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Weak nonmonotonic probabilistic logics ☆

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Abstract

We present an approach where probabilistic logic is combined with default reasoning from conditional knowledge bases in Kraus et al.'s System P, Pearl's System Z, and Lehmann's lexicographic entailment. The resulting probabilistic generalizations of default reasoning from conditional knowledge bases allow for handling in a uniform framework strict logical knowledge, default logical knowledge, as well as purely probabilistic knowledge. Interestingly, probabilistic entailment in System P coincides with probabilistic entailment under g-coherence from imprecise probability assessments. We then analyze the semantic and nonmonotonic properties of the new formalisms. It turns out that they all are proper generalizations of their classical counterparts and have similar properties as them. In particular, they all satisfy the rationality postulates of System P and some Conditioning property. Moreover, probabilistic entailment in System Z and probabilistic lexicographic entailment both satisfy the property of Rational Monotonicity and some Irrelevance property, while probabilistic entailment in System P does not. We also analyze the relationships between the new formalisms. Here, probabilistic entailment in System P is weaker than probabilistic entailment in System Z, which in turn is weaker than probabilistic lexicographic entailment. Moreover, they all are weaker than entailment in probabilistic logic where default sentences are interpreted as strict sentences. Under natural conditions, probabilistic entailment in System Z and lexicographic entailment even coincide with such entailment in probabilistic logic, while probabilistic entailment in System P does

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not. Finally, we also present algorithms for reasoning under probabilistic entailment in System Z and probabilistic lexicographic entailment, and we give a precise picture of its complexity.

Keywords: Probabilistic logic; Default reasoning from conditional knowledge bases; Entailment in System P; Entailment in System Z; Lexicographic entailment; Nonmonotonic probabilistic logics; Inconsistency handling; Algorithms; Computational complexity

1. Introduction

During the recent decades, reasoning about probabilities has started to play an important role in AI. In particular, reasoning about interval restrictions for conditional probabilities, also called conditional constraints [49], has been a subject of extensive research efforts. Roughly, a conditional constraint is of the form $(\psi|\phi)[l,u]$, where ψ and ϕ are events, and $[l,u]$ is a subinterval of the unit interval $[0,1]$. It encodes that the conditional probability of ψ given ϕ lies in $[l,u]$.

An important approach for handling conditional constraints is probabilistic logic, which has its origin in philosophy and logic, and whose roots can be traced back to already Boole in 1854 [12]. There is a wide spectrum of formal languages that have been explored in probabilistic logic, ranging from constraints for unconditional and conditional events to rich languages that specify linear inequalities over events (see especially the work by Nilsson [54,55], Fagin et al. [19], Dubois and Prade et al. [2,13,16,17], Frisch and Haddawy [21], and the author [48,49,51]; see also the survey on sentential probability logic by Hailperin [35]). The main decision and optimization problems in probabilistic logic are deciding satisfiability, deciding logical consequence, and computing tight logically entailed intervals. Recently, column generation techniques from operations research have been successfully used to solve large problem instances in probabilistic logic (see especially the work by Jaumard et al. [37] and Hansen et al. [36]).

Example 1.1 (*Eagles*). A simple collection of conditional constraints *KB* may encode the *strict logical knowledge* "all eagles are birds" and "all birds have feathers" as well as the *purely probabilistic knowledge* "birds fly with a probability of at least 0.95" (cf. Example 2.1). This collection of conditional constraints *KB* is satisfiable, and some logical consequences in probabilistic logic from *KB* are "all birds have feathers", "birds fly with a probability of at least 0.95", "all eagles have feathers", and "eagles fly with a probability between 0 and 1"; in fact, these are the tightest intervals that follow from *KB* (cf. Example 2.2). That is, we especially cannot conclude anything from *KB* about the ability to fly of eagles.

A closely related research area is default reasoning from conditional knowledge bases, which consist of a collection of strict statements in classical logic and a collection of defeasible rules, also called defaults. The former must always hold, while the latter are rules of the kind $\psi \leftarrow \phi$, which read as "generally, if ϕ then ψ". Such rules may have exceptions, which can be handled in different ways.

The literature contains several different proposals for default reasoning from conditional knowledge bases and extensive work on its desired properties. The core of these properties are the rationality postulates of System P by Kraus, Lehmann, and Magidor [40], which constitute a sound and complete axiom system for several classical model-theoretic entailment relations under uncertainty measures on worlds. They characterize classical model-theoretic entailment under preferential structures [40,64], infinitesimal probabilities [1,57], possibility measures [14], and world rankings [33,65]. As shown by Friedman and Halpern [20], many of these uncertainty measures on worlds are expressible as plausibility measures. The postulates of System P also characterize an entailment relation based on conditional objects [15]. A survey of the above relationships is given in [6,22].

Mainly to solve problems with irrelevant information, the notion of rational closure as a more adventurous notion of entailment was introduced by Lehmann [45,47]. It is equivalent to entailment in System Z by Pearl [58], to the least specific possibility entailment by Benferhat et al. [5], and to a conditional (modal) logic-based entailment by Lamarre [44]. Finally, mainly to solve problems with property inheritance from classes to exceptional subclasses, the maximum entropy approach to default entailment was proposed by Goldszmidt et al. [31]; lexicographic entailment was introduced by Lehmann [46] and Benferhat et al. [4]; conditional entailment was proposed by Geffner [24,26]; and an infinitesimal belief function approach was suggested by Benferhat et al. [7]. The following example due to Goldszmidt and Pearl [34] illustrates default reasoning from conditional knowledge bases.

Example 1.2 (*Penguins*). A conditional knowledge base *KB* may encode the *strict logical knowledge* "all penguins are birds" and the *default logical knowledge* "generally, birds fly", "generally, penguins do not fly", and "generally, birds have wings". Some desirable conclusions from *KB* [34] are "generally, birds fly" and "generally, birds have wings" (which both belong to *KB*), "generally, penguins have wings" (since the set of all penguins is a subclass of the set of all birds, and thus penguins should inherit all properties of birds), "generally, penguins do not fly" (since properties of more specific classes should override inherited properties of less specific classes), and "generally, red birds fly" (since "red" is not mentioned at all in *KB* and thus should be considered irrelevant to the ability to fly of birds).

There are several works in the literature on probabilistic foundations for default reasoning from conditional knowledge bases [1,11,31,57], on combinations of Reiter's default logic [63] with statistical inference [43,67], and on a rich first-order formalism for deriving degrees of belief from statistical knowledge including default statements [3]. However, there has been no work so far that extends probabilistic logic by the capability of handling defaults as in conditional knowledge bases.

In this paper, we try to fill this gap. We present extensions of probabilistic logic by defaults as in conditional knowledge bases under Kraus et al.'s System P [40], Pearl's System Z [58], and Lehmann's lexicographic entailment [46]. The new formalisms allow for expressing in a uniform framework *strict logical knowledge* and *purely probabilistic knowledge* from probabilistic logic, as well as *default logical knowledge* from default reasoning from conditional knowledge bases. Informally, strict logical knowledge represents sentences that must always hold, while purely probabilistic (resp., default logical) knowledge encodes sentences that may have exceptions, which is expressed in a quantitative (resp., qualitative) way.

Example 1.3 (*Ostriches*). Consider the *strict logical knowledge* “all ostriches are birds”, the *default logical knowledge* “generally, birds have legs” and “generally, birds fly”, and the *purely probabilistic knowledge* “ostriches fly with a probability of at most 0.05”. Obviously, some desired conclusions are “generally, birds have legs”, “generally, birds fly”, and “ostriches fly with a probability of at most 0.05”, since these sentences are explicitly stated above. Two other desired conclusions are “generally, ostriches have legs” (since the property of having legs of birds should be inherited down to the subclass of all ostriches) and “generally, red birds fly” (since the property of being red is not mentioned above, and thus it should be irrelevant to the ability to fly). But neither probabilistic logic nor default reasoning from conditional knowledge bases can produce all these desired conclusions, since the former cannot handle default logical knowledge, while the latter cannot deal with purely probabilistic knowledge. However, in the new formalisms of this paper, we can deal with all the above sentences. In particular, the probabilistic generalization of lexicographic entailment also produces all the above desired conclusions.

A companion paper [52] presents similar probabilistic generalizations of default reasoning from conditional knowledge bases. The formalisms in [52], however, are quite different from the ones in this paper, since they allow for handling *default purely probabilistic knowledge* rather than *(strict) purely probabilistic knowledge* in addition to strict logical knowledge and default logical knowledge. For example, the formalisms in [52] allow for expressing sentences of the form “*generally*, birds (and special birds) fly with a probability of at least 0.95” rather than “birds fly with a probability of at least 0.95”. Intuitively, the former means that being able to fly with a probability of at least 0.95 should apply to the class of all birds and all subclasses of birds, as long as this is consistent, while the latter says that being able to fly with a probability of at least 0.95 should only apply to the class of all birds. For this reason, the formalisms in [52] are generally much stronger than the ones here (cf. Section 8.1). Hence, they can be considered as *strong nonmonotonic probabilistic logics*, while the formalisms here are *weak nonmonotonic probabilistic logics*.

Interestingly, probabilistic reasoning in the probabilistic generalization of Kraus et al.’s System P in the present paper coincides with probabilistic reasoning under g-coherence from imprecise probability assessments in statistics (cf. Section 8.2).

The main contributions of this paper can be summarized as follows:

- We present combinations of probabilistic reasoning in probabilistic logic with default reasoning from conditional knowledge bases under Kraus et al.’s System P [40], Pearl’s System Z [58], and Lehmann’s lexicographic approach [46]. The resulting probabilistic formalisms, also called *weak nonmonotonic probabilistic logics*, allow for handling in a uniform framework strict logical knowledge and purely probabilistic knowledge from probabilistic logic, as well as default logical knowledge from conditional knowledge bases.
- We explore the nonmonotonic properties of the three weak nonmonotonic probabilistic logics. In particular, they all three satisfy the rationality postulates of System P

and have some Conditioning property. Furthermore, probabilistic entailment in System Z and probabilistic lexicographic entailment both satisfy the property of Rational Monotonicity and have some Irrelevance property, while probabilistic entailment in System P is lacking these two properties.

- We analyze the relationships between the three weak nonmonotonic probabilistic logics. It turns out that probabilistic entailment in System P is weaker than probabilistic entailment in System Z, which in turn is weaker than probabilistic lexicographic entailment. Furthermore, we show that all three formalisms are weaker than entailment in probabilistic logic from knowledge bases in which all the default sentences are simply interpreted as strict sentences.
- We show that probabilistic entailment in System Z and probabilistic lexicographic entailment coincide with entailment in probabilistic logic, whenever it is consistent to interpret all relevant default sentences as strict sentences, while probabilistic entailment in System P does not have this property. Furthermore, probabilistic entailment in Systems P and Z as well as probabilistic lexicographic entailment are proper generalizations of their classical counterparts.
- Finally, we present algorithms for computing tight intervals under probabilistic entailment in System Z and probabilistic lexicographic entailment, which are based on reductions to the standard tasks of deciding model existence and computing tight intervals under entailment in probabilistic logic. Furthermore, we draw a precise picture of the complexity of deciding logical consequence and of computing tight intervals under probabilistic entailment in System Z and probabilistic lexicographic entailment in general as well as restricted cases.

The rest of this paper is organized as follows. Section 2 recalls the main concepts from probabilistic logic, while Section 3 recalls entailment in Systems P and Z as well as lexicographic entailment from default reasoning from conditional knowledge bases. In Section 4, we introduce the novel probabilistic generalizations of entailment in System P, entailment in System Z, and lexicographic entailment. Section 5 explores the nonmonotonic properties of these new probabilistic formalisms, their relationships, and the relationships to their classical counterparts. In Sections 6 and 7, we provide algorithms for probabilistic reasoning under the new probabilistic formalisms, and we also analyze its computational complexity, respectively. Section 8 provides a comparison to related work. In Section 9, we finally summarize the main results and give an outlook on future research.

In order to not distract from the flow of reading, some technical details and proofs have been moved to Appendices A–E.

2. Probabilistic logic

In this section, we recall the main concepts from probabilistic logic (see especially the work by Nilsson [54,55], Fagin et al. [19], Dubois and Prade et al. [2,13,16,17], Frisch and Haddawy [21], and the author [48,49,51]). We define a propositional language of logical constraints and of Boolean combinations of conditional constraints, which are interpreted in probability distributions over a set of worlds. We also define probabilistic knowledge bases and the model-theoretic notions of satisfiability and logical entailment for probabilistic knowledge bases.

2.1. Syntax

We first formally define the syntax of logical constraints and Boolean combinations of conditional constraints as well as probabilistic knowledge bases.

We assume a set of *basic events* $\Phi = \{p_1, \ldots, p_l\}$ with $l \geqslant 1$. We use $\bot$ and $\top$ to denote *false* and *true*, respectively. We define *events* by induction as follows. Every element of $\Phi \cup \{\bot, \top\}$ is an event. If ϕ and ψ are events, then also $\neg\phi$ and $(\phi \wedge \psi)$. A *conditional event* is of the form $\psi|\phi$ with events ψ and ϕ. A *conditional constraint* is of the form $(\psi|\phi)[l, u]$ with a conditional event $\psi|\phi$ and real numbers $l, u \in [0, 1]$. We define *probabilistic formulas* by induction as follows. Every conditional constraint is a probabilistic formula. If F and G are probabilistic formulas, then also $\neg F$ and $(F \wedge G)$. Note that probabilistic formulas will especially be used for defining concepts around probability rankings (cf. Section 4.1). We use $(F \vee G)$ and $(F \Leftarrow G)$ to abbreviate $\neg(\neg F \wedge \neg G)$ and $\neg(\neg F \wedge G)$, respectively, where F and G are either two events or two probabilistic formulas, and we adopt the usual conventions to eliminate parentheses. A *logical constraint* is an event of the form $\psi \Leftarrow \phi$. A *probabilistic knowledge base* $KB = (L, P)$ consists of a finite set of logical constraints L and a finite set of conditional constraints P such that (i) $l \leqslant u$ for all $(\varepsilon)[l, u] \in P$, and (ii) $\varepsilon_1 \neq \varepsilon_2$ for any two distinct $(\varepsilon_1)[l_1, u_1], (\varepsilon_2)[l_2, u_2] \in P$.

Example 2.1 (*Eagles cont'd*). The strict logical knowledge "all eagles are birds" and "all birds have feathers", and the purely probabilistic knowledge "birds fly with a probability of at least 0.95" can be expressed by the probabilistic knowledge base $KB = (\{\mathit{bird} \Leftarrow \mathit{eagle}, \mathit{feathers} \Leftarrow \mathit{bird}\}, \{(\mathit{fly} \mid \mathit{bird})[0.95, 1]\})$.

2.2. Semantics

We next define the semantics of logical constraints and probabilistic formulas. To this end, we first define the semantics of events in *worlds*, which are truth assignments to the basic events. We then define the semantics of logical constraints and probabilistic formulas in probability distributions over such worlds. We also define the model-theoretic notions of satisfiability and logical entailment for this language and for probabilistic knowledge bases. We finally recall the relationship to model-theoretic logical entailment in ordinary propositional logic.

A *world* I associates with every basic event in Φ a binary truth value. We extend I by induction to all events as usual. We denote by $\mathcal{I}_\Phi$ the set of all worlds for Φ. A world I *satisfies* an event ϕ, or I is a *model* of ϕ, denoted $I \models \phi$, iff $I(\phi) = \mathbf{true}$. We say I *satisfies* a set of events L, or I is a *model* of L, denoted $I \models L$, iff I is a model of all $\phi \in L$. An event ϕ (resp., a set of events L) is *satisfiable* iff a model of ϕ (resp., L) exists. An event ψ is a *logical consequence* of ϕ (resp., L), denoted $\phi \models \psi$ (resp., $L \models \psi$), iff each model of ϕ (resp., L) is also a model of ψ. We use $\phi \not\models \psi$ (resp., $L \not\models \psi$) to denote that $\phi \models \psi$ (resp., $L \models \psi$) does not hold.

A *probabilistic interpretation* Pr is a probability function on $\mathcal{I}_\Phi$ (that is, a mapping $Pr : \mathcal{I}_\Phi \to [0,1]$ such that all $Pr(I)$ with $I \in \mathcal{I}_\Phi$ sum up to 1). The *probability* of an event ϕ in Pr, denoted $Pr(\phi)$, is the sum of all $Pr(I)$ such that $I \in \mathcal{I}_\Phi$ and $I \models \phi$. For events ϕ and ψ with $Pr(\phi) > 0$, we write $Pr(\psi|\phi)$ to abbreviate $Pr(\psi \wedge \phi) / Pr(\phi)$, and we define the *conditioning* of Pr on ϕ, denoted Pr_ϕ, by $Pr_\phi(I) = Pr(I) / Pr(\phi)$ for all $I \in \mathcal{I}_\Phi$ with $I \models \phi$, and by $Pr_\phi(I) = 0$ for all other $I \in \mathcal{I}_\Phi$. The *truth* of logical constraints and probabilistic formulas F in Pr, denoted $Pr \models F$, is inductively defined by (i) $Pr \models \psi \Leftarrow \phi$ iff $Pr(\psi \wedge \phi) = Pr(\phi)$, (ii) $Pr \models (\psi|\phi)[l,u]$ iff $Pr(\phi) = 0$ or $Pr(\psi|\phi) \in [l,u]$, (iii) $Pr \models \neg F$ iff not $Pr \models F$, and (iv) $Pr \models (F \wedge G)$ iff $Pr \models F$ and $Pr \models G$. Observe here that $Pr \models \psi \Leftarrow \phi$ iff $Pr \models (\psi|\phi)[1,1]$. We say Pr *satisfies* a logical constraint or probabilistic formula F, or Pr is a *model* of F, iff $Pr \models F$. We say Pr *satisfies* a set of logical constraints and probabilistic formulas $\mathcal{F}$, or Pr is a *model* of $\mathcal{F}$, denoted $Pr \models \mathcal{F}$, iff Pr is a model of all $F \in \mathcal{F}$. We say $\mathcal{F}$ is *satisfiable* iff a model of $\mathcal{F}$ exists. A logical constraint or probabilistic formula F is a *logical consequence* of $\mathcal{F}$, denoted $\mathcal{F} \mathrel{\|}\!\!\models F$, iff every model of $\mathcal{F}$ is also a model of F. A probabilistic knowledge base $KB = (L, P)$ is *satisfiable* iff $L \cup P$ is satisfiable. The notion of *logical entailment* for probabilistic knowledge bases $KB = (L, P)$ is defined as follows. A logical or conditional constraint F is a *logical consequence* of KB, denoted $KB \mathrel{\|}\!\!\models F$, iff $L \cup P \mathrel{\|}\!\!\models F$. A conditional constraint $(\psi|\phi)[l,u]$ is a *tight logical consequence* of KB, denoted $KB \mathrel{\|}\!\!\models_{tight} (\psi|\phi)[l,u]$, iff l (resp., u) is the infimum (resp., supremum) of $Pr(\psi|\phi)$ subject to all models Pr of $L \cup P$ with $Pr(\phi) > 0$. Note that here we define $[l,u]$ as the empty interval, denoted $[1,0]$, when $L \cup P \mathrel{\|}\!\!\models \bot \Leftarrow \phi$.

The following example illustrates the above notions of satisfiability, logical consequence, and tight logical consequence. Note that deciding satisfiability and logical consequence can be reduced to deciding the solvability of a system of linear constraints, while computing the interval of a tight logical consequence is reducible to solving two linear optimization problems; cf. especially [19,39,51].

