ELSEVIER

Artificial Intelligence 95 (1997) 115–154

Artificial Intelligence

Autoepistemic logic of knowledge and beliefs [1]

Teodor C. Przymusinski *

Department of Computer Science, University of California, Riverside, CA 92521, USA

Received November 1996; revised May 1997

Abstract

In recent years, various formalizations of nonmonotonic reasoning and different semantics for normal and disjunctive logic programs have been proposed, including autoepistemic logic, circumscription, *CWA*, *GCWA*, *ECWA*, epistemic specifications, stable, well-founded, stationary and static semantics of normal and disjunctive logic programs.

In this paper we introduce a simple nonmonotonic knowledge representation framework which isomorphically contains all of the above-mentioned nonmonotonic formalisms and semantics as special cases and yet is significantly more expressive than each one of these formalisms considered individually. The new formalism, called the *Autoepistemic Logic of Knowledge and Beliefs*, *AELB*, is obtained by augmenting Moore's autoepistemic logic, *AEL*, already employing the *knowledge operator*, $\mathcal{L}$, with an additional *belief operator*, $\mathcal{B}$. As a result, we are able to reason not only about formulae F which are *known* to be true (i.e., those for which $\mathcal{L}F$ holds) but also about those which are only *believed* to be true (i.e., those for which $\mathcal{B}F$ holds).

The proposed logic constitutes a powerful new formalism which can serve as a *unifying framework* for several major nonmonotonic formalisms. It allows us to better understand mutual relationships existing between different formalisms and semantics and enables us to provide them with simpler and more natural definitions. It also naturally leads to new, even more expressive, flexible and modular formalizations and semantics. © 1997 Published by Elsevier Science B.V.

Keywords: Nonmonotonic reasoning; Logics of knowledge and beliefs; Semantics of logic programs and deductive databases

* Email: teodor@cs.ucr.edu.

[1] The extended abstract of this paper appeared in: *Proceedings of the Twelfth National Conference on Artificial Intelligence (AAAI-94)*, Seattle, WA (1994), pp. 952–959. Partially supported by the National Science Foundation grant #IRI-9313061.

PII S0004-3702(97)00032-5

1. Introduction

Moore's autoepistemic logic, *AEL* [20], is obtained by augmenting classical propositional logic with a modal operator $\mathcal{L}$. The intended meaning of the modal atom $\mathcal{L}F$ in a stable autoepistemic expansion is "F is provable" or "F is logically derivable". Thus Moore's modal operator $\mathcal{L}$ can be viewed as a "knowledge operator" which allows us to reason about formulae *known* to be true in some stable expansion.[2] However, often times in addition to reasoning about statements which are known to be true we also need to reason about those statements that are only *believed* to be true, where what is believed or not believed is determined by some specific *nonmonotonic formalism*.

For example, consider a scenario in which:

- You rent a movie if you believe that you will neither go to a baseball game nor to a football game.
- You do not buy tickets to a game if you don't know that you will go to watch it.

We could describe this scenario as follows:[3]

$$\mathcal{B}\neg baseball \wedge \mathcal{B}\neg football \supset rent\_movie$$

$$\neg\mathcal{L}baseball \wedge \neg\mathcal{L}football \supset dont\_buy\_tickets.$$

Assuming that initially this is all you know and that your beliefs are based on minimal entailment (circumscription), you will likely rent a movie because you believe that you will not go to watch any games (i.e., both $\neg baseball$ and $\neg football$ hold in all minimal models) and you will not buy tickets because you don't know that you will go to any of the games (i.e., neither *baseball* nor *football* is provable).

Suppose now that you learn that you will either go to a baseball game or to a football game:

$$baseball \vee football.$$

In the new scenario you will no longer rent a movie (because $\neg baseball \wedge \neg football$ no longer holds in all minimal models) but you will not buy any tickets either because you don't know yet which game you are going to watch (i.e., neither *baseball* nor *football* is provable).

Finally, suppose that you eventually learn that you in fact go to a baseball game:

$$baseball.$$

Clearly, you no longer believe in not buying tickets because you now know that you are going to watch a specific game (i.e., *baseball* is provable).

Observe, that in the above example the roles played by the knowledge and belief operators are quite different and one cannot be substituted by the other. In particular, we cannot replace the premise $\mathcal{B}\neg baseball \wedge \mathcal{B}\neg football$ in the first implication by $\mathcal{L}\neg baseball \wedge \mathcal{L}\neg football$ because that would result in *rent_movie* not being true in first

[2] Moore's modal operator $\mathcal{L}$ is usually referred to as a "belief operator". In Section 2.1 we explain why we prefer to view it as a "knowledge operator".

[3] The second clause could be equivalently written as: $buy\_tickets \supset \mathcal{L}baseball \vee \mathcal{L}football$.

scenario.[4] Similarly, we cannot replace it by $\neg\mathcal{L}baseball \wedge \neg\mathcal{L}football$ because that would result in *rent_movie* being true in the second scenario.[5]

In order to be able to explicitly reason about beliefs, we introduce a new nonmonotonic formalism, called the *Autoepistemic Logic of Knowledge and Beliefs*, *AELB*, obtained by augmenting Moore's autoepistemic logic, *AEL*, already employing the *knowledge operator*, $\mathcal{L}$, with the additional *belief operator*, $\mathcal{B}$. As a result, we will be able to reason not only about formulae F which are *known* to be true (i.e., those for which $\mathcal{L}F$ holds) but also about those which are only *believed* to be true (i.e., those for which $\mathcal{B}F$ holds).

The resulting nonmonotonic knowledge representation framework turns out to be rather simple and yet quite powerful. We prove that several of the major nonmonotonic formalisms and semantics for normal and disjunctive logic programs are *isomorphically embeddable* into *AELB*. In particular this applies to autoepistemic logic [20]; propositional circumscription [15,18]; *CWA* [31]; *GCWA* [19]; *ECWA* [13]; epistemic specifications [10]; stable, well-founded, stationary and static semantics of normal and disjunctive logic programs [11,12,26,29,34]. At the same time the Autoepistemic Logic of Knowledge and Beliefs, *AELB*, is significantly more expressive and flexible than each one of these formalisms considered individually.

The proposed logic constitutes a powerful new formalism which can serve as a *unifying framework* for several major nonmonotonic formalisms. It allows us to better understand mutual relationships existing between different formalisms and semantics and enables us to provide them with simpler and more natural definitions. It also naturally leads to new, even more expressive and modular formalizations and semantics.

The paper is organized as follows:

- In Section 2 we introduce the *Autoepistemic Logic of Knowledge and Beliefs*, *AELB*, and establish its basic properties. We also show that both Moore's Autoepistemic Logic, *AEL*, and McCarthy's Circumscription are embeddable into *AELB*.
- Section 3 is devoted to a detailed study of the *Autoepistemic Logic of Beliefs*, *AEB*, a sub-logic of *AELB* whose theories are limited to formulae which do not use the knowledge operator $\mathcal{L}$. It turns out that the logic *AEB* has some very nice and regular properties. In particular, any such theory has the least static expansion which can be iteratively constructed as the fixed point of a natural minimal model operator.
- In Section 4 we demonstrate that normal and disjunctive logic programs, under all major semantics, can be equivalently translated into theories in the logic *AELB*. This allows us to better understand the meaning of different semantics and their mutual relations.
- In Section 5 we show that also Gelfond's epistemic specifications can be easily represented as special knowledge and belief theories in the logic *AELB*. As a byproduct we establish a simple duality relationship between the two operators appearing in Gelfond's logic.

---

[4] Because neither $\neg baseball$ nor $\neg football$ is provable.

[5] Because neither *baseball* nor *football* is provable.

- In Section 6 we illustrate how the meaning of theories in *AELB* can be adjusted by suitably changing the underlying nonmonotonic formalism on which the notion of belief is based. In previous sections we demonstrated that the semantics of *AELB* can be also changed by adding suitable axioms to the logic.
- Section 7 contains concluding remarks and a brief discussion of other applications of the logic *AELB* and of its relationship to other proposed logics.

2. Autoepistemic Logic of Knowledge and Beliefs

The language of the *Autoepistemic Logic of Knowledge and Beliefs, AELB*, is a propositional modal language, $\mathcal{K}_{L,B}$, with standard connectives $(\vee, \wedge, \supset, \neg)$, the propositional letter $\perp$ (denoting *false*) and two modal operators $\mathcal{L}$ and $\mathcal{B}$, called *knowledge* and *belief* operators, respectively. The atomic formulae of the form $\mathcal{L}F$ (respectively, $\mathcal{B}F$), where F is an arbitrary formula of $\mathcal{K}_{\mathcal{L},\mathcal{B}}$, are called *knowledge atoms* (respectively, *belief atoms*). The intended meaning of $\mathcal{L}F$ is "*F is known*" while the intended meaning of $\mathcal{B}F$ is "*F is believed*" (we make it precise later on in this section). Knowledge and belief atoms are jointly referred to as *introspective atoms*. Observe that arbitrary *nestings* of introspective atoms are allowed.

The formulae of $\mathcal{K}_{\mathcal{L},\mathcal{B}}$ in which neither $\mathcal{L}$ nor $\mathcal{B}$ occurs are called *objective* and the set of all such formulae is denoted by $\mathcal{K}$. Similarly, the set of all formulae of $\mathcal{K}_{\mathcal{L},\mathcal{B}}$ in which only $\mathcal{L}$ (respectively, only $\mathcal{B}$) occurs is denoted by $\mathcal{K}_{\mathcal{L}}$ (respectively, $\mathcal{K}_{\mathcal{B}}$). Any theory T in the language $\mathcal{K}_{\mathcal{L},\mathcal{B}}$ will be called an *autoepistemic theory of knowledge and beliefs*, or, briefly, a *knowledge and belief theory*.

Definition 2.1. (*Knowledge and belief theories*) By an *autoepistemic theory of knowledge and beliefs* or just a *knowledge and belief theory* we mean an arbitrary theory in the language $\mathcal{K}_{\mathcal{L},\mathcal{B}}$, i.e., a (possibly infinite) set of arbitrary clauses of the form

$$\begin{aligned}&B_1 \wedge \cdots \wedge B_m \wedge \mathcal{B}G_1 \wedge \cdots \wedge \mathcal{B}G_k \wedge \mathcal{L}H_1 \wedge \cdots \wedge \mathcal{L}H_s \supset \\ &\quad A_1 \vee \cdots \vee A_l \vee \mathcal{B}F_1 \vee \cdots \vee \mathcal{B}F_n \vee \mathcal{L}K_1 \vee \cdots \vee \mathcal{L}K_t\end{aligned}$$

where $m, n, k, l, s, t \geqslant 0$, the A_is and B_is are objective atoms and the F_is, G_is, H_is and K_is are arbitrary formulae of $\mathcal{K}_{\mathcal{L},\mathcal{B}}$.

Equivalently, a knowledge and belief theory consists of a set of arbitrary clauses of the form

$$\begin{aligned}&B_1 \wedge \cdots \wedge B_m \wedge \mathcal{B}G_1 \wedge \cdots \wedge \mathcal{B}G_k \wedge \mathcal{L}H_1 \wedge \cdots \wedge \mathcal{L}H_s \wedge \\ &\quad \neg\mathcal{B}F_1 \wedge \cdots \wedge \neg\mathcal{B}F_n \wedge \neg\mathcal{L}K_1 \wedge \cdots \wedge \neg\mathcal{L}K_t \supset A_1 \vee \cdots \vee A_l\end{aligned}$$

which say that if the B_is are true, the G_is are believed, the H_is are known, the F_is are not believed and the K_is are not known then one of the A_is is true.

By an *affirmative* knowledge and belief theory we mean any such theory all of whose clauses satisfy the condition that $l > 0$.

By a *rational* knowledge and belief theory we mean any such theory all of whose clauses satisfy the condition that $n = 0$.

In other words, *affirmative* knowledge and belief theories are precisely those theories that satisfy the condition that all of their clauses contain at least one positive *objective* atom. On the other hand, *rational* knowledge and belief theories are precisely those theories none of whose clauses contain any positive belief atoms $\mathcal{B}F_i$.

We assume the following two simple axiom schemata and one inference rule describing the arguably obvious properties of belief atoms:

(D) *Consistency Axiom*:

$$\neg\mathcal{B}\bot \tag{1}$$

(K) *Normality Axiom*: For any formulae F and G of the language $\mathcal{K}_{\mathcal{L},B}$:

$$\mathcal{B}(F \supset G) \quad \supset \quad (\mathcal{B}F \supset \mathcal{B}G) \tag{2}$$

(N) *Necessitation Inference Rule*: For any formula F of the language $\mathcal{K}_{\mathcal{L},B}$:

$$\frac{F}{\mathcal{B}F}. \tag{3}$$

The Consistency Axiom (D) states that falsity $\bot$ is *not* believed. The Normality Axiom (K) states that if we believe that F implies G and if we believe in F then we should believe in G as well. The Necessitation Rule (N) says that anything that is provable in the logic *AELB* is necessarily also believed.

Remark 2.2. Strictly speaking, the necessitation inference rule (N) is not needed because it is *automatically* satisfied in all "stable" theories (more precisely, in all static autoepistemic expansions that are defined in Section 2.2). Moreover, one can show that its omission results in the same "stable" theories.

Analogous axioms (D) and (K) and rule (N) could be as well assumed about the knowledge operator $\mathcal{L}$ but they are also *automatically* satisfied in all "stable" theories and thus can be safely omitted.

A modal logic is *normal* [22] if it includes the normality axiom (K) and is closed under the necessitation rule (N). In view of the above comments, one could equivalently define the underlying logic of *AELB* as a normal modal logic with two modal operators satisfying the ("no dead ends") Consistency Axiom (D). In the next subsection, we define static expansions of theories in this logic which provide a suitable meaning to the knowledge and belief atoms.

Definition 2.3. (*Formulae derivable from a knowledge and belief theory*). For any knowledge and belief theory T, we denote by $Cn_*(T)$ the smallest set of formulae of the language $\mathcal{K}_{\mathcal{L},B}$ which contains the theory T, all (substitution instances of) the axioms (D) and (K) and is closed under the Necessitation Rule (N) and under standard propositional consequence.

We say that a formula F is *derivable* from a knowledge and belief theory T in the logic *AELB* if F belongs to $Cn_*(T)$. We denote this fact by $T \vdash_* F$. Consequently, $Cn_*(T) = \{F \mid T \vdash_* F\}$.

We call a knowledge and belief theory T *consistent* if the theory $Cn_*(T)$ is consistent. Clearly, T is consistent if and only if $T \nvdash_* \bot$.

The following proposition will be frequently used in the sequel.

Proposition 2.4. *The following sentences and derivation rules are valid in the logic AELB for any formulae F and G of the language $\mathcal{K}_{\mathcal{L},\mathcal{B}}$:*

$$\mathcal{B}(F \wedge G) \equiv \mathcal{B}F \wedge \mathcal{B}G \tag{4}$$

$$\mathcal{B}F \supset \neg\mathcal{B}\neg F \tag{5}$$

$$\mathcal{B}(F \vee G) \supset \neg\mathcal{B}\neg F \vee \neg\mathcal{B}\neg G \tag{6}$$

$$\frac{F \equiv G}{\mathcal{B}F \equiv \mathcal{B}G} \tag{7}$$

The first sentence states that beliefs are distributive with respect to conjunction, i.e., that a conjunction of two formulae is believed if and only if each one of them is believed. The second sentence says that if a formula is believed then its negation is not believed. The last sentence says that if we believe in a disjunction of formulae $F \vee G$ then we either disbelieve $\neg F$ or we disbelieve $\neg G$.

