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Dynamic reasoning with qualified syllogisms

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Abstract

A *qualified syllogism* is a classical Aristotelean syllogism that has been "qualified" through the use of fuzzy quantifiers, likelihood modifiers, and usuality modifiers, e.g., "*Most* birds can fly; Tweety is a bird; therefore, it is *likely* that Tweety can fly." This paper introduces a formal logic **Q** of such syllogisms and shows how this may be employed in a system of nonmonotonic reasoning. In process are defined the notions of *path logic* and *dynamic reasoning system* (DRS). The former is an adaptation of the conventional formal system which explicitly portrays reasoning as an activity that takes place in time. The latter consists of a path logic together with a multiple-inheritance hierarchy. The hierarchy duplicates some of the information recorded in the path logic, but additionally provides an extralogical *specificity* relation. The system uses *typed* predicates to formally distinguish between properties and kinds of things. The effectiveness of the approach is demonstrated through analysis of several "puzzles" that have appeared previously in the literature, e.g., Tweety the Bird, Clyde the Elephant, and the Nixon Diamond. It is also outlined how the DRS framework accommodates other reasoning techniques—in particular, predicate circumscription, a "localized" version of default logic, a variant of nonmonotonic logic, and reason maintenance. Furthermore it is seen that the same framework accomodates a new formulation of the notion of *unless*. A concluding section discusses the relevance of these systems to the well-known *frame problem*. © 1997 Elsevier Science B.V.

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1. Introduction

1.1. Background

Modern logic was originated about a century ago, with its development aimed initially at investigating the foundations of mathematics. In process many key concepts were formulated, including in particular that of a formal logical system together with the associated notions of proof, consistency, completeness, syntax versus semantics, and so on. Several schools of thought arose concerning what one should take as a solid grounding for the mathematical disciplines. Among these, Intuitionism brought forth constructive mathematics (see [76]), which led to formulations of the notion of computability. This in turn played a key role in the development of computer science, and it revived anew the centuries old vision of a "thinking machine" (as in [178]). When the first computers were being invented, it was noted that these could be programmed not only for mathematical computations, but also logical reasoning. The subsequent quest to make computers perform ever more sophisticated types of mental activities then gave birth to the subdiscipline known as AI.

Shortly after computers were first being programmed to perform logical deductions, however, it was found that the notion of formal system which had served so well for exploring the foundations of mathematics would not be adequate for encoding all the salient aspects of natural human reasoning. Of critical importance was the discovery that everyday reasoning is oftentimes *nonmonotonic* in that the acquisition of new information may lead one to go back and retract old conclusions. This style of reasoning arises under conditions of incomplete information. If one has perfect knowledge regarding the situation at hand, then one's inferences can in principle be drawn with absolute certainty; but when the available information is less than complete, then one can draw conclusions only tentatively, keeping in mind that these may become invalid should additional information be obtained. Since the conventional type of formal system is *monotonic* in that the addition of new information (axioms) serves generally to expand the set of derivable conclusions (theorems), it does not allow for this newer kind of reasoning activity to be expressed.

Hence in the interests of creating machines with more human-like "intelligence", there ensued a concerted effort to identify and formalize various aspects of this nonmonotonic behavior. An early work of this genre is the 1969 paper by McCarthy and Hayes [104], now well-known for identifying the *frame problem.* That work contained the seeds of several later developments. In particular, it introduced the *situation calculus*, a new kind of formalism embodying a *history*—namely, a sequence of propositions expressed in a language admitting modal operators *consistent*, *normally*, and *probably*—together with a set of rules specifying how a history may be grown by adding new propositions. For example,

$$normally(\phi), consistent(\phi) \vdash probably(\phi)$$

says that if ϕ is normally true, and ϕ is consistent with what is known so far (i.e., with respect to the existing history), then one may add to the history the assertion that ϕ is probably true.

A decade later, four separate approaches to nonmonotonicity appeared almost simultaneously. First was Doyle's *truth maintenance system* [25]. This provided a mechanism for keeping track of logical dependencies in such a way that derivation steps can be retraced, and formerly held conclusions can be retracted. The resulting formalism bears a resemblance to the earlier notion of history, but also offers the novel feature that conclusions are held, not only because certain earlier propositions are held (are "in"), but also because certain other propositions are not held (are "out"). Such formalisms have more recently been dubbed *reason maintenance systems* reflecting the fact that it is the reasons for conclusions that are at issue, rather than their objective truth (e.g., see [163]).

Second was McCarthy's method of *circumscription* [100]. This provided a "rule of conjecture" by virtue of which one assumes that the only individuals x that have a certain property P are those that are explicitly required (i.e., as a consequence of the axioms) to have P. In effect, one "circumscribes" the collection of individuals that must have P by tentatively asserting that this is all there are. The kind of formalism one considers in circumscriptive reasoning thus represents a snapshot of the current state of affairs—it contains all conclusions that one might reasonably draw, given the present body of information. Nonmonotonicity enters when further individuals subsequently are found that also have property P. These are then added to the formalism, and conclusions (conjectures) are redrawn, yielding a potentially smaller set than before.

Third was Reiter's *default logic* [150]. This introduced derivation schemata such as

$$\frac{\mathrm{Bird}(x) : \mathcal{M}\mathrm{CanFly}(x)}{\mathrm{CanFly}(x)}$$

where $\mathcal{M}$ is a modal operator taken to express "is consistent". The rule is interpreted as saying that, if x is a bird, and it is consistent to assume that x can fly, then one may infer (by default) that x can fly. As with circumscription, the resulting formalism is a snapshot of the current knowledge and conjectures. Given that Tweety is a bird, and in the absence of any countervailing information, one concludes that Tweety can fly. If at a later time, however, it is learned that Tweety is a penguin, say, then the formalism would be expanded by adjoining this fact about Tweety as a new axiom. Then conclusions are redrawn based on this new, enlarged axiom set, and if there happens to be an axiom or theorem asserting that penguins cannot fly, CanFly(Tweety) now becomes inconsistent, and the earlier inference made by default is blocked.

Last is the *nonmonotonic logic* of McDermott and Doyle [107]. This is similar in spirit with Reiter's default logic, in that it too employs a modal operator $\mathcal{M}$ expressing "is consistent"; but the style of formalism is much different. Here $\mathcal{M}$ is made an explicit element of the language, so that the sense of the foregoing default rule is here expressed by the formula

$$\mathrm{Bird}(x) \wedge \mathcal{M}[\mathrm{CanFly}(x)] \rightarrow \mathrm{CanFly}(x)$$

The language for this system is provided with a Kripke-style "possible worlds" semantics, with $\mathcal{M}$ being interpreted as roughly equivalent with the classical modality "is possible". The intent is that, if the there is a possible world in which Tweety is a bird that can fly, then both conditions of the above inference hold, and one can apply classical Modus Ponens to infer that Tweety can fly. But if there are further propositions asserted, e.g., to the effect that Tweety is a penguin and penguins cannot fly, then there will be no possible world in which Tweety can fly. In this case the condition $\mathcal{M}[\text{CanFly}(x)]$ will fail to hold, and the inference cannot be employed. Again the formalism represents a snapshot of current knowledge, together with whatever conclusions may be inferred from that knowledge; and as new information is added, and conclusions are redrawn, a possibly smaller set of conclusions will result.

A later development was McCarthy's *formula circumscription* [101], a generalization of the earlier version which, to avoid confusion was here renamed *predicate circumscription*. McCarthy attributed the challenge to create logical formalisms adequate to "express the facts and non-monotonic reasoning concerning the ability of birds to fly" to Marvin Minsky, and he presented numerous examples illustrating how formula circumscription may be applied to this task.

While McCarthy and Hayes [104] had suggested the use of a modality expressing "probably", they also argued that it would be inappropriate to assign numerical probabilities to individual propositions. First, "it is not clear how to attach probabilities to statements containing quantifiers in a way that corresponds to the amount of conviction people have". Second, "the information necessary to assign numerical probabilities is not ordinarily available". To this one might also add the observation that humans themselves almost never reason in terms of numerical probabilities. Hence in the interests of developing more human-like reasoning systems, numerical probabilities need not (and perhaps should not) be explicitly portrayed.

This of course does not conflict with the earlier suggestion to include probability as a modality. In effect the proposal there was to employ a "qualitative" probability, rather than a numerical one. That suggestion notwithstanding, however, all four approaches described above avoid this issue altogether. Instead they in one way or another adopt the more straightforward reasoning strategy wherein, in the absence of any countervailing information, one simply assumes that a certain inference can be applied, while at the same time holding in reserve the option to revise this assumption should such information later be obtained. This approach is clearly a natural one, inasmuch as it is commonly employed in normal everyday reasoning. But by the same token, the probabilistic approach originally proposed is also quite natural. Indeed it is just as common as the nonprobabilistic variety.

Hence perhaps for this reason alone, the prospect of somehow applying probability theory to nonmonotonic reasoning has resisted being put to rest. Rich [152] expressed the view that default reasoning could be treated as a form of "likelihood reasoning", but did not develop it. Similar thoughts were expressed by Ginsberg [60], and then carried forward through [62] and the doctoral dissertation by Darwiche [21] (see also [22]). The latter presents a "symbolic generalization" of probability theory, based on the abstract notion of a "degree of support". This provides a qualitative theory of probability in the sense that it preserves the key elements of Bayesian probability theory, while removing the strict dependence on numerical evaluations. The same variety of qualitativeness is shared with the formulations of likelihood due to Nilsson [124], Halpern and Rabin [67], and Halpern and McAllester [66]. These systems are based on a Kripke-style possible-worlds semantics, however, taking likelihood as a reinterpretation of the underlying reachability relation, and hence yield a much different formalism than those based on probability theory.

In parallel with these studies was Pearl's development of the "ε-semantics" [132] and the System Z [133], which led to the award-winning doctoral dissertation by Geffner [59]. This line of thought exploits Adams' [1,2,4] presentation of the logical conditional as a conditional probability, and it uses Adams' notion of "high probability" as an interpretation of the (nonmonotonic) modifiers *typically* and *normally*. To wit, an assertion such as "*Typically* birds can fly" is represented by there being a high probability that an arbitrarily chosen bird will be able to fly. The idea of interpreting the logical conditional in this way dates back to Ramsey [149] and has been taken up in philosophical circles by Lewis [93] and Stalnaker [167]. Lewis's "triviality result", establishing certain limitations of this approach, has undergone subsequent investigation [27,108,110], and it plays a crucial role in the systems being presented here (Section 2.4). A related line of development deserving of mention is the study of "conditional objects" due to Dubois and Prade [32,33] based on their developments in possibility theory [29].

Adams [3] had further argued that "high probability" could be interpreted as *almost all*. Analogously with the above, an assertion to the effect that "*Almost all* A's are B's" may be represented by there being a high probability that an arbitrarily chosen A will be a B. This interpretation was not employed, however, in the Pearl–Geffner approach to nonmonotonicity.

A continuation of the Adams–Pearl line of thought has emerged in the works of Bacchus [9,10], Halpern [65], and Bacchus, Grove, Halpern, and Koller [11,12]. These papers develop formalisms that allow for expressions such as

$$\Pr[\mathrm{CanFly}(x) \mid \mathrm{Bird}(x)] \approx 1$$

taken as expressing "*Almost all* birds can fly", which in turn is taken as representing "*Typically* birds can fly", and expressions such as

$$\Pr[\mathrm{Hep}(x) \mid \mathrm{Jaun}(x)] \approx 0.8$$

taken as expressing "approximately 80% of patients with jaundice have hepatitis". A theory of "statistical syllogisms", featuring a similar probabilistic representation of the modifier *most*, has been proposed by Pollock [144]. Another related line of development are the foundational studies in probability and statistics due to Kyburg [88–90].

The main early papers on nonmonotonic reasoning have been reprinted in the collection [61]. An in-depth comparison of most of the foregoing formalisms, together with several others not considered here, may be found in [164]. This gives a summary of virtually all approaches to reasoning with incomplete information that were in print as of that date.

Reiter [150] had noted the possible relevance of the notion of "most" for nonmonotonic reasoning, and he regarded this concept as implicit in the use of default logic. The abovementioned works by Pearl, Halpern, Bacchus, and others take this another step by developing formalisms containing propositions which can be interpreted as more explicitly encoding the idea of "most". But none of these make the term fully explicit in the sense of directly introducing it into the syntax of the languages.

A body of work oriented toward the latter task has been advanced by L.A. Zadeh, the founder of the theory of fuzzy sets. The paper [199] presented a semantics for *fuzzy quantifiers*—modifiers such as *most*, *many*, *few*, *almost all*, etc.—and introduced the idea of reasoning with syllogistic arguments along the lines of "*Most* men are vain; Socrates is a man; therefore, it is *likely* that Socrates is vain", where vanity is given as a fuzzy predicate. In this and numerous succeeding publications (see Section 2.1) Zadeh has developed well-defined semantics also for *fuzzy probabilities* (e.g., *likely*, *very likely*, *uncertain*, *unlikely*, etc.) and fuzzy *usuality modifiers* (e.g., *usually*, *often*, *seldom*, etc.). In addition, he has argued at numerous conferences over the years that these modifiers offer an appropriate and intuitively correct approach to nonmonotonic reasoning.

The matter of exactly how these various modifiers are interrelated, however, and therefore of a concise semantics for such syllogisms, was not fully explored. Thus while a new methodology for nonmonotonic reasoning was suggested, it had not been completely developed. The present work grew initially out of an effort to realize this goal.

1.2. Overview

The first task, undertaken in Section 2, is to define the system **Q** for reasoning with *qualified syllogisms*. In effect, these are classical Aristotelean syllogisms that have been "qualified" through the use of fuzzy quantification, usuality, and likelihood. (The term "fuzzy likelihood" is here preferred over "fuzzy probability", taking the latter to mean a probability which is evaluated as a fuzzy number.) In contrast with the syllogisms originally considered by Zadeh, we here deal only with the case of fuzzy modifiers in application to crisp (nonfuzzy) predicates. Some examples are

Most birds can fly.
Tweety is a bird.

It is *likely* that Tweety can fly.

Usually, if something is a bird, it can fly.
Tweety is a bird.

It is *likely* that Tweety can fly.

Very few cats have no tails.
Felix is a cat.

It is *very unlikely* that Felix has no tail.

From a common-sense perspective, such arguments are certainly intuitively correct. System **Q** features a completely specified language, suitable for expressing such syllogisms, together with a semantics which validates them. The semantics is based on a probabilistic interpretation of the modifiers, and it adopts the earlier idea of interpreting logical conditionals as conditional probabilities. The system is unorthodox in that it employs two distinct semantic levels, the lower level being multivalent (probabilistic) and the upper being bivalent (classical). Briefly, a formula of the form *LikelyP* is assigned the value *true* just in case the probability value of P falls within a certain subinterval of $[0,1]$. Thus one obtains a qualitative version of probabilistic reasoning in the sense that the reasoner here deals directly only with the linguistic modifiers—no reference to numerical probabilities appears in the formal syntax.

Section 3 shows how a version of the classical modalities, *possibility* and *necessity*, can be defined in terms of fuzzy quantifiers. This builds on an idea put forth by Rescher [151]. A proposition may be deemed *necessary* if its probability of being true is 1, and *possible* if this probability is greater than 0. It turns out that not all the axioms for the well-known modal logics K, S4, and S5 (see [80]) can be expressed in the language of **Q** as presently construed, but those that can be are validated by the semantics for **Q**.

Section 4 introduces the notion of a *path logic*. In a system that offers fuzzy likelihood modifiers, there naturally arises the desire to have rules for modifier combination. For example, one might want such rules as

$$\frac{\begin{array}{l} \mathit{LikelyP} \\ \mathit{UnlikelyP} \end{array}}{\mathit{UncertainP}}$$

$$\frac{\begin{array}{l} \mathit{LikelyP} \\ \mathit{CertainlyP} \end{array}}{\mathit{CertainlyP}}$$

The former allows one to say that, if by some line of reasoning (or collection of evidence) one were to conclude *LikelyP*, and by another line (evidence) one were to conclude *UnlikelyP*, then one may combine these to conclude *UncertainP*. The latter rule asserts, in effect, that *certainly* dominates *likely*.

Such rules however cannot be expressed within the system **Q**. This is because that system's definition employs the conventional notions of formal system and semantics, wherein under any particular truth (or probability) valuation each formula must have a unique such value. Otherwise the valuation function would not be well-defined. In order that the first of the above rules be meaningful, however, each of the three different occurrences of P would need to have a different probability value (the subintervals associated with the three different likelihood modifiers should be nonoverlapping), and in the second rule, the two occurrences of P in the premises should have different probabilities, while the occurrence in the conclusion should have the same probability as the one in the second premise.

Upon reflection it becomes evident that, whenever one applies a rule of this kind, it is implicit that his reasoning is an activity which is taking place in *time*. To wit, the first example says that, if at a certain time t one has *LikelyP*, and at another time t' one has *UncertainP*, then at a later time t'' one may conclude *UncertainP*. Accordingly, in order to express such rules it is necessary to make this temporal aspect explicit in such a way as to distinguish between the different *occurrences* of P.

The notion of a *path logic* provides this capability. In effect this is a straightforward adaptation of the conventional notion of formal logical system, obtained by lending special semantic status to derivation paths. Formulas appearing in such a path can now be indexed by their location in the path, with each successive index representing the next *time step*. Then if the same formula occurs more than once, the different occurrences will be distinguished by having different indices. Probability valuations can then be defined as acting on indexed formulas in such a way that the different occurrences of P may have different values. The formula's index will be included, along with other extralogical items, in a *label*.

In addition to having a formula be appended to the path by deriving it from earlier formulas, a formula may be *received* from an external source. This choice of terminology reflects the view that a path logic may be regarded as a part of the "mind" of a digital (hardware or software) agent. As examples, formulas so received may represent input from a human user, data collected from a sensor, or messages received from another similar agent. Items added to the path might be new beliefs, facts (i.e., beliefs held with absolute certainty), or instructions to perform an action (e.g., turn right, log into website X, or send a message to agent Y). A belief may be retracted by turning its *status indicator* (also recorded in the formula's label) from "on" to "off". A formula with status set to "off" is no longer available for use in further inferences. This new formalism also allows that the language itself may grow, for example, by introducing new individual names or predicate symbols, and as well it admits addition of new inference rules. Collectively, these various features make a path logic amount to a propositional knowledge base which evolves over time. Nonmonotonicity occurs whenever a belief is retracted. A way in which such retractions may be negotiated through a modified version of Doyle's truth (or reason) maintenance system will be outlined in Section 5 and then taken up in greater detail in Section 7.

Shoham [157,158] has also explored the idea of making time an explicit feature of the formalism. But those works deal with formalizing one's reasoning *about* change, i.e., with modeling the way an agent might reason about temporal events in the external world, whereas the present concern is with modeling changes in the reasoner's own mental state. As such, the present work is more in spirit with, and in fact embodies some of the same ideas as, the "agent-oriented" system outlined by Shoham in [159]. A more recent approach to reasoning about change, and furthermore one incorporating probability theory, has been proposed by Hanks and McDermott [68]. There, as with Shoham's earlier works, the main concern is with robot motion planning.

The present work bears several elements in common with those of Perlis and his students [34,36–38,109,138]. These various writings develop a theory of "active logics" (formerly "step logics") which also portrays reasoning as an activity which is "situated" in time. Similarly as with path logics, reasoning is viewed as occurring in discrete time steps, and the present notion of a "path" turns out to be essentially identical with their notion of a sequence of formulas arranged along a "time line". Path logics employ a somewhat different style of formalism, however, and in the following are oriented toward a different set of problems. The present concern is with modeling multiple-inheritance reasoning about static domains (see the following), while there the concern is with reasoning about domains that are dynamically changing. Recent works have dealt with planning [35,125–127] and with inter-agent communication [63,64,140,147].