Example 2.2 (*Eagles cont'd*). Consider the probabilistic knowledge base $KB = (L, P)$ from Example 2.1. Then, it is easy to verify that the probabilistic interpretations Pr_1, Pr_2, and Pr_3 shown in Table 1 are models of KB. Hence, KB is satisfiable. Furthermore, some logical consequences of KB are given as follows:

$$KB \mathrel{\|}\!\!\models (\mathit{feathers} \mid \mathit{bird})[1,1], \quad KB \mathrel{\|}\!\!\models (\mathit{fly} \mid \mathit{bird})[0.95,1],$$

$$KB \mathrel{\|}\!\!\models (\mathit{feathers} \mid \mathit{eagle})[1,1], \quad KB \mathrel{\|}\!\!\models (\mathit{fly} \mid \mathit{eagle})[0,1].$$

Informally, "all birds have feathers", "birds fly with a probability of at least 0.95", "all eagles have feathers", and "eagles fly with a probability between 0 and 1". In fact, these are the tightest intervals that are logically entailed by KB, since $Pr_1(\mathit{feathers} \mid \mathit{bird}) = 1$, $Pr_1(\mathit{fly} \mid \mathit{bird}) = 1$, $Pr_1(\mathit{feathers} \mid \mathit{eagle}) = 1$, $Pr_1(\mathit{fly} \mid \mathit{eagle}) = 1$, $Pr_2(\mathit{fly} \mid \mathit{bird}) = 0.95$, and $Pr_3(\mathit{fly} \mid \mathit{eagle}) = 0$. Finally, observe that the strict logical property of having feathers is inherited from birds down to its subclass eagles, whereas the probabilistic property of being able to fly with a probability of at least 0.95 is *not* inherited from birds down to eagles.

Table 1
Some probabilistic interpretations Pr_1, Pr_2, and Pr_3

	eagle	*bird*	*feathers*	*fly*	Pr_1	Pr_2	Pr_3
I_1	**true**	**true**	**true**	**true**	1	0.95	0
I_2	**true**	**true**	**true**	**false**	0	0.05	0.05
I_3	**true**	**true**	**false**	**true**	0	0	0
I_4	**true**	**true**	**false**	**false**	0	0	0
I_5	**true**	**false**	**true**	**true**	0	0	0
I_6	**true**	**false**	**true**	**false**	0	0	0
I_7	**true**	**false**	**false**	**true**	0	0	0
I_8	**true**	**false**	**false**	**false**	0	0	0
I_9	**false**	**true**	**true**	**true**	0	0	0.95
I_{10}	**false**	**true**	**true**	**false**	0	0	0
I_{11}	**false**	**true**	**false**	**true**	0	0	0
I_{12}	**false**	**true**	**false**	**false**	0	0	0
I_{13}	**false**	**false**	**true**	**true**	0	0	0
I_{14}	**false**	**false**	**true**	**false**	0	0	0
I_{15}	**false**	**false**	**false**	**true**	0	0	0
I_{16}	**false**	**false**	**false**	**false**	0	0	0

Intuitively, the above notion of logical entailment of $(\psi|\phi)[l,u]$ from a probabilistic knowledge base $KB = (L, P)$ is based on the idea of performing a conditioning of every probability distribution Pr that satisfies $L \cup P$ on the premise ϕ. This result is more formally expressed by the following theorem.

Theorem 2.3. *Let $KB = (L, P)$ be a probabilistic knowledge base, and $(\psi|\phi)[l,u]$ be a conditional constraint. Then,* (a) $KB \Vdash (\psi|\phi)[l,u]$ *iff* $Pr_\phi(\psi) \in [l,u]$ *for all models Pr of $L \cup P$ with $Pr(\phi){>}0$; and* (b) $KB \Vdash_{tight} (\psi|\phi)[l,u]$ *iff* $l = \inf Pr_\phi(\psi)$ *(resp.,* $u = \sup Pr_\phi(\psi)$*) subject to all models Pr of $L \cup P$ with $Pr(\phi) > 0$.*

The following result shows that in probabilistic logic, a logical constraint $\psi \Leftarrow \phi$ has the same meaning as the conditional constraint $(\psi|\phi)[1,1]$.

Theorem 2.4. *Let $KB = (L, P)$ be a probabilistic knowledge base, and $(\psi|\phi)[1,1]$ be a conditional constraint. Then,* (a) $KB \Vdash (\psi|\phi)[1,1]$ *iff* $KB \Vdash \psi \Leftarrow \phi$*; and* (b) $(L, P \cup \{(\psi|\phi)[1,1]\})$ *has the same set of models as* $(L \cup \{\psi \Leftarrow \phi\}, P)$.

The next result says that model-theoretic logical entailment in probabilistic logic generalizes model-theoretic logical entailment in ordinary propositional logic.

Theorem 2.5. *Let $KB = (L, P)$ be a probabilistic knowledge base with $P = \emptyset$, and let $\psi \Leftarrow \phi$ be a logical constraint. Then, $KB \Vdash \psi \Leftarrow \phi$ iff $L \models \psi \Leftarrow \phi$.*

3. Default reasoning from conditional knowledge bases

In this section, we recall the following formalisms for default reasoning from conditional knowledge bases: Kraus et al.'s entailment in System P [40] (which is equivalent to several other formalisms; cf. Section 1), Pearl's entailment in System Z [34,58] (which is equivalent to Lehmann's rational closure [45,47], to the least specific possibility entailment by Benferhat et al. [5], and to a conditional (modal) logic-based entailment by Lamarre [44]), and Lehmann's lexicographic entailment [46] (a special case of Benferhat et al.'s lexicographic entailment [4]).

These formalisms for default reasoning from conditional knowledge bases all have in common that they can be defined in terms of world rankings (which are certain mappings from the set of all worlds to $\{0, 1, \ldots\} \cup \{\infty\}$), where entailment in System P can be expressed by a set of world rankings, while entailment in System Z and lexicographic entailment each have an associated unique world ranking.

Both Pearl's entailment in System Z and Lehmann's lexicographic entailment are more sophisticated than entailment in System P and show a nicer semantic behavior than the latter. The following example illustrates this aspect. Here, we use *p-entailment*, *z-entailment*, and *lex-entailment* to denote entailment in System P, entailment in System Z, and lexicographic entailment, respectively.

Example 3.1 (*Penguins cont'd*). Consider again the collection of strict and default logical sentences *KB* given in Example 1.2. Some default conclusions of *KB* under z- and *lex*-entailment compared to p-entailment are shown in Table 2. Differently from p-entailment, both z- and *lex*-entailment ignore irrelevant information. Furthermore, *lex*-entailment shows a correct property inheritance from birds to penguins, while p-entailment does not show any property inheritance at all, and z-entailment does not inherit the property of having wings from the class of all birds to the exceptional subclass of all penguins (and thus shows the problem of *inheritance blocking*). Finally, the default $\neg fly \leftarrow penguin$ is entailed by *KB* under all three notions of default entailment.

3.1. Preliminaries

We now formally define conditional knowledge bases as well as world and default rankings along with their admissibility with conditional knowledge bases.

Informally, a conditional knowledge base consists of a set of strict statements in classical logic and a set of defeasible rules (or defaults) of the form "$\psi \leftarrow \phi$", which informally read as "generally, if ϕ then ψ". Such rules may have exceptions, which can be handled in

Table 2
Some defaults entailed by *KB* under different semantics

	$fly \leftarrow red \wedge bird$	$wings \leftarrow penguin$	$\neg fly \leftarrow penguin$
p-entailment	$-$	$-$	$+$
z-entailment	$+$	$-$	$+$
lex-entailment	$+$	$+$	$+$

different ways. A *conditional rule* (or *default*) is an expression of the form $\psi \leftarrow \phi$, where ϕ and ψ are events. A *conditional knowledge base* $KB = (L, D)$ consists of a finite set of logical constraints L and a finite set of defaults D. The following example illustrates conditional knowledge bases.

Example 3.2 (*Penguins cont'd*). The strict logical knowledge "all penguins are birds" and the default logical knowledge "generally, birds fly", "generally, penguins do not fly", and "generally, birds have wings" is encoded by the conditional knowledge base $KB{=}(\{\mathit{bird} \Leftarrow \mathit{penguin}\}, \{\mathit{fly} \leftarrow \mathit{bird}, \neg\mathit{fly} \leftarrow \mathit{penguin}, \mathit{wings} \leftarrow \mathit{bird}\})$.

A world I *satisfies* a default $\psi \leftarrow \phi$, or I is a *model* of $\psi \leftarrow \phi$, denoted $I \models \psi \leftarrow \phi$, iff $I \models \psi \Leftarrow \phi$. We say I *verifies* $\psi \leftarrow \phi$ iff $I \models \phi \wedge \psi$. We say I *falsifies* $\psi \leftarrow \phi$ iff $I \models \phi \wedge \neg\psi$ (that is, $I \not\models \psi \leftarrow \phi$). We say I *satisfies* a set of events and defaults K, or I is a *model* of K, denoted $I \models K$, iff I satisfies every member of K. We say K is *satisfiable* iff a model of K exists. An event ϕ (resp., a default d) is a *logical consequence* of K, denoted $K \models \phi$ (resp., $K \models d$), iff every model of K is also a model of ϕ (resp., d). An event ϕ (resp., a default d) is a *logical consequence* of a conditional knowledge base $KB = (L, D)$, denoted $KB \models \phi$ (resp., $KB \models d$), iff $L \cup D \models \phi$ (resp., $L \cup D \models d$). A set of defaults D *tolerates* a default d *under* a set of logical constraints L iff $D \cup L$ has a model that verifies d. A set of defaults D is *under* L *in conflict* with a default $\psi \leftarrow \phi$ iff all models of $D \cup L \cup \{\phi\}$ satisfy $\neg\psi$.

A *world ranking* κ is a mapping $\kappa : \mathcal{I}_\Phi \rightarrow \{0, 1, \ldots\} \cup \{\infty\}$ such that $\kappa(I) = 0$ for at least one world I. It is extended to all events ϕ as follows. If ϕ is satisfiable, then $\kappa(\phi) = \min\{\kappa(I) \mid I \in \mathcal{I}_\Phi, I \models \phi\}$; otherwise, $\kappa(\phi) = \infty$. A world ranking κ is *admissible* with a conditional knowledge base $KB = (L, D)$ iff $\kappa(\neg\phi) = \infty$ for all $\phi \in L$, and $\kappa(\phi) < \infty$ and $\kappa(\phi \wedge \psi) < \kappa(\phi \wedge \neg\psi)$ for all defaults $\psi \leftarrow \phi \in D$.

Example 3.3 (*Penguins cont'd*). Table 3 shows the world rankings κ_1, κ_2, and κ_3. It is easy to verify that κ_1 and κ_2 are admissible with *KB* from Example 3.2. Note that κ_1 and κ_2 are the unique world rankings associated with *KB* in System Z and under lexicographic entailment, respectively (see Sections 3.3 and 3.4). But κ_3 is not admissible with *KB*, since L contains the logical constraint $\mathit{bird} \Leftarrow \mathit{penguin}$, but $\kappa_3(\mathit{penguin} \wedge \neg\mathit{bird}) = \min(\kappa_3(I_5), \kappa_3(I_6), \kappa_3(I_7), \kappa_3(I_8)) = 4 \neq \infty$. Moreover, D contains the default $\mathit{wings} \leftarrow \mathit{bird}$, but $\kappa_3(\mathit{bird} \wedge \mathit{wings}) = 0 = \kappa_3(\mathit{bird} \wedge \neg\mathit{wings})$.

A *default ranking* σ on a conditional knowledge base $KB = (L, D)$ maps each $d \in D$ to a nonnegative integer. It is *admissible* with *KB* iff each $D' \subseteq D$ that is under L in conflict with some $d \in D$ contains a default d' such that $\sigma(d') < \sigma(d)$.

Example 3.4 (*Penguins cont'd*). A default ranking σ on *KB* from Example 3.2 is given by $\sigma(\mathit{fly} \leftarrow \mathit{bird}) = \sigma(\mathit{wings} \leftarrow \mathit{bird}) = 0$ and $\sigma(\neg\mathit{fly} \leftarrow \mathit{penguin}) = 1$. It is not difficult to verify that σ is admissible with *KB*. Note that σ is in fact the default ranking associated with *KB* in System Z (see Section 3.3).

Table 3
Some world rankings κ_1, κ_2, and κ_3

	penguin	*bird*	*wings*	*fly*	κ_1	κ_2	κ_3
I_1	**true**	**true**	**true**	**true**	2	3	2
I_2	**true**	**true**	**true**	**false**	1	1	1
I_3	**true**	**true**	**false**	**true**	2	4	0
I_4	**true**	**true**	**false**	**false**	1	2	2
I_5	**true**	**false**	**true**	**true**	∞	∞	∞
I_6	**true**	**false**	**true**	**false**	∞	∞	4
I_7	**true**	**false**	**false**	**true**	∞	∞	∞
I_8	**true**	**false**	**false**	**false**	∞	∞	∞
I_9	**false**	**true**	**true**	**true**	0	0	0
I_{10}	**false**	**true**	**true**	**false**	1	1	1
I_{11}	**false**	**true**	**false**	**true**	1	1	1
I_{12}	**false**	**true**	**false**	**false**	1	2	2
I_{13}	**false**	**false**	**true**	**true**	0	0	0
I_{14}	**false**	**false**	**true**	**false**	0	0	0
I_{15}	**false**	**false**	**false**	**true**	0	0	0
I_{16}	**false**	**false**	**false**	**false**	0	0	0

3.2. Consistency and entailment in System P

We now describe the notions of consistency and entailment in Kraus et al.'s System P [40], which we call p-consistency and p-entailment, respectively. We define them in terms of world rankings (see especially [24,25] for the equivalence between entailment in System P and entailment under world rankings), and we then recall some important equivalent characterizations of them.

A conditional knowledge base *KB* is *p-consistent* iff there exists a world ranking κ on *KB* that is admissible with *KB*. It is *p-inconsistent* iff no such κ exists. A p-consistent conditional knowledge base *KB* *p-entails* a default $\psi \leftarrow \phi$ iff either $\kappa(\phi) = \infty$ or $\kappa(\phi \wedge \psi) < \kappa(\phi \wedge \neg\psi)$ for all world rankings κ admissible with *KB*.

The following result due to Geffner [24] shows that the notion of p-consistency is equivalent to the existence of admissible default rankings.

Theorem 3.5 (Geffner [24]). *A conditional knowledge base KB is p-consistent iff there exists a default ranking on KB that is admissible with KB.*

The next characterization of p-consistency is due to Goldszmidt and Pearl [32].

Theorem 3.6 (Goldszmidt and Pearl [32]). *A conditional knowledge base $KB = (L, D)$ is p-consistent iff an ordered partition $(D_0, \ldots, D_k)$ of D exists such that either* (a) *or* (b) *holds*:

(a) *Every D_i, $0 \leqslant i \leqslant k$, is the set of all $d \in \bigcup_{j=i}^{k} D_j$ tolerated under L by $\bigcup_{j=i}^{k} D_j$.*
(b) *For every i, $0 \leqslant i \leqslant k$, each $d \in D_i$ is tolerated under L by $\bigcup_{j=i}^{k} D_j$.*

The following characterization of the notion of p-entailment describes a reduction of p-entailment to p-consistency. This result is essentially due to Adams [1], who formulated it for $L = \emptyset$ and the notions of ε-consistency and ε-entailment (which are equivalent to p-consistency and p-entailment, respectively).

Theorem 3.7 (Adams [1]). *A p-consistent conditional knowledge base $KB = (L, D)$ p-entails a default $\psi \leftarrow \phi$ iff $(L, D \cup \{\neg\psi \leftarrow \phi\})$ is p-inconsistent.*

3.3. Entailment in System Z

We next recall Pearl's entailment in System Z [34,58], denoted z-entailment. In the sequel, let $KB = (L, D)$ be a p-consistent conditional knowledge base.

Entailment in System Z is linked to an ordered partition of D, a default ranking z on KB, and a world ranking κ^z. The *z-partition* of KB is the unique ordered partition $(D_0, \ldots, D_k)$ of D such that each D_i is the set of all $d \in \bigcup_{j=i}^{k} D_j$ tolerated under L by $\bigcup_{j=i}^{k} D_j$. We next define z and κ^z. For every $j \in \{0, \ldots, k\}$, each $d \in D_j$ is assigned the value j under z. The world ranking κ^z on all worlds I is defined by:

$$\kappa^z(I) = \begin{cases} \infty & \text{if } I \not\models L; \\ 0 & \text{if } I \models L \cup D; \\ 1 + \max_{d \in D:\, I \not\models d} z(d) & \text{otherwise.} \end{cases}$$

A *preference relation* on worlds I and I' is then defined as follows. We say that I is *z-preferable* to I' iff $\kappa^z(I) < \kappa^z(I')$. A model I of a set of events $\mathcal{F}$ (that is, I is a world that satisfies $\mathcal{F}$) is a *z-minimal model* of $\mathcal{F}$ iff no model of $\mathcal{F}$ is z-preferable to I. Note that even though the default ranking z and the world ranking κ^z are unique for a given p-consistent conditional knowledge base KB, there are generally several z-minimal models of a set of events $\mathcal{F}$.

We now use the above preference relation on worlds to define the notion of z-entailment as follows. A default $\psi \leftarrow \phi$ is a *z-consequence* of $KB = (L, D)$, denoted $KB \mid\!\sim^{z} \psi \leftarrow \phi$, iff ψ is true in all z-minimal models of $L \cup \{\phi\}$.

3.4. Lexicographic entailment

We finally recall Lehmann's lexicographic entailment [46], denoted *lex*-entailment. In the sequel, let $KB = (L, D)$ be a p-consistent conditional knowledge base.

We use the z-partition $(D_0, \ldots, D_k)$ of KB to define a *lexicographic preference relation* on worlds as follows. A world I is *lexicographically preferable* (or *lex-preferable*) to a world I' iff some $i \in \{0, \ldots, k\}$ exists such that $|\{d \in D_i \mid I \models d\}| > |\{d \in D_i \mid I' \models d\}|$ and $|\{d \in D_j \mid I \models d\}| = |\{d \in D_j \mid I' \models d\}|$ for all $i < j \leqslant k$. A model I of a set of events $\mathcal{F}$ is a *lexicographically minimal* (or *lex-minimal*) model of $\mathcal{F}$ iff no model of $\mathcal{F}$ is *lex*-preferable to I.

The lexicographic preference relation (which can also be expressed in terms of a unique world ranking) is then used as follows to define the notion of *lex*-entailment. A default $\psi \leftarrow \phi$ is a *lexicographic consequence* (or *lex-consequence*) of KB, denoted $KB \mid\!\sim^{lex} \psi \leftarrow \phi$, iff ψ is true in all *lex*-minimal models of $L \cup \{\phi\}$.

4. Weak nonmonotonic probabilistic logics

In this section, we present the new probabilistic formalisms, called *weak nonmonotonic probabilistic logics*, which allow for dealing with *strict logical knowledge*, *default logical knowledge*, and *purely probabilistic knowledge* in a uniform framework. To this end, we define a new semantics of probabilistic knowledge bases, where probabilistic logic is combined with Kraus et al.'s entailment in System P, Pearl's entailment in System Z, and Lehmann's lexicographic entailment.

The new semantics of probabilistic knowledge bases $KB = (L, P)$ is essentially obtained by defining probability and conditional constraint rankings, which generalize world and default rankings, respectively, for conditional knowledge bases. Under the new semantics, conditional constraints of the form $(\psi|\phi)[1, 1]$ and $(\psi|\phi)[0, 0]$ in P then behave as the defaults $\psi \leftarrow \phi$ and $\neg\psi \leftarrow \phi$, respectively.

Example 4.1 (*Ostriches cont'd*). The probabilistic knowledge base $KB = (L, P)$ in Table 4 encodes the strict logical knowledge "all ostriches are birds", the default logical knowledge "generally, birds have legs" and "generally, birds fly", and the purely probabilistic knowledge "ostriches fly with a probability of at most 0.05".

