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On conjectures in orthocomplemented lattices ☆

Enric Trillas [a,1], Susana Cubillo [b,*], Elena Castiñeira [b,2]

[a] *Dept. de Inteligencia Artificial, Univ. Politécnica de Madrid, 28660 Boadilla del Monte, Madrid, Spain*
[b] *Dept. Matemática Aplicada, Univ. Politécnica de Madrid, 28660 Boadilla del Monte, Madrid, Spain*

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Abstract

A mathematical model for conjectures in orthocomplemented lattices is presented. After defining when a conjecture is a consequence or a hypothesis, some operators of conjectures, consequences and hypotheses are introduced and some properties they show are studied. This is the case, for example, of being monotonic or non-monotonic operators.

As orthocomplemented lattices contain orthomodular lattices and Boolean algebras, they offer a sufficiently broad framework to obtain some general results that can be restricted to such particular, but important, lattices. This is, for example, the case of the structure's theorem for hypotheses. Some results are illustrated by examples of mathematical or linguistic character, and an appendix on orthocomplemented lattices is included.

Keywords: Orthocomplemented; Orthomodular and Boolean lattices; Conjectures; Consequences; Hypotheses and their structure

1. Introduction

1.1.

While the capability of conjecturing is one of the essential factors for the evolution of humankind, orderly conjecturing seems to be essential for scientific progress; *managing-conjectures* and *research* are extraordinarily interdependent terms. *Good-guesswork* and *rationality* might even be synonyms.

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* Corresponding author. Email: scubillo@fi.upm.es.
[1] Email: etrillas@fi.upm.es.
[2] Email: ecastineira@fi.upm.es.

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Conjectures are formed from preliminary information made explicit in some way or other; from information that is usually acquired by observation or experimentation and which constitutes the gross material on the basis of which a conjecture can be formed. The gross material must be debugged for it to be able to be considered as starting knowledge. The debugged knowledge is often made explicit as a set P of statements or premises and a minimum requirement, which cannot always be immediately met, is that there are no two premises p_i, p_j that are contradictory, that is, the conditional statement "If p_i, then not-p_j" cannot be considered true. In the other words, the set P of premises should not, ideally, be inconsistent; otherwise, the aforesaid set P can hardly be accepted as representing knowledge.

Taking such a body of knowledge P, we seek to arrive at a new statement q, where by either the knowledge P is inferred from them, they are inferred from P. In the former case, they are explanatory conjectures and in the latter case, they are inferrable conjectures. In principle, these statements q cannot be contradictory with any or, at least some, of the premises (in the latter case, it is necessary to consider what to do with the others). Such statement q is a conjecture and each one can verify:

(1) The statement "if q, then all the premises" is true, and q is a *hypothesis* of P.
(2) The statement "If all the premises, then q" is true, and q is a *consequence* of P.
(3) Neither (1) nor (2), and q is a *speculative conjecture* of P, a statement that is just not inconsistent with P.

This third type of conjectures will, in turn, have various relationships to P. Any process leading from P to a conjecture q is an induction; if, in particular, q is a hypothesis, it is an abduction or retrodiction, whereas if q is a consequence, it is an inference, which is a deduction if q can be attained by means of an algorithm or program. In this paper, we seek to formalise both a sufficiently general concept of conjecture and some particular concepts of hypothesis and consequence. For this purpose, we use a mathematical model which assumes that all the statements are elements of an orthocomplemented lattice, thus encompassing classical and quantic logical calculi. By means of the above model, a preliminary classification is attained of the conjectures and the consequences, whereas the relationship between the consequences of both a hypothesis and the premises is analysed. Furthermore, a characterization of the hypotheses is obtained, including a necessary and sufficient condition for their existence. This is merely a preliminary paper presenting a quite satisfactory provisional framework, in which conjectures, hypotheses and consequences appear as mathematical objects and in which the above concepts can be addressed as such. Questions such as the computability of the above objects, forms of aggregating preliminary information, stronger ways of defining the concept of consequence, or the impact of certainty factors of premises on the different types of conjectures are not addressed here; these issues will be the subject of subsequent papers concerning work now under way.

1.2.

In the following, L will be a complete orthocomplemented lattice (see Section 7), the three operations of L will be represented as $\cdot$, $+$ and $'$ (intersection, union and complementation, respectively), the least element of L as 0 and the greatest element as 1.

If P is a non-empty part of L, the infimum of P will be represented as $p_\wedge = \bigwedge P$ and the supremum as $p_\vee = \bigvee P$; both exist as L is complete. Obviously, if $P \subset Q$ are parts of L, then $q_\wedge \leqslant p_\wedge \leqslant p_\vee \leqslant q_\vee$.

The partial order of L will be denoted by the usual sign $\leqslant$, "$a \leqslant b$ if and only if $a \cdot b = a$ or $a + b = b$", where $a < b$ if $a \leqslant b$ and $a \neq b$. Accordingly, $0 \leqslant a \leqslant 1$ for every a of L, and any a for which $0 < a < 1$ will be called a contingent element of L. A pair of elements a, b of L may be comparable ($a \leqslant b$ or $b < a$) or incomparable, denoted, if applicable, as $a\, NC\, b$ or $b\, NC\, a$. Using $a \nleqslant b$, we denote that there is not $a \leqslant b$ and, hence, there may be $b < a$ or $a\, NC\, b$; obviously, $a \nleqslant b'$ is equivalent to $b \nleqslant a'$.

In any lattice L, $a \leqslant b$ if and only if there exists $x \in L$, such that $a = b \cdot x$; indeed, if $a \leqslant b$, then $a = a \cdot b$ and $x = a$, and if $a = b \cdot x$, $a \leqslant b$ follows. Similarly, $a \leqslant b$ if and only if there exists $y \in L$, such that $b = a + y$.

Obviously, $p_\wedge = 1$ is equivalent to $p = 1$ for any $p \in P$. In quite a few results, this is not restrictive but it is unusual; generally, although not expressly stated unless it is restrictive, $p_\wedge$ is assumed to be contingent. Let us denote the non-empty set of the parts P of L, such that $p_\wedge \neq 0$, as $\mathbb{P}_0(L)$, $L - \{0\}$ as L_0 and $L_0 - \{1\}$, that is, the set of contingent elements, as L_{01}. We will agree that $\emptyset \notin \mathbb{P}_0(L)$.

$P \in \mathbb{P}_0(L)$ implies that no element p of P is equal to 0 and that no pair of elements p_i, p_j of P exists such that $p_i \leqslant p'_j$, as if it, then $p_i \cdot p_j = 0$ and $p_\wedge = 0$; that is, there are no pairs of contradictory elements in P, P is not inconsistent.

2. Basic concepts

2.1.

Let $P \in \mathbb{P}_0(L)$. We will denote as:

$$\Phi_\vee(P) = \{q \in L;\ p_\vee \nleqslant q'\};$$
$$\Phi_\wedge(P) = \{q \in L;\ p_\wedge \nleqslant q'\};$$
$$C_\vee(P) = \{q \in L;\ p_\vee \leqslant q\};$$
$$C_\wedge(P) = \{q \in L;\ p_\wedge \leqslant q\};$$
$$H(P) = \{q \in L_0;\ q \leqslant p_\wedge\}.$$

It is clear that none of these sets contains 0; the latter only by definition. With regard to 1, it is in the first four, while it is in the last if and only if $1 = p_\wedge$; hence the new definition $H(P) = \{q \in L_{01};\ q \leqslant p_\wedge\}$ is better. We will write $H^*(P) = H(P) - \{p_\wedge\}$. Obviously, if $P = \{1\}$ then $\Phi_\vee(P) = \Phi_\wedge(P) = L_0$, $H(P) = L_{01}$ and $C_\vee(P) = C_\wedge(P) = \{1\}$. It is clear that if L is not finite, even if P is, the above sets can be not finite.

The choice of $P \in \mathbb{P}_0(L)$ for the above definitions was not made arbitrarily; remember that we seek to axiomatize the concepts commonly referred to as conjectures, consequences and hypotheses (represented initially as the sets Φ, C and H, respectively). Thus, this choice is justified by the following facts:

(1) $C_\wedge(P) = L$ *if and only if* $p_\wedge = 0$. Indeed, if $p_\wedge = 0$, as for every $q \in L$ is $0 \leqslant q$, then $L = C_\wedge(P)$. Reciprocally, if $C_\wedge(P) = L$, for every q of L is $p_\wedge \leqslant q$, therefore $p_\wedge = 0$.
(2) $\Phi_\wedge(P) = \emptyset$ *if and only if* $p_\wedge = 0$. Indeed, if $p_\wedge = 0$, as for every $q \in L$ is $0 \leqslant q'$, then $q \notin \Phi_\wedge(P)$; that is $\Phi_\wedge(P) = \emptyset$. Reciprocally, if $\Phi_\wedge(P) = \emptyset$, there exist no $q \in L$ such that $p_\wedge \nleqslant q'$; that is, for every $q \in L$, $p_\wedge \leqslant q'$ and hence $p_\wedge = 0$.
(3) $H(P) = \emptyset$ *if and only if* $p_\wedge = 0$. Indeed, if $H(P) = \{q \in L_0; q \leqslant p_\wedge\} = \emptyset$, then $(0, p_\wedge] = \emptyset$, that is $p_\wedge = 0$. The reciprocal is obvious.