Another recent work suggesting that reasoning might be portrayed as time-situated is Sandewall's [155].

Both path logic and active logic embody a view of reasoning that is expressed also by the methods of "belief revision" due to Gärdenfors [57,58], of "knowledge revision" due to Wrobel [189], and of "belief change" due to Friedman and Halpern [53–55]. A possibilistic treatment of some of these same issues has been studied by Dubois and Prade [31]. In these frameworks the totality of the reasoning agent's knowledge and beliefs is represented as a collection of propositions which evolves over a series of discrete time steps, with each step typically involving either an addition or a retraction of propositions in the knowledge base. As such, this constitutes a new approach to the problem of "logical omniscience" first discussed by Hintikka [77] and then taken up more recently by Fagin and Halpern [44] and Fagin, Halpern, Moses, and Vardi [45,46]. This problem is that when one uses a conventional formal logical system to represent an agent's knowledge, it is inherent in the formalism that if the reasoner knows certain facts about the world, then he also knows all possible logical consequences of those facts. Thus one is led to an unrealistic model of both human and digital reasoning. This difficulty is neatly avoided by the newer formalisms which represent knowledge as evolving *in* time. At each time step, the knowledge base contains only what the reasoner has either assumed or deduced as of that step, and nothing further.

Having thus explicated via path logic the temporal aspects of everyday reasoning, the present work turns next to the issue of reasoning with defaults. A particularly useful alternative to Reiter's style of default reasoning has arisen through the use of inheritance hierarchies. Here one employs a taxonomy for keeping track of the relation between different *kinds* of things—e.g., a penguin is a kind of bird, a bird is a kind of animal, etc. With each such kind there is an associated list of *properties*, and it is agreed that entities at lower levels inherit the properties of their higher-level kinds *by default*, unless such inheritance is *blocked* at the lower levels by countervailing properties. For example, a penguin might inherit all the properties of birds except that of being able to fly, since this is blocked by the fact that penguins have the property of being unable to fly. The underlying principle is that lower-level classifications are more *specific* than higher-level ones, and the more specific information takes priority. Early thoughts along these lines appeared in Fahlman's system NETL [47] (see also [173]). A closely related notion is that of a "frame" developed initially by Minsky [112] and later discussed by Hayes [71]. This amounts to a data structure representing a particular kind of things, comprised of a field containing the name of the kind, together with a collection of "slots" representing the properties of things of that kind. These various ideas have subsequently gone through several stages of development [39,40,43,119,175,176], with [119] being more fully developed as the basis for the commercial software package KEE [48]. These methods of knowledge representation have been found to harbor some rather unpleasant anomalies, however (see for example [17,18,81,177]), and efforts to resolve these problems have led to numerous further investigations (e.g., [16,78,79,83,154,156,162,168]).

It turns out that most of these difficulties stem from the need for dealing with *multiple inheritance*, i.e., allowing that an individual may be of more than one kind. Curiously, none of the works since those of Fahlman, Minsky, and Hayes have bothered to make the earlier distinction between properties and kinds. Instead, even though this distinction is intuitively implicit, the formalisms themselves treat all collections essentially as just kinds of things. This leads to a somewhat simpler style of formalism, but as will become evident here, this apparent simplification lies at the root of virtually all of the more troublesome anomalies.

The subject of multiple inheritance is taken up by the present work in Section 5. This introduces a new kind of formalism referred to as a *dynamic reasoning system* (DRS). Briefly, a DRS is a temporally evolving structure composed of a path logic and a *semantic network* which interact with one another in various ways. The subject of semantic networks has an extensive history, by some accounts dating back to the mid-1800s. For a recent survey see the edited collection by Lehmann [91]. In the present treatment, semantic networks will be inheritance hierarchies of the variety described above, including the use of *typed nodes* to distinguish properties from kinds, and allowing for multiple inheritance. The earlier notion of language for **Q** is here expanded to employ *typed predicate symbols*, so as to similarly distinguish formally between predicates representing properties versus kinds. Then it is agreed that, whenever formulas having certain specified forms are added to the path, an appropriate corresponding entry is made in the inheritance hierarchy. For example, if one were to add to the path a formula asserting that penguins are birds, then an entry to this effect would be made in the inheritance hierarchy, connecting a kind node representing "Penguin" to a kind node representing "Bird"; and if a formula stating that most birds can fly were to be added to the path, a link would be entered connecting the kind node for "Bird" to a property node representing "CanFly", and the link would be qualified with the label *most*. Restrictions are made to ensure that the hierarchy always remains "well-behaved", e.g., connections between kind nodes are always crisp (involve only non-fuzzy qualifiers) and loops and redundant paths are disallowed.

As such the inheritance hierarchy duplicates information already expressed by formulas in the path, but it does so in such a way as to provide a basis for defining the abovementioned specificity relation. Each kind node is assigned an *address* in such a way that nodes with higher addresses are lower in the hierarchy, and hence are more specific. The address is taken as a *specificity rank*, and the rank is recorded in the corresponding formula's label. Then inference rules can be devised for the path logic which reference the specificity ranks in such a way as to correctly perform the desired style of default reasoning. The above requirement for maintaining well-behavedness ensures that the specificity ranks will be well-defined.

The typing of predicates does not come without cost, however. This is because there is oftentimes a certain arbitrariness in deciding whether a given class of objects should be taken as representing a property versus a kind. For example, in the well-known Nixon Diamond problem it is a matter of choice how one thus regards the predicates "Quaker", "Republican", and "Pacifist" (see the discussion in Section 5). Nonetheless it turns out that by judicious use of these added extralogical distinctions, most of the anomalies that have vexed prior efforts can be avoided. This is demonstrated through detailed analysis of several "puzzles" that have arisen in the literature and which to a certain extent serve as test cases for multiple-inheritance reasoners. These include the time-worn examples of Tweety and Opus, Clyde the Elephant, and the Nixon Diamond, together with several others that have appeared more recently. The last example in the series illustrates a situation wherein the same predicate is taken as representing *both* a property and a kind.

Section 6 considers the common-sense notion of *unless* as it is found in natural language propositions of the form "if P then Q unless R". In effect this deals with logical inferences that have exceptions. This is closely tied up with default reasoning, and it appears in various guises in several of the works cited above. The type of reasoning represented by such propositions is already implicit in the DRS described in Section 5, inasmuch as exceptions may be deduced via the specificity relation in the multiple-inheritance hierarchy. For many implementations, however, it may be desired to make such reasoning more explicit. This can be done as follows. Agree that the label for a formula such as "*Most* Birds can fly" may contain an optional *exception list* having two different kinds of entries: exceptional individuals (e.g., Opus) and exceptional properties (e.g., being a penguin). Entries are made in the exception list only as they are deduced and entered into the path. This leads to a *procedural semantics* for the kind of syllogisms discussed above: apply an inference to an individual only if it is not known to be exceptional. One in this manner obtains a new and intuitively satisfying rendition of the normal use of the word "unless".

Section 7 reconsiders the four original approaches to nonmonotonic reasoning and shows how each, either in full or in some restricted sense, can be reformulated within the path-logic/DRS framework. The significance of this is that it shows how the formalisms studied here can be used to make explicit the temporal aspects of nonmonotonic reasoning that are merely implicit in the earlier studies. For predicate circumscription, one needs only that the path logic's language admit predicate variables. Then the circumscriptive "rule of conjecture" can be expressed as just another rule of inference. For default logic, there are the well-known difficulties that (i) in first-order systems it is undecidable whether a given formula is consistent with some given set of formulas, and (ii) even in simpler systems, where relative consistency is decidable, it typically is computationally intractable. Hence even though default logic is intuitively appealing, it forbids any direct implementation in all but the very simplest of cases. By appeal to the temporal features of path logics, however, one can devise a variant of default logic that avoids these difficulties. First define a notion of *local* or *n-ply consistency*, meaning free from contradiction for at least the next n reasoning steps. Then agree to apply defaults only under condition of consistency out to some reasonable n. Such a reasoning strategy would appropriately be used in conjunction with a reason maintenance system, so as to adequately deal with inconsistencies that may later be discovered beyond the nth future step. Reason maintenance can be implemented in the path logic formalism by allowing a formula's label to include information about how the formula was derived. This information can then be referenced in backtracking, looking for the sources of inconsistencies. A semblance of nonmonotonic logic can be crafted via the version of classical modalities defined in Section 3.

Section 8 takes up the frame problem, together with some of its relatives, from the perspective of a DRS. Here an important distinction is made between problems associated with reasoning about static (unchanging) domains versus dynamic (changing) ones. It is pointed out that the frame problem arises only in the context of the latter. Thus inasmuch as multiple-inheritance reasoning is always about static domains, the frame problem does not pertain. It is here outlined, however, how the notion of a path logic may be extended to the more complicated realm of dynamic domains and in this context how the frame problem and its relatives once again raise their troublesome heads. Section 9 outlines a few possible future developments.

This paper is the culmination of about 7 years of thinking on these various topics. The ideas have evolved through a rather lengthy series of conference articles discussing bits and pieces of the overall system, and in process they have undergone numerous modifications and expansions. This has included several false starts, a few blind alleys, and one or two outright blunders. A major modification for the system **Q**, appearing here for the first time, is a shift from a probabilistic semantics based on relative cardinalities to one based on Bayesian subjectivist probability theory (as in [23, 132]). This change was made in order to avoid certain limitations and mathematical complexities with the former approach. The earlier semantics is here reintroduced as an alternative, however, that can be useful for certain types of applications. There has also been some simplification of the syntax for **Q**, and several new topics have been added to the elaboration of the DRS. In particular, the material in Sections 6, 7, and 8 appear here for the first time. As such this paper is the first presentation of all these ideas in a single unified reasoning system. It is intended to supersede all prior works.

2. A logic of qualified syllogisms

2.1. Sources and related works

In this paper, the *fuzzy quantifiers* will include the crisp modifiers *all* and *no*, as well as the imprecise ones mentioned earlier. Similarly, the *usuality modifiers* will include *always* and *never*, and the *likelihood modifiers* will include *certainly* and *certainly not*. The fuzzy quantifiers discussed by Zadeh additionally included modifiers like "several" and "around 50", but these will not be considered here. In the literature, usuality modifiers are sometimes referred to as "frequency adjectives", and likelihood modifiers as "linguistic probabilities".

Following the initial discussion of fuzzy quantification in [199], the subject was developed at length by Zadeh and Bellman [206] and Zadeh [201, 202, 204]. A primary aim of those works was to provide a well-defined semantics for propositions such as "*Most* students are *young*", where *young* is given as a fuzzy subset of a collection of ages. Briefly, *most* is interpreted as a fuzzy subset S of the interval $[0, 1]$; there is computed a real number referred to as the *relative sigma count* of the number of students which are young; and the given proposition is interpreted as being equivalent with the assertion "$\Sigma\mathrm{count}(\mathit{young/student})$ is S". The *sigma count* of a fuzzy set, e.g., $\Sigma\mathrm{count}(\mathit{young})$, is a generalization of the notion of cardinality for crisp (non-fuzzy) sets, so that the relative sigma count is the corresponding generalization of the relative cardinalities of crisp sets; it is the sigma count of the number of students which are *young* divided by the total number of students. The resulting semantics based on sigma counts winds up being a natural and straightforward generalization to fuzzy sets of the simple probability measure that is based on relative cardinalities of crisp sets. Indeed, if one restricts the formula for the relative sigma count to only crisp sets, then one gets back the standard probability measure.

Several of these same modifiers had been studied earlier by Mostowski [118] under the heading of "generalized quantifiers", but employing a quite different semantics: each quantifier is there represented as a class of subsets of the underlying universe. Most subsequent work in this area has focused on cardinality quantifiers—e.g., *uncountably many*, *at least 10*, and *exactly 3* (see the collection by Barwise and Feferman [14])—with interest in modifiers like *few*, *most* and *many* only reemerging two decades later. Peterson [141] and Stump [169] undertook philosophical analyses of these terms, and Barwise and Cooper [13] continued the broader development of Mostowski's semantics for generalized quantifiers. The latter in turn sparked a new flurry of activity, leading to van Benthem's [180-182] and van Eijck's [185], as well as the edited collections by van Benthem and ter Meulen [184] and Gärdenfors [56]. A brief comparison of Mostowski's and Zadeh's semantics may be found in Thiele's [171]. Another related work is Kiesler's theory of "probability quantifiers" [82]. This allows for propositions of the form "the probability that P is true about individual a, is at least r", by taking r as the probability measure of the set of individuals about which P is true.

The fuzzy sets notion of usuality was introduced by Zadeh in [204] and then continued with [205]. Those papers additionally mention, but do not elaborate, an apparent connection between usuality and quantification. Nonetheless Zadeh's later papers clearly suggest that the two concepts might be modeled by essentially the same semantics. Usuality logic was termed "dispositional logic" by Zadeh, in reference to the fact that when one knows that a certain phenomenon "usually" occurs, then one is "disposed" to expect that it will continue to occur in the same way in the future. A way to employ usuality in nonmonotonic reasoning has been explored by Whalen and Schott [188].

Fuzzy likelihood was first touched upon by Zadeh in [199], discussed at length in [200], and then taken up again in [203]. In contrast with quantification and usuality, this represents likelihood modifiers as fuzzy subsets of the unit interval $[0, 1]$. Accordingly, the technical differences between these formulations of quantification versus likelihood make them unsuitable for direct implementation as a semantics for qualified syllogisms. An early attempt at formulating such syllogisms via probability theory is [113]. A selection of Zadeh's key papers are reprinted in [198].

An alternative approach to fuzzy quantification has been explored by Yager [190-197]. This semantics begins with the observation that the classical "for all", as in "*All* students are young", is logically equivalent with the conjunction "student-1 is young AND student-2 is young AND etc.," where "student-1," "student-2," etc. are all the students in the underlying domain of discourse. Similarly the classical "there exists", as in "*There exists* a student that is young", is logically equivalent with the disjunction "student-1 is young OR student-2 is young OR etc." Taking the fuzzy quantifiers like *most* and *few* as being intermediary between *for all* and *there exists*, Yager represents these as operators which are weakenings of the logical AND and/or strengthenings of the logical OR. These operators, known as "ordered weighted averages", or OWA operators, are shown to have many of the algebraic properties one would desire of fuzzy quantifiers.

Further investigations of the probabilistic interpretation of fuzzy quantifiers have been undertaken by Dubois and Prade [28,30], Amarger, Epenoy, and Grihon [5], and Amarger, Dubois, and Prade [6-8]. These works mostly consider just the case of fuzzy quantifiers applied to propositions involving only crisp predicates, and where the quantifiers are given as subintervals of $[0,1]$. The authors limit themselves to this simpler case for the purpose of studying the problem of inference chaining, but clearly reserve the option of generalizing to the case of fuzzy predicates and fuzzy quantifiers in the original sense of Zadeh. Some preliminary efforts in the latter direction have appeared in [30], however. The primary aim of these studies was to answer: For arbitrary quantifiers Q_1, Q_2, given that Q_1 A's are B's, and that Q_2 B's are C's, what can be said about how many A's might be C's? It has been shown that, when Q_1 and Q_2 are interpreted as subintervals of probabilities, an algorithm can be applied to compute upper and lower bounds for the probability that an A will be a C. It turns out that the interval determined by these upper and lower bounds is normally larger, and hence less precise, than the ones started with. This correlates with the normal intuition that imprecision accumulates as the reasoning chains increase in length. Similar ideas have been put forth by Narazaki and Turksen [120, 121].

Experimental investigations of the natural human use of usuality and likelihood modifiers have been reported in the Psychology literature. Mosteller and Youtz [117] summarize the results of 20 such studies, covering 52 different modifiers. Kuipers, Moskowitz, and Kassirer [87] studied the use of such modifiers in decision making under uncertainty. A psychological argument for regarding quantifiers as "fuzzy" notions has been put forth by Newstead [123].

Zimmer [207-210] undertook a series of experiments investigating the extent to which the natural human use of fuzzy quantifiers, usuality modifiers, and likelihood modifiers conforms to the standard fuzzy-sets interpretations. In [20, 186, 187, 211, 212] various combinations of Erev, Budescu, Forsyth, Rapaport, Wallsten, and Zwick report experiments studying the use of linguistic probabilities and the extent to which their actual meanings and rules of logical combination (in conjunctions and disjunctions) compare with various fuzzy-sets models. These collective works emphasize the potential usefulness of finding ways to replicate this type of reasoning on a computer.

2.2. Motivation

The first objective is to devise a logic which successfully captures the style of reasoning exemplified by the three syllogisms given in Section 1.2. The principal insights that led to the present rendition are as follows. First, it was noted that there is a natural connection between fuzzy quantification and fuzzy likelihood. To illustrate, the statement

Most birds can fly.

may be regarded as equivalent with

If x is a bird, then it is *likely* that x can fly.

The implicit connection is provided by the notion of a statistical sampling. In each case one is asserting

Given a bird randomly selected from the population of all birds,
there is a *high probability* that it will be able to fly.

Suppose we express this equivalence as

$$(\mathit{Most}\ x)(\mathrm{Bird}(x) \rightarrow \mathrm{CanFly}(x)) \leftrightarrow (\mathrm{Bird}(x) \rightarrow \mathit{Likely}\mathrm{CanFly}(x))$$

Then the first of the two syllogisms involving Tweety can be reduced to an application of this formula, together with the following syllogism:

$$\frac{\begin{array}{l}\mathrm{Bird}(x) \rightarrow \mathit{Likely}\mathrm{CanFly}(x)\\ \mathrm{Bird}(\mathrm{Tweety})\end{array}}{\mathit{Likely}\mathrm{CanFly}(\mathrm{Tweety})}$$

This follows because the left side of the equivalence is the first premise of the original syllogism, and the right side of the equivalence is the first premise of the above syllogism. A key observation to be made here is that the latter syllogism follows by instantiating x with Tweety and applying ordinary (classical) Modus Ponens. This suggests that the desired formulation of fuzzy quantification and fuzzy likelihood may be obtained merely by adjoining classical logic with an appropriate set of modifiers. It also suggests that the modifiers of interest may be introduced in the manner of either quantifiers or modal operators, and that the semantics for such a system could be based on some version of probability theory.

A second observation is that there is a similar connection between the foregoing two concepts and the concept of usuality. Based on the same idea of a statistical sampling, one has that

Usually, if something is a bird, then it can fly.

is equivalent with the former two assertions. Thus one should be able to include usuality modifiers along with quantifiers and likelihood modifiers in a similar extension of classical logic.

This is not to claim, of course, that such a system would capture the full range of meanings and nuances for these terms. In particular, an important distinction arises with regard to their temporal foci. To wit, usuality tends to be applied only in reference to past experiences, as in "It has *usually* been the case that a randomly selected bird can fly", while quantification tends to be applied in statements about the present state of affairs, as in "*Most* birds can fly", and likelihood is normally applied in reference to expectations about the future, "If at some future time I randomly select a bird, then it is *likely* that it will be able to fly". In the context of such distinctions, the above interrelations reflect an implicit presumption that the relevant state of affairs with the population of all birds tends to remain more or less constant throughout time. What was true in the past, is mostly true now, and may be expected to remain true in the foreseeable future. In effect, both quantification and likelihood are pragmatically rooted in our perceptions of usuality.

