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Abstract argumentation systems ☆

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Abstract

In this paper, we develop a theory of *abstract argumentation systems*. An abstract argumentation system is a collection of "defeasible proofs", called *arguments*, that is partially ordered by a relation expressing the difference in conclusive force. The prefix "abstract" indicates that the theory is concerned neither with a specification of the underlying language, nor with the development of a subtheory that explains the partial order. An unstructured language, without logical connectives such as negation, makes arguments not (pairwise) inconsistent, but (groupwise) incompatible. Incompatibility and difference in conclusive force cause defeat among arguments. The aim of the theory is to find out which arguments eventually emerge undefeated. These arguments are considered to be *in force*. Several results are established. The main result is that arguments that are in force are precisely those that are in the limit of a so-called *complete argumentation sequence*. © 1997 Elsevier Science B.V.

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1. Introduction

Various logics have been proposed to reason with symbolically represented forms of incomplete or uncertain information. One approach is influenced by (informal) argumentation theory. The idea is that argumentation theory contains the right concepts and ideas to reason with incomplete or uncertain information. These insights are formalized, and the formalism thus obtained is then used to represent and manipulate rule-based forms of incomplete or uncertain information on the computer. In a nutshell, this is the motivating thought behind theories of formal argument.

There exist a large number of formal argumentation systems.

(1) The American philosopher Pollock developed an argumentation system called OSCAR [31–33,35,36] (1985–1997). OSCAR has the possibility to reason with so-called *suppositional arguments*. A suppositional argument is a defeasible argument with conclusion q, in which a premise p is discarded to infer the implication $p \supset q$. (Cf. [3,40].)

(2) In 1986, the philosopher Donald Nute wrote a report, entitled "A non-monotonic logic based on conditional logic" [29]. In that report, Nute defines a simple and elegant system of defeasible reasoning, called LDR1, that he shows is readily implementable on the computer. [1] Two years later, in 1988, Nute [30] presented a revised and extended version of LDR1, called LDR. In LDR, adjudication among competing arguments is done via top-rules: one argument defeats another if and only if the antecedent of the top-rule of the first argument is strictly more specific than the antecedent of the top-rule of the second argument.

(3) In 1987, Loui [21] presented a system of defeat among arguments, in which defeat is defined recursively in terms of preference, interference, specificity, directness, d-shortness, and evidence. Later, in 1992, Loui presented, together with Simari [45], a mathematical treatment of defeasible reasoning in which a new defeasible consequence operator $\mid\!\sim$ is introduced. (Cf. [55], for a detailed analysis of both articles.) In 1991–1993, Loui et al. [24] developed NATHAN, a C implementation of rules for computing defeat among arguments. NATHAN is based on LMNOP (1993), a LISP-prototype developed by Costello, Loui and Merrill [24,45].

(4) In 1988, Horty and Thomason [16] presented a theory of mixed inheritance in nonmonotonic proof nets. Inheritance theory resembles symbolic argumentation, with an important difference that antecedents of rules consist of exactly one formula, and arguments are called "paths". Because paths are "one-dimensional" lines of reasoning, inheritance theory is not as involved as a theory in which arguments are trees. Perhaps in consequence thereof, it is right on a number of crucial aspects, especially when it comes to the resolution of complicated defeat and interference relations between paths.

[1] First, the program was called PROWIS (for programming with subjunctives), and was implemented in micro-PROLOG 3.1. Later (1988), the implementation was changed and renamed into d-PROLOG (for defeasible PROLOG).

(5) In 1988, Konolige [17] proposed a solution to the Yale Shooting Problem (YSP). Unlike most of the proposals, his treatment was not meant to promote an already existing formalism. Instead, it worked the other way around. By means of a particular problem—the YSP—Konolige discussed the issue of what should be involved when one reasons about events. The resulting discussion is extremely clear, and touches upon many important issues of defeasible argumentation in an almost casual manner. Meanwhile, the formalism more or less automatically emerges out of the discussion. Konolige's formalism ARGH (ARGumentation with Hypotheses) is based on McCarthy's situation calculus, where properties are attached to situations.

(6) Lin and Shoham [20] (1988, 1993) developed an argument system that captures a number of well-known nonmonotonic logics. Among others, the authors mention: Reiter's default logic, McDermott and Doyle's nonmonotonic logic I, Moore's autoepistemic logic, negation by failure, McCarthy's circumscription, Lifschitz' pointwise circumscription, various forms of inheritance, and Shoham's theory of chronological ignorance. It is possible to capture so many nonmonotonic logics, because the system in question does not have a component that is responsible for the resolution of argument conflicts.

(7) Recently, in 1995, the Thai researcher Dung [9] presented a mathematical argumentation theory, in which an argument is an abstract entity whose role is determined by its so-called *attack* relations to other arguments. Acceptability is a fundamental concept in Dung's theory. An argument is accepted by S if and only if S attacks all attackers of that argument. In Dung's theory, no special attention is paid to the internal structure of the arguments. (The latter is not meant as a value judgement.)

Why are there so many different argumentation systems? Our explanation is that the area is relatively young. No consensus has been reached yet on essential issues, such as the representation of arguments and their precise form of interaction. Each researcher, or group of researchers, has its own way to deal with these issues, which results in a large diversity of argumentation systems.

There are a number of problems with existing systems.

(a) A problem with OSCAR is that, in some cases, $\supset$-introduction produces fallacious arguments. These fallacious arguments unjustifiably defeat lines of reasoning that actually should emerge undefeated. This problem is further discussed and illustrated with a counterexample in [55].

(b) The mutual dependency relations between interfering arguments are difficult to handle for most systems. In Loui's system [21], it is possible to construct a cycle of three arguments in which each argument defeats its successor. With this cycle it remains unclear which of the three arguments should remain undefeated. This problem is caused by Loui's notion of defeat, that makes it possible that short arguments can be defeated by longer arguments that, in turn, may contain isomorphic copies of the shorter argument. These copies can, for the same reason, be defeated by other long arguments, thus constructing a chain of successive defeaters. If the ends fit, such a chain can be made into a cycle of three successively defeating arguments.

(c) Most systems have difficulties with a situation in which it is not clear which argument should win. One possibility is to keep up all arguments, and continue reasoning with them in different "worlds". This approach has become known as *credulous reasoning*. (Cf. [42]: multiple extensions; [7]: multiple contexts; [27]: multiple belief spaces; [28]: clusters of worlds.) A second possibility is to do away with every argument, on the basis of the idea that the arguments neutralize each other. This approach has become known as *skeptical reasoning*. (Cf. [21,30,36,45]: collective defeat. Horty and Thomason [16] introduce a mechanism of skeptical defeat that is slightly more refined.) A third possibility is to introduce three classes of arguments—"ultimately undefeated", "provisionally undefeated" and "ultimately defeated"—such that all arguments that are involved in an ambiguous conflict are considered as "provisionally undefeated". (Cf. [36].) Prakken uses a similar triple "preferred", "defensible", "defeated". (Cf. [39].) The problem is that none of these options is completely satisfactory. In [55] several examples are adduced and elaborated to show that, in some situations, the above approaches lack expressive power in order to describe counterbalanced or ambiguous argument conflicts. In a nutshell, the problem is that if, for instance, σ_1, σ_2, τ_1 and τ_2 are arguments such that σ_1 interferes with σ_2 and τ_1 interferes with τ_2, then all arguments "stay alive" in different worlds in the credulous approach, "eliminate" each other in the skeptical approach, or are placed in the grey pool of "provisionally defeated" arguments in Pollock's/Prakken's approach. It is not possible to formally express, for example, that σ_1 depends only on σ_2, or that σ_1 does not depend on τ_1.

(d) Almost all systems include a detailed and elaborated specification of defeat among arguments. A representative example is [21]. (Cf. point (3) above.) It is not only indicated which argument is stronger than another, but it is also indicated *why* an argument is stronger than another argument. However, for most of these detailed specifications of defeat it is possible to construct counterexamples for which an unintentional argument emerges undefeated. To obtain a counterexample, it suffices to construct two syntactically isomorphic but semantically different situation descriptions, such that an undefeated argument is right according to the first semantic interpretation but wrong according to the second semantic interpretation. (Cf. [51].) Thus, it can be shown that detailed and carefully specified definitions of defeat are often overspecified. An overspecified definition "takes sides" in cases where it is better to abstain. At the other side of the spectrum we find Lin and Shoham's argument system [20]. In [20], a logical hierarchy between arguments is absent. This lacuna makes it impossible to determine which arguments remain undefeated. It is reasonable to consider the latter as a problem.

In this paper, a theory of *abstract argumentation systems* is developed. An abstract argumentation system is a collection of defeasible proofs, called *arguments*, of varying conclusive force.

In the proposed theory, the above problems are solved in the following way.

(a) To avoid problems with suppositional arguments (cf. [36]), they are excluded from our theory. Our approach is to put the issue temporarily on the shelves until there is a better understanding of more simple forms of defeasible reasoning.

(b) The anomaly in Loui's system [21] is avoided by the stipulation that arguments cannot become stronger if they are made longer. This natural and intuitive principle is enforced in Definition 2.12.

(c) An abstract argumentation system can handle situations in which it is not clear which argument should win. Within an abstract argumentation system, it is possible to describe critical situations, not only with provisionally defeated arguments [33,36,45] and extensions [7,9,27,42], but also with a defeasible entailment relation $\mid\!\sim$ that is able to formally distinguish and express the various dependency relations across (groups of) arguments. The latter is demonstrated, among others, in Example 4.31.

(d) In abstract argumentation systems, potentially involved criteria of adjudication are put into an order of conclusive force. This order is supposed to contain all the relevant information concerning the relative strength among arguments. The order of conclusive force used is partial, which means that some pairs of arguments are incomparable. In some situations, even *most* arguments cannot be compared. This "inability" is not necessarily disadvantageous. Sometimes, there is simply not enough information to arrange a complete hierarchy among arguments. In such situations, incomparability is still better than overspecification and arbitrary conflict resolution.

That we speak of *abstract* argumentation systems in particular, is motivated by research of which the findings are reported in [51]. In that paper, we were interested in the feasibility of formulating general, but nontrivial, principles for determining the relative strength among arguments. This has been done because a notion of conclusive force lays the basis for a theory of defeat and, conversely, a theory of defeat presupposes a notion of conclusive force in which we can speak about the relative strength of arguments. The conclusion was that, in a formal theory that works on the basis of syntactical principles, it is difficult to formulate general but nontrivial principles that consistently pronounce upon the conclusive force of symbolically represented arguments. Conclusive force is not only determined by syntactical structure. Generally, further information is needed from the semantics of the domain of discourse to have a definite saying in which argument is stronger than the other. In any case, the whole issue seems to be sufficiently difficult to avoid it altogether, and to assume that an order on arguments already has been given. This assumption saves us from the responsibility of telling how and why a particular argument should overrule any other particular argument. The present paper thus attempts to understand the workings of defeat in its own right, freed from the involvements with specificity and conclusive force in which it was enmeshed in the earlier discussions.

In spite of these abstractions (or simplifications, for that matter), and given the impoverishment of the background language (there are, for instance, no genuine propositional connectives), the texture of the resulting formalism is surprisingly rich. This indicates that argument has a meta-theory that must be taken serious. I have done this by trying to embody the most interesting ideas in argument and present the material clearly. By doing so, I hope to convince mathematicians that argumentation theory has an interesting mathematics. Further, I invite non-logicians to work on argument. A number of computational and combinatorial problems formulated in this paper are not solved yet. I am sure that attempts to solve these problems will lead to new and interesting results.

2. Basic concepts

An abstract argumentation system is a static framework in which the basic notions of argumentation obtain a well-defined meaning. In and of itself, it does not prescribe how argumentation should be performed, which arguments are in force, or in what manner defeasible information should be manipulated. It merely attempts to provide a conceptual framework in which different ideas on argumentation can be worked out.

Definition 2.1. An *abstract argumentation system* is a triple

$$\mathcal{A} = (\mathcal{L}, R, \leqslant) \tag{1}$$

where $\mathcal{L}$ is a language, R is a set of rules of inference, and $\leqslant$ is a reflexive and transitive order on arguments. [2]

Definition 2.2 (*Language*). A *language* is a set $\mathcal{L}$, containing a distinguished element $\perp$.

The language $\mathcal{L}$ is not subject to particular constraints. Any set will do, provided it contains a distinguished element $\perp$ that represents a contradictory proposition. In particular, sentences do not need to be constructed with the help of logical connectives such as $\wedge$, $\vee$, $\neg$, $\rightarrow$, $>$, and $\equiv$. In the theory of abstract argumentation systems, we abstract away from the structure of the logical language. We do this because it is possible to produce a rich meta-theory, even on the basis of a poor language. (As this paper will show.) Refining $\mathcal{L}$ with logical connectives is always an option in a later stage. Elements of a language are called *sentences*. Every now and then we call them propositions, although it is generally known that sentences and propositions are not exactly the same. (Cf. [46, 58].) For the present, their distinction is irrelevant.

The rules R are given in terms of $\mathcal{L}$ and determine what inferences are possible.

Definition 2.3 (*Rules of inference*). Let $\mathcal{L}$ be a language. [3]

(1) A *strict rule of inference* is a formula of the form $\phi_1, \ldots, \phi_n \rightarrow \phi$ where $\phi_1, \ldots, \phi_n$ is a finite, possibly empty, sequence in $\mathcal{L}$ and ϕ is a member of $\mathcal{L}$.

[2] An abstract argumentation system is a (conservative) extension of the notion of proof system. (Cf. Definition 4.34.)

[3] If we are very precise, we should additionally stipulate that $\mathcal{L} \cap \{\rightarrow, \Rightarrow\} = \emptyset$. Otherwise, the meta-language would mix with the object language unintentionally.

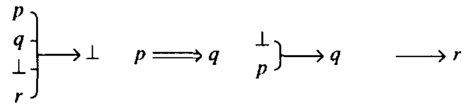

Fig. 1. Four rules.

(2) A *defeasible rule of inference* is a formula of the form $\phi_1, \ldots, \phi_n \Rightarrow \phi$ where $\phi_1, \ldots, \phi_n$ is a finite, possibly empty, sequence in $\mathcal{L}$ and ϕ is a member of $\mathcal{L}$.

A *rule of inference* is a strict or a defeasible rule of inference.

Rules of inference are meta-linguistic expressions. Although rules are defined in terms of $\mathcal{L}$, it is not so that they are elements of $\mathcal{L}$. Thus, $R \cap \mathcal{L} = \emptyset$. Because $\mathcal{L}$ has no *a priori* structure, the rules are supposed to be domain independent. (Just like the rules of inference of propositional logic are domain independent.) If one wishes to represent domain-specific rules (and one will in later stages), the language $\mathcal{L}$ must be refined with logical connectives. But domain-specific applications are not the primary concern of this paper. In contrast with other theories of defeasible argumentation, notably [30, 36], the theory has two kinds of rules, $\rightarrow$ and $\Rightarrow$.[4] Further, it does not know of so-called *undercutting* rules of inference.[5]

Example 2.4. Let $\mathcal{L} = \{p, q, r\} \cup \{\bot\}$. Then $p, q, \bot, r \rightarrow \bot$; $p \Rightarrow q$; $\bot, p \rightarrow q$; and $\rightarrow r$ are rules of inference constructed with elements from $\mathcal{L}$. (Cf. Fig. 1.) More examples of rules of inference are $p, q \rightarrow r$; $p \rightarrow \bot$; $p, q \Rightarrow r$; and $\Rightarrow q$.

Examples of non-rules are $p, q \Rightarrow$; $\rightarrow$; $\Rightarrow \bot, \bot, p$; and $p \Rightarrow q, r$. In other systems, the last two formulas would have been allowed, giving rise to rules with multiple consequents. In our system, these last two formulas would split into $\{\Rightarrow \bot, \Rightarrow \bot, \Rightarrow p\}$ and $\{p \Rightarrow q, p \Rightarrow r\}$, respectively. Rules with multiple consequents complicate the notion of argument considerably, and are therefore not included in our theory. Furthermore, we note that rules do not belong to the language, so that repeated use of $\rightarrow$ as well as $\Rightarrow$ is forbidden: $(p \rightarrow q) \rightarrow r$; $(p \rightarrow q) \rightarrow (p \Rightarrow q)$; $p \rightarrow (p \rightarrow (p \rightarrow (\ldots)))$ are invalid constructions.

The third component of an argumentation system, the order $\leqslant$, is a relation that is defined on arguments. (Cf. Definition 2.12.)

[4] Pollock has one: $\rightarrow$, while Nute has three: $\rightarrow$, $\Rightarrow$, and $?\!\!\rightarrow$.

[5] In Nute's logic of defeasible reasoning [30], there is an additional third type of rule of inference, called a *defeater*. In Pollock's theory [31,33], this third type of rule is called an *undercutting defeater*. In our theory, an undercutting defeater would look like $\phi_1, \ldots, \phi_n \ ?\!\!\rightarrow \phi$. In order the understand the function of an undercutting defeater, it is probably best to assume temporarily that $\mathcal{L}$ is closed under negation, i.e., that $\neg\phi \in \mathcal{L}$ whenever $\phi \in \mathcal{L}$, for every $\phi \in \mathcal{L}$. The meaning of an undercutting defeater is then defined as follows. If $\phi_1, \ldots, \phi_n \rightarrow \phi$ means that $\phi_1, \ldots, \phi_n$ is an absolute reason to derive ϕ, if $\phi_1, \ldots, \phi_n \Rightarrow \phi$ means that $\phi_1, \ldots, \phi_n$ is a good reason to derive ϕ, then $\phi_1, \ldots, \phi_n \ ?\!\!\rightarrow \phi$ means that $\phi_1, \ldots, \phi_n$ is a good reason not to derive $\neg\phi$. Thus, a defeater $\phi_1, \ldots, \phi_n \ ?\!\!\rightarrow \phi$ undercuts the rule $\phi_1, \ldots, \phi_n \Rightarrow \neg\phi$, without supporting ϕ. In Section 4.5, it will be argued that undercutting defeaters can always be defined on the basis of the two other types of rules, provided $\mathcal{L}$ is rich enough to account for strict and defeasible rules of inference.

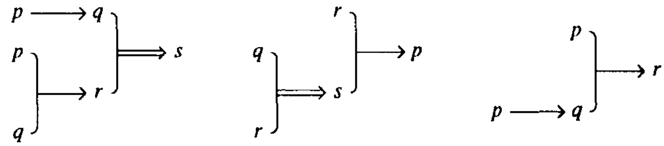

Fig. 2. Three arguments.

Arguments

Chaining rules together into trees, we get arguments.[6] In Fig. 2, three arguments are displayed. The premises of the first, second and third argument are $\{p,q\}$, $\{q,r\}$ and $\{p\}$, respectively. The conclusions are s, p, and r; the sentences are $\{p,q,r,s\}$, $\{p,q,r,s\}$, and $\{p,q,r\}$; the assumptions are $\{s\}$, $\{s\}$, and $\emptyset$; the lengths of all three arguments are equal to 3 (not 2); the sizes of the arguments are 6, 5, and 4, respectively. The first two arguments are defeasible; the third argument is strict.

Below, the notion of argument is defined precisely.

Definition 2.5 (*Argument*). Let R be a set of rules. An argument has *premises*, a *conclusion*, *sentences* (or propositions), *assumptions*, *subarguments*, *top-arguments*, a *length*, and a *size*. These are abbreviated by corresponding prefixes. An *argument* σ is

(1) a member of $\mathcal{L}$—in that case,

$$\begin{array}{lll} prem(\sigma) = \{\sigma\}, & conc(\sigma) = \sigma, & sent(\sigma) = \{\sigma\}, \\ asm(\sigma) = \emptyset, & sub(\sigma) = \{\sigma\}, & top(\sigma) = \{\sigma\}, \\ length(\sigma) = 1, & size(\sigma) = 1; & \end{array}$$

or

(2) a formula of the form $\sigma_1, \ldots, \sigma_n \rightarrow \phi$ where $\sigma_1, \ldots, \sigma_n$ is a finite, possibly empty, sequence of arguments, such that $conc(\sigma_1) = \phi_1, \ldots, conc(\sigma_n) = \phi_n$ for some rule $\phi_1, \ldots, \phi_n \rightarrow \phi$ in R, and $\phi \notin sent(\sigma_1) \cup \cdots \cup sent(\sigma_n)$—in that case,

$$\begin{aligned} & prem(\sigma) = prem(\sigma_1) \cup \cdots \cup prem(\sigma_n), \\ & conc(\sigma) = \phi, \\ & sent(\sigma) = sent(\sigma_1) \cup \cdots \cup sent(\sigma_n) \cup \{\phi\}, \\ & asm(\sigma) = asm(\sigma_1) \cup \cdots \cup asm(\sigma_n), \\ & sub(\sigma) = sub(\sigma_1) \cup \cdots \cup sub(\sigma_n) \cup \{\sigma\}, \\ & top(\sigma) = \{\tau_1, \ldots, \tau_n \rightarrow \phi \mid \tau_1 \in top(\sigma_1), \ldots, \tau_n \in top(\sigma_n)\} \cup \{\phi\}, \\ & length(\sigma) = max\{length(\sigma_1), \ldots, length(\sigma_n)\} + 1, \\ & size(\sigma) = size(\sigma_1) + \cdots + size(\sigma_n) + 1; \end{aligned}$$

or

[6] For philosophical considerations on exactly what is an argument, the reader is referred to [6] or [58]. Here, arguments are just formulas.