The inference rule states that if two formulae are known to be equivalent then so are their belief atoms In other words, the meaning of $\mathcal{B}F$ does not depend on the specific form of the formula F, e.g., the formula $\mathcal{B}(F \wedge \neg F)$ is equivalent to $\mathcal{B}(\bot)$ and thus it is false by (D).

Proof. *Proof of* (4). We first show that

$$\vdash_* \mathcal{B}(F \wedge G) \supset \mathcal{B}F \wedge \mathcal{B}G$$

Clearly, $\vdash_* (F \wedge G) \supset F$ and therefore, by the Necessitation Rule (N), we have $\vdash_* \mathcal{B}((F \wedge G) \supset F)$. From the Normality Axiom (K) we infer that $\vdash_* \mathcal{B}(F \wedge G) \supset \mathcal{B}F$. Similarly, $\vdash_* \mathcal{B}(F \wedge G) \supset \mathcal{B}G$. It follows that $\vdash_* \mathcal{B}(F \wedge G) \supset \mathcal{B}F \wedge \mathcal{B}G$.

We now show that

$$\vdash_* \mathcal{B}F \wedge \mathcal{B}G \supset \mathcal{B}(F \wedge G).$$

Clearly, $\vdash_* F \supset (G \supset F \wedge G)$ and therefore, by the Necessitation Rule (N), we have $\vdash_* \mathcal{B}(F \supset (G \supset F \wedge G))$. From the Normality Axiom (K) we infer that $\vdash_* \mathcal{B}F \supset \mathcal{B}(G \supset F \wedge G)$. Applying the Normality Axiom (K) again we conclude that $\vdash_* \mathcal{B}F \supset (\mathcal{B}G \supset \mathcal{B}(F \wedge G))$. This shows that $\vdash_* \mathcal{B}F \wedge \mathcal{B}G \supset \mathcal{B}(F \wedge G)$.

Proof of (5). Clearly $\vdash_* (F \wedge \neg F) \supset \bot$. By the Necessitation Rule (N) we get $\vdash_* \mathcal{B}((F \wedge \neg F) \supset \bot)$ and thus by the Normality Axiom (K), $\vdash_* \mathcal{B}(F \wedge \neg F) \supset \mathcal{B}\bot$. Consequently, the Consistency Axiom (D) implies $\vdash_* \neg\mathcal{B}(F \wedge \neg F)$. Using (4) we obtain $\vdash_* \neg\mathcal{B}F \vee \neg\mathcal{B}\neg F$ which is equivalent to $\vdash_* \mathcal{B}F \supset \neg\mathcal{B}\neg F$.

Proof of (6). By (5), $\vdash_* \mathcal{B}(F \vee G) \supset \neg\mathcal{B}(\neg F \wedge \neg G)$. From (4) we conclude that $\vdash_* \mathcal{B}(F \vee G) \supset \neg(\mathcal{B}\neg F \wedge \mathcal{B}\neg G)$. This proves that $\vdash_* \mathcal{B}(F \vee G) \supset \neg\mathcal{B}\neg F \vee \neg\mathcal{B}\neg G$.

Proof of (7). If $\vdash_* F \equiv G$ then, by necessitation, $\vdash_* \mathcal{B}(F \equiv G)$ and thus, by normality, $\vdash_* \mathcal{B}F \equiv \mathcal{B}G$. □

2.1. Intended meaning of knowledge and belief atoms

The modal operator $\mathcal{L}$ of Moore's Autoepistemic Logic, *AEL*, is usually referred to as a "belief operator" [20]. However, when restricted to *stable autoepistemic expansions* (which fully determine the semantics of *AEL*) it in fact plays the role of a "knowledge operator". Indeed, in any stable autoepistemic expansion T we have:

$$T \models \mathcal{L}F \quad \text{iff} \quad T \models F, \tag{8}$$

for any formula F. In other words, $\mathcal{L}F$ holds in the expansion T if and only if "*F is known*" in T, or, more precisely, if F is *logically derivable*[6] *from* T.

As we argued in the Introduction, in addition to reasoning about statements which are known to be true in "stable" expansions of knowledge and belief theories, we also need to reason about those statements that are only *believed* to be true, where what is believed or not believed is determined by some specific *nonmonotonic formalism*. Consequently, we want the new modal "belief operator" $\mathcal{B}$ of *AELB* to satisfy the condition that a formula F is believed in a "stable" expansion T if F is *nonmonotonically derivable* from T:

$$T \models \mathcal{B}F \quad \text{if} \quad T \models_{\text{nm}} F, \tag{9}$$

where $\models_{\text{nm}}$ denotes a specific nonmonotonic inference relation.

In general, different nonmonotonic inference relations, $\models_{\text{nm}}$, can be used, including various forms of predicate and formula *circumscription* [15, 18]. In this paper we select a specific nonmonotonic inference relation, namely a form of Minker's *Generalized Closed World Assumption GCWA* (see [13, 19]) or McCarthy's *predicate circumscription* [18] which says that a formula F is believed to be true if F is true in all minimal models of the theory, i.e., if F is *minimally entailed*. In other words, we require that the belief atoms $\mathcal{B}F$ satisfy the condition:

$$T \models \mathcal{B}F \quad \text{if} \quad T \models_{\min} F, \tag{10}$$

where $\models_{\min}$ is the minimal entailment operator defined below. Accordingly, beliefs considered in this paper are based on the principle of predicate minimization and thus can be called *minimal beliefs*.

We now give a precise definition of minimal models of knowledge and belief theories and the minimal entailment operator, $\models_{\min}$. In the next subsection we define *static* autoepistemic expansions of knowledge and belief theories which precisely enforce the meaning of the belief operator $\mathcal{B}$ discussed above.

[6] Observe, however, that since derivability in a static expansion is based not only on the formulae present in the original theory, i.e., on the initial knowledge, when we refer to the operator $\mathcal{L}F$ as a "knowledge operator" we have in fact in mind a form of *subjective knowledge*. Needless to say, subjective knowledge can also be viewed as a form of belief.

Throughout the paper we represent *models* as (consistent) *sets of literals*. An atom A is *true* in a model M if and only if A belongs to M. An atom A is *false* in a model M if and only if $\neg A$ belongs to M. A model M is *total* if for every atom A either A or $\neg A$ belongs to M. Otherwise, the model is called *partial*. Unless stated otherwise, all models are assumed to be total models. A (total) model M is *smaller* than a (total) model N if it contains fewer positive literals (atoms). For simplicity, when describing models we usually list *only* those of their members that are *relevant* to our considerations, typically those whose predicate symbols appear in the theory that we are currently discussing.

Definition 2.5. (*Minimal models*) By a *minimal model* of a knowledge and belief theory T we mean a model M of T with the property that there is *no* smaller model N of T which coincides with M on introspective atoms $\mathcal{B}F$ and $\mathcal{L}F$. If a formula F is true in all minimal models of T then we write

$$T \models_{\min} F$$

and say that F is *minimally entailed* by T.

For readers familiar with *circumscription*, this means that we are considering predicate circumscription $CIRC(T;\mathcal{K})$ of the theory T in which atoms from the objective language $\mathcal{K}$ are minimized while the introspective atoms $\mathcal{B}F$ and $\mathcal{L}F$ are fixed:[7]

$$T \models_{\min} F \quad \equiv \quad CIRC(T;\mathcal{K}) \models F.$$

In other words, minimal models are obtained by assigning *arbitrary* truth values to the introspective atoms and then *minimizing* objective atoms.

Remark 2.6. The main reason for using the *minimal model entailment* as a basis for beliefs is its simplicity and the fact that it plays a fundamental role in the semantics of logic programs and deductive databases (see [29]). However, various other nonmonotonic formalisms can be used to define the meaning of beliefs. In Section 6 we discuss some possible alternatives.

The reason why we minimize only objective atoms is that the objective atoms A represent *object-level* knowledge which, according to the closed world assumption, has to be minimized in order to arrive at minimal beliefs $\mathcal{B}A$. On the other hand, introspective atoms $\mathcal{B}F$ and $\mathcal{L}F$ intuitively describe *meta-level* knowledge, namely, a plausible rational *scenario*, which is not subject to minimization.

Example 2.7. Consider the following knowledge and belief theory T:

$$Car$$

$$Car \wedge \mathcal{B}\neg Broken \quad \supset \quad ShouldRun$$

Let us prove that T minimally entails $\neg Broken$, i.e., $T \models_{\min} \neg Broken$. Indeed, in order to find minimal models of T we need to assign an *arbitrary* truth value to the only belief

[7] The author is grateful to L. Yuan for pointing out the need to use circumscription that minimizes only objective rather than all propositional atoms [35].

atom $\mathcal{B}\neg Broken$, and then *minimize* the objective atoms *Broken*, *Car* and *ShouldRun*. We easily see that T has the following two minimal models (truth values of the remaining belief atoms are irrelevant and are therefore omitted):

$$M_1 = \{\mathcal{B}\neg Broken, Car, ShouldRun, \neg Broken\},$$
$$M_2 = \{\neg\mathcal{B}\neg Broken, Car, \neg ShouldRun, \neg Broken\}.$$

Since in both of them *Car* is true, and *Broken* is false, we deduce that $T \models_{\min} Car$ and $T \models_{\min} \neg Broken$.

2.2. Static autoepistemic expansions

Like Moore's autoepistemic logic, *AEL*, the autoepistemic logic of knowledge and beliefs, *AELB*, models the set of knowledge and beliefs that an ideally rational and introspective agent may hold given a set of premises T. It does so by defining the so-called *static autoepistemic expansions* T^* of T, which constitute plausible sets of such rational beliefs.

Definition 2.8. (*Static autoepistemic expansion*) A theory T^* is called a *static autoepistemic expansion* of a knowledge and belief theory T if it satisfies the following fixed-point equation:

$$T^* = Cn_*(T \cup \{\mathcal{L}F \mid T^* \models F\} \cup \{\neg\mathcal{L}F \mid T^* \not\models F\} \cup \{\mathcal{B}F \mid T^* \models_{\min} F\}),$$

where F ranges over all formulae of $\mathcal{K}_{L,B}$.

Definition 2.9. (*Static semantics*) By the (*skeptical*[8]) static semantics $Stat(T)$ of a knowledge and belief theory T we mean the set of all formulae that belong to all static autoepistemic expansions T^* of T.

Every theory has one inconsistent static expansion (which will typically be of no interest to us) and zero, one or more consistent static expansions. As we will show in the next section, a broad class of theories always has a consistent least (in the sense of inclusion) static expansion which therefore coincides with the static semantics.

It follows from the above definition that a static autoepistemic expansion T^* of a knowledge and belief theory T is built by augmenting T with:

- those knowledge atoms $\mathcal{L}F$ for which the formula F is logically implied by T^*,
- negations $\neg\mathcal{L}F$ of the remaining knowledge atoms $\mathcal{L}F$,
- those belief atoms $\mathcal{B}F$ for which the formula F is minimally entailed by T^*

and closing it under derivation in the logic *AELB*. Observe that since T^* appears on both sides of the equation the above definition represents a fixed-point equation. The first part of the definition is identical to the definition of *stable autoepistemic expansions*

[8] More generally, any class $\mathcal{S}$ of static expansions of a knowledge and belief theory T (in particular, a one-element class) naturally defines the corresponding (*credulous*) *static semantics* consisting of those formulae that belong to all expansions from the class $\mathcal{S}$.

in Moore's autoepistemic logic, *AEL*. However, as we will show, the addition of belief atoms $\mathcal{B}F$ results in a much more powerful nonmonotonic logic which contains, as special cases, several well-known nonmonotonic formalisms.

Note that negations $\neg\mathcal{B}F$ of the remaining belief atoms (i.e., those for which the formula F is *not* minimally entailed by T^*) are not *explicitly* added to the expansion, although, as we will see below, some of them will be forced in by the Consistency and Normality Axioms (1) and (2). However, their status in the expansion can be easily determined by using the knowledge operator $\mathcal{L}$, instead. Namely, from the formula (11) (see below) it follows immediately that for any static autoepistemic expansion T^* the following equivalences are true:

$$T^* \models \mathcal{B}F \quad \equiv \quad T^* \models \mathcal{L}\mathcal{B}F$$
$$T^* \not\models \mathcal{B}F \quad \equiv \quad T^* \models \neg\mathcal{L}\mathcal{B}F.$$

It is immediately clear from the above definition that, like in Moore's *AEL*, for any formula F of $\mathcal{K}_{L,B}$ and for any static autoepistemic expansion T^* we have:

$$T^* \models \mathcal{L}F \quad \text{iff} \quad T^* \models F. \tag{11}$$

In other words, $\mathcal{L}F$ holds in the static expansion T^* if and only if F is logically derivable from T^*.

Moreover, for any formula F of $\mathcal{K}_{L,B}$ and for any static autoepistemic expansion T^* we have:

$$T^* \models \mathcal{B}F \quad \text{if} \quad T^* \models_{\min} F, \tag{12}$$

thus a formula F is believed in a static expansion expansion T^* if it is minimally entailed by T^*.

Consequently, the definition of static expansions formally *enforces* the intended meaning of introspective atoms described in the previous subsection. In general, the converse of (12) does not hold because belief atoms may be forced in by the theory itself. For example, the theory $T = \{a \vee b, \mathcal{B}a\}$ has a consistent static expansion in which $\mathcal{B}a, \mathcal{B}(a \vee b), \mathcal{B}(\neg a \vee \neg b)$ hold and yet $T \not\models_{\min} a$. However, in *rational* knowledge and belief theories, i.e., those in which belief atoms do not occur positively, the converse is also true. Intuitively, rational theories are those theories in which beliefs $\mathcal{B}F$ cannot be explicitly stated but can be only implicitly inferred by virtue of minimal entailment of the underlying formulae F.

Theorem 2.10. (Meaning of knowledge and belief atoms) *Let T^* be a static autoepistemic expansion of a* rational *knowledge and belief theory T. For any formula F of $\mathcal{K}_{L,B}$ we have*

$$T^* \models \mathcal{B}F \quad \textit{iff} \quad T^* \models_{\min} F.$$
$$T^* \models \mathcal{L}F \quad \textit{iff} \quad T^* \models F.$$

Proof. If T^* is inconsistent then there is nothing to prove. Let M be any model of T^* and let M' be its modification obtained by making any belief atom $\mathcal{B}F$ true in M' if and only if $T^* \models_{\min} F$. It suffices to show that M' is also a model of $T^* = Cn_*(T \cup \{\mathcal{L}F \mid T^* \models F\} \cup \{\neg\mathcal{L}F \mid T^* \not\models F\} \cup \{\mathcal{B}F \mid T^* \models_{\min} F\})$.

Since belief atoms occur only negatively in clauses of T and since M' contains no more belief atoms than M, we immediately conclude that M' is also a model of T and therefore M' is also a model of $T' = T \cup \{\mathcal{L}F \mid T^* \models F\} \cup \{\neg\mathcal{L}F \mid T^* \not\models F\} \cup \{\mathcal{B}F \mid T^* \models_{\min} F\}$. In order to verify that M' is a model of $T^* = Cn_*(T')$ it suffices to note that all instances of the normality axiom $\mathcal{B}(F \supset G) \wedge \mathcal{B}F \supset (\mathcal{B}G)$ also hold in M'. □

Corollary 2.11. *If T is a* rational *knowledge and belief theory then for any formula F of $\mathcal{K}_{L,\mathcal{B}}$ we have*

$$Stat(T) \models \mathcal{B}F \quad \textit{iff} \quad Stat(T) \models_{\min} F.$$

$$Stat(T) \models \mathcal{L}F \quad \textit{iff} \quad Stat(T) \models F.$$

The above results precisely clarify the meaning of knowledge and belief atoms in static expansions of rational theories. We now return to the simple example discussed in the Introduction.