The system **Q** is an outgrowth of these various insights and reflections. In addition to the syllogisms illustrated in Section 1.2, it allows for expression of all similar syllogisms

Table 1
Interrelations across seven levels of the three kinds of modifiers

Quantification	Usuality	Likelihood
all	always	certainly
almost all	almost always	almost certainly
most	usually	likely
many/about half	frequently/often	uncertain/about 50-50
few/some	occasionally/seldom	unlikely
almost no	almost never/rarely	almost certainly not
no	never	certainly not

as represented by the lines of Table 1 (where the two "Tweety" examples are given by the third line, and the "Felix" example is given by first and last entry of the sixth line).

Table 1 is meant only for illustrative purposes and could be modified in various ways. In particular, one could increase (or decrease) the granularity of distinction between levels by adding (or removing) lines. For example, between the second and third lines one might insert *very many*, *very often*, *very likely*, and in parallel between the fifth and sixth lines insert *very few*, *very seldom*, *very unlikely*. What natural-language terms one puts in the various slots may also be at issue, partly because there is in some cases no completely appropriate choice. For example, it is not necessarily the case that *many* should be taken as synonymous with *about half*, or that *few* means the same as *some*. From the standpoint of the formalism presented below, the labels one puts in the various cells reflect a higher-level interpretation, however, which play no official role in deductions. For the sake of perspicuity they of course should in principle conform to at least some version of common sense. The terms "typically" and "normally", which appear frequently in the literature on nonmonotonic reasoning (Section 1), could here be taken as synonyms for *usually*.

2.3. Languages

We shall begin by defining the kind of languages to be employed. Let the modifiers in Table 1, in top-down then left-right order, be represented by $\mathcal{Q}_3, \ldots, \mathcal{Q}_{-3}$, $\mathcal{U}_3, \ldots, \mathcal{U}_{-3}$, $\mathcal{L}_3, \ldots, \mathcal{L}_{-3}$. As *symbols* select: an (*individual*) *variable*, denoted by x; countably infinitely many (*individual*) *constants*, denoted generically by $a, b, \ldots$; countably infinitely many unary *predicate symbols*, denoted generically by $\alpha, \beta, \ldots$; seven *logical connectives*, denoted by $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\dot{\rightarrow}$, $\ddot{\neg}$, and $\ddot{\vee}$; the abovementioned modifiers $\mathcal{Q}_i$, $\mathcal{U}_i$, and $\mathcal{L}_i$; and *parentheses* and *comma*, denoted as usual. Let the *formulas* be the members of the sets

$$F_1 = \{\alpha(x) \mid \alpha \text{ is a predicate symbol}\},$$

$$F_2 = F_1 \cup \{\neg P, (P \vee Q), (P \wedge Q) \mid P, Q \in F_1 \cup F_2\}, [2]$$

[2] This notation abbreviates the usual inductive definition, in this case the smallest class of formulas containing F_1 together with all formulas that can be built up from formulas in F_1 in the three prescribed ways.

$$F_3 = \{(P \to Q) \mid P, Q \in F_2\},$$

$$F_4 = \{\mathcal{L}_3(P \dot\to \mathcal{L}_i Q), \mathcal{L}_3(P \dot\to \mathcal{Q}_i Q), \mathcal{L}_3(P \dot\to \mathcal{U}_i Q),$$
$$\mathcal{Q}_3(P \dot\to \mathcal{L}_i Q), \mathcal{Q}_3(P \dot\to \mathcal{Q}_i Q), \mathcal{Q}_3(P \dot\to \mathcal{U}_i Q),$$
$$\mathcal{U}_3(P \dot\to \mathcal{L}_i Q), \mathcal{U}_3(P \dot\to \mathcal{Q}_i Q), \mathcal{U}_3(P \dot\to \mathcal{U}_i Q)$$
$$\mid P, Q \in F_2 \cup F_3, i = -3, \ldots, 3\},$$

$$F_5 = \{\mathcal{L}_i P, \mathcal{Q}_i P, \mathcal{U}_i P \mid P, Q \in F_2 \cup F_3,\ i = -3, \ldots, 3\},$$

$$F_6 = F_4 \cup F_5 \cup \{\ddot\neg P, (P \ddot\vee Q) \mid P, Q \in F_4 \cup F_5 \cup F_6\},$$

$$F_1' = \{P(a/x) \mid P \in F_1 \text{ and } a \text{ is an individual constant}\},$$

$$F_2' = \{P(a/x) \mid P \in F_2 \text{ and } a \text{ is an individual constant}\},$$

$$F_3' = \{P(a/x) \mid P \in F_3 \text{ and } a \text{ is an individual constant}\},$$

$$F_4' = \{\mathcal{L}_3(P \dot\to \mathcal{L}_i Q)(a/x) \mid \mathcal{L}_3(P \dot\to \mathcal{L}_i Q) \in F_4 \text{ and } a \text{ is an individual constant}\},$$

$$F_5' = \{\mathcal{L}_i P(a/x) \mid P \in F_5,\ a \text{ is an individual constant, and } i = -3, \ldots, 3\},$$

$$F_6' = F_5' \cup \{\ddot\neg P, (P \ddot\vee Q) \mid P, Q \in F_5' \cup F_6'\},$$

where $P(a/x)$ denotes the formula obtained from P by replacing every occurrence of the variable x with an occurrence of the constant a. As abbreviations take

$$(P \ddot\wedge Q) \quad \text{for} \quad \ddot\neg(\ddot\neg P \ddot\vee \ddot\neg Q)$$
$$(P \ddot\to Q) \quad \text{for} \quad (\ddot\neg P \ddot\vee Q)$$
$$(P \ddot\leftrightarrow Q) \quad \text{for} \quad ((P \ddot\to Q) \ddot\wedge (Q \ddot\to P))$$

Parentheses may be dropped when the intended grouping is clear; associativity is assumed to be to the right. Formulas without modifiers are *first-* or *lower-level* formulas, and those with modifiers are *second-* or *upper-level*. The members of the set $F_1 \cup F_1'$ are *elementary first-* or *lower-level* formulas, and the members of $F_4 \cup F_4' \cup F_5 \cup F_5'$ are *elementary second-* or *upper-level* formulas. A formula is *open* if it contains the variable x, and *closed* if not.

By a *language* L is meant any collection of symbols and formulas as described above. Part of the rationale for these definitions is as follows. A formula of the form $\mathcal{L}_1 P(a/x)$, applying a likelihood modifier to a closed formula, is intended to express "It is *likely* that the individual a satisfies the proposition P", and a formula of the form $\mathcal{L}_1 P$, where P is open, expresses "For an arbitrary individual x, it is *likely* that x satisfies P". A formula of the form $\mathcal{Q}_1 P$, P open, expresses "*For most* x, P is true about x"; similarly, $\mathcal{U}_1 P$, P open, expresses "*Usually* P is true about x". The language does not allow the intuitively meaningless application of $\mathcal{Q}$'s and $\mathcal{U}$'s to closed formulas. Since these languages provide only one individual variable, there is no need to identify it along with the modifier, e.g., as in $(\mathit{ForMost}\ x)P$; in other words, when P is open, the x is implicit. When expressing propositions informally, however, it will sometimes be convenient to make the x explicit.

In applying both to open and to closed formulas, likelihood behaves somewhat akin to a modality. By contrast, quantification and usuality apply only to open formulas, and thus behave as ordinary quantification. Since a modifier-free formula P is a lower-level formula, the convention is adopted that, in order to assert the proposition P at the second level, one writes $\mathcal{L}_3P$ (informally, *CertainlyP*). Further discussion of the idiosyncrasies of the foregoing definitions is best taken up in the context of the following presentation of the formal semantics (Section 2.4).

Languages differ from one another essentially only in their choice of individual constants and predicate symbols. As an example, the first of the foregoing syllogisms can be written in a language employing the individual constant a for Tweety and the predicate symbols α and β for Bird and CanFly—and for clarity writing these names instead of the symbols—as

$$\frac{\begin{array}{l}\mathcal{Q}_1(\mathrm{Bird}(x) \to \mathrm{CanFly}(x))\\ \mathcal{L}_3\mathrm{Bird}(\mathrm{Tweety})\end{array}}{\mathcal{L}_1\mathrm{CanFly}(\mathrm{Tweety})}$$

In words: For *most* x, if x is a Bird then x CanFly; it is *certain* that Tweety is a Bird; therefore it is *likely* that Tweety CanFly.

2.4. The Bayesian semantics

This section and the next define two alternative semantics for **Q**, one Bayesian and one non-Bayesian. The first will be the more general, but the second will be more useful for certain kinds of applications. In both semantics, an *interpretation* I for a language L will consist of a *likelihood mapping* l_I which associates each lower-level formula with a number in $[0,1]$, and a *truth valuation* v_I which associates each upper-level formula with a *truth value*, T or F. The subscript I will be dropped when the intended meaning is clear.

Here the definition of l is based on the Bayesian subjectivist theory of probability as described in [132, pp. 29–34]. A key feature of Bayesian theory is that it takes the notion of conditional probability as primitive. A *likelihood mapping* l_I *for* an interpretation I *of* a language L, will be any function defined on the lower-level formulas P of L, and the ordered pairs $(Q|P)$ of lower-level formulas of L, satisfying: for elementary P,

$$l(P) \in [0,1]$$

for ordered pairs $(Q|P)$ of formulas (elementary or not),

$$l(Q|P) \in [0,1]$$

and, for any P and Q (elementary of not),

$$l(\neg P) = 1 - l(P)$$
$$l(P \wedge Q) = l(Q|P)l(P)$$
$$l(P \vee Q) = l(P) + l(Q) - l(P \wedge Q)$$
$$l(P \to Q) = l(Q|P)$$
$$\text{if } l(P) = r, \text{ then for any } a,\ l(P(a/x)) = r$$
$$l(Q|P)l(P) = l(P|Q)l(Q)$$

The value $l(P)$ is here taken to be the Bayesian *degree of belief* (in the truth) of P. The value $l(Q|P)$ is taken to be the Bayesian *conditional probability*, which by definition is the degree of belief (in the truth) of P under the assumption that Q is known (to be true) with absolute certainty. Under this interpretation common sense would dictate that, if $l(P) = 0$, then $l(Q|P)$ should be undefined. The last of the above equations is a reconstrual of the familiar "inversion formula" (see [132, p. 32]) and ensures that $\wedge$ and $\vee$ are commutative. The second from the last line asserts that, if a formula P involving the variable x is held with a certain degree of belief, then in the absence of any special information about an individual a, the formula $P(a/x)$ will be held to the same degree. The issue of how to deal with the possibility of there being countervailing information about a plays a key role in the developments of Sections 4 and 5. The only thing left to make any such l a Bayesian probability function is to agree that "absolute certainty" will be represented by the value 1.

The "triviality result" established by Lewis [93] was mentioned in Section 1.1. In the present context this may be summarized as follows: if the lower level of the foregoing languages were expanded to allow for nesting of conditionals, e.g., as in $(P \to (Q \to R))$, then the foregoing semantics based on defining $l(P \to Q)$ as equal to $l(Q|P)$ could have at most 4 distinct likelihood values. This inherent limitation in the otherwise compelling idea of interpreting the logical conditional as conditional probability is the reason for defining the formula class F_3 as in Section 2.3; it admits a probabilistic conditional while avoiding triviality simply by ensuring that the nesting of such conditionals is disallowed.

To define the valuation mapping v, one must first select, for each $i = -3, \dots, 3$, a *likelihood interval* $\iota_i \subseteq [0, 1]$ in the manner of

$$\begin{aligned}
\iota_3 &= [1, 1] \quad (\textit{singleton } 1),\\
\iota_2 &= [\tfrac{4}{5}, 1),\\
\iota_1 &= [\tfrac{3}{5}, \tfrac{4}{5}),\\
\iota_0 &= (\tfrac{2}{5}, \tfrac{3}{5}),\\
\iota_{-1} &= (\tfrac{1}{5}, \tfrac{2}{5}],\\
\iota_{-2} &= (0, \tfrac{1}{5}],\\
\iota_{-3} &= [0, 0] \quad (\textit{singleton } 0).
\end{aligned}$$

These intervals then become associated with the corresponding modifiers. Their choice is largely arbitrary, but should in principle be guided either by intuition or experimental results (e.g., based on psychological studies of the kind cited in Section 2.1). The only formal requirement is that they be nonoverlapping and cover the interval $[0,1]$. Given such a set of intervals, the mapping υ is defined by, for all $i = -3,\dots,3$: for open lower-level P, Q, and with $\mathcal{M}$ being any of $\mathcal{L}$, $\mathcal{Q}$, or $\mathcal{U}$,

$$\upsilon(\mathcal{M}_3(P \dot{\rightarrow} \mathcal{M}_i Q) = T \quad \text{iff} \quad l(P \rightarrow Q) \in \iota_i$$

for closed lower-level P and Q,

$$\upsilon(\mathcal{L}_3(P \dot{\rightarrow} \mathcal{L}_i Q)) = T \quad \text{iff} \quad l(P \rightarrow Q) \in \iota_i$$

for open lower-level P and $\mathcal{M}$ being any of $\mathcal{L}$, $\mathcal{Q}$, or $\mathcal{U}$,

$$\upsilon(\mathcal{M}_i P) = T \quad \text{iff} \quad l(P) \in \iota_i$$

for closed lower-level P,

$$\upsilon(\mathcal{L}_i P) = T \quad \text{iff} \quad l(P) \in \iota_i$$

and for open or closed upper-level P and Q,

$$\upsilon(\ddot{\neg} P) = T \quad \text{iff} \quad \upsilon(P) = F$$

$$\upsilon(P \ddot{\vee} Q) = T \quad \text{iff} \quad \text{either } \upsilon(P) = T \text{ or } \upsilon(Q) = T$$

It is not difficult to verify that this provides a well-defined semantics for the languages in concern. Note that a second-level formula is either T or F, so that this part of the system is classical. This justifies introducing $\ddot{\wedge}$, $\ddot{\rightarrow}$, and $\ddot{\leftrightarrow}$ in the manner that is customary for classical logic, i.e., via the abbreviations given in Section 2.3. By contrast, at the lower level there is no similarly convenient syntactical way to express the definition of $l(P \vee Q)$ in terms of $l(P \wedge Q)$, so that there the two connectives must be defined separately.

Note also that these languages rule out compound modifiers such as *likely likely* and *for most x, likely*. This was done in order to confine the discussion to the simpler case. Such could be introduced, however, by extending to a semantics employing probabilities of probabilities. Pearl [132, pp. 357–372] explains a sense in which it is meaningful to do this, and that approach could be applied here to provide the needed interpretations. Such modifier combinations rarely occur in everyday discourse, however, so it is not a great loss to leave them out.

To illustrate this semantics, let us verify in detail that the foregoing syllogism regarding Tweety is *valid* in any such interpretation I, i.e. that if the premises of the syllogism are both T in I, then so also will be the conclusion. It will be seen that validity in this example is a direct result of associating $\mathcal{Q}_1$ (*most*) and $\mathcal{L}_1$ (*likely*) with the same likelihood interval. Suppose I is such that

$$\upsilon(\mathcal{Q}_1(\text{Bird}(x) \rightarrow \text{CanFly}(x)) = T$$

$$\upsilon(\mathcal{L}_3\text{Bird}(\text{Tweety})) = T$$

From the latter we obtain by definition of υ that

$$l(\text{Bird}(\text{Tweety})) = 1$$

which means that Bird(Tweety) is absolutely certain. From the former we obtain by definition of υ that

$$l(\text{Bird}(x) \rightarrow \text{CanFly}(x)) \in \iota_1$$

By definition of l, this gives

$$l(\text{Bird}(\text{Tweety}) \rightarrow \text{CanFly}(\text{Tweety})) \in \iota_1$$

whence

$$l(\text{CanFly}(\text{Tweety})|\text{Bird}(\text{Tweety})) \in \iota_1$$

In accordance with Bayesian theory, the latter means that the degree of belief in CanFly(Tweety), given that Bird(Tweety) is absolutely certain, is in ι_1. This, together with the above certainty about Tweety being a bird, yields that the degree of belief in CanFly(Tweety) must also be in ι_i. Then, by definition of l,

$$l(\text{CanFly}(\text{Tweety})) \in \iota_1$$

giving, by definition of V, that

$$\upsilon(\mathcal{L}_1\text{CanFly}(\text{Tweety})) = T$$

This is what we were required to show.

In general, it is a direct consequence of the definition of υ is that the semantics validates classical Modus Ponens at the upper level, namely

$$\frac{\begin{array}{l} P \mathrel{\dot{\rightarrow}} Q \\ P \end{array}}{Q}$$

for all $P, Q \in F_6 \cup F_6'$, and it is a direct consequence of the definitions of l and υ that the semantics validates the Substitution Rule

$$\frac{P}{P(a/x)}$$

for all $P \in F_6$ involving only $\mathcal{L}$'s (i.e., no $\mathcal{Q}$'s or $\mathcal{U}$'s).

A second-level proposition P will be *valid* if $\upsilon(P) = T$ for all such interpretations I. It is routine to verify that all second-level formulas having the form of tautologies of classical propositional calculus are valid. It is also easy to verify that the following are valid for all choices of i (and open P):

$$\mathcal{L}_i P \mathrel{\dot{\leftrightarrow}} \mathcal{Q}_i P$$

$$\mathcal{L}_i P \mathrel{\dot{\leftrightarrow}} \mathcal{U}_i P$$

$$\mathcal{Q}_i P \mathrel{\dot{\leftrightarrow}} \mathcal{U}_i P$$

The need for the single-dotted conditional $\dot{\rightarrow}$, introduced in F_4, arose from a desire to express certain propositions which could not otherwise be stated. For example, suppose one wishes to express the equivalence described informally in Section 2.2 as

$$(Most\ x)(\mathrm{Bird}(x) \rightarrow \mathrm{CanFly}(x)) \;\leftrightarrow\; (\mathrm{Bird}(x) \rightarrow Likely\mathrm{CanFly}(x))$$

The first $\rightarrow$ would naturally be here represented formally as the undotted $\rightarrow$, and the $\leftrightarrow$ would naturally be transcribed as the formal $\ddot{\leftrightarrow}$, but without the formal $\dot{\rightarrow}$, there would be no way to transcribe the second $\rightarrow$. Thus even though at the semantic level, $\rightarrow$ and $\dot{\rightarrow}$ are interpreted equivalently, at the syntactic level they play very distinct roles. In general, the semantics validates all formulas having the forms (for open P and Q):

$$\mathcal{Q}_i(P \rightarrow Q) \ddot{\leftrightarrow} \mathcal{Q}_3(P \dot{\rightarrow} \mathcal{L}_i Q)$$
$$\mathcal{U}_i(P \rightarrow Q) \ddot{\leftrightarrow} \mathcal{U}_3(P \dot{\rightarrow} \mathcal{L}_i Q)$$

Collectively, these express salient aspects of the interrelations between quantification, likelihood, and usuality.

