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Defeasible inheritance on cyclic networks ☆

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Abstract

In this paper, we are going to present a new notion of "extension" for defeasible inheritance networks that allows us to deal with cyclic nets. Horty has shown that cyclic nets need not have extensions in the sense of Touretzky. This paper presents a generalization of that notion of extension that can be applied to cyclic nets. The present proposal is inspired by a somewhat unexpected analogy between cyclic nets and "semantically closed" languages, i.e., languages containing their own truth predicate. Accordingly, this approach to defeasible inheritance networks with cycles shows similarities to the solution of semantic paradoxes put forth by Kripke. © 1997 Elsevier Science B.V.

Keywords: Cyclic networks; Defeasible inheritance; Theories of truth

1. Background and motivation

Defeasible inheritance networks were originally developed to gain a sound mathematical understanding of the way inheritance systems store, access, and manipulate taxonomic information with exceptions (a survey can be found in Thomason [21]). This paper is concerned with *direct* theories of inheritance that define a notion of *consequence* for inheritance networks in terms of the net itself. Alternatively, an *indirect* theory assigns meaning to inheritance networks by embedding them in a language already equipped with a well-understood semantics. For instance, an indirect approach was pursued, in the case of *strict* inheritance networks, by Hayes [6] (via an embedding into first-order logic), and in the case of defeasible inheritance networks by Etherington and Reiter [4] (using an embedding into default logic). However, the *direct* approach first introduced by Touretzky [23] has now become standard.

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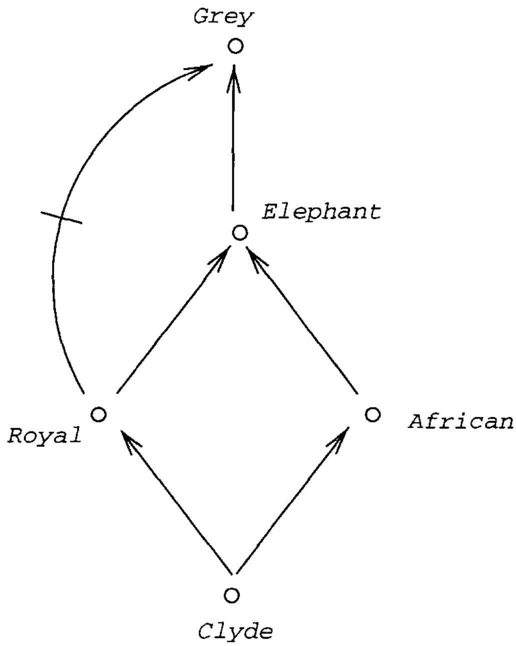

Fig. 1. The standard example of pre-emption.

An inheritance network can be identified with a collection of *signed links* (positive or negative) over a set of nodes. Links are of the form $n_1 \rightarrow n_2$ or $n_1 \not\rightarrow n_2$ respectively, where n_1 and n_2 are nodes in the net. Nodes are labelled by lexical items referring to categories of individuals—for convenience we will identify nodes with their labels. When the network is to be interpreted defeasibly, a link $n_1 \rightarrow n_2$ represents the fact that objects of category n_1 tend to be of category n_2, whereas a link of the form $n_1 \not\rightarrow n_2$ represents the fact that objects of category n_1 tend *not* to be of category n_2. A *path* over a net Γ is a sequence of links from Γ at most the last one of which is allowed to be negative. So both $n_1 \rightarrow n_2 \rightarrow n_3$ and $n_1 \rightarrow n_2 \not\rightarrow n_3$ are paths, while $n_1 \not\rightarrow n_2 \rightarrow n_3$ is not. A path is positive or negative according as its last link is positive or negative.

Theories of defeasible inheritance found in the literature rely on the three notions of *constructibility*, *conflict*, and *pre-emption*. Roughly speaking, a path is *constructible* relative to a net Γ if it can be obtained by chaining links from Γ. A path *conflicts* another path containing at least two links, if the first has the same endpoints but opposite sign as the second. So $n_1 \rightarrow n \rightarrow n_2$ is conflicted by $n_1 \rightarrow n' \not\rightarrow n_2$, and conversely. But perhaps the most important idea in defeasible inheritance is that of *pre-emption*. Pre-emption gives us a way to resolve conflicts between paths, based on the intuition that more specific information should override more generic information.

There are two ways to define pre-emption: *on-path* pre-emption of Touretzky [23] and Boutilier [2]; and *off-path* pre-emption of Sandewall [15], Horty, Thomason and Touretzky [8], and Stein [18, 19]. The latter has come to be prominent in the literature, and is used in this paper. Consider for instance the standard example of pre-emption

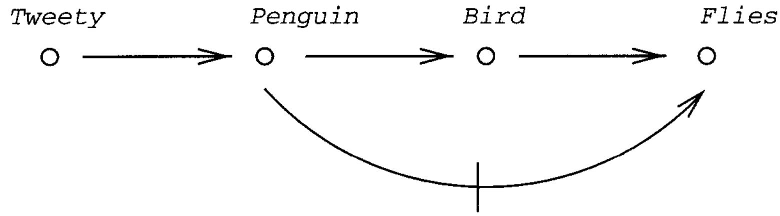

Fig. 2. Another example of pre-emption.

over a net represented in Fig. 1 (the example is due to Sandewall). The network tells us that Clyde is an African elephant and also that it is a Royal elephant. Of course, both African elephants and Royal elephants are elephants, but Royal elephants are not grey. Here the usual notion of off-path pre-emption precludes the conclusion that Clyde is grey, since information concerning Royal elephants is more specific than information concerning elephants. The notion of pre-emption captures this formally, using only topological properties of the network itself. It should be noted, however, that the notion of on-path pre-emption does not block the conclusion that Clyde is grey.

As another example, consider the network of Fig. 2. On this net, both on-path and off-path pre-emption give the same results. Indeed, although we are told that Tweety is a penguin, penguins are birds, and birds fly, the conclusion we naturally draw is that Tweety does not fly. The conclusion that Tweety flies is pre-empted by information to the effect that penguins don't fly. Since penguins are a kind of birds, information as to whether penguins fly is more specific than information about whether birds fly, and thus overrides it. We conclude that Tweety does not fly.

Once the notions of constructibility, conflict, and especially pre-emption have been defined, we can proceed with the definition of the *extensions* of a net Γ. Intuitively, an extension is a conflict-free set of paths that are supported by the net. There are essentially two ways to define extensions: *credulous* (see [23]) and *skeptical* (see [8]). Although the latter might be preferable on conceptual and computational grounds (as argued, among others, in [8,17]), the former is somewhat simpler to define, and therefore is adopted in this paper. The reader is referred to Horty's excellent survey [7] for details on these two kinds of extensions. An extension for a net Γ is credulous, roughly speaking, if it is a *maximal* conflict-free set of paths over Γ in which no path is pre-empted.

Extensions need not be unique. However, it is well known that if the underlying net Γ contains no cycles, then extensions always exist. This can be seen as follows. Given a path σ over Γ, define the *degree* of σ to be the length of the longest sequence of links (irrespective of their sign) from Γ having the same endpoints as σ. It is clear that this notion of degree makes sense only if Γ contains no cycles. Then, in the case of acyclic nets it is possible to show that extensions exist by means of an iterative process in which paths are considered in ascending order of their degrees (see [7] for a unified treatment of inheritance on acyclic nets).

Things are different in the case of networks with cycles. Such nets arise naturally in many situations, for instance whenever there are two mutually overlapping categories (see Fig. 3 for an example). In such nets, the presence of cycles is a cause for the

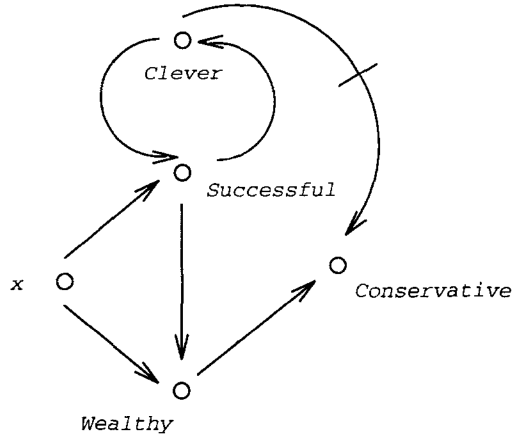

Fig. 3. A network with cycles.

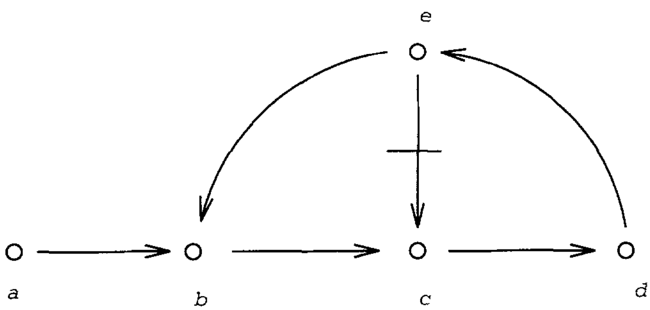

Fig. 4. A network with no credulous extension.

global character of the notion of specificity. For the purposes of the present discussion, let us say that a node n_1 is more specific than a node n_2 (relative to a given network), if there is a (positive) path from n_1 to n_2 but not vice versa. (Other notions of specificity can be found in the literature.) Then, as long as the net is acyclic, if a node n_1 is more specific than node n_2, then this character is preserved no matter how the network is extended. In this sense, specificity is a *local* property of nodes. Once cycles are allowed, however, n_1 might be more specific than n_2 relative to a certain net Γ, but not relative to a net Γ' extending Γ (because Γ' might introduce a (positive) path back from n_2 to n_1). In this sense, specificity is a global property of cyclic nets.

