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Constrained coalition formation on valuation structures: Formal framework, applications, and islands of tractability

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A B S T R A C T

Coalition structure generation is the problem of partitioning the agents of a given environment into disjoint and exhaustive coalitions so that the whole available worth is maximized. While this problem has been classically studied in settings where all coalitions are allowed to form, it has been recently reconsidered in the literature moving from the observation that environments often forbid the formation of certain coalitions. By following this latter perspective, a model for coalition structure generation is proposed where constraints of two different kinds can be expressed simultaneously. Indeed, the model is based on the concept of *valuation structure*, which consists of a set of *pivotal* agents that are pairwise incompatible, plus an *interaction graph* prescribing that a coalition C can form only if the subgraph induced over the nodes/agents in C is connected.

It is shown that valuation structures can be used to model a number of relevant problems arising in real-world application domains. Then, the complexity of coalition structure generation over valuation structures is studied, by assuming that the functions associating each coalition with its worth are given as input according to some compact encoding—rather than explicitly listing all exponentially-many associations. In particular, islands of tractability are identified based on the topological properties of the underlying interaction graphs and on suitable algebraic properties of the given worth functions. Finally, stability issues over valuation structures are studied too, by considering the *core* as the prototypical solution concept.

1. Introduction

1.1. Constrained coalition structure generation

Coalition structure generation is a fundamental problem in the study of coalition formation processes for multi-agent systems, which naturally occurs whenever agents can benefit form working together by forming coalitions (see, e.g., [59,31]). The problem is defined over a pair $\langle N, v\rangle$, referred to as a *coalitional game*, where $N = \{a_1, \dots, a_n\}$ is a set of agents and where v is a *valuation function* that, for each *coalition* C, i.e., non-empty set $C \subseteq N$ of agents, returns a real number $v(C)$ meant to express the worth that the members of C can jointly achieve by cooperating (see, e.g., [71,53]). The goal is to find

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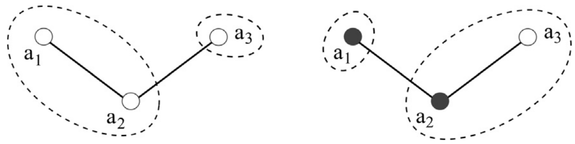

Fig. 1. Interaction graph in Example 1.2, with optimal coalition structures for the basic setting (left), and when a_1 and a_2 are pivotal agents (right).

an optimal *coalition structure*, i.e., a partition $\{C_1, \ldots, C_k\}$ of the agents into disjoint and exhaustive coalitions whose total value $\sum_{i=1}^{k} v(C_i)$ is maximized.

Example 1.1. Consider the coalitional game $\langle N, v\rangle$ where $N = \{a_1, a_2, a_3\}$ and where v is the valuation function such that:

$$\begin{cases} v(\{a_3\}) = 0 \\ v(\{a_1\}) = v(\{a_2\}) = v(\{a_2, a_3\}) = v(\{a_1, a_3\}) = 1 \\ v(\{a_1, a_2\}) = v(\{a_1, a_2, a_3\}) = 3 \end{cases}$$

Note that the optimal coalition structures are $\{\{a_1, a_2, a_3\}\}$ and $\{\{a_1, a_2\}, \{a_3\}\}$. Their associated value is $v(\{a_1, a_2, a_3\}) = v(\{a_1, a_2\}) + v(\{a_3\}) = 3$. $\lhd$

While coalition structure generation has been classically studied in the literature by assuming that all coalitions are allowed to form, in real-world applications it is often the case that some coalition structures are inadmissible, because they violate a number of *constraints* induced by the specific semantics of the application at hand (see, e.g., [59]).

Constraints on the coalition structures that are allowed naturally emerge in those settings where cooperation is guided by an underlying structure reflecting, for instance, physical limitation, legal banishments, and social relationships. In these cases, it is natural to assume that if two disconnected agents are not connected by intermediaries in a given coalition, then they might not be able to cooperate at all. In particular, following Myerson's influential work [51], this intuition has been often formalized (see, e.g., [15,70,9]) by equipping each coalitional game $\langle N, v\rangle$ with an undirected graph $G = (N, E)$, called *interaction* graph, defined over the set of the agents, and by considering a coalition C as a *feasible* one, only if the subgraph of G induced over the nodes in C is connected.

In fact, in addition to the "topological" constraints induced by the underlying interaction graphs, other kinds of constraints might occur in concrete domains when dealing with the coalition structure generation problem. For instance, the formation of certain coalitions might be prohibited by anti-trust laws or it might be subject to constraints on the coalition sizes. Settings of this kind have been also studied in the literature [26,65,56] and a general and unifying framework of these works has been proposed too [58]. In that framework, a coalitional game is equipped with two sets, $\mathcal{N}$ and $\mathcal{P}$, of "negative" and "positive" constraints respectively. A negative constraint $\mathbf{n} \in \mathcal{N}$ is a set of agents, and it prescribes that no coalition C such that $C \supseteq \mathbf{n}$ can be formed. Positive constraints are again formalized as set of agents, and they prescribe that for each feasible coalition C, there must exist a constraint $\mathbf{p} \in \mathcal{P}$ such that $C \supseteq \mathbf{p}$. As a matter of fact, however, the framework of [58] does not support the definition of interaction graphs and, despite its generality, it cannot simulate the topological constraints that are induced by them.

Example 1.2. Consider again the setting of Example 1.1 and the interaction graph reported on the left of Fig. 1. Note, for instance, that coalition $\{a_1, a_3\}$ is not allowed to form. Then, we claim that this simple scenario cannot be modeled via positive and negative constraints. Indeed, if a negative constraint $\mathbf{n} = \{a_1, a_3\}$ is considered, then the coalition $\{a_1, a_2, a_3\}$ would be not feasible precisely because of $\mathbf{n}$. More generally, because of the feasibility of $\{a_1, a_2, a_3\}$ and by the monotone semantics of negative constraints, no negative constraint can be defined at all.

Consider now the use of positive constraints. We know that $\{a_1\}$ is feasible and, hence, it must occur as a positive constraint. However, in absence of negative constraints, this immediately entails that $\{a_1, a_3\}$ is feasible, too. $\lhd$

As a matter of fact, topological constraints and positive/negative constraints have been separate worlds, so far. In the paper, we move from this observation and we propose to study a setting for coalition structure generation based on the concept of *valuation structure*, which basically consists of an interaction graph associated with certain kinds of negative constraints. In a nutshell, a set S of pairwise "incompatible" (*pivotal*) agents can be defined, so that every coalition C must satisfy the condition $|S \cap C| \leq 1$ in order to be a feasible one. Indeed, note that this is equivalent to having a negative constraint S', for each subset $S' \subseteq S$ with $|S'| = 2$.

Example 1.3. Assume that a_1 and a_2 are two pivotal agents in the setting of Example 1.1. Then, the feasible coalitions are further reduced to $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, and $\{a_2, a_3\}$, because the coalitions $\{a_1, a_2, a_3\}$ and $\{a_1, a_2\}$ would be no longer allowed to form. In this scenario, which is graphically illustrated on the right of Fig. 1, an optimal coalition structure is $\{\{a_1\}, \{a_2, a_3\}\}$ whose associated value is a_2. Hence, the incompatibility of a_1 and a_2 leads to reduce the total available worth. $\lhd$

1.2. Contribution

The intuition underlying our formalization is that pivotal agents in S possess some specific properties differentiating themselves from the remaining agents in $N \setminus S$. As an extreme case, a pivotal agent might well be an abstraction for some given parameter/object involved in the problem, i.e., it is not necessarily a "true" agent of the system. For instance, S might model a set of competing facilities to which the agents in $N \setminus S$ have to be connected. In fact, as the starting point of our analysis,

(1) We define a framework for equipping coalitional games with valuation structures and we show that constraints induced by pivotal agents, combined with the underlying interaction graphs, are capable of expressing a number of relevant problems arising in a number of real application scenarios.

Motivated by their relevance from the knowledge representation viewpoint, the paper then embarks on a systematic study of algorithmic and complexity issues arising with them. Prior to detailing these technical contributions, however, it is appropriate to recall that there is an extensive literature studying computational issues and proposing efficient solution algorithms for coalition structure generation, which we can partition in two groups based on the kinds of game encoding considered in the research.

Classically, valuation functions are viewed as "black boxes" that, on input a coalition C, return the value $v(C)$. In particular, encoding and representation issues are not taken into account. In this context, exact solution approaches or algorithms with worst case guarantee have been proposed (e.g., [62,22,57,60,50,66]); and heuristic methods have been defined, too (e.g., [64,46,65]). Orthogonally to these approaches and moving from the observation the naive encoding based on explicitly listing all possible coalitions, with attached their associated valuations, would require exponential space, efforts have been spent to define methods for representing valuation functions (hence, coalitional games) concisely. Then, coalition structure generation is more efficiently solved by applying optimization techniques to the compact representation directly [52]. In particular, algorithmic and complexity results have been derived for a number of well-known compact representations including *marginal contribution networks* [43,52], games with *synergy coalition groups* [17,52], *coalitional skill* games [5], games with *agent-type* representations [2,67], *weighted voting* games [29], and *graph* games [27,2,4].

Our analysis and our results are positioned within the latter group of works. Indeed, we study a number of computational problems related to coalition formation in the presence of valuation structures, by assuming that coalitional games are provided as input according to some compact encoding. In fact, in absence of valuation structures, it is well-known that coalition structure generation is **NP**-hard over a number of compact encodings including very simple ones, such as over graph games [2,4]. Surprisingly, however, the precise complexity of the problem was not pointed out in earlier literature. Our first technical result is to address this research question. Indeed,

(2) We show that coalition structure generation is complete for the polynomial time closure of **NP**, formally for the complexity class $\mathrm{F}\boldsymbol{\Delta}_2^{\mathbf{P}}$ [48]. The membership result is derived independently of the specific game encoding, while hardness is shown to hold even *(i)* over graph games and in absence of valuation structures, so that all coalitions are allowed to form, and *(ii)* over valuation structures whose underlying graphs are acyclic, so that very simple forms of cooperation can emerge only.

Motivated by the above bad news, in particular by focusing on the latter setting (*ii*), we then look at the class of valuation functions that are *independent of disconnected members* (short: `IDM`), i.e., such that two agents have no effect on each other's marginal contribution to their separator[1] w.r.t. the underlying interaction graph [69]. Indeed, in absence of valuation structures, coalition structure generation is already known [69] to be tractable for `IDM` functions over acyclic interaction graphs, or more generally over graphs having *bounded treewidth* [61]. The question is then whether this result can be extended to our more general setting where constraints on the coalitions allowed to form are expressed via valuation structures. This time, the answer is good news:

(3) First, we show that `IDM` functions defined over interaction graphs with bounded treewidth can be encoded in polynomial time into equivalent marginal contribution networks [43] again of bounded treewidth, and vice versa. This is interesting because marginal contribution networks are an encoding that generalizes graph games and that is capable of expressing any valuation function, though in general at the price of an exponential blow-up in the size of representation (cf. [43]). Moreover, note that `IDM` functions define a class of functions in terms of their algebraic (and topological) properties. Hence, our result is also interesting for it algebraically characterizes a well-known and influential representation scheme for coalitional games, and for it allows to smoothly extend on this scheme all results derived for `IDM` functions, and vice versa—when focusing on bounded treewidth instances.

[1] Formally, for each pair i, j of agents that are not directly connected in the graph, and for each $C \subseteq N$ with $i, j \notin C$, it holds that $v(C \cup \{i\}) - v(C) = v(C \cup \{i, j\}) - v(C \cup \{j\})$.

(4) Then, we generalize the tractability result of IDM functions over graphs having bounded treewidth [69] to the case where valuation structures are considered. Founding on the correspondence discussed at point (3) above, the result is obtained by recasting coalition structure generation over marginal contribution networks into a *constraint satisfaction problem (CSP)*, and by subsequently using structural decomposition methods for CSP instances (see, e.g., [36,35,41]). The approach uses novel ideas and methods that might be re-used in other application domains. Moreover, we show that it can be easily extended to deal with valuation structures in presence of arbitrary positive and negative constraints explicitly provided as input. The careful reader might note that alternative approaches would have been to use an encoding in *monadic second-order logic* by subsequently applying generalizations of Courcelle's theorem tailored to optimization problems [1], or to use general methods to solve combinatorial problems on graphs having bounded treewidth via dynamic programming [10]. However, these approaches would have asked to restrict the values returned by the valuation functions (to be given in unary) or the given interaction graphs (to have bounded degree).

(5) In addition, we show that the above results hold not only for IDM functions, but also for functions derived from IDM ones via "affine transformations"—whose specification is allowed as an additional feature in the formal definition of a valuation structure. Consider, for instance, a setting where in order to leave the *grand-coalition* N, a "splitting cost" c has to be paid. In this case, we would obtain an "adjusted" valuation function v' such that $v'(C) = v(C) - c$, for each $C \subset N$, and $v'(N) = v(N)$. In particular, note that such transformed functions are no longer IDM ones, so that specific solution approaches have been required to deal with them.

Finally, in order to provide a more comprehensive picture of the complexity issues arising with valuation structures, we complement the above research by addressing another fundamental problem for coalitional games, namely to determine how the worth that is obtained by forming an optimal coalition structure can be distributed over the agents in a way that is *stable* [53]. In particular,

(6) We consider the well-known concept of *core* and we position it within our setting. Then, we show that core-related problems can be solved in polynomial time for classes of IDM valuation functions over graphs having bounded treewidth, even in presence of valuation structures. Results are derived as an application of the methods developed to deal with coalition structure generation in the same setting.

1.3. Organization

The rest of the paper is organized as follows. Section 2 formalizes the concept of valuation structure and illustrates its applications. The coalition structure generation problem over valuation structures is defined in Section 3, where basic complexity results are also derived. Islands of tractability are singled out in Section 4, whereas Section 5 illustrates their specializations to the application scenarios we have previously pointed out. Stability issues are studied in Section 6, while a few final remarks are reported in Section 7.

2. Coalition structure generation revisited

In this section, we propose an extension of the coalition structure generation problem where optimization functions more general than the maximization of the social welfare are allowed, and where qualitative constraints on the coalition structures of interest can be expressed. The extension is based on the concept of "valuation structure", which is first discussed and subsequently exemplified with a number of concrete applications.[2]

2.1. Valuation structures: syntax and semantics

Let $\Gamma = \langle N, v\rangle$ be a coalitional game. Intuitively, the concept of valuation structure we are going to define on top of Γ plays the role to model certain kinds of constraints on the coalitions that are allowed to form plus an affine transformation on the basic valuation function v. In particular, because of the latter feature, throughout the paper N and v will not fully characterize a coalitional game in the conventional sense, because they do not suffice alone to describe the coalitional values in our extended setting.

Let $G = (N, E)$ be an undirected graph called *interaction graph*, whose nodes are the agents in N. Edges are viewed as sets of nodes, so that for each $e \in E$, $e \subseteq N$ and $|e| = 2$ hold. Let $S \subseteq N$ be a possibly-empty set of agents, called *pivotal agents*. Let $\alpha, \beta : S \mapsto \mathbb{R}$ be two real-valued functions, and let $x, y \in \mathbb{R}$ be real numbers.[3] Then, the tuple $\sigma = \langle G, S, \alpha, \beta, x, y\rangle$ is called a *valuation structure* for $\langle N, v\rangle$. The semantics of σ is next illustrated.

The role of the interaction graph G is to constrain the formation of the coalitions based on their connectivity properties, in that a non-empty set C of agents, called *coalition*, is allowed to form only when the subgraph induced over the nodes in C is connected. Moreover, we would like to forbid the formation of any coalition C containing more than one pivotal agent,

[2] A table summarizing the main notation we introduce in order to deal with valuation structures is reported in Appendix A.

[3] As usual, when moving to the complexity analysis, all numbers will be restricted to be rational ones and given in fractional form.

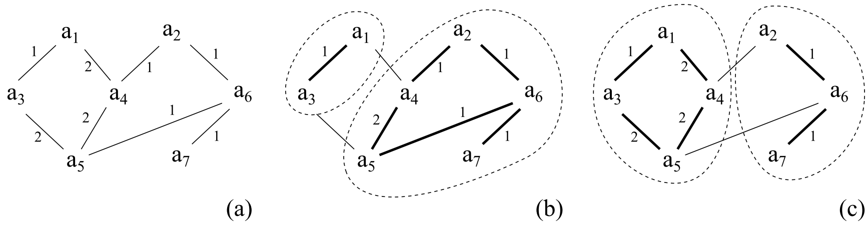

Fig. 2. (a) Coalitional game in Example 2.1; (b, c) Coalition structures in Example 2.3.

i.e., C is allowed only if $|C \cap S| \leq 1$. Whenever such connectivity and pivotal conditions are satisfied by C, we say that C is σ-feasible. The set of all σ-feasible coalitions in the game Γ is denoted by $\mathcal{F}_\sigma(\Gamma)$.

Example 2.1. Consider a coalitional game $\hat{\Gamma} = (\hat{N}, \hat{v})$ such that $\hat{N} = \{a_1, \ldots, a_7\}$. Moreover, consider the valuation structure $\hat{\sigma} = \langle \hat{G}, \{a_1, a_2\}, \hat{\alpha}, \hat{\beta}, \hat{x}, \hat{y} \rangle$ where $\hat{G}$ is the interaction graph depicted in Fig. 2(a). Then, $C_1 = \{a_1, a_3\}$ and $C_2 = \{a_5, a_6, a_7\}$ are two $\hat{\sigma}$-feasible coalitions. Instead, $\{a_1, a_2\}$ is not $\hat{\sigma}$-feasible because it contains two pivotal agents, and $\{a_5, a_7\}$ is not $\hat{\sigma}$-feasible because the subgraph of $\hat{G}$ induced over the coalition is not connected. $\lhd$

The functions α and β and the real numbers x and y are used to define an affine transformation val_σ of the valuation function v. Formally, if $C \in \mathcal{F}_\sigma(\Gamma)$ is a σ-feasible coalition, then we define its σ-value as follows:

$$val_\sigma(v, C) = \begin{cases} \alpha(a_i) \times v(C) + \beta(a_i) & \text{if } \{a_i\} = C \cap S, \\ x \times v(C) + y & \text{if } C \cap S = \emptyset \end{cases}$$

If v is understood from the context, then we just write $val_\sigma(C)$ in place of $val_\sigma(v, C)$.

Example 2.2. Consider again the setting of Example 2.1, and the graph $\hat{G} = (\hat{N}, \hat{E})$ in Fig. 2(a). Note that each edge $e \in \hat{E}$ is equipped with a weight $w_e \in \mathbb{R}$, so that we have, for instance, $w_{\{a_1,a_3\}} = 1$ and $w_{\{a_1,a_2\}} = 2$.[4] Moreover, assume that the valuation function $\hat{v}$ is such that if C is a $\hat{\sigma}$-feasible coalition, then its value $\hat{v}(C)$ is defined as the sum of the weights of the edges covered by C, i.e., $\hat{v}(C) = \sum_{e \in \hat{E}, e \subseteq C} w_e$. For instance, for the coalitions $C_1 = \{a_1, a_3\}$ and $C_2 = \{a_5, a_6, a_7\}$ presented in Example 2.1, we have $\hat{v}(C_1) = w_{\{a_1,a_3\}} = 1$, while $\hat{v}(C_2) = w_{\{a_5,a_6\}} + w_{\{a_6,a_7\}} = 1 + 1 = 2$.

Recall that $\hat{\sigma} = \langle \hat{G}, \{a_1, a_2\}, \hat{\alpha}, \hat{\beta}, \hat{x}, \hat{y} \rangle$ is the given valuation structure. Assume that $\hat{\alpha} : \{a_1, a_2\} \mapsto \{1\}$ and $\hat{\beta} : \{a_1, a_2\} \mapsto \{0\}$ are two constant functions over the domain $\{a_1, a_2\}$ of the pivotal agents, and that $\hat{x} = 0$ and $\hat{y} = -12$. Then, we have $val_{\hat{\sigma}}(C_1) = 1 \times \hat{v}(C_1) + 0 = 1$ and $val_{\hat{\sigma}}(C_2) = 0 \times \hat{v}(C_2) - 12 = -12$. More generally, each $\hat{\sigma}$-feasible coalition that does not include a pivotal agent gets -12 as its final $\hat{\sigma}$-value. Instead, the $\hat{\sigma}$-value of any other $\hat{\sigma}$-feasible coalition C, hence containing precisely one of the two pivotal agents, is just given by the expression $\sum_{e \in \hat{E}, e \subseteq C} w_e$. $\lhd$

A σ-feasible *coalition structure* for Γ is a set $\Pi = \{C_1, \ldots, C_k\}$ of σ-feasible coalitions such that $C_1 \cup \cdots \cup C_k = N$ and $C_i \cap C_j = \emptyset$, for each pair $i, j \in \{1, \ldots, k\}$ with $i \neq j$. The set of all possible coalition structures is denoted by $\mathcal{CS}_\sigma(\Gamma)$. Note that $\mathcal{CS}_\sigma(\Gamma) \neq \emptyset$, because each singleton coalition is σ-feasible, that is, $\{a_i\} \in \mathcal{F}_\sigma(\Gamma)$ holds, for each $a_i \in N$.

