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On the notion of concept I

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Abstract

It is well known that classical set theory is not expressive enough to adequately model categorization and prototype theory. Recent work on compositionality and concept determination showed that the quantitative solution initially offered by classical fuzzy logic also led to important drawbacks. Several qualitative approaches were thereafter tempted, that aimed at modeling membership through ordinal scales or lattice fuzzy sets. Most of the solutions obtained by these theoretical constructions however are of difficult use in categorization theory. We propose a simple qualitative model in which membership relative to a given concept f is represented by a function that takes its value in a finite abstract set A_f equipped with a total order. This function is recursively built through a stratification of the set of concepts at hand based on a notion of *complexity*. Similarly, the typicality associated with a concept f will be described using an ordering that takes into account the characteristic features of f. Once the basic notions of membership and typicality are set, the study of compound concepts is possible and leads to interesting results. In particular, we investigate the internal structure of concepts, and obtain the characterization of all smooth subconcepts of a given concept.

Keywords: Categorization; Concept; Extension; Intension; Typicality; Membership; Modular orders; Fuzzy sets; Formal concepts analysis

1. Introduction

In this paper we propose a new framework for the study of some basic notions classically used in categorization theory. In particular, we shall be concerned with the problem of finding a suitable theoretical apparatus to model the notions of *membership* and *typicality* that underlie prototype theory. It is well recognized since the work of Eleanor Rosch [18] that membership, for instance, is not an all-or-not matter: the classical set-theoretical or the two-value logic model are of therefore of little use to render count of most of the cognition process. This drove Zadeh and his followers [24] and [25] to propose a representation of concepts by fuzzy sets, membership being modeled through a real function with values in the unit interval. Such a representation nevertheless lead to counterintuitive results: see for instance the seminal papers of Kamp and Partee and of Osherson and Smith [12,16,17]. At a quite elementary level, for instance, it was observed that the membership degree relative to a compound concept could never be greater than the degree induced by any of its components, a result that cannot be accepted for both theoretical and experimental reasons. Even for elementary concepts, the representation of concepts as quantitative fuzzy sets poses problems: vague concepts like *to-be-an-adult* or *to-lie* are given continuous values in the unit interval, but what does it mean

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to qualify somebody as adult 'with degree .4837'? In particular, as observed by several authors (for instance [14]) there is no reason why the same set–the unit interval—should serve as a uniform criterion, being invariably referred to as a measure of membership whatever the concept at hand. True, in practice membership is often evaluated through statistical data, and the membership degree identified with a simple frequency. But the fact that, say, 87 individuals out of 100 consider a car seat as a piece of furniture by no means involves that, in an agent mind, the membership degree of a car-seat relative to the concept *to-be-a-piece-of-furniture* is equal .87.

These drawbacks led to various solutions which all aimed at replacing the primitive quantitative model by a qualitative one: thus, attention focused on ordinal scales and on lattice fuzzy sets—see for instance [11] or [25]. For a brief analysis of the most recent work on this area, the reader may refer to [14] or [3]. However, we consider that the solutions that were proposed are not fully adapted to model prototype theory, and that they cannot be easily exploited to address the classical questions raised by categorization theory.

In a different area, Peter Gärdenfors [9] or [8] proposed a geometrical model as a framework for concept theory: a concept is defined as a convex region of a multidimensional space, each dimension corresponding to a basic quality. Convexity is related with a notion of *betweenness* that is supposed to be meaningful for the relevant quality dimensions: if two objects are exemplars of a concept, such will be the case for any object that lies 'between' them. The typical instances of a concept are those which are located 'near the center' of the considered region. This *Geometry of Thought*, as the author calls it, provides interesting tracks in the analysis of concepts. However, it is mostly based on quantitative notions, which we find not best appropriate to model the cognition process. Furthermore, it does not seem that the distinction between vague and sharp concepts is fully taken into account.

For these reasons, we propose to revisit the basic notions linked with categorization theory and treat them from a qualitative point of view. Concerning membership, for instance, and rather than dealing with uniform gradation functions that take their values in the unit interval, we represent membership relative to a concept by a function *whose set of values depends on the chosen concept*. This set is endowed with a total order that can be used to evaluate to which degree a object falls under this concept. We think indeed that such a representation is the most adequate to model notions like: *object x plainly falls under the concept f*, *object x falls definitely not under the concept f* or *object x falls more than object y under the concept f*. These notions, which are the basis of categorization theory, are also the firsts one should deal with in order to understand the problems that arise with *vague* concepts: for instance, an agent may consider that an *elevator* is definitely less a *vehicle* than a *chairlift*, while being unable at the same time to attribute a precise numerical membership degree to any of these items. We propose in this paper an example of construction such an order, by making use of the set of *defining features* attached to the concepts at hand. Postulating the existence of such a set is part of most of the theories on categorization: see for instance [1,4,21,22] or more recently [2], where a concept is assimilated with *a set of properties which things that fall under the concept typically have or are believed to have*. These defining features, from the point of view of the agent, help understanding the chosen concept; they are *individually necessary and collectively sufficient* to decide whether or not an item is an exemplar of this concept. Given a vague concept f, we shall use this associated defining feature set to compare the f-membership of two items in the following way: an object x will be considered as falling less under f than an object y if it falls less than y under the f-defining features. The circularity of this definition will be avoided by attributing to each concept a complexity level: the *sharp* concepts, those for which membership is an all-or-not matter, will be given complexity level 0; at level 1, we shall rank all the vague concepts whose defining feature set only consists of sharp concepts; at level 2, we will have the vague concepts whose defining feature set consists of concepts that have complexity level equal to 0 or 1, and so on. This ranking will eventually render possible a recursive definition of membership, and, consequently, the construction of a membership order among the set of objects at hand.

Having represented concepts by means of order-functions poses the problem of finding an adequate representation of the notion of *typicality*. Since the work of E. Rosch, a considerable amount of study has been carried out on this notion, and it is now widely accepted that, relative to a given concept, objects may be classified following their *degree of typicality*. Although a precise and general definition of this typicality degree is still missing, one generally agrees on the fact that such a degree has to faithfully reflect the number of characteristic features attached to the concept at hand, together with the relative pertinence, or the frequency, of these features [15, Chapter 2]. Nevertheless the attempts at a rigorous construction are rare, and none of them seem to have gained general recognition. Besides, researchers in this domain restricted themselves to elementary cases, dealing with sharp concepts, for which membership is an all-or-not matter, or with concepts with sharp features. In particular, they did not seem to be concerned with situations in which the typicality relative to a concept depends on the membership relative to another concept: in order to determine the relative typicality of a *hen* as a *bird*, for instance, they would not consider that it is necessary to first evaluate its membership degree relative to the concept *to-fly*. We think on the contrary that typicality must be determined through membership, and that these two notions are correlated

We therefore propose the construction of a 'typicality order'—in fact a partial preorder—clear and easy to evaluate, that faithfully conforms with our intuition. This order is meant to reflect a particular agent's judgment at a precise time. It is based on the agent's choice, for each concept, of a an associated *characteristic feature set*, partially ordered through a *salience relation* that is meant to evaluate the relative importance of these features. The typicality of two items will be compared by investigating the characteristic features that apply to them, the way they apply to them, and their relative salience. Once we have completed the construction of the typicality order, it will be possible to define the typical instances of a concept as those that have maximal order, that is those that fall under all the characteristic features of this concept. This definition of typicality will then enable us to define the *intension* of a concept as the set of features that apply to all typical instances of the concept. Thus, the intension of a concept may be interpreted as the set of characteristic features that agents belonging to a well-defined cultural environment would generally agree to associate with this concept: it enlarges the more subjective notion of characteristic features sets.

A coherent theory of typicality must be able to correctly address the problem of compound concepts. We shall show that our formalism provides natural and intuitive answers concerning composed concepts, provided one departs from the idea that the logic of concepts boils down to a simple propositional calculus. Indeed we do not agree with the commonly admitted postulate following which the negation of a concept, the conjunction or the disjunction of two concepts should be again a concept: we do not consider that *not-to-be-an-apple* or (*to-be-an-apple*)&(*to-be-a-pear*) are concepts. Consequently, we believe that the treatment of such sentences, which clearly goes beyond the limits of the elementary concept theory we are dealing with, should be addressed only after a coherent logical framework for categorization has been proposed. In the present work, we shall therefore content ourselves with a language that only admits a single partial operator, the *determination connective*, which is meant to represent the determination of a principal concept by a secondary one: for instance, the concept *to-be-a-green-apple* is the determination of the principal concept *to-be-an-apple* by the secondary one, *to-be-green*. Concept determination is not *compositional*, except in some limit cases: this means that neither the membership, nor the typicality relative to a composed concept can be directly evaluated through a computation of the corresponding magnitudes of its components. However, it remains possible to determine the typical order, hence the typical instances of a composed concept, via the typicality orders induced by its components. This result is important as it can be considered as an answer to the compositionality problem.

1.1. Plan of this paper

After introducing in Section 2 the framework we are going to work in and recall the distinction between sharp and vague concepts, we shall introduce in Section 3 the membership orders and functions associated with elementary concepts. In Section 4, we shall present the determination connective and extend the membership order to compound concepts. We shall then turn to typicality, and build in Section 5 the typicality order associated with elementary and compound concepts. In Section 6, we show how the notion of smooth subconcepts can be formalized through the determination connective, and we propose an interpretation of our results in the language of Formal Concept Analysis. Section 7 is a conclusion in which we discuss our future work.

2. Concepts and objects

We denote by $\mathcal{O}$ the universe of discourse, which we may see as the set of all objects, real or fictive, that an agent has at his disposal. Together with this set, we suppose given a set $\mathcal{F}$ of concepts. These concepts constitute the elementary items on which the agent builds its reasoning process, and they reflect its knowledge on the world at a given time. A concept *applies* to an object if it describes a property that this object possesses, or if it is an attribute of this object. For instance, the concept *to-be-a-fruit* applies to the object an-apple. We will say indifferently that the concept f *applies* to the object x, that x *falls* under f, or that x is *an instance* of f. In the classical theory, where categories were modeled through set theory, membership relative to a concept was an all-or-none matter: an object could not partially fall under a concept. This perspective was also that of Frege [5], for whom concepts were defined as one-place predicates having a bivalent membership truth function. With prototype theory and the evidence that there existed *vague* concepts (e.g. *to-be-a-lie*, *to-be-an-adult*, *to-be-employed*, *to-be-a-sand-heap* etc.), it became clear that this primitive notion of concepts had to be enlarged and that membership was a question of degree, rather than an all-or-none matter. As observed in [12], "We all have strong intuitions that the concepts encoded by many natural-language predicates are vague; whether something is a chair, or is red, does not seem to be an all-or-none matter but a matter of degree; there may be some clear positive cases and some clear negative cases, but there are many unclear cases in between."

