

Anyone but him: The complexity of precluding an alternative[☆]

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Received 1 February 2006; received in revised form 18 January 2007; accepted 29 January 2007

Available online 7 February 2007

Abstract

Preference aggregation in a multiagent setting is a central issue in both human and computer contexts. In this paper, we study in terms of complexity the vulnerability of preference aggregation to destructive control. In particular, we study the ability of an election's chair to, through such mechanisms as voter/candidate addition/suppression/partition, ensure that a particular candidate (equivalently, alternative) does not win. And we study the extent to which election systems can make it impossible, or computationally costly (NP-complete), for the chair to execute such control. Among the systems we study—plurality, Condorcet, and approval voting—we find cases where systems immune or computationally resistant to a chair choosing the winner nonetheless are vulnerable to the chair blocking a victory. Beyond that, we see that among our studied systems no one system offers the best protection against destructive control. Rather, the choice of a preference aggregation system will depend closely on which types of control one wishes to be protected against. We also find concrete cases where the complexity of or susceptibility to control varies dramatically based on the choice among natural tie-handling rules.

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Keywords: Approval voting; Computational complexity; Computational resistance; Condorcet voting; Destructive control; Election systems; Plurality voting; Preference aggregation; Multiagent systems; Vote suppression

1. Introduction

Voting provides a broad framework for collective decision-making. The literature on voting is vast and active, and spans such areas as AI, complexity, economics, operations research, and political science. As noted by Conitzer, Lang, and Sandholm [9], voting has been proposed as a mechanism for use in decision-making in various computational

[☆] Supported in part by grants NSF-CCR-0311021, NSF-CCF-0426761, DFG-RO-1202/9-1, and DFG-RO-1202/9-3, a Friedrich Wilhelm Bessel Research Award, and the Alexander von Humboldt Foundation's TransCoop program. A preliminary version of this paper appeared in AAAI-05 [E. Hemaspaandra, L. Hemaspaandra, J. Rothe, Anyone but him: The complexity of precluding an alternative, in: Proceedings of the 20th National Conference on Artificial Intelligence, AAAI Press, 2005]. This work was done in part while the three authors were visiting Julius-Maximilians-Universität Würzburg, in part while the first two authors were visiting Heinrich-Heine-Universität Düsseldorf, and in part while the first author was on sabbatical at the University of Rochester.

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settings, including planning [13,14] and collaborative filtering [29]. Voting also may be useful in many large-scale computer settings. Examples of much recent interest include the (web-page) rank aggregation problem, and related issues of reducing “spam” results in web search and improving similarity search, for which the use of voting systems has been proposed [12,15]. In such an automated setting, it is natural to imagine decisions with thousands or millions of “voters” and “candidates”.

In Bartholdi, Tovey, and Trick’s seminal paper “How hard is it to control an election?” [3], the issue of constructive control of election systems is studied: How hard is it for a chair (who knows all voters’ preferences) to—through control of the voter or candidate set or of the partition structure of an election—cause a given candidate (equivalently, alternative) to be the (unique) winner?

To avoid any possible confusion, let us immediately stress that by “chair” we, following the use of Bartholdi, Tovey, and Trick [3], simply mean some entity that can carry out the given type of change to the structure of the election. Bartholdi, Tovey, and Trick probably chose to use the word “chair” because that role might typically be that of some authority responsible for organizing the election and determining its structure. Note that the chair is not necessarily one of the candidates, or the distinguished candidate c of our forthcoming problem specifications,¹ or one of the voters (although we do not exclude these possibilities). (In particular, “chair” here does not mean “a special, designated voter who if there is a tie is allowed to break the tie”—a meaning the term has in some other contexts, but not in this paper.)

To avoid a different potential source of confusion, let us also immediately focus on the fact that in the Bartholdi, Tovey, and Trick [3] model, which is also adopted here, the chair has complete information on the voters’ preferences. This is a natural assumption in many, though certainly not all, situations.² However, it is critical to note that since the case where complete information is available to the chair is a special subcase of the more general setting that allows information to be specified with any level of completeness, *lower bounds obtained in the complete information setting are inherited by any natural incomplete information model* (this point was made in this context even in the original control paper of Bartholdi, Tovey, and Trick [3]). This is worth stressing: All our NP-hardness results—obtained here in the model of complete information—are instantly inherited by any natural incomplete information model (e.g., models in which missing preferences are filled in by an adversary of the chair; models in which missing preferences are filled in as favorably for the chair’s goal as possible; models in which the chair must find an action that under at least k (k would be part of the input) of all possible completions of the unknown information allows the chair to obtain his/her desired outcome; models in which the chair must find an action that under at least 99 percent of all possible completions of the unknown information allows the chair to obtain his/her desired outcome; etc.). Another angle to approach this from would be to note that our lower-bound results are sufficiently strong that they show that even when the chair has all the information he or she could possibly ask for about preferences—namely, complete information—his/her task is nonetheless still NP-hard in many cases. Simply put, the complete-information model is actually, as a matter of complexity-theoretic basics, a more challenging one in which to prove NP-hardness lower bounds than a partial-information model would be (although the authors realize that this may seem counterintuitive).

Bartholdi, Tovey, and Trick studied plurality and Condorcet voting, and seven natural types of control: adding candidates, suppressing candidates, partition of candidates, run-off partition of candidates, adding voters, suppressing voters, and partition of voters. They found that in some cases there is *immunity* to constructive control (if his/her

¹ It certainly is the case that in our model the chair will be trying to make the distinguished candidate c win. And so there will be many natural situations in which c and the chair coincide. However, that need not always be the case. It is possible to envision situations in which c does not want c to win (e.g., elections for the chair of an academic department in which some faculty member out of duty would be willing to serve if elected but would far rather someone else serve; or elections for who gets “removed from the island” on some reality TV show) or situations, even when c does want c to win, in which the actor working to make c win is someone other than c (e.g., in particularly bare-knuckled academic faculty hiring, a department chairperson might set a vote date when certain faculty members are out of town, as an attempt to help a certain one of the external applicants).

² For example, in a computer science department, after endless discussions, most people know what each person’s position is on key issues. This is an example of a rather small-scale, intimate, private election. Of course, one might reasonably point out that large-scale public elections among humans are examples where, though polling data may give some information, typically no one will know all voters’ preferences. However, we note that, as mentioned in the paragraph to which this is a footnote, our lower-bound (NP-hardness) results for the case of complete information imply NP-hardness in the case of essentially any natural partial-information model. We also point out that large-scale elections among electronic agents might well have complete preference information available to the chair. (Even in cases of large-scale human elections, there are relatively broad sources of information, such as party affiliation information in voter registration records. Although this certainly isn’t perfectly predictive, it might well be useful in attempts, for example, to gerrymander district boundaries or even, and this is in some rough sense an example of control by adding voters, to target get-out-the-vote drives.)

candidate was not already the³ unique winner, no action of the specified type by the chair can make the candidate the unique winner), in some cases there is (*computational*) *resistance* to constructive control (it is NP-complete to decide whether the chair can achieve his/her desired outcome), and in some cases the system is (*computationally*) *vulnerable* to constructive control (there is a polynomial-time algorithm that will tell the chair how to achieve the desired outcome whenever possible⁴).

In this paper, we obtain results for each of their 14 cases (two preference aggregation systems, each under seven control schemes) in the setting of *destructive* control. In contrast with constructive control, in which a chair tries to ensure that a specified desirable candidate is the (unique) winner, in destructive control the chair tries to ensure that a specified detested candidate is not the (unique) winner. Regarding the naturalness of destructivity, the light-hearted title of this paper tries to reflect the fact that, in human terms, one often hears feelings expressed that focus strategically on precluding one candidate, and of course in other settings this also may be a goal. Regarding the reality of electoral control, from targeted “get-out-the-vote” advertisements of parties and candidates to (alleged) voter suppression efforts by independent groups, from the way a committee chair groups alternatives to any case where a faculty member hands out student course evaluations on a day some malcontent students are not in class, it is hard to doubt that the desire for electoral control—both destructive and constructive—is a real one.

Destruction has been previously studied by Conitzer, Lang, and Sandholm [9,10], but in the setting of election *manipulation*—in which some (coalition of) voters knowing all other voters’ preferences are free to shift their own preferences to affect the outcome. In contrast, in this paper we study destruction in the very different setting of electoral *control* [3]—where a chair, given fixed and unchangeable voter preferences, tries to influence the outcome via procedural/access means.

One might ask, “Why bother studying destructive control, since any rational chair would prefer to assert constructive control?” The answer is that it is plausible—and our results show it is indeed the case—that destructive control may be possible in settings in which constructive control is not. Informally put, destructive control may be easier for the chair to assert. For example, we prove formally that of the seven types of constructive control of Condorcet elections that Bartholdi, Tovey, and Trick [3] study, the four they showed not vulnerable to constructive control are all vulnerable to destructive control. The remaining three cases regarding Condorcet voting are vulnerable to constructive control [3], but we show that they are immune to destructive control. As another example, consider the ten types of constructive control of approval elections that the present paper studies. We will prove formally that among the seven we show not vulnerable to constructive control, five of those seven are vulnerable to destructive control and the remaining two of those seven are immune to destructive control. The remaining three cases regarding approval voting are vulnerable to constructive control but immune to destructive control.⁵

Table 1 summarizes our results on the complexity of destructively controlling Condorcet, plurality, and approval elections. We also when needed obtain, for comparative purposes, new results on the complexity of constructive control, and Table 1 displays those and also constructive control results of Bartholdi, Tovey, and Trick [3]. All entries

³ Really “a unique winner”, since there may be no winner at all, but we’ll usually write “the unique winner” when this is clear from context.

⁴ This is more like “computationally certifiably-vulnerable”, see Definition 3.1. Vulnerability as defined in [3] means one can quickly decide if there *exists* a way for the chair to achieve the desired outcome.

⁵ Savvy readers may wonder whether there is something very troubling in having a system be vulnerable to constructive control but immune to destructive control. After all, to ensure that the despised candidate *c* is not the unique winner we simply have to ask whether either at least one of the other candidates can be ensured to unique-win-or-tie-for-winner or it can be ensured that there are no winners. Put somewhat formally, this implies that for *strongly voiced* (i.e., systems for which whenever there is at least one candidate there will be at least one winner; so approval voting is strongly voiced but Condorcet voting is not) election systems destructive control polynomial-time disjunctively truth-table reduces [27] to constructive control (redefined to speak not of “unique winner” but to speak of “winner (possibly with others also winning)”), and so the destructive control problem can (within a polynomial factor) be no harder *computationally* than the (redefined) constructive control problem (this type of reduction is noted in a different setting by Conitzer and Sandholm [10]). Our brief explanation of why cases of such a form would not cause a paradox lies in the word “computational”: Although immunity is the most desirable case in terms of security from control, the *complexity* of recognizing whether a given candidate can be precluded from winning in immune cases will most typically be in P—after all, we can *never*, when immunity holds, change a given candidate from unique winner to not the unique winner, so the related decision problem is typically easy. (Technical side remark: We say “will most typically be in P/is typically” rather than “will be in P/is” because for impractical systems that—unlike those here—have winner-testing problems that are not in P, it is in concept possible that one can have immunity and yet also have the related language problem not belong to P.)

The disjunctive-truth-table connection mentioned above explains why, if $P \neq NP$, it is impossible for any strongly voiced election system to have computational resistance to destructive control hold for any problem that, when redefined to embrace ties, is vulnerable to constructive control.

Table 1
Summary of results

Control by	Plurality		Condorcet		Approval	
	Construct.	Destruct.	Construct.	Destruct.	Construct.	Destruct.
Adding Candidates	R	R	I	V	I	V
Deleting Candidates	R	R	V	I	V	I
Partition of Candidates	TE: R TP: R	TE: R TP: R	V	I	TE: V TP: I	TE: I TP: I
Run-off Partition of Candidates	TE: R TP: R	TE: R TP: R	V	I	TE: V TP: I	TE: I TP: I
Adding Voters	V	V	R	V	R	V
Deleting Voters	V	V	R	V	R	V
Partition of Voters	TE: V TP: R	TE: V TP: R	R	V	TE: R TP: R	TE: V TP: V

Results new to this paper are in boldface. Nonboldface results are due to Bartholdi, Tovey, and Trick [3]. Key: I = immune, R = resistant, V = vulnerable, TE = Ties-Eliminate, TP = Ties-Promote.

in boldface in Table 1 are new results obtained in this paper; the other results are due to Bartholdi, Tovey, and Trick [3]. (A preliminary version of this paper appeared as [23].) For each boldface “V” in the table, “certifiably-vulnerable” is in fact also achieved by our theorems. We mention in passing that for nonboldface “V”s in the table, “certifiably-vulnerable” can be seen directly from or by modifying the algorithms of Bartholdi, Tovey, and Trick [3].

