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On the complexity of core, kernel, and bargaining set ☆

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ABSTRACT

Coalitional games model scenarios where players can collaborate by forming coalitions in order to obtain higher worths than by acting in isolation. A fundamental issue of coalitional games is to single out the most desirable outcomes in terms of worth distributions, usually called solution concepts. Since decisions taken by realistic players cannot involve unbounded resources, recent computer science literature advocated the importance of assessing the complexity of computing with solution concepts. In this context, the paper provides a complete picture of the complexity issues arising with three prominent solution concepts for coalitional games with transferable utility, namely, the core, the kernel, and the bargaining set, whenever the game worth-function is represented in some reasonably compact form. The starting points of the investigation are the settings of graph games and of marginal contribution nets, where the worth of any coalition can be computed in polynomial time in the size of the game encoding and for which various open questions were stated in the literature. The paper answers these questions and, in addition, provides new insights on succinctly specified games, by characterizing the computational complexity of the core, the kernel, and the bargaining set in relevant generalizations and specializations of the two settings. Concerning the generalizations, the paper shows that dealing with arbitrary polynomial-time computable worth functions—no matter of the specific game encoding being considered—does not provide any additional source of complexity compared to graph games and marginal contribution nets. Instead, only for the core, a slight increase in complexity is exhibited for classes of games whose worth functions encode **NP**-hard optimization problems, as in the case of certain combinatorial games. As for specializations, the paper illustrates various tractability results on classes of bounded treewidth graph games and marginal contribution networks.

1. Introduction

Coalitional games were introduced by von Neumann and Morgenstern [59] in order to reason about scenarios where players can collaborate by forming coalitions with the aim of obtaining higher worths than by acting in isolation. In the Transferable Utility (TU) setting, coalition worths can be freely distributed amongst agents, while in the Non-Transferable Utility (NTU) setting coalitions are allowed to distribute worths only in some specified configurations, called consequences [62].

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In this paper, we consider only the classical TU setting, and thus by saying *game* we always mean hereafter *coalitional game with transferable utility*. Such a game can abstractly be modeled as a pair $\mathcal{G} = \langle N, v\rangle$, where N is a finite set of players, and v is a function associating with each coalition $S \subseteq N$ a certain worth $v(S) \in \mathbb{R}$ that players in S obtain by collaborating with each other. The outcome of $\mathcal{G}$ is an *imputation*, i.e., a vector of payoffs $(x_i)_{i\in N}$ meant to specify the distribution of the total worth $v(N)$ granted to each player in N. Imputations are required to be *efficient*, i.e., $\sum_{i\in N} x_i = v(N)$, and *individually rational*, i.e., $x_i \geqslant v(\{i\})$, for each $i \in N$. In the following, the set of all imputations of $\mathcal{G}$ is denoted by $X(\mathcal{G})$.

It is easily seen that, for any given coalitional game $\mathcal{G}$, the set $X(\mathcal{G})$ may even contain infinitely many payoff vectors. Therefore, a fundamental problem is to single out the most desirable ones in terms of appropriate notions of worth distributions, which are usually called *solution concepts*. Traditionally, this question was studied in economics and game theory with the aim of providing arguments and counterarguments about why such proposals are reasonable mathematical renderings of the intuitive concepts of fairness and stability. Well-known and widely-accepted solution concepts are the *Shapley value*, the *core*, the *kernel*, the *bargaining set*, and the *nucleolus* (see, e.g., [62]). Each solution concept defines a set of outcomes that are referred to with the name of the underlying concept. For instance, the "core of a game" is the set of those outcomes satisfying the conditions associated with the concept of core.

1.1. Coalitional games from the AI perspective: Complexity and representation issues

Solution concepts for coalitional games have been brought to the attention of the computer science community, by considering them from a computational point of view, in a seminal study by Megiddo [57]. There, it has been observed that the naïve approach of explicitly listing all associations of coalitions with their worths in the specification of coalitional games makes the "game theory approach" hardly applicable in practice, due to the exponential blow-up of the input representation w.r.t. the number of involved players. In fact, Megiddo [57] showed the importance of conceiving succinct representations of coalitional games, and taking into consideration computational complexity issues when analyzing classical solution concepts, with the aim of exhibiting efficient algorithms for their calculation. In particular, Megiddo [57] exhibited polynomial-time algorithms for computing the nucleolus and the Shapley value of *cost allocation games over trees*.

Another influential study on complexity issues related to coalitional games is due to Kalai and Zemel [48], who showed polynomial-time algorithms for computing an imputation in the core of *flow games*.

Deng and Papadimitriou [26] took a step further to use computational complexity in the analysis of coalitional games, by arguing that decisions taken by realistic agents cannot involve unbounded resources to support reasoning, and suggesting to formally capture the bounded rationality principle [76] by assessing the amount of resources needed to compute solution concepts in terms of their computational complexity [26,47]. In this context, Deng and Papadimitriou [26] were interested not only in exhibiting efficient algorithms, but also in characterizing those scenarios where such algorithms are unlikely to exist due to the inherent complexity of the solution concepts. In particular, they again noticed that computational questions are of interest whenever worth functions are encoded in some succinct way, e.g., when they are given in terms of polynomially computable functions over some combinatorial structure. However, to the end of assessing the intrinsic complexity of solution concepts, calling for succinct specifications is not only motivated by the practical difficulty of explicitly listing all associations of coalitions with their worths, but also because with an explicit encoding the input sizes are so large that complexity problems are trivially—and in fact artificially—easy. Coalitional games whose worth functions are encoded by means of some succinct representation mechanism are hereinafter called *compact games*.

Coalitional games gained popularity in the context of multi-agent systems and artificial intelligence research since the nineties, when they had been recognized by these research communities as very natural models to understand and reason about cooperative action. In particular, inspired by the approach of Deng and Papadimitriou [26], the questions of finding representation schemes to compactly encode worth functions and assessing over them the complexity of various solution concepts have motivated most of the research on coalitional games in the AI field. In fact, research works facing these questions can be classified into two main groups, depending on the kinds of representation schemes being adopted (cf. [1])[1]:

1. Representation schemes that are succinct, but not complete (i.e., such that there are coalitional games that cannot be captured by such representations). Complexity analysis have been conducted on several prominent schemes of this kind, including *graph games* [26], *traveling salesman games* [31], *flow games* [24], *matching games* [49], *facility location games* [37], *skill games* [9], *threshold games* [5,28,30], *minimum cost spanning tree games* [33], *combinatorial optimization games* (see [25] and the references therein), *games on combinatorial structures* [12], *voting games* [68], *games in multi-issue domains* [17], *linear production games* [63], *bin packing games* [32], *permutation games* [77], *path disruption games* [8], and *(vertex) connectivity games* [10].
2. Representation schemes that are complete, but not succinct (i.e., such that there are coalitional games requiring exponential space—in the worst case—to be encoded in such representations). Influential proposals thereof are *marginal contribution nets* [44], *read-once (and general) marginal contribution nets* [29], *games with synergies among coalitions* [18], and *multi-attribute games* [45].

[1] Details on the various representations mentioned here will be given in Section 3.1.

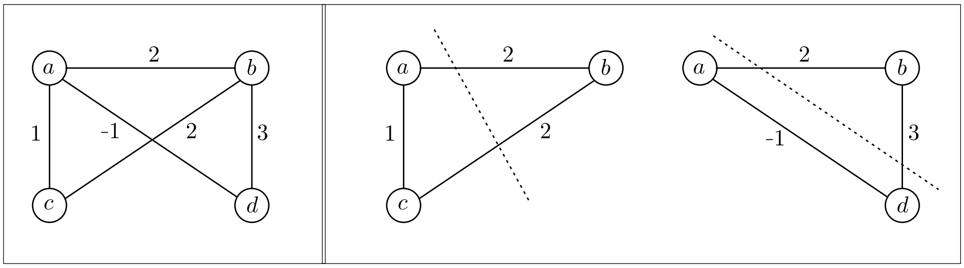

Fig. 1. The graph game in Example 1.1.

In this paper, we continue along the line of analyzing the computational complexity of reasoning about compact coalitional games, by focusing on three prominent solution concepts—the core, the kernel, and the bargaining set. The starting points of the investigation of our analysis are the settings of graph games and of marginal contribution nets, which are representative of the two perspectives discussed above. Various open questions were in fact stated in the literature about these classes of games. Here, we give an answer to these questions and, in addition, we provide new insights about compact games, by characterizing the computational complexity of the three solution concepts over relevant generalizations or specializations of the basic settings.

In order to clearly illustrate our contribution, a brief introduction to the classes of games considered throughout the paper is provided next.

1.2. Compactly specified games: Graph games and marginal contribution nets

Graph games. The setting of *graph games* is precisely the one analyzed by Deng and Papadimitriou [26]. In graph games, worths for coalitions over a set N of players are defined based on a weighted undirected graph $G = \langle (N, E), w\rangle$, whose nodes in N correspond to the players, and where the list w encodes the edge weighting function, so that $w(e) \in \mathbb{R}$ is the weight associated with the edge $e \in E$. Then, the worth of an arbitrary coalition $S \subseteq N$ is defined as the sum of the weights associated with the edges contained in S, i.e., as the value $v(S) = \sum_{e\in E | e\subseteq S} w(e)$.

Example 1.1. Consider the graph game (induced by the weighted graph) depicted on the left of Fig. 1 over the players in $\{a, b, c, d\}$. The subgraphs induced over the set of nodes $\{a, b, c\}$ and $\{a, b, d\}$ are reported on the right of the figure—for the moment, please ignore the dashed lines. It is easily seen that the coalition $\{a, b, c\}$ gets a worth $v(\{a, b, c\}) = 2 + 2 + 1 = 5$, while the coalition $\{a, b, d\}$ gets a worth $v(\{a, b, d\}) = 3 + 2 - 1 = 4$. Moreover, note that the graph encodes 2^4 coalition worths, via 5 weights only. In this representation, $O(n^2)$ weights succinctly encode the 2^n coalition worths, where n is the number of players.

Within the setting of graph games, Deng and Papadimitriou [26] characterized the intrinsic complexity of various tasks, mainly focusing on problems related to the core. For instance, they showed that checking whether the core is non-empty and whether a payoff vector belongs to the core are co-**NP**-complete problems. Moreover, they provided a polynomial-time computable closed-form characterization for the Shapley value, and showed that this value coincides with the (pre)nucleolus. Finally, they completed the picture of the complexity issues arising with graph games by showing the **NP**-hardness of deciding whether a payoff vector belongs to the bargaining set and by *conjecturing* the following two results for graph games:

(**C1**) *Deciding whether a payoff vector belongs to the kernel is* **NP**-*hard*; *and*
(**C2**) *Deciding whether a payoff vector belongs to the bargaining set is* $\mathbf{\Pi_2^P}$-*complete.*

Marginal contribution networks. A class of compact games that received considerable attention in the last few years is that of (games encoded via) *marginal contribution networks*, proposed by Ieong and Shoham [44].

A marginal contribution network (short: MC-net) M consists in a set $\{r_1, \ldots, r_n\}$ of rules involving a number of Boolean variables that represent (and thus will be called) players. For each $1 \leqslant i \leqslant n$, the rule r_i has the form $\{pattern_i\} \rightarrow value_i$, where $pattern_i$ is a conjunction that may include both positive and negative literals, and $value_i$ is the additive contribution associated with this pattern. A rule is said to apply to a coalition S if all the players whose literals occur positively in the pattern belong to S, and all the players whose literals occur negatively in the pattern do not belong to S. The value $v(S)$ for a coalition S in the coalition game induced by M is given by the sum of the values of all rules that apply to S. If no rule applies, then the value for the coalition is set to zero, by default.

Example 1.2. Consider the marginal contribution network consisting of the following three rules $\{a \wedge b\} \rightarrow 5$, $\{b\} \rightarrow 2$, and $\{a \wedge \neg b\} \rightarrow 3$, over the players in $\{a, b\}$. This network defines a coalitional game such that: $v(\{a\}) = 3$ (the third rule applies),

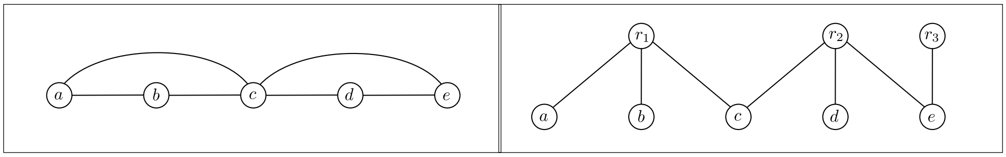

Fig. 2. The MC-net in Example 1.3: Its agent graph (left), and its incidence graph (right).

$v(\{b\}) = 2$ (the second rule applies), and $v(\{a,b\}) = 5 + 2 = 7$ (both the first and the second rules apply, but not the third one).

Ieong and Shoham [44] suggest to graphically represent the "structure" of player interactions in MC-nets via their associated *agent graphs*. The agent graph associated with a marginal contribution net M is the undirected graph $\mathtt{AG}(M)$ whose nodes are the players of the game, and where, for each rule $r_i : \{pattern_i\} \rightarrow value_i$, every pair of players (nodes) occurring in $pattern_i$ are connected by an edge in $\mathtt{AG}(M)$. The clique in the agent graph induced by these players weighs $value_i$.

Example 1.3. The agent graph associated with the marginal contribution network of Example 1.2 is defined over the nodes a and b, with an edge connecting them.

For a slightly more involved example, consider instead the marginal contribution network M consisting of the rules $r_1 : \{a \wedge b \wedge c\} \rightarrow 2$, $r_2 : \{c \wedge d \wedge e\} \rightarrow 1$, $r_3 : \{e\} \rightarrow -1$, over the players in $\{a,b,c,d,e\}$. Then, the agent graph $\mathtt{AG}(M)$ associated with M is the one depicted on the left of Fig. 2.

Note that, for any graph game $G = \langle (N,E), w\rangle$, there is an equivalent MC-net representation having the same size, and whose "structure" is preserved in that its associated agent graph coincides with G. Indeed, given any game specified via the graph $G = \langle (N,E), w\rangle$, we can just create the rule $\{i \wedge j\} \rightarrow w(e)$, for each edge $e = \{i,j\} \in E$ (cf. [44]). However, the converse is not true, since MC-nets allow to express any arbitrary coalitional game, while graph games are not fully expressive. For instance, graph games cannot model a scenario where a group of agents S has value 1 if and only if $|S| > |N|/2$ (see [44] for details on the expressiveness of the frameworks).

In the light of the above observations, hardness results for graph games immediately hold over marginal contribution networks. For instance, checking whether a payoff vector is in the core and checking whether the core is non-empty are co-**NP**-hard problems on MC-nets. Symmetrically, membership results for MC-nets also hold for graph games, even when they are established over some "structurally" restricted classes of games (because the structure is preserved). For instance, for marginal contribution networks associated with *acyclic* agent graphs or, more generally, with agent graphs having *bounded treewidth* [70], Ieong and Shoham [44] showed that deciding whether a payoff vector is in the core and deciding the non-emptiness of the core are feasible in polynomial time. Therefore, these feasibility results immediately apply to acyclic graph games (and, more generally, to graph games having bounded treewidth).

Moreover, Ieong and Shoham [44] proved that checking whether a payoff vector is in the core of a game encoded via marginal contribution networks is in co-**NP** (as for the class of graph games). However, they left as an intriguing open problem the following (which, in fact, does not follow from the corresponding result for graph games):

(**O1**) Is the problem of deciding core non-emptiness over MC-nets in co-**NP**?

1.3. Contributions

In this paper, we analyze the computational complexity of the core, the kernel, and the bargaining set over graph games and marginal contribution networks. In particular, we show that conjectures (**C1**) and (**C2**) by Deng and Papadimitriou [26] are correct, and we provide a positive answer to question (**O1**) by Ieong and Shoham [44]. In detail, as our main technical contributions, we show that:

- For graph games (and for marginal contribution networks), deciding whether a payoff vector is in the kernel is **NP**-hard, and actually $\boldsymbol{\Delta_2^P}$-complete;
- For graph games (and for marginal contribution networks), deciding whether a payoff vector is in the bargaining set is $\boldsymbol{\Pi_2^P}$-complete; and,
- For marginal contribution networks, deciding whether the core is non-empty is in co-**NP**.

These main achievements are summarized in Table 1. Note that the complexity of the problems of deciding the membership of a payoff vector in the core, in the kernel, and in the bargaining set (In-Core, In-Kernel, and In-BargainingSet, respectively) are now completely characterized for graph games and marginal contribution networks. Moreover, observe that the non-emptiness problem makes sense only for the core, given that the kernel and the bargaining set are always non-empty over coalitional games, unless there is no imputation at all (see, e.g., [62]).

Table 1
Summary of results. Hardness results on membership problems hold even if the payoff vector is actually an imputation.

Problem	Graph games	MC-nets
In-Core	co-**NP**-complete [26]	co-**NP**-complete [44]
In-Kernel	Δ_2^P-complete	Δ_2^P-complete
In-BargainingSet	Π_2^P-complete	Π_2^P-complete
Core-NonEmptiness	co-**NP**-complete [26]	co-**NP**-complete*

*Hardness shown by Ieong and Shoham [44].

Table 2
Summary of results for **P**, $\mathbf{wNP}^{opt}$, and $\mathbf{NP}^{opt}$-representations.

Problem	**P** and $\mathbf{wNP}^{opt}$-representations	$\mathbf{NP}^{opt}$-representations
In-Core	co-**NP**-complete	$\mathbf{D}^P$-complete
In-Kernel	Δ_2^P-complete	Δ_2^P-complete
In-BargainingSet	Π_2^P-complete	Π_2^P-complete
Core-NonEmptiness	co-**NP**-complete	Δ_2^P-complete

These achievements are only a part of the overall research reported in the paper. Indeed, we go beyond and study the computational issues arising in relevant *generalizations* and *specializations* of the settings of graph games and MC-nets.

Generalizations. We consider an abstraction of compact coalitional games that is based on assuming that the worth function is provided as an *oracle* operating over some given structure encoding the game. In particular, based on the computational properties of the oracle, we consider two different representation schemes:

(1) **P**-representations, where oracles encode functions computable in (*deterministic*) *polynomial time* w.r.t. the size of the game. Graph games and MC-nets are two notable examples of games that can be encoded via **P**-representations.
(2) $\mathbf{NP}^{opt}$-representations, where oracles are powerful enough to encode **NP**-hard *optimization* problems. Combinatorial optimization games [25] are the most notable class of games that can be encoded via $\mathbf{NP}^{opt}$-representations.

Moreover, we also consider the following restriction of the latter setting:

(3) *weak* $\mathbf{NP}^{opt}$-representations (short: $\mathbf{wNP}^{opt}$-representations), where the worth $v(N)$ associated with the whole set N of players is explicitly provided as an input in the game specification, or can be computed easily (read: polynomial time) from it, while there are no further requirements on $v(S)$, for any $S \subset N$. This setting, which comprises for instance the setting studied by Conitzer and Sandholm [18] of games with synergies among coalitions where the total worth is given as an input, naturally arises in those scenarios where players divide some given worth that is known beforehand to all of them.

A summary of the complexity results for the above three representations is reported in Table 2. It is easily seen that hardness results reported in Table 1 provide lower bounds for the complexity of the problems with **P**, $\mathbf{wNP}^{opt}$, and $\mathbf{NP}^{opt}$-representations. Notably and surprisingly, however, one may observe that nothing has to be paid for dealing not only with arbitrary **P**-representations, but even with the more powerful class of $\mathbf{wNP}^{opt}$-representations. This means, in particular, that as long as the worth $v(N)$ is easily computable or given in input, the cost of computing the worth of any coalition $S \subset N$ (which generally amounts to solving an **NP**-hard optimization problem) has no impact on the intrinsic complexity of the three solution concepts. An increase in complexity is instead exhibited for $\mathbf{NP}^{opt}$-representations, but just with the core.

In particular, note that membership results in Table 2 for $\mathbf{NP}^{opt}$-representations immediately entail membership results for many problems over well-studied classes of games for which no upper bound was known. For instance, this is the case for core-related problems over *traveling salesman games* [31], whose precise complexity is listed as an open problem by Okamoto [61].

Specializations. Finally, in the last part of the paper, by following the perspective adopted by Ieong and Shoham [44], we analyze the complexity of the solution concepts in “structurally restricted” marginal contribution networks. Our starting point of the investigation is the observation that the agent-graph encoding often obscures the actual intricacy of the game. For instance, classes of marginal nets with only one rule involving all the players of the game do not fall in the tractable classes analyzed by Ieong and Shoham [44], though all solution concepts can trivially be computed over them.

Table 3
Tractability over classes of games having bounded treewidth (btw).

Problem\btw classes	Graph games	MC-nets (agent graph)	MC-nets (incidence graph)
IN-CORE	**P** [44]	**P** [44]	**P** (small values)
IN-KERNEL	**P** (small values)	**P** (small values)	**P** (small values)
IN-BARGAININGSET	open	open	open
CORE-NONEMPTINESS	**P** [44]	**P** [44]	**P** (small values)

Motivated by this observation, our first contribution is to propose a novel encoding based on the *incidence graph* of a marginal contribution network,[2] and show that this encoding is always preferable to the agent graph one. Formally, the incidence graph $\mathtt{IG}(M)$ of a marginal contribution network M is a bipartite graph, whose set of nodes consists of the players and rules in M, and where there is an edge between a player p and a rule r if and only if p occurs in the pattern of r. As an example, Fig. 2 reports on the right the incidence graph associated with the MC-net of Example 1.3—note that $\mathtt{IG}(M)$ is acyclic, while $\mathtt{AG}(M)$ contains cycles. More generally, we shall observe that there are MC-nets whose incidence encodings are acyclic while the corresponding agent graphs have unbounded treewidth, and that if an agent graph has bounded treewidth, then the corresponding incidence representation has bounded treewidth too.

Based on this encoding, we then embark on a systematic study of the complexity of the solution concepts over classes of marginal contribution nets whose incidence graphs (and, hence, agent graphs) have treewidth bounded by some fixed constant. In particular, we establish several tractability results by showing that such concepts can be expressed in terms of optimization problems over *Monadic Second Order Logic* (*MSO*) formulae, and by subsequently applying Courcelle's Theorem [19] and its generalization to optimization problems due to Arnborg et al. [2].

A summary of our results is reported in Table 3. Note that, in order to get tractable classes via the logic-based approach of Arnborg et al. [2], it turns out that the values occurring in the network must be "small", that is, polynomially bounded in the size of the game or, equivalently, given in unary. Thus, tractability islands identified in this paper for the core (problems IN-CORE and CORE-NONEMPTINESS) are eventually incomparable to those singled out by Ieong and Shoham [44]. In particular, our islands lead to pseudo-polynomial algorithms when values are not small, and thus the question is open about whether (full) polynomial-time algorithms exist for MC-nets whose incidence graphs have bounded treewidth.

Note also that the logic-based approach allowed us to derive tractability results for the kernel of marginal contribution networks under the incidence graph encoding. Of course, such results immediately apply to the agent graph encoding—just recall from above that incidence encodings have been proved to be more general than agent graphs—and, in turn, to graph games—just recall from the previous section that any graph game can be encoded as a marginal contribution network having the same (agent graph) structure. Notably, these results concerning the kernel are the first structural tractability results exhibited in the literature for this solution concept.

