

Nondeterministic Asynchronous Dataflow in Isabelle/HOL

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Motivation

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Context:

- Stream Processing: programs that compute (possibly) unbounded sequences of data (streams)
- A common problem in the industry
- Frameworks: Google Cloud Dataflow, Apache Flink, and Timely Dataflow



- Why use frameworks?
 - Highly Parallel
 - Low latency (output as soon as possible)
 - Incremental computing (re-uses previous computations)

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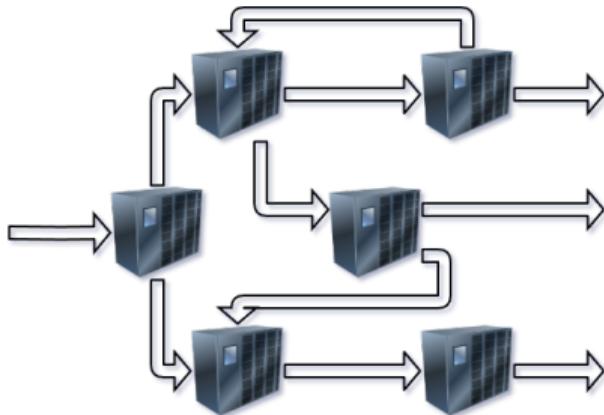
Our goal:

Mechanically Verify Timely Dataflow algorithms

A Good Foundation

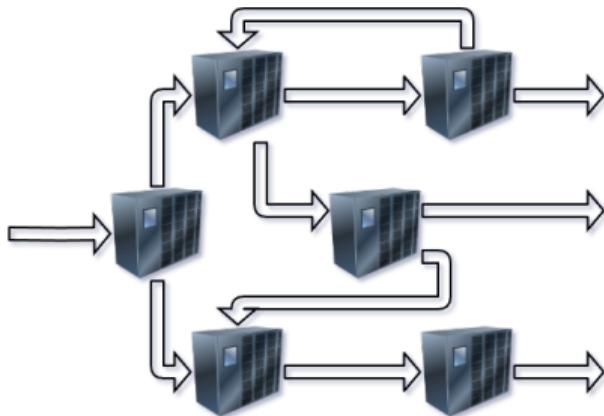
A Good Foundation

- Nondeterministic Asynchronous Dataflow

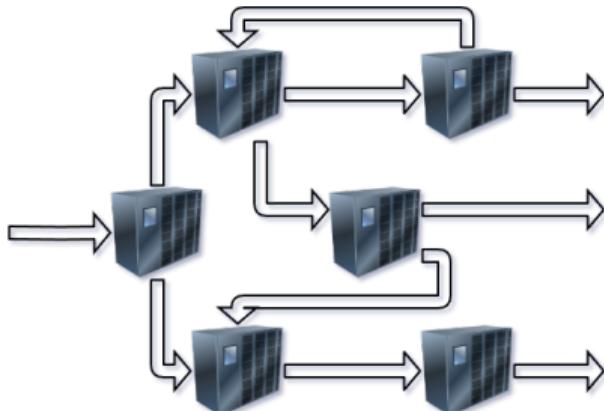


A Good Foundation

- Nondeterministic Asynchronous Dataflow
 - Dataflow: Directed graph of interconnected operators

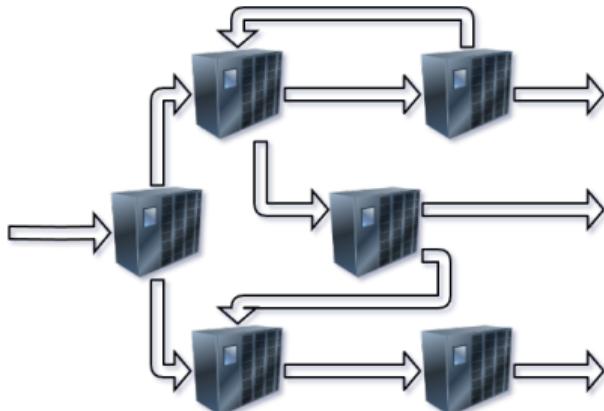


A Good Foundation



- Nondeterministic Asynchronous Dataflow
 - Dataflow: Directed graph of interconnected operators
 - Asynchronous:
 - Operators execute independently: processes without an orchestrator
 - Operators can freely communicate with the network (read/write); do silent computation steps
 - Networks are unbounded FIFO queues