Sharp concepts are defined as those for which membership is an all-or-not matter: an object simply falls or does not fall under such a concept, without the possibility of taking intermediate values. *To-be-a-human-being*, *to-be-a-tooth-brush*, *to-be-an-even-integer* may provide examples of sharp concepts. This definition has nevertheless to be understood as tightly related with a given agent's point of view, and we shall always consider that we work from a particular subjective perspective, and at a particular time: the same concept may appear as sharp to a non-expert agent while being considered as *vague* for an expert. For vague concepts, membership is indeed not an all-or-not-matter: such are for instance the concepts *to-be-a-lie*, *to-be-poor*, *to-be-employed*, *to-be-a-weapon-of-mass-destruction* or *to-be-a-mammal*. Indeed, politeness sometimes drives us to make compliments that, although not sincere, cannot be considered as real lies; to be poor or to be employed is clearly a matter of degree; a gun is more a WMD than a knife; and the platypus is and is not a mammal. Of course, opinions may differ whether a given concept should be considered as a sharp or a vague one, but, and this is the important point, it is well recognized that both kinds of concepts exist. An interesting suggestion of [1] is that, for noun concepts, the opposition between nominal and non-nominal categories reflects the duality between vague and sharp concepts: nominal categories can be defined through their defining features, and may therefore give rise to vague concepts, while non-nominal cannot. Non-nominal categories may be themselves divided between *natural kind* categories (e.g.: the category of *tigers* or of *games*) and *artifact* categories (e.g.: the category of *hammers*, *walls*, *cars*). Note that the distinction between nominal and natural kind concepts is far from being evident: a same concept may be considered as nominal for an expert, and as non-nominal for a non-expert agent. For instance, the concept *to-be-a-bird* is undoubtedly of a natural kind for a child, but it may turn later to a nominal one once the child has learnt that all and only those animals that have beak and feathers are to be considered as birds. In deciding whether the concept *to-be-a-bird* is or not a sharp concept, we have therefore to first analyze which of these two concepts we are referring to: an agent aware that birdhood may be defined through the sum of a certain number of conditions, will consider *to-be-a-bird* a vague concept: the octopus, for instance will be more a bird than the bat, since the octopus has a beak. On the other hand, for a child, *to-be-a-bird* is bond to be a sharp concept, and the penguin will simply not be a member of the category, while the bat will.

In the present work, we shall not address the problem of determining which concepts are vague and which are not. We shall only be concerned with the problem of finding an adequate model that correctly describes how the notion of membership is used in a given agent's behavior.

3. Membership for elementary concepts

In the original fuzzy logic model, a membership degree function is attributed to each concept, measuring how accurately this concept applies to the objects at hand. This degree however is not explicitly present in an agent's mind: this is so for example for young children, for whom notions like real numbers or unit interval are totally meaningless. Nevertheless, given a concept, the agent will be generally able to decide whether two objects have the same or different membership degrees, and which one, in the latter case, has higher degree: for instance, the agent may decide that the concept *to-be-a-piece-of-furniture* applies more to a car-seat than to a blackboard, without being able at the same time to attribute a numerical membership degree to any of these items. In other words, the agent associates with each concept f an implicit notion of a *membership order*. It is this order we now want to build.

We shall first deal with *elementary* concepts, leaving the case of compound concepts in the next section. In order to correctly define a suitable notion of membership for vague concepts, we start from the widely accepted theory following which each such concept f is given together with a *finite* auxiliary set Δ_f which, from the point of view of the agent, includes all the features that explain or illustrate f, helping differentiating it from its neighboring concepts. For instance, for the concept *to-be-a-bird*, the corresponding Δ_f may consist of the concepts *to-be-a-vertebrate*, *to-have-a-beak* and *to-have-feathers*; for the concept *to-be-a-tent*, it may list the features *to-be-a-shelter*, *to-be-made-of-cloth*. We interpret Δ_f as the set of *defining features* an agent or a group of agents would associate with f. The sets Δ_f may be seen as the outputs a dictionary or an encyclopedia would return when given vague concepts as inputs.

The elements of Δ_f are supposed to be *less complex* than the root concept f: in the agent's mind, they constitute an help for the understanding of f. This notion of complexity will be now given a precise meaning by attributing a *complexity level* $c(f)$ to the set $\mathcal{F}$ of concepts at hand in the following way:

- Sharp concepts are given complexity level 0.
- If Δ_f consists of sharp concepts, set $c(f) = 1$.
- If $c(g)$ has been defined for all concepts g of Δ_f, set $c(f) = 1 + Max(c(g))_{g\in\Delta_f}$.

We shall make the assumption that this procedure attributes a well-defined complexity level to every element of $\mathcal{F}$. In other words, our theory only applies to a set $\mathcal{F}$ that consists of concepts that either are sharp, or can be recursively defined through sharp concepts. Such concepts may be qualified as *constructible*. As a matter of fact, most of the elementary concepts one commonly deals with are constructible, with small complexity level, and we could have made the assumption that the set of concepts at hand solely consists of concepts f of level less than 3. However we find it more convenient to work in a more general framework, as the results are not more difficult to establish.

It may be the case that some elements of Δ_f are more important than others, when considered as a help for defining or illustrating f: for instance, given the concept *to-be-a-bird*, an agent may think that the feature *to-have-wings* is more *salient* than the feature *to-be-an-animal*, while both features may be part of the same set Δ_f. Thus, it is necessary to endow each set Δ_f with a (possibly empty) *salience relation* that reflects the relative importance of its elements as defining features of f. In its most general form, such a relation will be represented by a strict partial order $>_f$. This order has to be taken into account when comparing the f-membership of two items: an object x that falls under the most salient defining features of f will be considered a better instance of f than an object y that only falls under some non-salient defining feature of f.

We can now proceed to the construction of the membership preorder relation $\preceq^\mu_f$, which will be defined on the set of objects $\mathcal{O}$, and to the construction of the membership function φ_f, which will take its values in a finite totally ordered set $(A_f, <_f)$. We shall omit the subscripts when there is no ambiguity. We begin with the simplest case of sharp concepts:

Definition 1. For every elementary sharp concept f, A_f is the set $\{0, 1\}$, and φ_f the function: $\varphi_f(x) = 1$ if x falls under f and $\varphi_f(x) = 0$ otherwise. The associated membership preorder is defined by $x \preceq^\mu_f y$ if $\varphi_f(x) \leqslant \varphi_f(y)$.

The membership preorder and the membership function relative to an arbitrary elementary concept f will be now defined by induction on $c(f)$. This will be done in two steps.

3.1. The elementary membership order

Definition 2. Let f be an elementary concept, and suppose that the finite totally ordered sets $(A_g, <_g)$ and the membership functions φ_g have been defined for all elementary concepts g such that $c(g) < c(f)$. The relation $\preceq^\mu_f$ is then defined by:

$x \preceq^\mu_f y$ if for any concept h of Δ_f such that $\varphi_h(y) <_h \varphi_h(x)$, there exists a concept k of Δ_f, $k >_f h$, such that $\varphi_k(x) <_k \varphi_k(y)$.

The relation $\preceq^\mu_f$ thus compares the ways objects inherit the defining features of f, while taking into account the relative salience of these features. We will say that a preorder of this type is *induced* by the (ordered) set Δ_f. In the particular case where the salience order on Δ_f is empty, the relation boils down to: $x \preceq^\mu_f y$ if and only if $\varphi_h(x) \leqslant_h \varphi_h(y)$ for all h in Δ_f, that is if and only if no defining feature of f applies more to x than to y.

The hypothesis that, for $k \in \Delta_f$, the membership functions φ_k take their value in a totally ordered set guarantees the transitivity of the relation $\preceq^\mu_f$. More precisely we have the following result:

Lemma 1. *For any elementary concept f, the relation $\preceq^\mu_f$ is a partial preorder on $\mathcal{O}$.*

Proof. We have to prove that $\preceq_f^\mu$ is a reflexive and transitive relation. Reflexivity is immediate. For transitivity, suppose that x, y and z are three objects such that $x \preceq_f^\mu y$ and $y \preceq_f^\mu z$. We want to show that $x \preceq_f^\mu z$. Supposing that there exists a concept h of Δ_f such that $\varphi_h(z) < \varphi_h(x)$, we have to prove the existence of a concept $k \in \Delta_f$, k more salient than h, such that $\varphi_k(x) < \varphi_k(z)$. We make a proof by cases:

- Suppose first that $\varphi_h(x) \leqslant \varphi_h(y)$. Then we have $\varphi_h(z) < \varphi_h(y)$, and there exists therefore a concept k of Δ_f, $k >_f h$, such that $\varphi_k(y) < \varphi_k(z)$. We can suppose that k is maximal in Δ_f for this property (Δ_f is a finite set). If $\varphi_k(x) \leqslant \varphi_k(y)$, we get $\varphi_k(x) < \varphi_k(z)$ and we are done. If $\varphi_k(y) < \varphi_k(x)$, the hypotheses imply that there exists a concept g in Δ_f, $g >_f k$ such that $\varphi_g(x) < \varphi_g(y)$. We cannot have $\varphi_g(z) < \varphi_g(y)$, otherwise there would exist a concept l in Δ_f, $l >_f g$, such that $\varphi_l(y) < \varphi_l(z)$, which would contradict the maximality of k. We have therefore $\varphi_g(y) \leqslant \varphi_g(z)$ and it follows that $\varphi_g(x) < \varphi_g(z)$ as desired.
- Suppose now that we have $\varphi_h(y) < \varphi_h(x)$. There exists $k \in \Delta_f$, $k >_f h$, such that $\varphi_k(x) < \varphi_k(y)$. Again, we can suppose that k is maximal in Δ_f for these properties. If $\varphi_k(y) \leqslant \varphi_k(z)$, we get $\varphi_k(x) < \varphi_k(z)$, as desired. If on the contrary we have $\varphi_k(z) < \varphi_k(y)$, there exists a concept g in Δ_f, $g >_f k$, such that $\varphi_g(y) < \varphi_g(z)$. As before, the maximality of k implies that we necessarily have $\varphi_g(x) \leqslant \varphi_g(y)$. It follows that $\varphi_g(x) < \varphi_g(z)$, and the proof is complete. □

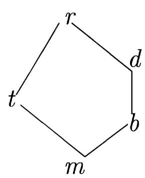
Let us denote by $\prec_f^\mu$ the relation: $x \prec_f^\mu y$ iff $x \preceq_f^\mu y$ and not $y \preceq_f^\mu x$. It follows from the above lemma that $\prec_f^\mu$ is a strict partial order on $\mathcal{O}$.

Example 1. Let f be the concept *to-be-a-bird*, and suppose that, from the point of view of an agent, its defining feature set is given by $\Delta_f = \{$*to-be-an-animal*, *to-have-two legs*, *to-lay-eggs*, *to-have-a-beak*, *to-have-wings*$\}$, all of these concepts being considered as sharp concepts for the agent. Suppose also that the salience order is given by: *to-lay-eggs* $>_f$ *to-have-two-legs*, *to-have-a-beak* $>_f$ *to-lay-eggs* and *to-have wings* $>_f$ *to-lay-eggs*.