For any $\Pi \in \mathcal{CS}_\sigma(\Gamma)$, by slightly abusing of notation, we define $val_\sigma(v, \Pi)$ (or, shortly, $val_\sigma(\Pi)$ if the valuation function v is clearly understood) as the value $\sum_{C \in \Pi} val_\sigma(v, C)$, which is called the σ-value of Π. Moreover, we denote by $opt_\sigma(\Gamma)$ the maximum σ-value that can be attained over all possible σ-feasible coalition structures. A coalition structure $\Pi^* \in \mathcal{CS}_\sigma(\Gamma)$ such that $val_\sigma(\Pi^*) = opt_\sigma(\Gamma)$ is said to be σ-*optimal* on Γ, and the set of all σ-optimal coalition structures is denoted by CS-$opt_\sigma(\Gamma)$.

Example 2.3. In the setting of Example 2.1 and Example 2.2, it clearly emerges that it is suboptimal to form a coalition without including one of the two pivotal agents. Therefore, $\hat{\sigma}$-optimal coalition structures have necessarily the form $\{A_1, A_2\}$ where $A_1 \cap A_2 = \emptyset$, $a_1 \in A_1$, $a_2 \in A_2$, and $A_1 \cup A_2 = \hat{N}$. Note also that, since A_1 and A_2 are $\hat{\sigma}$-feasible, the subgraph of $\hat{G}$ induced over A_i, with $i \in \{1, 2\}$, has to be connected. As an example, Fig. 2(b) illustrates the $\hat{\sigma}$-feasible coalition structure $\{\{a_1, a_3\}, \{a_2, a_4, a_5, a_6, a_7\}\}$, whose associated $\hat{\sigma}$-value is $1 + 6 = 7$. In fact, this is not $\hat{\sigma}$-optimal, because the $\hat{\sigma}$ value of $\{\{a_1, a_3, a_4, a_5\}, \{a_2, a_6, a_7\}\}$ is $7 + 2 = 9$—see Fig. 2(c). It can be checked that this latter coalition structure is indeed $\hat{\sigma}$-optimal. $\lhd$

Note that, if in the above example no pivotal agents were defined, then the coalition structure consisting of the grand-coalition (i.e., $\{\hat{N}\}$) would be the only optimal one. In particular, in this scenario, the constraints provided by the interaction

[4] Throughout the paper, we often exhibit exemplifications and discuss real-world application domains defined over graphs whose edges are equipped with weights. As in Example 2.2, such weights are used to succinctly define the underlying valuation function, in the spirit of the encoding for graph games [27].

graph would be immaterial, in the sense that $\{\hat{N}\}$ would be still optimal if the complete interaction graph (i.e., where every pair of agents is connected) were used in place of $\hat{G}$ in the definition of the structure $\hat{\sigma}$. In general this is not the case, as illustrated in the following example.

Example 2.4. Consider the game $\bar{\Gamma} = \langle \{a_1, a_2\}, \bar{v}\rangle$ such that $\bar{v}(\{a_1\}) = \bar{v}(\{a_2\}) = 0$ and $\bar{v}(\{a_1, a_2\}) = 1$. Consider the valuation structure $\bar{\sigma} = \langle \bar{G}, \emptyset, \bar{\alpha}, \bar{\beta}, \bar{x}, \bar{y}\rangle$ where $\bar{G} = (\{a_1, a_2\}, \emptyset)$. Thus, the only $\bar{\sigma}$-feasible coalitions are the singletons $\{a_1\}$ and $\{a_2\}$. It follows that there is precisely one $\bar{\sigma}$-optimal coalition structure, namely $\{\{a_1\}, \{a_2\}\}$, whose associated $\bar{\sigma}$-value is 0. However, if the graph where a_1 and a_2 are connected were used in place of $\bar{G}$, then the only optimal coalition structure would be $\{\{a_2, a_2\}\}$. So, $\bar{G}$ provides a true constraint w.r.t. the formation of optimal coalition structures. $\lhd$

Actually, by looking critically at the above example, the reader might have noticed that the setting is rather artificial and counter-intuitive. Indeed, the graph $\bar{G}$ tells us that we cannot directly benefit from the cooperation between a_1 and a_2, whereas $\bar{v}$ tells precisely the opposite, that is, it evidences that a synergy can emerge from their cooperation.

In fact, in order to study more natural scenarios, where worth functions suitably reflect the topological properties expressed by an underlying interaction graph, the concept of valuation functions that are *independent of disconnected members* was introduced in the literature. We recall from [69] that a valuation function v is independent of disconnected members (short: IDM) w.r.t. an interaction graph G if, for all agents a_i and a_j that are not connected in G and for each coalition C with $a_i, a_j \notin C$, the following holds:

$$v(C \cup \{a_i\}) - v(C) = v(C \cup \{a_i, a_j\}) - v(C \cup \{a_j\}). \tag{1}$$

Intuitively, the notion tells us that two agents have no effect on each other's marginal contribution to their vertex separator. In particular, if C_1 and C_2 are two coalitions and there are no edges between $C_1 \setminus C_2$ and $C_2 \setminus C_1$, then we have [69]:

$$v(C_1 \cup C_2) = v(C_1) + v(C_2) - v(C_1 \cap C_2). \tag{2}$$

Note that the valuation function $\hat{v}$ in Example 2.1 is an IDM function w.r.t. $\hat{G}$. Moreover, recall that in absence of pivotal agents the graph $\hat{G}$ might be relaxed with the complete interaction graph. It is instructive to point out that this is not by chance.

In order to formally state the result, observe that a valuation structure $\langle G, \emptyset, \alpha, \beta, x, y\rangle$ can be more compactly denoted by $\langle G\rangle$, because the functions α and β and the numbers x and y play no role. Moreover, if C is a coalition which is not necessarily $\langle G\rangle$-feasible and if $C_1, \dots, C_k$, with $k \geq 1$, are the connected components of the subgraph of G induced over the nodes in C, then the set $\{C_1, \dots, C_k\}$ is denoted by $C[G]$.

Theorem 2.5. *Let $\Gamma = \langle N, v\rangle$ be a coalitional game and let $\langle G_1\rangle$ and $\langle G_2\rangle$ be two valuation structures for Γ such that $G_i = (N, E_i)$, with $i \in \{1, 2\}$, and $E_1 \subseteq E_2$. Assume that v is an IDM function w.r.t. G_1 (hence, w.r.t. G_2). Then,*

(1) *If Π_2 is $\langle G_2\rangle$-optimal, then $\bigcup_{C \in \Pi_2} C[G_1]$ is $\langle G_1\rangle$-optimal.*
(2) *If Π_1 is $\langle G_1\rangle$-optimal, then Π_1 is $\langle G_2\rangle$-optimal.*

Proof. *(1)* Let Π_2 be any $\langle G_2\rangle$-optimal coalition structure, and consider the coalition structure $\Pi_1 = \bigcup_{C \in \Pi_2} C[G_1]$. We claim that Π_1 is $\langle G_1\rangle$-feasible. Indeed, if C' is a coalition in Π_1, then there is a coalition $C \in \Pi_2$ such that $C' \in C[G_1]$. That is, C' is the set of nodes of some connected component in the subgraph of G_1 induced over C. So, every coalition in Π_1 satisfies the connectedness condition enforced by the interaction graph G_1. Moreover, by construction of Π_1, it is immediate that $\bigcup_{C' \in \Pi_1} = N$ and $C' \cap C'' = \emptyset$, for each pair C', C'' of coalitions in Π_1. That is, Π_1 is a $\langle G_1\rangle$-feasible coalition structure.

We now claim that $val_{\langle G_2\rangle}(\Pi_2) = val_{\langle G_1\rangle}(\Pi_1)$. In fact, we show that, for each $C \in \Pi_2$, $val_{\langle G_2\rangle}(C) = val_{\langle G_1\rangle}(C[G_1])$ holds. To this end, let $C[G_1]$ be the set $\{C_1, \dots, C_k\}$ of $\langle G_1\rangle$-feasible coalitions, and observe that $val_{\langle G_1\rangle}(C[G_1]) = \sum_{i=1}^{k} val_{\langle G_1\rangle}(C_i) = \sum_{i=1}^{k} v(C_i)$. Then, recall that v is an IDM function w.r.t. G_1 and that for each pair C_h, C_j in $C[G_1]$ with $h \neq j$, there is no edge in G_1 connecting some node in C_h with some node in C_j, since C_h and C_j are precisely connected components in the subgraph of G_1 induced over the nodes in C. Thus, by Equation (2), $v(C_h \cup C_j) = v(C_h) + v(C_j)$. Moreover, observe that $C = C_1 \cup \cdots \cup C_k$. Hence, the above entails that $v(C) = \sum_{i=1}^{k} v(C_i)$. Finally, we observe that C is $\langle G_2\rangle$-feasible and, therefore, $val_{\langle G_2\rangle}(C) = v(C)$. So, $val_{\langle G_2\rangle}(\Pi_2) = val_{\langle G_1\rangle}(\Pi_1)$.

Armed with the above properties, let us assume, for the sake of contradiction, that Π_1 is not $\langle G_1\rangle$-optimal and let Π'_1 be a $\langle G_1\rangle$-optimal coalition structure with $val_{\langle G_1\rangle}(\Pi'_1) > val_{\langle G_1\rangle}(\Pi_1)$. Therefore, $val_{\langle G_2\rangle}(\Pi_2) = val_{\langle G_1\rangle}(\Pi_1) < val_{\langle G_1\rangle}(\Pi'_1)$. However, Π'_1 is trivially $\langle G_2\rangle$-feasible and, hence, $val_{\langle G_2\rangle}(\Pi'_1) = val_{\langle G_1\rangle}(\Pi'_1) > val_{\langle G_2\rangle}(\Pi_2)$ holds. However, by the optimality of Π_2, we know that $val_{\langle G_2\rangle}(\Pi_2) \geq val_{\langle G_2\rangle}(\Pi'_1)$. Contradiction.

(2) Assume that Π_1 is any $\langle G_1\rangle$-optimal coalition structure. Hence, Π_1 is $\langle G_2\rangle$-feasible too (and $val_{\langle G_1\rangle}(\Pi_1) = val_{\langle G_2\rangle}(\Pi_1)$). Assume, for the sake of contradiction, that Π_1 is not $\langle G_2\rangle$-optimal, and let Π_2 be a $\langle G_2\rangle$-optimal coalition structure with $val_{\langle G_2\rangle}(\Pi_2) > val_{\langle G_2\rangle}(\Pi_1)$. By point *(1)*, $\bigcup_{C \in \Pi_2} C[G_1]$ is $\langle G_1\rangle$-optimal and, by inspection in the proof, we also

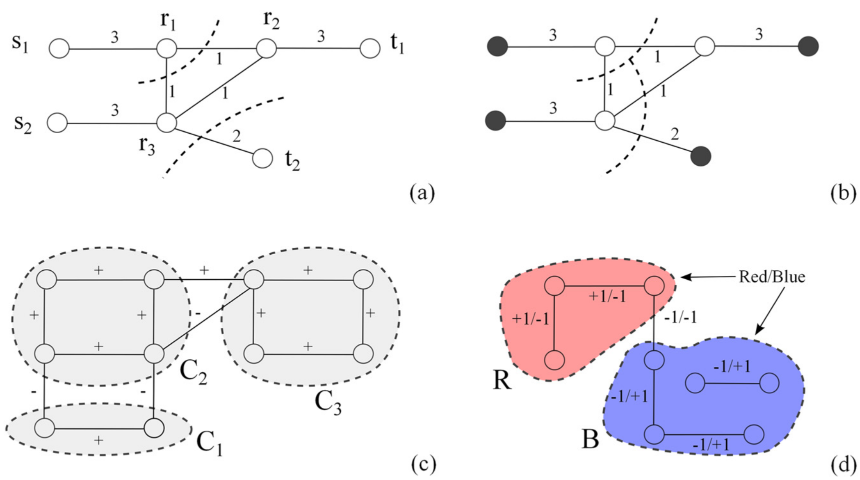

Fig. 3. Illustration of examples in Section 2.2.

know that $val_{\langle G_1\rangle}(\bigcup_{C\in\Pi_2} C[G_1]) = val_{\langle G_2\rangle}(\Pi_2)$. Therefore, $val_{\langle G_1\rangle}(\bigcup_{C\in\Pi_2} C[G_1]) > val_{\langle G_1\rangle}(\Pi_1)$, which is impossible by the optimality of Π_1. $\square$

2.2. Knowledge representation and applications

The concept of σ-optimal coalition structure illustrated in the above section is clearly more general than the counterpart where no valuation structure σ is considered. Indeed, given any coalitional game $\Gamma = \langle N, v\rangle$, by defining $\perp = \langle K_N\rangle$ as the valuation structure where K_N is the complete interaction graph over the set N of agents, we can immediately check that any coalition $S \subseteq N$ is $\perp$-feasible and that $val_\perp(v, S) = v(S)$ holds. Therefore, results in the literature where no valuation structures have been considered (see Section 1) can be viewed as results that hold over $\perp$-optimal coalition structures.

Our goal is then to analyze arbitrary valuation structures, and we start here by illustrating a number of application scenarios that naturally fit the basic setting and our more general one. Further applications will be discussed in Section 6. In the following, we denote by $+\infty$ any "sufficiently large" real number,[5] and by $-\infty$ its opposite. Moreover, for any real number $c \in \mathbb{R}$, we denote by $\mathbf{c}_D$ the real-valued function mapping each element in the domain D to the constant c—actually, the subscript will be hereinafter omitted, since this notation will be used for D coinciding with the set of pivotal agents only, and since this set will be clearly understood from the context.

It is worthwhile noticing that the tractability results we shall provide in the paper for instances having bounded treewidth immediately apply to the specific applications discussed below, in some cases allowing us to close tractability questions that have been left open in the literature—as we shall discuss in detail in Section 5.

2.2.1. Application to multicut problems

In the multicut problem, we are given an undirected graph $G = (N, E)$ where each edge $e \in E$ is associated with a positive real number $w_e \in \mathbb{R}^+$ (the weight of e), plus a set $P \subseteq N \times N$ of source-terminal pairs. A multicut is a set of edges $E' \subseteq E$ such that in the graph $(N, E \setminus E')$ each of the pairs in P is separated, i.e., for each $(s,t) \in P$ there is no path in $(N, E \setminus E')$ connecting s and t. The goal is to find a multicut whose edges have minimum total weight. For $|P| = 1$, the problem reduces to the famous "min-cut/max-flow" problem, which is of central significance in combinatorial optimization and which is known to be feasible in polynomial time (see, e.g., [47]). In general, the problem is **NP**-hard and a number of (in)approximability results are known for it (see, e.g., [68]).

Example 2.6. Consider the graph reported in Fig. 3(a), and assume that $\{(s_1,t_1), (s_2,t_2)\}$ is a given set of source-terminal pairs. Then, a solution to the multicut problem defined on these inputs is given by the multicut graphically illustrated in the same figure. Note that the total weight is $1 + 1 + 2 = 4$. $\lhd$

In order to encode the multicut problem in our setting, the set N of the nodes is transparently viewed as a set of agents. Moreover, we define a graph $G_P = (N, E_P)$ such that $E_P = E \cup \{\{s,t\} \mid (s,t) \in P\}$, i.e., the graph is obtained from G by adding an edge for each pair in P. Eventually, we let $w_{\{s,t\}} = -\sum_{e\in E} w_e - 1$, for each $(s,t) \in P$, and we define v_P as the function such that $v_P(C)$ is the sum of the weights of the edges in the subgraph of G_P induced over any coalition C. Note that $v_P(C) < 0$ holds if, and only if, C covers some pair in P.

[5] For instance, we can just set $+\infty \geq |N| \times \max_{S\in\mathcal{F}_\sigma(\Gamma)} |v(S)|$.

Consider now the (trivial) valuation structure $\perp$, and observe that given the definition of v_P, it is clearly suboptimal to form coalitions covering pairs from P. For instance, in the example illustrated in Fig. 3(a), a $\perp$-optimal coalition structure is $\{\{s_1, r_1\}, \{s_2, r_2, r_3, t_1\}, \{t_2\}\}$. Note that this coalition structure is induced by a multicut of minimum total weight. This is not by chance. Indeed,[6] in our reduction any multicut E' one-to-one corresponds to a $\perp$-feasible coalition structure Π for $\langle N, v_P\rangle$ such that $E' = \{e \in E \mid \forall C_j \in \Pi, |e \cap C_j| \leq 1\}$ and $val_\perp(v_P, \Pi) \geq 0$. In particular, note that for any coalition structure Π of this kind, $val_\perp(v_P, \Pi) = \sum_{e\in E} w_e - \sum_{e'\in E'} w_{e'}$ holds. Therefore, a multicut of minimum weight can be computed by algorithms for computing ($\perp$-)optimal coalition structures.

2.2.2. Application to multiway cut problems

Similarly to the multicut problem, the multiway cut problem takes as input a graph $G = (N, E)$, where each edge $e \in E$ is equipped with a weight $w_e \in \mathbb{R}^+$. Moreover, it takes as input a set $T \subseteq N$ of terminals. A multiway cut E' is a set of edges $E' \subseteq E$ such that in the graph $(N, E \setminus E')$ no pair of terminals from T are connected. The goal is to find a multiway cut whose edges have the minimum total weight. For $|T| = 2$ the problem is solvable in polynomial time (it again reduces to the min-cut problem), while it becomes **NP**-hard even for $|T| = 3$ (see [21]).

Example 2.7. Consider again the graph reported in Fig. 3(a), where all nodes in $\{t_1, t_2, s_1, s_2\}$ are now treated as terminals to be disconnected from each other. In order to end up with a multiway cut, we have in particular to disconnect s_2 and t_1, as it is done in Fig. 3(b). Note that the total weight of the cut becomes $1 + 1 + 2 + 1 = 5$. $\lhd$

The multiway cut problem nicely fits our general setting. Indeed, we can first define v_{mw} as the function such that $v_{mw}(C)$ is the weight of the edges in the subgraph of G induced over C—note that, in this case, we work on the original graph G. Then, we can consider the valuation structure $\sigma_{mw} = \langle G, T, \mathbf{1}, \mathbf{0}, 1, 0\rangle$ where terminals play the role of pivotal agents. Therefore, σ_{mw}-feasible coalition structures naturally correspond to multiway cuts (and vice-versa). Moreover, note that for each $C \subseteq N$, $v_{mw}(C) = val_{\sigma_{mw}}(C)$. That is, the affine transformation is immaterial, and similarly to the case of the multicut problem we get $val_{\sigma_{mw}}(v_{mw}, \Pi) = \sum_{e\in E} w_e - \sum_{e'\in E'} w_{e'}$, where E' is the cut and Π the corresponding coalition structure. Putting it all together, the multiway cut problem is reduced to the computation of a σ_{mw}-optimal coalition structure.

2.3. Application to k-clustering

Let us consider the framework for correlation clustering proposed by [6]. We have an undirected graph $G = (N, E)$ where each edge $e \in E$ has a label in $\ell_e \in \{+, -\}$. A cluster C is a non-empty subset of N such that the subgraph of G induced over C is connected.[7] A clustering is just a partition of N into disjoint and exhaustive clusters. For any cluster $C \subseteq N$, we denote by $E^+(C) = \{e \in E \mid e \subseteq C, \ell_e = +\}$ the set of its positive intra-cluster edges, and by $E^-(C) = \{e \in E \mid |e \cap C| = 1, \ell_e = -\}$ the set of negative inter-cluster edges with one end in C. Then, the correlation value of C is given by $v_{cc}(C) = |E^+(C)| + |E^-(C)|$, while the correlation value of a clustering Π is the sum of the correlation values of its clusters, i.e., the value $\sum_{C\in\Pi} v_{cc}(C)$. The goal is to find an optimal clustering, that is, a clustering having maximum correlation value.