A Good Foundation



- Nondeterministic Asynchronous Dataflow

- Dataflow: Directed graph of interconnected operators
- Asynchronous:
 - Operators execute independently: processes without an orchestrator
 - Operators can freely communicate with the network (read/write); do silent computation steps
 - Networks are unbounded FIFO queues
- Nondeterministic:
 - Operators can make nondeterministic choices

The Algebra for Nondeterministic Asynchronous Dataflow

- Bergstra et al. presents an algebra for Nondeterministic Asynchronous Dataflow
- Primitives:
sequential and parallel composition; feedback loop...
- 52 axioms
- A process calculus instance

Network Algebra for Asynchronous Dataflow*

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Main Contributions

- An Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
 - Operators as a shallow embedding as codatatypes
 - 51 axioms proved
- Executable via code extraction to Haskell

Isabelle/HOL Preliminaries

Isabelle/HOL

- Classical higher-order logic (HOL):
Simple Typed Lambda Calculus + axiom of choice + axiom of infinity + rank-1 polymorphism

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- Isabelle/HOL: Isabelle's flavor of HOL

Why Isabelle/HOL?

- Codatatypes: (possibly) infinite data structures (e.g., lazy lists, streams)
- Corecursion: always eventually produces some codatatype constructor
- Coinductive predicate: infinite number of introduction rule applications
- Coinduction: reason about coinductive predicates

Operators as a Codatatype

Operators

Operators in Isabelle/HOL

```
codatatype (inputs: 'i, outputs: 'o, 'd) op =  
  Read 'i ('d ⇒ ('i, 'o, 'd) op) | Write (('i, 'o, 'd) op) 'o 'd  
  Silent ('i, 'o, 'd) op | Choice (('i, 'o, 'd) op) cset
```

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- Type parameters:
inputs/output ports; data
- Operator's actions
- inputs/outputs:
Sets of used ports
- Possibly infinite trees

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Uncommunicative operators

$$\emptyset = \text{Choice } \{\}^c$$

$$\odot = \text{Silent } \odot$$

$$\otimes = \text{Choice } \{\otimes\}^c$$

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Uncommunicative operators

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More examples

$$\text{ex1} = \text{Choice } \{\text{Write ex1 1 42}, \emptyset\}^c$$

$$\text{ex2} = \text{Choice } \{\text{Write ex2 1 42}, \text{ex2}\}^c$$

$$\text{ex3} = \text{Choice } \{\text{Write ex3 1 42}, \text{Silent ex3}\}^c$$

An ideal equivalence relation

- Only equate operators that **behave** the same
- Useful reasoning principle

Operators Equivalences: Motivation

An ideal equivalence relation

- Only equate operators that **behave** the same
- Useful reasoning principle

- Milner's approach: Weak Bisimilarity



- Based on labeled transition systems (LTS)
- Labels (actions): Read, Write, Silent (τ)

Operators Equivalences: Label Transition System

Label Transition System

```
datatype ('i,'o,'d) IO = Inp 'i 'd | Out 'o 'd | Tau
```

inductive step where

```
step (Inp p x) (Read p f) (f x)
| step (Out q x) (Write op q x) op
| step Tau (Silent op) op
| op ∈c ops ⇒ step io op op' ⇒ step io (Choice ops) op'
```

Operators Equivalences: Weak Bisimilarity

Weakly Simulates

$$\text{estep } \text{Tau} = (\text{step } \text{Tau})^{==}$$

$$\text{estep } io = \text{step } io$$

$$\text{wstep } io = (\text{step } \text{Tau})^{**} \text{ OO } (\text{estep } io) \text{ OO } (\text{step } \text{Tau})^{**}$$

$$\text{wsim } R \text{ } op_1 \text{ } op_2 = (\forall io \text{ } op'_1. \text{ step } io \text{ } op_1 \text{ } op'_1 \longrightarrow (\exists op'_2. \text{ wstep } io \text{ } op_2 \text{ } op'_2 \wedge R \text{ } op'_1 \text{ } op'_2))$$

- A Relation R is a *weak simulation* if $\forall op_1 \text{ } op_2. \text{ } R \text{ } op_1 \text{ } op_2 \longrightarrow \text{wsim } R \text{ } op_1 \text{ } op_2$.