For control-by-partition problems—which will involve subelections—we distinguish between the models Ties-Eliminate (TE, for short) and Ties-Promote (TP, for short), which define what happens when there are ties among winners in a subelection (before the final election), namely, all participating candidates are eliminated and no one moves forward from that subelection (TE; to be very clear on this: In the TE model, a candidate moves forward from a subelection if and only if he/she uniquely wins that subelection), or all who tie for winner move forward (TP). Note that these models do not apply to Condorcet voting, under which when a winner exists he/she is inherently unique; so the TE/TP distinction is made only for plurality and approval voting. One purpose (see the discussion in the paragraph before Theorem 3.6) of introducing these tie-handling rules is to avoid the problem of the approach taken in [3], which in effect modifies the decision problem itself to avoid ties in subelections. In contrast, our two models leave the decision problem unchanged, while clearly defining how to handle such ties when they occur. We consider this general approach the more natural one. As to our two tie-handling models, a referee suggested that to him/her the TP model seems more natural than the TE model, and we pass on that comment to the reader. However, we point out that one could also see things the other way: Since in electoral control we seek to determine the complexity of making a candidate a unique winner, one could argue that it is most natural to apply in all subelections the same exacting standard, namely unique winning, that will be applied in the final election. And that is precisely what the TE model does. In this paper we always provide results in both models, TP and TE, and so fans of either model will find their model addressed, and even more importantly, readers interested in whether tie-handling models change results will find that a given control problem’s complexity sometimes changes dramatically depending on which tie-handling model is used.

The natural conclusion to draw from our results is that when selecting an election/preference aggregation system, one should at least be aware of the issue of the system’s vulnerability to control—and, beyond that, one’s choice of system will depend closely on which types of immunity or computational⁶ resistance one most values. Our results

⁶ We stress that the focus of this paper is on *computational* resistance to control (and we mention in passing that there is also a growing set of papers on the computational difficulty of manipulation and bribery, see for example [1,2,9,10,16,21] and the survey [17]). By having this computational focus, we in no way wish to detract from the fascinating results in the literature on social choice that touch on issues of when a candidate can be dethroned in various ways from what might be called an incentive-based point of view. For example, we mention the interesting recent work of Brams and Sanver [7] on “critical strategies under approval voting”, the focus of which is not on computational complexity issues but rather on incentive issues. Although a presentation of their results is beyond the scope of our paper we mention that, under appropriate assumptions about what strategies voters are allowed to hold and so on, they show that approval voting will always allow as a strong (protected even against any coalition of voters who would all benefit from a defection) Nash equilibrium the election of a single candidate who is a Condorcet winner, when one exists, and that various other voting systems lack this property and even lack some weaker properties.

also show that constructive and destructive control often differ greatly: A system immune to constructive control may be vulnerable to destructive control, and vice versa. Finally, our results show that the way in which ties in subelections are handled can have a significant effect: For both voting types where tie-handling rules are meaningful, we find cases where the complexity of or susceptibility to control varies dramatically based on the choice among natural tie-handling rules.

2. Preliminaries

We first define the three voting systems considered—plurality voting, Condorcet voting, and approval voting. Of course, there are many more voting systems than the three considered here; see, e.g., Brams and Fishburn [6] for a broad overview. The main reason for our choice of voting systems is that we wish to allow comparison with previous work. As mentioned earlier, Bartholdi, Tovey, and Trick [3] introduced and studied *constructive* control for plurality and Condorcet voting. A main contribution of this paper is to complement their results with the corresponding (and in the case of Condorcet voting, quite different) *destructive* control results. In addition, we study control for a third very natural and fundamental voting system, approval voting, and our results there give a quite different picture than those for plurality and Condorcet voting, as approval voting gives the only candidate-partition case we find where the choice of tie-handling rule makes a difference (see Section 3 for a detailed discussion of our results).

In *approval voting*, each voter votes “Yes” or “No” for each candidate. (So, for approval voting, a voter’s preferences are reflected by a 0–1 vector.) All candidates with the maximum number of “Yes” votes are winners. Approval voting has been proposed as a variant of plurality voting, see Brams and Fishburn [5].

Plurality and Condorcet voting are defined in terms of strict preferences. For them, an election is given by a *preference profile*, a pair (C, V) such that C is a set of candidates and V is the multiset (henceforth, we’ll just say set, as a shorthand) of the voters’ preference orders on C .⁷ We assume that the preference orders are irreflexive and antisymmetric (i.e., every voter has strict preferences over the candidates), complete (i.e., every voter ranks each candidate), and transitive.

A *voting system* is a rule for how to determine the winner(s) of an election. Formally, any voting system is defined to be a function mapping any given preference profile (or the analog with voters’ 0–1 vectors for the approval voting case) to society’s aggregate *choice set*, the set of candidates who have won the election.

In *plurality voting*, each candidate with the most “first preference among the candidates in the election” voters for him/her wins. In *Condorcet voting*, for each $c \in C$, c is a winner if and only if for each $d \in C$ with $d \neq c$, c defeats d by a strict majority of votes in a pairwise election between them based on the voters’ preferences.

The Condorcet Paradox observes that whenever there are at least three candidates, due to cyclic aggregate preference rankings Condorcet winners may not exist [8]. That is, the set of winners may be empty. However, a Condorcet winner is unique whenever one does exist. In the case of plurality and approval voting, due to ties, there may exist multiple winners. Regarding ties, we—following Bartholdi, Tovey, and Trick [3] to best allow comparison—focus in our control problems on creating a *unique* winner (constructive), and precluding a candidate from being the *unique* winner (destructive). (Ties in subelections, for the partition problems, are handled via the TE and TP rules described earlier.)

3. Results

The issue of control of an election by the authority conducting it (called the chair) can be studied under a variety of models and scenarios. For plurality and Condorcet voting, Bartholdi, Tovey, and Trick [3]—and for the rest of this section that paper will be referred to as “BTT92”—study constructive control by adding candidates, deleting candidates, partition of candidates, run-off partition of candidates, adding voters, deleting voters, and partition of voters. In their setting, the chair’s goal is to make a given candidate uniquely win the election. Analogously, we consider in turn the corresponding seven *destructive* control problems, where the chair’s goal is to preclude a given candidate from being the unique winner. For each of these control scenarios, we define the problem and present prior results and our results. (Formally, for each type of control one defines a decision problem and studies its computational

⁷ In various settings involving subelections, adding candidates, and resistance constructions, we will speak of an election (C', V) where the preferences of V are over some $C \supseteq C'$. In such cases, we intend the natural interpretation: For the purpose of that election one views the induced preference order (or approval vector) for the restriction to C' .

complexity.) To make comparisons as easy as possible, we in stating these control problems whenever possible exactly follow BTT92's wording for constructive control (except modified to the destructive case for the destructive cases), and when we diverge, we explain why and how.

As is common, we state our decision problems as “Given” instances, and a related Yes/No question. The language in each case is the set of all instances for which the answer is Yes. Since in each control scenario the “Given” instance is identical for the constructive and the destructive case, we state it just once and then state the corresponding two questions, one for constructive and one for destructive control.

3.1. Immunity, vulnerability, and resistance definitions, and Control by Adding Candidates

Given: A set C of qualified candidates and a distinguished candidate $c \in C$, a set D of possible spoiler candidates, and a set V of voters with preferences (in the approval case, the “preferences” will, as always for that case, actually be 0–1 vectors) over $C \cup D$.

Question (constructive): Is there a choice of candidates from D whose entry into the election would assure that c is the unique winner?

Question (destructive): Is there a choice of candidates from D whose entry into the election would assure that c is not the unique winner?

The above type of control captures the idea that the chair tries to enthrone the desired candidate c (in the constructive case) or to dethrone the despised candidate c (in the destructive case) by introducing new “spoiler” candidates.

V is formally a multiset. However, throughout this paper we assume—as is the standard approach in papers on the computational complexity of elections—that in the input the preferences are coded as a list (the ballots), one voter at a time, and in particular are not encoded as a multiset that uses binary numbers to code cardinalities.

With this first problem—Control by Adding Candidates—stated, now is a good time to define our notions of control. Our terminology will closely follow the notions in BTT92, to allow comparison.

Definition 3.1. We say that a voting system is *immune to control* in a given model of control (e.g., “destructive control via adding candidates”) if the model regards constructive control and it is never possible for the chair to by using his/her allowed model of control change a given candidate from being not a unique winner to being the unique winner, or the model regards destructive control and it is never possible for the chair to by using his/her allowed model of control change a given candidate from being the unique winner to not being a unique winner. If a system is not immune to a type of control, it is said to be *susceptible* to that type of control.

A voting system is said to be (*computationally*) *vulnerable to control* if it is susceptible to control and the corresponding language problem is computationally easy (i.e., solvable in polynomial time). If a system is not just vulnerable regarding some particular model of control but one can even produce in polynomial time the actual action of the chair to execute control the “best” way (namely, by adding or deleting the smallest number of candidates or voters for add/delete problems; for partition problems, any legal partition that works is acceptable), we say the system is (*computationally*) *certifiably-vulnerable to (that model of) control*.⁸

A voting system is said to be *resistant to control* if it is susceptible to control but the corresponding language problem is computationally hard (i.e., NP-complete).⁹

⁸ For the problems studied here, certifiably-vulnerable implies vulnerable (but we list both, since if one studied add/delete problems stated not in terms of “is there some subset” or “by adding/deleting at most k ” but rather in terms of “by adding/deleting exactly k ”, then for certain systems the implication need not hold).

⁹ It would be more natural to define resistance as meaning the corresponding language is (many-one) NP-hard. However, in this paper, we define resistance in terms of NP-completeness. One reason is that this matches the way the term is used by BTT92. More importantly, all the problems discussed in this paper have obvious NP upper bounds since testing whether a given candidate has won a given election for the systems considered here is obviously in P. So for the problems in this paper, NP-completeness and NP-hardness stand or fall together. We mention in passing that there are natural election systems whose complexity seems beyond NP. The first such case established was for the election system defined by Lewis Carroll in 1876 [11], where even the complexity of determining whether a given candidate has won is now known to be hard for parallel access to NP [22]. Other election systems whose winner complexity is hard for parallel access to NP include Kemeny and Young elections, see [20,25,33–35].

For general background on the theory of NP-completeness, see, e.g., [19,26].

As to what is known about Constructive Control by Adding Candidates, BTT92 shows that plurality is resistant and Condorcet is immune. Our results are:

Theorem 3.2. *Approval (voting) is immune to constructive control by adding candidates, and plurality, Condorcet, and approval (voting) are respectively resistant, vulnerable/certifiably-vulnerable, and vulnerable/certifiably-vulnerable to destructive control by adding candidates.*

So, though Condorcet and approval are immune to constructive control of this sort, they both are vulnerable to destructive control. This reverses itself for:

3.2. Control by Deleting Candidates

Given: A set C of candidates, a distinguished candidate $c \in C$, a set V of voters, and a positive integer $k < \|C\|$.

Question (constructive): Is there a set of k or fewer candidates in C whose disqualification would assure that c is the unique winner?

Question (destructive): Is there a set of k or fewer candidates in $C - \{c\}$ whose disqualification would assure that c is not the unique winner?

In this type of control, the chair seeks to influence the outcome of the election by suppressing certain candidates (other than c), in hopes that their voters now support c to ensure c 's victory (in the constructive case) or that they now support another candidate to ensure stopping c (in the destructive case).¹⁰

Note that this formalization of the destructive case is not a perfect analog of the constructive case of BTT92 in that we explicitly prevent deleting c , since otherwise any voting system in which the winners can efficiently be determined would be trivially vulnerable to this type of control.

Here, BTT92 establishes for constructive control resistance for plurality and vulnerability for Condorcet. Our results are:

Theorem 3.3. *Approval is vulnerable/certifiably-vulnerable to constructive control by deleting candidates. Plurality, Condorcet, and approval are respectively resistant, immune, and immune to destructive control by deleting candidates.*¹¹

We now handle jointly the two types of partition of candidates, since they yield identical results.

3.3. Control by Partition of Candidates

Given: A set C of candidates, a distinguished candidate $c \in C$, and a set V of voters.

Question (constructive): Is there a partition of C into C_1 and C_2 such that c is the unique winner in the sequential two-stage election in which the winners in the subelection (C_1, V) who survive the tie-handling rule move forward to face the candidates in C_2 (with voter set V)?

¹⁰ A reviewer, regarding the example of meta-search engines that our introduction gave to argue that some elections can have a large number of candidates, suggested the example wasn't interesting regarding control "as the election mechanism and the chair there are *indistinguishable*". We point out that in fact, even in that example, the chair and the election mechanism may, far from being indistinguishable, differ. The setting for this example is a meta-search engine that takes page-ranks from a number of search engines and aggregates them. Note that one might have an election mechanism implemented fairly by the meta-search engine. And one might have an evil external actor (who would in our terminology be the chair) trying to decide whether getting rid of some of the underlying search engines (who would in our terminology be the voters)—and depending on the setting this elimination could be by crashing their servers, by sabotaging their network connectivity to the meta-search engine, or even by buying their companies and shuttering them—he or she could get a more favorable overall outcome. This is a perfectly fair example of control by deletion of voters (a control type that will be defined in Section 3.6).

¹¹ For this and all other problems whose statements invoke a " k " bound, by immune we mean that for no election (and thus no k) can the chair's action ever cause change of the sort required to break immunity (i.e., taking someone who is not a unique winner and making him/her be a unique winner in the constructive cases, or taking someone who is a unique winner and making him/her no longer be a unique winner in the destructive cases), and by susceptible we mean "not immune" (under the definition just given).