(3) a formula of the form $\sigma_1, \ldots, \sigma_n \Rightarrow \phi$ where $\sigma_1, \ldots, \sigma_n$ is a finite, possibly empty, sequence of arguments, such that $conc(\sigma_1) = \phi_1, \ldots, conc(\sigma_n) = \phi_n$ for some rule $\phi_1, \ldots, \phi_n \Rightarrow \phi$ in R, and $\phi \notin sent(\sigma_1) \cup \cdots \cup sent(\sigma_n)$; for assumptions we have

$$asm(\sigma) = asm(\sigma_1) \cup \cdots \cup asm(\sigma_n) \cup \{\phi\};$$

premises, conclusions, and other attributes are defined as in (2).

Arguments of type (1) are *atomic* arguments; arguments of types (2) and (3) are *composite* arguments. Thus, atomic arguments are language elements. How these elements will be considered depends on the context.

Remark 2.6. From the above definition it follows that arguments may have identical sentences, as long as they do not occur more than once in the same branch. If an argument has a pair of identical sentences in the same branch, it is constructed with the help of a subargument and a subsubargument that support identical conclusions. This clearly is an elementary fallacy. (Cf. [15].) For example, Fig. 3 displays something that is not an argument because r occurs twice in the same branch.

Remark 2.7. Often, we abuse notation and write $\sigma_1, \ldots, \sigma_n \rightarrow \sigma$ to denote that σ is an argument, constructed from $\sigma_1, \ldots, \sigma_n$ using the rule $\phi_1, \ldots, \phi_n \rightarrow \phi$, with $conc(\sigma_1) = \phi_1, \ldots, conc(\sigma_n) = \phi_n$ and $conc(\sigma) = \phi$. The arguments $\sigma_1, \ldots, \sigma_n$ are referred to as the *immediate subarguments* of σ; the rule $\phi_1, \ldots, \phi_n \rightarrow \phi$ is referred to as the *top-rule* of σ. The same terminology is used for arguments that end with a defeasible rule of inference. Further, we omit parentheses wherever possible. With this convention, $((p \Rightarrow q) \Rightarrow r) \Rightarrow s$ is written as $p \Rightarrow q \Rightarrow r \Rightarrow s$, since the latter cannot be parsed as, e.g., $p \Rightarrow (q \Rightarrow r) \Rightarrow s$ or $p \Rightarrow (q \Rightarrow (r \Rightarrow s))$.

An argument has, besides subarguments, also *super-arguments*, and besides top-arguments, also *bottom-arguments*. (Super-arguments and bottom-arguments are used in Definition 6.9 and further.)

Definition 2.8 (*Super-argument, bottom-argument*). Let σ and τ be arguments. Instead of $\sigma \in sub(\tau)$ we often write $\sigma \sqsubseteq \tau$. Further, $sup(\sigma) = \{\rho \mid \sigma \in sub(\rho)\}$ and $bot(\sigma) = \{\rho \mid \sigma \in top(\rho)\}$.

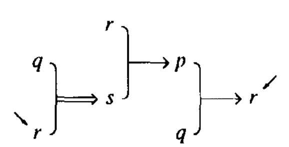

Fig. 3. A fallacy.

Example 2.9. Let $\mathcal{L} = \{p,q,r,s\} \cup \{\perp\}$ and

$$R = \{p \rightarrow q;\ p,q \rightarrow r;\ q,r \Rightarrow s;\ r,s \rightarrow p\}. \tag{2}$$

Let $\sigma = q,r \Rightarrow s$. Then

$$\begin{aligned}
&prem(\sigma) = \{q,r\}, && conc(\sigma) = s,\\
&sent(\sigma) = \{q,r,s\}, && asm(\sigma) = \{s\},\\
&sub(\sigma) = \{q,r\} \cup \{q,r \Rightarrow s\}, && sup(\sigma) = \{q,r \Rightarrow s;\ r,(q,r \Rightarrow s) \rightarrow p\},\\
&top(\sigma) = \{q,r \Rightarrow s\} \cup \{s\}, &&
\end{aligned}$$

$$\begin{aligned}
bot(\sigma) = \{&(p \rightarrow q),(p,q \rightarrow r) \Rightarrow s;\\
&(p \rightarrow q),r \Rightarrow s;\ q,(p,q \rightarrow r) \Rightarrow s;\ q,r \Rightarrow s\},
\end{aligned}$$

$$length(\sigma) = 2, \qquad size(\sigma) = 3.$$

Let $\tau = p$. Then

$$\begin{aligned}
&prem(\tau) = \{p\}, && conc(\tau) = p,\\
&sent(\tau) = \{p\}, && asm(\sigma) = \emptyset,\\
&sub(\tau) = \{p\}, &&
\end{aligned}$$

$$\begin{aligned}
sup(\tau) = &\{p,p \rightarrow q\}\\
&\cup \{p,q \rightarrow r;\ (p \rightarrow q),r \Rightarrow s;\ (p \rightarrow q),(p,q \rightarrow r) \Rightarrow s\},\\
&\cup \{(p \rightarrow q),(p,(p \rightarrow q) \rightarrow r) \Rightarrow s\},
\end{aligned}$$

$$\begin{aligned}
&top(\tau) = \{p\}, && bot(\tau) = \{r,(q,r \Rightarrow s) \rightarrow p\} \cup \{r,s \rightarrow p\} \cup \{p\},\\
&length(\tau) = 1, && size(\tau) = 1.
\end{aligned}$$

Definition 2.10. An argument σ is *strict* if $\sigma \in \mathcal{L}$ or $\sigma_1,\dots,\sigma_n \rightarrow \sigma$ where $\sigma_1,\dots,\sigma_n$ are strict arguments. An argument is *defeasible* if it is not strict.

With this terminology, an argument is either strict or defeasible, but never both strict and defeasible at the same time.

Definition 2.11. Let $\mathcal{L}$ be a language and let P be a subset of $\mathcal{L}$.

(1) An argument is *based* on P if the premises of that argument is a subset of P; a set of arguments is *based* on P if all its members are based on P.

(2) The set of all arguments that are based on P, is denoted by *arguments*(P); the set of all strict arguments that are based on P is denoted by *strict*(P); the set of all defeasible arguments that are based on P is denoted by *defeasible*(P).

(3) A member of $\mathcal{L}$ is *based* on P if it is the conclusion of an argument that is based on P; *strictly based* on P if it is the conclusion of a strict argument that is based on P; *defeasibly based* on P if it is the conclusion of a defeasible argument that is based on P.

Up to here, the theory does not much deviate from what is accustomed in classical proof theory. (Cf. [3,40,41].) There is a defeasible proof system, there are proofs, rules of inference, and other logical attributes. The only difference is that everything comes in two different sorts: there are strict rules and defeasible rules, strict arguments and defeasible arguments, strict conclusions and defeasible conclusions, and so forth.

A characteristic distinction between argumentation systems and proof systems is that arguments, unlike proofs, vary in conclusive force.

Conclusive force

The third component of an argumentation system is the order $\leqslant$. This order determines the relative difference in strength among arguments.

Definition 2.12 (*Order of conclusive force*). Let σ and τ be arguments. If $\sigma \leqslant \tau$, then τ is *as strong as* σ, and if $\sigma < \tau$, then τ is *stronger than* σ. An order of conclusive force satisfies, besides reflexivity and transitivity, three additional conditions.

(1) *Upwards well-founded.* There are no infinite chains $\sigma_1 < \sigma_2 < \cdots < \sigma_n < \cdots$.
(2) *Monotonically non-increasing.* If $\sigma \sqsubseteq \tau$, then $\tau \leqslant \sigma$, for all σ and τ.
(3) *Propagates through strict rules.* If $\sigma_1, \ldots, \sigma_n \rightarrow \sigma$, then $\sigma_i \leqslant \sigma$, for some $1 \leqslant i \leqslant n$.

The first condition ensures that defeat is a finite process. The second and third conditions ensure that the notion of conclusive force is distributed properly over arguments. All three conditions are essential to the proofs of subsequent results.

Throughout the paper, we assume that arguments do not vie with each other indefinitely, that arguments do not become stronger if they are made longer, and that there is no mysterious loss in conclusive force through strict rules of inference. These three conditions are our axioms of conclusive force, which we deliberately have chosen in order to derive interesting principles of defeat. At the same time, we claim that these three conditions are sufficiently general to ensure that no interesting distribution of conclusive force is excluded beforehand. This claim is further underpinned in [55].

In the theory of abstract argumentation systems, it is not indicated how the strength of arguments is determined. Note that it is unimportant to find an adequate criteria to determine the difference in strength among arguments. Many researchers have proposed general criteria for adjudicating between competing lines of argument. (Cf. [16,22,30, 37,39].) However, there is a "clash of intuitions" at this moment on the validity and relevance of such criteria. Most researchers agree on the principle of specificity, that says that arguments that are more "on point" are preferred. Poole introduced specificity in [37], after which alternative definitions were proposed by Nute [30], Loui [22], Poole [38], and Prakken [39]. Other criteria to compare arguments include directness [22], pre-emption [16], combined defeat [39], number of favorable factors with a *ceteris paribus* comparison [2], and accumulation of numerical strength [36]. In this paper, we do not go into this difficult matter. Accordingly, we do not commit to a specific method to compare arguments in conclusive force.

Example 2.13. Here are some examples of how the parameter $\leqslant$ might actually be defined.

(1) *Basic order.* For all σ and τ, we set $\sigma \leqslant \tau$ if and only if σ is defeasible or τ is strict. A basic order is indeed the most basic order that is in line with the ideas of defeasible argumentation. (It trivially obeys the conditions of Definition 2.12.)

(2) *Number of defeasible steps.* Not very realistic, but this order proves to be handy in examples and counterexamples: $\sigma \leqslant \tau$ if and only if $|\tau|_{\Rightarrow} \leqslant |\sigma|_{\Rightarrow}$, where $|\cdot|_{\Rightarrow}$ stands for the number of defeasible arrows in an argument. It is only upwards well-founded if the length of the arguments has an upper bound.

(3) *Weakest link.* With this order, an argument is as strong as its weakest link. Formally, this amounts to an extension of an order $\leqslant$ on R to an order $\leqslant$ on arguments, in a conservative manner. To ensure that arguments are as strong as their weakest link, the following additional conditions must be satisfied.
   (i) (As weak as the weakest link.) If σ is an argument with top-rule $\phi_1, \ldots, \phi_n \Rightarrow \phi$, then $\sigma \leqslant (\phi_1, \ldots, \phi_n \Rightarrow \phi)$.
   (ii) (As strong as the weakest link.) If $\sigma_1, \ldots, \sigma_n \Rightarrow \sigma$ is an argument with top-rule $\phi_1, \ldots, \phi_n \Rightarrow \phi$, and there is no $1 \leqslant i \leqslant n$ such that $\sigma_i \leqslant \sigma$, then $(\phi_1, \ldots, \phi_n \Rightarrow \phi) \leqslant \sigma$.

   Thus, conditions (i) and (ii) induce a ranking on arguments based on one of the rules. Condition (i) applies the principle of the weakest link; condition (ii) ensures that no argument is weaker than necessary. Definition 2.12 further ensures that the initial order on rules extends properly over arguments. Because the extended order compares arguments on the basis of rules, it follows that atomic arguments (i.e., arguments in $\mathcal{L}$) are incomparable with respect to this order.

(4) *Preferring the most specific argument.* For all σ and τ we have $\sigma < \tau$ if σ is defeasible and τ is strict, or σ and τ are both defeasible, but the premises of τ are based on the conclusions of subarguments of σ, i.e., $prem(\tau)$ is based on $conc \circ sub(\sigma)$. As it stands, this order does not fit in the shackles of Definition 2.12, but if we additionally assume that arguments cannot become arbitrary long, then all conditions will be met.

As we have argued in the introduction, we have chosen to take the actual definition of $\leqslant$ for granted. Working on a general and universally applicable criterion to correctly adjudicate among conflicting lines of argumentation is a difficult enterprise. The problem is that, as soon as the alleged criterion prefers the right argument on one benchmark problem, it is at fault with another. Recently, it has been argued that this complication cannot be avoided [17,51]. It is particularly difficult to adjudicate among competing arguments if one does not know what these arguments are about. Thus, syntactic principles to decide among competing arguments do not exist or will be extremely weak. In this connection, it may be of interest to note that, as far as existing literature is concerned, the emphasis is always on the flaws and the fallacies, i.e., on the negative aspect of the matter, and hardly ever on the matter of conclusive force itself. That is understandable, because it is easier to point at weak spots than to speak for the overall quality of an argument.

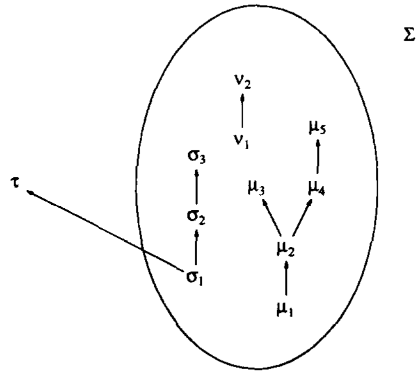

Fig. 4. Σ is undermined by τ because $\sigma_1 < \tau$.

The concept of undermining

An argument is said to undermine a set of arguments, if it dominates at least one element of that set. If a set of arguments is undermined by another argument, it cannot uphold or maintain all of its members in case of a conflict. The concept of undermining is important because it is one of the two building blocks of the theory. (The other is incompatibility.)

Because undermining is related to order, we recollect some essential (and possibly familiar) terminology.

Definition 2.14. Let Σ be a set of arguments. An element σ is a *$\leqslant$-least element* of Σ if, for every $\tau \in \Sigma$, we have $\sigma \leqslant \tau$. An element σ is a *$\leqslant$-minimal element* of Σ if, for every $\tau \in \Sigma$, we have $\sigma = \tau$ whenever $\tau \leqslant \sigma$. Similarly, the set Σ may have *$\leqslant$-greatest* or *$\leqslant$-maximal* elements.

With the present order, least elements as well as minimal elements are not uniquely defined, and may not exist. Moreover, minimal elements do not necessarily have to be least elements, and least elements do not necessarily have to be minimal elements. The same holds for greatest elements and maximal elements.

Definition 2.15. An argument τ is an *underminer* of a set of arguments Σ if $\sigma < \tau$, for some $\sigma \in \Sigma$. In this case, the set Σ is *undermined* by τ (see Fig. 4).

The concept of undermining causes a "paradox". On the one hand, a large set of arguments exhibits cogency, in the sense that it contains a large number of arguments. On the other hand, a large set of arguments is undermined more easily than a small one. If a set of arguments is large, there is a realistic chance that it will contain an argument that is weaker than some argument outside that set. Thus, many arguments adduce much evidence but are hard to maintain collectively. This fact will play an important role in the rest of the theory.

3. Compatibility

Compatibility is a generalization of consistency in propositional logic. The reason to use "compatibility" (instead of "consistency") is that we do not work in a conventional logical context. If the term "consistency" would be used here, it would invoke more than intended.

Definition 3.1 (*Contradiction*). An argument σ is *in contradiction* if $conc(\sigma) = \perp$.

An argument in contradiction is also called a contradictory argument.

Definition 3.2 (*Compatibility*). A subset P of $\mathcal{L}$ is *incompatible* if there exists a strict argument in contradiction, that is based on P. A subset of $\mathcal{L}$ is *compatible* if it is not incompatible. An incompatible subset of $\mathcal{L}$ is *minimally incompatible* if all proper subsets are compatible.

Example 3.3. Let $\mathcal{L} = \{p, q, r, s\} \cup \{\perp\}$ and let $R = \{p \Rightarrow r;\ p, q \rightarrow s;\ r, s \rightarrow \perp\}$. Then all subsets of $\{p, q, s\}$, $\{p, r\}$, and $\{q, r\}$ are compatible, while all supersets of $\{p, q, r\}$, and $\{r, s\}$ are incompatible. Moreover, both $\{p, q, r\}$, and $\{r, s\}$ are minimally incompatible.

Example 3.4. Let $\mathcal{L} = \{p, q\} \cup \{\perp\}$ and let $R = \{p, q \Rightarrow \perp\}$. Then all subsets of $\mathcal{L} - \{\perp\}$ are compatible. In particular, the set $\{p, q\}$ is compatible.

Compatibility naturally extends to sets of arguments. Thus, a set of arguments Σ is *compatible* if $conc(\Sigma)$ is compatible.

A minimally incompatible set (of formulas or arguments, that does not matter) is always finite. This follows from the definition of minimal incompatibility and the fact that arguments always have a finite number of premises. This observation is of importance to the proposition below.

The following proposition formulates an elementary result about the interaction between conclusive force and compatibility. The result will be used in Section 6.3.

Proposition 3.5. *Let $\leqslant$ be an order of conclusive force on arguments. Then every minimally incompatible set of arguments has a $\leqslant$-least element.*

Proof. Let Σ be a minimally incompatible set of arguments. We show that Σ has a least element with induction on the number of arguments in Σ. If Σ contains precisely one argument, then the proposition clearly follows. If Σ contains more than one argument, we reason as follows. According to Definition 3.2 there is a strict argument, τ, in contradiction based on the conclusions of Σ. Let $\phi_1, \ldots, \phi_n \rightarrow \phi$ be a smallest subargument of τ (which is evidently a rule). Because this rule is a subargument of an argument that is based on conclusions of Σ, the premises of $\phi_1, \ldots, \phi_n \rightarrow \phi$ are also based on the conclusions of Σ. Therefore, we may assume that $\Sigma = \Sigma' \cup \{\sigma_1, \ldots, \sigma_n\}$, such that $conc(\sigma_1) = \phi_1, \ldots, conc(\sigma_n) = \phi_n$. Let $\sigma =_{\text{def}} \sigma_1, \ldots, \sigma_n \rightarrow \phi$. It follows that the set

$\Sigma' \cup \{\sigma\}$ is minimally incompatible. (Here we assume the minimality of Σ.) Because $\Sigma' \cup \{\sigma\}$ is minimally incompatible and contains fewer elements than Σ, we may apply our induction hypothesis, which tells us that $\Sigma' \cup \{\sigma\}$ contains a least element. If σ is that least element, then we appeal to Definition 2.12(3), from which we may assume that, for some $1 \leqslant j \leqslant n$, we have $\sigma_j \leqslant \sigma$. Moreover, from Definition 2.12(2) we may assume that, for all $1 \leqslant i \leqslant n$, we have $\sigma \leqslant \sigma_i$, so that, for all $1 \leqslant i \leqslant n$, we have $\sigma_j \leqslant \sigma_i$. This suffices to conclude that σ_j is a least element of Σ. In the other case, if σ is not a least element of $\Sigma' \cup \{\sigma\}$, it must follow that some element σ' of Σ' is a least element of $\Sigma' \cup \{\sigma\}$. Again appealing to Definition 2.12(2), we know that, for all $1 \leqslant i \leqslant n$, we have $\sigma \leqslant \sigma_i$, so that σ' also is a least element of Σ. □

Proposition 3.5 tells us that, for every group of conflicting arguments in which every argument is essential to the conflict, there exists a weakest argument that is as weak as all the others. Obviously, such an argument is considered first when it comes to a resolution of the conflict.

4. Theory of warrant

The objective of a theory of warrant is to determine which arguments are in force, and which conclusions are warranted. There is no reasoning and, hence, no appeal to argumentation procedures and/or protocols for dispute.

4.1. Base set

Arguments are in force and conclusions are warranted relative to a set of basic information. Such a set of basic information is called a *base set*. In order to avoid conflicts between strict arguments, a base set must be compatible.

Definition 4.1. A *base set* is a finite compatible subset of $\mathcal{L}$.

A base set contains irreducible information. It is the point of departure in forward argumentation, and the final stopping place in backward justification. Examples of base sets are: sets of necessary and contingent facts, current fact situations [2], first principles in a case study, case descriptions and, in dialectics, initial concessions in a debate. That a base set must be finite is crucial to the theory of argumentation sequences. (Cf. Proposition 6.4.)