Example 2.12. Consider the knowledge and belief theory T discussed in the Introduction:

$$\mathcal{B}\neg baseball \wedge \mathcal{B}\neg football \supset rent\_movie$$

$$\neg\mathcal{L}baseball \wedge \neg\mathcal{L}football \supset dont\_buy\_tickets.$$

This theory has a unique (consistent) static autoepistemic expansion (we use obvious abbreviations):

$$T^* = Stat(T) = Cn_*(T \cup \{\mathcal{B}\neg bball, \mathcal{B}\neg fball, \neg\mathcal{L}bball, \neg\mathcal{L}fball, \mathcal{L}r\_movie, \mathcal{L}db\_tickets, \ldots\}),$$

in which you rent a movie, because you believe that you will not go to watch any games (i.e., both $\neg baseball$ and $\neg football$ hold in all minimal models) and you do not buy any tickets because you don't know that you will go to watch any of the games (i.e., neither *baseball* nor *football* are provable). (Here and in the rest of the paper we list only the "relevant" introspective atoms belonging to the expansion T^*, skipping, e.g., $\mathcal{B}dont\_buy\_tickets$, $\mathcal{L}\mathcal{B}\neg baseball$, etc.)

Suppose now that you learn that you either go to a baseball game or to a football game, i.e., suppose that we add the claus:

$$baseball \vee football$$

to T obtaining the theory T_2. Now T_2 has a unique (consistent) static autoepistemic expansion

$$T_2^* = Cn_*(T \cup \{\mathcal{B}(bball \vee fball), \mathcal{B}(\neg bball \vee \neg fball), \neg\mathcal{L}bball, \neg\mathcal{L}fball, \mathcal{B}\neg r\_movie, \mathcal{L}db\_tickets, \ldots\}),$$

in which you believe you should *not* rent a movie. Indeed, we know that $T_2^* \models$ *baseball* $\vee$ *football* and thus $T_2^* \models \mathcal{B}(baseball \vee football)$. From (6), we infer that $T_2^* \models \neg\mathcal{B}\neg baseball \vee \neg\mathcal{B}\neg football$. As a result *rent_movie* is false in all minimal models of T_2^* and consequently $T_2^* \models \mathcal{B}\neg rent\_movie$.

However, you still do not buy any tickets, because you don't know yet which game you are going to watch, i.e., neither *baseball* nor *football* are provable in T_2^*.

Finally, suppose that you learn that you actually go to watch a baseball game. After adding the clause

baseball

to T_2, the new theory T_3 has a unique (consistent) static autoepistemic expansion consisting of

$$T_3^* = Cn_*(T \cup \{\mathcal{B}bball, \mathcal{B}\neg fball, \mathcal{L}bball, \neg\mathcal{L}fball, \mathcal{B}\neg r\_movie, \mathcal{B}\neg db\_tickets, \ldots\}),$$

in which you still believe you should *not* rent any movies but you no longer believe in not buying tickets because you now know that you are going to watch a specific game. Indeed, $T_3^* \models$ *baseball* and thus $T_3^* \models \mathcal{B}baseball$. From (5) we obtain $T_3^* \models \neg\mathcal{B}\neg baseball$ and thus *rent_movie* is false in all minimal models of T_3^*. Consequently $T_3^* \models \mathcal{B}\neg rent\_movie$. Similarly, since $T_3^* \models \mathcal{L}baseball$ we deduce that $T_3^* \models_{\min} \neg dont\_buy\_tickets$ and therefore $T_3^* \models \mathcal{B}\neg dont\_buy\_tickets$.

As we can see, the static semantics assigned to the discussed knowledge and belief theories by their unique consistent static autoepistemic expansions seems to fully agree with their intended meaning. Observe, that we cannot replace the premise $\mathcal{B}\neg baseball \wedge \mathcal{B}\neg football$ in the first clause by $\mathcal{L}\neg baseball \wedge \mathcal{L}\neg football$ because that would result in *rent_movie* not being true in T^*. Similarly, we cannot replace it by $\neg\mathcal{L}baseball \wedge \neg\mathcal{L}football$ because that would result in *rent_movie* becoming true in T_2^*. We also cannot replace the premise $\neg\mathcal{L}baseball \wedge \neg\mathcal{L}football$ in the second implication by $\neg\mathcal{B}baseball \wedge \neg\mathcal{B}football$ or by $\mathcal{B}\neg baseball \wedge \mathcal{B}\neg football$, because it would no longer imply that we should not buy tickets in T_2^*. Thus the roles of the two operators are quite different and one cannot be substituted[9] by the other.

2.3. Embeddability of autoepistemic logic and circumscription

We conclude this section by showing that propositional circumscription and Moore's autoepistemic logic are isomorphically embeddable into the Autoepistemic Logic of Knowledge and Beliefs, *AELB*. In the following sections we will demonstrate that major semantics for normal and disjunctive logic programs (e.g., well-founded, stable, stationary and static semantics [11,12,26,29,34]) as well as Gelfond's epistemic specifications [10] are also embeddable into *AELB*.

Since the first part of the definition of static autoepistemic expansions is identical to the definition of *stable autoepistemic expansions* in Moore's autoepistemic logic, *AEL*,

[9] In this particular example, adding to the theory the axiom *PIA* discussed in Section 4.3 can make the two operators behave identically. However, the axiom *PIA* is very strong and in general its addition could lead to undesirable results.

it is not surprising that *AEL* is properly *embeddable* into the autoepistemic logic of knowledge and beliefs, *AELB*. However, the proof of this result is by no means trivial.

Theorem 2.13. (Embeddability of autoepistemic logic) *Let T be any autoepistemic theory in the language $\mathcal{K}_L$, i.e., any theory that does not use belief atoms $\mathcal{B}F$.*

- *For every consistent* static *autoepistemic expansion T^{**} of T in AELB its restriction $T^* = T^{**}|\mathcal{K}_L$ to the language $\mathcal{K}_L$ is a consistent stable expansion of T in AEL.*
- *Conversely, for every consistent* stable *autoepistemic expansion T^* of T in AEL there is a unique consistent static autoepistemic expansion T^{**} of T in AELB such that $T^* = T^{**}|\mathcal{K}_L$.*

Proof. Let T be any autoepistemic theory in the language $\mathcal{K}_L$, i.e., any theory that does not use belief atoms $\mathcal{B}F$. Suppose first that $T^{\diamond\diamond}$ is a consistent *static* autoepistemic expansion of T in *AELB* in the language $\mathcal{K}_{L,B}$, i.e.,

$$\begin{aligned} T^{\diamond\diamond} = Cn_*^{\mathcal{K}_{L,B}}\big(T \cup \{\mathcal{B}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models_{\min} F\} \\ \cup \{\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models F\} \\ \cup \{\neg\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \not\models F\}\big). \end{aligned}$$

Here, given a language $\mathcal{H}$ we denote by $Cn^{\mathcal{H}}$ (respectively, $Cn_*^{\mathcal{H}}$) the restriction of the operator Cn (respectively, Cn_*) to the language $\mathcal{H}$. Let $T^\diamond = T^{\diamond\diamond}|\mathcal{K}_L$ be the restriction of $T^{\diamond\diamond}$ to the language $\mathcal{K}_L$. Clearly, $T^\diamond$ is consistent. Moreover, it is easy to see that, due to the fact that $T^{\diamond\diamond}$ is obtained from $T^\diamond$ by adding some (introspective) atoms, for any formula F from $\mathcal{K}_L$ we have that $T^{\diamond\diamond} \models F$ if and only if $T^\diamond \models F$. This implies that

$$T^\diamond = Cn^{\mathcal{K}_L}(T \cup \{\mathcal{L}F \mid F \in \mathcal{K}_L \text{ and } T^\diamond \models F\} \cup \{\neg\mathcal{L}F \mid F \in \mathcal{K}_L \text{ and } T^\diamond \not\models F\})$$

and shows that $T^\diamond$ is a consistent stable expansion of T in *AEL* which completes the proof of the first part of the theorem.

Before proving the second part of Theorem 2.13 we will need some additional notation and two lemmas. Given a propositional language $\mathcal{H}$ let us denote by $\mathcal{B}(\mathcal{H})$ (respectively, $\mathcal{L}(\mathcal{H})$) its extension obtained by closing it under the belief (respectively, knowledge) operator.

Lemma 2.14. *Suppose that $\mathcal{H}$ is a propositional language and T and $T^\diamond$ are consistent theories in $\mathcal{H}$ such that*

$$\begin{aligned} T^\diamond = Cn_*^{\mathcal{H}}\big(T \cup \{\mathcal{B}F \mid \mathcal{B}F \in \mathcal{H} \textit{ and } T^\diamond \models_{\min} F\} \\ \cup \{\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H} \textit{ and } T^\diamond \models F\} \\ \cup \{\neg\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H} \textit{ and } T^\diamond \not\models F\}\big). \end{aligned}$$

There is a unique consistent extension $\mathcal{L}(T^\diamond)$ of the theory $T^\diamond$ in the language $\mathcal{L}(\mathcal{H})$ such that $\mathcal{L}(T^\diamond)|\mathcal{H} = T^\diamond$ and moreover

$$\begin{aligned}\mathcal{L}(T^\diamond) = Cn_*^{\mathcal{L}(\mathcal{H})}\big(T &\cup \{\mathcal{B}F \mid \mathcal{B}F \in \mathcal{L}(\mathcal{H}) \text{ and } \mathcal{L}(T^\diamond) \models_{\min} F\}\\ &\cup \{\mathcal{L}F \mid \mathcal{L}F \in \mathcal{L}(\mathcal{H}) \text{ and } \mathcal{L}(T^\diamond) \models F\}\\ &\cup \{\neg\mathcal{L}F \mid \mathcal{L}F \in \mathcal{L}(\mathcal{H}) \text{ and } \mathcal{L}(T^\diamond) \not\models F\}\big).\end{aligned}$$

Proof. The proof is similar to the proof of the result establishing the existence of a unique stable autoepistemic expansion of an objective theory [21] and thus it is only sketched in here.

Let $T_0 = T^\diamond$ and suppose that T_n was already constructed for some natural n. Let

$$T_{n+1} = Cn(T_n \cup \{\mathcal{L}F \mid T_n \models F\} \cup \{\neg\mathcal{L}F \mid T_n \not\models F\}),$$

where the Fs range over all formulae which involve at most n levels of nesting of the knowledge operator $\mathcal{L}$ and the standard closure Cn is taken over the language $\mathcal{L}(\mathcal{H})$. It is easy to see that the theory

$$\mathcal{L}(T^\diamond) = \bigcup_{n<\omega} T_n$$

satisfies the required condition. $\square$

Lemma 2.15. *Suppose that $\mathcal{H}$ is a propositional language and T and $T^\diamond$ are consistent theories in $\mathcal{H}$ such that*

$$\begin{aligned}T^\diamond = Cn_*^{\mathcal{H}}\big(T &\cup \{\mathcal{B}F \mid \mathcal{B}F \in \mathcal{H} \text{ and } T^\diamond \models_{\min} F\}\\ &\cup \{\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H} \text{ and } T^\diamond \models F\}\\ &\cup \{\neg\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H} \text{ and } T^\diamond \not\models F\}\big).\end{aligned}$$

There is a unique consistent extension $\mathcal{B}(T^\diamond)$ of the theory $T^\diamond$ in the language $\mathcal{B}(\mathcal{H})$ such that $\mathcal{B}(T^\diamond)|\mathcal{H} = T^\diamond$ and moreover

$$\begin{aligned}\mathcal{B}(T^\diamond) = Cn_*^{\mathcal{B}(\mathcal{H})}\big(T &\cup \{\mathcal{B}F \mid \mathcal{B}F \in \mathcal{B}(\mathcal{H}) \text{ and } \mathcal{B}(T^\diamond) \models_{\min} F\}\\ &\cup \{\mathcal{L}F \mid \mathcal{L}F \in \mathcal{B}(\mathcal{H}) \text{ and } \mathcal{B}(T^\diamond) \models F\}\\ &\cup \{\neg\mathcal{L}F \mid \mathcal{L}F \in \mathcal{B}(\mathcal{H}) \text{ and } \mathcal{B}(T^\diamond) \not\models F\}\big).\end{aligned}$$

Proof. The proof is analogous to the proof of the previous lemma and thus it is only sketched in here.

Let $T_0 = T^\diamond$ and suppose that T_n was already constructed for some natural n. Let

$$T_{n+1} = Cn_*(T_n \cup \{\mathcal{B}F \mid T_n \models_{\min} F\}),$$

where the Fs range over all formulae which involve at most n levels of nesting of the belief operator $\mathcal{B}$. It is easy to see that the theory

$$\mathcal{B}(T^\diamond) = \bigcup_{n<\omega} T_n$$

satisfies the required condition. $\square$

Suppose now that $T^\diamond$ is a consistent *stable* autoepistemic expansion of T in *AEL*. Since the language $\mathcal{H} = \mathcal{K}_L$ does not contain any belief atoms, from the definition of stable expansions it follows that

$$\begin{aligned} T^\diamond = Cn_*^{\mathcal{H}}\big(T \cup \{\mathcal{B}F \mid \mathcal{B}F \in \mathcal{H} \text{ and } T^\diamond \models_{\min} F\} \\ \cup \{\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H} \text{ and } T^\diamond \models F\} \\ \cup \{\neg\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H} \text{ and } T^\diamond \not\models F\}\big). \end{aligned}$$

Let $T_0 = T^\diamond$ and $\mathcal{H}_0 = \mathcal{H}$ and suppose that T_n and $\mathcal{H}_n$ are already defined. If n is even then $T_{n+1} = \mathcal{B}(T_n)$ and $\mathcal{H}_{n+1} = \mathcal{B}(\mathcal{H}_n)$, where $\mathcal{B}(T_n)$ is given by Lemma 2.15. If n is odd then $T_{n+1} = \mathcal{L}(T_n)$ and $\mathcal{H}_{n+1} = \mathcal{L}(\mathcal{H}_n)$, where $\mathcal{L}(T_n)$ is given by Lemma 2.14. Since $T_{n+1}|\mathcal{H}_n = T_n$, for every n, the sequence of theories $\{T_n\}$ is nondecreasing. Moreover, each theory T_n is consistent and satisfies

$$\begin{aligned} T_n = Cn_*^{\mathcal{H}_n}\big(T \cup \{\mathcal{B}F \mid \mathcal{B}F \in \mathcal{H}_n \text{ and } T_n \models_{\min} F\} \\ \cup \{\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H}_n \text{ and } T_n \models F\} \\ \cup \{\neg\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H}_n \text{ and } T_n \not\models F\}\big). \end{aligned}$$

Let $T^{\diamond\diamond} = \bigcup_{n<\omega} T_n$. Clearly, $T^{\diamond\diamond}$ is consistent and $T^\diamond = T^{\diamond\diamond}|\mathcal{K}_L$. It suffices therefore to show that $T^{\diamond\diamond}$ is a static autoepistemic expansion of T in *AELB* in the language $\mathcal{K}_{L,B}$, i.e., that it satisfies the equation

$$\begin{aligned} T^{\diamond\diamond} = Cn_*^{\mathcal{K}_{L,B}}\big(T \cup \{\mathcal{B}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models_{\min} F\} \\ \cup \{\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models F\} \\ \cup \{\neg\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \not\models F\}\big) \end{aligned}$$

and that $T^{\diamond\diamond}$ is a unique such static expansion of T. We first show that

$$\begin{aligned} T^{\diamond\diamond} \supseteq Cn_*^{\mathcal{K}_{L,B}}\big(T \cup \{\mathcal{B}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models_{\min} F\} \\ \cup \{\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models F\} \\ \cup \{\neg\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \not\models F\}\big). \end{aligned}$$

Suppose that $F \in \mathcal{K}_{L,B}$ and $T^{\diamond\diamond} \models F$. There must exist an n such that $T_n \models F$ and $\mathcal{L}F \in \mathcal{H}_n$. Then $T_n \models \mathcal{L}F$ and therefore $\mathcal{L}F \in T^{\diamond\diamond}$. Suppose that $F \in \mathcal{K}_{L,B}$ and $T^{\diamond\diamond} \not\models F$. There must exist an n such that $\mathcal{L}F \in \mathcal{H}_n$ and obviously $T_n \not\models F$. Then $T_n \models \neg\mathcal{L}F$ and therefore $\neg\mathcal{L}F \in T^{\diamond\diamond}$.

