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Generalized Region Connection Calculus ☆

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Abstract

The Region Connection Calculus (RCC) is one of the most widely referenced system of high-level (qualitative) spatial reasoning. RCC assumes a continuous representation of space. This contrasts sharply with the fact that spatial information obtained from physical recording devices is nowadays invariably digital in form and therefore implicitly uses a discrete representation of space. Recently, Galton developed a theory of discrete space that parallels RCC, but question still lies in that can we have a theory of qualitative spatial reasoning admitting models of discrete spaces as well as continuous spaces? In this paper we aim at establishing a formal theory which accommodates both discrete and continuous spatial information, and a generalization of Region Connection Calculus is introduced. GRCC, the new theory, takes two primitives: the mereological notion of part and the topological notion of connection. RCC and Galton's theory for discrete space are both extensions of GRCC. The relation between continuous models and discrete ones is also clarified by introducing some operations on models of GRCC. In particular, we propose a general approach for constructing countable RCC models as direct limits of collections of finite models. Compared with standard RCC models given rise from regular connected spaces, these countable models have the nice property that each region can be constructed in finite steps from basic regions. Two interesting countable RCC models are also given: one is a minimal RCC model, the other is a countable sub-model of the continuous space $\mathbb{R}^2$.

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1. Introduction

Qualitative Spatial Reasoning (QSR) is an important subfield of AI which has applications in areas such as Geographical Information Systems(GIS) [4,24,52], spatial query languages [10], natural languages [1] and many other fields. We invite the reader to consult [11] for an introduction and an overview of current trends.

This paper focus on one of the most important formalism for QSR, viz. the *Region Connection Calculus* (RCC). RCC was initially described by Randell, Cohn and Cui in [31, 32], which is intended to provide a logical framework for incorporating spatial reasoning into AI systems.

RCC takes regions rather than points as fundamental notion and is based on a single primitive (binary relation) **C** (for *connection*). Unlike other mereotopologies, RCC makes no distinction between closed, open, and semi-open regions and does not support the notion of boundaries. It is also a well-known result that regular connected topological spaces provide models of the RCC axioms by taking a region to mean a non-empty regular closed set and saying two regions are connected if they have common points [25]. But since each region is infinitely divisible, it has nothing to do with discrete spaces (in the sense that each region is a union of atomic regions). Randell, Cui and Cohn [32] suggest ways of atomic versions of RCC, but, as commented by Bennett [3], each of the alternatives seem more complex than is desirable and have not been worked out in detail. They also suggest in that paper that the problem lies with the definition of **P**, but a revised definition was not given.

On the other hand, discrete spaces are evidently important in implementations of spatial information systems, and their mereotopological aspects have only recently begun to be investigated [22,24,30,39,44]. As noted by Galton in [24], high-level qualitative approaches to handling spatial information are widely perceived as having little relevance to the domain of low-level quantitative data inhabited by "real-world" applications, and the congruity between the continuous space models favored by high-level approaches and the discrete, digital representations used at the lower level is one amongst many possible reasons for this.

Recently, Galton [24] attempts to bridge this gap by developing a high-level qualitative spatial theory of discrete space that parallels RCC, but questions still lie in, e.g., "Can we have a high-level theory of QSR which admits models of continuous spaces as well as discrete spaces?" and "What is the relation between continuous spaces (e.g., $\mathbb{R}^2$) and discrete spaces (e.g., $\mathbb{Z}^2$)?". In the present paper we try to answer these questions by introducing a generalized theory of RCC. The new theory, termed *Generalized Region Connection Calculus* (GRCC henceforth), is a subtheory of both RCC[1] and Galton's theory for discrete space.

[1] We are concerned in this paper with only 'strict' RCC [45], i.e., the extensionality of **C** is an axiom.

The original formulation of RCC is inspired by the earlier work of Whitehead [48] and Clarke [7,8]. These systems are all based on a single primitive, namely the concept of connection, and notions such as part are defined in terms of connection. This use of a single primitive relation is seen by Smith as problematic. In [42, p. 288], Smith puts: "The system has a single primitive, that of connection, in terms of which the notion of part is defined by means of what, intuitively, appears to be a logical trick. This means that the mereological and topological components of the resultant theories are difficult or impossible to separate formally. The power of the approach is thus reduced, since experiments in axiom-adjustment at different points in the theory cannot be carried out in controlled fashion".

This possible deficiency of the RCC theory, however, as Stell [45], as well as this paper, has shown, can be completely avoided. In [45], Stell introduces *Boolean connection algebras* (BCAs) and proves that these structures are equivalent to models of the RCC axioms.[2] Such an algebra is able to provide a neat separation of mereological and topological aspects of a set of regions. Moreover, by replacing the Boolean algebra by Łukasiewicz algebra, Roy and Stell [38] obtain a theory of vague spatial regions. The present paper is also strongly influenced by Stell's idea on the treatment of spatial regions.

In the GRCC theory proposed in this paper, however, we use two primitive notions: the mereological notion of part **P** and the topological notion of connection **C**. This treatment is not novel, the reader may consult for instance the work of Varzi [47] and Mosolo and Vieu [30] for more discussion. In [47], Varzi systematically examines three main ways of combining mereologies (as theories of parthood) and topologies (as theories of wholeness) to build general mereotopologies, namely unified theories of parts and wholes. Both work, particularly that of Mosolo and Vieu [30], also investigate the possibilities of characterizing atomicity in these mereotopologies.

Our mereology of GRCC (as well as the reformulated RCC theory) falls under the first account of Varzi's classifications, where mereology and topology form two independent (though mutually related) domains. Indeed, the mereological part of GRCC is the same as the *Closed Extensional Mereology* (**CEM**) [47] (with the additional requirement that the universe exists). The GRCC theory is then obtained by adding three additional axioms to the *Ground Mereotopology* **CEMT** [47]. The first requires a region a is connected to the (mereological) sum of two regions b, c if and only if it is connected to either one of b, c; the second stipulates in essence the universe is self-connected; the third requires there exist at least two different regions. The original RCC theory is then obtained by adding to GRCC an additional axiom which requires that any region other than the universe cannot be connected to all regions. Moreover, the theory of Galton for discrete space [24] is also an extension of the (atomistic) GRCC theory. Indeed, Galton's theory is obtained by adding to GRCC three more axioms. The first two stipulate that the Boolean algebra is atomic complete and the third requires two regions are connected if and only if there are two connected atoms contained respectively in these two regions.

[2] The fact that each RCC model leads to a Boolean algebra is also pointed out independently by Düntsch, Wang and McCloskey [17].

Eschenbach [22] also introduces a formal theory of *Closed Region Calculus* (CRC) based on two primitives: **P** for mereological notion of part and **DC** for the topological notion of disconnection or separation. CRC is similar to RCC and the 9-intersection calculus [19]. It provides the same terminology and justifies the same composition table, but differs with respect to the ontology. The main difference between CRC and RCC is that a finite set of regions that is explicitly represented in a spatial information system can be a model of CRC [22]. Our theory of GRCC is also a generalization of CRC in essence, it differs from CRC mainly in two aspects. The first is that GRCC is a first order theory while CRC is second order. The second lies in that the two theories use different topological primitives and different systems of axioms.

The fact that GRCC admits both continuous spaces and discrete spaces as models makes it possible to study the relation between these two kinds of spaces carefully. This is one of the main intentions of this paper. To reveal the relation between continuous spaces and discrete spaces, in particular that between the vector space $\mathbb{R}^2$ and the raster space $\mathbb{Z}^2$, we propose several basic operations on models of GRCC. These include the *sub-structures*, the *local-structures*, the *sums* and the *direct limits*. It is shown that the raster space $\mathbb{Z}^2$ with either 4-neighbors or 8-neighbors, which is at the same time a model of Galton's theory, is a sub-structure of the vector space $\mathbb{R}^2$. Our notion of sub-structures has intimately connection with the formal framework proposed by Worboys [52] for treating the notion of resolution and multi-resolution in geographic spaces. The operation 'sum' on models of GRCC is in a sense an inverse process of granulation of graphs [44].

More importantly, the operation of direct limits can be applied to construct countable RCC models from collections of finite GRCC models. Compared with standard RCC models given rise from regular connected spaces [25] and more generally, RCC models constructed from (dual) pseudo-complemented distributive lattices [45], these countable models have the nice property that regions in which can be finitely represented as aggregations of basic regions. In particular, this approach enables us to construct an RCC model which is minimal in the sense that each RCC model contains it as a sub-model. Combining with regular spatial partitioning (of the vector space $\mathbb{R}^2$), we also give (as far as spatial representation is concerned) a computationally tractable RCC model $\langle \mathcal{R}, \mathcal{C} \rangle$.[3] This model is a countable sub-model of the vector space $\mathbb{R}^2$, and its topological structure coincides with that of $\mathbb{R}^2$, namely, two regions are connected if and only if they have non-empty intersection. With a little adaption, regions in this model can be represented as (binary) quadtrees. This fact, together with the 4-*intersection*-based method (concerning the RCC5 relations) adopted by Winter in [50], leads to a hierarchical approach for determining RCC8 relations between regions in this model.

The present paper has a close relation with the recent GIScience 2002 work of Roy and Stell [39], where the authors show how the RCC theory can be modified so as to permit discrete spaces. Their work is also based on Stell's formulation of RCC as Boolean connection algebra [45]. They obtain their concept of *connection algebra* by (1) replacing the Boolean algebra with a more general kind of lattice, here is the dual pseudo-complemented distrib-

[3] The phrase "computationally tractable" is in the same sense of Worboys and Bofakos [53], where the authors describe a model regions in which can be uniquely represented as combinations of a finite number of atoms.

utive lattice (dual p-algebra); and (2) abandoning the condition that requires non-universe region should be disconnected from some other region. Our notion of GBCA that would be given in Section 3.2 is a special case of the connection algebra, and by definition, a connection algebra is a GBCA if and only if the ground lattice is Boolean.

While (non-Boolean) connection algebras allow regions that are boundaries of other regions, GRCC and RCC make no distinction between closed, open, and semi-open regions and therefore do not support the notion of boundaries. Moreover, as far as discrete spaces are concerned, it seems to us exactly these non-Boolean connection algebras are what Roy and Stell focused on. As a matter of fact, they give concrete examples of these non-Boolean (finite) connection algebras based on abstract cell complexes. Note that an abstract cell complexes is in fact a partially ordered set with certain properties. Recall the *Alexandrov topology* [26, p. 45] on a partially ordered set $\langle P, \leqslant\rangle$ is just the collection of all upper sets in P. It is worth noting that any connected[4] partially ordered set $\langle P, \leqslant\rangle$ also leads to such a non-Boolean connection algebra $\langle A, \mathbf{C}\rangle$: taking A as the lattice of closed sets, namely lower sets, of P, then A clearly is a dual p-algebra; for two elements $a, b \in A$, define $\mathbf{C}(a, b)$ if and only if $a \cap b \neq \emptyset$.

The structure of the paper is as follows. In Section 2 we present the mereological part of GRCC, which is indeed the *Closed Extensional Mereology* (**CEM**) [47] (with the additional requirement that the universe exists). We also show this mereology is equivalent to the theory of Boolean algebras. The theory of GRCC is then introduced in Section 3, where we also show generalized Boolean connection algebras (GBCAs) provide models of GRCC. Similar to Stell's construction of BCAs [45], we also give an explicit construction of GBCAs. In addition, it is shown that the theory GRCC admits connected spaces (with more than two regular closed sets) as its models. Section 4 considers consistency of GRCC with atomicity and shows Galton's theory of discrete space [24] is indeed an extension of (atomistic) GRCC. In Section 5, we introduce some operations on models of GRCC—sub-structures, sums, local-structures and direct limits. The relationship between discrete models and continuous ones is illustrated by one example. This suggests that some important discrete models are sub-structures of continuous ones. We also propose a general approach for constructing countable RCC models as direct limits of collections of finite models. In particular a minimal RCC model is obtained. In Section 6 we construct a countable computationally tractable RCC model and sketch a hierarchical approach for determining RCC8 relations in this model. Section 7 concludes this paper and points out some problems for further studies.

2. Mereology of GRCC

In this section we introduce the mereological part of our GRCC, which is indeed the *Closed Extensional Mereology* (**CEM**) [47] (with the additional requirement that the universe exists). We also show this mereology is equivalent to the theory of Boolean algebras.

[4] In the sense that P with the Alexandrov topology is connected.

Mereology is a theory of the binary ‘part-of’ relation, originally introduced by Leśniewski (see [27]) as an alternative to set theory. Recently, it has been used both in formal ontology, to model the generic part-whole relation, and in QSR, to model spatial inclusion between regions [1,32]. We invite the readers to consult [30,41,47] for detailed examination of various mereologies and mereotopologies. Our notation is in accord with that of [47].

In the following we assume a standard first-order language with identity supplied with a distinguished binary predicate constant, ‘**P**’, to be interpreted as the (possibly improper) parthood relation.

Recall that *Ground Mereology* **M** [47] is the first-order theory defined by the following three axioms:

$$\mathbf{P}(x,x), \tag{P1}$$

$$(\mathbf{P}(x,y)\wedge\mathbf{P}(y,x))\to x=y, \tag{P2}$$

$$(\mathbf{P}(x,y)\wedge\mathbf{P}(y,z))\to\mathbf{P}(x,z). \tag{P3}$$

Given (P1)–(P3), a number of mereological relations can be introduced. In particular, **M** supports the following mereological relations of proper part (**PP**), overlap (**O**), proper or partial overlap (**PO**), and disjointness (**DR**),

$$\mathbf{PP}(x,y)\Leftrightarrow_{\mathrm{def}}\mathbf{P}(x,y)\wedge\neg\mathbf{P}(y,x),$$

$$\mathbf{O}(x,y)\Leftrightarrow_{\mathrm{def}}\exists z[\mathbf{P}(z,x)\wedge\mathbf{P}(z,y)],$$

$$\mathbf{PO}(x,y)\Leftrightarrow_{\mathrm{def}}\mathbf{O}(x,y)\wedge\neg\mathbf{P}(x,y)\wedge\neg\mathbf{P}(y,x),$$

$$\mathbf{DR}(x,y)\Leftrightarrow_{\mathrm{def}}\neg\mathbf{O}(x,y).$$

The *Extensional Mereology* **EM** [47] is the extension of **M** obtained by adding the Supplementation Axiom [41]:

$$\neg\mathbf{P}(x,y)\to\exists z(\mathbf{P}(z,x)\wedge\neg\mathbf{O}(z,y)). \tag{P4}$$

It is interesting to note that in **EM** the binary relation **O** is extensional:

$$\mathbf{EM}\vdash\forall z(\mathbf{O}(z,x)\leftrightarrow\mathbf{O}(z,y))\leftrightarrow x=y. \tag{Oext}$$

The *Closed Extensional Mereology* **CEM** is the extension of **EM** obtained by adding the following axioms:[5]

$$\exists z\forall w(\mathbf{O}(w,z)\leftrightarrow(\mathbf{O}(w,x)\vee\mathbf{O}(w,y))), \tag{P5}$$

$$\mathbf{O}(x,y)\to\exists z\forall w(\mathbf{P}(w,z)\leftrightarrow(\mathbf{P}(w,x)\wedge\mathbf{P}(w,y))), \tag{P6}$$

$$\exists z(\mathbf{P}(z,x)\wedge\neg\mathbf{O}(z,y))\to\exists z\forall w(\mathbf{P}(w,z)\leftrightarrow(\mathbf{P}(w,x)\wedge\neg\mathbf{O}(w,y))), \tag{P7}$$

$$\exists z\forall x\mathbf{P}(x,z). \tag{P8}$$

[5] Note that Axiom (P8) is optional in the definition of **CEM** given by Varzi [47].

