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Research Note

Lattice-theoretic models of conjectures, hypotheses and consequences

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Abstract

Trillas, Cubillo and Castiñeira [Artificial Intelligence 117 (2000) 255–275] defined several interesting operators in orthocomplemented lattices. These operators give a quite general algebraic model for conjectures, consequences and hypotheses. We present some properties of conjectures, consequences and hypotheses in orthocomplemented lattices, which complement or improve the results by Trillas, Cubillo and Castiñeira. Furthermore, we introduce the graded versions of these notions in the setting of residuated lattices and derive some of their properties. These graded notions provide certain mathematical tools for modelling conjectures, consequences and hypotheses in the environment where uncertain and vague information is involved. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Roughly speaking, scientific research is a process in which one proposes certain hypotheses, makes some conjectures, and then verifies or refutes them by experiments

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or logical reasoning [9,16]. As a branch of Artificial Intelligence, Knowledge Discovery provides techniques with which computers can automatically find fundamental knowledge and principles that are original and useful [10]. Therefore, a fundamental issue of the theoretical aspect of Knowledge Discovery is to establish some suitable mathematical models for conjectures, consequences and hypotheses. In a sense, the whole subject of mathematical logic aims at studying various mathematical models of consequences (i.e., reasoning). On the other hand, not too much attention has been paid to conjectures and hypotheses in the community of mathematical logic. Trillas, Cubillo and Castiñeira [19] gave a quite general algebraic model for conjectures and hypotheses, as well as consequences. In their model, all statements or propositions of human thinking are represented as elements of an orthocomplemented lattice. They defined several interesting operators in orthocomplemented lattices. These operators formalize our intuitive notions of conjecture, consequence and hypothesis in an algebraic framework. Working in this model, they were able to present a nice classification of conjectures and consequences and to give a characterization of hypotheses.

Orthocomplemented lattices are very general algebraic structures, and so Trillas, Cubillo and Castiñeira's model provides us with a sufficiently broad framework in which we can describe mathematically conjectures and hypotheses. First, Boolean algebras are orthocomplemented lattices. Thus, the results obtained in [19] applies to Boolean (two-valued) logic. For example, Theorems 6.8 and 6.9 in [19] generalize Watanabe's structure theorem of hypotheses [20]; and Subsection 5.3 of [19] renders an abstract setting in which Reiter's Default Logic [17] can be reformulated. Second, orthocomplemented lattices contain orthomodular lattices. Orthomodular lattices are often seen as the algebraic counterpart of logic for quantum mechanics [2,15]. Perhaps, Trillas, Cubillo and Castiñeira's model can be used to depict some research activities in the progress of quantum mechanics. We can even anticipate that certain Knowledge Discovery techniques developed in the future based on the model proposed in [19] will help quantum physicists in their researches.

This paper is a continuation of [19], and the technical contribution is two-fold: (1) we improve and complement some main results in [19]; and (2) we propose a graded version of the model in [19] so that it is able to deal with uncertainty and vagueness. The main body of this paper is divided into two parts. In the first part (i.e., Section 2), we present some further properties of conjectures, consequences and hypotheses in orthocomplemented lattices. Theorem 2.2 of [19] establishes several inclusion relations between the operators of conjecture, consequence and hypothesis, but leaves the problem when these inclusions will become equalities open. We solve this problem and find several sufficient and necessary conditions under which the inclusions may be replaced by equalities (see Theorem 2.1). This is obviously a complement to Theorem 2.2 in [19]. Moreover, Theorem 2.2 in the present paper gives a representation for each of the consequence, conjecture and hypothesis operators in terms of the others; and Theorem 2.8 examines various iterations of these operators. This group of theorems thoroughly clarify the relation among these operators. Theorem 2.3 and Corollary 2.4 in [19] only show the up- and down-closedness of conjectures, consequences and hypotheses, and Theorem 2.3 of this paper further considers their closedness under union and meet of the underlying lattice and carefully describes their algebraic structures. Trillas, Cubillo and Castiñeira give the smallest and greatest elements in a set of consequences (cf. [19, Theorem 2.5]), whereas we are able to compute the same thing for conjectures and hypotheses (see Theorem 2.4). Under the condition that there is a bijection with elements of the considered premise set as its pre-fixed points, Theorem 2.7 in [19] demonstrates monotonicity of consequence, conjecture and hypothesis operators, and Theorem 2.5 of the present paper considerably weakens this condition. Furthermore, Theorems 2.6 and 2.7 show that these operators are homomorphisms with respect to set union and intersection. Note that monotonicity is implied by the substitution property of such homomorphisms. Finally, Theorem 2.9 in this paper generalizes Theorems 2.11 and 2.12 into the situation where infinite-ary operations on the lattice under consideration are allowed. In summary, the results in Section 2 of the present paper complement or improve those reported in [19].