This appears to be connected with the fact that cyclic networks need not have extensions in the standard sense of [7], as it was discovered by Horty. Consider the net of Fig. 4. According to the usual approaches to inheritance, this net cannot have any extensions: suppose for contradiction that Φ is an extension for the net. Then, clearly, either the path $a \to b \to c$ is in Φ or it isn't. If it is, then Φ must contain also the path

$$\sigma = a \to b \to c \to d \to e \to b$$

(since nodes d, e, and b have no incoming negative links, there is nothing to block the construction of σ); but then $a \to b \to c$ would be pre-empted in Φ, since Φ contains a node e, that is more specific than b (because the path σ is in Φ), and a direct link telling us that e's are not c's. This is impossible if Φ is an extension. If $a \to b \to c$ is not in Φ then σ can't be in Φ either, so that $a \to b \to c$ would not be pre-empted in Φ and so Φ cannot be an extension (because it would fail to contain a path that is constructible but neither conflicted nor pre-empted in Φ). The reason for this state of affairs seems to be that the path $a \to b \to c \to d \to e \to b$ pre-empts one of its initial segments.

This is a peculiar phenomenon, which bears a resemblance to the so-called *Liar paradox*. This paradox, as is well known, arises when considering the following sentence Λ: "Λ is not true". It is then impossible consistently to hold or deny that Λ is true. Philosophers and logicians have developed a number of solutions to the paradox, but one in particular is relevant here. Kripke [10] shows that it is possible to provide a semantics for a language containing its own truth predicate (and in which therefore Λ can be expressed), provided we give up *bivalence*; provided, that is, that we switch (for instance) to a three-valued setting. The desired semantics is then achieved by means of a fixpoint construction.

The situation is not too dissimilar with defeasible networks. If we construe paths as arguments, then in the context of the net of Fig. 4 the path $a \to b \to c \to d \to e \to b$ says of itself that it is not tenable. It is interesting to notice, however, that no explicit self-reference is anywhere in sight. We know from the discussion of Fig. 4 that we have to give up the Touretzky–Horty notion of extension, if we are to deal with cyclic networks.

In what follows we are going to provide a solution to this problem by defining a notion of extension according to which all nets have extensions. Such a solution is indeed inspired by the Kripke construction of a semantics for a language containing its own truth predicate. In that construction, given a three-valued truth schema such as the "strong Kleene" of [9] (in which $\neg\varphi$ is true, false or indeterminate according as φ is false, true, or indeterminate respectively), a sentence φ no longer needs explicitly to be counted as true in order to prevent $\neg\varphi$ from being counted as true; and similarly, it is only when explicitly counted as false that a sentence φ can no longer prevent $\neg\varphi$ from being counted as true.

The intuition behind the present approach to cyclic nets applies this idea to the definition of the concept of extension. An extension, according to present proposal, is a pair of sets of paths (the paths that are explicitly constructed and the ones that are explicitly ruled out), simultaneously satisfying a pair of fixpoint equations. It is then no longer necessary for a path to be explicitly constructed for it to *pre-empt* other paths. On the other hand, once a path has been explicitly ruled out as pre-empted or conflicted, it can no longer pre-empt other paths. Similar approaches have been pursued by Makinson and Schlechta (see [11,16]), but for other reasons.

There are other similarities, however, between the present approach to cyclic nets and the three-valued solution to the Liar paradox. Once we allow a language to contain its own truth predicate, there are many self-referential sentences that can be constructed, and not all as pathological as the Liar. For example, the following sentence T is known as the *truth-teller*: "T is true". There is a sense in which this sentence is as pathological as the Liar, in that it escapes the recursive clauses of a truth definition *à la* Tarski (see [5] for a recent discussion of truth and paradox). There is an important difference, however: contrary to what happens for Λ, it *is* possible consistently to hold that T is true, *and* it is possible consistently to deny that T is true (although not at the same time). That is, sentence T, contrary to Λ, does not force us to renounce bivalence.

Analogously, among the extensions introduced in this paper we single out a class of extensions referred to as "classical" in that they are the analog of a two-valued semantics for a formal language. In a classical extension any path not explicitly constructed is explicitly ruled out, so that there are no paths that fall "in between". Every extension in the sense of Touretzky–Horty is a classical extension in our sense. In particular, the Touretzky–Horty notion of extension is subsumed under the notion of extension presented here.

Every *acyclic* net has a classical extension. But these are not the only nets having classical extensions. Indeed, it is perhaps somewhat surprising that there are cyclic nets that exhibit a behavior similar to the truth-teller. Such nets are the $(2n)$-loops of Definition 30: that there are classical extensions for them is the import of Theorem 31. It is important to notice that the length (even or odd) of the cycles seems to play a crucial role in determining whether a net has a classical extension. (Every net has an extension, whether it contains odd or even length cycles, but such an extension will not, in general, be classical.)

Although the present approach draws its inspiration from the analogy with solutions of the Liar paradox given in philosophical logic, it is connected to other approaches in the literature. First of all, although (to the author's knowledge) there is no other work dealing with cycles in nonmonotonic inheritance networks, there are approaches dealing with cycles in *strict* settings.

Cycles in strict inheritance networks are dealt with by Thomason et al. [22]. When the network is construed strictly, cycles pose no particular problem (this is not to say that the semantics for such a network is trivial: on the contrary it is unexpectedly complex, but the complexity appears to be independent of whether the net contains cycles).

A different approach to cycles, although not in inheritance networks but in *terminological systems* is given by Nebel [12]. Nebel considers the case of a sequence of definitions, in which some lexical item is defined either directly in terms of itself or indirectly in terms of other items that in turn are defined in terms of it. Again, Nebel's construction takes place in a strict setting and proceeds by finding the least fixpoints of certain monotonic operators. Such fixpoints are then interpreted as providing the classical extension of the lexical items being defined. No paradoxical phenomena force the adoption of a "three-valued" approach.

A nonmonotonic extension of terminological systems is considered by Baader and Hollunder [1]: they propose a merge of terminological systems and a particular version of Reiter's default logic, but they do not address the particular problems deriving from cyclic representation formalisms.

On the other hand, there is an interesting connection between the present proposal and recent work in default logic by Papadimitriou and Sideri [13]. Building upon previous work by Etherington [3], they show how to associate with any default theory a particular graph, representing the logical dependencies among the defaults comprising the theory, and then establish that the theory must have at least one extension (in the sense of default logic, see for instance Reiter [14]), provided the associated graph contains no odd length cycles.

It seems that the same sort of phenomenon is at work in cyclic inheritance networks as in default theories: in both cases we have a sequence of what could be regarded as "inference rules", the firing of each one of which prevents triggering the next, and the firing of the last one of which prevents triggering the first (defaults are clearly a sort of inference rules: it is possible to construe paths through a net also as inference rules; for instance, this is the point of view of [7]). If the cycle has even length, it is possible consistently to partition the sequence in two alternating subsequences, containing the rules that are triggered and the rules that are pre-empted. If the cycle has odd length, no such partition is possible.

Moreover, the notion of extension employed in default logic is intrinsically "two-valued", in the sense that it contains the consequences of a maximal set of defaults whose justifications are consistent with the extension itself. (See either [14] or [13] for a technical definition of an extension as a solution for a certain fixpoint equation—or, equivalently, as the limit of a certain iterative process defined in terms of the extension itself.) In other words, the triggering of a default can only be prevented if its justification is explicitly refuted. In virtue of this maximality Papadimitriou and Sideri fail to consider a possibility that is available to the present approach to inheritance networks, namely that the subsequences of the rules mentioned in the previous might fail to be exhaustive. This would indeed give rise to a "three-valued" notion of extension for default logic, analogous to the one put forward here for inheritance networks, and according to which every default theory has an extension. Such a notion of extension has not yet appeared.

2. Graph-theoretical preliminaries

Let Γ be a finite inheritance network, i.e., a pair consisting of a finite set of nodes $\{n_1, \ldots, n_p\}$, along with a finite set of *signed* links between nodes of the form $n_i \rightarrow n_j$ or $n_i \not\rightarrow n_j$. So Γ is a directed graph with two sorts of edges. Since Γ contains finitely many links, there are also finitely many links between any two nodes: we can then assume with no loss in generality that given two nodes there is at most one positive and at most one negative link between the first and the second. A sequence of n nodes is a *trail* if either $n = 0$ or $n \geqslant 2$, and moreover each node in the sequence is connected by a link to the next one.

We use the following notational conventions. Lower-case letters of the Greek alphabet $\rho, \sigma, \tau, \ldots$ denote trails, and upper-case letters of the Greek alphabet $\Phi, \Psi, \ldots$ represent sets of trails. We use λ to refer to the empty trail. Negative links are represented by placing a bar over the end node (so for instance $n_i \not\rightarrow n_j$ is represented by $n_i\overline{n_j}$). Let $\rho = x_1 \ldots x_n a$ and $\sigma = a y_1 \ldots y_m$ be trails. Then the juxtaposition $\rho\sigma$ represents the sequence $x_1 \ldots x_n a y_1 \ldots y_m$. Moreover, if y is the end node of ρ, the notation ρx is shorthand for $\rho\langle y, x\rangle$. Similarly for $x\rho$.

