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RCC8 binary constraint network can be consistently extended ☆

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Abstract

The RCC8 constraint language developed by Randell et al. has been popularly adopted by the Qualitative Spatial Reasoning and GIS communities. The recent observation that RCC8 composition table describes only weak composition instead of composition raises questions about Renz and Nebel's maximality results about the computational complexity of reasoning with RCC8.

This paper shows that any consistent RCC8 binary constraint network (RCC8 network for short) can be consistently extended. Given Θ, an RCC8 network, and z, a fresh variable, suppose $x\mathbf{T}y \in \Theta$ and $\mathbf{T}$ is contained in the weak composition of $\mathbf{R}$ and $\mathbf{S}$. This means that we can add two new constraints $x\mathbf{R}z$ and $z\mathbf{S}y$ to Θ without changing the consistency of the network. The result guarantees the applicability to RCC8 of one key technique, (Theorem 5) of [J. Renz, B. Nebel, On the complexity of qualitative spatial reasoning: A maximal tractable fragment of the Region Connection Calculus. Artificial Intelligence 108 (1999) 69–123], which allows the transfer of tractability of a set of RCC8 relations to its closure under composition, intersection, and converse.

Keywords: Qualitative spatial reasoning; Computational complexity; Region connection calculus; Binary constraint network; Extensionality; Path-consistency

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1. Introduction

Constraint Satisfaction Problems (CSPs) have played a significant role in many areas of Artificial Intelligence such as vision, resource allocation in scheduling, and temporal and spatial reasoning [5,24]. Ladkin and Maddux formulate binary CSP concepts and methods using relation algebra, and this "clarifies the mathematics of binary constraint satisfaction methods, and allows problems with finite or potentially infinite domains to be handled in a uniform way". [10]

When formulating a problem as a binary CSP, we usually (implicitly) assume that the underlying relation algebra is a proper relation algebra [10] for the universe of the problem. For example, the well-known interval algebra [1] and point algebra [25] used in temporal reasoning are both proper relation algebras for the corresponding universe. This means that operations in these algebras, e.g., converse, intersection, composition, coincide with the usual set-theoretical operations.

The situation, however, is different in qualitative spatial reasoning (QSR), where a composition table usually describes only weak composition. Given a universe U, suppose $\mathcal{A} = \{\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n\}$ is a collection of jointly exhaustive and pairwise disjoint (JEPD) relations on U. For two relations $\mathbf{R}_i, \mathbf{R}_j$, the *weak composition* [6] of $\mathbf{R}_i$ and $\mathbf{R}_j$ is a subset of $\mathcal{A}$, written as $\mathbf{R}_i ; \mathbf{R}_j$, such that for any $1 \leqslant k \leqslant n$, $\mathbf{R}_k \in \mathbf{R}_i ; \mathbf{R}_j$ iff there exist $a, b, c \in U$ such that $a\mathbf{R}_i b$, $b\mathbf{R}_j c$ and $a\mathbf{R}_k c$ hold. Summarizing all weak compositions in an $n \times n$ table, we obtain the *weak composition table* of $\mathcal{A}$. If $\mathbf{R}_k \in \mathbf{R}_i ; \mathbf{R}_j$, we call $\langle \mathbf{R}_i, \mathbf{R}_k, \mathbf{R}_j \rangle$ a *composition triad*, and a composition triad $\langle \mathbf{R}_i, \mathbf{R}_k, \mathbf{R}_j \rangle$ is said to be *extensional* on U if $\mathbf{R}_k \subseteq \mathbf{R}_i \circ \mathbf{R}_j$, that is, for any two $a, c \in U$ with $a\mathbf{R}_k c$ we have a region b in U such that $a\mathbf{R}_i b$ and $b\mathbf{R}_j c$ hold, where $\circ$ is the set-theoretical relational composition on U. If all composition triads are extensional on U, then the weak composition table describes indeed composition, and we say it is *extensional*.

This paper is mainly concerned with the RCC8 constraint language, which was developed by Randell, Cohn, and Cui [4,17,18], and has now been popularly adopted by the QSR and GIS communities (see [21] for more information). Intuitively, spatial regions in RCC8 can be interpreted as nonempty regular closed subsets of some topological space. The composition table of RCC8 base relations has been independently established by Cui, Cohn, Randell [4] and Egenhofer [8]. This composition table, known as *RCC8 composition table*, however, is only a *weak* one. This is because spatial regions in RCC need not be one-piece and without holes [12,13]. For instance, given three disconnected regions o, p, q, let a be the union of o and p, and c the union of o and q. Then a partially overlaps c. According to the RCC8 composition table, **PO** (*partially overlap*) is in the cell specified by **EC** (*externally connected*) and **NTPP** (*non-tangential proper part*). But it is easy to see that there is no region b such that $\mathbf{EC}(a, b)$ and $\mathbf{NTPP}(b, c)$ hold at the same time. This suggests that $\mathbf{PO} \not\subseteq \mathbf{EC} \circ \mathbf{NTPP}$. As a matter of fact, Li and Ying [13] identify altogether 35 such non-extensional composition triads.

The above observation raises questions about Renz and Nebel's maximality results about the computational complexity of reasoning with RCC8 [20,22]. Indeed, if $\mathcal{U}$, the universe of an RCC8 binary constraint network (*RCC8 network* or simply *network* for short), is the collection of nonempty regular closed subsets of some topological space (or non-zero elements in a GRCC model [14]), *consistent base networks are even not nec-*

essarily path-consistent! For example, the RCC8 network $\Theta = \{x\mathbf{EC}y, y\mathbf{NTPP}z, x\mathbf{PO}y\}$ clearly has a solution in $\mathcal{U}$, but it is not path-consistent since $\mathbf{PO} \not\subseteq \mathbf{EC} \circ \mathbf{NTPP}$. Düntsch expresses the following concern:

> In the light of this it seems that some of the results in [76-78] [here [20,22]] are valid only in extensional interpretations of the weak RCC8 table such as the closed circles or areas bounded by closed Jordan curves, and not for RCC models. [5, footnote 1]

More important, the applicability of one key technique used in [20,22] to RCC8 becomes questionable now. To show that reasoning with RCC8 relations is in general NP-complete and to identify the boundary between tractability and NP-hardness, Renz and Nebel [20,22] use the following theorem to transfer tractability of a set of RCC8 relations $\mathcal{S}$ to its closure, $\widehat{\mathcal{S}}$, under composition, intersection, and converse, where $\mathbf{RSAT}(\mathcal{S})$ is the problem of deciding consistency of networks over $\mathcal{S}$.

Theorem 5 [22]. *Let $\mathcal{C}$ be a set of binary relations that is closed under composition, intersection, and converse. Then for any subset $\mathcal{S} \subseteq \mathcal{C}$ that contains the universal relation and the identity relation,*[1] *the problem $\mathbf{RSAT}(\widehat{\mathcal{S}})$ can be polynomially reduced to $\mathbf{RSAT}(\mathcal{S})$.*

This theorem suggests that, if $\mathcal{S}$ is a subset of $\mathcal{C}$ that contains the universal relation and the identity relation, then for any $\mathcal{T} \subseteq \mathcal{C}$ with $\mathcal{S} \subseteq \mathcal{T} \subseteq \widehat{\mathcal{S}}$, $\mathbf{RSAT}(\mathcal{S})$ is tractable if and only if $\mathbf{RSAT}(\mathcal{T})$ is. Renz and Nebel establish this reduction by constructing for each network Θ over $\widehat{\mathcal{S}}$ a network Θ' over $\mathcal{S}$, such that Θ' is consistent iff Θ is. This approach does not work for some calculi that use weak compositions (see Example 7.1 of this paper). Now since RCC8 uses weak composition instead of composition, its applicability to RCC8 becomes questionable.

This paper intends to remove all these doubts. To begin with, we address the ambiguity of the concept "path-consistency". This concept is usually defined as follows [22, p. 73, last paragraph]: A binary constraint network is *path-consistent* if and only if for any consistent instantiation of any two variables, there exists an instantiation of any third variable such that the three values taken together are consistent.

Note that this definition closely depends on the choice of universe. There is, however, another definition of path-consistency that is independent of the choice of universe. This definition is given by Ladkin and Maddux [10] in a more general manner using relation algebras. By this definition, a binary constraint network $\Theta = \{x_i\mathbf{R}_{ij}x_j\colon \mathbf{R}_{ij} \in \mathfrak{A}, 1 \leqslant i, j \leqslant n\}$ over an atomic relation algebra $\mathfrak{A}$ is *path-consistent* if and only if for any $1 \leqslant i, j, k \leqslant n$, $\mathbf{R}_{ii} \leqslant 1'$, $\mathbf{R}_{ij} = \mathbf{R}_{ji}^{\smallsmile}$ and $\mathbf{R}_{ij} \leqslant \mathbf{R}_{ik} ; \mathbf{R}_{kj}$, where $1'$, $^{\smallsmile}$ and ; are, respectively, the identity, the converse, and the composition of $\mathfrak{A}$. Under this interpretation, a consistent RCC8 network necessarily contains a path-consistent refinement (see Lemma 4.1 of this paper).