4.2. Defeat

Defeat is the most elementary notion of warrant, as it is the most elementary relation on arguments that involves compatibility as well as conclusive force. It lies at the basis of many dialectical concepts. The notion of enablement, for instance, can be stated in terms of defeat. Also other important dialectical concepts can be expressed with the help of this elementary notion.

Definition 4.2 (*Defeater*). Let P be a base set, and let σ be an argument. A set of arguments Σ is a *defeater* of σ if it is incompatible with this argument and not undermined by it; in this case σ is *defeated* by Σ, and Σ *defeats* σ. Σ is a *minimal defeater* of σ if all its proper subsets do not defeat σ.

Example 4.3. Consider the abstract argumentation system $\mathcal{A}$ with language $\mathcal{L} = \{p,q,r,s\} \cup \{\perp\}$ with rules

$$R = \{p \Rightarrow q, p \Rightarrow s, q \Rightarrow r, r \Rightarrow s\} \cup \{q,r,s \rightarrow \perp\}$$

and with the $\Rightarrow$-count order on arguments. (Cf. Example 2.13(2).) Let $\sigma = p \Rightarrow q \Rightarrow r$ and

$$\Sigma = \{p, p \Rightarrow q, p \Rightarrow s, p \Rightarrow q \Rightarrow r \Rightarrow s\}. \tag{3}$$

With this set of arguments,

- $\{p \Rightarrow q\}$ is not a defeater of σ, since it is compatible with σ,
- $\{p \Rightarrow s\}$ is not a defeater of σ, for the same reason,
- $\{p \Rightarrow q, p \Rightarrow s\}$ is a defeater of σ, since $q,r,s \rightarrow \perp$,
- $\{p, p \Rightarrow q, p \Rightarrow s\}$ is a defeater of σ, since $q,r,s \rightarrow \perp$,
- $\{p, p \Rightarrow q, p \Rightarrow s, p \Rightarrow q \Rightarrow r \Rightarrow s\}$ is not a defeater of σ, since it is undermined by σ.

Thus, Σ contains various defeaters of σ, without being a defeater of σ itself. The set Σ does not defeat σ, because it is undermined by σ. Small subsets of Σ also do not defeat σ, because they are not incompatible with σ. Maybe it is convenient to remember defeaters as the lean and mean elements of this theory. They must stay small in order not to be undermined by other arguments; meanwhile they must support enough conclusions to form a contradiction with the conclusions of other arguments.

4.3. Enablement

The basic idea behind enablement, is that enabled is "not defeated". An argument σ is enabled by a set of arguments Σ if and only if all subarguments of σ, including σ, are not defeated by subsets of Σ:

Definition 4.4. Let P be a base set, and let Σ be a set of arguments. An argument σ is *enabled by Σ on the basis of P*, written $P|_{\Sigma} \sim \sigma$, if

(1) the set P contains σ; or

(2) for some arguments $\sigma_1, \ldots, \sigma_n$ we have $P|_{\Sigma} \sim \sigma_1, \ldots, \sigma_n$ and $\sigma_1, \ldots, \sigma_n \rightarrow \sigma$; or

(3) for some arguments $\sigma_1, \ldots, \sigma_n$ we have $P|_{\Sigma} \sim \sigma_1, \ldots, \sigma_n$ and $\sigma_1, \ldots, \sigma_n \Rightarrow \sigma$ and Σ does not contain defeaters of σ.

(With $P|_{\Sigma} \sim \sigma_1, \ldots, \sigma_n$ we mean: $P|_{\Sigma} \sim \sigma_1$ and ... and $P|_{\Sigma} \sim \sigma_n$.)

Definition 4.4 consists of three clauses. The first clause simply ensures that every element of P, as a basic element, is enabled. The second clause enforces that the property of enablement propagates through strict arguments. The third clause forms the heart of Definition 4.4. It is similar to the second clause, upto the additional condition on Σ and σ. The rationale behind this additional condition is that σ is enabled by Σ only if all subsets of Σ are either compatible with σ or undermined by σ.

Example 4.5 (*Enablement*). Consider the abstract argumentation system $\mathcal{A}$ with language $\mathcal{L} = \{p, q, r\} \cup \{\perp\}$, with rules

$$R = \{p \Rightarrow q, q \Rightarrow r, p \Rightarrow s\} \cup \{q, s \rightarrow \perp\} \tag{4}$$

and with a basic order on arguments $\leqslant$. Let $P = \{p\}$. Let us, by way of example, see whether the argument $p \Rightarrow q \Rightarrow r$ is enabled by $\Sigma = \{p, p \Rightarrow q\}$ on the basis of P. Since $p \Rightarrow q \Rightarrow r$ ends with a defeasible rule, we are referred to Definition 4.4(3), which states that $p \Rightarrow q \Rightarrow r$ is enabled by $\Sigma = \{p, p \Rightarrow q\}$ on the basis of P if and only if $p \Rightarrow q$ is enabled by Σ, and Σ does not contain defeaters of $p \Rightarrow q \Rightarrow r$. Since the entire set Σ is compatible with $p \Rightarrow q \Rightarrow r$, the last condition is readily fulfilled. Hence, the problem whether $p \Rightarrow q \Rightarrow r$ is enabled by Σ, reduces to the problem whether its immediate subargument $p \Rightarrow q$ is enabled by Σ. Following the same line of reasoning as above, this problem reduces one step further (via Definition 4.4(3)) to the problem whether p is enabled by Σ. This, however, follows easily from Definition 4.4(1), since $p \in P$. Hence, the argument $p \Rightarrow q$ is enabled by Σ which, on its turn, implies that $p \Rightarrow q \Rightarrow r$ is enabled by Σ.

This example is pretty straightforward. An example that works out differently, is the following.

Example 4.6 (*Enablement*). Consider the abstract argumentation system $\mathcal{A}$ of Example 4.5, with $P = \{p\}$. Let us now determine whether $p \Rightarrow q$ is enabled by $\Sigma = \{p, p \Rightarrow q, p \Rightarrow s\}$ on the basis of P. Since $p \Rightarrow q$ ends with a defeasible rule, we are referred to Definition 4.4(3) once again, which states that $p \Rightarrow q$ is enabled by $\Sigma = \{p, p \Rightarrow q, p \Rightarrow s\}$ on the basis of P if and only if p is enabled by Σ, and Σ does not contain defeaters of $p \Rightarrow q$. But the singleton set $\Sigma_S =_{\text{def}} \{p \Rightarrow s\}$ is a defeater of $p \Rightarrow q$. Hence, the argument $p \Rightarrow q$ is not enabled by Σ. Note that this is so, even though $p \Rightarrow q$ is a member of Σ itself.

The notion of enablement obtains further meaning if it operates on sets.

Definition 4.7 (*Enablement operator*). Let P be a base set, and let Σ be a set of arguments. Then $enable_P(\Sigma)$ denotes the set of arguments $\{\sigma \mid P|_\Sigma \sim \sigma\}$.

The notation tries to express that the enablement operator works on sets of arguments, relative to a certain base set.[7] Before we discuss some of the properties of the enablement operator, we elaborate a bit further on the material of the previous examples.

[7] For a long time, I wavered between using $enable_P(\Sigma)$ or $enable_\Sigma(P)$. The notation $enable_\Sigma(P)$ is an alternative that is more in line with subsequent terminology (using the base set as the main argument), but turns away from the idea that enablement is an operator on sets of arguments. The last idea is, later on, required by the theory.

Example 4.8. Consider the abstract argumentation system $\mathcal{A}$ of Example 4.5, with $P = \{p\}$. Then

$$\begin{aligned}
&enable_P(p) = \{p, p \Rightarrow q, p \Rightarrow q \Rightarrow r, p \Rightarrow s\},\\
&enable_P(p, p \Rightarrow q) = \{p, p \Rightarrow q, p \Rightarrow q \Rightarrow r\},\\
&enable_P(p, p \Rightarrow s) = \{p, p \Rightarrow s\},\\
&enable_P(p, p \Rightarrow q, p \Rightarrow s) = \{p\},\\
&enable_P(p, p \Rightarrow q, p \Rightarrow q \Rightarrow r) = \{p, p \Rightarrow q, p \Rightarrow q \Rightarrow r\},\\
&enable_P(p, p \Rightarrow q, p \Rightarrow q \Rightarrow r, p \Rightarrow s) = \{p\}.
\end{aligned}$$

This example shows a number of characteristic properties of the enablement operator. To begin with, we see that, for every set of arguments Σ, if Σ is small, then $enable_P(\Sigma)$ is large. Conversely, if Σ is large, then $enable_P(\Sigma)$ is small. Further, we see that enablement is not reflexive, that is, we do not always have that $\Sigma \subseteq enable_P(\Sigma)$. Especially if Σ is large, it might disable (i.e., defeat) some of its own elements. This is because large sets usually are incompatible.

The following proposition will be used in the proof of Proposition 4.33.

Proposition 4.9. *Let P be a base set, and let Σ be a set of arguments that is compatible. Then Σ is included in $enable_P(\Sigma)$.* [8]

Proof. Let $\sigma \in \Sigma$. Since σ is compatible with every subset of $\Sigma - \{\sigma\}$, it is obviously enabled by Σ. □

Thus if Σ is compatible, then $\Sigma \subseteq enable_P(\Sigma)$. Further, if $\Sigma = arguments(P)$, then $enable_P(\Sigma) \subseteq \Sigma$. This pair of crossing inclusions suggests that, in some situations, typically those in which Σ is neither too big nor too small, the set Σ enables precisely itself. Later on, these sets, which are stable under the enablement operator, will be called extensions of P. Let us consider other non-properties of the enablement operator. Besides not being reflexive, the enablement operator is not idempotent as well. From this, it follows that it might be interesting to iterate the enablement operator to see what comes out. This will be done in a later stage of this paper.

The enablement operator has few regularities. In fact, the only regular property of enablement is that it reverses inclusion:

[8] A converse of this proposition—inclusion implies compatibility—holds as well, but its proof is more involved, and the result is not needed in the rest of the theory. Therefore, the proof that compatibility is fully expressible in terms of enablement is left as an "exercise". A related question is what is expressed by the inclusion $enable_P(\Sigma) \subseteq \Sigma$. We conjecture that this inclusion holds if and only if Σ contains all its underminers and is not extendible. (Not extendible: $\sigma \notin \Sigma$ implies incompatibility of $\Sigma \cup \{\sigma\}$.) If this would be so, then enablement is able to characterize, besides compatibility, also conclusive force. This in turn would imply that a theory of abstract argumentation could be defined in even more abstract terms. Instead of beginning with the notions of conclusive force and compatibility, it would then suffice to begin with the notion of enablement. In more concrete terms, this would mean that it is possible to talk about principles of defeat, without ever referring to contradictions and the conclusive force of arguments. [We leave the conjecture on the characterization of $enable_P(\Sigma) \subseteq \Sigma$ an open issue.]

Proposition 4.10. *Let P be a base set. Let Σ_1 and Σ_2 be two sets of arguments such that $\Sigma_1 \subseteq \Sigma_2$. Then $enable_P(\Sigma_2) \subseteq enable_P(\Sigma_1)$.*

Proof. Let us prove that, for every argument σ, we have $P|_{\Sigma_1} \sim \sigma$ whenever $P|_{\Sigma_2} \sim \sigma$. To this end, let σ be an argument with $P|_{\Sigma_2} \sim \sigma$. If $\sigma \in \mathcal{L}$ or, for some $\sigma_1, \ldots, \sigma_n$, we have $\sigma_1, \ldots, \sigma_n \rightarrow \sigma$, then there is nothing to prove as Σ_1 and Σ_2 are not involved here. If, for some $\sigma_1, \ldots, \sigma_n$, we have $\sigma_1, \ldots, \sigma_n \Rightarrow \sigma$, then we use the simple fact that if something holds for every subset of Σ_2, then it must hold for every subset of Σ_1 too. This suffices to establish the third case, and to conclude that $P|_{\Sigma_1} \sim \sigma$. Hence, the proposition follows. □

The fact that enablement reverses inclusion will be used repeatedly in the following section.

4.4. Inductive warrant

A theory of inductive warrant is based on the idea that defeaters can be defeated themselves, and that this hierarchy of defeat can be continued indefinitely. Thus, defeaters can be defeated by defeater-defeaters which, on their turn, can be defeated by defeater-defeater-defeaters, and so on. As opposed to enablement, which is about primary defeat, inductive warrant is able to deal with complex forms of defeat, such as cascaded defeat and reinstatement.

The basic idea behind inductive warrant is as follows. We begin with a fixed base set. The idea is that all arguments that are based on that particular base set are divided into different levels. At level 1, every argument is simply declared to be in force. At level 2, some arguments are defeated by arguments at level 1, so that level 2 contains precisely those arguments that are not defeated by arguments at level 1. At level 3, some arguments at level 1 that were defeaters of arguments at level 2, are defeated at level 2 themselves. Therefore, some of the arguments that were defeated at level 2 are reinstated at level 3. At level 4, some of the arguments that were reinstated at level 3 should be given up, as they are defeated by level-2 arguments and/or arguments that were reinstated at level 3. And so forth. The concept "level-n argument" is from [31,36].

If these ideas are formulated in the current vocabulary, we obtain the following definition.

Definition 4.11 (*Level-n arguments*). Let P be a base set. An argument σ is *in force at level* 1 *on the basis of* P if it is based on P. Let $n > 1$. An argument σ is *in force at level n on the basis of* P, written $P \mid\!\sim_n \sigma$, if

(1) the set P contains σ; or
(2) for some arguments $\sigma_1, \ldots, \sigma_m$ we have $P \mid\!\sim_n \sigma_1, \ldots, \sigma_m$ and $\sigma_1, \ldots, \sigma_m \rightarrow \sigma$; or
(3) for some arguments $\sigma_1, \ldots, \sigma_m$ we have $P \mid\!\sim_n \sigma_1, \ldots, \sigma_m$ and $\sigma_1, \ldots, \sigma_m \Rightarrow \sigma$ and every set of arguments Σ that is in force at level $n-1$ on the basis of P is not a defeater of σ.

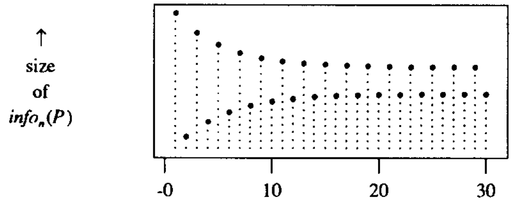

Fig. 5. Development of $\{info_n(P)\}_n$.

The expression $info_n(P)$ denotes the set of arguments $\{\sigma \mid P \mid\!\sim_n \sigma\}$ (see Fig. 5).

Inductive warrant can be expressed in terms of enablement. By means of applying the enablement operator n times on a base set, for some $n \geqslant 1$, we obtain precisely those arguments that are in force at level n.

Proposition 4.12 (Iteration of enablement). *Let P be a base set. Then, for every $n \geqslant 1$, $info_n(P) = enable^n_P(P)$.*

Proof. We prove this with induction on n. If $n = 1$, then both $info_1(P)$ and $enable^1_P(P)$ are equal to $arguments(P)$. Next, let us assume that the equality holds for $n - 1$. We must show that $info_n(P) = enable_P \circ info_{n-1}(P)$ holds. This can easily be verified with the help of Definitions 4.4 and 4.11. □

The previous proposition is a helpful device for computing inductive warrant.[9] Let us apply it immediately in the following example. In this example, we have taken a specific argumentation system that shows off some typical properties of inductive warrant.

Example 4.13. Consider the abstract argumentation system $\mathcal{A}$ with language $\mathcal{L} = \{p_i\}^n_{i=1} \cup \{\bot\}$ with rules

$$R = \{\Rightarrow p_i\}^n_{i=1} \cup \{p_i, p_{i+1} \to \bot\}^{n-1}_{i=1} \tag{5}$$

and with an order on arguments $\leqslant$ such that, for every $1 \leqslant i \leqslant n-1$, we have $\Rightarrow p_{i+1} < \Rightarrow p_i$. Let us, for the sake of brevity, put $\sigma_i =_{\text{def}} \Rightarrow p_i$, for every $1 \leqslant i \leqslant n$. Then $\sigma_n < \cdots < \sigma_1$, and every argument σ_i is incompatible with its immediate neighbor. Let $P = \emptyset$. In case $n = 8$,

[9] There is a relation with Dung's notion of acceptability [9]. An argument A is *acceptable* w.r.t. a set of arguments S if, for every argument B: if B attacks (i.e., defeats) A, then B is attacked (i.e., defeated) by S. (In our terminology: σ is accepted by Σ if every defeater of σ is defeated by a subset of Σ.) Dung's characteristic function F_{AF} is given by $F_{AF}(S) =_{\text{def}} \{A \mid A \text{ is acceptable w.r.t. } S\}$. It can be shown that $F_{AF} \equiv enable^2_P$. In words: if enablement is squared we obtain Dung's notion of acceptability. In more general terms this means that Dung's acceptability operator can be expressed in terms of our enablement operator. Whether the converse holds is unknown.

$$
\begin{aligned}
\mathit{info}_1(P) &= \{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_7,\sigma_8\},\\
\mathit{info}_2(P) &= \{\sigma_1\},\\
\mathit{info}_3(P) &= \{\sigma_1,\widehat{\sigma}_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_7,\sigma_8\},\\
\mathit{info}_4(P) &= \{\sigma_1,\widehat{\sigma}_2,\sigma_3\},\\
\mathit{info}_5(P) &= \{\sigma_1,\widehat{\sigma}_2,\sigma_3,\widehat{\sigma}_4,\sigma_5,\sigma_6,\sigma_7,\sigma_8\},\\
\mathit{info}_6(P) &= \{\sigma_1,\widehat{\sigma}_2,\sigma_3,\widehat{\sigma}_4,\sigma_5\},\\
\mathit{info}_7(P) &= \{\sigma_1,\widehat{\sigma}_2,\sigma_3,\widehat{\sigma}_4,\sigma_5,\widehat{\sigma}_6,\sigma_7,\sigma_8\},\\
\mathit{info}_8(P) &= \{\sigma_1,\widehat{\sigma}_2,\sigma_3,\widehat{\sigma}_4,\sigma_5,\widehat{\sigma}_6,\sigma_7\}.
\end{aligned}
$$

Further, for all $k \geqslant 1$, $\mathit{info}_{n+k}(P) = \mathit{info}_n(P)$.[10]

This example shows off nicely some typical properties of inductive warrant. First, we see that, for every $1 \leqslant i \leqslant n$, the argument σ_i is defeated and reinstated $(i-1)/2$ times if i is odd, and defeated $i/2$ times and reinstated $(i-2)/2$ times if i is even. So the arguments σ_i for which i is odd, emerge undefeated eventually. Further, we see that odd info's are big, while even info's are small. More precisely, we see that the odd info's, as a sequence of sets of arguments, are monotonically decreasing, while the even info's are monotonically increasing. Moreover, we see that every even info is contained in every odd info. These findings are not typical for the case at hand. They hold in general:

Proposition 4.14. *The sequence $\{\mathit{info}_n(P)\}_{n=1}^{\infty}$ is an alternating sequence of sets of arguments, such that $\{\mathit{info}_{2n}(P)\}_{n=1}^{\infty}$ is monotonically increasing, and $\{\mathit{info}_{2n-1}(P)\}_{n=1}^{\infty}$ is monotonically decreasing*:

$$\mathit{info}_2(P) \subseteq \mathit{info}_4(P) \subseteq \cdots \subseteq \mathit{info}_3(P) \subseteq \mathit{info}_1(P). \tag{6}$$

Proof. It suffices to prove that, for every $n \geqslant 1$, we have

$$
\begin{aligned}
&\mathit{info}_{2n}(P) \subseteq \mathit{info}_{2n-1}(P),\\
&\mathit{info}_{2n}(P) \subseteq \mathit{info}_{2n+2}(P),\\
&\mathit{info}_{2n+1}(P) \subseteq \mathit{info}_{2n-1}(P),\\
&\mathit{info}_{2n+2}(P) \subseteq \mathit{info}_{2n+1}(P).
\end{aligned}
$$

Let us prove this for $n = 1$. First, by using Proposition 4.12, we write $\mathit{info}_n(P) = \mathit{enable}^n_P(P)$. By Definition 4.4, we have $P \subseteq \mathit{enable}_P(P)$. Applying Proposition 4.10 three times, we get $\mathit{enable}^2_P(P) \subseteq \mathit{enable}_P(P)$, $\mathit{enable}^2_P(P) \subseteq \mathit{enable}^3_P(P)$, and $\mathit{enable}^4_P(P) \subseteq \mathit{enable}^3_P(P)$, respectively. Again by Definition 4.4, we have that $P \subseteq \mathit{enable}^2_P(P)$ holds. Applying Proposition 4.10 two times to this inclusion yields $\mathit{enable}^3_P(P) \subseteq \mathit{enable}_P(P)$, and $\mathit{enable}^2_P(P) \subseteq \mathit{enable}^4_P(P)$, respectively. This establishes the case for $n = 1$. For $n \geqslant 1$, the proof follows with an elementary induction argument. □

[10] The notation with the hat works as follows: $\{1,\widehat{2}\} = \{1\}$, $\{\widehat{a},b,\widehat{c}\} = \{b\}$, $\{1,\ldots,\widehat{i},\ldots,n\} = \{1,\ldots,i-1,i+1,\ldots,n\}$, $\{\widehat{x}\} = \emptyset$, and so on.

