Nondeterministic Asynchronous Dataflow in Isabelle/HOL

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14/05/2025

Motivation

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Context:

- Stream Processing: programs that compute (possibly) unbounded sequences of data (streams)
- A common problem in the industry
- Frameworks: Apache Flink, Google Cloud Dataflow, and Timely Dataflow

- Why use frameworks?
  - Highly Parallel
  - Low latency (output as soon as possible)
  - Incremental computing (re-uses previous computations)

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Our goal:

Mechanically Verify Timely Dataflow algorithms

A Good Foundation

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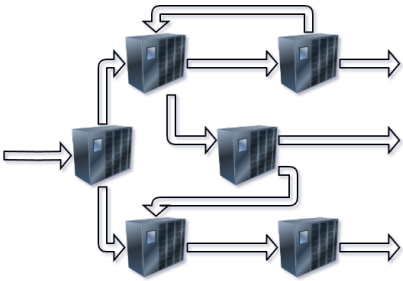

- Nondeterministic Asynchronous Dataflow

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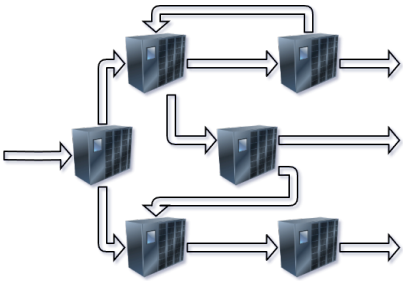

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  - Dataflow: Directed graph of interconnected operators

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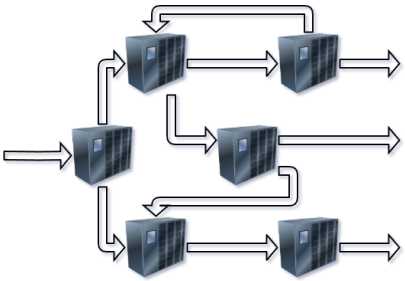

- Nondeterministic Asynchronous Dataflow
  - Dataflow: Directed graph of interconnected operators
  - Asynchronous:
    - Operators execute independently: processes without an orchestrator
    - Operators can freely communicate with the network (read/write); do silent computation steps
    - Networks are unbounded FIFO queues

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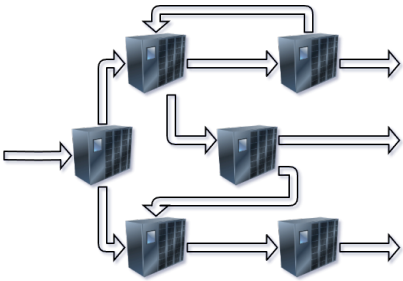

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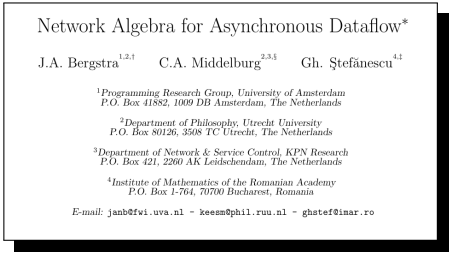
    - Networks are unbounded FIFO queues
  - Nondeterministic:
    - Operators can make nondeterministic choices

The Algebra for Nondeterministic Asynchronous Dataflow

- Bergstra et al. presents an algebra for Nondeterministic Asynchronous Dataflow
- Primitives:
  sequential and parallel composition; feedback loop...
- 52 axioms
- A process calculus instance



Network Algebra for Asynchronous Dataflow*

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Main Contributions

- A Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
  - Operators as a shallow embedding as codatatypes
  - 51 axioms proved
- Executable via code extraction to Haskell

Isabelle/HOL Preliminaries

Isabelle/HOL

- Classical higher-order logic (HOL):
  Simple Typed Lambda Calculus + axiom of choice + axiom of infinity + rank-1 polymorphism

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Why Isabelle/HOL?