Let r, m, t, b and d respectively stand for a robin, a mouse, a tortoise, a bat and a dragonfly, and let us compare their relative birdhood. In order to determine the induced membership order, we first build the following array:

	animal	*two-legs*	*lay-eggs*	*beak*	*wings*
robin	⋆	⋆	⋆	⋆	⋆
mouse	⋆				
tortoise	⋆		⋆	⋆	
bat	⋆	⋆			⋆
dragonfly	⋆		⋆		⋆

We readily check that $d \prec_f^\mu r$, $m \prec_f^\mu t$, and $m \prec_f^\mu b$. Note that we have $b \preceq_f^\mu d$, since the concept *to-have-two-legs* under which the bat falls, contrary to the dragonfly, is dominated by the concept *to-lay-eggs* that applies to the dragonfly and not to the bat. On the other hand, we do not have $d \preceq_f^\mu b$, as nothing compensates the fact that the dragonfly lays eggs and the bat does not. This yields $b \prec_f^\mu d$. We also remark that the tortoise and the bat are incomparable, that is, we have neither $b \preceq_f^\mu t$, nor $t \preceq_f^\mu b$. The strict f-membership order induced on these five elements is thus given by the following Hasse diagram:



We have therefore $m \prec_f^\mu b \prec_f^\mu d \prec_f^\mu r$ and $m \prec_f^\mu t \prec_f^\mu r$.

We can now precisely translate the notion of membership: an object x will be considered as *falling* under f if x is $\prec_f^\mu$-maximal in $\mathcal{O}$. We shall denote by $Ext\, f$, the *extension* of f, the set of all such objects.

We close this paragraph with a technical lemma:

Lemma 2. *The double inequality $x \preceq_f^\mu y$ and $y \preceq_f^\mu x$ holds if and only if $\varphi_h(x) = \varphi_h(y)$ for all concepts h of Δ_f.*

Proof. If $\varphi_h(x) = \varphi_h(y)$ for all concepts h of Δ_f, we have clearly $x \preceq_f^\mu y$ and $y \preceq_f^\mu x$. Conversely, suppose that $x \preceq_f^\mu y$ and $y \preceq_f^\mu x$. If we had not $\varphi_h(x) = \varphi_h(y)$ $\forall h \in \Delta_f$, there would exist a concept h of Δ_f such that $\varphi_h(x) \neq \varphi_h(y)$, and we could choose h with maximal salience for this property. We would have for instance $\varphi_h(x) <_h \varphi_h(y)$. But since $y \preceq_f^\mu x$, there would exist $k \in \Delta_f$, k more salient than h, such that $\varphi_k(y) <_k \varphi_k(x)$, thus contradicting the choice of h. $\square$

3.2. The membership function

It is clear that the ordering given by the relation $\preceq_f^\mu$ is not connected: given two objects x and y, it may well happen that neither $x \preceq_f^\mu y$, nor $y \preceq_f^\mu x$. It is nevertheless possible, starting from the strict partial order $\prec_f^\mu$, to build, a membership function φ_f that fairly translates the notion of a *degree of f-membership*. This function will satisfy $\varphi_f(x) < \varphi_f(y)$ whenever $x \prec_f^\mu y$: in a sense, this is the best one can hope (see [13] and her discussion on the impossibility for order relations to correctly represent vagueness). For this purpose, we shall proceed in a way that parallels, though in different context, a construction we proposed in [6].

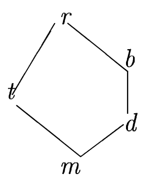
Given an object x, we say that x *initializes a membership chain of length n* if it is possible to find n objects $x_1, x_2, \ldots, x_n$ with last term $x_n \in Ext\, f$, such that $x \prec_f^\mu x_1 \prec_f^\mu x_2 \prec_f \cdots \prec_f^\mu x_n$. For instance, any element $x \in Ext\, f$ initializes a chain of length 0, and any object that does not fall under f initializes an membership-chain of strictly positive length $l \leqslant |A_f||\Delta_f|$. In a sense, the length of such a chain measures how distant x is from the set $Ext\, f$. Note that, given an object x, the existence and the length of such a chain is determined by the concepts and the objects the agent has at his disposal. Each link of a chain corresponds for this agent to a real (or a fictive) given object, together with some given concepts of the universe at hand.

Definition 3. The *membership distance* $\mu_f(x)$ of an object x to $Ext\, f$ is the maximal length of a membership chain initialized by x.

The distance of x to $Ext\, f$ is therefore an integer that is equal to 0 if and only if x falls under f. This measure will now be used for the definition of the membership function:

Definition 4. Let $\sim$ be the relation $x \sim y$ if $\mu_f(x) = \mu_f(y)$. Denote by $A_f = \mathcal{O}/\sim$ the associated quotient set and φ_f the canonical map from $\mathcal{O}$ onto A_f. Then the relation of total order $\leqslant_f$ on A_f is defined by $\varphi_f(x) \leqslant_f \varphi_f(y)$ if $\mu_f(x) \geqslant \mu_f(y)$.

Example 2. We take again example 1 and the Hasse diagram giving the membership order induced by the concept to-be-a-bird on the set {robin, tortoise, bat, mouse, dragonfly}:



Let us now compare the respective membership values taken by the membership function: we have $\mu_f(t) = 1$ because, to our knowledge, there exists no oviparous animal a that has a beak and satisfies $t \prec_f^\mu a \prec_f^\mu r$. Similarly, we have $\mu_f(d) = 1$, since, to our knowledge, there exists no animal a' such that $d \prec_f^\mu a' \prec_f^\mu r$. Since the bat falls under three out of the five elements of Δ_f we have necessarily $\mu_f(b) < 3$, and the inequality $b \prec_f^\mu d \prec_f^\mu r$ then yields $\mu_f(b) = 2$. As for the mouse, we have $m \prec_f^\mu b \prec_f^\mu d \prec_f r$, but this is not a chain of maximal length. For instance, noting that men have two legs, we also have the chain $m \prec_f^\mu h \prec_f^\mu b \prec_f^\mu d \prec_f^\mu r$, where h denotes a-human. This shows that $\mu_f(m) \geqslant 4$, and thus $\mu_f(m) = 4$ since we cannot have $\mu_f(m) > 5$. Putting all this together, we come finally to $\varphi_f(m) < \varphi_f(b) < \varphi_f(t) < \varphi_f(r)$ and $\varphi_f(t) = \varphi_f(fl)$.

Remark 1. It would be easy to normalize the set A_f and obtain an ordered set isomorphic to a finite subset of the unit interval [0, 1]. For example, we may take for φ_f the *f-membership degree* δ_f^μ defined by: $\delta_f^\mu = 1 - \frac{\mu_f}{N_f^\mu}$, where N_f^μ, the membership *width of* f, is the length of the longest f-membership chain initialized by an object. Such a solution however is misleading, as it attributes the same scale of values to different concepts, which we would have no reason to treat uniformly. Also, it may lead to fallacious comparisons and artificial problems, comparing for instance the degree of membership of an object x relative to a concept f with that of an object y relative to a concept g.

Remark 2. The greatest element of A_f is equal to $\varphi_f(x)$, where x is an arbitrary element of *Ext* f. Its least element is equal to $\varphi_f(z)$, z being any object such that $\mu_f(z) = N_f^\mu$.

We have now fully defined the notion of membership for concepts of arbitrary complexity. Observe that one easily recovers the characterization of Δ_f as a set of features that are *individually necessary and collectively sufficient for an object to be considered as an instance of* f: for this, we only need to assume that there exists at least an object that falls under all the elements of Δ_f.

Proposition 1. *Let f be an elementary concept and x an object. Then x falls under f if and only if x falls under every concept of Δ_f.*

Proof. It is clear that if an object x falls under all the elements of Δ_f, x is $\prec_f^\mu$-maximal and therefore falls under f. Conversely, suppose x does not fall under all the elements of Δ_f, and let y be an element that falls under the elements of Δ_f. Such an element exists by our assumption. We claim that we have $x \prec_f^\mu y$: indeed, since the elements $\varphi_h(y)$ are maximal in A_h for all concepts $h \in \Delta_f$ we have readily $x \preceq_f y$. Next, it is clear that we do not have $y \preceq_f x$, because, by the choice of x, there exists a concept $h \in \Delta_f$ such that $\varphi_h(x) <_h \varphi_h(y)$ while it is impossible to find a concept $g \in \Delta_f$ such that $\varphi_g(y) <_g \varphi_g(x)$. This yields $x \prec_f^\mu y$, as claimed, and we have shown that x is not $\prec_f^\mu$-maximal. □

Corollary 1. *It holds $x \prec_f^\mu y$ whenever y falls under f and x does not.*

Proof. Straightforward. □

4. Membership for compound concepts

It is sometimes possible to determine a concept f by a concept g, and get in this way a compound concept $g \star f$ that represents the *determination of f by g*. This determination is most often translated by an adjective-noun or an adjectived verb combination (e.g. the concepts *to-be-a-carnivorous-animal, to-be-a-flying-bird, to-be-a french-student*), but it can also be rendered by a noun-noun combination (e.g. *to-be-a-pet-fish, to-be-a-barnyard-bird*). Unlike ordinary conjunction, the connective $\star$ cannot be defined for arbitrary pairs of concepts (f, g): for instance, if g is the concept *to-fly*, f the concept *to-be-an-artefact* and h the concept *to-be-a-prime-number*, it makes sense, or at least it may make sense for some agent, to form the concept $g \star f$, which one would interpret as the concept *to-be-a-flying-artefact*; but it would be meaningless to try and form the concept $g \star h$ corresponding to the 'concept' *to-be-a-flying-prime-number*: in this case the determination connective simply cannot operate.

It is important to keep in mind that we consider only the conceptual combinations that are *intersective*: the objects that fall under the composed concept $g \star f$ are exactly the ones that both fall under f and under g. Thus, and to mention the most known examples, the determination connective cannot be used to form complex concepts like *to-be-a-brick-factory*, *to-be-a-criminal lawyer* or *to-be-a-topless-district*: indeed, a brick factory need not be a factory that is made out of bricks, a criminal lawyer not a layer that is a criminal, and a topless district not a district that is topless. The determination connective $\star$ operates on a *principal* concept to which it attributes some *secondary* properties usually expressed by an intersective (extensional) adjective or an instersective adjectived verb (see for instance [12] for the distinction between intersective and non-intersective adjectives). Typically, in the compound concept $g \star f$, the main concept f is defined through a predicate of the type *to-be-x*, while the accessory concept g is of the form *to-have-the-property-y*. We shall say that f is *g-determinable* when the concept $g \star f$ can be formed. This requires that the sets $Ext\, f$ and $Ext\, g$ have a non-empty intersection: there must exist elements of $\mathcal{O}$, that is real or fictive objects, that fall under f and under g.

Note that even when correctly defined, the determination connective does not necessarily enjoy the same properties as its analogue in propositional logic: unlike conjunction, indeed, it is not supposed to satisfy commutativity (for examples and discussion, and in particular the distinction between *games-that-are-sports* and *sports-that-are-games*, see [23]).