Note that in the presence of (P4), the entities whose (conditional) existence is asserted by (P5)–(P7) are unique. Thus, if the language has a description operator 'ι', **CEM** supports the following definitions:

$$x + y =_{\text{def}} \iota z \forall w(\mathbf{O}(w, z) \leftrightarrow (\mathbf{O}(w, x) \vee \mathbf{O}(w, y))) \quad \text{(sum)} \quad \text{(Dsum)}$$
$$x \times y =_{\text{def}} \iota z \forall w(\mathbf{P}(w, z) \leftrightarrow (\mathbf{P}(w, x) \wedge \mathbf{P}(w, y))) \quad \text{(product)} \quad \text{(Dprod)}$$
$$x - y =_{\text{def}} \iota z \forall w(\mathbf{P}(w, z) \leftrightarrow (\mathbf{P}(w, x) \wedge \neg\mathbf{O}(w, y))) \quad \text{(difference)} \quad \text{(Ddiff)}$$
$$\sim x =_{\text{def}} \iota z \forall w(\mathbf{P}(w, z) \leftrightarrow \neg\mathbf{O}(w, x)) \quad \text{(complement)} \quad \text{(Dcomp)}$$
$$\mathrm{u} =_{\text{def}} \iota z \forall x \mathbf{P}(x, z). \quad \text{(universe)} \quad \text{(Duni)}$$

We can also add infinitary closure conditions allowing the existence of sums and products of an infinite number of entities with the following 'fusion' axiom, where ϕ is a first-order formula and x is free in ϕ:

$$\exists x \phi(x) \to \exists z \forall w(\mathbf{O}(w, z) \leftrightarrow \exists x(\phi(x) \wedge \mathbf{O}(w, x))). \quad \text{(FUS)}$$

The theory of *General Extensional Mereology* **GEM** [47] is then the extension of **EM** obtained by adding the 'fusion' axiom. **GEM** is actually an extension of **CEM**.

The atomistic variant of **CEM** (**GEM**) (labelled **ACEM** (**AGEM**)) [47] is the extension of **CEM** by adding the axiom:

$$\forall x \exists y(\mathbf{P}(y, x) \wedge \exists z \neg\mathbf{PP}(z, y)). \quad \text{(P9)}$$

It is well known that every model of **(A)GEM** is isomorphic to an (atomic) complete quasi-Boolean algebra, i.e., a Boolean algebra with the zero element removed [46]. For **(A)CEM**, we have a similar result:

Theorem 2.1. *Every model of* **(A)CEM** *is isomorphic to an* (*atomic*) *quasi-Boolean algebra.*

Proof. Suppose $\langle R, \mathbf{P}\rangle$ is a model of **CEM** and $\{\mathrm{n}\}$ is a singleton set disjoint from R. Define binary operations $\vee$ and $\wedge$ on the set of $R \cup \{\mathrm{n}\}$ as follows:

$$x \vee y = \begin{cases} x + y & \text{if } x, y \in R, \\ x & \text{if } y = \mathrm{n}, \\ y & \text{if } x = \mathrm{n}, \end{cases} \qquad x \wedge y = \begin{cases} x \times y & \text{if } x, y \in R \text{ and } \mathbf{O}(x, y), \\ \mathrm{n} & \text{otherwise.} \end{cases}$$

Also, we can define the unary operation $'$ on the set $R \cup \{\mathrm{n}\}$ by $x' = \sim x$ for $x \in R - \{\mathrm{u}\}$, and by $\mathrm{u}' = \mathrm{n}$, and $\mathrm{n}' = \mathrm{u}$. Recall where u is the universe in R.

For $x, y \in R \cup \{\mathrm{n}\}$, we say $x \leqslant y$ if either $\mathbf{P}(x, y)$ or $x = \mathrm{n}$.

Now we are going to show that the structure $\langle R \cup \{\mathrm{n}\}; \mathrm{u}, \mathrm{n}, ', \wedge, \vee\rangle$ is a Boolean algebra.

We first show the structure $\langle R \cup \{\mathrm{n}\}, \leqslant\rangle$ is a partially ordered set. This is clear by (P1)–(P3).

Second, we demonstrate that $\vee$ and $\wedge$ defined on $R \cup \{\mathrm{n}\}$ are the supremum and infimum operations on $\langle R \cup \{\mathrm{n}\}, \leqslant\rangle$ respectively. To this end, note that we have the following facts:

1. $\forall x, y, z \in R \cdot \mathbf{P}(x + y, z) \Leftrightarrow \mathbf{P}(x, z) \wedge \mathbf{P}(y, z)$,
2. $\forall x, y, z \in R \cdot \mathbf{O}(y, z) \wedge \mathbf{P}(x, y \times z) \Leftrightarrow \mathbf{P}(x, y) \wedge \mathbf{P}(x, z)$.

Next we prove the distributivity:

$$\forall x, y, z \in R \cup \{\mathrm{n}\} \cdot x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

We need only to show the "$\leqslant$"-part. Without any loss of generality, we assume $x \wedge (y \vee z) \neq \mathrm{n}$, i.e., $\mathbf{O}(x, y+z)$. Suppose $\mathbf{O}(m, x \wedge (y \vee z))$ for some $m \in R$, we show either $\mathbf{O}(m, x \wedge y)$ or $\mathbf{O}(m, x \wedge z)$ holds. By $\mathbf{O}(m, x \wedge (y \vee z))$, we have some $r \in R$ such that $r \leqslant m$ and $r \leqslant x \wedge (y \vee z)$, hence $r \leqslant m$, $r \leqslant x$ and $r \leqslant y \vee z$. Note that by $r \leqslant y \vee z$ we have $\mathbf{O}(r, y \vee z)$, hence either $\mathbf{O}(r, y)$ or $\mathbf{O}(r, z)$ holds. Then there exists some $p \in R$ such that $p \leqslant r$ and $p \leqslant y$ or $p \leqslant r$ and $p \leqslant z$, thereby $p \leqslant m$ and $p \leqslant x \wedge y$ or $p \leqslant m$ and $p \leqslant x \wedge z$. Hence by definition we have either $\mathbf{O}(m, x \wedge y)$ or $\mathbf{O}(m, x \wedge z)$.

Lastly we show $x \wedge x' = \mathrm{n}$, $x \vee x' = \mathrm{u}$ hold for any $x \in R \cup \{\mathrm{n}\}$. Note that if x is either u or n, these two conditions hold by definition. So we suppose $x \in R - \{\mathrm{u}\}$. Since $\mathbf{P}(x', x')$ is always true for each $x \in R - \{\mathrm{n}\}$, we have by (Dcomp) and (P7) $\neg\mathbf{O}(x', x)$, i.e., $x \wedge x' = \mathrm{n}$. Moreover, for each $r \in R$, we have either $\mathbf{P}(r, x')$ or $\neg\mathbf{P}(r, x')$. Note that the latter is the same as $\mathbf{O}(r, x)$ by (Dcomp) and (P7). Accordingly, by (Dsum) and (P5), we have $\mathbf{O}(r, x \vee x')$ for each $r \in R$, therefore $x \vee x' = \mathrm{u}$.

So far we have proved that $\langle R \cup \{\mathrm{n}\}; \mathrm{u}, \mathrm{n}, ', \wedge, \vee\rangle$ is a Boolean algebra for each model $\langle R, \mathbf{P}\rangle$. If $\langle R, \mathbf{P}\rangle$ satisfies moreover (P9), clearly the Boolean algebra $\langle R \cup \{\mathrm{n}\}; \mathrm{u}, \mathrm{n}, ', \wedge, \vee\rangle$ is atomic.

On the other hand, for any Boolean algebra $A = \langle A; \bot, \top, ', \wedge, \vee\rangle$, let $R = A - \{\bot\}$. For any two elements $x, y \in R$, write $\mathbf{P}(x, y)$ if and only if $x \leqslant y$ in A. Then it is straightforward to verify that the structure $\langle R, \mathbf{P}\rangle$ is a model of **CEM**. In case that A is atomic, clearly the **CEM** model $\langle R, \mathbf{P}\rangle$ satisfies (P9). □

So models of the **CEM** axioms are equivalent to quasi-Boolean algebras. Suppose $A = \langle A; \bot, \top, ', \wedge, \vee\rangle$ is a Boolean algebra. Let $R = A - \{\bot\}$ and write $\mathbf{P}(x, y)$ if $x \leqslant y$ in A for any two elements x, y in R. Then we have $\mathrm{u} = \top$, and $x \times y = x \wedge y$, $x - y = x \wedge y'$ for all $x, y \in R$, and $\sim x = x'$ for all $x \in R - \{\top\}$. Moreover, for all $x, y \in R$, we have

$$\begin{array}{lll}
\mathbf{P}(x, y) & \text{iff} & x \leqslant y, \\
\mathbf{PP}(x, y) & \text{iff} & x < y, \\
\mathbf{O}(x, y) & \text{iff} & x \wedge y > \bot, \\
\mathbf{DR}(x, y) & \text{iff} & x \wedge y = \bot, \\
\mathbf{PO}(x, y) & \text{iff} & x \wedge y > \bot \text{ and neither } x \leqslant y \text{ nor } x \geqslant y.
\end{array}$$

In what follows we shall, for brevity, make no distinction between notations in **CEM** and the corresponding notations in Boolean algebras, e.g., the pair $\mathbf{P}(x, y)$ and $x \leqslant y$ for $x, y \in R$, or the pair $\sim x$ and x' for $x \in R - \{\mathrm{u}\}$.

3. The generalized Region Connection Calculus

Theories combining mereological notions and topological ones like those of "being connected with" or "being self-connected" have been called mereotopologies. In this section, we expand our language by adding a second distinguished predicate constant, ‘**C**’, the connection. Based on **C**, we have the following definitions:

$$\mathbf{DC}(x, y) \Leftrightarrow_{\text{def}} \neg\mathbf{C}(x, y),$$

$$\mathbf{EC}(x, y) \Leftrightarrow_{\text{def}} \mathbf{C}(x, y) \wedge \neg\mathbf{O}(x, y),$$

$$\mathbf{TPP}(x, y) \Leftrightarrow_{\text{def}} \mathbf{PP}(x, y) \wedge \exists z[\mathbf{EC}(z, x) \wedge \mathbf{EC}(z, y)],$$

$$\mathbf{NTPP}(x, y) \Leftrightarrow_{\text{def}} \mathbf{PP}(x, y) \wedge \neg\exists z[\mathbf{EC}(z, x) \wedge \mathbf{EC}(z, y)].$$

3.1. Ground Mereotopology CEMT

The *Ground Mereotopology* **CEMT** [47] is the extension of **CEM** obtained by adding the following axioms:

$$\mathbf{C}(x, x), \tag{C1}$$

$$\mathbf{C}(x, y) \to \mathbf{C}(y, x), \tag{C2}$$

$$\mathbf{P}(x, y) \to \forall z(\mathbf{C}(z, x) \to \mathbf{C}(z, y)). \tag{C3}$$

Since models of the **CEM** axioms are equivalent to quasi-Boolean algebras, a model of the **CEMT** axioms can be represented as $\langle R, \mathbf{P}, \mathbf{C}\rangle$ or simply $\langle R, \mathbf{C}\rangle$, where $R = A - \{\bot\}$ and $A = \langle A; \bot, \top, ', \wedge, \vee\rangle$ is a Boolean algebra, $\mathbf{P}(x, y)$ if and only if $x \leqslant y$ in R and **C** is a binary relation on R satisfying (C1)–(C3). Note that in A we have $x \wedge y = \bot$ if and only if $x \leqslant y'$.

Usually **CEMT** can be extended by adding one or more of the following important formulae:

$$\mathbf{C}(z, x + y) \leftrightarrow (\mathbf{C}(z, x) \vee \mathbf{C}(z, y)), \tag{C4}$$

$$\exists z\neg\mathbf{P}(z, x) \to \mathbf{C}(x, \sim x), \tag{C5}$$

$$\exists z\neg\mathbf{P}(z, x) \to (\exists z\neg\mathbf{C}(z, x)). \tag{C6}$$

The following theorem shows that, in **CEMT**, the ‘Complement Axiom’ (C5) is equivalent to extensionality for **EC**.

Theorem 3.1. *In* **CEMT**, *the following formulae are equivalent to* (C5):

$$\forall z(\mathbf{EC}(x, z) \leftrightarrow \mathbf{EC}(y, z)) \to x = y, \tag{C5$'$}$$

$$\forall z(\mathbf{EC}(x, z) \leftrightarrow \mathbf{EC}(y, z)) \leftrightarrow x = y. \tag{ECext}$$

Proof. Clearly (C5′) and (ECext) are equivalent in **CEMT**. We show (C5)↔(C5′) in **CEMT**.

Since **CEMT** is a first-order theory, we need only to show (C5) ↔ (C5′) are true in any **CEMT** model. Suppose $\langle R, \mathbf{C}\rangle$ is a **CEMT** model, where $R = A - \{\bot\}$ and A is a Boolean algebra.

Note that $\exists z\neg\mathbf{P}(z, y)$ if and only if y is not the universe.

Suppose (C5) holds in R and x, y are two elements in R satisfying $\mathbf{EC}(x, z)$ if and only if $\mathbf{EC}(y, z)$ for any $z \in R$. We now show $x = y$. Note that $\neg\mathbf{EC}(z, \mathrm{u})$ holds for any z in R.

If one of $x,\ y$ is the universe, we must have both are the universe. Now suppose neither is the universe. Since (C5) holds, we have $\mathbf{EC}(x, x')$ and $\mathbf{EC}(y, y')$. By assumption, we also have $\mathbf{EC}(y, x')$ and $\mathbf{EC}(x, y')$, hence $y \leqslant x$ and $x \leqslant y$ hold. Consequently we have $x = y$.

On the other hand, suppose (C5′) holds in R and x is an element in R other than the universe. Take $y = \mathrm{u}$, the universe. Then by $x \neq y$ we have some element z in R with $\mathbf{EC}(x, z)$. Since $z \leqslant x'$, we must have $\mathbf{EC}(x, x')$ by (C3).

This ends the proof. □

The following theorem shows the axiom (C6), which stipulates that each non-universal region is disconnected from some other region, is equivalent to extensionality for **C** in **CEMT**.