As a union of theories closed under the axioms (D) and (K), $T^{\diamond\diamond}$ is also closed under these axioms. Suppose that $F \in \mathcal{K}_{L,B}$ and $T^{\diamond\diamond} \models_{\min} F$. There must exist an n such that $\mathcal{B}F \in \mathcal{H}_n$. Moreover, as we show below, $T_n \models_{\min} F$ and therefore $T_n \models \mathcal{B}F$ which shows that $\mathcal{B}F \in T^{\diamond\diamond}$.

To see that $T_n \models_{\min} F$ suppose that M is a minimal model of T_n and let M' be any model of $T^{\diamond\diamond}$. Let N be an interpretation which coincides with M when restricted to the language $\mathcal{H}_n$ and coincides with M' otherwise. Since $T^{\diamond\diamond}$ is obtained from T_n by adding some (introspective) atoms, it is easy to see that N is a model of $T^{\diamond\diamond}$. If N was not a minimal model of $T^{\diamond\diamond}$ then there would exist a smaller model N' whose restriction to the language $\mathcal{H}_n$ would then be a smaller model of T_n which is not possible. Thus $N \models F$ which implies that $M \models F$ and shows that $T_n \models_{\min} F$.

In order to establish the opposite inclusion, it suffices to show that for every n:

$$\begin{aligned} T_n \subseteq Cn_*^{\mathcal{K}_{L,B}}\big(T \cup \{\mathcal{B}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models_{\min} F\} \\ \cup \{\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models F\} \\ \cup \{\neg\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \not\models F\}\big). \end{aligned}$$

This, however, follows easily by induction from the definition of T_n and from the following two observations:

- Due to the fact that $T^{\diamond\diamond}$ is obtained from $T^{\diamond}$ by adding some (introspective) atoms, for any formula F from $\mathcal{H}_n$ we have that $T^{\diamond\diamond} \models F$ if and only if $T_n \models F$.
- An argument similar to the one used in the previous paragraph establishes that for any formula F from $\mathcal{H}_n$ we have that $T^{\diamond\diamond} \models_{\min} F$ if and only if $T_n \models_{\min} F$.

This shows that $T^{\diamond\diamond}$ is a static expansion of T.

To complete the proof it suffices to prove that $T^{\diamond\diamond}$ is a unique static expansion of T extending $T^{\diamond}$. If this was not the case then we would be able to find a different static expansion T^* of T and the first n such that $T_n \subseteq T^*$ but $T_{n+1} \not\subseteq T^*$. However, the existence of such an n would violate the uniqueness of the extensions $\mathcal{L}(T^{\diamond})$ and $\mathcal{B}(T^{\diamond})$, guaranteed by the Lemmas 2.15 and 2.14. □

Theorem 2.13 shows that the restriction, $AELB|\mathcal{K}_L$, of the autoepistemic logic of knowledge and beliefs, *AELB*, to the language $\mathcal{K}_L$, i.e., its restriction to theories using only the knowledge operator $\mathcal{L}$, is *isomorphic* to Moore's autoepistemic logic, *AEL*. Thus, as its acronym suggests, *AELB* indeed constitutes an extension of Moore's *AEL* obtained by adding the belief operator $\mathcal{B}$.

The above result has a corollary showing that any consistent *objective* theory has a unique consistent static expansion.

Corollary 2.16. (Static expansions of objective theories) *Any consistent objective theory T, i.e., a theory which does not contain any introspective atoms $\mathcal{L}F$ and $\mathcal{B}F$, has a unique consistent static expansion $T^* = Stat(T)$.*

Moreover, an objective formula F is logically implied by T if and only if the knowledge atom $\mathcal{L}F$ belongs to T^*:

$$T \models F \;\equiv\; T^* \models F \;\equiv\; T^* \models \mathcal{L}F.$$

Similarly, an objective formula F is minimally entailed by T if and only if the belief atom $\mathcal{B}F$ belongs to T^*:

$$T \models_{\min} F \;\equiv\; T^* \models_{\min} F \;\equiv\; T^* \models \mathcal{B}F.$$

Proof. It is known that any consistent objective theory T has a unique consistent stable autoepistemic expansion T^* [21]. By the previous theorem, there is a unique static expansion T^{**} of T such that $T^* = T^{**}|\mathcal{K}_L$. If T had another consistent static expansion

T^{***} then its restriction to $\mathcal{K}_L$ would have to coincide with T^* and thus T^{***} would have to coincide with T^{**}.

In any static expansion $T^{**} \models F \equiv T^{**} \models \mathcal{L}F$, for any formula F. Moreover, since T is rational, it follows from Theorem 2.10 that $T^{**} \models_{\min} F \equiv T^{**} \models \mathcal{B}F$, for any formula F.

Let F be any objective formula. Clearly, if $T \models F$ then also $T^{**} \models F$. Suppose that $T^{**} \models F$ and $T \not\models F$. Then there is a model M of T in which F is false. Let N be any model of T^{**} and let N' be its modification obtained by replacing its valuation of objective atoms by the one from M. Since T^{**} differs from $Cn(T)$ only by the addition of some introspective atoms and since those atoms do not appear in T it follows that N' is also a model of T^{**} in which F is false which is impossible.

Clearly, if $T \models_{\min} F$ then also $T^{**} \models_{\min} F$. Suppose that $T^{**} \models_{\min} F$ and $T \not\models_{\min} F$. Then there is a minimal model M of T in which F is false. Let N be any model of T^{**} and let N' be its modification obtained by replacing its valuation of objective atoms by the one from M. Since T^{**} differs from $Cn(T)$ only by the addition of some introspective atoms and since those atoms do not appear in T it follows that N' is also a minimal model of T^{**} in which F is false which is impossible. $\square$

From the above proposition we immediately conclude that propositional circumscription (and thus also *CWA*, *GCWA* and *ECWA* [13,19,31]) is properly embeddable into *AELB*.

Corollary 2.17. (Embeddability of circumscription) *Propositional circumscription is properly embeddable into the autoepistemic logic of knowledge and beliefs, AELB. More precisely, if T is any consistent objective theory, i.e., a theory which does not contain any introspective atoms $\mathcal{L}F$ and $\mathcal{B}F$, then T has a unique consistent static expansion $T^* = Stat(T)$ and any objective formula F is logically implied by the circumscription CIRC(T) of T if and only if the belief atom $\mathcal{B}F$ belongs to T^*:*

$$CIRC(T) \models F \equiv T^* \models \mathcal{B}F \equiv Stat(T) \models \mathcal{B}F.$$

Proof. This follows immediately from the previous corollary and the fact that for an objective theory T we have $T \models_{\min} F \equiv CIRC(T) \models F$. $\square$

3. Autoepistemic logic of beliefs

While the restriction $AELB|\mathcal{K}_L$ of *AELB* to the language $\mathcal{K}_L$ is isomorphic to Moore's autoepistemic logic, *AEL*, the restriction $AELB|\mathcal{K}_B$ of *AELB* to the language $\mathcal{K}_B$, i.e., its restriction to theories using only the belief operator $\mathcal{B}$, constitutes an entirely new logic, which will be called the *Autoepistemic Logic of Beliefs* and will be denoted by *AEB*. We call theories restricted to the language $\mathcal{K}_B$ *belief theories*.

The following table illustrates the relationships between the three autoepistemic logics discussed in the paper:

AELB	Autoepistemic Logic of Knowledge and Beliefs	
AEL	Autoepistemic Logic (of Knowledge)	$AELB\|\mathcal{K}_L$
AEB	Autoepistemic Logic of Beliefs	$AELB\|\mathcal{K}_B$

It turns out that *AEB* has some quite natural and interesting properties. In particular, *every* belief theory T in *AEB* has the *least* (in the sense of inclusion) static expansion $\overline{T}$ which is called the *static completion* of T. Static completion $\overline{T}$ has an *iterative* definition as the *least fixed point* of a monotonic belief closure operator Ψ_T defined below. Although static completions may, in general, be inconsistent theories, we will show that they are in fact consistent for all *affirmative* belief theories. These properties of static expansions in the Autoepistemic Logic of Beliefs, *AEB*, sharply contrast with the properties of stable autoepistemic expansions in *AEL* which do not admit natural least fixed point definitions and, in general, do not have least elements. For completeness we include in this section some results previously established in the paper [29] which was limited to the study of logic programs.

The definition of static autoepistemic expansions in *AEB* is the same as in *AELB* except that it is restricted to the language of $\mathcal{K}_B$:

Definition 3.1. (*Static autoepistemic expansions in AEB*) A belief theory T^* is called a *static autoepistemic expansion* of a belief theory T in *AEB* if it satisfies the following fixed-point equation,

$$T^* = Cn_*(T \cup \{\mathcal{B}F \mid T^* \models_{\min} F\}),$$

where F ranges over all formulae of $\mathcal{K}_B$.

As shown by the following result, which is analogous to Theorem 2.13, the restriction to the language of $\mathcal{K}_B$ does not constitute any limitation because any static expansion T^* of a belief theory T in *AEB* can be uniquely extended to a static expansion T^{**} of T in *AELB* (and vice versa):

Theorem 3.2. (Embeddability of autoepistemic logic of beliefs) *Let T be any belief theory in the language $\mathcal{K}_B$, i.e., any theory that does not use knowledge atoms $\mathcal{L}F$.*

- *For every consistent* static *autoepistemic expansion T^{**} of T in AELB its restriction $T^* = T^{**}|\mathcal{K}_B$ to the language $\mathcal{K}_B$ is a consistent static expansion of T in AEB.*
- *Conversely, for every consistent* static *autoepistemic expansion T^* of T in AEB there is a unique consistent static autoepistemic expansion T^{**} of T in AELB such that $T^* = T^{**}|\mathcal{K}_B$.*

Proof. The proof is analogous to the proof of Theorem 2.13. Let T be any belief theory in the language $\mathcal{K}_B$, i.e., any theory that does not use knowledge atoms $\mathcal{L}F$. Suppose first that $T^{\diamond\diamond}$ is a consistent *static* autoepistemic expansion of T in *AELB* in the language $\mathcal{K}_{L,B}$, i.e.,

$$
\begin{aligned}
T^{\diamond\diamond} = Cn_*^{\mathcal{K}_{L,B}}\big(T \cup \{\mathcal{B}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models_{\min} F\} \\
\cup \{\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models F\} \\
\cup \{\neg\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \not\models F\}\big).
\end{aligned}
$$

As before, given a language $\mathcal{H}$ we denote by $Cn^{\mathcal{H}}$ (respectively, $Cn_*^{\mathcal{H}}$) the restriction of the operator Cn (respectively, Cn_*) to the language $\mathcal{H}$. Let $T^\diamond = T^{\diamond\diamond}|\mathcal{K}_B$ be the restriction of $T^{\diamond\diamond}$ to the language $\mathcal{K}_B$. Clearly, $T^\diamond$ is consistent. Moreover, it is easy to see that, due to the fact that $T^{\diamond\diamond}$ is obtained from $T^\diamond$ by adding some (introspective) atoms, for any formula F from $\mathcal{K}_B$ we have that $T^{\diamond\diamond} \models_{\min} F$ if and only if $T^\diamond \models_{\min} F$. This implies that

$$
T^\diamond = Cn_*^{\mathcal{K}_B}(T \cup \{\mathcal{B}F \mid F \in \mathcal{K}_B \text{ and } T^\diamond \models_{\min} F\})
$$

and shows that $T^\diamond$ is a consistent stable expansion of T in *AEB* which completes the proof of the first part of the theorem.

Suppose now that $T^\diamond$ is a consistent *static* autoepistemic expansion of T in *AEB*. As before, given a propositional language $\mathcal{H}$ we denote by $\mathcal{B}(\mathcal{H})$ (respectively, $\mathcal{L}(\mathcal{H})$) its extension obtained by closing it under the belief (respectively, knowledge) operator. Since the language $\mathcal{H} = \mathcal{K}_B$ does not contain any knowledge atoms, from the definition of static expansions it follows that

$$
\begin{aligned}
T^\diamond = Cn_*^{\mathcal{H}}\big(T \cup \{\mathcal{B}F \mid \mathcal{B}F \in \mathcal{H} \text{ and } T^\diamond \models_{\min} F\} \\
\cup \{\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H} \text{ and } T^\diamond \models F\} \\
\cup \{\neg\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H} \text{ and } T^\diamond \not\models F\}\big).
\end{aligned}
$$

Let $T_0 = T^\diamond$ and $\mathcal{H}_0 = \mathcal{H}$ and suppose that T_n and $\mathcal{H}_n$ are already defined. If n is odd then $T_{n+1} = \mathcal{B}(T_n)$ and $\mathcal{H}_{n+1} = \mathcal{B}(\mathcal{H}_n)$, where $\mathcal{B}(T_n)$ is given by Lemma 2.15. If n is even then $T_{n+1} = \mathcal{L}(T_n)$ and $\mathcal{H}_{n+1} = \mathcal{L}(\mathcal{H}_n)$, where $\mathcal{L}(T_n)$ is given by Lemma 2.14. Since $T_{n+1}|\mathcal{H}_n = T_n$, for every n, the sequence of theories $\{T_n\}$ is nondecreasing. Moreover, each theory T_n is consistent and satisfies

$$
\begin{aligned}
T_n = Cn_*^{\mathcal{H}_n}\big(T \cup \{\mathcal{B}F \mid \mathcal{B}F \in \mathcal{H}_n \text{ and } T_n \models_{\min} F\} \\
\cup \{\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H}_n \text{ and } T_n \models F\} \\
\cup \{\neg\mathcal{L}F \mid \mathcal{L}F \in \mathcal{H}_n \text{ and } T_n \not\models F\}\big).
\end{aligned}
$$

Let $T^{\diamond\diamond} = \bigcup_{n<\omega} T_n$. Clearly, $T^{\diamond\diamond}$ is consistent and $T^\diamond = T^{\diamond\diamond}|\mathcal{K}_B$. It suffices therefore to show that $T^{\diamond\diamond}$ is a static autoepistemic expansion of T in *AELB* in the language $\mathcal{K}_{L,B}$, i.e., that it satisfies the equation

$$
\begin{aligned}
T^{\diamond\diamond} = Cn_*^{\mathcal{K}_{L,B}}\big(T \cup \{\mathcal{B}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models_{\min} F\} \\
\cup \{\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \models F\} \\
\cup \{\neg\mathcal{L}F \mid F \in \mathcal{K}_{L,B} \text{ and } T^{\diamond\diamond} \not\models F\}\big)
\end{aligned}
$$

and that $T^{\diamond\diamond}$ is a unique such static expansion of T. The proof of this fact is identical to that given in the proof Theorem 2.13. □

Definition 3.3. (*Static semantics of belief theories*) By the *static semantics* $Stat(T)$ of a belief theory T in *AEB* we mean the set of all formulae that belong to all static autoepistemic expansions T^* of T in *AEB*.

