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Linguistic quantifiers modeled by Sugeno integrals

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Abstract

Since quantifiers have the ability of summarizing the properties of a class of objects without enumerating them, linguistic quantification is a very important topic in the field of high level knowledge representation and reasoning. This paper introduces a new framework for modeling quantifiers in natural languages in which each linguistic quantifier is represented by a family of fuzzy measures, and the truth value of a quantified proposition is evaluated by using Sugeno's integral. This framework allows us to have some elegant logical properties of linguistic quantifiers. We compare carefully our new model of quantification and other approaches to linguistic quantifiers. A set of criteria for linguistic quantification was proposed in the previous literature. The relationship between these criteria and the results obtained in the present paper is clarified. Some simple applications of the Sugeno's integral semantics of quantifiers are presented.

Keywords: High level knowledge representation and reasoning; Natural language understanding; Computing with words; Fuzzy logic; Quantifier; Fuzzy measure; Sugeno's integral

1. Introduction

First order logic increases the expressive power of propositional logic a lot through adding quantifiers. Classical first order logic only possesses two quantifiers, the universal quantifier (∀) and the existential quantifier (∃). However, these quantifiers are often too limited to express some properties of certain mathematical structures and to model certain knowledge stated in natural languages. This leads logicians and linguists to introduce the notion of generalized quantifiers.

As early as in 1957, Mostowski [29] proposed a general notion of generalized quantifier and showed that first order logic with a class of generalized quantifiers are not axiomatizable. This work together with others initiated the subject of model theoretic logics.

Barwise and Cooper [2] started the studies of generalized quantifiers in natural languages. Since then, a rich variety of generalized quantifiers in natural languages have been found, and their expressive power and logical properties have been thoroughly investigated from both semantic and syntactic aspects. In particular, van Benthem [36] viewed

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a generalized quantifier as a relation on the subsets of a universe of discourse, and systematically examined various relational behaviors of generalized quantifiers such as reflexivity, symmetry, transitivity, linearity and monotonicity and their roles in realizing certain inference patterns. For a recent review on the theory of generalized quantifiers in natural languages, we refer to [24].

It has been clearly realized in the artificial intelligence community that natural languages are suited to high level knowledge representation [26,33]. This is indeed one of the main motivations of computing with words [25,50,57]. However, classical logic is not adequate to face the essential uncertainty, vagueness and ambiguity of human reasoning expressed in natural languages. Consequently, the logical treatments and mathematical models of the concepts of uncertainty, vagueness and ambiguity is of increasing importance in artificial intelligence and related researches, and many logicians have proposed different logic systems as a formalization of reasoning under uncertainty, vagueness and ambiguity (see, for example, [3,44–47,49], [11, Chapter III.1], or [19, Chapter 7]).

Since quantifiers have the ability of summarizing the properties of a class of objects without enumerating them, linguistic quantification is a very important topic in the field of knowledge representation and reasoning. Quantifiers in natural languages are usually vague in some sense. Some representative examples of linguistic quantifiers with vagueness are [56]: *several, most, much, not many, very many, not very many, few, quite a few, large number, small number, close to five, approximately ten, frequently*. It is clear that two-valued logic is not suited to cope with vague quantifiers. There has been, therefore, increasing interest about logical treatment of quantifiers in human languages in fuzzy logic community. Indeed, sometimes fuzzy logic permits a more precise representation of the kind of quantifiers in various natural languages.

The first fuzzy set theoretic approach to linguistic quantifiers was described by Zadeh [55,56]. In his approach, linguistic quantifiers are treated as fuzzy numbers and they may be manipulated through the use of arithmetic for fuzzy numbers. The truth evaluation of a linguistically quantified statement is performed by computing the cardinality of the fuzzy set defined by the linguistic predicate in such a statement and then by finding the degree to which this cardinality is compatible with the involved quantifier. Since then, a considerable amount of literature [1,4–7,9,10, 12,15–18,30–32,42,43,51,52] has been devoted to the studies of linguistic quantifiers in the framework of fuzzy set theory. For example, in a series of papers [39–41], Yager proposed the substitution method for evaluating quantified propositions and the method based on OWA operators. For a survey, see [27,28].

On the other hand, fuzzy quantification models are employed in solving a great variety of problems from many different fields such as database querying [7,23], data mining and knowledge discovering [7,25], information fusion [21,25], group decision making and multiple-objective decision making [20,40], inductive learning [21], and optimization and control [22].

This paper introduces a new framework for modeling quantifiers in natural languages. In this framework, linguistic quantifiers are represented by Sugeno's fuzzy measures [35]. More precisely, a quantifier Q is seen as a family of fuzzy measures indexed by nonempty sets. For each nonempty set X, the quantifier Q limited to the discourse universe X is defined to be a fuzzy measure Q_X on X, and for any subset E of X, the quantity $Q_X(E)$ expresses the truth value of the quantified statement "*Q Xs are As*" when A is a crisp predicate and the set of elements in X satisfying A is E. As is well known, predicates in linguistically quantified statements are often vague too. In this general case, the truth value of a quantified proposition is then evaluated by using Sugeno's integral [35].

The advantage of this framework is that it allows us to have some elegant logical properties of linguistic quantifiers. For example, we are able to establish a prenex normal form theorem for linguistic quantifiers (see Corollary 34). It should be pointed out that this paper only deals with increasing quantifiers because fuzzy measures assume monotonicity. Thus, quantifiers such as *several, few, quite a few, small number, not many, not very many, close to five, approximately ten* cannot be modeled in our proposed setting.

This paper is arranged as follows. For convenience of the reader, in Section 2 we review some notions and results from the theory of Sugeno's fuzzy measures and integrals. In Section 3, (linguistic) quantifiers are formally defined in terms of fuzzy measure, and several operations of quantifiers are introduced. In Section 4, we construct a first order language with linguistic quantifiers and present semantics of such a logical language. In particular, the truth valuation of quantified formulas is given by using Sugeno's integrals. Section 5 is devoted to examine thoroughly logical properties of linguistic quantifiers. In particular, we prove a prenex normal form theorem for logical formulas with linguistic quantifiers. In Section 6, the notions of cardinal and numeric quantifiers are introduced so that we are able to establish a close link between the Sugeno integral semantics and the Zadeh's cardinality-based semantics of linguistic quantifiers. In Section 7, we present some simple applications to illustrate the utility of the results obtained in the current paper. In Section 8, our Sugeno integral approach to evaluation of quantified statements is compared with others. A set of criteria for linguistic quantification was proposed in the previous literature. The relationship between these criteria and the results obtained in the present paper is clarified. We draw conclusions and point out some problems for further studies in Section 9.

2. Fuzzy measures and Sugeno integrals

This is a preliminary section. In this section, we are going to review some notions and fundamental results needed in the sequel from the theory of fuzzy measures and Sugeno's integrals. For details, we refer to [35] or [11, Chapter 5].

The theory of fuzzy measures and integrals was originally proposed by Sugeno [35]. Fuzzy measure is a generalization of the notion of measure in mathematical analysis, and it relaxes the condition of additivity for usual measure and only assume monotonicity. Thus, fuzzy measures are very general, and probability measures, Zadeh's possibility measures, Shafer's belief functions among others [37] are shown to be special cases of fuzzy measures. Sugeno's integral is analogous to Lebesgue integral. The difference between them is that addition and multiplication in the definition of Lebesgue integral are replaced respectively by the operations "min" and "max" when Sugeno's integral is considered. Since its inception, the theory of Sugeno's measures and integrals has been applied in the fields of subjective evaluation, decision systems and pattern recognition, name a few.

A fuzzy measure on a set X is a function defined on some subsets of X. In general, the domain of a fuzzy measure can be taken to be a monotone family which is a set of subsets of X containing $\emptyset$ and X itself and closed under limits of monotone sequences of subsets of X. But in this paper we focus our attention on a special class of monotone family called Borel field.

Definition 1. [35, page 10] Let X be a nonempty set. A *Borel field* over X is a subset $\wp$ of 2^X satisfying the next conditions:

(1) $\emptyset \in \wp$;
(2) If $E \in \wp$, then $X - E \in \wp$; and
(3) If $E_n \in \wp$ for $1 \leqslant n < \infty$, then $\bigcup_{n=1}^{\infty} E_n \in \wp$.

A typical example of Borel field over a nonempty set X is the power set 2^X of X. Indeed, in this paper, we will mainly consider this special Borel field.

Definition 2. [35, Definition 2.3] If X is a nonempty set and $\wp$ is a Borel field over X, then $(X, \wp)$ is called a *measurable space*.

In order to define fuzzy measure, we need the notion of limit of set sequence. If $E_1 \subseteq \cdots \subseteq E_n \subseteq E_{n+1} \subseteq \cdots$, then the sequence $\{E_n\}$ is said to be increasing and we define

$$\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n,$$

and if $E_1 \supseteq \cdots \supseteq E_n \supseteq E_{n+1} \supseteq \cdots$, then the sequence $\{E_n\}$ is said to be decreasing and we define

$$\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n.$$

Both increasing and decreasing sequences of sets are said to be monotone.

Definition 3. [35, Definitions 2.2 and 2.4] Let $(X, \wp)$ be a measurable space. If a set function $m : \wp \to [0, 1]$ satisfies the following properties:

(1) $m(\emptyset) = 0$ and $m(X) = 1$;
(2) (Monotonicity) If $E, F \in \wp$ and $E \subseteq F$, then $m(E) \leqslant m(F)$; and

(3) (Continuity) If $E_n \in \wp$ for $1 \leqslant n < \infty$ and $\{E_n\}$ is monotone, then

$$m\Big(\lim_{n\to\infty} E_n\Big) = \lim_{n\to\infty} m(E_n),$$

then m is called a *fuzzy measure* over $(X, \wp)$, and $(X, \wp, m)$ is called a *fuzzy measure space*.

Intuitively, $m(E)$ expresses someone's subjective evaluation of the statement "x is in E" in a situation in which he guesses whether x is in E.

The continuity of fuzzy measure is often discarded. Obviously, a probability measure is a fuzzy measure. The notion of plausibility measure introduced in [13,14] or [19] (Section 2.8) is similar to that of fuzzy measure. The only difference between them is that the range of a plausibility measure can be any partially ordered set with top and bottom elements rather than the unit interval.

Let us first consider an example of fuzzy measure which will be used to define the existential quantifier.

Example 4. Let X be a nonempty set. If $\pi : X \to [0,1]$ is a mapping with $\sup_{x\in X} \pi(x) = 1$, then it is called a possibility distribution, and for each $E \subseteq X$, we define

$$\Pi_\pi(E) = \sup_{x\in E} \pi(x).$$

Then $\Pi_\pi(\cdot)$ is a fuzzy measure over $(X, 2^X)$ and it is called the possibility measure induced by π. It should be noted that a possibility measure is not continuous from top, that is, there is a decreasing sequence $\{E_n\}$ with

$$\Pi_\pi\Big(\lim_{n\to\infty} E_n\Big) < \lim_{n\to\infty} \Pi_\pi(E_n).$$

(1) Suppose that $x_0 \in X$ and

$$\pi_{x_0}(x) = \begin{cases} 1, & \text{if } x = x_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then π_{x_0} is a possibility distribution and $\Pi_{\pi_{x_0}}$ is the membership function of x_0, that is, for any $E \subseteq X$,

$$\Pi_{\pi_{x_0}}(E) = \begin{cases} 1, & \text{if } x_0 \in E, \\ 0, & \text{otherwise.} \end{cases}$$

(2) If we define $\pi(x) = 1$ for all $x \in X$, then it is easy to see that for any $E \subseteq X$,

$$\Pi_\pi = \begin{cases} 1, & \text{if } E \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

The notion of dual fuzzy measure is required when dealing with dual linguistic quantifier. We introduce such a notion in the next definition.