- Codatatypes: (possibly) infinite data structures (e.g., lazy lists, streams)
- Corecursion: always eventually produces some codatatype constructor
- Coinductive predicate: infinite number of introduction rule applications
- Coinduction: reason about coinductive predicates

Operators as a Codatatype

Operators

Operators in Isabelle/HOL

codatatype (inputs: $'i$, outputs: $'o$, $'d$) op =
Read $'i$ ($'d \Rightarrow$ ($'i$, $'o$, $'d$) op) | Write (($'i$, $'o$, $'d$) op) $'o$ $'d$
Silent ($'i$, $'o$, $'d$) op | Choice (($'i$, $'o$, $'d$) op) $cset$

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- Type parameters:
  inputs/output ports; data
- Operator's actions
- inputs/outputs:
  Sets of used ports
- Possibly infinite trees

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Uncommunicative operators

$\oslash$ = Choice $\{\}_c$

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Uncommunicative operators

$\oslash$ = Choice $\{\}_c$

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More examples

ex1 = Choice $\{$Write ex1 1 42, $\oslash\}_c$

ex2 = Choice $\{$Write ex2 1 42, ex2$\}_c$

ex3 = Choice $\{$Write ex3 1 42, Silent ex3$\}_c$

Operators Equivalences: Motivation

An ideal equivalence relation

- Only equate operators that **behave** the same
- Useful reasoning principle

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- Milner's approach: Weak Bisimilarity

- Based on labeled transition systems (LTS)
- Labels (actions): Read, Write, Silent (τ)

Operators Equivalences: Label Transition System

Label Transition System

datatype $('i, 'o, 'd)\ IO = \text{Inp}\ 'i\ 'd \mid \text{Out}\ 'o\ 'd \mid \text{Tau}$

inductive step **where**

$\quad \text{step}\ (\text{Inp}\ p\ x)\ (\text{Read}\ p\ f)\ (f\ x)$

$\mid \text{step}\ (\text{Out}\ q\ x)\ (\text{Write}\ op\ q\ x)\ op$

$\mid \text{step}\ \text{Tau}\ (\text{Silent}\ op)\ op$

$\mid op \in_c ops \implies \text{step}\ io\ op\ op' \implies \text{step}\ io\ (\text{Choice}\ ops)\ op'$

Operators Equivalences: Weak Bisimilarity

Weakly Simulates

$$\text{estep Tau} = (\text{step Tau})^{==}$$

$$\text{estep } io = \text{step } io$$

$$\text{wstep } io = (\text{step Tau})^{**} \text{ OO } (\text{estep } io) \text{ OO } (\text{step Tau})^{**}$$

$$\text{wsim } R\ op_1\ op_2 = (\forall io\ op_1'.\ \text{step } io\ op_1\ op_1' \longrightarrow (\exists op_2'.\ \text{wstep } io\ op_2\ op_2' \wedge R\ op_1'\ op_2'))$$

- A Relation R is a *weak simulation* if $\forall op_1\ op_2.\ R\ op_1\ op_2 \longrightarrow \text{wsim } R\ op_1\ op_2$.

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Weak Bisimilarity

coinductive wbisim (**infix** $\approx$ 40) **where**

wsim $(\approx)$ op_1 op_2 $\implies$ wsim $(\approx)$ op_2 op_1 $\implies$ $op_1 \approx op_2$

- R is a *weak bisimulation* when both R and its converse R^{-1} are weak simulations. Weak bisimilarity is the largest weak bisimulation.

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- R is a *weak bisimulation* when both R and its converse R^{-1} are weak simulations. Weak bisimilarity is the largest weak bisimulation.
- $\approx$ has a useful coinduction principle
- $\oslash \approx \otimes \approx \odot$ and ex1 $\approx$ ex2 $\approx$ ex3

Asynchronous Dataflow Operators

Auxiliary definitions

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Buffer functions

$\mathsf{BHD}\ p\ buf = \mathsf{hd}\ (buf\ p)$

$\mathsf{BTL}\ p\ buf = buf(p := \mathsf{tl}\ (buf\ p))$

$\mathsf{BENQ}\ p\ x\ buf = buf(p := buf\ p\ @\ [x])$

$buf_1 \gg buf_2 = (\lambda p.\ buf_2\ p\ @\ buf_1\ p)$

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choices function

$\mathsf{choices}\ \otimes = \{\}$

$\mathsf{Silent}\ op' \in_c \mathsf{choices}\ op \longleftrightarrow \mathsf{step}\ \mathsf{Tau}\ op\ op'$

$\mathsf{choices}\ (\mathsf{Choice}\ ops) = \bigcup_c\ (\mathsf{choices}\ `_c\ ops)$