Many artificial intelligence systems use a certain sort of uncertain reasoning to lead from evidences or clues to guesses and conclusions under condition of partial or vague information. Trillas, Cubillo and Castiñeira pointed out that the impact of certainty factors of premises on the different types of conjectures is an interesting topic for further study (see [19, p. 256, lines 35–39], also see [3,4]). The second part of this paper (i.e., Section 3) is devoted to introducing the graded versions of notions of conjecture, consequence and hypothesis in the framework of residuated lattices. These graded notions provide us with certain mathematical tools with which we can model conjectures, consequences and hypotheses in the environment where uncertain and vague information is involved. Various properties concerning these graded operators are also carried out in Section 3. To define graded notions of conjecture, consequence and hypothesis, an implication operator in the underlying lattice is needed, and it will be used to compare two elements (as the representations of two statements) of this lattice and to express the degree to which one statement entails the other. It turns out that all implication operators that one can reasonably imagine in orthocomplemented lattices are, to a certain extent, anomalous, at least in the sense that they do not share most of the fundamental properties which are satisfied by the implication in classical logic (see [5, p. 431, lines 21–25]). By contrast, the residuation operation in a residuated lattice [6,7] may serve as a natural interpretation of logical implication, and it possesses a lot of desirable properties. In particular, it retains many useful properties of implication in classical and intuitionistic logics. From this point of view, orthocomplemented lattices and residuated lattices are at the opposite extreme points of the spectrum of lattices. Moreover, residuated lattices have close links with various important logics. For example, if the multiplication adjoint to the residuation coincides with the meet, then residuated lattices reduce to Heyting algebras, which play an important role in the investigation of intuitionistic formal theories. Goguen, Pavelka, et al. [12–14,21–27] used residuated lattice as a tool to cope with inexact reasoning and as a basis of fuzzy logic. If we take the unit interval with usual multiplication, a special residuated lattice, as the set of truth values, a truth-functional system of probabilistic logic [18] may be seen as a residuated lattice-valued logic. (Note that a truth-functional version of probabilistic logic is slightly different from that in [11] which is a non-truth-functional system.) From the point of view of algebraic semantics, pure fragment of commutative linear logic [8] is a residuated lattice-valued logic. This is why we choose residuated lattices as our underlying lattices for defining graded operators of conjectures, consequences and hypotheses.

2. Some properties of conjectures, hypotheses and consequences in orthocomplemented lattices

An orthocomplemented lattice is a 6-tuple $(L, +, ., ', 1, 0)$, where $(L, +, .)$ is a lattice, it has the greatest element 1 and the least element 0, $+$ and . represent the union and meet operations respectively, and $'$ is a unary operation on L, called orthocomplementation, satisfying the following conditions:

(1) $a.a' = 0$,
(2) $a'' = a$, and
(3) $a \leqslant b$ implies $b' \leqslant a'$.

A typical example of orthocomplemented lattice is a Boolean algebra, say the Lindenbaum algebra of propositional logical formulas. Several orthocomplemented lattices that are not Boolean algebras are shown in Fig. 4(1)–(4) of [19]. In this section, we always assume that L is a complete orthocomplemented lattice. For any $P \subseteq L$, we write $\wedge P$ and $\vee P$ for the greatest lower bound and the least upper bound of P respectively. Thus, $a.b = \wedge\{a, b\}$ and $a + b = \vee\{a, b\}$ for all $a, b \in L$. As was done in [19], $\wedge P$ and $\vee P$ are often abbreviated to $p_\wedge$ and $p_\vee$ respectively. Let $a, b \in L$. Then a and b are said to be contradictory if $a \leqslant b'$, and they are incompatible if $a.b = 0$.

The aim of [19] is to establish an algebraic model of conjectures, consequences and hypotheses in the framework of orthocomplemented lattices. In such a model, an element of L will be used to denote a statement or a proposition. The main idea is to analyse the relations, among the elements of L, which depict the intuitive notions of conjectures, consequences and hypotheses. We denote $\boldsymbol{P}_0(L) = \{P \subseteq L\colon\ P \neq \phi \text{ and } p_\wedge \neq 0\}$. Each element in $\boldsymbol{P}_0(L)$ will represent a non-empty set of premises. The condition $p_\wedge \neq 0$ indicates that the premises in P are compatible.

For any $P \in \boldsymbol{P}_0(L)$, Trillas, Cubillo and Castiñeira [19] defined:

$$\Phi_\vee(P) = \{q \in L\colon\ p_\vee \nleqslant q'\},$$
$$\Phi_\wedge(P) = \{q \in L\colon\ p_\wedge \nleqslant q'\},$$
$$C_\vee(P) = \{q \in L\colon\ p_\vee \leqslant q\},$$
$$C_\wedge(P) = \{q \in L\colon\ p_\wedge \leqslant q\},$$
$$H(P) = \{q \in L\colon\ q \leqslant p_\wedge\}.$$

Intuitively, the operators Φ, C, H represent conjectures, consequences and hypotheses, respectively. The elements of $\Phi_\vee(P)$ are called conjectures of P, and $q \in \Phi_\vee(P)$ means that some $p \in P$ are not contradictory to q. $\Phi_\wedge(P)$ is the set of strict conjectures of P, and $q \in \Phi_\wedge(P)$ means that all $p \in P$ are not contradictory to q. The difference between $\Phi_\vee(P)$ and $\Phi_\wedge(P)$ as well as between $C_\vee(P)$ and $C_\wedge(P)$ is the combination of elements in the set P of premises. In $\Phi_\vee(P)$, these premises are combined with union (i.e., disjunction), whereas in $\Phi_\wedge(P)$ they are combined with meet (i.e., conjunction). The same thing happens in $C_\vee(P)$ and $C_\wedge(P)$. $C_\wedge(P)$ is the set of consequences of P, and $q \in C_\wedge(P)$ indicates that all premises in P together imply q. On the other hand, $C_\vee(P)$

is the set of loose consequences of P, and q is a loose consequence of P if some premise in P implies q. $H(P)$ is the set of hypotheses of P, and each hypothesis of P implies all statements in P. Furthermore, Trillas, Cubillo and Castiñeira [19] introduced some derived operators:

$$C(P) = \{q \in L\colon\ p_\wedge \leqslant q \leqslant p_\vee\},$$
$$\Phi(P) = \{q \in L\colon\ p_\wedge \nleqslant q' \nleqslant p_\vee\},$$
$$H^*(P) = H(P) - \{p_\wedge\},$$
$$\Phi^*_\wedge(P) = \Phi_\wedge(P) - \big(C_\wedge(P) \cup H(P)\big),$$
$$\Phi^*_\vee(P) = \Phi_\vee(P) - \Phi_\wedge(P),$$

where $P \in \boldsymbol{P}_0(L)$ and for $\Phi(P)$, it is required that $p_\vee \neq 1$. Intuitively, $C(P)$ are the restricted consequences of P, $H^*(P)$ is the set of proper hypotheses of P, $\Phi^*_\wedge(P)$ is the set of proper conjectures of P, $\Phi(P)$ is the set of strict and restricted conjectures of P, and $\Phi^*_\vee(P)$ are the loose conjectures of P. If we use L to denote the Lindenbaum algebra of propositional logical formulas, then the intuitive meanings of these operators are clear. For example, let P be a set of formulas. Then $\phi \in \Phi_\vee(P)$, i.e., ϕ is a conjecture of P if and only if each ψ in P does not implies the negation of ϕ. For more examples, we refer to [19, Sections 3 and 6.3]. These notions are also used to model Reiter's Default Reasoning [17] in Section 5.2 of [19].

The inclusion relations among $\Phi_\vee(P)$, $\Phi_\wedge(P)$, $C_\vee(P)$, $C_\wedge(P)$ and $H(P)$ are as follows (see [19, Theorem 2.2]):

(a) $C_\vee(P) \subseteq C_\wedge(P) \subseteq \Phi_\wedge(P) \subseteq \Phi_\vee(P)$.
(b) $H(P) \subseteq \Phi_\wedge(P)$.
(c) $P \subseteq C_\wedge(P)$.
(d) $C_\wedge(P) \cap H(P) = \{p_\wedge\}$.

An interesting problem is when these inclusion relations degenerate to equalities. Among other things, the following theorem gives an answer to this problem.

Theorem 2.1.

(1) *The following statements are equivalent*:
  (a) *P is a singleton, i.e., P has a unique element.*
  (b) $H(P) \cap C_\vee(P) \neq \phi$.
  (c) $P \subseteq C_\vee(P)$.
  (d) $P \subseteq H(P)$.
  (e) $\Phi_\vee(P) = \Phi_\wedge(P)$, *i.e.*, $\Phi^*_\vee(P) = \phi$.
  (f) $C_\wedge(P) = C_\vee(P)$.
(2) $P \cap C_\vee(P) \neq \phi$ *if and only if* $p_\vee \in P$.
(3) $P \cap H(P) \neq \phi$ *if and only if* $p_\wedge \in P$.
(4) $\Phi_\wedge(P) = C_\wedge(P)$ *if and only if* $p_\wedge$ *is atomic, where* $a \in L$ *is said to be atomic if for each* $b \in L$, *it always holds that* $a \leqslant b$ *or* $a \leqslant b'$.

(5) $\Phi_\wedge(P) = H(P)$ *if and only if* $P = \{1\}$.
(6) $P = C_\wedge(P)$ *if and only if* P *is a prime filter, i.e.,* $p_\wedge \in P$ *and* P *is upper-closed* (*i.e.,* $p \leqslant q$ *and* $p \in P$ *implies* $q \in P$).
(7) $\Phi_\wedge(P) = C_\wedge(P) \cup H(P)$, *i.e.,* $\Phi^*_\wedge(P) = \phi$ *if and only if* $p_\wedge$ *is sub-atomic, where* $a \in L$ *is said to be sub-atomic if* $a \nleqslant b$ *implies that* a *and* b' *are comparable, i.e.,* $a \leqslant b$ *or* $b' \leqslant a$.

Proof. Immediate. □

To clarify further the relation among the operators $\Phi_\vee, \Phi_\wedge, C_\vee, C_\wedge$ and H, we now consider how one of them can be described in terms of the others.

Theorem 2.2. *For any* $Q \subseteq L$, *we write* $Q' = \{q' \colon q \in Q\}$. *Then*

(1) $\Phi_\wedge(P) = L - (C_\wedge(P))'$, $C_\wedge(P) = L - (\Phi_\wedge(P))'$;
(2) $\Phi_\vee(P) = L - (C_\vee(P))'$, $C_\vee(P) = L - (\Phi_\vee(P))'$; *and*
(3) $C_\vee(P) = (H(P'))' \cup \{1\}$.

Proof. Straightforward. □

Now we consider the structures of $\Phi_\vee(P), \Phi_\wedge(P), C_\vee(P), C_\wedge(P)$ and $H(P)$. Recall that $X \subseteq L$ is called a filter if $a, b \in X$ implies $a.b \in X$, and $a \leqslant b$ and $a \in X$ imply $b \in X$. Dually, $X \subseteq L$ is called an ideal if $a, b \in X$ implies $a + b \in X$, and $a \leqslant b \in X$ implies $a \in X$.

Theorem 2.3.

