Verified Time-Aware Stream Processing

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What is this PhD/Status seminar about?

- Distributed Systems
  - Stream processing frameworks
    - Dataflow models
      - Time-Aware Computations
- Formal Methods
  - Verification using proof assistants
    - Isabelle proofs
      - Verified + executable + efficient code
- Formalization of Time-Aware Stream Processing

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Introduction

- Stream Processing: Abstraction for processing data when the input is not completely presented in the begging of the computation
- Dataflow Model:
  - Directed graph of interconnected operators that perform event-wise transformations
  - Examples: Apache Flink, Apache Samza, Apache Spark, Google Cloud Dataflow, and Timely Dataflow
  - Highly Parallel

- Time-Aware Computations
  - Timestamps: Metadata associating the data with some data collection
  - Watermarks: Metadata indicating the completion of a data collection

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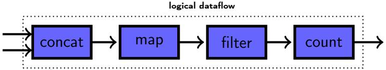

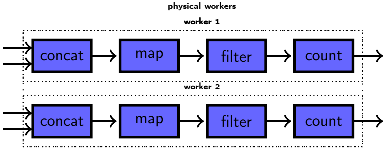

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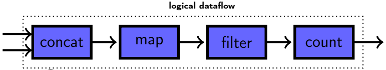

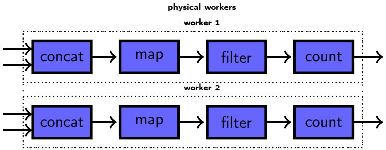

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Preliminaries

Isabelle/HOL

- Classical higher-order logic (HOL): Simple Typed Lambda Calculus + (Hilbert) axiom of choice + axiom of infinity + rank-1 polymorphism
- Isabelle: A generic proof assistant

- Isabelle/HOL: Isabelle's flavor of HOL
- All functions in Isabelle/HOL must be total

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Isabelle/HOL: (Co)datatypes

- Datatypes and Codatatypes

codatatype (lset: $'a$) *llist* = lnull: LNil | LCons (lhd: $'a$) (ltl: $'a$ *llist*)
  for map: lmap **where** ltl LNil = LNil

- Examples:
  - LNil
  - LCons *1* (LCons *2* (LCons *3* LNil))
  - LCons *0* (LCons *0* (LCons *0* (. . .)))
- Induction principle with lset assumption
  - If $x \in$ lset *lxs*, and if P holds for all lazy lists containing x, then P *lxs* is true
- Coinductive principle for lazy list equality:
  - Show that there is a pair of goggles that makes them to look the same, which implies that:
    - The first lazy list is empty iff second is
    - They have the same head
    - Their tail looks the same

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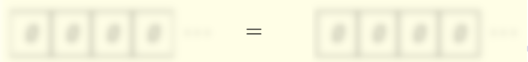

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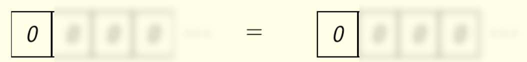

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0	0	0	0	···	=	0	0	0	0	···

Isabelle/HOL: Recursion and While Combinator

- Recursion

fun lshift $:: {}'a\ list \Rightarrow {}'a\ llist \Rightarrow {}'a\ llist$ ($infixr$ @@ 65) **where**
  lshift [] $lxs = lxs$
| lshift $(x \# xs)$ lxs $=$ LCons x (lshift xs lxs)

- While Combinator

definition while_option $:: ({}'a \Rightarrow bool) \Rightarrow ({}'a \Rightarrow {}'a) \Rightarrow {}'a \Rightarrow {}'a\ option$ **where**
while_option $b\ c\ s = \ldots$

- While rule for invariant reasoning (hoare-style):
  - There is something that holds before a step; that thing still holds after the step

Isabelle/HOL: Corecursion and Friends

- Corecursion is like recursion, but instead of always eventually reducing an argument it always eventually produces something
- Corec:

corec lapp :: $'a$ *llist* $\Rightarrow$ $'a$ *llist* $\Rightarrow$ $'a$ *llist* **where**
  lapp *lxs* *lys* = case *lxs* of LNil $\Rightarrow$ *lys* | LCons *x* *lxs'* $\Rightarrow$ LCons *x* (lapp *lxs'* *lys*)