It is important to point out that we generally cannot simply interpret KB in probabilistic logic, since then $(\psi|\phi)[1, 1]$ and $(\psi|\phi)[0, 0]$ in P have the meaning of the strict sentences $\psi \Leftarrow \phi$ and $\neg\psi \Leftarrow \phi$ in L, and not of the defaults $\psi \leftarrow \phi$ and $\neg\psi \leftarrow \phi$, respectively. The following example illustrates this aspect.

Example 4.2 (*Ostriches cont'd*). The probabilistic knowledge base $KB = (L, P)$ in Table 4 has the probabilistic interpretation Pr_1 in Table 6 as a model. This shows that KB is satisfiable. Some logical consequences of KB are given as follows:

$$KB \Vdash\!\!= (\mathit{legs} \mid \mathit{bird})[1, 1], \qquad KB \Vdash\!\!= (\mathit{fly} \mid \mathit{bird})[1, 1].$$

Table 4
Probabilistic knowledge base KB

$KB = (L, P)$	Type of knowledge
$L = \{\mathit{bird} \Leftarrow \mathit{ostrich}\}$	strict logical knowledge
$P = \{(\mathit{legs} \mid \mathit{bird})[1, 1], (\mathit{fly} \mid \mathit{bird})[1, 1],$	default logical knowledge
$(\mathit{fly} \mid \mathit{ostrich})[0, 0.05]\}$	purely probabilistic knowledge

Table 5
Tight conclusions from KB under logical and s-entailment, where $s \in \{\boldsymbol{lex}, \boldsymbol{z}, \boldsymbol{p}\}$

$(\psi\|\phi)$	$\Vdash\!\!=_{tight}$	$\Vdash\!\!\sim^{lex}_{tight}$	$\Vdash\!\!\sim^{z}_{tight}$	$\Vdash\!\!\sim^{p}_{tight}$
$(\mathit{legs} \mid \mathit{bird})$	$[1, 1]$	$[1, 1]$	$[1, 1]$	$[1, 1]$
$(\mathit{fly} \mid \mathit{bird})$	$[1, 1]$	$[1, 1]$	$[1, 1]$	$[1, 1]$
$(\mathit{legs} \mid \mathit{ostrich})$	$\mathbf{[1, 0]}$	$[1, 1]$	$\mathbf{[0, 1]}$	$\mathbf{[0, 1]}$
$(\mathit{fly} \mid \mathit{ostrich})$	$\mathbf{[1, 0]}$	$[0, 0.05]$	$[0, 0.05]$	$[0, 0.05]$
$(\mathit{fly} \mid \mathit{red} \wedge \mathit{bird})$	$[1, 1]$	$[1, 1]$	$[1, 1]$	$\mathbf{[0, 1]}$

Since $Pr_1(legs \mid bird) = Pr_1(fly \mid bird) = 1$, these conditional constraints are in fact tight logical consequences of *KB*. They are also the desired conclusions from *KB* (cf. Example 1.3). Some other tight logical consequences of *KB* are as follows:

$$KB \Vdash_{tight} (legs \mid ostrich)[1,0], \qquad KB \Vdash_{tight} (fly \mid ostrich)[1,0].$$

Here, the empty interval "$[1,0]$" is due to the fact that in probabilistic logic the ability to fly of birds is interpreted as strict logical knowledge, and inherited from birds to the subclass of ostriches. There, it is incompatible with the purely probabilistic knowledge that ostriches are able to fly with a probability of at most 0.05. Thus, our knowledge about ostriches is *locally inconsistent* in the sense that there exists no model *Pr* of $L \cup P$ with $Pr(ostrich) > 0$. This is why we obtain $(legs \mid ostrich)[1,0]$ and $(fly \mid ostrich)[1,0]$ rather than the desired tight conclusions $(legs \mid ostrich)[1,1]$ and $(fly \mid ostrich)[0,0.05]$ (cf. Example 1.3), respectively. Finally, another tight logical consequence of *KB* is given by $KB \Vdash_{tight} (fly \mid red \wedge bird)[1,1]$, which is also a desired tight conclusion from *KB* (cf. Example 1.3). Observe that for this last conclusion, probabilistic interpretations *Pr* are defined over the set of all truth assignments I to the basic events *ostrich*, *bird*, *legs*, *fly*, and *red*.

4.1. Preliminaries

We now define some probabilistic generalizations of concepts from default reasoning from Section 3.1. In particular, we define probability and conditional constraint rankings as well as their admissibility with probabilistic knowledge bases.

A probabilistic interpretation *Pr* *verifies* a conditional constraint $(\psi|\phi)[l,u]$ iff $Pr(\phi) > 0$ and $Pr \models (\psi|\phi)[l,u]$. We say that *Pr* *falsifies* $(\psi|\phi)[l,u]$ iff $Pr(\phi) > 0$ and $Pr \not\models (\psi|\phi)[l,u]$. A set of conditional constraints P *tolerates* a conditional constraint C *under* a set of logical constraints L iff $L \cup P$ has a model that verifies C. We say P is *under* L *in conflict* with C iff no model of $L \cup P$ verifies C.

In the sequel, we use $\alpha > 0$ to abbreviate the probabilistic formula $\neg(\alpha|\top)[0,0]$. Informally, a probabilistic interpretation *Pr* satisfies $\alpha > 0$ iff $Pr(\alpha) > 0$. A *probability ranking* κ is a function that associates with every probabilistic interpretation *Pr* on $\mathcal{I}_\Phi$ a value from $\{0,1,\ldots\} \cup \{\infty\}$ such that $\kappa(Pr) = 0$ for at least one *Pr*. It is extended to all logical constraints and probabilistic formulas F as follows. If F is satisfiable, then $\kappa(F) = \min\{\kappa(Pr) \mid Pr \models F\}$; otherwise, $\kappa(F) = \infty$. A probability ranking κ is *admissible* with a probabilistic knowledge base $KB = (L, P)$ iff $\kappa(\neg(\psi|\phi)[1,1]) = \infty$ for all $\psi \Leftarrow \phi \in L$, as well as $\kappa(\phi > 0) < \infty$ and $\kappa(\phi > 0 \wedge (\psi|\phi)[l,u]) < \kappa(\phi > 0 \wedge \neg(\psi|\phi)[l,u])$ for all $(\psi|\phi)[l,u] \in P$. Informally, the latter says that for every $(\psi|\phi)[l,u] \in P$, it holds that (i) $Pr(\phi) > 0$ and $\kappa(Pr) < \infty$ for some probabilistic interpretation *Pr*, and (ii) the minimal $\kappa(Pr)$ of all *Pr* verifying $(\psi|\phi)[l,u]$ is less than the minimal $\kappa(Pr)$ of all *Pr* falsifying $(\psi|\phi)[l,u]$.

Example 4.3 (*Ostriches cont'd*). Table 6 shows some probabilistic interpretations $Pr_1, \ldots, Pr_8$, and Table 7 gives their values under some probability rankings κ_1, κ_2, and κ_3. Observe that κ_3 is not admissible with $KB = (L, P)$ in Table 4, since $bird \Leftarrow ostrich$ is in L, but $\kappa_3(\neg(bird \mid ostrich)[1,1]) \leqslant \kappa_3(Pr_8) = 1 < \infty$. Moreover, $(fly \mid ostrich)[0,0.05]$ is in

Table 6
Some probabilistic interpretations $Pr_1, \ldots, Pr_8$

	ostrich	*bird*	*legs*	*fly*	Pr_1	Pr_2	Pr_3	Pr_4	Pr_5	Pr_6	Pr_7	Pr_8
I_1	**true**	**true**	**true**	**true**	0	0	0	1	0	0.5	0	0
I_2	**true**	**true**	**true**	**false**	0	1	0	0	0	0.5	0	0.5
I_3	**true**	**true**	**false**	**true**	0	0	0.05	0	1	0	0.5	0
I_4	**true**	**true**	**false**	**false**	0	0	0.95	0	0	0	0.5	0
I_5	**true**	**false**	**true**	**true**	0	0	0	0	0	0	0	0
I_6	**true**	**false**	**true**	**false**	0	0	0	0	0	0	0	0.5
I_7	**true**	**false**	**false**	**true**	0	0	0	0	0	0	0	0
I_8	**true**	**false**	**false**	**false**	0	0	0	0	0	0	0	0
I_9	**false**	**true**	**true**	**true**	1	0	0	0	0	0	0	0
I_{10}	**false**	**true**	**true**	**false**	0	0	0	0	0	0	0	0
I_{11}	**false**	**true**	**false**	**true**	0	0	0	0	0	0	0	0
I_{12}	**false**	**true**	**false**	**false**	0	0	0	0	0	0	0	0
I_{13}	**false**	**false**	**true**	**true**	0	0	0	0	0	0	0	0
I_{14}	**false**	**false**	**true**	**false**	0	0	0	0	0	0	0	0
I_{15}	**false**	**false**	**false**	**true**	0	0	0	0	0	0	0	0
I_{16}	**false**	**false**	**false**	**false**	0	0	0	0	0	0	0	0

Table 7
Values of $Pr_1, \ldots, Pr_8$ under some probability rankings κ_1, κ_2, and κ_3

	$(legs \mid bird)[1,1]$	$(fly \mid bird)[1,1]$	$(fly \mid ostrich)[0,0.05]$	κ_1	κ_2	κ_3
Pr_1	**true**	**true**	**true**	0	0	0
Pr_2	**true**	**false**	**true**	1	1	1
Pr_3	**false**	**false**	**true**	1	2	1
Pr_4	**true**	**true**	**false**	2	3	0
Pr_5	**false**	**true**	**false**	2	4	2
Pr_6	**true**	**false**	**false**	2	4	2
Pr_7	**false**	**false**	**false**	2	5	2
Pr_8	**true**	**false**	**true**	∞	∞	1

P, but $\kappa_3(ostrich > 0 \wedge \neg(fly \mid ostrich)[0, 0.05]) \leqslant \kappa_3(Pr_4) = 0 \leqslant \kappa_3(ostrich > 0 \wedge (fly \mid ostrich)[0, 0.05])$. Note that on $Pr_1, \ldots, Pr_8$, the rankings κ_1 and κ_2 coincide with the unique rankings associated with *KB* in probabilistic z- and *lex*-entailment (cf. Sections 4.3 and 4.4), respectively.

A *conditional constraint ranking* on a probabilistic knowledge base $KB = (L, P)$ is a mapping σ that associates with every conditional constraint $C \in P$ a nonnegative integer. If $P \neq \emptyset$, then σ is *admissible* with *KB* iff every $P' \subseteq P$ that is under L in conflict with some $C \in P$ contains some C' with $\sigma(C') < \sigma(C)$; if $P = \emptyset$, then σ is *admissible* with *KB* iff L is satisfiable. Notice that conditional constraint rankings σ are defined on the set of all conditional constraints in P and have values from $\{0, 1, \ldots\}$, while probability rankings κ are defined on the set of all probabilistic interpretations Pr on $\mathcal{I}_\Phi$ and have values from $\{0, 1, \ldots\} \cup \{\infty\}$.

Example 4.4 (*Ostriches cont'd*). A conditional constraint ranking σ for the probabilistic knowledge base *KB* in Table 4 is given by $\sigma((\mathit{legs} \mid \mathit{bird})[1,1]) = \sigma((\mathit{fly} \mid \mathit{bird})[1,1]) = 0$ and $\sigma((\mathit{fly} \mid \mathit{ostrich})[0,\ 0.05]) = 1$. It is not difficult to see that σ is admissible with *KB*. In fact, σ is the unique conditional constraint ranking that is associated with *KB* in probabilistic z-entailment (cf. Sections 4.3).

4.2. Probabilistic consistency and entailment in System P

We now define a semantics of probabilistic knowledge bases, where probabilistic logic is combined with System P [40]. More precisely, we generalize the notions of consistency and entailment in System P that are based on world rankings to probabilistic knowledge bases. We call these generalizations *probabilistic p-consistency* and *probabilistic p-entailment* (or simply ***p**-consistency* and ***p**-entailment*), respectively. Interestingly, these probabilistic notions of consistency and entailment coincide with the probabilistic notions of g-coherence and g-coherent entailment for imprecise probability assessments (cf. Section 8.2). In the following, we first define the probabilistic generalizations of consistency and entailment in System P, and we then give some equivalent characterizations of them.

In the sequel, let $KB = (L, P)$ be a probabilistic knowledge base. We say *KB* is ***p**-consistent* iff there exists a probability ranking κ that is admissible with *KB*. We then define the notion of **p**-entailment for **p**-consistent *KB* in terms of admissible probability rankings as follows. A conditional constraint $(\psi|\phi)[l,u]$ is a ***p**-consequence* of *KB*, denoted $KB \Vdash\!\sim^{p} (\psi|\phi)[l,u]$, iff $\kappa(\phi > 0) = \infty$ or $\kappa(\phi > 0 \wedge (\psi|\phi)[l,u]) < \kappa(\phi > 0 \wedge \neg(\psi|\phi)[l,u])$ for every probability ranking κ admissible with *KB*. We say $(\psi|\phi)[l,u]$ is a *tight **p**-consequence* of *KB*, denoted $KB \Vdash\!\sim^{p}_{tight} (\psi|\phi)[l,u]$, iff $l = \sup l'$ (resp., $u = \inf u'$) subject to $KB \Vdash\!\sim^{p} (\psi|\phi)[l',u']$.

The following result is a probabilistic generalization of Theorem 3.5. It says that the notion of **p**-consistency of a probabilistic knowledge base *KB* is equivalent to the existence of an admissible conditional constraint ranking. It is proved by showing that a probability ranking κ that is admissible with *KB* can be used to define a conditional constraint ranking σ that is admissible with *KB*, and vice versa.

Theorem 4.5. *A probabilistic knowledge base $KB = (L, P)$ is **p**-consistent iff there exists a conditional constraint ranking on KB that is admissible with KB.*

Based on this result, we also obtain a probabilistic generalization of Theorem 3.6, which says that the **p**-consistency of a probabilistic knowledge base $KB = (L, P)$ is equivalent to the existence of an ordered partition of P with certain properties.

Theorem 4.6. *A probabilistic knowledge base $KB = (L, P)$ is **p**-consistent iff there exists an ordered partition $(P_0, \ldots, P_k)$ of P such that either* (a) *or* (b) *holds*:

(a) *Every P_i, $0 \leqslant i \leqslant k$, is the set of all $F \in \bigcup_{j=i}^{k} P_j$ tolerated under L by $\bigcup_{j=i}^{k} P_j$.*
(b) *For every i, $0 \leqslant i \leqslant k$, each $F \in P_i$ is tolerated under L by $\bigcup_{j=i}^{k} P_j$.*

Example 4.7 (*Ostriches cont'd*). The probabilistic knowledge base $KB = (L, P)$ in Table 4 is $\boldsymbol{p}$-consistent, since condition (a) as well as condition (b) of Theorem 4.6 hold for the following ordered partition (P_0, P_1) of P:

$$(P_0, P_1) = \big(\{(\mathit{legs} \mid \mathit{bird})[1,1], (\mathit{fly} \mid \mathit{bird})[1,1]\}, \{(\mathit{fly} \mid \mathit{ostrich})[0, 0.05]\}\big).$$

More precisely, to see that (P_0, P_1) satisfies (b), observe that Pr_1 in Table 6 satisfies $L \cup P$ and verifies $(\mathit{legs} \mid \mathit{bird})[1,1]$ and $(\mathit{fly} \mid \mathit{bird})[1,1]$, while Pr_2 satisfies $L \cup P_1$ and verifies $(\mathit{fly} \mid \mathit{ostrich})[0, 0.05]$. To see that also (a) holds, observe that no Pr satisfies $L \cup P$ and also verifies $(\mathit{fly} \mid \mathit{ostrich})[0, 0.05]$ (cf. Example 4.2).

The following two theorems are a probabilistic generalization of Theorem 3.7. They say that the notion of $\boldsymbol{p}$-entailment for probabilistic knowledge bases can be expressed in terms of the notion of $\boldsymbol{p}$-consistency. The first theorem is on the notion of $\boldsymbol{p}$-consequence, while the second one is on tight $\boldsymbol{p}$-consequence.

Theorem 4.8. *Let $KB = (L, P)$ be a $\boldsymbol{p}$-consistent probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conditional constraint. Then, $KB \mathrel{\|\!\!\sim}^{p} (\beta|\alpha)[l, u]$ iff $(L, P \cup \{(\beta|\alpha)[p, p]\})$ is not $\boldsymbol{p}$-consistent for all $p \in [0, l) \cup (u, 1]$.*

Theorem 4.9. *Let $KB = (L, P)$ be a $\boldsymbol{p}$-consistent probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conditional constraint. Then, $KB \mathrel{\|\!\!\sim}^{p}_{tight} (\beta|\alpha)[l, u]$ iff*

(i) $(L, P \cup \{(\beta|\alpha)[p, p]\})$ *is not $\boldsymbol{p}$-consistent for all $p \in [0, l) \cup (u, 1]$, and*
(ii) $(L, P \cup \{(\beta|\alpha)[p, p]\})$ *is $\boldsymbol{p}$-consistent for all $p \in [l, u]$.*

The next two theorems show that $\boldsymbol{p}$-consistency and $\boldsymbol{p}$-entailment coincide with the probabilistic notions of g-coherence and g-coherent entailment, respectively, for imprecise probability assessments (cf. Section 8.2). They follow from Theorems 4.5 and 4.8 as well as similar characterizations of g-coherence and g-coherent entailment through conditional constraint rankings, presented in [10,11].

Theorem 4.10. *Let $KB = (L, P)$ be a probabilistic knowledge base. Then, KB is $\boldsymbol{p}$-consistent iff KB is g-coherent.*

Theorem 4.11. *Let $KB = (L, P)$ be $\boldsymbol{p}$-consistent, and let $(\beta|\alpha)[l, u]$ be a conditional constraint. Then, $KB \mathrel{\|\!\!\sim}^{p} (\beta|\alpha)[l, u]$ iff $KB \mathrel{\|\!\!\sim}^{g} (\beta|\alpha)[l, u]$.*

The following example illustrates the probabilistic notion of $\boldsymbol{p}$-entailment. In particular, it shows that $\boldsymbol{p}$-entailment does not realize an inheritance of default logical knowledge along subclass relationships. See Section 6 for algorithms for deciding $\boldsymbol{p}$-consistency and computing tight $\boldsymbol{p}$-consequences.

Example 4.12 (*Ostriches cont'd*). Consider again *KB* given in Table 4. Some tight $\boldsymbol{p}$-consequences of *KB* are shown in Table 5. More precisely, $(\mathit{legs} \mid \mathit{bird})[1,1]$, $(\mathit{fly} \mid$

bird)[1, 1], and (*fly* | *ostrich*)[0, 0.05] are tight $\boldsymbol{p}$-consequences of *KB*, as desired. Furthermore, (*legs* | *ostrich*)[0, 1] and (*fly* | *red* ∧ *bird*)[0, 1] are also tight $\boldsymbol{p}$-consequences of *KB*. But they differ from the desired ones (*legs* | *ostrich*)[1, 1] and (*legs* | *red* ∧ *bird*)[1, 1], respectively. Here, we observe that $\boldsymbol{p}$-entailment does not inherit default logical knowledge along subclass relationships.

4.3. Probabilistic entailment in System Z

We next extend Pearl's System Z [34,58] to $\boldsymbol{p}$-consistent probabilistic knowledge bases $KB = (L, P)$. The new notion of entailment in System Z, called *probabilistic* z*-entailment* (or simply z*-entailment*), is associated with an ordered partition of P, a conditional constraint ranking $\boldsymbol{z}$ on *KB*, and a probability ranking $\kappa^{\boldsymbol{z}}$.

The $\boldsymbol{z}$*-partition* of *KB* is the unique ordered partition $(P_0, \ldots, P_k)$ of P such that each P_i, $0 \leqslant i \leqslant k$, is the set of all $C \in \bigcup_{j=i}^{k} P_j$ tolerated under L by $\bigcup_{j=i}^{k} P_j$.