Finally, we point out that a characterization of the bargaining set in terms of optimization problems over Monadic Second Order Logic is missing in the paper. In fact, we leave as an open problem whether IN-BARGAININGSET remains intractable over acyclic games and when, moreover, small values are considered.

1.4. Organization

The rest of the paper is organized as follows. Preliminaries on computational complexity are reported in Section 2. An abstract framework for compact games is discussed in detail in Section 3. The complexity of the kernel, bargaining set, and core is analyzed in Sections 4, 5, and 6, respectively. Structural tractability results are discussed in Section 7 and, eventually, a few final remarks and discussions on some open research issues are reported in Section 8.

2. Preliminaries on computational complexity

In this section we recall some basic definitions about complexity theory, and refer the reader to the work by Johnson [46] for additional details.

2.1. The complexity of decision problems

Decision problems are maps from strings (encoding the input instance over a fixed alphabet, e.g., the binary alphabet $\{0,1\}$) to the set {"*yes*", "*no*"}. The class **P** is the set of decision problems that can be solved by a deterministic Turing machine in polynomial time with respect to the input size, that is, with respect to the length of the string that encodes the input instance. For a given input x, its size is usually denoted by $\|x\|$.

Throughout the paper, we shall often refer to computations carried out by *non-deterministic* Turing machines. We recall that these are Turing machines that, at some points of the computation, may not have one single next action to perform,

[2] This is inspired by the incidence graph encoding of constraint satisfaction problems (see, e.g., [41]).

but a *choice* between several possible next actions. A non-deterministic Turing machine answers a decision problem if, on any input x, (i) there is at least one sequence of choices leading to halt in an accepting state if x is a "yes" instance (such a sequence is called accepting computation path); and (ii) all possible sequences of choices lead to a rejecting state if x is a "no" instance. The class of decision problems that can be solved by non-deterministic Turing machines in polynomial time is denoted by **NP**.

Problems in **NP** enjoy a remarkable property: any "yes" instance x has a *certificate* for it being a "yes" instance, which has polynomial length and which can be checked in polynomial time (in the size $\|x\|$). As an example, deciding whether a Boolean formula Φ over the variables $X_1, \ldots, X_n$ is satisfiable, i.e., deciding whether there exists some truth assignment to these variables making Φ true, is a well-known problem in **NP**; in fact, any satisfying truth assignment for Φ is obviously a certificate that Φ is a "yes" instance, i.e., that Φ is satisfiable.

The class of problems whose complementary problems are in **NP** is denoted by co-**NP**. Of course, the class **P** is contained in both **NP** and co-**NP**. The class $\mathbf{D^P}$ is the class of problems that can be defined as a conjunction of two problems, one from **NP** and one from co-**NP**, while its complement co-$\mathbf{D^P}$ is the class of problems that can be defined as the disjunction of two problems, one from **NP** and one from co-**NP**. Thus, $\mathbf{D^P}$ and co-$\mathbf{D^P}$ are supersets of both **NP** and co-**NP**.

Throughout the paper, we shall also refer to a type of computation called computation with *oracles*. Intuitively, oracles are subroutines which are supposed to have unit cost.

The classes $\mathbf{\Sigma_k^P}$, $\mathbf{\Pi_k^P}$, and $\mathbf{\Delta_k^P}$, forming the *polynomial hierarchy*, are defined as follows: $\mathbf{\Sigma_0^P} = \mathbf{\Pi_0^P} = \mathbf{P}$ and for all $k \geqslant 1$, $\mathbf{\Sigma_k^P} = \mathbf{NP}^{\Sigma_{k-1}^P}$, $\mathbf{\Delta_k^P} = \mathbf{P}^{\Sigma_{k-1}^P}$, and $\mathbf{\Pi_k^P} = \text{co-}\mathbf{\Sigma_k^P}$ where co-$\mathbf{\Sigma_k^P}$ denotes the class of problems whose complementary problem is solvable in $\mathbf{\Sigma_k^P}$. Here, $\mathbf{\Sigma_k^P}$ (resp., $\mathbf{\Delta_k^P}$) models computability by a non-deterministic (resp., deterministic) polynomial-time Turing machine that may use an oracle in $\mathbf{\Sigma_{k-1}^P}$. Note that $\mathbf{\Sigma_1^P}$ coincides with **NP**, and that $\mathbf{\Pi_1^P}$ coincides with co-**NP**.

We conclude by recalling the notion of reducibility among decision problems. A decision problem A_1 is *polynomially reducible* to a decision problem A_2, denoted by $A_1 \leqslant_p A_2$, if there is a polynomial-time computable function h (called reduction) such that, for every x, $h(x)$ is defined and x is a "yes" instance of A_1 if and only if $h(x)$ is a "yes" instance of A_2. A decision problem A is *hard* for a class $\mathcal{C}$ of the polynomial hierarchy (at any level $k \geqslant 1$, i.e., beyond **P**) if every problem in $\mathcal{C}$ is polynomially reducible to A; if A is hard for $\mathcal{C}$ and belongs to $\mathcal{C}$, then A is said to be *complete* for $\mathcal{C}$. Thus, problems that are complete for $\mathcal{C}$ are the most difficult problems in $\mathcal{C}$. In particular, they cannot belong to some lower class in the hierarchy unless some collapse occurs.

Characterizing the precise computational complexity of a problem means understanding its sources of complexity. This information may allow us to identify tractable instances by limiting some source of intractability, and to design and analyze algorithms. For instance, note that any $\mathbf{\Sigma_2^P}$-complete problem exhibits two orthogonal sources of intractability. To give an idea of some practical consequence of this fact, assume we would like to solve such a problem on a standard (deterministic) machine. Then, the theory tells us that we cannot design any *flat-backtracking* algorithm for our problem, unless $\mathbf{P} = \mathbf{NP}$. Indeed, any algorithm with a search-space tree having a polynomial number of levels (and such that moving along the tree edges does not take exponential time) should solve a nested co-**NP**-hard problem to check whether a leaf node is a solution or not. That is, checking leaf-feasibility is an intractable problem as well, and hence it requires a nested call to a further backtracking procedure (or to another kind of procedure with an exponential-time worst case).

2.2. Complexity classes of functions

Often, we are interested in *search* problems where, for any given instance, a (non-Boolean) solution must be computed. The complexity classes of functions allow us to distinguish such problems according to their intrinsic difficulty, which is particularly relevant when their associated decision problems belong to the same complexity class.

Let a finite *alphabet* Σ with at least two elements be given. A (*partial*) *multi-valued function* $f : \Sigma^* \mapsto \Sigma^*$ associates no, one or several outcomes (*results*) with each input string. Let $f(x)$ stand for the set of possible results of f on the input string x; thus, we write $y \in f(x)$ if y is a value of f on the input string x. Define $dom(f) = \{x \mid \exists y(y \in f(x))\}$ and $graph(f) = \{\langle x, y\rangle \mid x \in dom(f),\ y \in f(x)\}$. If $x \notin dom(f)$, we say that f is undefined at x. The function f is *total* if $dom(f) = \Sigma^*$. The function f is single-valued if $|f(x)| = 1$, for each $x \in dom(f)$.

We say that a multi-valued function f is *polynomially balanced* if, for each x, the size of each result in $f(x)$ is polynomially bounded in the size of x. Then, the class **NPMV** is defined as the set of all multi-valued functions f such that (i) f is polynomially balanced and (ii) $graph(f)$ is in **NP**.

By analogy, the class $\mathbf{NPMV}_g$ (also known as **FNP** [64]) is defined as the class of all polynomially-balanced multi-valued functions f for which $graph(f)$ is in **P**. If we deal with (partial) *single-valued* functions, we get the corresponding classes **NPSV** and $\mathbf{NPSV}_g$, respectively [74].

A *transducer* is a (possibly, non-deterministic) Turing machine T on the alphabet Σ with a read-only input tape, a read–write work tape, and a write-only output tape. For any string $x \in \Sigma^*$, we say that T accepts x if T has an accepting computation-path on x. For each $x \in \Sigma^*$ accepted by T, we denote by $T(x)$ the set of all strings that are written by T on the output tape in its accepting computation-paths on input string x. Thus, every transducer is associated with some multi-valued function f (we say that T *computes* f) such that, for each $x \in \Sigma^*$, $f(x) = T(x)$ if x is accepted by T; otherwise, f is undefined at x (i.e., $x \notin dom(f)$).

The class **FP** consists of all functions that are computed by deterministic Turing transducers in polynomial time. The class **NPMV** can be equivalently characterized as the class of all multi-valued functions that can be computed by non-deterministic transducers in polynomial time.

It is worthwhile noting the difference between the two classes $\mathbf{NPMV}_g$ and **NPMV**. In fact, **NPMV** contains more complex functions than $\mathbf{NPMV}_g$ (assuming $\mathbf{P} \neq \mathbf{NP}$). For instance, consider the problem of computing the partial multi-valued function f_H that, given a graph G, outputs the Hamiltonian cycles of G (if any). This function is in $\mathbf{NPMV}_g$ since $graph(f_H)$ is polynomially balanced and decidable in deterministic polynomial time (for any pair $\langle G, C\rangle$, just check whether C is a Hamiltonian cycle of G). Let us consider now the weighted version of this problem, where the input graph G is edge-weighted, and the function values are the weights of the Hamiltonian cycles of G (if any). Then, this partial multi-valued function, say f_{WH}, belongs to **NPMV** but does not belong to $\mathbf{NPMV}_g$ (unless $\mathbf{P} = \mathbf{NP}$). Indeed, deciding whether a given pair $\langle G, w\rangle$ (graph, weight) belongs to $graph(f_{WH})$ is clearly **NP**-complete, as one needs to exhibit some Hamiltonian cycle having weight w in order to recognize that w is a correct function value.

To conclude, we notice that, like Turing machines, transducers can use oracles while computing. In particular, in this paper we will use the class $\mathbf{F\Delta_2^P}$, which contains those functions computable in polynomial time via a deterministic Turing transducer using an **NP** oracle.

3. A formal framework for compact representations

Graph games and marginal contribution networks are two prominent examples of compact representations for coalitional games, whose worth functions are defined in terms of some suitable (combinatorial) structure, instead of listing the worths of all coalitions. In fact, several other compact representations have been proposed in the literature, as we illustrated in Section 1. In this section, we shall provide a unifying framework for them, which will be the basis for our subsequent complexity analysis.

A compact representation $\mathcal{R}$ defines suitable encodings for a class of coalitional games, denoted by $\mathcal{C}(\mathcal{R})$. Formally, any representation $\mathcal{R}$ defines an encoding function $\xi^{\mathcal{R}}$ and a worth function $v^{\mathcal{R}}$ such that, for any coalitional game $\mathcal{G} \in \mathcal{C}(\mathcal{R})$, $\xi^{\mathcal{R}}(\mathcal{G})$ is the encoding of the game $\mathcal{G}$, and $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$ is the worth associated with the coalition S according to $\mathcal{G}$. Note that $v^{\mathcal{R}}$ is *total* and *single-valued*.

As an example, let us denote by $\mathcal{GG}$ the graph game representation. Then, any coalitional game $\mathcal{G} \in \mathcal{C}(\mathcal{GG})$ is encoded as a weighted graph $\xi^{\mathcal{GG}}(\mathcal{G})$, and the worth function $v^{\mathcal{GG}}(\xi^{\mathcal{GG}}(\mathcal{G}), S)$ is computed for every coalition S by taking the sum of the weights of all edges of $\xi^{\mathcal{GG}}(\mathcal{G})$ included in S. Similarly, for the case of the marginal contribution nets compact-representation $\mathcal{MCN}$, any game $\mathcal{G}$ is encoded by a set of rules $\xi^{\mathcal{MCN}}(\mathcal{G})$, and the function $v^{\mathcal{MCN}}(\xi^{\mathcal{MCN}}(\mathcal{G}), S)$ computes the worth of S as the sum of the values of those rules in $\xi^{\mathcal{MCN}}(\mathcal{G})$ that apply to coalition S.

Whenever a compact representation $\mathcal{R}$ is understood, we can just write $\mathcal{G}$ instead of $\xi^{\mathcal{R}}(\mathcal{G})$, and $v(S)$ instead of $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$. While doing so, we are identifying the game with its actual representation (by furthermore ignoring the actual procedure underlying the computation of the worth values). In fact, this is very popular in the literature. For instance, the naming "marginal contribution network" usually denotes a coalitional game rather than—more formally—its encoding. Similarly, the weighted graph underlying the definition of a graph game is usually identified with the game itself.

3.1. Computational-based classification of games representations

The framework we are going to illustrate is based on classifying any compact representation $\mathcal{R}$ on the basis of the computational properties of the associated worth function $v^{\mathcal{R}}$. In particular, two basic kinds of representation will be considered, corresponding to polynomial-time functions and to functions powerful enough to encode **NP**-hard optimization problems. A weak variant of this latter setting will also be discussed.

3.1.1. P-representations

Let us start with the computationally cheapest representation.

Definition 3.1. Let $\mathcal{R}$ be a compact representation for coalitional games. We say that $\mathcal{R}$ is a *polynomial-time compact representation* (short: **P**-representation) if $v^{\mathcal{R}}$ belongs to **FP**, i.e., if it is polynomial-time computable by a deterministic transducer.

Note that both $\mathcal{GG}$ and $\mathcal{MCN}$ are based on worth functions that are efficiently computable. Other polynomial-time compact representations that received considerable attention in the literature are reported below:

Minimum cost spanning tree games [57]: Let N be a set of players and consider the complete graph K_N on the node set $N \cup \{0\}$, where $0 \notin N$ is a distinguished (supply) node. Assume that the edges of K_N have associated nonnegative weights. Then, the function $v(S)$ returns, for each coalition $S \subseteq N$, the cost of a minimal spanning tree on the node set $S \cup \{0\}$. This cost, which can be computed in polynomial time (see, e.g., [20]), reflects the minimum cost of guaranteeing the connectivity of players in S with the supply node 0.

Flow games [48]: A flow network is given, that is, a directed graph $D = (N, E)$ with a source node $s \in N$, a sink node $t \in N$, and where each edge in E is associated with a weight denoting its capacity. Each player controls one of the directed edges in E, and the worth $v(S)$ of each coalition $S \subseteq E$ is the value of a maximum flow from s to t in the subnetwork induced by S. Note that this is a **P**-representation, since a maximum flow can be computed in polynomial time (see, e.g., [20]).

Linear production games (Owen's model) [63]: There are m resources and p final goods, such that the price of good j is denoted by v_j. A set N of players is given, and each player $i \in N$ is endowed with b_k^i units of resource k. Linear production technologies can be used to transform resources into goods. This is formalized via a matrix $A \in \mathbb{R}^{m \times p}$ such that $A_{k,j}$ is the units of resource of type k needed to produce one unit of good j. For each coalition $S \subseteq N$, its members share their resources and, thus, the value $v(S)$ is defined as the maximum market value of the outputs that can be jointly produced by the members of S, that is:

$$v(S) = \max \sum_j v_j x_j \quad \text{such that} \quad \begin{cases} \sum_j A_{k,j} x_j \leqslant \sum_{i \in S} b_k^i & \text{for each resource } k \\ x_j \geqslant 0 & \text{for each good } j \end{cases}$$

Note that $v(S)$ can be computed in polynomial time, since it is the solution of a linear program [65] (we stress here that values $A_{k,j}$'s in the linear program are constants, since those are values explicitly expressed in the input game representation).

Games in multi-issue domains [17]: In this representation scheme, a set N of players is concerned with a set T of independent issues that each coalition can address. In general, each issue $i \in T$ is formalized as a coalitional game having its own worth function $v_i : 2^N \mapsto \mathbb{R}$, which just concerns a set $C_i \subseteq N$ of players, i.e., $v_i(S_1) = v_i(S_2)$, for each pair of coalitions S_1 and S_2 such that $S_1 \cap C_i = S_2 \cap C_i$. Then, the worth $v(S)$ of a coalition $S \subseteq N$ is the sum of the contributions over all the issues, i.e., $v(S) = \sum_{i \in T} v_i(S)$. By assuming that, for each $i \in T$, $|C_i|$ is bounded by a fixed constant, all the functions can be represented in constant space and $v(S)$ can be computed in linear time.

Multi-attribute games [45]: We are given a set $N = \{1, \ldots, n\}$ of agents and a set $M = \{1, \ldots, m\}$ of attributes, with a matrix $A \in \mathbb{R}^{m \times n}$ where $A_{i,j}$ denotes the value of attribute i for agent j. Moreover, we are given the functions $a : \mathbb{R}^{m \times n} \times 2^N \mapsto \mathbb{R}^m$ and $w : \mathbb{R}^m \mapsto \mathbb{R}$ aggregating the attributes of the agents in any coalition $S \subseteq N$ into a single value per attribute, and grouping such aggregated values, respectively. Then, the worth $v(S)$ is defined as the value $w(a(A, S))$. This representation is capable to describe any coalitional game, and in fact the space needed to list all the associations of values determined by the function a is exponential in general (w.r.t. the number of players). Indeed, just notice that a is also a function of the given coalition. However, in some relevant cases (for instance, whenever a just sums the contributions of all the agents in the coalition), exponentially more succinct encodings can be used, yet preserving its polynomial-time computability.

Weighted voting games [58]: A set N of players is given, together with a non-negative value $q \in \mathbb{R}$. For each player $i \in N$, a non-negative weight w_i is additionally given, so that the worth $v(S)$ of each coalition $S \subseteq N$ is 1 if $\sum_{i \in S} w_i \geqslant q$; otherwise, $v(S) = 0$ holds.

Read-once (and general) marginal contribution nets [29]: The so-called general marginal contribution networks are generalizations of MC-nets, where the pattern of each rule can be an arbitrary Boolean formula (i.e., including conjunctions, negations, and disjunctions) over the variables representing the players of the game. Read-once nets restrict Boolean formulae to be read-once, in the sense that each variable can appear only once. Despite the increased expressive power w.r.t. MC-nets, the worth of any coalition can be still computed in polynomial time.

Skill games [9]: We are given a set N of agents, each one owning some skills. The agents have to perform certain tasks from a set T, each one requiring a set of skills in order to be accomplished. Thus, each coalition $S \subseteq N$ is capable of carrying out the set $T(S) \subseteq T$ of all those tasks whose required skills are owned by the players in S. A task value (increasing) function $u : 2^T \mapsto \mathbb{R}$ is additionally given, which maps any subset of the tasks a coalition achieves to a real value. The worth $v(S)$ of S is, then, the value returned by u over all the tasks that can be carried out by S, i.e., $v(S) = u(T(S))$. This general representation requires space exponential in $|T|$ (needed for encoding the function u). Succinct representations are obtained whenever $|T|$ is bounded by some fixed constant, or when the task value function u is of special forms (such as counting the total number of tasks accomplished by the coalition).

Matching games [75,49]: We are given an undirected graph $G = (N, E)$. The player set is N and the value of a coalition $S \subseteq N$ is defined as the size of a maximum matching, i.e., of a set of disjoint edges in the subgraph induced by the nodes in S. Note that this is a **P**-representation, since a maximum matching can be computed in polynomial time [34].

Path disruption games [8]: We are given a graph $G = (V, E)$ and two vertices $a, b \in V$, namely the source and the target vertices. The player set is $N = V \setminus \{a, b\}$ and the value of a coalition $S \subseteq N$ equals 1 if removing from G all the vertices in S results in the vertex a to be completely disconnected from b (i.e., there is no path between them). Otherwise the value of the coalition is 0.

(Vertex) Connectivity games [10]: We are given a graph $G = (V, E)$, and a partition of V in the three sets V_p (*primary* vertices), V_b (*backbone* vertices), and V_s (*standard* vertices). The player set is V_s and the value of a coalition $S \subseteq V_s$ equals 1 if every pair of primary vertices is connected in the subgraph of G induced over the vertices in $V_p \cup V_b \cup S$. Otherwise the value of the coalition is 0.

3.1.2. **NP**^{opt}*-representations*

In certain domains where worth functions reflect results of complex algorithmic procedures or combinatorial problems, **P**-representations are not adequate. In these cases, more powerful representations are needed, as illustrated in the following example.

Example 3.2. An undirected graph $G = (N, E)$ is given. The player set is N and the value of a coalition $S \subseteq N$ is defined as the size of a maximum clique in the subgraph induced by the nodes in S. Computing the size of a maximum clique is a well-known **NP**-hard problem (see, e.g., [35]). Therefore, under standard complexity assumptions, such a game cannot be encoded in any **P**-representation.

Note that the above class of games is just an instance of the very general framework of *combinatorial games* discussed by Deng et al. [25]. Roughly, in this framework, each coalition S is associated with a system L_S of (mixed-)integer linear inequalities, which can be built in polynomial time (w.r.t. the size of the game). Let $\Omega(L_S)$ denote the set of all feasible solutions of L_S. Then, the worth of S can be defined as the maximum over all these values, i.e., $v(S) = \max\{y \mid y \in \Omega(S)\}$. For instance, in the games of Example 3.2, given the graph $G = (N, E)$, the worth function can be written as follows:

$$v(S) = \max \sum_{v \in V} x_v \quad \text{such that} \quad \begin{cases} x_v + x_{v'} \leqslant 1 & \forall\{v, v'\} \subseteq N \text{ s.t. } \{v, v'\} \notin E \\ x_v \in \{0, 1\} & \forall v \in N \end{cases}$$

Inspired by this general framework, we propose the following **NP**^{opt}-representation scheme to capture games defined in terms of optimization problems (not necessarily specified via systems of mixed-integer linear inequalities).

Definition 3.3. Let $\mathcal{R}$ be a compact representation for coalitional games. We say that $\mathcal{R}$ is an *optimization non-deterministic polynomial-time compact representation* (short: **NP**^{opt}-representation) if the worth function $v^{\mathcal{R}}$ is such that, given a set S of players in a game $\mathcal{G} \in \mathcal{C}(\mathcal{R})$, $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) = \max\{w \mid w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)\}$, where $f^{\mathcal{R}}$ is an **NPMV** function (i.e., computable by a non-deterministic transducer in polynomial time) called the *feasibility* function (of $\mathcal{R}$).

Note that the role of $f^{\mathcal{R}}$ in the above definition is to provide the values associated with all possible feasible solutions, so that $v^{\mathcal{R}}$ will select the best one among them.[3] For instance, in the case of the games in Example 3.2, the feasibility function returns the set of all sizes of possible cliques: This function is clearly a multi-valued one, and a polynomial-time non-deterministic transducer may compute any value by just guessing (with polynomially many bits) a clique, and then returning its size as the output.