In general, the sequence $\{info_n(P)\}_{n=1}^{\infty}$ has no limit. This is because, generally, the even info's tend to remain proper subsets of the odd info's, even for large n. However, every sequence of sets of arguments does have an upper limit and a lower limit. [11] Accordingly, let us write $info_{\uparrow}(P)$ and $info_{\downarrow}(P)$ for $lim\ inf_n\ info_n(P)$ and $lim\ sup_n\ info_n(P)$, respectively. Then it follows from the previous proposition that $info_{\uparrow}(P)$ and $info_{\downarrow}(P)$ are equal to $lim_n info_{2n}(P)$ and $lim_n info_{2n-1}(P)$, respectively. [12] If $info_{\uparrow}(P) = info_{\downarrow}(P)$, then $lim_n\ info_n\ (P)$ exists, and equals either the upper or the lower limit of the original sequence.

With these definitions, the set *arguments*(P) is divided into three disjoint subclasses:

(1) *Ultimately undefeated arguments.* The class of ultimately undefeated arguments is equal to $info_{\uparrow}(P)$. Thus, an argument σ is ultimately undefeated if and only if, for some $n \geqslant 1$, we have $P \mid\!\sim_{n+k} \sigma$, for every $k \geqslant 1$.

(2) *Provisionally defeated arguments.* (Critical arguments.) The class of provisionally defeated arguments is equal to $info_{\downarrow}(P) - info_{\uparrow}(P)$. Here, an argument σ is provisionally defeated if and only if, for every $n \geqslant 1$, we have $P \mid\!\sim_{n+k} \sigma$, for some $k \geqslant 1$, and not $P \mid\!\sim_{n+l} \sigma$, for some other $l \geqslant 1$. A provisionally defeated argument is called also a *critical argument.* With critical arguments, it is not sure whether they are in force or not. [13]

(3) *Ultimately defeated arguments.* The class of ultimately defeated arguments is equal to *arguments*$(P) - info_{\downarrow}(P)$. An argument σ is ultimately defeated if and only if, for some $n \geqslant 1$, the entailment $P \mid\!\sim_{n+k} \sigma$ does no longer hold, for every $k \geqslant 1$.

The terms "ultimately (un)defeated" and "provisionally defeated" are from [31,36].

[11] The upper and lower limit of a sequence of sets of arguments are defined as follows. The upper limit $lim\ sup_n\ \Sigma_n$ of $(\Sigma_n)_{n=1}^{\infty}$ is defined as $\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}\Sigma_i$; the lower limit $lim\ inf_n\ \Sigma_n$ of $(\Sigma_n)_{n=1}^{\infty}$ is defined as $\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\Sigma_i$. In other words, the lower limit consists of all arguments that are contained in all but a finite number of terms, and the upper limit consists of all arguments that are contained in an infinite number of terms of this sequence.

[12] The limit $lim_n\ \Sigma_n$ of $(\Sigma_n)_{n=1}^{\infty}$ is defined only if this sequence is either monotonically increasing or monotonically decreasing. If $(\Sigma_n)_{n=1}^{\infty}$ is monotonically increasing, i.e., if, for every $n \geqslant 1$, we have $\Sigma_n \subseteq \Sigma_{n+1}$, then $lim_n\ \Sigma_n$ is defined as $\bigcup_{n=1}^{\infty}\Sigma_n$; if $(\Sigma_n)_{n=1}^{\infty}$ is monotonically decreasing, then $lim_n\ \Sigma_n$ is defined as $\bigcap_{n=1}^{\infty}\Sigma_n$. If $(\Sigma_n)_{n=1}^{\infty}$ is neither monotonically increasing nor monotonically decreasing, then $lim_n\ \Sigma_n$ is undefined, unless $lim\ inf_n\ \Sigma_n = lim\ sup_n\ \Sigma_n$. In that case, $lim_n\ \Sigma_n$ is defined to be equal to either the upper or lower limit of this sequence.

[13] Using the terminology of Prakken [39], these arguments are called *defensible*: "the main strong points of the present [Prakken's] framework seem to be that (...) the assessment of arguments is three-valued, in that it leaves room for arguments which are neither preferred [ultimately undefeated] nor defeated [ultimately defeated], but merely defensible [provisionally defeated]". In the theory on defeasible dialectics, it is indeed the case that arguments that are not ultimately defeated on the basis of P, can be defended successfully in an $\mathcal{A}$-debate on the basis of P. More in particular, the arguments that are ultimately undefeated can be defended successfully in finite depth, while arguments that are provisionally defeated (in Prakken's terminology: arguments that are defensible), can be defended successfully only in debates of infinite depth. (If the debate is carried out relative to a finite argumentation system, this means that parts of the debate are reiterated.) So Prakken's notion of defensible indeed corresponds to its intuitive meaning. Still, within the current setting, where dialectical considerations are not at issue, the more neutral "provisionally defeated" is probably a better choice.

Example 4.15 (*Critical arguments*). Consider the abstract argumentation system $\mathcal{A}$ with language $\mathcal{L} = \{p, q, r, s, t\} \cup \{\perp\}$, with rules

$$R = \{p \Rightarrow q, q \Rightarrow r, p \Rightarrow s, p \Rightarrow t\} \cup \{q, s \rightarrow \perp, r, t \rightarrow \perp\} \tag{7}$$

and with $\leqslant$ defined as $\sigma \leqslant \tau$ if and only if $|\tau|_{\Rightarrow} \leqslant |\sigma|_{\Rightarrow}$, where $|\sigma|_{\Rightarrow}$ stands for the number of defeasible arrows in σ. Let $P = \{p\}$. In this example, we determine the arguments that are ultimately undefeated, provisionally defeated, and ultimately defeated on the basis of P. To do so, we localize the points of conflict in *arguments*(P). These are $\{p \Rightarrow q, p \Rightarrow s\}$ and $\{p \Rightarrow q \Rightarrow r, p \Rightarrow t\}$. With the first set, none of the two arguments is better than the other, while, in the second set, the argument $p \Rightarrow t$ has one less defeasible arrow, compared to $p \Rightarrow q \Rightarrow r$. Therefore, the argument $p \Rightarrow t$ defeats $p \Rightarrow q \Rightarrow r$. With this information, it is not hard to find out which argument is in force on what level. If we do this for every argument, we obtain the following sequence.

$$\begin{aligned}
info_1(P) &= \{p, p \Rightarrow q, p \Rightarrow q \Rightarrow r, p \Rightarrow s, p \Rightarrow t\},\\
info_2(P) &= \{p, p \Rightarrow t\},\\
info_3(P) &= \{p, p \Rightarrow q, p \Rightarrow s, p \Rightarrow t\},\\
info_4(P) &= \{p, p \Rightarrow t\},\\
info_5(P) &= \{p, p \Rightarrow q, p \Rightarrow s, p \Rightarrow t\},\\
&\vdots
\end{aligned}$$

We see that

$$info_{2n}(P) = \{p, p \Rightarrow t\} \quad \text{and} \quad info_{2n+1}(P) = \{p, p \Rightarrow q, p \Rightarrow s, p \Rightarrow t\},$$
for every $n \geqslant 1$.

It follows that $info_{\uparrow}(P) = \{p, p \Rightarrow t\}$ and $info_{\downarrow}(P) = \{p, p \Rightarrow q, p \Rightarrow s, p \Rightarrow t\}$. Hence, with the terminology above, we conclude that arguments p and $p \Rightarrow t$ are ultimately undefeated, that the arguments $p \Rightarrow q$ and $p \Rightarrow s$ are provisionally defeated, and that the argument $p \Rightarrow q \Rightarrow r$ is ultimately defeated. To say $p \Rightarrow q$ and $p \Rightarrow s$ are provisionally defeated, we mark them as critical arguments, meaning that, on the basis of P, it cannot be determined whether $p \Rightarrow q$ should be in force at the expense of $p \Rightarrow s$, or conversely. Critical arguments will be considered further in Section 4.5.

Finally, there is the notion of inductive warrant for propositions:

Definition 4.16. Let P be a base set. Let $n \geqslant 1$. An element ϕ is *warranted at level n on the basis of P*, written $P \mid\!\sim_n \phi$, if $P \mid\!\sim_n \sigma$ and $conc(\sigma) = \phi$, for some argument σ. Accordingly, the set $warrant_n(P)$ is set equal to $conc \circ info_n(P)$.

Obviously, the above definition overloads our notion of defeasible entailment. That is, if $P \mid\!\sim_n \phi$, then either $\phi = \sigma$ for some argument σ, or $\phi = conc(\sigma)$, for some (other) argument σ. But as the first reading implies the second, the second reading prevails in this ambiguous case.

4.5. Warrant

A proposition is warranted if it is the conclusion of an argument that is in force. Whether an argument is in force is determined in a recursive definition. This definition, therefore, is more involved than its two predecessors. (Cf. Definitions 4.4 and 4.11.)

The definition below uses the defeasible entailment relation symbol (the crooked turnstile) $\mid\!\sim$. The symbol $\mid\!\sim$ is a relation between P and arguments based on P. If $P \mid\!\sim \sigma$ we say that σ *is in force on the basis of* P. Similarly, if Σ is a set of arguments such that $P \mid\!\sim \sigma$ for every $\sigma \in \Sigma$, we write $P \mid\!\sim \Sigma$, and we say that Σ *is in force on the basis of* P.

The definition of $\mid\!\sim$ differs from the definition of $|_{\Sigma} \sim$ and the definition of $\mid\!\sim_n$, in the following way. The enablement operator $|_{\Sigma} \sim$ (cf. Definition 4.4) and the level-n entailment operator $\mid\!\sim_n$ (cf. Definition 4.11) are binary (i.e., two-place) relations between base sets and arguments. For example, the expression $P \mid\!\sim_n \sigma$ is a binary relation $\mid\!\sim_n$ between the base set P and the argument σ.[14] Of course, there exist infinitely many such relations since, for every $n \geqslant 1$, we have another $\mid\!\sim_n$. But $\mid\!\sim_n$ itself is uniquely determined. Similarly, there exist many relations between P and σ of the type $|_{\Sigma} \sim$. Every Σ gives rise to a unique $|_{\Sigma} \sim$. The defeasible entailment relation $\mid\!\sim$ that will be defined next is also a binary relation. However, in contrast with the two former operators, it is not unique.

The definition of $\mid\!\sim$ is equal to the definition of $\mid\!\sim_n$ (cf. Definition 4.11), except for the third clause, which introduces recursion.

Definition 4.17. Let P be a base set. A relation $\mid\!\sim$ between P and arguments based on P is a *defeasible entailment relation* if, for every argument σ based on P, we have $P \mid\!\sim \sigma$ if and only if

(1) the set P contains σ; or

(2) for some arguments $\sigma_1, \dots, \sigma_n$ we have $P \mid\!\sim \sigma_1, \dots, \sigma_n$ and $\sigma_1, \dots, \sigma_n \to \sigma$; or

(3) for some arguments $\sigma_1, \dots, \sigma_n$ we have $P \mid\!\sim \sigma_1, \dots, \sigma_n$ and $\sigma_1, \dots, \sigma_n \Rightarrow \sigma$ and every set of arguments Σ that is in force on the basis of P is not a defeater of σ.

Recursion appears at "every set of arguments Σ that is in force on the basis of P". To find out whether σ is in force, it may happen that the definition leads to another set of arguments, Σ, for which we must answer a similar question. This recursive descent, this *regression*, may stop or may continue indefinitely. If it stops, then all questions are answered unambiguously and we are done. (This is formulated more precisely in Proposition 4.22.) If the regression continues indefinitely, there are two possibilities: The first possibility is that the impossibility to stop is caused by a circular dependency among defeaters. If defeaters depend on each other in this way, then the initial question comes back to σ after a finite number of steps. The second possibility is that the initial

[14] If we want to be very formal, the relation $\mid\!\sim_n$ must be written prefix instead of infix, i.e., $\mid\!\sim_n (P, \sigma)$, instead of $P \mid\!\sim_n \sigma$. We will not do this.

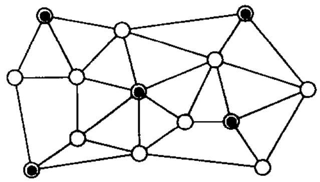

Fig. 6. A maximally independent set of vertices in a graph.

question invokes an infinite number of related questions that pertain to an infinite number of defeaters, each of which involves new arguments. In both cases, the argument σ is in force if and only if a collection of equivalent arguments is not in force. All three types of situations will be demonstrated in forthcoming examples.

Definition 4.18. A set of arguments Σ is an *extension* of P, if there exists a defeasible entailment relation $\mid\!\sim$ such that $\Sigma = \{\sigma \mid P \mid\!\sim \sigma\}$. The set $\{\sigma \mid P \mid\!\sim \sigma\}$ is *the extension generated by* $\mid\!\sim$ and denoted by $info_{\mid\sim}(P)$. The number of different extensions of P is the *degree* of P, written $deg(P)$. If $deg(P) = 1$, we cannot be mistaken about which defeasible entailment relation generates the extension, so that we write $info(P)$.

Illustration: the concept of maximal independence

The following problem lies at the basis of the recursive clause at Definition 4.17(3).

Suppose you are a jailer of n prisoners. Your task is to pick k prisoners, $k \leqslant n$, for a collective breath of air. The problem is that most prisoners have enemies. If two enemies are ventilated together, they will probably start a fight with all the unpleasant consequences which that may have.
Problem: select a large group of prisoners without enemies.

If prisoners are represented by points and their hostile relationship is represented by edges, the problem comes down to finding a maximal set of independent points in a graph (see Fig. 6).

The notion of a maximal set of independent points can be formally defined as follows. Let $G = (V, E)$ be a graph with vertices (points, nodes) V, and edges (lines) E. A subset I of V is called *independent* if there are no edges in E joining vertices of I in G. A subset of V is *maximally independent* all its proper supersets are not independent.

The notion of independency can also be defined in terms of a relation: a one-place relation (or predicate, or labeling) i on points of G is *maximally independent* if, for every point v in G, we have $i(v)$ if and only if

$$\text{for every neighbor } w \text{ of } v, \text{ not } i(w). \qquad (*)$$

Naturally, it turns out that a relation i is maximally independent if and only if the set of points labeled by i is maximally independent. This is nothing particular. The point is to note that a condition as simple as $(*)$ suffices to obtain, not only a set of independent vertices, but also a *maximal* set of independent vertices.

The definition of $\mid\!\sim$, Definition 4.17, emerges as soon as nodes represent sets of arguments and lines represent incompatibility. Thus, two sets of arguments are connected if and only if they are incompatible. Within an argument system, it is our objective to make as many arguments warranted, without making two incompatible sets of arguments warranted simultaneously. The latter is established by the third clause of Definition 4.17, that is basically an extended version of clause $(*)$ above.

The following proposition establishes a connection between enablement and warrant.

Proposition 4.19 (Fixed point of enablement). *Let P be a base set. Then Σ is an extension of P if and only if $enable_P(\Sigma) = \Sigma$.*

Proof. This proposition is an immediate consequence of Definitions 4.4 and 4.17. □

The next proposition establishes a connection between warrant and inductive warrant.

Proposition 4.20 (Mean of inductive warrant). *Let P be a base set, and let Σ be an extension of P. Then $info_\uparrow(P) \subseteq \Sigma$ and $\Sigma \subseteq info_\downarrow(P)$.*

Proof. At the end of Section 4.4 we have argued that $info_\uparrow(P) = lim_n\ info_{2n}(P)$ and $info_\downarrow(P) = lim_n\ info_{2n-1}(P)$. Therefore, it suffices to prove that, for every $n \geqslant 1$, we have $info_{2n}(P) \subseteq \Sigma$ and $\Sigma \subseteq info_{2n-1}(P)$. Let us prove this for $n = 1$. To begin with, it can easily be verified that

$$P \subseteq \Sigma \subseteq info_1(P) \tag{8}$$

since $info_1(P) = arguments(P)$. From Proposition 4.10 we know that applying the enablement operator to this chain of inclusions yields the reversed chain

$$enable_P \circ info_1(P) \subseteq enable_P(\Sigma) \subseteq enable_P(P). \tag{9}$$

Using Proposition 4.12 we get

$$info_2(P) \subseteq enable_P(\Sigma) \subseteq info_1(P). \tag{10}$$

With Proposition 4.19 we know that Σ is a fixed point of the enablement operator, so that Eq. (10) reduces to

$$info_2(P) \subseteq \Sigma \subseteq info_1(P). \tag{11}$$

This establishes the case for $n = 1$. Applying this procedure several times yields our desired result. □

The previous result perhaps suggests that inductive warrant is a good approximation of warrant. However, this need not be so. As soon as there is more than one extension, it follows from the previous proposition that $info_\uparrow(P)$ must be contained in every extension of P, and that every extension of P must be contained in $info_\downarrow(P)$. It is possible that these extensions have little in common. In such cases, it follows that $info_\uparrow(P)$ as well as $info_\downarrow(P)$ are poor approximations of single extensions.

Conjecture 4.21. *Let P be a base set, and let $\{\Sigma_i \mid i \in I\}$ be the collection of all extensions of P. Then*
(1) $\bigcap\{\Sigma_i \mid i \in I\} = info_{\uparrow}(P)$.
(2) $info_{\downarrow}(P) = \bigcup\{\Sigma_i \mid i \in I\}$.

The following conjecture would be a proposition if the above conjecture were to hold.

Conjecture 4.22. *A base set P has a unique extension if and only if no argument is provisionally defeated on the basis of P.*

Proof (*On the basis of Conjecture* 4.21). First, suppose that no argument is provisionally defeated. It follows immediately that $info_{\uparrow}(P) = info_{\downarrow}(P)$. With the help of Proposition 4.20 it may then be concluded that the extension of P is uniquely determined. Conversely, suppose that P has one extension. From Conjecture 4.21 it follows that

$$info_{\uparrow}(P) = \bigcap\{\Sigma_i \mid i \in I\} = \Sigma = \bigcup\{\Sigma_i \mid i \in I\} = info_{\downarrow}(P) \tag{12}$$

so that $info_{\downarrow}(P) - info_{\uparrow}(P) = \emptyset$. The latter means that there are no provisionally defeated arguments. □

Finally, there is the notion of warrant. Intuitively, a proposition is *warranted* in a situation if and only if an ideal reasoner starting from that situation would be justified in believing that proposition, had unlimited time and memory been available.[15]

Definition 4.23. Let P be a base set. An element ϕ is *warranted on the basis of P*, written $P \mid\!\sim \phi$, if $P \mid\!\sim \sigma$ and $conc(\sigma) = \phi$, for some argument σ. Accordingly, the set $warrant_{\mid\sim}(P)$ is set equal to $conc \circ info_{\mid\sim}(P)$.

In order to become acquainted with the material that has been presented thusfar, we will now present a number of detailed examples in which the relatively complex procedure of defeat can be seen at work. Let us start with elaborating a simple case.