Example 2.8. Consider the graph illustrated in Fig. 3(c), where each edge is associated with a label in $\{+, -\}$. It can be checked that an optimal correlation clustering is given by the set $\{C_1, C_2, C_3\}$. In particular, note that $v_{cc}(C_1) = 1 + 2$, $v_{cc}(C_2) = 4 + 3$, and $v_{cc}(C_3) = 4 + 1$, so that the overall value of the clustering is $3 + 6 + 5 = 14$. $\lhd$

Note that, in the basic correlation clustering framework, the number of clusters is not a-priori fixed. More recently, an extension of this framework has been studied, where the number of clusters is stipulated to be a small constant k [33]. For instance, in the case of Fig. 3(c), by fixing $k = 2$, we would look for a bipartition of the graph in two connected components maximizing the correlation value—interestingly, the problem is **NP**-hard even in this case [33]. Accordingly, we would single out the two components C_1 and $C_2 \cup C_3$, where $v_{cc}(C_2 \cup C_3) = 11$. In particular, the clustering $\{C_1, C_2, C_3\}$ is no longer admissible because it consists of 3 clusters.

In order to accommodate this k-clustering problem in our setting, we need not only to define suitable pivotal agents (as in the multiway cut problem), but also to use an affine transformation of the valuation function. Formally, for any set $R \subseteq N$ of nodes, we consider the valuation structure $\sigma_R = \langle G, R, \mathbf{1}, \mathbf{0}, 0, -\infty\rangle$. Intuitively, according to σ_R, it is always suboptimal to form a coalition that does not include an agent in R, so that any σ_R-feasible coalition structure Π such that $val_{\sigma_R}(v_{cc}, \Pi) \geq 0$ consists of exactly $|R|$ distinct coalitions (as R is the set of pivotal agents, there cannot be less than $|R|$ coalitions). Moreover, note that the set of possible coalitions is restricted to those that are feasible according to the graph G, as to avoid clusters formed by unconnected components. Therefore, Π is a clustering and $val_{\sigma_R}(v_{cc}, \Pi)$ is precisely its value. On the other hand, it is immediate to check that if Π is a clustering, then Π is also a σ_R-feasible coalition structure,

[6] For the sake of readability, we omit from this high-level discussion the formal statements and the proofs related to the correctness of the encodings. The formal treatment is in Appendix B.

[7] The connectivity condition is not necessary, but simplifies the presentation of the extension we shall next address.

for any set R including precisely one element for each cluster in Π. Hence, an optimal k-clustering can be computed as a coalition structure having the maximum σ_R-value over all σ_R-optimal coalition structures for any subset $R \subseteq N$ with $|R| = k$. In particular, since k is a given fixed constant [33], then polynomially many subsets R have to be considered.

2.4. Application to chromatic partitioning

In a number of graph-based applications, different weights might be associated to an edge in order to express the similarity between the linked nodes/objects w.r.t. different semantics relationships. The setting can be modeled by assuming that a set $\mathcal{C}$ of colors is given together with a graph $G = (N, E)$. Each edge $e \in E$ is associated with weights one-to-one corresponding to the colors. Accordingly, we denote by $w_e^c \in \mathbb{R}$ the weight of e under c. Then, we define a chromatic partition of G as a set $\{\langle C_1, c_1\rangle, \ldots, \langle C_h, c_h\rangle\}$ where $\{C_1, \ldots, C_h\}$ is a partition of N into disjoint and exhaustive coalitions and where $c_1, \ldots, c_h$ are h distinct colors taken from $\mathcal{C}$. The value of each element $\langle C_i, c_i\rangle$ is the sum of the weights under c_i of the edges covered by C, i.e., $\sum_{e\in E, e\subseteq C_i} w_e^{c_i}$, and the value of the partition is the sum of the values of its elements. The goal is to compute an optimal chromatic partition. Hence, the model is reminiscent of works on simultaneously labeling and partitioning (see, e.g., [12,20] and the references therein).

Example 2.9. Consider the graph illustrated in Fig. 3(d). We assume that two colors are available only, namely 'Red' and 'Blue'. Each edge e in the figure is associated with an expression of the form w_e^{Red}/w_e^{Blue} reporting the weight of e under Red/Blue. The figure also reports a chromatic partition $\{R, B\}$ whose overall value is 5. It is easy to see that this is indeed an optimal chromatic partition. $\lhd$

Note that for $\mathcal{C} = \{c_1\}$ chromatic partitioning is immaterial, as the only possible outcome is $\{\langle N, c_1\rangle\}$. For $\mathcal{C} = \{c_1, c_2\}$, the problem is intractable even when $w_e^{c_1} = w_e^{c_2}$, for each $e \in E$. Indeed, in this case, the problem reduces to computing a set $S \subseteq N$ maximizing the value $\sum_{e\in E, e\subseteq S} w_e^{c_1} + \sum_{e\in E, e\subseteq N\setminus S} w_e^{c_1}$, which is **NP**-hard (cf. [27]).

Let us now encode chromatic partitioning in our setting. Let $G_{\mathcal{C}}$ denote the graph $(N \cup \mathcal{C}, E \cup E_{\mathcal{C}})$, where colors are viewed as nodes and where $E_{\mathcal{C}}$ is obtained by including an edge between each node in N and each color in $\mathcal{C}$ (thus, $|E_{\mathcal{C}}| = |N| \times |\mathcal{C}|$). Intuitively, nodes in $\mathcal{C}$ will play the role of pivotal agents and we proceed along the line of the encoding for the k-clustering problem, with $k = |\mathcal{C}|$. Accordingly, we consider the valuation structure $\sigma_{ch} = \langle G_{\mathcal{C}}, \mathcal{C}, \mathbf{1}, \mathbf{0}, 0, -\infty\rangle$. In particular, by considering the interaction graph $G_{\mathcal{C}}$ and since each coalition including a node in $\mathcal{C}$ trivially satisfies the connectedness condition (due to the edges in $E_{\mathcal{C}}$), we are guaranteed that it will be always optimal to form σ_{ch}-feasible coalition structures consisting of k coalitions, each including one distinct of the available k colors. W.l.o.g, such coalitions have the form $\{C_1, \ldots, C_h, C_{h+1}, \ldots, C_k\}$ where $|C_i \cap \mathcal{C}| = 1$ holds, for each $i \in \{1, \ldots, h\}$ and where $C_j \subseteq \mathcal{C}$ holds, for each $j \in \{h+1, \ldots, k\}$. In fact, they one-to-one correspond to chromatic partitions $\{\langle C_1, C_1 \cap \mathcal{C}\rangle, \ldots, \langle C_h, C_h \cap \mathcal{C}\rangle\}$. Therefore, if v_{ch} is any valuation function such that, for each set of nodes $C \subseteq N \cup \mathcal{C}$ with $C \cap \mathcal{C} = \{c_i\}$, $v_{ch}(C) = \sum_{e\in E, e\subseteq C} w_e^{c_i}$ holds, then we can solve chromatic partitioning by computing a σ_{ch}-optimal coalition structure for $\langle N, v_{ch}\rangle$.

3. Coalition structure generation on valuation structures

An instance for the *coalition structure generation* problem on *valuation structures*, denoted in the following as CSG_{VAL}, is a pair (Γ, σ), where $\Gamma = (N, v)$ is a coalitional game, and $\sigma = \langle G, S, \alpha, \beta, x, y\rangle$ is a valuation structure on Γ. The goal is to compute a σ-optimal coalition structure $\Pi^* \in \text{CS-}opt_\sigma(\Gamma)$. The problem generalizes on the valuation structure σ the well-known coalition structure generation problem (short: CSG), which is just the special case where the trivial valuation structure $\perp$ is considered.

Based on the observations in Section 2.2, we already know that CSG_{VAL} has a number of different concrete applications. Here, we start its formal analysis by studying its computational complexity, i.e., we want to formally assess the amount of resources that are needed to compute a σ-optimal coalition structure. To carry out this analysis, we preliminary need to adopt a representation for the input. We proceed as follows:

- A trivial representation for Γ would just list all possible coalitions, with attached their associated valuations, hence requiring exponential space (w.r.t. $|N|$). However, more compact encodings can be obtained in some cases [49,45,27,30,43,17,14]. Just think, as an extreme case, that a valuation function v such that $v(C) = 0$, for each $C \subseteq N$, needs only constant space to be represented. Here, we do not commit ourselves to a specific representation strategy. Rather, by following [39], we assume that a *representation* $\mathcal{R}$ for coalitional games defines two functions $\xi^{\mathcal{R}}$ and $v^{\mathcal{R}}$, which are used to encode a class of games denoted by $\mathcal{C}(\mathcal{R})$. In particular, for each coalitional game $\Gamma \in \mathcal{C}(\mathcal{R})$, $\xi^{\mathcal{R}}(\Gamma)$ is the encoding of Γ, while $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C)$ is the value associated to the coalition C according to v. We require that this value can be computed in polynomial time w.r.t. $\|\Gamma\|$. Moreover, we assume as usual that all agents in N are listed in $\xi^{\mathcal{R}}(\Gamma)$, i.e., $\|\xi^{\mathcal{R}}(\Gamma)\| \geq |N|$ holds.
- The encoding of the valuation structure $\sigma = \langle G, S, \alpha, \beta, x, y\rangle$ is given by the encodings of its elements. We assume that α and β are represented by explicitly listing their entries, which in this case are polynomially many (in fact, we have $|S|$ entries), and we assume a standard encoding for the graph G in terms of an adjacency matrix, with size $\|G\|$. Eventually, the size of σ is denoted by $\|\sigma\|$.

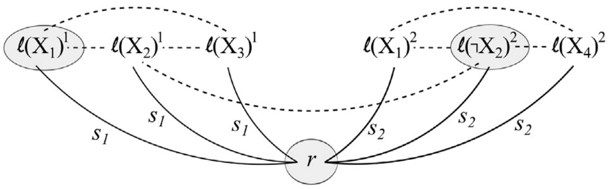

Fig. 4. Example reduction for the formula $\hat{\Phi}$ in the proof of Theorem 3.2—penalty edges are depicted with dashed lines.

3.1. Complexity in the general case

The complexity of the standard CSG problem has been already studied in the literature, and it has been shown to be **NP**-hard even when very specific kinds of encodings are considered, in particular, when coalitional games are given as *graph games* [69].

Here, we recall that according to the graph-game representation gg, the encoding $\xi^{\text{gg}}(\Gamma)$ of a coalitional game $\Gamma = \langle N, v\rangle \in \mathcal{C}(\text{gg})$ is given in terms of a graph (N, E) where each edge $e \in E$ is also equipped with a weight $w_e \in \mathbb{R}$. In particular, for each coalition $C \subseteq N$, the value $v(C) = v^{\text{gg}}(\xi^{\text{gg}}(\Gamma), C)$ is just given by the sum of the weights of the edges covered by the agents in C, i.e., $v(C) = \sum_{e \subseteq C, e \in E} w_e$. Note that most of the games we have considered so far are given in this representation, as in Example 2.2, Section 2.2.1, and Section 2.2.2. Moreover, observe that valuation functions for games in $\mathcal{C}(\text{gg})$ are independent on disconnected members w.r.t. the underlying weighted graphs on top of which they are defined (see, also, [69]).

The intractability of the CSG problem over graph games immediately leads to establish the intractability of our more general setting, where valuation structures can be possibly taken into account.

Theorem 3.1 *(cf. [27,69,4]).* CSG$_{\text{VAL}}$ *is* **NP**-*hard, even on the class* $\mathcal{C}(\text{gg})$.

Surprisingly, however, the precise complexity of the CSG problem has been not pointed out in earlier literature. In fact, our first contribution is to show that the problem is slightly more complex than just being **NP**-hard. Indeed, it emerges to be complete for the class F$\mathbf{\Delta}_2^{\mathbf{P}}$ consisting of all those *computation* problems that can be solved in polynomial time by possibly invoking, each time with a unitary cost, an oracle that solves a decision problem in the class **NP**. The reader is referred to, e.g., [54,48,44], for further background on these notions of complexity theory. Note that the membership in the class F$\mathbf{\Delta}_2^{\mathbf{P}}$ of the CSG problem has been recently shown in the literature [15]. Inspection in that proof reveals that it smoothly extends to the presence of valuation structures. Accordingly, we next focus on showing the F$\mathbf{\Delta}_2^{\mathbf{P}}$-hardness part only.

Theorem 3.2. CSG *(hence* CSG$_{\text{VAL}}$*) is* F$\mathbf{\Delta}_2^{\mathbf{P}}$-*complete. Hardness holds even on the class* $\mathcal{C}(\text{gg})$.

Proof. Let us show the hardness part. Let $\Phi = c_1 \wedge \ldots \wedge c_m$ be a Boolean formula in conjunctive normal form over the variables $X_1, \ldots, X_n$. That is, for each $i \in \{1, \ldots, m\}$, c_i is a disjunction of literals, where each literal is either a variable X_j or its negation $\neg X_j$. Let $s_1, \ldots, s_m$ be a list of positive weights associated with the corresponding clauses of Φ. For any (possibly partial) truth assignment τ, its *weight* is defined as the sum of the weights associated with the clauses satisfied by τ. Computing the maximum possible weight associated with any assignment is F$\mathbf{\Delta}_2^{\mathbf{P}}$-hard [48].

Based on Φ and $s_1, \ldots, s_m$, we build an encoding $\xi^{\text{gg}}(\Gamma)$ of a coalitional game $\Gamma = \langle N, v\rangle \in \mathcal{C}(\text{gg})$. In particular, $\xi^{\text{gg}}(v)$ is the graph (N, E) equipped with weights w_e, for each $e \in E$, such that:

- For each clause c_i and for each literal L in c_i, N includes a *literal node/agent* $\ell(L)^i$. Moreover, N includes the agent r, and no further agent is in N.
- For each pair $i, i' \in \{1, \ldots, m\}$ with $i \neq i'$, and for each variable X_j occurring in c_i such that $\neg X_j$ occurs in $c_{i'}$, the edge $\{\ell(X_j)^i, \ell(\neg X_j)^{i'}\}$ is in E. Moreover, for each clause c_i, with $i \in \{1, \ldots, m\}$, and for each pair of distinct literals L and L' in c_i, E includes the edge $\{\ell(L)^i, \ell(L')^i\}$. Finally, E includes the edge $\{r, \ell(L)^i\}$, for each literal L in any clause c_i, and no further edge is in E.
- Weights are such that $w_{\{r,\ell(L)^i\}} = s_i$, for each literal L occurring in the clause c_i. Moreover, for each edge $\{p, q\} \in E$ with $r \notin \{p, q\}$, hereinafter called *penalty* edge, we have that $w_{\{p,q\}} = -(\sum_{i=1}^{n} s_i + 1) \times (|E| + 1)$.

As an example, Fig. 4 reports the graph we can build for the formula $\hat{\Phi} = \hat{c}_1 \wedge \hat{c}_2$ over the variables $X_1, \ldots, X_4$, where $\hat{c}_1 = (X_1 \vee X_2 \vee X_3)$ and $\hat{c}_2 = (X_1 \vee \neg X_2 \vee X_4)$ are the clauses whose associated weights are s_1 and s_2, respectively.

Note that $v(S) = 0$ holds, for each singleton coalition $S \subseteq N$, i.e., with $|S| = 1$. Moreover, for an arbitrary coalition $S \subseteq N$, $v(S) > 0$ holds if, and only if, S does not cover any penalty edge while covering at least one non-penalty edge. By construction, we hence conclude that $v(S) > 0$ if, and only if, the following three conditions are satisfied:

(C1) $r \in S$, because at least one edge that is not a penalty one must be covered;

(C2) for each clause c_i, with $i \in \{1, \dots, m\}$, there is at most one literal L in c_i such that $\ell(L)^i \in S$, because any two such literals define agents that are connected with a penalty edge;
(C3) there is no variable X_j, with $j \in \{1, \dots, n\}$, and pair of indices $i, i' \in \{1, \dots, m\}$, such that $\ell(X_j)^i$ and $\ell(\neg X_j)^{i'}$ are in S, again because these literals define agents connected with a penalty edge.

Note that if Π^* is an optimal coalition structure in CS-$opt_\perp(\Gamma)$, then we have that $val_\perp(\Pi^*) \geq 0$. Now, let $\Delta \geq 0$ be any real number. Then, we claim: *for each* $\perp$*-feasible coalition structure* $\Pi \in \mathcal{CS}_\perp(\Gamma)$, $val_\perp(\Pi) \leq \Delta \Leftrightarrow$ *for each* $S \subseteq N$, $v(S) \leq \Delta$.

($\Rightarrow$) Assume that for each $\Pi \in \mathcal{CS}_\perp(\Gamma)$, $val_\perp(\Pi) \leq \Delta$. By contradiction, let $S^* \subseteq N$ be a coalition with $v(S^*) > \Delta$. Consider then the $\perp$-feasible coalition structure Π^* including S^* and a singleton coalition for each agent not in S^*. Then, $val_\perp(\Pi^*) = v(S^*) > \Delta$, which is impossible.
($\Leftarrow$) Assume that for each $S \subseteq N$, $v(S) \leq \Delta$. By condition (C1), for any two coalitions S_1 and S_2, $v(S_1) > 0$ and $v(S_2) > 0$ hold if, and only if, $S_1 \cap S_2 \supseteq \{r\}$. Therefore, if Π^* is a coalition structure in $\mathcal{CS}_\perp(\Gamma)$, then $val_\perp(\Pi^*) \leq \max_{S \subseteq N} v(S) \leq \Delta$ holds.

Because of the above result, the value of an optimal coalition structure precisely coincides with the maximum value associated to any coalition. That is,

$$\max_{\Pi \in \mathcal{CS}_\perp(\Gamma)} val_\perp(\Pi) = \max_{S \subseteq N} v(S). \tag{3}$$

To any coalition S with $v(S) > 0$, we now associate a (possibly partial) truth assignment τ_S such that X_j evaluates to true (resp., false) in τ_S if $\ell(X_j)^i$ (resp., $\ell(\neg X_j)^i$) occurs in S, for some $i \in \{1, \dots, m\}$. Because of the condition (C3) above, τ_S is well-defined. Moreover, by conditions (C1) and (C2), we have that $v(S)$ coincides with the weight associated to the assignment τ_S.

As an example, by considering the coalition $S = \{\ell(X_1^1), \ell(\neg X_2^2), r\}$ in the setting of Fig. 4 for the formula $\hat{\Phi}$, we get that the corresponding assignment τ_S is undefined on X_3 and X_4, while X_1 evaluates to true and X_2 evaluates false in it. Note that S is in fact a coalition getting the maximum overall value, and τ_S is a truth assignment having maximum weight (in fact, it is a satisfying assignment). This is not by chance. Indeed, the following result can be established: *S is such that $v(S) \geq v(\bar{S})$, for each $\bar{S} \subseteq N \Leftrightarrow \tau_S$ is a truth assignment having maximum weight.*

($\Rightarrow$) Assume, by contradiction, that τ' is a truth assignment with better weight than τ_S. Define S' as the coalition including r, and where, for each clause c_i satisfied by τ', there is precisely one agent $\ell(L)^i$ such that L is a literal in c_i evaluating to true in τ'. Note that the weight of $\tau_{S'}$ coincides with the weight of τ' itself. Thus, $v(S')$ coincides with the weight of τ', and we have $v(S') > v(S)$, which is impossible.
($\Leftarrow$) By contradiction, if S' is a coalition such that $v(S') > v(S)$, then we would have that $\tau_{S'}$ has a better weight than τ_S. Again, this is impossible.

The above result together with Equation (3) entails that the value of an optimal coalition structure $\Pi^* \in$ CS-$opt_\perp(\Gamma)$ coincides with the maximum possible weight associated with any assignment. The F$\mathbf{\Delta}_2^{\boldsymbol{P}}$-hardness is eventually established because the reduction is feasible in polynomial time. □

3.2. Structural restrictions: basic results

In the light of the above intractability result, it is sensible to single out classes of functions and valuation structures over which CSG$_{\text{VAL}}$ and CSG can be efficiently solved. A natural approach to identify such classes is to focus on interaction graphs enjoying suitable structural properties. In fact, a basic structural property of a graph is *acyclicity*, and we next consider the more general concept of bounded treewidth [61].

