, for any preferential inference relation C, the sets $C(\alpha)$ are closed with respect to Cn, that is $\mathrm{Cn}[C(\alpha)] = C(\alpha)$ for all formulae α. A formula α is $\mid\!\sim$-consistent, or C-consistent, iff $C(\alpha) \neq \mathcal{L}$. An inference relation C is said to be *consistency-preserving* iff $C(\alpha)$ is a consistent set for any consistent formula α. Thus a preferential inference relation C is consistency-preserving iff $C(\alpha) \neq \mathcal{L}$ whenever $\mathrm{Cn}(\alpha) \neq \mathcal{L}$.

1.2. Preferential models

A preferential structure is a triple $M = (S, <, l)$ where $<$ is an irreflexive and transitive relation defined on a set S (the set of "states"), and l (the "label function") is a mapping from S in the set of worlds W. For any state s, we say that s *satisfies* the formula α (written $s \models \alpha$) iff $l(s)$ does, and we denote by $\mathrm{Mod}_S(\alpha)$ the set of all such states s.

A preferential model, as defined in [11], is a preferential structure $(S, <, l)$ that satisfies the following condition of *smoothness*: given any formula α and any state s in $\mathrm{Mod}_S(\alpha)$ that is not minimal in $\mathrm{Mod}_S(\alpha)$, there exists a state t minimal in $\mathrm{Mod}_S(\alpha)$ such that $t < s$. This condition is always satisfied when the preferential structure is finite (i.e., when its set of states is finite).

A preferential model M determines a preferential inference relation $\mid\!\sim_M$ by

$$\alpha \mid\!\sim_M \beta \textit{ iff all minimal elements of } \mathrm{Mod}_S(\alpha) \textit{ satisfy } \beta. \qquad \text{(def)}$$

The interest of preferential models is that they represent all preferential relations. It was shown indeed in [11] that, for any preferential inference relation $\mid\!\sim$ defined on a language $\mathcal{L}$ (respectively, a logically finite language $\mathcal{L}$), there exists a preferential model (respectively, a finite preferential model) M such that $\mid\!\sim = \mid\!\sim_M$.

1.3. Injective models and faithfully representable inference relations

We shall call *injective* a preferential model in which the label function l is injective. We will be particularly interested in the case where the map l is a bijection. Then, the set of states can be identified with the set W of all worlds, and the model M is of the form $(W, <)$, where $<$ is a strict smooth partial order on W. A preferential inference relation that can be represented by a model of the form $(W, <)$ will be called a *faithfully representable* inference relation. Such an inference relation is clearly consistency-preserving. If C is a faithfully representable inference relation defined on a logically finite language, the model that represents C is unique, and we will refer to it as the *standard model* of C (cf. [3]), or as its *Shoham model* (cf. [17]).

When the language is logically finite, for any subset T of W we shall denote by χ_T the formula equal to the disjunction of all the complete formulae associated with the elements of T, that is the unique (up to classical equivalence) formula α such that $m \models \alpha$ iff $m \in T$. We have thus $\mathrm{Mod}(\chi_T) = T$.

In the case of a logically finite language, it will be convenient to identify a world with the sequence of positive literals that this world satisfies. Thus, in the language built on the propositional variables $p,\ q,\ r,\ s,\ t$, the world *prs* denotes the world that takes value 1 at $p,\ r$ and s, and takes value 0 at q and t. We will denote by $*$ the world that satisfies no positive literal.

1.4. The basic set of defaults associated with the preferential model $(W, <)$

Given any strict smooth partial order $<$ on the set W, there exists a set of formulae $\Delta_<$ that plays a prominent role in the study of faithfully representable inference relations. This set will be referred to as the *basic set of defaults associated with the model* $(W, <)$; it consists of all the formulae α that satisfy the following condition:

If α is satisfied by a world n,
then α is satisfied by all worlds m such that $m < n$. $(*)$

Note that the basic set of defaults associated with a preferential model $(W, <)$ is closed under conjunction and disjunction, and contains the tautologies as well as the contradictions of the language $\mathcal{L}$. When $\mathcal{L}$ is logically finite, we can define the basic set of defaults associated with a faithfully representable inference relation $\mid\!\sim$ as the set $\Delta_<$ associated with the unique model $(W, <)$ that represents $\mid\!\sim$. In this particular case of a logically finite language, we will always suppose fixed a representative of each class under classical equivalence, and choose the elements α of the basic set of defaults among the representatives of these equivalence classes. Thus in this case, we will view the basic set of defaults as a finite set consisting of the chosen representatives of the formulae α that satisfy $(*)$.

2. Poole systems

The definition and the principal properties of the inference relation associated with a Poole system can be found in [7,8,13]. We briefly recall some basic facts.

2.1. Definition and main properties of Poole systems

Let D be any subset of $\mathcal{L}$. We may identify this set with a Poole system without constraints $(D, \emptyset)$ and use this system to determine the conditions that would enable us to infer a formula β from a premiss α, holding true as much as possible from D. Clearly, such a formula β should be accepted in the simplest case where it is classically entailed by the conjunction of α and D, provided that the new information represented by α is consistent with the basic knowledge base D. In other words, if α is consistent with D, we consider that β follows from α modulo D if and only if $\alpha \to \beta \in \mathrm{Cn}(D)$. This idea leads to the following definition of the inference relation $\mid\!\sim_D$ *induced* by the set of defaults D: *β follows nonmonotonically from α modulo D iff $\alpha \to \beta$ is a classical consequence of every subset of D that is consistent with α and is maximal for that property*. This definition amounts to identifying the Poole system $(D, \emptyset)$ with its set of defaults, and to identify a default formula δ with the prerequisite-free normal Reiter style default: δ/δ. The set of all formulae β such that $\alpha \mid\!\sim_D \beta$ is then equal to the intersection of all the Reiter extensions of (α, D) as defined in [16]. Thus one has

$$\alpha \mid\!\sim_D \beta \quad \text{iff} \quad \beta \in \bigcap \mathrm{Cn}(\alpha, D_\alpha),$$

where the intersection is taken over all the subsets D_α of D that are maximally consistent with α. Clearly, if the language is logically finite, D may be assumed to be a finite set.

It is known that the inference relation $C_D = \mid\!\sim_D$ associated with a Poole system without constraints $(D, \emptyset)$ is a *preferential* inference relation (see, for instance, [7]), and it is immediate from its definition that this inference relation preserves consistency. We will refer to it as *the inference relation induced by the set of defaults D*. As noticed by

Alchourron and Makinson [2], the "limiting case" where D is a closed theory, that is where $D = \mathrm{Cn}(D)$, gives a trivial output: in this case, one has $C_D(\alpha) = \mathrm{Cn}(\alpha, D)$ for all formulae α consistent with D, and $C_D(\alpha) = \mathrm{Cn}(\alpha)$ for all other formulae [13, Observation 3.3.6].

Example 1 (*Defaults and axioms*). Suppose that D consists of the single default δ. One easily sees that the induced inference relation $\mid\!\sim_\delta$ is defined by

$\alpha \mid\!\sim_\delta \beta$ iff $\alpha \wedge \delta \vdash \beta$ when α is consistent with δ, and

$\alpha \mid\!\sim_\delta \beta$ iff $\alpha \vdash \delta$ when α is inconsistent with δ.

Thus in the principal case where α is consistent with δ, the set of formulae that can be inferred from α modulo δ coincides with the set of classical consequences of α in the axiomatic theory obtained by adding the axiom δ to the axioms of propositional calculus.

A set of defaults $D \subseteq \mathcal{L}$ induces a strict partial order $<_D$ on the set of worlds W, which is defined by $m <_D n$ iff $\mathrm{form}_D(n)$ is a strict subset of $\mathrm{form}_D(m)$. As observed independently by Makinson [13] and Poole [15], this order provides a semantic representation of the preferential relation $\mid\!\sim_D$ induced by D: one shows indeed that the structure $(W, <_D)$ is a preferential model that represents $\mid\!\sim_D$ (see [4] for an alternative proof), so that this inference relation is faithfully representable. It is worth pointing out that this model is particularly interesting because it is simple to build and corresponds to the intuitive idea that a *world m is less exceptional, or more normal, than a world n iff it satisfies a larger subset of formulae in D than n does*.

Example 2 (*Logical chains*). Let $D = \{\delta_0, \delta_1, \ldots, \delta_h\}$ be a set of nonequivalent formulae such that $\delta_h = \mathit{True}$ and, for all indices i, $0 \leqslant i \leqslant h-1$, $\delta_i \vdash \delta_{i+1}$. Such a set D—or the corresponding $(h+1)$-tuple $(\delta_0, \delta_1, \ldots, \delta_h)$—will be called a *logical chain*. Let us determine the inference relation induced by D. By definition, for any consistent formula α and any formula β, we have

$$\alpha \mid\!\sim_D \beta \quad \text{iff} \quad \beta \in \bigcap \mathrm{Cn}(\alpha, D_\alpha).$$

The only subset D_α of D that is maximally consistent with α is the set $\{\delta_i, \delta_{i+1}, \ldots, \delta_h\}$, where i is the first index such that α is consistent with δ_i. It follows that $\alpha \mid\!\sim_D \beta$ iff $\beta \in \mathrm{Cn}(\alpha, \delta_i)$ that is iff $\alpha \wedge \delta_i \vdash \beta$.

By the choice of D, the order of the associated preferential model $(W, <_D)$ is given by $m <_D n$ iff there exists an index i such that δ_i is satisfied by m and not by n. It is interesting to note that, when D is finite, this relation holds iff m satisfies a strictly greater number of elements of D than n: in other words, *given any finite logical chain D, the order induced by D is the* (*ranked*) *order $m <_D n$ iff* $\mathrm{Card}(\mathrm{form}_D(n)) < \mathrm{Card}(\mathrm{form}_D(m))$.

2.2. Faithfully representable inference relations and induced relations

We have seen that any induced inference relation is faithfully representable. A partial converse has been established in [4]; we present here a simple proof in the case of logically finite languages.

Theorem 3. *For any strict partial order $<$ defined on the set W of worlds of a logically finite language, one has $< = <_\Delta$, where Δ is the set of defaults associated with the preferential model $(W, <)$.*

Proof. (a) *Suppose first that it is not the case that $m < n$.* We have to prove that it is not the case that $m <_\Delta n$. Without loss of generality, one may suppose $m \neq n$. Let $\mathrm{Inf}(n)$ be the set of all worlds p such that $p = n$ or $p < n$, and δ_n the formula equal to $\chi_{\mathrm{Inf}(n)}$. We claim that δ_n is classically to an element of $\mathrm{form}_\Delta(n)$, but is equivalent to no element of $\mathrm{form}_\Delta(m)$. Indeed, observe that, up to classical equivalence, one has $\delta_n \in \Delta$, since condition $(*)$ is clearly satisfied. Moreover, n satisfies δ_n, and this formula cannot be satisfied by m since, otherwise, one would have $m < n$ or $m = n$, contradicting our assumption. This shows that there exists an element of $\mathrm{form}_\Delta(n)$ that is not in $\mathrm{form}_\Delta(m)$, so that one cannot have $m <_\Delta n$.