Other representations for coalitional games proposed in the literature and based on some **NP**-hard optimization problem are illustrated next. The reader may want to check that, for each of them, the associated feasibility function is computable in **NPMV**:

Games with synergies among coalitions [18]: A set N of players is given together with a set $W = \{(B_1, v(B_1)), \ldots, (B_n, v(B_n))\}$ of pairs where $B_i \subseteq N$ is a coalition and $v(B_i)$ is its associated worth (explicitly given in input). For any coalition $S \subseteq N$, the worth is then given by the maximum of $\sum_{(B_i, v(B_i)) \in W'} v(B_i)$ over all the subsets W' of W such that coalitions in W' form a partition of S. Conitzer and Sandholm [18] observed that this representation is capable of capturing any possible worth function, if the set W is allowed to contain exponentially may pairs (w.r.t. the number of players). In the relevant cases where W contains just polynomially many pairs (those that introduce "synergy"), the size of the input is exponentially more succinct, but computing the worth of any coalition becomes **NP**-hard.

Traveling salesman games [31]: Let N be a set of players and consider the complete graph K_N on the node set $N \cup \{0\}$, where $0 \notin N$ is a distinguished node. Assume that the edges of K_N have associated nonnegative weights. Then, the function $v(S)$ returns, for each coalition $S \subseteq N$, the cost of the minimum length traveling salesman tour visiting all nodes in $S \cup \{0\}$, i.e., the minimum cost of connecting 0 with all the players in S with a ring structure.

Facility location games [37]: A set N of customers needs a service which can be provided by connecting them to any facility taken from a given set F. Facilities are initially "close". Opening (or building) a facility $i \in F$ costs $f_i \geqslant 0$, and connecting customer j to it costs $c_{i,j} \geqslant 0$. As additional constraints, some facilities might be given that can handle just some fixed number of customers, and certain customers might be precluded to be assigned to certain facilities. For these games, $v(S)$ returns the minimum cost of providing the service to all players in S.

[3] Readers that are familiar with the theory of optimization problems might like to notice that the class of maximization functions in Definition 3.3 has the same computational power as the class **OptP** [52] of those functions that can be computed as the maximum over all the values written on the branches of an **NP** *metric* Turing machine, i.e., of a machine that halts on every branch and writes a number. These values play indeed the role of the feasible values returned by $f^{\mathcal{R}}$. Thus, Definition 3.3 might equivalently be restated by saying that $\mathcal{R}$ is an **NP**^{opt}-representation if $v^{\mathcal{R}}$ is computable in **OptP**. In the paper, we prefer to use the formulation based on a feasibility function, since we feel it is more intelligible to a wider audience, being based on standard Turing machines.

Combinatorial games over graphs [25]: A graph $G = (N, E)$ is given, and the worth $v(S)$, for each coalition $S \subseteq N$, is defined as the solution to some (typically, **NP**-hard) optimization problem over the subgraph induced by S. In the case of the games discussed in Example 3.2, the optimization problem is that of computing a clique of maximum cardinality. Among the other problems considered by [25], we mention here *minimum vertex cover*, *maximum independent sets*, *minimum edge cover*, and *minimum coloring*.

Bin-packing games [32]: A set $\{1, \ldots, n\}$ of items whose nonnegative sizes are $a_1, \ldots, a_n$, and a set $\{1, \ldots, m\}$ bins whose nonnegative sizes are $b_1, \ldots, b_m$ are given. It is assumed that every item fits every bin, that is, $a_i \leqslant b_j$, for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. The set N of players consists of all the items and all the bins. Let $S = A \cup B \subseteq N$ be a coalition, where $\emptyset \subseteq A \subseteq \{1, \ldots, n\}$ is a set of items, and $\emptyset \subseteq B \subseteq \{1, \ldots, m\}$ is a set of bins. The worth function associates with S the maximum total size of items in A that can be packed into bins in B.

Permutation games [77]: There is a set N of n players. Each player $i \in N$ has a job J_i and a machine M_i. Any machine can process at most one job, and $k_{i,j}$ is the cost for executing job J_i on machine M_j. For each coalition S, the worth function is defined by $v(S) = \min \sum_{i \in S} k_{i,\pi(i)}$, where the minimum is taken over all permutations $\pi : S \mapsto S$, and the value $\pi(i)$ indicates on which machine is executed the job of player i; that is, machine $M_{\pi(i)}$ processes job J_i.

In fact, some of the classes of games discussed above include functions defined in terms of minimization problems rather than maximization ones. In these cases, the underlying coalitional games are usually *cost-minimization* games, where players are not willing to maximize the worth they receive (as we presented them in the Introduction), but they are rather interested in minimizing the cost paid when carrying out certain tasks. Cost-minimization coalitional games come in the literature with their own specific definitions for the various solution concepts. However, such definitions are completely interchangeable with the (standard) ones for worth-maximization games, provided that each cost is viewed as the opposite of a worth (i.e., by inverting its algebraic sign). Therefore, as often done in the literature, we can just focus on one of the two settings only, without any loss of generality (thus, all the above classes of games actually fit Definition 3.3). In particular, we shall discuss the solution concepts as they are commonly defined for worth-maximization games and, accordingly, we have assumed optimization problems for $\mathbf{NP}^{opt}$-representations to be maximization ones.

We end the section by noticing that $\mathbf{NP}^{opt}$-representations include those cases where worth functions can be computed on the basis of the answer to some **NP**-complete *decision* problem, rather than as the optimal solution to an optimization problem.

Example 3.4 *(Graph games with legal coalitions).* Consider a slight modification of the setting of graph games in which only certain kinds of coalitions are allowed to form.

Let $G = \langle (N, E), w \rangle$ be a weighted graph. For a coalition $S \subseteq N$ of nodes, let an S-cut be a pair (S_1, S_2) such that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$, and let the weight of (S_1, S_2) be the sum of all the weights of the edges in E having one of their endpoints in S_1 and the other in S_2. For a threshold $\sigma \in \mathbb{R}$, we say that S is *legal* (*w.r.t.* σ) if the weight of the maximum S-cut is greater than σ.

As an example, consider again the graph game depicted on the left of Fig. 1 over the players in $\{a, b, c, d\}$, and let $\sigma = 4$ be the given input threshold. Then, the weight of the maximum cut for the coalition $\{a, b, c\}$ is the one illustrated in the figure, whose weight is 4. Hence $\{a, b, c\}$ is not legal. Instead, the coalition $\{a, b, d\}$ is legal, since the weight of the maximum cut for $\{a, b, d\}$ is 5.

Consider now the following worth function, where M is an arbitrarily large penalty assigned to coalitions that are not legal:

$$v(S) = \begin{cases} \sum_{e \in E | e \subseteq S} w(e) & \text{if } S \text{ is } \textit{legal} \\ -M & \text{otherwise} \end{cases}$$

To reformulate this function as a maximization problem, it suffices to consider the feasibility function f such that $f(S) = \{-M\} \cup \{\sum_{e \in E | e \subseteq S} w(e) \mid S \text{ is legal}\}$. In fact, f is multi-valued, and its maximum value is $\sum_{e \in E | e \subseteq S} w(e)$ if and only if S is legal. Moreover, f is computable by a non-deterministic transducer in polynomial time, given that deciding whether the weight of the maximum cut in a graph is greater than a given threshold is an **NP**-complete problem [35].

In fact, this example provides a general template to define several other instances of $\mathbf{NP}^{opt}$-representations, depending on the specific semantics underlying the notion of "legal" coalition (and, of course, on the worth assigned to such legal coalitions). For instance, in a planning environment, legal coalitions might be those that can achieve the desired goal. In a service composition scenario, legal coalitions might be such that a subset of their members can provide the whole set of tasks whose interplay form the desired service, while satisfying some budget constraints.

3.1.3. $\mathbf{wNP}^{opt}$-representations

As a special case of $\mathbf{NP}^{opt}$-representations, we next consider classes of games where the worth $v(N)$ associated with the whole set N, i.e., the total available worth, is easy to compute or is directly given as a part of the input instance. For instance, this is the case in the setting of games with synergies among coalitions [18], already described above.

Definition 3.5. Let $\mathcal{R}$ be a compact representation for coalitional games. We say that $\mathcal{R}$ is a *weak* $\mathbf{NP}^{opt}$-representation (short: $\mathbf{wNP}^{opt}$-representation) if the worth function $v^{\mathcal{R}}$ is such that, given a game $\mathcal{G} \in \mathcal{C}(\mathcal{R})$,

(1) $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), N)$ is computable in polynomial time, where N is the whole set of players in $\mathcal{G}$; and
(2) for each coalition $S \subset N$, $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) = \max\{w \mid w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)\}$, where the feasibility function $f^{\mathcal{R}}$ is in **NPMV**.

Note that **P**-representations are clearly a special case of **wNP**^{opt}-representations, where $f^{\mathcal{R}}$ is a (single-valued) polynomial-time function.

Remark 3.6. In fact, the notion of **wNP**^{opt}-representation can be further generalized by lifting the cost of computing $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), N)$ from polynomial time to **NPSV** without affecting any of our results (but two membership proofs, which would require a couple of straightforward modifications—an easy exercise for the interested reader). That is, such an extra power in the function defining the total worth comes for free, as far as the computational complexity of the solution concepts analyzed in this paper is concerned. However, for the sake of presentation and readability, we preferred to adopt the more familiar polynomial-time functions.

We just observe here that, by using **NPSV** functions instead, we could model classes of games where computing the total worth involves the solution of rather hard tasks. For example, one can imagine an extension of games with synergies among coalitions where, instead of having the total worth $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), N)$ as an input, this value is defined as the maximum payoff that any agent may obtain by playing some simple stochastic game [16], which can be easily seen to be an **NPSV** function (see, e.g., [79]).

3.2. Expressive power of game representations

In the previous section, we have classified game representations based on the intrinsic complexity of the associated worth functions. Here, we complete the picture by discussing the *expressive power* of game representations.

Let $\mathcal{R}$ be a game representation. If all coalitional games can be represented by $\mathcal{R}$, we say that this representation is *complete*. Of course, when analyzing a representation $\mathcal{R}$ (and comparing it with some other ones), one may firstly want to ask whether $\mathcal{R}$ is complete or not. For instance, $\mathcal{MCN}$ is a complete representation, while $\mathcal{GG}$ is not. However, there are games whose $\mathcal{MCN}$ encodings have size exponential in the number of players [44,29].

For comparison purposes, it is therefore relevant to look for a notion that captures expressiveness and succinctness of the specifications, as we formalize below.

Definition 3.7. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be a pair of game representations. We say that $\mathcal{R}_2$ is *at least as expressive* (*and succinct*) *as* $\mathcal{R}_1$, denoted by $\mathcal{R}_1 \precsim_e \mathcal{R}_2$, if there exists a function f in **FP** that translates a game $\xi^{\mathcal{R}_1}(\mathcal{G})$ represented in $\mathcal{R}_1$ into an equivalent game $\xi^{\mathcal{R}_2}(\mathcal{G})$ represented in $\mathcal{R}_2$, that is, into a game with the same players and the same worth function as the former one. More precisely, we require that $\xi^{\mathcal{R}_2}(\mathcal{G}) = f(\xi^{\mathcal{R}_1}(\mathcal{G}))$ and $v^{\mathcal{R}_1}(\xi^{\mathcal{R}_1}(\mathcal{G}), S) = v^{\mathcal{R}_2}(\xi^{\mathcal{R}_2}(\mathcal{G}), S)$, for each coalition of players S in the game $\mathcal{G}$.

Note that if $\mathcal{R}_1 \precsim_e \mathcal{R}_2$ holds, then any hardness result for the complexity of reasoning problems over $\mathcal{R}_1$ immediately apply to $\mathcal{R}_2$, as well as any membership result for problems over $\mathcal{R}_2$ immediately apply to $\mathcal{R}_1$. Thus, expressiveness relations can be used to derive further complexity results after the complexity of some representation has been characterized.[4]

It follows that it is convenient to prove hardness results for the least expressive representations and membership results for the most expressive ones. In particular, as far as the latter kind of representations is concerned, we consider the very general **NP**^{opt} and **wNP**^{opt} representations; for the former kind of representations, we focus on graph games because they are both interesting in their own and less expressive than many other representations. For instance, it can easily be shown that $\mathcal{GG} \precsim_e \mathcal{MCN}$, and thus hardness results for graph games immediately apply to marginal contribution nets (cf. [44]). In fact, we explicitly discuss results on marginal contribution nets, as they are a minimal natural extension of graph games that preserves their nice features (in terms of easiness of representation), but it is able to represent any coalitional game.

Of course, if one is interested in hardness results for some representation that is incomparable with graph games (w.r.t. the above notion of expressiveness), then our results cannot be directly applied, and *ad hoc* reductions should be devised.

4. The complexity of the kernel

The *kernel* is a solution concept introduced by Davis and Maschler [22]. To review its definition, we need to state some preliminary concepts and notations, which will be extensively used throughout the paper.

For any coalition $S \subseteq N$, let $|S|$ denote the cardinality of S, and let $\mathbb{R}^S$ be the $|S|$-dimensional real coordinate space, whose coordinates are labeled by the members of S; in particular, given a *payoff vector* $x \in \mathbb{R}^S$, x_i denotes the component associated with the player $i \in S$. A vector $x \in \mathbb{R}^S$ is called an S-feasible vector if $\sum_{i \in S} x_i = v(S)$. The value $\sum_{i \in S} x_i$ will be simply denoted by $x(S)$ in the following.

[4] We are implicitly assuming to talk about complexity results beyond **P** that are the subject of this paper, because for tractable classes inside **P** we should use translation functions below **FP**.

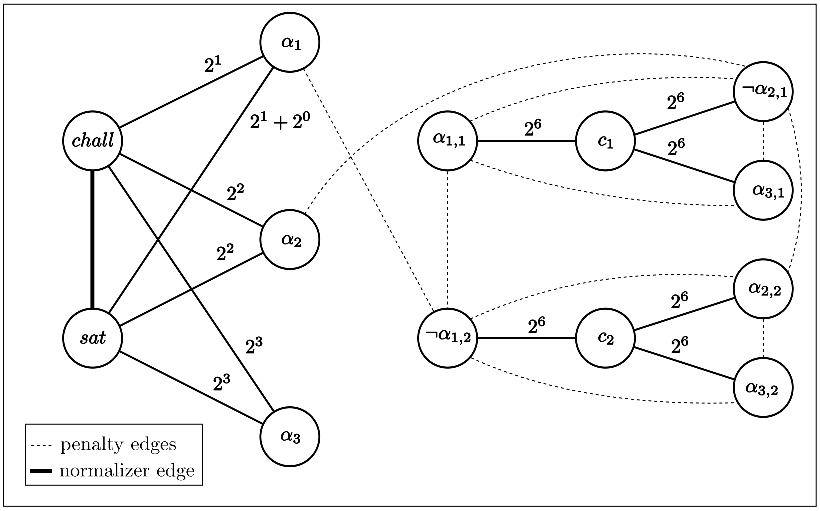

Fig. 3. The game $\boldsymbol{K}(\widehat{\phi})$, where $\widehat{\phi} = (\alpha_1 \vee \neg\alpha_2 \vee \alpha_3) \wedge (\neg\alpha_1 \vee \alpha_2 \vee \alpha_3)$.

For any pair of players i and j of a coalitional game $\mathcal{G} = \langle N, v\rangle$, we denote by $\mathcal{I}_{i,j}$ the set of all coalitions containing player i but not player j. The *excess* $e(S,x) = v(S) - x(S)$ of the generic coalition S at the imputation $x \in X(\mathcal{G})$ is a measure of the dissatisfaction of S at x. Define the *surplus* $s_{i,j}(x)$ of player i against player j at an imputation x as the value $s_{i,j}(x) = \max_{S\in\mathcal{I}_{i,j}} e(S,x) = \max_{S\in\mathcal{I}_{i,j}}(v(S) - x(S))$.

Intuitively, the surplus of player i against j at x is the highest payoff that player i can gain (or the minimal amount i can lose, if it is a negative value) without the cooperation of j, by forming coalitions with other players that are satisfied at x; thus, $s_{i,j}(x)$ is the weight of a possible threat of i against j. In particular, player i has more "bargaining power" than j at x if $s_{i,j}(x) > s_{j,i}(x)$; however, player j is immune to such threat whenever $x_j = v(\{j\})$, since in this case j can obtain $v(\{j\})$ even by operating on her own. We say that player i outweighs player j at x if $s_{i,j}(x) > s_{j,i}(x)$ and $x_j > v(\{j\})$. The kernel is then the set of all imputations where no player outweighs another one:

Definition 4.1. The *kernel* $\mathscr{K}(\mathcal{G})$ of a TU game $\mathcal{G} = \langle N, v\rangle$ is the set:

$$\mathscr{K}(\mathcal{G}) = \bigl\{x \in X(\mathcal{G}) \bigm| s_{i,j}(x) > s_{j,i}(x) \Rightarrow x_j = v\bigl(\{j\}\bigr),\ \forall i,j \in N,\ i \neq j\bigr\}.$$

Example 4.2. Let $\mathcal{G} = \langle N, v\rangle$ be a TU game with $N = \{a,b,c\}$, $v(\{a\}) = v(\{b\}) = v(\{c\}) = 0$, $v(\{a,b\}) = 20$, $v(\{a,c\}) = 30$, $v(\{b,c\}) = 40$, and $v(\{a,b,c\}) = 42$.

It is easily verified that the imputation x such that $x_a = 4$, $x_b = 14$, and $x_c = 24$ is in the kernel of $\mathcal{G}$. Indeed, we note first that every player in N receives in x a payoff strictly greater than what she is able to obtain by acting on her own. For this reason, in order for x to belong to $\mathscr{K}(\mathcal{G})$ it must be the case that $s_{i,j}(x) \leqslant s_{j,i}(x)$, for all distinct players i and j. By the definition of the worth function, the maximum excess that a coalition S including i and excluding j can achieve is obtained by the coalition $S \in \mathcal{I}_{i,j}$ such that $|S| = 2$. By this, $s_{i,j}(x) = s_{j,i}(x) = 2$ for all pairs of different players i, j. Thus, $x \in \mathscr{K}(\mathcal{G})$.

It is well known that $\mathscr{K}(\mathcal{G}) \neq \emptyset$, whenever $X(\mathcal{G}) \neq \emptyset$ (see, e.g., [62]). Thus, the non-emptiness problem is trivial for this concept. Instead, as discussed by Deng and Papadimitriou [26], it is of interest to ask for the computational complexity of deciding whether a given payoff vector belongs to the kernel. This problem was conjectured to be **NP**-hard by those same authors [26], even for graph games. In the rest of the section, we firstly confirm their conjecture by actually showing that the problem is $\mathbf{\Delta_2^P}$-hard. Then, we shall show that the corresponding membership result holds for any class of games $\mathcal{C}(\mathcal{R})$, where $\mathcal{R}$ is an arbitrary $\mathbf{NP}^{opt}$-representation.

4.1. Hardness for graph games

We show that checking whether an imputation belongs to the kernel is $\mathbf{\Delta_2^P}$-hard for graph games. The proof is based on a reduction from the problem of the lexicographically maximum satisfying assignment for Boolean formulae.

Let $\phi = c_1 \wedge \cdots \wedge c_m$ be a **3CNF** Boolean formula, that is, a Boolean formula in conjunctive normal form over the set $\{\alpha_1, \ldots, \alpha_n\}$ of variables that are lexicographically ordered (according to their indices), where each clause contains at most three literals (positive or negated variables). Assume, w.l.o.g., that there is a clause in ϕ containing at least two literals. Based on ϕ, we build in polynomial time the weighted graph $\boldsymbol{K}(\phi) = \langle (N_{\mathbf{K}}, E_{\mathbf{K}}), w\rangle$ such that (see Fig. 3 for an illustration):

- The set $N_{\mathbf{K}}$ of nodes (i.e., players) includes: a *variable player* α_i, for each variable α_i in ϕ; a *clause player* c_j, for each clause c_j in ϕ; a *literal player* $\ell_{i,j}$ (either $\ell_{i,j} = \alpha_{i,j}$ or $\ell_{i,j} = \neg\alpha_{i,j}$), for each literal ℓ_i ($\ell_i = \alpha_i$ or $\ell_i = \neg\alpha_i$, respectively) occurring in c_j; and, two special players "*chall*" and "*sat*".
- The set $E_{\mathbf{K}}$ consists of the following three types of edges.

(Positive edges): an edge $\{c_j, \ell_{i,j}\}$ with $w(\{c_j, \ell_{i,j}\}) = 2^{n+3}$, for each literal ℓ_i occurring in c_j; an edge $\{chall, \alpha_i\}$ with $w(\{chall, \alpha_i\}) = 2^i$, for each $1 \leqslant i \leqslant n$; an edge $\{sat, \alpha_i\}$ with $w(\{sat, \alpha_i\}) = 2^i$, for each $2 \leqslant i \leqslant n$; the edge $\{sat, \alpha_1\}$ with $w(\{sat, \alpha_1\}) = 2^1 + 2^0$.

("Penalty" edges): an edge $\{\ell_{i,j}, \ell_{i',j}\}$ with $w(\{\ell_{i,j}, \ell_{i',j}\}) = -2^{m+n+7}$, for each pair of literals ℓ_i and $\ell_{i'}$ occurring in c_j; an edge $\{\alpha_{i,j}, \neg\alpha_{i,j'}\}$ with $w(\{\alpha_{i,j}, \neg\alpha_{i,j'}\}) = -2^{m+n+7}$, for each variable α_i occurring positively in c_j and negated in $c_{j'}$; an edge $\{\alpha_i, \neg\alpha_{i,j}\}$ with $w(\{\alpha_i, \neg\alpha_{i,j}\}) = -2^{m+n+7}$, for each variable α_i occurring negated in c_j.

("Normalizer" edge): the edge $\{chall, sat\}$, for which we set $w(\{chall, sat\}) = 1 - \sum_{e \in E_{\mathbf{K}} | e \neq \{chall, sat\}} w(e)$.

Note that the size of $\boldsymbol{K}(\phi)$ (and in particular of the representation of all the weights) is polynomial in the number of variables and clauses of ϕ. Two crucial properties of the above construction are stated in the following lemma, whose proof is given in Appendix A.

Lemma 4.3. *Let $\boldsymbol{K}(\phi) = \langle (N_{\mathbf{K}}, E_{\mathbf{K}}), w\rangle$ be the graph game associated with the* **3CNF** *formula ϕ. Then:*

(A) $w(\{chall, sat\}) \geqslant D + 1$; *and,*
(B) $D + w(e) < 0$, *for each penalty edge $e \in E_{\mathbf{K}}$,*

where $D = \max_{\{chall, sat\} \not\subseteq S \subseteq N} v(S)$ denotes the maximum worth over all the coalitions not covering the edge $\{chall, sat\}$.

Based on the above properties, we can now prove the main result.

Theorem 4.4. *On the class $\mathcal{C}(\mathcal{GG})$ of graph games,* In-Kernel *is $\boldsymbol{\Delta_2^P}$-hard (even if the given payoff vector is an imputation).*

Proof. Let $\phi = c_1 \wedge \cdots \wedge c_m$ be a satisfiable **3CNF** formula over a set $\{\alpha_1, \ldots, \alpha_n\}$ of variables that are lexicographically ordered (according to their indices). Deciding whether α_1 (that is the lexicographically least significant variable) is true in the *lexicographically maximum satisfying assignment* for ϕ is a well-known $\boldsymbol{\Delta_2^P}$-complete problem [52].