A *tree decomposition* of a graph $G = (N, E)$ is a pair $\langle T, \chi \rangle$, where $T = (V, F)$ is a tree, and χ is a labeling function assigning to each vertex $p \in V$ a set of vertices $\chi(p) \subseteq N$, such that the following conditions are satisfied:

(1) for each node b of G, there exists $p \in V$ such that $b \in \chi(p)$;
(2) for each edge $\{b, d\} \in E$, there exists $p \in V$ such that $\{b, d\} \subseteq \chi(p)$; and,
(3) for each node b of G, the set $\{p \in V \mid b \in \chi(p)\}$ induces a connected subtree of T.

The *width* of $\langle T, \chi \rangle$ is the number $\max_{p \in V}(|\chi(p)| - 1)$. The *treewidth* of G, denoted by $tw(G)$, is the minimum width over all its tree decompositions. A graph G is acyclic if and only if $tw(G) = 1$. Deciding if a given graph has treewidth bounded by a fixed natural number k is known to be feasible in linear time [11].

Example 3.3. Consider the graph $\dot{G} = (\dot{N}, \dot{E})$ shown in Fig. 5(a), and note that it contains a cycle over the nodes/agents in $\{a_1, a_2, a_3\}$. In Fig. 5(b), a tree decomposition of $\dot{G}$ is reported whose width is 2. ◁

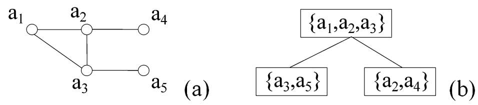

Fig. 5. Illustration of Example 3.3: An interaction graph and a tree decomposition having width 2.

By focusing on acyclic and, more generally, on nearly-acyclic interaction graphs we can significantly constraint the coalitions that are allowed to form. However, this is not yet sufficient to end up with an island of tractability.

Theorem 3.4. *CSG*$_{\text{VAL}}$ *is* F$\mathbf{\Delta}_2^{\mathbf{P}}$*-hard, even on structures* $\langle G\rangle$ *such that* G *is acyclic.*

Proof. Let $\Gamma = \langle N, v\rangle$ be a coalitional game such that $v(S) = 0$ holds, for each $S \subseteq N$ with $|S| = 1$. Based on Γ, we build a coalitional game $\Gamma' = \langle N \cup \{a_{|N|+1}\}, v'\rangle$ where $a_{|N|+1}$ is a fresh agent not in N and where v' is the valuation function such that $v'(S') = v(S' \setminus \{a_{|N|+1}\})$, for each $S' \subseteq N \cup \{a_{|N|+1}\}$, and $v'(\{a_{|N|+1}\}) = 0$. Moreover, we build the interaction graph $G' = (N \cup \{a_{|N|+1}\}, E')$ where $E' = \{\{a_{|N|+1}, a_i\} \mid a_i \in N\}$.

Note that $\mathcal{F}_{\langle G'\rangle}(\Gamma')$ consists of all the singleton coalitions plus all coalitions having the form $S \cup \{a_{|N|+1}\}$, for each $S \subseteq N$. Moreover, any coalition structure Π' in $\mathcal{CS}_{\langle G'\rangle}(\Gamma')$ has the form $\{C'_+\} \cup \Pi'_-$, where C'_+ is the coalition in Π' including $a_{|N|+1}$, and where Π'_- is a set of singleton coalitions. Therefore, it holds that $val_{\langle G'\rangle}(\Pi') = val_{\langle G'\rangle}(C'_+) = v'(C'_+) = v(C'_+ \setminus \{a_{|N|+1}\})$. In particular, by the properties of the valuation function v and by the construction of v', the following holds:

$$\max_{\Pi' \in \mathcal{CS}_{\langle G'\rangle}(\Gamma')} val_{\langle G'\rangle}(\Pi') = \max_{S \subseteq N} v(S). \tag{4}$$

In words, the computation of the value $\max_{S \subseteq N} v(S)$ is reduced to the computation of the value associated with $\langle G'\rangle$-optimal coalition structures in Γ'. Now, recall the proof of Theorem 3.2, by observing that the game built there based on the Boolean formula Φ precisely satisfies the conditions for our game Γ. Then, by Equation (4), the maximum value attained by any coalition in the game associated with Φ can be computed by solving CSG$_{\text{VAL}}$ on input $(\Gamma', \langle G'\rangle)$. In fact, we already know that computing such maximum value is F$\mathbf{\Delta}_2^{\mathbf{P}}$-hard (cf. Equation (3) in the proof of Theorem 3.2), so that the F$\mathbf{\Delta}_2^{\mathbf{P}}$-hardness of CSG$_{\text{VAL}}$ immediately follows. □

Intuitively, the hardness result can emerge because, over an acyclic interaction graph, we might still construct a "hard" valuation function where interactions are not properly taken into account. Indeed, we have already noticed that functions independent of disconnected members have been introduced in the literature precisely as classes of functions that adhere to the semantics of interaction graphs. However, it can be checked that the valuation function in the above proof does not satisfy this property. In fact, by applying the notion of treewidth over IDM valuation functions, the following tractability result—suitably restated within our setting and notation—was shown by [69] for the basic CSG problem (i.e., without taking into account valuation structures).

Theorem 3.5 *(cf. [69]). Let $h \geq 0$ be a fixed natural number. Let $\mathcal{R}$ be any representation for coalitional games, and let $\Gamma = \langle N, v\rangle \in \mathcal{C}(\mathcal{R})$ be a game such that v is a valuation function independent of disconnected members w.r.t. a graph G with $tw(G) \leq h$. Then,* CSG *can be solved on Γ in polynomial times (w.r.t. $\|\xi^{\mathcal{R}}(\Gamma)\|$ and $\|G\|$).*

Note that the result trivializes whenever the size $\|\xi^{\mathcal{R}}(\Gamma)\|$ of the encoding for the valuation function is exponential w.r.t. the number of the agents in N. Instead, interesting cases emerge for "succinct" encodings, such as for the graph game representation (for which tractability has been also independently derived by [4]). Moreover, note that by combining Theorem 3.5 with Theorem 2.5, we can obtain the following tractability result for valuation structures based on interaction graphs only.

Corollary 3.6. *Let $h \geq 0$ be a fixed natural number. Let $\mathcal{R}$ be any representation for coalitional games, let $\Gamma = \langle N, v\rangle \in \mathcal{C}(\mathcal{R})$ be a game such that v is a valuation function independent of disconnected members w.r.t. a graph G with $tw(G) \leq h$. Then,* CSG$_{\text{VAL}}$ *can be solved on $(\Gamma, \langle G\rangle)$ in polynomial time (w.r.t. $\|\xi^{\mathcal{R}}(\Gamma)\|$ and $\|G\|$).*

Proof. By Theorem 3.5, we can compute in polynomial time a solution Π^* to CSG on input Γ, i.e., a solution to CSG$_{\text{VAL}}$ on $(\Gamma, \langle K_N\rangle)$. Given Π^*, based on Theorem 2.5.*(1)*, a solution to CSG$_{\text{VAL}}$ on $(\Gamma, \langle G\rangle)$ can be computed in polynomial time, too. □

4. Islands of tractability for CSG$_{\text{VAL}}$

Our main technical achievement in the paper is to generalize Corollary 3.6 to arbitrary valuation structures (but still with bounded treewidth interaction graphs). The proof of the result is rather involved and it is based on two technical ingredients that are interesting in their own right:

(1) First, we show that, over bounded treewidth graphs, any IDM valuation function admits a very simple kind of succinct representation, in terms of the well-known encoding based on *marginal contribution networks* [43]. Moreover, no matter of the representation scheme originally adopted for the valuation function, an equivalent marginal contribution network can be computed in polynomial time.
(2) Second, we exhibit a polynomial-time algorithm solving $\mathrm{CSG}_{\mathrm{VAL}}$ over marginal contribution networks whose underlying interaction graphs have bounded treewidth. The algorithm is defined by means of a non-trivial encoding in terms of a *constraint satisfaction problem*, and by exploiting known structural tractability results in this latter setting.

The two ingredients are elaborated in Section 4.1 and Section 4.2, respectively.

4.1. Marginal contribution networks and IDM *functions*

A representation for coalitional games that received considerable attention in the last few years is based on *marginal contribution networks* [43].

A marginal contribution network (short: MC-net) M consists in a set of rules involving a number of Boolean variables that represent the agents. Each rule has the form $\{pattern\} \to value$, where *pattern* is a conjunction that may include both positive and negative literals, and *value* is the additive contribution associated with this pattern. A rule *applies* to a set C of agents if all the agents whose literals occur positively in the pattern belong to C, and all the players whose literals occur negatively in the pattern do not belong to C. In the following, we denote by mcn the representation for coalitional games such that for each $\Gamma = \langle N, v\rangle \in \mathcal{C}(\mathsf{mcn})$, $\xi^{\mathsf{mcn}}(\Gamma)$ is a marginal contribution network[8] and where, for each coalition C, $v^{\mathsf{mcn}}(\xi^{\mathsf{mcn}}(\Gamma), C)$ is given by the sum of the values of all rules that apply to C. If no rule applies, then the value for the coalition is set to zero, by default.

Example 4.1. Consider a coalitional game $\dot\Gamma = \langle \dot N, \dot v\rangle$, where $\dot N = \{a_1, \ldots, a_5\}$ and where $\dot v$ is such that: $\dot v(\{a_i\}) = 0$, for each $i \in \{1, \ldots, 5\}$; $\dot v(\{a_1, a_2\}) = \dot v(\{a_2, a_3\}) = \dot v(\{a_1, a_3\}) = 2$; $\dot v(\{a_1, a_2, a_3\}) = 5$; and $\dot v(C \cup \{a_4\}) = \dot v(C \cup \{a_5\}) = \dot v(C \cup \{a_4, a_5\}) = \dot v(C)$, for each $C \subseteq \{a_1, a_2, a_3\}$. An encoding for this game in terms of a marginal contribution network is given by the following set of rules:

$$\begin{array}{rcl}
\{a_1 \wedge a_2 \wedge \neg a_3\} & \to & 2 \\
\{\neg a_1 \wedge a_2 \wedge a_3\} & \to & 2 \\
\{a_1 \wedge \neg a_2 \wedge a_3\} & \to & 2 \\
\{a_1 \wedge a_2 \wedge a_3\} & \to & 5 \\
\{a_3 \wedge a_5\} & \to & 0 \\
\{a_2 \wedge a_4\} & \to & 0
\end{array}$$

For instance, $\dot v(\{a_1, a_2, a_4\}) = 2$ derives as the first rule and the last rule apply. $\lhd$

In order to model the interactions among the agents in marginal contribution network M, we define a graph $\mathrm{AG}(M)$, called the *agent graph* of M [43], whose nodes are the agents of the game, and where, for each rule $\{pattern\} \to value$ occurring in M, every pair of agents (nodes) occurring in *pattern* are connected by an edge in $\mathrm{AG}(M)$.

For instance, it can be checked that the agent graph associated with the marginal contribution network of Example 4.1 is the one illustrated in Fig. 5(a).

Our first result is that any function encoded via a marginal contribution network is independent of disconnected members w.r.t. its associated agent graph.

Theorem 4.2. *Let $\Gamma = \langle N, v\rangle$ be a game in $\mathcal{C}(\mathsf{mcn})$. Then, v is independent of disconnected members w.r.t. $\mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$.*

Proof. Let a_i and a_j be two agents in N that are not connected in the graph $\mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$ and let C be a coalition with $a_i, a_j \notin C$. We have to show that $v(C \cup \{a_i\}) - v(C) = v(C \cup \{a_i, a_j\}) - v(C \cup \{a_j\})$.

Let R_i be the set of all rules that apply to $C \cup \{a_i\}$ and that do not apply to C. Note that the agent a_i must positively occur in each rule in R_i. Moreover, note that $v(C \cup \{a_i\}) - v(C)$ precisely coincides with the sum of the values associated with the rules in R_i. Then, let $R_{i,j}$ be the set of all rules that apply to $C \cup \{a_i, a_j\}$ and that do not apply to $C \cup \{a_j\}$, by noticing similarly that $v(C \cup \{a_i, a_j\}) - v(C \cup \{a_j\})$ coincides with the sum of the values for them. Moreover, the agent a_i must positively occur in any rule taken from $R_{i,j}$. Based on the above properties, we can now show that $R_i = R_{i,j}$. Indeed, consider a rule r where agent a_i positively occurs. Since a_i and a_j are not connected in $\mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$, we conclude that r does not contain a_j. Hence, r is in R_i if, and only if, r is in $R_{i,j}$. $\square$

[8] We consider a standard encoding for a marginal contribution network, where all its rules are explicitly listed. Moreover, in order to guarantee that $\|\xi^{\mathsf{mcn}}(\Gamma)\| \geq |N|$ holds, we assume w.l.o.g. that each agent occurs at least in one rule. Indeed, for each agent $a_i \in N$, we can add the rule $\{a_i\} \to 0$ without altering any of the properties of the network.

We now show that the converse of the above result holds, too. Indeed, we show that, when considering IDM functions, we can focus w.l.o.g. on games encoded via marginal contribution networks. In particular, note that the marginal contribution network can be built in polynomial time whenever classes of bounded treewidth graphs are considered.

Theorem 4.3. *Let $\mathcal{R}$ be any representation for coalitional games, and let $\Gamma = \langle N, v\rangle \in \mathcal{C}(\mathcal{R})$ be a game such that v is a valuation function independent of disconnected members w.r.t. a graph G with $tw(G) = h$. Then, a marginal contribution network M (without negative literals) can be built in time polynomial w.r.t. $\|\xi^{\mathcal{R}}(\Gamma)\|$ and exponential w.r.t. h, such that $\mathrm{AG}(M) = G$ and $v^{\mathsf{mcn}}(\xi^{\mathcal{R}}(\Gamma), C) = v^{\mathsf{mcn}}(M, C)$, for each $C \subseteq N$.*

Proof. We have to build an encoding for the game $\Gamma = \langle N, v\rangle$ in terms of a marginal contribution network M. Let $C \subseteq N$ be a coalition, and let G_C denote the subgraph of G induced over the nodes in C. Throughout the proof, we shall say that the coalition C is *ℓ-sparse* if the number of nodes in C having less than $|C| - 1$ adjacent nodes in G_C is ℓ. Note that if C is 0-sparse, then G_C is actually a clique over the nodes in C.

The construction focuses precisely on 0-sparse coalitions (while arbitrary ℓ-sparse coalitions will play a role in the proof). Indeed, for each 0-sparse coalition $C \subseteq N$, we include in M a rule $r_C : \{pattern_C\} \to value_C$, such that $pattern_C$ is the conjunction containing all the agents in C and where: $value_C = v(C) - \sum_{C' \subset C} v(C')$. No further rule is in M. An example construction is illustrated below.

Example 4.4. Consider again the coalitional game $\dot{\Gamma}$ discussed in Example 4.1 and the interaction graph depicted in Fig. 5(a). The 0-sparse coalitions in this setting are the following ones: $\{a_1\}, \ldots, \{a_5\}$, $\{a_1, a_2\}$, $\{a_2, a_3\}$, $\{a_1, a_3\}$, $\{a_2, a_4\}$, $\{a_3, a_5\}$, and $\{a_1, a_2, a_3\}$. Then, the marginal contribution network consists of the following rules:

$$\begin{array}{rcl}
\{a_i\} & \to & \dot{v}(\{a_i\}) = 0, \forall i \in \{1, \ldots, 5\} \\
\{a_1 \wedge a_2\} & \to & \dot{v}(\{a_1, a_2\}) - \dot{v}(\{a_1\}) - \dot{v}(\{a_2\}) = 2 - 0 - 0 = 2 \\
\{a_2 \wedge a_3\} & \to & \dot{v}(\{a_2, a_3\}) - \dot{v}(\{a_2\}) - \dot{v}(\{a_3\}) = 2 - 0 - 0 = 2 \\
\{a_1 \wedge a_3\} & \to & \dot{v}(\{a_1, a_3\}) - \dot{v}(\{a_1\}) - \dot{v}(\{a_3\}) = 2 - 0 - 0 = 2 \\
\{a_2 \wedge a_4\} & \to & \dot{v}(\{a_2, a_4\}) - \dot{v}(\{a_2\}) - \dot{v}(\{a_4\}) = 0 - 0 - 0 = 0 \\
\{a_3 \wedge a_5\} & \to & \dot{v}(\{a_3, a_5\}) - \dot{v}(\{a_3\}) - \dot{v}(\{a_5\}) = 0 - 0 - 0 = 0 \\
\{a_1 \wedge a_2 \wedge a_3\} & \to & \dot{v}(\{a_1, a_2, a_3\}) - \sum_{C' \subset \{a_1, a_2, a_3\}} \dot{v}(C') = 5 - 6 = -1
\end{array}$$

Note that the encoding differs from the one discussed in Example 4.1. In particular, now there is no rule containing negated agents. $\lhd$

We start the proof by pointing out the following two properties of the construction:

- Assume that $tw(G) = h$. If C is a 0-sparse coalition, then $C \leq h + 1$ necessarily holds, by well-known results on tree decompositions. This means that $|\{C' | C' \subseteq C\}| \leq 2^{h+1}$. Therefore, $value_C$ can be built in polynomial time w.r.t. $\|\xi^{\mathcal{R}}(\Gamma)\|$ (and exponential w.r.t. h). Moreover, observe that the number of 0-sparse coalitions (hence cliques with $h + 1$ nodes at most in G) is polynomial w.r.t. the number of the nodes of the graph, and thus w.r.t. $\|\xi^{\mathcal{R}}(\Gamma)\|$ (again, still exponential w.r.t. h). Hence, the overall number of rules in M is still polynomial, and we conclude that M can be actually built in polynomial time.
- Each edge e in the graph G is actually a 0-sparse coalition. Therefore, there is a rule r_e in M and the two agents in e are connected in $\mathrm{AG}(M)$, too. On the other hand, if r_C is a rule in M (so that the agents in C form a clique in the graph $\mathrm{AG}(M)$), then we have that C is a 0-sparse coalition and hence all the agents in C already form a clique in G. We can thus conclude that $\mathrm{AG}(M) = G$ holds.

After these properties, it remains to be proven that the encoding is correct, that is, $v^{\mathsf{mcn}}(M, C) = v(C)$, for each $C \subseteq N$. The result will be proven by structural induction on the "sparseness" of the given coalition C.

Base Case: In the base case, we have to show that the encoding is correct, for each 0-sparse coalition $C \subseteq N$. In fact, even this case is not immediate, and it is proven by another structural induction, this time on the size of the coalition C.

Base Case: Assume that C is a 0-sparse coalition with $|C| = 1$. Then, only the rule r_C can apply to C, and hence we get by construction that $v^{\mathsf{mcn}}(M, C) = value_C = v(C)$.

Induction Step: Let $p > 0$ be a natural number, and assume that for each 0-sparse coalition $C' \subseteq N$ with $|C'| \leq p$, $v^{\mathsf{mcn}}(M, C') = v(C')$ holds. Let C be a 0-sparse coalition with $|C| = p + 1$. By the definition of the valuation function associated with a marginal contribution network, we get $v^{\mathsf{mcn}}(M, C) = \sum_{C' \subseteq C} value_{C'} = value_C + \sum_{C' \subset C} value_{C'}$, because all rules that apply to any subset $C' \subset C$ apply to C, too (recall that there is no negated agent in these rules). Note also that, for each $C' \subset C$, we have that C' is 0-sparse and $|C'| \leq p$. Therefore, we can apply the inductive hypothesis in order to conclude that $v^{\mathsf{mcn}}(M, C) = value_C + \sum_{C' \subset C} v(C')$. Eventually, recall that $value_C = v(C) - \sum_{C' \subset C} v(C')$. So, $value_C + \sum_{C' \subset C} v(C') = v(C)$ and we get that $v^{\mathsf{mcn}}(M, C) = v(C)$.

Induction Step: Assume now that for each ℓ-sparse coalition $C' \subseteq N$, $v^{\mathsf{mcn}}(M, C') = v(C')$. We will show that, for each $(\ell+1)$ sparse coalition $C \subseteq N$, $v^{\mathsf{mcn}}(M, C) = v(C)$. Again, we used a nested induction on the size of C.