Therefore, the case $p_\wedge = 0$ is singular. From an inconsistent set of premises, we get all consequences but neither conjectures nor hypotheses.

Note that:
- If $p_\wedge \nleqslant q'$, then for any $p \in P$ is $p \nleqslant q'$, as if for one of them $p \leqslant q'$ then $p_\wedge \leqslant p \leqslant q'$.
- If $p_\vee \nleqslant q'$ for some $p \in P$ it is $p \nleqslant q'$, as if for all $p \in P$ $p \leqslant q'$ then $p_\vee \leqslant q'$.
- If $p \leqslant q$ for all $p \in P$, then $p_\wedge \leqslant q$.
- If $p_\vee \leqslant q$, for all $p \in P$ is $p \leqslant q$ as it is $p \leqslant p_\vee$.

Theorem 2.1. *$q \in C_\wedge(P)$ if and only if $p_\wedge \in H(\{q\})$.*

Proof. Immediate. □

Theorem 2.2.
(a) $P \subset C_\wedge(P)$.
(b) $C_\vee(P) \subset C_\wedge(P) \subset \Phi_\wedge(P) \subset \Phi_\vee(P)$.
(c) $H(P) \subset \Phi_\wedge(P)$.
(d) $C_\wedge(P) \cap H(P) = \{p_\wedge\}$.
(e) $P \subset \Phi_\wedge(P)$.

Proof. (a) If $p \in P$ then $p_\wedge \leqslant p$. Hence $p \in C_\wedge(P)$.

(b) (1) If $q \in C_\vee(P)$ then $p_\vee \leqslant q$, and $p_\wedge \leqslant q$ follows from $p_\wedge \leqslant p_\vee$; hence $q \in C_\wedge(P)$.
(2) If $q \in C_\wedge(P)$, $p_\wedge = 0$ would follow from $p_\wedge \leqslant q'$; hence $p_\wedge \nleqslant q'$ and $q \in \Phi_\wedge(P)$.
(3) If $q \in \Phi_\wedge(P)$ then $p_\wedge \nleqslant q'$, and if $p_\vee \leqslant q'$, $p_\wedge \leqslant q'$ would follow from $p_\wedge \leqslant p_\vee$; hence, $p_\vee \nleqslant q'$ and $q \in \Phi_\vee(P)$.

(c) If $q \in H(P)$ then $q \leqslant p_\wedge$, and if $p_\wedge \leqslant q'$, $q \leqslant q'$ and $q = 0$ would follow; hence $p_\wedge \nleqslant q'$ and $q \in \Phi_\wedge(P)$.

(d) $q \in C_\wedge(P) \cap H(P)$ is equivalent to $q \leqslant p_\wedge \leqslant q$, that is, to $q = p_\wedge$.

(e) Follows from (a) and (b.2). □

Theorem 2.3. *If $q \leqslant r$, then*:
(a) *If $q \in C_\vee(P)$, then $r \in C_\vee(P)$.*
(b) *If $q \in C_\wedge(P)$, then $r \in C_\wedge(P)$.*
(c) *If $q \in \Phi_\vee(P)$, then $r \in \Phi_\vee(P)$.*
(d) *If $q \in \Phi_\wedge(P)$, then $r \in \Phi_\wedge(P)$.*
(e) *If $r \in H(P)$, then $q \in H(P)$.*

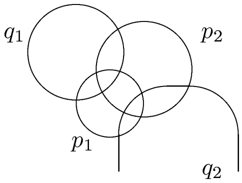

Fig. 1. Example.

Proof. Immediate. □

Corollary 2.4. *If $a \in L$, then*:
(a) *If $q \in C_\vee(P)$, then $a + q \in C_\vee(P)$.*
(b) *If $q \in C_\wedge(P)$, then $a + q \in C_\wedge(P)$.*
(c) *If $q \in \Phi_\vee(P)$, then $q + a \in \Phi_\vee(P)$.*
(d) *If $q \in \Phi_\wedge(P)$, then $q + a \in \Phi_\wedge(P)$.*
(e) *If $q \in H(P)$ and $a \cdot q \neq 0$, then $a \cdot q \in H(P)$.*

Theorem 2.5.
(a) $\bigwedge C_\wedge(P) = p_\wedge$ *and* $C_\wedge(P) \in \mathbb{P}_0(L)$.
(b) $\bigwedge C_\vee(P) = p_\vee$ *and* $C_\vee(P) \in \mathbb{P}_0(L)$.

Proof. (a) As all $q \in C_\wedge(P)$ verify $p_\wedge \leqslant q$, we have $p_\wedge \leqslant \bigwedge C_\wedge(P)$, and also, as $p_\wedge \in C_\wedge(P)$, $\bigwedge C_\wedge(P) \leqslant p_\wedge$. Then $\bigwedge C_\wedge(P) = p_\wedge$ and, hence, $C_\wedge(P) \in \mathbb{P}_0(L)$.

(b) All $q \in C_\vee(P)$ verify $p_\vee \leqslant q$, and it holds that $p_\wedge \leqslant p_\vee \leqslant \bigwedge C_\vee(P)$, and, as $p_\vee \in C_\vee(P)$ also $\bigwedge C_\vee(P) \leqslant p_\vee$. Then $\bigwedge C_\vee(P) = p_\vee$ and, hence, $C_\vee(P) \in \mathbb{P}_0(L)$. □

Generally, however, neither $\Phi_\wedge(P) \in \mathbb{P}_0(L)$ nor $\Phi_\vee(P) \in \mathbb{P}_0(L)$ can be assumed. Indeed, the Venn's diagram in Fig. 1 shows that with $P = \{p_1, p_2\}$, $q_1 \cdot q_2 = 0$, both q_1 and q_2 belong to $\Phi_\wedge(P)$ and, hence, to $\Phi_\vee(P)$.

It is clear that, generally, $H(P) \in \mathbb{P}_0(P)$ cannot be assumed either. Hence, if $C_\wedge$ is to be applied to $C_\wedge(P)$ and $C_\vee$ to $C_\vee(P)$, neither $\Phi_\wedge$ is to be applied to $\Phi_\wedge(P)$, nor $\Phi_\vee$ to $\Phi_\vee(P)$, nor H to $H(P)$. However, if $H(P) \in \mathbb{P}_0(L)$, then $H(H(P)) \subset H(P)$, as $q \in H(H(P))$ is equivalent to $q \leqslant \bigwedge H(P)$ and as $\bigwedge H(P) \leqslant p_\wedge$, then $q \leqslant p_\wedge$ because every $h \in H(P)$ verifies $h \leqslant p_\wedge$. In the particular case that $p_\wedge$ is an atom of the lattice, then $H(P) = \{p_\wedge\}$, $H(P) \in \mathbb{P}_0(P)$, and $H(H(P)) = H(\{p_\wedge\}) = \{p_\wedge\} = H(P)$.

Theorem 2.6.

$$\bigwedge \Phi_\vee(P) \leqslant \bigwedge \Phi_\wedge(P) \leqslant \bigwedge H(P).$$

Proof. Follows from Theorem 2.2. □

Whether the lattice L is finite or infinite, $\bigwedge H(P)$ must be either an atom or 0. If $\bigwedge H(P) = 0$, then $\bigwedge \Phi_\wedge(P) = \bigwedge \Phi_\vee(P) = 0$.

Theorem 2.7. *If $P, Q \in \mathbb{P}_0(L)$ and there exists a bijection $f : P \to Q$ such that for every $p \in P$ it holds that $p \leqslant f(p)$, then*:

(a) $\Phi_\wedge(P) \subset \Phi_\wedge(Q)$.
(b) $\Phi_\vee(P) \subset \Phi_\vee(Q)$.
(c) $C_\wedge(Q) \subset C_\wedge(P)$.
(d) $C_\vee(Q) \subset C_\vee(P)$.
(e) $H(P) \subset H(Q)$.

Proof. (a) If $q \in \Phi_\wedge(P)$ then $p_\wedge \nleqslant q'$, and if $q_\wedge \leqslant q'$, then

$$\bigwedge_{p\in P} p = p_\wedge \leqslant \bigwedge_{p\in P} f(p) = \bigwedge_{q\in Q} q = q_\wedge \leqslant q'$$

is proven, as $p \leqslant f(p)$ for every $p \in P$, which is absurd; hence, $q_\wedge \nleqslant q'$.