Let $\alpha = x_1 \ldots x_n$. A node x occurs in a trail α if $x = x_j$ for $1 \leqslant j \leqslant n$. Given α as above and a trail β we say that β is a prefix or initial segment (not necessarily proper) of α, written $\beta \sqsubseteq \alpha$, if and only if $\beta = x_1 \ldots x_m$ for some $m \leqslant n$. We also say that α is a subtrail of β if $\beta = \gamma\alpha\delta$ for some (possibly empty) trails γ and δ.

The definition given below fixes a technical meaning for the word "path", which we adopt throughout the paper. In particular, the definition depends on some antecedently fixed net Γ: so the only paths are the ones obtained by chaining links in Γ.

Definition 1. A trail is a *path* if and only if it contains at most one negative link, and such a link occurs as the last link in the trail. A path is *positive* or *negative* according as its last link is positive or negative. By Γ^* we refer to the set of all (positive as well as negative) paths over Γ.

From the definition it follows that we can have paths of any nonnegative number of nodes except one. If α and β are paths and α is a subsequence of β, we say that α is a *subpath* of β.

Definition 2. The *length* of a path σ, denoted by $\ell(\sigma)$, is the number of *links* in σ. E.g., if $\sigma = x_0 \ldots x_n$, then the length of σ is n. We also set $\ell(\lambda) = 0$.

Definition 3. If σ is a non-empty path, then σ^{i} and σ^{e} denote its initial and end node, respectively.

Definition 4. A path $\sigma \in \Gamma^*$ is *simple* if every node occurs at most once in it. Let S_Γ refer to the set of simple paths over Γ.

Definition 5. A path $\sigma \in \Gamma^*$ is a *cycle* if it is positive and has the form $x\rho x$, and $x\rho$ is simple. We refer to the set of cycles over Γ as C_Γ. A set of paths Φ is *cyclic* if some path in Φ contains a subpath that is a cycle, and Φ is *acyclic* otherwise; a net Γ is *cyclic* if and only if Γ^* is cyclic.

If Γ^* contains cycles, then Γ^* will be infinite, independently of whether the underlying network Γ itself is finite or not. (Observe that negative trails of the form $x\alpha\overline{x}$ do not give rise to an infinity of paths in Γ^*, since there is no chaining off of a negative link.) Given the infinity of Γ^* we are interested in defining a *finite* subset $\Gamma^{\#}$ of Γ^* with the property that $\Gamma^{\#}$ will contain a path from a node x to a node y if and only if Γ^* contains a path, of the same sign, from x to y. This will allow us to reduce questions about Γ^* to questions about $\Gamma^{\#}$. The basic idea in the construction of $\Gamma^{\#}$ is to take all the paths with no repetitions from Γ^* and "splice in" cycles in such a way as to "go around" each cycle at most once. While this idea is made precise below, it should be noted that this is by no means the only possible choice. In general, any finite subset of Γ^* that is closed under initial segments could be used in place of $\Gamma^{\#}$.

Lemma 6. *S_Γ and C_Γ are both finite.*

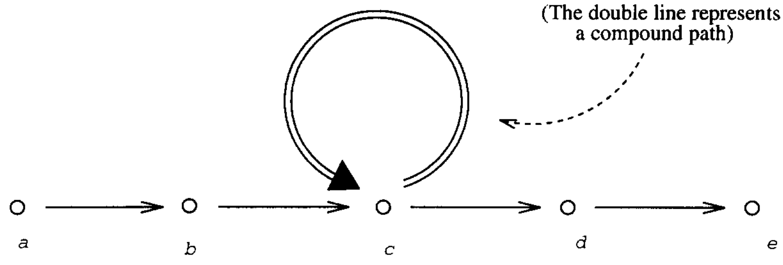

Fig. 5. A cycle is spliced into the path *abcde*.

The above lemma follows immediately from the finiteness of Γ. The following is a definition of "splicing"—the process that splices a cycle into a simple path that intersects the cycle.

Definition 7. Let ρ be a cycle with $\rho^{\mathrm{i}} = \rho^{\mathrm{e}} = x$, and let σ and τ be paths; we say that σ is *obtained from* τ *by splicing* ρ, written $\tau \leqslant_\rho \sigma$, if and only if τ is of the form $\alpha x \beta$, and σ is $\alpha \rho \beta$.

This notion of splicing can be generalized to capture the process by which a path σ can be turned into a path τ by successive splicings of cycles drawn from C_Γ, but in such a way that each cycle is used at most once.

Definition 8. Let $\mathcal{T}$ be the set of all sequences of paths of the form

$$\langle \lambda, \alpha_1, \ldots, \alpha_{k+1} \rangle,$$

such that:

(i) $\alpha_1 \in S_\Gamma$; and

(ii) there exist pairwise distinct paths $\rho_1 \ldots \rho_k \in C_\Gamma$ such that $\alpha_i \leqslant_{\rho_i} \alpha_{i+1}$, for all $i \leqslant k$.

$\mathcal{T}$, together with the "initial segment" relation on sequences of paths, constitutes a tree.

Lemma 9. *$\mathcal{T}$ is finite.*

Proof. First, observe that all branches in $\mathcal{T}$ are finite. Indeed, if N is the cardinality of C_Γ, then $N + 2$ is a bound on the length of the branches in $\mathcal{T}$. So, to show that $\mathcal{T}$ is finite it suffices to show that any sequence in the tree can only have finitely many immediate successors extending it.

Consider a sequence $\langle \lambda, \alpha_1, \ldots, \alpha_k \rangle$ in $\mathcal{T}$, and let $\rho_1 \ldots \rho_{k-1}$ be the corresponding cycles. We know that there are finitely many $\sigma \in C_\Gamma - \{\rho_1 \ldots \rho_{k-1}\}$, and for each such σ there are only finitely many τ such that $\alpha_k \leqslant_\sigma \tau$, since σ can be spliced in at finitely many nodes on α_k. It follows that there are only finitely many sequences $\langle \lambda, \alpha_1, \ldots, \alpha_{k+1} \rangle$ in $\mathcal{T}$. $\square$

Indeed, we can obtain a sharper bound in the following way: again consider a sequence $\langle \lambda, \alpha_1, \ldots, \alpha_k \rangle$. Let Q be the length of the longest path in S_Γ and P the length of the longest cycle in C_Γ. Then for any $k \leqslant N+2$ the length of α_k is bounded by $Q + kP$. Moreover, given a sequence as above, there are only $card(C_\Gamma) - k$ cycles available for splicing, and each can be spliced in at most $Q + kP$ nodes. If we now put

$$X = \prod_{k=0}^{N+2} (card(C_\Gamma) - k) \cdot (Q + kP),$$

then the branching factor is bounded by $\max(X, card(S_\Gamma))$.

Definition 10. Let $\Gamma^{\#}$ be the set of all paths α occurring in a sequence in $\mathcal{T}$.

It follows immediately from the definition that $\Gamma^{\#}$ is finite.

3. Constructing extensions

Since as mentioned in Section 1, extensions in the credulous sense of [23] do not necessarily exist for cyclic graphs, we define a new notion of extension that agrees with the definition of credulous extension for acyclic nets, and guarantees that cyclic nets also have extensions. In what follows, recall that any non-empty path σ that is not a link can be written in the form τx or $\tau\overline{x}$, where $\tau^{\mathrm{e}} \to x$ or (respectively) $\tau^{\mathrm{e}} \not\to x$ is a link.

Definition 11. Let Φ be a set of paths. A path σ is *constructible* in Φ, relative to Γ, if and only if either σ is the empty path; or σ is one of the links in Γ; or σ has the form τx or $\tau\overline{x}$, where $\tau \in \Phi$ and the link $\tau^{\mathrm{e}} \to x$ or, respectively, $\tau^{\mathrm{e}} \not\to y$ is in Γ.

Although links are never conflicted, they can in turn "conflict" longer, compound paths. This is justified by the intuition that a network Γ should always support at least those statements corresponding to the links, and is captured by the following definition.

Definition 12. Let Φ be a set of paths; we say that a path σ is *conflicted* in Φ if and only if $\ell(\sigma) \geqslant 2$, and Φ contains a path τ having the same endpoints as σ (i.e., $\tau^{\mathrm{i}} = \sigma^{\mathrm{i}}$ and $\tau^{\mathrm{e}} = \sigma^{\mathrm{e}}$), but opposite sign.

Definition 13. A positive path σx (of length $\geqslant 2$) is *pre-empted* in Φ (relative to Γ) if and only if $\sigma \in \Phi$, and there is a node v such that the link $v \not\to x$ is in Γ, and either $v = \sigma^{\mathrm{i}}$, or $\sigma^{\mathrm{i}}\tau_1 v \tau_2 \sigma^{\mathrm{e}} \in \Phi$, for some paths τ_1 and τ_2.

Similarly, a negative path $\sigma\overline{x}$ is *pre-empted* in Φ (relative to Γ) if and only if $\sigma \in \Phi$, and there is a node v such that the link $v \to x$ is in Γ, and either $v = \sigma^{\mathrm{i}}$, or $\sigma^{\mathrm{i}}\tau_1 v \tau_2 \sigma^{\mathrm{e}} \in \Phi$, for some paths τ_1 and τ_2.