Definition 5. Let $(X, \wp, m)$ be a fuzzy measure space. Then the dual set function $m^* : \wp \to [0,1]$ of m is defined by

$$m^*(E) = 1 - m(X - E)$$

for each $E \in \wp$.

It is easy to see that m^* is a fuzzy measure over $(X, \wp)$ too.

The following example gives the dual of possibility measure, necessity measure. It will be used in defining the universal quantifier.

Example 6. Let π be a possibility distribution over X and Π_π the possibility measure induced by π. Then the dual measure $N_\pi = \Pi_\pi^*$ of Π_π is given by

$$N_\pi(E) = \inf_{x\notin E}\big(1 - \pi(x)\big)$$

for each $E \subseteq X$, and it is called the necessity measure induced by π.

(1) It is interesting to note that $N_{\pi_{x_0}} = \Pi^*_{\pi_{x_0}} = \Pi_{\pi_{x_0}}$.

(2) If $\pi(x) = 1$ for every $x \in X$, then

$$N_\pi(E) = \begin{cases} 1, & \text{if } E = X, \\ 0, & \text{otherwise.} \end{cases}$$

We now turn to consider the Sugeno's integral of a function. It is not the case that every function is integrable. Those integrable functions are isolated by the following definition.

Definition 7. [35, Definition 3.3] Let $(X, \wp)$ be a measurable space, and let $h : X \to [0, 1]$ be a mapping from X into the unit interval. For any $\lambda \in [0, 1]$, we write

$$H_\lambda = \{x \in X\colon\ h(x) \geqslant \lambda\}.$$

If $H_\lambda \in \wp$ for all $\lambda \in [0, 1]$, then h is said to be *$\wp$-measurable*.

The following lemma demonstrates measurability of some composed functions. Indeed, we will see later that this measurability guarantees that truth values of quantifications of negation, conjunction and disjunction are well-defined in our Sugeno integral semantics.

Lemma 8. [35, Propositions 3.1 and 3.2] *If h, h_1 and $h_2 : X \to [0, 1]$ are all $\wp$-measurable functions, then so are $1 - h$, $\min(h_1, h_2)$ and $\max(h_1, h_2)$, where*

$$(1 - h)(x) = 1 - h(x),$$
$$\min(h_1, h_2)(x) = \min\big(h_1(x), h_2(x)\big)$$

and

$$\max(h_1, h_2)(x) = \max\big(h_1(x), h_2(x)\big)$$

for every $x \in X$.

Now we are able to present the key notion in this section.

Definition 9. [35, Definition 3.1 and page 19] Let $(X, \wp, m)$ be a fuzzy measure space. If $A \in \wp$ and $h : X \to [0, 1]$ is a $\wp$-measurable function, then the *Sugeno's integral* of h over A is defined by

$$\int_A h \circ m = \sup_{\lambda \in [0,1]} \min\big[\lambda, m(A \cap H_\lambda)\big],$$

where $H_\lambda = \{x \in X\colon\ h(x) \geqslant \lambda\}$ for each $\lambda \in [0, 1]$. In particular, $\int_A h \circ m$ will be abbreviated to $\int h \circ m$ whenever $A = X$.

The next lemma gives an alternative definition of Sugeno's integral for the case that the Borel field in a measurable space is taken to be the power set of the underlying set.

Lemma 10. [35, Theorem 3.1] *If the Borel field $\wp$ in the fuzzy measure space $(X, \wp, m)$ is the power set 2^X of X, then for any function $h : X \to [0, 1]$, we have*:

$$\int_A h \circ m = \sup_{F \in 2^X} \min\Big[\inf_{x \in F} h(x), m(A \cap F)\Big].$$

A simplified calculation method of Sugeno's integrals over finite sets is presented in the following lemma.

Lemma 11. [35, Theorem 4.1] *Let $X = \{x_1, \dots, x_n\}$ be a finite set, and let $h :\to [0,1]$ be such that $h(x_i) \leqslant h(x_{i+1})$ for $1 \leqslant i \leqslant n-1$ (if not so, rearrange $h(x_i)$, $1 \leqslant i \leqslant n$). Then*

$$\int_A h \circ m = \max_{i=1}^{n} \min\bigl[h(x_i), m(A \cap X_i)\bigr],$$

where $X_i = \{x_j\colon\ i \leqslant i \leqslant n\}$ for $1 \leqslant i \leqslant n$.

We now collect some properties of Sugeno's integrals needed in what follows.

Lemma 12. [35, Propositions 3.4 and 3.5, Theorems 3.2 and 3.5] *Suppose that $(X, \wp, m)$ is a fuzzy measure space.*

(1) *Let $a \in [0,1]$. Then it holds that*

$$\int a \circ m = a,$$

where "a" in the left-hand side is seen as a constant function $a : X \to [0,1]$ such that $a(x) = a$ for every $x \in X$.

(2) *Let $h, h' : X \to [0,1]$ be two $\wp$-measurable functions. If $h \leqslant h'$, then there holds*

$$\int h \circ m \leqslant \int h' \circ m.$$

Moreover, for any $\wp$-measurable functions $h_1, h_2 :\to [0,1]$, we have:

$$\int \max(h_1, h_2) \circ m \geqslant \max\left(\int h_1 \circ m, \int h_2 \circ m\right),$$

$$\int \min(h_1, h_2) \circ m \leqslant \min\left(\int h_1 \circ m, \int h_2 \circ m\right).$$

In particular, the first inequality becomes equality when m is a possibility measure.

(3) *Let $a \in [0,1]$ and let $h : X \to [0,1]$ be a $\wp$-measurable function. Then there hold*

$$\int \max(a, h) \circ m = \max\left(a, \int h \circ m\right),$$

$$\int \min(a, h) \circ m = \min\left(a, \int h \circ m\right),$$

where "a" in the left-hand side is as in (1).

It may be observed that fuzzy measures are defined over crisp sets before. To conclude this section, we introduce the notion of extension of fuzzy measure over fuzzy sets. Zadeh [54] introduced a natural extension of probability measure on fuzzy sets. In a similar manner, an extension of fuzzy measure on fuzzy sets can be defined.

Definition 13. [35, Definition 3.7] Let $(X, \wp, m)$ be a fuzzy measure space and let $\widetilde{\wp}$ be the set of fuzzy subsets of X with $\wp$-measurable membership functions. Then the extension $\widetilde{m}$ of m on $\widetilde{\wp}$ is defined by

$$\widetilde{m}(h) = \int h \circ m$$

for all $h \in \widetilde{\wp}$.

3. Fuzzy quantifiers

Many versions of fuzzy set theoretic definition of linguistic quantifier have been introduced in the previous literature. In this paper, we take a different starting point, and a linguistic quantifier will be represented by a family of fuzzy measures. We first give a formal definition of fuzzy quantifier in this new framework. To do this, a new notation is needed. For any measurable space $(X, \wp)$, we write $M(X, \wp)$ for the set of all fuzzy measures on $(X, \wp)$.

Definition 14. A fuzzy quantifier (or quantifier for short) consists of the following two items:
(i) for each nonempty set X, a Borel field $\wp_X$ over X is equipped; and
(ii) a choice function

$$Q:(X,\wp_X)\mapsto Q_{(X,\wp_X)}\in M(X,\wp_X)$$

of the (proper) class $\{M(X,\wp_X):(X,\wp_X)\text{ is a measurable space}\}$.

Intuitively, for a given discourse universe X, if the set of objects in X satisfying a (crisp) property A is E, then the quantity $Q_X(E)$ is thought of as the truth value of the quantified proposition "*Q Xs are As*".

For simplicity, $Q_{(X,\wp_X)}$ is often abbreviated to Q_X whenever the Borel field $\wp_X$ can be recognized from the context. In some applications, Q_X is allowed to be undefined or unspecified for some sets X.

To illustrate the above definition, let us consider some examples. The simplest quantifiers are the universal and existential quantifiers.

Example 15. The universal quantifier $\forall$ = "all" and the existential quantifier $\exists$ = "some" are defined as follows, respectively: for any set X and for any $E\subseteq X$,

$$\forall_X(E)=\begin{cases}1, & \text{if } E=X,\\ 0, & \text{otherwise;}\end{cases}$$

$$\exists_X(E)=\begin{cases}1, & \text{if } E\neq\emptyset,\\ 0, & \text{otherwise.}\end{cases}$$

The universal and existential quantifiers are crisp quantifiers because $\forall_X(E),\exists_X(E)\in\{0,1\}$ for all $E\subseteq X$.

This example shows that both the universal and existential quantifiers can be accommodated in our fuzzy measure definition of quantifiers. However, when the discourse universe X is infinite, Zadeh's cardinality approach to quantifiers cannot be used to treat the universal quantifier because it is possible that a proper subset E of X has the same cardinality as X.

Except the universal and existential quantifiers, some quantifiers frequently occurred in natural languages can also be defined well in terms of fuzzy measures. Let us see the following example.

Example 16. For any set X and for any $E\subseteq X$, we define

$$\textit{at least three}_X(E)=\begin{cases}1, & \text{if } |E|\geqslant 3,\\ 0, & \text{otherwise;}\end{cases}$$

"at least three" is an example of crisp generalized quantifier too. The following are three typical examples of fuzzy quantifiers. Suppose that X is a nonempty finite set. Then we define

$$\textit{many}_X(E)=\frac{|E|}{|X|},$$

$$\textit{most}_X(E)=\left(\frac{|E|}{|X|}\right)^{3/2},$$

$$\textit{almost all}_X(E)=\left(\frac{|E|}{|X|}\right)^{2}$$

for any subset E of X, where $|E|$ stands for the cardinality of E. The above definitions of quantifiers "many", "most" and "almost all" can be generalized to the case of infinite discourse universe X. Let $(X,\wp)$ be a measurable space, and let μ be a finite measure on $(X,\wp)$, that is, a mapping $\mu:\wp\to[0,\infty)$ such that

$$\mu\left(\bigcup_{n=1}^{\infty}E_n\right)=\sum_{n=1}^{\infty}\mu(E_n),$$

whenever $\{E_n\}_{n=1}^{\infty}$ is a countable pairwise disjoint sub-family of $\wp$. Then we may define

$$many_X(E) = \frac{\mu(E)}{\mu(X)},$$
$$most_X(E) = \left(\frac{\mu(E)}{\mu(X)}\right)^{3/2},$$
$$almost\ all_X(E) = \left(\frac{\mu(E)}{\mu(X)}\right)^2$$

for any $E \in \wp$.