(b) Proved similarly.

(c) If $q \in C_\wedge(Q)$, then $q_\wedge \leqslant q$, and $p_\wedge \leqslant q$ follows from $p_\wedge \leqslant q_\wedge$.

(d) Proved similarly.

(e) If $q \leqslant p_\wedge$, as $p_\wedge \leqslant q_\wedge$, then $q \leqslant q_\wedge$ also. □

Theorem 2.8. *If $P \subset Q$, where $P, Q \in \mathbb{P}_0(L)$, then*:

(a) $C_\wedge(P) \subset C_\wedge(Q)$.
(b) $C_\vee(Q) \subset C_\vee(P)$.
(c) $\Phi_\vee(P) \subset \Phi_\vee(Q)$.
(d) $\Phi_\wedge(Q) \subset \Phi_\wedge(P)$.
(e) $H(Q) \subset H(P)$.

Proof. All expressions follows because $q_\wedge \leqslant p_\wedge$ and $p_\vee \leqslant q_\vee$. □

Theorem 2.9. $C_\wedge(C_\wedge(P)) = C_\wedge(P)$, *for every* $P \in \mathbb{P}_0(L)$.

Proof.

$$C_\wedge\big(C_\wedge(P)\big) = \left\{q \in L;\ \bigwedge C(P) = p_\wedge \leqslant q\right\} = C_\wedge(P)$$

follows from Theorem 2.5(a). □

Theorem 2.10. *The mapping $C_\wedge : \mathbb{P}_0(L) \to \mathbb{P}_0(L)$ is a Tarski's Consequences Operator.*

Proof. Follows from Theorems 2.2(a), 2.8(a) and 2.9. □

With regard to $C_\vee$, it is certainly an application of $\mathbb{P}_0(L)$ in $\mathbb{P}_0(L)$, but it is not a Tarski's Consequences Operator. Indeed, it verifies Theorem 2.8(b) and if there were $P \subset C_\vee(P)$, then $p_\vee \leqslant p$ for every $p \in P$, and hence $p = p_\vee$; that is, all $p \in P$ would be equal. Thus, $P \subset C_\vee(P)$ if and only if P is composed of a single element $p \neq 0$, in which case $C_\vee(\{p\}) = C_\wedge(\{p\})$.

2.2.

For each $n \in \mathbb{N}$, let the set G_n be defined as

$$G_n = \{g : L^n \to L;\ g(x_1, \dots, x_n) \geqslant x_1 \cdot \cdots \cdot x_n, \forall (x_1, \dots, x_n) \in L^n\}.$$

In this section, let us consider finite subsets $P_n = \{p_1, \dots, p_n\} \in \mathbb{P}_0(L)$. For every $g \in G_n$, let us define:

$$\Phi_g(P_n) = \{q \in L;\ g(p_1, \dots, p_n) \nleqslant q'\},$$
$$C_g(P_n) = \{q \in L;\ g(p_1, \dots, p_n) \leqslant q\},$$

and also

$$C_{G_n}(P_n) = \{g(p_1, \dots, p_n);\ g \in G_n\}.$$

Theorem 2.11.

$$\Phi_\wedge(P_n) = \bigcap_{g \in G_n} \Phi_g(P_n).$$

Proof. If $p_\wedge \nleqslant q'$ and there were $g(p_1, \dots, p_n) \leqslant q'$ for some $g \in G_n$, then $p_\wedge \leqslant g(p_1, \dots, p_n) \leqslant q'$ which is absurd. Therefore, $g(p_1, \dots, p_n) \nleqslant q'$ for every $g \in G_n$ and, hence, $\Phi_\wedge(P_n) \subset \Phi_g(P_n)$ and thus

$$\Phi_\wedge(P_n) \subset \bigcap_{g \in G_n} \Phi_g(P_n).$$

Moreover, as the function $\bigwedge$ belongs to G_n, we have

$$\bigcap_{g \in G_n} \Phi_g(P_n) \subset \Phi_\wedge(P_n)$$

also. □

Theorem 2.12.

$$\bigcup_{g \in G_n} C_g(P_n) = C_\wedge(P_n) = C_{G_n}(P_n).$$

Proof. (a) It is evident from $g \geqslant \bigwedge$ that $C_g(P_n) \subset C_\wedge(P_n)$ for every $g \in G$. Thus

$$\bigcup_{g \in G} C_g(P_n) \subset C_\wedge(P_n).$$

On the other hand, $C_\wedge(P_n) \subset \bigcup_{g \in G} C_g(P_n)$ follows from $\bigwedge \in G$, hence equality holds.

(b) Obviously, $C_{G_n}(P_n) \subset C_\wedge(P_n)$ and if $q \geqslant p_\wedge$, it is sufficient to consider the mapping defined as $g(p_1, \dots, p_n) = q$ and if $(x_1, \dots, x_n) \neq (p_1, \dots, p_n)$, $g(x_1, \dots, x_n) = x_1 \cdot \cdots \cdot x_n$, and $q \in C_{G_n}(P_n)$ follows. □

3. Some examples

3.1.

Let L be the Boolean algebra associated with the random experiment of throwing a dice. This is a Boolean algebra with 2^6 elements, whose atoms p_i $(1 \leqslant i \leqslant 6)$ correspond with the statements "score i points in one throw". The statement "score 1, or 2, or 3, … or 6 in one throw" is represented in L by $p_1 + p_2 + \cdots + p_6$, which is the greatest element (1) of L. Let $P = \{p_1 + p_2 + \cdots + p_6\}$; then $\Phi_\wedge(P) = \Phi_\vee(P) = L_0$, $H(P) = L_{01}$, and $C_\wedge(P) = C_\vee(P) = \{p_1 + p_2 + \cdots + p_6\}$.

3.2.

Let L be an orthocomplemented lattice whose elements represent all the statements of a discourse involving the terms "midday", "eclipse" and "it is sunny", which are represented in L by m, e and s respectively. The set P is composed of three premises $p_1 = m$, $p_2 = m \cdot s'$ and $p_3 = (e \cdot s)' = e' + s'$; then $p_\wedge = m \cdot (m \cdot s') \cdot (e \cdot s)' = m \cdot s' = p_2 \neq 0$.

As $p_\wedge \leqslant s'$, we have $s' \in C_\wedge(P)$. However, $s \notin C_\wedge(P)$, as $m \cdot s' \leqslant s$ would imply $m \cdot s' = p_2 = 0$. Neither $s \in \Phi_\wedge(P)$ as $m \cdot s' \leqslant s'$; but $s \in \Phi_\vee(P)$ as

$$p_\vee = m + m \cdot s' + (e' + s') = m + e' + s'$$

is not less than or equal to s' (unless $m + e' \leqslant s'$). Thus, unless $m + e' \leqslant s'$ (which would be absurd), $s \in \Phi_\vee(P) - \Phi_\wedge(P)$. To have $s' \in H(P)$, it would be $s' \leqslant m \cdot s'$, that is, $s' = m \cdot s'$ which is equivalent to $s' \leqslant m$, not an acceptable tautology ($s' \to m = 1$ represents the true statement "if it is not sunny, then it is midday").

3.3.

If the above lattice L is a Boolean algebra with the premises $p_1 = m$, $p_2 = e$ and $p_3 = (e \cdot s)'$, then $p_\wedge = m \cdot e \cdot (e' + s') = m \cdot e \cdot s'$ and $p_\vee = 1$. Supposing $p_\wedge \neq 0$, then $s' \in C_\wedge(P)$ but $s \notin C_\wedge(P)$ as $m \cdot e \cdot s' \leqslant s$ implies $p_\wedge = 0$; neither is $s \in \Phi_\wedge(P)$. However, as $\Phi_\vee(P) = L_{01}$, it is sufficient that s is contingent for $s \in \Phi_\vee(P)$.

For having $s' \in H(P)$, $s' \leqslant m \cdot e \cdot s'$ would have to be verified, that is, $s' \leqslant m \cdot e$; otherwise $s' \notin H(P)$. It is $s \notin H(P)$ as otherwise would be verified $s \leqslant m \cdot e \cdot s'$, and then $s = 0$ hence $m \cdot e \cdot (e \cdot s)' = m \cdot e = 1$, which is not an acceptable tautology.

3.4.

Let $P = \{p_1, p_2\} \in \mathbb{P}_0(L)$. Obviously, all the following elements of L are in $C_\wedge(P)$: $p_1 \cdot p_2$, $p_1 + p_2$, $p_1' + p_1 \cdot p_2 = p_1 \to p_2$, $p_1' + p_2$.