Another general form validated by this semantics is

$$\mathcal{Q}_i(P \rightarrow Q) \ddot{\leftrightarrow} (\mathcal{L}_3 P \ddot{\rightarrow} \mathcal{L}_i Q)$$

This is established by:

$$\begin{aligned}
&\upsilon(\mathcal{Q}_i(P \rightarrow Q)) = T \\
&\quad \text{iff} \quad l(P \rightarrow Q) \in \iota_i \quad (\text{definition of } \upsilon) \\
&\quad \text{iff} \quad l(Q|P) \in \iota_i \quad (\text{definition of } l) \\
&\quad \text{iff} \quad l(P) = 1 \text{ implies } l(Q) \in \iota_i \quad (\text{Bayes conditional}) \\
&\quad \text{iff} \quad \upsilon(\mathcal{L}_3(P)) = T \text{ implies } \upsilon(\mathcal{L}_i Q) = T \quad (\text{definition of } \upsilon) \\
&\quad \text{iff} \quad \upsilon(\mathcal{L}_3 P \ddot{\rightarrow} \mathcal{L}_i Q) = T \quad (\text{classical propositional calculus})
\end{aligned}$$

One also has that negations behave in intuitively plausible ways. In particular, for all i, we have

$$\mathcal{L}_i P \ddot{\leftrightarrow} \mathcal{L}_i \neg\neg P$$
$$\mathcal{L}_i P \ddot{\leftrightarrow} \mathcal{L}_{-i} \neg P$$

(similarly for $\mathcal{Q}$'s and $\mathcal{U}$'s), and where $j_1, \ldots, j_6$ are the subscripts $\neq i$,

$$\ddot{\neg}\mathcal{L}_i P \ddot{\leftrightarrow} \mathcal{L}_{j_1} P \,\ddot{\vee} \cdots \ddot{\vee}\, \mathcal{L}_{j_6} P$$

The first of these expresses the usual property of double negations. By taking $i = 2$, the second expresses the common-sense proposition that *likely* is equivalent with *unlikely not*, and by taking $i = -2$, one has that *unlikely* is equivalent with *likely not*. Suppose that in the same schema, with $\mathcal{Q}$ in place of $\mathcal{L}$, we take $i = -3$. This gives

$$(ForNo\ x)P \ddot{\leftrightarrow} (ForAll\ x)\neg P$$

from which it follows by propositional calculus that

$$\ddot{\neg}(ForNo\ x)P \mathrel{\ddot{\leftrightarrow}} \ddot{\neg}(ForAll\ x)\neg P$$

Now suppose we introduce $(ForSome\ x)P$ as an abbreviation for $\ddot{\neg}(ForNo\ x)P$. Then we have

$$(ForSome\ x)P \mathrel{\ddot{\leftrightarrow}} \ddot{\neg}(ForAll\ x)\neg P$$

revealing *ForSome* and *ForAll* as probabilistic analogs of the classical $\exists$ and $\forall$. This seems intuitively correct, inasmuch as $(ForAll\ x)P$ is true iff the probability of P is 1, and $(ForSome\ x)P$ is true iff the probability of P is not 0.

The latter of the above three schemata conforms with ordinary intuition at the second (classical) level. There are, however, various related common-sense propositions which do not hold in the present formalism. For example, even though one can express both of

$$\ddot{\neg}Likely\ P \mathrel{\ddot{\leftrightarrow}} Unlikely\ P$$

$$(ForAll\ x)P \mathrel{\ddot{\rightarrow}} (ForMost\ x)P$$

neither of these are valid. It seems reasonable that one could obtain such expressions by introducing alternative versions of *likely* and *most*, e.g., a $Likely^2$ and a $ForMost^2$, but this has not yet been explored.

The daunting task of establishing a semantically complete axiomatization for **Q** also has yet to be attempted. For purposes of the reasoning systems discussed in this paper, however, even though having an axiomatization would be desirable, it is not actually required. One can still perform effective reasoning in spite of the system's being incomplete. Nevertheless, additional inference rules will surely be useful. In particular, one would likely want to adjoin the rules developed by Amarger et al. (Section 2.1) for chaining inferences involving fuzzy quantifiers.

The decision to employ only one individual variable was made in order to confine the analysis to the simplest case. Having now developed the one-variable case, it should not be difficult the extend this treatment to the use of multiple variables and n-ary predicates. Even in its present form, however, the system is not without utility. As will become evident in the sections that follow, it already provides a fairly rich semantics, and may well in itself lead to a new class of expert systems. Indeed virtually all real-world expert systems go little beyond ordinary propositional calculus, and in any case are never applied to more than one individual at a time.

2.5. The counting semantics

Whenever one uses a quantifier in everyday conversation, there is an implicit reference to an underlying domain of discourse. This observation evidently served as the basis for Zadeh's original formulation of fuzzy quantification (Section 2.1). For example, "*Most* birds can fly" refers to a domain of individuals which is presumed to include a collection of birds, and an assertion to the effect that there is a "high probability" that a randomly chosen bird will be able to fly (Section 2.2) is represented mathematically by the condition that a "large proportion" of birds are able to fly.

Unfortunately, the semantics developed in the preceding section does not reflect this type of meaning. While Bayesian theory insists on a purely subjectivist interpretation of probabilities as degrees of belief, however, there is nothing that rules out the statistical intuitions discussed earlier. Indeed the theory does not say anything about how one's degrees of belief are to be determined; it says only that they must be chosen in such a way that they conform to certain laws.

The present section develops an alternative semantics which explicitly portrays the role of the underlying domain. This *counting semantics* arises by restricting Zadeh's notion of "σ-count" to crisp predicates (see Section 2.1).

An *interpretation* I for a language L will now consist of: a *universe* U_I of *individuals* (here assume U_I is finite); assignment of a unique individual $a_I \in U_I$ to each individual constant a of L; assignment of a unique unary predicate α_I on U_I to each predicate symbol α of L; a *likelihood mapping* l_I which associates each lower-level formula with a number in $[0,1]$; and a *truth valuation* v_I which associates each upper-level formula with a *truth value*, T or F. As before, the subscript I will be dropped when the intended meaning is clear.

Given assignments for the individual constants and predicate symbols, the mappings l and v are defined in the following way. Observe that the assignments α_I induce the assignment of a unique subset P_I of U_I to each (open) formula in F_2 according to

$$(\neg P)_I = (P_I)^c$$

$$(P \vee Q)_I = P_I \cup Q_I$$

$$(P \wedge Q)_I = P_I \cap Q_I$$

For subsets $X \subseteq U$, define a *proportional size* σ by

$$\sigma(X) = |X|/|U|$$

where $|\cdot|$ denotes cardinality. Then l is defined by: for $P \in F_2$,

$$l(P) = \sigma(P_I)$$

for $(P \to Q) \in F_3$,

$$l(P \to Q) = \sigma(P_I \cap Q_I)/\sigma(P_I)$$

with l undefined if $\sigma(P_I) = 0$; and for $P \in F_2 \cup F_3$,

$$\text{if } l(P) = r, \text{ then } l(P(a/x)) = r$$

It is easy to see that σ is a probability function. These definitions merely replicate the standard way of defining probability where events are represented as subsets of a universe of alternative possibilities. The value $\sigma(P_I)$ is defined to be the probability that a randomly selected a_I in U_I will be in P_I. This means that, for each a and each open

$P \in F_2$, and given no additional information about a, $l(P(a/x))$ is the probability that $a_I \in P_I$. The definition of $l(P \to Q)$ is the traditional (non-Bayesian) way of defining conditional probability in terms of joint events (see [132, p. 31]). Thus the value of this ratio is, by definition, the probability that an individual a_I will be in Q_I, given that a_I is known to be in P_I.

Assuming this version of l, the corresponding v is defined exactly as in Section 2.4. It is a routine matter to verify that this semantics validates all the same syllogisms and formulas as were considered in Section 2.4. (This is not to say, however, that the two semantics are necessarily equivalent with respect to the given class of languages L, an issue which as yet remains unresolved.) To illustrate, the "Tweety" syllogism is established as follows. As before, assume that both premises have .value T. Letting Pr denote probability, we have

$$
\begin{aligned}
&v(Q_1(\mathrm{Bird}(x) \to \mathrm{CanFly}(x)) = T \\
&\quad \text{iff} \quad l(\mathrm{Bird}(x) \to \mathrm{CanFly}(x)) \in \iota_1 \quad \text{(definition of } v\text{)} \\
&\quad \text{iff} \quad \sigma(\mathrm{Bird}_I \cap \mathrm{CanFly}_I) \in \iota_1 \quad \text{(definition of } l\text{)} \\
&\quad \text{iff} \quad \forall a_I,\ \Pr(a_I \in \mathrm{Bird}_I) = 1 \text{ implies } \Pr(a_I \in \mathrm{CanFly}) \in \iota_i \quad \text{(non-Bayes cond.)} \\
&\quad \text{iff} \quad \forall a,\ l(\mathrm{Bird}(a)) = 1 \text{ implies } l(\mathrm{CanFly}(a)) \in \iota_i \quad \text{(discussion above)} \\
&\quad \text{iff} \quad \forall a,\ v(\mathcal{L}_3\mathrm{Bird}(a)) = T \text{ implies } v(\mathcal{L}_1\mathrm{CanFly}(a)) = T \quad \text{(definition of } v\text{)}
\end{aligned}
$$

Then taking the last line with Tweety as an instance of a and combining this with the second premise of the syllogism gives the desired result. The other items validated in Section 2.4 may be dealt with by similar techniques.

It would be easy to implement such a mapping σ in any database; one need only scan records and perform counts wherever appropriate. In other types of applications, however (e.g., many expert systems), the underlying universe will be such that it is not possible to count the numbers of objects that satisfy certain relations. For example, it is not known exactly how many birds there are in the world, nor how many of them can fly. Hence instead of basing the likelihood valuation l on actual counts, it would be more reasonable to define it in terms of estimates of sizes of populations. Such estimates might be arrived at by means of statistical samplings; alternatively, they might be subjective estimates of relative sizes, essentially educated guesses, not necessarily based on any deeper methodology. In the latter case one is nearing a return to the type of reasoning portrayed by the Bayesian semantics. The counting semantics would nonetheless be useful in this context, inasmuch as the principles of set theory can be used to help ensure that these estimates are selected in intuitively plausible ways. For example, if A's are known to always be B's, then in any valid interpretation the set of A's should be a subset of the set of B's. Such an approach might be characterized as subjective, but non-Bayesian.

The restriction to finite universes was made in order to define the counting semantics in terms of relative cardinalities. It seems reasonable that one could extend to infinite domains via an abstract measure-theoretic formulation of probability, as in Kolomogorov's treatise [84].

3. The classical modalities

The modern study of the modalities *possibly* and *necessarily* began in the 1920's with the works of C.I. Lewis (see [92]). This eventually grew into an extensive literature, with the majority of studies being based on the "possible worlds" semantics introduced by Kripke [85,86]. Under this interpretation, a proposition P is "necessary" if it is true in all possible worlds, and is "possible" if it is true in at least one possible world. A standard reference is [80]; more recent summaries and surveys include [170,179,183]. The modalities studied in those works are here termed "classical" to distinguish them from the lesser-known probabilistic variety.

The connection between probability theory and the classical modalities has been explored by Rescher [151]. There a semantics is presented wherein propositions P have two-place "truth values", $(V(P), \Pr(P))$, where $V(P)$ is the ordinary truth value of P (1 for true and 0 for false), and $\Pr(P)$ is the probability of P's being true. The two items are regarded as independent of one another, with the exception that, if $\Pr(P) = 1$, then $V(P)$ must be 1, and if $\Pr(P) = 0$, then $V(P) = 0$. A proposition of the form $Nec(P)$ (P is *necessary*) is assigned the V-value 1 if $\Pr(P) = 1$, and 0 if not. A proposition of the form $Pos(P)$ (P is *possible*) is assigned the V-value 0 if $\Pr(P) = 0$, and 1 if not. The Pr-values of $Nec(P)$ and $Pos(P)$ are defined to be the same as their V-values. Thus P is necessary iff its probability is 1, and P is possible iff its probability is not 0. Based on these ideas, Rescher defined the notion of an "M-tautology" and showed that the M-tautologies are just the theorems of the well-known modal logic S5.

As has become customary, *necessity* will be denoted by $\Box$ and *possibility* by $\Diamond$. There are two options for extending the system **Q** to accommodate expressions involving these modifiers: (i) add $\Box$ and $\Diamond$ to the symbol set, and formally introduce propositions of the form $\Box P$ and $\Diamond P$ into the set of allowable formulas, or (ii) employ $\Box$ and $\Diamond$ as a means of abbreviating formulas having certain forms. It will be more convenient to adopt the latter approach. Let us take

$$\Box P \quad \text{for} \quad \mathcal{L}_3 P$$

$$\Diamond P \quad \text{for} \quad \neg\neg\mathcal{L}_{-3} P$$

It is easily verified that

$$v(\Box P) = T \quad \text{iff} \quad l(P) = 1$$

$$v(\Diamond P) = T \quad \text{iff} \quad l(P) > 0$$

Thus these definitions are in accordance with the same intuitions as motivated Rescher.

The present aim is to determine the extent to which these versions of possibility and necessity satisfy the same properties as the versions encoded by the well-known systems K, S4, and S5 (the formulations used here are from van Benthem's monograph [183]). The language for these systems uses the connectives $\neg$, $\rightarrow$, $\Box$, and $\Diamond$, together with a collection $\mathrm{PL} = \{P', P'', \ldots\}$ of propositional letters, and consists of the set of formulas

$$F = \mathrm{PL} \cup \{\neg P, (P \rightarrow Q), \Box P, \Diamond P \mid P, Q \in \mathrm{PL} \cup F\}$$

For system K, the axioms are (i) all formulas having the form of classical tautologies, (ii) all "definitions" of the form

$$\Diamond P \leftrightarrow \neg\Box\neg P$$

and (iii) all formulas of the form

$$\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$$

The inference rules are Modus Ponens and Necessitation (from P infer $\Box P$). System S4 is obtained from K by adjoining the axioms schemata (iv) $\Box P \rightarrow P$ and (v) $\Box P \rightarrow \Box\Box P$. S5 is obtained from S4 by adjoining (vi) $\Diamond\Box P \rightarrow P$.

Section 2.4 mentioned that all propositional tautologies of **Q** are valid. This verifies item (i). The analog of item (ii) is

$$\Diamond P \ddot{\leftrightarrow} \ddot{\neg}\Box\neg P$$

A detailed proof of validity for formulas having this form is as follows.

$$\begin{aligned}
v(\ddot{\neg}\Box\neg P) = T \quad &\text{iff} \quad v(\ddot{\neg}\mathcal{L}_3\neg P) = T \quad (\text{definition of } \Box)\\
&\text{iff} \quad v(\mathcal{L}_3\neg P) = F \quad (\text{definition of } v)\\
&\text{iff} \quad l(\neg P) \notin \iota_3 \quad (\text{definition of } v)\\
&\text{iff} \quad l(\neg P) \neq 1 \quad (\text{definition of } \iota_3)\\
&\text{iff} \quad 1 - l(P) \neq 1 \quad (\text{definition of } l)\\
&\text{iff} \quad l(P) \neq 0\\
&\text{iff} \quad l(P) \notin \iota_{-3} \quad (\text{definition of } \iota_{-3})\\
&\text{iff} \quad v(\mathcal{L}_{-3}P) = F \quad (\text{definition of } v)\\
&\text{iff} \quad v(\ddot{\neg}\mathcal{L}_{-3}P) = T \quad (\text{definition of } v)\\
&\text{iff} \quad v(\Diamond P) = T \quad (\text{definition of } \Diamond)
\end{aligned}$$

In the same manner one may establish validity for

$$\Box P \ddot{\leftrightarrow} \ddot{\neg}\Diamond\neg P$$

The analog of axiom form (iii) is

$$\Box(P \rightarrow Q) \ddot{\rightarrow} (\Box P \ddot{\rightarrow} \Box Q)$$

the validity of which can also be established using methods similar to the above.

In Section 2.4 it was noted that Modus Ponens is valid for **Q**. Necessitation in the form stated above would not be meaningful for **Q**, since first-level formulas do not have truth values. However, the conventional sense of a propositional variable P standing alone is simply to assert (the truth of) P, and (as also discussed in Section 2.4) this may be captured in **Q** by the formula

$$\mathcal{L}_3 P$$

expressing "*CertainlyP*". Building on this one has, by the definitions, validity for

$$\frac{\mathcal{L}_3 P}{\Box P}$$

Using the same rationale, an analog for axiom (iv) would be $\Box P \dot{\rightarrow} \mathcal{L}_3 P$, which is also valid.

Thus we have captured all of K and most of S4. Unfortunately, no analogs of axioms (v) and (vi) are expressible in **Q**. This is clearly a limitation, but similarly as for the compound modifiers discussed in Section 2.4, it may be argued that this drawback is not severe. Propositions like "necessarily necessarily P" or "possibly necessarily P" almost never arise in everyday reasoning, and hence a capacity to express them would seldom be missed. Note nonetheless that if **Q** were expanded in the manner discussed previously, to a semantics accommodating probabilities of probabilities, then even these more complex propositions could also be expressed.

Last note that the present formulation allows for propositions that cannot be expressed in the classical systems. In particular, we have

$$\Box P \dot{\leftrightarrow} Q_3 P$$
$$\Diamond P \dot{\leftrightarrow} \dot{\neg} Q_{-3} P$$

which interrelate necessity and possibility with quantification. Also, where *ForSome* is as defined in Section 2.4, we have

$$(\mathit{ForSome}\ x) P \dot{\leftrightarrow} \Diamond P$$

which expresses the intuitive relationship between $\Diamond$ and $\exists$.

4. Path logics

The motivation and main ideas underlying the notion of a *path logic* were discussed in Section 1. As was mentioned, it is necessary not only that the path be allowed to grow over time, but also so should the language. For example, in order for a path logic to *receive* the proposition $(\mathit{Most}\ x)\mathrm{Bird}(x) \rightarrow \mathrm{CanFly}(x)$ from an external source, it is necessary that the language include predicate symbols for Bird and CanFly. Hence if it doesn't, these must be added. In addition it will be useful to allow that inference rules be added, e.g., some new rules for modifier combination. Accordingly a path logic will be formalized as a sequence of triples (language, path, rule set) indexed by the discrete time steps in which they are formed.

The general notion of path logic is intended to be completely unrestricted regarding both the kinds languages that may be employed and the ways in which the language may be modified. For example, one might start out with a first-order language, then at some point add modal operators, and at later point introduce second-order features. To illustrate the key ideas, however, if will be enough to consider only the kinds of languages defined in Section 2.3.

Let a *labeled formula* be a pair (P, λ), where P is a second-level formula as defined in Section 2.3, and λ is the *label*. In this section, λ will be a 4-tuple (i, fr, to, s), where i is an *index*, fr is a *from list*, to is a *to list*, and s is a *status indicator*. Each of these items will be described in detail below; here only note that the index i will always be a nonnegative integer. Further items will be added to the label in Sections 6 and 7.

By a *language* L is now meant a set of labeled formulas. Precisely, if P is a formula in a language as defined in Section 2.3, and λ is any admissible label for P, then (P, λ) is a formula of the corresponding language of the present type. In addition, we shall assume that languages include all formulas of the form $(\perp, \lambda)$, where $\perp$ is a special second-level symbol standing for *falsehood*. Let Λ denote the class of all such languages.

Inference rules are now defined only on labeled (second-level) formulas. In effect, rules are mappings from languages into themselves, stating how labeled formulas having certain forms may be derived from labeled formulas having certain other forms. As examples, two rules applicable to all languages in Λ would be the following versions of Modus Ponens and the Substitution Rule: for all second-level P and Q,

$$\frac{\begin{array}{l}(P \ddot{\rightarrow} Q, \lambda)\\ (P, \lambda')\end{array}}{(Q, \lambda'')}$$

where it is understood that the index in λ'' must be larger than the indices in λ and λ', and for second-level P involving only $\mathcal{L}$'s,

$$\frac{(P, \lambda)}{(P(a/x), \lambda')}$$

where the index of the conclusion must be larger than the index of the premise. Let Φ be the class of all inference rules definable on the languages in Λ.