Theorem 3.2. *In* **CEMT***, the following formulae are equivalent to* (C6):

$$\forall z(\mathbf{C}(z, x) \to \mathbf{C}(z, y)) \to \mathbf{P}(x, y), \tag{C6′}$$

$$\forall z(\mathbf{C}(z, x) \leftrightarrow \mathbf{C}(z, y)) \leftrightarrow x = y. \tag{Cext}$$

Proof. That (C6′) is equivalent to (Cext) is clear. We show (C6) ↔ (C6′) is true in each model of **CEMT**.

Let $\langle R, \mathbf{C}\rangle$ be a **CEMT** model, where $R = A - \{\bot\}$ and A is a Boolean algebra.

Suppose (C6) holds in R and x, y are two elements in R with $x \nleqslant y$. Then $x - y \in R$ and $(x - y)' = x' \vee y \neq \top$. Since (C6) holds in R we have some $z \in R$ with $\mathbf{DC}(z, x' \vee y)$. Clearly $z \leqslant x$ and $\mathbf{DC}(z, y)$ hold. Therefore we have an element $z \in R$ with $\mathbf{C}(z, x)$ but $\mathbf{DC}(z, y)$. As a result (C6′) is true in R.

On the other hand, suppose (C6′) holds in R and $y \in R$ is not the universe. If $\mathbf{C}(z, y)$ holds for any $z \in R$, then by (C6′) we have $x \leqslant y$ for any $x \in R$. This cannot be true since y is not the universe. Therefore we must have some $z \in R$ with $\mathbf{DC}(z, y)$. As a result, (C6) is true in R. □

Remark 3.1. In [30, p. 248], Masolo and Vieu note that in **GEMT** and **CEMT**, "Without (C4) [ours (C6′)], one could believe **P** is not constrained enough (with respect to the topological primitive **C**) to account for spatial inclusion. Thus, the possibility to add weaker axioms than (C4) should be explored". The above Theorem 3.1 shows that in any theory stronger than **CEMT** + (C5) (for instance the GRCC to be given in next subsection), (C5′) as well as (ECext) is a theorem. This suggests (C5) or but equivalently (C5′) can be used as a weaker axiom for constraining **P**. Note also that the counter-model given in [30, p. 249, Fig. 4.b] is misinterpreted for the authors only consider the atomic regions while any model of **CEMT** is a quasi-Boolean algebra.

Note that in **CEMT**, the axiom (C5) is equivalent to say the universe u is *self-connected*, namely, for any two regions x, y, we have $\mathbf{C}(x, y)$ if $x + y = \mathrm{u}$. The following theorem suggests that, in any **CEMT** model with a self-connected universe, the **NTPP** and **TPP** relations can be defined without explicitly using a quantifier.

Theorem 3.3. *Let* $\langle R, \mathbf{C}\rangle$ *be a* **CEMT** *model satisfying moreover* (C5)*, i.e.,* $\mathbf{C}(x, x')$ *for any* $x \neq \top$*, where* $R = A - \{\bot\}$ *and* A *is a Boolean algebra. Suppose* $y \in R$ *is not the universe. Then we have*

(1) $\mathbf{NTPP}(x, y)$ *if and only if* $\mathbf{DC}(x, y')$;
(2) $\mathbf{TPP}(x, y)$ *if and only if* $\mathbf{EC}(x, y')$ *and* $x \neq y$.

Proof. (1) Suppose $\mathbf{NTPP}(x, y)$. We show $\mathbf{DC}(x, y')$. Since $x < y$, we have either $\mathbf{EC}(x, y')$ or $\mathbf{DC}(x, y')$. If $\mathbf{EC}(x, y')$ holds, then by (C3) and $x < y$, we shall have $\mathbf{EC}(y, y')$ and $\mathbf{EC}(x, y')$. This cannot be true since $\mathbf{NTPP}(x, y)$. Therefore $\mathbf{DC}(x, y')$ holds.

On the other hand, suppose $\mathbf{DC}(x, y')$ we show $\mathbf{NTPP}(x, y)$. We have $x \leqslant y$ since $x \wedge y' = \bot$. Moreover, by (C5), x cannot be equal to y. If there exists some $m \in R$ such that $\mathbf{EC}(m, x)$ and $\mathbf{EC}(m, y)$, then $m \leqslant y'$. But by $\mathbf{DC}(x, y')$, we also have $\mathbf{DC}(x, m)$, a contradiction.

(2) Suppose $\mathbf{TPP}(x, y)$. Clearly we have $x \neq y$ and $x \wedge y' = \bot$ by $x < y$. Moreover, by (1) and $\neg\mathbf{NTPP}(x, y)$, we have $\mathbf{C}(x, y')$. Therefore $\mathbf{EC}(x, y')$.

On the other hand, suppose $\mathbf{EC}(x, y')$ holds and $x \neq y$. Then $x < y$ and, by (1), $\neg\mathbf{NTPP}(x, y)$. Therefore $\mathbf{TPP}(x, y)$ holds. □

By above theorems we also have the following theorem which shows, in any theory stronger than **CEMT** + (C5), the axiom (C6) is also equivalent to a variant of the '**NTPP**' axiom appeared in the original RCC theory [32].

Theorem 3.4. *In the theory obtained by adding* (C5) *to* **CEMT**, *the following formula* (NTPP) *is also equivalent to* (C6):

$$\exists z \neg \mathbf{P}(z, x) \to \exists z \mathbf{NTPP}(z, x). \tag{NTPP}$$

Proof. Note that $\exists z \neg \mathbf{P}(z, x) \leftrightarrow \exists z \neg \mathbf{P}(z, \sim x)$ and $\sim (\sim x) = x$. This follows from Theorem 3.2 and Theorem 3.3(1). □

Note that **CEMT** has a trivial model $\langle R, \mathbf{C}\rangle$ where $R = A - \{\bot\}$ and $A = \{\bot, \top\}$ which also satisfies (C4)–(C6).

3.2. Generalized Boolean connection algebras

Our theory of *Generalized Region Connection Calculus* (GRCC) is obtained by adding Axioms (C4), (C5) and (C7) to the Ground Mereotopology **CEMT** induced by **CEM**:

$$\exists x y \mathbf{PP}(x, y). \tag{C7}$$

This axiom is added to exclude the trivial model described in the end of above subsection.

We now justify that GRCC is indeed a generalization of the theory of Region Connection Calculus (RCC). Note that there are several equivalent ways of formulating the original RCC, we now follow the one in terms of Boolean connection algebras proposed by Stell [45]. BCAs permit a wealth of results from the theory of Boolean algebras to be applied to RCC. This notion can be adapted to provide models of GRCC.

Definition 3.1. Let $A = \langle A; \bot, \top, ', \vee, \wedge\rangle$ be a Boolean algebra with more than two elements, let R denote $A - \{\bot\}$, and let R_- denote $R - \{\top\}$. If $\mathbf{C}$ is a binary relation on R, then the structure $\langle A; \mathbf{C}\rangle$ is said to be a *generalized Boolean connection algebra* if it satisfies the following conditions.

A1. $\mathbf{C}$ is symmetric and reflexive.
A2. $\forall x \in R_- \cdot \mathbf{C}(x, x')$.
A3. $\forall x, y, z \in R \cdot \mathbf{C}(x, y \vee z)$ iff $\mathbf{C}(x, y)$ or $\mathbf{C}(x, z)$.

A generalized Boolean connection algebra (GBCA henceforth) $\langle A, \mathbf{C}\rangle$ is called a *Boolean connection algebra* (BCA) [45] if $\mathbf{C}$ satisfies additionally the following condition:

A4. $\forall x \in R_- \cdot \exists y \in R \cdot \neg\mathbf{C}(x, y)$.

We say two GBCAs (or BCAs) $\langle A_1, \mathbf{C}_1\rangle$ and $\langle A_2, \mathbf{C}_2\rangle$ are *isomorphic* if there is a Boolean isomorphism $f : A_1 \to A_2$ such that for any two elements $a, b \in A_1$, $\mathbf{C}_1(a, b)$ if and only if $\mathbf{C}_2(f(a), f(b))$.

We have shown in Section 2 quasi-Boolean algebras coincide with models of **CEM**, the mereological part of GRCC. Consequently, the equivalence between GBCAs and models of GRCC is clear. In the rest of this paper, *we shall make no distinction between a* GRCC *model and its corresponding* GBCA *structure*, and we shall often refer to a GRCC model (or its corresponding GBCA) by its ground set.

Note that Stell has shown that BCAs are equivalent to models of the RCC axioms [45]. Clearly, since models of **CEM** are equivalent to quasi-Boolean algebras, BCAs are also equivalent to models of GRCC + (C6). In another word, the original theory of RCC can be equivalently formulated as the extension of **CEMT** obtained by adding axioms (C4)–(C7). Consequently our theory of GRCC is indeed a generalization of RCC.

Remark 3.2. In the original formulation of RCC [31,32], the authors use the connection '$\mathbf{C}$' as the unique primitive relation. In particular, the parthood relation '$\mathbf{P}$' and the quasi-Boolean operations 'sum', 'difference', 'complement' and the 'universe' are defined in terms of '$\mathbf{C}$' as follows:[6]

$$\mathbf{P}(x, y) \Leftrightarrow_{\mathrm{def}} \forall z(\mathbf{C}(z, x) \to \mathbf{C}(z, y)),$$
$$x +' y =_{\mathrm{def}} \iota z \forall w(\mathbf{C}(w, z) \leftrightarrow (\mathbf{C}(w, x) \vee \mathbf{C}(w, y))),$$
$$x \times' y =_{\mathrm{def}} \iota z \forall w(\mathbf{C}(w, z) \leftrightarrow \exists v(\mathbf{P}(v, x) \wedge \mathbf{P}(v, y) \wedge \mathbf{C}(w, v))),$$

[6] In [31] and in the earlier work of Clarke, the complement definition is defined so that a region y connects with the complement of region x if and only if y is not a part of x. This brings the consequence that no region is connected with its complement. The revised definition of complement given below appeared in [32].

$$\sim' x =_{\text{def}} \iota z \forall w((\mathbf{C}(w,z) \leftrightarrow \neg \mathbf{NTPP}(w,x)) \wedge (\mathbf{O}(w,z) \leftrightarrow \neg \mathbf{P}(w,x))),$$
$$x -' y =_{\text{def}} \iota z \forall w(\mathbf{C}(w,z) \leftrightarrow \mathbf{C}(w, x \times (\sim y))),$$
$$\mathrm{u}' =_{\text{def}} \iota z \forall x \mathbf{C}(x,z),$$

where we use primes to distinguish the new operators from those defined in terms of '**P**'. This scarcity of primitives is usually recognized as a major attractive feature of the RCC theory. However, as noted by Smith [42], this formulation makes it difficult to separate formally the mereological and topological parts of the theory. Fortunately, the concept of Boolean connection algebras proposed by Stell [45], as well as the reformulation of RCC given above, suggests a clear separation of mereological part of the RCC from the topological part. It is worth noting that, in the reformulation of RCC given above, the two kinds of definitions of the quasi-Boolean operations are equivalent. Such a separation is very helpful for modifying the RCC theory to develop formal theories of, for example, discrete space (as this paper and [39] have shown) or vague spatial regions (see [38]).

Fig. 1 illustrates the hierarchy of binary relations defined in GRCC which is in accordance with the hierarchy of relations in RCC [32], where $\mathbf{\Phi i}$ denotes the inverse of $\mathbf{\Phi}$ for $\mathbf{\Phi} \in \{\mathbf{P}, \mathbf{PP}, \mathbf{TPP}, \mathbf{NTPP}\}$. Relations in $\{\mathbf{DC}, \mathbf{EC}, \mathbf{PO}, \mathbf{EQ}, \mathbf{TPP}, \mathbf{NTPP}, \mathbf{TPPi}, \mathbf{NTPPi}\}$ are identified as of particular importance [32]. This set of binary topological relations, known as RCC8 in the literature, has among others the following merits: RCC8 forms a JEPD set of relations and a composition table was derived which provides a basis for qualitative spatial reasoning [13,19]; RCC8 is one of the smallest sets of relations which makes topological distinctions rather than just mereological ones; The same set of relations has been independently identified as significant in the context of Geographical Information Systems (GIS) (see [19,20]).

It is worth noting that, for RCC8 relations, GRCC justifies the same composition table as that in RCC by Cui, Cohn and Randell [13]. For details we refer the reader to [28].

BCAs have some interesting properties. For example, given an arbitrary $a \in A - \{\bot, \top\}$ and any $\Phi \in \{\mathbf{DC}, \mathbf{EC}, \mathbf{PO}, =, \mathbf{TPP}, \mathbf{NTPP}, \mathbf{TPPi}, \mathbf{NTPPi}\}$, there exists a region $a^* \in A - \{\bot, \top\}$ such that $(a, a^*) \in \Phi$ [28].

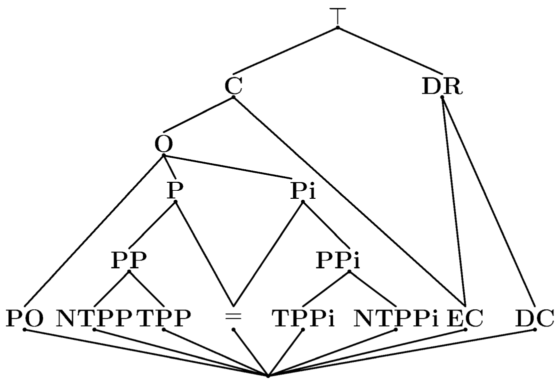

Fig. 1. Inclusion hierarchy of relations defined in GRCC and RCC.

The connection between BCA and Boolean algebra is an interesting problem. Clearly each BCA is an atomless Boolean algebra, however, the following question is still open:[7]

Question. If A is an atomless Boolean algebra, does there exist some connection relation $\mathbf{C}$ on A such that $\langle A, \mathbf{C}\rangle$ is a BCA?

Remark 3.3. It is claimed in [14] that if A is an atomless Boolean algebra, then there exists some connection relation $\mathbf{C}$ on A such that $\langle A, \mathbf{C}\rangle$ is a model of the RCC theory. But the proof given there is not correct. Düntsch's proof relies on the following statement:

Let X be a connected regular T_0 space. If B is an atomless subalgebra of the regular closed lattice of X, then B, together with the connection defined for all $x, y \in B - \{\emptyset\}$ as: $\mathbf{C}(x, y)$ iff $x \cap y \neq \emptyset$, is a BCA.

Unfortunately, this is not true. Let X be the unit closed disk in the real plane and let B be the subalgebra generated by all (closed) sectors. Then the connection defined as in above statement is a trivial relation. This is since that any two sectors are connected, namely, their intersection is non-empty.