Also, as $p_1 \cdot p_2 \leqslant (p_1 + p_2') \cdot (p_1' + p_2) = (p_1 \triangle p_2)'$ we have $(p_1 \triangle p_2)' \in C_\wedge(P)$ and, hence, $(p_1 \triangle p_2)' \in \Phi_\wedge(P)$ and $(p_1 \triangle p_2)' \in \Phi_\vee(P)$. But if $p_1 \triangle p_2 \neq 0$, then $(p_1 \triangle p_2)' \notin C_\vee(P)$; indeed, if $p_1 + p_2 \leqslant (p_1 \triangle p_2)' = (p_1 + p_2') \cdot (p_1' + p_2)$ then $p_2 \leqslant p_1 + p_2'$ or equivalently $(p_1 + p_2')' = p_1' \cdot p_2 \leqslant p_2'$ from which we deduce that $p_1' \cdot p_2 = 0$; in the same manner we deduce from $p_1 \leqslant p_1' + p_2$ that $p_1 \cdot p_2' = 0$, and we conclude that

$p_1 \triangle p_2 = 0$ which is absurd. Hence, if $p_1 \triangle p_2 \neq 0$, then $p_1 + p_2 \nleqslant (p_1 \triangle p_2)'$, that is, $(p_1 \triangle p_2)' \notin C_\vee(P)$. In the event that $p_1 \triangle p_2 = 0$, we have $(p_1 \triangle p_2)' = 1 \in C_\vee(P)$.

With regard to the symmetric difference, it holds that $p_1 \triangle p_2 \notin \Phi_\wedge(P)$ as $p_1 \cdot p_2 \leqslant (p_1 \triangle p_2)'$; and, hence, $p_1 \triangle p_2 \notin C_\vee(P)$ and $p_1 \triangle p_2 \notin C_\wedge(P)$. However, if $p_1 \triangle p_2 \neq 0$, as $p_1 + p_2 \nleqslant (p_1 \triangle p_2)'$, $p_1 \triangle p_2 \in \Phi_\vee(P)$ holds.

3.5.

If, taking P from Section 3.4, $p_1 \cdot p_2 \leqslant a$ (and consequently $a \neq 0$), then it must hold that obviously verified that

$$a \cdot p_1 \cdot p_2 + b \cdot p_1' \cdot p_2 + c \cdot p_1 \cdot p_2' + d \cdot p_1' \cdot p_2' \in C_\wedge(P),$$

for any $b, c, d \in L$. If the elements a, b, c, d are in $\{0, 1\} \subset L$, it must hold that $a = 1$ for the above element to be in $C_\wedge(P)$. In no case is such element in $H^*(P)$.

3.6.

Let $\mathcal{L}(\mathbb{R}^3)$ be the set of vector subspaces of $\mathbb{R}^3$ with the operations $+$ (sum of subspaces), $\cap$ (intersection of subspaces) and $\perp$ (the orthogonal complement). $(\mathcal{L}(\mathbb{R}^3), +, \cap, \perp)$ is an orthomodular lattice (see Section 7), whose greatest element is $\mathbb{R}^3$ and whose least element is the null vector $\bar{0}$. Let $P = \{\pi_{10}, \pi_{01}\}$, where π_{10} and π_{01}, respectively, are the coordinate planes *YZ* and *XZ*, that is, $\pi_{10} = \{x = 0\}$ and $\pi_{01} = \{y = 0\}$. Then $p_\wedge = \pi_{10} \cap \pi_{01} = \{x = 0,\ y = 0\}$ is the axis Z and, hence, $P \in \mathbb{P}_0(\mathcal{L}(\mathbb{R}^3))$; moreover, $p_\vee = \pi_{10} + \pi_{01} = \mathbb{R}^3$. Thus, we have

$$\Phi_\vee(P) = \{V \in \mathcal{L}(\mathbb{R}^3);\ p_\vee = \mathbb{R}^3 \not\subset V^\perp\} = \{V \in \mathcal{L}(\mathbb{R}^3);\ V \neq \bar{0}\} = \mathcal{L}(\mathbb{R}^3) - \{\bar{0}\}.$$

$$\begin{aligned}\Phi_\wedge(P) &= \{V \in \mathcal{L}(\mathbb{R}^3);\ V \not\subset p_\wedge^\perp = \{z = 0\}\} \\ &= \mathcal{L}(\mathbb{R}^3) - \{\{z = 0\}, \\ &\qquad\qquad \{l_{\alpha\beta} = \{\alpha x + \beta y = 0,\ z = 0\}, \alpha, \beta \in \mathbb{R}^2 - \{(0, 0)\}\}, \bar{0}\},\end{aligned}$$

that is, it is the set of all the subspaces of $\mathbb{R}^3$, except the coordinate plane XY, the lines in such plane and, of course, the vector $\bar{0}$.

$$\begin{aligned}C_\wedge(P) &= \{V \in \mathcal{L}(\mathbb{R}^3);\ p_\wedge = \{x = 0,\ y = 0\} \subset V\} \\ &= \{\{x = 0,\ y = 0\},\ \{\pi_{\alpha\beta} = \{\alpha x + \beta y = 0\}, (\alpha, \beta) \in \mathbb{R}^2 - \{\bar{0}\}\},\ \mathbb{R}^3\},\end{aligned}$$

that is, the axis Z, the bundle of planes generated by this axis and, of course, the greatest element $\mathbb{R}^3$.

$$C_\vee(P) = \{V \in \mathcal{L}(\mathbb{R}^3);\ p_\vee = \mathbb{R}^3 \subset V\} = \{\mathbb{R}^3\}.$$

$$\begin{aligned}H(P) &= \{V \in \mathcal{L}(\mathbb{R}^3) - \{\mathbb{R}^3, \bar{0}\};\ V \subset p_\wedge = \{x = 0,\ y = 0\}\} \\ &= \{x = 0,\ y = 0\} = \{p_\wedge\}.\end{aligned}$$

4. The sets $C(P)$ and $\Phi(P)$

4.1.

If $P \in \mathbb{P}_0(L)$, let $C(P) = \{q \in L_0;\ p_\wedge \leqslant q \leqslant p_\vee\}$. Obviously, $P \subset C(P) \subset C_\wedge(P)$ holds. Also, $\bigwedge C(P) = p_\wedge \leqslant p_\vee = \bigvee C(P)$, and $C(P) \in \mathbb{P}_0(L)$. It is clear that $1 \in C(P)$ implies $p_\vee = 1$.

Theorem 4.1. *If $P \subset Q$, then $C(P) \subset C(Q)$.*

Proof. From $q_\wedge \leqslant p_\wedge \leqslant p_\vee \leqslant q_\vee$ it follows that if $p_\wedge \leqslant r \leqslant p_\vee$, then $q_\wedge \leqslant r \leqslant q_\vee$. □

Theorem 4.2. *For every $P \in \mathbb{P}_0(L)$, $C(P) = C(C(P))$.*

Proof. As for every $P \in \mathbb{P}_0(L)$, $P \subset C(P)$ holds, $C(P) \subset C(C(P))$ follows from the above theorem. Reciprocally, if $q \in C(C(P))$ then $\bigwedge C(P) \leqslant q \leqslant \bigvee C(P)$ and then $p_\wedge \leqslant q \leqslant p_\vee$ or $q \in C(P)$. □

Hence, $C : \mathbb{P}_0(L) \to \mathbb{P}_0(L)$ is a Tarski's Consequences Operator. Obviously, $P \subset C(P) \subset C_\wedge(P) \subset \Phi_\wedge(P)$.

For example, if L is the hexagonal orthocomplemented lattice shown in Fig. 2 and if $P = \{b\}$, then $p_\wedge = p_\vee = b \neq 0$, $C_\wedge(P) = C_\vee(P) = \{1, b\}$, $C(P) = \{b\}$ and $\Phi_\wedge(P) = \Phi_\vee(P) = \{1, a, a', b\}$.

Theorem 4.3. *For every $P \in \mathbb{P}_0(L)$, $C(P) \cap C_\vee(P) = \{p_\vee\}$.*

Proof. The above intersection is the set $\{q \in L;\ p_\wedge \leqslant q \leqslant p_\vee \leqslant q\} = \{p_\vee\}$. □

4.2.

If $P \in \mathbb{P}_0(L)$ and $p_\vee \neq 1$, let $\Phi(P) = \{q \in L;\ p_\wedge \nleqslant q', q' \nleqslant p_\vee\}$. Obviously, $\Phi(P) \subset \Phi_\wedge(P)$.