Question (destructive): Is there a partition of C into C_1 and C_2 such that c is not the unique winner in the sequential two-stage election in which the winners in the subelection (C_1, V) who survive the tie-handling rule move forward to face the candidates in C_2 (with voter set V)?

3.4. Control by Run-off Partition of Candidates

Given: A set C of candidates, a distinguished candidate $c \in C$, and a set V of voters.

Question (constructive): Is there a partition of C into C_1 and C_2 such that c is the unique winner of the election in which those candidates surviving (with respect to the tie-handling rule) subelections (C_1, V) and (C_2, V) have a run-off with voter set V .

Question (destructive): Is there a partition of C into C_1 and C_2 such that c is not the unique winner of the election in which those candidates surviving (with respect to the tie-handling rule) subelections (C_1, V) and (C_2, V) have a run-off with voter set V .

These two types of control express settings—one via a cascading setup, and one via a run-off setup—in which the chair tries to, overall, partition the candidates in such a clever way that the favored candidate c is made the unique winner (in the constructive case) or that the hated candidate c fails to be the unique winner (in the destructive case). Here, BTT92 shows that for constructive control plurality is resistant (and their result on that holds in both our TE and TP models) and Condorcet is vulnerable. Our results are:

Theorem 3.4. *Approval is vulnerable/certifiably-vulnerable to constructive control by partition of candidates and run-off partition of candidates in model TE and immune to constructive control by partition of candidates and run-off partition of candidates in model TP. Plurality, Condorcet, and approval are, in models TE and TP, respectively resistant, immune, and immune to destructive control by partition of candidates and by run-off partition of candidates.*

So Condorcet, though vulnerable to constructive control, is immune to destructive control here. And, perhaps more interesting, for constructive control, approval changes from vulnerable to immune depending on the tie-handling rule.

We now turn to control of the voter set. The intuition behind seeking destructive control by adding or deleting voters is clear, e.g., getting out the vote and vote suppression.¹² We handle these two cases together as their results are identical.

¹² One referee reasonably wanted us to tell the reader something about the relative importance in US political elections of control via candidates versus control via voters, and so we pass on to the reader, verbatim, the referee's comment on this issue:

At least in the US, I think control by adding, deleting, and partitioning voters is less important than doing the same for candidates. True, before women got the vote in the 1920s and before the civil rights movement in the 1960s, voters were regularly disenfranchised. But now most manipulation today occurs through control of how and when candidates/alternatives are introduced and positioned, which in political science is often referred to as agenda control. Two good sources on this issue, one primarily empirical and the other theoretical, are W.H. Riker, *The Art of Political Manipulation* [32] and A.D. Taylor, *Social Choice and the Mathematics of Manipulation* [36].

Although we most certainly do not wish to disagree with the referee's expert comment, we do wish to mention that the importance of control-by-voters issues should not be overly discounted. First, the referee's comment is about US political elections, and we mention that our paper is relevant to voting issues not limited to *political* elections, but also in the broader setting of large-scale computer applications involving voting procedures, in which control of voters may be as important as candidate control, depending on the concrete situation. As to even US political elections, and as one example regarding actual in-law disenfranchisement, young people still lack the right to vote, and lest one dismiss this as a "natural" exclusion, we point out that the long-settled exclusion threshold regarding youth changed (in the 48 states that had a voting age over 18 at that time) in the early 1970s with the ratification of the 26th Amendment—a change, in turbulent times, that arguably had wide political implications. Even today, and still regarding in-law disenfranchisement, the exclusion of felons is a form of disenfranchisement that has political ramifications and is opposed by some. Also, as mentioned earlier, campaign ads, get-out-the-vote drives, spreading rumors that anyone who is wanted by the police will be arrested if he/she shows up to vote, and so on, though in some cases seeking to change who people vote for, also seek to encourage or discourage the decision to even exercise one's franchise at all.

3.5. Control by Adding Voters

Given: A set of candidates C and a distinguished candidate $c \in C$, a set V of registered voters, an additional set W of yet unregistered voters (both V and W have preferences over C), and a positive integer $k \leq \|W\|$.

Question (constructive): Is there a set of k or fewer voters from W whose registration would assure that c is the unique winner?

Question (destructive): Is there a set of k or fewer voters from W whose registration would assure that c is not the unique winner?

3.6. Control by Deleting Voters

Given: A set of candidates C , a distinguished candidate $c \in C$, a set V of voters, and a positive integer $k \leq \|V\|$.

Question (constructive): Is there a set of k or fewer voters in V whose disenfranchisement would assure that c is the unique winner?

Question (destructive): Is there a set of k or fewer voters in V whose disenfranchisement would assure that c is not the unique winner?

Here, BTT92 shows that for constructive control plurality is vulnerable and Condorcet is resistant. Our results are:

Theorem 3.5. *Approval is resistant to constructive control by adding voters and by deleting voters. Plurality, Condorcet, and approval are all vulnerable/certifiably-vulnerable to destructive control by adding voters and by deleting voters.*

So Condorcet and approval, though resistant to constructive control, are vulnerable to destructive control here.¹³ The final problem here results in a surprise.

3.7. Control by Partition of Voters

Given: A set of candidates C , a distinguished candidate $c \in C$, and a set V of voters.

Question (constructive): Is there a partition of V into V_1 and V_2 such that c is the unique winner in the hierarchical two-stage election in which the survivors of (C, V_1) and (C, V_2) run against each other with voter set V ?

Question (destructive): Is there a partition of V into V_1 and V_2 such that c is not the unique winner in the hierarchical two-stage election in which the survivors of (C, V_1) and (C, V_2) run against each other with voter set V ?

In this last type of control, the voter set is partitioned into two “subcommittees” that both separately select their “nominees”, who run against each other in the final decision stage. Unlike BTT92, we again distinguish between the two models Ties-Eliminate and Ties-Promote defined above. That is, in the Ties-Eliminate model, if two or more candidates tie for winning in a subcommittee’s election, no candidate is nominated by that subcommittee. In contrast, in the Ties-Promote model, all the candidates who tie for winning in a subcommittee’s election are nominated to run in the final decision stage.

We mention that both of our two tie-handling models, TE and TP, differ from the model adopted in BTT92, where they for vulnerability results about this problem adopt a third model in which ties are handled not by a tie-handling rule but rather by changing the decision problem itself to require the chair to find a partition that completely avoids

¹³ Note added in final revision: In a very recent paper by Procaccia, Rosenschein, and Zohar [30,31], electoral control is studied in a multi-winner setting. Their interesting model generalizes the BTT92 model by adding a utility function and some parameters. One of the many results of their paper is that approval voting is, in this more flexible model, resistant to constructive control by adding voters. This resistance result can also be shown from Theorem 3.5 along the lines of the way that more flexible models inherit lower bounds from more specific models. A natural reduction (namely, if one keeps in mind their model of tie-breaking and keeps all the inputs of the given BTT92-model instance exactly the same, and if in the target Procaccia–Rosenstein–Zohar-model election one sets their winner cardinality to 1, their utility-sum threshold to 1, the utility of the BTT92-model instance’s distinguished candidate to 1, and the utility of all other candidates to 0) embeds a BTT92-model instance (of constructive control by adding voters, under approval voting) into their setting, and thus the NP-hardness lower bound from Theorem 3.5 can be seen to be inherited for this.

ties in any subcommittee. We find our model the more natural, but for completeness we mention that they obtained for this case, in their tie model, a constructive-control vulnerability result for plurality. For Condorcet and constructive control, BTT92 proves that resistance holds. Our results are:

Theorem 3.6. *Approval is resistant to constructive control by partition of voters in models TE and TP, and vulnerable/certifiably-vulnerable to destructive control by partition of voters in models TE and TP. Plurality is vulnerable/certifiably-vulnerable to both constructive and destructive control by partition of voters in model TE, and is resistant to both constructive and destructive control by partition of voters in model TP. Condorcet is vulnerable/certifiably-vulnerable to destructive control by partition of voters.*

The most striking behavior here is that plurality voting varies between being vulnerable and being resistant, depending on the tie-handling rule. The loose intuition for this is that in TE, at most one candidate wins each subcommittee and in polynomial time we can explore every way this can happen. In contrast, under TP potentially any subset of candidates may move forward, and in this particular setting, that flexibility is enough to support NP-completeness. Also interesting is that both Condorcet and approval, while resistant to constructive control, are vulnerable to destructive control.

4. Proofs

In this section, we provide the proofs of the results stated in Section 3. Table 2 presents, for each of the seven control types considered, the corresponding main result from Section 3 as well as the specific theorems, corollaries, and examples from which this main result follows.

We first present the immunity and susceptibility results. Then we present the vulnerability results, and finally we present the resistance results. The proof techniques employed range from political-science-axiom-fueled arguments (for proving immunity), to designing efficient algorithms (for proving vulnerability), to the construction of NP-hardness reductions (for proving resistance).

4.1. Proving immunity and susceptibility

For each of the 39 boldfaced entries in Table 1, this section must establish immunity if the entry is a boldface “I” and must establish susceptibility if the entry is a boldface “R” or a boldface “V”. (Recall that the definitions of

Table 2
Overview of results yielding the main results

Control by	Main Result Stated as	Follows from
Adding Candidates	Thm. 3.2	Thm. 4.4, Cor. 4.7, Thm. 4.25, Cor. 4.32
Deleting Candidates	Thm. 3.3	Thm. 4.4, Cor. 4.7, Example 4.15, Thm. 4.26, Cor. 4.34
Partition of Candidates	Thm. 3.4	Cor. 4.7, Thm. 4.8, Example 4.14, Example 4.15, Thm. 4.26, Cor. 4.36
Run-off Partition of Candidates	Thm. 3.4	Cor. 4.7, Thm. 4.8, Example 4.14, Example 4.15, Thm. 4.26, Cor. 4.38
Adding Voters	Thm. 3.5	Example 4.11, Example 4.12, Example 4.13, Thm. 4.22, Thm. 4.23, Thm. 4.24, Thm. 4.43
Deleting Voters	Thm. 3.5	Example 4.9, Example 4.10, Example 4.11, Example 4.13, Thm. 4.22, Thm. 4.23, Thm. 4.24, Thm. 4.44
Partition of Voters	Thm. 3.6	Example 4.9, Example 4.10, Example 4.11, Example 4.16, Example 4.17, Thm. 4.18, Thm. 4.19, Thm. 4.20, Thm. 4.21, Cor. 4.42, Thm. 4.45, Thm. 4.46

For completeness, examples/theorems needed to establish susceptibility are listed even when they are invoked within a listed vulnerability or resistance theorem/corollary.

resistance and vulnerability require susceptibility, and so proving susceptibility is a first step toward proving resistance or vulnerability.)

4.1.1. Links between susceptibility cases

Rather than hand-proving each of the 39 cases just mentioned, it makes sense to extract connections between the cases. We start by stating four easy but useful dualities.

Theorem 4.1.

- (1) *A voting system is susceptible to constructive control by adding candidates if and only if it is susceptible to destructive control by deleting candidates.*
- (2) *A voting system is susceptible to constructive control by deleting candidates if and only if it is susceptible to destructive control by adding candidates.*
- (3) *A voting system is susceptible to constructive control by adding voters if and only if it is susceptible to destructive control by deleting voters.*
- (4) *A voting system is susceptible to constructive control by deleting voters if and only if it is susceptible to destructive control by adding voters.*

This theorem is easy to see, and so its proof is omitted.

We also have the following four implication results.

Theorem 4.2.

- (1) *If a voting system is susceptible to constructive control by partition of voters (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
- (2) *If a voting system is susceptible to constructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
- (3) *If a voting system is susceptible to constructive control by partition of voters in model TE, then it is susceptible to constructive control by deleting voters.*
- (4) *If a voting system is susceptible to destructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to destructive control by deleting candidates.*

Proof. We first prove parts (1) and (2). Let (C, V) be an election and $c \in C$ a candidate such that c is not the unique winner of (C, V) and such that c can be made the unique winner by partition of candidates, run-off partition of candidates, or partition of voters. Fix a partitioned election such that c is the unique winner of this election and let $D \subseteq C$ be the set of candidates that participate in the final round of the partitioned election. Then c is the unique winner of (D, V) . Thus, the voting system is susceptible to constructive control by deleting candidates.

For part (3), let (C, V) be an election and $c \in C$ a candidate such that c is not the unique winner of (C, V) and such that c can be made the unique winner by partition of voters in model TE. Let (V_1, V_2) be a partition of V such that c is the unique winner of the partitioned election. Since c makes it to the final run-off, and since we are in model TE, c is the unique winner of one of (C, V_1) and (C, V_2) . Without loss of generality, suppose c is the unique winner of (C, V_1) . Thus, the voting system is susceptible to constructive control by deleting voters.

For part (4), let (C, V) be an election and $c \in C$ a candidate such that c is the unique winner of (C, V) and such that c can be made to be not the unique winner by partition of candidates or run-off partition of candidates. Fix a partitioned election such that c is not the unique winner of this election and let $D \subseteq C$ be the set of candidates that participate in the final round of the partitioned election. If $c \in D$, then c is not the unique winner of (D, V) . If $c \notin D$, then c is not the unique winner of the subelection involving c . Thus, the voting system is susceptible to destructive control by deleting candidates. \square

Let us say that a voting system is *voiced* if in any election that has exactly one candidate, that candidate is always a (and thus, the unique) winner. Note that plurality, Condorcet, and approval are all voiced systems. For voiced systems, we have the following three additional results.