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Weak Bisimilarity

coinductive wbisim (**infix** ≈ 40) **where**

$\text{wsim } (\approx) \text{ } op_1 \text{ } op_2 \implies \text{wsim } (\approx) \text{ } op_2 \text{ } op_1 \implies op_1 \approx op_2$

- R is a *weak bisimulation* when both R and its converse R^{-1} are weak simulations.
Weak bisimilarity is the largest weak bisimulation.

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Weak Bisimilarity

coinductive `wbisim` (infix ≈ 40) **where**

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- R is a *weak bisimulation* when both R and its converse R^{-1} are weak simulations.
Weak bisimilarity is the largest weak bisimulation.
- \approx has a useful coinduction principle
- $\emptyset \approx \otimes \approx \odot$ and $\text{ex1} \approx \text{ex2} \approx \text{ex3}$

Asynchronous Dataflow Operators

Auxiliary definitions

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Buffer functions

BHD $p \text{ buf} = \text{hd} (\text{buf } p)$

BTL $p \text{ buf} = \text{buf}(p := \text{tl} (\text{buf } p))$

BENQ $p \times \text{buf} = \text{buf}(p := \text{buf } p @ [x])$

$\text{buf}_1 \gg \text{buf}_2 = (\lambda p. \text{buf}_2 p @ \text{buf}_1 p)$

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choices function

$\text{choices } \otimes = \{\}$

Silent $op' \in_c \text{choices } op \longleftrightarrow \text{step Tau } op \text{ } op'$

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Mapping ports functions

$\text{map_op} :: ('i_1 \Rightarrow 'i_2) \Rightarrow ('o_1 \Rightarrow 'o_2) \Rightarrow ('i_1, 'o_1, 'd) \text{ op} \Rightarrow ('i_2, 'o_2, 'd) \text{ op}$

$\text{projl} :: ('a + 'b) \Rightarrow 'a$

$\text{projr} :: ('a + 'b) \Rightarrow 'b$

$\curvearrowleft :: ('a + 'b) + 'c \Rightarrow 'a + 'b + 'c$

$\curvearrowright :: 'a + 'b + 'c \Rightarrow ('a + 'b) + 'c$

Equation of the identity operator

$$\text{id_op } buf = \text{Choice}$$
$$\cup_c$$

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$((((\lambda p. \text{Read } p (\lambda x. \text{id_op } (\text{BENQ } p \times buf)))) \backslash_c \mathfrak{U}_c) \\ \cup_c$

- \mathfrak{U}_c is the set of usable ports provide by its type

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$((\lambda p. \text{Write} (\text{id_op} (\text{BTL } p \ buf)) \ p \ (\text{BHD } p \ buf)) \backslash_c \ \{p \in_c \mathfrak{U}_c \mid buf \ p \neq []\}))$

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$((\lambda p. \text{Write} (\text{id_op} (\text{BTL } p buf)) p (\text{BHD } p buf)) \ `_c \ \{p \in_c \mathfrak{U}_c \mid buf \neq []\})$

- \mathfrak{U}_c is the set of usable ports provide by its type
- Stream delay!

Equation of the identity operator

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- \mathfrak{U}_c is the set of usable ports provide by its type
- Stream delay!