A *path logic* $\mathbf{L}$ *based on* Λ *and* Φ consists of a series of triples (L_i, π_i, ϕ_i) generated in the following way. Let L_0 be the language with no individual constants and no predicate symbols, let π_0 be the empty sequence, and let ϕ_0 be the empty set. Assume one has formed $(L_{n-1}, \pi_{n-1}, \phi_{n-1})$, for $n \geqslant 1$. Suppose π_{n-1} is the sequence of labeled formulas $(P_1, \lambda_1), \ldots, (P_{n-1}, \lambda_{n-1})$. The language L_n is formed from L_{n-1} by adding finitely many (and possibly no) new individual constants and predicate symbols. The rule set ϕ_n is formed from ϕ_{n-1} by possibly adding some new inference rules defined on L_n. The path π_n is formed by adding a new labeled formula (P_n, λ_n) to the path π_{n-1}, where P_n is either (i) arbitrarily chosen from L_n, i.e., is *received* from an external source, or (ii) derived from formulas in π_{n-1} by application of an inference rule in ϕ_n, with the proviso that formulas can only be used as premises if their status indicators s are set to *on*. The index i in label λ_n is just the integer n, indicating the formula's position in the sequence. In case P_n was received from without, the from-list fr is simply $\{rec\}$. In case P_n was derived from formulas $P_{i_1}, \ldots, P_{i_m}$ by applying inference rule φ, then $fr = \{\varphi, i_1, \ldots, i_m\}$. Also in this case, the to-list for each of the P_{i_j} is augmented by adding the integer n. The to-list for P_n is temporarily empty. The status indicator s for P_n is initially set to *on*.

The from-lists and to-lists allow one to trace through all derivations, either backwards or forwards, and are intended to be used for various forms of reason maintenance (see Sections 1, 5, and 7.4). A way in which these may be used in the context of modifier combination is discussed below. Examples illustrating the general use of labeled formulas appear in Section 5.

In an implementation, it might be convenient to expand the above definitions to allow also for adding *formula schemata* to the path. Let $S(A_1, \dots, A_n)$ be a schema involving the variables A_i, which are taken as ranging over all possible formulas. For example, $(A_1 \dot{\rightarrow} (A_2 \dot{\rightarrow} A_1))$ is a schema representing the general *form* of all formulas $(P \dot{\rightarrow} (Q \dot{\rightarrow} P))$. The schemata would be used in conjunction with a rule providing for schema instantiation:

$$\frac{S(A_1, \dots, A_n)}{(S(P_1, \dots, P_n / A_1, \dots, A_n), \lambda)}$$

for any formulas $P_1, \dots, P_n$. A typical use would be to initialize a path with schemata for the axioms of classical propositional calculus. Then the rule could be applied to derive individual axioms as they are desired, thus avoiding the need to input each such axiom from without.

The notions of interpretation defined in Sections 2.4 and 2.5 may be extended to path logics in the following manner. Let l be defined as before, but now from labeled first-level formulas into $[0, 1]$, and with the part of the definition regarding substitution instances removed. This means that there is now no intrinsic connection between the likelihood of a formula P and that of its instances $P(a/x)$. Let v be a valuation mapping for labeled second-level formulas that is defined in terms of l exactly as before, with the added provision that, for all λ, $v(\perp, \lambda) = F$. It follows that the truth of an instance $\mathcal{L}_i P(a/x)$ is no longer dependent on the truth of $\mathcal{L}_i P$.

Let I be an interpretation for a path logic **L**. When a formula is received into the path, it is natural to assume that its truth value is T. Then if all the rules being applied are truth-preserving, it is automatic that all the formulas in the path will be true. It turns out, however, that there will be occasion to allow rules which permit $\perp$ to be derived from true premises. A specific example appearing below is the rule for combining *Certainly* with *CertainlyNot*. It was here decided to regard this combination as contradictory, and this decision is made formal in a rule saying the from $(\mathcal{L}_3 P, \lambda)$ and $(\mathcal{L}_{-3} P', \lambda)$ one may infer $(\perp, \lambda'')$.

A rule application in **L** will be *valid with respect to I* if the conclusion in that application is T. Let us say that a sequence of formulas π generated in the manner of the foregoing definition of path is *correct*, and similarly that the rule applications involved in generating this path are *correct*. It follows that a rule application may be correct, but invalid. Let us say that a path is *valid in I* if all its formulas whose status indicators are *on* are true in I. The intention underlying these definitions is that one wants always that, with respect to a certain motivating interpretation, the path will remain valid. Thus, whenever a correct but invalid inference is detected by the appearance of a $\perp$, this signals an invalidity which must somehow be eradicated. Possible ways in which this might be done are (i) simply ignore the contradiction by switching the

Table 2
Sample rules for combining likelihood modifiers

	3	2	1	0	−1	−2	−3
3	3	3	3	3	3	3	*
2	3	2	2	2	2	0	−3
1	3	2	1	1	0	−2	−3
0	3	2	1	0	−1	−2	−3
−1	3	2	0	−1	−1	−2	−3
−2	3	0	−2	−2	−2	−2	−3
−3	*	−3	−3	−3	−3	−3	−3

status indicator of the derived $(\perp, \lambda)$ to *off*, (ii) invoke a Doyle-like form of reason maintenance (e.g., work backwards through from-lists to uncover one or more offending earlier propositions, then switch off the status indicators of all formulas from the derived falsehood back to the chosen culprit(s), and finally work forward through to-lists from all formulas just switched off, switching off all formulas which were derived as their consequences), or (iii) similarly as in (ii) except instead of switching all formulas off, possibly just revise their likelihood modifiers. A concrete example appears regarding the Nixon Diamond puzzle in Section 5. How best to deal with contradictions in general is reserved for future works.

The issue of modifier combination is akin to that of evidence combination. The situation in concern is where, at some point in a path, one has derived $\mathcal{L}_i P$, and at another point one has derived $\mathcal{L}_j P$, and it is desired to introduce a rule which says how one might combine these to deduce a formula $\mathcal{L}_k P$, where $\mathcal{L}_k$ is derived in some meaningful manner from $\mathcal{L}_i$ and $\mathcal{L}_j$. How one might go about this is somewhat arbitrary, since there is no established methodology for this kind of reasoning; here one must rely on one's own sense of what seems reasonable. A collection of intuitively plausible such rules is described in Table 2. This is read: given i and j as above, the corresponding k is the entry in the ith row and jth column. For example, the entry 0 in the number 1 (third) row and the number −1 (fifth) column of the table describes the rule

$$\frac{\begin{array}{l}\mathcal{L}_1(P, \lambda)\\ \mathcal{L}_{-1}(P, \lambda')\end{array}}{\mathcal{L}_0(P, \lambda'')}$$

which say that if P is both *likely* and *unlikely*, then P is *uncertain*. The "$*$" in the upper right and lower left corners indicate the contradictory situation where P is both *certainly* and *certainly not* the case. This represents a rule of the kind mentioned above, having a conclusion of the form $(\perp, \lambda'')$. In applying any such rule it is required that the status indicators of the premises, i.e., where they appear in the path, must be changed to *off*. Recall from the definition of path logic that the status of the rule's conclusion is always initially *on*. This ensures that only the more current information will be used in future derivations.

All applications of rules portrayed in Table 2, excepting the two yielding falsehood, will be valid with respect to any interpretation I whose likelihood mapping l satisfies: where $\mathcal{L}_k$ is derived from $\mathcal{L}_i$ and $\mathcal{L}_j$, if $l(\mathcal{L}_i, \lambda) = r_i$ and $l(\mathcal{L}_j, \lambda') = r_j$, then $r_k = l(\mathcal{L}_k, \lambda'')$ is

$$\begin{array}{ll} \text{undefined}, & \text{if } (r_i = 1 \text{ and } r_j = 0) \text{ or } (r_j = 1 \text{ and } r_i = 0) \\ 1, & \text{if } (r_i = 1 \text{ and } r_j \neq 0) \text{ or } (r_j = 1 \text{ and } r_i \neq 0) \\ 0, & \text{if } (r_i = 0 \text{ and } r_j \neq 1) \text{ or } (r_j = 0 \text{ and } r_i \neq 1) \\ \max[r_i, r_j], & \text{if } r_i, r_j \geqslant 0 \\ \min[r_i, r_j], & \text{if } r_i, r_j \leqslant 0 \\ (r_i + r_j)/2, & \text{otherwise} \end{array}$$

The intuition underlying this scheme is that the values 1 and 0 should be dominant. More exactly, except for the contradictory situations, if either value is 1 or 0, then that value dominates the combination; if both lean toward 1 or both toward 0 (relative to 1/2), then the value closest to that extremity dominates; otherwise the values are regarded as partly confirming and partly disconfirming, and so serve to cancel each other out.

The above combination scheme evinces a measure of common-sense appeal, but other equally plausible schemes can surely be devised. For example, one could use a set of rules based on the fuzzy sets interpretation of likelihood proposed originally by Zadeh [199,200,203]. Alternatively one might want to base the rules on the results of psychological studies of the kind cited in Section 2.1. Clearly, the choice of a particular combination scheme will be at least partially context-dependent, and different schemes will be more appropriate for different applications. In any case, the foregoing is intended only to lay down a general methodology for defining such modifier combination rules, and not to argue that any particular set of rules should be preferred over another.

5. Dynamic reasoning systems

As described in Section 1, reasoning is *nonmonotonic* when the discovery of new information causes one to go back and retract old conclusions. To illustrate, suppose we are given that Opus is a bird, where it is known that birds can fly; then we naturally conclude that Opus can fly. But next suppose we are given the additional information that Opus is a penguin, where it is known that penguins are exceptional among birds in that they cannot fly. Now we are compelled to go back and retract the former conclusion about Opus' flying ability, and affirm instead that he cannot.

The earlier discussion indicated that nonmonotonic reasoning already enjoys a history covering more than a quarter of a century and that it has grown into a complex and multifaceted field of research. The aim in this section is to lay out a method for formalizing a particular variety of nonmonotonicity, namely, that which arises in inheritance-based default reasoning. An outline of how these same methods may be extended to other types of nonmonotonic reasoning appears in Section 9.

Let the foregoing notion of a language L now be expanded to allow that predicate symbols are *typed* so as to distinguish between predicates representing *properties* of things and those representing *kinds*. In addition, allow that property-type predicate symbols may be assigned an integer *index*, the use of which will be explained below. The decision as to when a predicate should be treated as representing a kind versus a property is left to the system's user. This distinction will often be somewhat arbitrary, but also at least partially context-dependent. The user is thus required to intervene with his or her own measure of common sense. This shortcoming notwithstanding, predicate typing has the important advantage that when it is used properly, many of the anomalies afflicting earlier multiple-inheritance systems can be avoided. Several examples will appear shortly illustrating these facts. Kind and property predicate symbols are indicated, respectively, by superscripts (k) and (p).

A *multiple-inheritance hierarchy* ν will consist of a set of *nodes*, together with a set of *links* represented as ordered pairs of nodes. Nodes may be either *individual* nodes, *kind* nodes, or *property* nodes. A link of the form (individual node, kind node) will be an *element-of* link; one of the form (kind node, kind node) will be a *subset-of* link; and one of the form (individual node, property node) or of the form (kind node, property node) will be a *property-of* link. There will be no other types of links. An individual node may enter into element-of links with any number of different kind nodes. A kind node may enter into subset-of links with any number of other kind-nodes, as long as this will not create loops. In the following, individual nodes will be labeled with individual constants, kinds nodes will be labeled with kind-type predicate symbols, and property nodes will be labeled with property-type predicate symbols bearing indices. Subset-of and property-of links will be labeled with quantifiers or likelihood modifiers as appropriate.

Given such a ν, there is defined on the individual nodes and the kind nodes a *specificity relation* $>_s$ (read "more specific than") according to: (i) if $(\text{node}_1, \text{node}_2) \in \nu$ is either an element-of link or a subset-of link, then $\text{node}_1 >_s \text{node}_2$, and (ii) if $\text{node}_1 >_s \text{node}_2$ and $\text{node}_2 >_s \text{node}_3$, then $\text{node}_1 >_s \text{node}_3$. We shall also have a dual *generality relation* $>_g$ (read "more general than") defined by $\text{node}_1 >_g \text{node}_2$ iff $\text{node}_1 <_s \text{node}_2$. It follows that individual nodes are maximally specific and minimally general. It also follows that ν may have any number of maximally general nodes, and in fact that it need not be connected. A maximally general node is called a *root* node. A *path* in a hierarchy ν (not to be confused with the path in a path logic) will be a sequence $\text{node}_1, \ldots, \text{node}_n$ wherein node_1 is a root node and, for each $i = 1, \ldots, n-1$, the pair $(\text{node}_{i+1}, \text{node}_i)$ is either an element-of link or a subset-of link. Two distinct paths will form a *redundant pair* if they have some node in common beyond the first place where they differ. This means that they comprise two distinct paths to the common node(s). A path will be simply *redundant* (or *redundant in* ν) if it is a member of a redundant pair. A path having more than one occurrence of the same node is said to contain a *loop*. Provisions are made in the following to ensure that hierarchies with loops or redundant paths are not allowed. As is customary, the hierarchies will be drawn as directed graphs, with the upward direction being from more specific to less (less general to more), so that roots appear at the top and individuals appear at the bottom, and with property-of links extending horizontally from their associated individual or kind nodes.

In terms of the above specificity relation on ν, we can assign an *address* to each individual and kind node in the following manner. Let the addresses of the root nodes, in any order, be $(1), (2), (3), \ldots$. Then for, say the node with address (1), let the next most specific nodes in any order have the addresses $(1,1), (1,2), (1,3), \ldots$; let the nodes next most specific to the one with address $(1,1)$ have addresses $(1,1,1), (1,1,2), (1,1,3), \ldots$; and so on. Thus a address indicates the node's position in the hierarchy relative to some root node. Inasmuch as an individual or kind node may be more specific than several different root nodes, the same node may have more than one such address. Note that the successive initial segments of an address are the addresses of the nodes appearing in the path from the related root node to the node having that rank. Let $>$ denote the usual lexicographic order on addresses. We shall apply $>$ also to the nodes having those addresses. It is easily verified that $\text{node}_1 > \text{node}_2$ iff $\text{node}_1 >_s \text{node}_2$. For property and kind nodes, we shall use the term *specificity rank* (or just *rank*) synonymously with "address". Having thus defined specificity ranks for individual and kind nodes, let us agree that each property node inherits the rank of the individual or kind node to which it is linked. Thus for property nodes the rank is not an address.

We are now in a position to make the notion of dynamic reasoning system completely rigorous. By a *dynamic reasoning system* (DRS) will be meant a path logic, with language expanded as above, together with a sequence of multiple-inheritance hierarchies $\nu_0, \nu_1, \ldots$ constructed in tandem with the path-logic triples $(L_0, \pi_0, \phi_0), (L_1, \pi_1, \phi_1), \ldots$ as follows. Let ν_0 be the hierarchy with no nodes or links. Assume one has formed $(L_{n-1}, \pi_{n-1}, \phi_{n-1})$ and ν_{n-1}. Consider the new proposition P_n that is added to form path π_n. The hierarchy ν_n is formed in accordance with:

(1) If P_n is of the form $\mathcal{L}_3\alpha^{(k)}(a)$, and this formula has not previously appeared in the path π_{n-1}, then add an individual node for the individual constant a and a kind node for the predicate symbol $\alpha^{(k)}$ if these do not already exist in the hierarchy ν_{n-1}, and add an element-of link connecting them.

(2) If P_n is of the form $\mathcal{Q}_3(\alpha^{(k)}(x) \to \beta^{(k)}(x))$, and this formula has not previously appeared in the path π_{n-1}, then add kind nodes for the predicate symbols $\alpha^{(k)}$ and $\beta^{(k)}$ if these do not already exist in ν_{n-1}, add a subset-of link connecting them (i.e., from $\alpha^{(k)}$ to $\beta^{(k)}$), and label the link with the quantifier $\mathcal{Q}_3$, *but with the provisions*:
   (a) if this creates a loop in any path in ν, then the link just added must be removed,
   (b) if this creates a redundant path in ν then: consider the two paths in the redundant pair; let $\text{node}_1, \ldots, \text{node}_k$ and $\text{node}'_1, \ldots, \text{node}'_{k'}$ be their respective initial segments, leading from the root up to the first node they have in common beyond the place where they differ; if $k \leqslant k'$ ($k' < k$), remove from ν the link $(\text{node}_{k-1}, \text{node}_k)$ (the link $(\text{node}'_{k'-1}, \text{node}'_{k'})$).

(3) If P_n is of the form $\mathcal{L}_i\alpha^{(p)}(a)$ (or of the form $\mathcal{L}_i\neg\alpha^{(p)}(a)$), if this formula has not previously appeared in the path π_{n-1}, and if $\alpha^{(p)}$ does not bear an index, then add an individual node for the individual constant a if one does not already exist in ν_{n-1}, add a property node for the predicate symbol $\alpha^{(p)}$ (or $\neg\alpha^{(p)}$), add

the appropriate property-of link, and label this link with the likelihood modifier $\mathcal{L}_i$. In addition, if this is the kth appearance in ν of a property node for either the predicate symbol $\alpha^{(p)}$ or its negation, affix this symbol with the index k, both where it occurs on the node and where it appears in P_n.

(4) If P_n is of the form $\mathcal{Q}_i(\alpha^{(k)}(x) \to \beta^{(p)}(x))$ (or $\mathcal{Q}_i(\alpha^{(k)}(x) \to \neg\beta^{(p)}(x))$), if this formula has not previously appeared in the path π_{n-1}, and if $\beta^{(p)}$ does not bear an index, then add a kind node for $\alpha^{(k)}$ if one does not already exist in ν_{n-1}, add a property node for $\beta^{(p)}$ (or $\neg\beta^{(p)}$), add the appropriate property-of link, and label the link with the quantifier $\mathcal{Q}_i$. In addition, if this is the kth appearance in ν of a property node for either the predicate symbol $\beta^{(p)}$ or its negation, label this symbol with the index k, both where it occurs on the node and where it appears in P_n.

Note that here only property-of links are allowed to be fuzzy. Extensions permitting fuzzy subset-of links and/or fuzzy element-of links may be considered in future works. It is a consequence of this definition that any multiple-inheritance hierarchy ν constructed in this manner will have exactly one individual node for each individual constant appearing in the path, and exactly one kind node for each kind-type predicate symbol appearing the path, but may have any number of property nodes for each property-type predicate symbol appearing in the path. Each property node is linked to a unique individual or kind node.

The provisions in item (2) ensure that the associated specificity ranks are well-defined. It is a consequence of the above that an individual or kind node can appear in more than one path (and thus have more than one rank) only if those paths originate at different roots. This it will turn out is the only situation in which multiple inheritance can occur.

The integer indices on property nodes and property-type predicate symbols are used to keep track of which occurrences of the symbol in the path correlate with which occurrences of it in the hierarchy. In the following it is to be understood that the index attached to a particular predicate symbol will be "carried with" it through any succeeding applications of Modus Ponens or the Substitution Rule. More exactly, if Q is inferred from P and $P \dot{\dot{\to}} Q$ by Modus Ponens, and if the Q in the premise $P \dot{\dot{\to}} Q$ contains an occurrence of a property-type predicate symbol with index k, then the corresponding occurrence of that symbol in the conclusion Q also has index k. Similarly, if $P(a/x)$ is inferred from P by the Substitution Rule, and if the P in the premise contains an occurrence of a property-type predicate symbol with index k, then the corresponding occurrence of that symbol in the conclusion $P(a/x)$ also has index k. The stages in the evolution of a DRS may conveniently be described as four-tuples $(L_i, \pi_i, \phi_i, \nu_i)$.