The above notion of pre-emption is the usual notion of off-path pre-emption of [7,23]. It follows from the definition that links are never pre-empted.

In the above definition, if σ is pre-empted because $\sigma^{\mathrm{i}}\tau_1 \upsilon \tau_2 \sigma^{\mathrm{e}} \in \Phi$ (with $\upsilon \not\to x$ or $\upsilon \to x$ in Γ), then the compound path $\sigma^{\mathrm{i}}\tau_1 \upsilon \tau_2 \sigma^{\mathrm{e}}$ is called the *pre-empting path*; if σ is pre-empted because $\sigma^{\mathrm{i}} \not\to x$ or $\sigma^{\mathrm{i}} \to x$ is in Γ, then the link $\sigma^{\mathrm{i}} \not\to x$ or $\sigma^{\mathrm{i}} \to x$ itself is called the *pre-empting path*.

Definition 14. A set of paths Φ is *co-inductive* if it is closed under initial segments, i.e., $\rho \in \Phi$ whenever $\rho x \in \Phi$.

We are finally ready to introduce our new notion of extension. Recall that we are going to allow paths not explicitly constructed to pre-empt other paths from being constructed.

Definition 15. Let Φ be any co-inductive set of paths. An *extension* (for Φ) is a pair (Φ^+, Φ^-) of sets of paths from Φ simultaneously satisfying the following two fixpoint equations:

$$\begin{aligned}\Phi^+ = \{\sigma \in \Phi\colon\ & \sigma \text{ constructible in } \Phi^+ \wedge \\ & \sigma \text{ not conflicted in } \Phi^+ \wedge \\ & \sigma \text{ not pre-empted in } \Phi - \Phi^-\};\end{aligned}$$

$$\Phi^- = \{\tau \in \Phi\colon \text{ some prefix of } \tau \text{ is conflicted or pre-empted in } \Phi^+\}.$$

By abuse of language, a pair (Φ^+, Φ^-) is an extension for Γ if and only if it is an extension for $\Gamma^{\#}$.

The following theorem makes good on our promise that any defeasible inheritance network has an extension in the above sense.

Theorem 16. *Every finite co-inductive set Φ has an extension.*

Proof. Assume Φ is a finite and co-inductive set of paths. [1] Let Γ_Φ be the set of links in Φ, i.e., the set of length-one paths in Φ. We construct an extension for Φ in stages. For every $n \geqslant 0$ define Φ_n^+ and Φ_n^- inductively as follows:

- $\Phi_0^+ = \Gamma_\Phi$ and $\Phi_0^- = \emptyset$;
- Φ_{n+1}^+ = a maximal [2] conflict-free set of paths σ from Φ such that the following all hold:
  - (i) σ is constructible in Φ_n^+;
  - (ii) σ is not conflicted in Φ_n^+;
  - (iii) σ is not pre-empted in $\Phi - \Phi_n^-$.
- $\Phi_{n+1}^- = \{\tau \in \Phi\colon$ some prefix of τ is conflicted or pre-empted in $\Phi_{n+1}^+\}$.

[1] In Appendix A we show how to drop the assumption of finiteness.

[2] Such a maximal set of paths always exists. It is this maximality clause that gives the analogue of the "credulous" extensions. By dropping it, we would obtain the analogue of "flexible" extensions, whereas the analogue of "skeptical" extensions can be obtained by taking the set of all paths satisfying clauses (i)–(iii) and deleting any conflicting pairs.

We then set:

$$\Phi^+ = \bigcup_{n\in\mathbb{N}} \Phi_n^+; \qquad \Phi^- = \bigcup_{n\in\mathbb{N}} \Phi_n^-.$$

We now show, by induction on n, that $\Phi_n^+ \subseteq \Phi_{n+1}^+$ and $\Phi_n^- \subseteq \Phi_{n+1}^-$.

Base case $n = 0$. A link $x \to y$ or $x \not\to y$ in Γ_Φ is constructible, never conflicted, and never pre-empted; so $x \to y$ or $x \not\to y$, respectively, is in Φ_1^+. So $\Phi_0^+ = \Gamma_\Phi \subseteq \Phi_1^+$. Moreover, $\Phi_0^- = \emptyset \subseteq \Phi_1^-$.

Inductive step. Suppose $\sigma \in \Phi_{n+1}^+$. If σ is a link, $\sigma \in \Phi_{n+2}^+$, as above. If σ is not a link, then $\ell(\sigma) \geqslant 2$, and say $\sigma = \rho x$, with $\rho \in \Phi_n^+$ and $\rho^{\mathrm{e}} \to x \in \Gamma_\Phi$ (the case where σ is negative is similar). By the inductive hypothesis, we have $\rho \in \Phi_{n+1}^+$, which gives that σ is constructible in Φ_{n+2}^+ as well.

Now we show that σ is not pre-empted in $\Phi - \Phi_{n+1}^-$. So assume by way of contradiction that σ is pre-empted in $\Phi - \Phi_{n+1}^-$. This means that there is a node v such that $v \not\to x$ is in Γ, and either $v = \rho^{\mathrm{i}}$, or $\rho^{\mathrm{i}}\tau_1 v \tau_2 \rho^{\mathrm{e}} \in \Phi - \Phi_{n+1}^-$, for some paths τ_1 and τ_2. So we further distinguish two cases. If $v = \rho^{\mathrm{i}}$, then $\rho^{\mathrm{i}} \not\to x$ is in Γ, which would make σ already pre-empted in $\Phi - \Phi_n^-$, since Φ_n^- contains no direct links. The other case is $\rho^{\mathrm{i}}\tau_1 v \tau_2 \rho^{\mathrm{e}} \in \Phi - \Phi_{n+1}^-$. By the inductive hypothesis, $\Phi_n^- \subseteq \Phi_{n+1}^-$, whence $\rho^{\mathrm{i}}\tau_1 v \tau_2 \rho^{\mathrm{e}} \in \Phi - \Phi_n^-$. But this makes σ pre-empted in $\Phi - \Phi_n^-$, whence $\sigma \notin \Phi_{n+1}^+$, against assumption. This shows that $\sigma \in \Phi_{n+2}^+$.

Finally, σ is not conflicted in Φ_{n+1}^+, since Φ_{n+1}^+ is conflict-free. So, if still $\sigma \notin \Phi_{n+2}^+$, it is because it is conflicted in Φ_{n+2}^+. Let τ be the conflicting path: then τ is not a link (or else $\sigma \notin \Phi_{n+1}^+$): it follows that τ is conflicted in Φ_{n+1}^+ (by σ), which is impossible if τ is in Φ_{n+2}^+.

Now suppose $\sigma \in \Phi_{n+1}^-$: so σ extends some τ which is conflicted or pre-empted in Φ_{n+1}^+; by the inductive case for Φ_{n+1}^+, the conflicting or pre-empting path is also in Φ_{n+2}^+, whence $\sigma \in \Phi_{n+2}^-$ as required.

We now show that (Φ^+, Φ^-) is an extension. Although this is an immediate consequence of the finiteness of Φ, we give the following more general argument, which will be used in Appendix A. We need to establish, first, that if σ is constructible in Φ^+, but neither conflicted in Φ^+ nor pre-empted in $\Phi - \Phi^-$, then $\sigma \in \Phi^+$. So suppose σ is constructible in Φ^+ but neither conflicted in Φ^+ nor pre-empted in $\Phi - \Phi^-$. Then: (1) there is a stage p such that σ is constructible in all the Φ_m^+ for $m \geqslant p$ (this follows from the fact that the sequence we construct is increasing and Φ^+ is its limit); (2) at no stage n can σ be conflicted in Φ_n^+, or else it would be conflicted in the limit too; (3) suppose that for every n, σ is pre-empted in $\Phi - \Phi_n^-$, by some pre-empting path ρ_n. Since Φ is finite and $\Phi - \Phi_n^-$ *decreases* as n increases, there is some ρ_{n_0} such that ρ_{n_0} pre-empts σ and

$$\rho_{n_0} \in \Phi - \bigcup_{n\geqslant 0} \Phi_n^- = \Phi - \Phi^-,$$

which is impossible. It follows that σ is pre-empted in $\Phi - \Phi^-$ too, against assumption. So there is a q such that for every $n \geqslant q$, σ is not pre-empted in $\Phi - \Phi_n^-$. Now let $n^* = \max(p, q)$. Then σ is constructible in $\Phi_{n^*}^+$ by (1), but neither conflicted in $\Phi_{n^*}^+$

nor pre-empted in $\Phi - \Phi^-_{n^*}$ by (2) and (3): so $\sigma \in \Phi^+_{n^*+1} \subseteq \Phi^+$. Likewise, if some prefix of σ is conflicted or pre-empted in Φ^+, we obtain $\sigma \in \Phi^-$.