- Friendly function
  - Preserves productivity: it may consume at most one constructor to produce one constructor.

friend_of_corec lshift **where**
  xs @@ *lxs* = (case *xs* of
    [] $\Rightarrow$ (case *lxs* of LNil $\Rightarrow$ LNil | LCons *x* *lxs'* $\Rightarrow$ LCons *x* *lxs'*)
  | *x*#*xs'* $\Rightarrow$ LCons *x* (*xs'* @@ *lxs*))
  by (*auto split: list.splits llist.splits*) (*transfer_prover*)

lconcat *lxs* = case *lxs* of LNil $\Rightarrow$ LNil | LCons *xs* *lxs'* $\Rightarrow$ lshift *xs* (lconcat *lxs'*)

- Coinduction up to congruence: Coinduction for Lazy list equality can be extended to compare an entire finite prefix through a congruence relation

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Isabelle/HOL: (Co)inductive Predicates

- Inductive predicate
  - Finite number of introduction rule applications

inductive in_llist $:: {'a} \Rightarrow {'a}\ \mathit{llist} \Rightarrow \mathit{bool}$ **where**
  In_llist: in_llist x (LCons x lxs)
  | Next_llist: in_llist x lxs $\Rightarrow$ in_llist x (LCons y lxs)

in_llist 2 (LCons 1 (LCons (2 (...))))

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Lazy Lists Processors

Operator formalization

- Operator as a codatatype
  - Taking $'i$ as the input type, and $'o$ as the output type:
    $\textbf{codatatype}\ ('o, 'i)\ op = \text{Logic}\ (\text{apply}: ('i \Rightarrow ('o, 'i)\ op \times 'o\ list))$
  - Infinite trees: applying the selector apply "walks" a branch of the tree

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Execution formalization

- Produce function: applies the logic (co)recursively throughout a lazy list

  definition produce_1 *op lxs* = while_option ...

  corec produce **where**
  produce *op lxs* = (case produce_1 *op lxs* of None ⇒ LNil
  | Some (*op'*, *x*, *xs*, *lxs'*) ⇒ LCons *x* (*xs* @@ produce *op' lxs'*))

- produce_1 has an induction principle based on the while invariant rule

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- Produce function: applies the logic (co)recursively throughout a lazy list

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  corec produce **where**
  $\mathsf{produce}\ \mathit{op}\ \mathit{lxs} = (\mathtt{case}\ \mathsf{produce}_1\ \mathit{op}\ \mathit{lxs}\ \mathtt{of}\ \mathsf{None} \Rightarrow \mathsf{LNil}$
  $\mid \mathsf{Some}\ (\mathit{op}', x, \mathit{xs}, \mathit{lxs}') \Rightarrow \mathsf{LCons}\ x\ (\mathit{xs}\ @@\ \mathsf{produce}\ \mathit{op}'\ \mathit{lxs}'))$

- $\mathsf{produce}_1$ has an induction principle based on the while invariant rule

Operators: Count

- Example:

corec count_op **where** count_op P n =
  Logic ($\lambda e.$ `if` P e `then` (count_op P ($n + 1$), [n+1]) `else` (count_op P n, []))

$$\mathit{stream}_1 \quad = \quad \boxed{0}\,\boxed{3}\,\boxed{3}\,\boxed{6}\,\boxed{24}$$

$$\text{produce (count\_op } \mathit{is\_even}\ 0) \quad \mathit{stream}_1 \quad = \quad \boxed{1}\,\boxed{2}\,\boxed{3}$$

Sequential Composition

- Sequential composition: take the output of the first operator and give it as input to the second operator.