(b) *Suppose conversely that $m < n$*, and let us show that $m <_\Delta n$. It is clear that $\mathrm{form}_\Delta(n)$ is a subset of $\mathrm{form}_\Delta(m)$, since for any formula $\alpha \in \Delta$ satisfied by n, one has $m \models \alpha$ by condition $(*)$. It remains to prove that this inclusion is strict. But one does not have $n < m$, and a construction similar to the preceding one shows the existence of a formula δ_m that belongs to $\mathrm{form}_\Delta(m)$ and not to $\mathrm{form}_\Delta(n)$, whence the result. □

Note that Theorem 3 establishes two important results in the case of logically finite languages: the first one states that *any strict partial order $<$ on W is of inclusion type*: there exists a set D such that $< = <_D$. The second one is that the basic set of defaults Δ associated with $(W, <)$ may be taken for such a set D, and therefore provides an explicit construction for D by means of the condition $(*)$. Nevertheless, it should be emphasized that the set Δ is usually not the simplest of the sets that induce a given order on W. Since Δ is closed under conjunction and disjunction, it may be suspected that it is usually a "very large" set, a fact that is confirmed by the following observations:

Observation 4. *Any set D of formulae that satisfies $< = <_D$ is embedded in Δ.*

Proof. Let α be an element of D and n a world that satisfies α. If m is a world such that $m < n$, we have $m <_D n$, that is $\mathrm{form}_D(n) \subset \mathrm{form}_D(m)$. We have therefore $\alpha \in \mathrm{form}_D(m)$, so $m \models \alpha$. This shows that α satisfies condition $(*)$ and is therefore an element of Δ. □

The set Δ is therefore the largest subset D of $\mathcal{L}$ such that $< = <_D$. As we will show now, it is in fact large enough to "separate the worlds".

Observation 5. *Let $<$ be a strict partial order on the set W of worlds of a logically finite language, and Δ its associated basic set of defaults. Then two worlds agree on Δ iff they are equal.*

Proof. Suppose that there exists two worlds $m \neq n$ such that $\mathrm{form}_\Delta(m) = \mathrm{form}_\Delta(n)$. This equality shows that we do not have $m <_\Delta n$. By Theorem 3, this implies that $m \not< n$. As we saw in the proof of Theorem 3, there exists then a formula $\delta_n \in \mathrm{form}_\Delta(n)$ that is not an element of $\mathrm{form}_\Delta(m)$, contradicting our assumption. □

The property described in the observation above holds for some choices of injective models of the type $(W, <)$ even in the case of infinite languages, as can be seen in the following example:

Example 6. Let $\mathcal{L}$ be an arbitrary propositional language, δ a consistent formula of $\mathcal{L}$, and D the set $\mathrm{Cn}(\delta)$. Let us determine the basic set of defaults Δ associated with the model $(W, <_D)$. We first note that *$m <_D n$ iff δ is satisfied by m and not by n*: indeed, if $m <_D n$, one has $m \neq n$ and there exists a formula $\alpha \in D$ satisfied by n and not by m. The formula $\alpha \vee \delta$ is an element of D and is satisfied by n, hence by m since we have $m <_D n$. It follows that $m \models \delta$. Conversely, if $m \models \delta$ and $m \models \neg\delta$, one has $\mathrm{form}_D(n) \subset \mathrm{form}_D(m) = D$, whence $m <_D n$ as desired.

Let Δ be the basic set of defaults associated with $(W, <_D)$. *We claim that Δ is the set of all formulae that either imply δ, or are implied by δ.* Indeed, if α is an element of Δ such that $\delta \notin \mathrm{Cn}(\alpha)$, there exists a world n such that $n \models \alpha \wedge \neg\delta$. Let m be an arbitrary world that satisfies δ. Since δ is satisfied by m and not by n, we have $m <_D n$, and therefore m satisfies α. This shows that $\delta \vdash \alpha$, so that the elements of Δ either imply δ or are implied by δ. Conversely, any formula γ that implies δ is satisfied only by minimal worlds, and therefore satisfies condition $(*)$. Since, by Observation 4, we have $D \subseteq \Delta$, this shows that Δ is of the desired form.

Let us show now that Δ separates the worlds: suppose that there exists two world $m \neq n$ that agree on Δ, and let β be a formula satisfied by m and not by n. The formula $\beta \wedge \delta$ of Δ is not satisfied by n, and is therefore not satisfied by m. The formula $\neg\beta \vee \delta$ of Δ is satisfied by n, hence by m. We have therefore $m \models \neg\beta$, contradicting our assumption.

As a direct consequence of Theorem 3, we have the following

Corollary 7. *Any faithfully representable inference relation $\mid\!\sim$ defined on a logically finite language is induced by its associated basic set of defaults.*

Proof. Denote by $(W, <)$ the standard model of $\mid\!\sim$ and by Δ its associated basic set of defaults. It follows from Theorem 3 that we have $(W, <) = (W, <_\Delta)$, and the preferential relations represented by these models are therefore identical. □

The meaning of Corollary 7 is that, under some mild conditions, preferential reasoning *à la* Shoham is the same as reasoning *à la* Poole. But its interest is also to point out *the existence of a basic set of defaults that comes to conditionalize, implicitly or explicitly, the inference process of any agent that uses faithfully representable preferential logic on a finite language, and in particular of any agent that uses consistency-preserving rational logic,* as will be shown in the next section. This result is reminiscent of some well known theories on human behavior, is as much as they claim that human beings are determined in their judgments and actions by a set of primary affects.

The question naturally arises whether Theorem 3 and its corollary have an analogue in the case of infinite propositional languages. The answer is in general negative (see [4, Example 8]), and we do not know of any property that would characterize in this case the inference relations that can be induced by a set of defaults.

We have seen that it is possible to associate with any set of defaults a faithfully representable preferential inference relation; conversely, by Corollary 7, any such relation defined on a logically finite language is induced by a set of defaults. In this duality between default sets and faithfully representable inference relations, it is interesting to determine the default sets that correspond to the important sub-family of *rational* inference relations. This will be done in the next section, where we shall provide a characterization of all default sets whose induced relation is rational.

3. The analogue of rationality in the Poole–Shoham duality

3.1. The characteristic set of a rational inference relation

We shall turn our attention to a specific family of preferential inference relations that play a prominent role in the area of nonmonotonic reasoning, namely the preferential inference relations that satisfy the property of *rationality*. The aim of this property is to capture inference process that, whilst defeasible, stand as close as possible to monotonic consequence relations. Rational inference relations are characterized by the property of *Rational Monotony*, that reads, for all formulae α, β and γ:

RM. If $\alpha \mid\!\sim \beta$ and not $(\alpha \mid\!\sim \neg\gamma)$, then $\alpha \wedge \gamma \mid\!\sim \beta$.

An inference relation is said to be *rational* iff it is preferential and satisfies RM. It is well known that a preferential relation is rational iff it may be defined by an injective preferential model $(S, <)$, where S is a subset of W and $<$ is a smooth *modular* order on S, that is an order in which the inequality $m < n$ implies for any world p either $m < p$, or $p < n$. An order is modular iff there exists a totally ordered set $(T, <')$ and a “ranking” function κ from $(S, <)$ into $(T, <')$ such that $m < n$ iff $\kappa(m) <' \kappa(n)$ (see, for example, [3] or [12]). A preferential model $(S, <)$ with modular order $<$ is called a *ranked* model.

When the underlying language is supposed to be finite and the rational relation consistency-preserving, one may assume that the ranking function, defined on the whole set W, takes its value in a finite set of integers. The *height* h of such a rational relation (or of its standard model) is the number of elements of $\kappa(W)$. We will always suppose that the ranking function is normalized, i.e., that $\kappa(W)$ is the set $[0, h-1] = \{0, 1, \dots, h-1\}$.

Since rational consistency-preserving inference relations are known to be faithfully representable, they are induced, in the finite case, by their associated basic set of defaults. In fact, the additional hypothesis of rationality yields a more accurate result: let h be the height of such a relation $\mid\!\sim$, and, for any index i, $1 \leqslant i \leqslant h$, denote by ψ_i the (unique up to classical equivalence) formula such that $\mathrm{Mod}(\psi_i) = \kappa^{-1}[0, i-1]$. Clearly, one has $\psi_1 \vdash \psi_2 \vdash \cdots \vdash \psi_{h-1} \vdash \mathit{True}$. Let Ψ be the set $\{\psi_i \mid 1 \leqslant i \leqslant h\}$. This set, called the *characteristic set* of the rational relation, therefore has the structure of a logical chain, and it turns out that one has $\mid\!\sim = \mid\!\sim_\Psi$. *Thus any consistency-preserving rational inference* *relation defined on a logically finite language is induced by its characteristic set* (see [4] for details). As we saw in Example 2, relations C induced by such logical chains are easy to compute: they satisfy $C(\alpha) = \mathrm{Cn}(\alpha, \psi_i)$, where ψ_i is the first α-consistent element of the chain, and the order $<$ of their associated ranked model is given by $m < n$ iff $\mathrm{card}(\mathrm{form}_\Psi(n) < \mathrm{card}(\mathrm{form}_\Psi(m))$.

3.2. Stratified sets

It should not be inferred from the above considerations that, when a set of defaults D induces a rational inference relation, one has necessarily $m <_D n$ iff $\mathrm{card}(\mathrm{form}_D(n) < \mathrm{card}(\mathrm{form}_D(m))$. For instance, taking for $\mathcal{L}$ the language built on the two propositions p and q, and, for D, the set $\{p \wedge q, \neg q, \neg p \wedge q, q\}$, one easily checks that $<_D$ is the empty relation and is therefore modular. Note that although the world q satisfies two elements of D while p satisfies only one element, we do not have $q <_D p$.

From now on, we shall assume that the language is logically finite. By what precedes, any logical chain induces a rational consistency-preserving inference relation, and any consistency-preserving rational inference relation is induced by a logical chain. Nevertheless, there exist sets as in the above example that induce a rational inference relation without having the structure of logical chain. Our purpose is now to characterize these sets. To do so, we shall have first to look at the structure of the basic set of defaults associated with a modular order.

Lemma 8. *Let $\mathrel{|\!\sim}$ be a consistency-preserving rational inference relation defined on a logically finite language, $\Psi = \{\psi_i \mid 0 < i \leqslant h\}$ its characteristic set and Δ its basic set of defaults. Then a formula α is an element of Δ iff either $\alpha \vdash \psi_0$ or there exists an index i, $1 \leqslant i < h$, such that $\psi_i \vdash \alpha \vdash \psi_{i+1}$.*

Remark. For all indices i such that $1 \leqslant i < h$, let us denote by Δ_i the set of all formulae $\alpha \neq \psi_{i+1}$ such that $\psi_i \vdash \alpha \vdash \psi_{i+1}$. Set $\Delta_0 = \{\alpha \mid \alpha \vdash \psi_1,\ \alpha \neq \psi_1\}$ and $\Delta_h = \{\mathit{True}\}$. Then the above lemma shows that the sets Δ_i form a partition of Δ. Note that one has $\alpha \vdash \beta$ for all formulae $\alpha \in \Delta_i$ and $\beta \in \Delta_{i+1}$.

This remark will be used to characterize the sets D that induce a rational inference relation. We first need to introduce the notion of *determinable* subsets.

Definition 9. A subset X of a set of formulae Y is said to be determinable in Y iff there exists a world p such that $\mathrm{form}_Y(p) = X$.

Equivalently, X is determinable in Y iff X is consistent with the set $\{\neg\beta \mid \beta \in Y - X\}$. As we shall see, the notion of determinable subsets is fundamental to characterize the defaults sets that induce rational relations.

Definition 10. A subset D of a logically finite language is said to be stratified iff it is a disjoint union of sets D_i that satisfy the following conditions:

(1) $\alpha \vdash \beta$ for all formulae $\alpha \in D_i$ and $\beta \in D_{i+1}$.
(2) Any nonempty proper subset of D_i that is determinable in D_i is maximally so.