Example 4.24 (*Simple case*). Consider the abstract argumentation system $\mathcal{A}$ with language $\mathcal{L} = \{p, q, r, s\} \cup \{\bot\}$, with rules

$$R = \{p \Rightarrow q, p \Rightarrow r, r \Rightarrow s\} \cup \{q, r \rightarrow \bot\} \tag{13}$$

and with an order on arguments $\leqslant$ such that the argument $p \Rightarrow q$ is stronger than the argument $p \Rightarrow r$, i.e., $p \Rightarrow r < p \Rightarrow q$. Let $P = \{p\}$. Our goal is to determine which elements are warranted on the basis of P. According to Definition 4.23, this

[15] Our notion of defeasible entailment does not coincide with that of Simari and Loui [33,45]. In their theory, we have $P \mid\!\sim \phi$ if and only if ϕ is based on P. Thus, in contrast with our theory, their notion of defeasible entailment does not involve any notion of defeat and behaves monotonically.

amounts to finding the arguments that are in force on the basis of P. To begin with, Definition 4.17(1) together with $p \in P$ immediately yields $P \mathrel{|\!\sim} p$. So p, considered as an argument, is in force. Next, let us examine whether the argument $p \Rightarrow q$ is in force. According to Definition 4.17 we have $P \mathrel{|\!\sim} p \Rightarrow q$ if (i) $P \mathrel{|\!\sim} p$, (ii) the rule $p \Rightarrow q$ is in R, and (iii) every set of arguments Σ that is in force (on the basis of P) is not a defeater of $p \Rightarrow q$. Since conditions (i) and (ii) are clearly fulfilled, condition (iii) remains to be examined. We begin mentioning that $p \Rightarrow r$ and, hence, $p \Rightarrow r \Rightarrow s$ are strictly weaker than $p \Rightarrow q$. As a result, the only set of arguments Σ that is in force and such that the argument $p \Rightarrow q$ is not stronger than some member of Σ, is $\Sigma = \{p\}$. And this particular Σ clearly is compatible with $p \Rightarrow q$. Thus, condition (iii) is fulfilled as well. We may conclude that $p \Rightarrow q$ is in force on the basis of P, i.e., $P \mathrel{|\!\sim} p \Rightarrow q$. From this, it follows that the argument $p \Rightarrow r$ cannot be in force, since $\Sigma = \{p \Rightarrow q\}$ is a set of arguments that is in force, and such that $p \Rightarrow r$ is not stronger than some member of Σ. At the same time, the set Σ is incompatible with $p \Rightarrow r$. Therefore, the argument $p \Rightarrow r \Rightarrow s$ cannot be in force as well, as it fails to fulfill condition (iii). Thus, we end up with precisely one extension, namely $\mathit{info}(P) = \{p, p \Rightarrow q\}$.

The following example shows what happens if two arguments of equal force mutually exclude each other.

Example 4.25 (*Tie*). Consider the abstract argumentation system $\mathcal{A}$ with language $\mathcal{L} = \{p, q, r\} \cup \{\bot\}$, with rules

$$R = \{p \Rightarrow q, p \Rightarrow r;\ q, r \rightarrow \bot\} \tag{14}$$

and with a basic order on arguments. (Cf. Example 2.13.) Let $P = \{p\}$. Our goal is to determine which elements are warranted on the basis of P. According to Definition 4.23 it suffices to determine which arguments are in force on the basis of P. To begin with, Definition 4.17(1) together with $p \in P$ immediately yields $P \mathrel{|\!\sim} p$. So p, considered as an argument, is in force. Next, let us examine whether the argument $p \Rightarrow q$ is in force. According to Definition 4.17 we have $P \mathrel{|\!\sim} p \Rightarrow q$ if (i) $P \mathrel{|\!\sim} p$, (ii) the rule $p \Rightarrow q$ is in R, and (iii) every set of arguments Σ that is in force (on the basis of P) is not a defeater of $p \Rightarrow q$. Since conditions (i) and (ii) are clearly fulfilled, condition (iii) remains to be examined. To begin with, (iii) can be simplified by observing that (iii) holds if and only if every set Σ of defeasible arguments based on P that is in force, is compatible with $p \Rightarrow q$. Since $p \Rightarrow r$ is the only argument which may cause (iii) not to hold if it is in Σ, condition (iii) can be simplified further by stating that (iii) holds if and only if not $P \mathrel{|\!\sim} p \Rightarrow r$. Consequently, $P \mathrel{|\!\sim} p \Rightarrow q$ if not $P \mathrel{|\!\sim} p \Rightarrow r$. Because the argumentation system $\mathcal{A}$ considered here is symmetric in the variables q and r, the same line of reasoning goes through for $p \Rightarrow r$. Hence, $P \mathrel{|\!\sim} p \Rightarrow r$ if not $P \mathrel{|\!\sim} p \Rightarrow q$. Combining this with the previous result yields

$$P \mathrel{|\!\sim} p \Rightarrow q \text{ if and only if not } P \mathrel{|\!\sim} p \Rightarrow r.$$

As a result we have p is warranted and, moreover, q is warranted if and only if r is not warranted.

Example 4.25 shows that our notion of defeat is not skeptical: it says that one of the two arguments should be in force, without making further commitments. (So it is not credulous either.) With the above example, a skeptical adjudicator would simply abstain by saying that neither of the two arguments should be in force. This is a safe but inadequate procedure for dealing with conflicting arguments, as it provides no information about the arguments involved.[16] In the following example, for instance, the conclusion s would not emerge if our procedure of defeat was skeptical.

Example 4.26 (*Floating conclusion*). Consider the abstract argumentation system $\mathcal{A}$ as defined in Example 4.25, but now with rules

$$R = \{p \Rightarrow q, p \Rightarrow r;\ q, r \rightarrow \perp\} \cup \{q \rightarrow s, r \rightarrow s\}. \tag{15}$$

Because the addition of the rules $\{q \rightarrow s, r \rightarrow s\}$ to the set of rules of Example 4.25, (14) has no influence on the criteria on which we concluded that $P \mid\!\sim p \Rightarrow q$ if and only if not $P \mid\!\sim p \Rightarrow r$, we still have either $p \Rightarrow q$ or $p \Rightarrow r$ in force, but not both. If the argument $p \Rightarrow q$ is in force, Definition 4.17(2) together with $P \mid\!\sim p \Rightarrow q$ and $q \rightarrow s \in R$ yields $P \mid\!\sim p \Rightarrow q \rightarrow s$. Similarly, $P \mid\!\sim p \Rightarrow r$ and $r \rightarrow s \in R$ and Definition 4.17(2) would yield $P \mid\!\sim p \Rightarrow r \rightarrow s$. In any case, the element s is the conclusion of an argument that is in force in $\mathcal{A}$. Thus, the element s is warranted. However, we are unable to tell by which argument s actually is supported.[17]

Example 4.27 (*Explosion*). Consider the abstract argumentation system $\mathcal{A}$ with language

$$\mathcal{L} = \{a_1, \dots, z_1\} \cup \{a_2, \dots, z_2\} \cup \{\perp\} \tag{16}$$

with rules

$$R = \{\Rightarrow\phi \mid \phi \in \mathcal{L} - \{\perp\}\} \cup \{a_1, a_2 \rightarrow \perp, \dots, z_1, z_2 \rightarrow \perp\} \tag{17}$$

and with a basic order on arguments. (Cf. Example 2.13.) Let $P = \emptyset$. Like Example 4.25, there is a tie for each pair of incompatible (and equivalent) arguments:

$$\begin{aligned}
&P \mid\!\sim\ \Rightarrow a_1 \text{ if and only if not } P \mid\!\sim\ \Rightarrow a_2,\\
&P \mid\!\sim\ \Rightarrow b_1 \text{ if and only if not } P \mid\!\sim\ \Rightarrow b_2,\\
&\quad\vdots\\
&P \mid\!\sim\ \Rightarrow z_1 \text{ if and only if not } P \mid\!\sim\ \Rightarrow z_2.
\end{aligned}$$

Arguments are not crosswise related. Thus, the status of $\Rightarrow b_1$ does not depend on the status of, e.g., $\Rightarrow c_2$, or $\Rightarrow m_1$. It follows that every extension contains either $\Rightarrow a_1$ or

[16] In this context, it is interesting to note that Simari and Loui's notion of defeat is skeptical, and would not make mention of all the possible conclusions. (Cf. [45, p. 146].)

[17] In [26] the element s is said to be a *floating conclusion*, which means, in our vocabulary, that s is warranted, not by a fixed argument that is in force, but by a collection of arguments, all with conclusion s, of which it is only known that at least one of them is in force.

$\Rightarrow a_2$, either $\Rightarrow b_1$ or $\Rightarrow b_2$, and so forth. For every letter, there are two choices. Two choices for 26 letters yields $2^{26} = 67,108,864$ different extensions.

Example 4.28 (*Zombie path*). Consider the abstract argumentation system $\mathcal{A}$ with language $\mathcal{L} = \{p, q, r, s, t\} \cup \{\bot\}$, with rules

$$R = \{p \Rightarrow q, q \Rightarrow r, p \Rightarrow s, s \rightarrow t\} \cup \{q, s \rightarrow \bot;\ r, t \rightarrow \bot\} \tag{18}$$

and with an order of conclusive force $\leqslant$ on argument such that $p \Rightarrow s \ < \ p \Rightarrow q$ but $p \Rightarrow q \Rightarrow r \ < \ p \Rightarrow s \rightarrow t$. Let $P = \{p\}$. Let us determine which elements are warranted on the basis of P. This amounts to determining which arguments are in force on the basis of P. To begin with, it follows from Definition 4.17(1) that $P \mid\!\sim p$. Thus, the element p, considered as an argument, is in force. Next, let us determine whether the argument $p \Rightarrow q$ is in force. Since $p \Rightarrow q$ dominates the only argument it is incompatible with, namely, the argument $p \Rightarrow s$, it may be concluded that $P \mid\!\sim p \Rightarrow q$, at the expense of $p \Rightarrow s$. Let us now determine whether the argument $p \Rightarrow q \Rightarrow r$ is in force, and, since this is the heart of the example, let us do this in some detail. According to Definition 4.17, we have $P \mid\!\sim p \Rightarrow q \Rightarrow r$ if and only if (i) $P \mid\!\sim p \Rightarrow q$, (ii) the rule $q \Rightarrow r$ is in R, and (iii) every set of arguments Σ that is in force on the basis of P is not a defeater of $p \Rightarrow q \Rightarrow r$. Since (i) and (ii) are clearly fulfilled, condition (iii) remains to be examined. To begin with, we observe that (iii) can be invalidated only if $p \Rightarrow s \rightarrow t \in \Sigma$, since $p \Rightarrow s \rightarrow t$ is the only argument that is incompatible with $p \Rightarrow q \Rightarrow r$. However, the argument $p \Rightarrow q \Rightarrow r$ is not stronger than $p \Rightarrow s \rightarrow t$—as a matter of fact, we even have $p \Rightarrow q \Rightarrow r < p \Rightarrow s \rightarrow t$—so that, if $p \Rightarrow s \rightarrow t$ would be in force, the set $\Sigma = \{p \Rightarrow s \rightarrow t\}$ would invalidate condition (iii). Therefore, let us determine whether $p \Rightarrow s \rightarrow t$ is in force. According the definition, we have $P \mid\!\sim p \Rightarrow s \rightarrow t$ if and only if (i′) $P \mid\!\sim p \Rightarrow s$, and (ii′) the rule $s \rightarrow t$ is in R. In an earlier stage of this example, we already observed that $P \mid\!\sim p \Rightarrow q$ at the expense of $p \Rightarrow s$. Hence, the argument $p \Rightarrow s$ is not in force, and condition (i′) does not hold. This, in turn, implies that $p \Rightarrow s \rightarrow t$ cannot be in force. And because $p \Rightarrow s \rightarrow t$ is not in force, it can be concluded that there exists no set of arguments Σ that invalidates condition (iii). It follows that $P \mid\!\sim p \Rightarrow q \Rightarrow r$.

In recent literature, the argument $p \Rightarrow s \rightarrow t$ is called a *zombie path.* [18] A zombie path is, in our terminology, an argument that is not in force, but in some way still participates in the interplay of defeat among arguments. The idea, then, is that zombie paths would give the final blow to arguments that would otherwise only just survive the interaction with other arguments. The previous example, however, shows otherwise. For $p \Rightarrow s \rightarrow t$, it shows that, once one of its subarguments is defeated (here, $p \Rightarrow s$), the original argument does no longer stand against the argument $p \Rightarrow q \Rightarrow r$—even though $p \Rightarrow s \rightarrow t$ is stronger than $p \Rightarrow q \Rightarrow r$. This is not the case only with this example, but goes through in general. Thus, in general, our definition of defeat does not

[18] Cf. [26]. A path is a linear argument (that is, a tree with one branch only). Since Makinson and Schlechta consider only paths, this explains why they are using the term zombie *path* instead of "zombie argument" (which sounds rather awkward, anyway).

allow for "sleeping" arguments that interfere with other arguments nonetheless. In the present theory, arguments that are defeated remain defeated once and for all.[19]

In the following example, it will be shown what happens if arguments are equally strong.

Example 4.29 (*Lottery paradox*). An interesting paradox arises from the principle that the plausibility of two propositions necessarily implies the plausibility of their conjunction. But suppose that there is a lottery with n tickets. Let p_i denote the proposition that "ticket i will not win", where $1 \leqslant i \leqslant n$. Further, let $\sigma_i =_{\text{def}} \Rightarrow p_i$ denote the plausible argument supporting the proposition p_i. Obviously, any real argument supporting p_i has more structure than σ_i. But for the present analysis, all we need to know is that all σ_i are similar and highly plausible. Thus we ought, by the principle, to believe the conjunction of all arguments, which is equivalent to believing that no ticket will win. Evidently, the latter is impossible. So the natural principle of combining plausible arguments has a false consequence.

The lottery paradox, originally introduced by Kyburg in 1961, has often been studied, among others from the standpoint of nonmonotonic reasoning. By now, many researchers have presented satisfying solutions. Here is the paradox elaborated in the theory of abstract argumentation systems:

Consider the abstract argumentation system $\mathcal{A}$ with $\mathcal{L} = \{p_1, \ldots, p_n\} \cup \{\perp\}$, with

$$R = \{\Rightarrow p_1, \ldots, \Rightarrow p_n\} \cup \{p_1, \ldots, p_n \rightarrow \perp\} \tag{19}$$

and with a basic order on arguments $\leqslant$. Let $P = \emptyset$. The arguments in $\mathcal{A}$ are $\sigma_1, \ldots, \sigma_n$, and $\sigma_1, \ldots, \sigma_n \rightarrow \perp$. As can easily be verified,

$$\begin{gathered} P \mid\!\sim \{\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n\} \text{ if and only if not } P \mid\!\sim \sigma_i, \\ \text{for every } 1 \leqslant i \leqslant n. \end{gathered} \tag{20}$$

Accordingly, there are n extensions.

Suppose we have 10 independent arguments for a conclusion. Does that make the conclusion more justified than if we had just one?[20] It is natural to suppose that it does, but upon closer inspection, that becomes unclear.[21]

Example 4.30 (*Accrual of reasons*). Consider the abstract argumentation system $\mathcal{A}$ with $\mathcal{L} = \{p\} \cup \{q_1, \ldots, q_n\} \cup \{\perp\}$, with

$$R = \{\Rightarrow p\} \cup \{\Rightarrow q_1, \ldots, \Rightarrow q_n\} \cup \{p, q_1, \ldots, q_n \rightarrow \perp\} \tag{21}$$

and with an order on arguments $\leqslant$ such that $\Rightarrow q_i < \Rightarrow p$, for every $1 \leqslant i \leqslant n$. If $\sigma = \Rightarrow p$ and, for every $1 \leqslant i \leqslant n$, the argument τ_i is defined as $\tau_i = \Rightarrow q_i$, then all

[19] Pollock [33, p. 52] draws this conclusion on the basis of considerations on the accrual of reasons: "So it seems that if we are to reject the latter principle [the principle of accrual of reasons], then we should also conclude that arguments that survive rebuttal by conflicting arguments are not thereby diminished in strength."

[20] Cf. [33, Section D. "The accrual of reasons"].

[21] Pollock argues why.

arguments in $\mathcal{A}$ are $\sigma, \tau_1, \ldots, \tau_n$, and $\sigma, \tau_1, \ldots, \tau_n \rightarrow \perp$. As can easily be verified, we end up with one extension, namely $info(P) = \{\sigma\}$.

What the example says is that "the accrual of many poor arguments does not make a good one". We thereby range ourselves with Pollock [33], who argues in detail that the accrual of reasons is not a property of logical inference *a priori*, but an extra-logical principle that multiple support tends to confirm the justification of a proposition. (As Pollock points out, this principle is invalidated in, for example, a social community in which speakers tend to confirm each other's statements.)

Example 4.31 (*Complex case*). Consider the abstract argumentation system $\mathcal{A}$ with language $\mathcal{L} = \{p, q, r, s, t\} \cup \{\perp\}$, with rules

$$R = \{p \Rightarrow q, q \Rightarrow r, p \Rightarrow s, p \Rightarrow t\} \cup \{q, s \rightarrow \perp;\ r, t \rightarrow \perp\} \tag{22}$$

and with a basic order on arguments $\leqslant$. Let $P = \{p\}$. We are going to determine whether the argument $p \Rightarrow q \Rightarrow r$ is in force. According to Definition 4.17 we have $P \mid\!\sim p \Rightarrow q \Rightarrow r$ if (i) $P \mid\!\sim p \Rightarrow q$, (ii) the rule $q \Rightarrow r$ is in R, and (iii) every set of arguments Σ that is in force is not a defeater of $p \Rightarrow q \Rightarrow r$. Condition (iii) can be simplified by observing that (iii) holds if and only if not $P \mid\!\sim p \Rightarrow t$, since $p \Rightarrow t$ is the only (set of) argument(s) that is incompatible with $p \Rightarrow q \Rightarrow r$. In a similar way we may find that

(1) $P \mid\!\sim p \Rightarrow q \Rightarrow r$ if $P \mid\!\sim p \Rightarrow q$ and $q \Rightarrow r \in R$ and not $P \mid\!\sim p \Rightarrow t$;
(2) $P \mid\!\sim p \Rightarrow q$ if $P \mid\!\sim p$ and $p \Rightarrow q \in R$ and not $P \mid\!\sim p \Rightarrow s$;
(3) $P \mid\!\sim p$;
(4) $P \mid\!\sim p \Rightarrow s$ if $P \mid\!\sim p$ and $p \Rightarrow s \in R$ and not $P \mid\!\sim p \Rightarrow q$;
(5) $P \mid\!\sim p \Rightarrow t$ if $P \mid\!\sim p$ and $p \Rightarrow t \in R$ and not $P \mid\!\sim p \Rightarrow q \Rightarrow r$.

Because the rules $q \Rightarrow r$, $p \Rightarrow q$, $p \Rightarrow s$, and $p \Rightarrow t$ are all in R, clauses (1)-(5) reduce to

(1) $P \mid\!\sim p \Rightarrow q \Rightarrow r$ if $P \mid\!\sim p \Rightarrow q$ and not $P \mid\!\sim p \Rightarrow t$;
(2) $P \mid\!\sim p \Rightarrow q$ if $P \mid\!\sim p$ and not $P \mid\!\sim p \Rightarrow s$;
(3) $P \mid\!\sim p$;
(4) $P \mid\!\sim p \Rightarrow s$ if $P \mid\!\sim p$ and not $P \mid\!\sim p \Rightarrow q$;
(5) $P \mid\!\sim p \Rightarrow t$ if $P \mid\!\sim p$ and not $P \mid\!\sim p \Rightarrow q \Rightarrow r$.

Logically, this is equivalent to the exclusive disjunction of the following three expressions:

(I) $P \mid\!\sim p$ and $P \mid\!\sim p \Rightarrow q$ and $P \mid\!\sim p \Rightarrow q \Rightarrow r$,
(II) $P \mid\!\sim p$ and $P \mid\!\sim p \Rightarrow q$ and $P \mid\!\sim p \Rightarrow t$,
(III) $P \mid\!\sim p$ and $P \mid\!\sim p \Rightarrow s$ and $P \mid\!\sim p \Rightarrow t$.

Thus, (I) and not (II) and not (III); or (II) and not (I) and not (III); or (III) and not (II) and not (I). Which is precisely what we want. Accordingly, $deg(P) = 3$.

The following example demonstrates a highly important feature of the theory. It shows, all other things being equal, that if the ranking in conclusive force among arguments is refined, there becomes more clarity on exactly which arguments emerge victorious.

Example 4.32 (*Complex case, with a refined order*). Consider the abstract argumentation system $\mathcal{A}$ as defined in Example 4.31, but now with the order $\leqslant$ defined as $\sigma \leqslant \tau$ if and only if $|\tau|_{\Rightarrow} \leqslant |\sigma|_{\Rightarrow}$, where $|\sigma|_{\Rightarrow}$ stands for the number of defeasible arrows in σ. (Cf. Example 2.13.) With this new order $\leqslant$ we have more variety in conclusive force, more defeat, and hence a decreasing number of alternatives in comparison with Example 4.31.

The outcome is logically equivalent with the exclusive disjunction of (I) and (II):
(I) $P \mid\!\sim p$ and $P \mid\!\sim p \Rightarrow q$ and $P \mid\!\sim p \Rightarrow t$,
(II) $P \mid\!\sim p$ and $P \mid\!\sim p \Rightarrow s$ and $P \mid\!\sim p \Rightarrow t$.
In comparison with Example 4.31 the argument $p \Rightarrow q \Rightarrow r$ is now out of consideration. Hence, with this argumentation system, we have $deg(P) = 2$.