It follows from Theorem 3.2 that the static semantics of a belief theory T in *AEB* is just the restriction to the language $\mathcal{K}_B$ of the static semantics of T in *AELB*. By abuse of notation, we use the same symbol $Stat(T)$ to denote both of them. From Theorem 3.2 and Corollary 2.16 we immediately obtain:

Corollary 3.4. (Static expansions of objective theories in *AEB*) *Any consistent objective theory T, i.e., a theory which does not contain any introspective atoms $\mathcal{L}F$ and $\mathcal{B}F$, has a unique consistent static expansion T^* in AEB.*

Moreover, an objective formula F is minimally entailed by T if and only if the belief atom $\mathcal{B}F$ belongs to T^*:

$$T \models_{\min} F \;\equiv\; T^* \models_{\min} F \;\equiv\; T^* \models \mathcal{B}F. \qquad \square$$

From Theorem 3.2 and Corollary 2.17 we also conclude that propositional circumscription (and thus also *CWA*, *GCWA* and *ECWA* [13, 19, 31]) is properly embeddable into *AEB*.

Corollary 3.5. (Embeddability of circumscription into *AEB*) *Propositional circumscription is properly embeddable into the autoepistemic logic of beliefs, AEB. More precisely, if T is any consistent objective theory, i.e., a theory which does not contain any introspective atoms $\mathcal{L}F$ and $\mathcal{B}F$, then T has a unique consistent static expansion T^* in AEB and any objective formula F is logically implied by the circumscription $CIRC(T)$ of T if and only if the belief atom $\mathcal{B}F$ belongs to T^*:*

$$CIRC(T) \models F \;\equiv\; T^* \models \mathcal{B}F. \qquad \square$$

We now introduce the belief closure operator Ψ_T.

Definition 3.6. (*Belief closure operator*) For any belief theory T define the *belief closure* operator Ψ_T by the formula

$$\Psi_T(S) = Cn_*(T \cup \{\mathcal{B}F \mid S \models_{\min} F\}),$$

where S is an arbitrary belief theory and the Fs range over all formulae of $\mathcal{K}_B$.

Thus $\Psi_T(S)$ augments the theory T with all those belief atoms $\mathcal{B}F$ with the property that F is minimally entailed by S. We begin with the following easy observation.

Proposition 3.7. *A theory T^* is a static autoepistemic expansion of the belief theory T in AEB if and only if T^* is a fixed point of the operator Ψ_T, i.e., if $T^* = \Psi_T(T^*)$.*

Proof. Theory T^* is a fixed point of the operator Ψ_T if $T^* = \Psi_T(T^*) = Cn_*(T \cup \{\mathcal{B}F \mid T^* \models_{\min} F\})$ which is equivalent to T^* being a static expansion of T in *AEB*. $\square$

Consequently, in order to show that every belief theory has the least static expansion we need to prove that the operator Ψ_T has the least fixed point. We first establish the (restricted) *monotonicity* of the operator Ψ_T.

Proposition 3.8. (Monotonicity of the belief closure operator) *The operator Ψ_T is monotonic. More precisely, suppose that the theories V' and V'' are extensions of T obtained by adding some belief atoms $\mathcal{B}F$ to T and let $T' = Cn_*(V')$ and $T'' = Cn_*(V'')$.*

$$\text{If } T' \subseteq T'' \text{ then } \Psi_T(T') \subseteq \Psi_T(T'').$$

Proof. Suppose that $V' = T \cup \{\mathcal{B}F_s \mid s \in S'\}$, $V'' = T \cup \{\mathcal{B}F_s \mid s \in S''\}$, where S' and S'' are some sets of indices, and let $T' = Cn_*(V')$ and $T'' = Cn_*(V'')$. We have to show that $\Psi_T(T') \subseteq \Psi_T(T'')$. Since $Cn_*(V') \subseteq Cn_*(V'')$ we can clearly assume that $S' \subseteq S''$. It suffices to show that if $T' \models_{\min} F$ then $T'' \models_{\min} F$.

Suppose that $T' \models_{\min} F$ and let M be an arbitrary minimal model of T''. Since $V' \subseteq V''$ and V' and V'' differ only on the set of belief atoms and since minimal models do *not* minimize belief atoms, M is also a minimal model of V' and thus also a minimal model of T'. We conclude that $M \models F$ and therefore $T'' \models_{\min} F$. $\square$

Using Propositions 3.7 and 3.8 we can easily adapt the proof of the well-known Theorem of Tarski, ensuring the existence of least fixed points of monotonic operators, to obtain:

Theorem 3.9. (Least static expansion) *Every belief theory T in AEB has the* least *static expansion, namely, the least fixed point $\overline{T}$ of the monotonic belief closure operator Ψ_T.*

Moreover, the least static expansion $\overline{T}$ of a belief theory T can be constructed as follows. Let $T^0 = Cn_*(T)$ and suppose that T^α has already been defined for any ordinal number $\alpha < \beta$. If $\beta = \alpha + 1$ is a successor ordinal then define[10]

$$T^{\alpha+1} = \Psi_T(T^\alpha) = Cn_*(T \cup \{\mathcal{B}F \mid T^\alpha \models_{\min} F\}),$$

where F ranges over all formulae in $\mathcal{K}_B$. Else, if β is a limit ordinal then define

$$T^\beta = \bigcup_{\alpha<\beta} T^\alpha.$$

The sequence $\{T^\alpha\}$ is monotonically increasing and thus has a unique fixed point $\overline{T} = T^\lambda = \Psi_T(T^\lambda)$, for some ordinal λ.

Proof. From Proposition 3.8 it easily follows that the sequence $\{T^\alpha\}$ is monotonically increasing and thus has a unique fixed point $\overline{T} = T^\lambda = \Psi_T(T^\lambda)$, for some ordinal λ. This fixed point must therefore be the least fixed point of the operator Ψ_T. From Proposition 3.7 we infer that any fixed point of the operator Ψ_T is a static expansion of T. $\square$

[10] Since the sequence $\{T^\alpha\}$ is monotonically nondecreasing we can equivalently define $T^\beta = Cn_*(T^\alpha \cup \{\mathcal{B}F \mid T^\alpha \models_{\min} F\})$.

The existence of least static autoepistemic expansions of theories in *AEB* sharply contrasts with the properties of stable autoepistemic expansions in *AEL* which typically do not have least elements.

Definition 3.10. (*Static completion*) The least static expansion $\overline{T}$ of a belief theory T is called the *static completion* of T.

Since the static completion of a belief theory T is obtained by augmenting T with the *least possible* number of belief atoms $\mathcal{B}F$, it can be viewed as the *most skeptical* among static expansions (see [14]). Observe also that the *least* static autoepistemic expansion of T contains those and only those formulae which are true in *all* static autoepistemic expansions of T and thus, like Clark's *predicate completion comp*(P) of a logic program P, the static completion $\overline{T}$ of a belief theory T describes the *static semantics Stat*(T) of T.

Corollary 3.11. *For any belief theory T, the static completion $\overline{T}$ of T coincides with the static semantics Stat(T) of T in AEB.* □

The fact that the static semantics of belief theories in *AEB* can be constructed by means of the iterative minimal model procedure described in Theorem 3.9 is quite important. As a result, static semantics not only has an elegant fixed point characterization but it can simply be viewed as the *iterated minimal model semantics*. The last fact is important from the procedural point of view. Namely, once a suitable procedure is devised to compute the *minimal model* semantics, it can then be iteratively applied to compute the static semantics.

Remark 3.12. It is easy to see that if the belief theory is *finite* then the construction of the least static expansion will stop after *countably* many steps. However, the following powerful and somewhat surprising result obtained recently in [9] shows that for every finite belief theory T its least static expansion is in fact obtained by a *single iteration* of the operator Ψ_T.

Theorem 3.13. (Brass et al. [9]) *Let T be any finite belief theory. The least static expansion of T is always obtained by a single iteration of the operator Ψ_T. In other words, the static completion $\overline{T}$ of T coincides with $\Psi_T(T)$:*

$$\overline{T} = \Psi_T(T) = \Psi_T(\overline{T}). \qquad \square$$

The following theorem significantly extends Theorem 3.9 and provides a complete characterization of *all* static autoepistemic expansions of a belief theory T in the language $\mathcal{K}_B$.

Theorem 3.14. (Characterization theorem) *A theory T^* is a static autoepistemic expansion of a theory T in AEB if and only if T^* is the least static autoepistemic expansion $\overline{T'}$ of a theory $T' = T \cup \{\mathcal{B}F_s \mid s \in S\}$ satisfying the condition that $T^* \models_{\min} F_s$, for every $s \in S$. In particular, the least static autoepistemic expansion $\overline{T}$ of T is obtained when the set $\{\mathcal{B}F_s \mid s \in S\}$ is empty.*

Proof. ($\Rightarrow$) Suppose that T^* is a static expansion of T. Then

$$T^* = Cn_*(T \cup \{\mathcal{B}F \mid T^* \models_{\min} F\}).$$

Let $T' = T \cup \{\mathcal{B}F \mid T^* \models_{\min} F\}$. It suffices to show that T^* is the least static expansion $\overline{T'}$ of T'. Clearly, T^* is a static expansion of T' because

$$T^* = Cn_*(T' \cup \{\mathcal{B}F \mid T^* \models_{\min} F\}).$$

Since $T^* = Cn_*(T')$ it is also the least static expansion of T'.

($\Leftarrow$) Suppose now that T^* is the static completion $\overline{T \cup \{\mathcal{B}F_s \mid s \in S\}}$ of a belief theory $T' = T \cup \{\mathcal{B}F_s \mid s \in S\}$ which satisfies the condition that $T^* \models_{\min} F_s$, for every $s \in S$. Since T^* is the (least) static expansion of the belief theory $T \cup \{\mathcal{B}_{F_s} \mid s \in S\}$ we have

$$T^* = Cn_*(T \cup \{\mathcal{B}F_s \mid s \in S\} \cup \{\mathcal{B}F \mid T^* \models_{\min} F\}).$$

Since $T^* \models_{\min} F_s$, for every $s \in S$ we obtain

$$T^* = Cn_*(T \cup \{\mathcal{B}F \mid T^* \models_{\min} F\}),$$

which shows that T^* is a static expansion of T. $\square$

According to the above theorem, in order to find a static expansion T^* of a belief theory T one needs to:

- Select a set $\{\mathcal{B}F_s \mid s \in S\}$ of beliefs.
- Construct the static completion $T^* = \overline{T \cup \{\mathcal{B}F_s \mid s \in S\}}$ of the augmented belief theory $T' = T \cup \{\mathcal{B}F_s \mid s \in S\}$, e.g., by using the iterative fixed point definition from Theorem 3.9.
- Show that $T^* \models_{\min} F_s$, for every $s \in S$.

It is the first part, namely, the selection (or guessing) of the set of beliefs $\{\mathcal{B}F_s \mid s \in S\}$, that is most difficult. However, one can always choose the *empty* set to obtain the least static expansion (or static completion) $\overline{T}$ of the belief theory T.

Theorem 3.9 also immediately implies:

Corollary 3.15. (Greatest static expansion) *Every belief theory T in AEB has the* greatest static expansion *which is always an inconsistent theory.*

Proof. Let $T^* = \overline{T \cup \{\mathcal{B}(A \wedge \neg A)\}}$, where A is an arbitrary atom. It is easy to see that T^* is inconsistent. Indeed, since any theory logically implies $A \vee \neg A$ we infer that $\mathcal{B}(A \vee \neg A)$ belongs to T^*. Consequently, by the Consistency Axiom, $\neg\mathcal{B}(A \wedge \neg A)$ also belongs to T^* which shows that T^* is inconsistent. Since T^* is inconsistent, $T^* \models A \wedge \neg A$, which, by Theorem 3.9, implies that T^* is a stationary expansion of T. $\square$

It is time now to discuss some examples. For simplicity, unless explicitly needed, when describing static expansions we will ignore nested beliefs and list only those elements of the expansion that are "relevant" to our discussion, thus, for example, skipping such members of the expansion as $\mathcal{B}(F \vee \neg F)$, $\mathcal{B}\mathcal{B}(F \vee \neg F)$, etc.

Example 3.16. Consider the following belief theory T:

$$\begin{array}{lcl} Car \wedge \mathcal{B}\neg Broken & \supset & ShouldRun \\ Damaged \wedge \mathcal{B}\neg Fixed & \supset & Broken \\ Car & & \\ Damaged. & & \end{array}$$

In order to iteratively compute its static completion $\overline{T}$ we let $T^0 = Cn_*(T)$. Clearly, $T^0 \models Car \wedge Damaged$ and one easily checks that $T^0 \models_{\min} \neg Fixed$. Indeed, in order to find minimal models of T^0 we need to assign *arbitrary* truth values to belief atoms $\mathcal{B}\neg Broken$ and $\mathcal{B}\neg Fixed$ and then *minimize* the objective atoms *Fixed*, *ShouldRun*, *Broken*. We easily see that T^0 has the following four (classes of) minimal models (truth values of the remaining belief atoms are irrelevant and are therefore skipped):

$$M_1 = \{\mathcal{B}\neg Broken, \mathcal{B}\neg Fixed, Car, Damaged, ShouldRun, Broken, \neg Fixed\},$$
$$M_2 = \{\mathcal{B}\neg Broken, \neg\mathcal{B}\neg Fixed, Car, Damaged, ShouldRun, \neg Broken, \neg Fixed\},$$
$$M_3 = \{\neg\mathcal{B}\neg Broken, \mathcal{B}\neg Fixed, Car, Damaged, \neg ShouldRun, Broken, \neg Fixed\},$$
$$M_4 = \{\neg\mathcal{B}\neg Broken, \neg\mathcal{B}\neg Fixed, Car, Damaged, \neg ShouldRun, \neg Broken, \neg Fixed\}.$$

Since in all of them *Fixed* is false we deduce that $T^0 \models_{\min} \neg Fixed$. Consequently, if we define

$$T^1 = \Psi_T(T^0) = Cn_*(T \cup \{\mathcal{B}F \mid T^0 \models_{\min} F\}),$$

then

$$T^1 = Cn_*(T \cup \{\mathcal{B}Car, \mathcal{B}Damaged, \mathcal{B}\neg Fixed, \ldots\}).$$

We continue the iterative procedure by defining

$$T^2 = \Psi_T(T^1) = Cn_*(T \cup \{\mathcal{B}F \mid T^1 \models_{\min} F\}).$$

Since $T^1 \models Broken$ we conclude that

$$T^2 = Cn_*(T \cup \{\mathcal{B}Car, \mathcal{B}Damaged, \mathcal{B}\neg Fixed, \mathcal{B}Broken, \ldots\}).$$

By the Consistency Axiom (1), $T^2 \models \neg\mathcal{B}\neg Broken$ and therefore $T^2 \models_{\min} \neg ShouldRun$. Accordingly, if we define

$$T^3 = \Psi_T(T^2) = Cn_*(T \cup \{\mathcal{B}F \mid T^2 \models_{\min} F\}),$$

then we obtain

$$T^3 = Cn_*(T \cup \{\mathcal{B}Car, \mathcal{B}Damaged, \mathcal{B}\neg Fixed, \mathcal{B}Broken, \mathcal{B}\neg ShouldRun, \ldots\}).$$

It is easy to see that no other belief atom of the form $\mathcal{B}L$, where L is an objective literal, belongs to T^3 and that $T^3 = \Psi_T(T^3)$ is a fixed point of Ψ_T (we recall that for simplicity we ignore here nested beliefs). Consequently, $\overline{T} = T^3 =$

$Cn_*(T \cup \{\mathcal{B}Car, \mathcal{B}Damaged, \mathcal{B}\neg Fixed, \mathcal{B}Broken, \mathcal{B}\neg ShouldRun, \ldots\})$ is the static completion of T. The static semantics of T asserts therefore that the car is believed to be broken, unfixed and not in a running condition. Using Theorem 3.14 one easily verifies that T does not have any other (consistent) static expansions.