$$
\begin{array}{l}
\textbf{definition}\ \mathrm{fproduce}\ \mathit{op}\ \mathit{xs} = \mathrm{fold}\ (\lambda e\ (\mathit{op}, \mathit{out}). \\
\quad \texttt{let}\ (\mathit{op}', \mathit{out}') = \mathrm{apply}\ \mathit{op}\ e\ \texttt{in}\ (\mathit{op}', \mathit{out}\ @\ \mathit{out}'))\ \mathit{xs}\ (\mathit{op}, [\,]) \\
\textbf{corec}\ \mathrm{comp\_op}\ \textbf{where} \\
\quad \mathrm{comp\_op}\ \mathit{op}_1\ \mathit{op}_2 = \mathrm{Logic}\ (\lambda \mathit{ev}. \\
\qquad \texttt{let}\ (\mathit{op}_1', \mathit{out}) = \mathrm{apply}\ \mathit{op}_1\ \mathit{ev};\ (\mathit{op}_2', \mathit{out}') = \mathrm{fproduce}\ \mathit{op}_2\ \mathit{out} \\
\qquad \texttt{in}\ (\mathrm{comp\_op}\ \mathit{op}_1'\ \mathit{op}_2', \mathit{out}'))
\end{array}
$$

Sequential Composition: Correctness

- Correctness:

  $\text{produce } (\text{comp\_op } op_1\ op_2)\ lxs = \text{produce } op_2\ (\text{produce } op_1\ lxs)$

- Proof: coinduction principle for lazy list equality and produce_1 induction principle
  - Generalization: we must be able to reason about elements in arbitrary positions

    corec skip_op **where**
    skip_op $op\ n$ = Logic ($\lambda ev.$ `let` (op', out) = apply $op\ ev$ `in`
    `if` length $out < n$ `then` (skip_op op' (n − length out), [])
    `else` (op', drop $n\ out$))

  - Correctness

    produce (skip_op $op\ n$) lxs = ldropn produce $op\ lxs$

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  - Proof: Coinduction up to congruence for lazy list equality

Time-Aware Operators

Time-Aware Streams

- Time-Aware lazy lists

$\mathbf{datatype}\ ('t{::}order, 'd)\ event = \mathrm{DT}\ (\mathrm{tmp}{:}\ 't)\ (\mathrm{data}{:}\ 'd) \mid \mathrm{WM}\ (\mathrm{wmk}{:}\ 't)$

- Generalization to partial orders
  - Cycles
  - Operators with multiple inputs

Time-Aware Streams

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Monotone Time-Aware Streams

- Monotone: watermarks do not go back in time

$$
\begin{array}{l}
\textbf{coinductive}\ \mathsf{monotone} :: ('t{::}order,\ 'd)\ \mathit{event}\ \mathit{llist} \Rightarrow\ 't\ \mathsf{set} \Rightarrow \mathit{bool}\ \textbf{where} \\
\quad \mathsf{LNil}\!: \mathsf{monotone}\ \mathsf{LNil}\ W \\
\mid \mathsf{LConsR}\!: (\forall wm' \in W.\ \neg\ wm' \geq wm) \longrightarrow \mathsf{monotone}\ lxs\ (\{wm\} \cup W) \longrightarrow \\
\quad\ \mathsf{monotone}\ (\mathsf{LCons}\ (\mathsf{WM}\ wm)\ lxs)\ W \\
\mid \mathsf{LConsL}\!: (\forall wm \in W.\ \neg\ wm \geq t) \longrightarrow \mathsf{monotone}\ lxs\ W \longrightarrow \\
\quad\ \mathsf{monotone}\ (\mathsf{LCons}\ (\mathsf{DT}\ t\ d)\ lxs)\ W
\end{array}
$$

- Up to congruence coinduction principle
- Example:

$stream_2 =$

DT t_4 d	DT t_0 a	DT t_1 c	WM t_1	DT t_2 b
WM t_2	DT t_5 c	DT t_3 a	DT t_5 a	WM t_5 …

$$
\begin{tikzcd}
C_{t_4}\!: \{\ d\ \} & & C_{t_5}\!: \{\ a, c\ \} \\
C_{t_1}\!: \{\ c\ \} \arrow[u] & C_{t_2}\!: \{\ b\ \} \arrow[ur] & C_{t_3}\!: \{\ a\ \} \arrow[u] \\
& C_{t_0}\!: \{\ a\ \} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Productive Time-Aware Streams