Given the foregoing definitions, one can define a *Specificity Rule* by:

$$\begin{array}{ll} (\mathcal{L}_i P, \lambda) & \\ (\mathcal{L}_j \neg P, \lambda') & \\ \hline (\mathcal{L}_i P, \lambda''), & \text{if } \rho > \rho' \\ (\mathcal{L}_j \neg P, \lambda''), & \text{if } \rho' > \rho \end{array}$$

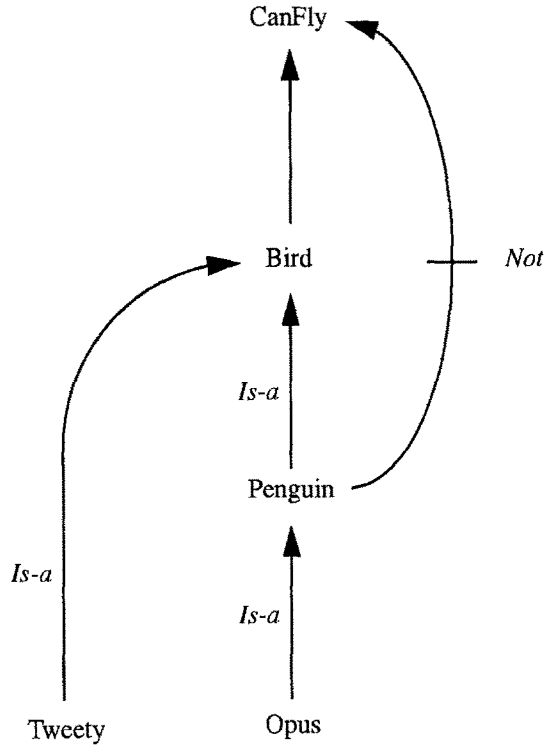

Fig. 1. Tweety can fly, but can Opus?

where P is of either the form $\alpha^{(p)}(x)$ or $\alpha^{(p)}(a)$, and assuming that k is the index of the occurrence of $\alpha^{(p)}$ in the first premise and that k' is the index of its occurrence in the second premise, ρ is one of the (possibly many) specificity ranks for the property node for $\alpha_k^{(p)}$ in the current ν, ρ' is one of the (possibly many) specificity ranks for the property node for $\alpha_{k'}^{(p)}$ in the current ν, ρ and ρ' are comparable by $<$, and where, as in prior rules, the index in λ'' is larger than the indices in λ and λ'. In addition, when applying this rule, the status of whichever premise does not wind up becoming the conclusion is changed to *off*. In effect the rule says that the property associated with the most specific node must always take priority over the less specific.

The remainder of this section is concerned with illustrating the various features of these definitions through a series of examples. We shall begin with the aforementioned situations regarding Tweety and Opus. This particular problem (or pair of problems) is customarily represented along the lines of Fig. 1 (adapted from Touretzky's [175]). Here Tweety clearly can fly, but it is not determined whether Opus can fly or not. On the latter point the diagram is either contradictory or ambiguous, depending on how one interprets it, and neither of these results seem completely satisfactory. Thus the brunt of most treatments of inheritance-based default reasoning has been to devise interpretations of this and similar such diagrams which will lead to intuitively correct results.

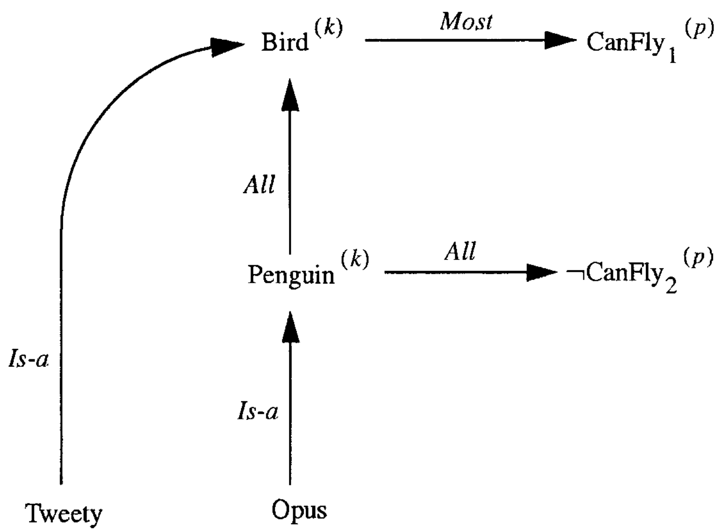

Fig. 2. Tweety *likely* can fly, and Opus *certainly* cannot.

One possible corresponding diagram resulting from the use of a DRS is shown in Fig. 2. This can be generated as follows. Let $(L_0, \pi_0, \phi_0, \nu_0)$ be as in the foregoing definitions, i.e., L_0 is the language with no individual constants or predicate symbols and $\pi_0 = \phi_0 = \nu_0 = \emptyset$ (empty set), let L_1 and all succeeding languages be the one obtained from L_0 by adjoining the individual constants Tweety and Opus and the predicate symbols $\text{Bird}^{(k)}$, $\text{Penguin}^{(k)}$, and $\text{CanFly}^{(p)}$, and let ϕ_1 and all succeeding inference rule sets consist of the versions of Modus Ponens (abbreviated MP) and the Substitution Rule (Sub) described in Section 4, together with the Specificity Rule (Spec) described above. Now let path π_1 be obtained from π_0 by inputting the labeled formula

$$\mathcal{L}_3\text{Bird}^{(k)}(\text{Tweety}) \qquad (1, \{rec\}, \emptyset, on)$$

Then in accordance with item (1) in the definition of DRS, ν_1 is formed from ν_0 by adding nodes for Tweety and $\text{Bird}^{(k)}$, together with an element-of link connecting them (labeled "is-a" in Fig. 2). Next form π_2 from π_1 by inputting

$$\mathcal{Q}_3(\text{Penguin}^{(k)}(x) \rightarrow \text{Bird}^{(k)}(x)) \qquad (2, \{rec\}, \emptyset, on)$$

which requires that we form ν_2 from ν_1 in accordance with DRS item (2). Here, since a node already exists for $\text{Bird}^{(k)}$, one need only add one for $\text{Penguin}^{(k)}$, together with the appropriate subset-of link. Next let us form π_3 by inputting

$$\mathcal{L}_3\text{Penguin}^{(k)}(\text{Opus}) \qquad (3, \{rec\}, \emptyset, on)$$

and form ν_3 from ν_2 by applying DRS item (1). Here we already have a node for $\text{Penguin}^{(k)}$, so we need only add one for Opus, together with with the appropriate element-of link. Next form π_4 by inputting

$$\mathcal{Q}_1(\text{Bird}^{(k)}(x) \rightarrow \text{CanFly}^{(p)}(x)) \qquad (4, \{rec\}, \emptyset, on)$$

and apply DRS item (3) to form ν_4 from ν_3. Since the occurrence of $\text{CanFly}^{(p)}$ does not bear an index, we must here add a node for $\text{CanFly}^{(p)}$, together with the appropriate property-of link. Moreover, since this is the first occurrence of $\text{CanFly}^{(p)}$ in the hierarchy, we must label this symbol with the index 1, both where it appears in ν_4 and where it appears in π_4, making the above labeled formula now read as

$$\mathcal{Q}_1(\text{Bird}^{(k)}(x) \to \text{CanFly}_1^{(p)}(x)) \qquad (4, \{rec\}, \emptyset, on)$$

Last, form π_5 by inputting

$$\mathcal{Q}_3(\text{Penguin}^{(k)}(x) \to \neg\text{CanFly}^{(p)}(x)) \qquad (5, \{rec\}, \emptyset, on)$$

and applying DRS item (4). Again the occurrence of $\text{CanFly}^{(p)}$ does not bear an index, so we must add a node for $\text{CanFly}^{(p)}$, together with the appropriate property-of link. Since this is the second occurrence of $\text{CanFly}^{(p)}$ in the hierarchy, we here label this symbol with the index 2, both in ν_5 and π_5, making the above read as

$$\mathcal{Q}_3(\text{Penguin}^{(k)}(x) \to \neg\text{CanFly}_2^{(p)}(x)) \qquad (5, \{rec\}, \emptyset, on)$$

This completes the formation of the hierarchy depicted in Fig. 2. At this point, the specificity ranks of the nodes are: $\text{Bird}^{(k)}$ has rank (1), Tweety has rank $(1,1)$, $\text{Penguin}^{(k)}$ has rank $(1,2)$, Opus has rank $(1,2,1)$, $\text{CanFly}_1^{(p)}$ inherits the rank (1) from $\text{Bird}^{(k)}$, and $\text{CanFly}_2^{(p)}$ inherits the rank $(1,2)$ from $\text{Penguin}^{(k)}$. Note that the order with which the above formulas are introduced is immaterial to the structure of the hierarchy, except that if formula (5) (i.e., the one with index 5) had preceded formula (4), then the subscripts on the two occurrences of $\text{CanFly}^{(p)}$ would be reversed.

To see how the hierarchy in Fig. 2 may be used for syllogistic and default reasoning, let us continue developing the above DRS as follows. The intention is to conduct all reasoning so as to preserve semantic validity with respect to the implicit interpretation (in effect, the hierarchy ν_5 itself). In some places where a deduction follows by standard techniques from classical propositional calculus, the necessary steps will here be condensed into a single step and justified as such (abbreviated PC). Recall from Section 2.4 that all formulas of the form

$$\mathcal{Q}_i(P \to Q) \mathrel{\dot{\leftrightarrow}} (\mathcal{L}_3 P \mathrel{\dot{\to}} \mathcal{L}_i Q) \qquad (*)$$

are valid under every interpretation. To illustrate how one does simple syllogistic reasoning in a DRS, let us first consider the situation of Tweety. One begins by inputting the appropriate instance of $(*)$, namely

$$\mathcal{Q}_1(\text{Bird}^{(k)}(x) \to \text{CanFly}_1^{(p)}(x)) \mathrel{\dot{\leftrightarrow}} (\mathcal{L}_3\text{Bird}^{(k)}(x) \mathrel{\dot{\to}} \mathcal{L}_1\text{CanFly}_1^{(p)}(x))$$
$$(6, \{rec\}, \emptyset, on)$$

Then from formulas (4) and (6), by propositional calculus, one has

$$\mathcal{L}_3\text{Bird}^{(k)}(x) \mathrel{\dot{\to}} \mathcal{L}_1\text{CanFly}_1^{(p)}(x) \qquad (7, \{\text{PC}, 4, 6\}, \emptyset, on)$$

At this step, in accordance with the definition of path logic (Section 4), the to-lists in the labels for formulas (4) and (6) must be changed from $\emptyset$ to $\{7\}$. Next apply the Substitution Rule to (7), yielding

$\mathcal{L}_3\text{Bird}^{(k)}(\text{Tweety}) \dashrightarrow \mathcal{L}_1\text{CanFly}_1^{(p)}(\text{Tweety})$ (8, {Sub, 7}, $\emptyset$, *on*)

and here change the to-list for (7) to {8}. Then Modus Ponens can be applied to (1) and (8), giving the desired result

$\mathcal{L}_1\text{CanFly}_1^{(p)}(\text{Tweety})$ (9, {MP, 1, 8}, $\emptyset$, *on*)

with the to-lists for (1) and (8) now reading {9}.

Simple default reasoning may be illustrated in the situation with Opus. Again using (∗), input

$\mathcal{Q}_3(\text{Penquin}^{(k)}(x) \rightarrow \text{Bird}^{(k)}(x)) \leftrightarrow(\mathcal{L}_3\text{Penguin}^{(k)}(x) \dashrightarrow \mathcal{L}_3\text{Bird}^{(k)}(x))$
(10, {*rec*}, $\emptyset$, *on*)

Then from (2) and (10), by propositional calculus, one has

$\mathcal{L}_3\text{Penguin}^{(k)}(x) \dashrightarrow \mathcal{L}_3\text{Bird}^{(k)}(x)$ (11, {PC, 2, 10}, $\emptyset$, *on*)

Here the to-lists in the labels for (2) and (10) are changed from $\emptyset$ to {11}. Now apply propositional calculus to (7) and (11), yielding

$\mathcal{L}_3\text{Penguin}^{(k)}(x) \dashrightarrow \mathcal{L}_1\text{CanFly}_1^{(p)}(x)$ (12, {PC, 7, 11}, $\emptyset$, *on*)

and changing the to-lists for (7) and (11) to {12}. Next apply the Substitution Rule, yielding

$\mathcal{L}_3(\text{Penguin}^{(k)}(\text{Opus}) \dashrightarrow \mathcal{L}_1\text{CanFly}_1^{(p)}(\text{Opus}))$ (13, {Sub, 12}, $\emptyset$, *on*)

with to-list for (12) changed to {13}. Then from (3) and (13), by Modus Ponens, one has

$\mathcal{L}_1\text{CanFly}_1^{(p)}(\text{Opus})$ (14, {MP, 3, 13}, $\emptyset$, *on*)

with to-lists for (3) and (13) changed to {14}. Now apply (∗) again to get

$\mathcal{Q}_3(\text{Penguin}^{(k)}(x) \rightarrow \neg\text{CanFly}_2^{(p)}(x)) \leftrightarrow(\mathcal{L}_3\text{Penguin}^{(k)}(x) \dashrightarrow \mathcal{L}_3\neg\text{CanFly}_2^{(p)}(x))$
(15, {*rec*}, $\emptyset$, *on*)

Then from (5) and (15) by propositional calculus one has

$\mathcal{L}_3\text{Penguin}^{(k)}(x) \dashrightarrow \mathcal{L}_3\neg\text{CanFly}_2^{(p)}(x))$ (16, {PC, 5, 15}, $\emptyset$, *on*)

(with to-lists for (5) and (15) now {16}) which, by Substitution, yields

$\mathcal{L}_3\text{Penguin}^{(k)}(\text{Opus}) \dashrightarrow \mathcal{L}_3\neg\text{CanFly}_2^{(p)}(\text{Opus}))$ (17, {Sub, 16}, $\emptyset$, *on*)

(with to-list for (16) now {17}). By Modus Ponens, (3) and (17) give

$\mathcal{L}_3\neg\text{CanFly}_2^{(p)}(\text{Opus})$ (18, {MP, 3, 17}, $\emptyset$, *on*)

(with to-list for (3) becoming {14, 18} and to-list for (17) becoming {18}). Now we are in a position to apply the Specificity Rule to (14) and (18). Since the rank of the node for $\neg\text{CanFly}_2^{(p)}$ is $>$ that of the node for $\text{CanFly}_1^{(p)}$, this yields

$\mathcal{L}_3\neg\text{CanFly}_2^{(p)}(\text{Opus})$ (19, {Spec, 14, 18}, $\emptyset$, *on*)

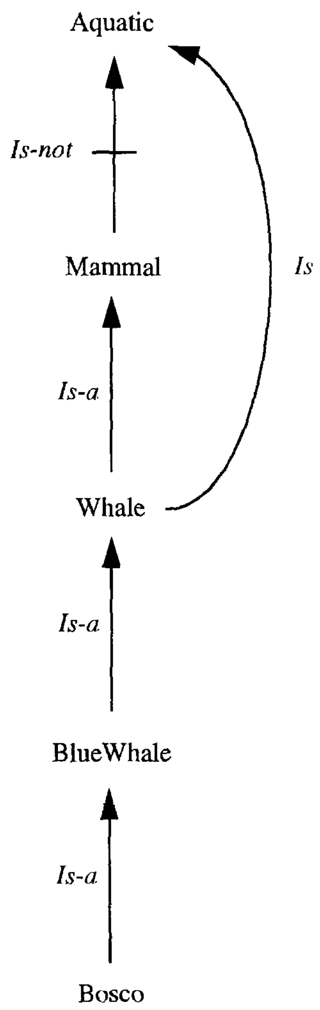

Fig. 3. Is Bosco aquatic, or not?

with to-list for (14) and (18) becoming {19}, and with the status indicator in the label for (14) being changed to *off*. Thus we arrive at the desired conclusion, and forbid any further use of the undesired alternative.

Having fully detailed this first example, the work will be shortened on the remaining examples by considering only the manners in which they differ from preceding examples. Fig. 3 is a puzzle adapted from Stein's [168]. This is similar to the foregoing case of Opus, in that it is not explicitly evident (in fact, again either contradictory or ambiguous) whether Bosco is aquatic. As shown in Fig. 4, this problem may resolved in essentially the same manner as for Opus. Here one need only take "aquatic" as a property rather than a kind. Then arguing upward through the kind nodes, one can conclude that Bosco is a mammal. Next, after the appropriate syllogistic arguments, together with the fact

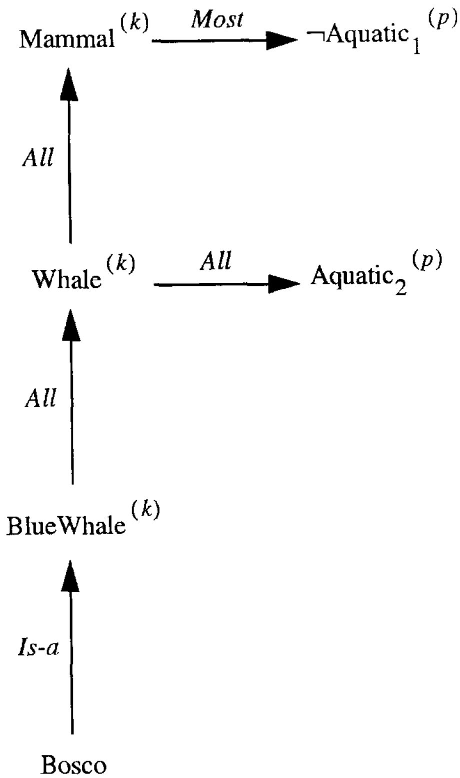

Fig. 4. Bosco is an aquatic mammal.

that $\text{Aquatic}_2^{(p)}$ is more specific than $\neg\text{Aquatic}_1^{(p)}$, the Specificity Rule gives that Bosco *certainly* is aquatic.

Fig. 5 is the well-known Nixon Diamond (taken from [177]). Here the problem is that Nixon inherits two conflicting properties, and the diagram does not say how this conflict should be resolved. One way in which this might be handled in a DRS is shown in Fig. 6. There "quaker" and "republican" are taken as kinds, "pacifist" is taken as a property, and for the sake of the illustration, the property-of links are qualified with the fuzzy quantifier *most*. As was pointed out in Section 1.2, the choice of whether a given predicate should be interpreted as representing a property versus a kind is oftentimes arbitrary, and this is surely the case here. As will be seen, however, the present choice has the advantage of leading to the intuitively desirable result.

Let us reason informally. Suppose that after the first four steps, the path is

Nixon is a quaker.	$(1, \{rec\}, \emptyset, on)$
Most quakers are pacificists$_1$.	$(2, \{rec\}, \emptyset, on)$
Nixon is a republican.	$(3, \{rec\}, \emptyset, on)$
Most republicans are not pacificists$_2$.	$(4, \{rec\}, \emptyset, on)$

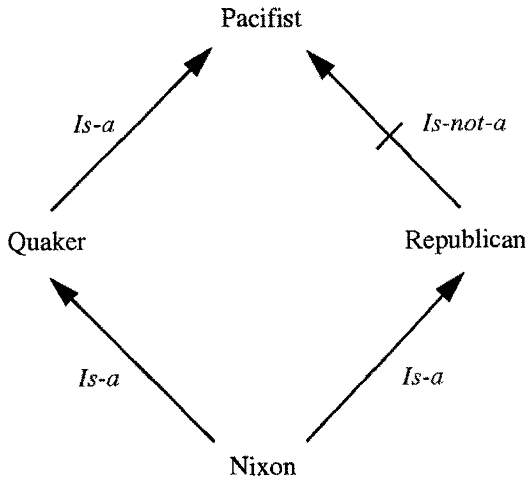

Fig. 5. *Nixon Diamond*: Is Nixon a pacifist, or not?