We proceed now to the definition of the membership preorder associated with a compound concept. Our construction is motivated by the fact that, unlike the elementary concepts, compound concepts are not usually associated in with a defining feature set: dictionaries or encyclopedias do not provide answers on queries about noun–noun or adjective–noun combinations. For this reason, we shall directly propose the construction of a composed membership preorder $\preceq^{\mu}_{g\star f}$ that naturally stems from the membership preorders $\preceq^{\mu}_{f}$ and $\preceq^{\mu}_{g}$ while taking into account the preeminence of the principal concept f.

Definition 5. Let f and g be elementary concepts and $\preceq^{\mu}_{f}$ and $\preceq^{\mu}_{g}$ their associated membership preorders. Suppose that f is g-determinable. Then the relation $\preceq^{\mu}_{g\star f}$ is defined by: $x \preceq^{\mu}_{g\star f} y$ if $x \preceq^{\mu}_{f} y$ and either $x \prec^{\mu}_{f} y$, or $x \preceq^{\mu}_{g} y$.

Thus, priority is given to the concept f, translating the fact that f is supposed so play the principal role in the composed concept.

It may be helpful to consider the relation $\preceq^{\mu}_{g\star f}$ as induced by a set of fictive defining features:

Lemma 3. *Let $>_f$ and $>_g$ be the salience orders on Δ_f and Δ_g. Denote by $\widetilde{\Delta}$ the set $\Delta_f \cup \Delta_g$ equipped with the 'salience' order $>$ that agrees with $>_f$ on Δ_f, agrees with $>_g$ on $\Delta_g - \Delta_f$, and satisfies $k > h$ for all k's in Δ_f and h's in $\Delta_g - \Delta_f$. Then $\preceq^{\mu}_{g\star f}$ is induced by $\widetilde{\Delta}$.*

In other words, $\preceq^{\mu}_{g\star f}$ agrees with the relation $\preceq$ defined by: $x \preceq_y$ iff for any concept h of $\widetilde{\Delta}$ such that $\varphi_h(y) <_h \varphi_h(x)$, there exists a concept k of $\widetilde{\Delta}$, $k > h$, such that $\varphi_k(x) <_k \varphi_k(y)$.

Proof. Suppose first that we have $x \preceq^{\mu}_{g\star f} y$, so that $x \preceq^{\mu}_{f} y$ and either $x \prec^{\mu}_{f} y$, or $x \preceq^{\mu}_{g} y$. We want to show that $x \preceq y$. Given any element h of $\widetilde{\Delta}$ such that $\varphi_h(y) <_h \varphi_h(x)$, we have to prove the existence of an element k of $\widetilde{\Delta}$, $k > h$, such that $\varphi_k(x) <_k \varphi_k(y)$. If $h \in \Delta_f$, there exists $k \in \Delta_f$, $k >_f h$, such that $\varphi_k(x) <_k \varphi_k(y)$, and, since $k > h$, we are done. Suppose therefore $h \in \Delta_g - \Delta_f$. If $x \prec^{\mu}_{f} y$, we do not have $y \preceq^{\mu}_{f} x$, and there exists therefore a concept $k \in \Delta_f$ such that $\varphi_k(x) <_k \varphi_k(y)$. But then, we have $k > h$, and, again, we are through. Finally, if $x \preceq^{\mu}_{g} y$, since $h \in \Delta_g$, there exists $k \in \Delta_g$, $k >_g h$ such that $\varphi_k(x) <_k \varphi_k(y)$. This implies $k > h$ and provides again the desired result.

Conversely, suppose that we have $x \preceq y$. To prove that $x \preceq^{\mu}_{g\star f} y$, we first show that $x \preceq^{\mu}_{f} y$: if $\varphi_h(y) <_h \varphi_h(x)$ for some concept h of Δ_f, there exists $k \in \widetilde{\Delta}$, $k > h$ such that $\varphi_k(x) <_k \varphi_k(y)$. But, by the construction of the preorder $>$, we have necessarily $k \in \Delta_f$ and $k >_f h$. This shows that $x \preceq^{\mu}_{f} y$. It remains to prove that either $x \prec^{\mu}_{f} y$, or $x \preceq^{\mu}_{g} y$. Suppose we have not $x \prec^{\mu}_{f} y$. We have then the double inequality $x \preceq^{\mu}_{f} y$ and $y \preceq^{\mu}_{f} x$. To prove that $x \preceq^{\mu}_{g} y$, let h be a concept in Δ_g such that $\varphi_h(y) <_h \varphi_h(x)$. Since $x \preceq y$, there exists a concept k of $\widetilde{\Delta}$, $k > h$, such that $\varphi_k(x) <_k \varphi_k(y)$. By Lemma 2, we cannot have $k \in \Delta_f$. We have therefore $k \in \Delta_g$ and $k >_g h$, which completes the proof. □

Clearly, the process of defining the complex relation $\preceq^{\mu}_{g\star f}$ from $\preceq^{\mu}_{f}$ and $\preceq^{\mu}_{g}$ can be iterated, and used to define the membership orders associated with the determination of arbitrary concepts.

Corollary 2. *The relation $\preceq^{\mu}_{g\star f}$ is reflexive and transitive on $\mathcal{O}$.*

Proof. Clear. □

We let $\prec^{\mu}_{g\star f}$ be the strict partial order induced by $\preceq^{\mu}_{g\star f}$. We have thus $x \prec^{\mu}_{g\star f} y$ iff $x \preceq^{\mu}_{f} y$ and either $x \prec^{\mu}_{f} y$ or $x \prec^{\mu}_{g} y$.

Lemma 4. *Let f be a g-determinable concept. Then*

- $\preceq^{\mu}_{g\star f} \subseteq \preceq^{\mu}_{f}$;
- $\prec^{\mu}_{f} \subseteq \prec^{\mu}_{g\star f}$.

Proof. Straightforward. □

As in the case of elementary concepts, we will say that an object x *falls* under the concept $g \star f$ if x is $\prec^{\mu}_{g\star f}$-maximal.

Let us again denote by $Ext(g \star f)$ the *extension* of the concept $g \star f$, that is the set of all objects that fall under $g \star f$. As we shall see, this set can be directly retrieved from the extensions $Ext\, f$ and $Ext\, g$:

Proposition 2. $Ext(g \star f) = Ext\, g \cap Ext\, f$.

Proof. Suppose $x \in Ext\, f \cap Ext\, g$. If x were not $\prec^{\mu}_{g\star f}$-maximal, there would exist an object z such that $x \prec^{\mu}_{g\star f} z$; therefore we would have either $x \prec^{\mu}_{f} z$, contradicting the hypothesis $x \in Ext\, f$, or $x \prec^{\mu}_{g} z$, contradicting the hypothesis $x \in Ext\, g$.

Conversely, suppose that is $x \prec^{\mu}_{g\star f}$-maximal. Clearly, x is then $\prec^{\mu}_{f}$-maximal, and therefore falls under f. Suppose we do not have $x \in Ext\, g$, and let z be any element of $Ext\, g \cap Ext\, f$ (such an element exists since f is g-determinable). Then we have $x \preceq^{\mu}_{g\star f} z$ and $x \prec^{\mu}_{g} z$, that is $x \prec^{\mu}_{g\star f} z$, contradicting the $\prec^{\mu}_{g\star f}$-maximality of x. □

The above proposition helps determining in which cases a compound concept may be a sharp concept. This clearly occurs in the particular case of the determination $f \star f$ of a sharp concept by itself, as readily follows from the definitions. In fact, the only case where a compound concept $g \star f$ is a sharp concept occurs when both f and g are sharp concepts having same extension:

Proposition 3. *$g \star f$ is a sharp concept if and only if both f and g are sharp concepts that satisfy $Ext\, f = Ext\, g$.*

Proof. Suppose first that $g \star f$ is a sharp concept. Then it is immediate that f is also a sharp concept, otherwise, we would find three objects x, y and z verifying $x \prec^{\mu}_{f} y \prec^{\mu}_{f} z$, that is $x \prec^{\mu}_{g\star f} y \prec^{\mu}_{g\star f}$, contradicting the sharpness of $g \star f$. Observe now that we have $Ext\, f \subseteq Ext\, g$. Indeed, if this were not the case, there would exist an object y such that $y \in Ext\, f$ and $y \notin Ext\, g$. But then we would have $x \prec^{\mu}_{g\star f} y \prec^{\mu}_{g\star f}$ for any objects x and z such that $x \notin Ext\, f$ and $z \in Ext\, g$, thus contradicting our sharpness hypothesis on $g \star f$. Observe also that if the converse inclusion $Ext\, g \subseteq Ext\, f$ did not hold, taking $u \notin Ext\, g$, $v \in Ext\, g - Ext\, f$ and $w \in Ext\, f$, we would have $u \prec^{\mu}_{g\star f} v \prec^{\mu}_{g\star f} w$, which would again contradict the sharpness of $g \star f$. Finally, to see the sharpness of g, suppose given three objects r, s and t such that $r \prec^{\mu}_{g} s \prec^{\mu}_{g} t$. Since $r \prec^{\mu}_{g} s$ and $Ext\, f \subseteq Ext\, g$, we cannot have $r \in Ext\, f$; similarly, we see that s cannot lie in $Ext\, f$. Since f is a sharp concept, we have therefore $r \preceq^{\mu}_{f} s \preceq^{\mu}_{f} t$. Putting all this together, we get $r \prec^{\mu}_{g\star f} s \prec^{\mu}_{g\star f} t$, contradicting again the sharpness of $g \star f$.

Conversely, suppose now that f and g are sharp concepts with $Ext\, f = Ext\, g$. We have to prove that $g \star f$ is sharp. But if this were not the case, we could find three objects x, y and z such that $x \prec^{\mu}_{g\star f} y \prec^{\mu}_{g\star f} z$. Being not a $\prec^{\mu}_{g\star f}$-maximal element, we would have $y \notin Ext(g \star f)$, hence $y \notin Ext\, f$. By the sharpness of f, this would imply $x \not\prec^{\mu}_{f} y$, and thus $x \prec^{\mu}_{g} y$. By the sharpness of g, this would finally lead to $y \in Ext\, g$, hence $y \in Ext\, f$, a contradiction. □

It follows from Proposition 2 that all objects that fall under $g \star f$ necessarily fall under f (and also under g). Hence, we may naturally ask whether the compound concept $g \star f$ could be considered as a *subconcept* of f. In the perspective of fuzzy logic, a concept k is a subconcept of h if and only $\varphi_k \leqslant \varphi_h$, that is, if and only if it holds, for all objects x, $\varphi_k(x) \leqslant \varphi_h(x)$. In our model, this first supposes an embedding of the ordered set $(A_h, \leqslant_h)$ in $(A_k, \leqslant_k)$. We shall see that such is indeed the case for the sets A_f and $A_{g\star f}$ and that the compound concept $g \star f$ may be therefore considered as a subconcept of f.