Identity operator with an empty buffer

$\mathcal{I} = \text{id_op} (\lambda_. \ [])$

Composition: Preliminaries

Composition operator type

```
comp_op :: ('o1 ⇒ 'i2 option) ⇒ ('i2 ⇒ 'd list) ⇒ ('i1, 'o1, 'd) op ⇒ ('i2, 'o2, 'd) op ⇒  
      ('i1 + 'i2, 'o1 + 'o2, 'd) op
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```

Sequential and parallel composition

$$op_1 \parallel op_2 = \text{comp_op } (\lambda_. \text{None}) (\lambda_. []) op_1 op_2$$
$$op_1 \bullet op_2 = \text{map_op projl projr } (\text{comp_op Some } (\lambda_. [])) op_1 op_2$$

Composition: equation

Composition operator

`comp_op wire buf op1 op2 =`

Composition: equation

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$\text{comp_op } \text{wire } \text{buf } op_1 \ op_2 = \text{Choice } (((\lambda \text{op}. \ \underline{\text{case}} \ \underline{\text{op}} \ \underline{\text{of}}}$
 $\text{Read } p \ f \Rightarrow \text{Read} (\text{Inl } p) (\lambda x. \ \text{comp_op } \text{wire } \text{buf } (f \ x) \ op_2)$

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| Write $op \ p \ x \Rightarrow (\underline{\text{case}} \ \text{wire } p \ \underline{\text{of}}$

None $\Rightarrow \text{Write} (\text{comp_op } \text{wire } \text{buf } op \ op_2) (\text{Inl } p) \ x$

| Some $q \Rightarrow \text{Silent} (\text{comp_op } \text{wire} (\text{BENQ } q \ x \ \text{buf}) \ op \ op_2))$

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Read $p \ f \Rightarrow \underline{\text{if}} \ p \in \text{ran } \text{wire}$

then $\text{Silent} (\text{comp_op } \text{wire} (\text{BTL } p \ \text{buf}) \ op_1 (f (\text{BHD } p \ \text{buf})))$

else $\text{Read} (\text{Inr } p) (\lambda x. \text{comp_op } \text{wire } \text{buf } op_1 (f \ x))$

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$\text{comp_op wire buf } op_1 \ op_2 = \text{Choice (((}\lambda op. \underline{\text{case}} \ op \ \underline{\text{of}}$

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(($\lambda op. \underline{\text{case}} \ op \ \underline{\text{of}}$

Read $p \ f \Rightarrow \underline{\text{if }} p \in \text{ran wire}$

then $\text{Silent} (\text{comp_op wire } (\text{BTL } p \ \text{buf}) \ op_1 (f (\text{BHD } p \ \text{buf})))$

else $\text{Read} (\text{Inr } p) (\lambda x. \text{comp_op wire buf } op_1 (f \ x))$

| Write $op \ p \ x \Rightarrow \text{Write} (\text{comp_op wire buf } op_1 \ op) (\text{Inr } p) \ x$

| Silent $op \Rightarrow \text{Silent} (\text{comp_op wire buf } op_1 \ op) \ \backslash_c \ \text{sound_reads wire buf } (\text{choices } op_2))$

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- Identity: \mathcal{I}
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$$\uparrow :: ('i + 'm, 'o + 'm, 'd) \ op \Rightarrow ('i, 'o, 'd) \ op$$

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Dummy source operator type

$$i :: ('i, 'o, 'd) \ op$$

Sink operator type

$$! :: ('i, 'o, 'd) \ op$$

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Transpose operator type

$$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd) \ op$$

Dummy source operator type

$$! :: ('i, 'o, 'd) \ op$$

Sink operator type

$$! :: ('i, 'o, 'd) \ op$$

Split operator type

$$\Lambda :: ('n, 'n + 'n, 'd) \ op$$

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$$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd) \ op$$

Dummy source operator type

$$! :: ('i, 'o, 'd) \ op$$

Sink operator type

$$! :: ('i, 'o, 'd) \ op$$