The above definitions for quantifiers "many", "most" and "almost all" seem to be largely arbitrary except the ordering: $almost\ all_X(E) \leqslant most_X(E) \leqslant many_X(E)$ for any $E \subseteq X$. One may naturally ask the question: how we can avoid this arbitrariness, or more generally, is there any general methodology for defining linguistic quantifiers? This is in a sense equivalent to the problem of how to derive the membership function of a fuzzy set, which is arguably the biggest open problem in fuzzy set theory. In the 1970s quite a few approaches to estimation of membership functions had been proposed, including psychological analysis, statistics and preference method. For a survey on these early works, see [11, Chapter IV.1]. However, no significant further progress has been made in more than 30 years. The current situation is that a theoretical foundation for membership function estimation is still missing. In particular, we do not have a mathematical theorem in fuzzy set theory which guarantees that the membership function of a fuzzy set can be approximated in some way, like the law of large numbers in probability theory.

Two kinds of fuzzy quantifiers are of special interest, and they are defined in the next definition. We will see later that they enjoy some very nice logical properties.

Definition 17. A quantifier Q is called a possibility quantifier (resp. necessity quantifier) if for any nonempty set X, and for any $E_1, E_2 \in \wp_X$,

$$Q_X(E_1 \cup E_2) = \max\big(Q_X(E_1), Q_X(E_2)\big)$$
$$\big(\text{resp. } Q_X(E_1 \cap E_2) = \min\big(Q_X(E_1), Q_X(E_2)\big)\big).$$

It is clear that the universal quantifier ($\forall$) and the existential quantifier ($\exists$) are respectively a necessity quantifier and a possibility quantifier. More generally, if for each set X, Q_X is a possibility (resp. necessity) measure induced by some possibility distribution on X, then Q is a possibility (resp. necessity) quantifier.

The following definition introduces a partial order between fuzzy quantifiers and three operations of fuzzy quantifiers.

Definition 18. Let Q, Q_1 and Q_2 be quantifiers. Then

(1) We say that Q_1 is stronger than Q_2, written $Q_1 \sqsubseteq Q_2$, if for any nonempty set X and for any $E \in \wp_X$, we have $Q_{1X}(E) \leqslant Q_{2X}(E)$.

(2) The dual Q^* of Q, and the meet $Q_1 \sqcap Q_2$ and union $Q_1 \sqcup Q_2$ of Q_1 and Q_2 are defined respectively as follows: for any nonempty set X and for any $E \in \wp_X$,

$$Q_X^*(E) \stackrel{\text{def}}{=} 1 - Q_X(X - E),$$
$$(Q_1 \sqcap Q_2)_X(E) \stackrel{\text{def}}{=} \min\big(Q_{1X}(E), Q_{2X}(E)\big),$$
$$(Q_1 \sqcup Q_2)_X(E) \stackrel{\text{def}}{=} \max\big(Q_{1X}(E), Q_{2X}(E)\big).$$

It may be observed that the meet and union operations of quantifiers are exactly the set-theoretic intersection and union operations, respectively, when quantifiers are imagined as fuzzy subsets of the given Borel field $\wp_X$ over an universe X of discourse. The intersection and union of fuzzy sets were first defined by Zadeh [53] in terms of "min" and "max" operations on membership functions. Afterwards, many different intersection and union operations of fuzzy sets have been proposed in the literature (see for example [11, Section II.1.B]). Indeed, all of these operations can be unified in the framework of t-norm and t-conorm, two notions initially introduced in the theory of probabilistic metric spaces [34]. This observation suggests the possibility of replacing the "min" and "max" operations in the defining equations of $Q_1 \sqcap Q_2$ and $Q_1 \sqcup Q_2$ by a general t-norm and t-conorm, respectively. When once such more general operations of quantifiers are adopted, the logical properties of linguistic quantifier related to the meet and union operations of quantifiers obtained in this paper should naturally be reexamined (cf. the remark after Proposition 31). On the other hand, in practical applications an important problem is how to choose suitable t-norms and t-conorms. There have been some experiments verifying the accurateness of various fuzzy set operations reported in the literature. For a short survey on the early works in this direction, see [11, Section IV.1.C].

Example 19. (1) The universal and existential quantifiers are the dual of each other: $\forall^* = \exists$ and $\exists^* = \forall$.

(2) For any nonempty finite set X and for any $E \subseteq X$,

$$(\textit{almost all})^*_X(E) = \frac{2|E|}{|X|} - \left(\frac{|E|}{|X|}\right)^2,$$

$$(\textit{at least three} \sqcap \textit{many})_X(E) = \begin{cases} \frac{|E|}{|X|}, & \text{if } |E| \geqslant 3, \\ 0, & \text{otherwise}, \end{cases}$$

$$(\textit{at least three} \sqcup \textit{many})_X(E) = \begin{cases} 1, & \text{if } |E| \geqslant 3, \\ \frac{2}{|X|}, & \text{if } |E| = 2, \\ \frac{1}{|X|}, & \text{if } |E| = 1, \\ 0, & \text{if } E = \emptyset. \end{cases}$$

Several algebraic laws for quantifier operations are presented in the next lemma. They show that quantifiers together with the operations defined above form a De Morgan algebra.

Lemma 20. (1) *For any quantifier Q, we have $\forall \sqsubseteq Q \sqsubseteq \exists$. In other words, the universal quantifier* ($\forall$) *is the strongest quantifier, and the existential quantifier* ($\exists$) *is the weakest one.*

(2) *For all quantifiers Q_1 and Q_2, it holds that $Q_1 \sqcap Q_2 \sqsubseteq Q_1$ and $Q_1 \sqsubseteq Q_1 \sqcup Q_2$.*

(3) *For all quantifiers Q_1 and Q_2, we have*:

(*Commutativity*) $Q_1 \sqcap Q_2 = Q_2 \sqcap Q_1$, $Q_1 \sqcup Q_2 = Q_2 \sqcup Q_1$.

(*Associativity*) $Q_1 \sqcap (Q_2 \sqcap Q_3) = (Q_1 \sqcap Q_2) \sqcap Q_3$, $Q_1 \sqcup (Q_2 \sqcup Q_3) = (Q_1 \sqcup Q_2) \sqcup Q_3$.

(*Absorption*) $Q_1 \sqcap (Q_1 \sqcup Q_2) = Q_1$, $Q_1 \sqcup (Q_1 \sqcap Q_2) = Q_1$.

(*De Morgan law*) $(Q_1 \sqcap Q_2)^* = Q_1^* \sqcup Q_2^*$, $(Q_1 \sqcup Q_2)^* = Q_1^* \sqcap Q_2^*$.

Proof. Immediate from Definition 18. □

4. A first order language with linguistic quantifiers and its semantics

We first construct a first order logical language $\mathbf{L_q}$ with linguistic quantifiers. The alphabet of our language $\mathbf{L_q}$ is given as follows:

(1) A denumerable set of individual variables: $x_0, x_1, x_2, \ldots$;

(2) A set $\mathbf{F} = \bigcup_{n=0}^{\infty} \mathbf{F}_n$ of predicate symbols, where $\mathbf{F}_n$ is the set of all n-place predicate symbols for each $n \geqslant 0$. It is assumed that $\bigcup_{n=1}^{\infty} \mathbf{F}_n \neq \emptyset$;

(3) Propositional connectives: $\sim$, $\wedge$; and

(4) Parentheses: (,).

The syntax of the language $\mathbf{L_q}$ is then presented by the following definition.

Definition 21. The set Wff of well-formed formulas is the smallest set of symbol strings satisfying the following conditions:

(i) If $n \geqslant 0$, $F \in \mathbf{F}_n$, and $y_1, \ldots, y_n$ are individual variables, then $F(y_1, \ldots, y_n) \in$ Wff;

(ii) If Q is a quantifier, x is an individual variable, and $\varphi \in$ Wff, then $(Qx)\varphi \in$ Wff; and

(iii) If $\varphi, \varphi_1, \varphi_2 \in$ Wff, then $\sim \varphi$, $\varphi_1 \wedge \varphi_2 \in$ Wff.

For the sake of simplicity, we introduce some abbreviations:

$$\varphi \vee \psi \stackrel{\text{def}}{=} \sim(\sim\varphi \wedge \sim\psi),$$
$$\varphi \to \psi \stackrel{\text{def}}{=} \sim\varphi \vee \psi,$$
$$\varphi \leftrightarrow \psi \stackrel{\text{def}}{=} (\varphi \to \psi) \wedge (\psi \to \varphi).$$

The notions of bound variable and free variable can be introduced in a standard way. We omit their detailed definitions here but freely use them in the sequel.

The semantics of our language $\mathbf{L_q}$ is given by the next two definitions.

Definition 22. An interpretation I of our logical language consists of the following items:
(i) A measurable space $(X, \wp)$, called the domain of I;
(ii) For each $n \geqslant 0$, we associate the individual variable x_i with an element x_i^I in X; and
(iii) For any $n \geqslant 0$ and for any $F \in \mathbf{F}_n$, there is a $\wp^n$-measurable function $F^I : X^n \to [0, 1]$.

For simplicity, in what follows we assume that the Borel field $\wp$ in the domain $(X, \wp)$ of an interpretation I is always taken to be the power set 2^X of X, and the Borel field $\wp_X$ equipped with a nonempty set X is also 2^X for any quantifier Q. This often shortens the presentation and proof of our results.

Definition 23. Let I be an interpretation. Then the truth value $T_I(\varphi)$ of a formula φ under I is defined recursively as follows:
(i) If $\varphi = F(y_1, \ldots, y_n)$, then

$$T_I(\varphi) = F^I\big(y_1^I, \ldots, y_n^I\big).$$

(ii) If $\varphi = (Qx)\psi$, then

$$T_I(\varphi) = \int T_{I\{./x\}}(\psi) \circ Q_X,$$

where X is the domain of I, $T_{I\{./x\}}(\psi) : X \to [0, 1]$ is a mapping such that

$$T_{I\{./x\}}(\varphi)(u) = T_{I\{u/x\}}(\varphi)$$

for all $u \in X$, and $I\{u/x\}$ is the interpretation which differs from I only in the assignment of the individual variable x, that is, $y^{I\{u/x\}} = y^I$ for all $y \neq x$ and $x^{I\{u/x\}} = u$;
(iii) If $\varphi = \sim\psi$, then

$$T_I(\varphi) = 1 - T_I(\psi),$$

and if $\varphi = \varphi_1 \wedge \varphi_2$, then

$$T_I(\varphi) = \min\big(T_I(\varphi_1), T_I(\varphi_2)\big).$$

The following proposition establishes a close link between the truth evaluation of quantified statement and the extension of fuzzy measure on fuzzy sets.

Proposition 24. *Let Q be a quantifier and x an individual variable, and let $\varphi \in$ Wff. Then for any interpretation I,*

$$T_I\big((Qx)\varphi\big) = \widetilde{Q_X}\big(T_I(\varphi)\big),$$

where $\widetilde{Q_X}$ is the extension of Q_X on fuzzy sets.

Proof. Immediate from Definitions 13 and 23(ii). □

In order to illustrate further the evaluation mechanism of quantified propositions, let us examine some simple examples.