Consider the graph game $\boldsymbol{K}(\phi) = \langle (N_{\mathbf{K}}, E_{\mathbf{K}}), w\rangle$, and the imputation x that assigns 0 to all players of $\boldsymbol{K}(\phi)$, but for *sat*, which receives 1. In fact, x is an imputation since $v(N) = 1$ because of the weight of the edge $\{chall, sat\}$ (recall that $w(\{chall, sat\}) = 1 - \sum_{e \in E_{\mathbf{K}} | e \neq \{chall, sat\}} w(e)$).

At first, observe that, by Definition 4.1 and since *sat* is the only player that receives in x a payoff strictly greater than her worth as a singleton coalition, $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ if and only if $\max_{S \in \mathcal{I}_{i,sat}} e(S, x) \leqslant \max_{S \in \mathcal{I}_{sat,i}} e(S, x)$, for each player $i \neq sat$. But we can notice that the following holds (the proof is given in Appendix A):

Property 4.4.(1). *For each player $i \notin \{sat, chall\}$, it holds that* $\max_{S \in \mathcal{I}_{i,sat}} e(S, x) \leqslant \max_{S \in \mathcal{I}_{sat,i}} e(S, x)$.

Hence, Property 4.4.(1) further restricts the coalitions of interest and, indeed, we can conclude that $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ if and only if

$$\max_{S \in \mathcal{I}_{chall,sat}} e(S, x) \leqslant \max_{S \in \mathcal{I}_{sat,chall}} e(S, x). \tag{1}$$

Now, we are going to characterize the structure of the two terms $\max_{S \in \mathcal{I}_{chall,sat}} e(S, x)$ and $\max_{S \in \mathcal{I}_{sat,chall}} e(S, x)$ occurring in Eq. (1) by establishing a connection with satisfying assignments for ϕ. In particular, for any truth assignment σ, we denote by $\sigma \models \phi$ the fact that σ satisfies ϕ, and by $\sigma(\alpha_i) = \mathtt{true}$ (resp., $\sigma(\alpha_i) = \mathtt{false}$) the fact that α_i evaluates to true (resp., false) in σ. Then, we can state the following two properties, whose proofs are given in Appendix A.

Property 4.4.(2). $\max_{S \in \mathcal{I}_{chall,sat}} e(S, x) = m \times 2^{n+3} + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i$.
Property 4.4.(3). $\max_{S \in \mathcal{I}_{sat,chall}} e(S, x) = m \times 2^{n+3} + \max_{\sigma \models \phi}(|\{\alpha_1 \mid \sigma(\alpha_1) = \mathtt{true}\}| + \sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i) - 1$.

We can now rewrite Eq. (1) in the light of the above two properties, and conclude that $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ if and only if

$$1 + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i \leqslant \max_{\sigma \models \phi} \left(\sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i + \big|\{\alpha_1 \mid \sigma(\alpha_1) = \mathtt{true}\}\big| \right),$$

that is, $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ if and only if α_1 is true in the lexicographically maximum satisfying assignment for ϕ. $\square$

Note that the above result can immediately be extended to all representations at least as expressive (and succinct) as graph games.

Corollary 4.5. *Let $\mathcal{R}$ be any compact representation such that $\mathcal{GG} \precsim_e \mathcal{R}$ (e.g., $\mathcal{R} = \mathcal{MCN}$). On the class $\mathcal{C}(\mathcal{R})$,* IN-KERNEL *is $\mathbf{\Delta_2^P}$-hard.*

Proof. From the $\mathbf{\Delta_2^P}$-hardness for graph games, we know that there is a polynomial-time reduction f_1 from any $\mathbf{\Delta_2^P}$ problem Υ to the IN-KERNEL problem for graph games. Moreover, recall from Section 3 that $\mathcal{GG} \precsim_e \mathcal{R}$ means that there exists a polynomial-time function f_2 that translates any graph game $\xi^{\mathcal{GG}}(\mathcal{G})$ into an equivalent game $f_2(\xi^{\mathcal{GG}}(\mathcal{G}))$ belonging to $\mathcal{C}(\mathcal{R})$, that is, into a game with the same worth function and, thus, the same kernel as the former one. Therefore, the composition of f_1 and f_2 is a polynomial-time reduction from any $\mathbf{\Delta_2^P}$ problem to the IN-KERNEL problem for games in $\mathcal{C}(\mathcal{R})$. $\square$

4.2. Membership results

Next, we show that the IN-KERNEL problem is in $\mathbf{\Delta_2^P}$, even when dealing with $\mathbf{NP}^{opt}$-representations. Before presenting this result, we illustrate some facts of $\mathbf{NP}^{opt}$-representations that will also be used to analyze the other solution concepts.

Lemma 4.6. *Let $\mathcal{R}$ be any compact representation for coalitional games, let $\mathcal{G}$ be a game in $\mathcal{C}(\mathcal{R})$ over a set N of players, and let $S \subseteq N$ be a coalition. Assume that the worth function $v^{\mathcal{R}}$ is such that $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) = \max\{w \mid w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)\}$, where $f^{\mathcal{R}}$ is a feasibility function computable in* **NPMV**. *Then*:

(1) *For any value h, checking whether $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) > h$ (i.e., whether there is a value $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$ such that $w > h$) is in* **NP**.
(2) *Computing the value $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$ is in $\mathbf{F\Delta_2^P}$.*

Proof. (1) For any value h, $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) > h$ holds if and only if there is a value $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$ such that $w > h$. This latter property can be verified by guessing in **NP** the value $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$ together with a certificate c that $\langle(\xi^{\mathcal{R}}(\mathcal{G}), S), w\rangle \in graph(f^{\mathcal{R}})$. Then, we can check in (deterministic) polynomial time that $w > h$ and, by exploiting c, that $\langle(\xi^{\mathcal{R}}(\mathcal{G}), S), w\rangle$ belongs to the graph of the function $f^{\mathcal{R}}$.

(2) The value $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$ can be computed by performing a binary search over the range of the possible values for the worth function by using an oracle, which checks whether, for any given value h in this range, $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) > h$ holds. Since $f^{\mathcal{R}}$ is computable in **NPMV**, the size of $v^{\mathcal{R}}(S)$ is polynomially bounded and, thus, the binary search converges in at most polynomially many steps. Finally, by point (1) above, note that the oracle belongs to **NP**. $\square$

Theorem 4.7. *Let $\mathcal{R}$ be any $\mathbf{NP}^{opt}$-representation. On the class $\mathcal{C}(\mathcal{R})$,* IN-KERNEL *is in $\mathbf{\Delta_2^P}$.*

Proof. Let $\mathcal{G} \in \mathcal{C}(\mathcal{R})$ be a coalitional game, and x a payoff vector. Recall from Definition 3.3 that for each coalition $S \subseteq N$, $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) = \max\{w \mid w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)\}$ holds, where $f^{\mathcal{R}}$ is a feasibility function computable in **NPMV**.

Note that, in order to check whether x is actually an imputation, we have to compute the worths associated to all the singleton coalitions plus the worth associated to the whole set of players. This is in $\mathbf{F\Delta_2^P}$, by Lemma 4.6(2).

Assume now that x is an imputation. For all pairs of distinct players i and j with $v(j) > x_j$, we have to check that $s_{i,j}(x) = s_{j,i}(x)$. Observe here that we can compute the value $s_{i,j}(x)$ by means of a binary search over the range of the possible values for the worth functions by using an oracle that, for any value h in this range, decides whether there is a coalition $S \in \mathcal{I}_{i,j}$ such that $e(S, x) > h$. Since the size of $v^{\mathcal{R}}(S)$ is polynomially bounded, the above binary search allows us to find the maximum excess in at most polynomially many steps. To complete the proof we need to show that deciding $e(S, x) > h$ is in **NP** for any given value h in this range. To this end, recall that $e(S, x) = v^{\mathcal{R}}(S) - x(S)$. Thus, $e(S, x) > h$ holds if and only if there is a feasible value $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$ such that $w > h + x(S)$. Based on this characterization, the check is easily seen to be in **NP** by Lemma 4.6(1). By this, the surplus computation is in $\mathbf{F\Delta_2^P}$. Thus, checking $s_{i,j}(x) = s_{j,i}(x)$ for all pairs of distinct players is in $\mathbf{\Delta_2^P}$. $\square$

From the above result and Corollary 4.5, we immediately get the following completeness result.

Corollary 4.8. *Let $\mathcal{R}$ be any $\mathbf{NP}^{opt}$-representation such that $\mathcal{GG} \precsim_e \mathcal{R}$ (e.g., $\mathcal{R} = \mathcal{MCN}$). On the class $\mathcal{C}(\mathcal{R})$,* IN-KERNEL *is $\mathbf{\Delta_2^P}$-complete.*

5. The complexity of the bargaining set

The concept of bargaining set was defined by Aumann and Maschler [3] (see also Maschler [55]). We start by recalling its formal definition.

Let $\mathcal{G} = \langle N, v\rangle$ be a coalitional game, and $x \in X(\mathcal{G})$ an imputation. Let $S \subseteq N$ be a coalition, and y an S-feasible payoff vector (i.e., $y(S) = v(S)$). The pair (y, S) is an *objection of player i against player j to x* if $i \in S$, $j \notin S$, and $y_k > x_k$ for all $k \in S$.

A *counterobjection to the objection* (y, S) *of i against j to x* is a pair (z, T) where $j \in T$, $i \notin T$, and z is a T-feasible payoff vector such that $z_k \geqslant x_k$ for all $k \in T \setminus S$ and $z_k \geqslant y_k$ for all $k \in T \cap S$. If there does not exist any counterobjection to (y, S), we say that (y, S) is a *justified objection*.

Definition 5.1. The *bargaining set* $\mathscr{B}(\mathcal{G})$ of a TU game $\mathcal{G}$ is the set of all imputations x to which there is no justified objection.

Example 5.2. Let $\mathcal{G} = \langle N, v\rangle$ be the TU game already illustrated in Example 4.2, that is, $N = \{a, b, c\}$, $v(\{a\}) = v(\{b\}) = v(\{c\}) = 0$, $v(\{a, b\}) = 20$, $v(\{a, c\}) = 30$, $v(\{b, c\}) = 40$, and $v(\{a, b, c\}) = 42$.

Consider the imputation x such that $x_a = 8$, $x_b = 10$, and $x_c = 24$. An objection of player c against player a to x is $((12, 28), \{b, c\})$. Player a can counterobject to this objection using $((8, 12), \{a, b\})$. Another objection of player c against player a to x is $((14, 26), \{b, c\})$. In this case, player a cannot counterobject. The reason is that coalition $\{a, b\}$ receives a payoff 20 and this is not sufficient for player a to counterobject since she needs at least 8 for herself and at least 14 for player b, in order to respond to the proposal of player c. Therefore the imputation x does not belong to $\mathscr{B}(\mathcal{G})$. The intuitive reason is that player a receives too much, according to this profile.

Consider now the imputation x' such that $x'_a = 4$, $x'_b = 14$, and $x'_c = 24$. We focus on the objections of player a against player c. We note that, in order to object, player a has to form the coalition $S = \{a, b\}$. The excess $e(S, x)$ of S at x is 2, hence players a and b have the possibility to distribute among themselves a payoff of 2 to make the objection. But player c can always counterobject to player a because she can form the coalition $T = \{b, c\}$ whose excess at x is 2 and hence she can always match the proposal made to player b by player a in order to object. A similar argument holds for every objection of every player against any other. Thus $x' \in \mathscr{B}(\mathcal{G})$.

It is well-known that $\mathscr{K}(\mathcal{G}) \subseteq \mathscr{B}(\mathcal{G})$ (hence, $\mathscr{B}(\mathcal{G}) \neq \emptyset$, whenever $X(\mathcal{G}) \neq \emptyset$), and that $\mathscr{C}(\mathcal{G}) \subseteq \mathscr{B}(\mathcal{G})$ (see, e.g., [62]). Thus, as in the case of the kernel, we shall just focus on the complexity of deciding whether a given payoff vector is in the bargaining set. The problem was conjectured to be $\mathbf{\Pi_2^P}$-complete for graph games by Deng and Papadimitriou [26]. In this section, we show that the conjecture is indeed correct. Also, we are able to generalize the result by showing that membership in $\mathbf{\Pi_2^P}$ holds for any class of games $\mathcal{C}(\mathcal{R})$, where $\mathcal{R}$ is an $\mathbf{NP}^{opt}$-representation scheme.

5.1. Hardness for graph games

It was suggested by Maschler [55] that computing the bargaining set might be intrinsically more complex than computing the core. The result presented in this section provides some fresh evidence that this is indeed the case for we shall show that checking whether a payoff vector is in the bargaining set is $\mathbf{\Pi_2^P}$-hard, even over graph games. The reduction is from the validity of quantified Boolean formulae.

Let $\boldsymbol{\Phi} = (\forall\boldsymbol{\alpha})(\exists\boldsymbol{\beta})\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an $\mathbf{NQBF}_{2,\forall}$ formula, i.e., a quantified Boolean formula over the variables $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_r\}$, where $\phi(\boldsymbol{\alpha}, \boldsymbol{\beta}) = c_1 \wedge \cdots \wedge c_m$ is a **3CNF** formula, and where each universally quantified variable $\alpha_k \in \boldsymbol{\alpha}$ occurs only in the two clauses $c_{i(k)} = (\alpha_k \vee \neg\beta_k)$ and $c_{\bar{i}(k)} = (\neg\alpha_k \vee \beta_k)$—intuitively, each variable α_k enforces the truth value of a corresponding variable β_k, which thus plays its role in the formula $\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Based on $\boldsymbol{\Phi}$, we define the weighted graph $\boldsymbol{BS}(\boldsymbol{\Phi}) = \langle (N_{\mathbf{BS}}, E_{\mathbf{BS}}), w\rangle$ such that (see Fig. 4 for a graphical illustration):

- The set $N_{\mathbf{BS}}$ of the nodes (i.e., players) includes: a *clause player* c_j, for each clause c_j; a *literal player* $\ell_{i,j}$, for each literal ℓ_i occurring in c_j; and, two special players "*chall*" and "*sat*".
- The set $E_{\mathbf{BS}}$ of edges includes three kinds of edges.

(Positive edges): an edge $\{c_j, \ell_{i,j}\}$ with $w(\{c_j, \ell_{i,j}\}) = 1$, for each literal ℓ_i occurring in the clause c_j; an edge $\{chall, \ell_{i,j}\}$ with $w(\{chall, \ell_{i,j}\}) = 1$, for each literal ℓ_i of the form α_i or $\neg\alpha_i$ (i.e., built over a universally quantified variable) occurring in c_j.

("Penalty" edges): an edge $\{\gamma_{i,j}, \neg\gamma_{i,j'}\}$ with $w(\{\gamma_{i,j}, \neg\gamma_{i,j'}\}) = -m - 1$, for each variable γ_i (either $\gamma_i = \alpha_i$ or $\gamma_i = \beta_i$) occurring in c_j and $c_{j'}$; an edge $\{\ell_{i,j}, \ell_{i',j}\}$ with $w(\ell_{i,j}, \ell_{i',j}) = -m - 1$, for each pair of literals ℓ_i and $\ell_{i'}$ occurring in c_j; an edge $\{chall, \ell_{i,j}\}$ with $w(\{chall, \ell_{i,j}\}) = -m - 1$, for each literal ℓ_i of the form β_i or $\neg\beta_i$ (i.e., built over an existentially quantified variable) occurring in c_j; an edge $\{chall, c_j\}$ with $w(\{chall, c_j\}) = -m - 1$, for each clause c_j.

("Normalizer" edge): the edge $\{chall, sat\}$ with $w(\{chall, sat\}) = n - 1 + m - \sum_{e\in E_{\mathbf{BS}} | e\neq\{chall,sat\}} w(e)$.

Note that the size of $\boldsymbol{BS}(\boldsymbol{\Phi})$ (and in particular of the representation of all the weights) is polynomial in the number of variables and clauses of $\boldsymbol{\Phi}$. Moreover, the properties summarized in the following lemma hold, whose proof is given in Appendix B.

Lemma 5.3. *Let* $\boldsymbol{BS}(\boldsymbol{\Phi}) = \langle (N_{\mathbf{BS}}, E_{\mathbf{BS}}), w\rangle$ *be the graph game associated with the* $\mathbf{NQBF}_{2,\forall}$ *formula* $\boldsymbol{\Phi}$. *Then*:

(A) $D \leqslant m$;

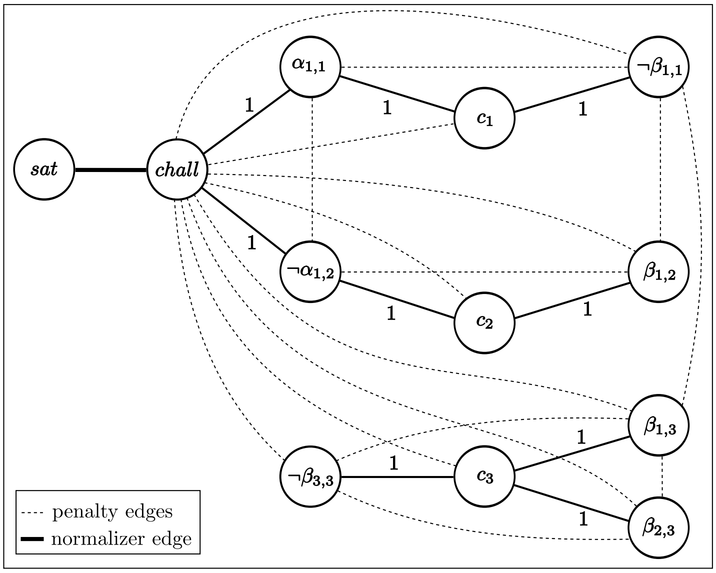

Fig. 4. The game $\boldsymbol{BS}(\widehat{\boldsymbol{\Phi}})$, where $\widehat{\boldsymbol{\Phi}} = (\forall\alpha_1)(\exists\beta_1, \beta_2, \beta_3)(\alpha_1 \vee \neg\beta_1) \wedge (\neg\alpha_1 \vee \beta_1) \wedge (\beta_1 \vee \beta_2 \vee \neg\beta_3)$.

(B) $w(\{chall, sat\}) > 2 \times m$;
(C) $D + w(e) < 0$, *for each penalty edge* $e \in E_{\mathbf{BS}}$; *and,*
(D) $m \geqslant 2 \times n$,

where $D = \max_{\{chall,sat\} \not\subseteq S \subseteq N} v(S)$ *denotes the maximum worth over all the coalitions not covering the edge* $\{chall, sat\}$.

Based on the above properties, we can now prove the following result.

Theorem 5.4. *On the class* $\mathcal{C}(\mathcal{GG})$ *of graph games,* In-BargainingSet *is* $\boldsymbol{\Pi_2^P}$*-hard (even if the given payoff vector is an imputation).*

Proof. Deciding the validity of $\mathbf{NQBF}_{2,\forall}$ formulae is $\boldsymbol{\Pi_2^P}$-complete [72]. Thus, given an $\mathbf{NQBF}_{2,\forall}$ formula $\boldsymbol{\Phi} = (\forall\boldsymbol{\alpha})(\exists\boldsymbol{\beta})\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$ (where $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$, $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_r\}$, and $\phi(\boldsymbol{\alpha}, \boldsymbol{\beta}) = c_1 \wedge \cdots \wedge c_m$), consider the graph game $\boldsymbol{BS}(\boldsymbol{\Phi}) = \langle (N_{\mathbf{BS}}, E_{\mathbf{BS}}), w\rangle$ and the imputation x that assigns m to *sat*, $n-1$ to *chall*, and 0 to all other players. Note that x is an imputation, since $v(N) = m + n - 1$ because of the weight of the edge $\{chall, sat\}$ (recall that $w(\{chall, sat\}) = m + n - 1 - \sum_{e \in E_{\mathbf{BS}} | e \neq \{chall,sat\}} w(e)$).

Beforehand, we note that the following properties hold on $\boldsymbol{BS}(\boldsymbol{\Phi})$ and x. The proofs of these properties are given in Appendix B.

Property 5.4.(1). *No player has a justified objection against a clause or a literal player.*

Property 5.4.(2). *No player has a justified objection against chall.*

Property 5.4.(3). *No player different from chall has a justified objection against sat.*

In the light of the properties above, we can limit our attention to the objections of *chall* against *sat*. Consider an objection (y, S) of *chall* against *sat* to x. In particular, y_{chall} must be greater than $x_{chall} = n - 1$ and $y_q > 0 = x_q$ for each $q \in S$ with $q \neq chall$. Thus, $y(S) = v(S) > n - 1$.

Then, because of Lemma 5.3(C), in order for such a coalition S to be such that $v(S) > n - 1$, no penalty edge must be covered by S. Moreover, given that $chall \in S$, we have that S must contain exactly one player per universally quantified variable so that $v(S) = n > n - 1$, i.e., $|S| = n + 1$ and $|S \cap \{\alpha_{k,i(k)}, \neg\alpha_{k,\bar{i}(k)}\}| = 1$, for each $1 \leqslant k \leqslant n$. By this, the objection y has to be a vector such that $y(S) = n$, $y_{chall} > n - 1$, and $y_q > 0$, for each $q \in S$ with $q \neq chall$. Note that the coalition S encodes an assignment σ_S for the variables in $\boldsymbol{\alpha}$ such that α_k evaluates to false (resp., true) in σ_S if $\alpha_{k,i(k)}$ (resp., $\neg\alpha_{k,\bar{i}(k)}$) occurs in S—note that, in this correspondence, the truth values are inverted with respect to the membership of the corresponding literal players in S.

Moreover, note that for each truth assignment σ for the variables in $\boldsymbol{\alpha}$, we may immediately build a coalition S and a vector y such that (y, S) is an objection of *chall* against *sat*, and $\sigma_S = \sigma$. Therefore, objections of *chall* against *sat* are in correspondence with truth assignments for universally quantified variables.

We are now ready to show that $\boldsymbol{\Phi}$ is valid if and only if x is in $\mathscr{B}(\boldsymbol{BS}(\boldsymbol{\Phi}))$:

($\Rightarrow$) Assume that $\boldsymbol{\Phi} = (\forall \boldsymbol{\alpha})(\exists \boldsymbol{\beta})\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is valid, and let (y, S) be any objection of *chall* against *sat* to x. We show that this objection is not justified, because *sat* has a counterobjection (z, T). Recall first that S encodes an assignment σ_S over the variables in $\boldsymbol{\alpha}$. Then, let σ be a satisfying assignment over the variables in $\boldsymbol{\Phi}$ such that α_k evaluates to true in σ if and only if it evaluates to true in σ_S; indeed, such a satisfying assignment σ exists since $\boldsymbol{\Phi}$ is valid. Based on σ, let us construct the coalition T such that $T \subseteq \{sat\} \cup \{\ell_{i,j} \mid \ell_i$ evaluates to true in $\sigma \wedge c_j$ is a clause where ℓ_i occurs$\} \cup \{c_1, \ldots, c_m\}$, $T \cap S = \emptyset$, $|T| = 2m + 1$, and $v(T) = m$. In particular, T can be such that $v(T) = m$, precisely because σ is a satisfying assignment and by construction of σ_S. Moreover, consider the vector z such that $z_{sat} = v(T) = x_{sat} = m$ and $z_q = x_q = 0$ for each $q \in T$ with $q \neq sat$ (and, hence, $q \notin S$). By construction (z, T) is a counterobjection to (y, S), which is therefore not justified in its turn.