Split operator type

$$\Lambda :: ('n, 'n + 'n, 'd) \ op$$

Merge operator type

$$\mathcal{V} :: ('n + 'n, 'n, 'd) \ op$$

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Dummy source operator type

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Sink operator type

$$! :: ('i, 'o, 'd) \ op$$

Split operator type

$$\Lambda :: ('n, 'n + 'n, 'd) \ op$$

Merge operator type

$$\mathcal{V} :: ('n + 'n, 'n, 'd) \ op$$

Copy operator type

$$\mathcal{C} :: ('n, 'n + 'n, 'd) \ op$$

Asynchronous Dataflow Operators

- Identity: \mathcal{I}
- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

Feedback operator type ($op \uparrow$)

$$\uparrow :: ('i + 'm, 'o + 'm, 'd) \text{ op} \Rightarrow ('i, 'o, 'd) \text{ op}$$

Transpose operator type

$$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd) \text{ op}$$

Dummy source operator type

$$! :: ('i, 'o, 'd) \text{ op}$$

Sink operator type

$$! :: ('i, 'o, 'd) \text{ op}$$

Split operator type

$$\Lambda :: ('n, 'n + 'n, 'd) \text{ op}$$

Merge operator type

$$\mathcal{V} :: ('n + 'n, 'n, 'd) \text{ op}$$

Copy operator type

$$\mathcal{C} :: ('n, 'n + 'n, 'd) \text{ op}$$

Equality test operator type

$$\mathcal{Q} :: ('n + 'n, 'n, 'd \text{ option}) \text{ op}$$

Asynchronous Dataflow Properties

Basic network algebra properties

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx \text{map_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx \text{map_op} \text{Inl} \text{Inl} op$$

$$B2_2: (\mathcal{I} :: (0, 0, 'd) op) \parallel op \approx \text{map_op} \text{Inr} \text{Inr} op$$

$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

$$B4_1: op \bullet \mathcal{I} \approx op$$

$$B4_2: \mathcal{I} \bullet op \approx op$$

$$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$$

$$B6: \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$$

$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx \text{map_op} \text{id} (\text{case_sum} \text{Inr} \text{Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx \text{map_op} \curvearrowleft \curvearrowleft (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map_op} \text{id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$F2: \mathcal{X} \uparrow \approx \mathcal{I}$$

$$R1: op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$$

$$R2: (op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$$

$$R3: op_1 \parallel (op_2 \uparrow) \approx (\text{map_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$$

$$R4: (op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$$

$$R5: \text{map_op} \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

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Basic network algebra properties

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$$R5: \text{map_op} \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

$$R6: (op \uparrow) \uparrow \approx (\text{map_op} \curvearrowleft \curvearrowleft op) \uparrow$$

Properties of equality test, merge, copy, split, source and sink operators

$$A1: (\mathcal{Q} \parallel \mathcal{I}) \bullet \mathcal{Q} \approx \text{map_op } \curvearrowright \text{id } ((\mathcal{I} \parallel \mathcal{Q}) \bullet \mathcal{Q})$$

$$A2: \mathcal{X} \bullet \otimes \approx \otimes \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\} \qquad \qquad A6: \otimes \bullet \mathcal{X} \approx \otimes \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A3_{\mathcal{Q}}: ((i :: (0, 'a, 'd) op) \parallel \mathcal{I}) \bullet \mathcal{Q} \approx (! :: (0 + 'a, 0, 'd) op) \bullet i$$