Theorem 4.3.

- (1) *If a voiced voting system is susceptible to destructive control by partition of voters (in model TE or TP), then it is susceptible to destructive control by deleting voters.*
- (2) *Each voiced voting system is susceptible to constructive control by deleting candidates.*
- (3) *Each voiced voting system is susceptible to destructive control by adding candidates.*

Proof. Fix a voiced voting system.

For part (1), suppose that our voting system is immune to destructive control by deleting voters. We will show that it is also immune to destructive control by partition of voters. Let (C, V) be an election such that c is the unique winner of (C, V) , and let (V_1, V_2) be an arbitrary partition of V . Then c is the unique winner of (C, V_1) and of (C, V_2) , and so c is the only candidate participating in the final run-off (both in model TP and in model TE). Since the voting system is voiced, c wins the final run-off, and is thus the unique winner of the partitioned election. It follows that the voting system is immune to destructive control by partition of voters.

For part (2), let $C = \{c, d\}$ and let V be an arbitrary set of voters with preferences over C . At least one of the candidates is not a unique winner of (C, V) . Without loss of generality, let c not be a unique winner of (C, V) . Since the voting system is voiced, c is the unique winner of $(\{c\}, V)$. It follows that the voting system is susceptible to constructive control by deleting candidates.

Part (3) follows immediately from part (2) of this theorem and part (2) of Theorem 4.1. \square

Since plurality, Condorcet, and approval are all voiced systems, we immediately have from Theorems 4.3 and 4.1 the following results that yield susceptibility results for four of Table 1's boldface "R" and boldface "V" entries.

Theorem 4.4. *Plurality, Condorcet, and approval are each susceptible to destructive control by adding candidates. Approval is susceptible to constructive control by deleting candidates.*

4.1.2. Immunity results

We start by proving the immunity results of Theorems 3.2, 3.3, and 3.4. These results are generally clear from the definitions. Bartholdi, Tovey, and Trick [3] observed¹⁴ that immunity to constructive control by adding candidates follows from the "unique" version of the Weak Axiom of Revealed Preference (denoted by Unique-WARP), which says that a unique winner among a collection of candidates always remains a unique winner among every subcollection of candidates that includes him/her. Theorem 4.6 states ways in which Unique-WARP influences a variety of destructive control scenarios.

Theorem 4.5. (*[3].*) *Any voting system that satisfies Unique-WARP is immune to constructive control by adding candidates.*

Theorem 4.6. *Any voting system that satisfies Unique-WARP is immune to destructive control by deleting candidates and (in both model TE and model TP) to destructive control by partition and run-off partition of candidates.*

Theorem 4.6 follows from Theorem 4.5 via Theorems 4.1 and 4.2 (and also is directly clear).

Bartholdi, Tovey, and Trick [3] note that Theorem 4.5 can be applied to show that Condorcet voting is immune to constructive control by adding candidates. We state further immunity results, via Theorems 4.5 and 4.6, as Corollary 4.7.

Corollary 4.7.

- (1) *Condorcet voting is immune to destructive control by deleting candidates, partition of candidates, and run-off partition of candidates.*

¹⁴ Their paper is somewhat nonspecific regarding the uniqueness issue and merely says WARP.

(2) Approval voting is immune to constructive control by adding candidates, and is immune to destructive control by deleting candidates and by partition and run-off partition of candidates (in both the TE and the TP models).

Proof. Both Condorcet and approval voting clearly satisfy Unique-WARP. The result now follows from Theorems 4.5 and 4.6. \square

Note that, unlike Condorcet and approval, plurality voting does not satisfy Unique-WARP, and we will see that immunity does not hold for plurality in any control scenario considered here.

We now state and prove the final two immunity results.

Theorem 4.8. *Approval voting is immune to constructive control by partition and run-off partition of candidates in model TP.*

Proof. It is easy to see that in approval voting, a candidate c is the unique winner if and only if there is a unique candidate with a maximum number of Yes votes and c is that candidate. In the TP model, this remains true even under the two partitioning schemes. \square

4.1.3. Susceptibility results

We now turn to proving susceptibility results for the 28 boldface “R” and boldface “V” boxes in Table 1.

Note that in concept each of the “R” and “V” claims made by Bartholdi, Tovey, and Trick [3] is asserting a susceptibility result, and around eight of those—via our Section 4.1.1 theorems—imply eight of the 28 susceptibility results that we need. However, Bartholdi, Tovey, and Trick [3] generally do not prove their susceptibility claims, and so we will prove susceptibility here for all our 28 cases.

Now, how does one prove susceptibility? One need simply give an example in each case. Alternatively, some cases we get indirectly from an earlier example via our Section 4.1.1 theorems. However, please note that even in those cases, there is implicitly a concrete example, as the theorems of Section 4.1.1 have constructive proofs. Also, we mention again that plurality, Condorcet, and approval are all voiced voting systems (in the sense defined in Section 4.1.1).

We first show that plurality is not immune to constructive or destructive control by partition of voters in model TE or to destructive control by deleting voters.

Example 4.9. Let us consider destructive control by partition of voters in model TE. Let $C = \{a, b, c\}$, and define V to consist of five voters with the following preferences:

$$\begin{aligned} v_1: a > b > c, & \quad v_2: a > b > c, \\ v_3: b > a > c, & \quad v_4: b > a > c, \\ v_5: c > a > b. \end{aligned}$$

Thus, c does not win in (C, V) . However, if we partition V into $V_1 = \{v_5\}$ and $V_2 = V - V_1$, c trivially wins the subelection (C, V_1) , but a and b tie for winner in the subelection (C, V_2) , so none of them proceeds to the run-off with c in model TE. It follows that c is the unique run-off winner. Thus, plurality voting is susceptible to constructive control by partition of voters in model TE.

For the destructive case, a is the unique plurality winner in the election (C, V') , where $V' = \{v_1, v_2, v_3, v_5\}$. Now, partitioning V' into $V'_1 = \{v_1, v_3\}$ and $V'_2 = \{v_2, v_5\}$ implies that none of the two subcommittees nominates a candidate in model TE, due to ties. In particular, a is not the unique run-off winner, and plurality voting thus is susceptible to destructive control by partition of voters in model TE.

By Theorem 4.3, this latter susceptibility claim implies that plurality is susceptible to destructive control by deleting voters (which is another of the 28 boldfaced “R”-or-“V” boxes we are handling).

We now prove that Condorcet voting is susceptible to destructive control by partition of voters and to destructive control by deleting voters.

Example 4.10. Let us consider destructive control by partition of voters. Let $C = \{a, b, c\}$, and define V to consist of seven voters with the following preferences:

$$\begin{array}{lll} v_1: c > a > b, & v_2: c > a > b, & v_3: c > b > a, \\ v_4: b > a > c, & v_5: b > c > a, & \\ v_6: a > b > c, & v_7: a > c > b. & \end{array}$$

Since in pairwise contests four voters prefer c to a and four voters prefer c to b , c is the Condorcet winner in the election (C, V) . However, partitioning V into $V_1 = \{v_1, v_7\}$ and $V_2 = V - V_1$ implies that there is no Condorcet winner in the subelection (C, V_1) , and b is the Condorcet winner in the subelection (C, V_2) . Thus, Condorcet voting is susceptible to destructive control by partition of voters.

By Theorem 4.3, this susceptibility claim implies that Condorcet is also susceptible to destructive control by deleting voters.

We now prove that approval voting is susceptible to destructive control by partition of voters in models TE and TP, to destructive control by deleting voters, and to constructive control by adding voters.

Example 4.11. Let $C = \{a, b, c\}$, and define V to consist of the following ten voters (specified by vectors from $\{0, 1\}^3$, with the first, second, and third bits specifying approval/disapproval for a , b , and c): $v_1 = v_2 = v_3 = v_4 = 001$, $v_5 = v_6 = v_7 = 100$, and $v_8 = v_9 = v_{10} = 010$. In (C, V) , c is the unique approval winner. But if V is partitioned into $V_1 = \{v_1, v_2, v_5, v_6, v_7\}$ and $V_2 = V - V_1$, then a and b are nominated by the subcommittees V_1 and V_2 , respectively, and tie for winner in the run-off. Thus, approval voting is susceptible to destructive control by partition of voters, both in model TE and in TP.

By Theorem 4.3, this susceptibility claim implies that approval voting is also susceptible to destructive control by deleting voters. And that claim itself, by Theorem 4.1, implies that approval voting is also susceptible to constructive control by adding voters.

The following example shows that both plurality voting and Condorcet voting are not immune to destructive control by adding voters.

Example 4.12. Let $C = \{a, b, c\}$. Define V to consist of one registered voter v with preference $c > a > b$, and define W to consist of one as yet unregistered voter w with preference $a > c > b$. Candidate c is the unique winner—both for plurality and Condorcet voting—in the election (C, V) , yet registration of w would assure that a and c tie in first-place votes in $(C, V \cup W)$, so c is not the unique plurality winner of this election. Similarly, c is no longer the Condorcet winner in $(C, V \cup W)$. Thus, both plurality and Condorcet voting are susceptible to destructive control by adding voters.

The following example shows that approval voting is not immune to destructive control by adding voters or to constructive control by deleting voters.

Example 4.13. Let us consider destructive control by adding voters. Let $C = \{a, b, c\}$. Define V to consist of one registered voter $v = 001$ (i.e., v approves of c and disapproves of a and b), and define W to consist of one unregistered voter $w = 100$. In (C, V) , c is the unique approval winner, yet registration of w would assure that a and c tie for winner in $(C, V \cup W)$, so c is not the unique plurality winner of this election. Thus, approval voting is susceptible to destructive control by adding voters.

By Theorem 4.1, this susceptibility claim implies that approval voting is also susceptible to constructive control by deleting voters (and indeed, as implicit in the proof of Theorem 4.1, this very same example works to show that).

We now show susceptibility for approval voting to constructive control by partition of candidates and run-off partition of candidates, both in model TE.

Example 4.14. Let $C = \{a, b, c\}$ be the candidate set. Let the voter set W consist of the two voters with vector representation $w_1 = 111$ and $w_2 = 110$, respectively. Then c loses to both a and b , who tie for winning in the election (C, W) . But if we partition C into $C_1 = \{a, b\}$ and $C_2 = \{c\}$, then no one moves forward from the subelection

(C_1, W) in model TE, so c wins overall. The same example works for the run-off partition of candidates case, since no one moves forward from the subelection (C_1, W) in model TE and c first wins the subelection (C_2, W) and then the run-off. Thus, approval voting is susceptible to constructive control by both partition of candidates in model TE and run-off partition of candidates in model TE.

Example 4.15 shows that plurality voting is susceptible to destructive control by partition and run-off partition of candidates (both in model TE and TP), and to destructive control by deleting candidates.

Example 4.15. Let us consider the partition cases. Let $C = \{a, b, c, d\}$ be the candidate set, and define the voter set V to consist of the following seven voters:

- 3 voters of the form $c > a > b > d$,
- 2 voters of the form $a > d > b > c$, and
- 2 voters of the form $b > d > a > c$.

Note that c is the unique plurality winner in the election (C, V) .

Now, partition the candidate set C into $C_1 = \{a, c\}$ and $C_2 = \{b, d\}$. Then a is the unique plurality winner in the subelection (C_1, V) . So c is dethroned in the partition of candidates setting. $C_1 = \{a, c\}$ and $C_2 = \{b, d\}$ also dethrones c in the run-off partition of candidates setting. Thus, plurality voting is susceptible to destructive control by partition and by run-off partition of candidates. Since each subelection has a unique winner (a in (C_1, V) and b in (C_2, V)), this is true regardless of the tie-handling rule.

By Theorem 4.2, these susceptibility claims imply that plurality voting is also susceptible to destructive control by deleting candidates.

The next example shows that plurality voting is susceptible to constructive and destructive control by partition of voters in model TP.

Example 4.16. Let $C = \{a, b, c\}$ be the candidate set, and define the voter set V to consist of the following eight voters:

- 3 voters (say u_1, u_2 , and u_3) of the form $a > c > b$,
- 2 voters (say v_1 and v_2) of the form $b > a > c$, and
- 3 voters (say w_1, w_2 , and w_3) of the form $c > a > b$.

For the constructive case, note that c is not the unique plurality winner in the election (C, V) , since a and c are tied for first place. Now, partition V into $V_1 = \{u_1, u_2, w_1, w_2, w_3\}$ and $V_2 = \{u_3, v_1, v_2\}$. Then c is the unique plurality winner in the subelection (C, V_1) , b is the unique plurality winner in the subelection (C, V_2) , and c wins the run-off against b . Thus, plurality voting is not immune to constructive control by partition of voters in model TP.