Conversely, we need to show that if $\sigma \in \Phi^+$ then any prefix of σ is constructible in Φ^+, but neither conflicted in Φ^+ nor pre-empted in $\Phi - \Phi^-$. So, let $\sigma \in \Phi^+$. If σ were conflicted by some $\rho \in \Phi^+$, then there would be an n such that both $\sigma, \rho \in \Phi^+_n$, which is impossible, since all Φ^+_n are conflict-free by construction. Similarly, if σ were not constructible in Φ^+, then it would not be constructible in any Φ^+_n, and hence it could not be in Φ^+. Finally, if σ were pre-empted by $\rho \in \Phi - \Phi^- = \bigcap_{n \geqslant 0}(\Phi - \Phi^-_n)$, then for every n we would have $\rho \in \Phi - \Phi^-_n$; then σ would be pre-empted in $\Phi - \Phi^-_n$ for every n, and hence never put into Φ^+, against hypothesis.

Finally, we need to show that $\sigma \in \Phi^-$ if and only if some prefix τ of σ is pre-empted or conflicted in Φ^+. We have: $\sigma \in \Phi^-$, iff for some n, $\sigma \in \Phi^-_n$; iff, by construction, σ is pre-empted or conflicted in Φ^+_n, iff σ is also pre-empted or conflicted in Φ^+ as well.

This shows that (Φ^+, Φ^-) is an extension for Φ. Moreover, given that Φ is finite (as when, e.g., $\Phi = \Gamma^{\#}$) then since Φ^+_n and Φ^-_n are subsets of Φ, this extension is reached at some finite stage. □

Let us consider the simple example of Fig. 2. Using the simplified node labelling `t = Tweety`, `p = Penguin`, `b = Bird`, `f = Flies`, let us see how an extension for this net can be obtained. First of all, let

$$\Phi = \{\mathtt{tp}, \mathtt{tpb}, \mathtt{tpbf}, \mathtt{tp\overline{f}}, \mathtt{pb}, \mathtt{pbf}, \mathtt{p\overline{f}}, \mathtt{bf}\},$$

so that Φ, the set of all paths over the net of Fig. 2, is finite and co-inductive. Then, the sets of paths obtained at each stage in the construction of Theorem 16 are as follows:

$$\begin{aligned}
\Phi^+_0 &= \{\mathtt{tp}, \mathtt{pb}, \mathtt{bf}, \mathtt{p\overline{f}}\},\\
\Phi^-_0 &= \emptyset,\\
\Phi^+_1 &= \Phi^+_0 \cup \{\mathtt{tpb}, \mathtt{tp\overline{f}}\},\\
\Phi^-_1 &= \{\mathtt{pbf}, \mathtt{tpbf}\},
\end{aligned}$$

after which a fixed point is reached. At the zeroth stage, only the links are explicitly constructed, and no paths are ruled out. At the next stage we consider all constructible paths, in this case all the length-two paths. Of these, `pbf` is constructible but conflicted in Φ^+_0 by the link $\mathtt{p\overline{f}}$. The other two constructible paths are `tpb` and $\mathtt{tp\overline{f}}$, and they are indeed explicitly constructed. To see this, first observe that neither is conflicted in Φ^+_0 (since there are no links `tf` or $\mathtt{t\overline{b}}$). Second, neither is conflicted in the complement of Φ^-_0. Indeed, $\Phi - \Phi^-_0 = \Phi$ given that Φ^-_0 is empty. Since a (positive) path can only be pre-empted if its final node has an incoming negative link, `tpb` cannot be pre-empted. Moreover, for $\mathtt{tp\overline{f}}$ to be pre-empted, one of two cases must occur: either the net contains a link $\mathtt{t} \to \mathtt{f}$ or there is a node x such that the net contains a link $x \to \mathtt{f}$ and x occurs on a path σ in Φ such that $\sigma^{\mathrm{i}} = \mathtt{t}$ and $\sigma^{\mathrm{e}} = \mathtt{p}$. Since neither case obtains, $\mathtt{tp\overline{f}}$ cannot be pre-empted. Now consider Φ^-_1: it contains two paths, `pbf` and `tpbf`: of these, the first is conflicted by the link $\mathtt{p\overline{f}}$, whereas the second is pre-empted by the path `tpb`. At this point, all paths in the net have been considered, and the construction reaches a fixed point.

As a further example, consider the net of Fig. 4, which has no credulous extension. The iterative process of Theorem 16 yields an extension (Φ^+, Φ^-) in which neither path $\sigma_1 = a \to b \to c$ nor $\sigma_2 = a \to b \to c \to d \to e \to b$ is in Φ^+. Clearly σ_2 cannot be pre-empted or conflicted (since there is no negative link incident upon b), so *a fortiori* it cannot be conflicted in Φ^+; so σ_2 is not in Φ^-. Since σ_2 pre-empts σ_1, it follows that σ_1 is pre-empted by a path in $\Phi - \Phi^-$ (for the appropriate path set Φ), and so it is not in Φ^+. Obviously, σ_2 cannot be in Φ^+, since its initial segment σ_1 is not. Also, since the only path pre-empting σ_1 is not in Φ^+, σ_1 is not in Φ^-. It also follows that neither path σ_1 nor σ_2 is in Φ^-. In particular, we have that σ_1 is neither in Φ^+ nor in Φ^-, and this witnesses the "three-valued" character of this notion of extension.

In order to show that this approach yields the same results as the classical one in the case of acyclic networks, we need to use the relation $\prec$, whose definition is given below.

Definition 17. Let α and β be non-empty paths, with $\beta = x_0 \dots x_n$; we say that α is *below* β, written $\alpha \prec \beta$, if and only if $\alpha^{\mathrm{i}} = \beta^{\mathrm{i}}$, and for some $k < n$, $\alpha^{\mathrm{e}} = \beta_k$.

The above definition requires that if $\alpha \prec \beta$, then the final node of α occurs as a node of β in a position other than the last one (although it might be repeated and occur as the last node too). The intuition behind the definition of $\prec$ is that if $\alpha \prec \beta$, then α is a "potentially" pre-empting path for β (depending on whether the net contains a node v on α such that the link $v \not\to \beta^{\mathrm{e}}$ is in Γ). Notice that if α is a proper initial segment of β then $\alpha \prec \beta$.

Recall that a relation R is well-founded over a set X if there are no infinite descending R-chains in X, or, equivalently, if any non-empty subset of X contains an R-minimal element.

Lemma 18. *Let Γ be a set of links, and Φ a co-inductive set of paths over Γ. If Φ is cyclic then $\prec$ is non-well-founded on Φ.*

Proof. First observe that if σ is a cycle, then we have $\sigma \prec \sigma$, since $\sigma^{\mathrm{i}} = \sigma^{\mathrm{i}}$, and $\sigma^{\mathrm{e}} = \sigma_k$, for some $k < \ell(\sigma)$. Similarly, if a cycle σ is a subpath of a path in Φ, we have $\tau\sigma \in \Phi$ for some τ, whence $\tau\sigma \prec \tau\sigma$. □

Remark. Under certain conditions it is possible to reverse the implications of the above theorem. Suppose for instance that there is a $\prec$-loop $\sigma_1 \prec \cdots \prec \sigma_n \prec \sigma_1$, and there are no nodes x, y such that $x \not\to y$ is in Γ, and y occurs on σ_i (in particular, all paths σ_i are positive). Then we can obtain a cycle as follows: start with σ_n^{e}, which occurs on σ_1; "follow" σ_1 to its end node σ_1^{e}, which lies on σ_2; follow σ_2 to its end node, ...: eventually we reach σ_n, which we follow to its end node σ_n^{e}, where we started. Since there are no incident negative links, the sequence of nodes encountered is actually a path in Γ^*. This shows that if the non-well-foundedness of $\prec$ on Γ^* derives from a positive loop, then Γ cannot be acyclic. To make a similar point in a different way, if

Γ is a net and Φ is the set of all *trails* over Γ, then $\prec$ is well-founded on Φ if and only if Γ is acyclic.

The following fact is easily established (and its proof omitted); it can be taken as further motivation for our choice of $\Gamma^{\#}$.

Lemma 19. *Suppose Γ is acyclic; then $\Gamma^* = \Gamma^{\#}$.*

Next, we show that our notion of extension agrees with that of credulous extension in the case of acyclic graphs. The relation $\prec$ will play a crucial role in establishing this. The notion of extension introduced below is classical in that it is the analogue of "two-valued", whereas extensions in general are "three-valued".

Definition 20. Let (Φ^+, Φ^-) be an extension for Φ; then (Φ^+, Φ^-) is *classical* if and only if $\Phi = \Phi^+ \cup \Phi^-$.

When (Φ^+, Φ^-) is a classical extension then $\Phi^+ = \Phi - \Phi^-$, and the two fixpoint equations defining extensions collapse into one. It follows that Φ^+ is a credulous extension of Γ in the sense of Horty [7], i.e., Φ^+ is the set of all paths that are constructible in Φ^+, but neither conflicted nor pre-empted in Φ^+.

Theorem 21. *Let Γ be acyclic and (Φ^+, Φ^-) be an extension for $\Gamma^{\#}$ (which, in this case, $= \Gamma^*$). Then (Φ^+, Φ^-) is classical.*

Proof. We need to show that $\Gamma^{\#} = \Phi^+ \cup \Phi^-$. Since Φ^+ and Φ^- are subsets of $\Gamma^{\#}$, it suffices to show that $\Gamma^{\#} - \Phi^- \subseteq \Phi^+$.

So assume that $\sigma \in \Gamma^{\#} - \Phi^-$, to show that $\sigma \in \Phi^+$. We proceed by induction on $\prec$. Suppose that $\sigma \in \Gamma^{\#} - \Phi^-$, and assume that the property holds for any $\rho \prec \sigma$.