To show that Theorem 5 in [22] really holds for RCC8 relations, we show that each consistent RCC8 network can be further extended at least one-shot. Suppose Θ is a consistent

[1] The reason that $\mathcal{S}$ should contain the identity relation is because we require any two spatial variables to be constrained by one and only one relation in a binary CSP (see [16]).

RCC8 network. This means, for any three RCC8 relations $\mathbf{R}$, $\mathbf{S}$, $\mathbf{T}$ with $\mathbf{T} = \mathbf{R}; \mathbf{S}$ and any constraint $x_i \mathbf{T} x_j \in \Theta$, the RCC8 network $\Theta' = \Theta \cup \{x_i \mathbf{R} z, z \mathbf{S} x_j\}$ is also consistent, where z is a fresh variable. This result guarantees the validity of the reduction method given in the proof of [22, Theorem 5] for RCC8.

Our proof of this statement is by construction. In an earlier paper, Li [11] gives an $O(n^3)$ algorithm to generate a realization in certain topological space for every path-consistent RCC8 base network. This construction can be further simplified and adapted for the present purpose. Indeed, we shall construct a canonical RCC8 model and show that every path-consistent RCC8 network has a one-shot extensible realization in this model (see Definition 5.2 and Proposition 5.1).

The rest of this paper proceeds as follows. In Section 2 we recall basic concepts of the RCC8 constraint language. Section 3 introduces a canonical RCC8 model. Section 4 describes our *One-shot Extensible Realization Algorithm*. We also show in this section that the model introduced in Section 3 is indeed a canonical model. Then, in Section 5 we show that this algorithm also generates a one-shot extensible realization. As a byproduct, Section 6 gives an affirmative answer to a conjecture made by Balbiani et al. [2] that every *infinite* path-consistent RCC8 base network is satisfiable. Further discussions and open questions are given in Section 7.

2. The RCC8 relation algebra

In this section we recall some basic concepts of the RCC8 constraint language.

2.1. RCC models and RCC8 relations

There are several equivalent formulations of the RCC theory. We here adopt the one using Boolean connection algebras given by Stell [23].

Definition 2.1 [23]. An RCC model is a Boolean algebra A containing more than two elements, together with a binary connection relation $\mathbf{C}$ on $A - \{\bot\}$ that satisfies the following conditions:

A1. $\mathbf{C}$ is reflexive and symmetric;
A2. $(\forall x \in A - \{\bot, \top\})\mathbf{C}(x, x')$;
A3. $(\forall xyz \in A - \{\bot\})\mathbf{C}(x, y \vee z) \leftrightarrow \mathbf{C}(x, y)$ or $\mathbf{C}(x, z)$;
A4. $(\forall x \in A - \{\bot, \top\})(\exists z \in A - \{\bot, \top\})\neg\mathbf{C}(x, z)$,

where $\bot$ and $\top$ are, respectively, the bottom and the top element of A, x' is the complement of x in A, $x \vee z$ is the least upper bound (lub) of x, and z in A.

In what follows, we also call any 2-tuple $\langle A, \mathbf{C}\rangle$ a *connection structure* provided that A is a Boolean algebra and $\mathbf{C}$ is a binary relation on $A - \{\bot\}$ that satisfies conditions A1 and A3 in Definition 2.1. A connection structure $\langle A, \mathbf{C}\rangle$ is called a GRCC model if it further satisfies condition A2 [14].

Table 1
RCC8 base relations and their topological interpretation

Relation	Interpretation	Topological interpretation
EQ(A,B)	A is *identical with* B	$A = B$
DC(A,B)	A is *disconnected from* B	$A \cap B = \emptyset$
EC(A,B)	A is *externally connected to* B	$A^\circ \cap B^\circ = \emptyset$ $A \cap B \neq \emptyset$
PO(A,B)	A *partially overlaps* B	$A^\circ \cap B^\circ \neq \emptyset$ $A \not\subseteq B$, $A \not\supseteq B$
TPP(A,B)	A is a *tangential proper part of* B	$A \subset B$, $A \not\subseteq B^\circ$
TPPi(A,B)	B is a *tangential proper part of* A	$B \subset A$, $B \not\subseteq A^\circ$
NTPP(A,B)	A is a *non-tangential proper part of* B	$A \subset B^\circ$
NTPPi(A,B)	B is a *non-tangential proper part of* A	$B \subset A^\circ$

Given a topological space X, we denote by $\mathsf{RC}(X)$ the complete Boolean algebra of regular closed subsets of X. We say two regions A and B in $\mathsf{RC}(X)$, that is, two nonempty regular closed sets of X, are *connected* if they have nonempty intersection. Denote this connectedness relation by $\mathbf{C}_X$. It is easy to verify that $\langle \mathsf{RC}(X), \mathbf{C}_X \rangle$ is a connection structure. If X happens to be a connected (connected regular, resp.) topological space, then this connection structure is also a GRCC (RCC, resp.) model [9,14,23].

Among others, there are eight JEPD relations that can be defined in the (G)RCC theory. These relations are known as RCC8 base relations, which we denote by $\mathcal{B}$. Table 1 gives topological interpretations of RCC8 base relations. To represent indefinite topological information, we often use disjunctions of RCC8 base relations. This results in $2^8 = 256$ different RCC8 relations altogether (including the empty relation and the universal relation). In what follows, we write $\mathcal{R}_8$ for the set of RCC8 relations. In general, an RCC8 relation is described by a set of RCC8 base relations. For example, the *overlap* relation **O** is just the set {**PO**, **EQ**, **TPP**, **NTPP**, **TPPi**, **NTPPi**}. If an RCC8 relation contains only one base relation, say **R**, we write it simply **R** rather than {**R**}. We also write "=" for **EQ**.

2.2. RCC8 composition table

The set of RCC8 relations $\mathcal{R}_8$ can be interpreted over the collection of closed disks in the Euclidean plane (see [5]). Under this interpretation, $\mathcal{R}_8$ forms a binary relation algebra, and we call this *RCC8 algebra*. In particular, $\mathcal{R}_8$ is closed under composition. Since this algebra is finite and contains 8 base relations, we can represent its composition by a 8×8 table, which specifies the composition of any two base RCC8 relations. Note that **EQ** is the identity relation; we often omit the row and column involving **EQ**. Table 2 gives the composition table of this algebra.

In [12], Li and Ying show that the collection of simple regions in the Euclidean plane is also a representation of RCC8 algebra. Moreover, they also show this representation is in a sense a maximal one.

The situation is rather different if the RCC8 relations are interpreted in an RCC model (a GRCC model, or a topological space). We first fix some notations.

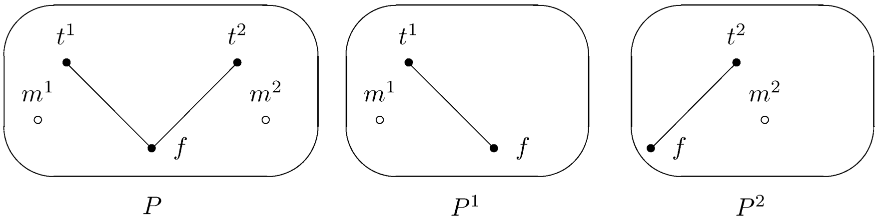

Fig. 1. A component of the canonical RCC8 model.

Given three RCC8 base relations $\mathbf{R}, \mathbf{S}, \mathbf{T}$, recall that $\langle \mathbf{R}, \mathbf{T}, \mathbf{S} \rangle$ is called a *composition triad* if $\mathbf{T}$ is in the cell specified by the ordered pair $(\mathbf{R}, \mathbf{S})$ in Table 2. It has been proved in [13] that for any RCC model R and any composition triad $\langle \mathbf{R}, \mathbf{T}, \mathbf{S} \rangle$, $\mathbf{T} \cap \mathbf{R} \circ \mathbf{S} \neq \emptyset$ and $\mathbf{R} \circ \mathbf{S} \subseteq \bigcup \{\mathbf{T}' \colon \mathbf{T}' \in \mathbf{R}; \mathbf{S}\}$, where '$\circ$' is the usual set-theoretical composition over R and ';' is the composition operation in RCC8 algebra specified by Table 2.