(1) $\Phi_\vee(P)$ *is prime, i.e., for any* $q_i \in L$ $(i \in I)$, *if* $\bigvee_{i \in I} q_i \in \Phi_\vee(P)$, *then* $q_{i_0} \in \Phi_\vee(P)$ *for some* $i_0 \in I$.
(2) $\Phi_\wedge(P)$ *is prime.*
(3) $C_\vee(P)$ *is closed under any intersection, i.e., for any* $q_i \in L$ $(i \in I)$, *if* $q_i \in C_\vee(P)$ *for every* $i \in I$, *then* $\bigwedge_{i \in I} q_i \in C_\vee(P)$. *So,* $C_\vee(P)$ *is a filter.*
(4) $C_\wedge(P)$ *is closed under any intersection.*
(5) $H(P)$ *is closed under any union, i.e., for any* $q_i \in L$ $(i \in I)$, *if* $q_i \in C_\vee(P)$ *for every* $i \in I$, *then* $\bigvee_{i \in I} q_i \in H(P)$. *So,* $H(P)$ *is an ideal.*

Proof. Immediate. □

If $a, b \in L$, and a and b are not comparable, i.e., $a \nleqslant b$ and $b \nleqslant a$, then we write $aNCb$. For any $x \in L$, we define:

$$NC(x) = \wedge\{y \in L \colon yNCx\}, \qquad G(x) = \wedge\{y \in L \colon y > x\}.$$

If $G(x) = x$, then L is said to be upper-dense at x. Theorem 2.5 in [19] tells us that $\wedge C_\wedge(P) = p_\wedge$ and $\wedge C_\vee(P) = p_\vee$. As a complement to this theorem, we have:

Theorem 2.4.

(1) $\wedge\Phi_\wedge(P) = NC((p_\wedge)').G((p_\wedge)')$. *In particular, if L is a chain, i.e., totally ordered* (*for any $a, b \in L$, $a \leqslant b$ or $b \leqslant b$*), *and L is upper-dense at $(p_\wedge)'$, then $\wedge\Phi_\wedge(P) = (p_\wedge)' = \vee P'$. Furthermore, if $P \neq \{1\}$, then $\Phi_\wedge(P) \in \boldsymbol{P}_0(L)$.*
(2) $\wedge\Phi_\vee(P) = NC((p_\vee)').G((p_\vee)')$.
(3) $\vee H(P) = p_\wedge$.

Proof. (1)

$$\begin{aligned}\wedge\Phi_\wedge(P) &= \wedge\{q \in L\colon\ p_\wedge \nleqslant q'\}\\ &= \wedge\{q \in L\colon\ q \nleqslant (p_\wedge)'\}\\ &= \wedge\{q \in L\colon\ qNC(p_\wedge)'\}.\wedge\{q \in L\colon\ q > (p_\wedge)'\}\\ &= NC\big((p_\wedge)'\big).G\big((p_\wedge)'\big).\end{aligned}$$

(2) is similar to (1), and (3) is obvious. □

The following theorem improves Theorem 2.7 in [19].

Theorem 2.5. *Let $P, Q \in \boldsymbol{P}_0(L)$.*

(1) *If for every $p \in P$, there exists $q \in Q$ such that $p \leqslant q$, then*

$$\Phi_\vee(P) \subseteq \Phi_\vee(Q), \qquad C_\vee(Q) \subseteq C_\vee(P).$$

(2) *If for every $p \in P$, there exists $q \in Q$ such that $q \leqslant p$, then*

$$\Phi_\wedge(Q) \subseteq \Phi_\wedge(P), \qquad C_\wedge(P) \subseteq C_\wedge(Q), \qquad H(Q) \subseteq H(P).$$

Proof. By an argument of comparing elements. □

Note that Theorem 2.7 in [19] requires that there exists a bijection $f : P \to Q$ such that $p \leqslant f(p)$ for every $p \in P$. The above theorem weakens this condition and so improves the original theorem.

The following three theorems discuss the interaction between operators $\Phi_\vee$, $\Phi_\wedge$, $C_\vee$, $C_\wedge$, H and other operations on sets of premises. For any $P, Q \subseteq L$, we write:

$$P.Q = \{p.q\colon\ p \in P \text{ and } q \in Q\}, \qquad P + Q = \{p + q\colon\ p \in P \text{ and } q \in Q\}.$$

Theorem 2.6. *Let $P, Q \neq \phi$ and $P \cup Q \in \boldsymbol{P}_0(L)$.*

(1) $C_\wedge(P).C_\wedge(Q) \subseteq C_\wedge(P \cup Q)$. *In particular, if L is distributive, i.e., L is a Boolean algebra, then $C_\wedge(P \cup Q) = C_\wedge(P).C_\wedge(Q)$.*
(2) $C_\vee(P \cup Q) = C_\vee(P) + C_\vee(Q)$.
(3) $H(P \cup Q) = H(P).H(Q)$.

Proof. (1)

$$\begin{aligned} C_\wedge(P).C_\wedge(Q) &= \{r_1.r_2\colon\ p_\wedge \leqslant r_1 \text{ and } q_\wedge \leqslant r_2\} \subseteq \{r\colon\ \wedge(P \cup Q) = p_\wedge.q_\wedge \leqslant r\} \\ &= C_\wedge(P \cup Q). \end{aligned}$$

Suppose that L is distributive. If $r \in C_\wedge(P \cup Q)$, then $p_\wedge.q_\wedge \leqslant r$, $r = p_\wedge.q_\wedge + r = (p_\wedge + r).(q_\wedge + r)$. Note that $p_\wedge + r \in C_\wedge(P)$ and $q_\wedge + r \in C_\wedge(Q)$. Thus, $r \in C_\wedge(P).C_\wedge(Q)$.

(2) Similar to (1), we can prove that $C_\vee(P) + C_\vee(Q) \subseteq C_\vee(P \cup Q)$. Conversely, if $r \in C_\vee(P \cup Q)$, then $r \geqslant \vee(P \cup Q) = p_\vee + q_\vee \geqslant p_\vee, q_\vee$ and $r \in C_\vee(P), C_\vee(Q)$. Furthermore, $r = r.r \in C_\vee(P).C_\vee(Q)$.