Example 3.17. Consider now the following belief theory T reflecting the anxieties of a guy living in Southern California who, while visiting Europe, was informed by a friend that apparently yet another disaster occurred in California. The European friend was not quite sure, however, whether it was an earthquake or fires:

$$\begin{array}{lcl} \mathcal{B}\neg Earthquake \wedge \mathcal{B}\neg Fires & \supset & Calm \\ \mathcal{B}(Earthquake \vee Fires) & \supset & Worried \\ \mathcal{B}Earthquake \wedge \mathcal{B}Fires & \supset & Panicked \\ \mathcal{B}\neg Calm & \supset & CallHome \\ Earthquake \vee Fires. & & \end{array}$$

In order to iteratively compute the static completion $\overline{T}$ we let $T^0 = Cn_*(T)$. Clearly, $T^0 \models Earthquake \vee Fires$ and one easily checks that $T^0 \models_{\min} \neg Earthquake \vee \neg Fires$. Consequently, if we define

$$T^1 = \Psi_T(T^0) = Cn_*(T \cup \{\mathcal{B}F \mid T^0 \models_{\min} F\}).$$

then

$$T^1 = Cn_*(T \cup \{\mathcal{B}(Eq \vee Fi), \mathcal{B}(\neg Eq \vee \neg Fi), \ldots\}).$$

Obviously, $T^1 \models Worried$. Moreover, by the Consistency and Conjunctive Belief Axioms, $T^2 \models \neg\mathcal{B}Earthquake \vee \neg\mathcal{B}Fires$ and $T^2 \models \neg\mathcal{B}\neg Earthquake \vee \neg\mathcal{B}\neg Fires$ and therefore $T^2 \models_{\min} \neg Calm$ and $T^2 \models_{\min} \neg Panicked$. Accordingly, if we continue the iterative procedure by defining

$$T^2 = \Psi_T(T^1) = Cn_*(T \cup \{\mathcal{B}F \mid T^1 \models_{\min} F\}).$$

then we obtain

$$T^2 = Cn_*(T \cup \{\mathcal{B}(Eq \vee Fi), \mathcal{B}(\neg Eq \vee \neg Fi), \mathcal{B}\neg Calm, \mathcal{B}\neg Panic, \mathcal{B}Worried, \ldots\}).$$

Clearly, $T^2 \models CallHome$ and thus if we now define

$$T^3 = \Psi_T(T^2) = Cn_*(T \cup \{\mathcal{B}F \mid T^2 \models_{\min} F\}),$$

then we obtain

$$T^3 = Cn_*(T \cup \{\mathcal{B}(Eq \vee Fi), \mathcal{B}(\neg Eq \vee \neg Fi), \mathcal{B}\neg Calm,$$
$$\mathcal{B}\neg Panic, \mathcal{B}Worried, \mathcal{B}Call, \ldots\}).$$

It is easy to see that no other belief atom of the form $\mathcal{B}L$, where L is an objective literal, belongs to T^3 and that $T^3 = \Psi_T(T^3)$ is a fixed point of Ψ_T (we again recall that for simplicity we ignore nested beliefs). Consequently, $\overline{T} = T^3 = Cn_*(T \cup \{\mathcal{B}(Eq \vee Fi), \mathcal{B}(\neg Eq \vee \neg Fi), \mathcal{B}\neg Calm, \mathcal{B}\neg Panic, \mathcal{B}Worried, \mathcal{B}Call, \ldots\})$ is the static completion of T. The static semantics of T asserts therefore that the guy is not calm, because he does not believe that none of the disasters took place, and thus he plans to call home and is worried. However, he is not exactly panicked because he has no reason to believe that both disasters struck. Using Theorem 3.14 one easily verifies that T does not have any other (consistent) static expansions.

Observe one more time that we cannot replace the belief atoms in the clause

$$\mathcal{B}\neg Earthquake \wedge \mathcal{B}\neg Fires \supset Calm$$

by the knowledge atoms

$$\neg\mathcal{L}Earthquake \wedge \neg\mathcal{L}Fires \supset Calm$$

because this would imply that the individual is calm. The original clause also cannot be replaced by the clause

$$\mathcal{L}\neg Earthquake \wedge \mathcal{L}\neg Fires \supset Calm$$

because this would preclude the individual to be calm that unless he has a *factual* knowledge that no disaster actually took place rather than, as intended, just believe that nothing happened based on the lack of information to the contrary.

Remark 3.18. According to Theorem 3.13, in both of the above examples static completions are in fact obtained in just *one* iteration of the belief operator. However, in order to show it, one has to deal with nested beliefs and thus the explanation becomes a bit more complex. That is why we decided to illustrate a weaker fact that fixed points are obtained in several steps of iteration.

Clearly, if we added $\neg Worried$ to the previous theory T we would obtain a belief theory whose static completion is inconsistent because it implies both *Worried* and $\neg Worried$. The new theory not only appears to describe contradictory information but it also demonstrates that the static completion $\overline{T}$ of a consistent belief theory T may in fact be inconsistent. [11]

It follows from Theorem 3.9 and from Corollary 3.15 that a belief theory T either has a *consistent* least static expansion (i.e., static completion) $\overline{T}$ or it does *not* have any consistent static expansions at all, in which case its least and greatest static expansions coincide. However, it turns out that static completions are always consistent for *affirmative* belief theories, introduced in Definition 2.1.

Theorem 3.19. (Consistency of static completions) *The static completion $\overline{T}$ of any affirmative belief theory T in AEB is always consistent.*

[11] The papers [2,3] deal with contradiction removal in belief theories.

Proof. We will prove by induction that T^{α}, as defined in Theorem 3.9, is consistent, for every α. To see that $T^0 = Cn_*(T)$ is consistent it suffices to take an interpretation of $\mathcal{K}_B$ in which all belief atoms are false and all objective atoms are true.

Suppose that we already proved that T^{α} is consistent, for any $\alpha < \beta$. If β is a limit ordinal then, by the Compactness Theorem, T^{β} must also be consistent as a union of an increasing sequence of consistent theories. If on the other hand $\beta = \alpha + 1$ then

$$T^{\beta} = Cn_*(T \cup \{\mathcal{B}F \mid T^{\alpha} \models_{\min} F\}).$$

Since T^{α} is consistent, the class of formulae F minimally entailed by T^{α} also constitutes a consistent theory. Define an interpretation M so that all the objective atoms and all the belief atoms $\mathcal{B}F$ such that $T^{\alpha} \models_{\min} F$ are true in M while all the remaining belief atoms are false. Clearly M is a model of $T \cup \{\mathcal{B}F \mid T^{\alpha} \models_{\min} F\}$ and since it also clearly satisfies axioms (1) and (4) it is a model of T^{β}. We conclude that T^{β} is consistent, which completes the inductive step. □

4. Logic programs as knowledge and belief theories

We already know that Propositional Circumscription, Moore's Autoepistemic Logic, *AEL*, and the Autoepistemic Logic of Beliefs, *AEB*, are all properly embeddable into the Autoepistemic Logic of Knowledge and Beliefs, *AELB*. We will now show that major semantics defined for normal and disjunctive *logic programs* are also easily embeddable into *AELB*. In the next section we will discuss yet another nonmonotonic formalism embeddable into *AELB*.

Recall that by a *disjunctive logic program* (or a disjunctive deductive database) P we mean a set of *informal* clauses of the form

$$A_1 \vee \cdots \vee A_l \leftarrow B_1 \wedge \cdots \wedge B_m \wedge not\, C_1 \wedge \cdots \wedge not\, C_n \tag{13}$$

where $l \geqslant 1$; $m, n \geqslant 0$ and A_i, B_i and C_i's are atomic formulae. If $l = 1$, for all clauses, then the program is called *normal* or *nondisjunctive*. As usual, we assume (see [23]) that the program P has been already *instantiated* and thus all of its clauses (possibly infinitely many) are propositional. This assumption allows us to restrict our considerations to a fixed *objective propositional language* $\mathcal{K}$. In particular, if the original (uninstantiated) program is finite and function-free then the resulting objective language $\mathcal{K}$ is also finite.

Clauses (13) are informal because the negation symbol $not\, C$ does not denote the *classical negation* $\neg C$ of C but rather a *nonmonotonic (common-sense) negation*. Moreover, the implication symbol $\rightarrow$ does not necessarily represent the standard *material implication* $\subset$. Various meanings can be associated with "$not\, C$" and "$\rightarrow$" leading, in general, to different semantics for logic programs. We will now show that many of the proposed semantics can be obtained by translating the informal disjunctive logic program P into a formal knowledge and belief theory $T(P)$ and thus assigning a specific meaning to the nonmonotonic negation $not\, C$ and the implication symbol $\rightarrow$. We argue therefore that the Autoepistemic Logic of Knowledge and Beliefs, *AELB*, constitutes a broad and flexible *semantic framework for logic programming* which not only enables us to reproduce virtually all major semantics recently introduced for logic programs but also allows us to introduce new semantics, analyze their properties and study their mutual relationships. For a more extensive treatment of this subject the reader is referred to [9,28,29].

4.1. Stable semantics of normal programs

Since Moore's autoepistemic logic, *AEL*, is isomorphic to the subset $AELB|\mathcal{K}_L$ of *AELB*, it follows from the results of Gelfond and Lifschitz [11] that stable semantics of normal logic programs can be obtained by means of a suitable translation of a logic program into a belief theory. Namely, for a normal logic program P consisting of clauses:

$$A \leftarrow B_1 \wedge \cdots \wedge B_m \wedge not\, C_1 \wedge \cdots \wedge not\, C_n$$

define $T_{\neg\mathcal{L}}(P)$ to be its translation into the (affirmative) knowledge and belief theory consisting of formulae:

$$B_1 \wedge \cdots \wedge B_m \wedge \neg\mathcal{L}C_1 \wedge \cdots \wedge \neg\mathcal{L}C_n \supset A.$$

The translation $T_{\neg\mathcal{L}}(P)$ is obtained therefore by replacing the nonmonotonic negation $not\, C$ by the negated knowledge atom $\neg\mathcal{L}C$ which gives it the intended meaning of "*C is not known to be true*". In addition, we replace the informal implication symbol $\rightarrow$ by the standard material implication $\supset$.

Theorem 4.1. (Embeddability of stable semantics) *There is a one-to-one correspondence between stable models $\mathcal{M}$ of the program P and consistent static autoepistemic expansions T^* of its translation $T_{\neg\mathcal{L}}(P)$ into belief theory. Namely, for any objective atom A we have*

$$\begin{aligned} A \in \mathcal{M} &\quad \textit{iff} \quad \mathcal{L}A \in T^* \\ \neg A \in \mathcal{M} &\quad \textit{iff} \quad \neg\mathcal{L}A \in T^*. \end{aligned}$$

Proof. Follows immediately from Theorem 2.13 and the results obtained in [11]. □

4.2. Stationary and well-founded semantics of normal programs

Similarly, since autoepistemic logic of beliefs, *AEB*, is isomorphic to the subset $AELB|\mathcal{K}_B$ of *AELB*, it follows from the results obtained in [29] that the stationary (or, equivalently, partial stable) semantics, and, in particular, the well-founded semantics, of normal logic programs can be obtained by means of a suitable translation of a logic program into a knowledge and belief theory. Namely, for a normal logic program P consisting of clauses

$$A \leftarrow B_1 \wedge \cdots \wedge B_m \wedge not\, C_1 \wedge \cdots \wedge not\, C_n$$

its translation into the (affirmative) belief theory $T_{\mathcal{B}\neg}(P)$ is given by the set of the corresponding clauses:

$$B_1 \wedge \cdots \wedge B_m \wedge \mathcal{B}\neg C_1 \wedge \cdots \wedge \mathcal{B}\neg C_n \supset A \tag{14}$$

obtained by replacing the nonmonotonic negation $not\, F$ by the belief atom $\mathcal{B}\neg F$ and by replacing the implication symbol $\rightarrow$ by the standard material implication $\supset$.

The translation, $T_{\mathcal{B}\neg}(P)$, gives therefore the following meaning to the nonmonotonic negation:

$$not\, F \stackrel{\text{def}}{\equiv} \mathcal{B}\neg F \equiv F \text{ is believed to be false} \equiv \neg F \text{ is minimally entailed} \tag{15}$$

and is patterned after the translation introduced earlier in [25].

Theorem 4.2. (Embeddability of stationary and well-founded semantics) *There is a one-to-one correspondence between stationary (or, equivalently, partial stable) models $\mathcal{M}$ of the program P and consistent static autoepistemic expansions T^* of its translation $T_{\mathcal{B}\neg}(P)$ into a knowledge and belief theory. Namely, for any objective atom A we have*

$$A \in \mathcal{M} \quad \textit{iff} \quad \mathcal{B}A \in T^*$$
$$\neg A \in \mathcal{M} \quad \textit{iff} \quad \mathcal{B}\neg A \in T^*.$$

Since the well-founded model $\mathcal{M}_0$ of the program P coincides with the least *stationary model of P [26], it corresponds to the static completion $\overline{T_{\mathcal{B}\neg}(P)}$ of $T_{\mathcal{B}\neg}(P)$, whose existence is guaranteed by Theorem 3.9.*

Moreover, (total) stable models $\mathcal{M}$ of P correspond to those consistent static autoepistemic expansions T^* of $T_{\mathcal{B}\neg}(P)$ that satisfy the condition that for all objective atoms A, either $\mathcal{B}A \in T^*$ or $\mathcal{B}\neg A \in T^*$.

Proof. Follows immediately from Theorem 3.2 and the results obtained in [29]. □

It is worth mentioning that for normal programs an analogous result applies to the translation $T_{\neg\mathcal{B}}(P)$ defined by

$$B_1 \wedge \cdots \wedge B_m \wedge \neg\mathcal{B}C_1 \wedge \cdots \wedge \neg\mathcal{B}C_n \supset A.$$

However, for disjunctive programs (discussed below) the two translations $T_{\mathcal{B}\neg}(P)$ and $T_{\neg\mathcal{B}}(P)$ lead, in general, to different results.

Example 4.3. It is easy to see that the belief theory T:

$$\begin{array}{lcl} \mathit{Car} \wedge \mathcal{B}\neg \mathit{Broken} & \supset & \mathit{ShouldRun} \\ \mathit{Damaged} \wedge \mathcal{B}\neg \mathit{Fixed} & \supset & \mathit{Broken} \\ \mathit{Car} & & \\ \mathit{Damaged}. & & \end{array}$$

considered in Example 3.16 can be viewed as a translation $T_{\mathcal{B}\neg}(P)$ of the logic program P given by

$$\begin{aligned} &ShouldRun &\leftarrow\ & Car \wedge not Broken\\ &Broken &\leftarrow\ & Damaged \wedge not Fixed\\ &Car && \\ &Damaged. && \end{aligned}$$

The unique static expansion (or static completion),

$$\overline{T} = Cn_*(T \cup \{\mathcal{B}Car, \mathcal{B}Damaged, \mathcal{B}\neg Fixed, \mathcal{B}Broken, \mathcal{B}\neg ShouldRun, \ldots\}).$$

of T corresponds therefore to the unique stationary (or stable) model

$$M = \{Car, Damaged, \neg Fixed, Broken, \neg ShouldRun, \ldots\}$$

of P which is also its unique well-founded model.

4.3. Semantics of disjunctive programs

As we have shown above, major semantics proposed for normal logic programs can be naturally captured by means of a suitable translation of logic programs into knowledge and belief theories in *AELB*. We now extend these results to the class of *disjunctive logic programs* (see [17] for an overview of disjunctive logic programming).