Example 4.13 (*Ostriches cont'd*). The z-partition of *KB* in Table 4 is given by the ordered partition (P_0, P_1) described in Example 4.7.

The conditional constraint ranking $\boldsymbol{z}$ and the probability ranking $\kappa^{\boldsymbol{z}}$ are defined as follows. For every $j \in \{0, \ldots, k\}$, each $C \in P_j$ is assigned the value j under $\boldsymbol{z}$. The probability ranking $\kappa^{\boldsymbol{z}}$ on all probabilistic interpretations *Pr* is then defined by:

$$\kappa^{\boldsymbol{z}}(Pr) = \begin{cases} \infty & \text{if } Pr \not\models L; \\ 0 & \text{if } Pr \models L \cup P; \\ 1 + \max_{C \in P:\, Pr \not\models C} \boldsymbol{z}(C) & \text{otherwise.} \end{cases}$$

The following lemma shows that $\boldsymbol{z}$ is a conditional constraint ranking on *KB* that is admissible with *KB*, and $\kappa^{\boldsymbol{z}}$ is a probability ranking that is admissible with *KB*.

Lemma 4.14. *Let* $KB = (L, P)$ *be a* $\boldsymbol{p}$*-consistent probabilistic knowledge base. Then,* (a) $\boldsymbol{z}$ *and* (b) $\kappa^{\boldsymbol{z}}$ *are both admissible with KB.*

We define a preference relation on probabilistic interpretations as follows. For probabilistic interpretations *Pr* and *Pr*′, we say *Pr* is $\boldsymbol{z}$*-preferable* to *Pr*′ iff $\kappa^{\boldsymbol{z}}(Pr) < \kappa^{\boldsymbol{z}}(Pr')$. A model *Pr* of a set of logical constraints and probabilistic formulas $\mathcal{F}$ is a $\boldsymbol{z}$*-minimal model* of $\mathcal{F}$ iff no model of $\mathcal{F}$ is $\boldsymbol{z}$-preferable to *Pr*.

We are now ready to define the notion of $\boldsymbol{z}$*-entailment*. A conditional constraint $(\psi|\phi)[l, u]$ is a $\boldsymbol{z}$*-consequence* of *KB*, denoted $KB \Vdash\!\sim^{z} (\psi|\phi)[l, u]$, iff every $\boldsymbol{z}$-minimal model of $L \cup \{\phi > 0\}$ satisfies $(\psi|\phi)[l, u]$. We say $(\psi|\phi)[l, u]$ is a *tight* $\boldsymbol{z}$*-consequence* of *KB*, denoted $KB \Vdash\!\sim^{z}_{tight} (\psi|\phi)[l, u]$, iff l (resp., u) is the infimum (resp., supremum) of $Pr(\psi|\phi)$ subject to all $\boldsymbol{z}$-minimal models *Pr* of $L \cup \{\phi > 0\}$.

The following example illustrates the probabilistic notion of z-entailment. In particular, it shows that z-entailment differs from $\boldsymbol{p}$-entailment in the sense that z-entailment realizes an inheritance of default logical properties from classes to non-exceptional subclasses. But z-entailment does not inherit default logical properties from classes to subclasses that are exceptional relative to some other property (and thus, like its classical counterpart, has the problem of inheritance blocking). Algorithms for computing tight intervals under z-entailment are given in Section 6.

Example 4.15 (*Ostriches cont'd*). Some tight conclusions under z-entailment from the probabilistic knowledge base *KB* in Table 4 are shown in Table 5. More precisely, we obtain the desired tight conclusions $(legs \mid bird)[1, 1]$, $(fly \mid bird)[1, 1]$, $(fly \mid ostrich)[0, 0.05]$, and $(fly \mid red \wedge bird)[1, 1]$. However, we also obtain the tight conclusion $(legs \mid ostrich)[0, 1]$ instead of the desired one $(legs \mid ostrich)[1, 1]$. Here, the interval "[0, 1]" is due to the fact that the default logical property of having legs is not inherited from birds to its exceptional subclass of ostriches.

The following theorem characterizes the notion of z-consequence in terms of the probability ranking κ^z (and thus relates z-entailment to $\boldsymbol{p}$-entailment).

Theorem 4.16. *Let $KB = (L, P)$ be a $\boldsymbol{p}$-consistent probabilistic knowledge base, and let $C = (\psi|\phi)[l, u]$ be a conditional constraint. Then, $KB \Vdash\!\sim^{z} C$ iff $\kappa^z(\phi > 0) = \infty$ or $\kappa^z(\phi > 0 \wedge C) < \kappa^z(\phi > 0 \wedge \neg C)$.*

4.4. Probabilistic lexicographic entailment

We finally define a generalization of Lehmann's lexicographic entailment [46] to $\boldsymbol{p}$-consistent probabilistic knowledge bases $KB = (L, P)$, which we call *probabilistic lexicographic entailment* (or simply ***lex**-entailment*). Note that, even though we do not use probability rankings here, the new notion of ***lex***-entailment can be easily expressed through a unique single probability ranking.

We use the z-partition $(P_0, \ldots, P_k)$ of *KB* to define a lexicographic preference relation on probabilistic interpretations as follows. For probabilistic interpretations Pr and Pr', we say Pr is *lexicographically preferable* (or ***lex**-preferable*) to Pr' iff some $i \in \{0, \ldots, k\}$ exists such that $|\{C \in P_i \mid Pr \models C\}| > |\{C \in P_i \mid Pr' \models C\}|$ and $|\{C \in P_j \mid Pr \models C\}| = |\{C \in P_j \mid Pr' \models C\}|$ for all $i < j \leqslant k$. A model Pr of a set of logical constraints and probabilistic formulas $\mathcal{F}$ is a *lexicographically minimal* (or ***lex**-minimal*) model of $\mathcal{F}$ iff no model of $\mathcal{F}$ is ***lex***-preferable to Pr.

We are now ready to define the notion of ***lex***-entailment as follows. A conditional constraint $(\psi|\phi)[l, u]$ is a ***lex**-consequence* of *KB*, denoted $KB \Vdash\!\sim^{lex} (\psi|\phi)[l, u]$, iff each ***lex***-minimal model of $L \cup \{\phi > 0\}$ satisfies $(\psi|\phi)[l, u]$. We say $(\psi|\phi)[l, u]$ is a *tight **lex**-consequence* of *KB*, denoted $KB \Vdash\!\sim^{lex}_{tight} (\psi|\phi)[l, u]$, iff $l = \inf Pr(\psi|\phi)$ (resp., $u = \sup Pr(\psi|\phi)$) subject to all ***lex***-minimal models Pr of $L \cup \{\phi > 0\}$.

In the following example, ***lex***-entailment realizes a correct inheritance of default logical properties, without showing the problem of inheritance blocking. See Section 6 for algorithms for computing tight intervals under ***lex***-entailment.

Example 4.17 (*Ostriches cont'd*). Consider again the probabilistic knowledge base *KB* given in Table 4. Some tight ***lex***-consequences are shown in Table 5. Observe that we obtain all the desired tight conclusions $(legs \mid bird)[1, 1]$, $(fly \mid bird)[1, 1]$, $(legs \mid ostrich)[1, 1]$, $(fly \mid ostrich)[0, 0.05]$, and $(fly \mid red \wedge bird)[1, 1]$.

5. Semantic properties

In this section, we explore the semantic properties of the new notions of ***p***-, ***z***-, and ***lex***-entailment, and give a comparison to logical entailment in probabilistic logic. We first describe their nonmonotonicity and nonmonotonic properties. We then explore the relationships between the formalisms and to their classical counterparts.

5.1. Nonmonotonicity

In the sequel, we denote by $\Vert\!\sim$ a generic notion of entailment for probabilistic knowledge bases $KB = (L, P)$, which relates KB to its entailed conditional constraints $(\psi|\phi)[l, u]$. The notion of logical entailment $\Vert\!\!\models$ has the following property of *inheritance of logical knowledge* (*L-INH*) along subclass relationships (recall that $KB \Vert\!\!\models F$ iff every model Pr of $L \cup P$ is also a model of F; see Section 2.2):

L-INH. If $KB \Vert\!\sim (\psi|\phi)[c, c]$ and $\phi \Leftarrow \phi^\star$ is valid, then $KB \Vert\!\sim (\psi|\phi^\star)[c, c]$,

for all events ψ, ϕ, and $\phi^\star$, all probabilistic knowledge bases *KB*, and all $c \in \{0, 1\}$. The notions of ***p***-, ***z***-, and ***lex***-entailment $\Vert\!\sim^{p}$, $\Vert\!\sim^{z}$, and $\Vert\!\sim^{lex}$ are nonmonotonic in the sense that they all do not satisfy *L-INH*. Here, ***p***-entailment completely fails *L-INH*, while ***z***- and ***lex***-entailment realize some weaker form of *L-INH*.

Notice that logical, ***p***-, ***z***-, and ***lex***-entailment *all do not have* the property of *inheritance of purely probabilistic knowledge* (*P-INH*) along subclass relationships:

P-INH. If $KB \Vert\!\sim (\psi|\phi)[l, u]$ and $\phi \Leftarrow \phi^\star$ is valid, then $KB \Vert\!\sim (\psi|\phi^\star)[l, u]$,

for all events ψ, ϕ, and $\phi^\star$, all probabilistic knowledge bases *KB*, and all $[l, u] \subseteq [0, 1]$ different from $[0, 0]$, $[1, 1]$, and $[1, 0]$. See [52] for entailment semantics that satisfy *P-INH* and restricted forms of *P-INH*. For example, under such entailment semantics, we can draw the conclusion $(\mathit{fly} \mid \mathit{eagle})[0.95, 1]$ from the probabilistic knowledge base $KB = (\{\mathit{bird} \Leftarrow \mathit{eagle}\}, \{(\mathit{fly} \mid \mathit{bird})[0.95, 1]\})$.

5.2. Nonmonotonic properties

We now explore the nonmonotonic behavior (especially related to the above property *L-INH*) of the probabilistic formalisms of this paper. We consider the *KLM postulates* [40], the property *Rational Monotonicity* (*RM*) [40], and the properties *Irrelevance* (*Irr*) and *Conditioning* (*Con*) (adapted from [7] and [61], respectively). An overview of the results on nonmonotonic properties is given in Table 8.

The rationality postulates of System P, namely, *Right Weakening* (*RW*), *Reflexivity* (*Ref*), *Left Logical Equivalence* (*LLE*), *Cut*, *Cautious Monotonicity* (*CM*), and *Or* proposed by Kraus, Lehmann, and Magidor [40], also called *KLM postulates*, are commonly regarded as being particularly desirable for any reasonable notion of nonmonotonic entailment. The following result shows that the notions of logical, ***p***-, ***z***-, and ***lex***-entailment all satisfy (probabilistic versions of) these postulates.

Table 8
Nonmonotonic properties of probabilistic formalisms

Property	$\Vdash$	$\Vert\!\sim^{lex}$	$\Vert\!\sim^{z}$	$\Vert\!\sim^{p}$
KLM postulates	Yes	Yes	Yes	Yes
Rational Monotonicity	Yes	Yes	Yes	**No**
Irrelevance	Yes	Yes	Yes	**No**
Conditioning	Yes	Yes	Yes	Yes

Theorem 5.1. *Every notion of entailment* $\Vert\!\sim$ *among* $\Vdash$, $\Vert\!\sim^{p}$, $\Vert\!\sim^{z}$, *and* $\Vert\!\sim^{lex}$ *satisfies the following properties for all probabilistic knowledge bases* $KB = (L, P)$, *all events* ε, ε', ϕ, *and* ψ, *and all real numbers* $l, l', u, u' \in [0, 1]$:

RW. *If* $(\phi|\top)[l, u] \Rightarrow (\psi|\top)[l', u']$ *is logically valid and* $KB \Vert\!\sim (\phi|\varepsilon)[l, u]$, *then* $KB \Vert\!\sim (\psi|\varepsilon)[l', u']$.
Ref. $KB \Vert\!\sim (\varepsilon|\varepsilon)[1, 1]$.
LLE. *If* $\varepsilon \Leftrightarrow \varepsilon'$ *is logically valid, then* $KB \Vert\!\sim (\phi|\varepsilon)[l, u]$ *iff* $KB \Vert\!\sim (\phi|\varepsilon')[l, u]$.
Cut. *If* $KB \Vert\!\sim (\varepsilon|\varepsilon')[1, 1]$ *and* $KB \Vert\!\sim (\phi|\varepsilon \wedge \varepsilon')[l, u]$, *then* $KB \Vert\!\sim (\phi|\varepsilon')[l, u]$.
CM. *If* $KB \Vert\!\sim (\varepsilon|\varepsilon')[1, 1]$ *and* $KB \Vert\!\sim (\phi|\varepsilon')[l, u]$, *then* $KB \Vert\!\sim (\phi|\varepsilon \wedge \varepsilon')[l, u]$.
Or. *If* $KB \Vert\!\sim (\phi|\varepsilon)[1, 1]$ *and* $KB \Vert\!\sim (\phi|\varepsilon')[1, 1]$, *then* $KB \Vert\!\sim (\phi|\varepsilon \vee \varepsilon')[1, 1]$.

Another desirable property is *Rational Monotonicity* (*RM*) [40], which describes a restricted form of monotony, and allows to ignore certain kinds of irrelevant knowledge. The next theorem shows that logical, ***z***-, and ***lex***-entailment all satisfy *RM*. Note that here $KB \Vert\!\not\sim C$ denotes that $KB \Vert\!\sim C$ does not hold.

Theorem 5.2. $\Vdash$, $\Vert\!\sim^{z}$, *and* $\Vert\!\sim^{lex}$ *satisfy the following property for all probabilistic knowledge bases* $KB = (L, P)$ *and all events* ε, ε', *and* ψ:

RM. *If* $KB \Vert\!\sim (\psi|\varepsilon)[1, 1]$ *and* $KB \Vert\!\not\sim (\neg\varepsilon'|\varepsilon)[1, 1]$, *then* $KB \Vert\!\sim (\psi|\varepsilon \wedge \varepsilon')[1, 1]$.

The notion of ***p***-entailment, however, generally does not satisfy the property *RM*, as the following example shows.

Example 5.3. Consider the following probabilistic knowledge base $KB = (L, P)$:

$$(L, P) = \big(\{bird \Leftarrow eagle\}, \{(fly \mid bird)[1, 1]\}\big).$$

Here, $(fly \mid bird)[1, 1]$ is a logical (resp., ***p***-, ***z***-, and ***lex***-) consequence of *KB*, and $(\neg eagle \mid bird)[1, 1]$ is not a logical (resp., ***p***-, ***z***-, and ***lex***-) consequence of *KB*. Observe now that $(fly \mid bird \wedge eagle)[1, 1]$ is a logical (resp., ***z***- and ***lex***-) consequence of *KB*, but $(fly \mid bird \wedge eagle)[1, 1]$ is not a ***p***-consequence of *KB*. Note that $(fly \mid bird \wedge eagle)[1, 1]$ is a tight logical (resp., ***z***- and ***lex***-) consequence of *KB*, while $(fly \mid bird \wedge eagle)[0, 1]$ is a tight ***p***-consequence of *KB*.

We next consider the property *Irrelevance* (*Irr*) adapted from [7]. Informally, *Irr* says that ε' is irrelevant to a conclusion "$P \Vert\!\sim (\psi|\varepsilon)[1, 1]$" when they are defined over disjoint sets of basic events. The following result shows that logical, $\boldsymbol{z}$-, and ***lex***-entailment all satisfy the property *Irr*.

Theorem 5.4. *$\Vdash$, $\Vdash\!\sim^{z}$, and $\Vdash\!\sim^{lex}$ satisfy the following property for all probabilistic knowledge bases $KB = (L, P)$ and all events ε, ε', and ψ:*

Irr. If $KB \Vdash\!\sim (\psi|\varepsilon)[1,1]$, and no basic event of KB and $(\psi|\varepsilon)[1,1]$ occurs in ε', then $KB \Vdash\!\sim (\psi|\varepsilon\wedge\varepsilon')[1,1]$.

The notion of ***p***-entailment, however, does not satisfy *Irr*. This is already clear from the tight ***p***-consequence $(\mathit{fly} \mid \mathit{red} \wedge \mathit{bird})[0,1]$ of *KB* in Table 4 (cf. Example 4.12). It is also shown by the following (less complex) example.

Example 5.5. Consider the following probabilistic knowledge base $KB = (L, P)$:

$$(L, P) = \big(\emptyset, \{(\mathit{fly} \mid \mathit{bird})[1,1]\}\big).$$

Here, $(\mathit{fly} \mid \mathit{bird})[1,1]$ is a logical (resp., ***p***-, $\boldsymbol{z}$-, and ***lex***-) consequence of *KB*. Observe now that $(\mathit{fly} \mid \mathit{red} \wedge \mathit{bird})[1,1]$ is a logical (resp., $\boldsymbol{z}$- and ***lex***-) consequence of *KB*, but $(\mathit{fly} \mid \mathit{red} \wedge \mathit{bird})[1,1]$ is not a ***p***-consequence of *KB*. Note that $(\mathit{fly} \mid \mathit{red} \wedge \mathit{bird})[1,1]$ is a tight logical (resp., $\boldsymbol{z}$- and ***lex***-) consequence of *KB*, while $(\mathit{fly} \mid \mathit{red} \wedge \mathit{bird})[0,1]$ is a tight ***p***-consequence of *KB*.

Finally, the properties *Conditioning* (*Con*) (adapted from [61]) and *Inclusion* (*Inc*) express that *KB* should entail all its own conditional constraints. The following result shows that logical, ***p***-, $\boldsymbol{z}$-, and ***lex***-entailment all satisfy *Con* and *Inc*. Obviously, *Con* implies *Inc*; conversely, *Inc* and *LLE* imply *Con*.

Theorem 5.6. *$\Vdash$, $\Vdash\!\sim^{p}$, $\Vdash\!\sim^{z}$, and $\Vdash\!\sim^{lex}$ satisfy the following properties for all probabilistic knowledge bases $KB = (L, P)$, all events ε, ϕ, and ψ, and all $l, u \in [0,1]$:*

Con. If $(\psi|\phi)[l,u] \in P$ and $\varepsilon \Leftrightarrow \phi$ is logically valid, then $KB \Vdash\!\sim (\psi|\varepsilon)[l,u]$.
Inc. If $(\psi|\phi)[l,u] \in P$, then $KB \Vdash\!\sim (\psi|\phi)[l,u]$.

5.3. Relationships between probabilistic formalisms

In this section, we investigate the relationships between the different probabilistic formalisms. The following theorem shows that logical entailment is stronger than ***lex***-entailment, and that the latter is stronger than $\boldsymbol{z}$-entailment, which in turn is stronger than ***p***-entailment. That is, the logical implications illustrated by the upper horizontal line of arrows in Fig. 1 hold between the probabilistic formalisms. Note that similar logical implications hold between their classical counterparts (which are illustrated by the lower horizontal line of arrows in Fig. 1).

Theorem 5.7. *Let $KB = (L, P)$ be a **p**-consistent probabilistic knowledge base, and let $C = (\psi|\phi)[l,u]$ be a conditional constraint. Then,*

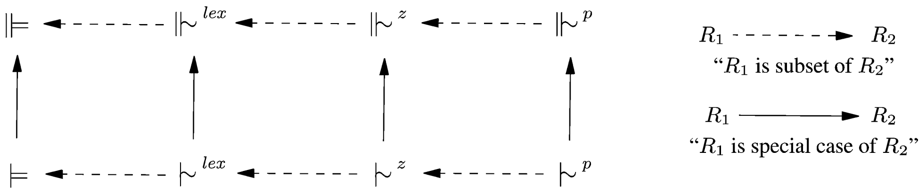

Fig. 1. Relationships between probabilistic and classical formalisms.