Note that there is only one (up to isomorphism) countable atomless Boolean algebra [6, p. 39, Proposition 1.4.5]. The BCA constructed in [45, Section 6.3], as well as the minimal BCA we shall introduced later, shows this question has affirmative answer for countable Boolean algebras.[8]

But for GBCA, the corresponding question is trivial. Given a Boolean algebra A, let $\mathbf{C} = (A - \{\bot\})^2$, then $\langle A, \mathbf{C}\rangle$ is a GBCA. We call this relation the trivial connection on A. These trivial GBCAs could be excluded by replacing Axiom (C7) by the following formulae

$$\exists xy \neg \mathbf{C}(x, y). \tag{C7$'$}$$

3.3. A construction for GBCAs

In [45], J.G. Stell also gives a construction of BCAs, hence a construction of models of RCC. This construction is still valid for GBCAs.

We recall some definitions and notations from [2,45].

A *pseudo-complemented distributive lattice* A is defined as a distributive lattice, A, equipped with a unary operation $^*: A \to A$, such that, for all $a \in A$, a^* is the pseudo-complement of a, namely, a^* is the greatest element of $\{x \in A \mid a \wedge x = \bot\}$. The set of *skeletal* elements of A is defined by $\mathcal{S}(A) = \{a \in A \mid a^{**} = a\} = \{a^* \mid a \in A\}$. It is well-known that $\mathcal{S}(A)$ is a Boolean algebra where $\bot$ and $\top$ are as in A, the complementation is the restriction of the pseudo-complement to $\mathcal{S}(A)$, and where the meet, $\sqcap$, and the join,

[7] Recently, Düntsch and Winter [16] have shown that every RCC model is isomorphic to a substructure of some $\langle X, \mathbf{C}_X\rangle$ for a connected weakly regular T_1 space X.

[8] The discussion of the case of countable Boolean algebras was suggested to us by one referee.

$\sqcup$, are defined by $x \sqcap y = x \wedge y$, and $x \sqcup y = (x \vee y)^{**}$ [2]. A lattice, A, is *connected* if it does not contain elements $a \neq \bot$ and $b \neq \bot$ such that $a \vee b = \top$ and $a \wedge b = \bot$. A pseudo-complemented distributive lattice A is *inexhaustible* if for every $b \in \mathcal{S}(A) - \{\bot\}$ there is some $a \in \mathcal{S}(A) - \{\bot\}$ such that $a^* \vee b = \top$ [45]. (Definition 26 in [45] is clearly misdefined for that it allows $a = \bot$ and therefore each pseudo-complemented distributive lattice will be inexhaustible.)

Let $\langle A; \bot, \top, {}^*, \vee, \wedge\rangle$ be a pseudo-complemented distributive lattice with $\langle \mathcal{S}(A); \bot, \top, {}', \sqcup, \sqcap\rangle$ as the Boolean algebra of its skeletal elements, and let the relation $\mathbf{C}$ on $\mathcal{S}(A) - \{\bot\}$ be defined by $\mathbf{C}(x, y)$ iff $x^* \vee y^* \neq \top$.

Stell shows that if A is connected and inexhaustible and if $\mathcal{S}(A)$ contains more than two elements, then $\langle \mathcal{S}(A); \mathbf{C}\rangle$ is a BCA [45]. Conversely, if $\mathcal{S}(A)$ contains more than two elements and if $\langle \mathcal{S}(A); \mathbf{C}\rangle$ is a BCA, then A must be connected and inexhaustible. In fact we have the following

Theorem 3.5. *Let $\langle A; \bot, \top, {}^*, \vee, \wedge\rangle$ be a pseudo-complemented distributive lattice with $\langle \mathcal{S}(A); \bot, \top, {}', \sqcup, \sqcap\rangle$ as its Boolean algebra of skeletal elements, and let the relation $\mathbf{C}$ on $\mathcal{S}(A) - \{\bot\}$ be defined by $\mathbf{C}(x, y)$ iff $x^* \vee y^* \neq \top$. Suppose that $\mathcal{S}(A)$ contains more than two elements. Then $\langle \mathcal{S}(A); \mathbf{C}\rangle$ is a* GBCA (BCA, *respectively*) *if and only if A is connected* (*connected and inexhaustible, respectively*).

Proof. The proof is routine and similar to that given in [45]. □

A topological space X is said to be *inexhaustible* if the lattice of its open sets is so. Then we have the following

Corollary 3.1 [25,45]. *Let X be a topological space, let R be the set of non-empty regular open sets of X, and assume that R contains more than two elements. Define the relation $\mathbf{C}$ on R by $\mathbf{C}(H, K)$ iff $\overline{H} \cap \overline{K} \neq \emptyset$. Define $H + K$ to be the interior of $\overline{H \cup K}$, define $H \times K$ to be $H \cap K$, and $\sim H$ to be the interior of $X - H$. Then $\langle R, \mathbf{C}\rangle$ is a model of the* GRCC (RCC, *respectively*) *if and only if X is connected* (*connected and inexhaustible, respectively*).

A dual construction of RCC models from a *pseudo-supplemented distributive lattice* is also given in [45]. A result dual to that in the above corollary is also valid, which asserts that a model of RCC can also be obtained from the non-empty regular closed sets of a topological space. This result appeared first in [25] for regular connected spaces. Similar results are valid for GRCC too.

Corollary 3.2 [25,45]. *Let X be a connected topological space, let R be the set of non-empty regular closed sets of X, and assume that R contains more than two elements. Define the relation $\mathbf{C}$ on R by $\mathbf{C}(H, K)$ iff $H \cap K \neq \emptyset$. Also define $H + K$ to be $H \cup K$, define $H \times K$ to be the closure of the interior of $H \cap K$, and $\sim H$ to be the closure of $X - H$. Then $\langle R, \mathbf{C}\rangle$ is a model of the* GRCC. *If X is moreover regular, then $\langle R, \mathbf{C}\rangle$ is a model of the* RCC.

For each connected topological space X, we call the relation defined above (for regular closed sets) the *canonical connection relation* on X and denote by $\langle X, \mathbf{C}_X\rangle$ or simply $\langle X, \mathbf{C}\rangle$, the corresponding GBCA. In particular, we shall refer the standard RCC model on $\mathbb{R}^2$ simply by $\langle \mathbb{R}^2, \mathbf{C}\rangle$.

4. Consistency of GRCC with atomicity

The RCC theory requests each region has a non-tangential proper part, consequently it does not support atomic regions. Nevertheless, atomicity is especially important if one seeks to bridge the gap between high-level qualitative approaches to handling spatial information and the domain of low-level quantitative data abounded in the real-world applications. Recently, Masolo and Vieu [30] and Galton [24] investigated the mereotopological properties of discrete spaces. In [30], the authors investigate the possibility of characterizing atomicity in various schemes of mereotopologies classified in [47], in particular they show **GEMT** is consistent with Axioms (P9) (the atomicity axiom) and (C5) (which is equivalent to say the universe is self-connected). In [24], Galton has outlined a theory of discrete space that parallels the RCC theory which is, as we shall show, an extension of the GRCC theory.

Since the corresponding BCA of each regular connected topological space X is complete, the 'fusion' axiom (FUS) is consistent with the RCC theory. However, if we interpret the 'fusion' operation in terms of '**C**', the resulting formula (FUS′) is inconsistent with the RCC theory:

$$\exists x\phi(x) \to \exists z\forall w(\mathbf{C}(w,z) \leftrightarrow \exists x(\phi(x) \wedge \mathbf{C}(w,x))). \tag{FUS′}$$

This is because that (NTPP) is a theorem in RCC and each region is the fusion of its non-tangential proper parts. But, as this section shall show, this '**C**'-based 'fusion' axiom is also consistent with the GRCC theory.

We now briefly introduce the mereotopology of discrete space developed by Galton [24].

Galton's theory of discrete space is motivated by a desire to bridge the gap between high-level qualitative approaches to spatial information and lower-level quantitative ones. The fundamental notion for a discrete space is that of *adjacency*. This is a relation on the minimal elements of the space, called by Galton *cells*. Two regions, identified as sets of cells, are connected if some cell in one region is adjacent (or equal) to some cell in the other. It is on this notion of connection that Galton founds his theory of discrete space. In the original formulation of his theory, Galton makes no assumption concerning further properties of the adjacency relation. However, for practical application, it is natural to assume the universe is self-connected, this means for any two regions a, b, we have $a + b = \mathrm{u} \to \mathbf{C}(a, b)$.

Suppose X is a nonempty set containing at least two elements and **A** is a binary reflexive symmetric relation on X satisfying $a \cup b = X \to \exists xy \in X(x \in a \wedge y \in b \wedge \mathbf{A}(x, y))$ for any two nonempty subsets $a, b \subseteq X$. This binary relation **A** is called an *adjacency* relation on X and the structure $\langle X, \mathbf{A}\rangle$ is called an *adjacency space*. Note that a region in $\langle X, \mathbf{A}\rangle$ is just a nonempty subset of X and two regions a, b are related as $\mathbf{C}(a, b)$ if and only if there exist two elements $x, y \in X$ with $x \in a$, $y \in b$ and $\mathbf{A}(x, y)$.

Under this natural assumption, we next show Galton's theory of adjacency spaces is indeed an extension of GRCC.

Theorem 4.1. *Galton's theory of adjacency spaces is equivalent to the theory obtained by adding* (P9), (FUS), (FUS′) *and the following axiom to* GRCC:

$$\mathbf{C}(a, b) \to \exists xy(\mathbf{AT}(x) \wedge \mathbf{AT}(y) \wedge \mathbf{P}(x, a) \wedge \mathbf{P}(y, b) \wedge \mathbf{C}(x, y)) \tag{C8}$$

where $\mathbf{AT}(x) \Leftrightarrow_{\mathrm{def}} \neg\exists z\mathbf{PP}(z, x)$.

Proof. Suppose $\langle X, \mathbf{A}\rangle$ is an adjacency space. Then Galton's model of discrete space is indeed a powerset algebra $(2^X, \mathbf{C})$ where two regions a, b, namely two nonempty subsets of X, are related as $\mathbf{C}(a, b)$ if and only if there exist two elements x, y in a, b respectively with $\mathbf{C}(\{x\}, \{y\})$. Roughly speaking, the adjacency relation $\mathbf{A}$ is just the restriction of the connection $\mathbf{C}$ on the class of singleton subsets of X. Clearly such a model fulfills the axioms of GRCC as well as (P9), (C8), (FUS) and (FUS′). Moreover, for any nonempty subset ϕ of 2^X, set $z_\phi = \cup\phi$. Then z_ϕ satisfies the condition specified in (FUS′), i.e., for any region w, w is connected with z_ϕ if and only if there exists some region $x \in \phi$ such that w is connected with x. But there may be regions other than z_ϕ satisfying above condition.

On the other hand, suppose $\langle B, \mathbf{C}\rangle$ is a GBCA which satisfies (P9), (C8), (FUS) and (FUS′). Then B is an atomic complete Boolean algebra with more than two elements. Note that it is a standard result that an atomic complete Boolean algebra is equivalent to a powerset algebra. Suppose X is a set with at least two elements such that $B \cong 2^X$. Define a binary relation $\mathbf{A}$ on X as $\mathbf{A}(x, y)$ if and only if $\mathbf{C}(\{x\}, \{y\})$ for any two elements x, y in X. Clearly $\mathbf{A}$ is an adjacency relation in the sense of Galton. Since $\langle B, \mathbf{C}\rangle$ satisfies (C8), clearly two regions a, b are related as $\mathbf{C}(a, b)$ if and only if there exist two elements $x, y \in X$ with $x \in a$, $y \in b$ and $\mathbf{A}(x, y)$. Moreover, the universe X is also self-connected since $\langle B, \mathbf{C}\rangle$ is a GBCA. □

In what follows we shall refer to this mereotopology as Galton's theory. Note that in this theory, the '**C**'-based fusion axiom is redundant. In fact, suppose $\langle B, \mathbf{C}\rangle$ is a GBCA which satisfies (P9), (C8) and (FUS) and ϕ is a nonempty set of regions. Set $z_\phi = \vee\phi$. Then it is routine to check z_ϕ satisfies the condition specified in (FUS′). But it is not clear whether or not (C8) is deducible from (P9) + (FUS) + (FUS′) in GRCC.

In [24], Galton also gives a reformulation of the RCC8 relations, but the definition of '**TPP**' is not right. Galton specifies two regions a, b are related as $\mathbf{TPP}(a, b)$ if and only if $\mathbf{EC}(a, X - b)$. Note that if the universe X is self-connected, this will lead to $\mathbf{TPP}(a, a)$ for any proper region a. Thus the definition should be replaced by, for instance, $\mathbf{TPP}(a, b)$ if and only if $\mathbf{PP}(a, b)$ and $\mathbf{EC}(a, X - b)$.

The following example shows that (FUS) is independent to (P9) + (C8) + (FUS′) in GRCC.

Example 4.1 (FUS) *is independent to* (P9) + (C8) + (FUS′) *in* GRCC. Take $X = \mathbb{N}$ and set B as the finite-cofinite subalgebra of 2^X, namely a subset a of X is in B if and only if either a or $X - a$ is finite. For two regions a, b (namely two nonzero elements) in B, define $\mathbf{C}(a, b)$ if and only if there exist $x \in a$ and $y \in b$ such that $x = y$ or $|x - y| \geqslant 3$. Clearly B is a non-complete atomic Boolean algebra and $\langle B, \mathbf{C}\rangle$ is a GBCA which satisfies (P9) and (C8). It is routine to check that $\langle B, \mathbf{C}\rangle$ also satisfies (FUS$'$). Suppose ϕ is a nonempty collection of regions, then we can take $z_\phi = \cup\phi$ or $z_\phi = \mathbb{N}$ according to whether $\cup\phi$ is finite or infinite. Now it is straightforward to show z_ϕ satisfies the condition specified prescribed in (FUS$'$). However, since B is non-complete, $\langle B, \mathbf{C}\rangle$ does not satisfy (FUS).

The following example shows (C8) and (FUS$'$) is independent to (P9) + (FUS) in GRCC.

Example 4.2 ((C8) *and* (FUS$'$) *is independent to* (P9) + (FUS) *in* GRCC). Take X as a set containing infinitely many elements and set $B = 2^X$. For two regions a, b in B, define $\mathbf{C}(a, b)$ if and only if either $a \cap b \neq \emptyset$ or $a \cup b$ is infinite. It is now routine to check that $\langle B, \mathbf{C}\rangle$ is indeed an atomic complete GBCA. That $\langle B, \mathbf{C}\rangle$ does not satisfy (C8) is clear by the definition of $\mathbf{C}$. As for (FUS$'$), take $Y \subset X$ such that both Y and $X - Y$ are infinite. Set $\phi = \{\{x\}\colon x \in Y\}$ and suppose z_ϕ is a region which satisfies the condition specified in (FUS$'$). Clearly z_ϕ cannot be finite since it connects with each element in ϕ. But if z_ϕ is infinite, by the definition of $\mathbf{C}$, it will connect with all regions. Note that a singleton set $\{x\}$ with $x \notin Y$ is disconnected from any $b \in \phi$. $\{x\}$ is therefore disconnected from z_ϕ, a contradiction. Consequently, such a region z_ϕ cannot exist and hence (FUS$'$) is not true in $\langle B, \mathbf{C}\rangle$.