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choices function

choices $\otimes = \{\}$

Silent $op' \in_c$ choices $op \longleftrightarrow$ step Tau $op\ op'$

choices (Choice ops) $= \bigcup_c$ (choices `$_c$ ops)

Mapping ports functions

map_op :: $('i_1 \Rightarrow 'i_2) \Rightarrow ('o_1 \Rightarrow 'o_2) \Rightarrow ('i_1, 'o_1, 'd)\ op \Rightarrow ('i_2, 'o_2, 'd)\ op$

projl :: $('a + 'b) \Rightarrow 'a$

projr :: $('a + 'b) \Rightarrow 'b$

$\curvearrowright$ :: $('a + 'b) + 'c \Rightarrow 'a + 'b + 'c$

$\curvearrowleft$ :: $'a + 'b + 'c \Rightarrow ('a + 'b) + 'c$

Identity

Equation of the identity operator

$\mathsf{id\_op}\ buf = \mathsf{Choice}$

$\bigcup_c$

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$$\begin{aligned}&\text{id\_op } buf = \text{Choice}\\&(((\lambda p.\ \text{Read } p\ (\lambda x.\ \text{id\_op (BENQ } p\ x\ buf))) \grave{}_c\ \mathfrak{U}_c)\\&\cup_c\end{aligned}$$

- $\mathfrak{U}_c$ is the set of usable ports provide by its type

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- $\mathfrak{U}_c$ is the set of usable ports provide by its type
- Stream delayer!

Identity

Equation of the identity operator

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- Stream delayer!

Identity operator with an empty buffer

$$\mathcal{I} = \mathsf{id\_op}\ (\lambda\_.\ [])$$

Composition: Preliminaries

Composition operator type

$$\text{comp\_op} :: ('o_1 \Rightarrow 'i_2\ option) \Rightarrow ('i_2 \Rightarrow 'd\ list) \Rightarrow ('i_1, 'o_1, 'd)\ op \Rightarrow ('i_2, 'o_2, 'd)\ op \Rightarrow ('i_1 + 'i_2, 'o_1 + 'o_2, 'd)\ op$$

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$('i_1 + 'i_2, 'o_1 + 'o_2, 'd)\ op$

Sequential and parallel composition

$op_1 \parallel op_2 =$ comp_op $(\lambda\_.\ \text{None})\ (\lambda\_.\ [])\ op_1\ op_2$

$op_1 \bullet op_2 =$ map_op projl projr (comp_op Some $(\lambda\_.\ [])\ op_1\ op_2$)

Composition: equation

Composition operator

comp_op *wire buf* op_1 op_2 =

Composition: equation

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comp_op *wire buf* op_1 op_2 = Choice

$\cup_c$

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$\grave{}_c$ choices op_1) $\cup_c$

Composition: equation

Composition operator

comp_op *wire buf* op_1 op_2 = Choice (((λ*op*. case *op* of
Read *p f* ⇒ Read (Inl *p*) (λ*x*. comp_op *wire buf* (*f x*) op_2)

$`_c$ choices op_1) $\cup_c$

Composition: equation

Composition operator

$$
\begin{aligned}
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&\ \mid \text{Write } \mathit{op}\ p\ x \Rightarrow (\underline{\text{case}}\ \mathit{wire}\ p\ \underline{\text{of}}\\
&\qquad \text{None} \Rightarrow \text{Write } (\text{comp\_op } \mathit{wire}\ \mathit{buf}\ \mathit{op}\ \mathit{op}_2)\ (\text{Inl } p)\ x\\
&\qquad \mid \text{Some } q \Rightarrow \text{Silent } (\text{comp\_op } \mathit{wire}\ (\text{BENQ } q\ x\ \mathit{buf})\ \mathit{op}\ \mathit{op}_2))\\
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&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{choices } \mathit{op}_2)))
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&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \grave{}_c\ \text{sound\_reads } \mathit{wire}\ \mathit{buf}\ (\text{choices } \mathit{op}_2)))
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&\qquad |\ \text{Some } q \Rightarrow \text{Silent } (\text{comp\_op } \mathit{wire}\ (\text{BENQ } q\ x\ \mathit{buf})\ \mathit{op}\ \mathit{op}_2))\\
&\ |\ \text{Silent } \mathit{op} \Rightarrow \text{Silent } (\text{comp\_op } \mathit{wire}\ \mathit{buf}\ \mathit{op}\ \mathit{op}_2))\ `_c\ \text{choices } \mathit{op}_1)\ \cup_c\\
&((\lambda \mathit{op}.\ \underline{\text{case}}\ \mathit{op}\ \underline{\text{of}}\\
&\quad \text{Read } p\ f \Rightarrow \underline{\text{if}}\ p \in \text{ran } \mathit{wire}\\
&\qquad \underline{\text{then}}\ \text{Silent } (\text{comp\_op } \mathit{wire}\ (\text{BTL } p\ \mathit{buf})\ \mathit{op}_1\ (f\ (\text{BHD } p\ \mathit{buf})))\\
&\qquad \underline{\text{else}}\ \text{Read } (\text{Inr } p)\ (\lambda x.\ \text{comp\_op } \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ (f\ x))\\
&\ |\ \text{Write } \mathit{op}\ p\ x \Rightarrow \text{Write } (\text{comp\_op } \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ \mathit{op})\ (\text{Inr } p)\ x\\
&\qquad\qquad `_c\ \text{sound\_reads } \mathit{wire}\ \mathit{buf}\ (\text{choices } \mathit{op}_2)))
\end{aligned}
$$