Base Case: Assume that C is a $(\ell+1)$-sparse coalition with $|C| = \ell+1$, with $\ell \geq 0$. Then, there are two agents/nodes a_1 and a_2 in C that are not connected by means of an edge in G. Consider the coalitions $C_1 = C \setminus \{a_1\}$, $C_2 = C \setminus \{a_2\}$, and $C_3 = C_1 \cap C_2$. It is immediately seen that $|C_1| = \ell$, $|C_2| = \ell$, $|C_3| \leq \ell$, and that C_i, for each $i \in \{1, 2, 3\}$, is (trivially) s_i-sparse for some $s_i \leq \ell$. Hence, $v^{\mathsf{mcn}}(M, C_i) = v(C_i)$, for each $i \in \{1, 2, 3\}$, holds by the inductive hypothesis, and we derive:

$$
\begin{aligned}
v^{\mathsf{mcn}}(M, C) &= \textstyle\sum_{C' \subseteq C_1} value_{C'} + \sum_{C' \subseteq C_2} value_{C'} - \sum_{C' \subseteq C_3} value_{C'} \\
&= v^{\mathsf{mcn}}(M, C_1) + v^{\mathsf{mcn}}(M, C_2) - v^{\mathsf{mcn}}(M, C_3) \\
&= v(C_1) + v(C_2) - v(C_3) \\
&= v(\{a_1\} \cup C_3) + v(\{a_2\} \cup C_3) - v(C_3).
\end{aligned}
$$

Since $C = \{a_1, a_2\} \cup C_3$ and since v is independent of disconnected members w.r.t. G, we conclude, by Equation (1) in Section 2.1, that $v(\{a_1\} \cup C_3) + v(\{a_2\} \cup C_3) - v(C_3) = v(C)$. Hence, $v^{\mathsf{mcn}}(M, C) = v(C)$.

Induction Step: Let $p \geq \ell+1$ be a natural number, and assume that for each $(\ell+1)$-sparse coalition $C'' \subseteq N$, with $|C''| \leq p$, $v^{\mathsf{mcn}}(M, C'') = v(C'')$ holds. Let C be a $(\ell+1)$-sparse coalition, with $|C| = p+1$, and let $\{a_1, \ldots, a_{\ell+1}\}$ be the set of the $\ell+1$ nodes in C having less than $|C|-1$ adjacent nodes in G_C. Consider the coalitions: C_1 consisting of a_1 plus all the nodes that are adjacent to it in G_C; $C_2 = C \setminus \{a_1\}$, and $C_3 = C_1 \cap C_2$. Note that $C_1 \setminus C_2 = \{a_1\}$ holds. Moreover, for each $i \in \{1, 2, 3\}$, $|C_i| \leq p$ and C_i is (trivially) a s_i-sparse coalition for some $s_i \leq p$. Hence, the inductive hypothesis can be applied on C_1, C_2, and C_3, by deriving that $v^{\mathsf{mcn}}(M, C_i) = v(C_i)$, for each $i \in \{1, 2, 3\}$. Eventually, we derive:

$$
\begin{aligned}
v^{\mathsf{mcn}}(M, C) &= \textstyle\sum_{C' \subseteq C_1} value_{C'} + \sum_{C' \subseteq C_2} value_{Sy'} - \sum_{C' \subseteq C_3} value_{C'} \\
&= v^{\mathsf{mcn}}(M, C_1) + v^{\mathsf{mcn}}(M, C_2) - v^{\mathsf{mcn}}(M, C_1 \cap C_2) \\
&= v(C_1) + v(C_2) - v(C_3).
\end{aligned}
$$

Now, we observe that there is no edge connecting a_1 (the only element of $C_1 \setminus C_2$) with a node in $C_2 \setminus C_1$. Therefore, we can apply Equation (2) in Section 2.1 and conclude that $v(C_1) + v(C_2) - v(C_1 \cap C_2) = v(C_1 \cup C_2)$, where $C_1 \cap C_2 = C_3$ and $C_1 \cup C_2 = C$. It follows that $v^{\mathsf{mcn}}(M, C) = v(C)$. $\square$

For any fixed constant $h > 0$, we shall denote by $\mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$ the restriction of the class $\mathcal{C}(\mathsf{mcn})$ to all those marginal contribution networks whose associated agent graphs have treewidth h at most. According to the above results, $\mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$ is representative of *all* coalitional games based on IDM valuation functions (w.r.t. interactions graphs having treewidth bounded by h). Moreover, observe that we can further assume that $\mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$ contains only networks with rules that do not involve negative literals. Indeed, the following is easily established.

Corollary 4.5. *Let $\Gamma = \langle N, v\rangle \in \mathcal{C}(\mathsf{mcn})$ be a game such that $tw(\mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))) \leq h$. Then, a marginal contribution network M containing no rule with a negated literal can be built in time polynomial w.r.t. $\|\xi^{\mathsf{mcn}}(\Gamma)\|$ and exponential w.r.t. h, such that $\mathrm{AG}(M) = \mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$ and $v^{\mathsf{mcn}}(M, C) = v^{\mathsf{mcn}}(\xi^{\mathsf{mcn}}(\Gamma), C)$, for each $C \subseteq N$.*

Proof. By Theorem 4.2, v is independent of disconnected members w.r.t. $\mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$. So, the result follows from Theorem 4.3 applied on Γ and the graph $\mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$. $\square$

4.2. Coalition structure generation over marginal contribution networks

The second technical ingredient we exhibit is a method to efficiently solve $\mathrm{CSG}_{\mathrm{VAL}}$ over the class $\mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$. The analysis is carried out by encoding $\mathrm{CSG}_{\mathrm{VAL}}$ in terms of a (weighted) constraint satisfaction problem (short: CSP), and by showing that CSP instances associated with games in $\mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$ can be solved in polynomial time.

Before embarking on this analysis, it is useful to recall, e.g., from [36] that the CSP-encoding approach is a powerful method that has already been exploited to show tractability results in a number of different areas of research. However, the issues that arise in the context of coalition structure generation under valuation structures have been not considered in earlier approaches, and the specific encoding algorithm is substantially more involved than those already available in the literature. In particular, a distinguishing feature of our proposal is the ability of handling "connectivity constraints" (induced by the underlying interactions graphs). This feature might turn out to be useful in different domains whenever a graph structure is given and solutions are required to "induce", according to some specific semantics, some connected substructure.

4.2.1. Constraint satisfaction problems

We start with some preliminaries on *constraint satisfaction*. The reader interested in expanding on this formalism is referred to [23].

A constraint satisfaction problem instance is a triple $\mathcal{I} = \langle Var, U, \mathbf{C}\rangle$, where Var is a finite set of variables, U is a finite domain of values, and $\mathbf{C} = \{C_1, C_2, \dots, C_q\}$ is a finite set of constraints. Each constraint C_v, for $1 \le v \le q$, is a pair (S_v, r_v), where $S_v \subseteq Var$ is a set of variables called the *constraint scope*, and r_v is a set of substitutions from variables in S_v to values in U indicating the allowed combinations of simultaneous values for the variables in S_v, called *tuples*. A substitution from a set of variables $V \subseteq Var$ to U is extensively denoted as the set of pairs of the form X/u, where $u \in U$ is the value to which $X \in V$ is mapped. A substitution θ *satisfies* a constraint C_v if its restriction to S_v, i.e., the set of all pairs $X/u \in \theta$ such that $X \in S_v$, occurs as a tuple in r_v. A *solution* to $\mathcal{I}$ is a substitution $\theta : Var \mapsto U$ for which q tuples $t_1 \in r_1, \dots, t_q \in r_q$ exist such that $\theta = t_1 \cup \dots \cup t_q$. Thus, a solution satisfies all the constraints in $\mathcal{I}$.

Encoding. Let $\Gamma = \langle N, v\rangle \in \mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$, with $N = \{a_1, \dots, a_n\}$, be a coalitional game and let $\sigma = \langle G, S, \alpha, \beta, x, y\rangle$ be an associated valuation structure where $G = \mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$. That is, interactions are constrained by the underlying agent graph (cf. Theorem 4.3). The CSP instance associated with Γ and σ, denoted by $CSP(\Gamma, \sigma) = \langle Var, U, \mathbf{C}\rangle$, is defined as follows:

- Variables are transparently viewed as the agents in N, that is, $Var = N$.
- The domain U contains an element of the form $\langle from_j, origin_\ell, level_h\rangle$ for each triple of indices $j, \ell, h \in \{1, \dots, n\}$. Moreover, U contains an element of the form $\langle origin_\ell, level_0\rangle$ for each index $\ell \in \{1, \dots, n\}$. No further constant is in U.
- The set $\mathbf{C}$ consists of the following three kinds of constraints.
  - (C1) For each rule $\gamma : \{pattern\} \to value$ in the encoding $\xi^{\mathsf{mcn}}(\Gamma)$, $\mathbf{C}$ contains the constraint $C_\gamma = (S_\gamma, r_\gamma)$ such that $S_\gamma = \{a_i \mid a_i \text{ occurs in } pattern\}$ and where r_γ contains all possible substitutions from S_γ to U. Note that these constraints are immaterial, and they will play a role only when equipped with a weighing function as we shall discuss later.
  - (C2) For each agent $a_i \in N$, $\mathbf{C}$ contains the constraint $C_i = (\{a_i\}, r_i)$ such that:
    1. If a_i is not a pivotal agent, i.e., $a_i \notin S$, then for each agent a_j connected with an edge to a_i in $\mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$, and for each pair of indices $\ell, h \in \{1, \dots, n\}$, the substitution $\{a_i/\langle from_j, origin_\ell, level_h\rangle\}$ is in r_i.
    2. The substitution $\{a_i/\langle origin_i, level_0\rangle\}$ is in r_i, and no further substitution occurs in it. Thus, $\{a_i/\langle origin_i, level_0\rangle\}$ is the only available substitution whenever a_i is a pivotal agent.
  - (C3) For each pair of agents a_i and a_j connected with an edge in $\mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$, $\mathbf{C}$ contains the constraint $C_{i,j} = (\{a_i, a_j\}, r_{i,j})$ defined as follows:
    1. For each $u \in U$, the substitution $\{a_i/\langle origin_i, level_0\rangle, a_j/u\}$ is in $r_{i,j}$.
    2. For each $u \in U$, for each $a_{j'} \neq a_j$ connected to a_i, and for each $\ell, h \in \{1, \dots, n\}$, the substitution $\{a_i/\langle from_{j'}, origin_\ell, level_h\rangle, a_j/u\}$ is in $r_{i,j}$.
    3. For each agent $a_k \neq a_i$ connected to a_j, for each $\ell, h \in \{1, \dots, n\}$, the substitution $\{a_i/\langle from_j, origin_\ell, level_{h+1}\rangle, a_j/\langle from_k, origin_\ell, level_h\rangle\}$ is in $r_{i,j}$.
    4. The substitution $\{a_i/\langle from_j, origin_j, level_1\rangle, a_j/\langle origin_j, level_0\rangle\}$ is in $r_{i,j}$, and no further substitution is in $r_{i,j}$.

Intuitively, elements in U are used to encode a number of spanning trees (of some underlying coalitions) defined over the interaction graph. In particular, the element $\langle origin_\ell, level_0\rangle$ can be assigned to the variable a_ℓ only, with the intended meaning that a_ℓ is the root of a spanning tree. Instead, if a node a_i is mapped to the constant $\langle from_j, origin_\ell, level_h\rangle$, then we intend that a_i belongs to a spanning tree rooted at node a_ℓ, that a_j is the father of a_i in this tree, and that a_i occurs at the level h of the tree (with level 0 being the root). More than one node can be defined as a root node, so that more than one spanning tree can be actually induced by a solution. However, we require that each pivotal agent is necessarily the root of a spanning tree, hence preventing that it occurs in the same tree (hence, component) associated with another pivotal agent.

Example 4.6. Consider again the coalitional game $\dot{\Gamma}$ and the encoding in terms of a marginal contribution network, say $\dot{M}$, defined in Example 4.4. Consider the valuation structure $\dot{\sigma} = \langle \mathrm{AG}(\dot{M}), \{a_3, a_5\}, \mathbf{1}, \mathbf{0}, 0, -\infty\rangle$, where a_3 and a_5 are pivotal agents.

A solution to the CSP associated with $\dot{\Gamma}$ and $\dot{\sigma}$ is given by the following substitution:

$$\begin{aligned}\dot{\theta} = \{\ & a_5/\langle origin_5, level_0\rangle,\\ & a_3/\langle origin_3, level_0\rangle,\\ & a_2/\langle from_3, origin_3, level_1\rangle,\\ & a_1/\langle from_3, origin_3, level_1\rangle,\\ & a_4/\langle from_2, origin_3, level_2\rangle\}.\end{aligned}$$

The reader can check that constraints of kind (C2) and (C3) are satisfied by $\dot{\theta}$. Moreover, the reader can observe in Fig. 6 that the CSP solution induces a $\dot{\sigma}$-feasible coalition structure, where a_3 and a_5 belong to different coalitions (and spanning trees). ◁

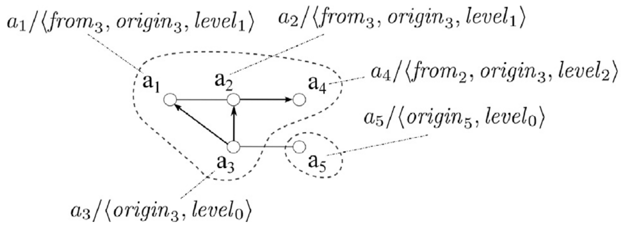

Fig. 6. Illustration of Example 4.6.

The crucial properties of the above correspondence from games to CSP instances are formalized below. Hereinafter, for any substitution θ from (any subset of) *Var* to U, and for any index $i \in \{1,\dots,n\}$, we denote by θ_i the set of all agents that are mapped via θ to a tuple containing the term '*origin$_i$*'. Moreover, we denote by Π_θ the coalition structure including precisely the coalition θ_i, for each $i \in \{1,\dots,n\}$ such that $\theta_i \neq \emptyset$.

For instance, in Example 4.6, we have that $\dot\theta_1 = \dot\theta_2 = \dot\theta_4 = \emptyset$, $\dot\theta_3 = \{a_3, a_2, a_1, a_4\}$, and $\dot\theta_5 = \{a_5\}$. Note that $\Pi_{\dot\theta} = \{\dot\theta_3, \dot\theta_5\}$ is a $\dot\sigma$-feasible coalition structure.

Lemma 4.7. *For each σ-feasible coalition structure Π in $\mathcal{CS}_\sigma(\Gamma)$, there is a solution θ to CSP(Γ,σ) such that $\Pi = \Pi_\theta$.*

Proof. Let Π be a σ-feasible coalition structure in $\mathcal{CS}_\sigma(\Gamma)$, and consider a substitution θ built as follows. For each coalition C in Π, observe that the subgraph G_C of $\mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$ induced over the nodes of C is connected. Let T_C be a spanning tree of G_C and let us root it at a node $a_\ell \in C$. In particular, if C contains a pivotal agent, then a_ℓ is precisely this (univocally determined) agent. Otherwise, a_ℓ is any arbitrary node in C. Then, the restriction of θ over the variables in C is built as follows. For the agent a_ℓ, we set $\theta(a_\ell) = \langle origin_\ell, level_0\rangle$. Moreover, for each agent $a_i \in C \setminus \{a_\ell\}$, let a_j be the parent of a_i, and let h be the number of edges occurring in the path connecting a_j and a_ℓ in T_C. Then, we set $\theta(a_i) = \langle from_j, origin_\ell, level_{h+1}\rangle$. Note that if the parent of a_i is a_ℓ, then we have that $\theta(a_i) = \langle from_\ell, origin_\ell, level_1\rangle$. Otherwise, i.e., if $a_j \neq a_\ell$, then we have that $h > 0$. Eventually, since the coalitions in Π are disjoint and they cover all the agents/variables in N, by iterating the above construction over all such coalitions, we get that the substitution θ is well defined, i.e., it maps each variable to a constant in U. In particular, by construction, $\Pi = \Pi_\theta$.

In order to conclude the proof, we claim that, for each coalition $C \in \Pi$, θ satisfies all the constraints where a variable in C occurs. Note that constraints of kind (C1) are trivially satisfied by any substitution. Thus, we have just to focus on the constraints of kind (C2) and (C3). Let a_i be a node in C, let a_ℓ be the root node of T_C and consider the following constraints:

(C2) Observe that $\theta(a_i)$ is either the tuple $\langle origin_i, level_0\rangle$ with $i = \ell$, or a tuple having the form $\langle from_j, origin_\ell, level_h\rangle$, for an index j such that a_j is connected to a_i, and where $h \in \{1,\dots,n\}$. In particular, whenever a_i is a pivotal agent, then we are guaranteed that $i = \ell$ holds, by construction. It follows that θ satisfies the constraint $C_i = (\{a_i\}, r_i)$.

(C3) Consider now a constraint of the form $C_{i,j}$, and let us distinguish the following cases (corresponding to the items in the definition of the elements in $r_{i,j}$).
1. If $\theta(a_i) = \langle origin_i, level_0\rangle$ (so that $i = \ell$ holds), then $C_{i,j}$ is satisfied no matter of the value of $\theta(a_j)$;
2. If $\theta(a_i) = \langle from_{j'}, origin_\ell, level_h\rangle$ with $j' \neq j$, then $C_{i,j}$ is trivially satisfied;
3. If $\theta(a_i) = \langle from_j, origin_\ell, level_{h+1}\rangle$ with $h \in \{1,\dots,n\}$, then by construction we are guaranteed that a_j is the parent of a_i in the spanning tree T_C and that $a_j \neq a_\ell$. Therefore, $\theta(a_j)$ has the form $\langle from_k, origin_\ell, level_h\rangle$ where a_k is the parent of a_j in T. It follows that θ satisfies $C_{i,j}$;
4. Finally, if $\theta(a_i) = \langle from_j, origin_\ell, level_1\rangle$, then we have to recall that $\ell = j$ actually holds. Therefore, $\theta(a_i) = \langle from_\ell, origin_\ell, level_1\rangle$ and, by construction, $\theta(a_j) = \theta(a_\ell) = \langle origin_\ell, level_0\rangle$. So, θ satisfies $C_{i,j}$.

Given that all constraints are satisfied by θ, we conclude that θ is a solution. □

Lemma 4.8. *For each solution θ to CSP(Γ,σ), Π_θ is a σ-feasible coalition structure in $\mathcal{CS}_\sigma(\Gamma)$.*

Proof. Let θ be a solution and consider the set Π_θ. We first show that, for each agent $a_i \in N$, there is a coalition $C \in \Pi_\theta$ such that $a_i \in C$. Indeed, as θ is solution, $\theta(a_i)$ is necessarily mapped to a tuple having a term of the form *origin$_\ell$* for some index $\ell \in \{1,\dots,n\}$. This means that $a_i \in \theta_\ell$ holds and that $\theta_\ell \in \Pi_\theta$. Moreover, by definition of Π_θ and given the universe U of the CSP instance, it is immediate to check that for each pair of coalitions C and C' in Π_θ, $C \cap C' \neq \emptyset$ holds. Therefore, Π_θ is a set of disjoint coalitions covering all the agents in N. To complete the proof, we have to show that each coalition C in Π_θ is a feasible one.

(Connectivity): Let C be a coalition in Π_θ, and assume that $a_\ell \in N$ is the agent such that $C = \theta_\ell$. First, note that each node a_i in C is mapped via θ to a tuple containing a term of the form *level$_h$*, where $h \in \{0,\dots,n\}$. In the following, we shall say that h is the level of a_i, and we prove that the subgraph of Π_θ induced over the nodes in C is connected.

In fact, we claim that for each node a_i in C whose level is $h+1$ with $h \geq 0$, a_i is connected to a node a_j in C whose level is h. To prove the claim, observe that because of the constraint $C_{i,j}$, $\theta(a_i) = \langle from_j, origin_\ell, level_{h+1}\rangle$ holds, where a_j is an agent connected to a_i. Therefore, in the base case where $j = \ell$, we have concluded. Otherwise, we are guaranteed that $\theta(a_j)$ has the form $\langle from_k, origin_\ell, level_h\rangle$, hence a_j is a node in C and its level is h. Finally, by a simple argument based on structural induction on the above observation, we can conclude that each node a_i whose level is $h+1$ is connected to a_ℓ by means of a path involving only nodes in C and whose associated levels are lower than h.