Given a stratified set D, any sequence (D_i) satisfying conditions (1) and (2) of the definition will be called a *stratification* of D. A stratified set therefore splits into graded sheaves D_i and the sequence (D_i) is analogous to a "logical multichain": by condition (1), any element of a sheaf dominates the elements that belong to a sheaf of higher rank. As for condition (2), note that it is equivalent to the condition

(2′) For all worlds m and n, if $\emptyset \subset \mathrm{form}(n) \cap D_i \subset \mathrm{form}(m) \cap D_i$, then $\mathrm{form}(m) \cap D_i = D_i$.

Thus, the trace on D_i of two worlds m and n are never comparable, except in the limit cases. If we denote by $<_i$ the order on W induced by D_i, we have thus $m <_i n$ iff either D_i is satisfied by m and not by n, or m satisfies at least an element of D_i while n satisfies no element of this set. Condition (2′) therefore shows that $<_i$ is a modular order of height $\leqslant 3$.

Example 11.

(a) The simplest nontrivial example of a stratified set is given by a logical chain $D = \{\delta_0, \delta_1, \ldots, \delta_h\}$ with $\delta_i \vdash \delta_{i+1}$ for all indices $i < h$. It is indeed immediate that the subsets $D_i = \{\delta_i\}$ satisfy conditions (1) and (2) of the definition. Another stratification of D is given by the partition $D_i = \{\delta_i\}$ if $i < n-1$ and $D_{n-1} = \{\delta_{n-1}, \delta_n\}$. Thus the stratification of a set, if it exists, is not necessarily unique.

(b) Any set D_i of cardinality $\leqslant 2$ satisfies condition (2) of the definition. It follows that the set $D = D_0 \cup D_1 \cup D_2$ is stratified, where $D_0 = \{p \wedge q, q \wedge r\}$, $D_1 = \{q, p \vee r\}$ and $D_2 = \{p \vee q \vee r\}$.

(c) Any closed subset D of $\mathcal{L}$ is stratified. Let us show indeed that such a set satisfies condition (2) of the definition. Suppose that X and Y are two nonempty determinable subsets of D such that $X \subset Y$. There exists then worlds m and n such that $\mathrm{form}_D(n) = X$ and $\mathrm{form}_D(m) = Y$. Let β be an element of Y that is not in X. We have $n \models \neg\beta$ and $m \models \beta$. We claim that $Y = D$: otherwise, there would exist a formula α of D such that $m \models \neg\alpha$. Since D is closed, D would contain the formula $\alpha \vee \neg\beta$, satisfied by n, and this would imply $\alpha \vee \neg\beta \in X$, hence $\alpha \vee \neg\beta \in Y$, leading to $m \models \alpha \vee \neg\beta$, a contradiction.
Since D satisfies condition (2) of the definition, the trivial sequence (D) provides a stratification of D, and this shows that *any closed set is stratified*.

Remark. If D is an union of sets D'_i that satisfy conditions (1) and (2) of Definition 10, then D is a stratified set: consider indeed the family (D_i) defined by $D_0 = D'_0$ and, for all $i > 0$, $D_i = D'_i - D'_{i+1}$. The sets D_i then form a partition of D for which condition (1) is clearly satisfied. To check condition (2), suppose that $X = \mathrm{form}(n) \cap D_i$ is a nonempty determinable subset of D_i strictly embedded in a set $Y = \mathrm{form}(m) \cap D_i$. Note that it follows from condition (1) that, up to classical equivalence, the set $D'_i \cap D'_{i+1}$ has at most one element. Let α be this element, if it exists. We have $n \models \alpha$, for n satisfies at least an element β of D'_i, and we know that $\beta \vdash \alpha$. Similarly, m satisfies α. It follows that $\mathrm{form}(n) \cap D'_i = X \subseteq \{\alpha\}$, $\mathrm{form}(m) \cap D'_i = Y \cup \{\alpha\}$, and thus that $\mathrm{form}(n) \cap D'_i \subset \mathrm{form}(m) \cap D'_i$. By condition (2), we have then $\mathrm{form}(m) \cap D'_i = D'_i$, showing that $Y = D_i$.

We are now ready to establish the main theorem of this section.

Theorem 12. *The inference relation induced by a set D is rational iff D is stratified.*

This theorem characterizes the sets D for which the induced inference relation is rational. As noticed before, such a relation is also induced by its characteristic set $\Psi = \{\psi_i,\ 1 \leqslant i \leqslant h\}$, where ψ_i is the formula defined by $m \models \psi_i$ iff $\kappa(m) < i$. The link between the set D and the chain Ψ is now given by the following observation, that shows how directly determine Ψ without computing the induced ranking κ:

Observation 13. *Let D be a stratified set and $(D_1, D_2, \ldots, D_h)$ be a stratification of D. For any index $i,\ 1 \leqslant i \leqslant h$, denote by η_i the conjunction of all elements of D_i, and by η'_i the disjunction of all elements of D_i. Set $\eta_{h+1} = \eta'_{h+1} = True$. Then the characteristic set Ψ of the rational inference relation induced by D is the set $\{\eta_i, \eta'_i \mid 1 \leqslant i \leqslant h+1\}$.*

Example 14. Let us determine the characteristic chains associated with the stratified sets of Example 11:

(a) If D is a logical chain $(\delta_1, \delta_2, \ldots, \delta_h)$, we have readily $\eta_i = \eta'_i = \delta_i$.

(b) If D is the stratified set $D_0 \cup D_1 \cup D_2$, with $D_0 = \{p \wedge q, q \wedge r\},\ D_1 = \{q, p \vee r\}$ and $D_2 = \{p \vee q \vee r\}$, we have

$$\eta_0 = p \wedge q \wedge r = \eta'_0, \qquad \eta_1 = q'(p \vee r),$$
$$\eta'_1 = p \vee q \vee r = \eta_2 = \eta'_2,$$

and the characteristic chain associated with D is

$$(p \wedge q \wedge r, q \wedge (p \vee r), p \vee q \vee r, True).$$

(c) If D is a closed set, there exists a formula $\delta \in D$ such that $D = \mathrm{Cn}(\delta)$ (recall we assumed the language to be finite). The associated characteristic set is then equal to $(\delta, True)$.

Remark. If D is a union of not necessarily disjoint sets D'_i that satisfy conditions (1) and (2) of Definition 10, the elements η_i of the characteristic set associated with D are equal to the conjunction of the elements of D'_i, and the elements η'_i are equal to their disjunction, as follows easily from the preceding remark.

As a straightforward consequence of Observation 13, we can explicitly compute the rational consequences of a formula α that are induced by a stratified set D:

Observation 15. *Let C be the rational inference relation induced by a stratified set D. Let $(D_j)_{1 \leqslant j < h}$ be a stratification of D, and $D_h = \{True\}$. For any consistent formula α, let i be the first index such that α is consistent with D_i. Then one has $C(\alpha) = \bigcap(\mathrm{Cn}(\alpha, \gamma) \mid \gamma \in D_{i-1})$ if there exists a formula of D_{i-1} that is consistent with α, and $C(\alpha) = \mathrm{Cn}(\alpha, D_i)$ otherwise.*

In the perspective of belief revision theory, this result may be analyzed as a generalization of Grove's theory of spheres [10]: when K is a belief set, any revision $*$ of K is determined by a system of embedded spheres K_i that are closed subsets of K. For any consistent formula α, the revision $K * \alpha$ of K by α is equal to the expansion $\mathrm{Cn}(\alpha,\ K_i)$ of K_i by α, where i is the first index such that $\neg\alpha \notin K_i$. The above observation shows that this system of embedded spheres may still be used in the general case where the set of beliefs is not closed under classical consequences: indeed, using the notations of Observation 13, we note that $\mathrm{Cn}(D_{i+1}) \subseteq \mathrm{Cn}(\eta_i') \subseteq \mathrm{Cn}(D_i)$, so that, in the light of Observation 15, these sets may be considered as the analogues of the spheres K_i's. The main difference is that these spheres are no longer subsets of the "set of beliefs" D.

Note, as an interesting consequence of Observation 15, that the sets $\bigcap(\mathrm{Cn}(\alpha, \gamma) \mid \gamma \in D_{i-1})$ and $\mathrm{Cn}(\alpha, D_i)$ do not depend on any particular stratification of D.

The study made in this subsection provides a satisfactory answer concerning the status of rationality in the Poole–Shoham duality between default sets and induced inference relations. It is possible and interesting to extend this study to some families of preferential inference relations that satisfy properties weaker than rationality, like *weak rationality* or *disjunctive rationality*. In this paper, our study of the statics aspects of induced inferences will be nevertheless restricted to the only case of plain rationality. Before turning to the dynamics study of induced inferences, we conclude this section with an application of our main results in the framework of *belief revision theory*.

3.3. Full meet base revision

This paragraph may be seen as an illustration of the results that we established in the study of rational induced relations. As we shall see, Theorem 12 and Observations 13 and 15 have interesting consequences in belief revision theory, leading to the definition and the study of a full meet base revision operator. The results we obtain in this paragraph have received an independent treatment in [5], where they have been established by the classical methods of belief theory.

Given an arbitrary set of formulae B, the *full meet revision* of B by a formula α is the set $B*\alpha = \bigcap \mathrm{Cn}(\alpha, B_\alpha)$, where the intersection is taken over all maximal α-consistent subsets B_α of B. In the case of a logically finite language, the revised set $B * \alpha$ is thus equal to the set of consequences $C_B(\alpha)$ inferred from α by the inference relation C_B induced from the set of defaults B. When the set B is closed under classical consequences, the full meet revision operator $*$ yields a trivial output, as results from the Alchourron–Makinson triviality theorem: in this case, $C_B(\alpha)$ is indeed equal to $\mathrm{Cn}(B, \alpha)$ if α is consistent with B, and to $\mathrm{Cn}(\alpha)$ otherwise. If now B is not a closed set, and is therefore a *belief base,* the results established in the previous section may be applied to study the properties of this full meet revision operator. Applying Theorem 12 and the formal correspondence established in [7] between nonmonotonic logic and the logic of theory change, we immediately get the following

Observation 16. *Let B be a belief base, K its closure and $\Diamond$ the revision on K defined by $K\Diamond\alpha = B * \alpha$. Then $\Diamond$ satisfies the extended set of AGM postulates iff B is stratified.*

The result stated in Observation 16 is in fact of little use in belief revision theory: indeed, it can be applied only when the theory K to be revised is given together with a set of "justifications" B. In particular, its iteration is meaningless, since no justification is provided for the revised theory $K\Diamond\alpha$. Note that the operator $*$ suffers from the same shortcoming: the result $B*\alpha$ of the revision of the base B by α is a closed theory, and the iteration of this revision provides a trivial output by the Alchourron–Makinson triviality theorem. Furthermore, the use of $*$ as a base revision operator violates the *categorial matching principle*, according to which "the representation of a belief state after a belief change has taken place should be of the same format as the representation of the belief state before change" [9]. This principle is therefore violated when, starting with a belief base B, or a theory K with known justification B, one obtains the closed theory $B*\alpha$ in which the track of any such justification is lost.

It is nevertheless possible to use Theorem 12 together with Observation 15 to define a full meet base revision operator $\Diamond$ that satisfies the principle of categorial matching, and the closure of which satisfies the extended set of AGM postulates. This operator is defined as follows:

Let B be a stratified set and $(B_j)_{0\leqslant j\leqslant s}$ a stratification of B. Without loss of generality, we can suppose that $True \in B$ and that $B_s = \{True\}$. Let α be a consistent formula, i the first index such that α is consistent with B_i, and $\beta_1, \beta_2, \ldots, \beta_n$ $(n \geqslant 0)$ the elements of B_{i-1} that are consistent with α. The *full meet base revision* of B by α is the set

$$B\Diamond\alpha = \{\alpha \wedge (\beta_1 \vee \beta_2 \vee \cdots \vee \beta_n)\} \bigcup_{k\geqslant i} (\alpha \wedge B_k) \cup \{True\},$$

where $\alpha \wedge B_k$ denotes the set $\{\alpha \wedge \gamma \mid \gamma \in B_k\}$.