But for a GBCA $\langle B, \mathbf{C}\rangle$, if B is finite, it must be a model of the Galton's theory. This is since any finite Boolean algebra is also atomic complete, and hence a power set algebra.

In what follows we describe an important example of discrete space.

In practice, the most useful discrete (raster) space is the digital plane $\mathbb{Z}^2$, which is defined as a rectangular array of points or pixels. Each point is addressed by a pair of integer valued coordinates (x, y). We briefly review here the concepts most relevant. More extensive and detailed treatments of digital topology can be found in [37].

Given a point in the plane, the neighboring points can be classified as 4-*neighbors* or 8-*neighbors*. The 4-neighbors of a point P are the vertically and horizontally adjacent points. Along with the diagonally adjacent points, they form the 8-neighbors (see Fig. 2). Then, in terms of Galton, we have two adjacency $\mathbf{A}_4$ and $\mathbf{A}_8$ and therefore two connections $\mathbf{C}_4$ and $\mathbf{C}_8$ respectively. The corresponding GBCAs are denoted by[9] $\langle\mathbb{Z}^2, \mathbf{C}_4\rangle$ and $\langle\mathbb{Z}^2, \mathbf{C}_8\rangle$ respectively.

These two GBCAs both can be realized in the vector space $\mathbb{R}^2$. In fact, let P_4 denote the set of all closed disks centered at each point in $\mathbb{Z}^2$ with radium $1/2$ and let P_8 be the set of all closed squares centered at each point in $\mathbb{Z}^2$ with length 1 (see Fig. 2). The subalgebras of $\langle\mathbb{R}^2, \mathbf{C}\rangle$ generated by P_4 and P_8 are denoted by B_4 and B_8 respectively. Clearly $\langle B_i, \mathbf{C}|_{B_i}\rangle$ is isomorphic to $\langle\mathbb{Z}^2, \mathbf{C}_i\rangle$ for $i = 4$ or $i = 8$.

[9] Strictly speaking, these GBCAs should be written as $\langle 2^{\mathbb{Z}^2}, \mathbf{C}_4\rangle$ and $\langle 2^{\mathbb{Z}^2}, \mathbf{C}_8\rangle$ respectively. But for simplicity we refer them as what follows.

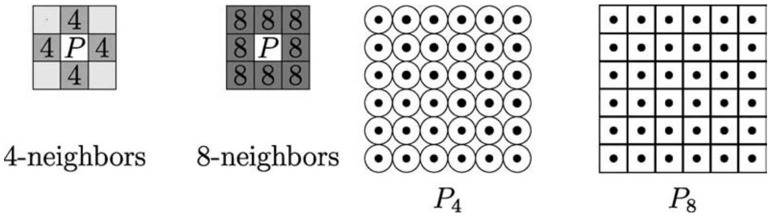

Fig. 2. $\mathbf{A}_4$ and $\mathbf{A}_8$ (left) and P_4 and P_8 (right).

The GBCA B_8 deserves a careful analysis. In this model, we interpret the raster elements as fields (instead of lattice points) and the raster is a regular subdivision of space into squares of equal size. Note that B_8 has the same topology as $\mathbb{R}^2$ in the sense that the connections are same. The so called *hybrid raster representation* [51] relates to this model closely. Each (bounded) region now is a (finite) union of elements in P_8, which is clearly a regular closed set in $\mathbb{R}^2$. The well-known 4-*intersection* or 9-*intersection* approach now can be applied to (simple connected) regions in B_8 to determine the RCC8 relations between them. This is in essence what has been explored by Winter [49,51]. Winter's approach contrasts sharply with the raster representation approach of Egenhofer and Sharma [21], the latter is based on digital topology and topological relations between regions in a raster representation are not the same as the RCC8 relations.

5. Operations on GBCAs

Formalisms of qualitative spatial representation such as RCC are mainly concerned with a single model, relation between these models are rarely investigated.[10] In this section we introduce several operations on models of GRCC, or in another word, on GBCAs. These conception, especially that of sub-structures and direct limits will shed light on the relation between continuous (vector) spaces and discrete (raster) spaces.

5.1. Sub-structures of GBCAs

Given a GBCA $\langle A, \mathbf{C}\rangle$, if B is a subalgebra of A with more than two elements, then we have a binary relation on $B - \{\bot\}$ obtained by restricting $\mathbf{C}$ on $B - \{\bot\}$. Write this binary relation as $\mathbf{C}|_B$, then it is straightforward to check that $\langle B, \mathbf{C}|_B\rangle$ is also a GBCA.

Definition 5.1 (*Sub-structure*). [11] Given $\langle A, \mathbf{C}\rangle$ a GBCA, if B is a subalgebra of A with more than two elements, then the GBCA $\langle B, \mathbf{C}|_B\rangle$ is called a *sub-structure*, or a *sub-GBCA* of $\langle A, \mathbf{C}\rangle$. If B happens to be a BCA, we say B is a sub-BCA of A.

[10] In [21], Egenhofer and Sharma compared topological relations between regions in $\mathbb{R}^2$ and $\mathbb{Z}^2$, but relation *between* these models is still left untouched.

[11] A similar definition is also given by Düntsch for Boolean Contact Algebra [14].

Note that in case $\langle A, \mathbf{C}\rangle$ is a finite GBCA, we have a corresponding adjacency space $\langle X, \mathbf{A}\rangle$ where X is the set of atoms of A and two atoms a, b are related as $\mathbf{A}(a, b)$ if and only if $\mathbf{C}(a, b)$. Suppose B is a subalgebra of A, then the set of atoms of B is a partition of X. We call the adjacency space corresponding to $\langle B, \mathbf{C}|_B\rangle$ a *sub-adjacency space* of $\langle X, \mathbf{A}\rangle$.

The following example shows that the digital plane (with either 4-adjacency or 8-adjacency) can be regarded as a sub-structure of the vector space $\mathbb{R}^2$.

Example 5.1 (*Continuous* (*vector*) *space* $\mathbb{R}^2$ *and discrete* (*raster*) *space* $\mathbb{Z}^2$). The continuous (vector) space of $\mathbb{R}^2$ are without question the most important model of the RCC theory. Recall we write this model simply by $\langle \mathbb{R}^2, \mathbf{C}\rangle$ and regions in this model are non-empty regular closed sets and two regions are connected if they have non-empty intersection.

On the other hand, the most useful discrete (raster) space is the digital plane $\mathbb{Z}^2$, which is defined as a rectangular array of points or pixels. Recall in Section 4 we have specified two adjacency $\mathbf{A}_4$ and $\mathbf{A}_8$ in the digital plane and obtained two GBCAs $\langle \mathbb{Z}^2, \mathbf{C}_4\rangle$ and $\langle \mathbb{Z}^2, \mathbf{C}_8\rangle$ respectively. Note that the GBCA $\langle B_i, \mathbf{C}|_{B_i}\rangle$ $(i = 4, 8)$ (given in Section 4) are two sub-GBCAs of the vector space $\langle \mathbb{R}^2, \mathbf{C}\rangle$. Now since $\langle B_i, \mathbf{C}|_{B_i}\rangle$ is isomorphic to $\langle \mathbb{Z}^2, \mathbf{C}_i\rangle$ for $i = 4$ or $i = 8$, we conclude that the digital plane with either 4-adjacency or 8-adjacency can be regarded as a sub-structure of the vector space $\langle \mathbb{R}^2, \mathbf{C}\rangle$.

Our notion of sub-structures has intimately connection with the formal framework proposed by Worboys [52] for treating the notion of resolution and multi-resolution in geographic spaces. We now recall some basic notions introduced in [52].

Let S be a set of locations (which may be, but does not have to be, a connected region of the Euclidean plane). A *resolution* R of S is a finite partition of S and an element $x \in R$ is called a *resel*. Such a resolution is any partition of the underlying set into a finite number of subsets which may arise from a pixellation of the space and be a regular square grid, or be formed from a triangulated irregular network (TIN). A resolution object (R-object) is defined in a similar way to a rough set as the two-stage set $\langle L, U\rangle$, where $L \subseteq U \subseteq R$. Let $\mathbf{R}$ be the set of all resolutions of a set S. Define a partial order $\leqslant$ on $\mathbf{R}$ as follows: for R_1 and R_2 belonging to $\mathbf{R}$, $R_1 \leqslant R_2$ if and only if $\forall x \in R_1, \exists y \in R_2$ such that $x \subseteq y$. Worboys then shows that $\langle \mathbf{R} \leqslant\rangle$ is indeed a lattice and any sublattice of $\mathbf{R}$ (not necessarily with the same top and bottom elements) will be termed a *resolution space*. Based on these notions, Worboys goes further to develop an approach to reasoning with imprecision about spatial entities and relationships resulting from finite resolution representations.

It seems to us that, in conjunction with the Łukasiewicz algebraic approach to spatial indeterminacy proposed by Roy and Stell [38], Worboys' approach can be fully reformulated in terms of our notion of sub-structures. A full working out of this theme will be the subject of another paper, here we only make some basic comparison.

Note that S is assumed to be finite throughout [52]. This restriction should be removed away if we want S to be a connected region of the Euclidean plane. Indeed, such an assumption is not essential. Of course, the structure $\langle \mathbf{R} \leqslant\rangle$ then cannot have a bottom, but this does not affect the definition of other notions, for instance resolution space. Moreover, if S is a finite set of locations, S can be interpreted as a GBCA which is a sub-structure of

Table 1
A comparison of notions in Worboys (1998) and notions in terms of sub-GBCA

Notions in Worboys (1998)	Notions in terms of sub-GBCA
set of locations S	GBCA $S = \langle S, \mathbf{C}\rangle$
resolution R	finite sub-GBCA B of S
$R_1 \leqslant R_2$	$B_2 \lhd B_1$
glb of R_1 and R_2	subalgebra generated by B_1 and B_2
lub of R_1 and R_2	subalgebra $B_1 \cap B_2$
R-object $\langle L, U\rangle$	vague region (L, U) in $B^{\rightarrow}$

some BCA arisen from a regular connected space. So, in what follows, we assume S is a bounded connected region of the Euclidean plane.

Now S, as a regular connected space, is naturally an RCC model. Write also $S = \langle S, \mathbf{C}\rangle$ the corresponding BCA. Then a resolution R of S is just a finite set of jointly exhaustive and pairwise disjoint (JEPD) regions in S. Namely, $R = \{s_1, \ldots, s_n\}$ satisfies $\bigvee_{i=1}^{n} s_i = \top$ and $s_i \wedge s_j = \bot$ for any $i \neq j$, where $\bigvee$, $\wedge$, $\top$, $\bot$ are interpreted as in the BCA S. There is a natural adjacency relation $\mathbf{A}$ on R defined as $\mathbf{A}(s_i, s_j)$ if and only if s_i and s_j are connected in the BCA S, namely, has nonempty intersection as two regular closed subsets of S. Clearly $\langle R, \mathbf{A}\rangle$ is an adjacency space in Galton's sense. Moreover the GBCA induced by this adjacency space is just a sub-GBCA of S. On the other hand, if B is a finite subalgebra of the BCA S, then $B = 2^R$ for some resolution R of S.

Furthermore, the collection of resolutions of S, $\mathbf{R}$, now corresponds to the collection of finite sub-structures of the BCA S, and two resolutions R_1, R_2 are related as $R_1 \leqslant R_2$ if and only if their corresponding sub-GBCAs, B_1 and B_2, are related as $B_2 \lhd B_1$, read as "B_2 is a sub-GBCA (or simply a subalgebra) of B_1". The greatest lower bound of R_1 and R_2 corresponds to the subalgebra of S generated by B_1 and B_2; and the least upper bound of R_1 and R_2 corresponds to $B_1 \cap B_2$.[12]

As for R-objects, note that any Boolean algebra B leads to a Łukasiewicz algebra $B^{\rightarrow}$ [38]. Then an R-object is just a vague region in $B^{\rightarrow}$.

Table 1 summarizes some basic correspondences between our theory of sub-GBCAs and the approach of Worboys.

5.2. Local structures and sums of GBCAs

In practice often there are demands for investigating the local property and, as a result, the discussion should be restricted on a small (bounded) region. The following definition of local structure is a proper reflection.

Given an element a in a Boolean algebra $\langle A; \bot, \top, ', \vee, \wedge\rangle$ with $a \neq \bot$. For any $x, y \leqslant a$, let $x^* = a \wedge x'$, and let $x \sqcup y = x \vee y$, $x \sqcap y = x \wedge y$. Then $\langle \downarrow a; \bot, a, ^*, \sqcup, \sqcap\rangle$ is also a Boolean algebra.

Given a GBCA $\langle A, \mathbf{C}\rangle$, a region a in GBCA $\langle A, \mathbf{C}\rangle$ is said to be *connected* if a cannot be divided into two disconnected parts, i.e., $\forall\, b, c \in A - \{\bot\}$, $b \vee c = a \rightarrow \mathbf{C}(b, c)$. For

[12] We here should loose the restriction that a GBCA contains more than two elements.

any $a \neq \bot$, we have a binary relation on $\downarrow a - \{\bot\}$ obtained by restricting $\mathbf{C}$ on $\downarrow a - \{\bot\}$. Write this binary relation as $\mathbf{C}|_{\downarrow a}$. If a is also a non-atomic connected region in $\langle A, \mathbf{C}\rangle$, then it is straightforward to check that $\langle \downarrow a, \mathbf{C}|_{\downarrow a}\rangle$ is also a GBCA.

Definition 5.2 (*Local structure*). Given $\langle A, \mathbf{C}\rangle$ a GBCA, let $a \in A$ be a connected non-atomic region other than $\top$. Then we call the GBCA $\langle \downarrow a, \mathbf{C}|_{\downarrow a}\rangle$ the *local structure* or *local GBCA* of A at a.

As is well known, the imprecision resulting from the resolution at which data are represented is an important component of spatial data quality. Worboys [52] introduces a formal framework for treating the notion of resolution and multi-resolution in geographic spaces. However, the cases that the universal region and some of its local parts (which may form a partition of the universe) are represented at different resolutions are rarely investigated. The situation is in a sense an inverse process of granulation for graphs [43].

Suppose $\langle X, \mathbf{A}\rangle$ is an adjacency space and $\{\langle A_x, \mathbf{C}_x\rangle\}_{x\in X}$ is a collection of GBCAs. We now show how to aggregate these local information. Set $A = \prod_X A_i$ to be the product algebra of all A_x. Then A is a Boolean algebra and each element $a \in A$ has form $(a_x)_{x\in X}$, in particular, $\bot = (\bot_x)_{x\in X}$ and $\top = (\top_x)_{x\in X}$, where $\bot_x$ and $\top_x$ are the bottom and top of the Boolean algebra A_x respectively. There are naturally two ways to associate A with a connection $\mathbf{C}$.