- Productive: always eventually allows the production
  - Batching operators: accumulate data until its completion
  - Data is always eventually completed by some watermark

coinductive productive **where**
  LFinite: lfinite *lxs* $\longrightarrow$ productive *lxs*
| EnvWM: $\neg$ lfinite *lxs* $\longrightarrow$ ($\exists u \in$ vimage WM (lset *lxs*). $u \geq t$) $\longrightarrow$
  productive *lxs* $\longrightarrow$ productive (LCons (DT *t d*) *lxs*)
| *SkipWM:* $\neg$ lfinite *lxs* $\longrightarrow$ productive *lxs* $\longrightarrow$
  productive (LCons (WM *t*) *lxs*)

  - Up to congruence coinduction principle

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$$
\begin{array}{l}
\textbf{coinductive}\ \mathsf{productive}\ \textbf{where} \\
\quad \mathsf{LFinite}\textit{:}\ \mathsf{lfinite}\ \mathit{lxs} \longrightarrow \mathsf{productive}\ \mathit{lxs} \\
\mid \mathsf{EnvWM}\textit{:}\ \neg\ \mathsf{lfinite}\ \mathit{lxs} \longrightarrow (\exists u \in \mathsf{vimage}\ \mathsf{WM}\ (\mathsf{lset}\ \mathit{lxs}).\ u \geq t) \longrightarrow \\
\quad\ \mathsf{productive}\ \mathit{lxs} \longrightarrow \mathsf{productive}\ (\mathsf{LCons}\ (\mathsf{DT}\ t\ d)\ \mathit{lxs}) \\
\mid \mathit{SkipWM}\textit{:}\ \neg\ \mathsf{lfinite}\ \mathit{lxs} \longrightarrow \mathsf{productive}\ \mathit{lxs} \longrightarrow \\
\quad\ \mathsf{productive}\ (\mathsf{LCons}\ (\mathsf{WM}\ t)\ \mathit{lxs})
\end{array}
$$

  - Up to congruence coinduction principle

Building Blocks: Batch Operator

- Building Blocks: reusable operators
  - Batching and incremental computations
- batch_op : produces batches of accumulated data

corec batch_op **where**
  batch_op *buf* = Logic (λ*ev*. case *ev* of DT *t d* $\Rightarrow$ (batch_op (*buf* @ [(*t*, *d*)]), [])
  | WM *wm* $\Rightarrow$ if $\exists(t, d) \in$ set *buf*. $t \leq wm$
    then let *out* = filter ($\lambda(t, \_)$. $t \leq wm$) *buf*;
          buf' = filter ($\lambda(t, \_)$. $\neg\, t \leq wm$) *buf*
        in (batch_op *buf'*, [DT *wm out*, WM *wm*])
    else (batch_op *buf*, [WM *wm*]))

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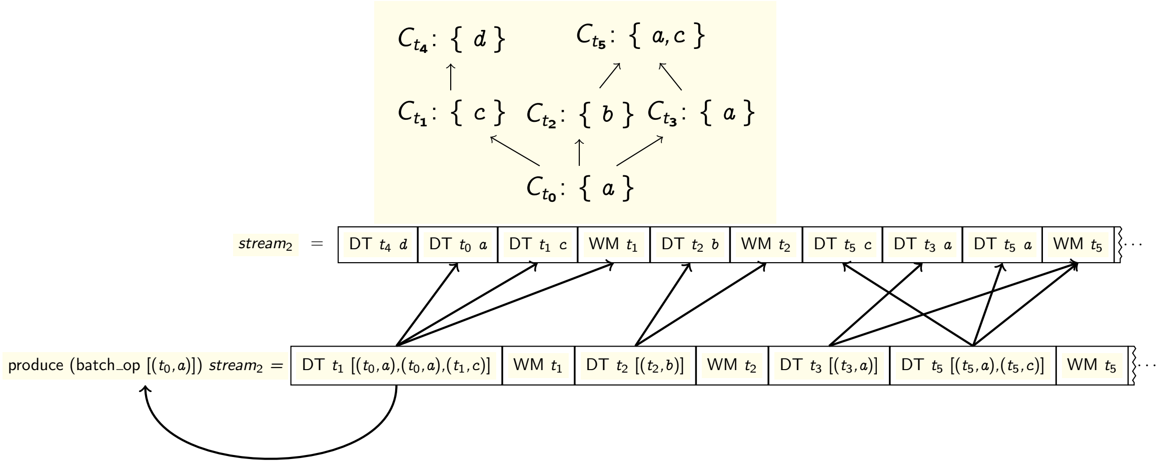
  batch_op buf = Logic (λev. `case` ev `of` DT t d $\Rightarrow$ (batch_op (buf @ [(t, d)]), [])
  | WM wm $\Rightarrow$ `if` $\exists(t, d) \in$ set buf. $t \leq wm$
    `then let` out = filter ($\lambda(t, \_)$. $t \leq wm$) buf;
              buf' = filter ($\lambda(t, \_)$. $\neg$ $t \leq wm$) buf
         `in` (batch_op buf', [DT wm out, WM wm])
    `else` (batch_op buf, [WM wm]))