For the destructive case, consider the election (C, V') with $V' = V \cup \{v_3, w_4\}$, where v_3 votes $b > a > c$ and w_4 votes $c > a > b$. In (C, V') , c is the unique plurality winner. Partition V' into $V'_1 = \{u_1, u_2, u_3, w_1, w_2\}$ and $V'_2 = \{v_1, v_2, v_3, w_3, w_4\}$. Then a is the unique plurality winner of the subelection (C, V'_1) , b is the unique plurality winner of the subelection (C, V'_2) , and a wins the run-off against b . So c is dethroned. Thus, plurality voting is not immune to destructive control by partition of voters in model TP.

Finally, we show that approval voting is susceptible to constructive control by partition of voters in models TE and TP.

Example 4.17. Let $C = \{a, b, c\}$ be the candidate set. Define the voter set V to consist of the following eight voters: $v_1 = v_2 = v_3 = 100$, $v_4 = v_5 = 010$, and $v_6 = v_7 = v_8 = 001$. In (C, V) , a and c are tied. Now, partition V into $V_1 = \{v_1, v_2, v_6, v_7, v_8\}$ and $V_2 = \{v_3, v_4, v_5\}$. Candidate c is the unique approval winner in the subelection (C, V_1) , b is the unique approval winner in the subelection (C, V_2) , and c wins the run-off against b . This works both in

model TE and TP, since ties do not occur in the subelections in our construction. So approval voting is susceptible to constructive control by partition of voters (both in TE and TP).

4.2. Proving vulnerability

The certifiably-vulnerable results (which here imply the vulnerable results) range from clear greedy algorithms to trickier algorithms based on characterizing the ways in which a candidate can be made to win (in the constructive case) or can be precluded from winning (in the destructive case). The more surprising of these have to do with the tie-handling cases of partition problems—where the chair can at times do shrewd things (e.g., shift voters counterintuitively to induce ties that kill off stronger candidates).

4.2.1. Partition of voters

We start with the “control by partition of voters” problems. For plurality voting, we here obtain the same results in the constructive and the destructive case, as stated in Table 1 and in Theorem 3.6. On the other hand, the question of whether resistance or vulnerability holds depends on which tie-handling rule is chosen.

Theorem 4.18. *In model TE, plurality voting is vulnerable/certifiably-vulnerable to constructive control by partition of voters.*

Proof. By Example 4.9, susceptibility holds.

Given a set of candidates C , a distinguished candidate $c \in C$, and a voter set V , we describe a polynomial-time algorithm for this problem. For any partition (V_1, V_2) of the voter set V , let $Nominees(C, V_i)$, $i \in \{1, 2\}$, denote the set of candidates who are nominated by the subcommittee V_i (with candidates C) for the run-off in model TE. To ensure that c is the unique winner, under the desired partition setup, we may without loss of generality focus on the following five cases (Cases 3 and 5 are not necessarily disjoint):

Case 1: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \emptyset$ due to $V_2 = \emptyset$.

Case 2: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \{c\}$.

Case 3: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \emptyset$ due to c and d (and possibly additional other candidates) tying, where $c \neq d$.

Case 4: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \{d\}$, where $c \neq d$.

Case 5: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \emptyset$ due to d and e (and possibly additional other candidates) tying, where $c \neq d \neq e \neq c$.

In Case 1, it clearly suffices to check whether c is an overall plurality winner. Note further that if Case 2 holds for some partition (V_1, V_2) , then c must be an overall plurality winner, and thus will also win via the partition (V, \emptyset) . We now argue that the same is true in Case 3. For any candidate i , let $score(i)$ denote the number of voters who rank i first-place in (C, V) . In Case 3, note that for all $e \in C - \{c\}$,

$$score(e) < score(c),$$

since c has strictly more first-place votes than e in (C, V_1) and e at best ties c for first-place votes in (C, V_2) . It follows that c must already be an overall plurality winner in Case 3, and thus will also win via the partition (V, \emptyset) .

So, our algorithm, after checking whether c is a plurality winner overall (thus catching Cases 1, 2, and 3), will by brute force check whether Case 4 or Case 5 can be made to hold for some partition of the voter set.

Given C , c , and V as above, our polynomial-time algorithm proceeds as follows. If c is a plurality winner of (C, V) , output (V, \emptyset) as a successful partition and halt; else if $\|C\| = 2$, then output “control impossible” (which in this context means that making c a unique winner is impossible) and halt. Otherwise, we first try to make Case 4 hold and then, if that fails, try to make Case 5 hold. These two tests are implemented by the two loops described below, and if they both fail, control is not possible.

Loop trying to make Case 4 hold: For each $d \in C$, $d \neq c$, such that c beats d in a pairwise plurality election by the voters in V , do the following: If it holds that, for each $e \in C$ with $c \neq e \neq d$,

$$\text{score}(e) \leq \text{score}(c) + \text{score}(d) - 2,$$

then output (V_1, V_2) as a successful partition and halt, where V_1 consists of all $\text{score}(c)$ voters whose first choice is c and exactly $\min(\text{score}(e), \text{score}(c) - 1)$ of the voters whose first choice is e , and where $V_2 = V - V_1$.

Loop trying to make Case 5 hold: If the loop trying to make Case 4 hold was not successful, then for each $d \in C$ and for each $e \in C$ such that $\|\{c, d, e\}\| = 3$ and $\text{score}(d) \leq \text{score}(e)$, do the following: If it holds that, for each $f \in C - \{c\}$,

$$\text{score}(f) \leq \text{score}(c) + \text{score}(d) - 1,$$

then output (V_1, V_2) as a successful partition and halt, where V_1 consists of all $\text{score}(c)$ voters whose first choice is c , of exactly $\text{score}(e) - \text{score}(d)$ of the voters whose first choice is e , and for all $f \in C - \{c, d, e\}$, exactly $\min(\text{score}(f), \text{score}(c) - 1)$ of the voters whose first choice is f , and where $V_2 = V - V_1$.

Otherwise (i.e., if the Case 5 loop was not successful either), c cannot win, so we output “control impossible” and halt. \square

We now make a general remark. In various cases, our polynomial-time algorithms have loops. In some cases, these loops can be collapsed or removed. Doing so

- improves the runtime and makes the algorithm look simpler, but
- makes it a bit harder to see that the algorithm is correct.

Since correctness is what we most care about, we do not collapse such loops. But let us explicitly mention the “look” of such collapses. In the proof of Theorem 4.18 above, the “For each $d \in C$, $d \neq c$, such that c beats d in a pairwise plurality election by the voters in V , do ...” loop trying to make Case 4 hold in the algorithm can safely be changed to: “If there exists some $d' \in C$, $d' \neq c$, such that c beats d' in a pairwise plurality election by the voters in V , then let d be some such d' for which $\text{score}(d)$ is maximized among all such $\text{score}(d')$ and do ...” This is a legal loop collapse, since if some d' works, then it works for all d'' that can pairwise beat c in a run-off whose $\text{score}(d'')$ is maximum. Again, this is just an example, and to have our correctness as unobscured as possible and as our focus is on the gap between P and NP-hard, we in general forgo such optimizations of the precise polynomial of the runtime.

We now turn to the destructive analog of Theorem 4.18.

Theorem 4.19. *In model TE, plurality voting is vulnerable/certifiably-vulnerable to destructive control by partition of voters.*

Proof. That susceptibility holds in this case has been shown in Example 4.9.

Given a set of candidates C , a distinguished candidate $c \in C$, and a voter set V , our polynomial-time algorithm for this control problem works as follows. If $C = \{c\}$, output “control impossible” and halt, as c must win; else if c already is not the unique plurality winner, output (V, \emptyset) as a successful partition and halt. Now, we check if every voter’s first choice is c or if $\|C\| = 2$, and if one of these two conditions is true, we output “control impossible” and halt, since c cannot help but win.

Again, let $\text{score}(i)$ denote the number of voters who rank candidate i first-place. Let d be a candidate who other than c got the most first-place votes, and let e be a candidate who other than c and d got the most first-place votes. We can certainly dethrone c if

$$\text{score}(c) \leq \text{score}(d) + \text{score}(e). \quad (4.1)$$

Namely, if Eq. (4.1) holds, we output (V_1, V_2) as a successful partition and halt, where V_1 consists of all $\text{score}(d)$ voters whose first choice is d and exactly $\text{score}(d)$ voters whose first choice is c (recall that in the current case we already know that $\text{score}(c) > \text{score}(d)$), and where $V_2 = V - V_1$. Then c and d will tie for winner in (C, V_1) , so no one will be nominated by the subcommittee V_1 in model TE, and e will tie or beat c in (C, V_2) , so c is not nominated by the subcommittee V_2 either.

On the other hand, if Eq. (4.1) is not satisfied, we have

$$\text{score}(c) > \text{score}(d) + \text{score}(e),$$

so in any partition (V_1, V_2) , c clearly will triumph in one of (C, V_1) or (C, V_2) . Thus, we now know it is impossible to make sure that c loses in both subcommittees. If c is nominated by both subcommittees (in model TE), c trivially is the unique winner of the final run-off. So, our algorithm now checks if it is possible for c to win in exactly one subcommittee, and yet can be made to not be the unique winner of the final run-off. For this to happen, it is (given the case we are in) a necessary and sufficient condition that there exists some candidate d such that:

- $d \neq c$,
- d ties or beats c in a pairwise plurality election, and
- for each candidate e , $c \neq e \neq d$, we have that $\text{score}(e) < \text{score}(c) + \text{score}(d) - 2$.

We can in polynomial time brute-force check whether the above three conditions hold for some candidate d , and if they do, let d' be some such candidate d and output (V_1, V_2) as a successful partition and halt, where V_1 consists of all $\text{score}(c)$ voters whose first choice is c and, for each candidate e with $c \neq e \neq d'$, of exactly $\min(\text{score}(c) - 1, \text{score}(e))$ voters whose first choice is e , and where $V_2 = V - V_1$. Finally, if the above three conditions cannot be satisfied for any d , output “control impossible” and halt. \square

We now prove that Condorcet voting is vulnerable/certifiably-vulnerable to destructive control by partition of voters.

Theorem 4.20. *Condorcet voting is vulnerable/certifiably-vulnerable to destructive control by partition of voters.*

Proof. By Example 4.10, susceptibility holds.

Given a set of candidates C , a distinguished candidate $c \in C$, and a voter set V , our polynomial-time algorithm for this control problem proceeds in three stages:

- (1) *Checking the trivial cases:* If $C = \{c\}$, output “control impossible” and halt, as c must win. Otherwise, if c already is not the Condorcet winner, output (V, \emptyset) as a successful partition and halt. Otherwise, if $\|C\| = 2$, output “control impossible” and halt, since in this case c is the Condorcet winner, so c is preferred by a strict majority of votes to the other candidate and thus will win at least one subcommittee and also the run-off.
- (2) *Loop:* Now, if none of the trivial cases applies, for each $a, b \in C$ with $\|\{a, b, c\}\| = 3$, we test whether we can make a tie or beat c in (C, V_1) and make b tie or beat c in (C, V_2) . For each voter, we will now focus just on the ordering of a, b , and c . We use the following notation. Denote the number of voters with order $c > a > b$ or $c > b > a$ by W_c , with order $a > b > c$ or $b > a > c$ by L_c , with order $a > c > b$ by S_a , and with order $b > c > a$ by S_b .

If $W_c - L_c > S_a + S_b$, then this a and b are hopeless, so move on to consider the next a and b in the loop. Otherwise, we have

$$W_c - L_c \leq S_a + S_b. \quad (4.2)$$

Output (V_1, V_2) as a successful partition and halt, where V_1 contains all the S_a voters with order $a > c > b$, and also $\min(W_c, S_a)$ voters contributing to W_c , and where $V_2 = V - V_1$.

In (C, V_1) , a ties or beats c , since a gets S_a votes and c gets $\min(W_c, S_a)$ votes. And in (C, V_2) , b ties or beats c , since there are $S_b + L_c$ voters who prefer b to c , and there are $W_c - \min(W_c, S_a)$ voters who prefer c to b . Thus, to prove that the construction works, we need that

$$S_b + L_c \geq W_c - \min(W_c, S_a),$$

which is equivalent to

$$S_b + \min(W_c, S_a) \geq W_c - L_c. \quad (4.3)$$

But if $S_a \leq W_c$ then Eq. (4.3) is implied by Eq. (4.2), and if $S_a > W_c$ then Eq. (4.3) follows immediately from the fact that $S_b + L_c \geq 0$. Thus, b indeed ties or beats c in (C, V_2) .

- (3) *Termination*: If in no loop iteration did we find an a and b that allowed us to output a partition of voters dethroning c , then output “control impossible” and halt.

This completes the proof of Theorem 4.20. \square

We now prove that approval voting is vulnerable/certifiably-vulnerable to destructive control by partition of voters in models TE and TP.

Theorem 4.21. *Approval voting is vulnerable/certifiably-vulnerable to destructive control by partition of voters in models TE and TP.*

Proof. That susceptibility holds in this case is shown by Example 4.11.