If any proper initial segment ρ of σ is in Φ^- then so is σ; consequently all proper initial segments of σ are in $\Gamma^{\#} - \Phi^-$. Since any initial segments of σ are $\prec$-below σ, it follows by the inductive hypothesis that all initial segments are in Φ^+. Hence, σ is constructible in Φ^+. Moreover, if σ were conflicted in Φ^+ then $\sigma \in \Phi^-$, against the hypothesis.

It remains to show that σ is not pre-empted in $\Gamma^{\#} - \Phi^-$. So assume by way of contradiction that σ is pre-empted in $\Gamma^{\#} - \Phi^-$, and let ρ be the pre-empting path. Then $\rho \prec \sigma$, and by the inductive hypothesis $\rho \in \Phi^+$. This makes σ pre-empted in Φ^+, whence $\sigma \in \Phi^-$, against the hypothesis.

Since σ is constructible in Φ^+, but neither conflicted in Φ^+ nor pre-empted in $\Gamma^{\#} - \Phi^-$, it follows $\sigma \in \Phi^+$, as required. $\square$

4. Non-well-founded networks

In Section 3 we have shown how to obtain extensions of networks that may contain cycles, and that these extensions coincide with the credulous extensions in the case of acyclic nets. In this section we explore these extension a little more closely and show that a network can have extensions that are pointwise $\subseteq$-smaller than one another. In contrast, acyclic networks can only have credulous extensions that are $\subseteq$-incomparable: the proof of this fact in [7] crucially employs the hypothesis of acyclicity.

Definition 22. Let Φ be a finite set of paths; the *well-founded part* of Φ, $\mathrm{WF}(\Phi)$, is defined as follows:

$$\Psi_0 = \emptyset;$$

$$\Psi_{n+1} = \{\sigma \in \Phi\colon\ (\forall \rho \in \Phi)[\rho \prec \sigma \Rightarrow \rho \in \Psi_n]\};$$

$$\mathrm{WF}(\Phi) = \bigcup_{n \in \mathbb{N}} \Psi_n.$$

Of course, if Φ contains k paths, then $\mathrm{WF}(\Phi) = \bigcup_{n \leqslant k} \Psi_n$.

Lemma 23. *Let Φ be a finite set of paths; then $\prec$ is well-founded on Φ if and only if $\Phi = \mathrm{WF}(\Phi)$.*

Proof. In one direction, observe that if $\rho \in \Psi_n$, then any descending $\prec$-chain from ρ is of length at most n. Consequently, if $\prec$ is not well-founded on Φ there is a path that is never put in Ψ_n, because not all of its $\prec$-predecessors are in Ψ_n. Conversely, if $\prec$ is well-founded on Φ then any descending $\prec$-chains from ρ are finite and without repetitions; moreover, since Φ is finite the length of such chains is bounded by some n. Hence, $\rho \in \Psi_{n+1}$. $\square$

If $\prec$ is well-founded on Φ we shall also say that Φ itself is well-founded. Similarly, if $\prec$ is not well-founded on Φ, we say that Φ itself is non-well-founded. The following theorem shows that it is the well-foundedness of $\prec$ on acyclic nets that makes the difference.

Theorem 24. *Let Φ be a finite set of paths with the property that if $\rho \in \Phi$ is conflicted or pre-empted by $\sigma \in \Phi$, then $\rho \in \mathrm{WF}(\Phi)$ if and only if $\sigma \in \mathrm{WF}(\Phi)$. Let (Φ^+, Φ^-) be an extension for Φ, and put $\Psi^+ = \Phi^+ \cap \mathrm{WF}(\Phi)$, and similarly $\Psi^- = \Phi^- \cap \mathrm{WF}(\Phi)$. Then (Ψ^+, Ψ^-) is a classical extension for* $\mathrm{WF}(\Phi)$.

Proof (*Sketch*). Using the hypothesis on Φ it is immediate to verify that (Φ^+, Φ^-) is an extension for $\mathrm{WF}(\Phi)$. Arguing as in the proof for acyclic Γ, using the fact that if a path is in $\mathrm{WF}(\Phi)$ then so are its initial segments, it is possible to establish by $\prec$-induction that $(\mathrm{WF}(\Phi) - \Psi^-) \subseteq \Psi^+$. $\square$

The extensions we have considered so far are all *minimal*, in the sense that of any two, neither one is extended by the other. We now show that there are non-minimal extensions. First, we want to look a little closer at the ways in which a net might fail to be well-founded. When both Γ and Γ^* are finite, $\prec$ can be non-well-founded only if there are $\prec$-loops. The notion of $\prec$-loop is defined below.

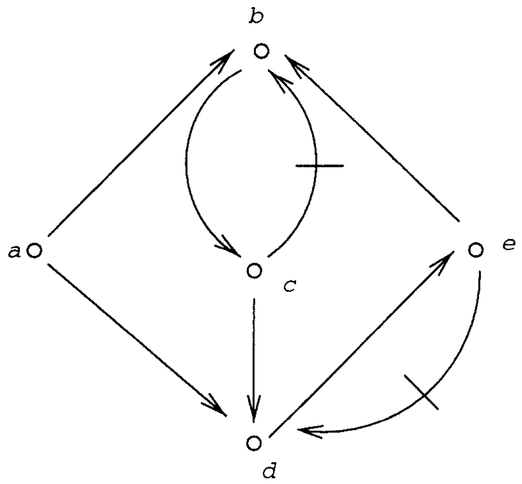

Fig. 6. A loop of two paths.

Definition 25. A finite set of paths $\{\sigma_1, \ldots, \sigma_n\}$ is a *$\prec$-loop* if $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_n \prec \sigma_1$. A $\prec$-loop of cardinality n is called an n-loop.

Definition 26. Let Φ be a set of paths; then $\mathsf{C}(\Phi)$ is the *closure* of Φ under the "initial segment" relation, i.e., the smallest set of paths having Φ as a subset and containing the initial segments of any path in it. Of course, by definition, $\mathsf{C}(\Phi)$ is co-inductive.

Definition 27. Let x and y be nodes occurring on a path σ; we say that x occurs in σ *properly after* y if σ has the form $\alpha y \beta x \gamma$, for some non-empty β.

The path $\sigma = abcdeb$ of Fig. 4 is a loop of cardinality 1, since σ is incident on a node of σ in a position other than the last one. Another loop is given by the two paths $abcd$ and $adeb$ in Fig. 6, since each one of them has the same initial node as the other, but its end node lies properly on the other (i.e., lies on the other but it is not the last node occurrence).

The following definition singles out certain pairs of sets of paths as being *sound*. Sound pairs can be used to construct non-minimal extensions.

Definition 28. Let Φ be a set of paths closed under initial segments, and $\Psi^+, \Psi^- \subseteq \Phi$; then (Ψ^+, Ψ^-) is *sound* for Φ if the following conditions all hold:

(i) for every path $\sigma \in \Psi^+$:
  (a) σ is constructible in Ψ^+;
  (b) σ not conflicted in $\Psi^+ \cup \Gamma_\Phi$, where Γ_Φ is the set of links in Φ;
  (c) σ is not pre-empted in $\Phi - \Psi^-$;

(ii) every path in Φ^- has a prefix which is either conflicted or pre-empted in Ψ^+.

The point of the above definition is that if (Ψ^+, Ψ^-) is sound then it can be used as a starting point in the construction of an extension for Φ in such a way that no paths are lost at the next iteration. In other words, if (Υ^+, Υ^-) is obtained from (Ψ^+, Ψ^-) in the same way as (Φ_1^+, Φ_1^-) is obtained from (Φ_0^+, Φ_0^-) in Theorem 16, then $\Psi^+ \subseteq \Upsilon^+$ and $\Psi^- \subseteq \Upsilon^-$.

Theorem 29. *Let (Ψ^+, Ψ^-) be sound for a co-inductive set Φ of paths; then there is an extension for Φ extending (Ψ^+, Ψ^-).*

Proof (*Sketch*). Let (Φ_n^+, Φ_n^-) be a sequence generated as in Theorem 16, except that (Ψ^+, Ψ^-) is used as a starting point, i.e.:

$$(\Phi_0^+, \Phi_0^-) = (\Psi^+, \Psi^-).$$

Show that the sequence is increasing by induction on n. The case of $n = 0$ follows immediately from the assumptions, and the case of $n > 0$ is established similarly to the corresponding case of Theorem 16. Let (Φ^+, Φ^-) be the limit of this sequence, and show that (Φ^+, Φ^-) is an extension. $\square$

Consider the net Γ of Fig. 6, together with the corresponding $\Gamma^\#$. The paths *abcd* and *adeb* pre-empt each other (relative to the net given in the figure). In particular, *abcd* pre-empts *adeb*, since (1) it has the same initial node as the latter, (2) is incident on a node of the latter, viz., d, occupying a non-final position, and (3) there is a negative link $c \not\rightarrow b$. Similarly, *adeb* pre-empts *abcd*, since (1) it has the same initial node, (2) it is incident on a node of the latter, viz., b, occupying a non-final position, and (3) there is a negative link $e \not\rightarrow d$. Now let (Ψ^+, Ψ^-) be a pair of sets of paths over the given net, where Ψ^+ contains *abcd* and its initial segments and Ψ^- contains all paths from $\Gamma^\#$ having *adeb* as a prefix. Then (Ψ^+, Ψ^-) is a sound starting point relative to $\Gamma^\#$. Therefore, there is an extension that explicitly constructs the paths in Ψ^+ and explicitly pre-empts the paths in Ψ^-, and such an extension is non-minimal (neither *abcd* nor *adeb* is in the well-founded part of the net, so neither one is in the least extension). This gives rise to a taxonomy of these loops.