Recall that a composition triad $\langle \mathbf{R}, \mathbf{T}, \mathbf{S} \rangle$ is said to be *extensional* if $\mathbf{T} \subseteq \mathbf{R} \circ \mathbf{S}$ [13]. A simple examination of RCC8 composition table then shows that not all composition triads are extensional. This was first observed by Bennett [3] for regions involving the universe. More instances were then found in [6]. Later, Li and Ying [13] performed an exhaustive examination of the extensionality of RCC8 composition table and identified 35 triads that cannot be explained extensionally. This also explains why Düntsch et al. [5,6] call this a *weak* table. Table 2 summarizes the result, where we add a superscript $^\times$ to each cell entry that cannot be interpreted extensionally in the standard RCC model $\mathsf{RC}(\mathbb{R}^2)$.

3. A canonical RCC8 model

Our canonical RCC8 model[2] is a topological space that contains countably many homeomorphic disjoint components. We begin by introducing the basic component of the space.

Let $P = \{f, t^1, t^2, m^1, m^2\}$ be a partially ordered set with $f < t^1, t^2$ and all other pairs are not comparable (see Fig. 1). We refer to these points in order: the *false* point (f), the *left true* point (t^1), the *right true* point (t^2), the *left middle* point (m^1), and the *right middle* point (m^2) of P.

Consider the lower topology $\mathcal{T}$ of $(P, \leqslant)$. It has a (minimal) base containing five open sets, viz. ${\uparrow} f = \{f, t^1, t^2\}$, ${\uparrow} t^1 = \{t^1\}$, ${\uparrow} t^2 = \{t^2\}$, ${\uparrow} m^1 = \{m^1\}$, and ${\uparrow} m^2 = \{m^2\}$. Clearly, $\mathsf{RC}(P)$, the regular closed algebra of P, contains 16 elements, namely

- $\emptyset$, $\{m^1\}$, $\{m^2\}$, $\{m^1, m^2\}$;
- $\{f, t^1\}$, $\{f, t^1, m^1\}$, $\{f, t^1, m^2\}$, $\{f, t^1, m^1, m^2\}$;
- $\{f, t^2\}$, $\{f, t^2, m^1\}$, $\{f, t^2, m^2\}$, $\{f, t^2, m^1, m^2\}$;
- $\{f, t^1, t^2\}$, $\{f, t^1, t^2, m^1\}$, $\{f, t^1, t^2, m^2\}$, and $P = \{f, t^1, t^2, m^1, m^2\}$.

[2] A canonical RCC8 model is a model that supports the definition of RCC8 relations and any consistent RCC8 network has a realization in it (see also [19]).

Table 2
Extensional composition table for RCC8 relations

;	DC	EC	PO	TPP	NTPP	TPPi	NTPPi
DC	DC EC PO TPP NTPP = TPPi NTPPi	DC EC$^{\times}$ PO$^{\times}$ TPP$^{\times}$ NTPP	DC EC PO TPP NTPP	DC EC$^{\times}$ PO$^{\times}$ TPP$^{\times}$ NTPP	DC EC PO TPP NTPP	DC	DC
EC	DC EC$^{\times}$ PO$^{\times}$ TPPi$^{\times}$ NTPPi	DC EC$^{\times}$ PO$^{\times}$ = TPP TPPi	DC EC PO TPP NTPP	EC PO$^{\times}$ TPP$^{\times}$ NTPP	PO$^{\times}$ TPP$^{\times}$ NTPP	DC EC	DC
PO	DC EC PO TPPi NTPPi	DC EC PO TPPi NTPPi	DC EC PO TPP TPPi = NTPP NTPPi	PO TPP NTPP	PO TPP NTPP	DC EC PO TPPi NTPPi	DC EC PO TPPi NTPPi
TPP	DC	DC EC	DC EC PO TPP NTPP	TPP NTPP	NTPP	DC = EC$^{\times}$ PO$^{\times}$ TPP TPPi	DC EC$^{\times}$ PO$^{\times}$ TPPi$^{\times}$ NTPPi
NTPP	DC	DC	DC EC PO TPP NTPP	NTPP	NTPP	DC EC$^{\times}$ PO$^{\times}$ TPP$^{\times}$ NTPP	DC EC PO TPP TPPi = NTPP NTPPi
TPPi	DC EC$^{\times}$ PO$^{\times}$ TPPi$^{\times}$ NTPPi	EC PO$^{\times}$ TPPi$^{\times}$ NTPPi	PO TPPi NTPPi	PO$^{\times}$ TPP TPPi =	PO$^{\times}$ TPP$^{\times}$ NTPP	TPPi NTPPi	NTPPi
NTPPi	DC EC PO TPPi NTPPi	PO$^{\times}$ TPPi$^{\times}$ NTPPi	PO TPPi NTPPi	PO$^{\times}$ TPPi$^{\times}$ NTPPi	PO = TPP NTPP TPPi NTPPi	NTPPi	NTPPi

For convenience, we set

$$P^1 = \{f, t^1, m^1\}, \qquad P^2 = \{f, t^2, m^2\},$$
$$Q = \{f, t^1, t^2\}, \qquad Q^1 = \{f, t^1\}, \qquad Q^2 = \{f, t^2\}.$$

We call P^1 and P^2 respectively the left and the right *branch* of P. Note that by definition we have $P^1\mathbf{EC}P^2$, $P^l\mathbf{NTPP}P$ and $\{m^l\}\mathbf{NTPP}P^l$ $(l = 1, 2)$ in the connection structure $\langle \mathsf{RC}(P), \mathbf{C}_P\rangle$.

Now we define our canonical model. Suppose that, for each ordered pair (i, j) of positive integers, P_{ij} is a homeomorphic copy of P and $P_{ij} \cap P_{mk} \neq \emptyset$ if and only if $i = m$ and $j = k$. Set $\mathfrak{P} = \bigcup\{P_{ij}\colon\ i, j \in \mathbb{N}^+\}$, where $\mathbb{N}^+$ is the set of positive integers. A set

$A \subseteq \mathfrak{P}$ is open iff $A \cap P_{ij}$ is open for each (i, j). Clearly, the collection of open sets forms a topology on $\mathfrak{P}$. Our canonical model is then the connection structure associated with $\mathfrak{P}$, i.e., $\langle \mathsf{RC}(\mathfrak{P}), \mathbf{C}_{\mathfrak{P}} \rangle$. For simplicity, we write $\mathfrak{P}$ for this model. It is clear that the RCC8 relations can be defined in this model. Note also that a region contains a false point if and only if it contains at least one true point in the same component.

Notice that for each (i, j), P_{ij}, P^1_{ij} and P^2_{ij} are all regions in this model. We call P_{ij} the (i, j)-component of $\mathfrak{P}$, and call P^1_{ij} (P^2_{ij}, resp.) accordingly the *right* (*left*, resp.) branch of the (i, j)-component of $\mathfrak{P}$.

Our construction of a realization of a consistent RCC8 base network only involves special regions in $\mathfrak{P}$.

Definition 3.1. A region A in $\mathfrak{P}$ is said to be *normal* if

$$m^l_{ij} \in A \Leftrightarrow t^l_{ij} \in A \quad (1 \leqslant i, j \leqslant n,\ l = 1, 2).$$

For a normal region A, set $\widehat{A} = \bigcup\{P_{ij}\colon\ f_{ij} \in A\}$. It is easy to see $A\mathbf{NTPP}\widehat{A}$ if $A \subset \widehat{A}$. The following proposition summarizes some basic properties about normal regions in this model.

Proposition 3.1. *Given two normal regions A, B in $\mathfrak{P}$,*

(1) *$A\mathbf{DC}B$ iff $A \cap B = \emptyset$. Particularly, $P_{ij}\mathbf{DC}P_{mk}$ iff $(i, j) \neq (m, k)$.*
(2) *$A\mathbf{EC}B$ iff $\emptyset \neq A \cap B \subseteq \{f_{ij}\colon i, j \in \mathbb{N}^+\}$.*
(3) *$A\mathbf{PO}B$ iff $A \nsubseteq B$, $B \nsubseteq A$ and $A \cap B$ is normal.*
(4) *$A\mathbf{TPP}B$ iff $A \subset B$ and $\widehat{A} = \bigcup\{P_{ij}\colon\ f_{ij} \in A\} \nsubseteq B$.*
(5) *$A\mathbf{NTPP}B$ iff $A \subset B$ and $\widehat{A} = \bigcup\{P_{ij}\colon\ f_{ij} \in A\} \subseteq B$.*

In the next section we shall show that the model $\mathfrak{P}$ is indeed a canonical RCC8 model, i.e., any consistent RCC8 network has a realization in $\mathfrak{P}$. In fact, if we restrict the model on $\mathfrak{Q} = \bigcup\{Q_{ij}\colon\ i, j \in \mathbb{N}^+\}$, then the sub-model $\mathfrak{Q}$ already has this property. But to show that each consistent RCC8 network has a one-shot extensible realization, we require each P_{ij} to contain m^1_{ij} and m^2_{ij}. This requirement particularly entails each normal region has a non-tangential proper part. (See the fifth line and the last two lines of Table 4.)