We describe two polynomial-time algorithms for these two control problems, one for TE and one for TP. Given a set of candidates C , a distinguished candidate $c \in C$, and a voter set V , both algorithms again proceed in the following three phases:

- (1) *Checking the trivial cases*: If $C = \{c\}$, output “control impossible” and halt, as c must win. Otherwise, if c already is not the unique winner, output (V, \emptyset) as a successful partition and halt. Otherwise, if $\|C\| = 2$, output “control impossible” and halt, since in this case c is the unique winner, so c will win in at least one subcommittee and will also win the run-off.
- (2) *Loop*: In this phase, if none of the trivial cases applies, we try to find a pair of candidates, a and b , that allows us to determine a successful partition of voters. This phase is described below, separately for TE and TP.
- (3) *Termination*: If in no loop iteration did we find an a and b that allowed us to output a partition of voters dethroning c , then output “control impossible” and halt.

The two algorithms differ only in the second phase. To describe one loop iteration for some pair of candidates, a and b , we use the following notation: For each voter in V , we focus just on his/her approval of a , b , and c , represented (in that order) as a vector from $\{0, 1\}^3$. Denote the number of voters with preference 001 by W_c , with 110 by L_c , with 100 by S_a , with 010 by S_b , with 101 by S_{ac} , and with 011 by S_{bc} . (Voters with preference 000 or 111 need not be considered, since they do not affect the difference of Yes votes among a , b , and c .)

Loop in model TE: For each $a, b \in C$ with $\|\{a, b, c\}\| = 3$, we test whether we can make a tie or beat c in (C, V_1) and make b tie or beat c in (C, V_2) .

If $W_c - L_c > S_a + S_b$, then this a and b are hopeless, so move on to consider the next a and b in the loop. Otherwise, we have

$$W_c - L_c \leq S_a + S_b. \quad (4.4)$$

Output (V_1, V_2) as a successful partition and halt, where V_1 contains all voters contributing to S_{ac} and S_a , and also $\min(W_c, S_a)$ voters contributing to W_c , and where $V_2 = V - V_1$.

In (C, V_1) , a ties or beats c , since a gets

$$S_a - \min(W_c, S_a) \geq 0$$

more Yes votes than c . And in (C, V_2) , b ties or beats c , since b receives

$$S_b + L_c - (W_c - \min(W_c, S_a))$$

more Yes votes than c . So, for the construction to work, we must argue that

$$S_b + L_c + \min(W_c, S_a) - W_c \geq 0.$$

That is, we need

$$W_c - L_c \leq \min(W_c, S_a) + S_b. \quad (4.5)$$

If $W_c < S_a$, Eq. (4.5) follows trivially from the fact that $0 \leq L_c + S_b$. And if $W_c \geq S_a$, Eq. (4.5) follows immediately from Eq. (4.4).

Loop in model TP: For each $a, b \in C$ with $\|\{a, b, c\}\| = 3$, we test whether we can make a strictly beat c in (C, V_1) and make b strictly beat c in (C, V_2) .

If $W_c - L_c > S_a + S_b - 2$ or $S_a = 0$ or $S_b = 0$, then this a and b are hopeless, so move on to consider the next a and b in the loop. Otherwise, we have

$$W_c - L_c \leq S_a + S_b - 2 \quad (4.6)$$

and $S_a > 0$ and $S_b > 0$, and output (V_1, V_2) as a successful partition and halt, where V_1 contains all voters contributing to S_{ac} and S_a , and also $\min(W_c, S_a - 1)$ voters contributing to W_c , and where $V_2 = V - V_1$.

In (C, V_1) , a (strictly) beats c , since a gets

$$S_a - \min(W_c, S_a - 1) > 0$$

more Yes votes than c . And in (C, V_2) , b (strictly) beats c , since b has

$$S_b + L_c - (W_c - \min(W_c, S_a - 1))$$

more Yes votes than c . So, for the construction to work, we must argue that

$$S_b + L_c + \min(W_c, S_a - 1) - W_c > 0.$$

That is, we need

$$W_c - L_c < \min(W_c, S_a - 1) + S_b. \quad (4.7)$$

If $W_c \leq S_a - 1$, Eq. (4.7) reduces to $0 < L_c + S_b$, which follows from the fact that in the current case $S_b > 0$. And if $W_c > S_a - 1$, Eq. (4.7) follows immediately from Eq. (4.6). \square

4.2.2. Adding and deleting voters, destructive case

We now turn to proving the vulnerability results for destructive control by adding and by deleting voters for each of plurality, Condorcet, and approval voting. We start with plurality voting.

Theorem 4.22. *Plurality voting is vulnerable/certifiably-vulnerable to destructive control both by adding voters and by deleting voters.*

Proof. By Examples 4.9 and 4.12, susceptibility holds.

In a nutshell, for the adding voters case, we give a “smart greedy” algorithm, and for the deleting voters case, we give a “dumb greedy” algorithm. In both cases, we prove only that plurality voting is certifiably-vulnerable to destructive control, since this implies vulnerability. Recall that no “ k ” is specified in the corresponding control problems, as in this setting the chair seeks to determine in polynomial time the smallest number of voters needed to be added or deleted to execute control.

In the adding voters case, we are given a set C of candidates, a distinguished candidate c , a set V of registered voters, and an additional set W of as yet unregistered voters (both V and W have preferences over C). If c already is not a unique plurality winner in the election (C, V) , adding no voters accomplishes our goal, and we are done. Otherwise, sort all candidates in C distinct from c by how many votes each needs to tie c . Let d_i denote the i th candidate in the ordering thus obtained, and let $\text{diff}(d_i)$ denote d_i ’s deficit of first-place votes needed to tie c . Thus, the order is such that $\text{diff}(d_1) \leq \text{diff}(d_2) \leq \dots \leq \text{diff}(d_{\|C\|-1})$. For $i = 1, 2, \dots, \|C\| - 1$, if the number of unregistered voters whose first choice is d_i is greater than or equal to $\text{diff}(d_i)$, then add $\text{diff}(d_i)$ of these unregistered voters to ensure that d_i ties c (and c thus is not the unique winner) and halt. If in no iteration of this for-loop was some candidate able to dethrone c , output “control impossible” and halt.

In the deleting voters case, we are given a set C of candidates, a distinguished candidate c , and a set V of voters with preferences over C . If $C = \{c\}$, then output “control impossible” and halt; else if c already is not the unique plurality winner in the election (C, V) , deleting no voters accomplishes our goal, and we are done. Now, if every candidate other than c gets zero first-place votes, then output “control impossible” and halt. Otherwise, let d be the candidate closest to c in first-place votes, and let $\text{diff}(d)$ denote d ’s deficit of first-place votes needed to tie c . Then deleting $\text{diff}(d)$ voters whose first choice is c assures that c is not the unique winner, and this is the fewest deletions that can achieve that. \square

Theorem 4.23. *Condorcet voting is vulnerable/certifiably-vulnerable to destructive control both by adding voters and by deleting voters.*

Proof. By Examples 4.10 and 4.12, susceptibility holds.

We again prove only certifiable vulnerability, since this here implies vulnerability.

In the adding voters case, we are given a set C of candidates, a distinguished candidate c , a set V of registered voters, and an additional set W of as yet unregistered voters (both V and W have preferences over C). If $C = \{c\}$, then output “control impossible” and halt; else if c already is not a Condorcet winner in the election (C, V) , adding no voters accomplishes our goal, and we are done. Otherwise, for each candidate $i \neq c$, call i *lucky* if and only if the surplus of c relative to i (denoted by $\text{surplus}(c, i)$, which is defined as the number of registered voters who prefer c to i minus the number of registered voters who prefer i to c) is less than or equal to the number of unregistered voters who prefer i to c . If there is at least one lucky candidate, then let d be a lucky candidate such that the surplus of c relative to d is minimum, and add $\text{surplus}(c, d)$ unregistered voters who prefer d to c . If there exists no lucky candidate, output “control impossible” and halt.

In the deleting voters case, we are given a set C of candidates, a distinguished candidate c , and a set V of voters with preferences over C . If $C = \{c\}$, then output “control impossible” and halt; else if c already is not a Condorcet winner in the election (C, V) , deleting no voters accomplishes our goal, and we are done. Otherwise, find a candidate d who comes closest to c (i.e., relative to whom the surplus of c is minimum), and delete $\text{surplus}(c, d)$ voters from V who prefer c to d . Now c and d tie, so c is dethroned. \square

Theorem 4.24. *Approval voting is vulnerable/certifiably-vulnerable to destructive control both by adding voters and by deleting voters.*

Proof. That susceptibility holds in this case is shown by Examples 4.11 and 4.13.

As before, we prove only certifiable vulnerability, since this here implies vulnerability.

In the adding voters case, we are given a set C of candidates, a distinguished candidate c , a set V of registered voters, and an additional set W of as yet unregistered voters (both V and W have preferences over C). If $C = \{c\}$, then output “control impossible” and halt. Otherwise, if c already is not the unique approval winner in the election (C, V) , adding no voters accomplishes our goal, and we are done. Otherwise, for each candidate $i \neq c$, again define $\text{surplus}(c, i)$ to be the number of Yes votes for c in V minus the number of Yes votes for i in V . Among all candidates j other than c (if any) such that there exist at least $\text{surplus}(c, j)$ voters in W who vote Yes for j and No for c , let d be any such j for which $\text{surplus}(c, j)$ is minimum, and add $\text{surplus}(c, d)$ unregistered voters who vote Yes for d and No for c . If no j satisfying the above conditions exists, then output “control impossible” and halt.

In the deleting voters case, we are given a set C of candidates, a distinguished candidate c , and a set V of voters with preferences over C . If $C = \{c\}$, then output “control impossible” and halt. Otherwise, if c already is not the unique approval winner in the election (C, V) , deleting no voters accomplishes our goal, and we are done. Otherwise, let d be a candidate among $C - \{c\}$ for whom $\text{surplus}(c, d)$ is minimum, and delete $\text{surplus}(c, d)$ voters from V who vote Yes for c and No for d (such voters must exist, as they are what is causing the surplus in the first place). \square

4.2.3. Adding candidates, destructive case, Condorcet and approval voting

Next, we prove that both Condorcet and approval voting are certifiably-vulnerable (and thus vulnerable) to destructive control by adding candidates.

Theorem 4.25. *Both Condorcet and approval voting are vulnerable/certifiably-vulnerable to destructive control by adding candidates.*

Proof. That susceptibility holds in this case is shown by Theorem 4.4.

We again prove only certifiable vulnerability, since this here implies vulnerability. We are given a set C of qualified candidates and a distinguished candidate $c \in C$, a set D of possible spoiler candidates, and a set V of voters with preferences (in the approval case, the “preferences” are 0–1 vectors) over $C \cup D$.

For Condorcet voting, if c already is not the Condorcet winner, adding no candidates accomplishes our goal, and we are done. Otherwise, if any spoiler candidate ties or beats c , add one such candidate and halt. Otherwise, output “control impossible” and halt.

For approval voting, if c already is not the unique approval winner in the election (C, V) , adding no candidates accomplishes our goal, and we are done. Otherwise, if there exists a spoiler candidate d who ties or beats c among the voters in V in Yes votes, add one such spoiler candidate and halt. Otherwise, output “control impossible” and halt. \square

4.2.4. Deleting candidates, partition and run-off partition of candidates, constructive case, approval voting

Finally, we show the vulnerability results for approval voting for constructive control by deleting candidates, and by partition of candidates and run-off partition of candidates, both in model TE.

Theorem 4.26. *Approval voting is vulnerable/certifiably-vulnerable to constructive control by deleting candidates, partition of candidates in model TE, and run-off partition of candidates in model TE.*

Proof. That susceptibility holds in this case is shown by Theorem 4.4 and Example 4.14.

As in the previous proofs, we only show certifiable vulnerability, which again implies vulnerability. Thus, no “ k ” is specified in the control problem corresponding to the deleting candidates case, and in all three cases we are given a set C of candidates, a distinguished candidate c , and a set V of registered voters. We now describe a polynomial-time algorithm for each of the three constructive control problems considered.

In the deleting candidates case, if c already is the unique approval winner in the election (C, V) , deleting no candidates accomplishes our goal, and we are done. Otherwise, delete every candidate other than c who has at least as many Yes votes as c has in V and halt.

In the partition of candidates case, if c already is the unique approval winner, then output (\emptyset, C) as a successful partition and halt. Otherwise, for each candidate $a \in C$, let y_a denote the number of Yes votes cast for a in V , and let $Y = \max\{y_a \mid a \in C\}$.

Now, if there exists *exactly* one $a \in C - \{c\}$ such that $y_a = Y$, then output “control impossible” and halt, since c cannot be made the unique winner in this case. On the other hand, if there exist at least two distinct candidates in $C - \{c\}$ whose number of Yes votes is Y , then output (C_1, C_2) with $C_1 = C - \{c\}$ and $C_2 = \{c\}$ as a successful partition and halt. This works, since in subelection (C_1, V) all candidates are eliminated.

Note that the same algorithm also works for the run-off partition of candidates case in model TE. \square

4.3. Proving resistance

The resistance results are based on clear containments in NP, plus (polynomial-time many-one) reductions establishing NP-hardness.

The following lemma says that for the voting systems considered here (though that may be different in general), whenever the corresponding control problem is NP-hard, immunity cannot hold unless $P = NP$.