We showed in [5] that *this set $B\Diamond\alpha$ does not depend on the choice of the stratification* (B_i), so that the operator $\Diamond$ is well-defined (the proof of this result is rather long and tedious, and we do not incorporate it here, as belief revision is not the subject of this paper). It is now easy to prove that $B\Diamond\alpha$ is a stratified base, as the sequence $B'_0 = \{\alpha \wedge (\beta_1 \vee \beta_2 \vee \cdots \vee \beta_n)\}$, $B'_1 = \alpha \wedge B_i$, $B'_2 = \alpha \wedge B_{i+1}, \ldots, B'_{s-i+1} = \{\alpha\}$ satisfies conditions (1) and (2) of Definition 10.

For all consistent formulae α, the full meet base revision operator $\Diamond$ therefore associates a stratified base $B\Diamond\alpha$ with any stratified base B. Using Observation 15, one readily checks that the full meet revision operator $*$ and the full meet base revision operator $\Diamond$ are connected by the equality $\mathrm{Cn}(B\Diamond\alpha) = B*\alpha$, showing that the closure-extension of $\Diamond$ is an AGM revision.

Example 17.

(a) Let B be the logical chain $\delta_0 \vdash \delta_1 \vdash \cdots \vdash \delta_{h-1} \vdash True$. For any consistent formula α, one has $B\Diamond\alpha = \delta_i \wedge \alpha \vdash \delta_{1+1} \wedge \alpha \vdash \cdots \vdash \{\alpha\} \vdash \{True\}$, where i is the first index such that α is consistent with δ_i.

(b) If $B = \{p, q, True\}$, we can use the stratification $B = \{p, q\} \cup \{True\}$ to compute the revised base $B\Diamond\alpha$. In the particular case where $\alpha = \neg(p \wedge q)$, we find

$$B\Diamond\neg(p \wedge q) = \{\neg(p \wedge q) \wedge (p \vee q), \neg(p \wedge q), True\},$$

which is equal, up to classical equivalence, to the set

$$\{(p \wedge \neg q) \vee (\neg p \wedge q), \neg(p \wedge q), True\}.$$

This result conforms with our intuition that revising B by the formula $\neg(p \wedge q)$ exactly means that one at least of the elements p and q has to be discarded.

We close this section with a list of some elementary properties of the full meet base operator $\Diamond$, to which we shall return in Section 6. A detailed analysis and a discussion on the meaning of these properties in the framework of belief revision is made in [5]. We denote by B a stratified base such that $\mathit{True} \in B$, and by α a consistent formula.

FM1. $B\Diamond\alpha$ is a consistent stratified base.

FM2. $\alpha \in B\Diamond\alpha$.

FM3. $B\Diamond\alpha = (\alpha \wedge B) \cup \{\mathit{True}\}$ if α is consistent with B.

FM4. $(B\Diamond\alpha)\Diamond\alpha = B\Diamond\alpha$.

FM5. $B\Diamond(\alpha \wedge \beta) = \beta \wedge (B\Diamond\alpha) \cup \{\mathit{True}\}$ if β is consistent with $B\Diamond\alpha$.

FM6. For all $\beta \in B$, $\alpha \wedge \beta \in B\Diamond\alpha$ or $\alpha \wedge \beta \vdash B\Diamond\alpha$.

FM7. If β is consistent with α, $B\Diamond\alpha\Diamond\beta = \{\beta\} \cup B\Diamond(\alpha \wedge \beta)$.

4. The dynamics of induced systems

So far, the results we have established concern the correspondence between default sets and inference relations in the Poole–Shoham duality. They can be considered as a study of the *statics* of default systems. From now on, we shall adopt a different point of view and study the effect of a perturbation of a set D on its induced inference relation.

In studying the dynamics of induced systems, it is useful to consider a set of defaults $D \subseteq \mathcal{L}$ as representing a basic information that completely specifies the inference process of an agent. This process, symbolized by the inference relation $\mid\!\sim_D$ induced by D, enables the agent to jump from a premiss α to the conclusion β in all cases where $\alpha \to \beta$ is classically entailed by the maximal α-consistent subsets of D. The role of the set D, analyzed as a set of data generating an inference relation, is thus reminiscent of that played by belief states (or, more precisely, by belief bases) in belief revision theory. Nevertheless this analogy is purely formal. In revision theory, the agent disposes of a set of beliefs K and of a revision operator $*$: in the presence of some piece of information α, the agent proceeds to a revision of her original beliefs, so that α becomes part of the new set of beliefs $K * \alpha$. The transformation $K \to K * \alpha$ is required to obey some elementary principles, like the AGM extended set of postulates. In this perspective, a conclusion β will be inferred from α iff β is a member of the revised set of beliefs $K * \alpha$. In other words, β is a consequence of α iff it is not possible to incorporate α in the agent's belief without incorporating β as well. In this framework, the subjectivity of the agent is present at two different stages. First, for the agent, the language $\mathcal{L}$ in its whole reduces to a collection of pieces of information that are evaluated comparatively to the belief base K: in order to decide whether β follows from α, the agent has to pretend that the information encoded by α is more reliable than the information encoded by K. Next, it should be emphasized that it is the agent that chooses the revision process $*$ in the context given by the set K and the formula α at hand. Quite different from this is the framework of induced systems $(D, \mid\!\sim_D)$, where the set D is not supposed to reflect the current beliefs of the agent but, rather, is meant to encode some basic information that will be used as a *criterion* by the agent to build up her beliefs or determine her behavior. In this perspective, where $(D, \mid\!\sim_D)$ may be considered as an expert system that enables the agent to draw conclusions modulo D via the inference process $\mid\!\sim_D$, any perturbation of the set D has to be studied, first of all, through its effect on the induced relation $\mid\!\sim_D$, and the dynamics of induced systems thus fundamentally differs from the classical theories of belief revision, where additional pieces of information are considered only through their effect on the set D.

In this paper, we shall not discuss the general problem of the influence, on the induced inference relation, of arbitrary changes in the original default set; we will only deal with some elementary dynamical problems. First, we shall discuss the problem of *rationalizing* an inference relation, and present some solutions that can be used to transform into a rational relation the preferential inference relation induced by an arbitrary set of defaults. Then we will restrict our attention to stratified sets of defaults and introduce a *full meet revision operator* analogue to the one introduced in the preceding section. As we shall see, this operator will prove to be an important tool in the problem of *base revision*, where one has to change the original set of defaults in order to take into account a new information. In the last part of this section, we shall deal with the problem of *conditional revision*, which occurs when a set of defaults has to be revised in order to entail a particular conditional that was not initially part of the induced relation.

5. Rationalizing an inference process

Since rational inference relations play a central role in the field of nonmonotonic reasoning, it is natural to expect from an agent a *rational behavior*, i.e., an inference process in which the rule of rational monotony is satisfied. As we saw, the behavior of an agent is fully determined by a set of defaults which we will consider as representing all the available information, and rationalizing the agent's behavior therefore amounts to correcting a default set that primitively came to induce an irrational inference process.

We are therefore considering the following situation: starting with a set of defaults $D \subseteq \mathcal{L}$, we want to transform it into a set D_r so that the induced inference relation $\mid\!\sim_{D_r}$ is rational. In looking for a solution of this problem, we have to take into account some elementary principles that may determine our choice between different possible rationalizations. Before all, it seems clear that a *minimality principle* has to be respected during the rationalization process. This *minimal change* principle may be differently interpreted, as one may focus either on the default set itself, or on the induced inference relation. In the present framework, as noted in the beginning of this section, priority is given to the induced relation, and changes on the default set can be justified only through their effect on the induced relation. We shall therefore first take into account the principle:

R-1 Minimal change of inferences. The revised inference relation should differ as little as possible from the original one.

For practical reasons, it may also be desirable to keep as much as possible from the original set of defaults. For instance, a rationalization procedure $D \to D_r$ will be acceptable only if it keeps invariant the sets D that already induce a rational inference relation, that is the stratified sets. Thus, together with R-1, we might take into consideration the principle:

R-2 Minimal change of information. The information encoded in the revised set of defaults should differ as little as possible from the original one.

Another principle can be added to R-1 and R-2, according to which no belief should be lost in the rationalizing process: by this, we mean that is a conclusion β could be initially entailed from the premiss α through the set of defaults D, this conclusion should remain valid when D changes into D_r, so that, in the presence of α, the agent will still believe in it. This can be put in the following *beliefs-preservation* principle:

R-3 Conditionals are preserved in the revision process.

Note that in any revision process $D \to D_r$, this condition is fulfilled if and only if the original induced inference relation $\mid\!\sim_D$ is a sub-relation of the new relation $\mid\!\sim_{D_r}$, and that this amounts to require the inclusion $<_D \subseteq <_{D_r}$.

Finally, in the particular case of rationalization, one may consider the *reliability* of the chosen rationalization operator: by this we mean that if two sets of defaults D and D' induce the same inference relation (in which case we will say that they are *equivalent*), then the rationalized sets D_r and D'_r should also induce the same rational relation. This can be put into a fourth principle:

R-4 Equivalence of default sets is preserved through rationalization.

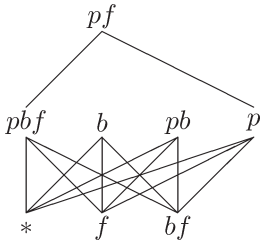
Principles R-1 to R-4 may be used in defining a set of postulates for rationalization. In this paper, we shall only present and compare two process of rationalization: the first one has the feature of being fairly simple and intuitive, while the second one seems to be the best solution to the rationalization problem with respect to the above principles.

5.1. *Chaining the set D*

Since the rationalized set D_r is supposed to induce a rational inference relation, this set must be stratified, and the problem therefore boils down to transform the original set D into a stratified set. As the simple examples of such sets are provided by logical chains, one may require the set D_r itself to be a logical chain. In this perspective, the rationalization problem amounts to construct a suitable logical chain from an arbitrary set D. The most natural way to build such a chain may be given by the following procedure, evoked in [4]:

For any integer $k \leqslant \operatorname{card} D$, denote by γ_k the formula $\gamma_k = \bigvee(\alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_k})$, where the disjunction is taken over all k-tuples of different elements of D (it may be the case that $g_k = \mathit{False}$). Set $c(D) = \{\gamma_{\operatorname{card} D},\ \gamma_{\operatorname{card} D-1}, \ldots, \gamma_1, \mathit{True}\}$. This set clearly has the structure of a logical chain with $1 + \operatorname{card} D$ elements, that can be directly computed from the set D. It thus provides a rationalization of the inference relation induced by D.

Example 18 (*Penguins*). We compute the chain $c(D)$ in the case where D is the set $\{\alpha, \beta, \gamma\}$ with $\alpha = p \to b$, $\beta = b \to f$ and $\gamma = p \to \neg f$. The order $<$ induced by D is



We have

$$\gamma_3 = \alpha \wedge \beta \wedge \gamma = \neg p \wedge (b \to f),$$

$$\gamma_2 = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \vee (\beta \wedge \gamma) = \neg p \vee b \vee \neg f,$$

$$\gamma_1 = \alpha \vee \beta \vee \gamma = \mathit{True},$$

and the chain $c(D)$ is therefore equal to $(\neg p \vee (b \to f), \neg p \vee b \vee \neg f, \mathit{True})$. The ranked model associated with the rational inference relation induced by $c(D)$ is

pf	2
pbf *b* *pb* *p*	1
$*$ *f* *bf*	0

One easily checks that the induced order $<_{c(D)}$ satisfies $m <_{c(D)} n$ iff m satisfies more formulae in D than n, that is iff $\operatorname{card}(\operatorname{form}_D(n)) < \operatorname{card}(\operatorname{form}_D(m))$. This interesting property holds in the general case:

Observation 19. *If $<_{c(D}$ denotes the order induced by $c(D)$ on W, one has for any world m, $m <_{c(D)} n$ iff* $\operatorname{card}(\operatorname{form}_D(n)) < \operatorname{card}(\operatorname{form}_D(m))$.