Example 25. We first consider the simplest case of quantification. For any quantifier Q and for any $\varphi \in \mathrm{Wff}$, if I is an interpretation with the domain being a singleton $X = \{u\}$, then for any quantifier Q,

$$T_I\big((Qx)\varphi\big) = T_I(\varphi).$$

This means that quantification degenerates on a singleton discourse universe.

Example 26. We now consider the strongest and weakest quantifiers. This example shows that the Sugeno integral evaluation of universally and existentially quantified statements coincide with the standard way, and so gives a witness for reasonableness of Sugeno integral semantics of linguistic quantification. Let Q be a quantifier and x an individual variable, and let $\varphi \in \mathrm{Wff}$. Then for any interpretation I with domain X, we have

$$\begin{aligned} T_I\big((\exists x)\varphi\big) &= \int T_{I\{u/x\}}(\varphi) \circ \exists_X = \sup_{F \subseteq X} \min\Big[\inf_{u \in F} T_{I\{u/x\}}(\varphi), \exists_X(F)\Big] \\ &= \sup_{\emptyset \neq F \subseteq X} \inf_{u \in F} T_{I\{u/x\}}(\varphi) = \sup_{u \in X} T_{I\{u/x\}}(\varphi). \end{aligned}$$

Similarly, it holds that

$$T_I\big((\forall x)\varphi\big) = \inf_{u \in X} T_{I\{u/x\}}(\varphi).$$

To conclude this section, in the case of finite discourse universe we give a necessary and sufficient condition under which the truth value of a quantified proposition is bound by a given threshold value from up or below. This condition is very useful in some real applications (see Example 43 below).

Proposition 27. *Let X be a finite set, let I be an interpretation with X as its domain, and let $\lambda \in [0, 1]$. Then for any quantifier Q and $\varphi \in \mathrm{Wff}$, we have*:

(i) $T_I((Qx)\varphi) \geqslant \lambda$ *if and only if*

$$Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \lambda\big\}\big) \geqslant \lambda.$$

(ii) $T_I((Qx)\varphi) \leqslant \lambda$ *if and only if*

$$Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi) > \lambda\big\}\big) \leqslant \lambda.$$

Proof. (i) If $Q_X(\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \lambda\}) \geqslant \lambda$, then we obtain

$$\begin{aligned} T_I\big((Qx)\varphi\big) &= \sup_{\mu \in [0,1]} \min\big(\mu, Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \mu\big\}\big)\big) \\ &\geqslant \min\big(\lambda, Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \lambda\big\}\big)\big) \geqslant \lambda. \end{aligned}$$

Conversely, if

$$T_I\big((Qx)\varphi\big) = \sup_{F \subseteq X} \min\Big(\inf_{u \in F} T_{I\{u/x\}}(\varphi), Q_X(F)\Big) \geqslant \lambda,$$

then there exists $F_0 \subseteq X$ such that

$$\min\Big(\inf_{u \in F_0} T_{I\{u/x\}}(\varphi), Q_X(F_0)\Big) \geqslant \lambda$$

because X is finite. Then it holds that $\inf_{u \in F_0} T_{I\{u/x\}}(\varphi) \geqslant \lambda$ and $Q_X(F_0) \geqslant \lambda$. Furthermore, we have $T_{I\{u/x\}}(\varphi) \geqslant \lambda$ for every $u \in F_0$ and $\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \lambda\} \supseteq F_0$. This yields

$$Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \lambda\big\}\big) \geqslant Q_X(F_0) \geqslant \lambda.$$

(ii) We first prove the "if" part. If $\mu \leqslant \lambda$, then it is obvious that

$$\min\big[\mu, Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \mu\big\}\big)\big] \leqslant \lambda,$$

and if $\mu > \lambda$, then $\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \mu\} \subseteq \{u \in X\colon T_{I\{u/x\}}(\varphi) > \lambda\}$,

$$Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \mu\big\}\big) \leqslant Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi) > \lambda\big\}\big) \leqslant \lambda,$$

and we also have

$$\min\big[\mu, Q_X\big(\{u \in X:\ T_{I\{u/x\}}(\varphi) \geqslant \mu\}\big)\big] \leqslant \lambda.$$

Thus,

$$T_I\big((Qx)\varphi\big) = \sup_{\mu\in[0,1]} \min\big[\mu, Q_X\big(\{u \in X:\ T_{I\{u/x\}}(\varphi) \geqslant \mu\}\big)\big] \leqslant \lambda.$$

For the "only if" part, suppose that

$$\mu_0 = \inf\{T_{I\{u/x\}}(\varphi):\ T_{I\{u/x\}}(\varphi) > \lambda\}.$$

Since X is finite, it holds that $\mu_0 > \lambda$ and $\{u \in X:\ T_{I\{u/x\}}(\varphi) > \lambda\} = \{u \in X:\ T_{I\{u/x\}}(\varphi) \geqslant \mu_0\}$. Consequently,

$$\begin{aligned}\min\big[\mu_0, Q_X\big(\{u \in X:\ T_{I\{u/x\}}(\varphi) > \lambda\}\big)\big] &= \min\big[\mu_0, Q_X\big(\{u \in X:\ T_{I\{u/x\}}(\varphi) \geqslant \mu_0\}\big)\big] \\ &\leqslant \sup_{\mu\in[0,1]} \min\big[\mu, Q_X\big(\{u \in X:\ T_{I\{u/x\}}(\varphi) \geqslant \mu\}\big)\big] \\ &= T_I\big((Qx)\varphi\big) \leqslant \lambda.\end{aligned}$$

Finally, from $\mu_0 > \lambda$ we know that $Q_X(\{u \in X:\ T_{I\{u/x\}}(\varphi) > \lambda\}) \leqslant \lambda$. □

5. Logical properties of linguistic quantifiers

The main purpose of this section is to establish various logical properties of linguistic quantifiers. In order to present these properties in a compact way, several meta-logical notions are needed.

Definition 28. Let $\varphi \in \mathrm{Wff}$ and $\Sigma \subseteq \mathrm{Wff}$.

(1) If for any interpretation I, $T_I(\varphi) \geqslant \frac{1}{2}$, then φ is said to be fuzzily valid and we write $\models^{Fuz} \varphi$.

(2) If for any interpretation I,

$$\inf_{\psi\in\Sigma} T_I(\psi) \leqslant T_I(\varphi),$$

then φ is called a consequence of Σ and we write $\Sigma \models \varphi$. In particular, if $\emptyset \models \varphi$, that is, $T_I(\varphi) = 1$ for each interpretation I, then φ is said to be (absolutely) valid and we write $\models \varphi$.

(3) If $\varphi \models \psi$ and $\psi \models \varphi$, that is, for any interpretation I, $T_I(\varphi) = T_I(\psi)$, then we say that φ and ψ are equivalent and write $\varphi \equiv \psi$.

We first consider the question how the consequence relation is preserved by quantifiers and the question when can quantifiers distribute over conjunctions and disjunctions. The following proposition answers these two questions.

Proposition 29. (1) *Suppose that $\varphi_1, \varphi_2 \in \mathrm{Wff}$. Then $\varphi_1 \models \varphi_2$ if and only if for any quantifier Q and for any individual variable x, $(Qx)\varphi_1 \models (Qx)\varphi_2$ always holds.*

(2) *For any quantifier Q and for any $\varphi_1, \varphi_2 \in \mathrm{Wff}$, we have*:

$$(Qx)(\varphi_1 \wedge \varphi_2) \models (Qx)\varphi_1 \wedge (Qx)\varphi_2,$$

$$(Qx)\varphi_1 \vee (Qx)\varphi_2 \models (Qx)(\varphi_1 \vee \varphi_2).$$

(3) *If Q is a possibility quantifier, then for any $\varphi_1, \varphi_2 \in \mathrm{Wff}$, we have*:

$$(Qx)\big(\varphi_1(x) \vee \varphi_2(x)\big) \equiv (Qx)\varphi_1(x) \vee (Qx)\varphi_2(x).$$

Proof. (1) The "only if" part is a direct corollary of Lemma 12(2). For the "if" part, let I be an interpretation and X be the domain of I. We want to show that $T_I(\varphi_1) \leqslant T_I(\varphi_2)$. We set $Q_X = \Pi_{\pi_{x^I}}$, where x^I is the assignment of I to x, and $\Pi_{\pi_{x^I}}$ is the fuzzy measure induced by the singleton possibility distribution π_{x^I} (see Example 4(1)). Then from $(Qx)\varphi_1 \models (Qx)\varphi_2$ and a simple calculation we know that

$$T_I(\varphi_1) = \int T_{I\{./x\}}(\varphi_1) \circ \Pi_{\pi_{x^I}} = T_I\big((Qx)\varphi_1\big) \leqslant T_I\big((Qx)\varphi_2\big) = \int T_{I\{./x\}}(\varphi_2) \circ \Pi_{\pi_{x^I}} = T_I(\varphi_2).$$

(2) is immediate from (1).

(3) For any interpretation I, let X be the domain of I, then it follows that

$$\begin{aligned}
&T_I\big((Qx)(\varphi_1 \vee \varphi_2)\big)\\
&\quad = \sup_{\lambda\in[0,1]} \min\big[\lambda, Q_X\big(\big\{u \in X\colon \max\big(T_{I\{u/x\}}(\varphi_1), T_{I\{u/x\}}(\varphi_2)\big) \geqslant \lambda\big\}\big)\big]\\
&\quad = \sup_{\lambda\in[0,1]} \min\big[\lambda, Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi_1) \geqslant \lambda\big\} \cup \big\{u \in X : T_{I\{u/x\}}(\varphi_2) \geqslant \lambda\big\}\big)\big]\\
&\quad = \sup_{\lambda\in[0,1]} \min\big[\lambda, \max\big(Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi_1) \geqslant \lambda\big\}\big), Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi_2) \geqslant \lambda\big\}\big)\big)\big]\\
&\quad = \sup_{\lambda\in[0,1]} \max\big\{\min\big[\lambda, Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi_1) \geqslant \lambda\big\}\big)\big], \min\big[\lambda, Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi_2) \geqslant \lambda\big\}\big)\big]\big\}\\
&\quad = \max\big\{\sup_{\lambda\in[0,1]} \min\big[\lambda, Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi_1) \geqslant \lambda\big\}\big)\big], \sup_{\lambda\in[0,1]} \min\big[\lambda, Q_X\big(\big\{u \in X\colon T_{I\{u/x\}}(\varphi_2) \geqslant \lambda\big\}\big)\big]\big\}\\
&\quad = T_I\big((Qx)\varphi_1 \vee (Qx)\varphi_2\big). \qquad \square
\end{aligned}$$

Second, we see how the consequence relation between two quantified propositions depends on the strength of involved quantifiers.