Theorem 4.4. *For every $P \in \mathbb{P}_0(L)$ such that $p_\vee \neq 1$, $P \subset \Phi(P)$.*

Proof. If $p \in P$ and $p_\wedge \leqslant p'$, this would imply $p_\wedge = 0$; if $p' \leqslant p_\vee$, then necessarily $1 = p + p' \leqslant p + p_\vee$, and $1 = p_\vee$. Then, $p_\wedge \nleqslant p'$ and $p' \nleqslant p_\vee$. □

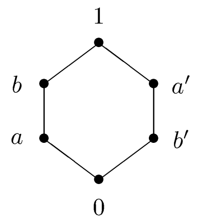

Fig. 2. Hexagonal lattice.

In particular, $p_\vee \leqslant \bigvee \Phi(P)$ and $\bigwedge \Phi(P) \leqslant p_\wedge$.

Theorem 4.5. *If $P, Q \in \mathbb{P}_0(L)$ and $P \subset Q$, then $\Phi(Q) \subset \Phi(P)$.*

Proof. If $r \in \Phi(Q)$, then $q_\wedge \nleqslant r'$ and $r' \nleqslant q_\vee$. If $p_\wedge \leqslant r'$, then $q_\wedge \leqslant r'$ would follow from $q_\wedge \leqslant p_\wedge$, which is absurd; if $r' \leqslant p_\vee$, then $r' \leqslant q_\vee$ would follow from $p_\vee \leqslant q_\vee$, which is absurd. Hence, $p_\wedge \nleqslant r'$ and $r' \nleqslant p_\vee$; that is, $r \in \Phi(P)$. □

Generally, it cannot be said that if $P \in \mathbb{P}_0(L)$ then $\Phi(P) \in \mathbb{P}_0(L)$; the example used above in the case of $\Phi_\wedge(P)$, and shown in Fig. 1, is valid for this purpose. There, $P = \{p_1, p_2\}$, $p_\wedge \neq 0$, and as $q_1, q_2 \in \Phi(P)$ it holds that $q_1 \cdot q_2 = 0$, hence $\bigwedge \Phi(P) = 0$. Generally Φ is not a $\mathbb{P}_0(L) \to \mathbb{P}_0(L)$ operator.

Neither is $C_\wedge(P) \subset \Phi(P)$ generally, as it is shown in the example given in Fig. 2 with $P = \{b\}$. In this case it holds that $\Phi(P) = \{a, b\}$, while $C_\wedge(P) = \{1, b\}$.

Theorem 4.6. *If $P \in \mathbb{P}_0(L)$ and $p_\vee \neq 1$, then $C(P) \subset \Phi(P)$.*

Proof. If $q \in C(P)$ and $q \notin \Phi(P)$, then $p_\wedge \leqslant q'$ ó $q' \leqslant p_\vee$. In the first case, as $p_\wedge \leqslant q$, $p_\wedge = 0$ would follow; in the second case, as $q \leqslant p_\vee$, $1 = p_\vee$ would follow. □

Hence, we have the chain $P \subset C(P) \subset \Phi(P) \subset \Phi_\wedge(P)$.

Theorem 4.7.

$$\Phi(P) - C(P) = \left\{q \in L;\ p_\wedge \nleqslant q', q' \nleqslant p_\vee, p_\wedge \nleqslant q\right\}.$$

Proof. Obvious. □

Note. If P is reduced to a single premise, it is not generally the case that $\Phi(P)$ is reduced to a single element; however, $C(\{c\}) = \{c\}$ is verified.

4.3.

For each $n \in \mathbb{N}$, let it be the set

$$A_n = \left\{g : L^n \to L;\ x_1 \cdot \cdots \cdot x_n \leqslant g(x_1, \ldots, x_n) \leqslant x_1 + \cdots + x_n,\ \forall (x_1, \ldots, x_n) \in L^n\right\}$$

and we consider finite subsets $P_n = \{p_1, \ldots, p_n\} \in \mathbb{P}_0(L)$. For every $g \in A_n$, let us define:

$$C_{A_n}(P_n) = \left\{g(p_1, \ldots, p_n);\ g \in A_n\right\}.$$

Theorem 4.8. $C_{A_n}(P_n) = C(P_n)$.

Proof. As in part (b) of the proof of Theorem 2.12. □

5. Classification of the set $\Phi_\vee(P)$

5.1.

The set $H_\vee(P) = \{q \in L_{01};\ q \leqslant p_\vee\}$ has not been considered, as $C_\wedge(P) \cap H_\vee(P) = \{q \in L_{01};\ p_\wedge \leqslant q \leqslant p_\vee\}$ is not, generally, empty (unless $p_\wedge = 1$), and it would be pointless to consider numerous explanatory elements of P that can also be inferred from P.

Theorem 5.1. *If $q \in C_\wedge(P)$, then $q' \notin C_\wedge(P)$.*

Proof. If $p_\wedge \leqslant q'$ and also $p_\wedge \leqslant q$, $p_\wedge = 0$ would follow; then $p_\wedge \nleqslant q'$, as is known. □

If, in particular, $q \in C_\vee(P)$, then $q' \notin C_\wedge(P)$ and also $q' \notin C_\vee(P)$. And if $q \in C(P)$, then $q' \notin C(P)$ follows from $C(P) \subset C_\wedge(P)$.

Theorem 5.2. *$q \in \Phi_\wedge(P)$ if and only if $q' \notin C_\wedge(P)$.*

Proof. Obviously, $p_\wedge \nleqslant q'$ is equivalent to $q' \notin C_\wedge(P)$. □

As a corollary it holds that $q \notin C_\wedge(P)$ if and only if $q' \in \Phi_\wedge(P)$.

Theorem 5.3. *$q \in \Phi_\wedge(P) - C_\wedge(P)$ if and only if $q' \in \Phi_\wedge(P) - C_\wedge(P)$.*

Proof. $p_\wedge \nleqslant q'$ and $p_\wedge \nleqslant q$, is equivalent to $p_\wedge \nleqslant q'$ and $p_\wedge \nleqslant (q')'$. □

Theorem 5.4.

$$\Phi_\vee(P) - \Phi_\wedge(P) = \bigl\{q \in L_{01}; q \in \Phi_\vee(P) \text{ and } q' \in C_\wedge(P)\bigr\}.$$

Proof. Follows from Theorem 5.2. □

Theorem 5.5.

$$\Phi_\wedge(P) - \bigl(C_\wedge(P) \cup H(P)\bigr) = \bigl\{q \in L_{01};\ p_\wedge\, NC\, q,\ p_\wedge \nleqslant q'\bigr\}.$$

Proof. The elements of the above set verify $p_\wedge \nleqslant q'$ and do not verify $p_\wedge \leqslant q$ and $q \leqslant p_\wedge$. □

Theorem 5.6.

$$C_\wedge(P) = C_\vee(P) \cup C(P) \cup NC_\vee(P),$$

where $NC_\vee(P) = \{q \in C_\wedge(P); q\, NC\, p_\vee\}$.

Proof. Obviously,

$$C_\vee(P) \cup C(P) \cup NC_\vee(P) \subset C_\wedge(P).$$

Also, $q \in C_\wedge(P) - (C_\vee(P) \cup C(P))$, verifies neither $p_\vee \leqslant q$ nor $q \leqslant p_\vee$; hence, $q\, NC\, p_\vee$. $\square$

Consequently, $NC_\vee(P) \cap [C_\vee(P) \cup C(P)] = \emptyset$.

Theorem 5.7. *Supposing $p_\wedge \neq 1$, if $q \in H(P)$ then $q' \notin H(P)$.*

Proof. If $q \leqslant p_\wedge$ and $q' \leqslant p_\wedge$, then $q + q' = 1 \leqslant p_\wedge$, that is, $p_\wedge = 1$. $\square$

Theorem 5.8. *If $h_1, h_2 \in H(P)$ are such that $h_1 \leqslant h_2'$, then h_1 and h_2 are in $H^*(P)$.*

Proof. Let us suppose that $h_1 = p_\wedge$. $h_2 \leqslant p_\wedge'$ follows from $p_\wedge \leqslant h_2'$ and, hence, $h_2 = 0$, which is absurd. $\square$

Example. Contradictory hypotheses do indeed exist. Let L be the Boolean algebra of 2^4 elements with atoms a_1, a_2, a_3 and a_4. If $p = a_1 + a_2 + a_3$ and $P = \{p\}$, then $p_\wedge = p \neq 0$, and $H^*(P) = \{a_1, a_2, a_3, a_1 + a_2, a_1 + a_3, a_2 + a_3\}$. Obviously, $a_2' = a_1 + a_3 + a_4$, thus $a_1 \leqslant a_2'$. It is clear that if $h \in H^*(P) \cup \{p\}$, then $h' \notin H^*(P) \cup \{p\}$, as $p_\wedge = p \neq 1$.