(a) $KB \Vdash\!\sim^{p} C$ *implies* $KB \Vdash\!\sim^{z} C$.
(b) $KB \Vdash\!\sim^{z} C$ *implies* $KB \Vdash\!\sim^{lex} C$.
(c) $KB \Vdash\!\sim^{lex} C$ *implies* $KB \Vdash C$.

In general, none of the converse implications holds, as Table 5 immediately shows. However, if $L \cup P$ has a model where the conditioning event ϕ has a positive probability, then logical, z-, and ***lex***-entailment of $(\psi|\phi)[l,u]$ from KB all coincide. Roughly, in this special case, it is consistent to transform all defaults $\beta \leftarrow \alpha$ in P that are relevant to a conclusion of $(\psi|\phi)[l,u]$ from KB into strict logical constraints $\beta \Leftarrow \alpha$ in L. This important result is expressed by the following theorem.

Theorem 5.8. *Let* $KB = (L, P)$ *be a* ***p****-consistent probabilistic knowledge base, and let* $C = (\psi|\phi)[l,u]$ *be a conditional constraint such that* $L \cup P$ *has a model* Pr *with* $Pr(\phi) > 0$. *Then,* $KB \Vdash C$ *iff* $KB \Vdash\!\sim^{lex} C$ *iff* $KB \Vdash\!\sim^{z} C$.

The following example shows that ***p***-entailment, however, generally does not coincide with logical entailment when $L \cup P$ has a model Pr with $Pr(\phi) > 0$.

Example 5.9. Consider $KB = (L, P) = (\{bird \Leftarrow eagle\}, \{(fly \mid bird)[1,1]\})$. Here, $L \cup P$ has a model Pr with $Pr(eagle) > 0$, and $(fly \mid eagle)[1,1]$ is a logical (resp., z- and ***lex***-) consequence of KB, but $(fly \mid eagle)[1,1]$ is not a ***p***-consequence of KB. Note that $(fly \mid eagle)[1,1]$ is a tight logical (resp., z- and ***lex***-) consequence of KB, while $(fly \mid eagle)[0,1]$ is a tight ***p***-consequence of KB.

5.4. Relationships to classical formalisms

Finally, we explore the relationships between ***p***-, z-, and ***lex***-entailment and their classical counterparts. The following result shows that ***p***-, z-, and ***lex***-entailment for ***p***-consistent probabilistic knowledge bases generalize their classical counterparts for ***p***-consistent conditional knowledge bases. Here, the operator γ on conditional constraints, sets of conditional constraints, and conditional knowledge bases replaces each conditional constraint $(\psi|\phi)[1,1]$ by the default $\psi \leftarrow \phi$. By Theorems 2.4 and 2.5, logical entailment in probabilistic logic similarly generalizes its classical counterpart. All this is illustrated by the vertical arrows in Fig. 1.

Theorem 5.10. *Let* $KB = (L, \{(\psi_i|\phi_i)[1,1] \mid i \in \{1,\ldots,n\}\})$ *be a* ***p****-consistent probabilistic knowledge base, and let* $(\beta|\alpha)[1,1]$ *be a conditional constraint. Then,*

(a) $KB \,\|\!\!\sim^{p} (\beta|\alpha)[1,1]$ *iff* $\gamma(KB) \,|\!\!\sim^{p} \beta \leftarrow \alpha$.
(b) $KB \,\|\!\!\sim^{z} (\beta|\alpha)[1,1]$ *iff* $\gamma(KB) \,|\!\!\sim^{z} \beta \leftarrow \alpha$.
(c) $KB \,\|\!\!\sim^{lex} (\beta|\alpha)[1,1]$ *iff* $\gamma(KB) \,|\!\!\sim^{lex} \beta \leftarrow \alpha$.

6. Algorithms

In this section, we provide algorithms for the main reasoning problems in weak non-monotonic probabilistic logics.

6.1. Overview

The main decision and optimization problems of probabilistic reasoning in weak non-monotonic probabilistic logics are summarized as follows:

$\boldsymbol{p}$-CONSISTENCY: Given a probabilistic knowledge base KB, decide whether KB is $\boldsymbol{p}$-consistent.
S-CONSEQUENCE: Given a $\boldsymbol{p}$-consistent probabilistic knowledge base KB and a conditional constraint $(\beta|\alpha)[l,u]$, decide whether $KB \,\|\!\!\sim^{s} (\beta|\alpha)[l,u]$ holds, for some fixed semantics $s \in \{\boldsymbol{p}, \boldsymbol{z}, \boldsymbol{lex}\}$.
TIGHT S-CONSEQUENCE: Given a $\boldsymbol{p}$-consistent probabilistic knowledge base KB and a conditional event $\beta|\alpha$, compute $l, u \in [0,1]$ such that $KB \,\|\!\!\sim^{s} (\beta|\alpha)[l,u]$, for some fixed semantics $s \in \{\boldsymbol{p}, \boldsymbol{z}, \boldsymbol{lex}\}$.

The basic idea behind the algorithms below for solving the above decision and optimization problems is to perform a reduction to the following standard decision and optimization problems in model-theoretic probabilistic logic:

POSITIVE PROBABILITY: Given a probabilistic knowledge base $KB = (L, P)$ and an event α, decide whether $L \cup P$ has a model Pr such that $Pr(\alpha) > 0$.
LOGICAL CONSEQUENCE: Given a probabilistic knowledge base KB and a conditional constraint $(\beta|\alpha)[l,u]$, decide whether $KB \,\|\!\!= (\beta|\alpha)[l,u]$ holds.
TIGHT LOGICAL CONSEQUENCE: Given a probabilistic knowledge base KB and a conditional event $\beta|\alpha$, compute $l, u \in [0,1]$ such that $KB \,\|\!\!=_{tight} (\beta|\alpha)[l,u]$.

The problems POSITIVE PROBABILITY and LOGICAL CONSEQUENCE can be reduced to the problem of deciding whether a system of linear constraints is solvable, while TIGHT LOGICAL CONSEQUENCE is reducible to computing the optimal solutions of two linear optimization problems; cf. especially [19,39,51].

Since the notions of $\boldsymbol{p}$-consistency and $\boldsymbol{p}$-entailment coincide with the notions of g-coherence and g-coherent entailment (cf. Section 8.2), existing algorithms for deciding g-coherence and computing tight intervals under g-coherent entailment can be used for solving $\boldsymbol{p}$-CONSISTENCY and TIGHT $\boldsymbol{p}$-CONSEQUENCE, respectively. Such algorithms are shown in Figs. 2 and 4, respectively. Here, the one in Fig. 2 also computes the $\boldsymbol{z}$-partition of KB, if KB is $\boldsymbol{p}$-consistent; it is similar to the algorithm for deciding

Algorithm $\boldsymbol{p}$-consistency (essentially Biazzo et al. [9])

Input: probabilistic knowledge base $KB = (L, P)$.
Output: z-partition of KB, if KB is $\boldsymbol{p}$-consistent; *nil* otherwise.

1. **if** $P = \emptyset$ **then if** L is satisfiable **then return** () **else return** *nil*;
2. $R := P$;
3. $i := -1$;
4. **repeat**
5. $\quad i := i + 1$;
6. $\quad D[i] := \{(\psi|\phi)[l, u] \in R \mid L \cup R \cup \{\phi > 0\}$ is satisfiable$\}$;
7. $\quad R := R \setminus D[i]$
8. **until** $R = \emptyset$ **or** $D[i] = \emptyset$;
9. **if** $R = \emptyset$ **then return** $(D[0], \ldots, D[i])$ **else return** *nil*.

Fig. 2. Algorithm *$\boldsymbol{p}$-consistency*.

Algorithm tight-$\boldsymbol{p}$-consequence (essentially Biazzo et al. [9])

Input: $\boldsymbol{p}$-consistent probabilistic knowledge base KB=(L, P), conditional event $\beta|\alpha$.
Output: interval $[l, u] \subseteq [0, 1]$ such that $KB \,\|\!\!\sim^{p}_{tight} (\beta|\alpha)[l, u]$.

1. $R := P$;
2. **repeat**
3. $\quad \Delta := \{(\psi|\phi)[l, u] \in R \mid L \cup R \cup \{\bot \Leftarrow \alpha\} \cup \{\phi > 0\}$ is satisfiable$\}$;
4. $\quad R := R \setminus \Delta$
5. **until** $\Delta = \emptyset$;
6. compute $l, u \in [0, 1]$ such that $L \cup R \,\|\!\!=_{tight} (\beta|\alpha)[l, u]$;
7. **return** $[l, u]$.

Fig. 3. Algorithm *tight-$\boldsymbol{p}$-consequence*.

ε-consistency in default reasoning by Goldszmidt and Pearl [32]. The algorithm in Fig. 4 is based on the result that the notion of $\boldsymbol{p}$-entailment from KB coincides with logical entailment from a unique subbase of KB. The decision problem $\boldsymbol{p}$-CONSEQUENCE can be solved in a similar way.

In the next subsection, we provide algorithms for solving the optimization problems TIGHT z- and TIGHT ***lex***-CONSEQUENCE. The decision problems z- and ***lex***-CONSEQUENCE can be solved in a similar way.

6.2. Tight z- and *lex*-consequence

We now give algorithms for solving TIGHT z- and TIGHT ***lex***-CONSEQUENCE. In the sequel, let $KB = (L, P)$ be a $\boldsymbol{p}$-consistent probabilistic knowledge base, and let $(P_0, \ldots, P_k)$ be its z-partition. We first give some preparatory definitions.

For $G, H \subseteq P$, we say G is *z-preferable* to H iff some $i \in \{0, \ldots, k\}$ exists such that $P_i \subseteq G$, $P_i \nsubseteq H$, and $P_j \subseteq G$ and $P_j \subseteq H$ for all $i < j \leqslant k$. We say G is ***lex**-preferable* to H iff some $i \in \{0, \ldots, k\}$ exists such that $|G \cap P_i| > |H \cap P_i|$ and $|G \cap P_j| = |H \cap P_j|$ for all $i < j \leqslant k$. For $\mathcal{D} \subseteq 2^P$ and $s \in \{z, \boldsymbol{lex}\}$, we say G is *s-minimal* in $\mathcal{D}$ iff $G \in \mathcal{D}$ and no $H \in \mathcal{D}$ is s-preferable to G.

Algorithm tight-z-consequence

Input: $\boldsymbol{p}$-consistent probabilistic knowledge base $KB{=}(L, P)$, conditional event $\beta|\alpha$.
Output: interval $[l, u] \subseteq [0, 1]$ such that $KB \Vert\!\!\sim^{z} (\beta|\alpha)[l, u]$.
Notation: $(P_0, \ldots, P_k)$ denotes the z-partition of KB.

1. $R := L$;
2. **if** $R \cup \{\alpha > 0\}$ is unsatisfiable **then return** $[1, 0]$;
3. $j := k$;
4. **while** $j \geqslant 0$ **and** $R \cup P_j \cup \{\alpha > 0\}$ is satisfiable **do begin**
5. $\quad R := R \cup P_j$;
6. $\quad j := j - 1$
7. **end**;
8. compute $l, u \in [0, 1]$ such that $R \Vert\!\!\models_{tight} (\beta|\alpha)[l, u]$;
9. **return** $[l, u]$.

Fig. 4. Algorithm *tight-z-consequence*.

The following theorem shows how TIGHT s-CONSEQUENCE, where $s \in \{z, \boldsymbol{lex}\}$, can be reduced to POSITIVE PROBABILITY and TIGHT LOGICAL CONSEQUENCE. The key idea behind this reduction is that there exists a set $\mathcal{D}^s_\alpha(KB) \subseteq 2^P$ such that $KB \Vert\!\!\sim^{s} (\beta|\alpha)[l, u]$ iff $L \cup H \Vert\!\!\models (\beta|\alpha)[l, u]$ for all $H \in \mathcal{D}^s_\alpha(KB)$.

Theorem 6.1. *Let $KB = (L, P)$ be a $\boldsymbol{p}$-consistent probabilistic knowledge base, and let $\beta|\alpha$ be a conditional event. Let $s \in \{z, \boldsymbol{lex}\}$. Let $\mathcal{D}^s_\alpha(KB)$ be the set of all s-minimal elements in $\{H \subseteq P \mid L \cup H \cup \{\alpha > 0\}$ is satisfiable$\}$. Then, l (resp., u) such that $KB \Vert\!\!\sim^{s}_{tight} (\beta|\alpha)[l, u]$ is given as follows*:

(a) *If $L \cup \{\alpha > 0\}$ is unsatisfiable, then $l = 1$ (resp., $u = 0$).*
(b) *Otherwise, $l = \min c$ (resp., $u = \max d$) subject to $L \cup H \Vert\!\!\models_{tight} (\beta|\alpha)[c, d]$ and $H \in \mathcal{D}^s_\alpha(KB)$.*

For $s = z$ (resp., $s = \boldsymbol{lex}$), Algorithm *tight-s-consequence* (see Fig. 4 (resp., 5)) computes tight intervals under s-entailment. Step 2 checks whether $L \cup \{\alpha > 0\}$ is unsatisfiable. If this is the case, then $[1, 0]$ is returned by Theorem 6.1(a). Otherwise, we compute $\mathcal{D}^s_\alpha(KB)$ along the z-partition of KB in steps 3–7 (resp., 3–15), and the requested tight interval using Theorem 6.1(b) in step 8 (resp., 16–20).

7. Computational complexity

In this section, we draw a precise picture of the computational complexity of the decision and optimization problems described in Section 6.1.

7.1. Complexity classes

We assume some basic knowledge about the complexity classes P, NP, and co-NP. We now briefly describe some other complexity classes that occur in our results; see especially [23,38,56] for further background.

Algorithm tight-*lex*-consequence

Input: ***p***-consistent probabilistic knowledge base $KB=(L, P)$, conditional event $\beta|\alpha$.
Output: interval $[l, u] \subseteq [0, 1]$ such that $KB \Vdash\!\sim^{lex}_{tight} (\beta|\alpha)[l, u]$.
Notation: $(P_0, \ldots, P_k)$ denotes the z-partition of KB.

```
R := L;
if R ∪ {α > 0} is unsatisfiable then return [1, 0];
𝓗 := {∅};
for j := k downto 0 do begin
  n := 0;
  𝓗′ := ∅;
  for each G ⊆ D_j and H ∈ 𝓗 do
    if R ∪ G ∪ H ∪ {α > 0} is satisfiable then
      if n = |G| then 𝓗′ := 𝓗′ ∪ {G ∪ H}
        else if n < |G| then begin
          𝓗′ := {G ∪ H};
          n := |G|
        end;
  𝓗 := 𝓗′;
end;
(l, u) := (1, 0);
for each H ∈ 𝓗 do begin
  compute c, d ∈ [0, 1] such that R ∪ H ||=_tight (β|α)[c, d];
  (l, u) := (min(l, c), max(u, d))
end;
return [l, u].
```

Fig. 5. Algorithm *tight-**lex**-consequence*.

The class P^{NP} contains all decision problems that can be solved in deterministic polynomial time with an oracle for NP. The class $\mathrm{P}^{\mathrm{NP}}_{\|}$ contains the decision problems in P^{NP} where all oracle calls must be first prepared and then issued in parallel. The relationship between these complexity classes is described by the following inclusion hierarchy (note that all inclusions are currently believed to be strict):

$$\mathrm{P} \subseteq \mathrm{NP},\ \text{co-NP} \subseteq \mathrm{P}^{\mathrm{NP}}_{\|} \subseteq \mathrm{P}^{\mathrm{NP}}.$$

To classify problems that compute an output value, rather than a Yes/No-answer, function classes have been introduced. In particular, FP and $\mathrm{FP}^{\mathrm{NP}}$ are the functional analogs of P and P^{NP}, respectively.

7.2. Overview of complexity results

In the complexity analysis, we consider the decision and optimization problems s-CONSEQUENCE and TIGHT s-CONSEQUENCE, where $s \in \{z, \boldsymbol{lex}\}$. We assume that KB as well as $(\beta|\alpha)[l, u]$ contain only rational numbers.

The complexity results are compactly summarized in Tables 9–10. In detail, the problems z-CONSEQUENCE and ***lex***-CONSEQUENCE are complete for the classes $\mathrm{P}^{\mathrm{NP}}_{\|}$

Table 9
Complexity of z- and $\boldsymbol{lex}$-CONSEQUENCE

Problem	Complexity
z-CONSEQUENCE	$\mathrm{P}_{\parallel}^{\mathrm{NP}}$-complete
$\boldsymbol{lex}$-CONSEQUENCE	P^{NP}-complete

Table 10
Complexity of TIGHT z- and $\boldsymbol{lex}$-CONSEQUENCE

Problem	Complexity
TIGHT z-CONSEQUENCE	$\mathrm{FP}^{\mathrm{NP}}$-complete
TIGHT $\boldsymbol{lex}$-CONSEQUENCE	$\mathrm{FP}^{\mathrm{NP}}$-complete

and P^{NP}, respectively, whereas the problems TIGHT z-CONSEQUENCE and TIGHT $\boldsymbol{lex}$-CONSEQUENCE are both complete for the class $\mathrm{FP}^{\mathrm{NP}}$.

The hardness results often hold even in the restricted *literal-Horn case*, where *KB* and $\beta|\alpha$ are both literal-Horn. Here, a conditional event $\psi|\phi$ (resp., logical constraint $\psi \Leftarrow \phi$) is *literal-Horn* iff ψ is a basic event (resp., ψ is either a basic event or the negation of a basic event) and ϕ is either $\top$ or a conjunction of basic events. A conditional constraint $(\psi|\phi)[l,u]$ is *literal-Horn* iff the conditional event $\psi|\phi$ is literal-Horn. A probabilistic knowledge base $KB = (L, P)$ is *literal-Horn* iff every member of $L \cup P$ is literal-Horn.

Note that the problems $\boldsymbol{p}$-CONSISTENCY, $\boldsymbol{p}$-CONSEQUENCE and TIGHT $\boldsymbol{p}$-CONSEQUENCE are complete for NP, co-NP, and $\mathrm{FP}^{\mathrm{NP}}$, respectively, in the general case and also in restricted cases. This is immediate by similar complexity results for g-coherence and g-coherent entailment [9] and the equivalence of these notions to $\boldsymbol{p}$-consistency and $\boldsymbol{p}$-entailment, respectively; cf. Section 8.2. Similarly, also the problems POSITIVE PROBABILITY, LOGICAL CONSEQUENCE, and TIGHT LOGICAL CONSEQUENCE in probabilistic logic are complete for NP, co-NP, and $\mathrm{FP}^{\mathrm{NP}}$, respectively, in the general case and also in restricted cases; cf. especially [51].

7.3. Detailed complexity results

The following two theorems show that the problems z- and $\boldsymbol{lex}$-CONSEQUENCE are complete for the classes $\mathrm{P}_{\parallel}^{\mathrm{NP}}$ and P^{NP}, respectively. Here, hardness for $\mathrm{P}_{\parallel}^{\mathrm{NP}}$ and P^{NP} follows from Theorem 5.10 and $\mathrm{P}_{\parallel}^{\mathrm{NP}}$- and P^{NP}-hardness of deciding z- and *lex*-entailment, respectively, in classical default reasoning [18].

Theorem 7.1. *Given a $\boldsymbol{p}$-consistent probabilistic knowledge base KB, and a conditional constraint $(\beta|\alpha)[l,u]$, deciding whether $KB \Vdash\!\sim^{z} (\beta|\alpha)[l,u]$ is $\mathrm{P}_{\parallel}^{\mathrm{NP}}$-complete.*

Theorem 7.2. *Given a $\boldsymbol{p}$-consistent probabilistic knowledge base KB, and a conditional constraint $(\beta|\alpha)[l,u]$, deciding whether $KB \Vdash\!\sim^{lex} (\beta|\alpha)[l,u]$ is P^{NP}-complete. Hardness holds even if KB and $\beta|\alpha$ are literal-Horn.*

The next two theorems show that TIGHT s-CONSEQUENCE, where $s \in \{z, \boldsymbol{lex}\}$, is $\mathrm{FP}^{\mathrm{NP}}$-complete. Hardness holds by a polynomial reduction from the $\mathrm{FP}^{\mathrm{NP}}$-complete *traveling salesman cost* problem [56].