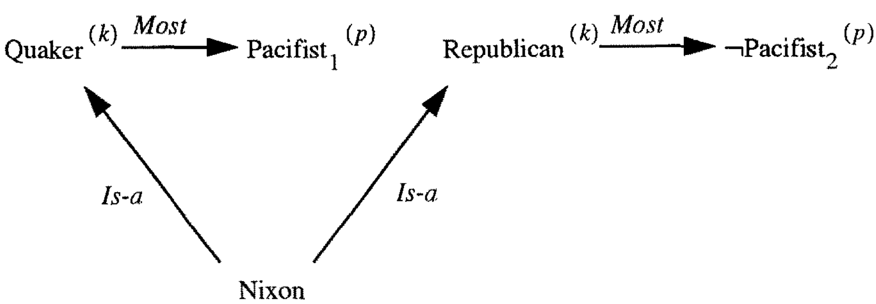

Fig. 6. It is *uncertain* whether Nixon is a pacifist.

This creates the figure as shown. Then from (1) and (2), one may argue syllogistically (here condensing several steps into one) to conclude

Nixon *likely* is a pacifist$_1$. $(5, \{\mathrm{Syll}, 1, 2\}, \emptyset, on)$

and from (3) and (4), one may argue syllogistically to conclude

Nixon is *likely* not a pacifist$_2$. $(6, \{\mathrm{Syll}, 3, 4\}, \emptyset, on)$

Now recall from Section 2.4, that all formulas of the form

$$\mathcal{L}_i P \dot{\leftrightarrow} \mathcal{L}_{-i} \neg P$$

are valid. Taking $i = -2$, this gives

Nixon is *unlikely* to be a pacifist$_2$ iff Nixon is *likely* not a pacifist$_2$.

$(7, \{rec\}, \emptyset, on)$

Then from (6) and (7) one can derive

Nixon is *unlikely* to be a pacifist_2. $(8, \{\text{PC}, 6, 7\}, \emptyset, on)$

Let the rule set for our DRS now be expanded to include the modifier combination rules described by Table 2 (Section 4). Then by the rule corresponding to row 2, column −2, from (5) and (8) one obtains

It is *uncertain* whether Nixon is a pacifist_0. $(9, \{\text{Tbl2}(2, -2), 5, 8\}, \emptyset, on)$

At this step, the status indicators for (5) and (8) must be changed to *off*. Note that here the index of the predicate "pacifist" has been given as 0. This action would naturally be a part of the general definition of the modifier combination rules, inasmuch as, after using such a rule, the index becomes irrelevant. The use of 0 as a "dummy index" prevents the introduction of this formula into the path from leading also to the entry of new nodes and/or links into the associated inheritance hierarchy (i.e., as would be prescribed by the definition of DRS).

This conclusion regarding Nixon corresponds to the "skeptical" inheritances of Horty, Thomason, and Touretzky [79], and clearly conforms with ordinary common sense. A variation, perhaps more in keeping with Fig. 5, would have both occurrences of the modifier *Most* in Fig. 6 replaced by *All*. Then arguments such as the above would yield, on the one hand, that Nixon *certainly* is a pacifist, and on the other that he is *certainly not* a pacifist, from which modifier combination (row 3, column −3) would yield the contradiction

$\perp$ $(9, \{\text{Tbl2}(3, -3), 5, 8\}, \emptyset, on)$

again with status of (5) and (8) changed to *off*. This would naturally lead next to application of some form of reason maintenance along the lines described in Section 4. The simplest approach would be to just turn the status of (9) to *off*, thereby disallowing its use in any future derivations. Another would be to backtrack through the from-lists to weed out one or more earlier "culprit" formulas, turn their status indicators *off*, and then forward chain through to-lists making further status modifications as appropriate to all conclusions based on those culprits. Here one might want, for example, to disable the formula which says that all quakers are pacifists (and also the above contradiction $\perp$), leaving only the possibility that Nixon is (*certainly*) not a pacifist inasmuch as he is a republican. A third approach would be to go back through the from-lists and alter the quantifiers, e.g., change one or more occurrences of *all* to *most*, and then forward chain through to-lists to make appropriate modifier changes. The effect here would be to disable the above contradictory conclusion $\perp$ and introduce a qualified statement about Nixon's pacifism. The details of how and when to apply these various techniques have yet to be worked out.

Note that the Specificity Rule would not be applicable in the Nixon example, since the rankings of Pacifist_1 and Pacifist_2 in Fig. 6 are not comparable by $<$. To wit, if Quaker and Republican are assigned ranks (1) and (2), respectively, then these are inherited by Pacifist_1 and Pacifist_2, respectively, and neither is $>$ than the other; and it would make no difference if Republican were ranked (1) and Quaker with (2). Note also that Nixon has two ranks, $(1, 1)$ and $(2, 1)$, corresponding to the two possible inheritance paths.

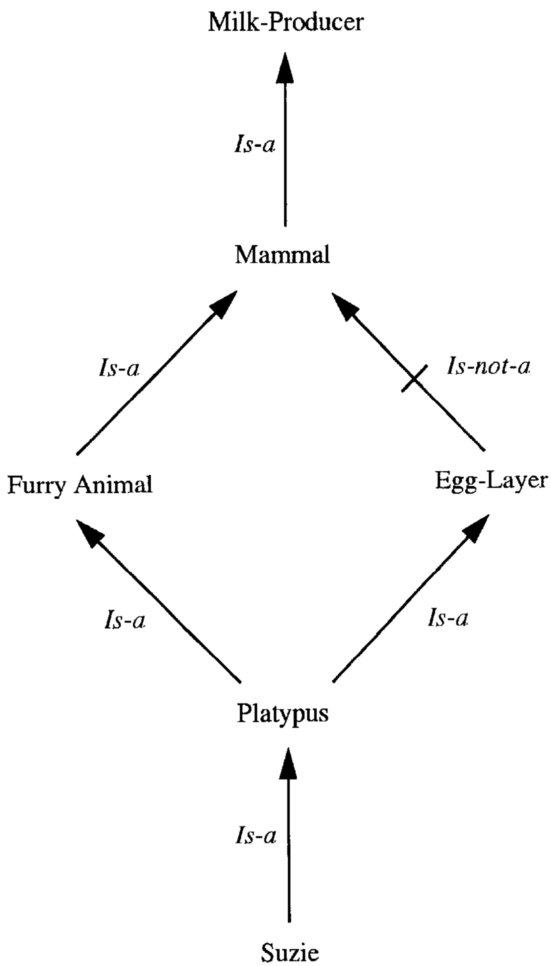

Fig. 7. Is Suzie a mammal, hence a milk producer? Does she lay eggs?

Fig. 7 is also adapted from Stein's [168]. Here it is indeterminate whether Susie is a mammal, and hence also whether she is a milk-producer. One of the several possible ways this might be dealt with in a DRS is shown in Fig. 8. The various decisions as to what should be properties versus kinds seem at least reasonable. Given this hierarchy, the techniques of the foregoing examples can be applied to infer that Suzie is *certainly* a mammal, that she *certainly* produces milk, and she *certainly* lays eggs. It is of interest, that even though Fig. 7 bears a structure similar to Fig. 5, the corresponding solution via predicate typing leads to two very different kinds of inheritance hierarchies, one involving multiple inheritance, and one not.

Another problem arising in the literature is depicted in Fig. 9 (from [177]). Here one has a contradiction arising from the "redundancy" in there being two alternative paths from Clyde to Elephant. A simple solution using the current methods is Fig. 10, yielding

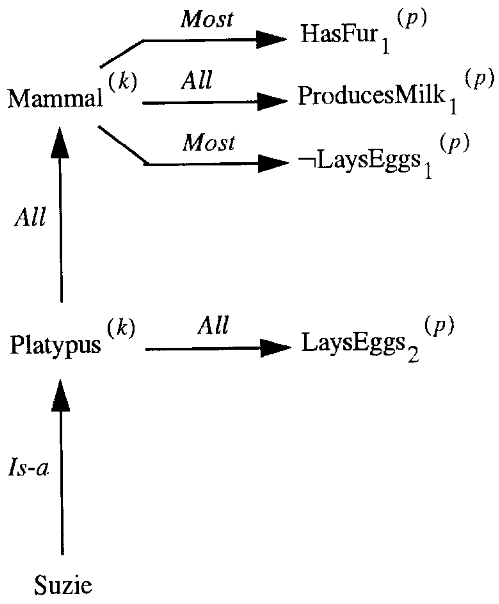

Fig. 8. Susie is a mammal that produces milk and lays eggs.

the conclusion that Clyde is *likely* not gray. The significant feature here is the manner in which the definition of DRS implicitly prevents the redundancy. If after inputting the formulas needed to create this hierarchy, one next introduces a formula saying

Clyde is an Elephant.

DRS item (2b) requires that the corresponding (redundant) link be removed immediately after adding it. Alternatively, if this formula were input prior to, say,

Royal elephants are elephants.

then DRS item (2b) would allow the link to be added, but when the link corresponding to the latter is added, creating a 2-step path from Clyde to Elephant, then the same item would now require that the 1-step link from Clyde to Elephant, corresponding to the former formula, be removed. Thus, in any case, one winds up with the same hierarchy as shown in Fig. 10.

Last consider the somewhat more complex Expanded Nixon Diamond shown in Fig. 11 (also from [177]). In this case it is indeterminate both whether Nixon is a pacifist and whether he is anti-military. A possible rendition of this in a DRS is Fig. 12. Here the significant feature is that certain predicates are taken as representing both properties and kinds, namely, "pacifist" and "football fan". After inputting the formulas needed to create this (disconnected) hierarchy, one would then also want to input the formulas

$$Q_3\text{Pacifist}^{(k)}(x) \dot{\leftrightarrow} Q_3\text{Pacifist}^{(p)}(x)$$

$$Q_3\text{FootballFan}^{(k)}(x) \dot{\leftrightarrow} Q_3\text{FootballFan}^{(p)}(x)$$

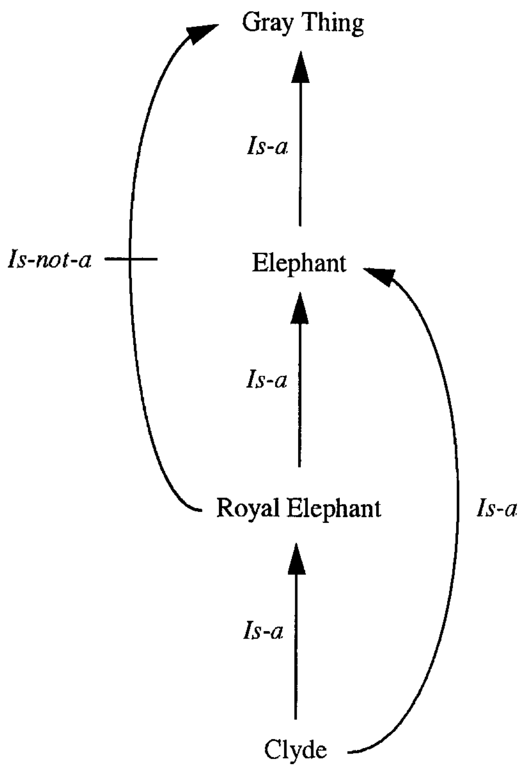

Fig. 9. Is Clyde gray, or not?

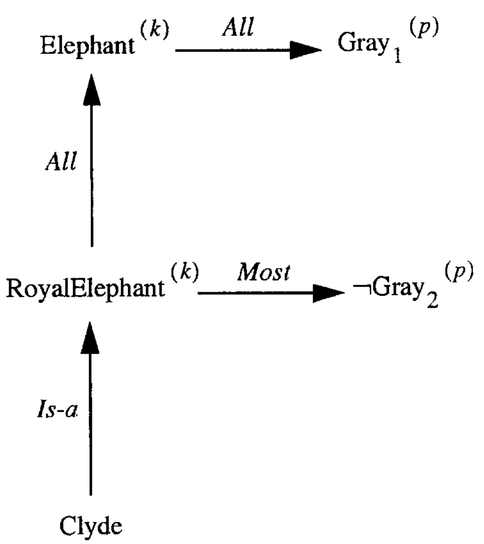

Fig. 10. Clyde is *likely* not gray.

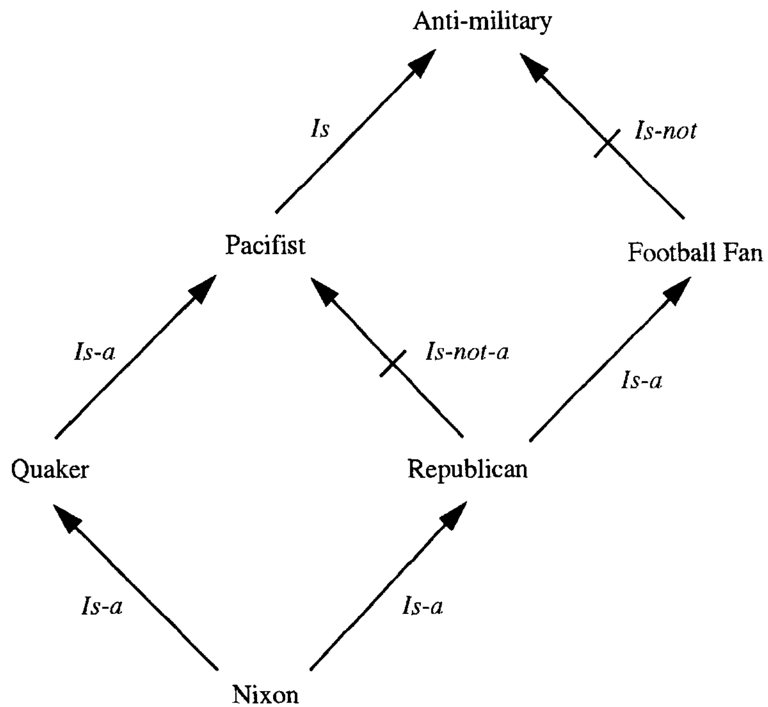

Fig. 11. *Expanded Nixon Diamond*: Is Nixon anti-military, or not?

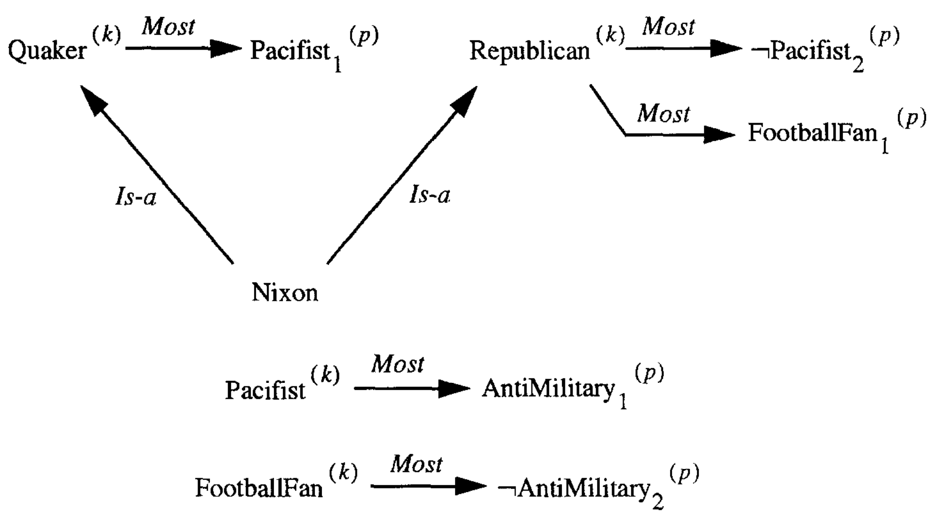

Fig. 12. It is *uncertain* whether Nixon is anti-military.

establishing the necessary formal connections between the predicate symbols of the two types. Then reasoning along the lines of the foregoing Nixon Diamond example will lead to the propositions that Nixon, on the one hand, is *likely* to be anti-military, and on the other hand, is *unlikely* to be anti-military, which through modifier combination yields that it is *uncertain* whether Nixon is anti-military or not. This again seems to comply with ordinary common sense.

Note that instead of the above two formulas it might have been preferable to input something along the lines of, e.g.,

$$Q_3(\mathrm{Pacifist}^{(k)}(x) \dot{\leftrightarrow} \mathrm{Pacifist}^{(p)}(x))$$

in order to more exactly capture the intention that $\mathrm{Pacifist}^{(k)}$ and $\mathrm{Pacifist}^{(p)}$ denote the same underlying predicate, but such formulas are not expressible in the present language. One possible remedy for this would be to introduce a binary $\leftrightarrow_b$ at the lower linguistic level, defined semantically by

$$l(P \leftrightarrow_b Q) = \begin{cases} 1, & \text{if either } l(P) = l(Q) = 1 \text{ or } l(P) = l(Q) = 0, \\ 0, & \text{if not.} \end{cases}$$

Then in accordance with the definitions in Section 4, one would have the second-level formula

$$Q_3(\mathrm{Pacifist}^{(k)}(x) \leftrightarrow_b \mathrm{Pacifist}^{(p)}(x))$$

which does capture the intended interrelation. It is noteworthy also that a full set of binary (classical) connectives could be introduced at the lower level using similar techniques.

6. The notion of "unless"

The matter of formulating the everyday notion of "unless" dates back to Sandewall's [153] and has appeared in various guises throughout the literature (see Section 7.5). The problem is to devise a system that allows for propositions such as

$$\mathrm{Bird}(x) \dot{\rightarrow} \mathrm{CanFly}(x) \ \textit{Unless}\ \mathrm{Penguin}(x)$$

or possibly

$$\mathrm{Bird}(x) \dot{\rightarrow} \mathrm{CanFly}(x) \ \textit{Unless}\ x = \mathrm{Opus}$$

which state that individuals having certain properties, or possibly certain specific individuals, are exceptions to the "rule" that birds can fly. The intention is that such propositions be used in syllogisms of the form, for open P, Q, R,

$$\begin{array}{l} P \dot{\rightarrow} Q \ \textit{Unless}\ R \\ P(a/x) \\ \dot{\neg} R(a/x) \\ \hline Q(a/x) \end{array}$$

where R is a disjunction of any number of possible exceptions having either the form $\alpha^{(p)}(x)$ or $x = a$.

It happens that this style of reasoning is already implicit in the methods of inheritance hierarchies described in Section 5. In effect, exceptions are nodes that block defaults based on specificity, as in the example of Opus. In order to fully portray the above notion of "unless", however, the exceptions need to be explicitly stated along with the relevant inferences. Note that these same considerations arise in the context of the fuzzy modifiers. For example, one might want a rule along the lines of

Most birds can fly.
Tweety is a bird.
Tweety is not exceptional.

It is *likely* that Tweety can fly.

Here it will be outlined how such rules might be formulated within a DRS by extending that framework in an appropriate way. Let formula labels include as a sixth item an *exception list*, denoted *ex*, which may contain any number of property-type predicate symbols and any number of individual constants. Introduce as a new inference rule: for open P, Q,

$$\frac{\begin{array}{l}(\mathcal{Q}_i(P \mathrel{\ddot{\rightarrow}} Q), \lambda)\\ (P(a/x), \lambda')\\ a \text{ is not exceptional as per } ex_\lambda\end{array}}{(\mathcal{L}_i Q(a/x), \lambda'')}$$

where the third premise is validated by (i) verifying that for no individual constant $a' \in ex_\lambda$ is $a = a'$, and (ii) searching backwards through the path and verifying, for each $\alpha^{(p)} \in ex_\lambda$, that no formula of the form $\alpha^{(p)}(a)$ has formerly been derived. In effect, one here adopts a procedural semantics for logical inference.