(3) Similar to (2). □

Theorem 2.7.

(1) $\Phi_\vee(\bigcup_{i\in I} P_i) = \bigcup_{i\in I} \Phi_\vee(P_i)$. *In particular,* $\Phi_\vee(P) = \bigcup_{p\in P} \Phi_\vee(\{p\})$.
(2) $C_\vee(\bigcup_{i\in I} P_i) = \bigcap_{i\in I} C_\vee(P_i)$, *and* $C_\vee(P) = \bigcap_{p\in P} C_\vee(\{p\})$.
(3) $H(\bigcup_{i\in I} P_i) = \bigcap_{i\in I} H(P_i)$, *and* $H(P) = \bigcap_{p\in P} H(\{p\})$.

Proof. Similar to the proof of Theorem 2.9 below. □

We use L_0 to denote $L - \{0\}$. The following theorem deals with various iterations of operators $\Phi_\vee$, $\Phi_\wedge$, $C_\vee$, $C_\wedge$ and H. It is worth noting that Theorem 2.8(4) gives a representation of $\Phi_\vee$ in terms of $\Phi_\wedge$ and $C_\vee$, and Theorem 2.8(7) gives a representation of $C_\wedge$ in terms of $C_\vee$ and H.

Theorem 2.8.

(1) $\Phi_\vee(\Phi_\wedge(P)) = \Phi_\vee(C_\vee(P)) = \Phi_\vee(C_\wedge(P)) = L_0$.
(2) $C_\vee(\Phi_\vee(P)) = C_\vee(C_\wedge(P)) = C_\vee(\Phi_\wedge(P)) = \{1\}$.
(3) $\Phi_\vee(H(P)) = \Phi_\wedge(P)$ *if* $H(P) \in \boldsymbol{P}_0(L)$.
(4) $\Phi_\wedge(C_\vee(P)) = \Phi_\vee(P)$.
(5) $\Phi_\wedge(C_\wedge(P)) = \Phi_\wedge(P)$.
(6) $C_\wedge(C_\vee(P)) = C_\vee(P)$.
(7) *If* $H(P) \in \boldsymbol{P}_0(L)$, *then* $C_\vee(H(P)) = C_\wedge(P)$.
(8) $H(C_\wedge(P)) = H(P)$.

Proof. Straightforward. □

For the case of a finite set P, Trillas, Cubillo and Castiñeira [19] introduced several new operators:

$$\begin{aligned} G_n = \big\{&\text{mapping } g \text{ from } L^n \text{ into } L\colon\ g(x_1,\ldots,x_n) \geqslant x_1,\ldots,x_n \\ &\qquad\qquad\qquad\qquad\qquad \text{for all } x_1,\ldots,x_n \in L\big\}, \\ \Phi_g(P_n) = \{&q \in L\colon\ g(p_1,\ldots,p_n) \nleqslant q'\}, \end{aligned}$$

$$C_g(P_n) = \{q \in L\colon\ g(p_1, \ldots, p_n) \leqslant q\}, \quad \text{and}$$
$$C_{G_n}(P_n) = \{g(p_1, \ldots, p_n)\colon\ g \in G_n\}$$

for any $P_n = \{p_1, \ldots, p_n\} \in P_0(L)$ and for any $g \in G_n$.

The above operators are generalizations of Φ, C and H, respectively. In fact, if we define:

$$\vee_n(x_1, \ldots, x_n) = x_1 + \cdots + x_n, \quad \text{and}$$
$$\wedge_n(x_1, \ldots, x_n) = x_1, \ldots, x_n$$

for every $x_1, \ldots, x_n \in L$, then

$$\Phi_\vee(P_n) = \Phi_{\vee_n}(P_n), \qquad \Phi_\wedge(P_n) = \Phi_{\wedge_n}(P_n),$$
$$C_\vee(P_n) = C_{\vee_n}(P_n), \qquad C_\wedge(P_n) = C_{\wedge_n}(P_n).$$

In addition, it is easy to see that $C_{G_n}(P_n) = \{q \in L\colon\ p_1, \ldots, p_n \leqslant q\} = C_\wedge(P_n)$.

The operators Φ_g and C_g can be further generalized to the case of infinite-ary g. Let $\alpha \in On$ (the class of ordinals). Then an α-ary operation on L is a mapping from L^α into L. For any α-ary operation $g : L^\alpha \to L$, and for any α-tuple $\bar{x} = \{x_\beta\colon\ \beta < \alpha\} \in L^\alpha$, we can define:

$$\Phi_g(\bar{x}) = \{q \in L\colon\ g(\bar{x}) \nleqslant q'\}, \quad \text{and}$$
$$C_g(\bar{x}) = \{q \in L\colon\ g(\bar{x}) \leqslant q\}.$$

Now we can see that the above operators $\Phi_g(\bar{x})$ and $C_g(\bar{x})$ are generalizations of $\Phi_\vee(P), \Phi_\wedge(P), C_\vee(P)$ and $C_\wedge(P)$, respectively, even for the case of infinite P. Let $P \in \boldsymbol{P}_0(L)$ and $\alpha = |P|$ (the cardinality of P). We set

$$\vee_\alpha(\{x_\beta\colon\ \beta < \alpha\}) = \bigvee_{\beta<\alpha} x_\beta, \quad \text{and}$$
$$\wedge_\alpha(\{x_\beta\colon\ \beta < \alpha\}) = \bigwedge_{\beta<\alpha} x_\beta$$

for all $x_\beta \in L$ $(\beta < \alpha)$. Moreover, P may be enumerated as an α-tuple $\overline{p} = \{p_\beta\colon\ \beta < \alpha\}$. Then

$$\Phi_\vee(P) = \Phi_{\vee_\alpha}(\overline{p}), \qquad \Phi_\wedge(P) = \Phi_{\wedge_\alpha}(\overline{p}),$$
$$C_\vee(P) = C_{\vee_\alpha}(\overline{p}), \qquad C_\wedge(P) = C_{\wedge_\alpha}(\overline{p}).$$