Proposition 30. *Let Q_1 and Q_2 be two quantifiers. Then $Q_1 \sqsubseteq Q_2$ if and only if for any $\varphi \in \mathrm{Wff}$, $(Q_1x)\varphi \models (Q_2x)\varphi$ always holds.*

Proof. The "only if" part is obvious. For the "if" part, we only need to show that for any set X and for any $E \subseteq X$, $Q_{1X}(E) \leqslant Q_{2X}(E)$. We can find some $P \in \mathbf{F}_n$ with $n > 0$ because it is assumed that $\bigcup_{n=1}^{\infty} \mathbf{F}_n \neq \emptyset$. Let $\varphi = P(x, \dots, x)$ and consider the interpretation I in which the domain is X and

$$P^I(u_1, \dots, u_n) = \begin{cases} 1, & \text{if } u_1 = \cdots = u_n \in E,\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$T_I\big((Q_1x)\varphi\big) = \sup_{F\subseteq X} \min\big[\inf_{u\in F} P^I(u, \dots, u), Q_X(F)\big] = \sup_{F\subseteq E} Q_X(F) = Q_{1X}(E)$$

because

$$\inf_{u\in F} P^I(u, \dots, u) = \begin{cases} 1, & \text{if } F \subseteq E,\\ 0, & \text{otherwise.}\end{cases}$$

A similar calculation leads to $T_I((Q_2x)\varphi) = Q_{2X}(E)$. Since $(Q_1x)\varphi \models (Q_2x)\varphi$, we have $T_I((Q_1x)\varphi) \leqslant T_I((Q_2x)\varphi)$, and this completes the proof. $\square$

Third, we show that a proposition with the meet or union of two quantifiers can be transformed to the conjunction or disjunction, respectively, of the propositions with component quantifiers.

Proposition 31. *For any quantifiers Q_1, Q_2, individual variable x and $\varphi \in \mathrm{Wff}$, we have*:

(1) $((Q_1 \sqcap Q_2)x)\varphi \equiv (Q_1x)\varphi \wedge (Q_2x)\varphi$;
(2) $((Q_1 \sqcup Q_2)x)\varphi \equiv (Q_1x)\varphi \vee (Q_2x)\varphi$.

Proof. (1) For any interpretation I with domain X, it is clear that

$$T_I\big(\big((Q_1 \sqcap Q_2)x\big)\varphi\big) \leqslant T_I\big((Q_1x)\varphi \wedge (Q_2x)\varphi\big).$$

Conversely, we have

$$\begin{aligned}
T_I\big((Q_1x)\varphi \wedge (Q_2x)\varphi\big) &= \min\big[T_I\big((Q_1x)\varphi\big), T_I\big((Q_2x)\varphi\big)\big] \\
&= \min\Big[\sup_{F_1\subseteq X}\min\big(\inf_{u\in F_1} T_{I\{u/x\}}(\varphi), Q_{1X}(F_1)\big), \sup_{F_2\subseteq X}\min\big(\inf_{u\in F_2} T_{I\{u/x\}}(\varphi), Q_{2X}(F_2)\big)\Big] \\
&= \sup_{F_1,F_2\subseteq X}\min\big(\inf_{u\in F_1} T_{I\{u/x\}}(\varphi), \inf_{u\in F_2} T_{I\{u/x\}}(\varphi), Q_{1X}(F_1), Q_{2X}(F_2)\big) \\
&= \sup_{F_1,F_2\subseteq X}\min\big(\inf_{u\in F_1\cup F_2} T_{I\{u/x\}}(\varphi), Q_{1X}(F_1), Q_{2X}(F_12)\big) \\
&\leqslant \sup_{F_1,F_2\subseteq X}\min\big(\inf_{u\in F_1\cup F_2} T_{I\{u/x\}}(\varphi), Q_{1X}(F_1\cup F_2), Q_{2X}(F_1\cup F_2)\big) \\
&\leqslant \sup_{F\subseteq X}\min\big(\inf_{u\in F} T_{I\{u/x\}}(\varphi), Q_{1X}(F), Q_{2X}(F)\big) \\
&= \sup_{F\subseteq X}\min\big(\inf_{u\in F} T_{I\{u/x\}}(\varphi), (Q_1\sqcap Q_2)_X(F)\big) \\
&= T_I\big(\big((Q_1\sqcap Q_2)x\big)\varphi\big).
\end{aligned}$$

(2) Let

$$F_{X,\lambda} = \big\{u\in X\colon\ T_{I\{u/x\}}(\varphi)\geqslant\lambda\big\}.$$

Then it follows that

$$\begin{aligned}
T_I\big(\big((Q_1\sqcup Q_2)x\big)\varphi\big) &= \sup_\lambda \min\big[\lambda, (Q_1\sqcup Q_2)_X(F_{X,\lambda})\big] \\
&= \sup_\lambda \min\big[\lambda, \max\big(Q_{1X}(F_{X,\lambda}), Q_{2X}(F_{X,\lambda})\big)\big] \\
&= \sup_\lambda \max\big[\min\big(\lambda, Q_{1X}(F_{X,\lambda})\big), \min\big(\lambda, Q_{2X}(F_{X,\lambda})\big)\big] \\
&= \max\big[\sup_\lambda \min\big(\lambda, Q_{1X}(F_{X,\lambda})\big), \sup_\lambda \min\big(\lambda, Q_{2X}(F_{X,\lambda})\big)\big] \\
&= \max\big(T_I\big((Q_1x)\varphi\big), T_I\big((Q_2x)\varphi\big)\big) \\
&= T_I\big((Q_1x)\varphi\vee(Q_2x)\varphi\big). \qquad \square
\end{aligned}$$

We pointed out in the remark after Definition 18 that more general meet and union operations of quantifiers may be defined by using a t-norm and t-conorm in the places of "min" and "max", respectively. The above proposition can be generalized to such general meet and union of quantifiers. To this end, we have to consider a generalization of Sugeno's integral, which is obtained by replacing "min" in Definition 9 with a t-norm (see [38] for a detailed discuss on such generalized Sugeno's integrals).

An inference scheme with linguistic quantifiers discussed in [25, Section 7] is as follows:

$$\frac{(Q_1x)\varphi_1,\ldots,(Q_nx)\varphi_n}{(Qx)\varphi = ?}.$$

The above proposition enables us to give a solution to this inference problem. Indeed, from the above proposition we know that the following inference is valid:

$$\frac{(Q_1x)\varphi_1,\ldots,(Q_nx)\varphi_n}{((Q_1\sqcap\cdots\sqcap Q_n)x)(\varphi_1\vee\cdots\vee\varphi_n)}.$$

However, it should be pointed out that such a solution is usually not optimal; more precisely, there may be some logical formula ψ of the form $(Qx)\varphi$ such that

$$\models \psi \to \big((Q_1\sqcap\cdots\sqcap Q_n)x\big)(\varphi_1\vee\cdots\vee\varphi_n),$$

but

$$\models \big((Q_1\sqcap\cdots\sqcap Q_n)x\big)(\varphi_1\vee\cdots\vee\varphi_n) \to \psi$$

does not hold.

The next proposition indicates that quantifiers do not shed any influence on bound variables.

Proposition 32. *For any quantifier Q and for any $\varphi, \psi \in$ Wff, if individual variable x is not free in ψ, then we have*:

$$(Qx)\varphi \wedge \psi \equiv (Qx)(\varphi \wedge \psi);$$
$$(Qx)\varphi \vee \psi \equiv (Qx)(\varphi \vee \psi).$$

Proof. We only demonstrate the first equivalence relation, and the second is similar. For any interpretation I, it holds that

$$T_I\big((Qx)(\varphi \wedge \psi)\big) = \int T_{I,x}(\varphi \wedge \psi) \circ Q_X = \int \min\big[T_{I\{u/x\}}(\varphi), T_{I\{u/x\}}(\psi)\big] \circ Q_X,$$

where X is the domain of I. Since x is not free in ψ, it is easy to see that $T_{I\{u/x\}}(\psi) = T_I(\psi)$ for every $u \in X$. With Lemma 12(3), we obtain

$$T_I\big((Qx)(\varphi \wedge \psi)\big) = \min\bigg[\int T_{I\{u/x\}}(\varphi) \circ Q_X, T_I(\psi)\bigg] = \min\big[T_I\big((Qx)\varphi\big), T_I(\psi)\big] = T_I\big((Qx)\varphi \wedge \psi\big). \qquad \square$$

To conclude this section, we observe the function of dual quantifiers. It is worth noting that the fuzzy validity $\models^{Fuz}$ in the following proposition cannot be strengthened by the stricter validity $\models$.

Proposition 33. *For any quantifier Q and for any $\varphi \in$ Wff, it holds that*

$$\models^{Fuz} \sim (Qx)\varphi \leftrightarrow (Q^*x) \sim \varphi,$$

where Q^* is the dual of Q.

Proof. We set

$$\psi_1 = \sim (Qx)\varphi \to (Q^*x) \sim \varphi$$

and

$$\psi_2 = (Q^*x) \sim \varphi \to \sim (Qx)\varphi.$$

Then for each interpretation I with domain X, we have

$$T_I\big(\sim (Qx)\varphi \leftrightarrow (Q^*x) \sim \varphi\big) = \min\big[T_I(\psi_1), T_I(\psi_2)\big],$$

and it suffices to show that $T_I(\psi_1) \geqslant \frac{1}{2}$ and $T_I(\psi_2) \geqslant \frac{1}{2}$. Furthermore, it holds that

$$\begin{aligned} T_I(\psi_1) &= \max\big[T_I\big((Qx)\varphi\big), T_I(Q^*x) \sim \varphi)\big] \\ &= \max\bigg[\int T_{I\{u/x\}}(\varphi) \circ Q_X, \int \big(1 - T_{I\{u/x\}}(\varphi)\big) \circ Q_X^*\bigg]. \end{aligned}$$

If

$$\int T_{I\{u/x\}}(\varphi) \circ Q_X \geqslant \frac{1}{2},$$

then it is obvious that $T_I(\psi_1) \geqslant \frac{1}{2}$. Otherwise, with Lemma 10 we obtain

$$\sup_{F \subseteq X} \min\big[\inf_{x \in F} T_{I\{u/x\}}(\varphi), Q_X(F)\big] = \int T_{I\{u/x\}}(\varphi) \circ Q_X < \frac{1}{2}.$$

For any $\varepsilon > 0$, let $F(\varepsilon) = \{u \in X\colon\ T_{I\{u/x\}}(\varphi) < \frac{1}{2}\}$. Then for all $u \in F(\varepsilon)$, $1 - T_{I\{u/x\}}(\varphi) \geqslant \frac{1}{2} - \varepsilon$, and

$$\inf_{u \in F(\varepsilon)} \big(1 - T_{I\{u/x\}}(\varphi)\big) \geqslant \frac{1}{2} - \varepsilon.$$

On the other hand, since $X - F(\varepsilon) = \{u \in X\colon\ T_{I\{u/x\}}(\varphi) \geqslant \frac{1}{2} + \varepsilon\}$, it follows that

$$\inf_{u \in X - F(\varepsilon)} T_{I\{u/x\}}(\varphi) \geqslant \frac{1}{2} + \varepsilon.$$

Note that

$$\min\Big[\inf_{x\in X-F(\varepsilon)} T_{I\{u/x\}}(\varphi), Q_X\big(X-F(\varepsilon)\big)\Big] \leqslant \int T_{I\{u/x\}}(\varphi)\circ Q_X < \frac{1}{2}.$$

We know that $Q_X(X-F(\varepsilon)) < \frac{1}{2}$. This yields $Q_X^*(F(\varepsilon)) = 1 - Q_X(X-F(\varepsilon)) > \frac{1}{2}$ and

$$\int \big(1-T_{I\{u/x\}}(\varphi)\big)\circ Q_X^* \geqslant \min\Big[\inf_{u\in F(\varepsilon)}\big(1-T_{I\{u/x\}}(\varphi)\big), Q_X^*(F(\varepsilon))\Big] \geqslant \frac{1}{2} - \varepsilon.$$

Let $\varepsilon \to 0$. Then it holds that

$$\int \big(1-T_{I\{u/x\}}(\varphi)\big)\circ Q_X^* \geqslant \frac{1}{2}$$

and $T_I(\psi_1) \geqslant \frac{1}{2}$.