Lemma 5. *For all x, $\mu_f(x) \leqslant \mu_{g\star f}(x)$.*

Proof. Suppose that for an object x, one has $m = \mu_{g\star f}(x) < \mu_f(x) = n$, and let $x \prec_f^{\mu} x_1 \prec_f^{\mu} x_2 \prec_f \cdots \prec_f^{\mu} x_n$ be an x-membership f-chain of maximal length ($x_n \in Ext\, f$). Note that $x_{n-1} \notin Ext\, f$. Let z be an element of $Ext(g \star f)$. By Lemma 4, we have $x_i \prec_{g\star f}^{\mu} x_{i+1}$ for $i < n$, and it follows that $x \prec_{g\star f}^{\mu} x_1 \prec_{g\star f}^{\mu} x_2 \prec_{g\star f} \cdots \prec_{g\star f}^{\mu} x_{n-1} \prec_{g\star f} z$, is a membership $g \star f$-chain of length n, contradicting the hypothesis. □

This lemma shows that the range of the function μ_f is embedded in that of the function $\mu_{g\star f}$. Identifying the sets A_f and $A_{g\star f}$ with these ranges then yields $A_{g\star f} \subseteq A_f$. Since we have, for all x, $\mu_f(x) \leqslant \mu_{g\star f}(x)$, it follows, from this identification and from the very definition of φ, that $\varphi_{g\star f}(x) \leqslant \varphi_f(x)$, showing that $g \star f$ is a subconcept of f.

5. Typicality for elementary concepts

How can our model render count of the fact that, inside a category, there exist elements that are more *typical* than others? It is because of this evidence that the classical view on categorization had to be given up: a concept could not be defined anymore by its extension or its associated membership function, because such a definition would not explain the typicality effect. Membership orders and functions may be accurate enough to tell us that the penguin and the robin are equally birds, or that the mouse is less a bird than the bat, but they will be unable to account for the fact that, as a bird, the robin is more typical than the penguin.

We have therefore to complete and extend the formalism proposed in the preceding sections. The auxiliary set Δ_f through which an agent constructs the f-membership order is not sufficient on its own to fully capture the information encoded by f: it becomes necessary to add a supplementary set, which will consist of the features that, from the agent's point of view, are bond to apply to all *typical* instances of the concept at hand, whatever significance this term may carry for the agent.

Together with each elementary concept f, we will therefore suppose given a finite set of *characteristic features*, χ_f, the elements of which consist of concepts that complete and illustrate the core information that may be provided by Δ_f. Note that $\chi_f \cap \Delta_f$ need not be the empty set. In particular, χ_f may include most, if not all, of the most salient elements of Δ_f. For instance, if f denotes the concept *to-be-a-fruit*, we may take for χ_f the set consisting of the elements *to-grow-on-trees*, *to-be-sweet*, *to-be-raw-edible*, *to-yield-juice* and *to-have-a-seed*, while Δ_f will consist of the two concepts *to-be-a-vegetable* and *to-heave-seeds*.

As was the case for Δ_f, the characteristic set χ_f will be equipped with a (possibly empty) strict partial *salience order*, meant to compare the pertinence of two different characteristic features.

Again, characteristic sets and salience orders are purely subjective items: they reflect the *Weltanschauung* of a given agent at a given time. For example, the characteristic features associated with the concept *to-be-a-bird* may consist, for an agent, of the set {*to-fly, to-have-feathers, to-live-in-the-trees, to-sing, to-have-wings*}, in which the properties of *flying* or of *having wings* are given maximal salience. For another agent, however, the characteristic set of the same concept may consist of the elements {*to-fly, to-have-feathers, to-be-oviparous, to build nests, to-have-a-beak*}, the property of *having wings* being considered as more salient than that of *flying*.

The idea of associating with every elementary concept its set of 'characteristic' features is due to Smith et al. [22], and the term is theirs. The authors took care to distinguish this set from the set of *necessary features*, this latter consisting of all the *essential* properties associated with the concept at hand (see also [4] for the distinction between *necessary features* and *defining features*). For instance, *to-fly* is considered as a characteristic feature of the concept *to-be-a-bird* but it is not an essential one, as it is possible to conceive birds that do not fly. On the contrary, *to-have-a beak* or *to-be-warm-blooded* are necessary features for this concept, as each of these properties is an essential one: one cannot conceive a bird that would not have a beak or that would be cold-blooded.

Typicality relative to a concept f may be defined for arbitrary objects, and not only for elements $Ext\, f$: for instance, Rosch and Mervis [19] included non-members of categories in their typicality rating lists. Nevertheless, as observed in [10], *typicality carries with it the assumption of a range restricted to category members*. This is the position that we will adopt.

5.1. Typicality order for elementary concepts

In order to grasp the notion of *more or less typical* objects, we propose the following definition, which parallels that of the membership order:

Definition 6. Let f be an elementary concept, and x and y two elements of $Ext\, f$. We shall say that x is at most as f-typical as y, written $x \preceq^{\tau}_{f} y$, if for any concept h of χ_f such that $\varphi_h(y) <_h \varphi_h(x)$, there exists a concept k of χ_f, k more salient than h, such that $\varphi_k(x) <_k \varphi_k(y)$.

Note that in the general case, it may well happen that a less typical object x falls under a feature $h \in \chi_f$ that does not apply to y, provided there exists a more salient feature $k \in \chi_f$ that applies to y and not to x.

It is important to observe that the above definition relates typicality relative to a concept with the characteristic features of this concept, and only with them: it does not take into account the *normality* or the *abnormality* of the objects at hand. To judge the typicality of a dog for instance, we do not have to examine whether it is blind, has a wounded paw or lives with an artificial heart: there exists a clear distinction between the notions of typicality and normality, and prototype theory only deals with the first one.

Proposition 4. *For all elementary concepts f, the relation $\preceq^{\tau}_{f}$ is a partial preorder on $Ext\, f$.*

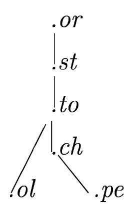
Proof. Analogue to that of Lemma 5.1. □

The strict partial order induced by $\preceq^{\tau}_{f}$ will be denoted by $\prec^{\tau}_{f}$. We have therefore $x \prec^{\tau}_{f} y$ iff $x \preceq_f y$ and not $y \preceq_f x$.

Example 3. Let us take for f the concept *to-be-a-fruit* and suppose that for a given agent the set χ_f consists of the concepts *to-grow-on-trees*(*tr*), *to-be-sweet*(*sw*), *to-be-raw-edible*(*ra*), *to-yield-juice*(*ju*), *to-have-a-skin*(*sk*), together with a salience order $>$ given by: $ra > tr$, $ra > ju$, $tr > ju$, $tr > sw$, $sk > ju$ and $sk > sw$. Consider the following six fruits: a chestnut (*ch*), an olive (*ol*), a pepper (*pe*), a strawberry (*st*), an orange (*or*) and a tomato (*to*). For the sake of simplicity, we suppose our agent attributed them only extremal membership values:

	tr	*sw*	*ra*	*ju*	*sk*
ch	⋆				⋆
ol	⋆		⋆		
pe			⋆		⋆
st		⋆	⋆		
to			⋆	⋆	⋆
or	⋆	⋆	⋆	⋆	⋆

The induced typicality order is then given by the following Hasse diagram:



An important case that deserves to be mentioned occurs when the salience order on χ_f is a *total* order, a situation we find in particular when the characteristic set is reduced to a single element—*to-fly* being for instance taken as the unique characteristic feature of *to-be-a-bird*.

Proposition 5. *Suppose the salience order $>_f$ on χ_f is a total order. Then $\preceq^{\tau}_{f}$ is a total preorder on $Ext\, f$. More precisely, given two exemplars x and y of $Ext\, f$, we have either $x \prec^{\tau}_{f} y$, or $y \preceq^{\tau}_{f} x$, or $\varphi_h(x) = \varphi_h(y)$ $\forall h \in \chi_f$.*

Proof. Suppose indeed that the equality $\varphi_h(x) = \varphi_h(y)$ is not satisfied for all elements of χ_f. Let k be the most salient concept of χ_f such that $\varphi_k(x) \neq \varphi_k(y)$. We have for instance $\varphi_k(x) <_k \varphi_k(y)$. This yields $x \prec_f^\tau y$: indeed, for all concepts h of χ_f such that $\varphi_h(y) <_h \varphi_h(x)$, k is more salient than h and we have $\varphi_k(x) <_k \varphi_k(y)$. Note that we cannot have $y \prec_f^\tau x$, because, by a proof similar to that of Lemma 2, this would yield $\varphi_h(x) = \varphi_h(y)$ for all $h \in \chi_f$. This shows that $x \prec_f^\tau y$ as desired. □

Notation. For any subset X of *Ext* f, we shall denote by $X_{\tau(f)}$ the set of $\prec_f^\tau$-maximal elements of X; the finiteness of χ_f implies that $X_{\tau(f)} \neq \emptyset$ whenever $X \neq \emptyset$.

Remark 3. Contrary to what happened in the case of the membership order, the typicality order as we defined it does not require the preliminary use of a specific gradation function. The construction of a typicality gradation would be of course possible, defining first a *typicality distance* τ_f, a *typicality width* N_f^τ, and a subsequent *typicality degree* δ_f^τ similarly to what was done in Section 3.2. It is interesting to note that, in the case where χ_f is given a *total* salience order, the resulting typicality degree faithfully translates the (total) typicality order $\prec_f^\tau$: in this case, we have indeed $x \prec_f^\tau y$ iff $\delta_f^\tau(x) < \delta_f^\tau(y)$.

5.2. Typical elements

We now come to the definition of the *typical instances* of an elementary concept f. We shall make the assumption that there always exists an exemplar of f that falls under all the elements of χ_f.

Proposition 6. *Let f be an elementary concept and x an element of Ext f. Then the two following properties are equivalent*:

- *x falls under all the elements of χ_f*;
- *x is $\prec_f^\tau$-maximal in Ext f.*

Proof. Analogue to that of Proposition 1. □

We shall say that an object x is f-typical if it satisfies the properties of Proposition 6. For instance, and provided that *to-fly* is a characteristic feature of birds, a hen will not be considered as a typical bird: indeed, it does not plainly fall under the concept *to-fly*.

Denoting by *Typ* f the set of f-typical objects, we have readily $\mathit{Typ}\, f = (\mathit{Ext}\, f)_{\tau(f)} = \bigcap_{g \in \chi_f} \mathit{Ext}\, g$.

Lemma 6. *Given an f-typical element z, one has $x \prec_f^\tau z$ for any instance x of f that is not typical.*

Proof. Straightforward from Proposition 6. □

Example 4. Consider the concept *to-be-a-tree*, and suppose that its characteristic set includes the concept *to-have-deciduous-leaves*. Suppose, on the other hand, that the concept *to-be-a-conifer* includes in its characteristic set the concept *to-have-evergreen-needles*. It follows from our definition that the larch is atypical as a conifer, while typical as a tree. In our framework, typicality relative to a category does not depend from membership relative to a subcategory: for somebody ignoring that the larch is a conifer, its typicality as a tree makes no doubts. Similarly, a flying ostrich will be considered as typical as a bird.

The classical notion of *intension* can now be recovered from the typical instances of f:

Definition 7. The *intension Int* f of a concept f is the set of all concepts g for which $\mathit{Typ}\, f \subseteq \mathit{Ext}\, g$.

Note that $f \in \mathit{Int}\, f$ and that $\chi_f \subseteq \mathit{Int}\, f$.