Definition 30. An n-loop $\Phi = \{\sigma_1, \ldots, \sigma_n\}$ is *complete* with respect to Γ if and only if the following conditions hold:

(1) no σ in $\mathsf{C}(\Phi)$ is conflicted in $\mathsf{C}(\Phi)$;
(2) for any i such that $0 < i < n$, σ_i is the unique path in $\mathsf{C}(\Phi)$ pre-empting an initial segment (not necessarily proper) of σ_{i+1};
(3) σ_n is the unique path in $\mathsf{C}(\Phi)$ pre-empting an initial segment of σ_1.

In other words, a loop $\{\sigma_1, \ldots, \sigma_n\}$, with $n > 1$, is complete if for every i $(0 < i < n)$ there are nodes $x_i \in \sigma_i$ and $y_i \in \sigma_{i+1}$ such that y_i occurs in σ_{i+1} properly after σ_i^{e}; and $x_i \not\rightarrow y_i$ is in Γ, if σ_{i+1} is positive, and $x_i \rightarrow y_i$ is in Γ otherwise (and similarly for σ_n and σ_1). In the special case where $n = 1$ we say that the loop $\{\sigma_1\}$ is complete if and only if there is $\tau \sqsubset \sigma_1$ such that σ_1 pre-empts τ in Γ, i.e., if and only if $\sigma_1 \prec \tau$ and for some node $x \in \sigma_1$ the link $x \not\rightarrow \tau^{\mathrm{e}}$ is in Γ.

Consider again the paths *abcd* and *adeb* of Fig. 6. As we observed, they form a complete loop, since each one of them pre-empts the other relative to the given net.

Observe also that *abcde* and *adebc* form a loop as well, but one in which the uniqueness condition fails, since both *abcde* and *abcd* pre-empt *adeb*.

There is a difference between complete $(2n)$-loops and complete $(2n+1)$-loops: the former, but not the latter, can be consistently partitioned in two sets Φ^+ and Φ^- that are included in some extension (for a co-inductive set containing the loop as a subset), provided no pre-emption relations hold other than the ones explicitly mentioned in the definition of a complete loop. The point is that with $(2n)$-loops we can pick a path σ_i as a member of Φ^+: this in turn will force us to put σ_{i+1} and σ_{i-1} in Φ^-; in turn, we will have to put σ_{i+2} and σ_{i-2} in Φ^+; and so on. At the end these choices will fit in together. In the case of a $(2n+1)$-loop, however, we will find ourselves having to put the same path in Φ^+ and in Φ^-, which is of course impossible. Thus, $(2n+1)$-loops cannot be partitioned by any extension. If we want to pursue the analogy with the theory of truth, we can say that $(2n+1)$-loops behave like "liars" whereas $(2n)$-loops behave like "truth-tellers".

Theorem 31. *Let Γ be a net and $\Phi = \{\sigma_1, \ldots, \sigma_{2n}\}$ a complete $(2n)$-loop over Γ. Then there is a classical extension for $\mathsf{C}(\Phi)$.*

Proof. Let $\Psi^\clubsuit = \{\sigma_{2i+1}\colon i < n\}$ and $\Psi^\spadesuit = \{\sigma_{2i+2}\colon i < n\}$. Now define a starting point (Φ_0^+, Φ_0^-) by setting, say, $\Phi_0^+ = \mathsf{C}(\Psi^\clubsuit)$ and $\Phi_0^- = \{\tau\colon \tau \text{ extends a path in } \Psi^\spadesuit\}$. (Here $\Psi^\spadesuit$ and $\Psi^\clubsuit$ could have been switched.)

We check that (Φ_0^+, Φ_0^-) is sound for $\mathsf{C}(\Phi)$, in the sense of Definition 28. Let $\tau \in \Phi_0^+$, and say that τ is an initial segment of σ_{2i+1}. Obviously, τ is constructible in Φ_0^+. Since Φ is complete, τ is not conflicted. It remains to show that τ is not pre-empted in $\Phi - \Phi_0^-$.

Assume by way of contradiction that τ is pre-empted in $\Phi - \Phi_0^-$. If $i > 0$, then the pre-empting path must be σ_{2i}, and if $i = 0$ then the pre-empting path is σ_{2n}. In either case the pre-empting path is not in $\Phi - \Phi_0^-$.

Now let $\tau \in \Phi_0^-$; then τ extends σ_{2i+2} for some $i < n$. We need to show that τ has a prefix that is either conflicted or pre-empted in Φ_0^+. But this follows immediately, since τ extends σ_{2i+2}, which is pre-empted by $\sigma_{2i+1} \in \Phi_0^+$.

So (Φ_0^+, Φ_0^-) is sound for $\mathsf{C}(\Phi)$. By Theorem 29 there is an extension (Φ^+, Φ^-) extending (Φ_0^+, Φ_0^-). We need to check that such an extension is classical for $\mathsf{C}(\Phi)$.

Let τ be a path in $\mathsf{C}(\Phi)$ but not in Φ^-. If τ is in Φ then $\tau = \sigma_k$ for some positive $k \leqslant 2n$, and given that τ is not in Φ^- then k must be odd, so that $\tau \in \Phi_0^+ \subseteq \Phi^+$. If τ is not in Φ, it is a proper initial segment of a path in Φ, and we further distinguish the following cases:

(i) $\tau \sqsubset \sigma_{2i+1}$ for some $i < n$. Then again $\tau \in \Phi^+$ by construction.

(ii) $\tau \sqsubset \sigma_{2i+2}$ for some $i < n$. We show by induction on the length that any $\tau \sqsubset \sigma_{2i+2}$ such that $\tau \notin \Phi^-$ is in Φ^+. Since (Φ^+, Φ^-) is an extension, all links are in Φ^+, which takes care of the base case. For the inductive step: if some prefix of τ were in Φ^-, then so would τ; it follows that every prefix of τ is outside Φ^- and hence by the inductive hypothesis in Φ^+; so τ is constructible in Φ^+. Moreover, because of the completeness hypothesis, τ cannot be conflicted in Φ^+. Finally, if τ were pre-empted, given the completeness hypothesis and the fact that $\tau \sqsubset \sigma_{2i+2}$, it would have to be pre-empted by $\sigma_{2i+1} \in \Phi^+$, which would put τ in Φ^-, against assumption.

Thus, τ is constructible in Φ^+, but neither conflicted in Φ^+ nor pre-empted in $\mathsf{C}(\Phi) - \Phi^-$. Since (Φ^+, Φ^-) is an extension, $\tau \in \Phi^+$, as required. □

On the other hand, there is no way to partition a complete $(2n+1)$-loop in two subsets, each one containing all and only the paths that pre-empt an initial segment of a path in the other one. So we cannot obtain a sound starting point for a $(2n+1)$-loop. Moreover, such a loop, as already mentioned, cannot intersect either Φ^+ or Φ^- if (Φ^+, Φ^-) is to be an extension: if some σ_i belongs to Φ^+ or Φ^-, then σ_{i+1} must be in Φ^- or Φ^+, respectively. Eventually, we come back full circle, having to put σ_i in Φ^- or Φ^+, respectively, which contradicts the assumption that (Φ^+, Φ^-) is an extension.

This is the reason why Horty's network in Fig. 4 has no credulous extensions. On the other hand, the 2-loop in Fig. 6 has three extensions: a minimal one in which neither path is constructed, and two non-minimal ones in which one path is constructed and the other pre-empted.

Appendix A. Related issues

At several points in this paper we have used the hypothesis that the sets of paths we deal with are finite. In this appendix we take up the problem of how to extend the present approach to infinite sets of paths. Such an extension, although not of immediate interest for the purposes of implementation, bears some mathematical interest. The treatment in this appendix is meant more as an indication of future research than as a report on acquired results, and is therefore somewhat more informal than the preceding.

Infinite sets of paths can arise in a number of ways: for instance, if the net Γ contains infinitely many links, then obviously Γ^*, the set of all paths over Γ, will not be finite, independently of whether Γ itself contains cycles or not. Alternatively, if Γ contains cycles, then again Γ^* will not be finite, independently of whether Γ contains infinitely many links. This is why in the above treatment of inheritance over finite but cyclic nets we had to isolate a finite subset $\Gamma^\#$ of Γ^*, which is still, in some sense, representative of all of Γ^*.

In what follows we sketch the beginnings of a possible treatment of inheritance using infinite sets of paths. The hypothesis that Φ is finite is used in the proof of Theorem 16 and in the definition of $\mathrm{WF}(\Phi)$ (see Definition 22). Accordingly, we indicate how to reformulate theorem and definition to take into account the possibility that Φ might now be infinite.