$$A3_{\mathcal{V}}: ((i :: (0, 'a, 'd) op) \parallel \mathcal{I}) \bullet \mathcal{V} \approx \text{map_op Inr id } \mathcal{I}$$

$$A4: \otimes \bullet ! \approx ! \parallel ! \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\} \qquad \qquad A8: i \bullet \otimes \approx i \parallel i \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A5: \mathcal{C} \bullet (\mathcal{C} \parallel \mathcal{I}) \approx \text{map_op id } \curvearrowright (\mathcal{C} \bullet (\mathcal{I} \parallel \mathcal{C}))$$

$$A7: \mathcal{C} \bullet (! \parallel \mathcal{I}) \approx \text{map_op id Inr } \mathcal{I} \qquad \qquad A9: i \bullet ! \approx \mathcal{I}$$

$$A10: \mathcal{Q} \bullet \mathcal{C} \approx (\mathcal{C} \parallel \mathcal{C}) \bullet (\text{map_op } \curvearrowright \curvearrowright (\text{map_op } \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I})) \bullet (\mathcal{Q} \parallel \mathcal{Q})$$

$$A11: \mathcal{C} \bullet \mathcal{Q} \approx \mathcal{I} \qquad \qquad \qquad A12: i \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$A16: ! \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$A13: i \approx i \parallel i$$

$$A17: ! \approx ! \parallel !$$

$$A14: \text{map_op id Inl } (\otimes :: (0 + 0, 0, 'd) op) \approx \mathcal{I} \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$A15: \otimes \approx \text{map_op } \curvearrowright \curvearrowright (\text{map_op } \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \bullet (\otimes \parallel \otimes) \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$A18: \text{map_op Inl id } (\otimes :: (0, 0 + 0, 'd) op) \approx \mathcal{I} \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A19: \otimes \approx (\otimes \parallel \otimes) \bullet \text{map_op } \curvearrowright \curvearrowright (\text{map_op } \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$F3: (\text{map_op id Inr } \otimes) \uparrow \approx ! \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$F4: (\text{map_op Inr id } \otimes) \uparrow \approx i \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$F5: ((\mathcal{I} \parallel \mathcal{C}) \bullet \text{map_op } \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet (\mathcal{I} \parallel \mathcal{Q})) \uparrow \approx ! \bullet i$$

Properties of equality test, merge, copy, split, source and sink operators

$$A1: (\mathcal{Q} \parallel \mathcal{I}) \bullet \mathcal{Q} \approx \text{map_op } \curvearrowleft \text{id } ((\mathcal{I} \parallel \mathcal{Q}) \bullet \mathcal{Q})$$

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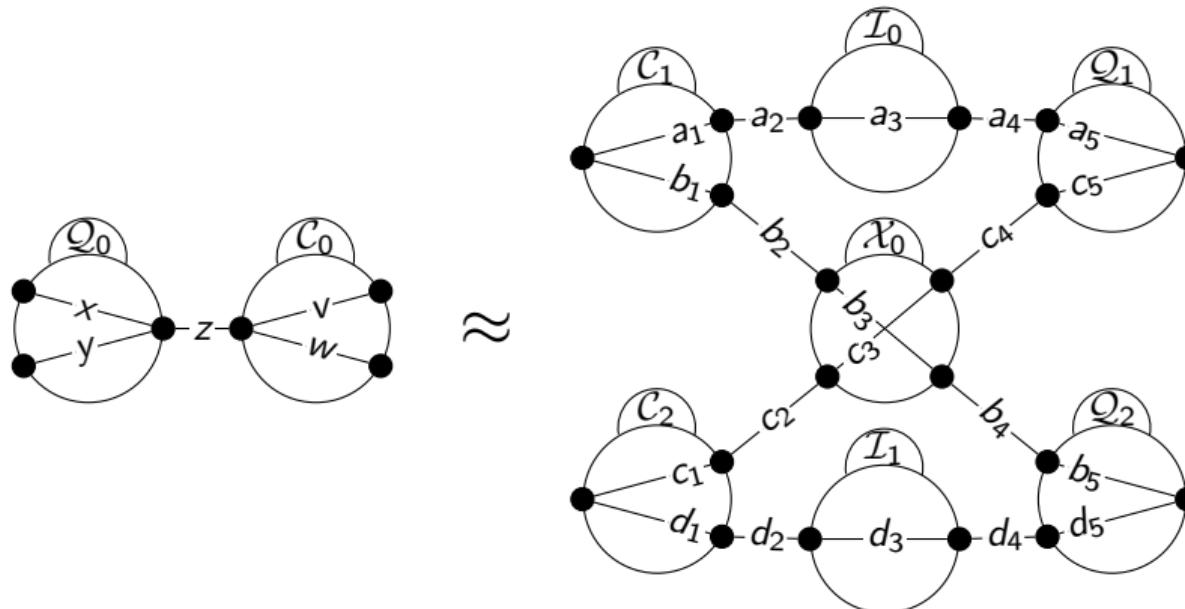
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- \approx coinduction principle
- Buffers generalization
- Buffers invariant

Related Work

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- A Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
 - Operators as a shallow embedding as codatatypes
 - 51 axioms proved
- Executable via code extraction to Haskell

Conclusion

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We conjecture it also does not hold in the process algebra from Bergstra et al.

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- Future work:
 - Brock–Andersen Time anomaly
 - Formalize Timely Dataflow infrastructure
 - Verify Timely Dataflow algorithms

Questions, comments and suggestions