We now turn to consider ψ_2. It is clear that

$$\begin{aligned} T_I(\psi_2) &= \max\Big[1-\int\big(1-T_{I\{u/x\}}(\varphi)\big)\circ Q_X^*, 1-\int T_{I\{u/x\}}(\varphi)\circ Q_X\Big] \\ &= 1-\min\Big[\int\big(1-T_{I\{u/x\}}(\varphi)\big)\circ Q_X^*, \int T_{I\{u/x\}}(\varphi)\circ Q_X\Big]. \end{aligned}$$

To show that $T_I(\psi_2) \geqslant \frac{1}{2}$, it suffices to prove

$$\min\Big[\int\big(1-T_{I\{u/x\}}(\varphi)\big)\circ Q_X^*, \int T_{I\{u/x\}}(\varphi)\circ Q_X\Big] \leqslant \frac{1}{2}.$$

If

$$\int T_{I\{u/x\}}(\varphi)\circ Q_X \leqslant \frac{1}{2},$$

we are done. Otherwise, we have

$$\sup_{F\subseteq X}\min\Big[\inf_{u\in F} T_{I\{u/x\}}(\varphi), Q_X(F)\Big] = \int T_{I\{u/x\}}(\varphi)\circ Q_X > \frac{1}{2}.$$

Therefore, there exists $F_0 \subseteq X$ such that

$$\inf_{u\in F_0} T_{I\{u/x\}}(\varphi) > \frac{1}{2}$$

and $Q_X(F_0) > \frac{1}{2}$. Now, for any $u \in F_0$, $T_{I\{u/x\}}(\varphi) > \frac{1}{2}$, $1 - T_{I\{u/x\}}(\varphi) < \frac{1}{2}$, and $Q_X^*(X-F_0) = 1 - Q_X(F_0) < \frac{1}{2}$. We are going to show that

$$\int\big(1-T_{I\{u/x\}}(\varphi)\big)\circ Q_X^* = \sup_{F\subseteq X}\min\Big[\inf_{u\in F}\big(1-T_{I\{u/x\}}(\varphi)\big), Q_X^*(F)\Big] \leqslant \frac{1}{2}.$$

To this end, we only need to demonstrate that for each $F \subseteq X$, $Q_X^*(F) \leqslant \frac{1}{2}$ or

$$\inf_{u\in F}\big(1-T_{I\{u/x\}}(\varphi)\big) \leqslant \frac{1}{2}.$$

In fact, if $F \subseteq X - F_0$, then $Q_X^*(F) \leqslant Q_X^*(X-F_0) < \frac{1}{2}$, and if $F \nsubseteq X - F_0$, then there exists $u_0 \in F \cap F_0$ and

$$\inf_{u\in F}\big(1-T_{I\{u/x\}}(\varphi)\big) \leqslant 1 - T_{I\{u_0/x\}}(\varphi) < \frac{1}{2}.$$

This completes the proof. □

Combining the above results we obtain a prenex normal form theorem for logical formulas with linguistic quantifiers.

Corollary 34 *(Prenex Normal Form). For any* $\varphi \in \mathrm{Wff}$, *there exists* $\psi \in \mathrm{Wff}$ *satisfying the following two conditions*:

(i) ψ *is in the form of*

$$(Q_1 y_1)\dots(Q_n y_n)M,$$

where $n \geqslant 0$, $Q_1, \dots, Q_n$ *are quantifiers, and* $M \in \mathrm{Wff}$ *does not contain any quantifier*; *and*

(ii) $\models^{Fuz} \varphi \leftrightarrow \psi$.

Proof. Immediate from Propositions 31–33. □

6. Cardinal quantifiers

The studies on generalized quantifiers has traditionally been based on cardinalities [2,29,36]. The reason is that when a quantified proposition is considered, what really concerns us is how many individuals satisfies the proposition, and it is usually irrelevant to know what are they concretely. Consequently, most methods of evaluating linguistic quantification in the previous literature are based on cardinalities of fuzzy sets too. Our fuzzy measure and Sugeno integral approach to linguistic quantifiers is in a sense presented in a wider framework in which no cardinality condition is required. This section aims at clarifying the relationship between the semantics proposed in Section 4 and the cardinality-based semantics of linguistic quantifiers. To this end, we have to introduce a special class of quantifiers.

Definition 35. A quantifier Q is called a cardinal quantifier if for any two nonempty sets X and X', for any bijection $f : X \to X'$, and for any $E \in \wp_X$ with $f(E) \in \wp'_{X'}$,

$$Q_X(E) = Q_{X'}\big(f(E)\big),$$

where $f(E) = \{f(x)\colon x \in E\}$.

In the above definition, the requirement that $f : X \to X'$ is a bijection implies that X and X' has the same cardinality, and E and $f(E)$ also has the same cardinality. Consider the following two quantified statements with the same quantifier Q: $\varphi =$ "*Q Xs are As*" and $\varphi' =$ "$Q\ X's$ *are* $A's$". Suppose E is the set of elements in X that satisfy the property A, that is, $E = \{x \in X\colon x \text{ satisfies } A\}$. Similarly, let $E' = \{x \in X'\colon x \text{ satisfies } A'\}$. Then the condition in the above definition means that φ and φ' are equivalent provided X and X' has the same cardinality and so do A and A', no matter what are elements of X, X', A and A'.

The notion of generalized quantifier introduced in [45] is similar to cardinal quantifier with the only difference that the set of truth values was allowed to be any complete lattice in [45].

The notion of cardinal quantifier has an equivalent and simplified definition given in terms of cardinal numbers. For any cardinal number α, we write $m(\alpha)$ for the set of all (increasing) functions

$$f : \{\text{cardinal number } \beta\colon\ \beta \leqslant \alpha\} \to [0, 1]$$

such that $\beta_1 \leqslant \beta_2$ implies $f(\beta_1) \leqslant f(\beta_2)$.

Definition 36. A numeric quantifier is a choice function

$$q : \alpha \mapsto q_\alpha \in m(\alpha)$$

of the class $\{m(\alpha)\colon\ \alpha \text{ is a cardinal number}\}$.

For any cardinal quantifier Q, we define:

$$|Q| : \alpha \mapsto |Q|_\alpha \quad \text{for any cardinal number } \alpha,$$

$$|Q|_\alpha(\beta) \overset{\text{def}}{=} Q_X(E) \quad \text{for any } \beta \leqslant \alpha,$$

where X is some set with $|X| = \alpha$ and E is some subset of X with $|E| = \beta$. From Definition 35 we know that Q is well-defined, and it is a numeric quantifier. Thus, cardinal quantifier Q is represented by a numeric quantifier $|Q|$. Conversely, for each numeric quantifier q, we put

$$[q]_X(E) \stackrel{\text{def}}{=} q_{|X|}\big(|E|\big)$$

for any nonempty set X, and for any $E \subseteq X$. It is easy to see that $[q]$ is a cardinal quantifier, and thus numeric quantifier q induces a cardinal quantifier $[q]$.

Example 37. The universal quantifier ($\forall$) is not a cardinal quantifier because for an infinite set X, it is possible that X has a proper subset E with $|E| = |X|$. The existential quantifier ($\exists$) is a cardinal quantifier, and for any cardinal numbers α, β with $\alpha \leqslant \beta$,

$$|\exists|_\alpha(\beta) = \begin{cases} 1, & \text{if } \beta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The above example shows that the universal quantifier cannot be treated by the usual approach based on cardinality when the discourse universe is infinite. This exposes indeed one of the advantages of our fuzzy measure approach to linguistic quantifiers.

In order to give another way of evaluating the truth value of a proposition with cardinal quantifier and to compare fuzzy measure semantics of linguistic quantifiers with others, we need the notion of consistency between two fuzzy sets.

Definition 38. Let X be a nonempty set, and let A, B be two fuzzy subsets of X. Then the consistency index of A and B is defined as

$$Con(A, B) = \sup_{x \in X} \min\big[A(x), B(x)\big].$$

The notion of consistency was originally introduced by Zadeh in [55], and it have also been used by other authors to serve as similarity measures between two fuzzy sets [58]. Moreover, the quantity $1 - Con(A, \overline{B})$ was introduced as inclusion grade of fuzzy set A in fuzzy set B (see [11, page 25]), where $\overline{B}$ is the complement of B, that is, $\overline{B}(x) = 1 - B(x)$ for each $x \in X$.

A concept of fuzzy cardinality of fuzzy set is also needed.

Definition 39. Let X be a nonempty set, and let A be a fuzzy subset of X. Then the fuzzy cardinality of A is defined to be the fuzzy subset $FC(A)$ of {cardinal number α: $\alpha \leqslant |X|$} with

$$FC(A)(\alpha) = \sup\big\{\lambda \in [0, 1]:\ |A_\lambda| = \alpha\big\}$$

for any $\alpha \leqslant |X|$, where $A_\lambda = \{x \in X:\ A(x) \geqslant \lambda\}$ is the λ-cut of A.

We give a simple example to illustrate the notion of fuzzy cardinality.

Example 40. We adopt the Zadeh's notation for fuzzy sets, that is, a fuzzy subset A of a set X is denoted by

$$A = \sum_{x \in X} A(x)/x.$$

Let $X = \{a_1, a_2, \ldots, a_{10}\}$. We consider the following fuzzy subset A of X:

$$A = 0.9/a_1 + 0.4/a_2 + 0.7/a_4 + 0.1/a_5 + 0.82/a_6 + 0.83/a_7 + 0.9/a_8 + 1/a_9 + 1/a_{10}.$$

Then a routine calculation yields

$$FC(A) = 1/2 + 0.9/4 + 0.83/5 + 0.82/6 + 0.7/7 + 0.4/8 + 0.1/9.$$

It is easy to see that if A is a crisp set and $|A| = \alpha$ then $FC(A) = 1/\alpha$ (in Zadeh's notation). Conversely, if X is a finite set, A is a fuzzy subset of X and $FC(A) = 1/\alpha$, then A is a crisp set and $|A| = \alpha$. This indicates that $FC(\cdot)$ is a reasonable fuzzy generalization of the concept of cardinality for crisp sets.

In the literature, there are mainly two ways of defining cardinalities of fuzzy sets. The first gives a (non-fuzzy) real number as the cardinality of a fuzzy set A; for example, if A is a fuzzy subset of a finite set X, then its sigma-count is defined by

$$\Sigma Count(A) = \sum_{x \in X} A(x)$$

(see [56]). In the second way, the cardinality of a fuzzy set A is defined to be a fuzzy subsets of nonnegative integers, and typical examples are Zadeh's $FGCount(\cdot)$, $FLCount(\cdot)$ and $FECount(\cdot)$ [56] and Ralescu's $card(\cdot)$ [31]. Obviously, $FC(\cdot)$ in Definition 39 is also given in the second way, but here we are concerned not only finite sets and infinite cardinalities are allowed too. Moreover, even for the case of finite cardinalities, it is easy to observe that $FC(\cdot)$ is different from $FGCount(\cdot)$, $FLCount(\cdot)$, $FECount(\cdot)$ and $card(\cdot)$. We are not going to give a careful comparison of them.