(Pivotal agents): Let a_i and a_j be two pivotal agents in S. Because of the constraints of kind (C2), it must be the case that $\theta(a_i) = \langle origin_i, level_0\rangle$ and $\theta(a_j) = \langle origin_j, level_0\rangle$. Thus, by construction of Π_θ, a_i and a_j belongs to distinct coalitions in Π_θ.

Given that the above two properties hold, we conclude that Π_θ is σ-feasible. □

4.2.2. Weighted CSPs

The next ingredient we need is to equip CSP instances with weights as to properly encode the goal of maximizing the σ-value. Note that this is not immediate in our setting, because the original valuation function v is mapped, by the affine transformation provided by the valuation structure σ, into a novel function val_σ that do not preserve the properties of v. In particular, even if v is an IDM function, this is not necessarily the case for val_σ and, hence, the results derived in the previous section would not apply on it. For this reason, we are forced to work on the original valuation function v, and to deal with the affine transformation with a more sophisticated encoding.

Before detailing the encoding, we recall that a *weighted* CSP (short: WCSP) instance consists of a tuple $\langle \mathcal{I}, w_{r_1}, \ldots, w_{r_q}\rangle$, where $\mathcal{I} = \langle Var, U, \mathbf{C}\rangle$ with $\mathbf{C} = \{C_1, C_2, \ldots, C_q\}$ is a CSP instance, and where, for each tuple $t_v \in r_v$, $w_{r_v}(t_v) \in \mathbb{R}$ denotes the weight associated with t_v. For a solution $\theta = t_1 \cup \cdots \cup t_q$ to $\mathcal{I}$, we define $w(\theta) = \sum_{v=1}^{q} w_{r_v}(t_v)$ its associated weight. Then, a solution to $\langle \mathcal{I}, w_{r_1}, \ldots, w_{r_q}\rangle$ is a solution[9] θ to $\mathcal{I}$ such that $w(\theta) \geq w(\theta')$, for each solution θ' to $\mathcal{I}$.

Given the game $\Gamma = \langle N, v\rangle \in \mathcal{C}_{\mathbf{TW}\text{-}h}(\text{mcn})$ and $\sigma = \langle G, S, \alpha, \beta, x, y\rangle$, we define $WCSP(\Gamma, \sigma)$ as the weighted CSP instance whose underlying CSP instance is $CSP(\Gamma, \sigma)$ and where each constraint relation r_γ of kind (C1), which is in fact associated with the rule $\gamma : \{pattern\} \rightarrow value$, is equipped with the function w_{r_γ} defined as follows.

First, we say that a rule γ is *active w.r.t.* a tuple $t \in r_\gamma$ if there is an agent $a_i \in \{a_1, \ldots, a_n\}$, called the witness, such that t_i (i.e., the set of agents mapped via t to a constant with a term $origin_i$) coincides with the set of the agents occurring in *pattern* (i.e., with the scope S_γ)—recall from Corollary 4.5 that we are assuming all agents to occur positively in *pattern*. Then, for each tuple $t \in r_\gamma$,

- if γ is not active w.r.t. t, then we set $w_{r_\gamma}(t) = 0$;
- otherwise, i.e., if γ is active w.r.t. t and a_i is the associated witness (which is, in fact, unique), then we distinguish two cases:
  - if a_i is a pivotal agent in S, then we set $w_{r_\gamma}(t) = value \times \alpha(a_i)$;
  - if a_i is not a pivotal agent, then we set $w_{r_\gamma}(t) = value \times x$.

Concerning constraints of kind (C2), for each constraint relation r_i, with $a_i \in N$,

- we set $w_{r_i}(a_i/u) = 0$, for each constant $u \in U \setminus \{\langle origin_i, level_0\rangle\}$;
- for the constant $\langle origin_i, level_0\rangle$, we distinguish two cases:
  - if a_i is a pivotal agent in S, then we set $w_{r_i}(\{a_i/\langle origin_i, level_0\rangle\}) = \beta(a_i)$;
  - if a_i is not a pivotal agent, then we set $w_{r_i}(\{a_i/\langle origin_i, level_0\rangle\}) = y$.

All other constraint relations of kind (C3) are equipped with the constant function assigning 0 to each substitution, and we hence shall get rid of them.

We now show that the proposed weighting scheme leads to equip coalition structures with their associated σ-value.

Lemma 4.9. *For each solution θ to $CSP(\Gamma, \sigma)$, $w(\theta) = val_\sigma(\Pi_\theta)$.*

Proof. Let Π_θ be the coalition structure associated with θ (cf. Lemma 4.7 and Lemma 4.8). In the following, for any set of nodes/variables W, we denote by $\theta[W]$ the restriction of the substitution θ over the elements in W.

Let a_i be an agent such that $\theta_i \in \Pi_\theta$, and let R_i denote the set of rules in $\xi^{\text{mcn}}(\Gamma)$ that apply to θ_i. Then, we first claim that:

$$val_\sigma(\theta_i) = \sum_{\gamma \in R_i} w_{r_\gamma}(\theta[S_\gamma]) + w_{r_i}(\theta[S_i]). \tag{5}$$

[9] Note that one can dually interpret weights as costs and look at minimization problems, rather than maximization ones.

In order to prove the claim, let us consider the definition of $val_\sigma(\theta_i)$, by distinguishing two cases, depending on whether a_i is a pivotal agent:

(1) In the case where a_i is a pivotal agent, then $val_\sigma(\theta_i) = v(\theta_i) \times \alpha(a_i) + \beta(a_i)$;
(2) otherwise, we have $val_\sigma(\theta_i) = v(\theta_i) \times x + y$.

Let us focus on case (1). In this case, we have that $\theta(a_i) = \langle origin_i, level_0\rangle$, and hence $w_{r_i}(\theta[S_i]) = \beta(a_i)$ holds, by definition of the weighting function. Moreover, consider any rule $\gamma \in R_i$ with $\gamma : \{pattern\} \rightarrow value$. Let A be the set of agents occurring (positively) in *pattern*, and consider the tuple $t \in r_\gamma$ such that $\theta \supseteq t$ (hence, $A = S_\gamma$). Since γ applies to θ_i, it must be the case that $t_i = A$. Hence, γ is active w.r.t. t and we have that $w_{r_\gamma}(\theta[S_\gamma]) = value \times \alpha(a_i)$. Now, recall that the worth of any coalition is just given by the sum of the values associated to the rules that apply to it. Therefore, $v(\theta_i)$ is the sum of the values of all rules in R_i. Hence, $\sum_{\gamma\in R_i} w_{r_\gamma}(\theta[S_\gamma]) + w_{r_i}(\theta[S_i]) = v(\theta_i) \times \alpha(a_i) + \beta(a_i)$.

Let us focus on case (2). Then, we apply the same line of reasoning as above, by replacing the value $\alpha(a_i)$ (resp., $\beta(a_i)$), with x (resp., y). Eventually, we derive that $\sum_{\gamma\in R_i} w_{r_\gamma}(\theta[S_\gamma]) + w_{r_i}(\theta[S_i]) = v(\theta_i) \times x + y$.

At this point, we know that Equation (5) is correct. Moreover, we recall that $val_\sigma(\Pi_\theta)$ is just given by the sum of $val_\sigma(\theta_i)$ over each coalition $\theta_i \in \Pi_\theta$. That is,

$$val_\sigma(\Pi_\theta) = \sum_{\theta_i\in\Pi_\theta}\sum_{\gamma\in R_i} w_{r_\gamma}(\theta[S_\gamma]) + \sum_{\theta_i\in\Pi_\theta} w_{r_i}(\theta[S_i]). \tag{6}$$

On the other hand, if R denotes the set of all the rules, we can write:

$$w(\theta) = \sum_{\gamma\in R} w_{r_\gamma}(\theta[S_\gamma]) + \sum_{a_j\in N} w_{r_j}(\theta[S_j]). \tag{7}$$

In order to conclude the proof, we have to show that the right-hand sides of Equation (6) and Equation (7) coincide. To this end, consider first an agent $a_j \in N$ such that θ_j is not in Π_θ. This means that $\theta(a_j) \neq \langle origin_j, level_0\rangle$ and, hence, $w_{r_j}(\theta[S_j]) = 0$ holds. That is, $\sum_{\theta_i\in\Pi_\theta} w_{r_i}(\theta[S_i]) = \sum_{a_j\in N} w_{r_j}(\theta[S_j])$.

We will now prove that $\sum_{\theta_i\in\Pi_\theta}\sum_{\gamma\in R_i} w_{r_\gamma}(\theta[S_\gamma]) = \sum_{\gamma\in R} w_{r_\gamma}(\theta[S_\gamma])$. In fact, note that any rule γ that belongs to a set R_i cannot belong to a set R_j for some agent $a_j \neq a_i$. Therefore, we have just to take care of those rules γ that do not apply to any coalition in Π_θ, and we shall show that $w_{r_\gamma}(\theta[S_\gamma]) = 0$ holds for them. Indeed, consider the tuple $\theta[S_\gamma] \in r_\gamma$, and just note that such rules γ are not active w.r.t. $\theta[S_\gamma]$. □

4.2.3. Proof of the main result and extensions

The structure of a CSP instance $\mathcal{I}$ is often represented in the literature by its associated primal graph $\mathrm{PG}(\mathcal{I})$ defined over the variables in *Var* and where two variables are connected with an edge if, and only if, they occur in the same scope of some constraint. While looking at the CSP instance $CSP(\Gamma,\sigma) = \langle Var, U, \mathbf{C}\rangle$, it is immediate to check that two variables occur in the same scope of some constraint if, and only if, the associated agents occurs in some rule of the underlying marginal contribution network. Therefore, the graph associated with the constraints coincides with the agent graph underlying Γ.

Fact 4.10. $\mathrm{PG}(CSP(\Gamma,\sigma)) = \mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$.

Moreover, over bounded treewidth instances the construction of $WCSP(\Gamma,\sigma)$ can be efficiently carried out.

Lemma 4.11. *Let $h \geq 0$ be a fixed natural number. Let $\Gamma = \langle N, v\rangle$ be a game in $\mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$, and let $\sigma = \langle \mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma)), S, \alpha, \beta, x, y\rangle$ be a valuation structure. Then, $WCSP(\Gamma,\sigma)$ can be built in polynomial time (w.r.t. $\|\xi^{\mathsf{mcn}}(\Gamma)\| + \|\sigma\|$).*

Proof. We have to build $CSP(\Gamma,\sigma) = \langle Var, U, \mathbf{C}\rangle$. Clearly, *Var* and U can be build in polynomial time as $|N| \leq \|\xi^{\mathsf{c\text{-}mcn}}(\Gamma)\|$. Constraints of the form (C2) and (C3) can be built in polynomial time, as they are defined over two variables at most. Concerning the constraints of kind (C1), observe that we have one constraint (S_γ, r_γ), for each rule $\gamma : \{pattern\} \rightarrow value$ in the encoding, where $|S_\gamma| \leq |N|$. In particular, we observe that $|r_\gamma| \leq n^{|S_\gamma|}$ and to conclude we just we claim that $|S_\gamma| \leq h$. Indeed, if $|S_\gamma| = p > h$, then the agent graph $\mathrm{AG}(\xi^{\mathsf{mcn}}(\Gamma))$ would contain a clique over $p > h$ agents (in S_γ), and the treewidth of this graph would be at least p, hence greater than h. In order to conclude the proof, we can just notice that the weighting functions can be also built in polynomial time. In particular, we explicitly list the weights in the encoding, with their size being bounded by $\|\sigma\|$.[10] □

By Lemma 4.7, Lemma 4.8, Lemma 4.9, Fact 4.10, Lemma 4.11, Theorem 4.3, and given that solutions to (weighted) constraint satisfaction are known to be computable in polynomial time on classes of instances whose associated primal graphs

[10] This is the standard encoding in the results we shall use about the complexity of WCSPs [40].

have treewidth bounded by some fixed constant (see, e.g., [23,40,34]), we derive our main result generalizing Corollary 3.6 to arbitrary valuation structures.

Theorem 4.12. *Let $h \geq 0$ be a fixed natural number. Let $\mathcal{R}$ be any representation for coalitional games, let $\Gamma = \langle N, v\rangle \in \mathcal{C}(\mathcal{R})$ be a game such that v is a valuation function independent of disconnected members w.r.t. a graph G with $tw(G) \geq h$. Let $\sigma = \langle G, S, \alpha, \beta, x, y\rangle$ be a valuation structure. Then,* CSG$_{\text{VAL}}$ *can be solved on (Γ, σ) in polynomial time (w.r.t. $\|\xi^{\mathcal{R}}(\Gamma)\|$ and $\|\sigma\|$).*

Interestingly, the flexibility of the CSP encoding allow us to easily generalize the above result to handle the constraints studied by [58] (see Section 1). In order to accommodate the extension, the CSP instance associated with a game $\Gamma = \langle N, v\rangle \in \mathcal{C}(\mathsf{mcn})$, a valuation structure σ, and sets $\mathcal{P}$ and $\mathcal{N}$ of positive and negative constraints, respectively, is obtained from the instance $CSP(\Gamma, \sigma) = \langle Var, U, \mathbf{C}\rangle$ defined in Section 4.2.1 as follows.

First, we define $U_{\mathcal{P}}$ as the set obtained from U by replacing the constant $origin_i$, for each $a_i \in N$, with the *marked* constants $origin_i^{\mathbf{p}}$, for each positive constraint $\mathbf{p} \in \mathcal{P}$. We do not modify the set of variables, and we say that a substitution θ is *consistent* if, for each pair of variables a_j, a_h in its domain, $\theta(a_j) = origin_i^{\mathbf{p}}$ and $\theta(a_j) = origin_i^{\bar{\mathbf{p}}}$ implies that $\mathbf{p} = \bar{\mathbf{p}}$. Moreover, for each agent $a_i \in N$ and $\mathbf{p} \in \mathcal{P}$, we denote by $\theta_i^{\mathbf{p}}$ the coalition including all agents that are mapped by θ to the constant $origin_i^{\mathbf{p}}$. Eventually, if θ is a substitution with image contained in $U_{\mathcal{P}}$, we denote by $unmark(\theta)$ the substitution obtained by stripping off all the markings.

Second, we build a set $\mathbf{C}_{\mathcal{P}}$ of constraints as follows. For each constraint $(S, r) \in \mathbf{C}$, the constraint $(S, r_{\mathcal{P}})$ is in $\mathbf{C}_{\mathcal{P}}$, where $r_{\mathcal{P}}$ is obtained by including all the possible consistent (marked) tuples that can be built from the tuples in r. Clearly enough, all the properties of $CSP(\Gamma, \sigma) = \langle Var, U, \mathbf{C}\rangle$ are preserved with this transformation. In particular, an assignment θ is a solution to the resulting CSP instance if, and only if, $\Pi_{unmark(\theta)}$ is a σ-feasible coalition structure (cf. Lemma 4.7 and Lemma 4.8).

Eventually, the following two groups of constraints are added in $\mathbf{C}_{\mathcal{P}}$:

(C4) For each $\mathbf{p} \in \mathcal{P}$ with $\mathbf{p} \subseteq N$, $\mathbf{C}_{\mathcal{P}}$ contains the constraint $C_{\mathbf{p}} = (\mathbf{p}, r_{\mathbf{p}})$ having the agents in $\mathbf{p}$ as scope and whose relation $r_{\mathbf{p}}$ contains all the substitutions t such that $t_i^{\mathbf{p}} = \mathbf{p}$, for some a_i. Moreover, it contains all substitutions t' for which no constant in the image is marked with $\mathbf{p}$. Note that since in any solution θ each agent is eventually mapped to a constant in $U_{\mathcal{P}}$ with marking, say $\bar{\mathbf{p}}$, taken from $\mathcal{P}$, then the corresponding constraint $C_{\bar{\mathbf{p}}}$ forces the formation of a coalition $\theta_i^{\bar{\mathbf{p}}}$ covering $\bar{\mathbf{p}}$. Hence, $\bar{\mathbf{p}}$ witnesses that θ_i satisfies the positive constraints.

(C5) For each $\mathbf{n} \in \mathcal{N}$ with $\mathbf{n} \subseteq N$, $\mathbf{C}_{\mathcal{P}}$ contains the constraint $C_{\mathbf{n}} = (\mathbf{n}, r_{\mathbf{n}})$ having the agents in $\mathbf{n}$ as scope and whose relation $r_{\mathbf{n}}$ contains all possible substitutions t from $\mathbf{n}$ to $U_{\mathcal{P}}$ for which there is *no* agent $a_i \in N$ with $t_i = \mathbf{n}$. Note that in this case, we do not take care at all of the markings, as none of the formed coalitions can cover a negative constraint.

By putting it all together, we derive that an assignment θ is a solution to the resulting CSP instance if, and only if, $\Pi_{unmark(\theta)}$ is a σ-feasible coalition structure (cf. Lemma 4.7 and Lemma 4.8) and moreover each coalition in $\Pi_{unmark(\theta)}$ covers at least one positive constraint, while there is no coalition in $\Pi_{unmark(\theta)}$ covering a negative constraint. Eventually, constraints of kind (C4) and (C5) can be equipped with 0 as a weight, while each tuple t in a constraint of the form $r_{\mathcal{P}}$ is equipped with the weight associated with the tuple $unmark(t)$ in the original constraint r. Therefore, Lemma 4.9 trivially holds on the modified scenario, too. The only difference is now that the primal graph of the CSP will no longer coincide with the underlying agent graph. Rather, it can be obtained from the agent graph by adding the edges associated with such novel constraints. Hence, we can derive that CSG$_{\text{VAL}}$, under constraints $\mathcal{P}$ and $\mathcal{N}$, can be solved on in polynomial time on valuation functions v that are independent of disconnected members w.r.t. graphs G such that $G_{\mathcal{P},\mathcal{N}}$ have treewidth bounded by some fixed natural number. In particular, by $G_{\mathcal{P},\mathcal{N}}$ we denote the graph obtained by adding to G an edge between each pair of agents involved in a positive or negative constraint.

Remark. As pointed out in the Introduction, the semantics of a set S of pivotal agents can be recast in terms of the set of negative constraints $\{\{a_i, a_j\} \mid a_i, a_j \in S\}$. However, the resulting graph would contain a clique over the agents in S, thereby obscuring the intricacy of the setting. Our results deal instead with pivotal agents as first-class citizens, and the set S does not influence the treewidth of the underlying graph.

5. Back to the applications

Now that we have established Theorem 4.12, we turn back to the applications of Section 2.2 by discussing their complexity over structures having bounded treewidth. The analysis comes as a simple corollary of our general result, though in some cases we actually close tractability questions that have been left open in the literature.

5.1. Multicut problems

The problem of identifying islands of tractability for the multicut problem has been intensively studied in the literature and a number of results have been derived by exploiting the structural properties of the graphs on top of which the problem is defined.

For classes of graphs G having bounded treewidth, it has been shown [8] that the problem is tractable if the size of the set P of the source-terminal pairs to be disconnected is constant (see, also, [42]). The result does not prescribe any restriction on the weights associated with the edges. Instead, if we deal with unitary weights only (so that the problem reduces to finding the multicut consisting of the minimum possible number of edges), then we know that tractability holds over the instances for which G_P, i.e., the graph derived from G by adding an edge between each pair in P, has bounded treewidth [37]. Note that, while this latter result focuses on unitary weights (i.e., undirected graphs only), it is actually incomparable with the result by [8]. Indeed, if the size of P is a constant, then the treewidth of G_P is within a constant from the treewidth of G. However, G_P might have bounded treewidth even when the size of P is not constant.

It is interesting to point out that the result by [37] is based on encoding the multicut problem as a monadic second-order (MSO) formula over the structure G_P, and by exploiting a generalization of Courcelle's Theorem [19] tailored to optimization problems [1]. Motivated by the fact that the resulting algorithm is unpractical (the running time is non-elementary in terms of the number of quantifier alternations of the MSO formula), a direct solution approach has been proposed more recently by [55]. However, it was open so far whether the result by [37] can be extended to arbitrary weighted graphs.[11] Here, we close this question by providing the following positive answer. In the proof, note that the only technical ingredient we exploit is the connection between the multicut problem and the problem of computing $\perp$-optimal coalitions structures, which was not pointed out in earlier literature—in particular, Theorem 4.12 plays no role.