Given a set of paths Φ and a path $\sigma \in \Phi$, we define the rank of σ relative to Φ as follows:

$$\mathrm{rk}_\Phi(\sigma) = \sup\{\mathrm{rk}_\Phi(\rho) + 1\colon\ \rho \in \Phi \text{ and } \rho \prec \sigma\},$$

if such a sup exists, and $\mathrm{rk}_\Phi(\sigma) = \infty$ otherwise (in which case we say that $\mathrm{rk}_\Phi(\sigma)$ is undefined).

Notice that if Φ is well-founded then $\mathrm{rk}_{\Phi}(\sigma)$ exists and is defined for every $\sigma \in \Phi$, whereas if Φ is not well-founded we will have $\mathrm{rk}_{\Phi}(\sigma) = \infty$ for some $\sigma \in \Phi$. We will show that indeed there are nets containing paths of transfinite rank.

We begin by defining $\Phi_n = \{\sigma_m\colon m \leqslant n\}$, where $\sigma_m = x_1, \ldots, x_m$. Then $\mathrm{rk}_{\Phi_n}(\sigma_n) = n$ for every n. Next, we remark that for any set of paths Φ it is possible to obtain a set of paths Ψ disjoint from, but topologically equivalent to Φ, simply by renaming its nodes. So for each n, let Ψ_n be a copy of Φ_n, but with the property that Ψ_n has no nodes in common with any Ψ_m, for $m < n$. Each Ψ_n contains a path σ_n such that $\mathrm{rk}_{\Psi_n}(\sigma_n) = n$.

Now put $\Psi = \bigcup_{n \geqslant 0} \Psi_n$. By the disjointness hypothesis, it follows that also $\mathrm{rk}_{\Psi}(\sigma_n) = n$. Now pick nodes a, b, c not occurring in Ψ, and for each n create links $a \to \sigma_n^{\mathrm{i}}$ and $\sigma_n^{\mathrm{e}} \to b$, and refer to the path $a\sigma_n b$ as τ_n.

Then each τ_n has rank $= n$ relative to $\Psi \cup \{\tau_n\colon n \geqslant 0\}$, and moreover all the paths τ_n have the same endpoints. If we now put $\tau = abc$ and $\Omega = \Psi \cup \{\tau_n\colon n \geqslant 0\} \cup \{\tau\}$, we have $\mathrm{rk}_{\Omega}(\tau) = \omega$. Notice that Ω is well-founded.

Now that we know that there are paths with infinite rank, we can extend the definition of $\mathrm{WF}(\Phi)$ to infinite sets of paths. On such sets, in general we can still define WF, but we will have to iterate the definition up the ordinal

$$\alpha^{\Phi} = \sup\{\mathrm{rk}_{\Phi}(\sigma)\colon\ \sigma \in \Phi \text{ and } \mathrm{rk}_{\Phi}(\sigma) \neq \infty\}.$$

(It's easy to see that such an ordinal will not only be countable, as is obvious, but since $\prec$ is definable in first-order arithmetic, it will be bounded by ϵ_0. The reader is referred to Takeuti [20] for the details.)

In order to extend the definition of $\mathrm{WF}(\Phi)$ (Definition 22) to infinite sets of paths, we add a clause

$$\Psi_{\lambda} = \bigcup_{\alpha<\lambda} \Psi_{\alpha},$$

for λ a limit ordinal, and define $\mathrm{WF}(\Phi)$ by taking the union over α^{Φ}. Then Lemma 23 goes through without the hypothesis on Φ.

All is left to show is how to modify the proof of Theorem 16 to allow for infinite sets of paths. First of all we extend the construction into the transfinite, by taking unions at limit stages, i.e., we set, for λ a limit ordinal:

$$\Phi_{\lambda}^{+} = \bigcup_{\alpha<\lambda} \Phi_{\alpha}^{+},$$

and

$$\Phi_{\lambda}^{-} = \bigcup_{\alpha<\lambda} \Phi_{\alpha}^{-}.$$

We carry out the construction up to stage α^{Φ}, i.e., we set

$$\Phi^{+} = \bigcup_{\alpha<\alpha^{\Phi}} \Phi_{\alpha}^{+},$$

and similarly for Φ^{-}.

The argument to the effect that the sequence of sets of paths obtained in this way is increasing goes through as before. In the proof of Theorem 16 the hypothesis of finiteness is used in showing that if σ is not pre-empted in $\Phi - \Phi^-$ then it cannot be pre-empted in any $\Phi - \Phi_n^-$ for any n.

In the present context, in order to carry out the proof that if σ is not pre-empted in $\Phi - \Phi^-$ then it cannot be pre-empted in any $\Phi - \Phi_\alpha^-$ for any α, we proceed as follows. First we show that if $\max(\ell(\sigma), \mathrm{rk}_\Phi(\sigma)) = \alpha$ then $\sigma \in \Phi^+$ if and only if $\sigma \in \Phi_\alpha^+$, and similarly for Φ^-: this can be shown by induction on α or, which comes to the same thing, by $\prec$-induction. It follows that if σ were pre-empted by ρ in $\Phi - \Phi_\alpha^-$—where $\alpha = \max(\ell(\sigma), \mathrm{rk}_\Phi(\sigma))$ as above—then it would be pre-empted in $\Phi - \Phi^-$ as well, as desired.

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References

[1] F. Baader and B. Hollunder, Computing extensions of terminological default theories, in: G. Lakemeyer and B. Nebel, eds., *Foundations of Knowledge Representation and Reasoning*, Lecture Notes in Computer Science **810** (Springer, Berlin, 1994).

[2] C. Boutilier, On the semantics of stable inheritance reasoning, Tech. Rept. KRR-TR-89-11, Department of Computer Science, University of Toronto, Toronto, Ont. (1989); also in: *Proceedings IJCAI-89*, Detroit, MI (1989).

[3] D. Etherington, Formalizing nonmonotonic reasoning systems, *Artif. Intell.* **31** (1987) 41–85.

[4] D. Etherington and R. Reiter, On inheritance hierarchies with exceptions, in: *Proceedings AAAI-83*, Washington, DC (1983) 104–108.

[5] A. Gupta and N.D. Belnap, *The Revision Theory of Truth* (MIT Press, Cambridge, MA, 1993).

[6] P.J. Hayes, The logic of frames, in: D. Metzing, ed., *Frame Conceptions and Text Understanding* (Walter de Gruyter, Berlin, 1979) 46–61; reprinted in: R.J. Brachman and H.J. Levesque, eds., *Readings in Knowledge Representation* (Morgan Kaufmann, Los Altos, CA, 1985).

[7] J.F. Horty, Some direct theories of nonmonotonic inheritance, in: D.M. Gabbay, C.J. Hogger and J.A. Robinson, eds., *Handbook of Logic in Artificial Intelligence and Logic Programming* **3**: *Nonmonotonic Reasoning and Uncertain Reasoning* (Oxford University Press, Oxford, 1994).

[8] J.F. Horty, R.H. Thomason and D.S. Touretzky, A skeptical theory of inheritance in nonmonotonic semantic networks, *Artif. Intell.* **42** (1990) 311–348.

[9] S.C. Kleene, *Introduction to Metamathematics* (Van Nostrand Reinhold, Princeton, NJ, 1952).

[10] S. Kripke, Outline of a theory of truth, *J. Philos.* **72** (1975) 690–716; now reprinted in: R. Martin, ed., *Recent Essays on Truth and the Liar Paradox* (Clarendon Press, Oxford, 1984) 53–81.

[11] D. Makinson and K. Schlechta, Floating conclusions and zombie paths: two deep difficulties in the "directly skeptical" approach to defeasible inheritance networks, *Artif. Intell.* **48** (1991) 199–209.

[12] B. Nebel, *Reasoning and Revision in Hybrid Representation Systems*, Lecture Notes in Artificial Intelligence **422** (Springer, Berlin, 1990).

[13] C.H. Papadimitriou and M. Sideri, Default theories that always have extensions, *Artif. Intell.* **69** (1994) 347–357.

[14] R. Reiter, A logic for default reasoning, *Artif. Intell.* **13** (1980) 81–132.

[15] E. Sandewall, Non-monotonic inference rules for multiple inheritance with exceptions, *Proc. IEEE* **74** (1986) 1345–1353.

[16] K. Schlechta, Directly skeptical inheritance cannot capture the intersection of extensions, *J. Logic Comput.* **3** (1993) 455–467.

[17] G. Simonet and R. Ducournau, On Stein's paper: resolving ambiguity in nonmonotonic inheritance hierarchies, *Artif. Intell.* **71** (1994) 183–193.

[18] L. Stein, Resolving ambiguity in nonmonotonic reasoning, PhD Thesis, Computer Science Department, Brown University (1990).

[19] L. Stein, Resolving ambiguity in nonmonotonic inheritance hierarchies, *Artif. Intell.* **55** (1992) 259–310.

[20] G. Takeuti, *Proof Theory* (North-Holland, Amsterdam, 2nd ed., 1987).

[21] R.H. Thomason, NETL and subsequent path-based inheritance theories, *Comput. Math. Appl.* **23** (1992) 179–204.

[22] R.H. Thomason, J.F. Horty and D.S. Touretzky, A calculus for inheritance in monotonic semantic nets, in: Z.W. Ras and M. Zemankove, eds., *Methodologies for Intelligent Systems* (Elsevier, Amsterdam, 1987) 280–287.

[23] D.S. Touretzky, *The Mathematics of Inheritance Systems* (Morgan Kaufmann, Los Altos, CA, 1986).