Now we are able to present the main result of this section. It clarifies the relation between our Sugeno integral semantics of linguistic quantifiers and the usual approach based on cardinality.

Proposition 41. *If Q is a cardinal quantifier, then for any $\varphi \in$ Wff and for any interpretation I, we have*

$$T_I\big((Qx)\varphi\big) = Con\big(FC\big(T_I(\varphi)\big), |Q|_{|X|}\big),$$

where $T_I(\varphi)$ is considered as a fuzzy subset of X with $T_I(\varphi)(u) = T_{I\{u/x\}}(\varphi)$ for all $u \in X$, and $|Q|$ is the numeric quantifier induced by Q.

Proof.

$$\begin{aligned}
T_I\big((Qx)\varphi\big) &= \sup_{\lambda \in [0,1]} \min\big[\lambda, Q_X\big(\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \lambda\}\big)\big] \\
&= \sup_{\lambda \in [0,1]} \min\big[\lambda, |Q|_{|X|}\big(\big|\{u \in X\colon T_{I\{u/x\}}(\varphi) \geqslant \lambda\}\big|\big)\big] \\
&= \sup_{\lambda \in [0,1]} \min\big[\lambda, |Q|_{|X|}\big(\big|\big(T_I(\varphi)\big)_\lambda\big|\big)\big] \\
&= \sup_{\alpha \leqslant |X|} \sup_{\lambda \in [0,1] \text{ s.t } |(T_I(\varphi))_\lambda| = \alpha} \min\big[\lambda, |Q|_{|X|}(\alpha)\big] \\
&= \sup_{\alpha \leqslant |X|} \min\Big[\sup_{\lambda \in [0,1] \text{ s.t } |(T_I(\varphi))_\lambda| = \alpha} \lambda, |Q|_{|X|}(\alpha)\Big] \\
&= \sup_{\lambda \leqslant |X|} \min\big[FC\big(T_I(\varphi)\big)(\alpha), |Q|_{|X|}(\alpha)\big] \\
&= Con\big(FC\big(T_I(\varphi)\big), |Q|_{|X|}\big). \qquad \square
\end{aligned}$$

7. Some simple applications

To illustrate the utility of the Sugeno integral semantics of linguistic quantifiers, in this section we present three simple examples.

The first example is concerned with the weather in a week.

Example 42. Recall from Example 16 that the quantifier $Q =$ "many" is defined on a finite set X as follows:

$$Q_X(E) = \frac{|E|}{|X|}$$

for each $E \subseteq X$, where $|E|$ is the cardinality of E. Let $P_1 =$ "to be cloudy" and $P_2 =$ "to be cold" be two (fuzzy) linguistic predicates. Consider the interpretation I in which the domain is

$X = \{$Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday$\}$

Table 1
Truth table of linguistic predicates P_1 and P_2

	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
P_1^I	0.1	0	0.5	0.8	0.6	1	0.2
P_2^I	1	0.9	0.4	0.7	0.3	0.4	0

Table 2
Health condition of 10 students

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
$H(x)$	0.73	0.1	0.95	1	0.84	0.67	0.7	0.9	1	0.81

and the truth values of P_1 and P_2 are given in Table 1. Then the logical formula

$$\varphi = (Qx)\big(P_1(x) \wedge \sim P_2(x)\big)$$

means that "many days (in this week) are cloudy but not cold", and its truth value under I is

$$T_I(\varphi) = \max_{E \subseteq X}\Big[\min_{u \in E} \min\big(P_1^I(u), 1 - P_2^I(u)\big)\Big] = \frac{3}{7} \approx 0.43.$$

The next example demonstrates applicability of Proposition 27 in health data summarization.

Example 43. Consider a set $X = \{s_1, s_2, \ldots, s_{10}\}$ consisting of 10 students. The respective health condition of these students is indicated by Table 2, where the symbol H stands for the linguistic predicate "to be healthy". We want to find a suitable linguistic quantifier from $Q_1 =$ "some", $Q_2 =$ "at least three", $Q_3 =$ "many", $Q_4 =$ "most", $Q_5 =$ "almost all" and $Q_6 =$ "all" to summarize the above data (see Examples 15 and 16 for the definitions of these quantifiers). In other words, we hope to have a high level description of the whole health condition of this group of students. It is required that such a summarization should hold with a high truth value, say, $\geqslant 0.7$. From the above table, we have

$$E \stackrel{\text{def}}{=} \big\{x \in X\colon\ H(x) \geqslant 0.7\big\} = \{s_1, s_3, s_4, s_5, s_7, s_8, s_9, s_{10}\}$$

and

$$Q_{4X}(E) = \left(\frac{|E|}{|X|}\right)^{3/2} = \left(\frac{4}{5}\right)^{3/2} > 0.7 > \left(\frac{4}{5}\right)^{2} = Q_{5X}(E).$$

With Proposition 27 we know that

$$T_I\big((Q_4x)H(x)\big) > 0.7 > T_I\big((Q_5x)H(x)\big),$$

where I is the interpretation given according Table 2. So, what we can say is "most students are healthy".

We finally consider the problem of soft database queries. This kind of problems were handled in [23] by employing Zadeh's approach [56] to linguistic quantifiers. However, here we adopt the Sugeno integral semantics developed in the previous sections.

Example 44. Suppose that a record or entity is a list of attributes, a file is a collection of records of the same type, and a database is a collection of files of various types. Formally, a type is a tuple $T = (x_1, x_2, \ldots, x_k)$ of attribute names. For each $i \leqslant k$, we assume that the set of possible values of the attribute x_i is V_i. Then a record of type T can be written as

$$R = (x_1 : a_1, x_2 : a_2, \ldots, x_k : a_k),$$

where $a_i \in V_i$ is the value of attribute x_i in this record for every $i \leqslant k$. On the other hand, a (soft) query of type T may be formulated as follows:

$$q = \text{“}\textit{find all records such that } Q \textit{ of the attributes out of}$$
$$\{x_1 \textit{ is } F_1,\ x_2 \textit{ is } F_2,\ \ldots,\ x_k \textit{ is } F_k\} \textit{ match}\text{”},$$

where Q is a linguistic quantifier, and for any $i \leqslant k$, F_i is a given linguistic predicate which represents a (soft) constraint on the attribute x_i and which can be defined as a fuzzy subset of V_i. We usually write the types of record R and query q as $\mathit{Type}(R)$ and $\mathit{Type}(q)$, respectively. The above query can be expressed as a formula in our logical language $\mathbf{L_q}$:

$$\varphi = (Qx)\big(\mathit{value}(x) \in \mathit{constraint}(x)\big),$$

where x is an individual variable, both "*value*" and "*constraint*" are unary functions, and $\in$ is a binary predicate. We can imagine that a record R and a query q give an interpretation I of the formula φ: the discourse universe is the type $T = \{x_1, x_2, \ldots, x_k\}$, and

$$\mathit{value}^I(x_i) = a_i,$$
$$\mathit{constraint}^I(x_i) = F_i,$$
$$\in^I (a_i, F_i) = F_i(a_i)$$

for any $i \leqslant k$, for any $a_i \in V_i$ and for any fuzzy subset F_i of V_i. Now the degree to which a record R matches the query q is calculated as the truth value of the formula φ:

$$\mathit{Match}(q, R) = T_I(\varphi) = \int h \circ Q_T,$$

where $h(x_i) = F_i(a_i)$ for every $i \leqslant k$. Furthermore, a database can be defined as

$$D = \bigcup_{j=1}^{m} F_j,$$

where F_j is a file of type T_j, that is, a set of some records with type T_j for each $j \leqslant m$. Let λ be a pre-specified threshold. If the matching degree of a record R and the query q exceeds λ, then R is introduced into the output file of q. Therefore, the output file of query q is given as

$$\mathit{OUTPUT}(q) = \big\{R \in D\colon\ \mathit{Type}(R) = \mathit{Type}(q) \text{ and } \mathit{Match}(q, R) \geqslant \lambda\big\}.$$

Let us now consider a concrete query

$$q = \text{“}\textit{find all records such that almost all of the attributes out of}$$
$$\{x_1 \textit{ is small},\ x_2 \textit{ is small},\ x_3 \textit{ is big},\ x_4 \textit{ is around } 4,\ x_5 \textit{ is big},\ x_6 \textit{ is very big},$$
$$x_7 \textit{ is small},\ x_8 \textit{ is small},\ x_9 \textit{ is very small},\ x_{10} \textit{ is very big}\} \textit{ match}\text{”},$$

where the type of q is $T = \{x_1, x_2, \ldots, x_{10}\}$. Assume that $V_1 = V_2 = \cdots = V_{10} = \{1, 2, \ldots, 8\}$ and

$$\text{“}\textit{small}\text{”} = 1/1 + 0.8/2 + 0.3/3,$$
$$\text{“}\textit{very small}\text{”} = 1/1 + 0.64/2 + 0.09/3,$$
$$\text{“}\textit{big}\text{”} = 0.3/6 + 0.8/7 + 1/8,$$
$$\text{“}\textit{very big}\text{”} = 0.09/6 + 0.64/7 + 1/8,$$
$$\text{“}\textit{around } 4\text{”} = 0.3/2 + 08/3 + 1/4 + 0.8/5 + 0.3/6.$$

Furthermore, we consider the following record

$$R = (x_1 : 7, x_2 : 2, x_3 : 8, x_4 : 3, x_5 : 6, x_6 : 6, x_7 : 1, x_8 : 2, x_9 : 2, x_{10} : 7).$$

The interpretation I determined by the record R and the query q is then given by Table 3.

Using Lemma 11 it is routine to compute the matching degree of the record R to the query q:

$$\mathit{Match}(q, R) = \int h \circ \mathit{almost\ all}_T = \min\big[0.64, \mathit{almost\ all}_T\big(\{x_2, x_3, x_4, x_7, x_8, x_9, x_{10}\}\big)\big] = 0.49,$$

where the quantifier "almost all" is defined according to Example 16.

Table 3
The interpretation I determined by record R and query q

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$h(x_i) = F_i(a_i)$	0	0.8	1	0.8	0.3	0.09	1	0.8	0.64	0.64

8. Discussion

The main aim of this section is to clarify further the relationship between the results obtained in this paper and related works.

Zadeh [56] distinguished two kinds of fuzzy quantifiers, absolute quantifiers and relative quantifiers. An absolute quantifier refers to a (fuzzily defined) number such as "at least three", "few" and "not very many", while a relative quantifier refers to a proportion such as "most" and "at most half". Then truth evaluation of quantified statements is performed by calculating (absolute and relative) cardinalities of fuzzy sets. In this paper, however, evaluation of quantification is directly done over subsets (not their cardinalities) of the discourse universe. This is different from the way considered in [56] and enables us not need to distinguish absolute and relative quantifiers.