In particular, we can extend the transformation $T_{\mathcal{B}\neg}(P)$ to any disjunctive logic program P consisting of clauses

$$A_1 \vee \cdots \vee A_l \leftarrow B_1, \ldots, B_m, not\, C_1, \cdots, not\, C_n$$

by translating it into the (affirmative) autoepistemic theory consisting of formulae

$$B_1 \wedge \cdots \wedge B_m \wedge \mathcal{B}\neg C_1 \wedge \cdots \wedge \mathcal{B}\neg C_n \supset A_1 \vee \cdots \vee A_l.$$

It follows from the results obtained in [29] that this transformation immediately leads to the *static semantics* of disjunctive logic programs:

Theorem 4.4. (Embeddability of static semantics) *There is a one-to-one correspondence between static expansions P^* of the disjunctive logic program P, as defined in* [29], *and consistent static autoepistemic expansions T^* of its translation $T = T_{\mathcal{B}\neg}(P)$ into belief theory. Namely, a formula F belongs to P^* if and only if $\mathcal{B}F$ belongs to T^*.*

Proof. Immediate consequence of Theorem 3.2 and the results obtained in [29]. □

Example 4.5. Consider the following disjunctive logic program P describing the state of mind of a person planning a trip to either Australia or Europe:

$$\begin{aligned} &Goto\_Australia \vee Goto\_Europe && \\ &Goto\_Both &\leftarrow\ & Goto\_Australia \wedge Goto\_Europe\\ &Save\_Money &\leftarrow\ & not\, Goto\_Both\\ &Cancel\_Reservation &\leftarrow\ & not\, Goto\_Australia\\ &Cancel\_Reservation &\leftarrow\ & not\, Goto\_Europe \end{aligned}$$

and its translation into the (affirmative) belief theory $T = T_{\mathcal{B}\neg}(P)$:

$$\begin{array}{lcl}
Goto_Australia \vee Goto_Europe & & \\
Goto_Australia \wedge Goto_Europe & \supset & Goto_Both \\
\mathcal{B}\neg Goto_Both & \supset & Save_Money \\
\mathcal{B}\neg Goto_Australia & \supset & Cancel_Reservation \\
\mathcal{B}\neg Goto_Europe & \supset & Cancel_Reservation.
\end{array}$$

Let $T^0 = Cn_*(T)$ and assume obvious abbreviations. Clearly, in all minimal models of T^0 the disjunctions $GA \vee GE$ and $\neg GA \vee \neg GE$ hold true. Therefore,

$$T^0 \models_{\min} GA \vee GE, \quad T^0 \models_{\min} \neg GA \vee \neg GE \quad \text{and} \quad T^0 \models_{\min} \neg GB$$

and, consequently,

$$T^1 = \Psi_T(T^0) = Cn_*(T \cup \{\mathcal{B}(GA \vee GE), \mathcal{B}(\neg GA \vee \neg GE), \mathcal{B}\neg GB, \ldots\}).$$

Now $T^1 \models_{\min} SM$ and thus

$$T^2 = \Psi_T(T^1) = Cn_*(T \cup \{\mathcal{B}(GA \vee GE), \mathcal{B}(\neg GA \vee \neg GE), \mathcal{B}\neg GB, \mathcal{B}SM, \ldots\}).$$

It is easy to see that there is a minimal model of T^2 in which $\mathcal{B}\neg Goto\_Australia$ is true and thus also *Cancel_Reservation* is true. But there is also is a minimal model of T^2 in which both $\mathcal{B}\neg Goto\_Australia$ and $\mathcal{B}\neg Goto\_Europe$ are false and thus also *Cancel_Reservation* is false. Consequently,

$$T^2 \not\models_{\min} CR \quad \text{and} \quad T^2 \not\models_{\min} \neg CR.$$

This leads to the conclusion that $T^3 = \Psi_T(T^2) = T^2$ is a fixed point and therefore the static completion $\overline{T}$ of T is given by

$$\overline{T} = Cn_*(T \cup \{\mathcal{B}(GA \vee GE), \mathcal{B}(\neg GA \vee \neg GE), \mathcal{B}\neg GB, \mathcal{B}SM, \ldots\}).$$

It establishes that the individual is expected to travel either to Australia or to Europe but is not expected to do both trips and thus will save money. One easily verifies that T does not have any other (consistent) static expansions.

It is important to stress that *Cancel_Reservation* is *not* a logical consequence of the static semantics $\overline{T}$ of the previously considered (translated) program $T = T_{\mathcal{B}\neg}(P)$. This follows from the fact that the static completion $\overline{T}$ does *not* infer [12] $\mathcal{B}\neg Goto\_Australia \vee \mathcal{B}\neg Goto\_Europe$ even though it derives $\mathcal{B}(\neg Goto\_Australia \vee \neg Goto\_Europe)$. This reflects the notion that from the fact that a disjunction $F \vee G$ is believed to be true, one does *not* necessarily want to conclude that either F is believed or G is believed. In this particular case, we do not want to cancel our reservations to either Australia or to Europe until we find out precisely *which* one of them we will actually *not* visit. In other

[12] However, by the Consistency Axiom (1), $\overline{T}$ implies the weaker formula: $\neg\mathcal{B}(Goto\_Australia \wedge Goto\_Europe) \equiv \neg\mathcal{B}Goto\_Australia \vee \neg\mathcal{B}Goto\_Europe$.

words, we usually do *not* want to assume that the belief operator $\mathcal{B}$ is *distributive* with respect to disjunctions. However, one could easily ensure distributivity of beliefs with respect to disjunctions by assuming the following:

Disjunctive Belief Axiom. For any formulae F and G:

$$(DBA) \qquad \mathcal{B}(F \vee G) \equiv \mathcal{B}F \vee \mathcal{B}G.$$

This optional axiom states that our beliefs are distributive with respect to disjunctions. Note that, due to the Conjunctive Belief Axiom (4), our beliefs are always distributive with respect to conjunctions.

Example 4.6. When augmented with the Disjunctive Belief Axiom, *DBA*, the static semantics of the (translated) program $T = T_{\mathcal{B}\neg}(P)$ from Example 4.5:

$$\begin{array}{lll} Goto\_Australia \vee Goto\_Europe & & \\ Goto\_Australia \wedge Goto\_Europe & \supset & Goto\_Both \\ \mathcal{B}\neg Goto\_Both & \supset & Save\_Money \\ \mathcal{B}\neg Goto\_Australia & \supset & Cancel\_Reservation \\ \mathcal{B}\neg Goto\_Europe & \supset & Cancel\_Reservation. \end{array}$$

implies *Cancel_Reservation*. Indeed, belief theory T has a unique static expansion (static completion) $\overline{T}$ of T given by

$$\overline{T} = Cn_*(T \cup \{\mathcal{B}(GA \vee GE), \mathcal{B}(\neg GA \vee \neg GE), \mathcal{B}\neg GB, \mathcal{B}SM, \mathcal{B}CR, \ldots\}),$$

because now the axiom (DBA) implies $\mathcal{B}\neg Goto\_Australia \vee \mathcal{B}\neg Goto\_Europe$.

Observe also that the definition of static expansions and static completions carefully distinguishes between these formulae F which are *known to be true* in the expansion T^* (i.e., those for which $T^* \models F$), and those formulae F which are only *believed* (i.e., those for which $T^* \models \mathcal{B}F$). This important distinction not only increases the *expressiveness* of the language but is in fact quite crucial for many forms of reasoning. However, if we wanted to ensure that a formula F is always true whenever it is believed to be true we could use the following:

Belief Closure Axiom.

$$(BCA) \qquad \mathcal{L}\mathcal{B}F \supset F \quad \text{for any formula } F.$$

This optional axiom states that if a formula F is believed to be true (in a given expansion T^*), i.e., if $T^* \models \mathcal{B}F$ then F is in fact true (in T^*), i.e., $T^* \models F$. This is a powerful rule which, in essence, erases the distinction between facts *believed* to be true (in the expansion) and those which are actually *true*.

Static semantics for disjunctive programs has a number of important advantages but it is not the only semantics for disjunctive programs that can be derived by means of a suitable translation of a logic program into the autoepistemic logic of knowledge and beliefs, *AELB*. For example, we can also extend the transformation $T_{\neg\mathcal{L}}(P)$ to any disjunctive logic program P consisting of clauses

$$A_1 \vee \cdots \vee A_l \leftarrow B_1, \ldots, B_m, not\, C_1, \ldots, not\, C_n$$

by translating it into the (affirmative) autoepistemic theory consisting of formulae

$$B_1 \wedge \cdots \wedge B_m \wedge \neg\mathcal{L}C_1 \wedge \cdots \wedge \neg\mathcal{L}C_n \supset A_1 \vee \cdots \vee A_l.$$

It follows from the results obtained in [27] and from Theorem 2.13 that this transformation augmented with the following:

Positive Introspection Axiom. For any objective atom A:

$$(PIA) \qquad A \supset \mathcal{L}A$$

produces the *disjunctive stable semantics* originally defined in [12,26]. This optional and rather strong axiom states that if A holds in some model of a given theory then A is known to be true, i.e., $\mathcal{L}A$ holds.

Theorem 4.7. (Embeddability of disjunctive stable semantics) *There is a natural one-to-one correspondence between disjunctive stable models of a disjunctive program P and consistent static expansions $T^\diamond$ of its translation $T_{\neg\mathcal{L}}(P)$ into belief theory augmented with the Positive Introspection Axiom, (PIA).*

Proof. By Theorem 2.13 there is a one-to-one correspondence between stable autoepistemic expansions and consistent static expansions of $T_{\neg\mathcal{L}}(P)$. The claim now follows from the results obtained in [27]. □

Example 4.8. When augmented with augmented with the Positive Introspection Axiom, (PIA), the program $T = T_{\neg\mathcal{L}}(P)$ from Example 4.5,

$$\begin{array}{lcl}
Goto_Australia \vee Goto_Europe & & \\
Goto_Australia \wedge Goto_Europe & \supset & Goto_Both \\
\neg\mathcal{L}Goto_Both & \supset & Save_Money \\
\neg\mathcal{L}Goto_Australia & \supset & Cancel_Reservation \\
\neg\mathcal{L}Goto_Europe & \supset & Cancel_Reservation.
\end{array}$$

has two static expansions,

$$T_1^* = Cn_*(T \cup \{GA, \neg\mathcal{L}GE, \neg\mathcal{L}GB, SM, CR, \ldots\}),$$
$$T_2^* = Cn_*(T \cup \{GE, \neg\mathcal{L}GA, \neg\mathcal{L}GB, SM, CR, \ldots\}),$$

which correspond to the so-called *perfect models* of this stratified disjunctive logic program [24].

Several other semantics proposed for disjunctive programs can be obtained in a similar way (see e.g. [8]) thus demonstrating the expressive power and modularity of *AELB*.

4.4. Combining knowledge and belief: Mixing stable and well-founded negation

As we have seen in Theorems 4.1 and 4.2, both stable and well-founded (partial stable) negation in logic programs can be obtained by translating the nonmonotonic negation $not\,C$ into introspective literals $\neg\mathcal{L}C$ and $\mathcal{B}\neg C$, respectively. However, the existence of both types of introspective literals in *AELB* allows us to *combine both types of negation* in one belief theory consisting of formulae of the form

$$B_1 \wedge \cdots \wedge B_m \wedge \neg\mathcal{L}C_1 \wedge \cdots \wedge \neg\mathcal{L}C_k \wedge \mathcal{B}\neg C_{k+1} \wedge \cdots \wedge \mathcal{B}\neg C_n \supset A_1 \vee \cdots \vee A_l.$$

Such a belief theory may be viewed as representing a more *general disjunctive logic program* which permits the simultaneous use of both types of negation. In such logic programs, the first k negative premises represent *stable negation* and the remaining ones represent the *well-founded negation*. The ability to use both types of negation significantly increases the expressibility of logic programs. For instance, the example

$$\begin{aligned} \mathcal{B}\neg baseball \wedge \mathcal{B}\neg football &\supset rent\_movie \\ \neg\mathcal{L}baseball \wedge \neg\mathcal{L}football &\supset dont\_buy\_tickets. \end{aligned}$$

discussed in the Introduction is a special case of such a generalized logic program.

4.5. Adding strong negation to logic programs

The negation operator $not\,A$ used in logic programs does not represent the *classical negation*, but rather a nonmonotonic negation by default. Gelfond and Lifschitz pointed out [12] that in logic programming, as well as in other areas of nonmonotonic reasoning, it is often useful to use *both* the nonmonotonic negation and a different negation, $-A$, which they called "classical negation" but which can perhaps more appropriately be called "strong negation" [1]. They also extended the stable model semantics to the class of *extended logic programs* with strong negation.

It is easy to add strong negation to the autoepistemic logic of knowledge and beliefs, *AELB*. All one needs to do is to augment the original objective language $\mathcal{K}$ with new *objective* propositional symbols "$-A$" with the intended meaning that "*$-A$ is the strong negation of A*", or, equivalently, "*$-A$ is the opposite of A*" and assume the following *strong negation axiom* schema:

$$(SNA) \qquad A \wedge -A \supset false, \text{ or, equivalently, } -A \supset \neg A.$$

Observe that, as opposed to classical negation $\neg$, the law of excluded middle $A \vee -A$ is not assumed. As pointed out by Bob Kowalski, the proposition A may describe the property of being "*good*" while proposition $-A$ describes the property of being "*bad*". The strong negation axiom states that things cannot be both good and bad. We do not assume, however, that things must always be either good or bad.

Since this method of defining strong negation applies to *all* belief theories, it applies, in particular, to normal and disjunctive logic programs (see also [1]). Moreover, the following theorem shows that the resulting general framework provides a strict *generalization* of the original approach proposed by Gelfond and Lifschitz.

Theorem 4.9. (Embeddability of extended stable semantics) *There is a one-to-one correspondence between stable models $\mathcal{M}$ of an extended logic program P with strong negation, as defined in* [12], *and consistent static autoepistemic expansions T^* of its translation $T_{\neg\mathcal{L}}(P)$ into belief theory in which strong negation of an atom A is translated into* $-A$.

Proof. Easily follows from Theorem 2.13 and the results obtained in [29]. □

The reader is referred to [4–6] for a much more thorough discussion of strong negation as well as explicit negation in belief theories. The notion of strong negation is used in the next section.

5. Epistemic specifications as knowledge and belief theories

Epistemic specifications were recently introduced in [10] using a fairly involved language of belief sets and world views which includes two operators, $\mathbf{K}F$ and $\mathbf{M}F$, called belief and possibility operators, respectively. As an illustration of the expressive power of the Autoepistemic Logic of Minimal Beliefs, *AELB*, we now demonstrate that epistemic specifications can be also *isomorphically embedded* as a proper subset of *AELB*, and thus, in particular, epistemic specifications can be defined entirely in the language of classical propositional logic.

We show that Gelfond's belief operator $\mathbf{K}F$ can be defined as $\mathcal{LB}F$ and thus have the intended meaning "F is known to be believed". On the other hand, the possibility operator $\mathbf{M}F$ is proved to be equivalent to $\neg\mathbf{K}\neg F$, or, equivalently, to $\neg\mathcal{LB}\neg F$. The translation provides therefore an example of a *nested use* of the belief and knowledge operators, $\mathcal{B}$ and $\mathcal{L}$ (see also the axiom (*GCWA*) in Section 4.3).