($\Leftarrow$) Let σ be an assignment over the variables in $\boldsymbol{\alpha}$ witnessing that $\boldsymbol{\Phi}$ is not valid. Let (y, S) be an arbitrary objection of *chall* against *sat* to x such that $\sigma_S = \sigma$. We claim that (y, S) is justified. Indeed, assume for the sake of contradiction that (y, S) is not justified and let (z, T) be a counterobjection. Since $sat \in T \setminus S$ and since we must have $z_{sat} \geqslant x_{sat} = m$, because of Lemma 5.3(A) we actually have that $z_{sat} = x_{sat}$. In fact, since m is the maximum available payoff over all the coalitions not including both *chall* and *sat*, $z(T) = v(T) = m$ and the fact that (z, T) is a counterobjection to (y, S) together entail that $S \cap T = \emptyset$. Also, these entail that T contains all clause players, exactly one literal player per clause, so that $|T| = 2m+1$. Observe now that T encodes a satisfying truth value assignment σ_T for $\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$, because $v(T) > 0$ and hence T does not cover any penalty edge. More precisely, we let each variable γ_i (either $\gamma_i = \alpha_i$ or $\gamma_i = \beta_i$) to evaluate true (resp., false) in σ_T if $\gamma_{i,j}$ (resp., $\neg\gamma_{i,j}$) occurs in T, for some clause c_j. Let $\sigma_T^{\boldsymbol{\alpha}}$ denote the restriction of σ_T over the variables in $\boldsymbol{\alpha}$. Then, since $S \cap T = \emptyset$ and given the definition of σ_S, we have that $\sigma_S = \sigma_T^{\boldsymbol{\alpha}}$. Clearly, this contradicts the fact that $\sigma = \sigma_S$ witnesses that $\boldsymbol{\Phi}$ is not valid. $\square$

Again, by the same argument used for the proof of Corollary 4.5, the above result extends to compact representations more expressive than graph games.

Corollary 5.5. *Let $\mathcal{R}$ be any* $\mathbf{NP}^{opt}$*-representation such that $\mathcal{GG} \precsim_e \mathcal{R}$ (e.g., $\mathcal{R} = \mathcal{MCN}$). On the class $\mathcal{C}(\mathcal{R})$,* IN-BARGAININGSET *is* $\mathbf{\Pi_2^P}$*-hard.*

5.2. Membership results

Checking whether a payoff vector belongs to the bargaining set has already been argued by Deng and Papadimitriou [26] to be in $\mathbf{\Pi_2^P}$ for graph games. Indeed, they observed that one may decide that an imputation x is not in the bargaining set by firstly guessing in **NP** the objection (y, S), and then by calling a co-**NP** oracle, by which to check that there is no counterobjection (z, T) to (y, S). However, to apply this argument one typically assumes real values to have a fixed-precision. In fact, Greco et al. [39] recently proved that there always exist suitable objections and counterobjections that may be guessed in polynomial time, so that IN-BARGAININGSET was formally proved to be in $\mathbf{\Pi_2^P}$. In particular, it was shown that the problem remains in $\mathbf{\Pi_2^P}$, for any arbitrary worth function that can be computed in polynomial time (thus, for classes of games based on a **P**-representation scheme) and for certain kinds of worth functions computable in non-deterministic polynomial time. Our main achievement in this section is to show that membership in $\mathbf{\Pi_2^P}$ (surprisingly) holds on any arbitrary $\mathbf{NP}^{opt}$-representation. To this end, we need to devise some novel technical machinery.

First, we shall provide a useful characterization of a player i having a justified objection against some player j. The result is in the spirit of one of Maschler's [54] and connects the existence of a justified objection of i against j to some algebraic conditions to hold on coalitions that i and j (may) belong to.

Lemma 5.6. *Let $\mathcal{G}$ be a coalitional game, and let x be an imputation of $\mathcal{G}$. Then, player i has a justified objection against player j to x through a coalition $S \in \mathcal{I}_{i,j}$ if and only if there exists a vector $y \in \mathbb{R}^S$ such that:*

(1) $y(S) = v(S)$;
(2) $y_k > x_k$, *for each* $k \in S$; *and,*
(3) $v(T) < y(T \cap S) + x(T \setminus S)$, *for each* $T \in \mathcal{I}_{j,i}$.

Proof. ($\Rightarrow$) Assume that player i has a justified objection against player j to x through a coalition $S \in \mathcal{I}_{i,j}$. Let (y, S) be such justified objection and note that, by definition, $y(S) = v(S)$ and $y_k > x_k$ for each $k \in S$ hold. Assume now, for the sake of contradiction, that (3) above does not hold. Let $\bar{T} \in \mathcal{I}_{j,i}$ be such that $v(\bar{T}) \geqslant y(\bar{T} \cap S) + x(\bar{T} \setminus S)$. Based on $\bar{T}$, we can build a vector $z \in \mathbb{R}^{\bar{T}}$ such that: (i) $z_k = x_k + \delta$ for each $k \in \bar{T} \setminus S$; and, (ii) $z_k = y_k + \delta$ for each $k \in \bar{T} \cap S$, where

$$\delta = \frac{v(\bar{T}) - y(\bar{T} \cap S) - x(\bar{T} \setminus S)}{|\bar{T}|} \geqslant 0.$$

Note that $z(\bar{T}) = v(\bar{T})$ holds by construction. Hence, $(z, \bar{T})$ is a counterobjection to (y, S), a contradiction.

($\Leftarrow$) Assume that there is a vector $y \in \mathbb{R}^S$ such that all the three conditions above hold. Consider the pair (y, S) (with $S \in \mathcal{I}_{i,j}$) and note that due to (1) and (2), (y, S) is in fact an objection of player i against player j. We now claim that

(y, S) is justified. Indeed assume, for the sake of contradiction, that a counterobjection $(z, \bar{T})$ (with $\bar{T} \in \mathcal{I}_{j,i}$) exists such that: $z(\bar{T}) = v(\bar{T})$, $z_k \geqslant x_k$ for each $k \in \bar{T} \setminus S$, and $z_k \geqslant y_k$ for each $k \in \bar{T} \cap S$. In this case, it holds that $v(\bar{T}) = z(\bar{T}) \geqslant y(T \cap S) + x(T \setminus S)$, which contradicts (3). $\square$

Second, we need to recall the following beautiful result on families of convex sets. Remember that a set of points $S \subseteq \mathbb{R}^n$ is said convex if, for every pair of points $p, q \in S$, (each point in) the straight line connecting p and q belongs to S as well.

Proposition 5.7 *(Helly's Theorem [43,69]). Let $\mathcal{C} = \{c_1, \ldots, c_m\}$ be a finite family of convex sets in $\mathbb{R}^n$, where $m > n$. If $\bigcap_{i=1}^m c_i = \emptyset$, there is a family $\mathcal{C}' \subseteq \mathcal{C}$, such that $|\mathcal{C}'| = n+1$ and $\bigcap_{c_i \in \mathcal{C}'} c_i = \emptyset$.*

With the above two ingredients, we can now prove our main result about the bargaining set:

Theorem 5.8. *Let $\mathcal{R}$ be any* $\mathbf{NP}^{opt}$*-representation. On the class $\mathcal{C}(\mathcal{R})$,* IN-BARGAININGSET *is in* $\mathbf{\Pi_2^P}$.

Proof. Let $\mathcal{G} \in \mathcal{C}(\mathcal{R})$ be a coalitional game, and x a payoff vector. Checking whether x is an imputation is in $\mathbf{\Delta_2^P}$ (see the proof of Theorem 4.7).

Assume now that x is an imputation, and consider the complementary problem of deciding whether $x \notin \mathscr{B}(\mathcal{G})$. We are going to show that this is in $\mathbf{\Sigma_2^P}$. In the light of Lemma 5.6, $x \notin \mathscr{B}(\mathcal{G})$ if and only if there exist two players i and j, and a coalition $S \in \mathcal{I}_{i,j}$ such that the following set is non-empty:

$$\begin{aligned} W(i,j,S) = \big\{ y \in \mathbb{R}^S \;\big|\; & y(S) = v^{\mathcal{R}}\big(\xi^{\mathcal{R}}(\mathcal{G}), S\big) \wedge \\ & y_k > x_k,\ \forall k \in S \wedge \\ & v^{\mathcal{R}}\big(\xi^{\mathcal{R}}(\mathcal{G}), T\big) < y(T \cap S) + x(T \setminus S),\ \forall T \in \mathcal{I}_{j,i} \big\} \end{aligned}$$

Recall from Definition 3.3 that $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) = \max\{w \mid w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)\}$, where $f^{\mathcal{R}}$ is a *feasibility* function computable in **NPMV**. Thus, we can solve the problem by first guessing in **NP** two players i and j, a coalition $S \in \mathcal{I}_{i,j}$, a value w_S, and a certificate c_S that $\langle(\xi^{\mathcal{R}}(\mathcal{G}), S), w_S\rangle \in graph(f^{\mathcal{R}})$. Note that, by exploiting c_S, we can then check in (deterministic) polynomial time that $\langle(\xi^{\mathcal{R}}(\mathcal{G}), S), w_S\rangle$ belongs to the graph of the function $f^{\mathcal{R}}$, and thus that $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) \geqslant w_S$. After this first phase, we have then to check that (i) $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) \leqslant w_S$, and that (ii) $W(i,j,S)$ is non-empty. To complete the proof, it suffices to show that both checks are in co-**NP**.

Indeed, concerning (i), the complementary problem of checking that $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) > w_S$ is in **NP** by Lemma 4.6(1). Concerning (ii), consider the complementary problem of checking the emptiness of $W(i,j,S)$, which can be equivalently described by the following inequalities:

$$\begin{aligned} W(i,j,S) = \big\{ y \in \mathbb{R}^S \;\big|\; & y(S) = v^{\mathcal{R}}\big(\xi^{\mathcal{R}}(\mathcal{G}), S\big) = w_S \wedge \\ & y_k > x_k,\ \forall k \in S \wedge \\ & w_T < y(T \cap S) + x(T \setminus S),\ \forall T \in \mathcal{I}_{j,i}, \forall w_T \in f^{\mathcal{R}}\big(\xi^{\mathcal{R}}(\mathcal{G}), T\big) \big\} \end{aligned}$$

Observe now that $W(i,j,S)$ is defined by the intersection of a finite number of convex regions of $\mathbb{R}^S$, which are associated to the various (in)equalities. Hence, by Helly's Theorem, if $W(i,j,S)$ is empty, then there exists a subsystem of (at most) $|S|+1$ inequalities that does not admit a solution. Note that we can always add to this subsystem the equality $y(S) = w_S$ plus the $|S|$ inequalities of the form $y_k > x_k$ (for each $k \in S$), as they can be built in polynomial time. Possible remaining inequalities associated with any coalition $T \in \mathcal{I}_{j,i}$ can instead be obtained in non-deterministic polynomial time. Indeed, we can guess in **NP** $m \leqslant |S|+1$ coalitions $T_1, \ldots, T_m$ from $\mathcal{I}_{j,i}$ and, for each $h \in \{1, \ldots, m\}$, a value w_{T_h} and a certificate c_{T_h} that $w_{T_h} \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), T_h)$. Then, we can check in polynomial time via c_{T_h} that w_{T_h} is actually a value returned by $f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), T_h)$. Thus, after these steps in **NP**, checking whether $W(i,j,S)$ is empty is reduced to checking whether a system of polynomially-many linear (in)equalities admits a solution. This latter task is in polynomial time, by standard arguments in linear programming (see, e.g., [65]). $\square$

From the above theorem and Corollary 5.5, we get the following completeness result.

Corollary 5.9. *Let $\mathcal{R}$ be any* $\mathbf{NP}^{opt}$*-representation such that $\mathcal{GG} \precsim_e \mathcal{R}$ (e.g., $\mathcal{R} = \mathcal{MCN}$). On the class $\mathcal{C}(\mathcal{R})$,* IN-BARGAININGSET *is* $\mathbf{\Pi_2^P}$*-complete.*

6. The complexity of the core

The concept of the *core* goes back to the work by Edgeworth [27] and it was formalized by Gillies [36].

Let $\mathcal{G} = \langle N, v\rangle$ be a coalitional game, and let x be an imputation taken from the set $X(\mathcal{G})$ of all imputations of $\mathcal{G}$. The pair (y, S) is an *objection to* x if y is an S-feasible payoff vector such that $y_k > x_k$ for all $k \in S$—in this case, the coalition S is also said to *block* x via y.

Definition 6.1. The *core* $\mathscr{C}(\mathcal{G})$ of a coalitional game $\mathcal{G} = \langle N, v\rangle$ is the set of all imputations x to which there is no objection; that is,

$$\mathscr{C}(\mathcal{G}) = \left\{x \in X(\mathcal{G}) \;\middle|\; \nexists S \subseteq N \text{ and } y \in \mathbb{R}^S \text{ such that } y(S) = v(S) \text{ and } y_k > x_k, \forall k \in S\right\}$$

Thus, an imputation x in the core is "stable" precisely because there is no coalition whose members will receive a higher payoff than in x by leaving the grand coalition, i.e., the set of all players in the game.

It is easily seen that Definition 6.1 can be equivalently restated as the set of all solutions satisfying the following inequalities (see, e.g., [62]):

$$\sum_{i\in S} x_i \geqslant v(S), \quad \forall S \subseteq N,\ S \neq \emptyset \tag{2}$$

$$\sum_{i\in N} x_i \leqslant v(N) \tag{3}$$

In particular, the last inequality, combined with its opposite in (2), enforces the efficiency of solutions; moreover, inequalities in (2) over singleton coalitions enforce their individual rationality.

Example 6.2. Let $\mathcal{G} = \langle N, v\rangle$ be the TU game already illustrated in Example 4.2, that is, $N = \{a, b, c\}$, $v(\{a\}) = v(\{b\}) = v(\{c\}) = 0$, $v(\{a, b\}) = 20$, $v(\{a, c\}) = 30$, $v(\{b, c\}) = 40$, and $v(\{a, b, c\}) = 42$. Consider the imputation x such that: $x_a = 4$, $x_b = 14$, and $x_c = 24$. Since $v(\{b, c\}) = 40 > 38 = x(\{b, c\})$, $x \notin \mathscr{C}(\mathcal{G})$.

In fact, we can show that $\mathscr{C}(\mathcal{G})$ is empty. To this end, consider coalitions $S_1 = \{a, b\}$, $S_2 = \{a, c\}$ and $S_3 = \{b, c\}$ and the worths associated with them by the worth function. To be in $\mathscr{C}(\mathcal{G})$, an imputation x has to satisfy the following three conditions:

$$x_a + x_b \geqslant 20$$
$$x_a + x_c \geqslant 30$$
$$x_b + x_c \geqslant 40$$

Summing up these inequalities we obtain that $2x_a + 2x_b + 2x_c \geqslant 90$, implying that $x_a + x_b + x_c \geqslant 45$. Thus, the core of $\mathcal{G}$ is empty because the grand coalition would need as much worth as 45 in order to satisfy the claims of S_1, S_2 and S_3, while the total available worth is just $v(N) = 42$.

Consider, instead, the game $\mathcal{G}' = \langle N, v'\rangle$ whose worth function v' is the same as that of $\mathcal{G}$ except for the grand coalition for which $v'(N) = 45$. Then, it is easily checked that the imputation x' such that $x'_a = 5$, $x'_b = 15$, and $x'_c = 25$ is in $\mathscr{C}(\mathcal{G}')$.

In the rest of this section, we shall analyze the complexity of the core and, in particular, of the problems IN-CORE and CORE-NONEMPTINESS.

6.1. Complexity of IN-CORE

Recall from the Introduction that the IN-CORE problem consists in deciding whether, given a game $\mathcal{G}$ and a payoff vector x, it is the case that x belongs to $\mathscr{C}(\mathcal{G})$. In the setting of graph games, the problem has been proven to be co-**NP**-hard [26]. By the same argument used for the proof of Corollary 4.5, we easily obtain the following.

Proposition 6.3. *Let $\mathcal{R}$ be any compact representation such that $\mathcal{GG} \precsim_e \mathcal{R}$ (e.g., $\mathcal{R} = \mathcal{MCN}$). On the class $\mathcal{C}(\mathcal{R})$,* IN-CORE *is* co-**NP**-*hard.*

Moreover, a hardness result for the class $\mathbf{D}^{\boldsymbol{P}}$ can be obtained in the case where $\mathbf{NP}^{opt}$-representations are considered. Note that, in the result below, we do not reveal the hardness of any specific $\mathbf{NP}^{opt}$ representation proposed in the literature, since such analysis is outside the scope of the paper. Rather, we show that hardness for $\mathbf{D}^{\boldsymbol{P}}$ can emerge under the conditions in Definition 3.3. The proof is routine, and it is reported for the sake of completeness only.

Proposition 6.4. *There exists an* $\mathbf{NP}^{opt}$*-representation* $\mathcal{R}$ *such that, on the class* $\mathcal{C}(\mathcal{R})$, IN-CORE *is* $\mathbf{D}^{\boldsymbol{P}}$-*hard.*

Proof. Let $G = (N, E)$ be a graph. For a given value k, deciding whether the size of the maximum clique over the nodes of G equals k is a $\mathbf{D^P}$-complete problem [66]. Consider the worth function v such that $v(S) = 0$, for each $S \subset N$, and where $v(N)$ is the size of the maximum clique in G. Note that the function satisfies the condition in Definition 3.3. Now, consider a payoff vector x such that $x_i = k/|N|$, for each $i \in N$. It is immediate to check that x belongs to $\mathscr{C}(\mathcal{G})$ (in fact, that x is an imputation) if and only if the size of the maximum clique equals k. $\square$

We conclude the analysis of In-Core by showing that the lower bounds in the above two propositions are tight. In fact, Greco et al. [39] recently showed that this problem is in co-**NP** for any arbitrary worth function that can be computed in polynomial time (thus, for classes of games based on a **P**-representation scheme) and for certain kinds of worth functions computable in non-deterministic polynomial time. We next generalize this result, by showing that In-Core is in co-**NP** over those $\mathbf{NP}^{opt}$-representations where the worth associated with the set of all players is explicitly given as an input or can be computed in polynomial time (i.e., over $\mathbf{wNP}^{opt}$-representations), while it is in $\mathbf{D^P}$ over arbitrary $\mathbf{NP}^{opt}$-representations.

Theorem 6.5. *Let $\mathcal{R}$ be any compact representation. On the class $\mathcal{C}(\mathcal{R})$, In-Core is*

(1) *in* co-**NP**, *if $\mathcal{R}$ is a $\mathbf{wNP}^{opt}$-representation*; *and,*
(2) *in $\mathbf{D^P}$, if $\mathcal{R}$ is an $\mathbf{NP}^{opt}$-representation.*

Proof. Let $\mathcal{R}$ be any compact representation. Let $\mathcal{G}$ be a game in $\mathcal{C}(\mathcal{R})$, over the set N of players. Let x be a given payoff vector, and recall that the problem In-Core is to check whether x belongs to $\mathscr{C}(\mathcal{G})$. Consider the complementary problem of deciding whether $x \notin \mathscr{C}(\mathcal{G})$ holds, i.e., whether (i) $\sum_{i \in N} x_i \neq v^{\mathcal{R}}(N)$ or (ii) there is a coalition $S \subset N$ such that $\sum_{i \in S} x_i < v^{\mathcal{R}}(S)$. We shall show that conditions (i) and (ii) can be checked in **NP** with **P**-representations and $\mathbf{wNP}^{opt}$-representations, and that they can be checked in co-$\mathbf{D^P}$ with $\mathbf{NP}^{opt}$-representations.

(1) In the case where $\mathcal{R}$ is a $\mathbf{wNP}^{opt}$-representation, (i) is checkable in polynomial time, since $v^{\mathcal{R}}(N)$ is in polynomial time. We next show that (ii) can be checked in **NP**. Recall from Definition 3.5 that $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) = \max\{w \mid w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)\}$, where $f^{\mathcal{R}}$ is a feasibility function computable in **NPMV**. Therefore, for any coalition S, $\sum_{i \in S} x_i < v^{\mathcal{R}}(S)$ holds if and only if there is a value $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$ such that $\sum_{i \in S} x_i < w$. We now claim that the existence of a coalition S and a value $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$ with $\sum_{i \in S} x_i < w$ can be checked in **NP**. Indeed, since $f^{\mathcal{R}}$ is computable in **NPMV**, we can guess in **NP** not only the coalition S, but also the value w and a certificate c that $\langle(\xi^{\mathcal{R}}(\mathcal{G}), S), w\rangle \in graph(f^{\mathcal{R}})$. By exploiting c, we can then check in (deterministic) polynomial time that $\langle(\xi^{\mathcal{R}}(\mathcal{G}), S), w\rangle$ belongs to the graph of the function $f^{\mathcal{R}}$, i.e., that $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$ actually holds. Finally, we can check that $\sum_{i \in S} x_i < w$ holds.

(2) $\mathbf{NP}^{opt}$-representations generalize $\mathbf{wNP}^{opt}$-representations just in that the worth associated with the whole set N of players is not given in the input, but rather is of the form $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), N) = \max\{w \mid w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), N)\}$. Thus, from the proof of (1) above, we already know that condition (ii) can be checked in **NP**, even if $\mathcal{R}$ is an $\mathbf{NP}^{opt}$-representation. Consider now condition (i) and notice that $\sum_{i \in N} x_i \neq v^{\mathcal{R}}(N)$ if and only if at least one of the two following conditions holds: (i′) there is no feasible value $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), N)$ such that $\sum_{i \in N} x_i = w$; (i″) there actually is a feasible value $w^* \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), N)$ such that $w^* > \sum_{i \in N} x_i$. From Lemma 4.6(1), we know that (i″) is in **NP**. We now claim that (i′) is in co-**NP**. Indeed, we can solve the complementary problem of checking whether there actually is a feasible value $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), N)$ such that $\sum_{i \in N} x_i = w$ by guessing in **NP** the value w together with a certificate c, and then checking in polynomial time, by using c, that $\langle(\xi^{\mathcal{R}}(\mathcal{G}), N), w\rangle$ belongs to the graph of the worth function $f^{\mathcal{R}}$, and eventually that $\sum_{i \in N} x_i = w$ holds. By putting it all together, $x \notin \mathscr{C}(\mathcal{G})$ holds if and only if (i′) or (i″) or (ii) holds. Deciding whether (i″) or (ii) holds is in **NP** (just notice that the problem is the disjunction of two problems in **NP**), while deciding whether (i′) holds is in co-**NP**. Thus, the complement of In-Core is in co-$\mathbf{D^P}$. $\square$

As for the kernel and the bargaining set, we therefore immediately get the following result.