The intension of f is the set of features that apply to all typical instances of f. For instance, the concepts *to-sing, to-fly, to-be-oviparous, to-have wings* are elements of the intension of *to-be-a-bird*: all typical birds sing, fly, are oviparous and have wings. On the contrary, *to-be-black* or *to-build-nests* are not elements of $\mathit{Int}\, f$, because there exists typical birds that are not black, like the robin, and typical birds that do not build nests, like the cuckoo. We can characterize the elements of $\mathit{Int}\, f$ as describing features that apply to the *good exemplars* of f, or as properties that are *generally expected* from f: 'birds generally fly', indeed, exactly means that the birds that do not fly must be considered as atypical. In this sense, we may interpret the elements of $\mathit{Int}\, f$ as being *induced* by f, and analyze the link between f and g as representing a relation of *inference*, which could be approached through the ordinary tools of non-monotonic logics. It is this perspective that we shall develop in a forthcoming paper, where the properties of typical induction, studied as a non-monotonic inference relation, will be studied and compared to the existing systems.

Proposition 7. $\mathit{Typ}\, f = \bigcap_{g \in \mathit{Int}\, f} \mathit{Ext}\, g$.

The proposition says that an object is f-typical if and only if it falls under all the concepts of $\mathit{Int}\, f$.

Proof. If x is f-typical, it falls under every concept of $\mathit{Int}\, f$ by definition of $\mathit{Int}\, f$. Conversely, if an object falls under every concept of $\mathit{Int}\, f$, it falls under f and it also falls under every concept of χ_f. It is therefore f-typical. □

In a sense, the sets $\mathit{Int}\, f$ and $\mathit{Ext}\, f$ taken together characterize the concept f. Since the set $\mathit{Int}\, f$ is dual of the set $\mathit{Typ}\, f$, we can equivalently characterize a concept by the sets $\mathit{Typ}\, f$ and $\mathit{Ext}\, f$, which are subsets of $\mathcal{O}$. Given two *sharp* concepts f and g, we can therefore consider them as *equivalent*, (written $f \equiv g$) if they have same intension and same extension. In the case of *vague* concepts, it is difficult to define such a notion of equivalence without also requiring that the membership functions of f and g are isomorphic. The question then naturally arises of finding 'natural' concepts f and g with $\mathit{Ext}\, f = \mathit{Ext}\, g$, $\mathit{Typ}\, f = \mathit{Typ}\, g$, but such that for some pair (x, y), one has $\varphi_f(x) <_f \varphi_f(y)$ and $\varphi_g(y) \leqslant_g \varphi_g(x)$. In the absence of such an evidence, we shall extend our notion of equivalence to elementary vague concepts, and write $f \equiv g$ whenever f and g have same extension and same intension.

6. Smooth subconcepts and concept determination

In this final section, we propose to investigate the internal structure of concepts. The importance of this study appears in most of the work dealing with categorization-level and hierarchies. We shall introduce the notion of *smooth* subconcepts, and show that all these concepts are obtained through a specific kind of determination.

6.1. Subconcepts

As observed at the end of Section 4, if g is a subconcept of f, we must have $\varphi_g \leqslant \varphi_f$, and, consequently, $\mathit{Ext}\, g \subseteq \mathit{Ext}\, f$.

On the contrary, even if g is a subconcept of f, there may well exist no relationship between the sets $\mathit{Typ}\, f$ and $\mathit{Typ}\, g$. As a matter of fact, the typicality orders respectively associated with f and g are most often incomparable, and the corresponding typical sets may have an empty intersection: for instance, and as long as we consider penguins as exceptional birds, we will not be ready to accept as an exemplar of a typical bird any exemplar of a typical penguin.

The subconcepts g for which $\mathit{Typ}\, g \subseteq \mathit{Typ}\, f$ are therefore rather exceptional. We shall qualify these as *smooth* subconcepts:

Definition 8. A subconcept g of f is said to be *smooth* if it satisfies $\mathit{Typ}\, g \subseteq \mathit{Typ}\, f$.

Thus, g is a smooth subconcept of f if any typical exemplar of g may be considered as typical relative to f. Note that this condition is equivalent to $\chi_f \subseteq \mathit{Int}\, g$, which can also be expressed by $\mathit{Int}\, f \subseteq \mathit{Int}\, g$.

Example 5. The fact that we consider robins as typical birds means that any typical exemplar of a robin is a typical exemplar of a bird. Thus, if g is the concept *to-be-a-robin* and f the concept *to-be-a-bird*, we have $\mathit{Typ}\, g \subseteq \mathit{Typ}\, f$: g is a smooth subconcept of f.

We shall show in the next section how it is possible to characterize the smooth subconcepts of a given concept f. In fact we shall establish a representation theorem characterizing the smooth f-subconcepts as the determinations of f by concepts that, in a way do not contradict f. But before addressing this problem, we have to study the typicality associated with compound concepts.

6.2. The typicality of compound concepts

It is clear that the instances of a composed concept $g \star f$ that we may intuitively consider as typical of this concept, cannot be retrieved from the typical instances of f and the typical instances of g. A typical walking bird has nothing to do with a typical bird, and nothing to do with a typical walking animal either. More generally, the attributes that are typically induced by a compound concept cannot be retrieved by the intensions of its components: green apples are bound to be sour, but *to-be-sour* is not a member of the intension of *to-be-an-apple*, nor a member of the intension of *to-be-green*. Thus, no simple formula will enable us to deduce the typical elements of $g \star f$ from the typical elements of f and the typical elements of g.

What conditions would we require to consider an object as *typical* relative to a compound concept of $g \star f$? This question was experimentally addressed by [20], who showed that the *context* in which a concept appears affects the typicality of its instances. For instance, for somebody that works in the context *to-live-in-a-barnyard*, a chicken may be considered as a typical bird, although, relative to the concept *to-be-a-bird* it is not. In our framework, the concept *to-be-a-bird* taken in the context *to-live-in-a-barnyard* is simply represented by the determination (*to-live-in-a-barnyard*)$\star$(*to-be-a-bird*). Studying contextual typicality then amounts to determining the set $Typ(g \star f)$. For this, we need to define a suitable typicality order $\prec^{\tau}_{g\star f}$. Such an order will characterize the $g \star f$-typical elements as maximal elements of $Ext(g \star f)$. Note that the primary role played by f in the composition $g \star f$ implies that the $g \star f$-typical objects will be expected, before all, to be as typical as possible *relative to* f. Making use of the typical-order relation $\prec_f$, this amounts to saying that $Typ(g \star f)$ should be a subset of $Ext(g \star f)_{\tau f}$, and we should have therefore $Typ(g \star f) \subseteq (Ext\, g \cap Ext\, f)_{\tau(f)}$. But this sole condition is not sufficient to ensure $(g \star f)$-typicality. To see this, consider the following example: take for f the concept *to-be-French* and for g the concept *to-live-in-U.S.A*. Consider the case of Mr Dupont, a French traveler arrested on his arrival at Kennedy airport for drug traffic in 1998, condemned to 10 years jail, and since then detained in Red Onion prison, Virginia. Although typical as far as *to-be-a-frenchman* is concerned, and therefore an element of $(Ext\, g \cap Ext\, f)_{\tau(f)}$, Mr Dupont is definitely not a prototype of the concept *to-be-a-Frenchman-living-in-the-States*. For instance, it is clear that Mr Dupont is less representative of this latter concept than Mr Martin, a French student who is now completing his PhD in Berkeley University of California: this latter on the contrary may be seen as a typical relative to the concept $(g \star f)$. Thus, the order induced by $g \star f$ has to be chosen in such a way that it guarantee the typicality of all elements of $((Ext\, g \cap Ext\, f)_{\tau(f)})_{\tau(g)})$. As we shall see now, a construction analogue to that of the membership order for compound concepts will do the job.

Lemma 7. *Let f and g be two elementary concepts wit associated typicality preorders $\preceq^{\tau}_f$ and $\preceq^{\tau}_g$. Suppose f is g-determinable, and define the relation $\preceq^{\tau}_{g\star f}$ on $Ext(g \star f)$ by:*

$x \preceq^{\tau}_{g\star f} y$ if $x \preceq^{\tau}_f y$ and either $x \prec_f y$, or $x \preceq^{\tau}_g y$.

Then $\preceq^{\tau}_{g\star f}$ is a preorder on $Ext(g \star f)$.

Proof. Analogue to that of Lemma 2: $\preceq^{\tau}_{g\star f}$ is induced by the set $\widetilde{\chi} =_{def} \chi_f \cup \chi_g$ equipped with the adequate salience order . □

The above construction provides a typicality preorder for the determination of an elementary concept f by an elementary concept g. It is clear that it can be recursively used for any compound concept $k \star h$, where k and h are arbitrary concepts.

Let $\prec^{\tau}_{g\star f}$ be the strict partial order associated with the relation $\preceq^{\tau}_{g\star f}$, that is: $x \prec^{\tau}_{g\star f} y$ if and only if $x \preceq^{\tau}_f y$ and either $x \prec^{\tau}_f y$ or $x \prec^{\tau}_g y$. For any subset X of $Ext(g \star f)$, we let $X_{\tau(g\star f)}$ be the set of $\prec^{\tau}_{g\star f}$-maximal elements of X. We shall denote by $Typ(g \star f)$ the set of $\prec^{\tau}_{g\star f}$-maximal elements of $Ext(g \star f)$: $Typ(g \star f) = (Ext(g \star f))_{\tau(g\star f)}$. These elements will be considered as *typical* relative to the concept $g \star f$.

It has been objected (J.A. Hampton, *personal communication*) that the model we propose for compound typicality may lead to counterintuitive results: for instance, given the concept *to-be-an-Antarctic-bird* and following our construction, the Antarctic gull will be shown to be more typical than the penguin, provided that *to-fly* is part of the characteristic set of *to-be-a-bird*. However, when referring to Antarctic birds, people will most often consider the penguin as more typical than the gull. This apparent contradiction comes from the fact that, in people's mind, the category of Antarctic birds is altogether perceived as *opposed* to the category of usual, or European, birds. When referring to a member of the first category, one implicitly excludes the specimen that also lie in the second. This phenomena is even emphasized when typicality is concerned: a typical Antarctic bird is implicitly expected be much different from a usual bird. Thus the concept *to-be-an-Antarctic-bird* cannot be simply analyzed as the determination of a concept by another, which would be the case for instance for the concept *to-be-a-black-bird*: it appears to convey more information than would do a simple intersective combination.

It is possible, as we did in the case of elementary concepts, to define the *intension* $Int(g \star f)$ of a compound concept $g \star f$ as the set of all concepts h that apply to all elements of $Typ(g \star f)$: $Int(g \star f) = \{h; Typ(g \star f) \subseteq Ext\, h\}$. However, and contrary to what happened in the elementary case, there exist no duality between the sets $Int(g \star f)$ and $Typ(g \star f)$: the latter cannot be retrieved from the former, as was the case for elementary concepts (see Proposition 7). This comes from the fact that one may well find no object falling under all the elements of $\chi_f \cup \chi_g$. For this reason, we have to refine of definition of equivalent concepts.