Suppose that g_i is an α-ary operation on L for every $i \in I$. Then we define $\bigvee_{i\in I} g_i$ and $\bigwedge_{i\in I} g_i$ to be two α-ary operations on L such that

$$\Big(\bigvee_{i\in I} g_i\Big)(\bar{x}) = \bigvee_{i\in I} g_i(\bar{x}),$$
$$\Big(\bigwedge_{i\in I} g_i\Big)(\bar{x}) = \bigwedge_{i\in I} g_i(\bar{x})$$

for any $\bar{x} \in L^\alpha$.

Theorem 2.9.

(1) $\Phi_{\bigvee_{i\in I} g_i}(\bar{x}) = \bigcup_{i\in I} \Phi_{g_i}(\bar{x})$.
(2) $C_{\bigvee_{i\in I} g_i}(\bar{x}) = \bigcap_{i\in I} C_{g_i}(\bar{x})$.

Proof. (1) First, we know that

$$q \in \Phi_{\bigvee_{i\in I} g_i}(\bar{x}) \quad \text{iff} \quad \bigvee_{i\in I} g_i(\bar{x}) = \Big(\bigvee_{i\in I} g_i\Big)(\bar{x}) \nleqslant q'$$
$$\text{iff} \quad q \nleqslant \Big(\bigvee_{i\in I} g_i(\bar{x})\Big)' = \bigwedge_{i\in I} \big(g_i(\bar{x})\big)', \quad \text{and}$$
$$q \in \bigcup_{i\in I} \Phi_{g_i}(\bar{x}) \quad \text{iff} \quad \exists i \in I \text{ such that } q \in \Phi_{g_i}(\bar{x})$$
$$\text{iff} \quad \exists i \in I \text{ such that } g_i(\bar{x}) \nleqslant q'$$
$$\text{iff} \quad \exists i \in I \text{ such that } q' \nleqslant \big(g_i(\bar{x})\big)'.$$

Therefore, it suffices to show that $q \nleqslant \bigwedge_{i\in I}(g_i(\bar{x}))'$ if and only if $q \nleqslant (g_i(\bar{x}))'$ for some $i \in I$. In fact, if it is not the case that there exists $i \in I$ with $q \nleqslant (g_i(\bar{x}))'$, then for all $i \in I$, $q \leqslant (g_i(\bar{x}))'$ and $q \leqslant \bigwedge_{i\in I}(g_i(\bar{x}))'$. Conversely, if $q \leqslant \bigwedge_{i\in I}(g_i(\bar{x}))'$, then it is obvious that $q \leqslant (g_i(\bar{x}))'$ for all $i \in I$.

(2) We only need to note that $\bigvee_{i\in I} g_i(\bar{x}) \leqslant q$ if and only if $g_i(\bar{x}) \leqslant q$ for every $i \in I$. □

Through this section, it is required that a set P of premises satisfies the condition of $p_\wedge \neq 0$. This design decision is to exclude contradiction from the premises. A much weaker assumption for this purpose would be: it is impossible that $p \leqslant q'$ for all $p, q \in P$. We need the requirement of $p_\wedge \neq 0$ in Theorem 2.1, but the other theorems in this section are still valid when it is replaced by the above weaker one.

3. Conjectures, hypotheses and consequences in residuated lattices

A residuated lattice is a 7-tuple $(L, +, ., 0, 1, \otimes, \to)$ where

(i) $(L, +, .)$ is a bounded lattice with the least element 0 and the greatest element 1;
(ii) $(\otimes, \to)$ is an adjoint couple on L, i.e., $\otimes$ and $\to$ satisfy the following conditions:
  (M) $\otimes$ is a binary operation on L and it is isotone in both the two variables,
  (R) $\to$ is a binary operation on L and it is antitone in the first and isotone in the second variable, and
  (c) for all $a, b, c \in L$, $a \otimes b \leqslant c$ if and only if $a \leqslant b \to c$; and
(iii) $(L, \otimes, 1)$ is a commutative monoid.

The operation $\otimes$ is called multiplication, and $\to$ is called residuation. For more details, we refer to [6,7,13]. A typical example of residuated lattice is the unit interval equipped with one of the following three pairs of multiplication and residuation operations: for all $a, b \in [0, 1]$,

(1) Heyting operators: $a \otimes b = \min(a, b)$,

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \\ b & \text{otherwise.} \end{cases}$$

(2) Probabilistic operators: $a \otimes b = a \times b$,

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \\ b/a & \text{otherwise.} \end{cases}$$

(3) Lukasiewicz operators: $a \otimes b = \max(0, a + b - 1)$, $a \to b = \min(1, 1 - a + b)$.