Composition: equation

Composition operator

$$
\begin{aligned}
&\text{comp\_op } \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ \mathit{op}_2 = \text{Choice } (((\lambda \mathit{op}.\ \underline{\text{case}}\ \mathit{op}\ \underline{\text{of}}\\
&\quad \text{Read } p\ f\ \Rightarrow \text{Read } (\text{Inl } p)\ (\lambda x.\ \text{comp\_op } \mathit{wire}\ \mathit{buf}\ (f\ x)\ \mathit{op}_2)\\
&\ |\ \text{Write } \mathit{op}\ p\ x \Rightarrow (\underline{\text{case}}\ \mathit{wire}\ p\ \underline{\text{of}}\\
&\qquad \text{None} \Rightarrow \text{Write } (\text{comp\_op } \mathit{wire}\ \mathit{buf}\ \mathit{op}\ \mathit{op}_2)\ (\text{Inl } p)\ x\\
&\quad\ |\ \text{Some } q \Rightarrow \text{Silent } (\text{comp\_op } \mathit{wire}\ (\text{BENQ } q\ x\ \mathit{buf})\ \mathit{op}\ \mathit{op}_2))\\
&\ |\ \text{Silent } \mathit{op} \Rightarrow \text{Silent } (\text{comp\_op } \mathit{wire}\ \mathit{buf}\ \mathit{op}\ \mathit{op}_2))\ `_c\ \text{choices } \mathit{op}_1)\ \cup_c\\
&((\lambda \mathit{op}.\ \underline{\text{case}}\ \mathit{op}\ \underline{\text{of}}\\
&\quad \text{Read } p\ f \Rightarrow \underline{\text{if}}\ p \in \text{ran } \mathit{wire}\\
&\qquad \underline{\text{then}}\ \text{Silent } (\text{comp\_op } \mathit{wire}\ (\text{BTL } p\ \mathit{buf})\ \mathit{op}_1\ (f\ (\text{BHD } p\ \mathit{buf})))\\
&\qquad \underline{\text{else}}\ \text{Read } (\text{Inr } p)\ (\lambda x.\ \text{comp\_op } \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ (f\ x))\\
&\ |\ \text{Write } \mathit{op}\ p\ x \Rightarrow \text{Write } (\text{comp\_op } \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ \mathit{op})\ (\text{Inr } p)\ x\\
&\ |\ \text{Silent } \mathit{op} \Rightarrow \text{Silent } (\text{comp\_op } \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ \mathit{op}))\ `_c\ \text{sound\_reads } \mathit{wire}\ \mathit{buf}\ (\text{choices } \mathit{op}_2)))
\end{aligned}
$$

Asynchronous Dataflow Operators

- Identity: $\mathcal{I}$
- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

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$\uparrow :: ('i + 'm, 'o + 'm, 'd)\ op \Rightarrow ('i, 'o, 'd)\ op$

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Dummy source operator type

$\mathsf{i} :: ('i, 'o, 'd)\ op$

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Transpose operator type

$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd)\ op$

Dummy source operator type

i $:: ('i, 'o, 'd)\ op$

Sink operator type

! $:: ('i, 'o, 'd)\ op$

Asynchronous Dataflow Operators

- Identity: $\mathcal{I}$
- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

Feedback operator type ($op \uparrow$)