Definition 9. Two concepts f and g are said to be equivalent, written $f \equiv g$, if $Ext\, f = Ext\, g$ and $Typ\, f = Typ\, g$.

The following result provides an upper and a lower bound for the set $Typ(g \star f)$:

Proposition 8. $((Ext(g \star f))_{\tau(f)})_{\tau(g)} \subseteq Typ(g \star f) \subseteq (Ext(g \star f))_{\tau(f)}$.

Proof. We first prove the second embedding. Let x be an element of $Typ(g \star f)$. If $x \notin (Ext(g \star f))_{\tau(f)}$, there exists an element $y \in Ext(g \star f)$ such that $x \prec^{\tau}_{f} y$, and we have then $x \prec^{\tau}_{g\star f} y$, contradicting the choice of x. To prove the first inclusion, let x be an element of $((Ext(g \star f))_f)_{\tau(g)}$. If x were not $\prec^{\tau}_{g\star f}$-maximal in $Ext(g \star f)$, there would exist an element y in $Ext(g \star f)$ such that $x \prec^{\tau}_{g\star f} y$. Since $x \in (Ext(g \star f))_{\tau(f)}$, we cannot have $x \prec^{\tau}_{f} y$, and it follows that we must have $x \prec^{\tau}_{g} y$. But, from the second embedding, we have $y \in (Ext(g \star f))_{\tau(f)}$, and since x was chosen in $((Ext(g \star f))_{\tau(f)})_{\tau(g)}$, the latter inequality is impossible. □

In the particular case where the salience order on χ_f is *total*, the above proposition takes a simpler form:

Proposition 9. *Let f be g-determinable elementary concept with totally ordered associated characteristic set. Then* $Typ(g \star f) = ((Ext(g \star f))_{\tau(f)})_{\tau(g)}$.

Proof. We only have to check that $Typ(g \star f)$ is a subset of $((Ext(g \star f))_{\tau(f)})_{\tau(g)}$. Let therefore x be an $g \star f$-typical element. By the above proposition, we know that $x \in (Ext(g \star f))_{\tau(f)}$. Suppose x where not $\prec^{\tau}_{g}$-maximal in this set. Then we would have $x \prec^{\tau}_{g} z$ for some element z of $(Ext(g \star f))_{\tau(f)}$. But since the salience order on χ_f is total, the $\prec^{\tau}_{f}$-maximality of z implies $x \preceq^{\tau}_{f} z$ by Proposition 5. We have then $x \prec^{\tau}_{g\star f} z$, which contradicts the choice of x. □

In this particular case, the proposition shows that, in order to get the typical elements associated with $g \star f$, one has to simply choose among the most f-typical exemplars of $Ext\, g \star f$ those that are also the most g-typical.

Note that the typical representatives of the concept $g \star f$ are generally not chosen among the elements of $Typ\, f$. Proposition 9 shows that the definition of $\preceq^{\tau}_{g\star f}$ is coherent with the intuitive definition of $Typ(g \star f)$, which has to include $((Ext\, g \cap Ext\, f)_f)_g$ as a subset.

Example 6. Let us take again Example 4 and take for f the concept *to-be-a-tree* and for g the concept *to-be-a-conifer*. By what we saw, the larch may be considered as f-typical, so it is an element of $(Ext(g \star f))_{\tau(f)}$. It is also $\prec_g$-maximal in this set, so the larch may be considered as $g \star f$-typical. It cannot be considered as $f \star g$-typical, though, because it is not an element of $(Ext(g \star f))_{\tau(g)}$: the pine tree, for example, is more g-typical than the larch.

Proposition 10. *Let f be a g-determinable concept and denote by $\widetilde{(\prec_f^\tau)}$ the restriction of $\prec_f^\tau$ to the set $Ext(g \star f)$. Then*

- $\preceq_{g\star f}^\tau \subseteq \preceq_f^\tau$;
- $\widetilde{(\prec_f^\tau)} \subseteq \prec_{g\star f}^\tau$.

Proof. The embedding $\preceq_{g\star f}^\tau \subseteq \preceq_f^\tau$ directly follows from the definition of $\preceq_{g\star f}^\tau$. For the second part of the proposition, suppose that x and y are elements of $Ext\, g \cap Ext\, f$ with $x \prec_f^\tau y$. We have then $x \preceq_{g\star f}^\tau y$. If we had $y \preceq_{g\star f}^\tau x$, this would imply $y \preceq_f^\tau x$ by the first part of the proposition, contradicting the choice of x and y. This shows that $x \prec_{g\star f}^\tau y$ as desired. □

Remark 4. If x is at most as typical as y relative to the concept $g \star f$, x will also be at most as typical as y relative to the concept f. This conclusion does not hold for arbitrary subconcepts of f: the larch is less typical as a conifer than the pine tree, while more typical as a tree.

Proposition 10 enables us to compare the typicality orders induced by f and $g \star f$; it cannot be used however to compare the resulting *degrees* (see Remark 3). On one hand, we may have $\delta_f^\tau(x) > \delta_{g\star f}^\tau(x)$, as happens for instance when x is f-typical and falls under g without being $\prec_g$-maximal in $(Ext(g \star f))_f$; on the other hand, we may also have $\delta_f^\tau(x) < \delta_{g\star f}^\tau(x)$, as is for the case when x is $g \star f$-typical but not f-typical (e.g. a striped apple). Thus our representation of typicality orders neither predicts, nor contradicts the well-known *conjunction effect*, [17] pp. 198–199, following which an object that falls under a compound concept $g \star f$ will be often considered a 'better example' of the category $g \star f$ than of the category f alone.

For an illustration of the conjunction effect, let us study the example of the *striped apple* that was discussed by [16]:

Example 7. Let f be the concept *to-be-an-apple*, and g that of *to-have-stripes*. We suppose that N_f^τ, the *typicality width* of f (see Remark 3) is equal to 5. We may then attribute the value 6 to $N_{g\star f}^\tau$. We take for x a particular kind of apple that we suppose sour, without a stem, and with regular stripes on its surface. We let x_1 be a second apple, similar to x except that it is not sour, and x_2 be a third apple similar to x_1 except that it has a stem. Finally, we denote by x_3 a typical apple. Note that x_2 is $g \star f$-typical but is not f-typical. We have now $x \prec_{g\star f} x_1 \prec_{g\star f} x_2$, showing that the $(g \star f)$ typicality distance of x is equal to 2, and its corresponding degree to $2/3$. On the other hand, we have $x \prec_f x_1 \prec_f x_2 \prec_f x_3$: indeed, x_2 cannot be considered a typical apple because typical apples are not striped. It follows that the f-typicality distance of x is equal to 3 and its corresponding degree to $2/5$. We have therefore $\delta_{g\star f}^\tau(x) > \delta_f^\tau(x)$: the degree of x relative to the compound concept $g \star f$ is strictly greater than its degree relative to f.

As a first property of the determination connective, we observe it satisfies idempotence. It also satisfies associativity, provided the concepts $h \star (g \star f)$ and $(h \star g) \star f$ are defined:

Proposition 11. *The following equivalences hold*:

- $f \star f \equiv f$;
- $h \star (g \star f) \equiv (h \star g) \star f$.

Proof. We have readily $Ext(f \star f) = Ext\, f$ and $Typ(f \star f) = Typ\, f$, which proves the first part of the proposition. To prove the second equivalence, we have to check that $Typ(h \star (g \star f)) = Typ((h \star g) \star f)$. Let therefore x be a $\prec_{h\star(g\star f)}^\tau$-maximal element of $Ext\, h \cap Ext\, g \cap Ext\, f$. If x were not $(h \star g) \star f$-maximal, there would exist y such that $x \prec_{(h\star g)\star f}^\tau y$, that is $x \preceq_f^\tau y$ and either $x \prec_f^\tau y$ or $x \prec_{h\star g}^\tau y$. In the first case, we get $x \prec_{g\star f}^\tau y$, and therefore $x \prec_{h\star(g\star f)}^\tau y$, contradicting the choice of x. In the second case, we get $x \preceq_f^\tau y$, $x \preceq_g^\tau y$ and either $x \prec_g^\tau y$, or $x \prec_h^\tau y$. But the inequalities $x \preceq_f^\tau y$ and $x \prec_g^\tau y$ yield $x \prec_{g\star f}^\tau y$, that is $x \prec_{h\star(g\star f)}^\tau y$, and the same conclusion is obtained if we suppose that $x \preceq_f^\tau y$, $x \preceq_g^\tau y$ and $x \prec_h^\tau y$. In any case, this contradicts the choice of x.

Conversely, suppose that x is an element of *Typ*$((h \star g) \star f)$. If x were not $\prec^{\tau}_{h\star(g\star f)}$-maximal, there would exist an element y such that $x \prec^{\tau}_{h\star(g\star f)} y$. We would then have $x \preceq^{\tau}_{g\star f} y$ and either $x \prec^{\tau}_{g\star f} y$ or $x \prec^{\tau}_{h} y$. In the first case, we would have $x \preceq^{\tau}_{f} y$ and either $x \prec^{\tau}_{f} y$ or $x \prec^{\tau}_{g} y$, both leading to $x \prec^{\tau}_{(h\star g)\star f} y$. In the second case we have $x \preceq^{\tau}_{g\star f} y$ and $x \prec^{\tau}_{h} y$, which amounts to $x \prec^{\tau}_{h} y$, $x \preceq^{\tau}_{f} y$ and either $x \prec^{\tau}_{f} y$ or $x \preceq^{\tau}_{g} y$. If $x \prec^{\tau}_{f} y$, we get directly $x \prec^{\tau}_{(h\star g)\star f} y$. If $x \preceq^{\tau}_{g} y$ and $x \prec^{\tau}_{h} y$, we get $x \prec^{\tau}_{h\star g} y$, which, together with $x \preceq^{\tau}_{f} y$, leads again to $x \prec^{\tau}_{(h\star g)\star f} y$, a contradiction. □

We shall now establish a necessary and sufficient condition for $g \star f$ to be a *smooth* subconcept of f. Recall that h is a smooth subconcept of f if *Typ* $h \subseteq$ *Typ* f. We first make a definition:

Definition 10. A concept g is *exceptional* for f if *Typ* $f \cap$ *Ext* $g = \emptyset$.

A concept g that is exceptional for f applies to no typical instances of f. For instance, the concept *to-be-poisonous* is exceptional for the concept *to-be-a-fruit*: indeed, no typical fruit is poisonous.

Proposition 12. *Let f be an elementary g-determinable concept. Then the following conditions are equivalent*:

1. *g is not exceptional for f*;
2. *Typ*$(g \star f) \subseteq$ *Typ* f;
3. *Typ*$(g \star f) = ($*Typ* $f \cap$ *Ext* $g)_{\tau(g)}$.