For two regions $a, b \in A$, define $\mathbf{C}_1(a, b)$ if and only if (i) $a \wedge b \neq \bot$ or (ii) there exist two different $x, y \in X$ with $\mathbf{A}(x, y)$, $a_x \neq \bot_x$ and $b_y \neq \bot_y$; define $\mathbf{C}_2(a, b)$ if and only if (i) $\mathbf{C}_1(a, b)$ or (ii) there exists some $x \in X$ such that $\mathbf{C}_x(a_x, b_x)$. A routine check will show $\langle A, \mathbf{C}_1\rangle$ and $\langle A, \mathbf{C}_2\rangle$ are two GBCAs, we call these, respectively, the *weak sum* and the *strong sum* of $\{\langle A_x, \mathbf{C}_x\rangle\}_{x\in X}$ *via* the adjacency space $\langle X, \mathbf{A}\rangle$.

For each $x \in X$, write x^* the region in A with $(x^*)_x = \top_x$ and $(x^*)_y = \bot_y$ for any $y \neq x$. Clearly $X^* = \{x^*\colon x \in X\}$ forms a JEPD set of regions in A. This X^* is, roughly speaking, a resolution of A (see Section 5.1). Note that for any two $x^*, y^* \in X^*$, we have $\mathbf{C}_1(x^*, y^*)$ if and only if $\mathbf{C}_2(x^*, y^*)$. Therefore X^* inherits an adjacency $\mathbf{A}^*$, from either $\langle A, \mathbf{C}_1\rangle$ or $\langle A, \mathbf{C}_2\rangle$, which is defined as follows: $\mathbf{A}^*(x^*, y^*)$ if and only if $\mathbf{C}_1(x^*, y^*)$. Such an adjacency space $\langle X^*, \mathbf{A}^*\rangle$ is isomorphic to the one $\langle X, \mathbf{A}\rangle$ and corresponds to a sub-GBCA of both $\langle A, \mathbf{C}_1\rangle$ and $\langle A, \mathbf{C}_2\rangle$. Moreover, in the strong sum $\langle A, \mathbf{C}_2\rangle$, since each x^* is clearly a non-atomic connected region other than the top, the local GBCA of $\langle A, \mathbf{C}_2\rangle$ at x^* is isomorphic to the GBCA $\langle A_x, \mathbf{C}_x\rangle$.

Combining the notion of sum of GBCAs with the notion of resolution will be helpful for constructing new adjacency spaces from old ones. Suppose $\langle X, \mathbf{A}\rangle$ is an adjacency space and $\mathcal{Y} = \{Y_1, \ldots, Y_n\}$ is a resolution of X and each Y_i is a connected region. Then we have a sub-adjacency space $\langle \mathcal{Y}, \mathcal{A}\rangle$ and a local adjacency space $\langle Y_i, \mathbf{A}_i\rangle$ for each Y_i. Now we have two new adjacency relations $\mathbf{A_1}$ and $\mathbf{A_2}$ on X defined as follows: $\mathbf{A}_1(x, y)$ if and only if there exist two different resels Y_i and Y_j in $\mathcal{Y}$ such that $x \in Y_i$ and $y \in Y_j$ and $\mathcal{A}(Y_i, Y_j)$; and $\mathbf{A}_2(x, y)$ if and only if either $\mathbf{A}_1(x, y)$ or x and y are adjacent in a local adjacency space $\langle Y_i, \mathbf{A}_i\rangle$. We call $\langle X, \mathbf{A}_1\rangle$ and $\langle X, \mathbf{A}_2\rangle$ respectively the weak and strong sum of $\{\langle Y_i, \mathbf{A}_i\rangle\colon i = 1, \ldots, n\}$ via $\langle \mathcal{Y}, \mathcal{A}\rangle$. These two notions are corresponding ones of their GBCA counterparts. Both these two spaces have in common with the original space in some but not every respects. These two spaces can be taken as abstractions of the original space. Although we may lost some information, but, from the weak sum we can regain the sub-adjacency space and, from the strong one we can also regain each local space.

In above we have shown how to construct new GBCA form old ones by introducing several operations. The so constructed GBCA is usually not a Boolean connection algebra. The next subsection, however, introduces an approach to construct a BCA from a sequence of GBCAs.

5.3. Direct limits of GBCAs

The following simple proposition will be useful in the construction of direct limits.

Proposition 5.1. *Let A be a Boolean algebra. Suppose $\{A_k\colon\ k \in \mathbb{N}\}$ is a collection of subalgebras of A and $A_k \subseteq A_{k'}$ for each pair $(k, k') \in \mathbb{N}^2$ with $k \leqslant k'$. Then $\bigcup_{n\in\mathbb{N}} A_k$ is also a subalgebra of A.*

Let A be a Boolean algebra. Suppose $\{\langle A_k, \mathbf{C}_k\rangle\}_{k\in\mathbb{N}}$ is a collection of GBCAs which satisfies the following conditions:

(1) A_k is a subalgebra of A for $k \in \mathbb{N}$;
(2) $\langle A_k, \mathbf{C}_k\rangle$ is a sub-GBCA of $\langle A_{k'}, \mathbf{C}_{k'}\rangle$ for $k \leqslant k' \in \mathbb{N}$.

Write $A_\omega = \bigcup_{k\in\mathbb{N}} A_k$. By Proposition 5.1, it is also a subalgebra of A. Define a binary relation $\mathbf{C}_\omega$ on $A_\omega - \{\bot\}$ as follows:

$$\forall x, y \in A_\omega,\ \mathbf{C}_\omega(x, y) \quad \text{iff} \quad \exists k \in \mathbb{N} \text{ s.t. } x, y \in A_k \text{ and } \mathbf{C}_k(x, y).$$

In another word, $\mathbf{C}_\omega = \bigcup_{k\in\mathbb{N}} \mathbf{C}_k$. It is routine to check that $\langle A_\omega, \mathbf{C}_\omega\rangle$ is a GBCA. We call $\langle A_\omega, \mathbf{C}_\omega\rangle$ the *direct limit* of $\{\langle A_k, \mathbf{C}_k\rangle\}_{k\in\mathbb{N}}$.

The following proposition shows $\langle A_k, \mathbf{C}_k\rangle$ is a sub-GBCA of $\langle A_\omega, \mathbf{C}_\omega\rangle$, namely $\mathbf{C}_k = \mathbf{C}_\omega|_{B_k}$, for each $k \in \mathbb{N}$.

Proposition 5.2. *Suppose $\langle A_\omega, \mathbf{C}_\omega\rangle$ is the direct limit of $\{\langle A_k, \mathbf{C}_k\rangle\}_{k\in\mathbb{N}}$. Then, for any two $a, b \in B_n$, we have $\mathbf{C}_\omega(x, y)$ if and only if $\mathbf{C}_n(x, y)$.*

Proof. We need only to show the 'necessity' part. Suppose $a, b \in B_n$ and $\mathbf{C}_\omega(x, y)$. Then by the definition of $\mathbf{C}_\omega$, we have some m such that $a, b \in B_m$ and $\mathbf{C}_m(x, y)$. If $m < n$, then $\mathbf{C}_n(x, y)$ holds for $\mathbf{C}_m = \mathbf{C}_n|_{B_m}$; if $m \geqslant n$, then $\mathbf{C}_n(x, y)$ holds for $\mathbf{C}_n = \mathbf{C}_m|_{B_n}$. □

In another word, this proposition suggests, for any two $a, b \in B_n$, $\neg\mathbf{C}_\omega(x, y)$ if and only if $\neg\mathbf{C}_n(x, y)$.

The following theorem characterizes when does a direct limit be a Boolean connection algebra.

Theorem 5.1. *Let A be a Boolean algebra. Suppose $\langle A_\omega, \mathbf{C}_\omega\rangle$ is the direct limit of $\{\langle A_k, \mathbf{C}_k\rangle\}_{k\in\mathbb{N}}$, where $\{\langle A_k, \mathbf{C}_k\rangle\}_{k\in\mathbb{N}}$ satisfies conditions* (1) *and* (2) *given above. Then $\langle A_\omega, \mathbf{C}_\omega\rangle$ is a Boolean connection algebra if and only if for each $k \in \mathbb{N}$ and each $x \in A_k - \{\bot, \top\}$, there exists some $k' \geqslant k$ and some $y \in A_{k'} - \{\bot, \top\}$ such that $\neg\mathbf{C}_{k'}(x, y)$.*

Proof. This is clear since the GBCA $\langle A_\omega, \mathbf{C}_\omega \rangle$ is a BCA if and only if it satisfies Axiom (C6), namely, for each $x \in A_\omega - \{\bot, \top\}$, there exists some $y \in A_\omega - \{\bot, \top\}$ such that x is disconnected from y. □

Given in advance a Boolean connection algebra $\langle A, \mathbf{C} \rangle$, this construction can be applied to obtain a countable sub-BCA of $\langle A, \mathbf{C} \rangle$. First we take a finite sub-GBCA of $\langle A, \mathbf{C} \rangle$, noted as $\langle A_1, \mathbf{C}_1 \rangle$. Then, since $\langle A, \mathbf{C} \rangle$ is a BCA, for each region a in $A_1 - \{\bot, \top\}$, we can choose some region $\bar{a} \in A - \{\bot, \top\}$ such that $\neg\mathbf{C}(a, \bar{a})$. Set A_2 as the sub-Boolean algebra generated by $A_1 \cup \{\bar{a}\colon a \in A_1 - \{\bot, \top\}\}$. Then A_2 is also finite since A_1 is so. Write $\langle A_2, \mathbf{C}_2 \rangle$ the corresponding sub-GBCA of $\langle A, \mathbf{C} \rangle$. Continuing this procedure, we shall obtain a collection of finite sub-GBCAs $\{\langle A_k, \mathbf{C}_k \rangle\}_{k \in \mathbb{N}}$ of $\langle A, \mathbf{C} \rangle$ which satisfies, in addition to Conditions (1) and (2) given in the definition of direct limit, the following condition:

(3) For each $k \in \mathbb{N}$ and each $x \in A_k - \{\bot, \top\}$, there exists some $y \in A_{k+1} - \{\bot, \top\}$ such that $\neg\mathbf{C}_{k+1}(x, y)$.

Then by Theorem 5.1, the direct limit of $\{\langle A_k, \mathbf{C}_k \rangle\}_{k \in \mathbb{N}}$ of $\langle A, \mathbf{C} \rangle$, noted as $\langle A_\omega, \mathbf{C}_\omega \rangle$, is a sub-BCA of $\langle A, \mathbf{C} \rangle$. Clearly A_ω is countable.

Remark 5.1. Note that the Boolean algebra of regular closed sets of any topological space is complete. Each standard model of the RCC theory induced by a regular connected space (for instance the Euclidean plane $\mathbb{R}^2$) is an atomless complete BCA. These standard BCAs cannot be countable since there exists (up to isomorphism) only one countable atomless Boolean algebra [6, p. 39, Proposition 1.4.5], which is, however, non-complete. Consequently, regions in these standard BCAs cannot be finitely represented as aggregations of basic regions in a predefined countable collection (in the case of $\mathbb{R}^2$, these basic regions could be taken from for instance squares, triangles, disks or other semi-algebraic regions). However, in the countable BCAs constructed as above, every region is an aggregations of atomic regions in a certain finite sub-GBCA. Thus as far as spatial representation is concerned, these non-standard models are computationally tractable.

In next subsection, we shall construct a minimal Boolean connection algebra using the approach described above.

5.4. A minimal BCA

Let $\Sigma = \{0, 1\}$ and let Σ^* be the set of finite strings over Σ. We denote by $|s|$ the length of a string $s \in \sigma^*$. For $s,\ t \in \Sigma^*$, we write $s \prec t$ if s is an initial segment of t, namely s starts t.

Recall ε is the empty string in Σ^*. Now for each string $s \in \Sigma^*$, we associate a left-closed-and-right-open sub-interval of $[0, 1)$ as follows: Take

$$x_\varepsilon = [0, 1);$$

$$x_0 = [0, 1/2), \quad x_1 = [1/2, 1);$$

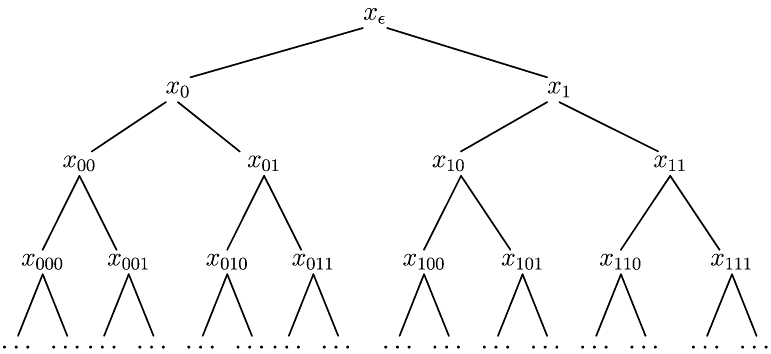

Fig. 3. The infinite complete binary tree T_ω.

$$x_{00} = [0, 1/4), \quad x_{01} = [1/4, 1/2),$$

$$x_{10} = [1/2, 3/4), \quad x_{11} = [3/4, 1); \quad \text{and so on.}$$

In general, suppose x_s has been defined for a string $s \in \{0,1\}^*$, we define x_{s0} to be the first half left-closed-and-right-open sub-interval of x_s, and x_{s1} the second half.

Write $X_n = \{x_s\colon\ s \in \Sigma^* \text{ with } |s| = n\}$ for each $n \geqslant 1$ and write $X_\omega = \{x_s\colon\ s \in \Sigma^*\}$. Denote by B_n the subalgebra of the powerset algebra $2^{[0,1)}$ generated by X_n and, by B_ω, the subalgebra generated by X_ω. Clearly B_n contains 2^n atoms and B_ω is a countable atomless Boolean algebra. Moreover, we have B_n is a subalgebra of B_m for $n < m$ and B_n is a subalgebra of B_ω for each n. It is also clear that $B_\omega = \bigcup_{n\in\mathbb{N}} B_n = \{\bigcup_{i=1}^{n-1} x_{s_i}\colon\ s_i \in \Sigma^*, n \in \mathbb{N}\}$.

Denote $Y_n = \{x_s\colon\ s \in \Sigma^* \text{ with } |s| \leqslant n\}$ for each $n \geqslant 1$, namely $Y_n = \bigcup_{k=1}^{n} X_k$. Then, for each $n \in \mathbb{N}$, Y_n (with the ordering of set inclusion) can be visualized as a complete binary tree T_n of height n. Similarly, X_ω, with the ordering of set inclusion, can be visualized as an infinite complete binary tree T_ω (see Fig. 3).

We now define inductively a binary relation $\mathbf{A}_\omega$ on X_ω:

(i) $\mathbf{A}_\omega(x_s, x_s)$ for each $s \in \Sigma^*$;
(ii) $\mathbf{A}_\omega(x_{s0}, x_{s1})$ and $\mathbf{A}_\omega(x_{s1}, x_{s0})$ for each $s \in \Sigma^*$;
(iii) $\mathbf{A}_\omega(x_{s1}, x_{t1})$ if $\mathbf{A}_\omega(x_s, x_t)$.