Now let $(L, +, ., \otimes, \to)$ be a complete residuated lattice, and let 1 and 0 be the greatest and least elements of L, respectively. For any $a \in L$, its complementation is defined by $a' = a \to 0$. The contradiction in L is defined to be $Con = \bigvee_{x \in L - \{0\}} (x \to x') \in L$. The intuitive meaning of *Con* is that there is a statement which is not false and implies its negation. Furthermore, the weak complementation of $a \in L$ is defined by $a^{\perp} = a \to Con$. Clearly, $a' \leqslant a^{\perp}$. We now are ready to introduce the graded operators of conjectures, consequences and hypotheses in residuated lattices. As what was done in Section 2, we here also only consider sets P of premises in $\boldsymbol{P}_0(L) = \{P \subseteq L\colon\ P \neq \phi \text{ and } p_{\wedge} \neq 0\}$. For any $P \in \boldsymbol{P}_0(L)$, and for any $q \in P$, we define:

$$\begin{aligned} &\Phi_{\vee}(P)(q) = (p_{\vee} \to q')^{\perp}, \\ &\Phi_{\wedge}(P)(q) = (p_{\wedge} \to q')^{\perp}, \\ &C_{\vee}(P)(q) = p_{\vee} \to q, \\ &C_{\wedge}(P)(q) = p_{\wedge} \to q, \quad \text{and} \\ &H(P)(q) = q \to p_{\wedge}. \end{aligned}$$

Thus, for each $P \in \boldsymbol{P}_0(L)$, $\Phi_{\vee}(P), \Phi_{\wedge}(P), C_{\vee}(P), C_{\wedge}(P), H(P) \in L^L$ (the set of all mappings from L into itself), and $\Phi_{\vee}, \Phi_{\wedge}, C_{\vee}, C_{\wedge}$ and H are all mappings from 2^L (the power set of L, i.e., the set of all subsets of L) into L^L. The above definitions are the graded versions of Trillas, Cubillo and Castiñeira's corresponding operators. For example, $\Phi_{\wedge}(P)(q)$ expresses the degree to which q is a strict conjecture of P, and it may be seen as the truth value of the proposition "it is a contradiction that all premises in P together implies the negation of q". Let L be the unit interval with the probabilistic operators. Then it follows that $Con = 0$ and

$$a^{\perp} = a' = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, suppose that P is the interval $[1/3, 1/2]$, and each value in P may be used to indicate the belief degree of a certain statement. For any $\lambda \in [0, 1]$, by a routine calculation we have

$$\Phi_{\vee}(P)(\lambda) = \Phi_{\wedge}(P)(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0, \\ 1 & \text{otherwise,} \end{cases} \qquad C_{\vee}(P)(\lambda) = \begin{cases} 1 & \text{if } \lambda \geqslant 1/2, \\ 2\lambda & \text{otherwise,} \end{cases}$$

$$C_\wedge(P)(\lambda) = \begin{cases} 1 & \text{if } \lambda \geqslant 1/3, \\ 3\lambda & \text{otherwise,} \end{cases} \quad \text{and}$$
$$H(P)(\lambda) = \begin{cases} 1 & \text{if } \lambda \leqslant 1/3, \\ 1/3\lambda & \text{otherwise.} \end{cases}$$

Intuitively, for instance, if $\lambda = 2/3$, then the belief degree of the assertion that a statement whose belief degree is λ may be seen as a hypothesis of some premises with belief degrees in the interval $P = [1/3, 1/2]$ will be $1/2$.

We now turn to present some interesting properties of the operators introduced above.

Theorem 3.1.

(1) *For any* $p \in P$, $C_\wedge(P)(p) = 1$.
(2) *For any* $q \in L$,

$$C_\vee(P)(q) \leqslant C_\wedge(P)(q) \leqslant \Phi_\wedge(P)(q) \leqslant \Phi_\vee(P)(q), \quad \textit{and}$$
$$H(P)(q) \leqslant \Phi_\wedge(P)(q).$$

(3) $C_\wedge(P)(q).H(P)(q) = 1$ *if and only if* $q = p_\wedge$.

Proof. We only show that $C_\wedge(P)(q) \leqslant \Phi_\wedge(P)(q)$. Similarly, $H(P)(q) \leqslant \Phi_\wedge(P)(q)$ can be proven, and the others are obvious.

$$\begin{aligned} C_\wedge(P)(q) \otimes (p_\wedge \to q') &= (p_\wedge \to q) \otimes \big(p_\wedge \to (q \to 0)\big) \\ &= (p_\wedge \to q) \otimes \big(q \to (p_\wedge \to 0)\big) \\ &\leqslant p_\wedge \to (p_\wedge \to 0) \\ &= p_\wedge \to (p_\wedge)' \leqslant Con, \quad \text{and} \end{aligned}$$
$$C_\wedge(P)(q) \leqslant (p_\wedge \to q') \to Con = (p_\wedge \to q')^\perp = \Phi_\wedge(P)(q). \qquad \square$$

It is clear that the above theorem is a graded generalization of Theorem 2.2 in [19]. One may conceive that it is natural to define $\Phi_\vee(P)$ and $\Phi_\wedge(P)$ in the following way:

$$\Phi_\vee(P)(q) = (p_\vee \to q')', \quad \text{and} \quad \Phi_\wedge(P)(q) = (p_\wedge \to q')'$$

for every $q \in L$. However, with these modified definitions, $C_\wedge(P)(q) \leqslant \Phi_\wedge(P)(q)$ and $H(P)(q) \leqslant \Phi_\wedge(P)(q)$ do not hold any more.

Theorem 3.2.