The proof is straightforward, observing that a world m satisfies a number k of elements of D iff it satisfies $\gamma_k \wedge \neg\gamma_{k+1}$, that is iff $\operatorname{form}_{c(D)}(m) = \{\gamma_i \mid 1 \leqslant i \leqslant k\}$.

The chaining procedure described above provides an easy way to rationalize the inference relation induced by a set D, by directly reorganizing this set into a logical chain. As shown by Observation 19, this procedure can be performed even when the original induced order is not explicitly given, and it only requires the knowledge, for each world, of the number of elements of D satisfied by this world. Thus the rationalized inference relation can be directly computed from the value, for each world m, of $\operatorname{card}(\operatorname{form}_D(m)$; in this sense, it is, by far, the most natural and the simplest of the rationalization procedures we know of.

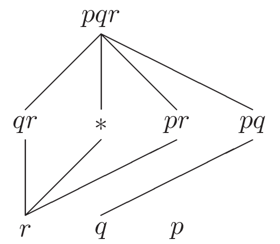
Example 20. Consider, in the propositional language built on the three variables p, q and r, the set $D = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ with $\alpha_1 = (p \vee r) \wedge \neg q$, $\alpha_2 = \neg p$, $\alpha_3 = q \wedge \neg r$, $\alpha_4 = \neg \vee q \vee r$, $\alpha_5 = \neg q \vee (\neg p \wedge r)$ and $\alpha_6 = \neg r \wedge (p \vee q)$. We write the corresponding Boolean matrix, where the intersection of the ith line and the jth column is equal to 1 iff the corresponding world satisfies the corresponding formula:

m	α_1	α_2	α_3	α_4	α_5	α_6	$\mathrm{card}_D\mathrm{form}(m)$
$*$	0	1	0	1	1	0	3
p	1	0	0	0	1	1	3
q	0	1	1	1	0	1	4
r	1	1	0	1	1	0	4
pq	0	0	1	1	0	1	3
pr	1	0	0	1	1	0	3
qr	0	1	0	1	1	0	3
pqr	0	0	0	1	0	0	1

Applying Observation 19, we can directly display the ranked model associated with $\mathrel{|\!\sim}_{c(D)}$:

pqr	2
$*\ p\ \ pq\ \ pr\ \ qr$	1
$q\ r$	0

Let us now briefly test this chaining procedure with respect to the principles discussed in the last paragraph: since the set $c(D)$ is built up from D by conjunction and disjunction, we can consider that the principle R-2 of minimal change of information is fairly satisfied. Furthermore, it readily follows from Observation 19 that $<_D \subseteq <_{c(D)}$, so that beliefs are preserved through the chaining process, which meets the requirements of principle R-3. Nevertheless, the above chaining procedure does not satisfy the principle R-1 of minimal change in the induced relation. Computing the order $<_D$ induced by D, one gets indeed the model



showing that the minimality of p is lost in the chaining process.

Even more, this rationalization process suffers from an important shortcoming, as it may be the case that two sets D and D' induce the same inference relation, while the rational relations induced by $c(D)$ and $c(D')$ differ. This may even occur when D is already a stratified set:

Example 21. Let $\mathcal{L}$ be built on the two propositional variables p and q, and $D = \{p \wedge q, \neg q, \neg p \wedge q, q\}$. We have $\mathrm{form}_D(*) = \mathrm{form}_d(p) = \{\neg q\}$, $\mathrm{form}_D(q) = \{q, \neg p \wedge q\}$ and $\mathrm{form}_D(pq) = \{q, p \wedge q\}$. It follows that the induced order $<_D$ is the empty relation, so that $\mid\!\sim_D$ is the classical consequence $\vdash$. Nevertheless, the order induced by $c(D)$ is not empty: one has indeed $q <_{c(D)^*}$.

The failure of R-1 and of R-4 shows that, in spite of its attractive simplicity, the chaining procedure $D \to c(D)$ cannot be considered as an ideal solution to the rationalization problem. It seems that the best compromise between simplicity and efficiency may be found in the *rational closure* process that we present now.

5.2. Rational closure

The construction that Lehmann and Magidor proposed in [12] to extend an arbitrary set of conditional assertions can be successfully applied to rationalize induced inference relations, and provides a rationalization operator $D \to D_{r.c.}$ that meets all the requirements of principles R-1, R-3 and R-4. This construction is performed in the following way (cf. Section 5.7 of [12] for details):

Let $(W, <)$ be the standard model associated with a faithfully representable inference relation $\mid\!\sim$. For any world m, define the *height of* m as the length of a maximal sequence $m_1 < m_2 < \cdots < m_k < m$. The rational closure of $<$ is then the order $<_{r.c}$ defined by $m <_{r.c} n$ iff the height of m is strictly less than that of n. The relation $<_{r.c}$ is then a modular order that extends $<$. The rational inference relation $\mid\!\sim_{r.c}$ represented by the model $(W, <_{r.c})$ is called the *rational closure of* $\mid\!\sim$. Its induced ranking $\kappa_{r.c}$ satisfies $\kappa_{r.c}(m) = \mathrm{height}(m)$.

Given an arbitrary set of defaults D, we define the rational closure $D_{r.c}$ of D to be the logical chain associated with the rational closure of the relation $\mid\!\sim_D$ induced by D. We have thus $D_{r.c} = \{\delta_0, \delta_1, \ldots, \delta_h = \mathit{True}\}$ where, for all indices i, δ_i is the formula such that $m \models \delta_i$ iff m is of height less or equal to i.

Example 20 (*Continued*). Computing the height of each world from the model $(W, <_D)$ of Example 20, we get the following ranked model for the rational closure of $<_D$:

pqr	2
$*\ qr\ \ pr\ \ pq$	1
$p\ \ q\ r$	0

The characteristic associated chain is

$$D_{r.c} = \{(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r), \neg p \vee \neg q \vee \neg r, \mathit{True}\}.$$

It follows from the general properties of rational closure that the rationalization process it generates fully matches the requirements of the principles R-1 and R-3 of minimal belief change and belief preservation. As it also satisfies the principle R-4—two equivalent default sets induce the same order on W, and their rational closures therefore coincide—this rational closure process provides quite an acceptable solution to the problem of rationalizing the inference induced by an arbitrary set of defaults D. It seems, though, that some improvement could be found in the construction of the rationalized default set $D_{r.c}$: we defined very simply this set to be the logical chain associated with the rational closure of the original relation $\mid\!\sim_D$, but any set inducing this relation and "closer" to D would provide a better solution. In this sense, we shall consider the rationalization problem as completely solved only once we will be able to find an rationalization operator $D \to D_r$ such that D_r induces the rational closure of $\mid\!\sim_D$ and agrees with D whenever D is stratified.

6. Base revision

When considering a set of formulae D as the knowledge base of an expert system that determines the inference process $\mid\!\sim_D$ of an agent, one has to analyze the problem of adding or retracting an information in a different perspective than in the classical theory of belief change. Basically, no problem is encountered if one wishes to incorporate in D a new element α representing a piece of information or a rule that has to be taken into account: indeed, since D may be quite an arbitrary set, generally not closed under classical consequence and not even consistent, it is enough to replace it by the set $D \cup \{\alpha\}$ or the set $D - \{\alpha\}$. Nevertheless, we already noted in the last paragraph that first place must be given to the induced inference relation, and it is through this relation that changes in D have to be studied and evaluated. Some elementary principles therefore have to be observed in this perspective, and, before all, that of *rationality preservation*: if the inference relation induced by the original set D is rational, so must be the inference relation induced by the revised set $D * \alpha$. Note, however, that violations of this principle are quite conceivable, as it is always possible to rationalize the relation induced by the revised set through its rational closure. This could lead to a stronger *rational closure preservation* principle, requiring the equality $D_{r.c} * \alpha = (D * \alpha)_{r.c}$: the revision by α of the rationalized set $D_{r.c}$ agrees with the rationalization of the revised set $D * \alpha$. In this study, though, we shall only consider stratified sets, on which we shall apply the *full meet base revision operator* $\Diamond$. This operator was defined as follows: for any stratified set D with stratification (D_j) and any consistent formula α, let i be the first index such that D_i is α-consistent, and $\beta_1, \beta_2, \ldots, \beta_n$ $(n \geqslant 0)$ the elements of D_{i-1} that are consistent with α. The full meet base revision of D by α is the set

$$D\Diamond\alpha = \{\alpha \wedge (\beta_1 \vee \beta_2 \vee \cdots \vee \beta_n)\} \bigcup_{k \geqslant i} (\alpha \wedge D_k) \cup \{True\}.$$

In order to incorporate some new data α into a stratified set D, it is quite natural to consider the set $D\Diamond\alpha$. The operator $\Diamond$ may thus be used in the framework of induced systems as a tool to revise a set of defaults by a formula α. As $\Diamond$ maps stratified sets into stratified sets, the principle of rationality preservation is satisfied. The explanation and the justification of properties FM1–FM7 remain quite relevant to the perspective adopted in this section, where D is not anymore seen as a belief base but, rather, as a set of items generating an inference process. It turns out, though, that in the framework of induced systems, the best results are obtained with the help of a *tail* added to the operator $\Diamond$:

The *tailed operator* $\blacklozenge$ is defined, for any stratified set D, by $D\blacklozenge\alpha = (D\Diamond\alpha) \cup (\alpha \vee D)$. Clearly, if a set D_i satisfies condition (2) of Definition 10, so does the set $\alpha \vee D_i$, so that $D\blacklozenge\alpha$ is a stratified set. The revision of D by α obtained through the operator $\blacklozenge$ enjoy properties similar to that of $\Diamond$. In particular, one has readily $\mathrm{Cn}(D\blacklozenge\alpha) = \mathrm{Cn}(D\Diamond\alpha) = D^*\alpha$, so that $\blacklozenge$ induces an AGM revision. We want to investigate the effect of this revision operation on the inference relation induced by D.

Let C_D be the rational inference relation induced by the stratified set D, and $C_{D\blacklozenge\alpha}$ the rational inference relation induced by the full meet base revision $D\blacklozenge\alpha$ of D by the consistent formula α. Our aim is to investigate the link existing between these to relations. To do so, it is convenient to make use of the associated characteristic sets.