Batch Operator: Soundness

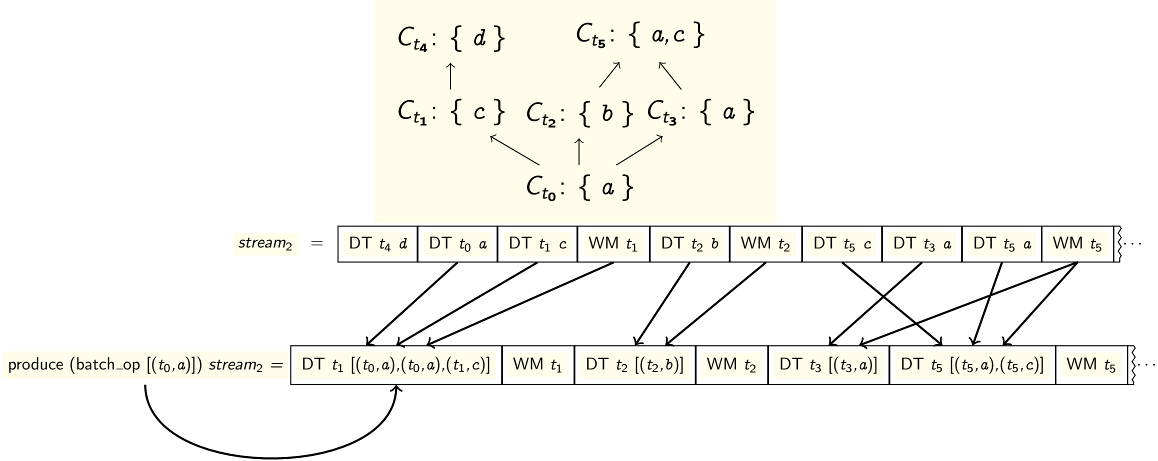
- Given a monotone time-aware stream



- Proof: lset induction, produce_1 induction, and generalization with skip_op

Batch Operator: Completeness

- Given a monotone and productive time-aware stream
- if not finite



- Proof: induction over the position (nat) of the element in the input, and soundness of batch_op

Batch Operator: Monotone and productive preservation

- The operators must preserve monotone and productive, so we can compose it with something that needs these properties!

$$\text{monotone } lxs\ W \longrightarrow \text{monotone (produce (batch\_op } buf)\ lxs)\ W \tag{1}$$

$$\text{productive } lxs \longrightarrow \text{productive (produce (batch\_op } buf)\ lxs) \tag{2}$$