The two examples above suggest that, if the hierarchy among arguments becomes more determined, the number of extensions will decrease accordingly. This is indeed a general phenomenon.

Proposition 4.33. *Let $\mathcal{A}_1 = (\mathcal{L}, R, \leqslant_1)$ and $\mathcal{A}_2 = (\mathcal{L}, R, \leqslant_2)$ be abstract argumentation systems, such that $\leqslant_1$ is a* refinement *of $\leqslant_2$—i.e., $\sigma \leqslant_1 \tau$ whenever $\sigma \leqslant_2 \tau$. Then, for every base set P, we have $deg_1(P) \leqslant deg_2(P)$.*

Proof. We will show this by proving that every extension of P in $\mathcal{A}_1$ is an extension of P in $\mathcal{A}_2$ as well. Suppose that Σ is an extension of P in $\mathcal{A}_1$. With Proposition 4.19 it follows that Σ is a fixed point of the P-enablement operator in $\mathcal{A}_1$: $enable_1^P(\Sigma) = \Sigma$.[22] Further, the fact that $\leqslant_1$ is able to compare more arguments than $\leqslant_2$, implies that Σ enables more arguments in $\mathcal{A}_1$ than in $\mathcal{A}_2$: $enable_2^P(\Sigma) \subseteq enable_1^P(\Sigma)$. This is easy to verify. Together with the previous fixed point equation, this gives $enable_2^P(\Sigma) \subseteq \Sigma$. In order to prove the reversed inclusion $\Sigma \subseteq enable_2^P(\Sigma)$, we note that Σ is compatible, since it is an extension in $\mathcal{A}_1$. The compatibility of Σ carries over to $\mathcal{A}_2$, so that, together with Proposition 4.9, the reversed inclusion follows. Combining the two inclusions yields $enable_2^P(\Sigma) = \Sigma$, which means that Σ is an extension of P in $\mathcal{A}_2$. □

The above proposition distinguishes defeasible argumentation from nonmonotonic reasoning. In nonmonotonic reasoning, there is little variety in conclusive force: there are proofs, and there are nonmonotonic proofs. Maybe there are a few extra principles that prefer some nonmonotonic proofs above other nonmonotonic proofs (like specificity), but usually that is it. It follows that the hierarchy $\leqslant$ among "arguments" in nonmonotonic reasoning systems contains, as a rule, chains of length at most 3. (If the logic is extraordinarily refined and sophisticated, perhaps 4.) It follows with Proposition 4.33 that nonmonotonic logics often produce multiple extensions in the presence of conflicting nonmonotonic proofs. Argumentation systems work different. On the one hand, an argumentation system can order its arguments as crude as a nonmonotonic logic. (Cf. Example 2.13, basic order.) On the other hand, one might imagine an argumentation system in which the conclusive force of arguments is expressed by a number ranging

[22] Since we are labeling the enablement operators with 1 and 2, the P has been pushed up above temporarily.

from 0.000 to 1.000 (three-place precision). In this way, it is very likely that arguments always differ in conclusive force, because their strength is expressed by an absolute number that assumes values on a finely divided graduated scale. With such a notion of conclusive force, the chance that an arbitrary base set gives rise to precisely one extension is comparatively high.

Example: negation by failure

Negation by failure (NBF) is the principle by which a proposition may be derived as soon as we fail to prove its negation. (Cf. [1,4,18].) Within the present formalism, it is relatively simple to define an abstract argumentation system that works according to the negation by failure principle.

Nonmonotonic reasoning by negation by failure is always defined relative to a fixed proof system.

Definition 4.34. A *proof system* $\mathcal{P}$ is a pair $(\mathcal{L}, R)$ where $\mathcal{L}$ is a language, and R is a set of strict rules of inference.

Example 4.35 (*Negation by failure*). Let $\mathcal{P} = (\mathcal{L}, R)$ be a proof system in which $\mathcal{L}$ is closed under negation. The corresponding NBF argumentation system, then, is $\mathcal{A} = (\mathcal{L}, R', \leqslant)$ with

$$R' = R \cup \{\Rightarrow\phi \mid \phi \in \mathcal{L}\} \cup \{\phi, \neg\phi \rightarrow \perp\} \tag{23}$$

and with $\leqslant$ the standard basic order of conclusive force. Then, for every base set P, we have $P \mathrel{|\!\sim} \phi$ if and only if $P \vdash \phi$ or $P \nvdash \neg\phi$.

This example nicely demonstrates the relative simplicity of NBF. Among the defeasible arguments, there is no difference in conclusive force, so that all improvable propositions are supported by arguments of the same type. Among others, this means for example that if there are n irrefutable, improvable and pairwise incompatible propositions, then applying NBF yields n different extensions. A situation in which each argument can only live in its "own" extension, screened from its competitors. In this way, the present theory of abstract argumentation systems positions NBF as a rather crude way to deal with plausible but improvable propositions.

Degenerate argumentation systems

It has already been observed that a defeasible entailment relation is sometimes ambiguously determined. In some situations, there is more than one valid way to define $\mathrel{|\!\sim}$. In addition it may also happen, at least in theory, that, for some argumentation system and for some base set, there is no defeasible entailment relation at all. I.e., there may exist a pair $\mathcal{A}, P$ for which there exists no relation $\mathrel{|\!\sim}$ that satisfies all three conditions of Definition 4.17. Let us call a pair $\mathcal{A}, P$ for which such a negative event is the case *degenerate*. The question whether there exist degenerate argumentation systems is equivalent with the question whether there exist base sets without extensions. The answer to this question is unknown to us. Although we spent considerable effort in constructing a degenerate argumentation system, we failed to do so. On the other hand, we were unable to substantiate our findings in a theorem which states that degenerate argumentation systems do not exist. At this point, the existence of degenerate argumentation systems is an open problem.

5. Completeness

In this section, we consider sets of arguments in more detail. As opposed to one single argument, a set of arguments may satisfy additional interesting properties. Sets of arguments that satisfy these additional properties, may act as "information carrier" in the theory. (We will come back to this point in Section 5.1.)

5.1. Argument structures

The theory will further develop along the framework of argument structures. [23] An argument structure is a structured set of arguments. [24] From an epistemic point of view, it might be conceived as a description of a part of the world. Therefore, terms like *possible world*, *scenario*, *situation*, *state of affairs*, or even *model* suit equally well. Because an argument structure is not closed under any sort of argumentation whatsoever, it generally is a *partial* and *unfinished* description of a state of the world.

Definition 5.1. A set of arguments Σ is an *argument structure* if it is compatible and contains all subarguments of its members.

If a set of arguments is compatible, it is not self-contradictory. If it contains all subarguments of its members, it may be considered to contain all intermediate stages of argumentation. Further, argument structures are not closed with respect to argumentation—not even with respect to *strict* argumentation. The reason for not automatically incorporating strict continuations of arguments into our notion of argument structure is that, at all times, we would like to keep a full saying over the arguments that are included in an argument structure.

[23] Our notion of argument structure is taken from [20]. In fact, their notion of argument structure coincides with ours upto one minor point. Lin and Shoham exchange compatibility for *consistency* and stipulate further that argument structures are *closed* with respect to strict argumentation. (In the reading of Lin and Shoham, consistency is defined as not having an argument supporting ϕ and another supporting $\neg\phi$, for some ϕ. Instead of the contradiction, the negation is the elementary connective in their formalism.) The problem with strict closure, however, is that the inclusion of a single defeasible argument may incorporate a large number of other arguments as well. We do not want this. Instead of a theoretical closure operator, which already implies something about the properties on argumentation, we just want to have a simple device for holding arguments.

[24] In consequence, our notion of argument structure has nothing to do with that of [45]. Their notion of argument structure represents what we would call an argument. (Maybe, it is interesting to note that most of the technical choices in the 1992 paper were done by Simari, not by Loui. In actual fact, Loui tends to agree with the present definition.)

Example 5.2. Consider the abstract argumentation system $\mathcal{A}$ with language $\mathcal{L} = \{p, q, r\} \cup \{\perp\}$, with rules

$$R = \{p \Rightarrow q, p \Rightarrow r\} \cup \{q, r \rightarrow \perp\} \tag{24}$$

and with a basic order on arguments $\leqslant$. Let $P = \{p\}$. Then $\Sigma_1 = P$ is an argument structure, like $\Sigma_2 = \{p, p \Rightarrow q\}$ and $\Sigma_3 = \{p, p \Rightarrow r\}$. The set $\Sigma_4 = \{p, p \Rightarrow q, p \Rightarrow r\}$ is not an argument structure because it is incompatible, and $\Sigma_5 = \{p \Rightarrow q\}$ is not an argument structure because it does not contain all subarguments of its members.

Like (partial) models in (modal) mathematical logic, argument structures are the information carriers of the theory. They speak for or against (individual) arguments. It is also possible that an argument structure says nothing about an argument. In that case, the argument in question is neither contained in, nor contradicted by, the argument structure. In other cases, the argument structure is able to distinguish the argument as an element that either does or does not hold.

Definition 5.3. Let τ be an argument. An argument structure Σ is able to *distinguish* τ if $\tau \in \Sigma$ whenever Σ is compatible with τ.

Thus, an argument structure that is able to distinguish τ is either explicitly "for" τ, or else explicitly "against" τ. In the example above, Σ_1 is able to distinguish p, while both Σ_2 and Σ_3 are able to distinguish p, $p \Rightarrow q$, and $p \Rightarrow r$.

Obviously, an argument structure that is able to distinguish every argument [based on P], is maximally compatible [within *arguments*(P)].

The notion of argument structure is the vehicle on which the theory is further developed. More specifically, in the following sections it will be shown that the set of all arguments that are in force is an argument structure itself, and can be approximated in a constructive manner by a sequence of argument structures.

5.2. Complete argument structures

The most interesting argument structures should be those that contain not only many arguments, but also many *strong* arguments. To this end, maximally compatible argument structures appear to be right candidates. Maximally compatible argument structures may be thought of as obtained by extending compatible argument structures monotonically, until further extension is impossible. That is, without giving up arguments earlier included and without sacrificing compatibility. However, there is nothing that assures us that maximally compatible argument structures contain relatively many strong arguments.

The notion of complete argument structure is defined in such a way, that it incorporates strong arguments. For complete argument structures, the only criterion for enlargement is preservation of compatibility. This means that arguments earlier included may be given up in order to give place to other (presumably stronger) arguments. This idea is embodied by the notion of appendability.

The basic idea behind appendability is as follows. Given an argument structure Σ, we consider an argument σ having its proper subarguments in Σ. If σ is not in Σ, it potentially provides new and better information relative to Σ. However, this need not be so. If σ is incompatible with Σ and is too weak to undermine Σ, then σ is not a valuable piece of new information that should be added to Σ. On the other hand, if σ is compatible with Σ, it is new information that must be appended to Σ. Also if σ is incompatible with Σ, it may be appended to Σ, provided the argument σ is strong enough to defeat some member of Σ. If that is the case, σ provides information that is even better than what is in Σ, and σ should be appended to Σ as well.

Definition 5.4. Let Σ be an argument structure. An argument σ is *appendable* to Σ if
(1) the argument σ already is in Σ; or
(2) for some arguments $\sigma_1, \dots, \sigma_n$ we have $\sigma_1 \in \Sigma, \dots, \sigma_n \in \Sigma$ and $\sigma_1, \dots, \sigma_n \to \sigma$; or
(3) for some arguments $\sigma_1, \dots, \sigma_n$ we have $\sigma_1 \in \Sigma, \dots, \sigma_n \in \Sigma$ and $\sigma_1, \dots, \sigma_n \Rightarrow \sigma$, and Σ does not contain defeaters of σ.

All subsets of Σ must be examined for defeaters. To see this, suppose that σ is incompatible with Σ but that, at the same time, $\sigma' < \sigma$, for some $\sigma' \in \Sigma$. On the basis of this information, it would be wrong to say that σ can be appended to Σ at the expense of σ', because σ' might be irrelevant to the conflict. That is, the difference $\Sigma - \{\sigma'\}$ might still be incompatible with σ. For this reason, it is necessary to consider more subsets of Σ. Large subsets of Σ are not likely to invalidate Definition 5.4(3) because there is a fair chance that such subsets contain an element σ' that is stronger than σ. Similarly, small subsets of Σ are not likely to invalidate Definition 5.4(3), because they are often compatible with σ. Somewhere in between, however, there might be a defeater $\Sigma_S \subseteq \Sigma$ that invalidates the third clause. In effect, this interplay between conclusive force and compatibility necessarily involves all subsets of Σ. (Cf. also Example 4.3.) This partially explains the relative complexity of Definition 5.4(3).

Definition 5.5. Let Σ be an argument structure. Then *append*(Σ) denotes the collection of arguments that are appendable to Σ.

Also here, the enablement operator takes part in the theory. If the enablement operator is indexed with argument structures, rather than with base sets alone, then *append*(Σ) can be defined as $enable_\Sigma(\Sigma)$. This notation also establishes a correspondence between the notion of extension and completeness. (Cf. Proposition 5.11.)

Example 5.6 (*Appendability*). Consider the abstract argumentation system $\mathcal{A}$ with language

$$\mathcal{L} = \{p, q, r\} \cup \{q_1, r_1, r_2, r_3\} \cup \{\bot\} \tag{25}$$

with rules

$$\begin{aligned} R = \{&p \Rightarrow q, q \Rightarrow r, p \Rightarrow q_1, q_1 \Rightarrow r_1, p \Rightarrow r_2, p \to r_3\} \\ &\cup \{r, r_1 \to \bot;\ r, r_2 \to \bot;\ r, r_3 \to \bot\} \end{aligned} \tag{26}$$

and with the $\Rightarrow$-count order on arguments. (Cf. Example 4.32.) Let

$$\Sigma = \{p, p \Rightarrow q, p \Rightarrow q \Rightarrow r\} \cup \{p \Rightarrow q_1\}. \tag{27}$$

The set Σ is an argument structure. If $\sigma = p \Rightarrow q_1 \Rightarrow r_1$, $\tau = p \Rightarrow r_2$ and $\rho = p \rightarrow r_3$, then τ and ρ are appendable to Σ while σ is not. Let us see why this is so. The argument $\rho = p \rightarrow r_3$ is appendable to Σ simply because $p \in \Sigma$ and we may unconditionally append strict continuations of members of Σ [cf. Definition 5.4(2)]. If we want to check whether τ is appendable to Σ, item (3) applies. If Σ_S is a subset of Σ that is incompatible with τ, then $p \Rightarrow q \Rightarrow r$ must be in Σ_S. Since $p \Rightarrow r_2$ is stronger than $p \Rightarrow q \Rightarrow r$ we conclude that $p \Rightarrow r_2$ is stronger than some member of Σ_S. Hence, item (3) holds for $p \Rightarrow r_2$, so that $p \Rightarrow r_2$ is appendable to Σ. Finally, σ is not appendable to Σ because $\Sigma_S =_{\text{def}} \Sigma$ is a defeater of σ.

Definition 5.7. An argument structure Σ is *complete* if every argument that is appendable to Σ already is in Σ.

Because more information is not possible, a complete argument structure is *saturated* or *finished*. Again, from the epistemic perspective, a complete argument structure might be considered a complete description of a part of the world (cf. the discussion preceding Definition 5.1). This idea is elaborated in the following proposition. This proposition states that a complete argument structure is complete in the sense that it distinguishes every argument that can be generated.

Proposition 5.8. *Let P be a base set. A complete argument structure Σ with prem(Σ) = P is able to distinguish every element in arguments(P).*

Proof. Let Σ be a complete argument structure with *prem*$(\Sigma) = P$, and let $\tau \in$ *arguments*(P). We will have to prove that $\tau \in \Sigma$ whenever Σ is compatible with τ. However, let us prove something stronger. Let us prove that every element of every finite sequence $\tau_1, \ldots, \tau_n$ that is, as a whole, compatible with Σ, is contained in Σ. This stronger statement ensures that the induction argument below will go through. More precisely, the last proposition will be proven with induction on the construction of $\tau_1, \ldots, \tau_n$. Without loss of generality, we may prove the proposition for single arguments only (but then turning to multiple arguments in the induction step). Suppose Σ is compatible with τ. We distinguish three cases: (i) $\tau \in \mathcal{L}$; (ii) $\tau_1, \ldots, \tau_n \rightarrow \tau$ for some $\tau_1, \ldots, \tau_n$; or (iii) $\tau_1, \ldots, \tau_n \Rightarrow \tau$ for some $\tau_1, \ldots, \tau_n$. If (i) $\tau \in \mathcal{L}$, then $\tau \in P$. Because $P = prem(\Sigma)$, we immediately conclude that $\tau \in \Sigma$. If (ii) $\tau_1, \ldots, \tau_n \rightarrow \tau$, then $\tau_1, \ldots, \tau_n$ is, as a whole, compatible with Σ as well. Using our induction argument, it follows that $\tau_1 \in \Sigma, \ldots, \tau_n \in \Sigma$. By definition, this means that τ is appendable to Σ. Since Σ is complete, it follows that $\tau \in \Sigma$. If (iii) $\tau_1, \ldots, \tau_n \Rightarrow \tau$, then $\tau_1 \in \Sigma, \ldots, \tau_n \in \Sigma$ as in (ii). Since Σ is compatible with τ, it follows immediately that every subset Σ_S of Σ is also compatible with τ. Hence, τ is appendable to Σ, so that $\tau \in \Sigma$. Since, in all three cases, it is proven that $\tau \in \Sigma$, the proof is completed. $\square$

Intuitively, this proposition says that complete argument structures are either "for" or "against" every argument that can be generated from the base set. This was the basic idea of completeness in [20].[25] Does the converse hold? That is, are argument structures that are able to distinguish many arguments automatically complete? The answer is "no". The construction of an elementary counterexample is left to the reader.

For a proper understanding of the theory, it is important to know that not every argument structure is contained in a complete argument structure. In fact, if an argument structure contains arguments that can be defeated by other arguments rooting in the same premises, these arguments must be given up first before this argument structure eventually might be completed. From the fact that not every argument structure is contained in a complete argument structure, it also follows that not every argument structure that is maximal with respect to set inclusion, is a complete argument structure. (Addition of weak arguments shows this.) Thus, maximality is not precisely the same as completeness. However, we do have the following proposition.

Proposition 5.9. *Within every class of argument structures that coincide on atomic arguments, complete argument structures are $\subseteq$-maximal.*

Proof. Let Σ_1 and Σ_2 be two argument structures with $prem(\Sigma_1) = prem(\Sigma_2)$, such that Σ_1 is complete and $\Sigma_1 \subseteq \Sigma_2$. Let us prove that $\Sigma_2 \subseteq \Sigma_1$ with induction on the construction of the members of Σ_2. Let $\sigma \in \Sigma_2$. If (i) the argument σ is an atomic argument, then $\sigma \in prem(\Sigma_2)$, and hence $\sigma \in prem(\Sigma_1)$, since Σ_1 and Σ_2 are supposed to have equal premises. If (ii) for some arguments $\sigma_1, \ldots, \sigma_n$ we have $\sigma_1, \ldots, \sigma_n \to \sigma$, then all of $\sigma_1, \ldots, \sigma_n$ are in Σ_2, because Σ_2 is an argument structure. Using our induction hypothesis, we may conclude that all of $\sigma_1, \ldots, \sigma_n$ are in Σ_1 so that, by the completeness of Σ_1, we have $\sigma \in \Sigma_1$. If (iii) for some arguments $\sigma_1, \ldots, \sigma_n$ we have $\sigma_1, \ldots, \sigma_n \Rightarrow \sigma$, then all of $\sigma_1, \ldots, \sigma_n$ are in Σ_1, as above. Because Σ_1 is a complete argument structure, it suffices to show that σ is appendable to Σ_1. This readily follows from the fact that $\sigma \in \Sigma_2$, $\Sigma_1 \subseteq \Sigma_2$, and Σ_2 is compatible. $\square$

From this proposition it follows that complete argument structures with equal premises are incomparable with respect to set inclusion. It is possible for complete argument structures to include one another, but from Proposition 5.9 it then follows that, in such cases, these argument structures must have different premises.

Complete argument structures are extensions of their premises:

Proposition 5.10. *Let P be a base set, and let Σ be a complete argument structure with $prem(\Sigma) = P$. Then Σ is an extension of P.*

Proof. Let Σ be a complete argument structure with $prem(\Sigma) = P$. We must prove that Σ is an extension of P, which means that we must find a defeasible entailment relation

[25] In the literature, there is at least one alternative notion of completeness, namely that of [20]. In Lin and Shoham's theory, an argument structure T is *complete about* ϕ if either $\phi \in T$ or $\neg\phi \in T$. Our notion of completeness (cf. Definition 5.7) is based on ideas that are different than Lin and Shoham's ideas on completeness.