Due to the space limitation, we assume familiarity with epistemic specifications. Let G be a database describing Gelfond's epistemic specification. Define $T(G)$ to be its translation into autoepistemic logic of minimal beliefs, *AELB*, obtained by

(i) Replacing, for all *objective* atoms A, the classical negation symbol $\neg A$ by the strong negation symbol $-A$. We assume that the objective language $\mathcal{K}$ was first augmented with strong negation atoms $-A$ as described in Section 4.5.

(ii) Eliminating Gelfond's "possibility" operator $\mathbf{M}$ by replacing every expression of the form $\mathbf{M}F$ by the expression $\neg\mathbf{K}\neg F$, where $\mathbf{K}$ is Gelfond's "belief" operator.

(iii) Finally, eliminating Gelfond's "belief" operator $\mathbf{K}$ by replacing every expression of the form $\mathbf{K}F$ by the autoepistemic formula $\mathcal{LB}F$.

The substitution (i) is motivated by the fact that in his paper Gelfond uses the classical negation symbol $\neg A$ when in fact he refers to *strong negation* $-A$. The substitution allows us to reserve the standard negation symbol $\neg A$ for true classical negation. The substitution (ii) is motivated by the fact that Gelfond's "possibility" operator $\mathbf{M}F$ can now be shown to be *equivalent* to $\neg\mathbf{K}\neg F$, and, vice versa, $\mathbf{K}F$ can be shown to be equivalent to $\neg\mathbf{M}\neg F$. The last substitution (iii) leads to a complete translation of epistemic specifications into an autoepistemic theory of knowledge and belief. It replaces $\mathbf{K}F$ by the formula $\mathcal{LB}F$ with the intended meaning "*F is known to be believed*". Equivalently, its intended meaning can be described by "*F is known to be true in all minimal models*".

The following result shows that epistemic specifications are isomorphically embeddable into the autoepistemic logic of minimal beliefs, *AELB*.

Theorem 5.1. (Embeddability of epistemic specifications) *Epistemic specifications are isomorphically embeddable into the autoepistemic logic of minimal beliefs, AELB. More precisely, there is a one-to-one correspondence between world views V of an epistemic specification G and static autoepistemic expansions T^* of its translation $T(G)$ into AELB. Moreover, there is a one-to-one correspondence between belief sets B of a world view V and minimal models $\mathcal{M}$ of the corresponding static expansion T^* of $T(G)$.*

Proof. The limited size of this paper does not allow us to provide all the details involved in the definition of epistemic specifications. Consequently, the proof of this theorem will appear elsewhere [30]. □

Remark 5.2. It is important to point out that formulae in epistemic specifications can contain existential quantifiers. However, since existential quantification in epistemic specifications is defined by means of substituting all terms of the corresponding Herbrand universe for the quantified variables, it is completely equivalent to a (possibly infinite) quantifier-free theory. Consequently, the non-existence of quantifiers in *AELB* does not hinder in any way the generality of the above result.

Gelfond's paper contains several interesting examples of epistemic specifications which now can be easily translated into the language of *AELB*.

6. Modifying the notion of belief

The proposed formalism of Autoepistemic Logic of Knowledge and Belief is quite flexible and allows various extensions and modifications. We have already seen that one can often ensure desired meaning of autoepistemic theories by adding suitable axioms to the logic *AELB*. Below we show that the meaning of theories in *AELB* can also be adjusted by suitably changing the underlying nonmonotonic formalism on which the notion of belief is based.

In our approach we used the minimal model semantics $T \models_{\min} F$ or the *Generalized Closed World Assumption GCWA* [19] to define the meaning of our beliefs $\mathcal{B}F$. In other words, F is believed if F is true in all minimal models of the expansion. As illustrated by the following example, by using the weak minimal model semantics $T \models_{\mathrm{wmin}} F$ or the *Weak Generalized Closed World Assumption WGCWA* [32,33] instead and thus requiring that F is believed if F is true in all *weakly minimal models* of T, one can ensure that disjunctions are treated *inclusively* rather than *exclusively*. Due to the limited size of the paper the reader is referred to the above listed publications for the definition of weakly minimal models.

Example 6.1. Consider the following (translated) positive program T:

$$\begin{array}{l} A \vee B \\ A \wedge B \supset C. \end{array}$$

Let $T^0 = Cn_*(T)$. Clearly, in all *weakly* minimal models of T^0 the disjunction $A \vee B$ holds. On the other hand, while the disjunction $\neg A \vee \neg B$ is true in all minimal models of T^0 it is not true in all weakly minimal models of T^0. Therefore,

$$T^0 \models_{\mathrm{wmin}} A \vee B \quad \text{and} \quad T^0 \models_{\mathrm{min}} \neg A \vee \neg B \quad \text{and yet} \quad T^0 \not\models_{\mathrm{wmin}} \neg A \vee \neg B.$$

As a result:

$$\begin{array}{l} \{\mathcal{B}(A \vee B), \mathcal{B}(\neg A \vee \neg B), \mathcal{B}\neg C\} \subseteq T^1 = \Psi_T(T^0) \\ \text{but only } \{\mathcal{B}(A \vee B)\} \subseteq T^1_{\mathrm{w}} = \Psi^{\mathrm{w}}_T(T^0), \end{array}$$

where by the index "w" we indicate the fact that we are using *WGCWA* instead of *GCWA* in the definition of static expansions and static completion. It is easy to see that $T^1 = \Psi_T(T^1)$ is a fixed point and therefore

$$\overline{T} = \overline{T \cup \{\mathcal{B}(A \vee B), \mathcal{B}(\neg A \vee \neg B), \mathcal{B}\neg C\}}.$$

Similarly, $T^1_{\mathrm{w}} = \Psi^{\mathrm{w}}_T(T^1_{\mathrm{w}})$ is a fixed point and therefore

$$\overline{T}^{\mathrm{w}} = \overline{T \cup \{\mathcal{B}(A \vee B)\}}.$$

We conclude that under *GCWA* we can derive that both $A \vee B$ and $\neg A \vee \neg B$ as well as $\neg C$ are believed, whereas *WGCWA* only allows us to believe $A \vee B$.

Both *GCWA* and *WGCWA* are very natural nonmonotonic formalisms which seem to closely correspond to the intuitive meaning of negation in logic programs and deductive databases. However, they also share an important feature which in some applications domains may be viewed as a drawback, namely the fact that they both minimize only *positive* literals (atoms) thus leading to immediate asymmetry between positive and negative literals. If this feature of *GCWA* and *WGCWA* is undesirable, one can use some other nonmonotonic formalism, naturally leading to a different notion of belief and thus to a different semantics. In particular, one can use a suitable form of predicate or formula circumscription which minimizes those and only those predicates (formulae) whose minimization is desired.

7. Conclusion

We introduced an extension, *AELB*, of Moore's autoepistemic logic, *AEL*, and showed that it provides a powerful and general knowledge representation framework unifying several well-known nonmonotonic formalisms and semantics for normal and disjunctive logic programs. It allows us to compare different formalisms, better understand mutual relationships existing between them and introduce simpler and more natural definitions of some of them.

Other applications of *AELB* include contradiction removal, abduction and diagnosis [3,4]. In [9] semantic and syntactic characterizations of static expansions are obtained in special classes of belief theories which are then used in the implementation of a prototype interpreter for disjunctive logic programming. In [8] these characterizations of static expansions are used to establish a close relationship between the static semantics and the disjunctive well-founded semantics, *D-WFS* [7], of disjunctive programs.

The proposed formalism significantly differs from other formalisms based on the notion of minimal beliefs. In particular, it it is different from the circumscriptive autoepistemic logic introduced in [25] and the logic of minimal beliefs and negation as failure proposed in [16]. The proposed approach is also quite flexible by allowing various extensions and modifications, including the use of a different formalism defining the *meaning of beliefs* and introduction of *additional axioms*. By using such modifications one may be able to tailor the formalism to fulfill the needs of different application domains.

Acknowledgments

The author is grateful to Jose Alferes, Roland Bol, Stefan Brass, Juergen Dix, Michael Gelfond, Vladimir Lifschitz, Victor Marek, Luis Pereira, Halina Przymusinska, Mirek Truszczynski and two anonymous referees for their helpful comments. The author is especially grateful to Grigorij Schwartz for his extensive remarks.

References

[1] J.J. Alferes and L.M. Pereira, On logic program semantics with two kinds of negation, in: K. Apt, ed., *Proceedings International Joint Conference and Symposium on Logic Programming* (MIT Press, Cambridge, MA, 1992) 574–588.

[2] J.J. Alferes, L.M. Pereira and T.C. Przymusinski, Belief revision in logic programming and nonmonotonic reasoning, in: *Progress in AI, Proceedings International Artificial Intelligence Conference (EPIA-95)*, Madeira, Portugal (1995) 41–56.

[3] J.J. Alferes, L.M. Pereira and T.C. Przymusinski, Belief revision in logic programming and nonmonotonic reasoning, *Fund. Inform.* 28 (1-2) (1996) 1-2.

[4] J.J. Alferes, L.M. Pereira and T.C. Przymusinski, 'Classical' negation in non-monotonic reasoning, *J. Automated Reasoning*, to appear.

[5] J.J. Alferes, L.M. Pereira and T.C. Przymusinski, Symmetric negation in non-monotonic reasoning, in: *Proceedings International Conference on Artificial Intelligence and Mathematics (AI&Math-96)* (1996) 1–4.

[6] J.J. Alferes, L.M. Pereira and T.C. Przymusinski, Strong and explicit negation in nonmonotonic reasoning and logic programming, in: J.J. Alferes, L.M. Pereira and E. Orlowska, eds., *Logic in Artificial Intelligence: Proceedings European Workshop (JELIA-96)*, Lecture Notes in Artificial Intelligence, Vol. 1126 (Springer, Berlin, 1996) 143–163.

[7] S. Brass and J. Dix, Disjunctive semantics based upon partial and bottom-up evaluation, in: L. Sterling, ed., *Proceedings 12th International Conference on Logic Programming* (MIT Press, Cambridge, MA, 1995).

[8] S. Brass, J. Dix, I. Niemela and T.C. Przymusinski, A comparison of the static and disjunctive well-founded semantics, in: *Proceedings International Logic Programming Symposium (ILPS-97)*, to appear.

[9] S. Brass, J. Dix and T.C. Przymusinski, Super logic programs, in: *Proceedings International Conference on Principles of Knowledge Representation and Reasoning (KR-96)*, Vancouver, BC (Morgan Kaufmann, San Mateo, CA, 1996) 529–541.

[10] M. Gelfond, Logic programming and reasoning with incomplete information, Tech. Rept., University of Texas at El Paso, TX (1992).

[11] M. Gelfond and V. Lifschitz, The stable model semantics for logic programming, in: R. Kowalski and K. Bowen, eds., *Proceedings 5th Logic Programming Symposium* (MIT Press, Cambridge, MA, 1988) 1070–1080.

[12] M. Gelfond and V. Lifschitz, Logic programs with classical negation, in: *Proceedings 7th International Logic Programming Conference*, Jerusalem, Israel (MIT Press, Cambridge, MA, 1990) 579–597.

[13] M. Gelfond, H. Przymusinski and T.C. Przymusinski, On the relationship between circumscription and negation as failure, *Artificial Intelligence* 38 (1989) 75–94.

[14] J.F. Horty, R.H. Thomason and D.S. Touretzky, A skeptical theory of inheritance in non-monotonic semantic nets, in: *Proceedings AAAI-87*, Seattle, WA (Morgan Kaufmann, Los Altos, CA, 1987).

[15] V. Lifschitz, Computing circumscription, in: *Proceedings IJCAI-85*, Los Angeles, CA (Morgan Kaufmann, Los Altos, CA, 1985) 121–127.

[16] V. Lifschitz, Minimal belief and negation as failure, *Artificial Intelligence* 70 (1994) 53–72.

[17] J. Lobo, J. Minker and A. Rajasekar, *Foundations of Disjunctive Logic Programming* (MIT Press, Cambridge, MA, 1992).

[18] J. McCarthy, Circumscription—a form of non-monotonic reasoning, *Artificial Intelligence* 13 (1980) 27–39.

[19] J. Minker, On indefinite data bases and the closed world assumption, in: *Proceedings 6th Conference on Automated Deduction* (Springer, New York, 1982) 292–308.

[20] R.C. Moore, Semantic considerations on nonmonotonic logic, *Artificial Intelligence* 25 (1985) 75–94.

[21] W. Marek, Stable theories in autoepistemic logic, *Fund. Inform.* 12 (1989) 243–254.

[22] W. Marek and M. Truszczynski, *Non-Monotonic Logic* (Springer, Berlin, 1994).

[23] H. Przymusinska and T.C. Przymusinski, Semantic issues in deductive databases and logic programs, in: R. Banerji, ed., *Formal Techniques in Artificial Intelligence* (North-Holland, Amsterdam, 1990) 321–367.

[24] T.C. Przymusinski, On the declarative semantics of deductive databases and logic programs, in: J. Minker, ed., *Foundations of Deductive Databases and Logic Programming* (Morgan Kaufmann, Los Altos, CA, 1988) 193–216.

[25] T.C. Przymusinski, Autoepistemic logics of closed beliefs and logic programming, in: A. Nerode, W. Marek and V.S. Subrahmanian, eds., *Proceedings 1st International Workshop on Logic Programming and Non-Monotonic Reasoning*, Washington, DC (MIT Press, Cambridge, MA, 1991) 3–20.

[26] T.C. Przymusinski, Stable semantics for disjunctive programs, *New Generation Computing J.* 9 (1991) 401–424; extended abstract appeared as: Extended stable semantics for normal and disjunctive logic programs, in: *Proceedings 7th International Logic Programming Conference*, Jerusalem, Israel (MIT Press, Cambridge, MA, 1990) 459–477.

[27] T.C. Przymusinski, Three-valued nonmonotonic formalisms and semantics of logic programs, *Artificial Intelligence* 49 (1991) 309–343; extended abstract appeared as: Three-valued non-monotonic formalisms and logic programming, in: *Proceedings 1st International Conference on Principles of Knowledge Representation and Reasoning (KR-89)*, Toronto, Ont. (Morgan Kaufmann, Los Altos, CA, 1989) 341–348.

[28] T.C. Przymusinski, Semantics of normal and disjunctive logic programs: a unifying framework, in: J. Dix, L.M. Pereira and T.C. Przymusinski, eds., *Proceedings Workshop on Non-Monotonic Extensions of Logic Programming at the 11th International Logic Programming Conference (ICLP-95)*, Santa Margherita Ligure, Italy, 1994 (Springer, Berlin, 1995) 43–67.

[29] T.C. Przymusinski, Static semantics for normal and disjunctive logic programs, *Ann. Math. Artif. Intell.* (Special Issue on Disjunctive Programs) 14 (1995) 323–357.

[30] T.C. Przymusinski, Epistemic specifications as theories of knowledge and belief, University of California at Riverside, in preparation.

[31] R. Reiter, On closed-world data bases, in: H. Gallaire and J. Minker, eds., *Logic and Data Bases* (Plenum Press, New York, 1978) 55–76.

[32] A. Rajasekar, J. Lobo and J. Minker, Weak generalized closed world assumption, *J. Automated Reasoning* 5 (1989) 293–307.

[33] K. Ross and R. Topor, Inferring negative information from disjunctive databases, *J. Automated Reasoning* 4 (1988) 397–424.

[34] A. van Gelder, K.A. Ross and J.S. Schlipf, The well-founded semantics for general logic programs, *J. ACM* (1990) to appear; preliminary abstract appeared in: *Proceedings 7th ACM Symposium on Principles of Database Systems* (1988) 221–230.

[35] L. Yuan and J. You, Autoepistemic circumscription and logic programming, Tech. Rept. TR92-18, Department of Computing Science, University of Alberta, Edmonton, Alta. (1992); also: *J. Automated Reasoning*, to appear.