This clearly has the desired effect. It remains only to establish a procedure for making entries into the exception list. A simple approach would be to have exceptions be already present in the list for any formula that is received by the DRS from outside the system. For example, the human user could provide some such exceptions in advance. In addition to this, however, one would also want exceptions to be entered into the list whenever they are "discovered" as a result of reasoning with the semantic network. This would require higher-level mechanisms that "observe" the ongoing reasoning process and "notice" when an exceptional situation has appeared. Exactly how to do this is another item reserved for future works.

7. Other varieties of nonmonotonicity

While the notion of a DRS was devised initially to formulate default reasoning with fuzzy modifiers, it can be used also to portray other types of reasoning. This section outlines how one may formulate versions of the earlier varieties of nonmonotonic reasoning (Section 1) within this frame. This makes explicit the temporal aspects that were only implicit in the earlier formalisms.

7.1. Circumscription

McCarthy's "predicate" circumscription [100] and "formula" circumscription [101] were described in Section 1.1. These ideas soon attracted a following, and further versions were proposed. Minker and Perlis [111] developed a method of "protected" circumscription, Lifschitz [96] introduced "pointwise" circumscription, Perlis [135] considered circumscribing "with sets", and in [136] introduced the idea of "autocircumscription". Some general problems of self-reference arising in systems which allow circumscriptive axioms were discussed by Perlis [134], and a solution to these and related difficulties subsequently appeared as [41,42]. Other relevant works are [95,97,98,139]. A concise summary may be found in [137] and a recent history in [103].

This section considers only predicate circumscription. It has yet to be determined if the others can be handled in a similar way. Let α be an n-ary predicate symbol, and let $P(\alpha)$ be a first-order formula containing the elementary formula $\alpha(\bar{x})$, where $\bar{x}$ stands for the sequence $x_1, \ldots, x_n$ of individual variables. Let $\dot{\alpha}$ be an n-ary predicate variable, and let $P(\dot{\alpha}/\alpha)$ be the expression obtained from $P(\alpha)$ by replacing all occurrences of α with occurrences of $\dot{\alpha}$. Then, considered as a rule of inference, *Predicate Circumscription* may be expressed as

$$\frac{P(\alpha)}{P(\dot{\alpha}/\alpha) \wedge \forall\bar{x}(\dot{\alpha}(\bar{x}) \rightarrow P) \rightarrow \forall\bar{x}(P \rightarrow \dot{\alpha}(\bar{x}))}$$

McCarthy described this as a nonmonotonic "rule of conjecture". It has the effect of saying that, if $\bar{a}^1, \ldots, \bar{a}^m$ are all the n-ary sequences of individual constants from the language of a first-order theory T for which the formulas $P(\bar{a}^i/\bar{x})$ $(i = 1, \ldots, m)$ are *known* to be true, then after applying the rule, these will be the only n-ary sequences about which P *can possibly* be true. Knowing that a formula P is true is expressed formally by its being derivable in T. The rule is validated by restricting the semantics for T to a collection of *minimal models.* At the semantic level, circumscription says that the *only* models in which $P(\alpha)$ is valid are the smallest ones in which it *must* be valid.

This style of reasoning harbors a temporal element in that, if after applying the above rule, one discovers new individuals about which $P(\alpha)$ is true, then the foregoing rule application becomes invalid, and can only be revalidated by expanding the minimal models to include the new individuals. Nonmonotonicity arises in that some conclusions derived from the earlier rule application may no longer be valid in the newer models. Such earlier conclusions are thus retracted simply by virtue of the fact that the newer models render them invalid.

This type of reasoning can be described in a DRS framework as follows. "Discovering" new individuals would be represented by the system's "receiving" some new propositions of the form $P(\alpha)(\bar{a}/\bar{x})$. In this step, the language would be expanded to include the new individual constants. Then in this context, the rule of circumscription would be reapplied, yielding a "conjecture" identical to the above, except with a label indicating a later time step. By having a different label, this formula can be provided with a different set of models, and retraction of the earlier conjecture would be accomplished by switching its status indicator to *off*. Then to-lists would need to be traced in order to also retract any conclusions that may have been based on the conjecture thus retracted.

7.2. Default logic

The formulation of default logic due to Reiter [150] has inspired several alternative versions that similarly employ the more conventional style of formalism, i.e., that do not use inheritance hierarchies. These include Fisher [50], Pollock [142–144], Poole [145,146], Nute [128–131], Li and You [94], Simari and Loui [161], Thirunarayan and Kifer [172].

Despite the intuitive appeal of Reiter's version, it has the well-known drawback that in all but the very simplest cases, it is either impossible, or very difficult, to implement. This is due to its use of the operator $\mathcal{M}$, meaning "is consistent". In a system offering the semantic richness of first-order logic, for example, $\mathcal{M}$ is impossible to implement because, in that context, whether an arbitrarily selected set of formulas is consistent is formally undecidable. And in simpler systems, like propositional calculus, even though consistency is decidable, it is normally computationally intractable.

An inference rule similar in spirit with Reiter's default could nonetheless be implemented in a DRS, if one were to replace the operator $\mathcal{M}$ with one representing a kind of "local consistency". Let (L, π, ϕ) be a path-logic triple in a DRS; let the *length* of π be its number of formula occurrences; and let an *extension* of π be any path which includes π as an initial segment, which is comprised exclusively of formulas in L, and which employs applications only of rules in ϕ. Suppose the length of π is m. Then say that a formula Q is *n-ply consistent with* π if, for no extension π' of π having length $n+m$, do we have either (i) $Q \in \phi'$, or (ii) $Q(a/x) \in \phi'$, for any individual constant a of L. Let $\mathcal{M}_n Q$ represent "Q is n-ply consistent with the current path". Then the desired rule may be written

$$\frac{\begin{array}{l} P \xrightarrow{\cdot\cdot} Q \\ P \\ \mathcal{M}_n Q \end{array}}{Q}$$

Such would best be used in conjunction with a suitable reason maintenance system. Then, if after drawing a conclusion Q based on it's n-ply consistency, it is later discovered that Q is in fact inconsistent, there would be a way for it to be retracted. This type of reasoning would be useful for robot motion planning. The robot could be programmed to look ahead some number of steps in order to determine whether a particular action may reasonably be expected to lead to the desired goal, and if nothing to the contrary is detected, it would be allowed to go ahead and make the considered step.

7.3. Nonmonotonic logic

The system of nonmonotonic logic introduced by McDermott and Doyle [107] was problematic in that it did not provide a completely specified semantics. McDermott [105] attempted to fill this void, but with limited success. A modification which effectively solved this problem was the "autoepistemic logic" developed by Moore [114, 115]. The collection [122] adds further results along this line, and recent progress is reviewed in [116].

Section 1 noted that the original system is characterized by having propositions of the form

$$P \ddot{\wedge} \mathcal{M}Q \ddot{\rightarrow} Q$$

where again $\mathcal{M}$ is interpreted as "is consistent" but is now taken more exactly as a reinterpretation of the classical modality "is possible". This suggests that a semblance of nonmonotonic logic may be formulated within the type of extension of **Q** described in Section 3. Here one simply takes the above $\mathcal{M}$ as another notation for the probabilistic $\Diamond$. The extent to which this gives the desirable features of the later systems has yet to be explored.

7.4. Truth maintenance

The subject of truth (or reason) maintenance has been mentioned several times throughout this paper, and it has been explained how a simple variety of this could be employed in a DRS. Doyle's truth maintenance system [25] is unique among the early proposals in that it explicitly addressed the temporal aspect of natural human reasoning. That system employed a somewhat richer and more complex style of reasoning than has been considered here, however. There each formula is associated with both an *in list* and an *out list*. At any given point in time, a formula is "in" (believed) if (i) all the formulas in its in-list are "in", and (ii) all the formulas in its out list are "out" (not believed).

Nonetheless, this richer logical structure can be formulated via an expanded notion of DRS. The in-list and out-list for a formula can be given as additional items in the label. Then let a formula be "in" it all the members of it in-list are present in the path with status indicators "on", and let it be "out" if some member of its out-list either is not currently in the path or is present in the path with status "off". Note that, in the context of a full-fledged DRS, if one allows for formulas to be retracted, then one must also provide for removal of any corresponding items in the associated semantic net.

7.5. "Unless" revisited

Each of circumscription, default logic, and nonmonotonic logic embody aspects of "unless". McCarthy [100, 101] explained how circumscription may be used in conjunction with a predicate *ab* expressing "abnormality". For example,

$$\mathrm{Bird}(x) \wedge \neg ab(x) \rightarrow \mathrm{CanFly}(x)$$

says that if x is a bird, and x is not abnormal, then x can fly. Suppose there are two individuals, Tweety and Opus, asserted to be birds, and suppose we additionally have $ab(\text{Opus})$. Then predicate circumscription with respect to ab yields a formula which says that the *only* bird that can fly is Tweety. Thus, in so many words, the above inference says that a bird can fly *unless* it is explicitly known to be abnormal.

In default logic, the modal operator $\mathcal{M}$ embodies the notion of *unless* in the sense of expressing *unless inconsistent*, where inconsistent means leading to a contradiction. Thus as was seen, if it is known that Opus cannot fly, then it would be inconsistent to assume that he can, in which case the default would not be applied. In so many words, this says that it is permitted to infer that Opus can fly *unless* it is explicitly known that he can't. The $\mathcal{M}$ in nonmonotonic logic may be interpreted similarly, but where "inconsistent" is synonymous with "impossible".

The formulation of *unless* given in Section 6 bears many similarities with these, and may be thus construed as a further, roughly equivalent, alternative. Its advantage is that it leads to a much more straightforward implementation.

8. The frame problem

As mentioned in Section 1.1, the *frame problem* was defined by McCarthy and Hayes [104] in the context of an elaboration of McCarthy's *situation calculus* [99]. The aim of this calculus was to formalize the sense in which an intelligent agent (human or digital) may keep track of, reason about, and possibly affect the changes taking place in its surrounding environment. Informally a *situation* is a description of the state of affairs that obtain at a particular point in time, and successive changes in the environment are represented as transitions from one situation to the next. The word *frame* is taken more or less synonymously with "situation", but carries the added connotation of being a frame of reference. It may also be regarded analogously with a frame in a movie film. Formally a frame consists of a collection of properties, called *fluents*, which tend to persist in time but are subject to change. To illustrate, suppose situation s has a blue block sitting on the left side of a table, and situation s' is obtained by moving the block to the right. Consider the fluents "color" and "position". One would say that in the transition from situation s to situation s', the position fluent has changed, while the color fluent (barring any unknown intermediary actions, such as painting the block red) has remained unchanged. A concise restatement of the problem appears in Hayes' [73, p. 125] as follows:

> One feels that there should be some economical and principled way of succinctly saying what changes an action makes, without having to explicitly list all the things it doesn't change as well; yet there doesn't seem to be any other way to do it. *That* is the frame problem.

This has subsequently come to be recognized as one of the key problems in the field of AI. Both McCarthy and Hayes have returned to the topic, the former in [102] and the latter in [69,73,74]. Moreover, Hayes [70,72] has tied it to a new realm of applications known as "naive physics". The problem has become the subject of several workshops [19, 51, 52, 148], and a recent survey of the main issues and contributors has appeared in the review [174] of [52].

An outgrowth of these studies has been the identification of a number of related problems and subproblems. In particular, the *qualification problem* [100, 101] is exemplified by the parenthetical statement in the above example, to wit, that it is generally impossible to list all the exceptions which might invalidate a certain inference (or prevent a certain action). Others are the *ramification problem*, that it is generally impossible to specify all the potential consequences of a given inference (or action), and the *persistence problem*, that of how to represent which fluents endure, and which do not endure, as the system moves from each situation to the next.

Proper analysis of these problems requires first of all that a clear distinction be made between the changes of *state* effected by the internal processes of the reasoning agent and the changes of *situation* that occur in the agent's external environment. This internal-external distinction has been explored to some depth in the literature on "situation theory" [15, 24]. From this vantage it is clear that none of the above problems arise in the context of multiple-inheritance reasoning as it is described in the foregoing sections. This is because one is in these cases always reasoning about an essentially static universe. Even though both the path and inheritance hierarchy can grow over time, the underlying domain of discourse remains static. In other words, we have here been modeling the evolution of an agent's *knowledge* about an unchanging universe.

There is nothing inherent in the notion of DRS, however, which requires that this always be the case. Indeed propositions being added to the path can be instructions for the DRS to perform certain actions (e.g., turn right, send a message to agent X), or they can be internal representations of information received concerning changes in the external environment (e.g., sensor data).

In the case of a human agent interacting with the real world, it would be natural to assume that both state and situation transitions occur along a time continuum. When developing a representational formalism suitable for implementation in a digital agent, however, it is more natural to assume that at least the internal state changes occur in discrete time steps. The length of a step might be the time required to perform one logical inference, or to perform some other kind of primitive action. In making this assumption one thereby implicitly agrees that the agent must forsake a certain variety of omniscience, inasmuch as an all-knowing agent would be one whose internal states exactly mirror the external situations.

In addition one must also forsake the kind of omniscience which claims a complete understanding of all past and potential future events (see Section 1). This of course is justified, inasmuch as humans themselves do not have such perfect awareness. Hence it would be unreasonable to expect this of any digital agents. McDermott [106] in particular has argued along these lines to suggest that the frame problem itself may not actually need to be completely resolved; and indeed this view may be seen to underly virtually all studies in nonmonotonic reasoning, inasmuch as they aim to create agents which reason and make successful decisions in spite of having only incomplete information about the external world. From this standpoint, a perfect resolution of the frame problem may be viewed as a cognitive ideal, i.e., something to be strived toward in an effort to create ever more sophisticated digital agents, but not something one would expect to ever fully achieve.

Here it will be outlined how these various problems might be construed within the context of a DRS. In order to do so, it will be necessary to decide what should be meant by a *transition* in the agent's internal state. It would be natural to allow that a state transition involve changes in any number of (propositions representing) fluents. For purposes of portraying this within the framework of a DRS, however, it will be simpler to agree that a state transition involves only one such change. Then a *state* may be given as an initial segment of a path, and a *state transition* will entail a single reasoning step, either (i) moving from a path of some length n to one of length $n+1$, or (ii) going back into the path and changing some proposition's status indicator from "on" to "off", or possibly (iii) one action of each of these types made in immediate succession (see below). Note that there is no loss of generality in this, since multiple changes can be easily represented as a sequence of individual changes.

Given these assumptions, the persistence problem amounts to that of representing each relevant fluent by a proposition in the path. In the blocks world example given above, state n might consist of the propositions:

The block is on the left. (with status "on")

The block is blue. (status "on")

in which case state $n+1$ would consist of

The block is on the left. (with status now "off")

The block is blue. (status "on")

The block is on the right. (status "on")

In this manner, the fact that the position fluent has changed is represented by the addition of a new proposition, together with a change in the status for an earlier proposition, and the fact that the color fluent has endured is represented by the fact that the status of the second proposition in the path has remained unchanged. This procedure is similar to the one employed by the STRIPS planning system [49]. The persistence problem is solvable in this context to the extent that it is possible to list all the fluents pertinent to the given situation. To the extent that some fluents may be left out, this constitutes a further impingement on the agent's omniscience.

The qualification problem harkens the discussion in Section 6. In that context, this amounts to the problem of including all relevant exceptions in a proposition's exception list. Here omniscience might be compromised because certain exceptions are omitted or overlooked.

The ramification problem amounts to that of exploring all possible extensions of a given path. Since paths are potentially infinite, and are also potentially infinite in number, solving this problem is not generally possible. One is able to explore out only as far as some finite number of possible future steps. Thus omniscience is limited here insofar as this is taken to include prescience.

Last let us consider the frame problem itself. Inasmuch as the persistence problem has to do with how an agent may keep track of what changes and what remains the same as the agent's attention moves from one situation to the next, it is an important part of the frame problem. In addition to this, however, the latter has also to do with how the agent may know what may change, and what may continue to persist, as a result of a particular action. In effect, the critical aspect of the frame problem is that it has to do with endowing the agent with an ability to predict what will occur in the future, rather than to merely keep track of what has occurred in the past.

The earliest, and most straightforward, way of dealing with this is to introduce what are known as *frame axioms*. For example, one might add a proposition asserting that merely moving a block will not change its color. In order to assert such principles in a DRS, however, it would be more appropriate to use *frame rules*. This in fact was the original method proposed by Hayes [69]. Here one could add

$$\frac{(\text{Blue}(\text{Block-1}), \lambda)}{(\text{Blue}(\text{Block-1}), \lambda')}$$

Then if at a later time it is learned that Block-1 has been painted red, a reason maintenance operation could be invoked to resolve the contradiction. This of course does not "solve" the frame problem, but merely shows how it may be expressed within the DRS framework. The crux of the problem is that it is typically impossible to list all the frame rules that may be relevant to a given domain.

9. Concluding remarks

This paper has introduced the notion of a *dynamic reasoning system* (DRS) and explained how this may be used to formulate various forms of nonmonotonic reasoning. A secondary theme has been to introduce the logic **Q** of *qualified syllogisms* and to show how this may be employed in a new type of default reasoning. The use of a qualified syllogism to conclude that a default is *likely* true is equally as natural as the style of reasoning in which one simply assumes that a default holds as long as no countervailing information is explicitly known. A virtue of the DRS formalism is that it allows for both types of reasoning to be represented within the same frame.

In process this work has left open numerous items for further investigation. Several have been mentioned at various places throughout the preceding sections. A further manner in which the notion of a DRS might be expanded would be to include not just one type of language, but a variety of different types of languages, each of which is equipped with its own set of inference rules. One would expect in this case that some of the languages would be interrelated, and that there would be axioms and inference rules expressing these interrelations. For example, one language might be of the ordinary first-order variety, another might be second-order, and a third might be a temporal logic. In addition, as shown by the literature on conceptual graphs (e.g., Sowa's [165, 166]), there are numerous reasons to consider semantic networks having features beyond those provided by a simple multiple-inheritance hierarchy.

The DRS is a simple, yet general, framework which can serve as the basis for much more sophisticated reasoning systems. In particular, for creating more "intelligent" or "autonomous" agents, one will need a layer of higher-level controls capable of guiding the agent toward specific *goals*. Such would entail an algorithm for deciding which inference rules to apply when, exactly how reason maintenance should be carried out, the conditions under which new symbols need to be added to the languages, etc. This might be accomplished in part through such methods as described by Shoham [159], namely, implementing a logic of beliefs, obligations, and abilities. But to this one may also wish to add a capacity for reasoning with expectations, intentions, and desires (see [26] for a start in this direction). In principle, just as with human agents, our digital agents should be able to choose between several different reasoning strategies as the situation demands.

Another issue here left open is how to prescribe rules for agent interaction, again for purposes of achieving goals. The notion of an "agent society" has been advanced by Shoham and Tennenholtz [160], and the notion of a DRS could serve well for formulating this idea. Each agent in the society would be given as a distinct DRS. This provides for both the receiving and sending of messages, reasoning about the messages thus received, and performing various other types of actions.

The ultimate objective of AI has traditionally been to create agents with ever more human-like types of minds, where by "mind" is here meant a memory together with a reasoning mechanism. Whether we should actually want our digital agents to be human-like, however, has recently become a subject for debate [75]. Perhaps a more realistic aim would be to first identify among humans those mental activities that it would be desirable to replicate, and then seek to make only these manifest in our machines. Such would surely include reasoning and goal-seeking activities, and there conceivably could be use for agents that respond with emotions like desire and fear. On the other hand, one probably would not want to replicate the human propensity for behaving irrationally or making mistakes. While the present notion of DRS is, in this light, only a very rudimentary kind of "mind", it does show potential for being developed in these directions. This of course will also require gaining a deeper understanding of mind itself.

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