Lemma 22. *The characteristic set associated with $D\blacklozenge\alpha$ is equal to $\Psi\blacklozenge\alpha$, where Ψ is the characteristic set associated with D.*

Proof. We use the notations of Observation 13: if η_i is the conjunction of the elements of D_i, and η_i' their disjunction $(1 \leqslant i \leqslant h)$, setting $\eta_{h+1} = \eta_{h+1}' = \mathit{True}$, we have $\Psi = \{\eta_i, \eta_i' \mid 1 \leqslant i \leqslant h+1\}$. Let i be the first index such that α is consistent with D_i, and β_1, $\beta_2, \ldots, \beta_n$ $(n \geqslant 0)$ the α-consistent elements of D_{i-1}. We have then

$$D\blacklozenge\alpha = \{\alpha \wedge (\beta_1 \vee \beta_2 \vee \cdots \vee \beta_n)\bigcup_{k \geqslant i}(\alpha \wedge D_k) \cup (\alpha \vee D) \cup \{\mathit{True}\}.$$

Note that the formula $\alpha \wedge (\beta_1 \vee \beta_2 \vee \cdots \vee \beta_n)$ is equal modulo classical equivalence to the formula $\alpha \wedge \eta_{i-1}'$. Using the remark made after Example 14, we see that the characteristic set associated with $D\blacklozenge\alpha$ is

$$\left\{\alpha \wedge \eta_{i-1}', \alpha \wedge \eta_i, \alpha \wedge \eta_i', \alpha \wedge \eta_{i+1}, \ldots, \alpha\right\} \cup \left\{\alpha \vee \eta_j, \alpha \vee_j' \mid 1 \leqslant j \leqslant h+1\right\}.$$

This set is precisely equal to $\Psi\blacklozenge\alpha$. $\Box$

We can now easily establish the link existing between the relations C_D and $C_{D\blacklozenge\alpha}$, respectively, induced by D and by $D\blacklozenge\alpha$. By the above lemma, we have $C_{D\blacklozenge\alpha} = C_{\Psi\blacklozenge\alpha}$. Let γ be an arbitrary formula. If γ is inconsistent with α, we have $C_{D\blacklozenge\alpha}(\gamma) = \mathrm{Cn}(\gamma, \alpha \vee \eta_i)$ where i is the first index such that η_i is consistent with γ. Thus, in this case, we have $C_{D\blacklozenge\alpha}(\gamma) = C_D(\gamma)$. If now γ is consistent with α, let j be the first index such that γ is consistent with $\alpha \wedge \eta_j$. We have therefore $C_{D\blacklozenge\alpha}(\gamma) = \mathrm{Cn}(\gamma, \alpha \wedge \eta_j)$. But j is the first index such that the element η_j of Ψ is consistent with $\alpha \wedge \gamma$. It follows that $\mathrm{Cn}(\gamma, \alpha \wedge_j) = \mathrm{Cn}(\alpha \wedge \gamma,\ \eta_j = C_\Psi(\gamma) = C_D(\gamma)$. We have therefore proven the

Theorem 23. *For all formulae γ, one has $C_{D\blacklozenge\alpha}(\gamma) = C_D(\alpha \wedge \gamma)$ if γ is consistent with α, and $C_{D\blacklozenge\alpha}(\gamma) = C_D(\gamma)$ otherwise.*

This result shows that the process of revision by the tailed operator enjoys two properties that make it well adapted to the framework of induced systems. The first one is that the consequences of a premiss γ will remain unchanged after any revision by an information α that γ contradicts. From the knowledge base $D = \{p \to b, b \to f\}$ for instance, an agent will infer that penguin generally fly $(p \mid\!\sim_D f)$, and there is no reason why this conclusion should be invalidated after the agent learns that penguin do not exist $(\neg p)$. A second interesting feature of tailed revision is that, as shown by Theorem 23, the consequences of a premiss γ are preserved through the revision by α whenever $\neg\alpha$ was not part of these consequences: indeed, if $\neg\alpha \notin C_D(\gamma)$, we have, by rationality, $C_D(\gamma) \subseteq C_D(\alpha \wedge \gamma)$, that is $C_D(\gamma) \subseteq C_{D\blacklozenge\alpha}(\gamma)$. Thus the revision of a stratified default set by the tailed operator $\blacklozenge$ provides an optimal solution to the problem of minimal change, at the level of the knowledge base as well as that of the induced inference relation.

We shall now turn our attention to the last problem of the dynamics of default reasoning and devote the next sections to the study of *conditional revision*.

7. Conditional change

In the preceding section, the revision of a default set D was considered in the perspective where a formula α had to be incorporated in the knowledge base D. In this *base revision problem*, the attention was focused before all on the set of defaults, and the solutions had to meet some requirements—the principle of minimal change, or of categorial matching, for instance—analogous to those that govern the theory of belief change. Quite different is the problem of *conditional revision*, that occurs when it appears desirable to force a change of the agent's behavior in some specific situation: for instance, we might require the agent to accept the conditional $p \wedge b \mid\!\sim f$, so that in her beliefs, penguin birds will normally fly, although such a conditional was not entailed from the initial set of defaults. Alternatively, we could require that the agent gives up a conditional primitively induced by a set of defaults. The purpose of conditional revision is therefore to modify, through a modification of the set D, the inference relation induced by D. This problem splits into two parts. In the first one, one wishes to perform a *conditional contraction* in order to retract a given conditional $\alpha \mid\!\sim \beta$, and one builds a set $D \div (\alpha \mid\!\sim \beta)$ that will not entail this conditional. In the second one, one wishes to perform a *conditional revision*: given a set of defaults D and a conditional $\alpha \mid\!\sim \beta$ that is not induced by D, one tries to build a set $D * (\alpha \mid\!\sim \beta)$ that will entail the conditional $\alpha \mid\!\sim \beta$. In both cases, the role played by the set D is secondary, as is the link between D and the revised sets $D \div (\alpha \mid\!\sim \beta)$ or $D * (\alpha \mid\!\sim \beta)$; our guiding line is the success of the operation: the conditional $\alpha \mid\!\sim \beta$ has to be entailed by $D * (\alpha \mid\!\sim \beta)$, and should not be entailed by $D \div (\alpha \mid\!\sim \beta)$. Apart from this *success postulate*, both principles R-1 (minimal change of inferences) and R-3 (belief preservation) will be determinant when comparing the value of different solutions. Thus, in the case of conditional contraction, principle R-1 commands that no new conditional should be added and that a minimal number of conditionals should be discarded, so that the revised inference relation should be a sub-relation of the original one, maximal among the sub-relations that do not entail $\alpha \mid\!\sim \beta$; similarly, in the case of conditional revision, it follows from R-1 and R-2 that the revised inference relation should contain the original one and be minimal among those that entail $\alpha \mid\!\sim \beta$.

For simplicity, we will limit our study to *rational* induced relations. Since the sets of defaults are taken into account only in as much as they generate an inference relation, we will always suppose that these sets have the structure of a logical chain. As a matter of fact, we will see that solutions to the chain conditional revision problem are not easily applicable to the general case of stratified sets.

7.1. Conditional contraction

By what precedes, the result of a contraction of a rational inference C should be a rational sub-relation of C. It is therefore important to compare the characteristic chain of C with that of a sub-relation:

Lemma 24. *Let C and C' be two rational inference relations with associated characteristic sets Ψ and Ψ'. Then C' is a sub-relation of C iff Ψ' is a subset of Ψ.*

Let us now examine the problem of conditional contraction: we suppose given a set $D = \{\delta_0, \delta_1, \ldots, \delta_n = \mathit{True}\}$ such that $\delta_i \vdash \delta_{i+1}$ for all indices $i < n$, and a conditional $\alpha \mid\!\sim \beta$ induced by D. We have thus $\beta \in C(\alpha)$, where C the rational relation induced by D. Our aim is to build a set D' that has the structure of a logical chain $(\delta'_0, \delta'_1, \ldots, \mathit{True})$ with induced relation C' that satisfies $C' \subseteq C$ and $\beta \notin C'(\alpha)$. Clearly, this problem admits no solution in the case where $\alpha \vdash \beta$, and we shall therefore suppose that β is not a classical consequence of α. By Lemma 24, we can see that this problem amounts to choose a subset D' of D for which $\alpha \mid\!\sim_{D'} \beta$ does not hold. Such subsets D' are easy to determine:

Observation 25. *Let $D' = (\delta'_k)$ be a sub-chain of $D = (\delta_l)$, and $\alpha \mid\!\sim \beta$ a conditional entailed by D and not entailed by D'. If i is the first index such that δ_i is consistent with α and j the first index such that δ_j is consistent with $\alpha \wedge \neg\beta$, one has $j > i$ and $\delta'_k \neq \delta_s$ for all indices k and s such that $i - 1 < s < j$.*

This result states that, in order not to entail the conditional $\alpha \mid\!\sim \beta$ primitively entailed by D, it is necessary to retract from D at least the sub-chain $\delta_i \vdash \delta_{i+1} \vdash \cdots \vdash \delta_{j-1}$.

Proof. Let $<$ be the modular order induced on W by D. Since we have $\alpha \mid\!\sim_D \beta$, any $<$-minimal world that satisfies α must satisfy β. By the choice of i, the minimal worlds that satisfy α have rank i, and any world of rank i therefore satisfies $\neg\alpha \vee \beta$. This shows that $\delta_i \vdash \neg\alpha \vee \beta$, hence that $i < j$. Suppose now that there exists elements δ'_k of D' such that $\delta'_k \in \{\delta_i, \delta_{i+1}, \ldots, \delta_{j-1}\}$, and let r be the first index such that $\delta'_r = \delta_s \in \{\delta_i, \delta_{i+1}, \ldots, \delta_{j-1}\}$. By the choice of i, $r = \mathrm{Min}(t \mid \delta'_t$ is consistent with $\alpha)$, so that the $<'$-minimal worlds that satisfy α satisfy δ'_r. Since, by the choice of j, δ'_r is inconsistent with $\alpha \wedge \neg\beta$, such minimal worlds satisfy β. This leads to $\alpha \mid\!\sim_{D'} \beta$, contradicting our hypothesis. □

By what precedes, any solution to the conditional contraction problem necessarily boils down to choose a "best" element among the family of the subsets D' of $D - \{\delta_i, \delta_{i+1}, \ldots, \delta_{j-1}\}$. It turns out that the maximum element of this family, the whole set $D - \{\delta_i, \delta_{i+1}, \ldots, \delta_{j-1}\}$, does the job (as do, in fact, all the elements of this family):

indeed, the chain associated with the set $D' = D - \{\delta_i, \delta_{i+1}, \ldots, \delta_{j-1}\}$ is the chain $\delta_0 \vdash \delta_1 \vdash \cdots \vdash \delta_{i-1} \vdash \delta_j \vdash \delta_{j+1} \vdash \cdots \vdash$ *True*. The first element of this chain that is consistent with α is the element δ_j, which is consistent with $\alpha \wedge \beta$. It follows that there exists a world m, $<'$-minimal among those that satisfy α, such that m satisfies $\neg\beta$. This is equivalent to say that one does not have $\alpha \mid\!\sim_{D'} \beta$.

We can now summarize our results, observing that by the definition of i and j, we have $i = j$ iff C does not entail $\alpha \mid\!\sim \beta$.

Theorem 26. *Let C be a rational inference relation, and α and β two formulae such that $\beta \notin \mathrm{Cn}(\alpha)$. Then there exists a greatest rational sub-relation C' of C such that $\beta \notin C'(\alpha)$. If C is induced by the logical chain $\delta_0 \vdash \delta_1 \vdash \cdots \vdash$ True, C' is induced by the sub-chain $\delta_0 \vdash \delta_1 \vdash \cdots \vdash \delta_{i-1} \vdash \delta_j \vdash \delta_{j+1} \vdash \cdots$ True, where i is the first index such that δ_i is consistent with α, and j is the first index such that δ_j is consistent with $\alpha \wedge \neg\beta$.*

If D is the set $\{\delta_0, \delta_1, \ldots, \mathit{True}\}$, the characteristic chain of the relation C' will be referred to as the *contraction of D by the conditional* $\alpha \mid\!\sim \beta$, and denoted by $D \div (\alpha \mid\!\sim \beta)$. Note that $D = D \div (\alpha \mid\!\sim \beta)$ iff D does not entail $\alpha \mid\!\sim \beta$.