Theorem 5.1. *Let $h \geq 0$ be a fixed natural number. On classes of graphs G and source-terminal pairs P with $tw(G_P) \leq h$, the multicut problem can be solved in polynomial time.*

Proof. Recall from Section 2.2.1 that the multicut problem can be reduced to computing a $\perp$-optimal coalition structure for the game $\langle N, v_P \rangle$ (see Theorem B.1 in Appendix B for the formal correspondence). Recall also that v_P is the function such that $v_P(C)$ is the sum of the weights of the edges in the subgraph of G_P induced over any coalition C. Therefore, $\langle N, v_P \rangle \in \mathcal{C}(\text{gg})$ holds, i.e., the valuation function can be encoded as graph game. Then, we recall from [69] that valuation functions for games in $\mathcal{C}(\text{gg})$ are independent of disconnected members w.r.t. the underlying graphs. So, v_P is an $\mathtt{IDM}$ function w.r.t. G_P, and we can apply Theorem 3.5, in order to conclude that a $\perp$-optimal coalition structure for $\langle N, v_P \rangle$ can be computed in polynomial time. □

5.2. Multiway cut (and k-clustering) problems

Similarly to the multicut problem, a number of structural tractability results for the multiway cut problems are known, but the picture is still not completely clear. The tractability of the problem over classes of graphs G that are trees has been established in the early nineties by [16]. However, a practical dynamic programming algorithm (with a linear running time) followed many years later [18].

In a very influential paper on cut problems [21], it has been claimed that standard dynamic programming methods can be used to show that the multicut problem remains tractable when moving from trees to graphs having bounded treewidth. However, no algorithm has been actually reported there, and no formal proof of the result has been reported in the subsequent literature. Here, we show that the claim of [21] is correct.

Theorem 5.2. *Let $h \geq 0$ be a fixed natural number. On classes of graphs G such that $tw(G) \leq h$, the multiway cut problem can be solved in polynomial time.*

Proof. Recall from Section 2.2.1 that the multiway cut problem can be reduced to computing a σ_{mw}-optimal coalition structure for $\langle N, v_{mw} \rangle$ (see Theorem B.2 in Appendix B for the formal correspondence). Observe also that $\langle N, v_{mw} \rangle$ can be encoded as a graph game, because $v_{mw}(C)$ is just the sum of the weights of the edges in the subgraph of G induced over any coalition C. Moreover, recall that $\sigma_{mw} = \langle G, T, \mathbf{1}, \mathbf{0}, 1, 0 \rangle$. Given these ingredients, we are in the position of applying Theorem 4.12 on $(\langle N, v_{mw} \rangle, \sigma_{mw})$. □

The encoding we have discussed for the k-clustering problem in Section 2.2.1 founds on the encoding for the multiway cut problem. Indeed, we have observed that an optimal k-clustering can be computed as a coalition structure having the maximum σ_R-value (for $\langle N, v_{cc} \rangle$) over all σ_R-optimal coalition structures for any subset $R \subseteq N$ with $|R| = k$, where

[11] In fact, the techniques used in that paper can be adapted to show tractability whenever the weights are "small", i.e., when each of them can be encoded in logarithmic space.

$\sigma_R = \langle G, R, \mathbf{1}, \mathbf{0}, 0, -\infty\rangle$ is the valuation structure that solves the problem of finding the multiway cut of minimum weight where R is considered as the set of terminals. So, polynomially many subsets R have to be considered and, by applying Theorem 5.2, the problem is solvable in polynomial time.

5.3. Chromatic clustering

We conclude our analysis by considering the chromatic clustering problem. Recall from Section 2.2 that the problem has been reduced to computing a σ_{ch}-optimal coalition structure for the game $\langle N, v_{ch}\rangle$, where $\sigma_{ch} = \langle G_{\mathcal{C}}, \mathcal{C}, \mathbf{1}, \mathbf{0}, 0, -\infty\rangle$ and where v_{ch} is any valuation function such that, for each set of nodes $C \subseteq N \cup \mathcal{C}$ with $C \cap \mathcal{C} = \{c_i\}$, $v_{ch}(C) = \sum_{e\in E, e\subseteq C} w_e^{c_i}$ (see Theorem B.4 in Appendix B). Moreover, recall that $G_{\mathcal{C}}$ is the graph where colors are viewed as nodes and where an edge is added between each node in N and each color in $\mathcal{C}$. Then, the following can be established.

Theorem 5.3. *Let $h \geq 0$ be a fixed natural number. On classes of graphs G and sets $\mathcal{C}$ of colors such that $tw(G_{\mathcal{C}}) \leq h$, the chromatic clustering problem can be solved in polynomial time.*

Proof. Similarly to the cases discussed above, given the encoding we have proposed and analyzed, the tractability of chromatic clustering follows for graphs $G_{\mathcal{C}}$ having bounded treewidth, provided we can show that v_{ch} is independent of disconnected members. So, in the light of Theorem 4.2, we will complete the proof by showing that v_{ch} can be encoded as a marginal contribution network M such that $\mathrm{AG}(M) = G_{\mathcal{C}}$. Indeed, the network can be built as follows. For each color $c_i \in \mathcal{C}$ and for each edge $e = \{a, b\} \in E$, we include in M the rule: $a \wedge b \wedge c_i \rightarrow w_e^{c_i}$, and no further rule is included in M. It is immediate to check that the encoding is correct and that $\mathrm{AG}(M) = G_{\mathcal{C}}$ (w.l.o.g., we assume there are no isolated nodes in G). □

6. Stability issues under valuation structures

Computing an optimal coalition structure Π^* is generally not enough in applications where agents collaborate within the same environment. Indeed, another fundamental problem for coalitional games is to determine how the worth that is obtained by forming the structure Π^* can be subsequently distributed over the agents in a way that is *stable*. This problem has been largely studied in the literature, and several approaches have been proposed founding on well-known solution concepts, such as the *core*, the *kernel*, the *bargaining set*, the *nucleolus*, and the *Shapley value* (see, e.g., [53]).

In the following, we shall focus on the *core*, which is arguably the most influential and considered stability concept, and we shall position it within our setting where coalitional games are equipped with valuation structures. In particular, in Section 6.1, we formalize this concept together with some natural reasoning problems that are related to it. Eventually, Section 6.2 is devoted to analyze the complexity of such problems, with the usual intended goal of identifying possibly large islands of tractability.

6.1. Solution concepts and computational problems

Let $\Gamma = \langle N, v\rangle$ be a coalitional game, where $N = \{a_1, \ldots, a_n\}$. A worth distribution in Γ can be simply viewed as a n-dimensional *payoff vector* $\boldsymbol{x} \in \mathbb{R}^n$ whose i-th component, denoted by x_i, is the worth received by agent a_i, for each $i \in \{1, \ldots, n\}$. For any coalition $S \subseteq N$, $x(S)$ is hereinafter used as a shorthand for $\sum_{a_i \in S} x_i$.

Let σ be a valuation structure for Γ, and assume that the agents organize themselves in the σ-feasible coalition structure $\Pi \in \mathcal{F}_\sigma(\Gamma)$. According to Π, the worth to be divided over the agents in a coalition $C \in \Pi$ is given by $val_\sigma(v, C)$. Therefore, the "output" of the game can be viewed as a pair $(\Pi, \boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{R}^n$ is a payoff vector that is *efficient* w.r.t. Π, that is, $x(C) = val_\sigma(v, C)$ holds, for each $C \in \Pi$. In the following, we denote by $\mathcal{E}_\sigma(v, \Pi)$ (or, shorty, by $\mathcal{E}_\sigma(\Pi)$ if v is understood) the set of all such efficient payoff vectors. Note that $\mathcal{E}_\sigma(v, \Pi)$ can contain infinitely many payoff vectors, and so the fundamental problem arises of singling out the most desirable ones among them.

The concept of the core goes back to the work by [28] and it was formalized by [32]. Here, we consider its natural generalization to the setting where valuation structures are taken into account. Intuitively, a pair $(\Pi, \boldsymbol{x})$, with $\boldsymbol{x} \in \mathcal{E}_\sigma(v, \Pi)$, belongs to the core in that it is "stable" because there is no σ-feasible coalition whose members will receive a higher payoff than in $\boldsymbol{x}$ by leaving the current coalition structure. We refer to this notion[12] as the *coalition structure core* of Γ w.r.t. σ, formally defined as the set

$$\text{CS-}core_\sigma(\Gamma) = \{(\Pi, \boldsymbol{x}) \mid \Pi \in \mathcal{CS}_\sigma(\Gamma), \boldsymbol{x} \in \mathcal{E}_\sigma(\Pi), \text{and } x(C) \geq val_\sigma(C),\ \forall C \in \mathcal{F}_\sigma(\Gamma)\}.$$

The following property is useful to simplify our reasoning on this concept.[13]

[12] Whenever $\sigma = \langle G\rangle$ holds, the concept reduces to the notion of the core discussed, for instance, by [13,24,25,15] in order to deal with the restrictions imposed by an underlying interaction graph.

[13] The property is well-known for games without valuation structures and its extension in our setting is straightforward (for instance, by inspecting the results in [15]).

Theorem 6.1. *Let $\Gamma = \langle N, v\rangle$ be a coalitional game, let σ be a valuation structure, and let $(\Pi, \boldsymbol{x})$ be in CS-core$_\sigma(\Gamma)$. Then, $\Pi \in$ CS-opt$_\sigma(\Gamma)$. Moreover, $(\Pi^*, \boldsymbol{x})$ is in CS-core$_\sigma(\Gamma)$, for each σ-optimal coalition structure $\Pi^* \in$ CS-opt$_\sigma(\Gamma)$.*

Example 6.2. Consider again the coalitional game $\dot{\Gamma} = \langle\{a_1, \dots, a_5\}, \dot{v}\rangle$ and the valuation structure $\dot{\sigma} = \langle \mathtt{AG}(\dot{M}), \{a_3, a_5\}, \mathbf{1}, \mathbf{0}, 0, -\infty\rangle$ defined in Example 4.4 and Example 4.6, respectively. Moreover, consider the coalition structure $\dot{\Pi} = \{\{a_5\}, \{a_1, a_2, a_3, a_4\}\}$, and the payoff vector $\boldsymbol{x} \in \mathbb{R}^5$ such that $x_1 = x_2 = x_3 = \frac{5}{3}$ and $x_4 = x_5 = 0$. Note that $x_5 = val_{\dot{\sigma}}(\{a_5\}) = \dot{v}(\{a_5\}) = 0$ and $x(\{a_1, a_2, a_3, a_4\}) = x_1 + x_2 + x_3 + x_4 = val_{\dot{\sigma}}(\{a_1, a_2, a_3, a_4\}) = \dot{v}(\{a_1, a_2, a_3, a_4\}) = 5$. Hence, $\boldsymbol{x}$ belongs to the set $\mathcal{E}_{\dot{\sigma}}(\dot{\Pi})$ of the payoff vectors that are efficient w.r.t. $\dot{\Pi}$. In fact, $(\dot{\Pi}, \boldsymbol{x})$ belongs to CS-*core*$_{\dot{\sigma}}(\dot{\Gamma})$, as it can be checked that $x(C) \geq val_{\dot{\sigma}}(C)$ holds, for each σ-feasible coalition C. Note that, by Theorem 6.1, this entails that $\dot{\Pi}$ belongs to CS-*opt*$_{\dot{\sigma}}(\dot{\Gamma})$.

Consider now the game $\ddot{\Gamma} = \langle\{a_1, \dots, a_5\}, \ddot{v}\rangle$ where the valuation function $\ddot{v}$ is considered such that $\ddot{v}(C) = \dot{v}(C)$, for each $C \subseteq N$ with $|C \cap \{a_1, a_2, a_3\}| \neq 3$, and $\ddot{v}(C) = 2$ for the remaining coalitions C. Note that the modification does not alter the optimality of $\dot{\Pi}$, which belongs to CS-*opt*$_{\dot{\sigma}}(\ddot{\Gamma})$. Moreover, observe that if a vector $\boldsymbol{x}$ is efficient w.r.t. $\dot{\Pi}$, then it holds that $x_1 + x_2 + x_3 + x_4 = 2$ and $x_5 = 0$. Therefore, this vector cannot simultaneously satisfy the stability conditions required to be in the core. In particular, just observe that the following set of inequalities is not satisfiable:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 2 & \\ x_1 + x_2 \geq val_{\dot{\sigma}}(\ddot{v}, \{a_1, a_2\}) & \text{where} \quad val_{\dot{\sigma}}(\ddot{v}, \{a_1, a_2\}) = \ddot{v}(\{a_1, a_2\}) = 2 \\ x_1 + x_3 \geq val_{\dot{\sigma}}(\ddot{v}, \{a_1, a_3\}) & \qquad\qquad val_{\dot{\sigma}}(\ddot{v}, \{a_1, a_3\}) = \ddot{v}(\{a_1, a_3\}) = 2 \\ x_2 + x_3 \geq val_{\dot{\sigma}}(\ddot{v}, \{a_2, a_3\}) & \qquad\qquad val_{\dot{\sigma}}(\ddot{v}, \{a_2, a_3\}) = \ddot{v}(\{a_2, a_3\}) = 2 \\ x_4 \geq val_{\dot{\sigma}}(\ddot{v}, \{a_4\}) & \qquad\qquad val_{\dot{\sigma}}(\ddot{v}, \{a_4\}) = \ddot{v}(\{a_4\}) = 0 \end{cases}$$

Hence, there is no pair of the form $(\dot{\Pi}, \boldsymbol{x})$ in the coalition structure core of $\ddot{\Gamma}$ w.r.t. $\dot{\sigma}$. More generally, it can be checked that CS-*core*$_{\dot{\sigma}}(\ddot{\Gamma}) = \emptyset$. Indeed, this follows by Theorem 6.1 and since $\dot{\Pi}$ belongs to CS-*opt*$_{\dot{\sigma}}(\ddot{\Gamma})$. ◁

6.2. Complexity analysis

With the above concepts in place, we can now state the main reasoning (computation) problem we shall consider in the subsequent analysis:

- CS-Core-Find: Given a pair (Γ, σ), compute an element in CS-*core*$_\sigma(\Gamma)$, or decide that CS-*core*$_\sigma(\Gamma) = \emptyset$.

The problem is easily seen to be intractable, formally **NP**-hard, as it inherits the results that are known to hold for coalitional games without coalitional structures [15]. Motivated by the bad news, we next consider `IDM` functions evaluated w.r.t. interaction graphs having bounded treewidth. Hence, because of the results in Section 4.1, we shall hereinafter study CS-Core-Find and CS-Approx-Core-Find on the class $\mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$ of all marginal contribution networks whose agent graphs have treewidth h at most, where h is a fixed constant. To prove the tractability on this class, we start by analyzing the complexity of two tasks playing a key role in the computation of core-related questions.

First, we consider the problem of computing a coalition getting the maximum value over all possible coalitions. This problem is conceptually even more foundational when compared with the coalition structure generation problem, and technically it can be viewed as a kind of special case. Therefore, the following comes with no surprise.

Theorem 6.3. *Let $h \geq 0$ be a fixed natural number. Let $\Gamma = \langle N, v\rangle \in \mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$ and let $\sigma = \langle G, S, \alpha, \beta, x, y\rangle$ be a valuation structure. Then, computing a σ-feasible coalition C^* such that $val_\sigma(C^*) \geq val_\sigma(C)$, for each $C \in \mathcal{F}_\sigma(\Gamma)$, is feasible in polynomial time (w.r.t. $\|\xi^{\mathsf{mcn}}(\Gamma)\|$ and $\|\sigma\|$).*

Proof. Based on σ, we first build in polynomial time two different settings:

(1) For each $a_i \in S$, let $\sigma_i = \langle G, S, \alpha_i, \beta_i, 0, 0\rangle$ be the valuation structure where α_i (resp., β_i) is the function such that $\alpha_i(a_i) = \alpha(a_i)$ (resp., $\beta_i(a_i) = \beta(a_i)$); and $\alpha_i(a_j) = 0$ (resp., $\beta_i(a_j) = 0$), for each $a_j \in S \setminus \{a_i\}$. Let Π be a σ_i-feasible coalition structure, and note that $val_{\sigma_i}(v, \Pi) = val_{\sigma_i}(v, C_i)$, where C_i is the coalition in Π such that $a_i \in C_i$. In its turn, $val_{\sigma_i}(v, C_i)$ coincides with $val_\sigma(v, C_i)$, by construction of σ_i. In particular, check that C_i is trivially a σ-feasible coalition. On the other hand, if C_i is a σ-feasible coalition, then we can build a σ_i-feasible coalition structure Π consisting of C_i and all other agents as singleton coalitions. Again, we have that $val_{\sigma_i}(v, \Pi) = val_\sigma(v, C_i)$. Therefore, the value of any σ_i-optimal coalition structure coincides with the maximum possible value of any σ-feasible coalition including agent a_i. In the following, we shall denote this optimal value by vc_i^*, and we explicitly remark here that, because of Theorem 4.12 and the above characterization, vc_i^* can be computed in polynomial time.

(2) For each agent $a_j \in N \setminus S$, let $\sigma_j = \langle G, S \cup \{a_j\}, \alpha_j, \beta_j, 0, 0\rangle$ be the valuation structure such that $\alpha_j(a_i) = \beta_j(a_i) = 0$, for each $a_i \in S$, and $\alpha_j(a_j) = x$ and $\beta_j(a_j) = y$. By exploiting the same line of reasoning as in the point (1) above, we can derive that the value of any σ_j-optimal coalition coincides with the maximum possible value of any σ-feasible coalition including agent a_j and excluding all agents in S, hereinafter denoted by vc_j^*. In particular, observe that a_j is

not a pivotal agent in the original valuation structure, so that the affine transformation is determined by the parameters x and y. Instead, a_j plays the role of a pivotal agent in σ_j, and the functions α_j and β_j are accordingly defined as to guarantee the correctness of the valuation. Eventually, we remark that vc_j^* can be computed in polynomial time, because of the above characterization and again by Theorem 4.12.

Let now C^* be a σ-feasible coalition such that $val_\sigma(v, C^*) \geq val_\sigma(v, C)$, for each $C \in \mathcal{F}_\sigma(\Gamma)$, and let us distinguish two cases. First, there can exist a pivotal agent $a_i \in S$ such that $a_i \in C^*$, in which case we can apply the construction in (1) for deriving that $vc_i^* = val_\sigma(v, C^*)$. Otherwise, we have that $C^* \cap S = \emptyset$, in which case we can apply the construction in (2) for deriving that $vc_j^* = val_\sigma(v, C^*)$, for each agent $a_j \in C^*$. Since the values of the form vc_i^* and vc_j^* correspond to valuations of σ-feasible coalitions, by putting the above ingredients together, we conclude that:

$$\max\{\max_{a_i \in S} vc_i^*, \max_{a_j \in N \setminus S} vc_j^*\} = val_\sigma(v, C^*).$$

In order to conclude the proof, note that the left side of the above equation involves polynomially-many values, and recall that each of them can be computed in polynomial time. Therefore, $val_\sigma(v, C^*)$ (and a σ-feasible coalition where this value is attained) can be overall computed in polynomial time, too. □

Let us now move to the second task. Let $\Gamma = \langle N, v\rangle$ be a coalitional game with $n = |N|$, let σ be a valuation structure, and let $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ be a payoff vector. For each feasible coalition $C \in \mathcal{F}_\sigma(\Gamma)$, we define the *excess* of C at $\hat{\boldsymbol{x}}$ (in Γ w.r.t. σ) as the value $e(\hat{\boldsymbol{x}}, C, \Gamma, \sigma) = \hat{x}(C) - val_\sigma(v, C)$ (shortly denoted as $e(\hat{\boldsymbol{x}}, C)$, when Γ and σ are clearly understood). Intuitively, this is a measure of the satisfaction of the agents in C when they receive the payoff $\hat{x}(C)$. Indeed, the agents in C are satisfied by the given payoff distribution and would not like to deviate from $\hat{\boldsymbol{x}}$ if, and only if, $e(\hat{\boldsymbol{x}}, C) \geq 0$. The problem of deciding whether the minimum possible excess is non-negative has been already studied in the literature for classes of games encoded via marginal contribution networks. In particular, it has been shown to be feasible in polynomial time over the class $\mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$, in absence of valuation structures and for any fixed natural number h [43]. Based on Theorem 6.3, we now extend this result to arbitrary valuation structures.