Corollary 6.6. *Let $\mathcal{R}$ be any $\mathbf{wNP}^{opt}$-representation such that $\mathcal{GG} \precsim_e \mathcal{R}$ (e.g., $\mathcal{R} = \mathcal{MCN}$). On the class $\mathcal{C}(\mathcal{R})$,* In-Core *is* co-**NP**-complete.

6.2. *Complexity of* Core-NonEmptiness

Recall from Example 6.2 that the core of a game can be empty, even if its set of imputations is not. Therefore, it makes sense to analyze the complexity of the Core-NonEmptiness problem of deciding whether, given a game $\mathcal{G}$, $\mathscr{C}(\mathcal{G}) \neq \emptyset$. This problem is known to be co-**NP**-hard for graph games [26]. Thus, by following exactly the same line of reasoning as in the proof of Corollary 4.5, the following can be established.

Proposition 6.7. *Let $\mathcal{R}$ be any compact representation such that $\mathcal{GG} \precsim_e \mathcal{R}$ (e.g., $\mathcal{R} = \mathcal{MCN}$). On the class $\mathcal{C}(\mathcal{R})$,* Core-NonEmptiness *is* co-**NP**-hard.

In fact, for the case of graph games, the precise complexity of the non-emptiness problem for the core is known, as Deng and Papadimitriou [26] have shown that this problem is co-**NP**-complete. However, it was left open whether the membership still holds for marginal contribution networks [44] and, possibly, for more general compact representations. In this section, we positively answer this question and show that membership in co-**NP** generally holds over $\mathbf{wNP}^{opt}$-representations.

In order to get this result, we have to identify some succinct certificate that the core of a game is empty. Indeed, it was observed that the "obvious" certificate of non-emptiness of the core is exponential in size [44]. Our crucial observation is that the technical machinery that we need is Helly's Theorem, which we have already used while analyzing the complexity of the bargaining set. By exploiting Helly's Theorem and the characterization of the core based on the linear inequalities in (2) and (3), we can indeed show the following.

Theorem 6.8. *Let $\mathcal{R}$ be any compact representation. On the class $\mathcal{C}(\mathcal{R})$,* Core-NonEmptiness *is*

(1) *in* co-**NP***, if $\mathcal{R}$ is a $\mathbf{wNP}^{opt}$-representation*; *and,*
(2) *in $\mathbf{\Delta_2^P}$, if $\mathcal{R}$ is an $\mathbf{NP}^{opt}$-representation.*

Proof. Let $\mathcal{R}$ be any compact representation, and let $\mathcal{G}$ be a game in $\mathcal{C}(\mathcal{R})$, over the set N of players. Recall that Core-NonEmptiness is the problem of deciding whether $\mathscr{C}(\mathcal{G})$ is not empty, and consider the complementary problem of checking whether $\mathscr{C}(\mathcal{G})$ is empty. We shall show that this latter problem is in **NP** for $\mathbf{wNP}^{opt}$-representations, and that it is in $\mathbf{\Delta_2^P}$ for arbitrary $\mathbf{NP}^{opt}$-representations (recall, here, that $\mathbf{\Delta_2^P}$ is closed under complement).

Let $w \in \mathbb{R}$ be any real number. For each coalition $S \subset N$, let $\varphi_{S,w}$ denote the convex set $\{x \in \mathbb{R}^N \mid x(S) \geqslant w\}$; and (for $S = N$) let $\varphi_{N,w}$ denote the convex set $\{x \in \mathbb{R}^N \mid x(N) = w\}$. By the characterization of the core in terms of the inequalities (2) and (3), we have that $\mathscr{C}(\mathcal{G}) = \bigcap_{\emptyset \neq S \subseteq N} \varphi_{S,v^{\mathcal{R}}(S)}$.

(1) Recall first that, in any $\mathbf{wNP}^{opt}$-representation $\mathcal{R}$, for each coalition $S \subset N$, $v^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S) = \max\{w \mid w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)\}$ holds, where $f^{\mathcal{R}}$ is a feasibility function computable in **NPMV**. Let (S, w) be a pair formed by a coalition S and a value w. The pair (S, w) is valid if either $S \subset N$ and $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$, or $S = N$ and $w = v^{\mathcal{R}}(N)$. Let VP be the set of all valid pairs. Note that $\varphi_{S,w} \supseteq \varphi_{S,v^{\mathcal{R}}(S)}$. Then, $\mathscr{C}(\mathcal{G}) = \bigcap_{(S,w) \in VP} \varphi_{S,w}$. If $\mathscr{C}(\mathcal{G})$ is empty then, by Helly's Theorem, there exists a family $\mathcal{E} \subseteq VP$ of $|N| + 1$ valid pairs (one-to-one associated with the above convex sets) such that $\bigcap_{(S,w) \in \mathcal{E}} \varphi_{S,w} = \emptyset$.

In order to complete the proof, we now claim that this latter condition can be checked in **NP**. To this end, observe that the system of the $|N| + 1$ linear inequalities defining the convex sets associated with $\mathcal{E}$ can be built in polynomial time. Indeed, we can guess in **NP** the set $\mathcal{E}$ together with a suitable certificate $c_{S,w}$, for each $(S, w) \in \mathcal{E}$. Then, for each $(S, w) \in \mathcal{E}$ such that $S \subset N$, we can use the certificate $c_{S,w}$ to actually check in polynomial time that $w \in f^{\mathcal{R}}(\xi^{\mathcal{R}}(\mathcal{G}), S)$, and hence to build the corresponding linear equality defining the convex set $\varphi_{S,w}$; for each $(S, w) \in \mathcal{E}$ such that $S = N$, we check in polynomial time that $w = v^{\mathcal{R}}(N)$ and we build the corresponding linear equality. Finally, we check in polynomial time that this system of (in)equalities has no solutions by standard techniques of linear programming.

(2) In the case where $\mathcal{R}$ is an $\mathbf{NP}^{opt}$-representation, we can apply the same line of reasoning as before, by just taking into account the additional complexity for computing $v^{\mathcal{R}}(N)$, which now may be a harder task. From Lemma 4.6(2), we know that this task is in $\mathbf{F\Delta_2^P}$. The statement then follows, since the remaining computation is in co-**NP** by (1) above (and since co-**NP** is contained in $\mathbf{\Delta_2^P}$). $\square$

The membership result (1) above and Proposition 6.7 imply that the non-emptiness problem for the core is co-**NP**-complete for all **P**-representations and all $\mathbf{wNP}^{opt}$-representations at least as expressive as graph games. Therefore, we can now state the precise complexity of Core-NonEmptiness even over marginal contribution networks, which was left open by Ieong and Shoham [44].

Corollary 6.9. *Let $\mathcal{R}$ be any $\mathbf{wNP}^{opt}$-representation such that $\mathcal{GG} \precsim_e \mathcal{R}$ (e.g., $\mathcal{R} = \mathcal{MCN}$). On the class $\mathcal{C}(\mathcal{R})$,* Core-NonEmptiness *is* co-**NP**-complete.

For the sake of completeness, note that the membership result in Theorem 6.8(2) can be tight in some cases, i.e., there exist $\mathbf{NP}^{opt}$-representations over which Core-NonEmptiness is $\mathbf{\Delta_2^P}$-hard. In fact, this is a trivial observation since worth functions satisfying Definition 3.3 can encode $\mathbf{\Delta_2^P}$-complete problems.[5] However, it tells us nothing about the hardness of specific $\mathbf{NP}^{opt}$ representations proposed in the literature. And, in fact, analyzing such representations to check whether they are any easier than the general case appears to be an interesting avenue of further research.

[5] Indeed, they have the power of **OptP** functions (see the note on p. 1886), and it is known that every function in $\mathbf{\Delta_2^P}$ decomposes into an **OptP** function, followed by a polynomial-time calculation [52].

6.3. A closer look at Helly's Theorem

As we have seen, Helly's Theorem guarantees the existence of "small" infeasibility certificates for problems based on families of convex subsets. Actually, our historically first proof of the computational complexity of core non-emptiness [53] was based on a different approach that exploited the fact that the core of any coalitional game is a *Polyhedron* of $\mathbb{R}^n$. We believe that this proof may be of independent interest for the reader and thus we describe it in the rest of this section. Indeed, it provides a nice geometrical and constructive view of "small" emptiness certificates for polyhedra, and thus a geometrical interpretation of Helly's Theorem, for those contexts where convex sets are given by linear inequalities like those involved in the core non-emptiness problem.[6]

6.3.1. Preliminaries on polyhedral sets

We begin by giving some useful definitions and facts about polyhedral sets. We refer the interested reader to textbooks on the subject for further reading (see, e.g., [42,14]).

Let $n > 0$ be any natural number. A *Polyhedral Set* (or *Polyhedron*) P of $\mathbb{R}^n$ is the intersection of a finite set $\mathcal{S}$ of closed halfspaces of $\mathbb{R}^n$. Note that in this paper we always assume, unless otherwise stated, that $n > 0$. We denote this polyhedron by $\mathrm{Pol}(\mathcal{S})$.

Recall that a *hyperplane* H of $\mathbb{R}^n$ is a set of points $\{x \in \mathbb{R}^n \mid a^T x = b\}$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The closed *halfspace* H^+ is the set of points $\{x \in \mathbb{R}^n \mid a^T x \geqslant b\}$. We say that these points *satisfy* H^+. We denote the points that do not satisfy this halfspace by H^-, i.e., $H^- = \mathbb{R}^n \setminus H^+ = \{x \in \mathbb{R}^n \mid a^T x < b\}$. Note that H^- is an open halfspace. We say that H *determines* H^+ and H^-. Define the *opposite* of H^+ as the set of points $\bar{H}^+ = \{x \in \mathbb{R}^n \mid a'^T x \geqslant b'\}$, where $a' = -1 \cdot a$ and $b' = -1 \cdot b$. Note that $\bar{H}^+ = H^- \cup H$, since it is the set of points $\{x \in \mathbb{R}^n \mid a^T x \leqslant b\}$.

Let $P = \mathrm{Pol}(\mathcal{S})$ be a polyhedron and H a hyperplane. We say that H *cuts* P if both H^+ and H^- contain points of P, and we say that H *passes through* P, if there is a non-empty touching set $C = H \cap P$. Furthermore, we say that H *supports* P, or that it is a *supporting hyperplane* for P, if H does not cut P, but passes through P, i.e., it just touches P (in the set C).

Moreover, we say that H^+ is a *supporting halfspace* for P if H is a supporting hyperplane for P and $P \subseteq H^+$. Note that $P \subseteq \mathrm{Pol}(\mathcal{S}')$ for any set of halfspaces $\mathcal{S}' \subseteq \mathcal{S}$, since the latter polyhedron is obtained from the intersection of a smaller set of halfspaces than P. We say that such a polyhedron is a *supporting polyhedron* for P.

Recall that, for any set $A \subseteq \mathbb{R}^n$, its dimension $\dim(A)$ is the dimension of its affine hull. For instance, if A consists of two points, or it is a segment, its affine hull is a line and thus $\dim(A) = 1$. By definition, $\dim(\emptyset) = -1$, while single points have dimension 0. For any hyperplane H, $\dim(H) = n - 1$, while the intersection C of any pair of (non-parallel) hyperplanes H_1 and H_2 has dimension $n - 2$.

Every hyperplane H has precisely two normal vectors, while its associated halfspace H^+ has one normal vector, that is, the normal vector of H that belongs to H^+. The *dihedral angle* $\delta(H_1^+, H_2^+)$ between two non-parallel halfspaces H_1^+ and H_2^+ is the smallest angle between the corresponding normal vectors. For such halfspaces, in this paper we always consider rotations whose axis is the (affine) subspace $C = H_1 \cap H_2$ of dimension $n - 2$, so we avoid to explicitly mention rotation axes, hereafter. Note that such rotations are uniquely identified by their amount (angle) of rotation, which is the one available degree of freedom. Formally, define the result of a *rotation* of a halfspace H_1^+ towards a halfspace H_2^+ of an angle $-\pi < \alpha < \pi$ to be the halfspace H_3^+ such that $\delta(H_3^+, H_2^+) = \delta(H_1^+, H_2^+) - \alpha$, and $H_i \cap H_j = C$, $\forall i, j \in \{1, 2, 3\}$, $i \neq j$ (the points on the axis C of rotation are fixed).

A set $F \subseteq P$ is a *face* of P if either $F = \emptyset$, or $F = P$, or if there exists a supporting hyperplane H_F of P such that F is their touching set, i.e., $F = H_F \cap P$. In the latter case, we say that F is a *proper face* of P. A *facet* of P is a proper face of P having the largest possible dimension, that is, whose dimension is $\dim(P) - 1$.

The following facts are well known [42]:

1. For any facet F of P, there is a halfspace $H^+ \in \mathcal{S}$ such that $F = H^+ \cap P$. We say that H^+ *generates* F.
2. For any proper face F of P, there is a facet F' of P such that $F \subseteq F'$.
3. If F and F' are two proper faces of P and $F \subset F'$, then $\dim(F) < \dim(F')$.

6.3.2. Separating polyhedra from a few supporting halfspaces

We start with a pair of technical results. Since coalitions correspond to the inequalities (2) and hence to the associated halfspaces of $\mathbb{R}^n$, we hereafter slightly abuse notation and use these terms interchangeably.

Lemma 6.10 *(Roof Lemma). Let H_1^+, H_2^+, and H_3^+ be three halfspaces such that $H_i \cap H_j = C \neq \emptyset$, $\forall i, j \in \{1, 2, 3\}$, $i \neq j$, and such that H_3^+ may be obtained by rotating H_1^+ towards H_2^+ by $\beta < 0$, with $\delta(H_2^+, H_1^+) - \beta < \pi$. Then, H_1^+ is a supporting halfspace for $H_2^+ \cap H_3^+$, i.e., for* $\mathrm{Pol}(\{H_2^+, H_3^+\})$.

[6] This section is self-contained and its results are not used in the subsequent sections. Thus, it can be skipped without troubles by a reader who is not interested in a deeper understanding of the existence of "small" infeasibility certificates.

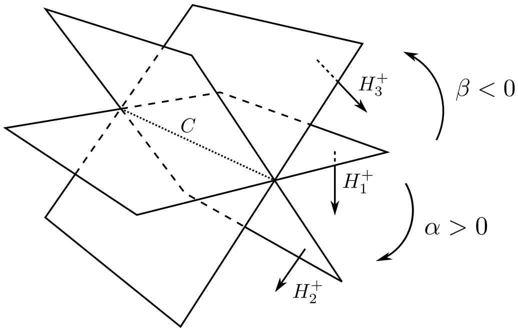

Fig. 5. Rotations of halfspaces in Lemma 6.10.

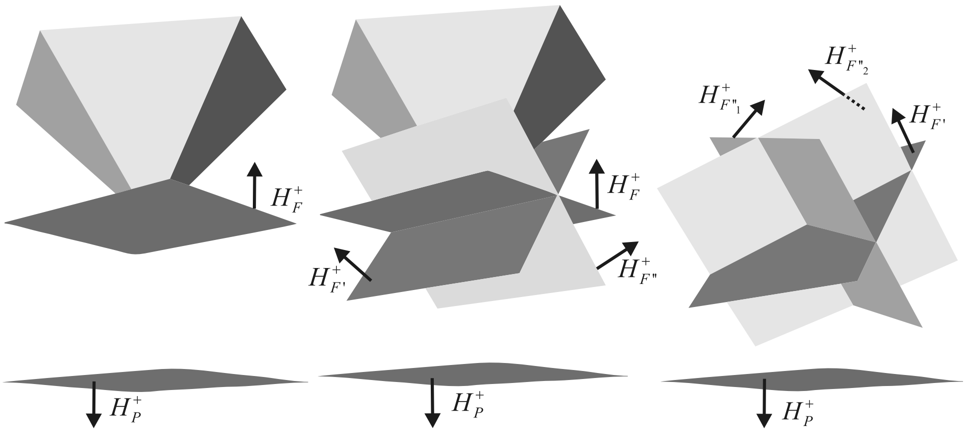

Fig. 6. Construction of a certificate of emptiness for the core.

Proof. Note that H_2^+ may be obtained by rotating H_1^+ towards H_2^+ by their dihedral angle $0 < \alpha = \delta(H_2^+, H_1^+) < \pi$. Thus, all points A of $H_1 \setminus C$ that belong to H_2^+ are in H_3^-, because H_3 is obtained by rotating those points (in the opposite direction) by the angle $\beta < 0$. Symmetrically, all points B of $H_1 \setminus C$ belonging to H_3^+ are in H_2^-, see Fig. 5, for a three-dimensional illustration. Moreover, observe that all points in $H_1 \setminus C$ are involved in the rotations and thus belong to either A or B. It follows that $(H_1 \setminus C) \cap (H_2^+ \cap H_3^+) = \emptyset$, whence H_1 is a supporting hyperplane. Finally, since $\delta(H_2^+, H_1^+) - \beta < \pi$, it is easy to see that some point in $H_2^+ \cap H_3^+$ is in H_1^+, and thus H_1^+ is in fact a supporting halfspace for $\mathrm{Pol}(\{H_2^+, H_3^+\})$. $\square$

We next show that, whenever a full-dimensional polyhedron $P = \mathrm{Pol}(\mathcal{S})$ of $\mathbb{R}^n$ is separated from some hyperplane H_P, there exists a subset of at most n halfspaces corresponding to facets of P that define a larger polyhedron of $\mathbb{R}^n$ (a rough approximation of P), which is still separated from H_P. We first give the proof idea, with the help of Fig. 6. For the sake of intuition, imagine that H_P^+ is the inequality (3) associated with the grand coalition, while the set $\mathcal{S}$ corresponds to the other inequalities (2), whence in this case the core is empty. If there is a facet F of P that is parallel to H_P, we are trivially done, because its associated halfspace already provides the desired separated polyhedron, succinctly described by just one inequality. Otherwise such a face F has a smaller dimension but, from Fact 2, there exists a facet F' of P such that $F \subset F'$. In the three-dimensional example shown in Fig. 6, F is the vertex at the bottom of the diamond, H_F^+ is the halfspace (anti-)parallel to H_P that contains F (but it is not associated with any inequality generating P), and F' is some facet on its "dark side". Let $C = H_F \cap H_{F'}$, and consider the rotation of H_F^+ towards $H_{F'}^+$ of a negative angle β (we go on the opposite direction w.r.t. $H_{F'}^+$) that first touches P, say $H_{F''}$. As illustrated in Fig. 6, where the face F'' is an edge of the diamond, F'' properly includes F and its dimension is at least $d > \dim(F)$. From the above lemma, H_F^+ is a supporting halfspace of the polyhedron $\mathrm{Pol}(H_{F'}^+, H_{F''}^+)$, called its roof, which contains P and is separated from H_P. However, we are not satisfied because we would like that such a polyhedron is described by (at most n) halfspaces *taken from* $\mathcal{S}$, and $H_{F''}^+$ does not belong to $\mathcal{S}$, in general (as in our example, where it does not generate a facet of P). Then, we proceed inductively, by observing that $H_{F''}^+$ is a supporting halfspace for $\mathrm{Pol}(H_{F''_1}^+, H_{F''_2}^+)$—its roof, whose faces have higher dimension than F''. In the running example, they are both facets of the diamond, and hence the property immediately holds. In general, the procedure may continue, encountering each time at least one facet and one more face with a higher dimension than the current one. The formal proof follows.

Lemma 6.11. *Let $P = \mathrm{Pol}(\mathcal{S})$ be a polyhedron of $\mathbb{R}^n$ with $\dim(P) = n$, and H_F^+ a supporting halfspace of P whose touching set is F. Then, there exists a set of halfspaces $\mathcal{H}_F \subseteq \mathcal{S}$ such that $|\mathcal{H}_F| \leqslant n - \dim(F)$, H_F^+ is a supporting halfspace of $\mathrm{Pol}(\mathcal{H}_F)$, and their touching set C is such that $F \subseteq C$.*

Proof. The proof is by induction. *Base case*: If $\dim(F) = n - 1$ we have that the touching face $F = H_F^+ \cap P$ is a facet of P. Thus, from Fact 1, F is generated by some halfspace $H^+ \subseteq \mathcal{S}$ such that $H^+ \cap P = F$, as for H_F. Since $\dim(F) = \dim(H) = \dim(H_F) = n - 1$, it easily follows that, in fact, $H = H_F$ holds. Thus, H_F^+ is trivially a supporting halfspace of H^+, and this case is proved: just take $\mathcal{H}_F = \{H^+\}$ and note that $|\mathcal{H}_F| = 1$.

Inductive step: By the induction hypothesis, the property holds for any supporting halfspace $H_{F'}^+$ of P such that its touching face F' has a dimension $d \leqslant \dim(F') \leqslant n - 1$, for some $d > 0$. We show that it also holds for any supporting halfspace H_F^+ of P, whose touching face F has a dimension $\dim(F) = d - 1$.

Since F is not a facet, from Fact 2 there exists a facet F' of P such that $F \subset F'$. Let $C = H_F \cap H_{F'}$, where $H_{F'}^+ \in \mathcal{S}$ is the halfspace that generates the facet F'. Note that $F \subseteq C$. Let $H_{F''}^+$ be the halfspace that first touches the polyhedron P obtained by rotating H_F^+ towards $H_{F'}^+$ by some negative angle β (i.e., we are going in the opposite direction w.r.t. $H_{F'}^+$). Then, $H_{F''}^+$ is a supporting halfspace of P, and the touching set F'' includes C. Since $\dim(P) = n$, we have that $\delta(H_F^+, H_{F'}^+) - \beta < \pi$ and $F \subset F''$ (as the latter face contains some point of the polyhedron outside the axis $C \supseteq F$), and thus the dimension $\dim(F'')$ of this face is strictly greater than $\dim(F)$, from Fact 3. Moreover, from Lemma 6.10, H_F^+ is a supporting halfspace of $\mathrm{Pol}(H_{F'}^+, H_{F''}^+)$, with $C = H_{F'} \cap H_{F''}$ as touching set.

By the induction hypothesis, since both $\dim(F')$ and $\dim(F'')$ are at least d, we know that there are two sets $\mathcal{H}_{F'} \subseteq \mathcal{S}$ and $\mathcal{H}_{F''} \subseteq \mathcal{S}$ such that: $H_{F'}^+$ is a supporting halfspace of $\mathrm{Pol}(\mathcal{H}_{F'})$, with $F' \subseteq C'$, where C' is their touching set; and $H_{F''}^+$ is a supporting halfspace of $\mathrm{Pol}(\mathcal{H}_{F''})$, with $F'' \subseteq C''$, where C'' is their touching set. In particular, $\mathrm{Pol}(\mathcal{H}_{F'}) \subseteq H_{F'}^+$ and $\mathrm{Pol}(\mathcal{H}_{F''}) \subseteq H_{F''}^+$, respectively. Let $\mathcal{H}_F = \mathcal{H}_{F'} \cup \mathcal{H}_{F''}$. Then, we get $\mathrm{Pol}(\mathcal{H}_F) \subseteq \mathrm{Pol}(\{H_{F'}^+, H_{F''}^+\}) \subseteq H_F^+$.