Example 18 (*Continued*). We have seen that the chain $D = \{\delta_0, \delta_1, \delta_2\}$ with $\delta_0 = \neg p \wedge (b \to f)$, $\delta_1 = \neg p \vee b \vee \neg f$ and $\delta_2 = \mathit{True}$ induced the following ranking

	pf			2
pbf	*b*	*pb*	*p*	1
$*$ *f*	*bf*			0

Note that D entails the conditional $p \wedge f \mid\!\sim b$, so that, for any agent using the expert system $(D, \mid\!\sim_D)$, penguins that fly are normally birds. Let us determine the contraction $D \div (p \wedge f \mid\!\sim b)$ of D by $p \wedge f \mid\!\sim b$: with the above notation, we have $i = 1$ and $j = 2$. The result of this contraction is therefore the set $D \div (p \wedge f \mid\!\sim b) = \{\delta_0, \delta_2\}$, inducing the ranked model

pf	*pbf*	*b*	*pb*	*p*	1
$*$	*f*	*bf*			0

It is possible to directly display the ranking associated with the revised default set $D' = D \div (\alpha \mid\!\sim \beta)$, without computing explicitly the elements of this chain: note indeed that we have $i = \mathrm{Min}(\kappa(m) \mid m \models \alpha)$ and $j = \mathrm{Min}(\kappa(m) \mid m \models \alpha \wedge \neg\beta)$ where κ is the ranking associated with D. If κ' is the ranking associated with D', we have $\kappa'(m) = \mathrm{Min}(k \notin \{i, i+1, \ldots, j-1\} \mid m \models \delta_k)$. *The ranked model associated with D' is therefore obtained from the original model by putting at the same κ'-rank all the worlds m such that* $i \leqslant \kappa(m) < j$. In the above example, for instance, worlds of rank 1 and of rank 2 move at the same level.

7.2. Conditional revision

In this section, we consider the problem of *conditional revision*, dual to the one of conditional contraction: given a conditional $\alpha \mid\!\sim \beta$ that is not entailed by a logical chain D, we want to perform changes on D in order to get a chain D' that entails $\alpha \mid\!\sim \beta$. Consider for instance the set D of the preceding example. The ranked model associated with D was

	pf			2
pbf	b	pb	p	1
	$*\ f$	bf		0

which did not entail $p \wedge b \mid\!\sim f$ since the world pb, minimal in $\mathrm{Mof}(p \wedge b)$, does not satisfy f. If it appears desirable that, in the agent's beliefs, penguin-birds fly, one has to modify the knowledge base D into a set D' that yields the conditional $p \wedge b \mid\!\sim b$.

Let C and C' be the rational relations induced by D and the set D' we are looking for. By hypothesis, we have $\beta \notin C(\alpha)$, and C' has the property that $\beta \in C'(\alpha)$. In view of the belief preservation principle, C must be a sub-relation of C'. Nevertheless, this will be impossible if $\neg\beta \in C(\alpha)$, since, in this case, C' would not be consistency-preserving. It will be therefore necessary to first perform a *conditional contraction*, in order to retract the conditional $\alpha \mid\!\sim \neg\beta$ from C.

We suppose this has been done, so that, possibly after a contraction, the set D does not entail the conditional $\alpha \mid\!\sim \neg\beta$. It is then possible to consider the family of rational extensions of C that entail $\alpha \mid\!\sim \beta$, and to choose for C' a minimal element of this family. By Lemma 24 indeed, this amounts to build a minimal chain $D' = (\delta'_i)$ extending $D = (\delta_j)$, with $\beta \in \mathrm{Cn}(\alpha, \delta'_j)$, where j is the first index such that α is consistent with δ'_i. The procedure we propose is the following one: denote by i the first index such that α is consistent with δ_i. We simply define the conditional revision $D * (\alpha \mid\!\sim \beta)$ of D by $(\alpha \mid\!\sim \beta)$ to be the set $D' = (\delta_0, \delta_1, \ldots, \delta_{i-1}, \delta_{i-1} \vee (\delta_i \wedge (\neg\alpha \vee \beta), \delta_i, \delta_{i+1}, \ldots, \mathit{True})$ that is obtained from D by adding the single formula $\delta' = \delta_{i-1} \vee (\delta_i \wedge (\neg\alpha \vee \beta)$. Observe that the hypothesis $\neg\beta \notin C(\alpha) = \mathrm{Cn}(\alpha, \delta_i)$ implies that δ' is the first element of the chain D' that is consistent with α. Since we have readily $\beta \in \mathrm{Cn}(\alpha, \delta')$, we see that, as desired, the conditional $\alpha \mid\!\sim \beta$ is entailed by the revised set D'. Note that the sets D and D' coincide iff $\beta \in \mathrm{Cn}(\alpha, \delta_i)$ that is iff D already entailed the conditional $\alpha \mid\!\sim \beta$.

It is easy to describe the semantics of the rational relation induced by the set D' defined as above: the revised ranking κ' is given by $\kappa'(m) = \kappa(m)$ if $\kappa(m) \leqslant i - 1$, $\kappa'(m) = i$ if $\kappa(m) = i$ and $m \models \neg\alpha \vee \beta$, and $\kappa'(m) = \kappa(m) + 1$ otherwise. Thus, a new rank has simply been added between the original ranks i and $i + 1$, which consists of all worlds m that satisfy $\neg\beta$ and had minimal κ-rank in $\mathrm{mod}(\alpha)$.

Example 18 (*Continued*). Consider the chain $D = \{\delta_0, \delta_1, \delta_2\}$ of Example 18, with $\delta_0 = \neg p \wedge (b \to f)$, $\delta_1 = \neg p \vee b \vee \neg f$ and $\delta_2 = \mathit{True}$, inducing the ranking

	pf			2
pbf	*b*	*pb*	*p*	1
	$*$ *f*	*bf*		0

To perform the revision of D by $p \wedge b \mid\!\sim f$, we determine the first index i such that δ_i is consistent with $p \wedge b$. We find $i = 1$. The revised chain is therefore

$$\begin{aligned} D' &= D * (p \wedge b \mid\!\sim f) \\ &= (\neg p \wedge (b \to f), \neg p \wedge (b \to f) \vee [(\neg p \vee \neg b \vee \neg f) \wedge (\neg p \vee \neg b \vee f)], \\ &\quad \neg p \vee b \vee f, \mathit{True}), \end{aligned}$$

that is

$$D' = (\neg p \wedge (\neg b \vee f), \neg p \vee [(b \vee \neg f) \wedge (\neg b \vee f)], \neg p \vee b \vee f, \mathit{True}).$$

The induced κ' ranking is

	pf		3
	pb		2
p	*b*	*pbf*	1
$*$	*f*	*bf*	0

The world *pb*, that satisfies $\neg f$, had κ-rank 1 and was minimal in mod$(p \wedge b)$. In the new ranking, it is given rank 2. The conditional $p \wedge b \mid\!\sim f$ is now entailed by D'.

It is clear that there exists several minimal solutions to the conditional revision problem. Instead of adding to D the formula $\delta' = \delta_{i-1} \vee (\delta_i \wedge (\neg\alpha \vee \beta))$, we could have considered for instance the set $D'' = D \cup \{\delta_{i-1} \vee (\delta_i \wedge \beta)\}$, that also entails the conditional $\alpha \mid\!\sim \beta$. The choice of δ' is nevertheless imposed by the principle R-2 of minimal change in the default set: one shows indeed easily that if δ'' *is a formula such that the chain* $(\delta_0, \ldots, \delta_{i-1}, \delta'', \delta_1, \delta_{i+1}, \ldots, \mathit{True})$ *induces the conditional* $\alpha \mid\!\sim \beta$, then $\delta'' \vdash \delta'$. In this sense, the solution we presented here may be considered as the best solution to conditional revision.

So far, we supposed that the initial set of defaults D did not entail the conditional $\alpha \mid\!\sim \neg\beta$, and mentioned that it was always possible to retract this conditional if this condition was not satisfied. By this, we mean that we define the revision of D by $(\alpha \mid\!\sim \beta)$ as the set $(D \div (\alpha \mid\!\sim \neg\beta)) * (\alpha \mid\!\sim \beta)$. Using the results of the preceding section, we have $D \div (\alpha \mid\!\sim \neg\beta) = (\delta_0, \delta_1, \ldots, \delta_{i-1}, \delta_j, \delta_{j+1}, \ldots, \mathit{True})$, where i is the first index such that δ_i is consistent with α, and j the first index such that δ_j is consistent with $\alpha \wedge \beta$. Note that δ_j becomes the first element of the contracted chain $D \div (\alpha \mid\!\sim \neg\beta)$ that is consistent with α. Recall that, as observed before, we have $D = D \div (\alpha \mid\!\sim \neg\beta)$ iff D does not entail $\alpha \mid\!\sim \neg\beta$. We finally proceed to the revision of $D \div (\alpha \mid\!\sim \neg\beta)$ by $\alpha \mid\!\sim \beta$

as described above, and get the chain $D * (\alpha \mid\!\sim \beta)$ equal to $(\delta_0, \delta_1, \ldots, \delta_{i-1}, \delta_{i-1} \vee (\delta_j \wedge (\neg\alpha \vee \beta)), \delta_j, \delta_{j+1}, \ldots, \mathit{True})$. This proves the

Theorem 27. *Given two formulae α and β such that β is consistent with α, there exists a unique conditional revision process that fully satisfies the principles* R-1, R-2 *and* R-3 *and transforms a logical chain D into a logical chain $D * (\alpha \mid\!\sim \beta)$ inducing the conditional $\alpha \mid\!\sim \beta$. If i is the first index such that δ_i is consistent with α, and j the first index such that δ_j is consistent with $\alpha \wedge \beta$, the chain $D * (\alpha \mid\!\sim \beta)$ is equal to $(\delta_0, \delta_1, \delta_{i-1}, \delta_{i-1} \vee (\delta_j \wedge (\neg\alpha \vee \beta)), \delta_j, \delta_{j+1}, \ldots, \mathit{True})$.*

The semantics induced by $D * (\alpha \mid\!\sim \beta)$ is easily described, using first the semantics of conditional contractions.

Example 28. Consider, in the language built on the three variables p, q, r, the chain $D = (\delta_0, \delta_1, \delta_2, \mathit{True})$, where $\delta_0 = p \wedge q \wedge r, \delta_1 = q \wedge (p \vee r), \delta_2 = p \vee q \vee r$. The associated ranked model is

$*$	3
p q pr r	2
pq qr	1
pqr	0

We want to revise D by the conditional $\neg p \wedge r \mid\!\sim \neg q$. With the above notations, we find $i = 1$ and $j = 2$. The revised chain $D * (\neg p \wedge r \mid\!\sim \neg q)$ is therefore equal

$$(p \wedge q \wedge r, (p \wedge q \wedge r) \vee ((p \vee q \vee r) \wedge (p \vee \neg r \vee \neg q), (p \vee q \vee r), \mathit{True}).$$

We thus have

$$D * (\neg p \wedge r \mid\!\sim \neg q) = (p \wedge q \wedge r, p \vee (q \wedge \neg) \vee (\neg q \wedge r), p \vee q \vee r), \mathit{True}).$$

To find the associated model, we first display the model of $D \div (\neg p \wedge r \mid\!\sim q)$, which is obtained by putting at the same rank worlds that had rank 1 and 2:

$*$	2
p q pr r pq qr	1
pqr	0

We then note that there exits one world that minimally satisfies $\neg p \wedge r$, and satisfies q, which is the world qr. This world is then given a new rank 2, and the model associated with $D * (\neg p \wedge r \mid\!\sim \neg q)$ is therefore equal to

$*$	3
qr	2
p q pr r pq	1
pqr	0

8. Conclusion and further work

In studying the relations between a set of formulae and its induced inference relation, we were able to establish interesting results in several important domains that either have to do with the statics of induced systems (e.g., our characterization of the analogue of rationality in the Poole–Shoham duality), or concern problems linked to the dynamics of these systems, like those of rationalization or of conditional revision. The solutions that we proposed are effective and adequate, but it is clear that some improvements have to be found in several points. Let us mention two of them: first, the problem of rationalization through rational closure cannot be considered as settled, since it is far from satisfying the principle of base-minimal change. An ideal solution would provide, from an arbitrary set D, a stratified set D_r that 1) would induce the rational closure of the inference relation induced by D, and 2) would minimally differ from D. In the solution we presented, the rationalized set was taken to be a logical chain, which, in fact, had little to do with the original set D. The second improvement that can be searched in the dynamics of induced systems is the possibility to extend to arbitrary stratified sets the constructions of conditional contractions and revisions that were made in Section 7 for logical chains. It would be interesting to know if, in the general framework of stratified sets, a unique solution could still be proposed for this revision problem.