Theorem 7.3. *Given a* $\boldsymbol{p}$*-consistent probabilistic knowledge base KB, and a conditional event* $\beta|\alpha$*, computing* $l, u \in [0,1]$ *such that* $KB \Vdash\!\sim^{z}_{tight} (\beta|\alpha)[l,u]$ *is* $\mathrm{FP}^{\mathrm{NP}}$*-complete. Hardness holds even if KB and* $\beta|\alpha$ *are literal-Horn, and* $L = \emptyset$*.*

Theorem 7.4. *Given a* $\boldsymbol{p}$*-consistent probabilistic knowledge base KB, and a conditional event* $\beta|\alpha$*, computing* $l, u \in [0,1]$ *such that* $KB \Vdash\!\sim^{lex}_{tight} (\beta|\alpha)[l,u]$ *is* $\mathrm{FP}^{\mathrm{NP}}$*-complete. Hardness holds even if KB and* $\beta|\alpha$ *are literal-Horn, and* $L = \emptyset$*.*

8. Related work

In this section, we give a comparison to the related works on probabilistic default reasoning [52] and on probabilistic reasoning under g-coherence [8,27–29].

8.1. Strong nonmonotonic probabilistic logics

A companion paper [52] presents similar probabilistic generalizations of Pearl's entailment in System Z, Lehmann's lexicographic entailment, and Geffner's conditional entailment [24,26]. These formalisms, however, are quite different from the ones in this paper, since they allow for handling *default purely probabilistic knowledge* rather than *(strict) purely probabilistic knowledge* in addition to strict logical knowledge and default logical knowledge. More precisely, the formalisms in [52] are an extension of a variant of probabilistic logic (in which purely probabilistic conditional constraints are interpreted as default sentences) by defaults as in conditional knowledge bases, while the formalisms here are an extension of probabilistic logic (in which purely probabilistic conditional constraints are interpreted as strict sentences) by defaults as in conditional knowledge bases. For example, the formalisms in [52] interpret the purely probabilistic conditional constraint $(\mathit{fly} \mid \mathit{bird})[0.95, 1]$ as "*generally*, birds (and special birds) fly with a probability of at least 0.95", while the formalisms here interpret $(\mathit{fly} \mid \mathit{bird})[0.95, 1]$ as "birds fly with a probability of at least 0.95". Roughly, the former means that being able to fly with a probability of at least 0.95 should apply to the class of all birds and all subclasses of birds, as long as this does not create any inconsistencies, while the latter says that being able to fly with a probability of at least 0.95 should only apply to the class of all birds. That is, the formalisms in [52] interpret purely probabilistic conditional constraints in a much stronger way than the formalisms here. For this reason, they are generally much stronger than the formalisms here. This is why the formalisms in [52] can be considered as *strong nonmonotonic probabilistic logics*, while the formalisms here are *weak nonmonotonic probabilistic logics*. The former are especially useful where logical entailment in probabilistic logic is too weak, for example, in probabilistic logic programming [50,51] and probabilistic ontology reasoning in the Semantic Web [30]. Other applications are deriving degrees of belief from statistical knowledge and degrees of belief, handling inconsistencies in probabilistic knowledge bases, and probabilistic belief revision.

In particular, in reasoning from statistical knowledge and degrees of belief, the probabilistic generalization of Lehmann's lexicographic entailment in [52], which we call here *strong* ***lex****-entailment*, shows a similar behavior as reference-class reasoning [41,42,60,62] in a number of uncontroversial examples. Furthermore, it also avoids many drawbacks of reference-class reasoning [52]. In particular, it can handle complex scenarios and even purely probabilistic subjective knowledge as input. Moreover, conclusions are drawn in a global way from all the available knowledge as a whole. The following example illustrates the use of strong ***lex***-entailment for reasoning from statistical knowledge and degrees of belief.

Example 8.1. Suppose that we have the *statistical knowledge* "all penguins are birds", "between 90% and 95% of all birds fly", "at most 5% of all penguins fly", and "at least 95% of all yellow objects are easy to see". Furthermore, suppose that our *belief* is "Sam is a yellow penguin". What do we then conclude about Sam's property of being easy to see? Under reference-class reasoning, which is a machinery for dealing with such statistical knowledge and degrees of belief, we conclude "Sam is easy to see with a probability of at least 0.95". This is also exactly what we obtain using the notion of strong ***lex***-entailment from [52]:

The above statistical knowledge can be represented by the probabilistic knowledge base $KB = (L, P) = (\{bird \Leftarrow penguin\}, \{(fly \mid bird)[0.9, 0.95], (fly \mid penguin)[0, 0.05], (easy\_to\_see \mid yellow)[0.95, 1]\})$, where conditional constraints $(\psi|\phi)[l, u]$ in P now informally read as "generally, the probability of ψ given ϕ is in $[l, u]$". This KB is strongly ***p***-consistent [52], and under strong ***lex***-entailment from KB, we obtain the tight conclusion $(easy\_to\_see \mid yellow \wedge penguin)[0.95, 1]$, as desired.

Note that KB is also satisfiable and ***p***-consistent. However, under every semantics among logical and (weak) ***p***-, ***z***-, and ***lex***-entailment from KB, we obtain the tight conclusion $(easy\_to\_see \mid yellow \wedge penguin)[0, 1]$, rather than the desired one.

8.2. Probabilistic reasoning under g-coherence

Another related formalism is probabilistic reasoning under g-coherence. It is an approach to reasoning with imprecise probability assessments, which has been extensively explored especially in the field of statistics, and which is based on the coherence principle of de Finetti and suitable generalizations of it (see, for example, the work by Biazzo and Gilio [8], Gilio [27,28], and Gilio and Scozzafava [29]), or on similar principles that have been adopted for lower and upper probabilities (Pelessoni and Vicig [59], Vicig [66], and Walley [68]).

Interestingly, the notions of ***p***-consistency and ***p***-entailment for probabilistic knowledge bases coincide with the notions of g-coherence and g-coherent entailment, respectively, for imprecise probability assessments (cf. Theorems 4.10 and 4.11). We now recall the main concepts from probabilistic reasoning under g-coherence. We start by defining (precise) probability assessments and their coherence. We then define imprecise probability assessments and the notions of g-coherence and g-coherent entailment for them and for probabilistic knowledge bases.

A *probability assessment* (L, A) on a set of conditional events $\mathcal{E}$ consists of a set of logical constraints L, and a mapping A that assigns to each $\varepsilon \in \mathcal{E}$ a real number in $[0, 1]$. Informally, L describes logical relationships, while A represents probabilistic knowledge.

For $\{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\} \subseteq \mathcal{E}$ with $n \geqslant 1$ and n real numbers $s_1, \ldots, s_n$, let the mapping $G : \mathcal{I}_\Phi \to \mathbf{R}$ be defined as follows. For every $I \in \mathcal{I}_\Phi$:

$$G(I) = \sum_{i=1}^{n} s_i \cdot I(\phi_i) \cdot \big(I(\psi_i) - A(\psi_i|\phi_i)\big).$$

In the framework of betting criterion, G can be interpreted as the random gain corresponding to a combination of n bets of amounts $s_1 \cdot A(\psi_1|\phi_1), \ldots, s_n \cdot A(\psi_n|\phi_n)$ on $\psi_1|\phi_1, \ldots, \psi_n|\phi_n$ with stakes $s_1, \ldots, s_n$. In detail, to bet on $\psi_i|\phi_i$, one pays an amount of $s_i \cdot A(\psi_i|\phi_i)$, and one gets back the amount of s_i, 0, and $s_i \cdot A(\psi_i|\phi_i)$, when $\psi_i \wedge \phi_i$, $\neg\psi_i \wedge \phi_i$, and $\neg\phi_i$, respectively, turns out to be true. The following notion of *coherence* now assures that it is impossible (for both the gambler and the bookmaker) to have sure (or uniform) loss. A probability assessment (L, A) on a set of conditional events $\mathcal{E}$ is *coherent* iff for every $\{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\} \subseteq \mathcal{E}$ with $n \geqslant 1$ and for all real numbers $s_1, \ldots, s_n$, the following holds:

$$\max_{I \in \mathcal{I}_\Phi, I \models L \cup \{\phi_1 \vee \cdots \vee \phi_n\}} \sum_{i=1}^{n} s_i \cdot I(\phi_i) \cdot \big(I(\psi_i) - A(\psi_i|\phi_i)\big) \geqslant 0.$$

An *imprecise probability assessment* (L, A) on a set of conditional events $\mathcal{E}$ consists of a set of logical constraints L and a mapping A that assigns to each $\varepsilon \in \mathcal{E}$ an interval $[l, u] \subseteq [0, 1]$, $l \leqslant u$. We say (L, A) is *g-coherent* iff a coherent precise probability assessment $(L, A^\star)$ on $\mathcal{E}$ exists with $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$. The imprecise probability assessment $[l, u]$ on a conditional event γ, denoted $\{(\gamma, [l, u])\}$, is called a *g-coherent consequence* of (L, A) iff $A^\star(\gamma) \in [l, u]$ for every coherent precise probability assessment $A^\star$ on $\mathcal{E} \cup \{\gamma\}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$. It is a *tight g-coherent consequence* of (L, A) iff l (resp., u) is the infimum (resp., supremum) of $A^\star(\gamma)$ subject to all coherent precise probability assessments $A^\star$ on $\mathcal{E} \cup \{\gamma\}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$. Observe that for $\varepsilon = \beta|\alpha$ such that $L \models \neg\alpha$, every $\{(\varepsilon, [l, u])\}$ with $l, u \in [0, 1]$ is a g-coherent consequence of (L, A), and $\{(\varepsilon, [1, 0])\}$ is the unique tight g-coherent consequence of (L, A).

We now recall the concepts of g-coherence and g-coherent entailment for probabilistic knowledge bases from [10,11]. Every imprecise probability assessment $IP = (L, A)$, where L is finite, and A is defined on a finite set of conditional events $\mathcal{E}$, can be represented by the following probabilistic knowledge base:

$$KB_{IP} = \big(L, \big\{(\psi|\phi)[l, u] \mid \psi|\phi \in \mathcal{E},\ A(\psi|\phi) = [l, u]\big\}\big).$$

Conversely, each probabilistic knowledge base $KB = (L, P)$ can be expressed by the following imprecise probability assessment $IP_{KB} = (L, A_{KB})$ on $\mathcal{E}_{KB}$:

$$A_{KB} = \big\{\big(\psi|\phi, [l, u]\big) \mid (\psi|\phi)[l, u] \in KB\big\},$$
$$\mathcal{E}_{KB} = \big\{\psi|\phi \mid \exists\, l, u \in [0, 1]\colon\ (\psi|\phi)[l, u] \in KB\big\}.$$

A probabilistic knowledge base KB is *g-coherent* iff IP_{KB} is g-coherent. For g-coherent probabilistic knowledge bases KB and conditional constraints $(\psi|\phi)[l, u]$, we say $(\psi|\phi)[l, u]$ is a *g-coherent consequence* of KB, denoted $KB \Vdash\!\sim^{g} (\psi|\phi)[l, u]$, iff $\{(\psi|\phi, [l, u])\}$ is a g-coherent consequence of IP_{KB}. We say $(\psi|\phi)[l, u]$ is a *tight g-coherent consequence* of KB, denoted $KB \Vdash\!\sim^{g}_{tight} (\psi|\phi)[l, u]$, iff $\{(\psi|\phi, [l, u])\}$ is a tight g-coherent consequence of IP_{KB}.

9. Summary and outlook

We have presented approaches to weak nonmonotonic probabilistic logics, which are combinations of probabilistic logic with default reasoning in Kraus et al.'s System P, Pearl's System Z, and Lehmann's lexicographic entailment. The new formalisms allow for handling in a uniform framework strict and default logical knowledge as well as purely probabilistic knowledge (for example, such as the strict logical knowledge "all ostriches are birds", the default logical knowledge "generally, birds have legs" and "generally, birds fly", and the purely probabilistic knowledge "ostriches fly with a probability of at most 0.05"). Interestingly, probabilistic entailment in System P coincides with probabilistic entailment under g-coherence from imprecise probability assessments. We have then analyzed the semantic and nonmonotonic properties of the new formalisms. We have shown that they all are proper generalizations of their classical counterparts, and they have similar properties as them. In particular, they all satisfy the rationality postulates of System P and a Conditioning property. Moreover, probabilistic entailment in System Z and probabilistic lexicographic entailment both satisfy the property of Rational Monotonicity and an Irrelevance property, while probabilistic entailment in System P does not. We have also analyzed the relationships between the new formalisms. Here, probabilistic entailment in System P is weaker than probabilistic entailment in System Z, which in turn is weaker than probabilistic lexicographic entailment. Moreover, they all are weaker than entailment in probabilistic logic where default sentences are interpreted as strict sentences. Whenever this does not create any inconsistencies, both probabilistic entailment in System Z and probabilistic lexicographic entailment even coincide with such entailment in probabilistic logic, while probabilistic entailment in System P does not. Finally, we have also presented algorithms for reasoning under probabilistic entailment in System Z and probabilistic lexicographic entailment, and given a precise picture of its computational complexity.

In the same spirit as a companion paper [52], this paper has shed light on exciting novel formalisms for probabilistic reasoning with conditional constraints beyond probabilistic logic. Differently from the formalisms in [52], however, the ones here are a "conservative" integration of probabilistic logic with conditional knowledge bases (cf. Section 8.1). That is, they allow for handling in a uniform framework logical and conditional constraints as in probabilistic logic as well as defaults as in conditional knowledge bases. Hence, they are especially useful for reasoning about degrees of belief and defaults (as in, for example, medical or fault diagnosis). An implementation of reasoning in weak (and also strong) nonmonotonic probabilistic logics is available as a part of the system NMPROBLOG [53].

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Appendix A. Proofs for Section 2

Proof of Theorem 2.3. Recall that $KB \mathrel{|}\!\!\models (\psi|\phi)[l,u]$ iff every model Pr of $L \cup P$ is also a model of $(\psi|\phi)[l,u]$. The latter is equivalent to $Pr(\psi|\phi) \in [l,u]$ for every model Pr of $L \cup P$ with $Pr(\phi) > 0$, which in turn is equivalent to $Pr_\phi(\psi) \in [l,u]$ for every model Pr of $L \cup P$ with $Pr(\phi) > 0$. This argument also shows that $KB \mathrel{|}\!\!\models_{tight} (\psi|\phi)[l,u]$ iff l (resp., u) is the infimum (resp., supremum) of $Pr_\phi(\psi)$ subject to all models Pr of $L \cup P$ with $Pr(\phi) > 0$. $\square$

Proof of Theorem 2.4. The two statements of the theorem follow immediately from the observation that probabilistic interpretations Pr satisfy a logical constraint $\psi \Leftarrow \phi$ iff they satisfy the conditional constraint $(\psi|\phi)[1,1]$. $\square$

Proof of Theorem 2.4. Recall that $KB \mathrel{|}\!\!\models \psi \Leftarrow \phi$ iff every model Pr of $L \cup P = L$ is also a model of $\psi \Leftarrow \phi$. Consider now any model $I \in \mathcal{I}_\Phi$ of L. Let the probabilistic interpretation Pr be defined by $Pr(I) = 1$ and $Pr(J) = 0$ for all other $J \in \mathcal{I}_\Phi$. Then, Pr is a model of L, and thus also satisfies $\psi \Leftarrow \phi$. That is, I is a model of $\psi \Leftarrow \phi$. Conversely, consider any model Pr of L. Hence, every $I \in \mathcal{I}_\Phi$ with $Pr(I) > 0$ is a model of L, and thus also of $\psi \Leftarrow \phi$. That is, Pr is a model of $\psi \Leftarrow \phi$. $\square$

Appendix B. Proofs for Section 4

Proof of Theorem 4.5. We first suppose that $P = \emptyset$. Recall that the empty mapping σ on such P is admissible with KB iff L is satisfiable. The latter is equivalent to the existence of a probability ranking κ that is admissible with KB, since every probability ranking κ satisfies $\kappa(Pr) = 0 < \infty$ for at least one probabilistic interpretation Pr. In the following, we assume that $P \neq \emptyset$.

($\Leftarrow$) Assume that there exists a conditional constraint ranking σ on KB that is admissible with KB. Let the probability ranking κ on all Pr be defined as follows:

$$\kappa(Pr) = \begin{cases} \infty & \text{if } Pr \not\models L; \\ 0 & \text{if } Pr \models L \cup P; \\ 1 + \max_{C \in P:\, Pr \not\models C} \sigma(C) & \text{otherwise.} \end{cases}$$

We now show that κ is indeed a probability ranking and that κ is also admissible with KB. Let $C \in P$ such that $\sigma(C)$ is minimal. Since σ is admissible with KB, it follows that C is tolerated by P under L. Hence, in particular, there exists a model Pr of $L \cup P$. Thus, $\kappa(Pr) = 0$. We next show that $\kappa(\neg F) = \infty$ for all $F \in L$. Observe that $\kappa(Pr) = \infty$ for all Pr such that $Pr \not\models F$ (that is, $Pr \models \neg F$) for some $F \in L$. Thus, $\kappa(\neg F) = \infty$ for all $F \in L$. We finally show that $\kappa(\phi > 0) < \infty$ and $\kappa(\phi > 0 \wedge C) < \kappa(\phi > 0 \wedge \neg C)$ for all $C = (\psi|\phi)[l,u] \in P$. Since $\{C' \in P \mid \sigma(C') \geqslant \sigma(C)\}$ tolerates C under L, it holds that

$\kappa(\phi > 0) < \infty$ and $\kappa(\phi > 0 \wedge C) \leqslant \sigma(C)$. Since $Pr \not\models C$ for all models Pr of $\phi > 0 \wedge \neg C$, it holds that $\sigma(C) < \kappa(\phi > 0 \wedge \neg C)$. In summary, $\kappa(\phi > 0) < \infty$ and $\kappa(\phi > 0 \wedge C) < \kappa(\phi > 0 \wedge \neg C)$ for every $C = (\psi|\phi)[l, u] \in P$. This shows that κ is admissible with *KB*.

($\Rightarrow$) Let κ be a probability ranking admissible with *KB*. We define the conditional constraint ranking σ on *KB* by $\sigma(C) = \kappa(\phi > 0 \wedge C)$ for all $C = (\psi|\phi)[l, u] \in P$. We now show that σ is admissible with *KB*. Suppose $P' \subseteq P$ is in conflict with $C = (\psi|\phi)[l, u] \in P$ under L. Towards a contradiction, let $\sigma(C') \geqslant \sigma(C)$ for all $C' \in P'$. Let Pr be a model of L such that $\sigma(C) = \kappa(Pr)$ and $Pr \models \phi > 0 \wedge C$. Assume now $Pr \not\models C'$ for some $C' = (\beta|\alpha)[r, s] \in L'$. Then, $\kappa(\alpha > 0) < \infty$ and $\kappa(\alpha > 0 \wedge \neg C') \leqslant \sigma(C) \leqslant \sigma(C') = \kappa(\alpha > 0 \wedge C')$. But this contradicts κ being admissible with *KB*. Thus, Pr is a model of P'. But this contradicts P' being in conflict with C under L. Hence, $\sigma(C') < \sigma(C)$ for some $C' \in P'$. Thus, σ is admissible with *KB*. $\square$

Proof of Theorem 4.6. Immediate by Theorem 4.5 and the fact that the existence of an admissible conditional constraint ranking on *KB* is equivalent to the existence of an ordered partition $(P_0, \ldots, P_k)$ of P such that either (a) or (b) holds. $\square$

Proof of Theorem 4.8. ($\Rightarrow$) Suppose that $(L, P \cup \{(\psi|\phi)[p, p]\})$ is $\boldsymbol{p}$-consistent for some $p \in [0, l) \cup (u, 1]$. By Theorem 4.5, there exists a probability ranking that is admissible with *KB* such that $\kappa(\phi > 0) < \infty$ and $\kappa(\phi > 0 \wedge (\psi|\phi)[p, p]) < \kappa(\phi > 0 \wedge \neg(\psi|\phi)[p, p])$. Since $\kappa(\phi > 0 \wedge \neg(\psi|\phi)[l, u]) \leqslant \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$ and $\kappa(\phi > 0 \wedge \neg(\psi|\phi)[p, p]) \leqslant \kappa(\phi > 0 \wedge (\psi|\phi)[l, u])$, it follows $\kappa(\phi > 0) < \infty$ and $\kappa(\phi > 0 \wedge \neg(\psi|\phi)[l, u]) \leqslant \kappa(\phi > 0 \wedge (\psi|\phi)[l, u])$. That is, $KB \,\|\!\!\sim^{p} (\psi|\phi)[l, u]$ does not hold.