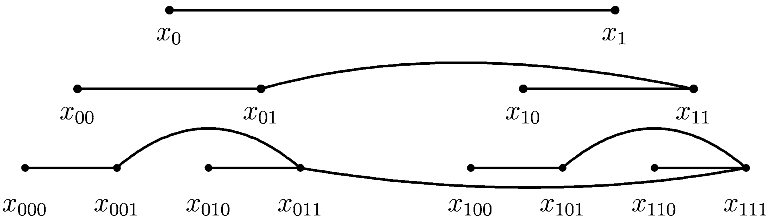

Fig. 4. An illustration of adjacency relations $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$.

Clearly, $\mathbf{A}_\omega$ is a reflexive and symmetric binary relation on X_ω. Recall X_n is the set of atoms of B_n. Denote $\mathbf{A}_n = \mathbf{A}_\omega|_{X_n}$, then $\langle X_n, \mathbf{A}_n\rangle$ is an adjacency space (see Fig. 4 for illustration). Denote by $\mathbf{C}_n$ the connection relation induced by $\mathbf{A}_n$. Then $\langle B_n, \mathbf{C}_n\rangle$ is a finite GBCA. Consider the direct limit of the collection of finite GBCAs $\{\langle B_n, \mathbf{C}_n\rangle\}_{n\in\mathbb{N}}$. Write this countable GBCA by $\langle B_\omega, \mathbf{C}_\omega\rangle$. Recall where $\mathbf{C}_\omega = \bigcup_{k\in\mathbb{N}} \mathbf{C}_k$. Then for any two elements $S = \bigcup_{i=1}^n x_{s_i}$ and $T = \bigcup_{i=1}^m x_{t_i}$ in $B_\omega - \{\perp\}$, $\mathbf{C}_\omega(S, T)$ if and only if there exist $s, t \in \Sigma^*$ with $\mathbf{A}_\omega(x_s, x_t)$ and $x_s \subseteq S,\ x_t \subseteq T$.

In what follows we give a characterization of the binary relation $\mathbf{A}_\omega$:

Proposition 5.3. *For $s \neq t \in \Sigma^*$, $\mathbf{A}_\omega(x_s, x_t)$ if and only if $|s| = |t|$ and there exist some string $s_1 \in \Sigma^*$ and some $n \geqslant 0$ with $\{s, t\} = \{s_1 0 \underbrace{1\cdots1}_n,\ s_1 1 \underbrace{1\cdots1}_n\}$.*

Proof. Suppose $|s| = |t|$ and $s = s_1 0 \underbrace{1\cdots1}_n$, $t = s_1 1 \underbrace{1\cdots1}_n$. Then by item (i) in the definition of $\mathbf{A}_\omega$, we have $\mathbf{A}_\omega(s_1, s_1)$; by (ii), we have $\mathbf{A}_\omega(s_1 0, s_1 1)$; by (iii), we have $\mathbf{A}_\omega(s, t)$.

On the other hand, suppose $\mathbf{A}_\omega(s, t)$. By using induction on n, the larger one of $|s|$ and $|t|$, we can easily show $|s| = |t|$.

Now using induction on $n = |s| = |t|$. Suppose the statement holds for any $k \leqslant n$. Suppose $s \neq t$ and $|s| = |t| = n + 1$. Write s_1 the longest string which is a common initial segment of s and t. If the last symbol of either s or t is not 1, then by the definition of $\mathbf{A}_\omega$ we should have $\{s, t\} = \{s_1 0, s_1 1\}$; and otherwise suppose s_2 is the longest string with form $1\cdots1$ which is a common final segment of s and t, then $|s_2| \geqslant 1$ and s, t have form $s_1 s' s_2$, $s_1 t' s_2$ respectively. By Item (iii), we should have $\mathbf{A}_\omega(s_1 s', s_1 t')$. Note that $|s_2| \geqslant 1$. By the induction assumption, we shall have $\{s_1 s', s_1 t'\} = \{s_1 0, s_1 1\}$. This ends the proof. □

By above proposition we have the following corollary:

Corollary 5.1. (i) *Suppose $\mathbf{A}_\omega(x_s, x_t)$. If s_1 and t_1 are initial segments of s, t respectively and $|s_1| = |t_1|$, then $\mathbf{A}_\omega(x_{s_1}, x_{t_1})$;*

(ii) *Suppose $\mathbf{A}_\omega(x_s, x_t)$. If s_1 and t_1 are final segments of s, t respectively and $|s_1| = |t_1|$, then $\mathbf{A}_\omega(x_{s_1}, x_{t_1})$;*
(iii) *Suppose $\mathbf{A}_\omega(x_s, x_t)$. Then $\mathbf{A}_\omega(x_{s's}, x_{s't})$ for any $s' \in \Sigma^*$;*
(iv) *Suppose $\mathbf{A}_\omega(x_{s0}, x_t)$. Then $t = s0$ or $t = s1$;*
(v) *For any string s with $|s| > 1$, there exists s' with same length such that $\mathbf{A}_\omega(s, s')$ does not hold.*

Proof. The first four items follow directly from Proposition 5.3. As for (v), suppose $s = a_1 \cdots a_{|s|}$, let $s' = (1 - a_1)\cdots(1 - a_{|s|})$. Since $|s| > 1$, we clearly have $\neg\mathbf{A}_\omega(s, s')$ by Proposition 5.3. □

Next we show $\langle B_\omega, \mathbf{C}_\omega\rangle$ is indeed a Boolean connection algebra.

Proposition 5.4. *$\langle B_\omega, \mathbf{C}_\omega\rangle$ is a Boolean connection algebra.*

Proof. We first show that the collection of finite GBCAs $\{\langle B_n, \mathbf{C}_n\rangle\}_{n\in\mathbb{N}}$ satisfies the following condition: For each $k \in \mathbb{N}$ and each $S \in B_k - \{\bot, \top\}$, there exists some string s with length k such that $\neg\mathbf{C}_{k+1}(S, x_{s0})$.

Note that $S \neq x_\varepsilon$ and $S \in B_k$. There exists some string s with length k such that $x_s \nsubseteq S$. We claim that $x_s \cap S = \emptyset$. This is because that x_s is an atom in B_k. By Item (iv) of Corollary 5.1 and the definition of $\mathbf{C}_{k+1}$, x_{s0} cannot be connected with x_t for any string $t \neq s$ with length k. Then $\neg\mathbf{C}_{k+1}(S, x_{s0})$ by Axiom (C4) of GRCC.

Next, by Theorem 5.1, $\langle B_\omega, \mathbf{C}_\omega\rangle$, as the direct limit of $\{\langle B_n, \mathbf{C}_n\rangle\}_{n\in\mathbb{N}}$, is a countable BCA. □

In what follows we write $\mathbf{NTPP}_\omega$ the non-tangential proper part relation in $\langle B_\omega, \mathbf{C}_\omega\rangle$.

Proposition 5.5. (i) *For any two strings s, t with same length, $\mathbf{A}_\omega(x_s, x_t)$ if and only if $\mathbf{C}_\omega(x_s, x_t)$;*

(ii) *For each string s, x_s is self-connected, namely, for all non-empty $S, T \in B_\omega$, if $S \cup T = x_s$, then $\mathbf{C}_\omega(S, T)$;*
(iii) *For any two strings s, t with $t \neq \varepsilon$, $\mathbf{NTPP}_\omega(x_s, x_t)$ if and only if $t \prec s$ and $s \neq t\underbrace{1\cdots1}_{n-1}$ for any $n \geqslant 1$. In particular, we have $\mathbf{NTPP}_\omega(x_{s0}, x_s)$;*
(iv) *For any nonempty $S \in B_\omega$ and any string $s \neq \varepsilon$, $\mathbf{NTPP}_\omega(S, x_s)$ if and only if $S \subseteq x_s - x_{s\underbrace{1\cdots1}_{n}}$ for some $n \geqslant 1$;*
(v) *For any string s, $\mathbf{NTPP}_\omega(x_{s0}, x_{s0} \cup x_{s\underbrace{1\cdots1}_{n}})$ for any $n \geqslant 1$.*

Proof. (i) Suppose $|s| = |t| = k$. If $\mathbf{A}_\omega(x_s, x_t)$, then $\mathbf{C}_\omega(x_s, x_t)$ by the definition of $\mathbf{C}_\omega$. On the other hand, if $\mathbf{C}_\omega(x_s, x_t)$, then there exists some k' with $\mathbf{C}_{k'}(x_s, x_t)$. Clearly $k' \geqslant k$ since x_s is an atom of B_k. Note that $\mathbf{C}_k = \mathbf{C}_{k'}|_{B_k}$ since $\langle B_k, \mathbf{C}_k\rangle$ is a sub-GBCA of $\langle B_{k'}, \mathbf{C}_{k'}\rangle$. We also have $\mathbf{C}_k(x_s, x_t)$, hence $\mathbf{A}_k(x_s, x_t)$ and $\mathbf{A}_\omega(x_s, x_t)$.

(ii) Suppose S, T are nonempty and $S \cap T = \emptyset$, $S \cup T = x_s$. Note that for any $n \geqslant 1$, $S \in B_n$ if and only if $T \in B_n$. We can take $k > |s|$ with $S, T \in B_k - B_{k-1}$. Recall X_k is the set of atoms of B_k. Since $S, T \notin B_{k-1}$, there must exist some string t with length $k - |s| - 1$ such that neither $x_{st0}, x_{st1} \subseteq S$ nor $x_{st0}, x_{st1} \subseteq T$ can hold. Therefore either $x_{st0} \subseteq S$, $x_{st1} \subseteq T$ or $x_{st0} \subseteq T$, $x_{st1} \subseteq S$ holds. By $\mathbf{A}_\omega(x_{st0}, x_{st1})$, we have $\mathbf{C}_\omega(S, T)$.

(iii) We first show $\mathbf{NTPP}_\omega(x_{s0}, x_s)$. Suppose $|s| > 0$. Note that $\neg\mathbf{A}_\omega(x_{s0}, x_{ta})$ for any t other than s with $|t| = |s|$ and any symbol $a \in \{0, 1\}$. By item (i) of this proposition, we have $\neg\mathbf{C}_\omega(x_{s0}, x_{ta})$, hence $\neg\mathbf{C}_\omega(x_{s0}, x_\varepsilon - x_s)$. Therefore there cannot exist a region which is externally connected to both x_{s0} and x_s, namely $\mathbf{NTPP}_\omega(x_{s0}, x_s)$ holds.

Suppose $t \prec s$ and $s \neq t\underbrace{1\cdots1}_{n-1}$ for any $n \geqslant 1$. Then $x_s \subseteq x_{t\underbrace{1\cdots1}_{n-1}0}$ for some $n \geqslant 1$. By above observation, we have $\mathbf{NTPP}_\omega(x_s, x_t)$. On the other hand, if $\mathbf{NTPP}_\omega(x_s, x_t)$, then clearly $t \prec s$ and $t \neq s$. Suppose $s = t\underbrace{1\cdots1}_{n}$ for some $n \geqslant 1$. Note that $t \neq \varepsilon$. t has form t_1a for some $t_1 \in \Sigma^*$ and some $a \in \Sigma$. Set $s' = t_1(1-a)\underbrace{1\cdots1}_{n}$. Then $\mathbf{A}_\omega(x_s, x_{s'})$, hence

$\mathbf{C}_\omega(x_s, x_{s'})$. Note that $x_{s'} \cap x_t = \emptyset$. We have $x_{s'}$ is externally connected to both x_s and x_t. This contradicts with the assumption that $\mathbf{NTPP}_\omega(x_s, x_t)$.

(iv) For $s \neq \varepsilon$ and some $n \geqslant 1$, we show $\mathbf{NTPP}_\omega(x_s - x_{s\underbrace{1\cdots1}_n}, x_s)$. Write $k = |s| + n$. Then for each atom x_t in B_k which is contained in $x_s - x_{s\underbrace{1\cdots1}_n}$, by (iii), we have $\mathbf{NTPP}(x_t, x_s)$. Consequently, we also have $\mathbf{NTPP}_\omega(x_s - x_{s\underbrace{1\cdots1}_n}, x_s)$.

On the other hand, suppose $S \not\subseteq x_s - x_{s\underbrace{1\cdots1}_n}$ for any $n \geqslant 1$. Then there exists some $k \geqslant 1$ such that $x_{s\underbrace{1\cdots1}_k} \subseteq S$. By (iii) again, this shows $\mathbf{NTPP}_\omega(S, x_s)$ cannot hold.

(v) This follows from the fact that, for any nonempty $S \in B_\omega$, S is externally connected to x_{s0} if and only if S contains $x_{s\underbrace{1\cdots1}_n}$ for some $n \geqslant 1$. □

By above proposition, $\mathbf{NTPP}_\omega(x_{s0}, x_s)$ holds and there cannot exist another region $S \in B_\omega$ such that both $\mathbf{NTPP}_\omega(x_{s0}, S)$ and $\mathbf{NTPP}_\omega(S, x_s)$ hold. This observation shows that $\mathbf{NTPP}_\omega \circ \mathbf{NTPP}_\omega \neq \mathbf{NTPP}_\omega$. Hence gives a negative answer to a question raised by Düntsch et al. ([15, p.405]). In general, if we write inductively $\mathbf{NTPP}_\omega^{n+1} = \mathbf{NTPP}_\omega \circ \mathbf{NTPP}_\omega^n$, then $\mathbf{NTPP}_\omega, \mathbf{NTPP}_\omega^2, \ldots, \mathbf{NTPP}_\omega^n, \ldots$, is a strict decreasing chain [29, Theorem 4.1].

We now show that $\langle B_\omega, \mathbf{C}_\omega\rangle$ is a 'minimal' BCA in the sense that each BCA contains B_ω as a sub-BCA.