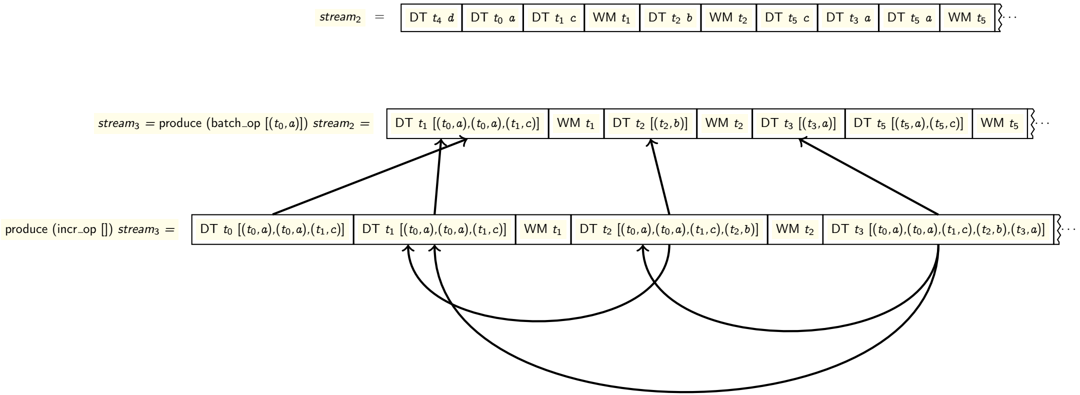
- Proof: coinduction up to congruence

Building Blocks: Incremental Operator

- Incremental computations
- incr_op : produces accumulated batches of accumulated data

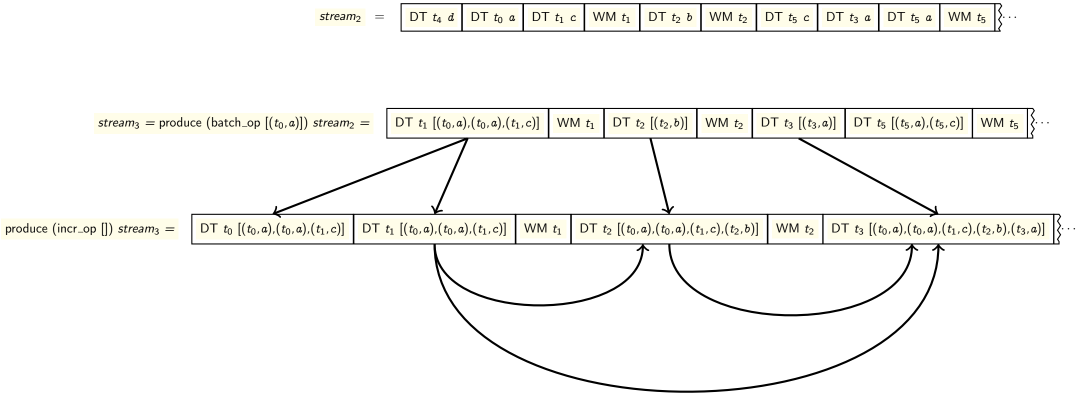
$$\begin{array}{l}
\textbf{corec}\ \mathsf{incr_op}\ \textbf{where} \\
\quad \mathsf{incr_op}\ \mathit{buf} = \mathsf{Logic}\ (\lambda\ \mathit{ev}.\ \mathtt{case}\ \mathit{ev}\ \mathtt{of}\ \mathsf{DT}\ \mathit{wm}\ \mathit{batch} \Rightarrow \\
\qquad \mathtt{let}\ \mathit{out} = \mathsf{map}\ (\lambda t.\ \mathsf{DT}\ t\ (\mathit{buf}\ @\ \mathit{batch}))\ (\mathsf{remdups}\ (\mathsf{map}\ \mathsf{fst}\ \mathit{batch})) \\
\qquad \mathtt{in}\ (\mathsf{incr_op}\ (\mathit{buf}\ @\ \mathit{batch}),\ \mathit{out}) \\
\quad \mid \mathsf{WM}\ \mathit{wm} \Rightarrow\ (\mathsf{incr_op}\ \mathit{buf},\ [\mathsf{WM}\ \mathit{wm}]))
\end{array}$$

Incremental Operator: Soundness



- Proof: produce$_1$ induction, and generalization with skip_op

Incremental Operator: Completeness



- Proof: induction over the position (nat) of the element in the input

Incremental Operator: Monotone and productive preservation

$$\text{monotone } \mathit{lxs}\ W \longrightarrow \text{monotone (produce (incr\_op []) } \mathit{lxs})\ W \tag{3}$$

$$\text{productive } \mathit{lxs} \longrightarrow \text{productive (produce (incr\_op []) } \mathit{lxs}) \tag{4}$$