Recall that $F \subseteq C$. Moreover, $F \subset F'$ and $F \subset F''$, and thus $F \subseteq (F' \cap F'') \subseteq (C' \cap C'')$, since $F' \subseteq C'$ and $F'' \subseteq C''$. It follows that the touching set of H_F^+ and $\mathrm{Pol}(\mathcal{H}_F)$ includes F. Indeed, this touching set includes $(C' \cap C'') \cap H_F \supseteq (C' \cap C'') \cap (H_F \cap H_{F'} \cap H_{F''}) = (C' \cap C'') \cap C$, and from the above observations $(C' \cap C'') \cap C \supseteq F$.

Finally, note that $|\mathcal{H}_F| \leqslant |\mathcal{H}_{F'}| + |\mathcal{H}_{F''}| = 1 + |\mathcal{H}_{F''}|$, because $\dim(F') = n - 1$ and the base case applies. Moreover, $\dim(F'') > \dim(F) = d - 1$ and thus, by the induction hypothesis, we obtain $|\mathcal{H}_F| \leqslant 1 + n - \dim(F'') \leqslant 1 + n - d = n - \dim(F)$. □

Let $\mathcal{G} = \langle N, v\rangle$ be a coalitional game. A coalition set $\mathcal{S} \subseteq 2^N$ is a *certificate of emptiness* (or *infeasibility certificate*) for the core of $\mathcal{G}$ if the intersection of $\mathrm{Pol}(\mathcal{S})$ with the grand coalition halfspace (3) is empty. In fact, this definition is motivated by the following observation. Let P be the polyhedron of $\mathbb{R}^n$ obtained as the intersection of all halfspaces (2). Since $\mathcal{S}$ is a subset of all possible coalitions, $P \subseteq \mathrm{Pol}(\mathcal{S})$. Therefore, if the intersection of $\mathrm{Pol}(\mathcal{S})$ with the grand coalition halfspace (3) is empty, the intersection of this halfspace with P is empty, as well.

Theorem 6.12. *Let $\mathcal{G} = \langle N, v\rangle$ be a coalitional game. If the core of $\mathcal{G}$ is empty, there is a certificate of emptiness $\mathcal{S}$ for it such that $|\mathcal{S}| \leqslant |N|$.*

Proof. Let $n = |N|$ and P be the polyhedron of $\mathbb{R}^n$ obtained as the intersection of all halfspaces (2). Since we are not considering the feasibility constraint (3), there is no upper-bound on the values of any variable x_i, and thus it is easy to see that $P \neq \emptyset$ and $\dim(P) = n$.

Let H_P^+ be the halfspace defined by the grand coalition inequality (3). If the core of $\mathcal{G}$ is empty, the whole set of inequalities has no solution, that is, $P \cap H_P^+ = \emptyset$.

Let $\bar{H}_F^+$ be the halfspace parallel to H_P^+ that first touches P, that is, the smallest relaxation of H_P^+ that intersect P. Consider the opposite H_F^+ of $\bar{H}_F^+$, as shown in Fig. 6, on the left. By construction, $H_P^+ \cap H_F^+ = \emptyset$, $H_F = \bar{H}_F$ is a supporting hyperplane of P, and H_F^+ is a supporting halfspace of P. Let F be the touching set of H_F with P, and let $d = \dim(F)$. (In Fig. 6, F is the vertex at the bottom of the diamond P.) From Lemma 6.11, there is a set of halfspaces $\mathcal{S}$ associated with inequalities from (2), with $|\mathcal{S}| \leqslant n - d$, and such that H_F^+ is a supporting halfspace for $\mathrm{Pol}(\mathcal{S})$. It follows that $H_P^+ \cap \mathrm{Pol}(\mathcal{S}) = \emptyset$, whence $\mathcal{S}$ is a certificate of emptiness for the core of $\mathcal{G}$. Finally, note that the largest cardinality of $\mathcal{S}$ is n, and corresponds to the case $\dim(F) = 0$, that is, to the case where the face F is just a vertex. Therefore the maximum cardinality of the certificate is n. In our three-dimensional example, the certificate is $\{H_{F'}^+, H_{F_1''}^+, H_{F_2''}^+\}$, as shown in Fig. 6, on the right. □

7. Tractable classes of graph games

Many **NP**-hard problems in different application areas ranging, e.g., from AI [67] and Database Theory [11] to Game Theory [21], are known to be efficiently solvable when restricted to instances whose underlying structures can be modeled via *acyclic* graphs or *nearly-acyclic* ones, such as those graphs having *bounded treewidth* [70]. Indeed, on these kinds of instances, solutions can usually be computed via dynamic programming, by incrementally processing the acyclic (hyper)graph, according to some of its topological orderings. In this section, we shall show that (near) acyclicity is also a key for the tractability of coalitional games represented in terms of marginal contribution nets.

Our results are established by showing that the various solution concepts studied in this paper can be expressed in terms of optimization problems over *Monadic Second Order Logic (MSO)* formulae, and by subsequently applying Courcelle's Theorem [19] and its generalization to optimization problems due to Arnborg, Lagergren, and Seese [2].

For the sake of completeness, we start by reviewing the concepts of treewidth and Monadic Second Order Logic.

7.1. Treewidth and Monadic Second Order logic

Treewidth. A *tree decomposition* of a graph $G = (N, E)$ is a pair $\langle T, \chi\rangle$, where $T = (V, F)$ is a tree, and χ is a labeling function assigning to each vertex $p \in V$ a set of vertices $\chi(p) \subseteq N$, such that the following conditions are satisfied:

(1) for each node b of G, there exists $p \in V$ such that $b \in \chi(p)$;
(2) for each edge $(b, d) \in E$, there exists $p \in V$ such that $\{b, d\} \subseteq \chi(p)$; and,
(3) for each node b of G, the set $\{p \in V \mid b \in \chi(p)\}$ induces a connected subtree of T.

The *width* of $\langle T, \chi\rangle$ is the number $\max_{p\in V}(|\chi(p)| - 1)$. The *treewidth* of G, denoted by $tw(G)$, is the minimum width over all its tree decompositions. A graph G is acyclic if and only if $tw(G) = 1$. Deciding if a given graph has treewidth bounded by a fixed natural number k is known to be feasible in linear time [13].

A *finite structure* $\mathcal{A}$ consists of a domain A and relations $R_1, \ldots, R_k$ of arities $a_1, \ldots, a_k$, respectively. Each relation R_i consists of a set of tuples $(e_1, \ldots, e_{a_i})$, where $e_j \in A$, for each $1 \leqslant j \leqslant a_i$. The *size* of $\mathcal{A}$, denoted by $\|\mathcal{A}\|$, is the value $\|\mathcal{A}\| = |A| + \sum_{j=1}^{k} |R_j| \times a_j$. Note that a graph $G = (N, E)$ can be viewed as a finite structure whose domain is N, and where E is a binary relation encoding its edges.

The *Gaifman graph* of a finite structure $\mathcal{A}$ is the undirected graph $G(\mathcal{A})$ whose vertices are the elements of the domain of $\mathcal{A}$, and where there is an edge between the elements e and e' if and only if there is a tuple of some relation of $\mathcal{A}$ where e and e' jointly occur. The treewidth of $\mathcal{A}$, denoted by $tw(\mathcal{A})$, is the treewidth of its Gaifman graph, i.e., $tw(\mathcal{A}) = tw(G(\mathcal{A}))$.

MSO. A First Order logic formula is made up of relation symbols, individual variables (usually denoted by lowercase letters), the logical connectives $\vee$, $\wedge$, and $\neg$, and the quantifiers $\exists$ and $\forall$. Monadic Second Order (MSO) enhances the expressiveness of first order logic by allowing the use of set variables (usually denoted by uppercase letters), of the membership relation $\in$, and of the quantifiers $\exists$ and $\forall$ over set variables. In addition, it is often convenient to use symbols like $\subseteq$, $\subset$, $\cap$, $\cup$, and $\rightarrow$ with their usual meaning, as abbreviations. When an MSO formula ϕ is evaluated over a finite structure $\mathcal{A}$, the relation symbols of ϕ are interpreted as the corresponding relations of $\mathcal{A}$ and the variables of ϕ range over the domain A of $\mathcal{A}$. The fact that an MSO formula ϕ holds over $\mathcal{A}$ is denoted by $\mathcal{A} \models \phi$.

Example 7.1. Let $G = (N, E)$ be an undirected graph (interpreted as a finite structure). Then, the fact that G is 3-*colorable* can be expressed via the following MSO formula:

$$\begin{aligned}
\exists R, B, Y,\quad & R \cup B \cup Y = N \wedge \\
& R \cap B = \emptyset \ \wedge\ R \cap Y = \emptyset \ \wedge\ B \cap Y = \emptyset \wedge \\
& \forall x,\ x \in B \rightarrow \big(\forall y, \{x, y\} \in E \rightarrow \neg(y \in B)\big) \wedge \\
& \forall x,\ x \in R \rightarrow \big(\forall y, \{x, y\} \in E \rightarrow \neg(y \in R)\big) \wedge \\
& \forall x,\ x \in Y \rightarrow \big(\forall y, \{x, y\} \in E \rightarrow \neg(y \in Y)\big)
\end{aligned}$$

In particular, note that the formula checks whether there exists a partition of the nodes in N into three disjoint sets of nodes R, B, and Y, which respectively correspond to the nodes that are colored red, blue, and yellow. Moreover, the formula checks that for each node, all its adjacent nodes are colored with a different color.

The relationship between treewidth and MSO is illustrated next.

Proposition 7.2 *(Courcelle's Theorem [19]). Let ϕ be a fixed* MSO *sentence, let k be a fixed constant, and let $\mathcal{C}_k$ be a class of finite structures having treewidth bounded by k. Then, for each finite structure $\mathcal{A} \in \mathcal{C}_k$, deciding whether $\mathcal{A} \models \phi$ holds is feasible in linear time* (*w.r.t.* $\|\mathcal{A}\|$).

For instance, from the above theorem and Example 7.1, we can immediately conclude that 3-colorability is a property that can be checked in polynomial time on classes of graphs having bounded treewidth, while, on arbitrary classes of graphs, the problem is known to be **NP**-complete (see, e.g., [35]).

An important generalization of MSO formulae to *optimization* problems was presented by Arnborg et al. [2]. Next, we state a simplified definition of these kinds of problems.

Let $\mathcal{A}$ be a finite structure over the domain A, and let w be a list of weights associated with the elements in A, such that $w(e)$ is a rational number for each $e \in A$. The pair $\langle \mathcal{A}, w \rangle$ is hereinafter called a *weighted finite structure*, and its size $\|\langle \mathcal{A}, w \rangle\|$ is defined as the size of $\mathcal{A}$ plus all the values (numerators and denominators) in w.

Let $\phi(\bar{X})$ be an MSO formula over $\mathcal{A}$, where $\bar{X}$ is the set of free variables occurring in ϕ. For an interpretation $\mathcal{I}$ mapping variables in $\bar{X}$ to subsets of A, we denote by $\phi[\mathcal{I}]$ the MSO formula (without free variables) where each variable $X \in \bar{X}$ is replaced by $\mathcal{I}(X)$.

A *solution* to ϕ over $\langle \mathcal{A}, w \rangle$ is an interpretation $\mathcal{I}$ such that $\mathcal{A} \models \phi[\mathcal{I}]$ holds. The *cost* of $\mathcal{I}$ is the value $\sum_{X \in \bar{X}} \sum_{e \in \mathcal{I}(X)} w(e)$. A solution of minimum cost is said *optimal*.

Example 7.3. Let $G = (N, E)$ be an undirected graph (interpreted as a finite structure). Then, the property that a set X of nodes is a *vertex cover*, i.e., a set such that each edge in E has at least one endpoint incident on it, can be expressed via the following MSO formula where X is its free variable:

$$\mathit{vertexCover}(X) \equiv X \subseteq N \wedge \big(\forall y,\, y \in N \rightarrow (y \in X) \vee \big(\exists x \in X,\ \{x, y\} \in E\big)\big)$$

By considering a list w of weights assigning 1 to each node in N, we have that an optimal solution to *vertexCover* over $\langle G, w \rangle$ is a minimum-cardinality vertex cover.

The fact that, over bounded treewidth structures, deciding the existence of a solution to a fixed MSO sentence is feasible in polynomial time is the well-known result by Courcelle. The result below evidences that not only the decision problem, but even the associated problem of *computing* a solution of minimum cost is feasible in polynomial time on such structures.

Theorem 7.4 *(Simplified from Arnborg et al. [2]). Let ϕ be a fixed* MSO *sentence, let k be a fixed constant, and let $\mathcal{C}_k$ be a class of finite structures having treewidth bounded by k. Then, for each weighted finite structure $\langle \mathcal{A}, w \rangle$ such that $\mathcal{A} \in \mathcal{C}_k$, computing an optimal solution to ϕ over $\langle \mathcal{A}, w \rangle$ is feasible in polynomial time (w.r.t. $\|\langle \mathcal{A}, w \rangle\|$).*

For instance, from the above theorem and Example 7.3, we can immediately conclude that computing a minimum-cardinality vertex cover is feasible in polynomial time on classes of graphs having bounded treewidth whereas, on arbitrary classes of graphs, it is **NP**-hard (see, e.g., [4]). A further example is discussed below.

Example 7.5. A Boolean formula F in conjunctive normal form over a set V of variables can be interpreted as a structure where $cl(z)$ (resp., $var(z)$) means that z is a clause (resp., a variable) in F, and where $pos(x, c)$ (resp., $neg(x, c)$) means that x occurs in a positive (resp., negative) literal in the clause c. Then, the property that a set X of variables is a model for F can be expressed via the following MSO formula:

$$\mathit{model}(X) \equiv X \subseteq V \wedge \forall c, cl(c) \rightarrow \exists x, \big(\big(x \in X \wedge pos(x, c)\big) \vee \big(x \notin X \wedge neg(x, c)\big)\big)$$

By considering a list w of weights assigning 1 to each variable in V, we have that an optimal solution to *model* over $\langle F, w \rangle$ is a minimum-sized model, i.e., a satisfying truth assignment whose number of variables evaluating to true is the minimum possible one over all possible satisfying truth assignments.

7.2. MSO encoding for marginal contribution networks

Seminal results on the tractability of special classes of marginal contribution nets have been discussed by Ieong and Shoham [44], who showed that In-Core and Core-NonEmptiness are tractable on classes of MC-nets whose associated agent graphs have bounded treewidth. However, as observed in the Introduction, since for each rule $\{\mathit{pattern}\} \rightarrow \mathit{value}$ the subgraph induced by the players occurring together in *pattern* is a clique in the agent graph, by using this representation one may overvalue the actual intricacy of the game. For instance, classes of MC-nets with one rule only involving all the players of the game do not fall in the tractable classes analyzed by Ieong and Shoham [44], as the associated agent graph consists of a clique over all the players (whose treewidth is thus not bounded by any fixed constant). After this observation, we next explore the possibility of isolating larger islands by encoding MC-nets in terms of suitable finite structures.

Let M be a marginal contribution network whose sets of players and rules are P and R, respectively. We encode M in terms of a finite structure $\mathcal{A}_M$ defined over the universe $P \cup R$, over the unary relations *player* and *rule*, and over the binary relations *pos* and *neg*. Tuples in the relations are defined as follows: for each player $p \in P$, the tuple (p) is in *player*; for each rule $r \in R$, the tuple (r) is in *rule*; for each rule r of M having the form $\{\mathit{pattern}\} \rightarrow \mathit{value}$, and for each player p occurring positively (resp., negatively) in *pattern*, the tuple (p, r) is in *pos* (resp., *neg*). For a marginal contribution network M, we define its *incidence graph* $\mathtt{IG}(M)$ as the Gaifman graph of the structure $\mathcal{A}_M$. Note that $\mathtt{IG}(M)$ is a bipartite graph, whose set of nodes consists of the players and the rules in M, and where there is an edge between a player p and a rule r if and only if p occurs in the pattern of r.

Example 7.6. Consider again the marginal contribution network M of Example 1.3 (on p. 1880), which consists of the rules $r_1 : \{a \wedge b \wedge c\} \rightarrow 2$, $r_2 : \{c \wedge d \wedge e\} \rightarrow 1$, $r_3 : \{e\} \rightarrow -1$, over the players in $\{a, b, c, d, e\}$. Then, the finite structure $\mathcal{A}_M$ associated with M is such that: $\mathit{players} = \{(a), (b), (c), (d), (e)\}$, $\mathit{rules} = \{(r_1), (r_2), (r_3)\}$, $\mathit{pos} = \{(a, r_1), (b, r_1), (c, r_1), (c, r_2), (d, r_2), (e, r_2), (e, r_3)\}$, and $\mathit{neg} = \{\}$.

Fig. 2 reports, on the right, the incidence graph associated with the MC-net of Example 1.3. Note that $\mathtt{IG}(M)$ is acyclic, while the agent graph $\mathtt{AG}(M)$, reported on the left of the figure, contains cycles.

For each natural number $n > 0$, let M_n denote the network over the players $p_1, \ldots, p_n$ and just containing the rule $\{p_1 \wedge \cdots \wedge p_n\} \to 0$. Then, it is easily seen that $tw(\mathtt{IG}(M_n)) = 1$, i.e., the incidence graph is acyclic, while $tw(\mathtt{AG}(M_n)) = n-1$. Therefore, the class of all such MC-nets has bounded treewidth (actually, treewidth 1) if incidence graphs are considered, while it has unbounded treewidth if agent graphs are considered. In fact we next show that, as long as one is interested in identifying structurally restricted classes of tractable MC-nets, the incidence graph encoding is always preferable to the agent-graph one, as classes of games whose agent graphs have bounded treewidth are such that their associated incidence graphs have bounded treewidth too (and we have just observed that the inverse does not hold).

Proposition 7.7. *For any MC-net M, $tw(\mathtt{IG}(M)) \leqslant tw(\mathtt{AG}(M)) + 1$.*

Proof. Let M be an MC-net, and let $\langle T, \chi\rangle$ be a tree decomposition of $\mathtt{AG}(M)$, where $T = (V, F)$ is a tree. Note that for each rule $r : \{pattern\} \to value$ in M, the subgraph of $\mathtt{AG}(M)$ induced over the set P_r of all the players occurring in *pattern* is a clique. Thus, there is a vertex $v_r \in V$ such that $P_r \subseteq \chi(v_r)$—in case where several vertices cover the variables in P_r, let v_r denote an arbitrarily chosen one enjoying the property.

Based on $\langle T, \chi\rangle$, we build a pair $\langle T', \chi'\rangle$ where T' is the tree obtained from T by creating a novel vertex w_r, for each rule r of M, and by connecting w_r to the vertex v_r. Moreover, let us define $\chi'(v) = \chi(v)$, for each $v \in T$, and $\chi'(w_r) = P_r \cup \{r\}$, for each vertex $w_r \in T' \setminus T$. It is immediate to check that $\langle T', \chi'\rangle$ is a tree decomposition of $\mathtt{IG}(M)$. To this end, note that all the edges and nodes of $\mathtt{IG}(M)$ are covered in the vertices of the form w_r, that the connectedness condition over the player-nodes is guaranteed by the connectedness condition on $\langle T, \chi\rangle$, and that the connectedness condition over rule-nodes is guaranteed by the fact that each such node exactly occurs in one vertex. The result follows by the fact that the width of $\langle T', \chi'\rangle$ is bounded by the width of $\langle T, \chi\rangle$ plus 1. □

7.3. Complexity results

Now that the framework for encoding the structure of marginal contribution networks has been illustrated, a natural question arises, that is, whether bounded treewidth over incidence graph encodings is a guarantee for the tractability of the solution concepts studied in the paper. In the rest of the section, we shall provide an answer to this question. The answer is positive, at least for those cases where the values occurring in the network are polynomially bounded in the number of players and rules.

For any natural number k, let $\mathcal{MCN}_k$ be the class of marginal contribution nets such that, for each $M \in \mathcal{MCN}_k$, $tw(\mathtt{IG}(M)) \leqslant k$ and the value of each rule of M is polynomially bounded by the size of $\mathcal{A}_M$. Moreover, for each marginal contribution network M, let $\|M\|$ denote its size, naturally measured as the size of $\mathcal{A}_M$ plus the number of bits necessarily to encode all the values in the rules of M (in binary).

We next provide our results concerning the tractability of the problems IN-CORE, CORE-NONEMPTINESS and IN-KERNEL over the class $\mathcal{MCN}_k$. Note that the results pertaining to the core are incomparable to those by Ieong and Shoham [44]. Indeed, on the one hand, we consider classes of marginal contribution networks that are larger in terms of their structure but, on the other hand, we restrict ourselves to deal with “small” values. In particular, these islands of tractability lead to pseudo-polynomial algorithms when values are not small, and the question is open about whether (full) polynomial-time algorithms exist for MC-nets with bounded-treewidth incidence graphs.

Theorem 7.8. *On the class $\mathcal{MCN}_k$, IN-CORE is feasible in polynomial time.*

Proof. Let M be a marginal contribution network in $\mathcal{MCN}_k$, and let x be a payoff vector. We have to check whether x belongs to the core of the game induced by M. Assume w.l.o.g. that x is an imputation. Indeed, over marginal contribution networks, this latter property can be checked in polynomial time. Let $\langle \mathcal{A}_M, w\rangle$ be the weighted relational structure where $\mathcal{A}_M$ is the relational encoding of M, and where $w(r) = -value$, for each rule $r : \{pattern\} \to value$, and $w(p) = x_p$, for each player p in M.

Consider the following MSO formulae:

$$rules(X) \equiv \forall r, r \in X \to r \in rule$$

$$players(Y) \equiv \forall p, p \in Y \to p \in player$$

$$active(X, Y) \equiv \forall p, \forall r, \big((p, r) \in pos \wedge r \in X \to p \in Y\big) \wedge \big((p, r) \in neg \wedge r \in X \to p \notin Y\big)$$

$$maxActive(X, Y) \equiv active(X, Y) \wedge \big(\forall X', X' \supset X \to \neg active\big(X', Y\big)\big)$$

$$excess(W) \equiv \exists X, Y\colon\ rules(X) \wedge players(Y) \wedge X \cup Y = W \wedge maxActive(X, Y)$$

and note that an interpretation $\mathcal{I}$ such that $\mathcal{A}_M \models excess[\mathcal{I}]$ is just a mapping from the set W (of the free variables in the formula) to a disjoint union of two sets X and Y of rules and players, respectively, where X consists of the rules of M that apply to coalition Y. The cost of $\mathcal{I}$ is therefore the value $\sum_{r\in X} w(r) + \sum_{p\in Y} w(p) = x(Y) - v^{\mathcal{MCN}}(Y)$, where $v^{\mathcal{MCN}}$ is the worth function encoded via the network M.

From the above observation, it follows that there is a one-to-one correspondence between optimal solutions to the formula *excess* and the sets Y^* of players such that $x(Y^*) - v^{\mathcal{MCN}}(Y^*) \leqslant x(Y) - v^{\mathcal{MCN}}(Y)$, for each coalition Y. Thus, the imputation x belongs to the core of the game if and only if the cost of any optimal solution is non-negative.