4. $\mathfrak{P}$ is a canonical RCC8 model

Recall that an RCC8 network is said to be *consistent* if it has a realization in some topological space. Recently, Düntsch and Winter [7] have shown that each RCC model can be isomorphically embedded into certain canonical model over a topological space. This shows that, if an RCC8 network has a realization in an RCC model, then it has a realization in a topological space, i.e., it is consistent. But does each consistent RCC8 network have a realization in an RCC model? This is answered affirmatively in Li [11]. Suppose R is an arbitrary RCC model. He also shows that an RCC8 network is consistent if and only if it has a realization in R [11].

Note also that an RCC8 network is consistent if and only if it has a consistent refinement of all relations to the base relations. As a consequence, to show $\mathfrak{P}$ is a canonical RCC8 model, we need only to consider networks of RCC8 base relations.

To begin with, we first show that each consistent RCC8 base network is necessarily path-consistent in the sense of [10].

Lemma 4.1. *Suppose $\Theta = \{x_i \mathbf{R}_{ij} x_j\colon\ 1 \leqslant i, j \leqslant n\}$ is a consistent RCC8 base network. Then Θ is path-consistent, i.e., for any $1 \leqslant i, j, k \leqslant n$, $\mathbf{R}_{ii} = \mathbf{EQ}$, $\mathbf{R}_{ij} = \mathbf{R}_{ji}^{\sim}$, and $\mathbf{R}_{ik} \subseteq \mathbf{R}_{ij} ; \mathbf{R}_{jk}$, where '$;$' is the composition in RCC8 algebra.*

Proof. Recall that Θ is consistent if and only if it has a realization in any RCC model [11]. The path-consistency of Θ then follows directly from the fact that $\mathbf{R}_{ij} \circ \mathbf{R}_{jk} \subseteq \bigcup \mathbf{R}_{ij} ; \mathbf{R}_{jk}$ holds in any RCC model R [13]. $\square$

Next we give an algorithm that generates a one-shot extensible realization for any path-consistent RCC8 base network.

Given $\Theta = \{x_i \mathbf{R}_{ij} x_j\colon\ 1 \leqslant i, j \leqslant n\}$ a path-consistent RCC8 base network with n different spatial variables, without loss of generality, we assume $\mathbf{R}_{ij} \neq \mathbf{EQ}$ for any $i \neq j$. Now we show that Θ has a realization in $\mathsf{RC}(\mathfrak{P})$, that is, there exist regions $X_i^* \in \mathsf{RC}(\mathfrak{P})$ $(1 \leqslant i \leqslant n)$ such that $X_i^* \mathbf{R}_{ij} X_j^*$ holds for any $1 \leqslant i, j \leqslant n$.

Table 3 describes our algorithm for constructing these X_i^*. Recall that $P_{ij} = \{f_{ij}, t_{ij}^1, t_{ij}^2, m_{ij}^1, m_{ij}^2\}$ is a homeomorphic copy of P, and for $(i, j) \neq (m, k)$, P_{ij} and P_{mk} are disjoint. Recall also that $P_{ij}^1 = \{f_{ij}, t_{ij}^1, m_{ij}^1\}$ and $P_{ij}^2 = \{f_{ij}, t_{ij}^2, m_{ij}^2\}$.

In what follows, we give a simple description of the algorithm.

To begin with, we set $X_i = P_{ii}^1 \cup \bigcup\{P_{ik}\colon\ 1 \leqslant k \leqslant n, k \neq i\}$ in step 1. Clearly all X_i are pairwise disjoint. Our strategy is then to modify the spatial scenario step by step.

We consider in step 2 how to realize the **EC** or **PO** constraints in this spatial scenario. Intuitively, if $\mathbf{R}_{ij}$ is **EC**, $X_i^* \cap X_j^*$ should contain and only contain some *false* points. To this ends, we cut the right branch of P_{ij} from X_i and add it to X_j; dually, we also cut the right branch of P_{ji} from X_j and add it to X_i. Note then that the revised X_i and X_j will meet at two points f_{ij} and f_{ji}. Indeed, we have $X_i' \cap X_j' = \{f_{ij}, f_{ji}\}$, and X_i' is externally connected to X_j'.

If $\mathbf{R}_{ij}$ is **PO**, $X_i^* \cap X_j^*$ should contain some interior points. This time we add to X_j the right branch of P_{ij} and add to X_i the right branch of P_{ji}. Note then that $X_i' \cap X_j' = P_{ij}^2 \cup P_{ji}^2$. Therefore X_i' partially overlaps X_j'.

Table 3
One-shot extensible realization algorithm

Step 1. Set $X_i = P_{ii}^1 \cup \bigcup\{P_{ik}\colon\ 1 \leqslant k \leqslant n, k \neq i\}$.
Step 2. Set $X_i' = (X_i - \{t_{ik}^2, m_{ik}^2\colon\ \mathbf{R}_{ik} = \mathbf{EC}\}) \cup \bigcup\{P_{ki}^2\colon\ \mathbf{R}_{ki} \in \{\mathbf{EC}, \mathbf{PO}\}\}$.
Step 3. Set $X_i'' = X_i' \cup \bigcup\{X_k'\colon\ \mathbf{R}_{ki} \in \{\mathbf{TPP}, \mathbf{NTPP}\}\}$.
Step 4. Set $X_i''' = X_i'' \cup \bigcup\{P_{mk}\colon\ f_{mk} \in X_j'',\ \mathbf{R}_{ji} = \mathbf{NTPP}\}$.
Step 5. Set $X_i^* = X_i''' - \{t_{ik}^2, m_{ik}^2\colon\ \mathbf{R}_{ik} = \mathbf{TPP}\}$.

To sum up, after step 2, for any two different spatial variables x_i, x_j, we have (i) $\mathbf{R}_{ij} = \mathbf{EC}$ iff $X'_i \cap X'_j = \{f_{ij}, f_{ji}\}$; (ii) $\mathbf{R}_{ij} = \mathbf{PO}$ iff $X'_i \cap X'_j = P^2_{ij} \cup P^2_{ji}$; (iii) $\mathbf{R}_{ij} \notin \{\mathbf{EC}, \mathbf{PO}\}$ iff $X'_i \cap X'_j = \emptyset$; and (iv) $f_{ii} \in X'_j$ iff $i = j$.

Next, in step 3, we consider how to realize the proper part constraints in the spatial scenario. Our intuition is simple: when $\mathbf{R}_{ji}$ is either **TPP** or **NTPP**, we should merge X'_j in X'_i. This results in the following equation:

$$X''_i = X'_i \cup \bigcup \{X'_k\colon\ \mathbf{R}_{ki} \in \{\mathbf{TPP}, \mathbf{NTPP}\}\}.$$

After step 3 we have, for $i \neq j$, $X''_i \subset X''_j$ if and only if $\mathbf{R}_{ij}$ is either **TPP** or **NTPP**; and $X''_i \cap X''_j = \emptyset$ if and only if $\mathbf{R}_{ij}$ is **DC**. What's even better, if $\mathbf{R}_{ij}$ is either **EC** or **PO**, the relation between X''_i and X''_j is the same as that between X'_i and X'_j, namely that $X''_i \mathbf{R}_{ij} X''_j$ still holds. We give proofs to these claims to illustrate where the path-consistency condition is used.

Proof. (1) If $\mathbf{R}_{ij} = \mathbf{TPP}$ or $\mathbf{R}_{ij} = \mathbf{NTPP}$, then for any k with $\mathbf{R}_{ki} \in \{\mathbf{TPP}, \mathbf{NTPP}\}$, we have $\mathbf{R}_{kj} \in \{\mathbf{TPP}, \mathbf{NTPP}\}$ since $\mathbf{R}_{kj} \subseteq \mathbf{R}_{ki}; \mathbf{R}_{ij}$ and $\mathbf{R}_{kj}$ is a base relation. This shows $X'_k \subseteq X''_j$. As a result, we have $X''_i = X'_i \cup \bigcup\{X'_k\colon\ \mathbf{R}_{ki} \in \{\mathbf{TPP}, \mathbf{NTPP}\}\} \subseteq \bigcup\{X'_k\colon\ \mathbf{R}_{kj} \in \{\mathbf{TPP}, \mathbf{NTPP}\}\} \subset X''_j$.