Lemma 4.27. *For each voting system for which winners can be tested in polynomial time, if the control problem corresponding to one of the settings considered here is NP-hard, then the system cannot be immune to control in this setting unless $P = NP$.*

Proof. Consider any voting system for which winner-testing (“Is c a winner?”) can be done in polynomial time. Suppose that the decision problem associated with any one of the control scenarios defined in Section 3 is NP-hard. Then, as mentioned in Footnote 5, if immunity were to hold, the associated decision problem would be in P, which would imply $P = NP$. \square

However, proving immunity and susceptibility under assumptions regarding P-versus-NP is obviously less attractive than proving immunity and susceptibility unconditionally. In particular, the ideal first step toward proving resistance results is to prove, via theorems or examples, susceptibility to the corresponding types of control. We have done that in Section 4.1, and will invoke items from that section here.

4.3.1. Plurality voting

We whenever possible try to achieve multiple resistance results via a single proof. For example, with a single proof we establish the key part of all seven resistance results for plurality voting: destructive control by adding, deleting, partition (TE and TP), and run-off partition (TE and TP) of candidates,¹⁵ and by partition of voters (TP). We now provide this proof, which is achieved via one general construction that yields the reductions, each from the NP-complete problem Hitting Set, see Garey and Johnson [19].

Hitting set

Given: A set $B = \{b_1, b_2, \dots, b_m\}$, a family $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of subsets S_i of B , and a positive integer k .

Question: Does \mathcal{S} have a hitting set of size at most k ? That is, is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i , $S_i \cap B' \neq \emptyset$?

We now present our general construction for the destructive control problems related to plurality voting.

Construction 4.28 (*Construction of an Election from a Hitting Set Instance*). Given a triple (B, \mathcal{S}, k) , where $B = \{b_1, b_2, \dots, b_m\}$ is a set, $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ is a family of subsets S_i of B , and $k \leq m$ is a positive integer, we construct the following election:

- The candidate set is $C = B \cup \{c, w\}$.
- The voter set V is defined as follows:
 - There are $2(m - k) + 2n(k + 1) + 4$ voters of the form $c > w > \dots$, where “ \dots ” means that the remaining candidates follow in some arbitrary order.
 - There are $2n(k + 1) + 5$ voters of the form $w > c > \dots$.
 - For each i , $1 \leq i \leq n$, there are $2(k + 1)$ voters of the form $S_i > c > \dots$, where “ S_i ” denotes the elements of S_i in some arbitrary order.
 - Finally, for each j , $1 \leq j \leq m$, there are two voters of the form $b_j > w > \dots$.

We now show that the election (C, V) constructed above has some useful properties needed to establish resistance to destructive control for plurality voting in the seven settings mentioned. For every candidate d , let $\text{score}(d)$ denote the number of voters who rank d first in a given election.

Claim 4.29. If B' is a hitting set of \mathcal{S} of size k , then w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$.

Proof. If B' is a hitting set of \mathcal{S} of size k , then in the election $(B' \cup \{c, w\}, V)$, we have

$$\begin{aligned} \text{score}(c) &= 2(m - k) + 2n(k + 1) + 4, \\ \text{score}(w) &= 2n(k + 1) + 5 + 2(m - k), \quad \text{and} \\ \text{score}(b_j) &\leq 2n(k + 1) + 2 \quad \text{for each } j. \end{aligned}$$

It follows that w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$. \square

Claim 4.30. Let $D \subseteq B \cup \{w\}$. If c is not the unique plurality winner of election $(D \cup \{c\}, V)$, then there exists a set $B' \subseteq B$ such that

- (1) $D = B' \cup \{w\}$,
- (2) w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$, and
- (3) B' is a hitting set of \mathcal{S} of size less than or equal to k .

¹⁵ Our constructions ensure that the distinguished candidate is never tied for winner in any subelection in the image of the NP-hardness reduction. Thus, these results hold both in the Ties-Eliminate and Ties-Promote models.

Proof. Let $D \subseteq B \cup \{w\}$ and suppose that c is not the unique plurality winner of election $(D \cup \{c\}, V)$. We show the three properties stated in the claim.

First note that for all $b \in D \cap B$, $\text{score}(b) < \text{score}(c)$ in $(D \cup \{c\}, V)$. Since c is not the unique plurality winner of $(D \cup \{c\}, V)$, it follows that $w \in D$ and $\text{score}(w) \geq \text{score}(c)$. Let $B' \subseteq B$ be such that $D = B' \cup \{w\}$. Then $D \cup \{c\} = B' \cup \{c, w\}$. Since $\text{score}(w)$ is odd and $\text{score}(c)$ is even, it follows that w is the unique plurality winner of $(B' \cup \{c, w\}, V)$. This proves the first two properties stated.

To prove the third property, note that in $(B' \cup \{c, w\}, V)$, we have

$$\begin{aligned} \text{score}(w) &= 2n(k+1) + 5 + 2(m - \|B'\|) \quad \text{and} \\ \text{score}(c) &= 2(m-k) + 2n(k+1) + 4 + 2(k+1)\ell, \end{aligned}$$

where ℓ is the number of sets in \mathcal{S} that are not hit by B' (i.e., that have an empty intersection with B'). Since $\text{score}(c) \leq \text{score}(w)$, it follows that

$$2(m-k) + 2(k+1)\ell \leq 1 + 2(m - \|B'\|),$$

which implies $(k+1)\ell + \|B'\| - k \leq 0$. So $\ell = 0$. Thus, B' is a hitting set of \mathcal{S} of size at most k , which proves the third property. \square

Next, we show that Construction 4.28 yields a polynomial-time many-one reduction from Hitting Set to Destructive Control by Adding Candidates for plurality voting.

Claim 4.31. *\mathcal{S} has a hitting set of size less than or equal to k if and only if destructive control by adding candidates can be executed for the election with qualified candidates $\{c, w\}$, spoiler candidates B , distinguished candidate c , and voter set V .*

Proof. If \mathcal{S} has a hitting set of size less than or equal to k , then since $k \leq m$, \mathcal{S} has a hitting set of size k . Thus, the implication from left to right follows from Claim 4.29. The implication from right to left follows from Claim 4.30. \square

So from this and Theorem 4.4 we have the following.

Corollary 4.32. *Plurality voting is resistant to destructive control by adding candidates.*

By a similar argument, Hitting Set can be reduced to Destructive Control by Deleting Candidates for plurality voting.

Claim 4.33. *\mathcal{S} has a hitting set of size at most k if and only if the election with candidate set C , distinguished candidate c , and voter set V can be destructively controlled by deleting at most $m - k$ candidates.*

Proof. Let B' be a hitting set of \mathcal{S} of size k . By Claim 4.29, c is not the unique plurality winner of the election $(B' \cup \{c, w\}, V)$. Since $B' \cup \{c, w\} = C - (B - B')$, $\|B\| = m$, and $\|B'\| = k$, the right-hand side of the equivalence follows.

For the converse, let $D \subseteq B \cup \{w\}$ be such that $\|D\| \leq m - k$, and suppose that c is not the unique plurality winner of $(C - D, V)$. Since $c \in C - D$, it follows from Claim 4.30 that $(C - D) - \{c\} = B' \cup \{w\}$, where B' is a hitting set of \mathcal{S} of size less than or equal to k . \square

So from this and Example 4.15 we have the following.

Corollary 4.34. *Plurality voting is resistant to destructive control by deleting candidates.*

Now, we show that Construction 4.28 also yields a polynomial-time many-one reduction from Hitting Set to Destructive Control by Partition of Candidates for plurality voting.

Claim 4.35. *\mathcal{S} has a hitting set of size at most k if and only if the election with candidate set C , distinguished candidate c , and voter set V can be destructively controlled by partition of candidates (both in model TE and TP).*

Proof. Let B' be a hitting set of \mathcal{S} of size k . Partition C into $C_1 = B' \cup \{c, w\}$ and $C_2 = B - B'$. By Claim 4.29, w is the unique plurality winner of (C_1, V) , and c thus cannot win the election (C, V) .

For the converse, suppose that there exists a partition of candidates such that c is not the unique plurality winner of the two-stage election corresponding to that partition. Then, certainly, there exists a set $D \subseteq B \cup \{w\}$ such that c is not the unique plurality winner of $(D \cup \{c\}, V)$. By Claim 4.30, \mathcal{S} has a hitting set of size at most k . \square

So from this and Example 4.15 we have the following.

Corollary 4.36. *Plurality voting is resistant to destructive control by partition of candidates (both in model TE and TP).*

The same argument works for proving that plurality voting is resistant to destructive control by run-off partition of candidates, again by a reduction from Hitting Set.

Claim 4.37. *\mathcal{S} has a hitting set of size at most k if and only if the election with candidate set C , distinguished candidate c , and voter set V can be destructively controlled by run-off partition of candidates (both in model TE and TP).*

Proof. Let B' be a hitting set of \mathcal{S} of size k . Partition C into $C_1 = B' \cup \{c, w\}$ and $C_2 = B - B'$. By Claim 4.29, w is the unique plurality winner of (C_1, V) , and c thus cannot win the election (C, V) .

For the converse, suppose that there exists a partition of candidates such that c is not the unique plurality winner in the run-off election corresponding to that partition. Then, certainly, there exists a set $D \subseteq B \cup \{w\}$ such that c is not the unique plurality winner of $(D \cup \{c\}, V)$. By Claim 4.30, \mathcal{S} has a hitting set of size at most k . \square

So from this and Example 4.15 we have the following.

Corollary 4.38. *Plurality voting is resistant to destructive control by run-off partition of candidates (both in model TE and TP).*

Finally, we show that plurality voting is resistant to both constructive and destructive control by partition of voters in the TP model. To this end, we reduce from the Hitting Set problem restricted to instances where $n(k+1)+1 \leq m-k$. We first define this restriction and prove that it still is NP-complete.

Restricted Hitting Set

Given: A set $B = \{b_1, b_2, \dots, b_m\}$, a family $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of subsets S_i of B , and a positive integer k such that $n(k+1)+1 \leq m-k$.

Question: Does \mathcal{S} have a hitting set of size at most k ? That is, is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i , $S_i \cap B' \neq \emptyset$?

Theorem 4.39. *Restricted Hitting Set is NP-complete.*

Proof. Restricted Hitting Set clearly is in NP. To show that it is NP-hard, we reduce Hitting Set to Restricted Hitting Set. Let $(\hat{B}, \hat{\mathcal{S}}, k)$ be a Hitting Set instance, where

$$\hat{B} = \{b_1, b_2, \dots, b_{\hat{m}}\},$$

$$\hat{\mathcal{S}} = \{\hat{S}_1, \hat{S}_2, \dots, \hat{S}_n\},$$

$\hat{S}_i \subseteq \hat{B}$ for each i , $1 \leq i \leq n$, and $k+1 \leq \hat{m}$. Define an instance of Restricted Hitting Set (B, \mathcal{S}, k) , where

$$\begin{aligned}
B &= \hat{B} \cup \{a_{i,j} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq k+1\}, \\
S_i &= \hat{S}_i \cup \{a_{i,1}, a_{i,2}, \dots, a_{i,k+1}\}, \quad \text{for } 1 \leq i \leq n, \quad \text{and} \\
\mathcal{S} &= \{S_1, S_2, \dots, S_n\}.
\end{aligned}$$

It is immediate that $\hat{\mathcal{S}}$ has a hitting set of size k if and only if \mathcal{S} has a hitting set of size k .

Let $m = \|B\| = \hat{m} + n(k+1)$. Since $k+1 \leq \hat{m}$, we have

$$n(k+1) + k + 1 \leq n(k+1) + \hat{m} = m,$$

i.e., $n(k+1) + 1 \leq m - k$. \square

Claim 4.40. *In the election (C, V) from Construction 4.28, if $n(k+1) + 1 \leq m - k$ then for every partition of V into V_1 and V_2 , c is a plurality winner of (C, V_1) or of (C, V_2) .*

Proof. For a contradiction, suppose that c is a winner of neither (C, V_1) nor (C, V_2) . For each $U \subseteq V$ and for each $i \in C$, let $\text{score}_U(i)$ denote the number of first-place votes that i has in (C, U) . Let $x \in B \cup \{w\}$ be a winner of (C, V_1) , and let $y \in B \cup \{w\}$ be a winner of (C, V_2) . Then

$$\text{score}_{V_1}(x) + \text{score}_{V_2}(y) \geq \text{score}_V(c) + 2. \quad (4.8)$$

Since c 's score in (C, V) is greater than that of any other candidate, we have $x \neq y$. It follows that

$$\begin{aligned}
\text{score}_{V_1}(x) + \text{score}_{V_2}(y) &\leq \text{score}_V(w) + \text{score}_V(b_i) \\
&\leq 2n(k+1) + 5 + 2n(k+1) + 2 \\
&\leq 2n(k+1) + 5 + 2(m-k) \\
&= \text{score}_V(c) + 1,
\end{aligned}$$

which contradicts Eq. (4.8). Thus, c is a winner of (C, V_1) or of (C, V_2) . \square

We now show that Construction 4.28 also provides a reduction from Restricted Hitting Set both to Constructive Control by Partition of Voters and to Destructive Control by Partition of Voters in the Ties-Promote model for plurality voting.

Claim 4.41. *In the election (C, V) from Construction 4.28, if $n(k+1) + 1 \leq m - k$ then the following three statements are equivalent:*

- (1) \mathcal{S} has a hitting set of size at most k .
- (2) V can be partitioned such that w is the unique plurality winner in the TP model.
- (3) V can be partitioned such that c is not the unique plurality winner in the TP model.