($\Leftarrow$) Suppose that $KB \,\|\!\!\sim^{p} (\psi|\phi)[l, u]$ does not hold. That is, $\kappa(\phi > 0) < \infty$ and $\kappa(\phi > 0 \wedge (\psi|\phi)[l, u]) \geqslant \kappa(\phi > 0 \wedge \neg(\psi|\phi)[l, u])$ for some probability ranking κ admissible with *KB*. Let Pr be a model of L such that $Pr \models \phi > 0 \wedge \neg(\psi|\phi)[l, u]$ and $\kappa(Pr) = \kappa(\phi > 0 \wedge \neg(\psi|\phi)[l, u])$. We define $p \in [0, l) \cup (u, 1]$ by $p = Pr(\psi|\phi)$. It then follows that $\kappa(\phi > 0 \wedge \neg(\psi|\phi)[l, u]) = \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$. Moreover, it holds that $\kappa(\phi > 0 \wedge (\psi|\phi)[q, q]) \geqslant \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$ for all $q \in [0, l) \cup (u, 1]$. In summary, it thus follows that ($\star$) $\kappa(\phi > 0) < \infty$ and $\kappa(\phi > 0 \wedge \neg(\psi|\phi)[p, p]) \geqslant \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$. We now show that $KB' = (L, P \cup \{(\psi|\phi)[p, p]\})$ is $\boldsymbol{p}$-consistent. We define the conditional constraint ranking σ on *KB* by (i) $\sigma(C) = \kappa(\alpha > 0 \wedge C)$ for all $C = (\beta|\alpha)[r, s] \in P$ such that $\kappa(\alpha > 0 \wedge C) < \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$, (ii) $\sigma((\psi|\phi)[p, p]) = \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$, and (iii) $\sigma(C) = \kappa(\alpha > 0 \wedge C) + 1$ for all $C = (\beta|\alpha)[r, s] \in P$ with $\kappa(\alpha > 0 \wedge C) \geqslant \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$. We now show that σ is admissible with KB'. It is sufficient to show that every $C \in P$ is tolerated by $P_C = \{C' \in P \cup \{(\psi|\phi)[p, p]\} \mid \sigma(C') \geqslant \sigma(C)\}$ under L. By the proof of Theorem 4.5, it follows that σ restricted to P is admissible with *KB*. Thus, it is sufficient to show that every $C = (\beta|\alpha)[r, s] \in P$ is tolerated by $P_C = \{C' \in P \cup \{(\psi|\phi)[p, p]\} \mid \sigma(C') \geqslant \sigma(C)\}$ under L, where either (a) $\kappa(\alpha > 0 \wedge C) < \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$, or (b) $C = (\psi|\phi)[p, p]$. Towards a contradiction, assume first that some $C = (\beta|\alpha)[r, s] \in P$ with (a) is not tolerated by P_C under L. Let Pr be a model of L such that $Pr \models \alpha > 0 \wedge C$ and $\kappa(Pr) = \kappa(\alpha > 0 \wedge C)$. Let $C' = (\beta'|\alpha')[r', s'] \in P_C$ such that $Pr \not\models C'$ and (a.i) $\kappa(\alpha' > 0 \wedge C') < \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$, or (a.ii) $C' = (\psi|\phi)[p, p]$, or (a.iii) $\kappa(\alpha' > 0 \wedge C') \geqslant \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$. It then holds $\kappa(\alpha' > 0) < \infty$ and $\kappa(\alpha' > 0 \wedge \neg C') \leqslant \kappa(Pr) = \sigma(C)$. Furthermore, it holds (a.i) $\sigma(C) \leqslant \sigma(C') = \kappa(\alpha' > 0 \wedge C')$, or (a.ii) $\sigma(C) < \sigma(C') = \kappa(\alpha' > 0 \wedge C')$, or (a.iii) $\sigma(C) + 1 < \sigma(C') = \kappa(\alpha' > 0 \wedge C') + 1$. But in (a.ii) this contradicts ($\star$) and in (a.i) and (a.iii) this contradicts κ being admissible with *KB*. Hence, *Pr* is a model of P_C. But this contradicts C not being tolerated by P_C under L. Assume next that (b) $C = (\psi|\phi)[p, p]$ is not tolerated by P_C under L. Let *Pr* be a model of L such that $Pr \models \phi > 0 \wedge C$ and $\kappa(Pr) = \kappa(\phi > 0 \wedge C)$. Let $C' = (\beta|\alpha)[r, s] \in P_C$ such that $Pr \not\models C'$. Observe that $\kappa(\alpha > 0 \wedge C') \geqslant \kappa(\phi > 0 \wedge (\psi|\phi)[p, p])$. Thus, $\kappa(\alpha > 0) < \infty$ and $\kappa(\alpha > 0 \wedge \neg C') \leqslant \kappa(Pr) = \sigma(C) < \sigma(C') = \kappa(\alpha > 0 \wedge C') + 1$. But this contradicts κ being admissible with *KB*. Hence, *Pr* is a model of P_C. But this contradicts C not being tolerated by P_C under L. In summary, σ is admissible with KB'. That is, KB' is $\boldsymbol{p}$-consistent, where $p \in [0, l) \cup (u, 1]$. $\square$

Proof of Theorem 4.9. Immediate by Theorem 4.8. $\square$

Proof of Lemma 4.14. Towards a contradiction, assume that z is not admissible with *KB*. That is, some $P' \subseteq P$ is under L in conflict with some $C \in P$, and P' contains no C' with $z(C') < z(C)$. Thus, $P' \subseteq P_C = \{C' \in P \mid z(C') \geqslant z(C)\}$. Since P_C tolerates C under L, also P' tolerates C under L. But this contradicts P' being under L in conflict with C. Hence, (a) z is admissible with *KB*, and by the "$\Leftarrow$"-part of the proof of Theorem 4.5, also (b) κ^z is admissible with *KB*. $\square$

Proof of Theorem 4.16. Suppose first that $L \not\models \bot \Leftarrow \phi$. Then, $\kappa^z(\phi > 0) < \infty$, and $\kappa^z(\phi > 0 \wedge C) < \kappa^z(\phi > 0 \wedge \neg C)$ iff all z-minimal models *Pr* of L with $Pr(\phi) > 0$ satisfy C. Assume next that $L \models \bot \Leftarrow \phi$. Then, $\kappa^z(\phi > 0) = \infty$, and all z-minimal models *Pr* of L with $Pr(\phi) > 0$ satisfy C. $\square$

Appendix C. Proofs for Section 5

While the proofs of the results in Section 4 are often similar to the proofs of their classical counterparts in default reasoning, the proofs for Section 5 require some genuinely probabilistic reasoning. In particular, in the proof of Theorem 5.1, we use the following notations and preliminary results. For probabilistic knowledge bases $KB = (L, P)$ and events α such that $L \not\models \neg\alpha$, we denote by $P_\alpha(KB)$ the set of all subsets $P_n = \{(\psi_i|\phi_i)[l_i, u_i] \mid i \in \{1, \ldots, n\}\}$ of P such that every model *Pr* of $L \cup P_n$ with $Pr(\phi_1 \vee \cdots \vee \phi_n \vee \alpha) > 0$ satisfies $Pr(\alpha) > 0$. For $KB = (L, P)$ and α such that $L \models \neg\alpha$, we define $P_\alpha(KB) = \{\emptyset\}$. For events α and $\boldsymbol{p}$-consistent probabilistic knowledge bases $KB = (L, P)$, we denote by KB_α the probabilistic knowledge base $(L, P^\star)$, where $P^\star$ is the greatest element in $P_\alpha(KB)$. Then, the following result says that probabilistic p-entailment of $(\beta|\alpha)[l, u]$ from *KB* can be reduced to logical entailment of $(\beta|\alpha)[l, u]$ from KB_α. It follows immediately from a similar result for g-coherent entailment in [11] and the equivalence of probabilistic p-entailment and g-coherent entailment, by Theorem 4.11.

Theorem C.1. *Let $KB = (L, P)$ be a $\boldsymbol{p}$-consistent probabilistic knowledge base, let $(\beta|\alpha)[l, u]$ be a conditional constraint, and let KB_α be defined as above. Then,*

(a) $KB \,\|\!\!\sim^{p} (\beta|\alpha)[l,u]$ *iff* $KB_\alpha \,\|\!\!= (\beta|\alpha)[l,u]$.

(b) $KB \,\|\!\!\sim^{p}_{tight} (\beta|\alpha)[l,u]$ *iff* $KB_\alpha \,\|\!\!=_{tight} (\beta|\alpha)[l,u]$.

Proof of Theorem 5.1. It is easy to verify the result for $\|\!\!=$. In the following, we prove the result for $\|\!\!\sim^{p}$, $\|\!\!\sim^{z}$, and $\|\!\!\sim^{lex}$.

RW. Assume first $KB \,\|\!\!\sim^{s} (\phi|\varepsilon)[l,u]$, where $s \in \{\boldsymbol{z}, \boldsymbol{lex}\}$. That is, $Pr \models (\phi|\varepsilon)[l,u]$ for all s-minimal models Pr of $L \cup \{\varepsilon > 0\}$. Since $(\phi|\top)[l,u] \Rightarrow (\psi|\top)[l',u']$ is logically valid, $Pr \models (\psi|\varepsilon)[l',u']$ for all s-minimal models Pr of $L \cup \{\varepsilon > 0\}$. That is, $KB \,\|\!\!\sim^{s} (\psi|\varepsilon)[l',u']$. Assume next $KB \,\|\!\!\sim^{p} (\phi|\varepsilon)[l,u]$. That is, by Theorem C.1, $KB_\varepsilon \,\|\!\!= (\phi|\varepsilon)[l,u]$. Thus, $KB_\varepsilon \,\|\!\!= (\psi|\varepsilon)[l',u']$. That is, $KB \models^{g} (\psi|\varepsilon)[l',u']$.

Ref. Every probabilistic interpretation Pr satisfies $(\varepsilon|\varepsilon)[1,1]$. This shows that $KB \,\|\!\!\sim^{s} (\varepsilon|\varepsilon)[1,1]$ for all $s \in \{\boldsymbol{p}, \boldsymbol{z}, \boldsymbol{lex}\}$.

LLE. Assume first $KB \,\|\!\!\sim^{s} (\phi|\varepsilon)[l,u]$, where $s \in \{\boldsymbol{z}, \boldsymbol{lex}\}$. That is, $Pr \models (\phi|\varepsilon)[l,u]$ for all s-minimal models Pr of $L \cup \{\varepsilon > 0\}$. Since $\varepsilon \Leftrightarrow \varepsilon'$ is logically valid, $Pr \models (\phi|\varepsilon')[l,u]$ for all s-minimal models Pr of $L \cup \{\varepsilon' > 0\}$. That is, $KB \,\|\!\!\sim^{s} (\phi|\varepsilon')[l,u]$. Suppose next $KB \,\|\!\!\sim^{p} (\phi|\varepsilon)[l,u]$. That is, by Theorem C.1, $KB_\varepsilon \,\|\!\!= (\phi|\varepsilon)[l,u]$. Since $\varepsilon \Leftrightarrow \varepsilon'$ is logically valid, $KB_{\varepsilon'} \,\|\!\!= (\phi|\varepsilon')[l,u]$. That is, $KB \models^{g} (\phi|\varepsilon')[l,u]$.

Cut. Assume first $KB \,\|\!\!\sim^{s} (\varepsilon|\varepsilon')[1,1]$ and $KB \,\|\!\!\sim^{s} (\phi|\varepsilon \wedge \varepsilon')[l,u]$, where $s \in \{\boldsymbol{z}, \boldsymbol{lex}\}$. That is, $Pr \models (\varepsilon|\varepsilon')[1,1]$ and $Pr \models (\phi|\varepsilon \wedge \varepsilon')[l,u]$ for all s-minimal models Pr of $L \cup \{\varepsilon' > 0\}$ and $L \cup \{\varepsilon \wedge \varepsilon' > 0\}$, respectively. It thus follows $Pr \models (\phi|\varepsilon')[l,u]$ for all s-minimal models Pr of $L \cup \{\varepsilon' > 0\}$. That is, $KB \,\|\!\!\sim^{s} (\phi|\varepsilon')[l,u]$. Suppose next $KB \models^{g} (\varepsilon|\varepsilon')[1,1]$ and $KB \,\|\!\!\sim^{p} (\phi|\varepsilon \wedge \varepsilon')[l,u]$. That is, by Theorem C.1, $KB_{\varepsilon'} \,\|\!\!= (\varepsilon|\varepsilon')[1,1]$ and $KB_{\varepsilon \wedge \varepsilon'} \,\|\!\!= (\phi|\varepsilon \wedge \varepsilon')[l,u]$. By Theorem C.1, it is then easy to see that $KB_{\varepsilon'} = KB_{\varepsilon \wedge \varepsilon'}$. Thus, $KB_{\varepsilon'} \,\|\!\!= (\phi|\varepsilon')[l,u]$. That is, $KB \,\|\!\!\sim^{p} (\phi|\varepsilon')[l,u]$.

CM. Assume first $KB \,\|\!\!\sim^{s} (\varepsilon|\varepsilon')[1,1]$ and $KB \,\|\!\!\sim^{s} (\phi|\varepsilon')[l,u]$, where $s \in \{\boldsymbol{z}, \boldsymbol{lex}\}$. That is, $Pr \models (\varepsilon|\varepsilon')[1,1]$ and $Pr \models (\phi|\varepsilon')[l,u]$ for all s-minimal models Pr of $L \cup \{\varepsilon' > 0\}$. It follows that $Pr \models (\phi|\varepsilon \wedge \varepsilon')[l,u]$ for all s-minimal models Pr of $L \cup \{\varepsilon \wedge \varepsilon' > 0\}$. That is, $KB \,\|\!\!\sim^{s} (\phi|\varepsilon \wedge \varepsilon')[l,u]$. Suppose next $KB \,\|\!\!\sim^{p} (\varepsilon|\varepsilon')[1,1]$ and $KB \,\|\!\!\sim^{p} (\phi|\varepsilon')[l,u]$. That is, by Theorem C.1, $KB_{\varepsilon'} \,\|\!\!= (\varepsilon|\varepsilon')[1,1]$ and $KB_{\varepsilon'} \,\|\!\!= (\phi|\varepsilon')[l,u]$. Thus, $KB_{\varepsilon'} \,\|\!\!= (\phi|\varepsilon \wedge \varepsilon')[l,u]$. By Theorem C.1, it is easy to see that $KB_{\varepsilon'} = KB_{\varepsilon \wedge \varepsilon'}$. Thus, $KB_{\varepsilon \wedge \varepsilon'} \,\|\!\!= (\phi|\varepsilon \wedge \varepsilon')[l,u]$. That is, $KB \,\|\!\!\sim^{p} (\phi|\varepsilon \wedge \varepsilon')[l,u]$.

Or. Assume first $KB \,\|\!\!\sim^{s} (\phi|\varepsilon)[1,1]$ and $KB \,\|\!\!\sim^{s} (\phi|\varepsilon')[1,1]$, where $s \in \{\boldsymbol{z}, \boldsymbol{lex}\}$. That is, $Pr \models (\phi|\varepsilon)[1,1]$ and $Pr \models (\phi|\varepsilon')[1,1]$ for all s-minimal models Pr of $L \cup \{\varepsilon > 0\}$ and $L \cup \{\varepsilon' > 0\}$, respectively. It then follows $Pr \models (\phi|\varepsilon \vee \varepsilon')[1,1]$ for all s-minimal models Pr of $L \cup \{\varepsilon \vee \varepsilon' > 0\}$. That is, $KB \,\|\!\!\sim^{s} (\phi|\varepsilon \vee \varepsilon')[1,1]$. Suppose next $KB \,\|\!\!\sim^{p} (\phi|\varepsilon)[1,1]$ and $KB \,\|\!\!\sim^{p} (\phi|\varepsilon')[1,1]$. That is, by Theorem C.1, $KB_\varepsilon = (L, P_\varepsilon) \,\|\!\!= (\phi|\varepsilon)[1,1]$ and $KB_{\varepsilon'} = (L, P_{\varepsilon'}) \,\|\!\!= (\phi|\varepsilon')[1,1]$. By Theorem C.1, it is then easy to see that $P_{\varepsilon \vee \varepsilon'} \supseteq P_\varepsilon$ and $P_{\varepsilon \vee \varepsilon'} \supseteq P_{\varepsilon'}$. Hence, $KB_{\varepsilon \vee \varepsilon'} \,\|\!\!= (\phi|\varepsilon)[1,1]$ and $KB_{\varepsilon \vee \varepsilon'} \,\|\!\!= (\phi|\varepsilon')[1,1]$, where $KB_{\varepsilon \vee \varepsilon'} = (L, P_{\varepsilon \vee \varepsilon'})$. It thus follows $KB_{\varepsilon \vee \varepsilon'} \,\|\!\!= (\phi|\varepsilon \vee \varepsilon')[1,1]$. That is, $KB \,\|\!\!\sim^{p} (\phi|\varepsilon \vee \varepsilon')[1,1]$. $\Box$

Proof of Theorem 5.2. Assume first $KB \,\|\!\!= (\psi|\varepsilon)[1,1]$ and $KB \,\|\!\!\not\models \neg(\varepsilon'|\varepsilon)[1,1]$. In particular, $Pr \models (\psi|\varepsilon)[1,1]$ for all models Pr of $L \cup P \cup \{\varepsilon > 0\}$. Hence, $Pr \models (\psi|\varepsilon \wedge \varepsilon')[1,1]$ for all models Pr of $L \cup P \cup \{\varepsilon \wedge \varepsilon' > 0\}$. That is, $KB \,\|\!\!= (\psi|\varepsilon \wedge \varepsilon')[1,1]$. Assume next $KB \,\|\!\!\sim^{s} (\psi|\varepsilon)[1,1]$ and $KB \,\|\!\!\not\sim^{s} (\neg\varepsilon'|\varepsilon)[1,1]$, $s \in \{\boldsymbol{z}, \boldsymbol{lex}\}$. That is, $Pr \models (\psi|\varepsilon)[1,1]$ for all s-minimal models Pr of $L \cup \{\varepsilon > 0\}$, and $Pr \models \neg(\varepsilon'|\varepsilon)[0,0]$ for some s-minimal models Pr

of $L \cup \{\varepsilon > 0\}$. So, $Pr \models (\psi|\varepsilon\wedge\varepsilon')[1,1]$ for all s-minimal models Pr of $L \cup \{\varepsilon \wedge \varepsilon' > 0\}$. That is, $KB \mathrel{\|\!\!\sim}^{s} (\psi|\varepsilon\wedge\varepsilon')[1,1]$. $\Box$