(2) If $\mathbf{R}_{ij} = \mathbf{DC}$, then for any m, k with $\mathbf{R}_{mi}, \mathbf{R}_{kj} \in \{\mathbf{TPP}, \mathbf{NTPP}\}$ or $m = i$ or $k = j$, we have $\mathbf{R}_{mk} = \mathbf{DC}$ by path-consistency. This shows $X'_m \cap X'_k = \emptyset$ for any two such m, k. As a result, we have $X''_i \cap X''_j = \emptyset$, i.e., $X''_i \mathbf{DC} X''_j$.

(3) If $\mathbf{R}_{ij} = \mathbf{EC}$, then for any m, k with $\mathbf{R}_{mi}, \mathbf{R}_{kj} \in \{\mathbf{TPP}, \mathbf{NTPP}\}$ or $m = i$ or $k = j$, we have $\mathbf{R}_{mk} \in \{\mathbf{DC}, \mathbf{EC}\}$ by path-consistency. This shows $X'_m \cap X'_k$ contains at most some false points for any two such m, k. Since $X'_i \cap X'_j \neq \emptyset$, we know $X''_i \cap X''_j \neq \emptyset$. As a result, we have $X''_i \mathbf{EC} X''_j$.

(4) If $\mathbf{R}_{ij} = \mathbf{PO}$, then $X''_i \cap X''_j \supseteq X'_i \cap X'_j = P^2_{ij} \cup P^2_{ji}$. Note that $\mathbf{R}_{ik} \in \{\mathbf{PO}, \mathbf{EC}, \mathbf{DC}\}$ for any k with $\mathbf{R}_{kj} \in \{\mathbf{TPP}, \mathbf{NTPP}\}$. We have $f_{ii} \notin X''_j$. This shows $X''_i \not\subseteq X''_j$. Similarly, we can show $X''_j \not\subseteq X''_i$. This suggests $X''_i \mathbf{PO} X''_j$. □

Furthermore, we can show that $X''_i \mathbf{TPP} X''_j$ holds, provided that $\mathbf{R}_{ij}$ is either **TPP** or **NTPP**. Note that in this case, we have $P^1_{ii} \subset X''_i$, $P_{ii} \cap X''_j = P^1_{ii}$ and $P^1_{ii} \mathbf{EC} P^2_{ii}$. This suggests P^2_{ii} is externally connected to both X''_i and X'_j. Hence, by definition, $X''_i \mathbf{TPP} X''_j$.

So in the next step we should differentiate **NTPP** from **TPP**. Note that for any i, m, k $(1 \leqslant i, m, k \leqslant n)$, we have either $P^1_{mk} \subset X''_i$ or $P^2_{mk} \subset X''_i$ or $P_{mk} \cap X''_i = \emptyset$. When only $P^1_{mk} \subset X''_i$ ($P^2_{mk} \subset X''_i$, resp.) holds, we say X''_i contains only the left (right, resp.) branch of the (m, k)-component P_{mk} of $\mathfrak{P}$.

Suppose $\mathbf{R}_{ji} = \mathbf{NTPP}$. To make X''_j a non-tangential proper part of X''_i, for any component P_{mk}, if X''_i contains (at least) one branch of P_{mk}, we should include the whole component in the revised ith region, i.e.

$$X'''_i = X''_i \cup \bigcup \{P_{mk}\colon\ f_{mk} \in X''_j,\ \mathbf{R}_{ji} = \mathbf{NTPP}\}.$$

We can show that $X'''_i \mathbf{R}_{ij} X'''_j$ holds for any pair (i, j). In other words, $\{X'''_i\colon\ 1 \leqslant i \leqslant n\}$ is already a realization of Θ.

The last step is a technical modification. This procedure will guarantee that any consistent RCC8 base network can be consistently extended at least one-shot.

Consider the composition triad $\langle \mathbf{EC}, \mathbf{TPP}, \mathbf{TPP} \rangle$. Suppose Θ is an RCC8 base network that contains two variables, x_1, x_2, and $\mathbf{R}_{12} = \mathbf{TPP}$. Clearly Θ is consistent. Note that by our construction $\{X_1''' = P_{11}^1 \cup P_{12}, X_2''' = X_1''' \cup P_{21} \cup P_{22}^1\}$ is a realization of Θ in the model $\mathfrak{P}$. But for a region Z in $\mathfrak{P}$, Z is externally connected to X_1''' only if $t_{11}^2 \in Z$. This shows that there cannot exist a region Z such that $X_1''' \mathbf{EC} Z \mathbf{TPP} X_2'''$.

To obtain a one-shot extensible realization, we in step 5 cut the right branch of P_{ik} from X_i''' if $\mathbf{R}_{ik} = \mathbf{TPP}$, i.e., the final form of ith region is

$$X_i^* = X_i''' - \left\{t_{ik}^2, m_{ik}^2 \colon \mathbf{R}_{ik} = \mathbf{TPP}\right\}.$$

The reader may rightly conclude that such a modification doesn't change the RCC8 relation between two regions. Furthermore, if $\mathbf{R}_{ij} = \mathbf{TPP}$, taking $Z = P_{ij}^2 \cup P_{jj}^1$, then $X_i^* \mathbf{EC} Z \mathbf{TPP} X_j^*$ holds.

5. One-shot extensibility

Given a universe U, suppose $\mathcal{A} = \{\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n\}$ is a collection of JEPD relations on U. For $\mathbf{R}, \mathbf{S}, \mathbf{T} \in \mathcal{A}$, recall we say $\langle \mathbf{R}, \mathbf{T}, \mathbf{S} \rangle$ is a composition triad if $\mathbf{T}$ is contained in $\mathbf{R}; \mathbf{S}$, the weak composition of $\mathbf{R}$ and $\mathbf{S}$ (see p. 2).

Definition 5.1 (*one-shot extensibility*). Suppose $\mathcal{A} = \{\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n\}$ is a collection of JEPD relations on a universe U. A composition triad $\langle \mathbf{R}, \mathbf{T}, \mathbf{S} \rangle$ is said to have *one-shot extensibility* if for any consistent network Θ over $\mathcal{A}$, and any constraint $x\mathbf{T}y$ in Θ, the new network $\Theta \cup \{x\mathbf{R}z, z\mathbf{S}y\}$ is also consistent.

This concept is closely related to the following one:

Definition 5.2 (*one-shot extensible realization*). Given $\mathcal{A}$, U as above, suppose Θ is a consistent network over $\mathcal{A}$, a realization $\{a_1, a_2, \ldots, a_n\}$ of Θ in U is called a *one-shot extensible realization* if the following condition is satisfied:

> For any composition triad $\langle \mathbf{R}, \mathbf{T}, \mathbf{S} \rangle$, if $a_i \mathbf{T} a_j$ for some $1 \leqslant i, j \leqslant n$, then there exists $b \in U$ such that $a_i \mathbf{R} b$ and $b \mathbf{S} a_j$ hold.

It is now easy to see that if each consistent network over $\mathcal{A}$ has a one-shot extensible realization in U, then all composition triads have one-shot extensibility.

In the rest of this section we show that *all RCC8 composition triads have one-shot extensibility*. We prove this by showing that each consistent RCC8 base network has a one-shot extensible realization.

For a consistent RCC8 base network Θ, applying the One-shot Extensible Realization Algorithm, we obtain a realization $\{X_i^* \colon 1 \leqslant i \leqslant n\}$ of Θ in the canonical model $\mathfrak{P}$. We now show that $\{X_i^* \colon 1 \leqslant i \leqslant n\}$ is a one-shot extensible realization of Θ.

To this ends, we construct for each composition triad $\langle \mathbf{R}, \mathbf{T}, \mathbf{S}\rangle$ and each pair (i, j) with $\mathbf{R}_{ij} = \mathbf{T}$ a region Z such that $X_i^* \mathbf{R} Z$ and $Z \mathbf{S} X_j^*$. Note that if we have showed that a triad $\langle \mathbf{R}, \mathbf{T}, \mathbf{S}\rangle$ satisfies this condition, then so does its inverse triad $\langle \mathbf{S}^\smile, \mathbf{T}^\smile, \mathbf{R}^\smile\rangle$. Furthermore, if either $\mathbf{R}$ or $\mathbf{S}$ is $\mathbf{EQ}$, then the triad $\langle \mathbf{R}, \mathbf{T}, \mathbf{S}\rangle$ clearly satisfies this condition. So we need only check 101 triads (see also [13, p. 139]). Table 4 (for extensional triads) and Table 5 (for non-extensional triads) summarize the construction results, recall where $\widehat{X} = \bigcup\{P_{mk}\colon\ f_{mk} \in X\}$ and $\mathcal{B}$ is the set of RCC8 base relations.