Proof. Suppose first that g is not exceptional for f, and let x be an element of *Typ*$(g \star f)$. Denote by z an element of *Typ* $f \cap$ *Ext* g. If x were not an element of *Typ* f, we would have $x \prec_f z$ (Lemma 6); since $z \in$ *Ext*$(g \star f)$, this would contradict the $\prec_f$-maximality of x in this set. This shows that $1 \Rightarrow 2$. Conversely, suppose that we have *Typ*$(g \star f) \subseteq$ *Typ* f and let z be an element of *Typ*$(g \star f)$. Then z is an element of *Ext* $g \cap$ *Ext* f and therefore of *Ext* g. We have thus $z \in$ *Typ* $f \cap$ *Ext* g, showing that this set is not empty, and hence that g is not exceptional for f. We have therefore $1 \Leftrightarrow 2$ as desired.

Supposing now *Typ*$(g \star f) \subseteq ($*Typ* $f)$, let us show that *Typ*$(g \star f) = ($*Typ* $f \cap$ *Ext* $g)_{\tau(g)}$. Let x be an element of *Typ*$(g \star f)$. If x were not $\prec^{\tau}_{g}$-maximal in *Typ* $f \cap$ *Ext* g, there would exist $y \in ($*Typ* $f \cap$ *Ext* $g)$ with $x \prec^{\tau}_{g} y$. Since $x \preceq^{\tau}_{f} y$, this would yield $x \prec^{\tau}_{g\star f} y$, contradicting the choice of x. We have therefore proven that *Typ*$(g \star f) \subseteq ($*Typ* $f \cap$ *Ext* $g)_{\tau(g)}$. To prove the converse inclusion, take an element x of $($*Typ* $f \cap$ *Ext* $g)_{\tau(g)}$. If x were not $\prec^{\tau}_{g\star f}$-maximal, we would have $x \prec^{\tau}_{g\star f} y$ for some element $y \in$ *Ext* $g \cap$ *Ext* f. By the $\prec^{\tau}_{f}$-maximality of x, this would imply $y \in$ *Typ* f and $x \prec^{\tau}_{g} y$, thus contradicting the choice of x. □

Example 8. Any typical exemplar of the concept *to-be-a-black-bird* is also a typical exemplar of the concept *to-be-a-bird*, because there exist typical birds that are black. The conclusion would be of course different if we were to look for typical *pink* birds: the American flamingo, for instance, can be considered as a typical pink bird, but not as a typical bird.

Corollary 3. *Typ*$(g \star f) = ($*Typ* $f)_{\tau(g)}$ *for all* $g \in$ *Int* f.

Proof. Since $g \in$ *Int* f, we have *Typ* $f \subseteq$ *Ext* g, and the result immediately follows from Proposition 12. □

The corollary means that, g being a characteristic feature of f, the typical exemplars of $g \star f$ will be the most g-typical among the typical exemplars of f.

We finally mention as a particular case the only example where the set *Typ*$(g \star f)$ can be directly recovered from the sets *Typ* f and *Typ* g.

Proposition 13. *Typ*$(g \star f) =$ *Typ* $g \cap$ *Typ* f *if and only if* *Typ* $f \cap$ *Typ* $g \neq \emptyset$.

For instance, a typical black olive is nothing but a typical olive that is typically black.

Proof. Straightforward. □

The above results now provide an easy characterization of the smooth subconcepts of a given concept. Recall that a subconcept h of f is smooth if $\mathit{Typ}\, h \subseteq \mathit{Typ}\, f$.

Theorem 1. *A concept h is a smooth subconcept of f if and only if there exists a concept g, not exceptional for f, such that $h \equiv g \star f$.*

Proof. We already notice that $(g \star f)$ is a subconcept of f. If g is not exceptional for f, we have $\mathit{Typ}(g \star f) \subseteq \mathit{Typ}\, f$ by Proposition 12, and it follows that $g \star f$ is a smooth subconcept of f. Conversely, let h be a smooth subconcept of f. Then f is h-determinable. Note that h is not exceptional for f, since f applies to any typical instance of h. We claim that $h \equiv h \star f$. Indeed, observe first that $\mathit{Ext}(h \star f) = \mathit{Ext}\, h \cap \mathit{Ext}\, f = \mathit{Ext}\, h$, since $\mathit{Ext}\, h \subseteq \mathit{Ext}\, f$. Next, since h is smooth, and by Proposition 13, we have $\mathit{Typ}(h \star f) = \mathit{Typ}\, h \cap \mathit{Typ}\, f = \mathit{Typ}\, h$. This shows that the concepts h and $h \star f$ have same intention and same extension, and they are therefore equivalent. □

6.3. Concepts and formal concept analysis

The determination connective $\star$ provides interesting results concerning the structure of the lattice of formal subconcepts. Let us first recall the basic definitions of Formal Concept Analysis [7].

A *formal context* is a triple (G, M, I) where G is a set of objects, M a set of attributes and I a binary relation between these two sets: the property $(g, m) \in I$ is to be read as "the object g has the attribute m". A *formal concept* of the formal context (G, M, I) is then defined as a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ such that

- $B = \{m \in M \mid (g, m) \in I\ \forall g \in A\}$;
- $A = \{g \in G \mid (g, m) \in I\ \forall m \in B\}$.

The set B is therefore the set of all attributes of M that are shared by all objects of A; similarly, A is the set of all objects that have in common all the attributes of B. In the terminology of FCA, A is called the *extent* of the formal concept (A, B), and B is its *intent*.

If (A, B) and (A', B') are formal concepts, one has clearly $A \subseteq A'$ iff $B' \subseteq B$. When this is the case, (A, B) is called a *formal subconcept* of (A', B'), and (A', B') a *formal superconcept* of (A, B). This will be denoted by $(A, B) \leqslant (A', B')$. The set $\mathcal{L}(G, M, I)$ of formal concepts of a given formal context (G, M, I) is then partially ordered through the relation $\leqslant$. An important result in FCA is that this ordered set $(\mathcal{L}(G, M, I), \leqslant)$ has the structure of a *complete lattice*.

In our framework, the formal context we are working in is the triple $(\mathcal{O}, \mathcal{F}, I)$, where $\mathcal{O}$ is the set of objects that form the universe of discourse of a given agent, $\mathcal{F}$ his set of concepts and I the relation $(x, h) \in I$ iff x *falls under* h. A *formal concept* then consists of a couple (A, B) where A is a set of objects, B a set of (individual) concept such that

- $f \in B$ if and only if f applies to all elements of A;
- $x \in A$ if and only if x falls under all concepts of B.

In particular, Proposition 7 shows that for any elementary concept f, $(\mathit{Typ}\, f, \mathit{Int}\, f)$ is a formal concept.

We can also define another class of formal concepts, using the notion of extension: let indeed $\mathit{Ess}\, f$, the *essence* of f, be defined as the set of all concepts g that apply to the elements of $\mathit{Ext}\, f$. Then, it appears that $(\mathit{Ext}\, f, \mathit{Ess}\, g)$ is a formal concept:

Proposition 14. *For each concept f, denote by $\mathit{Ess}\, f$ the set of concepts g such that $\mathit{Ext}\, f \subseteq \mathit{Ext}\, g$. Then one has $\mathit{Ext}\, f = \bigcap_{g \in \mathit{Ess}\, g} \mathit{Ext}\, g$.*

Proof. If x is an element of $\mathit{Ext}\, f$, x falls under every concept of $\mathit{Ess}\, f$ by definition of $\mathit{Ess}\, f$. Conversely, if x falls under every element of $\mathit{Ess}\, f$, x must fall under f, since $f \in \mathit{Ess}\, f$.

Elements of $Ess\, f$ may be seen as *essential* in the sense that they necessarily apply to all objects that fall under f. It does not mean however that $Ess\, f$ should be identified with the set of *core* properties of f, which determine the meaning of f: to take Fodor's example [4], each of the concepts *to-have-a-backbone* and *to-have-a-heart* is part of the essence of the other, but none can be considered as a core feature of the other.

The above result shows that for all concepts f $(Ext\, f, Ess\, f)$ is a formal concept. This enables us to reinterpret the notion of smooth subconcepts in the framework of Formal Concept Analysis:

Proposition 15. *Given two elementary concepts f and h, h is a smooth subconcept of f if and only if $(Ext\, h, Ess\, h) \leqslant (Ext\, f, Ess\, f)$ and $(Typ\, h, Int\, h) \leqslant (Typ\, f, Int\, f)$.*

Proof. Immediate. □

Let us write $h \preceq f$ whenever h is a smooth subconcept of f. This clearly yields an order relation in the quotient set $\mathcal{F}/\equiv$, because one has $h \equiv f$ whenever $h \preceq f$ and $f \preceq h$. We do not obtain a lattice structure, as was the case in Formal Concept Analysis for the relation $\leqslant$, but it is possible to find a greatest lower bound for $\star$-commuting concepts, and this greatest lower bound turns out to be their mutual determination.

Theorem 2. *Let f and g be mutually determinable concepts. Then f and g admit common lower bounds if and only if $Typ\, f \cap Typ\, g \neq \emptyset$. When this condition is satisfied, they admit a greatest lower bound, which, up to equivalence, is equal to the determination of f by g.*

Proof. If f and g admit a common lower bound, there exists a concept k such that $k \preceq f$ and $k \preceq g$. By definition, we have then $Typ\, k \subseteq Typ\, f \cap Typ\, g$, and this latter set is therefore non-empty. Conversely, suppose that we have $Typ\, f \cap Typ\, g \neq \emptyset$. Then it follows from Proposition 13 that $Typ(g \star f) = Typ\, g \cap Typ\, f$. As we also have $Ext(g \star f) = Ext\, g \cap Ext\, f$, we see that $g \star f \preceq f$ and $g \star f \preceq g$. The determination of f by g is therefore a lower bound of the set $\{f, g\}$. Note that we have $g \star f \equiv f \star g$. Suppose now that, k is a smooth subconcept of f and of g, that is $k \preceq f$ and $k \preceq g$. We have then $Exp\, k \subseteq (Exp\, f \cap Exp\, g)$ and $(Typ\, k \subseteq Typ\, f \cap Typ\, g)$. This shows that $k \preceq g \star f$. We have therefore proven that, up to equivalence, $g \star f$ is the greatest lower bound of the set $\{f, g\}$. □

7. Conclusion and future work

The present work is a first an attempt at setting a suitable framework for the study of categorization and typicality problems. It is essentially centered on the basic notions of membership and typicality, these notions being defined through the defining feature set and the characteristic set attached to a concept. We obtained in this way a coherent theory for elementary and compound concepts. This preliminary work will be completed in a forthcoming paper, in which we shall show how the study of non-monotonic logics applies to categorization and prototype theory, providing interesting and non-trivial results. We shall indeed reinterpret the notions of membership and typicality in the framework of *inference relations*: thus, the set $Ess\, f$ will be considered as a set of *consequences* of f, and the relation $g \in Ess\, f$, denoted $f \vdash g$, will play the role of a consequence relation. Similarly, typical inference will be denoted by $f \mid\!\sim g$, and will be studied as a non-monotonic inference relation. We shall show that, relative to the determination connective $\star$, the relation $\vdash$ behaves like the classical monotonic consequence relation, while $\mid\!\sim$ satisfies properties analogous to those of *rational* inference relations. The tools recently developed in the study of preferential and rational inference relations will therefore appear as a useful complement for the study of classical problems in categorization theory, leading in particular to interesting results concerning the theory of contextual inference.

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