Remark. The hypotheses a_1, a_2, a_3 are pairwise contradictory and verify $a_1 + a_2 + a_3 = p_\wedge$, that is, they are exhaustive.

Theorem 5.9.
(a) *$H(P) \subset H(\{p\})$, for every $p \in P$.*
(b) *If $p \in P$, then $p \in H(P)$ if and only if $H(P) = H(\{p\})$.*

Proof. (a) Follows from $\{p\} \subset P$.
(b) If $H(P) = H(\{p\})$, then as $\bigwedge\{p\} = p$ we have $p_\wedge = p$ and, hence, $p \in H(\{p\})$. Reciprocally, if $p \in H(P)$, then $p_\wedge = p$ follows from $p \leqslant p_\wedge \leqslant p$, hence $q \leqslant p_\wedge$ is equivalent to $q \leqslant p$. $\square$

Theorem 5.10. *If there is more than one element in $P \in \mathbb{P}_0(L)$ and there exists $p* \in P$ such that $p \leqslant p*$ for every $p \in P$, then*:
(a) *$\Phi_\wedge(P) = \Phi_\wedge(P - \{p*\})$.*
(b) *$\Phi_\vee(P) = \Phi_\vee(\{p*\})$.*
(c) *$C_\wedge(P) = C_\wedge(P - \{p*\})$.*
(d) *$C_\vee(P) = C_\vee(\{p*\})$.*
(e) *$H(P) = H(P - \{p*\})$.*

Proof. All are immediate, as $p_\wedge = \bigwedge P = \bigwedge(P - \{p*\})$ and $p_\vee = \bigvee P = p*$. $\square$

The theorem is valid in particular if $1 \in P$.
If $P \in \mathbb{P}_0(L)$, let $P' = \{p'; p \in P\}$. Then

$$\bigwedge P' = \bigwedge_{p \in P} p' = \Bigl(\bigvee_{p \in P} p\Bigr)' = 0$$

if and only if $p_\vee = \bigvee P = 1$, and hence, if $p_\vee \neq 1$ then $P' \in \mathbb{P}_0(L)$.

Theorem 5.11. *If $P \in \mathbb{P}_0(L)$ and $p_\vee \neq 1$, then $q \in H(P)$ if and only if $q' \in C_\vee(P')$.*

Proof. $q \leqslant p_\wedge$ if and only if

$$q' \geqslant p'_\wedge = \Bigl(\bigwedge_{p\in P} p\Bigr)' = \bigvee_{p\in P} p' = \bigvee P'. \quad \square$$

5.2.

The above results lead to a partition of the set $\Phi_\vee(P)$, whose elements we will call *conjectures* of P. As $q \in \Phi_\vee(P)$ means $p_\vee \nleqslant q'$, it is clear that not all the $p \in P$ will be contradictory with q; some $p \in P$ are not contradictory with q. Moreover, as $q \in \Phi_\wedge(P)$ means $p_\wedge \nleqslant q'$, it is clear that no $p \in P$ can be contradictory with q; we will say that $\Phi_\wedge(P)$ is the set of *strict conjectures* of P. We will say that $C_\wedge(P)$ is the set of *consequences* of P, $C_\vee(P)$ is the set of *loose consequences* of P, and $C(P)$ are the *restricted consequences* of P. We will say that $H(P)$ is the set of *hypotheses* of P, and $H^*(P)$ is the set of *proper hypotheses* of P. We will say that $\Phi_\wedge(P) - (C_\wedge(P) \cup H(P)) = \Phi^*_\wedge(P)$ is the set of *proper conjectures* of P, $\Phi(P)$ the set of *strict and restricted conjectures*, and $\Phi_\vee(P) - \Phi_\wedge(P) = \Phi^*_\vee(P)$ will be the *loose conjectures* of P. Fig. 3 shows all the sets of conjectures considered and the obtained classification of $\Phi_\vee(P)$.

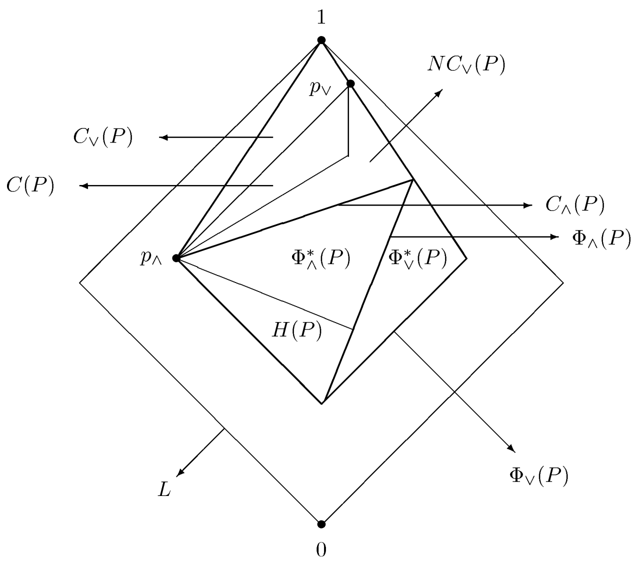

Fig. 3. Classification of $\Phi_\vee(P)$.

5.3.

This paper facilitates a theoretical framework in which Reiter's *Default Logic* can be reformulated. In fact, if $P = \{p_1, \ldots, p_n\}$, a *default rule*

$$P;\, p_{n+1} / q,$$

can be read as:

> "*If* $P \in \mathbb{P}_0(L)$, $q \in \Phi_\wedge(P)$, $p_{n+1} \neq 0$ and p_{n+1} is not incompatible with $p_\wedge$, *then* conclude q by default",

because it holds that $p_1 \cdot \cdots \cdot p_n \cdot p_{n+1} \neq 0$, it follows from $p_1 \cdot \cdots \cdot p_n \cdot p_{n+1} \leqslant q'$ that $p_\wedge \leqslant q'$ and hence $q \in \Phi_\wedge(\{p_1, \ldots, p_n, p_{n+1}\})$. For short, q can be conjectured from the information given by the *consistent* couple $(P;\, p_{n+1})$.

6. Existence and form of hypotheses

6.1.

Let $P \in \mathbb{P}_0(L)$. For every $h \in H(P)$, as $h \neq 0$, it makes sense to consider $C_\wedge(\{h\}) = C_\vee(\{h\})$ which we will be denote as $C(h)$.

Theorem 6.1. *For every $h \in H(P)$, $C_\wedge(P) \subset C(h) \subset \Phi_\wedge(\{h\}) \subset \Phi_\wedge(P)$.*

Proof. If $p_\wedge \leqslant q$, it follows from $h \leqslant p_\wedge$ that $h \leqslant q$. If $h \leqslant q$ and also $h \leqslant q'$ then $h = 0$; hence $h \nleqslant q'$. If $h \nleqslant q'$ and $p_\wedge \leqslant q'$, it follows from $h \leqslant p_\wedge$ that $h \leqslant q'$ would be verified, which is absurd; hence, $p_\wedge \nleqslant q'$. □

That is, any hypothesis of the premises has no fewer consequences than the premises have, and such consequences are strict conjectures of the hypotheses and also of the premises.

Therefore, in order to ascertain that some $h \in L_0$ *is not* a hypothesis for P, it will suffice to find a $q \in C_\wedge(P)$ such that $q \notin C(h)$ or an $r \in C(h)$ such that $r \notin \Phi_\wedge(\{h\})$, etc. That is, in practice, to *falsify* a hypothesis h.

With regard to the restricted consequences of $h \in H(P)$, we have that $C(\{h\}) = \{h\}$ and it is no longer the case that $C(P) \subset C(\{h\})$; it holds that $C(P) \cap C(\{h\}) = \emptyset$ if $h < p_\wedge$, and if $h = p_\wedge$, then $C(P) \cap C(\{h\}) = \{p_\wedge\}$. That is, the restricted Consequences Operator is of no interest for analysing the consequences of a hypothesis of P.

Theorem 6.2. *For every $h \in H(P)$, $C(h) - C_\wedge(P) \subset H^*(P) \cup \Phi^*_\wedge(P)$.*

Proof. $q \in C(h) - C_\wedge(P)$ is equivalent to $h \leqslant q$ and $p_\wedge \nleqslant q$, where $h \leqslant p_\wedge$. If $q \leqslant p_\wedge$ then $q \in H(P)$, but if $q = p_\wedge$, then $q \in C_\wedge(P)$, which is absurd; hence, in this case, $q \in H^*(P)$. If $q \nleqslant p_\wedge$, then:

- If $p_\wedge \leqslant q$, then $q \in C_\wedge(P)$. Hence $p_\wedge \nleqslant q$.

- If $p_\wedge \leqslant q'$, where $h \leqslant p_\wedge$, then $h \leqslant q'$ and $h = 0$. Hence $p_\wedge \nleqslant q'$.