(1) *If* $q \leqslant r$, *then*

$$\Phi_\vee(P)(q) \leqslant \Phi_\vee(P)(r), \qquad \Phi_\wedge(P)(q) \leqslant \Phi_\wedge(P)(r).$$

(2) $C_\vee(P)(\bigwedge_{i\in I} q_i) = \bigwedge_{i\in I} C_\vee(P)(q_i)$, $C_\wedge(P)(\bigwedge_{i\in I} q_i) = \bigwedge_{i\in I} C_\wedge(P)(q_i)$.
(3) $H(P)(\bigvee_{i\in I} q_i) = \bigwedge_{i\in I} H(P)(q_i)$.

Proof. Easy. $\square$

Theorem 3.3.

(1) *If* $P \subseteq Q$, *then*

$$\Phi_\vee(P)(q) \leqslant \Phi_\vee(Q)(q), \qquad \Phi_\wedge(Q)(q) \leqslant \Phi_\wedge(P)(q),$$
$$C_\wedge(P)(q) \leqslant C_\wedge(Q)(q).$$

(2) $C_\vee(\bigcup_{i\in I} P_i)(q) = \bigwedge_{i\in I} C_\wedge(P_i)(q)$.
(3) $H(\bigcup_{i\in I} P_i)(q) = \bigwedge_{i\in I} H(P_i)(q)$.

Proof. Easy. □

Theorem 3.4.

(1) $\Phi_\vee(P)(q) = (C_\vee(P)(q'))^\perp$, $\Phi_\wedge(P)(q) = (C_\wedge(P)(q'))^\perp$.
(2) $C_\vee(P)(q) \leqslant (\Phi_\vee(P)(q'))^\perp$, $C_\wedge(P)(q) \leqslant (\Phi_\wedge(P)(q'))^\perp$.
(3) $H(P)(q) \leqslant C_\vee(P')(q')$.

Proof. Straightforward. □

Suppose that $\varphi : 2^L \to L^L$ be a mapping. Then its natural extension $\overline{\varphi} : L^L \to L^L$ is defined as follows: for any $A \in L^L$ and for any $x \in L$,

$$\overline{\varphi}(A)(x) = \bigvee_{\lambda\in L} [\lambda \otimes \varphi(A_\lambda)(x)],$$

where $A_\lambda = \{x \in L\colon\ A(x) \geqslant \lambda\} \subseteq L$ is the λ-cut of A for each $\lambda \in L$. For simplicity, $\overline{\varphi}$ is often abbreviated to φ.

Theorem 3.5. *For any* $P \subseteq L$, $C_\wedge(C_\wedge(P)) = C_\wedge(P)$.

Proof. From Theorems 3.1(1) and 3.3(1) we know that $P \subseteq C_\wedge(P)$ and $C_\wedge(P)(q) \leqslant C_\wedge(C_\wedge(P)_\lambda)(q)$. Then

$$\begin{aligned} C_\wedge\big(C_\wedge(P)\big)(q) &= \bigvee_{\lambda\in L}\big[\lambda \otimes C_\wedge\big(C_\wedge(P)_\lambda\big)(q)\big] \geqslant \bigvee_{\lambda\in L}\big[\lambda \otimes C_\wedge(P)(q)\big] \\ &= \Big(\bigvee_{\lambda\in L}\lambda\Big) \otimes C_\wedge(P)(q) = 1 \otimes C_\wedge(P)(q) = C_\wedge(P)(q). \end{aligned}$$

Conversely, for any $\lambda \in L$, we have

$$\begin{aligned} C_\wedge(P)_\lambda &= \{q \in L\colon\ C_\wedge(P)(q) = p_\wedge \to q \geqslant \lambda\} = \{q \in L\colon\ \lambda \otimes p_\wedge \leqslant q\}, \\ \wedge C_\wedge(P)_\lambda &= \lambda \otimes p_\wedge, \end{aligned}$$

$$\begin{aligned} C_\wedge\big(C_\wedge(P)_\lambda\big)(q) &= \wedge C_\wedge(P)_\lambda \to q = (\lambda \otimes p_\wedge) \to q \\ &= \lambda \to (p_\wedge \to q), \quad \text{and} \end{aligned}$$

$$\lambda \otimes C_\wedge\big(C_\wedge(P)_\lambda\big)(q) \ \leqslant\ \lambda \otimes [\lambda \to (p_\wedge \to q)] \leqslant p_\wedge \to q = C_\wedge(P)(q).$$

Therefore, it follows that

$$C_\wedge\big(C_\wedge(P)\big)(q) = \bigvee_{\lambda \in L}\big[\lambda \otimes C_\wedge\big(C_\wedge(P)_\lambda\big)(q)\big] \leqslant C_\wedge(P)(q). \qquad \square$$

4. Concluding remarks

Trillas, Cubillo and Castiñeira [19] introduced the operators $\Phi_\vee, \Phi_\wedge, C_\vee, C_\wedge$ and H in orthocomplemented lattices as an algebraic model of conjectures, consequences and hypotheses. In this framework, they presented a classification of conjectures and consequences and gave a characterization of hypotheses.

In the first part of this paper, we obtain some new properties of conjectures, consequences and hypotheses in orthocomplemented lattices. These results complement or improve some theorems in [19]. In the second part of this paper, to deal with conjectures, consequences and hypotheses in the environment where uncertain and vague information is essential, we employ residuated lattices as our mathematical model and introduce graded versions of operators $\Phi_\vee, \Phi_\wedge, C_\vee, C_\wedge$ and H. Some properties of these graded operators are derived.

Alchourron, Gärdenfors and Makinson [1] proposed a theory of belief revision. An interesting topic for further study is to establish an algebraic generalization of this theory within orthocomplemented lattices as well as residuated lattices, in the spirit of [19] and this paper.

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