Theorem 6.4. *Let $h \geq 0$ be a fixed natural number. Let $\Gamma = \langle N, v\rangle \in \mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$ with $n = |N|$, let $\sigma = \langle G, S, \alpha, \beta, x, y\rangle$ be a valuation structure, and let $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ be a payoff vector. Then, computing a σ-feasible coalition C^* such that $e(\hat{\boldsymbol{x}}, C^*, \Gamma, \sigma) \leq e(\hat{\boldsymbol{x}}, C, \Gamma, \sigma)$, for each $C \in \mathcal{F}_\sigma(\Gamma)$, is feasible in polynomial time (w.r.t. $\|\xi^{\mathsf{mcn}}(\Gamma)\|$, $\|\sigma\|$, and $\|\hat{\boldsymbol{x}}\|$).*

Proof (*Sketch*). Let $\xi^{\mathsf{mcn}}(\Gamma)$ be the marginal contribution network encoding the coalitional game Γ. For each w, consider the marginal contribution network M_w defined as follows. For each rule $\{pattern\} \rightarrow value$ in $\xi^{\mathsf{mcn}}(\Gamma)$, M_w includes the rule $\{pattern\} \rightarrow w \times value$. For each agent $a_i \in N$, M_w includes the rule $\{a_i\} \rightarrow -\hat{x}_i$. No further rule is in M_w. Note that the network M_w induces a valuation function v_w such that $v_w(C) = -\hat{x}(C) + w \times v(C)$, for each $C \in \mathcal{F}_\sigma(\Gamma)$. The idea is then to apply the construction reported in the proof of Theorem 6.3 on the valuation structure $\langle G, S, \mathbf{1}, \beta, 1, y\rangle$ and the valuation function v_w, where w is defined as follows. When we are in charge of dealing with case (1), for each agent a_i in S, we define $w = \alpha(a_i)$. Instead, when we deal with (2), for each agent $a_j \in N \setminus S$, we define $w = x$. Then, by inspecting the proof of Theorem 6.3, we derive that in the case (1), vc_i^* coincides with the minimum excess computed over any coalition including agent a_i; while in the case (2), vc_j^* coincides with the minimum excess computed over any coalition that includes agent a_j and does not include any agent in S. In particular, note that the maximum values vc_i^* and vc_j^* coincide with minimum excesses because the valuation functions are opposite in sign. Eventually, by computing the maximum value over all agents $a_i \in S$ and $a_j \in N \setminus S$, we end up with the minimum possible excess in polynomial time. □

Now, we have all ingredients in place to prove the main result of this section. We further stress that the result is given for marginal contribution networks having bounded treewidth but, because of the characterizations derived in Section 4.1, it actually holds on any `IDM` function w.r.t. an interaction graph having bounded treewidth.

Theorem 6.5. *Let $h \geq 0$ be a fixed natural number. Let $\Gamma = \langle N, v\rangle \in \mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$ and let $\sigma = \langle G, S, \alpha, \beta, x, y\rangle$ be a valuation structure. Then,* CS-Core-Find *can be solved on (Γ, σ) in polynomial time (w.r.t. $\|\xi^{\mathcal{R}}(\Gamma)\|$ and $\|\sigma\|$).*

Proof. Let us consider CS-Core-Find. We are given a game $\Gamma = \langle N, v\rangle \in \mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$ and we have to compute an element in CS-$core_\sigma(\Gamma)$, or decide that this set is empty. First, we compute a σ-optimal coalition structure $\Pi^* \in$ CS-$opt_\sigma(\Gamma)$, which is feasible in polynomial time by Theorem 4.12. Because of Theorem 6.1, Core-Find can be reduced to computing a payoff vector $\boldsymbol{x}$ such that $x(C) = val_\sigma(v, C)$, for each $C \in \Pi^*$; and $x(C) \geq val_\sigma(v, C)$, for each $C \in \mathcal{F}_\sigma(\Gamma)$ (or decide that no vector of this kind exist). Note that $\boldsymbol{x}$ is just a vector that satisfies a system of linear (in)equalities defined over $|N|$ variables and possibly exponentially-many (in)equalities. It is well-known that a system of this kind can be solved in polynomially many steps, where each step consists of asking to a *separation oracle* whether a given vector is a solution and, if not, to report an (in)equality that is violated [63]. In particular, the whole computation is feasible in polynomial time, provided the existence of a separation oracle whose invocations take polynomial time. In our case, an oracle of this kind is as follows. Given a vector $\boldsymbol{x}$, we can trivially check whether $x(C) = val_\sigma(v, C)$, for each $C \in \Pi^*$. If some of the (polynomially-many) equalities is violated, then it is reported as output. Otherwise, we need to check whether $x(C) \geq val_\sigma(v, C)$, for each $C \in \mathcal{F}_\sigma(\Gamma)$. To this end, we compute a σ-feasible coalition C^* getting the minimum possible excess (at $\boldsymbol{x}$) over all possible feasible coalitions. Note that, by Theorem 6.4, this task is feasible in polynomial time. If $e(\boldsymbol{x}, C^*, \Gamma, \sigma) \geq 0$, then no inequality is violated and we can return $(\Pi^*, \boldsymbol{x})$ as an element in the core. Otherwise, i.e., if $e(\boldsymbol{x}, C^*, \Gamma, \sigma) < 0$, then we know that the inequality $x(C^*) \geq val_\sigma(v, C^*)$ is violated by $\boldsymbol{x}$ and it is returned as output by the oracle. □

7. Conclusion

The coalition structure generation problem has been reconsidered in this paper, by taking into account constraints on the allowed coalitions defined in terms of "valuation structures". A clear picture of the complexity issues arising in this setting has been depicted, by singling out islands on tractability based on the properties of the underlying valuation functions and of the underlying interaction graphs. Notably, the results have been formulated in a way that is independent of the specific scheme being adopted to represent the valuations functions. In particular, they hold for any given compact encoding. A number of concrete applications have been discussed, too.

Our work paves the way for further investigations. First, it would be interesting to study further kinds of constraints that are likely to occur in real-world application domains, in addition to those associated with pivotal agents and interaction graphs. With this respect, inspired by constrained clustering methods (see, e.g., [72]), we envisage that general forms of *cannot-link* constraints can be profitably incorporated in the concept of valuation structure, in order to prescribe that some pivotal agent a in S is "incompatible" with some proper subset of $S \setminus \{a\}$—currently, our setting accommodates only the case where a is "incompatible" with all the remaining pivotal agents in $S \setminus \{a\}$. Further extensions can be defined to deal with a-priori knowledge on the sizes of the coalitions that are allowed to form [7] or with labeling mechanisms more sophisticated than those discussed in Section 2.4 (inspired, e.g., to [12,20]).

Orthogonally to the above directions, it would be relevant to extend the analysis of the questions related to worth distribution problems in the presence of valuation structures to further solution concepts. In the paper, we have focused on the core and, hence, we mainly dealt with stability issues. Other solution concepts proposed in the literature, for instance the *Shapley value*, look at worth distribution problems by taking into account fairness issues, too. In fact, it is well-known that computing the Shapley value is an intractable problem in general, formally #P-complete. However, recent research evidenced that, on some classes of games and when certain structural restrictions are considered, the value can be computed in polynomial time (see, e.g., [38,3]). An interesting question is therefore whether the Shapley value retains this desirable computational property on IDM functions and in presence of valuation structures.

Finally, from the practical viewpoint, the most natural avenue of further research is to embark on the implementation of the approaches we have exhibited to single out islands of tractability and to conduct experimental activity to assess their scalability on synthetic benchmarks and real-world instances.

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Appendix A. Notation

Table A.1 reports the notation used consistently throughout the paper.

Appendix B. Results in Section 2.2

Theorem B.1. *Let $E' \subseteq E$ be a set of edges. Then, the following properties hold:*

(1) *If E' is a multicut, then there is a $\perp$-feasible coalition structure Π for $\langle N, v_P \rangle$ with $E' = \{e \in E \mid \forall C_j \in \Pi, |e \cap C_j| \leq 1\}$ and $val_\perp(\Pi) = \sum_{e \in E} w_e - \sum_{e' \in E'} w_{e'} \geq 0$.*
(2) *If Π is a $\perp$-feasible coalition structure for $\langle N, v_P \rangle$ with $val_\perp(\Pi) \geq 0$, then $E' = \{e \in E \mid \forall C_j \in \Pi, |e \cap C_j| \leq 1\}$ is a multicut and $val_\perp(\Pi) = \sum_{e \in E} w_e - \sum_{e' \in E'} w_{e'}$.*

Proof. *(1)* Assume that E' is a multicut, and let $C_1, \ldots, C_k$ be the connected components in the graph $(N, E \setminus E')$. If $e \in E$ is an edge such that $|e \cap C_j| \leq 1$ holds, for each $j \in \{1, \ldots, k\}$, then e belongs to the multicut E'. That is, by letting $\Pi = \{C_1, \ldots, C_k\}$, it clearly holds that $E' = \{e \in E \mid \forall C_j \in \Pi, |e \cap C_j| \leq 1\}$. Moreover, Π exhaustively covers all the nodes in N, and $C_i \cap C_j = \emptyset$ holds, for each pair of distinct indices $i, j \in \{1, \ldots, k\}$. Therefore, Π is a $\perp$-feasible coalition structure for $\langle N, v_P \rangle$.

Table A.1
Notation used in the paper.

Notation	Description
$\Gamma = \langle N, v\rangle$	coalitional game
$C \subseteq N$	coalition
$G = (N, E)$	undirected graph, where agents in N are viewed as nodes
$\sigma = \langle G, S, \alpha, \beta, x, y\rangle$	valuation structure
$\mathcal{F}_\sigma(\Gamma)$	the set of all σ-feasible coalitions in Γ
$val_\sigma(v, C)$	the σ-value of C w.r.t. v
$val_\sigma(C)$	shorthand for $val_\sigma(v, C)$, when v is clearly understood
$\Pi = \{C_1, \ldots, C_k\}$	coalition structure, with $C_1, \ldots, C_k \in \mathcal{F}_\sigma(\Gamma)$
$\mathcal{CS}_\sigma(\Gamma)$	the set of all σ-feasible coalition structures for Γ
$val_\sigma(v, \Pi)$	the σ-value of Π w.r.t. v
$val_\sigma(\Pi)$	shorthand for $val_\sigma(v, \Pi)$, when v is clearly understood
$opt_\sigma(\Gamma)$	the maximum possible σ-value over $\mathcal{CS}_\sigma(\Gamma)$
CS-$opt_\sigma(\Gamma)$	the set of all σ-optimal coalition structures on Γ
$\langle G\rangle$	shorthand for valuation structures $\langle G, \emptyset, \alpha, \beta, x, y\rangle$
K_N	the complete graph over N
$\perp$	shorthand for the valuation structure $\langle K_N\rangle$
$\mathbf{c}$	the constant function mapping each element to $c \in \mathbb{R}$
$\mathcal{C}(\mathcal{R})$	the class of all games encoded via the representation $\mathcal{R}$
$\xi^{\mathcal{R}}(\Gamma)$	the encoding of Γ according to $\mathcal{R}$
$v^{\mathcal{R}}(\xi^{\mathcal{R}}(\Gamma), C)$	the value of C according to v as returned by $\mathcal{R}$
$\\| \cdot \\|$	the function returning the size of the encoding
$tw(\cdot)$	the function returning the treewidth of a graph
gg	the graph-game encoding
mcn	the encoding via marginal contribution networks
$\mathrm{AG}(M)$	the agent graph of the marginal contribution network M
$\mathcal{C}_{\mathbf{TW}\text{-}h}(\mathsf{mcn})$	the restriction of $\mathcal{C}(\mathsf{mcn})$ to agent graphs having treewidth at most h
$\mathcal{I} = \langle Var, U, \mathbf{C}\rangle$	CSP instance
$CSP(\Gamma, \sigma)$	the CSP instance associated with Γ and σ
$\mathrm{PG}(\mathcal{I})$	the primal graph of the CSP instance $\mathcal{I}$
$\mathcal{E}_\sigma(v, \Pi)$	the set of all efficient vectors for Π w.r.t. v
$\mathcal{E}_\sigma(\Pi)$	shorthand for $\mathcal{E}_\sigma(v, \Pi)$, when v is clearly understood
CS-$core_\sigma(\Gamma)$	the coalition structure core of Γ w.r.t. σ

Moreover, it holds that $val_\perp(\Pi) = \sum_{j=1}^{k} v_P(C_j)$, where $v_P(C_j)$ is the sum of the weights of the edges in the subgraph of G_P induced over C_j. In fact, every pair in P is disconnected by E', so that its endpoints belong to different components. This entails that $v_P(C_j)$ is also the sum of the weights of the edges in the subgraph of G induced over C_j. Hence, $val_\perp(\Pi)$ is just the sum of all the weights of the edges in G, except those belonging to E'. Formally, $val_\perp(\Pi) = \sum_{e\in E} w_e - \sum_{e'\in E'} w_{e'}$.

(2) Assume that Π is a $\perp$-feasible coalition structure such that $E' = \{e \in E \mid \forall C_j \in \Pi, |e \cap C_j| \leq 1\}$ and $val_\perp(\Pi) \geq 0$. By definition of the valuation function, this entails that for each source-terminal pair in P, its endpoints belongs to distinct coalitions of Π. That is, the set E' of the edges that are not entirely covered by any coalition in Π forms a multicut. In particular, Π is precisely the coalition structure associated with E' according to point *(1)*, and hence $val_\perp(\Pi) = \sum_{e\in E} w_e - \sum_{e'\in E'} w_{e'}$. □

Theorem B.2. *Let $E' \subseteq E$ be set of edges. Then, the following properties hold:*

(1) *If E' is a multiway cut, then there is a σ_{mw}-feasible coalition structure Π for $\langle N, v_{mw}\rangle$ such that $E' = \{e \in E \mid \forall C_j \in \Pi, |e \cap C_j| \leq 1\}$ and $val_{\sigma_{mw}}(\Pi) = \sum_{e\in E} w_e - \sum_{e'\in E'} w_{e'}$.*
(2) *If Π is a σ_{mw}-feasible coalition structure for $\langle N, v_{mw}\rangle$, then $E' = \{e \in E \mid \forall C_j \in \Pi, |e \cap C_j| \leq 1\}$ is a multiway cut and $val_{\sigma_{mw}}(\Pi) = \sum_{e\in E} w_e - \sum_{e'\in E'} w_{e'}$.*

Proof. *(1)* If E' is a multiway cut and $C_1, \ldots, C_k$ are the connected components in the graph $(N, E \setminus E')$, then $|C_j \cap T| \leq 1$ holds, for each $j \in \{1, \ldots, k\}$. Hence, $\Pi = \{C_1, \ldots, C_k\}$ is a σ_{mw}-feasible coalition structure. Eventually, we trivially have that $E' = \{e \in E \mid \forall C_j \in \Pi, |e \cap C_j| \leq 1\}$ and $val_{\sigma_{mw}}(\Pi) = \sum_{e\in E} w_e - \sum_{e'\in E'} w_{e'}$.

(2) If Π is a σ_{mw}-feasible coalition structure, then $|C_j \cap T| \leq 1$ holds, for each $C_j \in \Pi$. Therefore, the set $E' = \{e \in E \mid \forall C_j \in \Pi, |e \cap C_j| \leq 1\}$ is a multiway cut. Again, by definition of the valuation function, we have $val_{\sigma_{mw}}(\Pi) = \sum_{e\in E} w_e - \sum_{e'\in E'} w_{e'}$. □

Theorem B.3. *Let Π be a clustering. Then, the following properties hold:*

(1) *If Π is a k-clustering, then there is a set $R \subseteq N$ with $|R| = k$ and such that Π is a σ_R-feasible coalition structure for $\langle N, v_{cc}\rangle$ with $val_{\sigma_R}(\Pi) = \sum_{C_i\in\Pi} v_{cv}(C_i) \geq 0$.*

(2) *If there is a set $R \subseteq N$ of agents with $|R| = k$ and such that Π is a σ_R-feasible coalition structure for $\langle N, v_{cc}\rangle$ with $val_{\sigma_R}(\Pi) \geq 0$, then Π is a k-clustering and $val_{\sigma_R}(\Pi) = \sum_{C_i \in \Pi} v_{cc}(C_i)$.*

Proof. *(1)* Let Π be a k-clustering. Thus, Π has the form $\{C_1, \ldots, C_k\}$ where $C_1 \cup \cdots \cup C_k = N$ and $C_i \cap C_j = \emptyset$, for each pair of distinct indices i and j. For each $i \in \{1, \ldots, k\}$, let r_i be an agent in C_i, and define $R = \{r_1, \ldots, r_k\}$. Then, Π is a σ_R-feasible coalition structure, where $\sigma_R = \langle G, R, \mathbf{1}, \mathbf{0}, 0, -\infty\rangle$. In particular, observe that the subgraph induced over C_i is connected, for each $i \in \{1, \ldots, k\}$, by definition of cluster. Hence, both pivotal and connectedness conditions hold. Moreover, $val_{\sigma_R}(\Pi) = \sum_{C_i \in \Pi} v_{cc}(C_i)$ holds, by definition of σ_R and since there is no coalition in Π that does not include an agent in R.

(2) Assume that Π is a σ_R-feasible coalition structure with $|R| = k$ and $val_{\sigma_R}(\Pi) \geq 0$. Since $|R| = k$, we have that $|\Pi| \geq k$ holds. Moreover, if the inequality were strict, then we would have $val_{\sigma_R}(\Pi) < 0$, because of the fact that $\sigma_R = \langle G, R, \mathbf{1}, \mathbf{0}, 0, -\infty\rangle$. Therefore, $|\Pi| = k$. Moreover, Π is a k-clustering because the connectedness condition holds on each of its coalitions (again because of the σ_R-feasibility), and we have $val_{\sigma_R}(\Pi) = \sum_{C_i \in \Pi} v_{cc}(C_i)$. $\square$

Theorem B.4. *The following properties hold:*

(1) *If $\{(C_1, c_1), \ldots, (C_h, c_h)\}$ is a chromatic partition, then $\Pi = \{C_1 \cup \{c_1\}, \ldots, C_h \cup \{c_h\}\} \cup \{\{c'\} \mid c' \in \mathcal{C} \setminus \{c_1, \ldots, c_h\}\}$ is a σ_{ch}-feasible coalition structure for $\langle N, v_{ch}\rangle$ such that $val_{\sigma_{ch}}(\Pi) = \sum_{C_i \in \Pi} \sum_{e \in E, e \subseteq C_i} w_e^{c_i}$.*

(2) *If Π is a σ_{ch}-feasible coalition structure for $\langle N, v_{ch}\rangle$ with $val_{\sigma_{ch}}(C_i) \neq -\infty$, for each $C_i \in \Pi$, then there is a chromatic partitioning $\{(C_1, c_1), \ldots, (C_h, c_h)\}$ such that $\Pi = \{C_1 \cup \{c_1\}, \ldots, C_h \cup \{c_h\}\} \cup \{\{c'\} \mid c' \in \mathcal{C} \setminus \{c_1, \ldots, c_h\}\}$ and $val_{\sigma_{cp}}(\Pi) = \sum_{C_i \in \Pi} \sum_{e \in E, e \subseteq C_i} w_e^{c_i}$.*

Proof. *(1)* If $\{(C_1, c_1), \ldots, (C_h, c_h)\}$ is a chromatic partition, then $c_i \neq c_j$ for each $i, j \in \{1, \ldots, h\}$ with $i \neq j$. Therefore, Π satisfies the pivotal condition. Moreover, for each $i \in \{1, \ldots, h\}$, we have that each agent in C_i is connected to c_i in $G_{\mathcal{C}}$. Hence, the connectivity condition is satisfied too, and Π is a σ_{ch}-feasible coalition structure. Eventually, since there is no coalition in Π that does not contain a pivotal agent, we immediately get $val_{\sigma_{cp}}(\Pi) = \sum_{C_i \in \Pi} \sum_{e \in E, e \subseteq C_i} w_e^{c_i}$.

(2) If Π is a σ_{ch}-feasible coalition structure with $val_{\sigma_{cp}}(C_i) \neq -\infty$, for each $C_i \in \Pi$, then Π includes no coalition without pivotal agents. Thus, there is a chromatic partitioning $\{(C_1, c_1), \ldots, (C_h, c_h)\}$ such that $\Pi = \{C_1 \cup \{c_1\}, \ldots, C_h \cup \{c_h\}\} \cup \{\{c'\} \mid c' \in \mathcal{C} \setminus \{c_1, \ldots, c_h\}\}$ and $val_{\sigma_{cp}}(\Pi) = \sum_{C_i \in \Pi} \sum_{e \in E, e \subseteq C_i} w_e^{c_i}$. $\square$

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