We give two examples for illustration. Note first that, by our construction, $t^1_{mk} \in X_i^*$ if and only if either $m = i$ or $\mathbf{R}_{mi} \in \{\mathbf{TPP}, \mathbf{NTPP}\}$.

Example 5.1. (1) (Line 5 of Table 4.) Suppose $X_i^* \mathbf{T} X_j^*$ with $\mathbf{T} \notin \{\mathbf{EQ}, \mathbf{TPPi}, \mathbf{NTPPi}\}$. Set $Z = \{m^1_{ji}\}$. Then $X_i^* \mathbf{DC} Z$ and $Z \mathbf{NTPP} X_j^*$.

(2) (Line 8 of Table 5.) Suppose $X_i^* \mathbf{TPP} X_j^*$. Set $Z = P^2_{ij} \cup P^1_{jj}$. Then $X_i^* \mathbf{EC} Z$ and $Z \mathbf{TPP} X_j^*$.

Proof. (1) Note that $X_i^* \not\supseteq X_j^*$ since $\mathbf{T} \notin \{\mathbf{EQ}, \mathbf{TPPi}, \mathbf{NTPPi}\}$. This suggests $t^1_{ji} \notin X_i^*$, or equivalently, $P^1_{ji} \not\subseteq X_i^*$. Since X_i^* is a normal region, we have $X_i^* \{\mathbf{DC}, \mathbf{EC}\} P^1_{ji}$.[3] Now by $Z = \{m^1_{ji}\} \mathbf{NTPP} P^1_{ji}$ and $P^1_{ji} \subset X_j^*$, we have $X_i^* \mathbf{DC} Z$ and $Z \mathbf{NTPP} X_j^*$.

(2) Note that by $X_i^* \mathbf{TPP} X_j^*$, we have $P^2_{ij} \not\subseteq X_i^*$ and $P_{jj} \cap X_i^* = \emptyset$. This shows $X_i^* \mathbf{EC} Z$ since $P^1_{ij} \subset X_i^*$ and $P^1_{ij} \mathbf{EC} P^2_{ij}$. That $Z \subset X_j^*$ is also clear. Now, by P^2_{jj} externally connected to both Z and X_j^*, we have $Z \mathbf{TPP} X_j^*$. $\square$

We summarize the above result as a proposition.

Proposition 5.1. *Each consistent RCC8 base network has a one-shot extensible realization in* $\mathfrak{P}$.

This proposition leads to the following theorem.

Theorem 5.1. *All RCC8 composition triads have one-shot extensibility.*

We now arrive at the main result of this paper.

Theorem 5.2. *Given a consistent RCC8 network* $\Theta = \{x_i \mathbf{R}_{ij} x_j\colon\ 1 \leqslant i, j \leqslant n\}$, *suppose* $\mathbf{M}$ *and* $\mathbf{N}$ *are two RCC8 relations, and suppose* $\mathbf{R}_{mk} \subseteq \mathbf{M}; \mathbf{N}$ *for some* $1 \leqslant m, k \leqslant n$. *Then the RCC8 network* $\Theta' = \Theta \cup \{x_m \mathbf{M} z, z \mathbf{N} x_k\}$ *is also consistent, where* z *is a fresh variable.*

Proof. Since Θ is consistent, it can be consistently refined to an RCC8 base network $\Theta^* = \{x_i \mathbf{R}^*_{ij} x_j\colon\ 1 \leqslant i, j \leqslant n\}$. Suppose $\mathbf{R}^*_{mk} = \mathbf{T} \in \mathcal{B}$. Then by $\mathbf{R}^*_{mk} \in \mathbf{R}_{mk} \subseteq \mathbf{M}; \mathbf{N}$, there exist two RCC8 base relations $\mathbf{R}, \mathbf{S}$ such that $\mathbf{R} \in \mathbf{M}$, $\mathbf{S} \in \mathbf{N}$ and $\langle \mathbf{R}, \mathbf{T}, \mathbf{S}\rangle$ is a composition

[3] Note that if $\mathbf{T} = \mathbf{EC}$ or $\mathbf{T} = \mathbf{TPP}$, we have $P^2_{ji} \subset X_i^*$. In these two cases we have $P^1_{ji} \mathbf{EC} X_i^*$.

Table 4
Case I: Triads without superscript $^{\times}$

Triad	$X_i^*\mathbf{T}X_j^*$	Z
$\langle\mathbf{DC},\mathbf{T},\mathbf{DC}\rangle$	$\mathbf{T}\in\mathcal{B}$	$P_{(n+1)(n+1)}$
$\langle\mathbf{DC},\mathbf{T},\mathbf{EC}\rangle$	$\mathbf{T}\in\{\mathbf{DC},\mathbf{NTPP}\}$	P_{jj}^2
$\langle\mathbf{DC},\mathbf{T},\mathbf{PO}\rangle$	$\mathbf{T}\notin\{\mathbf{EQ},\mathbf{TPPi},\mathbf{NTPPi}\}$	P_{jj}
$\langle\mathbf{DC},\mathbf{T},\mathbf{TPP}\rangle$	$\mathbf{T}\in\{\mathbf{DC},\mathbf{NTPP}\}$	P_{jj}^1
$\langle\mathbf{DC},\mathbf{T},\mathbf{NTPP}\rangle$	$\mathbf{T}\notin\{\mathbf{EQ},\mathbf{TPPi},\mathbf{NTPPi}\}$	$\{m_{ji}^1\}$
$\langle\mathbf{DC},\mathbf{DC},\mathbf{TPPi}\rangle$	$\mathbf{T}=\mathbf{DC}$	$X_j^*\cup P_{(n+1)(n+1)}$
$\langle\mathbf{DC},\mathbf{DC},\mathbf{NTPPi}\rangle$	$\mathbf{T}=\mathbf{DC}$	$\widehat{X_j^*}$
$\langle\mathbf{EC},\mathbf{T},\mathbf{EC}\rangle$	$\mathbf{T}\in\{\mathbf{DC},\mathbf{TPP},\mathbf{TPPi}\}$	$P_{ii}^2\cup P_{jj}^2$
$\langle\mathbf{EC},\mathbf{T},\mathbf{PO}\rangle$	$\mathbf{T}\notin\{\mathbf{EQ},\mathbf{TPPi},\mathbf{NTPPi}\}$	$P_{ii}^2\cup P_{ji}^1\cup P_{(n+1)(n+1)}$
$\langle\mathbf{EC},\mathbf{EC},\mathbf{TPP}\rangle$	$\mathbf{T}=\mathbf{EC}$	P_{ij}^2
$\langle\mathbf{EC},\mathbf{NTPP},\mathbf{TPP}\rangle$	$\mathbf{T}=\mathbf{NTPP}$	$P_{ii}^2\cup P_{jj}^1$
$\langle\mathbf{EC},\mathbf{NTPP},\mathbf{NTPP}\rangle$	$\mathbf{T}=\mathbf{NTPP}$	P_{ii}^2
$\langle\mathbf{EC},\mathbf{T},\mathbf{TPPi}\rangle$	$\mathbf{T}\in\{\mathbf{DC},\mathbf{EC}\}$	$P_{ii}^2\cup X_j^*$
$\langle\mathbf{EC},\mathbf{DC},\mathbf{NTPPi}\rangle$	$\mathbf{T}=\mathbf{DC}$	$P_{ii}^2\cup\widehat{X_j^*}$
$\langle\mathbf{PO},\mathbf{T},\mathbf{PO}\rangle$	$\mathbf{T}\in\mathcal{B}$	$P_{ii}^1\cup P_{jj}^1\cup P_{(n+1)(n+1)}$
$\langle\mathbf{PO},\mathbf{T},\mathbf{TPP}\rangle$	$\mathbf{T}\in\{\mathbf{PO},\mathbf{NTPP}\}$	$P_{ij}^2\cup P_{jj}^1$
$\langle\mathbf{PO},\mathbf{TPP},\mathbf{TPP}\rangle$	$\mathbf{T}=\mathbf{TPP}$	$P_{ii}^1\cup P_{jj}^1$
$\langle\mathbf{PO},\mathbf{PO},\mathbf{NTPP}\rangle$	$\mathbf{T}=\mathbf{PO}$	P_{ji}
$\langle\mathbf{PO},\mathbf{T},\mathbf{NTPP}\rangle$	$\mathbf{T}\in\{\mathbf{TPP},\mathbf{NTPP}\}$	$P_{ij}\cup P_{ji}$
$\langle\mathbf{PO},\mathbf{T},\mathbf{TPPi}\rangle$	$\mathbf{T}\notin\{\mathbf{EQ},\mathbf{TPP},\mathbf{NTPP}\}$	$P_{ii}^1\cup X_j^*\cup P_{(n+1)(n+1)}$
$\langle\mathbf{PO},\mathbf{T},\mathbf{NTPPi}\rangle$	$\mathbf{T}\notin\{\mathbf{EQ},\mathbf{TPP},\mathbf{NTPP}\}$	$P_{ii}^1\cup\widehat{X_j^*}\cup P_{(n+1)(n+1)}$
$\langle\mathbf{TPP},\mathbf{T},\mathbf{TPP}\rangle$	$\mathbf{T}\in\{\mathbf{TPP},\mathbf{NTPP}\}$	$X_i^*\cup P_{jj}^1$
$\langle\mathbf{TPP},\mathbf{NTPP},\mathbf{NTPP}\rangle$	$\mathbf{T}=\mathbf{NTPP}$	$X_i^*\cup P_{ji}$
$\langle\mathbf{TPP},\mathbf{T},\mathbf{TPPi}\rangle$	$\mathbf{T}\in\{\mathbf{DC},\mathbf{TPP},\mathbf{TPPi}\}$	$X_i^*\cup X_j^*$
$\langle\mathbf{TPP},\mathbf{DC},\mathbf{NTPPi}\rangle$	$\mathbf{T}=\mathbf{DC}$	$X_i^*\cup\widehat{X_j^*}$
$\langle\mathbf{TPP},\mathbf{NTPPi},\mathbf{NTPPi}\rangle$	$\mathbf{T}=\mathbf{NTPPi}$	$X_i^*\cup P_{(n+1)(n+1)}$
$\langle\mathbf{NTPP},\mathbf{NTPP},\mathbf{TPP}\rangle$	$\mathbf{T}=\mathbf{NTPP}$	$\widehat{X_i^*}\cup P_{jj}^1$
$\langle\mathbf{NTPP},\mathbf{NTPP},\mathbf{NTPP}\rangle$	$\mathbf{T}=\mathbf{NTPP}$	$\widehat{X_i^*}$
$\langle\mathbf{NTPP},\mathbf{T},\mathbf{NTPPi}\rangle$	$\mathbf{T}\in\mathcal{B}$	$\widehat{X_i^*}\cup\widehat{X_j^*}\cup P_{(n+1)(n+1)}$
$\langle\mathbf{TPPi},\mathbf{TPP},\mathbf{TPP}\rangle$	$\mathbf{T}=\mathbf{TPP}$	P_{ii}^1
$\langle\mathbf{TPPi},\mathbf{NTPP},\mathbf{NTPP}\rangle$	$\mathbf{T}=\mathbf{NTPP}$	P_{ii}^1
$\langle\mathbf{NTPPi},\mathbf{PO},\mathbf{NTPP}\rangle$	$\mathbf{T}=\mathbf{PO}$	$\{m_{ij}^2\}$
$\langle\mathbf{NTPPi},\mathbf{T},\mathbf{NTPP}\rangle$	$\mathbf{T}\in\{\mathbf{TPP},\mathbf{NTPP}\}$	$\{m_{ii}^1\}$