$\uparrow :: ('i + 'm, 'o + 'm, 'd)\ op \Rightarrow ('i, 'o, 'd)\ op$

Transpose operator type

$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd)\ op$

Dummy source operator type

$\mathsf{i} :: ('i, 'o, 'd)\ op$

Sink operator type

$! :: ('i, 'o, 'd)\ op$

Split operator type

$\Lambda :: ('n, 'n + 'n, 'd)\ op$

Asynchronous Dataflow Operators

- Identity: $\mathcal{I}$
- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

Feedback operator type ($op\uparrow$)

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Transpose operator type

$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd)\ op$

Dummy source operator type

$\mathsf{i} :: ('i, 'o, 'd)\ op$

Sink operator type

$! :: ('i, 'o, 'd)\ op$

Split operator type

$\Lambda :: ('n, 'n + 'n, 'd)\ op$

Merge operator type

$\mathcal{V} :: ('n + 'n, 'n, 'd)\ op$

Asynchronous Dataflow Operators

- Identity: $\mathcal{I}$
- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

Feedback operator type ($op \uparrow$)

$\uparrow :: ('i + 'm, 'o + 'm, 'd)\ op \Rightarrow ('i, 'o, 'd)\ op$

Transpose operator type

$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd)\ op$

Dummy source operator type

$\mathsf{i} :: ('i, 'o, 'd)\ op$

Sink operator type

$! :: ('i, 'o, 'd)\ op$

Split operator type

$\Lambda :: ('n, 'n + 'n, 'd)\ op$

Merge operator type

$\mathcal{V} :: ('n + 'n, 'n, 'd)\ op$

Copy operator type

$\mathcal{C} :: ('n, 'n + 'n, 'd)\ op$

Asynchronous Dataflow Operators

- Identity: $\mathcal{I}$
- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

Feedback operator type ($op \uparrow$)

$\uparrow :: ('i + 'm, 'o + 'm, 'd)\ op \Rightarrow ('i, 'o, 'd)\ op$

Transpose operator type

$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd)\ op$

Dummy source operator type

$\mathsf{i} :: ('i, 'o, 'd)\ op$

Sink operator type

$! :: ('i, 'o, 'd)\ op$

Split operator type

$\Lambda :: ('n, 'n + 'n, 'd)\ op$

Merge operator type

$\mathcal{V} :: ('n + 'n, 'n, 'd)\ op$

Copy operator type

$\mathcal{C} :: ('n, 'n + 'n, 'd)\ op$

Equality test operator type

$\mathcal{Q} :: ('n + 'n, 'n, 'd\ option)\ op$

Asynchronous Dataflow Properties

Basic network algebra properties

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \text{map\_op } \curvearrowright \curvearrowright (op_1 \parallel op_2) \parallel op_3$
B2_1: $op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \text{map\_op Inl Inl } op$
B2_2: $(\mathcal{I} :: (0, 0, 'd)\ op) \parallel op \approx \text{map\_op Inr Inr } op$
B3: $(op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$
B4_1: $op \bullet \mathcal{I} \approx op$
B4_2: $\mathcal{I} \bullet op \approx op$
B5: $(op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$
B6: $\mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$
B7: $\mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$
B8: $(\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \text{map\_op id (case\_sum Inr Inl) } \mathcal{I}$
B9: $\mathcal{X} \approx \text{map\_op } \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map\_op id } \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$
B10: $(op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$
F1: $\mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$
F2: $\mathcal{X} \uparrow \approx \mathcal{I}$
R1: $op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$
R2: $(op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$
R3: $op_1 \parallel (op_2 \uparrow) \approx (\text{map\_op } \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$
R4: $(op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$
R5: $\text{map\_op Inl Inl } ((op :: ('i + 0, 'o + 0, 'd)\ op) \uparrow) \approx op$
R6: $(op \uparrow) \uparrow \approx (\text{map\_op } \curvearrowright \curvearrowright op) \uparrow$