Proof. To show that the first statement implies the second statement, let B' be a hitting set of \mathcal{S} of size k . Partition V into V_1 and V_2 , where V_1 consists of one voter of the form $w > c > \dots$ and for every $b \in B'$ one voter of the form $b > w > \dots$, and where $V_2 = V - V_1$. Then the candidates in $B' \cup \{w\}$ are the winners of (C, V_1) and move forward to the run-off in the TP model, and c is the winner of (C, V_2) . By Claim 4.29, w is the unique plurality winner of the final election $(B' \cup \{c, w\}, V)$.

Clearly, if w is the unique plurality winner for some partition of V in the TP model, then c cannot be the unique plurality winner of this election for the same partition. Thus, the second statement implies the third statement.

Finally, we show that the third statement implies the first statement. Suppose there is a partition of V such that c is not the unique plurality winner of the election in the TP model. By Claim 4.40, c is a winner of one of the subelections and will thus participate in the final run-off. It follows that c is not the unique winner of a run-off election involving c , i.e., c is not the unique winner in $(D \cup \{c\}, V)$, for some $D \subseteq B \cup \{w\}$. By Claim 4.30, \mathcal{S} has a hitting set of size at most k . This completes the proof. \square

Theorem 4.39, Claim 4.41, and Example 4.16 have the following corollary.

Corollary 4.42.

- (1) *Plurality voting is resistant to constructive control by partition of voters in model TP.*
- (2) *Plurality voting is resistant to destructive control by partition of voters in model TP.*

4.3.2. Approval voting, constructive case, voter control

For approval voting, our reductions proving resistance are from the NP-complete problem Exact Cover by Three-Sets (X3C, for short), see Garey and Johnson [19].

Exact Cover by Three-Sets (X3C)

Given: A set $B = \{b_1, b_2, \dots, b_m\}$, where $m = 3k$ for a positive integer k , and a family $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of subsets S_i of B with $\|S_i\| = 3$ for each i .

Question: Does \mathcal{S} have an exact cover for B ? That is, is there a subfamily $\mathcal{S}' \subseteq \mathcal{S}$ such that every element of B occurs in exactly one set in \mathcal{S}' ?

Theorem 4.43. *Approval voting is resistant to constructive control by adding voters.*

Proof. That susceptibility holds in this case is shown by Example 4.11.

Given an instance (B, \mathcal{S}) of X3C, where $B = \{b_1, b_2, \dots, b_m\}$, $m = 3k$, $k > 1$, $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$, and $S_i \subseteq B$ with $\|S_i\| = 3$ for each i , $1 \leq i \leq n$, construct the following instance of Constructive Control by Adding Voters for approval voting:

- The candidate set is $C = B \cup \{w\}$, where w is the distinguished candidate.
- V consists of $k - 2$ registered voters who each approve of b_1, b_2, \dots, b_m and disapprove of w .
- W consists of n unregistered voters: For each i , $1 \leq i \leq n$, there is one voter in W who approves of w and the three candidates in S_i , and who disapproves of all other candidates.

We claim that \mathcal{S} contains an exact cover for B if and only if w can be made the unique approval winner by adding at most k voters.

For the left to right direction, simply add the k voters from W that correspond to the exact cover for B . Then w has k Yes votes and every $b \in B$ has $(k - 2) + 1 = k - 1$ Yes votes, so w is the unique approval winner.

For the right to left direction, suppose that w can be made the unique approval winner by adding at most k voters. Then we clearly need to add exactly k voters and every $b \in B$ can gain at most one Yes vote. Since each voter in W casts three Yes votes for candidates in B , it follows that every $b \in B$ gains exactly one Yes vote. Thus, the k added voters correspond to an exact cover for B . \square

Theorem 4.44. *Approval voting is resistant to constructive control by deleting voters.*

Proof. That susceptibility holds in this case is shown by Example 4.13.

Let an instance (B, \mathcal{S}) of X3C be given, where $B = \{b_1, b_2, \dots, b_m\}$, $m = 3k$, $k > 0$, $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$, and $S_i \subseteq B$ with $\|S_i\| = 3$ for each i , $1 \leq i \leq n$. For each j , $1 \leq j \leq m$, let

$$\ell_j = \|\{S_i \in \mathcal{S} \mid b_j \in S_i\}\|.$$

Construct the following election:

- The candidate set is $C = B \cup \{w\}$, where w is the distinguished candidate.
- The voter set V consists of the following voters:
 - For each i , $1 \leq i \leq n$, there is one voter in V who approves of all candidates in S_i and who disapproves of all other candidates.
 - There are n voters v_1, v_2, \dots, v_n in V such that, for each i , $1 \leq i \leq n$, v_i approves of w , and v_i approves of b_j if and only if $i \leq n - \ell_j$.

Note that the election (C, V) has the property that all candidates have n Yes votes.

We claim that S contains an exact cover for B if and only if w can be made the unique approval winner by deleting at most k voters.

For the left to right direction, simply delete the k voters from V that correspond to an exact cover for B . Then every $b \in B$ loses one Yes vote, leaving w the unique approval winner.

For the right to left direction, suppose that w can be made the unique approval winner by deleting at most k voters. Without loss of generality, we may assume that none of the deleted voters approves of w . So, we assume that only voters corresponding to S_i 's have been deleted. For w to have become the unique winner, every $b \in B$ must have lost at least one Yes vote. It follows that the deleted voters correspond to a cover, and since the cover has size at most k , this must be an exact cover for B . \square

Theorem 4.45. *Approval voting is resistant to constructive control by partition of voters in model TP.*

Proof. That susceptibility holds in this case is shown by Example 4.17.

Let an instance (B, S) of X3C be given, where $B = \{b_1, b_2, \dots, b_m\}$, $m = 3k$, $k > 0$, $S = \{S_1, S_2, \dots, S_n\}$, and $S_i \subseteq B$ with $\|S_i\| = 3$ for each i , $1 \leq i \leq n$. We modify the construction from the proof of Theorem 4.44. As in that proof, for each j , $1 \leq j \leq m$, let

$$\ell_j = \|\{S_i \in S \mid b_j \in S_i\}\|.$$

Now, define the following election:

- The candidate set is $C = B \cup \{w, x, y\}$, where w is the distinguished candidate.
- The voter set V consists of the following voters:
 - For each i , $1 \leq i \leq n$, there is one voter in V who approves of y and of all elements of S_i and who disapproves of all other candidates.
 - There are n voters v_1, v_2, \dots, v_n in V such that, for each i , $1 \leq i \leq n$, v_i approves of w , v_i disapproves of x , v_i disapproves of y , and v_i approves of b_j if and only if $i \leq n - \ell_j$.
 - There are $k + 1$ voters in V who approve of x and disapprove of all other candidates.
 - Finally, there are $k + 2$ voters in V who disapprove of x and approve of all other candidates.

Note that this election has the property that all candidates other than x have $n + k + 2$ Yes votes.

We claim that S contains an exact cover for B if and only if w can be made the unique approval winner by partition of voters in model TP.

For the left to right direction, if S contains an exact cover for B , then let V_2 consist of the k voters corresponding to the sets in the cover and of all the $k + 1$ voters who approve of only x , and let $V_1 = V - V_2$. Then

- w is the unique approval winner of (C, V_1) ,
- x is the unique approval winner of (C, V_2) , and
- w wins the run-off against x .

For the right to left direction, suppose that w can be made the unique approval winner by partition of voters in model TP. Since w is the unique winner in the run-off, and since every candidate other than x is tied with w (each having $n + k + 2$ Yes votes in V), the only candidates that can participate in the run-off are w and x . Since we are in the TP model, w must be the unique winner of one of the subelections and x must be the unique winner of the other subelection.

Let (V_1, V_2) be a partition of V such that w is the unique winner of (C, V_1) and such that x is the unique winner of (C, V_2) . As in the proof of Theorem 4.44, it follows that the voters corresponding to S_i 's that are not in V_1 (i.e., that are in V_2) correspond to a cover. Since x is the unique winner of (C, V_2) and x has $k + 1$ Yes votes, y can have at most k Yes votes in V_2 . It follows that there are at most k voters corresponding to S_i 's in V_2 . Thus, there are exactly k such voters, and these voters correspond to an exact cover. \square

Note that the previous construction won't work for the TE model, since in that model, w also wins the election if two or more candidates are tied for first place in V_2 . In the proof of the next theorem, we will adapt the construction from the proof of Theorem 4.45.

Theorem 4.46. *Approval voting is resistant to constructive control by partition of voters in model TE.*

Proof. That susceptibility holds in this case is shown by Example 4.17.

Let an instance (B, \mathcal{S}) of X3C be given, where $B = \{b_1, b_2, \dots, b_m\}$, $m = 3k$, $k > 0$, $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$, and $S_i \subseteq B$ with $\|S_i\| = 3$ for each i , $1 \leq i \leq n$. We modify the construction from the proof of Theorem 4.45. As in that proof, for each j , $1 \leq j \leq m$, let

$$\ell_j = \|\{S_i \in \mathcal{S} \mid b_j \in S_i\}\|.$$

Now, define the following election:

- The candidate set is $C = B \cup \{w, x, y\} \cup \{z_1, \dots, z_n\}$, where w is the distinguished candidate.
- The voter set V consists of the following voters:
 - For each i , $1 \leq i \leq n$, there is one voter in V who approves of y and of all elements of S_i and who disapproves of all other candidates.
 - For each i , $1 \leq i \leq n$, there is one voter in V who approves of y and z_i and who disapproves of all other candidates.
 - There are n voters v_1, v_2, \dots, v_n in V such that, for each i , $1 \leq i \leq n$, v_i approves of w , v_i disapproves of x , v_i disapproves of y , v_i approves of b_j if and only if $i \leq n - \ell_j$, and v_i approves of z_j if and only if $i \neq n$.
 - There are $n + k$ voters in V who approve of x and who disapprove of all other candidates.

Note that this election has the property that all candidates other than x and y have n Yes votes.

We claim that \mathcal{S} contains an exact cover for B if and only if w can be made the unique approval winner by partition of voters in model TE.

For the left to right direction, if \mathcal{S} contains an exact cover for B , then let V_2 consist of the k voters corresponding to the sets in the cover and of all the $n + k$ voters who approve of only x and for each i , $1 \leq i \leq n$, of the voter who approves of only y and z_i . Let $V_1 = V - V_2$. Then w is the unique approval winner of (C, V_1) , and x and y are tied for first place in (C, V_2) with $n + k$ Yes votes each. Since we are in model TE, no candidates are nominated by (C, V_2) , and w wins the run-off (and thus the election) by default.

For the right to left direction, suppose that w can be made the unique approval winner by partition of voters in model TE. Since we are in model TE, w must be the unique winner of one of the subelections. Let (V_1, V_2) be a partition of V such that w is the unique winner of (C, V_1) . As in the proof of Theorem 4.44, it follows that the voters corresponding to S_i 's that are not in V_1 (i.e., that are in V_2) correspond to a cover.

Suppose that there are more than k voters that correspond to S_i 's in V_2 . Note that for each i , $1 \leq i \leq n$, the voter that approves of only y and z_i must also be in V_2 (for if it weren't, z_i would have at least as many Yes votes in V_1 as w). It follows that y has more than $n + k$ Yes votes in V_2 . But then y is the unique approval winner in V_2 , since no other candidate has more than $n + k$ Yes votes in V . Since y beats w in the run-off, this contradicts the fact that w wins the election. It follows that there are at most k voters corresponding to S_i 's in V_2 . Thus, there are exactly k such voters, and these voters correspond to an exact cover. \square

5. Conclusions

In this paper, we studied the computational resistance and vulnerability of three voting systems—plurality, Condorcet, and approval voting—to destructive control by an election's chair in each of seven control scenarios: candidate addition, suppression, partition, and run-off partition, and voter addition, suppression, and partition. We classified each case as immune, vulnerable, or computationally resistant. We also studied the analogous constructive control cases and fully resolved those that were not considered by Bartholdi, Tovey, and Trick [3].

We identified cases where a system immune to constructive control still can be vulnerable to destructive control (e.g., Condorcet voting for control by adding candidates), and vice versa (e.g., approval voting for control by deleting

candidates). We saw that, among the systems studied, none is globally superior to the others. Rather, when choosing a voting system, one's choice will (along of course with consideration of all the many social-choice benefits/weaknesses of the given system, and other applicable factors) be influenced by the types of control against which protection is most desired. (We mention in passing that very recent work by the authors has shown that there exists an artificial voting system that is resistant to each of the twenty types of control discussed in the present paper [24].) Finally, we saw that tie-handling can have a significant effect: For those control types that involve partitions of the candidate or voter set, we studied two natural tie-handling rules, and we found cases in which the complexity of the corresponding control problem varies crucially depending on which tie-handling rule is adopted.

Acknowledgements

We thank Jeroen Snippe for helpful comments, and we thank Klaus Wagner and his group for hosting a visit during which this work was done in part. We thank the three *Artificial Intelligence* referees—two anonymous referees and S. Brams as an open referee—for their careful readings and helpful comments, which much improved this paper.

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