Moreover, q is contingent, as if $q = 0$, then $h = 0$, and if $q = 1$, then $q \in C_\wedge(P)$. Hence, $q \in H^*(P) \cup \Phi^*_\wedge(P)$ (see Theorem 5.5). □

Given $h \in H(P)$, let us denote $\Delta(P; h) = \{q \in L; h \leqslant q, p_\wedge \nleqslant q\}$. Note that if $h = p_\wedge$, then $\Delta(P; h) = \emptyset$.

Theorem 6.3.

$$\Delta(P; h) = \big\{q \in L; h \leqslant q < p_\wedge\big\} \cup \big\{q \in L; h \leqslant q \; NC \; p_\wedge\big\}.$$

Proof. Obvious. □

Hence, $\Delta(P; h) = [h, p_\wedge) \cup \{q \in L; h \leqslant q \; NC \; p_\wedge\}$.

Theorem 6.4. *$C(h) = C_\wedge(P) \cup \Delta(P; h)$, and is a disjoint union.*

Proof.

$$\begin{aligned} C(h) &= \{q \in L; h \leqslant q\} = \{q \in L; p_\wedge \leqslant q\} \cup \{q \in L; h \leqslant q, p_\wedge \nleqslant q\} \\ &= C_\wedge(P) \cup \Delta(P; h). \end{aligned}$$

Obviously, $C_\wedge(P)$ and $\Delta(P; h)$ have no common elements. □

Corollary 6.5. *$C(h) = \{h\}$ if and only if $C_\wedge(P) = \{h\}$ and $\Delta(P; h) = \emptyset$.*

Proof. As $p_\wedge \in C(h)$, $h = p_\wedge$ follows from $C(h) = \{h\}$, this implies that $\Delta(P; h) = \emptyset$ and, hence, $C_\wedge(P) = C(h) = \{h\}$. The reciprocal is immediate. □

As $1 \in C(h)$, then $C(h) = \{h\}$ if and only if $h = p_\wedge = 1$.

Corollary 6.6.

$$C(h) - C_\wedge(P) = \Delta(P; h).$$

Proof. Immediate. □

Hence, $C(h) = C_\wedge(P)$ if and only if $\Delta(P; h) = \emptyset$.

Theorem 6.7. *$C_\wedge(P) = C(h)$ if and only if $h = p_\wedge$.*

Proof. If $C_\wedge(P) = C(h)$, $p_\wedge \leqslant h$ follows; hence, $h = p_\wedge$. Reciprocally, if $h = p_\wedge$ then

$$C(h) = C(\{p_\wedge\}) = \{q \in L_0; p_\wedge \leqslant q\} = C_\wedge(P). \quad □$$

Hence, $C_\wedge(P) \subset C(h)$ and $C_\wedge(P) \neq C(h)$ if and only if $h \in H^*(P)$.

Note that $h = p_\wedge$ if and only if $\Delta(P; h) = \emptyset$, and, hence $\Delta(P; h) \neq \emptyset$ if and only if $h \in H^*(P)$.

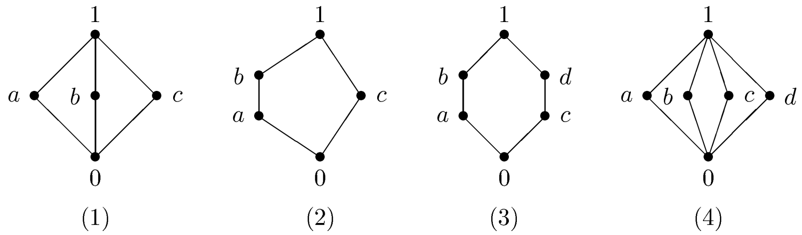

Fig. 4. Lattices.

6.2.

For every $P \in \mathbb{P}_0(L)$, set $A(P) = \{a \in L_{01};\ p_\wedge NC\, a,\ a \cdot p_\wedge \neq 0\}$ and $B(P) = \{a \in L_{01};\ p_\wedge \nleqslant a,\ a \cdot p_\wedge \neq 0\}$. Obviously, $A(P) \subset B(P)$ and neither 1, nor $p_\wedge$, nor $p'_\wedge$ are members of both sets. As $p_\wedge \cdot a \neq 0$ implies $p_\wedge \nleqslant a'$, it is clear that $\Phi^*_\wedge(P) \subset A(P) \subset B(P)$.

In the hexagonal orthocomplemented lattice shown in Fig. 4(3), for example, if $P = \{b\}$, then $p_\wedge = b \neq 0$ and $B(P) = \{a\}$, but $A(P) = \emptyset$. In the rhombic orthocomplemented lattice shown in Fig. 4(4), if $P = \{a\}$ then $p_\wedge = a \neq 0$ and $B(P) = \emptyset$.

Theorem 6.8.

$$H^*(P) = \big\{a \cdot p_\wedge;\, a \in B(P)\big\} \doteq p_\wedge B(P).$$

Proof. If $B(P) = \emptyset$, it follows that $H^*(P) = \emptyset$, since if there exists some $h \in H^*(P)$, then $h \in L_{0,1}$, $h \leqslant p_\wedge$ and $h \neq p_\wedge$; thus, $p_\wedge \nleqslant h$ and also $h \cdot p_\wedge = h \neq 0$; that is $h \in B(P)$, which is absurd. Hence $H^*(P) = \emptyset$ and $H^*(P) = p_\wedge B(P)$.

If $B(P) \neq \emptyset$, then as $a \cdot p_\wedge \leqslant p_\wedge$ and $a \cdot p_\wedge = p_\wedge$ if and only if $p_\wedge \leqslant a$, and it is impossible that $a \cdot p_\wedge = 0$, it is clear that $a \cdot p_\wedge \in H^*(P)$, hence $p_\wedge B(P) \subset H^*(P)$. Reciprocally, if $h \in H^*(P)$, then $h \neq 0$, $h \leqslant p_\wedge$ and $h \neq p_\wedge$, whereby there exists $a \in L_{0,1}$ such that $h = a \cdot p_\wedge$, and such a verifies $p_\wedge \nleqslant a$, as if $p_\wedge \leqslant a$, we would have $h = p_\wedge$; also $a \cdot p_\wedge \neq 0$, since if it were 0, then $h = 0$. Hence, $a \in B(P)$ and $H^*(P) \subset p_\wedge B(P)$. □

Note that, consequently, it holds that $H^*(P) = \emptyset$ if and only if $B(P) = \emptyset$.

Theorem 6.9. *If L is an orthomodular lattice, then $H^*(P) = p_\wedge A(P)$.*

Proof. It suffices to prove that if L is orthomodular, then $p_\wedge A(P) = p_\wedge B(P)$. As $A(P) \subset B(P)$, then $p_\wedge A(P) \subset p_\wedge B(P)$.

Reciprocally, if $B(P) = \emptyset$ it is clear that $p_\wedge B(P) = p_\wedge A(P) = \emptyset$. If $B(P) \neq \emptyset$ and $a \in B(P)$, it follows that $p_\wedge \nleqslant a$, that is, either $a \leqslant p_\wedge$ and $a \neq p_\wedge$, or $a\, NC\, p_\wedge$. In the latter case, $a \in A(P)$. In the other case, let $h = a + p'_\wedge$; then:

(1) $h \neq 0$ as $h \geqslant a \neq 0$;
(2) $h \neq 1$, as if $h = 1$ we would have $p_\wedge = p_\wedge \cdot 1 = p_\wedge(a + p'_\wedge) = a$ as $a \leqslant p_\wedge$, and L is orthomodular;

(3) $p_\wedge \nleqslant h$, because if $p_\wedge \leqslant h$, then as $p'_\wedge \leqslant a + p'_\wedge = h$, $1 = p_\wedge + p'_\wedge \leqslant h$ follows, which is impossible;

(4) $h \nless p_\wedge$, as, otherwise, we would have $p'_\wedge \leqslant a + p'_\wedge < p_\wedge$ and $p'_\wedge = 0$. Moreover, as L is orthomodular, $p_\wedge \cdot h = p_\wedge \cdot (a + p'_\wedge) = a \neq 0$. Hence, $h \in A(P)$ and $h \cdot p_\wedge = a \cdot p_\wedge \in p_\wedge A(P)$. □

Thus, in the hexagonal lattice shown in Fig. 4(3), if $P = \{b\}$, then $p_\wedge = b$, $B(P) = \{a\}$, and we have that $H^*(P) = \{a\} = b \cdot B(P) = \{a \cdot b\}$. However, $H^*(P) \neq b \cdot A(P)$ as $A(P) = \emptyset$, which is not surprising since the lattice is not orthomodular.