Basic network algebra properties

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \text{map\_op}\ \curvearrowright\curvearrowright\ (op_1 \parallel op_2) \parallel op_3$
B2_1: $op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \text{map\_op Inl Inl}\ op$
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B8: $(\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \text{map\_op id (case\_sum Inr Inl)}\ \mathcal{I}$
B9: $\mathcal{X} \approx \text{map\_op}\ \curvearrowright\curvearrowright\ (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map\_op id}\ \curvearrowleft\ (\mathcal{I} \parallel \mathcal{X})$
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R1: $op_2 \bullet (op_1\uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1)\uparrow$
R2: $(op_1\uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I}))\uparrow$
R3: $op_1 \parallel (op_2\uparrow) \approx (\text{map\_op}\ \curvearrowleft\curvearrowleft\ (op_1 \parallel op_2))\uparrow$
R4: $(op_1 \bullet (\mathcal{I} \parallel op_2))\uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1)\uparrow$
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Basic network algebra properties

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \text{map\_op}\ \curvearrowright\ \curvearrowright\ (op_1 \parallel op_2) \parallel op_3$

B2_1: $op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \text{map\_op Inl Inl } op$

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B4_2: $\mathcal{I} \bullet op \approx op$

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B6: $\mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$

B7: $\mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$

B8: $(\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \text{map\_op id (case\_sum Inr Inl) } \mathcal{I}$

B9: $\mathcal{X} \approx \text{map\_op}\ \curvearrowright\ \curvearrowright\ (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map\_op id}\ \curvearrowleft\ (\mathcal{I} \parallel \mathcal{X})$

B10: $(op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$

F1: $\mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$

F2: $\mathcal{X} \uparrow \approx \mathcal{I}$

R1: $op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$

R2: $(op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$

R3: $op_1 \parallel (op_2 \uparrow) \approx (\text{map\_op}\ \curvearrowleft\ \curvearrowleft\ (op_1 \parallel op_2)) \uparrow$

R4: $(op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$

R5: $\text{map\_op Inl Inl } ((op :: ('i + 0, 'o + 0, 'd)\ op) \uparrow) \approx op$

R6: $(op \uparrow) \uparrow \approx (\text{map\_op}\ \curvearrowright\ \curvearrowright\ op) \uparrow$

Basic network algebra properties

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R1: $op_2 \bullet (op_1\uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1)\uparrow$

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R3: $op_1 \parallel (op_2\uparrow) \approx (\text{map\_op}\ \curvearrowleft\ \curvearrowleft\ (op_1 \parallel op_2))\uparrow$

R4: $(op_1 \bullet (\mathcal{I} \parallel op_2))\uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1)\uparrow$

R5: $\text{map\_op Inl Inl } ((op :: ('i + 0, 'o + 0, 'd)\ op)\uparrow) \approx op$

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Basic network algebra properties

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R4: $(op_1 \bullet (\mathcal{I} \parallel op_2))\uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1)\uparrow$

R5: $\text{map\_op Inl Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op)\uparrow) \approx op$

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B10: $(op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$

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R3: $op_1 \parallel (op_2\uparrow) \approx (\text{map\_op}\ \curvearrowleft\ \curvearrowleft\ (op_1 \parallel op_2))\uparrow$

R4: $(op_1 \bullet (\mathcal{I} \parallel op_2))\uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1)\uparrow$

R5: $\text{map\_op Inl Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op)\uparrow) \approx op$

R6: $(op\uparrow)\uparrow \approx (\text{map\_op}\ \curvearrowright\ \curvearrowright\ op)\uparrow$

Properties of equality test, merge, copy, split, source and sink operators

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ (op_1 \parallel op_2) \parallel op_3$

B2_1: $op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \mathsf{map\_op\ Inl\ Inl}\ op$

B2_2: $(\mathcal{I} :: (0, 0, 'd)\ op) \parallel op \approx \mathsf{map\_op\ Inr\ Inr}\ op$

B3: $(op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$

B4_1: $op\vdash \bullet \mathcal{I} \approx op\vdash$ B4_2: $\mathcal{I} \bullet \dashv op \approx \dashv op$

B5: $(op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$

B6: $\mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$ B7: $\mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$

B8: $(\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \mathsf{map\_op\ id\ (case\_sum\ Inr\ Inl)}\ \mathcal{I}$

B9: $\mathcal{X} \approx \mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ (\mathcal{X} \parallel \mathcal{I}) \bullet \mathsf{map\_op\ id}\ \curvearrowleft\ (\mathcal{I} \parallel \mathcal{X})$