Theorem 5.2. *For each BCA $\langle A, \mathbf{C}\rangle$, A contains a sub-BCA isomorphic to $\langle B_\omega, \mathbf{C}_\omega\rangle$.*

Proof. Take $x \in A - \{\bot, \top\}$, and write $x_0 = x$, $x_1 = x'$. For $a \in \{0, 1\}$, since A is a BCA, there exists $x_{a0} \in A - \{\bot, \top\}$ with $\mathbf{NTPP}(x_{a0}, x_a)$. Write $x_{a1} = x_a \wedge (x_{a0})'$ for $a \in \{0, 1\}$. Denote by A_2 the subalgebra of A generated by $\{x_{ab}\colon a, b \in \{0, 1\}\}$, clearly $\langle A_2, \mathbf{C}|_{A_2}\rangle$ is isomorphic to $\langle B_2, \mathbf{C}_2\rangle$ given above. Continuing this procedure infinitely, for each $n \geqslant 2$, we shall get a GBCA, noted by $\langle A_n, \mathbf{C}|_{A_n}\rangle$, which is isomorphic to $\langle B_n, \mathbf{C}_n\rangle$. By Proposition 5.1, we have $A_\omega = \bigcup_{n\geqslant 2} A_n$ is also a subalgebra of A, which is isomorphic to B_ω. It is now routine to check that $\langle A_\omega, \mathbf{C}|_{A_\omega}\rangle$ is isomorphic to $\langle B_\omega, \mathbf{C}_\omega\rangle$. □

6. Regular spatial partitioning and GBCAs

In Section 5.1, we have shown that discrete space $\langle \mathbb{Z}^2, \mathbf{C}_8\rangle$ is a sub-structure of the continuous space $\langle \mathbb{R}^2, \mathbf{C}\rangle$. Combining this fact with regular spatial partitioning (of the continuous space $\mathbb{R}^2$), we now construct a countable RCC model (hence is also a 'continuous' model) which is a sub-structure of $\langle \mathbb{R}^2, \mathbf{C}\rangle$. Combining with Winter's approach for determining RCC5 relations [50], we also propose a hierarchical approach for determining RCC8 relations between two regions in such a model.

Suppose that we have a uniform regular partition of the real plane $\mathbb{R}^2$ of level n as $Z_n \equiv \{[\frac{i}{2^n}, \frac{i+1}{2^n}] \times [\frac{j}{2^n}, \frac{j+1}{2^n}]\colon i, j \in \mathbb{Z}\}$ for each $n \in \mathbb{N}$. Clearly, Z_n is a collection of regular closed sets in $\mathbb{R}^2$. Write $\mathcal{R}_n = \langle \mathcal{R}_n, \mathbf{C}_{\mathcal{R}_n} \rangle$ for the substructure of the BCA $\langle \mathbb{R}^2, \mathbf{C} \rangle$ generated by Z_n. It is routine to check that the connection relation $\mathbf{C}|_{\mathcal{R}_n}$, the restriction of $\mathbf{C}$ on subalgebra $\mathcal{R}_n$, is the same as that induced by the 8-adjacency $\mathbf{A}_8$ on Z_n. Write $\mathcal{R} \equiv \bigcup_{n \in \mathbb{N}} \mathcal{R}_n$ and write $\mathcal{C}$ for the restriction of $\mathbf{C}$ on $\mathcal{R}$. Then by Proposition 5.1, we know $\langle \mathcal{R}, \mathcal{C} \rangle$ is also a sub-structure of $\langle \mathbb{R}^2, \mathbf{C} \rangle$. Note that each basic element $[\frac{i}{2^n}, \frac{i+1}{2^n}] \times [\frac{j}{2^n}, \frac{j+1}{2^n}]$ of $\langle \mathcal{R}, \mathcal{C} \rangle$ has an **NTPP** part $[\frac{4i+1}{2^{n+2}}, \frac{4i+3}{2^{n+2}}] \times [\frac{4j+1}{2^{n+2}}, \frac{4j+3}{2^{n+2}}]$. $\mathcal{R} = \langle \mathcal{R}, \mathcal{C} \rangle$ is a countable BCA.

Note that each Z_n is infinite and as a result the construction of $\mathcal{R}$ is not computing feasible. There are two ways to amend this weakness. The first solution is more direct: Choose $Z'_n = \{[\frac{i}{2^n}, \frac{i+1}{2^n}] \times [\frac{j}{2^n}, \frac{j+1}{2^n}]\colon -n \leqslant i, j \leqslant n-1\} \cup \{\overline{\mathbb{R}^2 - [-n, n]^2}\}$, then the subalgebra generated by $\bigcup_{n \in \mathbb{N}} Z'_n$ is the same as that generated by $\bigcup_{n \in \mathbb{N}} Z_n$, namely, the subalgebra $\mathcal{R}$.

However, practical applications are always restricted to a finite area. We can also restrict the universe region to a square S, e.g., $[0, 1]^2$. A similar construction can be done for S. In this case the square S is recursively decomposed into four equal-sized quadrants and at each construction step only finite basis elements are concerned.

We adopt the second modification and assume $S = [0, 1]^2$.

Let $\Sigma = \{1, 2, 3, 4\}$ and let Σ^* be the set of finite strings over Σ. We denote by $|s|$ the length of a string $s \in \Sigma^*$. For each $s \in \Sigma^*$, we recursively associate to s a sub-square of $[0, 1]^2$ as follows:

$$x_\varepsilon = [0, 1]^2;$$
$$x_1 = [0, 1/2] \times [1/2, 1], \quad x_2 = [1/2, 1]^2, \quad x_3 = [0, 1/2]^2,$$
$$x_4 = [1/2, 1] \times [0, 1/2].$$

In general, if x_s has been defined for some string s, we define x_{s1}, x_{s2}, x_{s3} and x_{s4} in order as the top left, top right, bottom left and bottom right sub-square of x_s respectively.

Write $X_n = \{x_s\colon s \in \Sigma^* \text{ with } |s| = n\}$ for each $n \geqslant 1$ and write $X_\omega = \{x_s\colon s \in \Sigma^*\}$. Denote by $\mathcal{S}_n$ the subalgebra of $\Omega([0, 1]^2)$, the complete Boolean algebra of regular closed subsets of $[0, 1]^2$, generated by X_n and, by $\mathcal{S}_\omega$, the subalgebra generated by X_ω. Clearly $\mathcal{S}_n$ contains 4^n atoms and $\mathcal{S}_\omega$ is a countable atomless Boolean algebra. Moreover, we have $\mathcal{S}_n$ is a subalgebra of $\mathcal{S}_m$ for $n < m$ and $\mathcal{S}_n$ is a subalgebra of $\mathcal{S}_\omega$ for each n. It is also clear that $\mathcal{S}_\omega = \bigcup_{n \in \mathbb{N}} \mathcal{S}_n = \{\bigcup_{i=1}^{n-1} x_{s_i}\colon s_i \in \Sigma^*, n \in \mathbb{N}\}$.

Denote $Y_n = \{x_s\colon s \in \Sigma^* \text{ with } |s| \leqslant n\}$ for each $n \geqslant 1$, namely $Y_n = \bigcup_{k=1}^{n} X_k$. Then, for each $n \in \mathbb{N}$, Y_n (with the ordering of set inclusion) can be visualized as a complete quadtree T_n of height n. Similarly, X_ω, with the ordering of set inclusion, can be visualized as an infinite complete quadtree T_ω. In what follows we refer in a natural way to a node in T_ω by x_s.

Recall $\langle [0, 1]^2, \mathbf{C} \rangle$ is the standard BCA associated to the regular connected space $[0, 1]^2$. Consider the substructures of $\langle [0, 1]^2, \mathbf{C} \rangle$ on $\mathcal{S}_n$ and $\mathcal{S}_\omega$. We write also these sub-GBCAs $\mathcal{S}_n$ and $\mathcal{S}_\omega$ respectively. Then $\mathcal{S}_\omega$ is the direct limit of $\{\mathcal{S}_n\}_{n \in \mathbb{N}}$. Moreover, $\mathcal{S}_\omega$ is indeed a countable BCA for that each basic region has an **NTPP** part. A region X in $\mathcal{S}_\omega$ can be represented as a (binary) region quadtree in Morton order (with digits out of $\{1, \ldots, 4\}$) [40]. See Fig. 5 for an illustration, where a node x_s is a leaf in X (illustrated as a black one) if and only if $x_s \subseteq X$ and no ancestor of x_s is contained in X. Consequently, $\mathcal{S}_\omega$ is just the Boolean algebra of all region quadtrees.

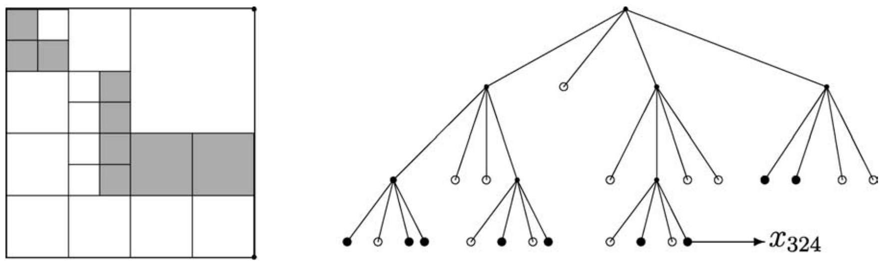

Fig. 5. A region in $\mathcal{S}_\omega$ (left) and its quadtree representation (right).

Note also that each sub-GBCA $\mathcal{S}_n$ is equivalent to an adjacency space on X_n, the set of atoms of $\mathcal{S}_n$, where two nodes x_s and x_t of length $|s| = |t| = n$ are adjacent if and only if they are connected, namely $x_s \cap x_t \neq \emptyset$. Clearly this adjacency relation is the same as $\mathbf{A}_8$. For a node x_s, write $N_s = \{x_t\colon\ s \neq t \text{ and } \mathbf{A}_8(x_s, x_t)\}$ for the set of 8-neighbors of x_s. Then N_s contains at most 8 nodes, these neighbors are referred in order as, NW-neighbor, N-neighbor, NE-neighbor, W-neighbor, E-neighbor, SW-neighbor, S-neighbor, SE-neighbor. There are algorithms for determining these neighbors. For example, suppose $s = a_1a_2 \cdots a_n \in \Sigma^*$. Then x_t, the N-neighbor of x_s, is determined as follows: if $a_i \leqslant 2$ for all i, then x_t does not exist; if otherwise, suppose $1 \leqslant k \leqslant n$ is the largest number such that $a_k > 2$, write $b_i = a_i$ if $i < k$, $b_k = a_k - 2$ and $b_i = a_i + 2$ if $i > k$. Then x_t is the N-neighbor of x_s, where $t = b_1b_2 \cdots b_n$.

We now sketch a hierarchical approach to determine RCC8 relations between regions in $\mathcal{S}_\omega$. Our approach is based on the hierarchical approach for determining RCC5 relations proposed by Winter [50].

Using the Egenhofer's 4-*intersection* model [18,19], Winter [50] proposed a hierarchical spatial reasoning method to determine the RCC5 topological relations between two region quadtrees. The hierarchical approach starts at the root level with total uncertainty about a topological relation and refines the results incrementally level by level. The process stops immediately when the refined information is sufficient to answer a given query. Combining with the Winter's approach, we now can determine hierarchically the RCC8 topological relations.

We begin with the connection $\mathbf{C}$. In what follows, we say a node x_s *belonging to* a region X if x_s or one of its ancestors x_t is a leaf in X.

Suppose X and Y are two regions in $\mathcal{S}_\omega$. We pose the query: "Does X connect with Y?". We suppose the depth of X, d, is not smaller than that of Y. Recall in Proposition 5.2, we have shown X and Y are connected in $\mathcal{S}_\omega$ if and only if they are connected in $\mathcal{S}_d$. Consequently, to determine the connectedness of X and Y, considering the adjacency relations between two nodes with lengths lesser than d is enough.

We examine the two quadtrees level by level. Within each level the tree nodes are referred to in Morton order (with digits out of $\{1, \ldots, 4\}$), e.g., the rightmost node in level 3 of Fig. 5 is noted as x_{324}. For a node x_s of level $n \leqslant d$, if x_s is a node belonging to X and $N_s \cup \{x_s\}$ contains a node belonging to Y, then X is connected with Y. The process stops. Do this for all nodes of level n alphabetically, if the results are all "NO", turn to level $n + 1$. The process will stop at some level $m \leqslant d$.

Next, combining with Winter's approach, we can determine the RCC8 relation between X and Y as follows. For two region quadtrees X, Y, we first apply Winter's approach to determine the RCC5 relation between X, Y, and refine the relation to an RCC8 relation. If $\mathbf{PO}(X, Y)$ or $\mathbf{EQ}(X, Y)$, then this is already an RCC8 relation. If $\mathbf{DR}(X, Y)$, then we pose the query: "Does X connect with Y?". Answer "YES" will give $\mathbf{EC}(X, Y)$ and "NO", $\mathbf{DC}(X, Y)$. If $\mathbf{PP}(X, Y)$, we ask "Does X connect with the complement quadtree of Y?". Answer "Yes" will indicate $\mathbf{TPP}(X, Y)$ and "NO", $\mathbf{NTPP}(X, Y)$ (see Theorem 3.3). The case of $\mathbf{PP}(Y, X)$ is similar.

7. Conclusions and further work

This paper is mainly concerned with a unified treatment of two formal systems of spatial reasoning: the theory of RCC premised on continuous-space models introduced in [32] and that of Galton premised on discrete-space models [24]. The latter is developed to bridge the gap between high-level qualitative approaches to spatial information and low-level quantitative ones. The theory developed in this paper, GRCC, admits continuous models as well as discrete ones. The relation between continuous models and discrete ones is also clarified by introducing some operations on models of GRCC. In particular, we have proposed a general approach for constructing countable RCC models as direct limits of collections of finite models. More importantly, we construct two interesting RCC models: one is a minimal RCC model, the other is a countable sub-model of the continuous space $\mathbb{R}^2$. Compared with standard RCC models given rise from regular connected spaces, these countable models have the nice property that each region can be constructed in finite steps from basic regions.

To conclude this paper, we would like to mention some interesting problems for the further studies:

- This paper only describes relation between different models. But relation between reasoning and representation in different models is still not clear. One may wonder that "Can we use reasoning and representation in a (usually more simple) model to approximately solve a corresponding problem in another (usually more complex) model?". For example, can we approximately deduce a certain property in $\mathbb{R}^2$ by reasoning in the countable sub-model $\mathcal{R}$?
- The complexity of reasoning with the RCC theory have been investigated by Renz and Nebel in a series of paper [33,35,36] (see also [34]). In particular, a maximum tractable fragment of the RCC8 theory was identified in [35]. Renz [33] also showed that any path-consistent atomic network has a canonical model in any n dimensional Euclidean space. The problem here is "Are these results hold in the countable RCC model $\mathcal{R}$?". Note that, in this model, each region can be constructed in finite steps from basic regions, here are squares.[13]

[13] In [29], we have shown any path-consistent atomic network has a realization in any RCC model. This suggests the complexity results of Renz and Nebel also hold for countable RCC models.

- The topic of regions with indeterminate boundaries has now become popular in the context of GIS as well as in AI (see [5,9,12,38], also see SVUG 2001, the first COSIT workshop on Spatial Vagueness, Uncertainty and Granularity). Roy and Stell [38] develop an algebraic framework to spatial indeterminacy using Łukasiewicz algebras which provides a generalization of the "egg-yolk" approach of Cohn and Gotts [12]. Based on this framework, they extend RCC relations from crisp regions to indeterminate ones, semantically and syntactically. Similar to RCC, [38] assumes that the indeterminate regions relate to continuous space. Clearly, techniques analogous to those proposed in this paper can be applied to these Łukasiewicz models. In conjunction with this algebraic approach, it seems that the formal framework of resolution and multi-resolution developed by Worboys can be fully reformulated in terms of substructures. This will be worked out in another paper.

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