- Proof: coinduction up to congruence

Compositional Reasoning

- batch_op and incr_op can be composed

  definition incr_batch_op *buf1 buf2* = comp_op (batch_op *buf1*) (incr_op *buf2*)

- Soundness, completeness, and monotone and productive preservation

Case Study

Histogram

- A histogram count the elements of a collection
- Incremental histogram: timestamps smaller or equal
- $H_{t_5} = C_{t_0} + C_{t_0} + C_{t_1} + C_{t_2} + C_{t_5}$
- paths to t_5: $\{t_0, t_2\}$ and $\{t_0, t_3\}$

$$\begin{tikzcd}
C_{t_4}\colon \{\, d \,\} & & & C_{t_5}\colon \{\, a, c \,\} & \\
C_{t_1}\colon \{\, c \,\} \arrow[u] & & C_{t_2}\colon \{\, b \,\} \arrow[ur] & & C_{t_3}\colon \{\, a \,\} \arrow[ul] \\
& & C_{t_0}\colon \{\, a \,\} \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

$$\begin{tikzcd}
H_{t_4}\colon \{\, a, c, d \,\} & & & H_{t_5}\colon \{\, a, a, a, a, b, c \,\} & \\
H_{t_1}\colon \{\, c,\ a \,\} \arrow[u] & & H_{t_2}\colon \{\, b,\ a \,\} \arrow[ur] & & H_{t_3}\colon \{\, a,\ a \,\} \arrow[ul] \\
& & H_{t_0}\colon \{\, a \,\} \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Histogram Operator

corec map_op **where** map_op f = Logic (λ *ev.* case *ev* of
  WM *wm* $\Rightarrow$ (map_op f, [WM *wm*]) | DT t d $\Rightarrow$ (map_op f, [DT t (f t d)]))

definition incr_coll t xs = mset ...

definition incr_hist_op *buf1* *buf2* =
 comp_op (incr_batch_op *buf1* *buf2*) (map_op incr_coll)

Histogram Operator: Soundness, Completeness, Monotone and Productive Preservation

- Given a monotone and productive time-aware stream

$$stream_2 = \boxed{\text{DT } t_4\ d} \boxed{\text{DT } t_0\ a} \boxed{\text{DT } t_1\ c} \boxed{\text{WM } t_1} \boxed{\text{DT } t_2\ b} \boxed{\text{WM } t_2} \boxed{\text{DT } t_5\ c} \boxed{\text{DT } t_3\ a} \boxed{\text{DT } t_5\ a} \boxed{\text{WM } t_5} \cdots$$

$$\text{produce (incr\_hist\_op [] []) } stream_2 = \boxed{\text{DT } t_0 \text{ (mset } [a])} \boxed{\text{DT } t_1 \text{ (mset } [a,c])} \boxed{\text{WM } t_1} \boxed{\text{DT } t_2 \text{ (mset } [a,b])} \boxed{\text{WM } t_2} \boxed{\text{DT } t_3 \text{ (mset } [a,a])} \boxed{\text{DT } t_5 \text{ (mset } [a,a,a,b,c])} \boxed{\text{WM } t_5} \cdots$$

- Proof: soundness, completeness, monotone and productive preservation of incr_batch_op

Efficient Histogram Operator

- Efficient histogram operator $\mathrm{incr\_batch\_op}'$ for timestamp in total order
  - State of the operator: last computed histogram, and buffer of newly accumulated data
- Equivalent $\mathrm{incr\_batch\_op}$ only for monotone time-aware stream (equivalence relation)

Join

- Use the `sum` type to represent two stream as one
- Partial order for the `sum`: left compares with left, right compares with right
- Defined using `incr_batch_op`
- Soundness, Completeness, Monotone
  - WIP: Productive

Next Steps

Next Steps

- Feedback loop
- Exit argument
- Connect to the Isabelle-LLVM refinement framework

Questions, comments and suggestions