triad, i.e., $\mathbf{T}\in\mathbf{R};\mathbf{S}$. By Theorem 5.1, $\langle\mathbf{R},\mathbf{T},\mathbf{S}\rangle$ has one-shot extensibility, so $\Theta'^* = \Theta^*\cup\{x_m\mathbf{R}z, z\mathbf{S}x_k\}$ is by definition consistent. Clearly Θ' is also consistent because Θ'^* is a refinement of Θ'. $\square$

Table 5
Case II: Triads with superscript $^{\times}$

Triad	$X_i^* \mathbf{T} X_j^*$	Z
$\langle \mathbf{DC}, \mathbf{T}, \mathbf{EC} \rangle$	$\mathbf{T} \in \{\mathbf{EC}, \mathbf{PO}\}$	P_{jj}^2
$\langle \mathbf{DC}, \mathbf{T}, \mathbf{TPP} \rangle$	$\mathbf{T} \in \{\mathbf{EC}, \mathbf{PO}\}$	P_{jj}^1
$\langle \mathbf{DC}, \mathbf{TPP}, \mathbf{EC} \rangle$	$\mathbf{T} = \mathbf{TPP}$	P_{jj}^2
$\langle \mathbf{DC}, \mathbf{TPP}, \mathbf{TPP} \rangle$	$\mathbf{T} = \mathbf{TPP}$	P_{jj}^1
$\langle \mathbf{EC}, \mathbf{T}, \mathbf{EC} \rangle$	$\mathbf{T} \in \{\mathbf{EC}, \mathbf{PO}\}$	$P_{ii}^2 \cup P_{jj}^2$
$\langle \mathbf{EC}, \mathbf{PO}, \mathbf{TPP} \rangle$	$\mathbf{T} = \mathbf{PO}$	$P_{jj}^1 \cup P_{ji}^1$
$\langle \mathbf{EC}, \mathbf{PO}, \mathbf{NTPP} \rangle$	$\mathbf{T} = \mathbf{PO}$	P_{ji}^1
$\langle \mathbf{EC}, \mathbf{TPP}, \mathbf{TPP} \rangle$	$\mathbf{T} = \mathbf{TPP}$	$P_{ij}^2 \cup P_{jj}^1$
$\langle \mathbf{EC}, \mathbf{TPP}, \mathbf{NTPP} \rangle$	$\mathbf{T} = \mathbf{TPP}$	P_{ij}^2
$\langle \mathbf{TPP}, \mathbf{T}, \mathbf{TPPi} \rangle$	$\mathbf{T} \in \{\mathbf{EC}, \mathbf{PO}\}$	$X_i^* \cup X_j^*$
$\langle \mathbf{TPP}, \mathbf{T}, \mathbf{NTPPi} \rangle$	$\mathbf{T} \in \{\mathbf{EC}, \mathbf{PO}\}$	$X_i^* \cup \widehat{X_j^*}$
$\langle \mathbf{TPP}, \mathbf{TPPi}, \mathbf{NTPPi} \rangle$	$\mathbf{T} = \mathbf{TPPi}$	$X_i^* \cup \widehat{X_j^*}$
$\langle \mathbf{TPPi}, \mathbf{PO}, \mathbf{TPP} \rangle$	$\mathbf{T} = \mathbf{PO}$	$P_{ij}^2 \cup P_{ji}^2$
$\langle \mathbf{TPPi}, \mathbf{PO}, \mathbf{NTPP} \rangle$	$\mathbf{T} = \mathbf{PO}$	P_{ji}^2
$\langle \mathbf{TPPi}, \mathbf{TPP}, \mathbf{NTPP} \rangle$	$\mathbf{T} = \mathbf{TPP}$	P_{ij}^1

By this theorem, we have

Theorem 5.3. *Let $\mathcal{S}$ be any subset of RCC8 relations that contains the universal relation. Given $\mathbf{S}, \mathbf{T} \in \mathcal{S}$ with $\mathbf{R} = \mathbf{S}; \mathbf{T} \notin \mathcal{S}$, suppose Θ is a consistent network such that for any constraint $x\mathbf{M}y \in \Theta$, $\mathbf{M} \in \mathcal{S}$ or $\mathbf{M} = \mathbf{R}$. Define Θ' to be the following constraint network: If $x\mathbf{M}y \in \Theta$ and $\mathbf{M} \in \mathcal{S}$, then $x\mathbf{M}y \in \Theta'$; otherwise, add $x\mathbf{S}z$ and $z\mathbf{T}y$ to Θ', where z is a fresh variable. Then Θ' is also consistent.*

This theorem justifies the applicability of Theorem 5 in [22] to RCC8 algebra. Recall that Renz and Nebel [22] termed **RSAT** the problem of deciding consistency of RCC8 networks.

Theorem 5.4. *For any subset $\mathcal{S}$ of RCC8 relations that contains the universal relation and the identity relation, the problem $\mathbf{RSAT}(\widehat{\mathcal{S}})$ can be polynomially reduced to $\mathbf{RSAT}(\mathcal{S})$, where $\widehat{\mathcal{S}}$ is the closure of $\mathcal{S}$ under composition, intersection, and converse.*

Proof. Same as that given by Renz and Nebel in the proof of [22, Theorem 5]. The idea is to construct for each network over $\widehat{\mathcal{S}}$ a new network over $\mathcal{S}$ such that the two networks are equivalent, i.e., one is consistent iff the other is. Notice that by Theorem 5.3 of this paper the new constraint network Θ' constructed in Renz and Nebel's proof is consistent if and only if Θ is. $\square$

As a corollary, we have the following important result, which provides a key technique for identifying the boundary of tractability when reasoning with RCC8 algebra [20,22].