It should be finally emphasized that the present work only dealt with logically finite languages: in an arbitrary propositional language, it is possible to apply these results when dealing with finite sets of defaults by considering an adequate finite sub-language, but, in the general case, nothing is known about the relationship between the structure of a set of defaults and that of its induced inference relation. Among the open problems in this area remains that, evoked in a preceding paper, of characterizing all the defaults sets that induce a rational inference relation.

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Appendix A. Proofs of lemmas and theorems

Lemma 8. *Let $\mid\sim$ be a consistency-preserving rational inference relation defined on a logically finite language, $\Psi = \{\psi_i \mid 0 < i \leqslant h\}$ its characteristic set and Δ its basic set of defaults. Then a formula α is an element of Δ iff $\alpha \vdash \psi_0$ or there exists an index i, $1 \leqslant i < h$, such that $\psi_i \vdash \alpha \vdash \psi_{i+1}$.*

Proof. Suppose first that a formula α satisfies the property stated in the lemma and let us prove that $\alpha \in \Delta$, i.e., that property $(*)$ holds for α. Clearly we can suppose that α is consistent. Note that it follows from the definition of ψ_i that a world p satisfies ψ_i iff $\kappa(p) < i$, where κ is the ranking function associated with $\mid\sim$. Let n be a world that satisfies α. If $\alpha \vdash \psi_1$, we have $n \models \psi_1$, n is of rank 0, hence n is minimal and the condition $(*)$ is trivially satisfied. If α is such that $\psi_i \vdash \alpha \not\Updownarrow \psi_{i+1}$, n is of rank $\kappa(n) \leqslant i$. For any world $m < n$, we have $\kappa(m) < i$, showing that m satisfies ψ_i, hence $m \models \alpha$. This shows that α satisfies condition $(*)$ and is therefore an element of Δ.

Conversely, let α be an element of Δ. If α is a contradiction, we have trivially $\alpha \vdash \psi_1$. If α is consistent, let i be rank of a world n of maximal rank that satisfies α. Any world m satisfying α has therefore a rank $\kappa(m) \leqslant i$, and therefore satisfies ψ_{i+1}. This shows that $\alpha \vdash \psi_{i+1}$. If now a world p satisfies ψ_i, we have $\kappa(p) < i = \kappa(n)$, that is $p < n$. By condition $(*)$, we have $p \models \alpha$, and this shows that $\psi_i \vdash \alpha$, completing the proof of the lemma. □

Theorem 12. *The inference relation induced by a set D is rational iff D is stratified.*

Proof. Suppose first that the inference relation induced by D is rational, and let us prove that D is a stratified set. Denote by $\Psi = \{\psi_1, \psi_2, \ldots, \psi_h\}$ the characteristic set of the induced inference relation and by Δ its basic set of defaults. For each index i, $0 < i < h$, let Δ_i be the subset of Δ consisting of all formulae $\alpha \in \Delta$ such that $\alpha \neq \psi_{i+1}$ and $\psi_i \vdash \alpha \vdash \psi_{i+1}$. Set $\Delta_0 = \{\alpha \in \Delta \mid \alpha \vdash \psi_1 \text{ and } \alpha \neq \psi_i\}$ and $\Delta_h = \{\mathit{True}\}$. We know from Observation 4 that $D \subseteq \Delta$. For each index i, let D_i be the set $D \cap \Delta_i$ (it may be the case that $D_i = \emptyset$). The set D is then a disjoint union of the sets D_i's, and condition (1) of Definition 10 is clearly satisfied. To check condition (2), suppose first that $i > 0$, and let X be a nonempty strict subset of D_i determinable in D_i. There exists a world n such that $\mathrm{form}(n) \cap D_i = X$. Since n satisfies elements of Δ_i, we have $\kappa(n) \leqslant i$, where κ is the ranking function associated with the rational relation induced by D. Note that, X being a strict subset of D_i, we do not have $n \models D_i$, so that n does not satisfy ψ_i. This shows that $\psi(n) = i$, and it follows from condition (1) that $\mathrm{form}_D(n) = X \bigcup_{j>i} D_j$. If X were not a maximal strict determinable subset of D_i, there would exist a strict subset Y of D_i with $X \subset Y$, and a world m such that $\mathrm{form}(m) \cap D_i = Y$. As noticed above, this would imply $\kappa(m) = i$ and $\mathrm{form}_D(m) = Y \bigcup_{j>i} D_j$. We would then have $m <_D n$, contradicting the fact that m and n have same rank i. This shows that condition (2) of the definition is satisfied for $i > 0$. If now X is a strict nonempty determinable subset of D_0, and n a world such that $\mathrm{form}(n) \cap D_0 = X$, we see that n has rank 0 and is therefore $<_D$-minimal. One concludes, that X is maximal among the determinable strict subsets of D_0.

Conversely, let us prove now that the inference relation induced by a stratified set D is rational. To do so, we shall show that $<_D$ is a modular order. We denote by (D_i) a stratification of D.

Let n, m and p be elements of W such that $m <_D n$ and not $m <_D p$. We have to check that $p <_D n$. Suppose first that $\mathrm{form}_D(n) = \emptyset$. Since we do not have $m <_D p$, the set $\mathrm{form}_D(p)$ is not empty, and it follows that $p <_D n$ as desired. We are therefore left with the case where $\mathrm{form}_D(n)$ is not empty. Let j be the first index such that $\mathrm{form}(n) \cap D_j \neq \emptyset$. By condition (1) of Definition 10, we have readily $\mathrm{form}(n) \cap D = \mathrm{form}(n) \cap D_j \bigcup_{k>j} D_k$. This set is a strict subset of $\mathrm{form}_D(m)$, as results from the inequality $m <_D n$. By condition (1) again, we have then $\mathrm{form}(n) \cap D_J \subset \mathrm{form}(m) \cap D_j$. Applying now condition (2), we see that $\mathrm{form}(m) \cap D_j = D_j$, showing that $m \models D_j$. Since we do not have $m <_D p$, we have either $\mathrm{form}_D(m) = \mathrm{form}_D(p)$, in which case we get $p <_D n$ as desired, or

$\mathrm{form}_D(p) \nsubseteq \mathrm{form}_D(m)$. In the latter case, there exists an element α of D that is satisfied by p and not by m. Clearly, $\alpha \notin \bigcup_{k \geqslant j} D_k$ and, by condition (1), $\alpha \vdash D_k$ for all $k \geqslant j$. Since $p \models \alpha$, we have then $\{\alpha\} \bigcup_{k \geqslant j} D_k \subseteq \mathrm{form}_D(p)$. As m does not satisfy α, and $m <_D n$, n does not satisfy α either, and $\mathrm{form}_D(n)$ is therefore a strict subset of $\{\alpha\} \bigcup_{k \geqslant j} D_k$. This leads to $\mathrm{form}_D(n) \subset \mathrm{form}_D(p)$, showing that $p <_D n$ as desired. □

Observation 13. *Let D be a stratified set and $(D_1, D_2, \ldots, D_h)$ be a stratification of D. For any index i, $1 \leqslant i \leqslant h$, denote by η_i the conjunction of all elements of D_i, and by η'_i the disjunction of all elements of D_i. Set $\eta_{h+1} = \eta'_{h+1} = True$. Then the characteristic set Ψ of the rational inference relation induced by D is equal the set $\{\eta_i, \eta'_i \mid 1 \leqslant i \leqslant h+1\}$.*

Proof. Let Ψ' be the set $\{\eta_i, \eta'_i \mid 1 \leqslant i \leqslant h\}$. Clearly, Ψ' has the structure of a logical chain. To show equality (modulo classical equivalence) between the two logical chains Ψ and Ψ', it is enough to show that they induce the same inference relation, and this amounts to show that check that the induced orders $<_\Psi$ and $<_{\Psi'}$ are equal. Since we have $<_\Psi = <_D$, we have therefore to prove that the orders $<_D$ and $<_{\Psi'}$ coincide, that is that the inequality $m <_D n$ holds iff there exists a formula in Ψ' that is satisfied by m and not by n.

Suppose first that $m <_D n$. Then there exists a formula $\alpha \in D$ that is satisfied by m and not by n. Let i be the first index such that $\alpha \in D_i$. If m satisfies the whole set D_i, we see that the formula $\eta_i \in \Psi'$ is satisfied by m and not by n. If now $\mathrm{form}(m) \cap D_i$ is a strict subset of D_i, it follows from condition (2) together with the inequality $m <_D n$ that $\mathrm{form}(n) \cap D_i = \emptyset$, and thus that the element η'_i of Ψ' is satisfied by m and not by n.

Suppose now that there exists an element of Ψ' that is satisfied by m and not by n. This element is of the form η_i or η'_i. In the first case, m satisfies the whole set D_i, which is not satisfied by n. Condition (1) then implies that $\mathrm{form}_D(m) \supseteq \bigcup_{j \geqslant i} D_j$ while $\mathrm{form}_D(n) \subset \bigcup_{j \geqslant i} D_j$. It follows that $m <_D n$. In the second case, we have $\mathrm{form}(m) \cap D_i \neq \emptyset$, $\mathrm{form}(n) \cap D_i = \emptyset$, and we conclude similarly that $m <_D n$. □

Lemma 24. *Let C and C' be two rational inference relations with associated characteristic sets Ψ and Ψ'. Then C' is a sub-relation of C iff Ψ' is a subset of Ψ.*

Proof. It is immediate that $C' \subseteq C$ whenever $\Psi' \subseteq \Psi$. Conversely, suppose that C' is a sub-relation of C. Denote by κ and Ψ' the associated rankings and by $<$ and $<'$ the associated modular orders. Note that $<'$ is a sub-relation of $<$. Let $(\psi'_0, \psi'_1, \ldots, True)$ and $(\psi_0, \psi_1, \ldots, True)$ be the characteristic chains associated with C' and C. Since any $<$-minimal world is $<'$-minimal, we have $\psi_0 \vdash \psi'_0$. Let ψ'_i be any element of Ψ', and j the first index such that $\psi'_i \vdash \psi_j$. We claim that these formulae are equal modulo classical equivalence, that is that $\psi_j \vdash \psi'_i$. If $j = 0$, this results from the fact that $\psi_0 \vdash \psi'_0 \vdash \psi'_i$. If $j > 0$, there exists a world p that satisfies $\psi'_i \wedge \neg\psi_{j-1}$. This world has κ-rank $\kappa(p) = j$. Note that, for any world m, $\kappa'(m) > i$ implies $\kappa(m) > j$: indeed, if $\kappa'(m) > i$, we have $p <' m$, hence $p < m$, yielding $\kappa(m) > j$. If now a world q satisfies ψ_j, we have, by what precedes, $\kappa'(q) \leqslant i$, so that q satisfies ψ'_i. This shows that $\psi_j \vdash \psi'_i$ and the proof of the lemma is complete. □

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