B10: $(\dashv op_1 \parallel \dashv op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2\vdash \parallel op_1\vdash)$

F1: $\mathcal{I}\uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ F2: $\mathcal{X}\uparrow \approx \mathcal{I}$

Table 1 Basic network algebra properties

R1: $\mathsf{Inr} \succ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr} \succ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow op_2 \bullet (op_1\uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1)\uparrow$

R2: $\mathsf{Inr} \succ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr} \succ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow (op_1\uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I}))\uparrow$

R3: $\mathsf{Inr} \succ \mathsf{inputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr} \succ \mathsf{outputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow op_1 \parallel (op_2\uparrow) \approx (\mathsf{map\_op}\ \curvearrowleft\ \curvearrowleft\ (op_1 \parallel op_2))\uparrow$

R4: $\mathsf{Inr} \succ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr} \succ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{inputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{outputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow (\dashv op_1 \bullet (\mathcal{I} \parallel op_2))\uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1\vdash)\uparrow$

R5: $\mathsf{Inr} \succ \mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr} \succ \mathsf{outputs}\ op = \{\} \Longrightarrow \mathsf{map\_op\ Inl\ Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op)\uparrow) \approx op$

R6: $\mathsf{Inr} \succ \mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr} \succ \mathsf{outputs}\ op = \{\} \Longrightarrow \mathsf{Inr} \succ \mathsf{Inl} \succ \mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr} \succ \mathsf{Inl} \succ \mathsf{outputs}\ op = \{\} \Longrightarrow (op\uparrow)\uparrow \approx (\mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ op)\uparrow$

A10

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \mathsf{map\_op}\ \curvearrowright\curvearrowright\ (op_1 \parallel op_2) \parallel op_3$

B2_1: $op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \mathsf{map\_op\ Inl\ Inl}\ op$

B2_2: $(\mathcal{I} :: (0, 0, 'd)\ op) \parallel op \approx \mathsf{map\_op\ Inr\ Inr}\ op$

B3: $(op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$

B4_1: $op\vdash \bullet \mathcal{I} \approx op\vdash$ B4_2: $\mathcal{I} \bullet \dashv op \approx \dashv op$

B5: $(op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$

B6: $\mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$ B7: $\mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$

B8: $(\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \mathsf{map\_op\ id\ (case\_sum\ Inr\ Inl)}\ \mathcal{I}$

B9: $\mathcal{X} \approx \mathsf{map\_op}\ \curvearrowright\curvearrowright\ (\mathcal{X} \parallel \mathcal{I}) \bullet \mathsf{map\_op\ id}\ \curvearrowleft\ (\mathcal{I} \parallel \mathcal{X})$

B10: $(\dashv op_1 \parallel \dashv op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2\vdash \parallel op_1\vdash)$

F1: $\mathcal{I}\uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ F2: $\mathcal{X}\uparrow \approx \mathcal{I}$

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Well-Behaved Operators

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \mathsf{map\_op}\ \curvearrowright\curvearrowright\ (op_1 \parallel op_2) \parallel op_3$

B2_1: $op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \mathsf{map\_op\ Inl\ Inl}\ op$

B2_2: $(\mathcal{I} :: (0, 0, 'd)\ op) \parallel op \approx \mathsf{map\_op\ Inr\ Inr}\ op$

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B9: $\mathcal{X} \approx \mathsf{map\_op}\ \curvearrowright\curvearrowright\ (\mathcal{X} \parallel \mathcal{I}) \bullet \mathsf{map\_op\ id}\ \curvearrowleft\ (\mathcal{I} \parallel \mathcal{X})$

B10: $(\dashv op_1 \parallel \dashv op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2\vdash \parallel op_1\vdash)$

F1: $\mathcal{I}\uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ F2: $\mathcal{X}\uparrow \approx \mathcal{I}$

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Related Work

Related Work

- A Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
  - Operators as a shallow embedding as codatatypes
  - 51 axioms proved
- Executable via code extraction to Haskell

Conclusion

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- Isabelle/HOL has a good tool set to formalize and reason about stream processing:
  - Codatatypes, coinductive predicates, corecursion with friends, reasoning up to friends (congruence),
    - Coinduction up to congruence principle is automatically derived for codatatypes (but not for coinductive principles)
- Next step: Feedback loop

Questions, comments and suggestions