Proof of Theorem 5.4. Assume that ($\star$) no atom of KB and $(\psi|\varepsilon)[1,1]$ occurs in ε'. Suppose first $KB \mathrel{\|\!\!=} (\psi|\varepsilon)[1,1]$. That is, $Pr \models (\psi|\varepsilon)[1,1]$ for all models Pr of $L \cup P \cup \{\varepsilon > 0\}$. Hence, $Pr \models (\psi|\varepsilon \wedge \varepsilon')[1,1]$ for all models Pr of $L \cup P \cup \{\varepsilon \wedge \varepsilon' > 0\}$. That is, $KB \mathrel{\|\!\!=} (\psi|\varepsilon \wedge \varepsilon')[1,1]$. Assume next $KB \mathrel{\|\!\!\sim}^{s} (\psi|\varepsilon)[1,1]$, where $s \in \{z, \boldsymbol{lex}\}$. That is, $Pr \models (\psi|\varepsilon)[1,1]$ for all s-minimal models Pr of $L \cup \{\varepsilon > 0\}$. By ($\star$), it follows that $Pr \models (\psi|\varepsilon \wedge \varepsilon')[1,1]$ for all s-minimal models Pr of $L \cup \{\varepsilon \wedge \varepsilon' > 0\}$. That is, $KB \mathrel{\|\!\!\sim}^{s} (\psi|\varepsilon \wedge \varepsilon')[1,1]$. $\Box$

Proof of Theorem 5.6. Assume that $(\psi|\phi)[l,u] \in P$ and $\varepsilon \Leftrightarrow \phi$ is logically valid. Clearly, $KB \mathrel{\|\!\!=} (\psi|\varepsilon)[l,u]$. Since KB is $\boldsymbol{p}$-consistent, the conditional constraint ranking z exists, and $(\psi|\phi)[l,u]$ is tolerated by $\{C \in P \mid z(C) \geqslant z((\psi|\phi)[l,u])\}$ under L. Hence, every s-minimal model Pr of $L \cup \{\varepsilon > 0\}$ satisfies $(\psi|\varepsilon)[l,u]$, where $s \in \{z, \boldsymbol{lex}\}$. Hence, $KB \mathrel{\|\!\!\sim}^{s} (\psi|\varepsilon)[l,u]$. Since $(L, P \cup \{(\psi|\varepsilon)[p,p]\})$ is not $\boldsymbol{p}$-consistent for all $p \in [0,l) \cup (u,1]$, it also follows $KB \mathrel{\|\!\!\sim}^{p} (\psi|\varepsilon)[l,u]$. $\Box$

Proof of Theorem 5.7. (a) Suppose $KB \mathrel{\|\!\!\sim}^{p} C$. By Theorem 4.8, $\kappa(\phi > 0) = \infty$ or $\kappa(\phi > 0 \wedge C) < \kappa(\phi > 0 \wedge \neg C)$ for every probability ranking κ admissible with KB. By Lemma 4.14, κ^{z} is admissible with KB. Hence, $\kappa^{z}(\phi > 0) = \infty$ or $\kappa^{z}(\phi > 0 \wedge C) < \kappa^{z}(\phi > 0 \wedge \neg C)$. By Theorem 4.16, it thus holds $KB \mathrel{\|\!\!\sim}^{z} C$.

(b) Suppose $KB \mathrel{\|\!\!\sim}^{z} C$. That is, every z-minimal model Pr of $L \cup \{\phi > 0\}$ satisfies C. Since every $\boldsymbol{lex}$-minimal model Pr of $L \cup \{\phi > 0\}$ is also a z-minimal model of $L \cup \{\phi > 0\}$, it follows that every $\boldsymbol{lex}$-minimal model Pr of $L \cup \{\phi > 0\}$ satisfies C. That is, $KB \mathrel{\|\!\!\sim}^{lex} C$.

(c) Suppose $KB \mathrel{\|\!\!\sim}^{lex} C$. That is, every $\boldsymbol{lex}$-minimal model Pr of $L \cup \{\phi > 0\}$ satisfies C. Assume first $Pr(\phi) = 0$ for every model Pr of $L \cup P$. Then, $KB \mathrel{\|\!\!=} C$ trivially holds. Assume next $L \cup P \cup \{\phi > 0\}$ is satisfiable. Thus, Pr is a $\boldsymbol{lex}$-minimal model of $L \cup \{\phi > 0\}$ iff it is a model of $L \cup P \cup \{\phi > 0\}$. Hence, every model of $L \cup P \cup \{\phi > 0\}$ satisfies C. That is, $KB \mathrel{\|\!\!=} C$. $\Box$

Proof of Theorem 5.8. The existence of some model Pr of $L \cup P \cup \{\phi > 0\}$ implies that a probabilistic interpretation Pr is a model of $L \cup P \cup \{\phi > 0\}$ iff it is a $\boldsymbol{lex}$-minimal model of $L \cup \{\phi > 0\}$ iff it is a z-minimal model of $L \cup \{\phi > 0\}$. $\Box$

Proof of Theorem 5.10. (a) A conditional constraint ranking σ on KB is admissible with KB iff the default ranking $\sigma \circ \gamma^{-1}$ on $\gamma(KB)$ is admissible with $\gamma(KB)$.

(b), (c) Observe that $(P_0, \ldots, P_k)$ is the z-partition of KB iff $(\gamma(P_0), \ldots, \gamma(P_k))$ is the classical z-partition of $\gamma(KB)$. Furthermore, every s-minimal model Pr of $L \cup \{\alpha > 0\}$ satisfies $(\beta|\alpha)[1,1]$ iff every classical s-minimal model I of $L \cup \{\alpha\}$ satisfies β, where $s \in \{z, \boldsymbol{lex}\}$. $\Box$

Appendix D. Proofs for Section 6

Proof of Theorem 6.1. (a) If $L \cup \{\alpha > 0\}$ is unsatisfiable, then $KB \mathrel{\|\!\!\sim}^{s}_{tight} (\beta|\alpha)[1,0]$.

(b) Assume $L \cup \{\alpha > 0\}$ is satisfiable. It is sufficient to show that Pr is an s-minimal model of $L \cup \{\alpha > 0\}$ iff Pr is a model of $L \cup H \cup \{\alpha > 0\}$ for some $H \in \mathcal{D}^s_\alpha(KB)$:

($\Rightarrow$) Let Pr be an s-minimal model of $L \cup \{\alpha > 0\}$. Let $H' = \{C \in P \mid Pr \models C\}$. Clearly, $Pr \models L \cup H' \cup \{\alpha > 0\}$. We now show that $H' \in \mathcal{D}^s_\alpha(KB)$. Suppose not. That is, some $H'' \subseteq P$ exists such that $L \cup H'' \cup \{\alpha > 0\}$ is satisfiable and that H'' is s-preferable to H'. Thus, a model Pr' of $L \cup H'' \cup \{\alpha > 0\}$ exists. As H'' is s-preferable to H', the model Pr' of $L \cup \{\alpha > 0\}$ is s-preferable to Pr. But this contradicts Pr being an s-minimal model of $L \cup \{\alpha > 0\}$. Thus, $H' \in \mathcal{D}^s_\alpha(KB)$.

($\Leftarrow$) Let Pr be a model of $L \cup H' \cup \{\alpha > 0\}$ for some $H' \in \mathcal{D}^s_\alpha(KB)$. Clearly, Pr is a model of $L \cup \{\alpha > 0\}$. We now show that Pr is an s-minimal model of $L \cup \{\alpha > 0\}$. Suppose not. That is, there exists a model Pr' of $L \cup \{\alpha > 0\}$ that is s-preferable to Pr. Thus, $\{C \in P \mid Pr' \models C\} \subseteq P$ is s-preferable to H'. But this contradicts H' being a member of $\mathcal{D}^s_\alpha(KB)$. Hence, Pr is an s-minimal model of $L \cup \{\alpha > 0\}$. □

Appendix E. Proofs for Section 7

The proofs of Theorems 7.1–7.4 are similar to the proofs of related complexity results in [52]. We first give some preparatory definitions as follows. In the sequel, let $KB = (L, P)$ be a $\boldsymbol{p}$-consistent probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conditional constraint. Let n denote the cardinality of P. For the following definitions, let $L \cup \{\alpha > 0\}$ be satisfiable. An ordered partition $(P_0, \ldots, P_k)$ of P is *admissible* with KB iff for each $i \in \{0, \ldots, k\}$ and each $(\psi|\phi)[r, s] \in P_i$, the set $L \cup \{\phi > 0\} \cup \bigcup\{P_j \mid j \geqslant i\}$ is satisfiable. The *weight* of an ordered partition $(P_0, \ldots, P_k)$ of P is defined as $\sum_{i=0}^{k} i \cdot |P_i|$. Let $w_{\min}$ denote the least weight w of all ordered partitions of P that are admissible with KB. As in classical default reasoning, the z-partition of KB is the unique ordered partition $(P^\star_0, \ldots, P^\star_k)$ of P that is admissible with KB and that has the weight $w_{\min}$. Let $j_{\min}$ denote the least $j \in \{0, \ldots, k{+}1\}$ such that $L \cup \bigcup\{P^\star_i \mid i \geqslant j\} \cup \{\alpha > 0\}$ is satisfiable. Let $\boldsymbol{n}_{\min} = (|P' \cap P^\star_0|, \ldots, |P' \cap P^\star_k|)$ for some $P' \in \mathcal{D}^{lex}_\alpha(KB)$.

Proof of Theorem 7.1. Let $KB = (L, P)$. We first prove membership in $\mathrm{P}^{\mathrm{NP}}_{\|}$. By Theorem 6.1, it holds that $KB \,\|\!\sim^z (\beta|\alpha)[l, u]$ iff either (i) or (ii) holds:

(i) $L \cup \{\alpha > 0\}$ is unsatisfiable.
(ii) $L \cup \{\alpha > 0\}$ is satisfiable, and $L \cup \bigcup\{P^\star_i \mid i \geqslant j_{\min}\} \,\|\!\!= (\beta|\alpha)[l, u]$.

Deciding whether $L \cup \{\alpha > 0\}$ is satisfiable can be done with one NP-oracle call. If $L \cup \{\alpha > 0\}$ is satisfiable, then we compute the least weight $w_{\min} \in \{0, \ldots, n(n{-}1)/2\}$ and the value $j_{\min} \in \{0, \ldots, n + 1\}$, which can both be done in deterministic polynomial time with $O(\log n)$ calls to an NP-oracle. Finally, we decide whether $L \cup \bigcup\{P^\star_i \mid i \geqslant j_{\min}\} \,\|\!\!= (\psi|\phi)[l, u]$, which can be done with one NP-oracle call. Since four rounds of parallel NP oracle queries can be replaced by a single round of NP queries, this means that the problem is in $\mathrm{P}^{\mathrm{NP}}_{\|}$.

Hardness for $\mathrm{P}_{\|}^{\mathrm{NP}}$ is proved by a polynomial reduction from the following $\mathrm{P}_{\|}^{\mathrm{NP}}$-complete problem [18]. Given a p-consistent conditional knowledge base $KB' = (L', D')$ and a default $\delta \leftarrow \gamma$, decide whether $KB' \mid\!\sim^{z} \delta \leftarrow \gamma$.

We define $KB = (L', \{(\psi|\phi)[1,1] \mid \psi \leftarrow \phi \in D'\})$ and $\beta|\alpha = \delta|\gamma$. By Theorem 5.10, $KB' \mid\!\sim^{z} \delta \leftarrow \gamma$ iff $KB \parallel\!\sim^{z} (\beta|\alpha)[1,1]$. $\square$

Proof of Theorem 7.2. Let $KB = (L, P)$. We first prove P^{NP}-membership. By Theorem 6.1, it holds that $KB \parallel\!\sim^{lex} (\beta|\alpha)[l,u]$ iff either (i) or (ii) holds:

(i) $L \cup \{\alpha > 0\}$ is unsatisfiable.
(ii) $L \cup \{\alpha > 0\}$ is satisfiable, and $L \cup P' \models (\beta|\alpha)[l,u]$ for all $P' \in \mathcal{D}_{\alpha}^{lex}(KB)$.

Deciding whether $L \cup \{\alpha > 0\}$ is satisfiable can be done with one NP-oracle call. If $L \cup \{\alpha > 0\}$ is satisfiable, then we compute the least weight $w_{\min} \in \{0, \ldots, n(n-1)/2\}$, which can be done in deterministic polynomial time with $\mathrm{O}(\log n)$ calls to an NP-oracle. Moreover, we compute the vector $\boldsymbol{n}_{\min} \in \{0, \ldots, n\}^k$. This can be done with k rounds of binary search, where each round runs in deterministic polynomial time with $\mathrm{O}(\log n)$ calls to an NP-oracle. Finally, we decide whether $L \cup P' \models (\beta|\alpha)[l,u]$ for all $P' \in \mathcal{D}_{\alpha}^{lex}(KB)$, which can be done with one call to an NP-oracle. In summary, the problem is in P^{NP}.

Hardness for P^{NP} is proved by a polynomial reduction from the following P^{NP}-complete problem [18]. Given a p-consistent conditional knowledge base $KB' = (L', D')$, where L' is a finite set of literal-Horn logical constraints and D' is a finite set of *literal-Horn* defaults (which are of the form $\psi \leftarrow \phi$, where ψ is either a basic event or the negation of a basic event, and ϕ is either $\top$ or a conjunction of basic events), and $\delta \leftarrow \gamma$ is a literal-Horn default, decide whether $KB' \mid\!\sim^{lex} \delta \leftarrow \gamma$.

We now construct $KB = (L, P)$ and $C = (\beta|\alpha)[l,u]$ as stated in the theorem such that $KB' \mid\!\sim^{lex} \delta \leftarrow \gamma$ iff $KB \parallel\!\sim^{lex} C$. We define KB and C by $L = L'$ and

$$P = \big\{(p|\phi)[1,1] \mid p \leftarrow \phi \in D',\ p \in \Phi\big\} \cup \big\{(p|\phi)[0,0] \mid \neg p \leftarrow \phi \in D',\ p \in \Phi\big\},$$

$$C = \begin{cases} (p|\gamma)[1,1] & \text{if } \delta = p \text{ and } p \in \Phi, \\ (p|\gamma)[0,0] & \text{if } \delta = \neg p \text{ and } p \in \Phi. \end{cases}$$

Notice that KB and C are literal-Horn. By a slight generalization of Theorem 5.10, $KB' \mid\!\sim^{lex} \delta \leftarrow \gamma$ iff $KB \parallel\!\sim^{lex} C$. $\square$

Proof of Theorems 7.3 and 7.4. Let $KB = (L, P)$. We first prove membership in $\mathrm{FP}^{\mathrm{NP}}$. Let $s \in \{\boldsymbol{z}, \boldsymbol{lex}\}$. The interval $[l,u] \subseteq [0,1]$ such that $KB \parallel\!\sim^{s} (\beta|\alpha)[l,u]$ can be computed by a variant of Algorithm tight-entailment-opt in [51], which can be done in $\mathrm{FP}^{\mathrm{NP}}$. Rather than checking the existence of some model Pr of $L \cup P$ with $Pr(\alpha) > 0$, we check the existence of some $P' \in \mathcal{D}_{\alpha}^{s}(KB)$ and some model Pr of $L \cup P'$ with $Pr(\alpha) > 0$. Once the z-partition of KB, the value $j_{\min}$, and the vector $\boldsymbol{n}_{\min}$ are computed (which can be done in $\mathrm{FP}^{\mathrm{NP}}$ by the proofs of Theorems 7.1 and 7.2) guessing and verifying $P' \in \mathcal{D}_{\alpha}^{s}(KB)$ is in NP, and thus does not increase the complexity. Hence, the new algorithm can be done in $\mathrm{FP}^{\mathrm{NP}}$.

Hardness for $\mathrm{FP}^{\mathrm{NP}}$ is shown by a polynomial reduction from the $\mathrm{FP}^{\mathrm{NP}}$-complete *traveling salesman cost* problem [56]. Given a set of $n \geqslant 1$ cities $V = \{1, 2, \ldots, n\}$ and a nonnegative integer distance $d_{i,j} = d_{j,i}$ between any two cities i and j, we have to compute the smallest length d of a tour through all the cities, that is, the minimum of $\sum_{i=1}^{n} d_{\pi(i),\pi(\sigma(i))}$ subject to all permutations π, where $\sigma(n) = 1$, and $\sigma(i) = i + 1$ for all $i < n$. Without loss of generality, we can assume $n \geqslant 3$.

Let s be the sum of all $d_{i,j}$ with $i, j \in V$ and $i < j$. We now construct $KB = (L, P)$ and $\beta|\alpha$ as stated in the theorem such that the smallest length d of a tour is $s \cdot l$, where l is given by $KB \Vdash\!\sim^{z}_{tight} (\beta|\alpha)[l, 1]$ (and also $KB \Vdash\!\sim^{lex}_{tight} (\beta|\alpha)[l, 1]$).

Let $E = \{\{i, j\} \subseteq V \mid i \neq j\}$ and $w_{\{i,j\}} = d_{i,j}/s$ for all $\{i, j\} \in E$. The set of basic events Φ is defined as $\Phi_1 \cup \Phi_2$, where $\Phi_1 = \{p_{i,j} \mid i, j \in V\}$ and $\Phi_2 = \{p\} \cup \{p_e \mid e \in E\}$. We then define a set of literal-Horn conditional constraints $P_1 = P_{1,1} \cup P_{1,2} \cup P_{1,3}$ that describes the set of all permutations of the members in V as follows:

$$P_{1,1} = \big\{(p_{i,j} \mid p_{i,k})[0, 0] \mid i, j, k \in V,\ j < k\big\},$$
$$P_{1,2} = \big\{(p_{i,j} \mid p_{k,j})[0, 0] \mid i, j, k \in V,\ i < k\big\},$$
$$P_{1,3} = \big\{(p_{i,j} \mid \top)[1/n, 1/n] \mid i, j \in V\big\}.$$

Roughly speaking, each world I with $Pr_1(I) > 0$ for some model Pr_1 of P_1 corresponds to a permutation of the members in V, and vice versa. We next define a set of literal-Horn conditional constraints $P_2 = P_{2,1} \cup P_{2,2} \cup P_{2,3}$ that associates each such permutation with its tour length, and the predicate symbol p with the sum of all such tour lengths as follows:

$$P_{2,1} = \big\{(p_{e_1} \mid p_{e_2})[0, 0] \mid e_1, e_2 \in E,\ e_1 \neq e_2\big\},$$
$$P_{2,2} = \big\{(p_{\{i,j\}} \mid p_{u,i} \wedge p_{\sigma(u),j})[w_{\{i,j\}}, w_{\{i,j\}}] \mid u \in V,\ \{i, j\} \in E\big\},$$
$$P_{2,3} = \big\{(p \mid p_e)[1, 1] \mid e \in E\big\}.$$

We finally define $KB = (L, P) = (\emptyset, P_1 \cup P_2)$. Observe that KB and $p|\top$ are literal-Horn and that L is empty. As proved in [9], KB is $\boldsymbol{p}$-consistent. This shows in particular that $L \cup P$ has a model Pr with $Pr(\top) > 0$. Hence, by Theorem 5.8, $KB \Vdash_{tight} (p|\top)[l, 1]$ iff $KB \Vdash\!\sim^{z}_{tight} (p|\top)[l, 1]$ (iff $KB \Vdash\!\sim^{lex}_{tight} (p|\top)[l, 1]$). As shown in [51], $KB \Vdash_{tight} (p|\top)[l, 1]$ iff $s \cdot l$ is the smallest length of a tour through all the cities. In summary, $KB \Vdash\!\sim^{z}_{tight} (p|\top)[l, 1]$ (iff $KB \Vdash\!\sim^{lex}_{tight} (p|\top)[l, 1]$) iff $s \cdot l$ is the smallest length of a tour through all the cities. $\square$

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