In [25], a linguistic quantifier is defined as a constraint on probability values, and it can be viewed as an alternative form of imprecise probability. From the angle of mathematical representation, linguistic quantifiers dealt with in [25] can be identified with the relative quantifiers in [56], both being defined to be fuzzy subsets of the unit interval. More essentially, however, interpretations of quantifiers and reasoning mechanics with quantifiers in [25,56] are quite different: in [25], a quantifier is treated in a probabilistic way and using the voting model, and reasoning with quantifiers is conducted by Bayesian method.

The idea of representing linguistic quantifiers by fuzzy measures and Sugeno's integrals was originally introduced by the author in [42], where some basic logical properties of linguistic quantifiers were presented, including a version of Proposition 33. This paper is a continuation of [42], and indeed some part of the present paper can be seen as an elaboration of [42]. In 1994, Bosc and Lietard [4–7] proposed independently the idea of evaluating linguistically quantified propositions with Sugeno's integrals. However, the motivations of [42] and [4–7] are quite different: the main aim of [42] is to establish a first order logic with fuzzy quantifiers, whereas in [4–7], a close connection between Prade and Yager's approaches to the evaluation of fuzzy quantified statements and Sugeno and Choquet's integrals [8] was found. The different motivations leads to some big differences between [42] (and this paper) and [4–7]. First, the discourse universes considered in [4–7] are required to be finite. Indeed, in almost all previous approaches [1,4–7,9, 10,12,15–18,20–23,25,27,28,30–33,39–41,56] to fuzzy quantification, only finite discourse universes are considered because they are mainly application-oriented and usually in real applications finite universes are enough. In the current paper, however, infinite discourse universes are allowed. This enables us to have some deeper logical properties of fuzzy quantifiers. Second, in [4–7] a fuzzy (absolute) quantifier Q over the universe X is still defined to be a fuzzy sets of integers according to Zadeh [56]. Then a fuzzy measure m_Q on X is induced by

$$m_Q(E) = Q\big(|E|\big)$$

for each $E \subseteq X$, where $|E|$ is the cardinality of E, and Sugeno's integrals with respect to m_Q are used to evaluate propositions quantified by Q. (In fact, the process that m_Q is induced from Q is an inverse of the process after Definition 36 that a numeric quantifier is induced from a cardinal quantifier Q.) Thus, the fuzzy quantifiers dealt with in [4–7] are all cardinal in the sense of Definition 36.

Usually, two types of fuzzy quantified statements are identified, namely, type I proposition of the form "*Q Xs are As*" and type II statement of the form "*Q Ds are As*", where X is the discourse universe, Q is a fuzzy quantifier, and A and D are two fuzzy predicates over X [27,28,56]. In this paper, for an elegant logical theory of fuzzy quantifiers, we only consider type I propositions, and each type II proposition may be translated to a type I proposition in the standard ways:

"*Q Ds are As*" $\Leftrightarrow$ "*Q Xs* (*Xs* are *Ds implies Xs* are *As*)"

or

"*Q Ds are As*" $\Leftrightarrow$ "*Q Xs* are *Ds and As*".

The choice of translation methods is determined by the strength of quantifier Q. In general, for stronger quantifiers such as the universal quantifier ($\forall$) we adopt the first method, and for weaker quantifiers such as the existential quantifier ($\exists$) we use the second one.

A list of desirable properties for fuzzy quantification was presented in [1,9,15–18]. Here, we briefly examine the relation between the most relevant ones and the results obtained in this paper. *Independence of order in the elements of the referential* [1,15] is similar to the definition of cardinal quantifier (Definition 35). It is obvious that many quantifiers do not enjoy this property. Example 25 shows that the Sugeno integral semantics of quantifiers satisfies the property of *induced operators* [1,15]. Furthermore, Example 26 shows that this semantics also satisfies *coherence with fuzzy logic* and the property of *correct generalization* [1,9,15,32]. Delgado, Sanchez and Vila [9] proposed that quantification evaluation must be not too "strict". Precisely, this criterion requires that for any quantifier Q there is a fuzzy predicate A such that the evaluation value of sentence "*Q Xs are As*" is neither 0 nor 1 provided Q is not a crisp quantifier. It is easy to see that such a criterion is satisfied by the Sugeno integral semantics of quantifiers. Indeed, let Q be a quantifier and X a set such that there exists $E_0 \subseteq X$ with $0 < Q_X(E_0) < 1$. For any unary predicate symbol P, we consider an interpretation I with X as its domain such that

$$P^I(u) = \begin{cases} \lambda, & \text{if } u \in E_0, \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < \lambda < 1$. Then $0 < T_I((Qx)P(x)) = \min(\lambda, Q_X(E_0)) < 1$. Propositions 29(1) and 30 indicates that both *quantifier monotonicity* [1,9,15,32] and *local monotonicity* are valid in the framework presented in this paper. *Convexity* [1,15]. is then a direct corollary of local monotonicity. *Type I to type II transformation* [1,15,32]. is already discussed in the above paragraph. The property of *decomposition* [1] is given in Proposition 31(1). The property of *external negation* [16] cannot be considered in the Sugeno integral semantics of quantifiers because the complement $\overline{Q}$ of a quantifier Q, defined by $\overline{Q_X}(E) = 1 - Q_X(E)$ for any $E \subseteq X$, does not fulfil the monotonicity in Definition 3 and it is not a quantifier. The property of *antonym* [16] is not true in general. Proposition 33 presents a version of *duality* weaker than that required in [16]. In order to make its practical application possible, it was pointed out in [9,32] that the evaluation method of quantification should be efficient in the sense of computational complexity. Suppose that the domain of interpretation I is a finite set and its cardinality is n. At the first glance, we may think that the evaluation of the truth value $T_I((Qx)\varphi)$ of the quantified formula $(Qx)\varphi$ under I will run in exponential time with complexity $O(2^n)$ (cf. Definitions 9 and 23(ii) and Lemma 10). Fortunately, Lemma 11 provides us with an evaluation algorithm of quantification with complexity $O(n \log n)$.

Continuity or smoothness in behavior [1,18] means that a small perturbation in the membership values of fuzzy properties D or A does not give rise to a drastic change of the evaluation of fuzzy quantified proposition. This property warrants insensibility of fuzzy quantification to noise. A similar problem for compositional rule of fuzzy inference was addressed by the author in [48,52] where linguistic quantifiers were not involved. To give such a property, we first need to introduce a notion of distance between fuzzy sets. Let A and B be two fuzzy subsets of X. Then the distance between A and B is defined to be

$$d(A, B) = \sup_{x \in X} \left| A(x) - B(x) \right|.$$

The following proposition exposes the continuity for our evaluation mechanism of fuzzy quantified statements.

Proposition 45. *For quantifier Q, for any $\varphi, \psi \in$ Wff, and for any interpretation I, we have*:

$$\left| T_I\big((Qx)\varphi\big) - T_I\big((Qx)\psi\big) \right| \leqslant d\big(T_I(\varphi), T_I(\psi)\big).$$

Proof. Note that for any $a, b, c, a_i, b_i \in [0, 1]$ ($i \in J$), it holds that

$$\left|\min(a, c) - \min(b, c)\right| \leqslant |a - b|,$$
$$\left|\inf_{i \in J} a_i - \inf_{i \in J} b_i\right| \leqslant \inf_{i \in J} |a_i - b_i|$$

and

$$\left|\sup_{i \in J} a_i - \sup_{i \in J} b_i\right| \leqslant \sup_{i \in J} |a_i - b_i|.$$

Then it follows that

$$\begin{aligned}\big|T_I\big((Qx)\varphi\big) - T_I\big((Qx)\psi\big)\big| &= \Big|\sup_{F\subseteq X}\min\Big[\inf_{u\in F} T_{I\{u/x\}}(\varphi), Q_X(F)\Big] - \sup_{F\subseteq X}\min\Big[\inf_{u\in F} T_{I\{u/x\}}(\psi), Q_X(F)\Big]\Big| \\ &\leqslant \sup_{F\subseteq X}\sup_{u\in F}\big|T_{I\{u/x\}}(\varphi) - T_{I\{u/x\}}(\psi)\big| \\ &= \sup_{x\in X}\big|T_{I\{u/x\}}(\varphi) - T_{I\{u/x\}}(\psi)\big| \\ &= d\big(T_I(\varphi), T_I(\psi)\big). \qquad \square\end{aligned}$$

The continuity given in the above proposition is with respect to the change of truth values of statements bound by the same quantifier. Indeed, we also have continuity with respect to perturbation of quantifiers. Let Q_1 and Q_2 be two quantifiers and X a nonempty set. The distance of Q_1 and Q_2 on X is defined to be

$$d_X(Q_1, Q_2) = \sup_{E\subseteq X}\big|Q_{1X}(E) - Q_{2X}(E)\big|.$$

It is worth noting that this definition of distance between quantifiers is similar to variation distance between probability measures [19, page 108].

Proposition 46. *For any quantifiers Q_1 and Q_2, for any $\varphi \in$ Wff, and for any interpretation I, we have*:

$$\big|T_I\big((Q_1x)\varphi\big) - T_I\big((Q_2x)\varphi\big)\big| \leqslant d_X(Q_1, Q_2),$$

where X is the domain of I.

Proof. Similar to Proposition 45. $\square$

It should be pointed out that continuity cannot be expressed in our logical language $\mathbf{L_q}$ given in Section 4.

9. Conclusion

In this paper, we present the Sugeno integral semantics of linguistic quantifiers in which a quantifier is represented by a family of fuzzy measures [35] and the truth value of a quantified proposition is computed by using Sugeno's integral [35]. Several elegant logical properties of linguistic quantifiers are derived, including a prenex normal form theorem. This semantics is compared with the usual cardinality-based approaches (see for example [9,12,31,56]) to linguistic quantifiers.

Three simple applications to data summarization and database queries were presented in Section 7. Nevertheless, more applications are anticipated in other fields such as information fusion [21,25], decision making [20,40] and inductive learning [21].

For further theoretical studies, we wish to examine carefully various model theoretic properties of the logic with linguistic quantifiers and want to see whether it is axiomatizable. The mathematical tool used in this paper for aggregating truth values is the Sugeno integral. As mentioned after Definition 18 and Proposition 31, a generalization of Sugeno's integral was proposed in the previous literature by replacing the operation "min" in Definition 9 with a general t-norm, and this kind of generalized Sugeno's integrals can also be used in our semantics of linguistic quantifiers. It seems that quantifications interpreted in terms of Sugeno's integrals with t-norms allow us to aggregate data, information and knowledge in a "softer" and "more flexible" way. So, a careful examination of linguistic quantifiers modeled by generalized Sugeno's integrals would be another interesting topic. In addition, the Choquet integral is widely used as an aggregation operator in economics, game theory and multi-criteria decision making. It is reasonable to expect that the Choquet integral can also be used to evaluate the truth values of linguistically quantified propositions. Indeed, as pointed out in the last section, Bosc and Lietard [4–7] already noticed a connection between Prade and Yager's representation methods of linguistic quantifiers [30,39,41] and the Choquet integral. Thus, a systematic development of the Choquet integral semantics of linguistic quantifiers would also be an interesting topic.

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