Corollary 5.1. **RSAT**$(\mathcal{S})$ *is in* ***P*** (***NP****-hard, resp.*) *if and only if* **RSAT**$(\widehat{\mathcal{S}})$ *is.*

6. Realization of infinite path-consistent RCC8 base network

Nebel [15] has shown that any finite RCC8 path-consistent base network is consistent. This result cannot be directly extended to the situation when the network involves infinitely many spatial variables. As a matter of fact, no proof is known for the infinite situation. In a recent paper, Balbiani et al. [2] phrased it formally as a conjecture:

Conjecture. *Let Θ be an infinite path-consistent RCC8 base network. Then Θ is satisfiable.*

We now give an affirmative answer to this conjecture. The One-shot Extensible Realization Algorithm described in Table 3 can be applied to generalize a realization of any infinite path-consistent RCC8 base network Θ. Indeed, if our aim is simply to generalize such a realization, the algorithm as well as the canonical space can be simplified.

This time we choose our basic component as $Q = \{f, t^1, t^2\}$ and our canonical space now is $\mathfrak{Q} = \bigcup\{Q_{ij}\colon\ i, j \in \mathbb{N}^+\}$. The algorithm is given in Table 6. Justification of the correctness of this algorithm is similar to the one for finite networks.

7. Further discussions and open problems

In [20], Renz gave a complete analysis for maximal tractable fragments of RCC8. Using Theorem 5 of [22], he first found more intractable subsets of RCC8 and then, based on these hardness results, he identified three candidates of maximal tractable subsets. To prove the tractability of these candidates, he also proposed a general approach for proving tractability of **RSAT** problems. This *reduction-by-refinement* approach was then used in (re-)proving tractability of the three candidates.

Since this new approach is sufficient for deciding tractability of the three candidates, it seems that the applicability of [22, Theorem 5] to RCC8 is irrelevant to the complexity results obtained in [20,22]. This is, however, not true. One key point is that the aim here is to identify the boundary between tractability and intractability. Renz's new approach

Table 6
General realization algorithm

Step 1. Set $X_i = Q^1_{ii} \cup \bigcup\{Q_{ik}\colon\ k \neq i\}$.
Step 2. Set $X'_i = (X_i - \{t^2_{ik}\colon\ \mathbf{R}_{ik} = \mathbf{EC}\}) \cup \bigcup\{Q^2_{ki}\colon\ \mathbf{R}_{ki} \in \{\mathbf{EC}, \mathbf{PO}\}\}$.
Step 3. Set $X''_i = X'_i \cup \bigcup\{X'_k\colon\ \mathbf{R}_{ki} \in \{\mathbf{TPP}, \mathbf{NTPP}\}\}$.
Step 4. Set $X^*_i = X''_i \cup \bigcup\{Q_{mk}\colon\ f_{mk} \in X''_j, \mathbf{R}_{ji} = \mathbf{NTPP}\}$.

Table 7
The definition of relations in $\mathcal{A}$ (left) and its weak composition (right)

a	eq	b	$\Leftrightarrow$	$a = b$;
a	e	b	$\Leftrightarrow$	$a - b = 1$;
a	w	b	$\Leftrightarrow$	$b - a = 1$;
a	d	b	$\Leftrightarrow$	$\lvert a - b\rvert > 1$.

;	e	w	d	eq
e	d	eq	w,d	e
w	eq	d	e,d	w
d	w,d	e,d	e,w,d,eq	d
eq	e	w	d	eq

cannot guarantee the "maximality" of these candidates; that is, suppose $\mathcal{S}$ is one of these candidates and $\mathbf{T}$ is any new RCC8 relation, this approach cannot tell us whether or not the new subset $\mathcal{S} \cup \{\mathbf{T}\}$ is intractable. More than $3 \times 76 = 228$ cases[4] should be checked, but only a few are known to be intractable [20,22]. Without proving that the rest are all intractable, we cannot be sure that these candidates are indeed maximal. This will be of course a tedious and difficult work. It would be more elegant, as suggested in [20], if we could prove the applicability of Theorem 5 of [22] to RCC8 and use this result to transfer intractability.

In this paper, the applicability of Theorem 5 of [22] to RCC8 was proved by showing that any consistent RCC8 network can be consistently extended at least one-shot (Theorem 5.2). Suppose $\mathbf{T} \subseteq \mathbf{R};\mathbf{S}$, $x\mathbf{T}y$ is in the network Θ, and z is a fresh variable. This means that we can add two new constraints $x\mathbf{R}z$ and $z\mathbf{S}y$ to Θ without changing the consistency of the network. One may wonder whether this result holds in general. From the proof for Theorem 5.2, one can conclude that, for a set of JEPD relation $\mathcal{A}$, Theorem 5.2 holds if and only if Theorem 5.1 holds. Now our question can be phrased as follows.

Question 1. Suppose U is a nonempty set, $\mathcal{A}$ is a set of JEPD relations on U that is closed under converse and contains the identity relation. Suppose $\langle \mathbf{R}, \mathbf{T}, \mathbf{S}\rangle$ is a composition triad of $\mathcal{A}$, does it have one-shot extensibility? That is, for any consistent network Θ over $\mathcal{A}$ which includes a constraint $x\mathbf{T}y$, is $\Theta \cup \{x\mathbf{R}z, z\mathbf{S}y\}$ consistent? where z is a fresh variable.

Note that if a composition triad $\langle \mathbf{R}, \mathbf{T}, \mathbf{S}\rangle$ is extensional on U (see p. 2), i.e., $\mathbf{T} \subseteq \mathbf{R} \circ \mathbf{S}$, then it has one-shot extensibility. As for RCC8 algebra, by Theorem 5.1, we know that if U is the canonical RCC8 model $\mathfrak{P}$ and $\mathcal{A}$ is the RCC8 base relations, then Question 1 has an affirmative answer for all composition triads.[5] The following example, however, shows that this is not true in general.

Example 7.1. Set the universe to be the set of integer numbers $\mathbb{Z}$. Consider the following set of JEPD relations $\mathcal{A} = \{\mathsf{e}, \mathsf{w}, \mathsf{d}, \mathsf{eq}\}$ on $\mathbb{Z}$ (see Table 7). Note that $\mathsf{d} = \mathsf{e};\mathsf{e}$, hence $\langle \mathsf{e}, \mathsf{d}, \mathsf{e}\rangle$ is a composition triad. Consider the following consistent network

$$\Theta = \{x_1\mathsf{e}x_2, x_2\mathsf{e}x_3, x_3\mathsf{e}x_4, x_1\mathsf{d}x_4\}.$$

[4] There are 76 RCC8 relations that are not in either of these candidates.

[5] The same conclusion then can be applied to any RCC model R. This is because an RCC8 network is consistent if and only if it has a realization in R [11].

Notice that $x_1 \mathsf{d} x_4$ is in Θ. We now have the following extension of Θ:

$$\Theta' = \{x_1 \mathsf{e} x_2, x_2 \mathsf{e} x_3, x_3 \mathsf{e} x_4, x_1 \mathsf{d} x_4, x_1 \mathsf{e} z, z \mathsf{e} x_4\}.$$

Θ' is, however, inconsistent. In fact, Θ' is even not path-consistent. This suggests that $\langle \mathsf{e}, \mathsf{d}, \mathsf{e} \rangle$ has no one-shot extensibility, hence the answer to Question 1 is in general negative.

This example also suggests that the reduction method proposed in the proof of Theorem 5 of [22] cannot work for all calculi using weak compositions. For example, set $\mathcal{S} = \{\mathsf{e}, \mathsf{eq}, *\}$, then $\mathsf{d} \in \widehat{\mathcal{S}}$, where $*$ is the universal relation over $\mathcal{A}$. The above network Θ is a consistent network over $\widehat{\mathcal{S}}$, but the corresponding extension, namely Θ', is inconsistent.

It will be an interesting problem to identify sufficient conditions under which every constraint triads have one-shot extensibility. There is another related and more difficult problem.

Question 2. Suppose U is a nonempty set, $\mathcal{A}$ is a set of JEPD relations on U that is closed under converse and contains the identity relation. Given a subset $\mathcal{S} \subseteq 2^{\mathcal{A}}$ that contains the universal relation and the identity relation, whether and when can the problem $\mathbf{RSAT}(\widehat{\mathcal{S}})$ be polynomially reduced to $\mathbf{RSAT}(\mathcal{S})$?

It is clear that an affirmative answer to Question 1 will lead to an affirmative answer to Question 2. But it is still not clear whether there exists a polynomial reduction from $\mathbf{RSAT}(\widehat{\mathcal{S}})$ to $\mathbf{RSAT}(\mathcal{S})$ in general.

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