$\mid\!\sim$ such that $info_{\mid\sim}(P) = \Sigma$. To this end, we define $P \mid\!\sim \sigma$ if and only if $\sigma \in \Sigma$. We then show that this definition of $\mid\!\sim$ obeys the rules of Definition 4.17 as follows. First, suppose $P \mid\!\sim \sigma$. By definition, this means $\sigma \in \Sigma$. It follows that either (i) the argument σ is in $prem(\Sigma)$; or (ii) for some arguments $\sigma_1 \in \Sigma, \ldots, \sigma_n \in \Sigma$ we have $\sigma_1, \ldots, \sigma_n \rightarrow \sigma$; or (iii) for some arguments $\sigma_1 \in \Sigma, \ldots, \sigma_n \in \Sigma$ we have $\sigma_1, \ldots, \sigma_n \Rightarrow \sigma$ and Σ does not contain defeaters of σ. If these three cases are translated back in terms of entailment, we arrive at our desired result. Conversely, suppose that either (i) the set P contains σ; or (ii) for some arguments $\sigma_1, \ldots, \sigma_n$ we have $P \mid\!\sim \sigma_1, \ldots, P \mid\!\sim \sigma_n$ and $\sigma_1, \ldots, \sigma_n \rightarrow \sigma$; or (iii) for some arguments $\sigma_1, \ldots, \sigma_n$ we have $P \mid\!\sim \sigma_1, \ldots, P \mid\!\sim \sigma_n$ and $\sigma_1, \ldots, \sigma_n \Rightarrow \sigma$ and every set of arguments Σ that is in force on the basis of P is not a defeater of σ. Translating all entailments in terms of membership in Σ, and using the completeness of Σ, it follows that $\sigma \in \Sigma$. By definition, this means that $P \mid\!\sim \sigma$. □

A converse of this proposition holds as well:

Proposition 5.11. *Let P be a base set, and let Σ be an extension of P. Then Σ is a complete argument structure with $prem(\Sigma) = P$.*

Proof. Since Σ is an extension of P, there is a relation $\mid\!\sim$ between P and arguments on P such that $\Sigma = info_{\mid\sim}(P)$. Thus, for every argument σ, we have $\sigma \in \Sigma$ if and only if $P \mid\!\sim \sigma$. Let us now prove that Σ is a complete argument structure with $prem(\Sigma) = P$. There are three statements to be proved: (a) the set Σ is an argument structure, (b) the argument structure Σ is complete, and (c) we have $prem(\Sigma) = P$.

(a) To prove that Σ is an argument structure, we must prove that (1) the set Σ is closed under subarguments, and that (2) the set Σ is compatible. Point (1) immediately follows from Definition 4.17. To prove (2), assume to the contrary that Σ is incompatible. By the definition of incompatibility (Definition 3.2) there must be a finite incompatible subset Σ_S of Σ. It can easily be verified that, without loss of generality, we may assume that all elements in Σ_S are of the form $\sigma_1, \ldots, \sigma_n \Rightarrow \sigma$. Since Σ_S is finite, it has a $\leqslant$-least element σ (cf. Proposition 3.5). Because $\Sigma_S - \{\sigma\}$ is a defeater of σ that is in force, the argument σ should according to Definition 4.17(3) not be in force. But it is, as $\sigma \in \Sigma_S$ and $\Sigma_S \subseteq \Sigma$, so that, after all, Σ cannot be incompatible.

(b) To prove that Σ is complete, we must prove that every argument appendable to Σ already is in Σ. So let σ be an argument that is appendable to Σ. In proving that $\sigma \in \Sigma$, we follow Definition 5.4. If (i) the argument σ already is in Σ, we are done. If (ii) the argument σ satisfies condition (2) or condition (3) of the definition of appendability, then σ immediately satisfies condition (2) or condition (3) of Definition 4.17, since every subset Σ_S of Σ simply is a set of arguments Σ that is in force. Because, in these two cases, σ either satisfies condition (2) or condition (3) of Definition 4.17, σ is in force, i.e., $\sigma \in \Sigma$.

(c) To prove that $prem(\Sigma) = P$, it suffices to refer to Definition 4.17, from which it follows that every argument in Σ has its premises in P.

Conditions (a), (b), and (c) together yield that Σ is an argument structure. □

From Proposition 5.11 it follows that every extension of P is a complete argument structure; and the latter implies that the set of elements that are warranted on the basis of P, is compatible. Another consequence of Proposition 5.11 is that extensions are maximal argument structures.

6. Construction of complete argumentation structures

Extensions are important but problematic sets. They are defined by means of fixed point definitions, which are not constructive. In the first instance, this would imply that extensions cannot be constructed by adding arguments to an existing argument structure. The latter is right: it is indeed hard to formulate a constructive procedure on the basis of a fixed point definition. But from the previous section we know that extensions coincide with complete argument structures, so that we might try to find a procedure for constructing complete argument structures. The aim of this section is to do just that.

6.1. Elementary argumentation steps

An elementary argumentation step is an elementary transition from one argument structure to the next, involving the application of at most one rule of inference.

An elementary argumentation step is defined with the help of the notion of direct construction.

Definition 6.1. Let Σ be an argument structure. An argument is *directly constructed* from Σ if all its proper subarguments are in Σ. An argument structure is *directly constructed* from Σ if all its members are directly constructed from Σ.

As an example of arguments that are directly constructed from a particular argument structure, we might take the arguments that are appendable to that argument structure. (This fact readily follows from the definition of appendability.)

Definition 6.2. An *elementary argumentation step* from Σ_1 to Σ_2 is a pair of argument structures Σ_1, Σ_2 such that Σ_2 is directly constructed from Σ_1 and $\Sigma_2 - \Sigma_1$ consists of precisely one element.

Let Σ_1, Σ_2 be an elementary argumentation step. We call an argument *kept* if it is in $\Sigma_1 \cap \Sigma_2$, *deleted* if it is in $\Sigma_1 - \Sigma_2$, and *new* if it is in $\Sigma_2 - \Sigma_1$. With this terminology, it clearly follows that every argument in Σ_1 is either kept or deleted, and that every argument in Σ_2 is either kept or new. In contrast with arguments that are new, there is no upper bound on the number of arguments that may be deleted per argumentation step.

6.2. Argumentation sequences

Beginning with a base set, it is the intention to work from one argument structure to the next, always including one argument at a time. Sometimes, when weak arguments are exchanged for better ones, arguments are collectively deleted. In the process of including and deleting arguments, the final objective is to increase the net amount of information eventually.

Definition 6.3. An *argumentation sequence* $(\Sigma_n)_{n=1}^{\infty}$ is a collection of argument structures $\{\Sigma_n\}_{n=1}^{\infty}$ such that, for every $n \geqslant 1$, the pair Σ_n, Σ_{n+1} is an elementary argumentation step. The sequence $(\Sigma_n)_{n=1}^{\infty}$ *begins with* P if $P = \Sigma_1$.

Knowing that a base set contains a finite number of arguments, and using the definition of an elementary argumentation step, it follows with an easy induction argument that every term of an argumentation sequence is a finite argument structure. Because this fact will be used later on, it is listed as a proposition.

Proposition 6.4. *Let $(\Sigma_n)_{n=1}^{\infty}$ be an argumentation sequence. Then, for every $n \geqslant 1$, the term Σ_n is a finite argument structure.*

Proof. Follows from Definitions 4.1, 6.2, and 6.3. □

In the course of argumentation, an argument may be included and deleted several times. Eventually, there are three possibilities. Either an argument is definitely included, it is definitely deleted, or it is included and deleted an indefinite number of times. Let us define an argument to be in the limit of an argumentation sequence if, at some point, it is definitely included in every subsequent term of that sequence. This is equivalent with saying that the argument is contained in all but a finite number of terms.

Definition 6.5. Let $(\Sigma_n)_{n=1}^{\infty}$ be an argumentation sequence. This sequence has a *limit*, denoted by $lim_n\ \Sigma_n$. An argument σ is in $lim_n\ \Sigma_n$ if it is in all but a finite number of terms of $(\Sigma_n)_{n=1}^{\infty}$.

It is in order to mention that the limit could have been defined otherwise. For example, we could have defined an argument to be in the limit of an argumentation sequence if and only if it is contained in an infinite number of terms of the sequence. This kind of limit contains exactly those arguments that cannot be defeated. Our notion of limit additionally contains those arguments that are definitely established in the course of argumentation.

The following result ensures that, even if we take limits, our theory remains within the framework of argument structures.

Proposition 6.6. *Let $(\Sigma_n)_{n=1}^{\infty}$ be an argumentation sequence, beginning with a base set P. Then $lim_n\ \Sigma_n$ is an argument structure containing P.*

Proof. There are two statements to be proved: (a) the limit of the argumentation sequence $(\Sigma_n)_{n=1}^{\infty}$ is closed under subarguments, and (b) the limit of $(\Sigma_n)_{n=1}^{\infty}$ is compatible.

(a) To prove that $lim_n \Sigma_n$ is closed under subarguments, let σ be an arbitrary argument in $lim_n \Sigma_n$. By definition, σ is included in all but a finite number of terms of the sequence $(\Sigma_n)_{n=1}^{\infty}$. Since every term of $(\Sigma_n)_{n=1}^{\infty}$ is an argument structure, it follows that every subargument of σ is included in all but a finite number of terms of the sequence $(\Sigma_n)_{n=1}^{\infty}$. So, by definition, every subargument of σ is in $lim_n \Sigma_n$.

(b) To prove that $lim_n \Sigma_n$ is compatible, suppose it is not. By definition of incompatibility there must be a finite, incompatible subset Σ of $lim_n \Sigma_n$. Now it can easily be seen that, due to the finiteness of Σ, the entire set Σ is included in all but a finite number of terms of $(\Sigma_n)_{n=1}^{\infty}$. Hence, all but a finite number of terms of $(\Sigma_n)_{n=1}^{\infty}$ is incompatible, which is in contradiction with the compatibility of each individual term of $(\Sigma_n)_{n=1}^{\infty}$.

From (a) and (b) we conclude that the limit of every argumentation sequence is an argument structure. □

The previous proposition ensures that the following definition makes sense. It singles out those argumentation sequences that approximate complete argument structures.

Definition 6.7. An argumentation sequence is *complete* if its limit is complete.

The concept of a limit is clear: it contains arguments that, in this sequence, remain undefeated. However, if an argument remains undefeated in an argumentation sequence, it may not be concluded that it cannot be defeated whatsoever. If there exists a collection of strong counter-arguments but the argumentation sequence fails to incorporate them, the argument in question is saved from being defeated. Nevertheless, it can be proven that this undesired circumstance will not occur. Thus, if an argument emerges victorious in a complete argumentation sequence, we are allowed to conclude that it really cannot be defeated. This is stated in the next proposition.

Proposition 6.8. *Let $(\Sigma_n)_{n=1}^{\infty}$ be a complete argumentation sequence, beginning with a base set P. Then $lim_n \Sigma_n$ is an extension of P.*

Proof. Let $lim_n \Sigma_n = \Sigma$, and let us begin proving that $prem(\Sigma) = P$. First, from Proposition 6.6 it follows that Σ is an argument structure containing P. Moreover, the argumentation sequence begins with $\Sigma_1 = P$, and in the course of argumentation, only arguments outside $\mathcal{L}$ are produced. Hence, the set $prem(\Sigma)$ contains no more than what is in P. Now, according to Proposition 5.10, we are allowed to conclude that Σ is an extension of P. □

The remainder of this paper is concerned with the specification of an elementary argumentation step, in such a way that we are able to prove that it brings forward a complete argumentation sequence.

6.3. Specification of an elementary argumentation step

The aim of this section is to formulate a specific *elementary argumentation step* (cf. Section 6.1).

Let Σ be an argument structure and let σ be appendable to Σ (cf. Definition 5.4). Of course, we are not going to combine Σ and σ just anyhow. Our aim is to construct a successor of Σ, denoted by $\Sigma + \sigma$, such that

(i) the set $\Sigma + \sigma$ is an argument structure,
(ii) the transition from Σ to $\Sigma + \sigma$ is an elementary argumentation step,
(iii) the argument structure $\Sigma + \sigma$ contains σ,
(iv) from Σ, as much as possible will be kept,
(v) if arguments must be given up, the $\leqslant$-weakmost arguments will be given up first.

Conditions (i)–(v) form a collection of *rationality postulates*. The formulation of these postulates is inspired by a similar approach in the theory of belief revision.[26] We now try to construct a successor argumentation structure that meets the above five conditions.

The contents of $\Sigma + \sigma$ mainly depends on the compatibility of Σ and σ:

(a) If Σ is compatible with σ, then $\Sigma + \sigma$ may straightforwardly be defined as $\Sigma \cup \{\sigma\}$.
(b) If Σ is incompatible with σ, some elements of Σ need to be given up to maintain compatibility. It is reasonable to drop relatively weak elements (that contribute to the conflict) first.

(By the way, (a) and (b) are not in order for complete argument structures, because complete argument structures have no arguments that are appendable.)

Definition 6.9 (*M-successor*). Let Σ be an argument structure, and let σ be an argument that is appendable to Σ. Let $\{\Sigma_i \mid i \in I\}$ be the collection of subsets of Σ that are minimally incompatible with σ, and let $\{\sigma_i \mid i \in I\}$ be a set of $\leqslant$-least elements for this family. (In a moment, it will be shown that these elements exist.) Then

$$\Sigma + \sigma \stackrel{\text{def}}{=} \Sigma - sup(\sigma_i \mid i \in I) \cup \{\sigma\} \tag{28}$$

is an *M-successor* of Σ.[27] (The *sup* operator is defined in Definition 2.8.) Depending on the choice of the least elements, an argument structure may have more than one M-successor. By writing $\Sigma + \sigma$ we mean to denote a set that obeys the equation above.

In the previous definition it is assumed that, for every $i \in I$, the set Σ_i has a least element. That this is indeed the case, can be seen as follows. From Proposition 3.5 it follows that $\Sigma_i \cup \{\sigma\}$ has a least element. Moreover, this least element cannot be equal

[26] Cf. work of Gärdenfors and Makinson [11].

[27] Why M-successor (instead of A-successor, B-successor, C-successor)? The only reason to use "M" is that it is easily pronounceable in combination with the word successor. If this (pragmatic) reason remains unmentioned, one might suspect that the letter "M" has a deeper meaning, which is not the case. The motivation to use a letter at all, is that the notion "successor" alone has insufficient recognizability to profile itself as a special theoretical concept, with a designated function.

to σ, because σ is assumed to be appendable to Σ. Thus, we conclude that the least element of $\Sigma_i \cup \{\sigma\}$ is also the least element of Σ_i.

The following proposition states that every M-successor keeps an argument structure. ("Keeps" in the sense of our official terminology.) This proposition is needed in order to prove that every M-successor *is* an argument structure.

Proposition 6.10. *Let Σ be an argument structure. Then*

$$\{\sigma \mid \sigma \textit{ is kept in the elementary argumentation step } \Sigma, \Sigma + \sigma\} \tag{29}$$

is an argument structure.

Proof. By definition, the set of arguments that is kept in moving from Σ to $\Sigma + \sigma$, is equal to $\Sigma \cap (\Sigma + \sigma)$. We must prove that this set is an argument structure. There are two statements to be proved: (a) the set $\Sigma \cap (\Sigma + \sigma)$ is closed under subarguments, and (b) the set $\Sigma \cap (\Sigma + \sigma)$ is compatible.

(a) We must prove that, for every argument τ, if τ is kept in moving from Σ to $\Sigma + \sigma$, then every subargument of τ is kept as well. As it turns out, it is easier to prove the converse, that is, it is easier to prove that, for every argument τ, if τ is deleted in moving from Σ to $\Sigma + \sigma$, then every super-argument of τ is deleted as well. This statement, however, is an immediate consequence of Definition 6.9.

(b) We must prove that $\Sigma \cap (\Sigma + \sigma)$ is compatible. This follows immediately from the fact that $\Sigma \cap (\Sigma + \sigma)$ is a subset of Σ, and Σ is compatible.

We conclude that the set of arguments that is kept in moving from Σ to one of its M-successors, is an argument structure. □

Proposition 6.11. *Let Σ be an argument structure. Then every M-successor of Σ is an argument structure.*

Proof. Let σ be an argument that is appendable to the argument structure Σ, and let $\Sigma + \sigma$ be an M-successor of Σ. There are two statements to be proved: (a) the set $\Sigma + \sigma$ is closed under subarguments, and (b) the set $\Sigma + \sigma$ is compatible.

(a) We will have to prove that $\Sigma + \sigma$ contains all subarguments of its members. Suppose that $\tau \in \Sigma + \sigma$. Then (1) the argument τ is in Σ, or (2) the argument τ is equal to σ. If (1) we have $\tau \in \Sigma$, then apparently τ is kept. Since Σ is an argument structure, $sub(\tau) \subseteq \Sigma$. By Proposition 6.10 we may now conclude that $sub(\tau)$ is kept, i.e., $sub(\tau) \subseteq \Sigma + \sigma$. If (2) we have $\tau = \sigma$, then we must prove that $sub(\sigma) \subseteq \Sigma + \sigma$. By Definition 6.9 the argument σ must be appendable to Σ. Let us follow Definition 5.4 to prove (2). If (i) the argument σ already is in Σ, then it can easily be seen that $\Sigma + \sigma$ equals Σ, so that nothing has to be proved. If (ii) for some $\sigma_1, \ldots, \sigma_n$ we have $\sigma_1, \ldots, \sigma_n \to \sigma$ and $\sigma_1 \in \Sigma, \ldots, \sigma_n \in \Sigma$, then it follows readily from Definition 6.9 that $\Sigma \cup \{\sigma\}$ is compatible so that $\Sigma + \sigma$ is equal to $\Sigma \cup \{\sigma\}$, and therefore contains all subarguments of σ. If (iii) for some $\sigma_1, \ldots, \sigma_n$ we have $\sigma_1, \ldots, \sigma_n \Rightarrow \sigma$ and $\sigma_1 \in \Sigma, \ldots, \sigma_n \in \Sigma$ and Σ does not contain defeaters of σ. In this case, it suffices to prove that $\{\sigma_1, \ldots, \sigma_n\}$ is kept in constructing $\Sigma + \sigma$ from Σ, because $sub(\sigma) = sub(\sigma_1) \cup \cdots \cup sub(\sigma_n) \cup \{\sigma\}$

and σ already is in $\Sigma + \sigma$. Suppose to the contrary that one of $\sigma_1, \ldots, \sigma_n$ is removed, say σ_i for some $1 \leqslant i \leqslant n$. Then it follows from Proposition 6.14 that $\sigma_i < \sigma$.[28] However, since $\sigma_i \sqsubseteq \sigma$ we also have $\sigma \leqslant \sigma_i$ so that, by transitivity of $\leqslant$, we have $\sigma_i < \sigma_i$, which is absurd. Because this holds for all $1 \leqslant i \leqslant n$, we are allowed to conclude that $\{\sigma_1, \ldots, \sigma_n\}$ is kept. Hence, by Proposition 6.10, we have $sub(\sigma_1) \cup \cdots \cup sub(\sigma_n)$ is kept, and hence $sub(\sigma) \subseteq \Sigma + \sigma$. Thus, in both cases, $sub(\tau) \subseteq \Sigma + \sigma$, which was to be proved.

(b) To show that $\Sigma + \sigma$ is compatible, suppose it is not. Let Σ_S be a smallest incompatible subset of $\Sigma + \sigma$. Since $(\Sigma + \sigma) - \{\sigma\}$ is contained in Σ, the set Σ_S also is a smallest subset of Σ that is incompatible with σ. By definition of $\Sigma + \sigma$, the set Σ_S is equal to Σ_i for some $i \in I$ in Definition 6.9 and should therefore contain a smallest representative σ_i. But the argument σ_i is by definition not in $\Sigma + \sigma$, which contradicts our earlier findings that $\sigma_i \in \Sigma_S$ and $\Sigma_S \subseteq \Sigma + \sigma$. We must conclude that $\Sigma + \sigma$ does not contain a (smallest) incompatible subset Σ_S and, hence, that $\Sigma + \sigma$ must be compatible.