To conclude the proof, observe now that for any fixed constant k, by Theorem 7.4, an optimal solution (and thus its associated cost) can be computed in polynomial time (w.r.t. $\|\langle \mathcal{A}_M, w\rangle\|$) on each class $\mathcal{C}_k$ of structures such that every element $\mathcal{A}_M \in \mathcal{C}_k$ has treewidth bounded by k, i.e., such that $tw(\mathtt{IG}(M)) \leqslant k$. Recall now that the size $\|\langle \mathcal{A}_M, w\rangle\|$ is defined as the size of $\mathcal{A}_M$ plus all the values in w (or, equivalently plus the number of bits that are necessary to encode the weights in unary). Whenever the value of each rule of M is polynomially bounded by the size of $\mathcal{A}_M$, we have that an optimal solution can be computed in polynomial time w.r.t. $\|M\|$ (i.e., w.r.t. the size of $\mathcal{A}_M$ plus the number of bits that are necessary to encode all the values in the rules of M in binary). Thus, on the class $\mathcal{MCN}_k$, In-Core is feasible in polynomial time. □

The proof of the above results provides us with a key ingredient to show the tractability of Core-NonEmptiness on the class $\mathcal{MCN}_k$. Indeed, Ieong and Shoham [44] observed that Core-NonEmptiness can be solved in polynomial time via the ellipsoid method, provided that a polynomial time *separation oracle* exists for the problem of checking whether a given imputation x is not in the core. A separation oracle is any procedure that either confirms that x belongs to the core, or reports a coalition S witnessing that this is not the case (i.e., such that $x(S) - v^{\mathcal{MCN}}(S) < 0$). Computing an optimal solution to the MSO formula *excess* in the proof of Theorem 7.8 immediately provides us with such a separation oracle, which is computable in polynomial time on the class $\mathcal{MCN}_k$. Thus, the following is established.

Theorem 7.9. *On the class $\mathcal{MCN}_k$,* Core-NonEmptiness *is feasible in polynomial time.*

We now conclude the analysis by considering the kernel.

Theorem 7.10. *On the class $\mathcal{MCN}_k$,* In-Kernel *is feasible in polynomial time.*

Proof. Let x be a given payoff vector. Since we can check in polynomial time whether x is an imputation, by Definition 4.1 we have just to show that checking whether $s_{i,j}(x) > s_{j,i}(x) \Rightarrow x_j = v(\{j\})$ holds is feasible in polynomial time (on the class $\mathcal{MCN}_k$), for each pair of players $i \neq j$. We shall prove that $s_{i,j}(x)$ can be computed in polynomial time for any pair of players, which clearly entails the tractability result. In particular, recall that $s_{i,j}(x) = \max_{S\in\mathcal{I}_{i,j}} e(S,x)$, so that $s_{i,j}(x) = \max_{S\in\mathcal{I}_{i,j}}(v^{\mathcal{MCN}}(S) - x(S)) = -\min_{S\in\mathcal{I}_{i,j}}(x(S) - v^{\mathcal{MCN}}(S))$. Thus, it can equivalently be shown that we can compute in polynomial time the value $\min_{S\in\mathcal{I}_{i,j}}(x(S) - v^{\mathcal{MCN}}(S))$.

To this end, given a marginal contribution network $M \in \mathcal{MCN}_k$, consider the relational structure $\mathcal{A}_M^+$ obtained from $\mathcal{A}_M$ by adding the unary relations *in* and *out*, with $in = \{(i)\}$ and $out = \{(j)\}$. Note that $\mathcal{A}_M$ and $\mathcal{A}_M^+$ have precisely the same Gaifman graph (thus, the modification does not alter their structural properties).

Consider now the MSO formulae in the proof of Theorem 7.8, with the following change:

$$\begin{aligned} excess^+(W) \equiv\ & \exists X, Y\colon\ rules(X) \wedge players(Y) \wedge X \cup Y = W \wedge maxActive(X,Y) \wedge \\ & (\forall p,\ p \in in \rightarrow p \in Y) \wedge (\forall p,\ p \in out \rightarrow p \notin Y) \end{aligned}$$

An interpretation $\mathcal{I}$ such that $\mathcal{A}_M^+ \models excess[\mathcal{I}]$ is now a mapping from the set W to a disjoint union of two sets X and Y of rules and players, respectively, where, in particular, X consists of the rules of M that apply to coalition Y, and where Y contains player i and does not contain player j. Thus, by considering the weighting function w in the proof of Theorem 7.8, we have that optimal solutions one-to-one correspond to coalitions $S \in \mathcal{I}_{i,j}$ minimizing the expression $(x(S) - v^{\mathcal{MCN}}(S))$ over all the possible coalitions in $\mathcal{I}_{i,j}$. By Theorem 7.4, an optimal solution (and thus its associated cost) can be computed in polynomial time on each class $\mathcal{C}_k^+$ of structures such that, for each $\mathcal{A}_M^+ \in \mathcal{C}_k^+$, $tw(\mathcal{A}_M^+) = tw(\mathtt{IG}(M)) \leqslant k$ holds, and such that the value of each rule of M is polynomially bounded by the size of $\mathcal{A}_M^+$. The tractability on $\mathcal{MCN}_k$ immediately follows from the fact that, given any marginal net $M \in \mathcal{MCN}_k$, the corresponding structure $\mathcal{A}_M^+ \in \mathcal{C}_k^+$ can be built in polynomial time, and it is such that $tw(\mathcal{A}_M) = tw(\mathcal{A}_M^+)$. □

By combining the above three results with Proposition 7.7 and with the fact that any graph game can be encoded as a marginal contribution network having the same agent graph structure, we get the following tractability results for graph games.

Corollary 7.11. *Let $\mathcal{GG}_k$ be any class of graph games such that, for each $\mathcal{G} \in \mathcal{GG}_k$, $tw(\mathcal{G}) \leqslant k$ and all the weights in $\mathcal{G}$ are polynomially bounded in the number of its nodes. Then, on the class $\mathcal{GG}_k$,* In-Core, Core-NonEmptiness, *and* In-Kernel *are feasible in polynomial time.*

We leave the section by stressing that the running time of the algorithms exploited by Courcelle [19] and Arnborg et al. [2] to evaluate MSO formulae exponentially depends on the treewidth of the underlying structure. Thus, bounded treewidth is an actual key for tractability only if the given domain exhibit instances with low treewidth. As an example, such instances were subject of study in a number of papers dealing with (*wireless*) *communication networks* [51,15], and *middleware strategies* defined on top of such networks (where tree-like structures often emerge [50]). Therefore, results in this section are likely to be of practical interest for those classes of (graph) games exploited in the literature to model cooperation as a coordination paradigm in communication networks [71].

8. Conclusions

In this paper, we have provided a complete picture of the complexity issues arising from the notions of the core, kernel, and bargaining set on compactly specified coalitional games. Our results confirm two conjectures by Deng and Papadimitriou [26] concerning graph games, and positively answer an open question regarding marginal contribution networks posed by Ieong and Shoham [44]. In addition, we have studied the complexity of these concepts on $\mathbf{NP}^{opt}$-representations, and on classes of marginal contribution networks whose incidence-graph encodings have bounded treewidth.

Our research leaves open some specific technical issues, which might be addressed in further research. In particular, it would be interesting to assess (1) whether hardness results for $\mathbf{NP}^{opt}$-representation can be established for some specific formalism proposed in the literature, (2) whether tractability results for the core and the kernel still hold in case of arbitrary—possibly very large—weights, and (3) whether the bargaining set is tractable over classes of marginal networks whose incidence-graph encodings have bounded treewidth.

Moreover, our research opens the way to further investigate more general questions regarding coalitional games. The first natural question is whether some of the techniques described in this paper are applicable to other core-based solution concepts, such as the *nucleolus* [73], the *least core* [56], or the *cost of stability* [6]. With this respect, note that the complexity of the nucleolus over succinctly specified games has been already studied by Greco et al. [38], who provided hardness results that still hold over graph games (and, hence, marginal contribution networks). Moreover, the complexity of the least core and of the cost of stability has been recently studied by Greco et al. [40], with hardness results being provided which are specific to the oracle setting. In fact, the question of whether such hardness results still hold over graph games and/or marginal contribution networks was not addressed by Greco et al. [40], and cannot be answered by a straightforward adaptation of the techniques illustrated in this paper.

Another avenue of further research concerns the application of the method developed in Section 7 to other relevant game representations and to other problems arising with coalitional games. On the one hand, concerning the application to other game representations, it would be interesting to extend the concept of incidence graph as to capture the interactions in general MC-nets, i.e., in networks where patterns can be arbitrary Boolean formulae [29]. On the other hand, concerning the application to other problems for coalitional games, one may consider for instance the *coalition structure generation problem*. In fact, this problem has recently been studied over coalitional skill games by Bachrach et al. [7], who showed that it is feasible in polynomial time over instances whose underlying structures have bounded treewidth. The result has been established by encoding coalition structure generation in terms of a *constraint satisfaction problem (CSP)* [23], and exploiting well-known results on the tractability of CSP instances having bounded treewidth. As CSPs can be straightforwardly encoded, in their turn, in terms of Monadic Second Order Logic (in fact, even in First Order Logic without negation and disjunction), the existence of pseudo-polynomial algorithms directly follows from the line of reasoning exploited in Section 7. However, in order to assess whether such tractability results still hold in case of arbitrary weights, direct solution approaches are needed. Similarly, our techniques might be used to assess whether coalition structure generation over graph games and MC-nets [60] remains feasible in pseudo-polynomial time when the underlying structures have bounded treewidth. However, extending the results to get (full) polynomial time algorithms would require new methods. As an example, tractability results on certain classes of minor-free graphs have recently been singled out by Voice et al. [78].

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Appendix A. Proofs in Section 4

Lemma 4.3. *Let $\boldsymbol{K}(\phi) = \langle (N_{\mathbf{K}}, E_{\mathbf{K}}), w\rangle$ be the graph game associated with the* **3CNF** *formula ϕ. Then*:

(A) $w(\{chall, sat\}) \geqslant D + 1$; *and,*
(B) $D + w(e) < 0$, *for each penalty edge $e \in E_{\mathbf{K}}$,*

where $D = \max_{\{chall,sat\} \not\subseteq S \subseteq N} v(S)$ denotes the maximum worth over all the coalitions not covering the edge $\{chall, sat\}$.

Proof. Let $P = \sum_{e \in E_{\mathbf{K}} | e \neq \{chall, sat\}, w(e) > 0} w(e)$ be the sum of all the positive edges, but the normalizer one, in $\boldsymbol{K}(\phi)$. Let us firstly observe that:

$$P \leqslant 3 \times m \times 2^{n+3} + 2 \times \sum_{i=1}^{n} 2^i + 2^0 \leqslant 2^{m+n+5} + 2^{n+2} + 2^0 \leqslant 2^{m+n+6}$$

Thus, $2^{m+n+7} \geqslant 2 \times P$ holds. Moreover, observe that $\boldsymbol{K}(\phi)$ contains at least one penalty edge, since w.l.o.g. there is a clause in ϕ containing at least two literals. Hence, $w(\{chall, sat\}) = 1 - \sum_{e \in E_{\mathbf{K}} | e \neq \{chall, sat\}} w(e) \geqslant 1 - P + 2^{m+n+7}$. It follows that $w(\{chall, sat\}) \geqslant 1 - P + 2 \times P = 1 + P$. Eventually, $P \geqslant D$ holds by definition of D and, therefore, $w(\{chall, sat\}) \geqslant 1 + D$, which proves (A).

As for (B), given that $D > 0$ and $P \geqslant D$, we may note that $2^{m+n+7} \geqslant 2 \times P$ implies $2^{m+n+7} > D$. □

Property 4.4.(1). *For each player $i \notin \{sat, chall\}$, it holds that* $\max_{S \in \mathcal{I}_{i,sat}} e(S, x) \leqslant \max_{S \in \mathcal{I}_{sat,i}} e(S, x)$.

Proof. Let S be an arbitrary coalition in $\mathcal{I}_{i,sat}$ with $i \neq chall$ (and $i \neq sat$), and consider the coalition $T = \{chall, sat\} \in \mathcal{I}_{sat,i}$. Note that $e(T, x) = v(T) - x(T) = v(\{chall, sat\}) - 1$ while $e(S, x) = v(S)$. By Lemma 4.3(A), we know that $v(\{chall, sat\}) \geqslant v(S) + 1$. Thus, $e(T, x) \geqslant e(S, x)$ holds for all coalitions $S \in \mathcal{I}_{i,sat}$ with $i \neq chall$. □

Property 4.4.(2). $\max_{S \in \mathcal{I}_{chall,sat}} e(S, x) = m \times 2^{n+3} + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i$.

Proof. Let us firstly note that $\max_{S \in \mathcal{I}_{chall,sat}} e(S, x) = \max_{S \in \mathcal{I}_{chall,sat}} v(S)$, by construction of the imputation x. Let S_* be the coalition getting maximum worth over all the coalitions in $\mathcal{I}_{chall,sat}$. Because of Lemma 4.3(B) and since $v(\{chall\}) = 0$, S_* cannot cover any penalty edge, for otherwise S_* would not be a coalition with maximum worth amongst those belonging to $\mathcal{I}_{chall,sat}$. Thus, (i) for each variable player $\alpha_i \in S_*$, no literal player of the form $\neg\alpha_{i,j}$ is in S_*; (ii) for each clause player $c_j \in S_*$, at most one literal player of the form $\ell_{i,j}$ is in S_*; and, (iii) for each variable α_i, S_* contains no pairs of literal players of the form $\alpha_{i,j}$ and $\neg\alpha_{i,j'}$.

It follows that the worth of S_* is such that: $v(S_*) = |C| \times 2^{n+3} + \sum_{\alpha_i \in S_*} 2^i$, where C is the set of the clause players $c_j \in S_*$ for which exactly one literal player $\ell_{i,j}$ is in S_*; in particular, recall that 2^i is the weight associated with the edge $\{chall, \alpha_i\}$, while 2^{n+3} is the weight associated with each edge of the form $\{c_j, \ell_{i,j}\}$. Now, let, $\widehat{\sigma}$ be a truth assignment such that $\widehat{\sigma}(\alpha_i) = \mathtt{true}$ (resp., $\widehat{\sigma}(\alpha_i) = \mathtt{false}$) if $\alpha_{i,j}$ (resp., $\neg\alpha_{i,j}$) occurs in S_* for some clause c_j. Note that $\widehat{\sigma}$ may be a partial assignment, over a set of variables $\widehat{\alpha} \subseteq \{\alpha_1, \dots, \alpha_n\}$; however, because of (iii) above, $\widehat{\sigma}$ is non-contradictory and satisfies all the clauses whose players are in C. Eventually, since ϕ is satisfiable and since $2^{n+3} > \sum_{i=1}^{n} 2^i$, because of (ii), S_* will certainly contain all the m clause players (i.e., $\widehat{\sigma}$ is a satisfying assignment for ϕ). That is, $v(S_*) = m \times 2^{n+3} + \sum_{\alpha_i \in S_*} 2^i$.

Observe now that if $\alpha_{i,j}$ is in S_*, then α_i is in S_* as well, since this leads to maximize the worth of S_*. Moreover, if $\neg\alpha_{i,j}$ is in S_*, then α_i is not in S_* because of (i). Thus, the assignment σ_{S_*} such that $\sigma_{S_*}(\alpha_i) = \mathtt{true}$ (resp., $\sigma_{S_*}(\alpha_i) = \mathtt{false}$) if α_i occurs (resp., not occurs) in S_* coincides with $\widehat{\sigma}$ when restricted over the domain of the variables in $\widehat{\alpha}$. Therefore, σ_{S_*} is a satisfying assignment, and we have:

$$v(S_*) = m \times 2^{n+3} + \sum_{\alpha_i | \sigma_{S_*}(\alpha_i) = \mathtt{true}} 2^i \leqslant m \times 2^{n+3} + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i.$$

We conclude the proof by showing that the above inequality cannot be strict. Indeed, assume, for the sake of contradiction, that a satisfying assignment $\bar{\sigma}$ exists for ϕ such that $v(S_*) < m \times 2^{n+3} + \sum_{\alpha_i | \bar{\sigma}(\alpha_i) = \mathtt{true}} 2^i$. Based on $\bar{\sigma}$, we can build a coalition $\bar{S}$ such that: (a) $\{chall, c_1, \dots, c_m\} \subseteq \bar{S}$; (b) $\alpha_i \in \bar{S}$, for each α_i such that $\bar{\sigma}(\alpha_i) = \mathtt{true}$; (c) exactly one literal $\ell_{i,j}$ is in $\bar{S}$, for each clause c_j that is satisfied by $\ell_{i,j}$ according to the truth values defined in $\bar{\sigma}$; (d) no further player is in $\bar{S}$.

Given that $\bar{\sigma}$ is a satisfying assignment, no penalty edge is covered by $\bar{S}$. In particular, $v(\bar{S}) = m \times 2^{n+3} + \sum_{\alpha_i \in \bar{S}} 2^i$ and, hence, $v(\bar{S}) = m \times 2^{n+3} + \sum_{\alpha_i | \bar{\sigma}(\alpha_i) = \mathtt{true}} 2^i$. But, this is not possible since we would have a coalition $\bar{S} \in \mathcal{I}_{chall,sat}$ such that $v(\bar{S}) > v(S_*) = \max_{S \in \mathcal{I}_{chall,sat}} v(S)$. □

Property 4.4.(3). $\max_{S \in \mathcal{I}_{sat,chall}} e(S, x) = m \times 2^{n+3} + \max_{\sigma \models \phi}(|\{\alpha_1 \mid \sigma(\alpha_1) = \mathtt{true}\}| + \sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i) - 1$.

Proof. The property can be proven precisely along the same line of reasoning as in the proof of Property 4.4.(1). The differences are that: $x(S) = 1$ holds for each S with $sat \in S$; and that the weight associated with the edge $\{sat, \alpha_i\}$ is 2^i, for each $2 \leqslant i \leqslant n$, while it is $2^1 + 2^0$ for the case where $i = 1$. In particular, $|\{\alpha_1 \mid \sigma(\alpha_1) = \mathtt{true}\}|$ precisely encodes the fact that unitary weight has to be added to any assignment where α_1 evaluates to true. □

Appendix B. Proofs in Section 5

Lemma 5.3. *Let $\boldsymbol{BS}(\boldsymbol{\Phi}) = \langle (N_{\mathbf{BS}}, E_{\mathbf{BS}}), w \rangle$ be the graph game associated with the* $\mathbf{NQBF}_{2,\forall}$ *formula $\boldsymbol{\Phi}$. Then*:

(A) $D \leqslant m$;

(B) $w(\{chall, sat\}) > 2 \times m$;
(C) $D + w(e) < 0$, *for each penalty edge* $e \in E_{\mathbf{BS}}$; *and*,
(D) $m \geqslant 2 \times n$,

where $D = \max_{\{chall,sat\} \not\subseteq S \subseteq N} v(S)$ *denotes the maximum worth over all the coalitions not covering the edge* $\{chall, sat\}$.

Proof. The fact (A) that $D \leqslant m$ is immediate by construction. Moreover, the weight of each penalty edge is $-m-1$ and, hence, (C) holds, too. Eventually, (D) (i.e., $m \geqslant 2 \times n$) is also immediate since $\boldsymbol{\Phi}$ is an $\mathbf{NQBF}_{2,\forall}$ formula.

Let us, hence, focus on (B) by observing that $\sum_{e \in E_{\mathbf{BS}} | e \neq \{chall,sat\}} w(e) \leqslant -m-1$, for $E_{\mathbf{BS}}$ contains more penalty edges than positive ones. Thus, $w(\{chall, sat\}) = n - 1 + m - \sum_{e \in E | e \neq \{chall,sat\}} w(e) \geqslant n - 1 + m + (m+1) > 2 \times m$. □

Property 5.4.(1). *No player has a justified objection against a clause or a literal player.*

Proof. Indeed, any clause player c_j receives 0 in x and is such that $v(\{c_j\}) = 0$. Therefore, she can counterobject to any objection through the singleton coalition $\{c_j\}$. Similarly, any literal player $\ell_{i,j}$ receives 0 in x and is such that $v(\{\ell_{i,j}\}) = 0$, and can counterobject through $\{\ell_{i,j}\}$. □

Property 5.4.(2). *No player has a justified objection against chall.*

Proof. Assume that a player $p \in N_{\mathbf{BS}}$ wants to object against *chall* through a coalition S. If $v(S) < 0$ then p cannot object against anyone. Indeed, she has to propose an S-feasible vector y such that $y_k > x_k$ for all players $k \in S$, and hence $v(S) = y(S) > x(S)$; however $x(S) \geqslant 0$ by definition, so player p cannot fulfill this requirements if $v(S) < 0$. Then, because of Lemma 5.3(C), S does not include any penalty edges. In particular, for each universally quantified variable α_k, it holds that $|S \cap \{\alpha_{k,i(k)}, \neg\alpha_{k,\bar{i}(k)}\}| \leqslant 1$, where $c_{i(k)}$ and $c_{\bar{i}(k)}$ are the two clauses where this variable occurs (recall that $\boldsymbol{\Phi}$ is an $\mathbf{NQBF}_{2,\forall}$ formula).

Consider, then, the coalition $T \subseteq \{chall\} \cup \{\alpha_{k,i(k)}, \neg\alpha_{k,\bar{i}(k)} \mid 1 \leqslant k \leqslant n\}$ such that $|T| = n + 1$, $T \cap S = \emptyset$, and $|T \cap \{\alpha_{k,i(k)}, \neg\alpha_{k,\bar{i}(k)}\}| = 1$, for each $1 \leqslant k \leqslant n$. Note that $v(T) = n$ and $x(T) = x_{chall} = n - 1$. Then, consider the vector z such that $z_{chall} = x_{chall} = n - 1$ and $z_q = \frac{1}{n} > x_q = 0$ for each $q \in T$ with $q \neq chall$, and observe that $z(T) = v(T)$. Eventually, since $T \cap S = \emptyset$, (z, T) is a counterobjection to any objection of p against *chall* through S. □

Property 5.4.(3). *No player different from chall has a justified objection against sat.*

Proof. Suppose that a player $p \neq chall$ has an objection (y, S) against *sat* to x. Since $sat \notin S$, it is the case that $\{chall, sat\} \not\subseteq S$ and hence, by Lemma 5.3(A), that $v(S) \leqslant m$. Then, we claim that $(z, \{sat, chall\})$ is a counterobjection to (y, S) of p against *sat*, where z is a feasible distribution that assigns m to *sat* and $w(\{chall, sat\}) - m$ to *chall*. In fact, by Lemma 5.3(B), *chall* receives a payoff strictly greater than m (i.e., $z_{chall} > m$). Then, note that $z_{sat} = x_{sat}$ and let us distinguish two cases:

1. $chall \notin S$: in this case, we have that $z_{chall} > m \geqslant n - 1 = x_{chall}$ (recall that $m \geqslant 2 \times n$, by Lemma 5.3(D));
2. $chall \in S$: in this case, we have that $z_{chall} > m$ while $y_{chall} \leqslant v(S) \leqslant m$, and hence $z_{chall} > y_{chall}$.

It follows that, in both cases, $(z, \{sat, chall\})$ is a counterobjection to (y, S). □

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