It should be pointed out that as last theorem holds, in particular, if L is a Boolean algebra, Theorems 6.8 and 6.9 generalize the one given in [8] on the structure of hypothesis.

6.3.

Let us return to the lattice of the vector subspaces of a vector space, specifically, let $(\mathcal{L}(\mathbb{R}^n), +, \cap, \perp)$ and let us denote the coordinate hyperplanes as $\pi_i = \{x_i = 0\}$. Given the set of premises $P = \{\pi_i;\ i = 4, \ldots, n\}$, then

$$p_\wedge = \bigcap_{i=4}^{n} \pi_i = \{x_i = 0;\ \forall i = 4, \ldots, n\}$$

is isomorphic to $\mathbb{R}^3$ and, hence, the set of proper hypotheses will be isomorphic to all the proper vector subspaces of $\mathbb{R}^3$, that is, to all the planes and the straight lines of $\mathbb{R}^3$:

$$H^*(P) = \left\{ \left\{ \pi_{\bar\alpha},\ \bar\alpha \in \mathbb{R}^3 - \{\bar 0\} \right\} \cup \left\{ l_{\bar\alpha\bar\beta},\ \bar\alpha, \bar\beta \in \mathbb{R}^3 - \{\bar 0\} \right\} \right\},$$

where for every $\bar\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 - \{\bar 0\}$, we have

$$\pi_{\bar\alpha} = \left\{ \begin{array}{l} x_i = 0,\ \forall i = 4, \ldots, n \\ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \end{array} \right\},$$

and for every $\bar\alpha = (\alpha_1, \alpha_2, \alpha_n)$ and every $\bar\beta = (\beta_1, \beta_2, \beta_3)$ of $\mathbb{R}^3 - \{\bar 0\}$,

$$l_{\bar\alpha\bar\beta} = \left\{ \begin{array}{l} x_i = 0,\ \forall i = 4, \ldots, n \\ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \\ \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0 \end{array} \right\}.$$

If $h \in H^*(P)$ is isomorphic to a plane of $\mathbb{R}^3$, that is, $h = \pi_{\bar\alpha}$ for some $\bar\alpha = (\alpha_1, \alpha_2, \alpha_n) \in \mathbb{R}^3 - \{\bar 0\}$ then the hyperplane

$$a = \{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0\} \in A(P),$$

as it is not comparable with $p_\wedge$, and also $p_\wedge \cap a = h$. If $h \in H^*(P)$ is isomorphic to a straight line of $\mathbb{R}^3$, that is, $h = l_{\bar\alpha\bar\beta}$ for some $\bar\alpha = (\alpha_1, \alpha_2, \alpha_n)$ and $\bar\beta = (\beta_1, \beta_2, \beta_3)$ of $\mathbb{R}^3 - \{\bar 0\}$, it would suffice to consider

$$a = \left\{ \begin{array}{l} \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \\ \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0 \end{array} \right\} \in A(P),$$

to obtain h as the intersection of $p_\wedge$ with a. Moreover, in both cases, there are infinite elements $a \in A(P)$ that give h when intersecting them with $p_\wedge$.

Appendix A. Note on orthocomplemented lattices

A.1.

Let L be a lattice with $+$ (union) and $\cdot$ (intersection) operations, greatest element 1 and least element 0. The partial order $\leqslant$ of the lattice is defined by "$a \leqslant b$ *iff* $a \cdot b = a$ *iff* $a + b = b$", and, therefore, is $0 \leqslant a \leqslant 1$ for every $a \in L$.

A lattice L is orthocomplemented if it has a unary operation $' : L \to L$ such that:

(1) $a \cdot a' = 0$.
(2) $a \leqslant a''$.
(3) $(a + b)' = a' \cdot b'$.
(4) $'(L) = L$, for any a, b of L ($'$ is an orthocomplementation).

It follows from the above four properties that:

(5) If $a \leqslant b$, then $b' \leqslant a'$.
(6) $(a \cdot b)' = a' + b'$.
(7) $a'' = a$.
(8) If $a \leqslant b'$, then $a \cdot b = 0$.
(9) $a + a' = 1$, for any a, b of L.

The operation $a \to b = a' + a \cdot b$ translates into L the statement "If a, then b" and it is, generally, $a \to b \leqslant a' + b$. However, if L is distributive, it is

$$a' + a \cdot b = (a' + a) \cdot (a' + b) = 1 \cdot (a' + b) = a' + b.$$

A lattice L verifying the modular law, that is, "If $a \leqslant c$, then $(a+b) \cdot c = a + b \cdot c$", is said to be a modular lattice. An *orthomodular* lattice is an orthocomplemented lattice verifying: "If $a \leqslant c$, then $c \cdot (c' + a) = a$", a property which is weaker than the modular property. Thus, any orthocomplemented lattice that verifies the modular law, is an orthomodular lattice. A distributive and orthocomplemented lattice is a Boolean algebra, which is a particular case of an orthomodular lattice.

The rhombic lattice (Fig. 4(1)) is modular but not distributive and admits no orthocomplementation. The pentagonal lattice (Fig. 4(2)) is not modular and, hence, not distributive; the application $'$ given by $0' = 1$, $1' = 0$, $a' = c$, $b' = c$ and $c' = b$, verifies laws (1), (2) and (3), but does not verify (4), as $'(L) = \{0, 1, c, b\} \neq L$; hence, it is not orthocomplemented; the sublattice $'(L)$ is a Boolean algebra. The hexagonal lattice (Fig. 4(3)) is not modular and is orthocomplemented with $0' = 1$, $1' = 0$, $a' = d$, $b' = c$, $c' = b$, $d' = a$. Finally, the lattice shown in Fig. 4(4) is modular and non-distributive, admitting the following orthocomplementations:

- $0' = 1,\ 1' = 0,\ a' = b,\ b' = a,\ c' = d,\ d' = c,$
- $0' = 1,\ 1' = 0,\ a' = d,\ b' = c,\ c' = b,\ d' = a,$
- $0' = 1,\ 1' = 0,\ a' = c,\ b' = d,\ c' = a,\ d' = b.$

Distributive orthocomplemented lattices, or Boolean algebras, are univocally orthocomplemented.

A.2.

$a \in L$ is *contradictory* with $b \in L$ if $a \leqslant b'$ (which is equivalent to $a \to b' = 1$). This is a symmetric relation, as it is equivalent to $b \leqslant a'$; hence, it suffices that $a \nleqslant b'$ or $b \nleqslant a'$ for a and b not to be contradictory. The only self-contradictory element of an orthocomplemented lattice is $a = 0$.

$a \in L$ is *incompatible* with $b \in L$ if $a \cdot b = 0$. This is obviously a symmetric relation and, also, the only self-incompatible element is $a = 0$. If $a \leqslant b'$, then $a \cdot b \leqslant b \cdot b' = 0$ and $a \cdot b = 0$: contradiction implies incompatibility. However, the reverse does not generally hold: with the orthocomplementation of the above hexagonal lattice, it is $b \cdot d = 0$ and not it is $b \leqslant d'$ as $d' = a$. A sufficient condition for a and b not to be contradictory is, therefore, that $a \cdot b \neq 0$.

Inequality $a \cdot b + a \cdot b' \leqslant a$, is always verified for any a, b of L; b is said *to commute* with a, if $a \cdot b + a \cdot b' = a$. This relation is not generally symmetric: in the above hexagonal lattice it is $b \cdot a' + b \cdot a'' = b$ (a' commutes with b), but $a' \cdot b + a' \cdot b' = b' \neq a'$ (b does not commute with a'). If L is distributive, any pair a, b of L commutes:

$$a = a \cdot 1 = a \cdot (b + b') = a \cdot b + a \cdot b'.$$

Theorem A.1. *If two elements are incompatible, then they are contradictory if and only if one commutes with the other.*

Proof. If $a \cdot b = 0$ and, for example, $a = a \cdot b + a \cdot b'$ then $a = a \cdot b'$, or $a \leqslant b'$. Reciprocally, if $a \leqslant b'$, then $a \cdot b + a \cdot b' = 0 + a = a$; as also $b \leqslant a'$,

$$b \cdot a + b \cdot a' = 0 + b = b$$

follows. □

Accordingly, the relations of contradiction and incompatibility coincide in a Boolean algebra. In fact, a set L with $+$ and $'$ operations, in which the $\cdot$ operation is defined as $x \cdot y = (x' + y')'$, is certain to be a Boolean algebra if it verifies the laws:

- $x + y = y + x$,
- $x + (y + z) = (x + y) + z$,
- $x = x \cdot y + x \cdot y'$.

In the non-distributive modular lattice shown in Fig. 4(4), it is $a \cdot b = 0$ and it is not $a \leqslant b'$ with the second orthocomplementation, as $b' = c$.

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