We have proven that $\Sigma + \sigma$ is closed under subarguments and is compatible. Thus, the set $\Sigma + \sigma$ is an argument structure. □

Proposition 6.12. *Let Σ be an argument structure. Then moving from Σ to one of its M-successors is an elementary argumentation step.*

Proof. Let $\Sigma + \sigma$ be an M-successor of Σ. It follows from Proposition 6.11 that $\Sigma + \sigma$ is an argument structure. Further, there are two statements to be proved: (a) the set $\Sigma + \sigma$ is directly constructed from Σ, and (b) the set $(\Sigma + \sigma) - \Sigma$ contains at most one argument. Both statements follow immediately from Definition 6.9. □

Proposition 6.13. *Let $\{\Sigma_n\}_{n=1}^{\infty}$ be a sequence of sets of arguments such that $P = \Sigma_1$ is a base set and, for every $n \geqslant 1$, the set Σ_{n+1} is an M-successor of Σ_n. Then $\{\Sigma_n\}_{n=1}^{\infty}$ is an argumentation sequence $(\Sigma_n)_{n=1}^{\infty}$ beginning with the base set P.*

Proof. This proposition is an immediate consequence of Definition 6.3 and Proposition 6.12. □

Thus far, we have proven that Definition 6.9 meets the minimal requirements for constructing an argumentation sequence: the procedure amounts to an iteration of the M-successor construction. The question remains, however, if such a successor sequence will increase the net amount of information eventually. If the arguments that are deleted will outnumber the arguments that are appended, an argumentation sequence would be merely a complicated method of collecting and discharging information, without carrying us one step further. The following three propositions, however, tell progress from motion. They show that, although argumentation is nonmonotonic on the spot, it is monotonic in the long run.

[28] We readily admit that the use of forward references distracts from the global outline. However, Proposition 6.14 will be used a second time, and we think that it comes out at its best right before Proposition 6.15. The results are not circularly dependent.

The first proposition states that new and deleted arguments are always comparable, and that new arguments are always stronger than deleted arguments.

Proposition 6.14. *Let Σ be an argument structure. Let $\Sigma + \sigma$ be an M-successor of Σ, and let τ be an argument that is deleted by σ. Then $\tau < \sigma$.*

Proof. If the argument τ is deleted by σ, then it is in Σ, but not in $\Sigma + \sigma$. So τ must be in $sup(\sigma_i \mid i \in I)$, as defined in Definition 6.9. Suppose that $\tau \in sup(\sigma_i)$, for some $i \in I$. Because Σ_i is a subset of Σ that is incompatible with σ, condition (3) of Definition 5.4 together with the appendability of σ tells us that σ undermines Σ_i. Moreover, it follows from Definition 6.9 that σ_i is a least element of Σ_i, so that σ must be stronger than σ_i, i.e., $\sigma_i < \sigma$. Because σ_i is a subargument of τ, also $\tau \leqslant \sigma_i$ (Definition 2.12), so that by transitivity of $\leqslant$ we have $\tau < \sigma$. □

Endless chains of successively deleting arguments do not exist:

Proposition 6.15. *Let $(\Sigma_n)_{n=1}^{\infty}$ be an argumentation sequence such that, for every $n \geqslant 1$, the argument structure Σ_{n+1} is an M-successor of Σ_n, and let $\sigma_{n_1}, \sigma_{n_2}, \sigma_{n_3}, \dots,$ $\sigma_{n_k}, \dots$ be a sequence of arguments with, for every $k \geqslant 1$, $\sigma_{n_k} \in \Sigma_{n_k}$ and σ_{n_k} deleted by $\sigma_{n_{k+1}}$ in constructing $\Sigma_{n_{k+1}}$. Then this sequence is finite.*

Proof. By Proposition 6.14 it follows that, for every $k \geqslant 1$, the argument $\sigma_{n_{k+1}}$ is stronger than σ_{n_k}. The proposition now follows from Definition 2.12(1). □

Finally, the third proposition ensures that the terms of an argumentation sequence grow in due course.

Proposition 6.16. *Let $(\Sigma_n)_{n=1}^{\infty}$ be an argumentation sequence such that, for every $n \geqslant 1$, the argument structure Σ_{n+1} is an M-successor of Σ_n. Then $lim_n\ \Sigma_n$ is finite if and only if, for some $n \geqslant 1$, the argument structure Σ_n is complete.*

Proof. First, suppose that, for some $n \geqslant 1$, the argument structure Σ_n is complete. This means that no argument is appendable to Σ_n, so that Σ_n has no M-successors, other than itself. Because every term of an argumentation sequence is a finite argument structure (cf. Proposition 6.4), we conclude that Σ_n is finite, and hence that $lim_n\ \Sigma_n$ is finite as well. Conversely, suppose that $lim_n\ \Sigma_n$ is finite. To obtain a contradiction, let us assume that every term of $(\Sigma_n)_{n=1}^{\infty}$ is not complete. Because $lim_n\ \Sigma_n$ is finite we have, for some $n \geqslant 1$, that $lim_n\ \Sigma_n$ is in Σ_{n+k}, for every $k \geqslant 1$. Because, in particular, the set Σ_n is not complete, there is some argument σ_1 that is not in Σ_n, but is included in Σ_{n+1}. That is, $\Sigma_n + \sigma_1 = \Sigma_{n+1}$. Because σ_1 is not in Σ_n, it is not in $lim_n\ \Sigma_n$. Hence, σ_1 must be deleted by another argument σ_2 in a later stage. But σ_2 is also not a member of $lim_n\ \Sigma_n$, so that σ_2 must be deleted by yet another argument, say, σ_3. In this way, we are constructing an infinite sequence $\{\sigma_n\}_{n=1}^{\infty}$ of mutually deleting arguments, which is in contradiction with Proposition 6.15. Hence, there must be some $n \geqslant 1$ such that Σ_n is complete. □

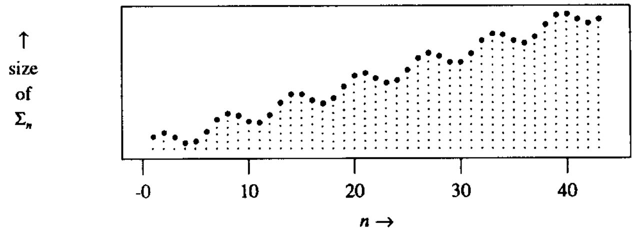

Fig. 7. Characteristic development of an argumentation sequence.

An alternative formulation of the proposition is that the limit of an M-argumentation sequence with incomplete terms contains infinitely many arguments.

Fig. 7 suggests how the elements of an argumention sequence grow and shrink in the process of argumentation. In practice, the undulation is not that even, but depends on the choice of the first arguments to develop. Thus, an argumentation sequence is locally nonmonotonically increasing, but globally monotonically increasing.

7. Finite argumentation systems

In this section a conclusive result will be established for finite argumentation systems.

Definition 7.1. An argumentation system $\mathcal{A}$ is *finite* if it enables at most a finite number of arguments.

Proposition 7.2. *An argumentation system $\mathcal{A}$ is finite if either its language $\mathcal{L}$, or the corresponding set of rules R is finite.*

Proof. In both cases, the length of the arguments is bounded by either $|\mathcal{L}|$ or $|R|$, respectively. From there, it is not difficult to see that the number of possible arguments is also finite. $\square$

Proposition 7.3. *Let $\mathcal{A}$ be a finite argumentation system. Then every argumentation sequence $(\Sigma_n)_{n=1}^{\infty}$ such that, for every $n \geqslant 1$, the argument structure Σ_{n+1} is an M-successor of Σ_n, is complete.*

Proof. Because $\mathcal{A}$ is finite, it follows that $lim_n\ \Sigma_n$ is finite. Our desired result now immediately follows from Proposition 6.16. $\square$

Proposition 7.4. *Let $\mathcal{A}$ be a finite argumentation system, and let $(\Sigma_n)_{n=1}^{\infty}$ be an argumentation sequence, starting with a base set P such that, for every $n \geqslant 1$, the set Σ_{n+1} is an M-successor of Σ_n. Then $lim_n\ \Sigma_n$ is an extension of P.*

Proof. This proposition is an immediate result of Propositions 6.8 and 7.3. $\square$

The last result is not concerned with computational efficiency. In particular, it does not provide an upper bound on the number of elementary argumentation steps that need to be performed before a definite outcome is established. One might imagine that, in the worst case, if weak arguments are developed first, there is a considerable turnover in arguments, a process in which the right argument emerges only after a large number of argumentation steps. Still, the proposition ensures that warranted propositions emerge victorious in the long run.

8. Infinite argumentation systems

Insofar as the finite case is concerned, our theory is completed. However, the following example shows that, in the general case, a successor sequence might develop itself in the wrong direction.

Example 8.1. Consider the abstract argumentation system $\mathcal{A}$ with language $\mathcal{L} = \{p_n\}_{n=1}^{\infty} \cup \{q_1, q_2\} \cup \{\perp\}$, with rules

$$R = \{p_n \Rightarrow p_{n+1}\}_{n=1}^{\infty} \cup \{q_1 \Rightarrow q_2\} \cup \{p_3, q_2 \rightarrow \perp\} \tag{30}$$

and with the $\Rightarrow$-count order on arguments $\leqslant$ (cf. Example 4.32). Note that $\mathcal{A}$ enables an infinite number of arguments. It follows that $\mathcal{A}$ is not a finite argumentation system. Let $P = \{p_1, q_1\}$. Now, there are p-arguments and q-arguments clashing at level 3 and 2, respectively. Therefore, the only p-arguments that are in force are p_1 and $p_1 \Rightarrow p_2$. This is because $q_1 \Rightarrow q_2$ is stronger than $p_1 \Rightarrow p_2 \Rightarrow p_3$ while both arguments are incompatible due to the rule $p_3, q_2 \rightarrow \perp$. However, let $(\Sigma_n)_{n=1}^{\infty}$ be the argumentation sequence with, for every $n \geqslant 1$, the term $\Sigma_n = \{p_1 \Rightarrow \cdots \Rightarrow p_n\}$. As easily can be verified, for every $n \geqslant 1$, the argument structure Σ_{n+1} is an M-successor of Σ_n. So it follows from Proposition 6.13 that $(\Sigma_n)_{n=1}^{\infty}$ is an argumentation sequence. Obviously, the argument $q_1 \Rightarrow q_2$ is not in $lim_n\ \Sigma_n$. At the same time, $q_1 \Rightarrow q_2$ is appendable to $lim_n\ \Sigma_n$. It follows that the sequence in question is not complete.

Although this example was meant to show what might happen in the infinite case, it also offers a first insight in the (new) problem of how to construct complete argument structures in the presence of an infinite number of arguments. Working on a solution for this problem, however, falls beyond the objective of this paper.

9. Summary

We have developed a theory of abstract argumentation systems that is capable of dealing with a number of important problems of defeasible reasoning. The notion of *defeat* is the starting point of this theory, and the notion of *argument structure* is the vehicle on which the theory is further developed.

To maintain insight in the often complex relations between competing arguments, the theory is developed progressively. Defeat is defined in terms of *undermining* (a concept arising from *conclusive force*) and *incompatibility* (a concept arising from *contradiction*). With the help of the notion of defeat, three consequence operators were introduced, of increasing complexity.

- $|_{\Sigma} \sim$. *Enablement* is a more sophisticated form of defeat that takes into account the effects of defeaters on subarguments and sets of (sub-)arguments.
- $\mid\!\sim_n$. *Inductive warrant* involves iterated defeat, such as cascaded defeat and reinstatement. Later on, it was proven that inductive warrant on level n is equal to enablement, n times iterated.
- $\mid\!\sim$. *Warrant* is a relation between competing arguments. As such, it often does not give more information than a description of the mutual dependency between arguments, without pronouncing which argument actually is in force. In consequence, the relation $\mid\!\sim$ usually defines more than one fixed point, or, in terms of nonmonotonic reasoning, more than one *extension*. Later on, it was proven that all extensions reside between the upper and lower limit of the sets of odd and even level-n arguments, respectively.

Thus, transparent and adequate criteria for adjudication among competing arguments were formulated, beginning with the relatively elementary notion of defeat.

In the rest of the paper we were concerned with the formulation of methods that lead to the construction of extensions. (Extensions are defined by means of fixed point definitions, which are not constructive.) We have shown that, under certain conditions, it is possible to construct extensions with the help of an *argumentation sequence*. An argumentation sequence is an infinite series of argument structures of which any two succeeding elements are connected via an *elementary argumentation step*. In an elementary argumentation step, at most one new argument may be added and an unlimited number of arguments may be deleted. Within the notion of elementary argumentation step there is room for different types of successors and thus for different types of argumentation sequences. In the last section we have proven that, in finite argumentation systems, extensions can be constructed by means of *M-successor sequences*. For the infinite case we have not found a solution yet.

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Appendix A. Rebutting and undercutting defeaters

The present formalism, as we have developed it here, is ruled by a simple notion of defeat. This rule says that an argument σ is defeated by a group of arguments Σ if and only if Σ is in force, the argument σ is not stronger than any member of Σ, and σ is incompatible with Σ. John Pollock, however, argues for (at least) two notions of defeat, concerned with contradiction and vitiation, respectively. If one argument contradicts the other then, according to Pollock, we have a case of rebutting defeat. If, on the other hand, one argument merely vitiates (an inference of) another argument, then we have a case of undercutting defeat, for it merely undercuts the other argument, without really contradicting its conclusion. However, in spite of all this, we think that rebutting and undercutting defeaters are two sides of the same coin, and that one can well speak of a single and uniform notion of defeat.

Before we can define rebutting and undercutting defeaters in a formal manner, we will have to be able to represent rules of inference (which are meta-linguistic notions) in the object language. Therefore, we temporarily assume that every language of every abstract argumentation system is closed under negation ($\neg$), conjunction ($\wedge$), implication ($\supset$), and defeasible implication ($>$). Thus, the implication symbols $\supset$ and $>$ are the counterparts of $\rightarrow$ and $\Rightarrow$, respectively.

We are now able to define rebutting and undercutting defeaters in a formal fashion. Let σ and τ be arguments with conclusions ϕ and ψ, respectively, such that $\sigma < \tau$. Let $\phi_1, \ldots, \phi_n \Rightarrow \phi$ be the last rule of σ.

(1) *Rebutting defeat.* If $\psi = \neg\phi$, then τ is said to be a rebutting defeater of σ. Thus, the conclusion of a rebutting defeater contradicts the conclusion of the argument it defeats.

(2) *Undercutting defeat.* If $\psi = \neg(\phi_1 \wedge \cdots \wedge \phi_n > \phi)$, i.e., if ψ is the negation of the last rule of σ in the object language, then τ is said to be an undercutting defeater of σ. Thus, the conclusion of an undercutting defeater contradicts the last inference of the argument it defeats.

At this point, it may be in order to recall Pollock's original argument.

> Prima facie reasons for which the only defeaters are rebutting defeaters would be analogous to normal defaults in default logic. Experience in using prima facie reasons in epistemology indicates that there are no such prima facie reasons. Every prima facie reason has associated defeaters that are not rebutting defeaters, and these are the most important kinds of defeaters for understanding any complicated reasoning. Defeaters that are not rebutting defeaters attack a prima facie reason without attacking its conclusion. They accomplish this by instead attacking the connection between the premises and the conclusion. For instance, "x looks red" is a prima facie reason for "x is red". But if I know not only that x looks red but also that x is illuminated by red lights and red lights can make things look red when they are not, then it is unreasonable for me to infer that x is red. Consequently, "x is illuminated by red lights and red lights can make things look red when they are not" is a defeater, but it is not a reason for thinking that x is not red, so it is not a rebutting defeater. Instead, it attacks the connection between "x looks red" and "x is red", giving us a reason for doubting that x wouldn't look red unless it were red. (See Pollock [33].)

Pollock's example can be formalized as follows. Let r stand for "x is red", l for "x looks red", and i for "x is illuminated by red light". According to Pollock, the two prima facie reasons become $l \Rightarrow r$ and $i \Rightarrow \neg(l > r)$, where the first is stronger than the second, so that the second rule is an undercutting defeater of the first rule.

We think, however, that this formalization is not entirely correct, particularly at the point where prima facie reasons are translated into formal expressions. At this point, prima facie rules are translated into defeasible rules of inference, where defeasible conditionals would be appropriate. For instance, the prima facie rule that things that look red usually are red, should not be presented as a defeasible rule of inference $l \Rightarrow r$, but as a defeasible conditional $l > r$ instead. The reason why this should be so is that rules of inference are meant to be general patterns of reasoning, to be used as templates in various inferences with matching form but different content. Conditional implications, on the other hand, whether they are defeasible or not, are statements in a language, that express—but not perform—a certain rule of thumb. Thus, conditional implications are elements of a language, merely denoting or referring to the application of a certain inference. In the running example, this inference would be $l, l > r \Rightarrow r$. Thus, from the information that "x is red" and the prima facie reason that "things that look red usually are red", we defeasibly infer that "x is red". Obviously, the general pattern here is $\phi, \phi > \psi \Rightarrow \psi$, a defeasible counterpart of modus ponens.

If the considerations above are taken seriously into account, the rules $l \Rightarrow r$ and $i \Rightarrow \neg(l > r)$ should become $l > r$ and $i > \neg(l > r)$, respectively. Moreover, from Pollock's example it follows that $l > r$ is not only a statement that expresses a defeasible implication, but also a statement that is defeasible in itself. Hence we should write $>(l > r)$, stating that "normally, things that look red, usually are red", where $>\phi$ is an abbreviation of *true* $> \phi$. Here, the corresponding inference is $>(l > r) \Rightarrow l > r$ which, on its turn, is an instantiation of the general pattern $(>\phi) \Rightarrow \phi$. Now if $\{>(l > r), i > \neg(l > r)\}$ is background knowledge, we observe l, and we come to know that i, we know

$$P = \{>(l > r), i > \neg(l > r)\} \cup \{l, i\} \tag{A.1}$$

so that $\sigma = l, (>(l > r) \Rightarrow l > r) \Rightarrow r$ is an argument based on P that argues for r, and $\tau = (i, i > (\neg(l > r))) \Rightarrow \neg(l > r)$ is an argument based on P that argues against the prima facie reason $l > r$. Writing this a bit differently, we get $\sigma = l, \sigma_1 \Rightarrow r$, where $\sigma_1 = >(l > r) \Rightarrow l > r$. We see that τ contradicts a subargument σ_1 of σ, so that τ disables σ if it defeats σ_1. If arguments are judged on the basis of specificity, then we have $\sigma_1 < \tau$, because the statement that enables τ (namely, i) is more specific than the statement that enables σ_1 (namely, *true*). In that case, if $\sigma_1 < \tau$, we have τ is in force and σ_1 is not. From this it follows that the argument σ cannot be in force as well.

To conclude our digression on rebutting and undercutting defeaters, let us consider these notions in the general setting of our framework, and show that they are nothing but special cases of one simple notion of defeat.

(1) For rebutting defeat, the general situation is that we have two arguments σ and τ with conclusions ϕ and $\neg\phi$, respectively, such that $\sigma < \tau$. If $\phi, \neg\phi \rightarrow \perp$ is among our rules of inference—which is reasonable to assume—then defeat among σ and τ can be handled adequately by Definition 4.17. It is interesting to note that, as soon as negation is at our disposal, there is always "binary" defeat among arguments.

(2) For undercutting defeat, the situation is somewhat more complicated. To begin with, suppose that the implication $\phi_1 \wedge \cdots \wedge \phi_n > \phi$ is at issue. In the general case, we then have an argument v for $\phi_1 \wedge \cdots \wedge \phi_n > \phi$, arguments $\sigma_1, \ldots, \sigma_n$ for $\phi_1, \ldots, \phi_n$, respectively, and an argument τ for $\neg(\phi_1 \wedge \cdots \wedge \phi_n > \phi)$, i.e., against $\phi_1 \wedge \cdots \wedge \phi_n > \phi$. If $\phi_1, \ldots, \phi_n \rightarrow \phi_1 \wedge \cdots \wedge \phi_n$ is among our rules of inference—which, again, is reasonable to assume—then $\sigma_1, \ldots, \sigma_n$ obviously assemble into an argument σ for $\phi_1 \wedge \cdots \wedge \phi_n$. Moreover, if $\phi_1 \wedge \cdots \wedge \phi_n, \phi_1 \wedge \cdots \wedge \phi_n > \phi \Rightarrow \phi$ is a valid rule of inference, then σ and v assemble into an argument $\sigma, v \Rightarrow \phi$ for ϕ. Now, if τ is stronger v, then τ defeats v, and hence $\sigma, v \Rightarrow \phi$. If we avail ourselves of Pollock's vocabulary, then τ is an undercutting defeater of the argument $\sigma, v \Rightarrow \phi$. At the same time, also this case can be handled adequately by Definition 4.17.

Finally, to avoid possible misunderstandings, it must be said that we have not been arguing against the existence of two types of defeaters. The distinction between rebutting and undercutting defeaters is a real one, and does make sense (as John Pollock has shown us). We only wanted to allege our reasons for not incorporating these notions in our theory.

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