Nondeterministic Asynchronous Dataflow in Isabelle/HOL

Rafael Castro G. Silva, Laouen Fernet and Dmitriy Traytel

rasi@di.ku.dk

Department of Computer Science
University of Copenhagen

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Motivation

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Context:

- Stream Processing: programs that compute (possibly) unbounded sequences of data (streams)
- A common problem in the industry
- Frameworks:
  Apache Flink, Apache Samza, Apache Spark, Google Cloud Dataflow, and Timely Dataflow
- Why use frameworks?
  - Highly Parallel
  - Low latency (output as soon as possible)
  - Incremental computing (re-uses previous computations)

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Our goal:

Mechanically Verify Timely Dataflow algorithms

A Good Foundation

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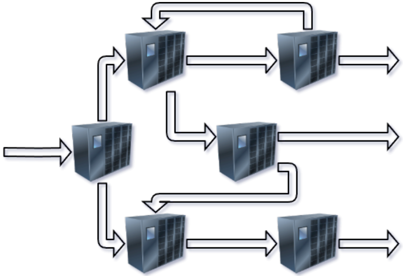

- Nondeterministic Asynchronous Dataflow

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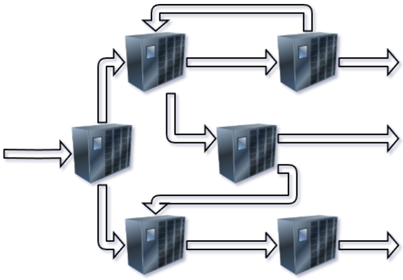

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    - Dataflow: Directed graph of interconnected operators

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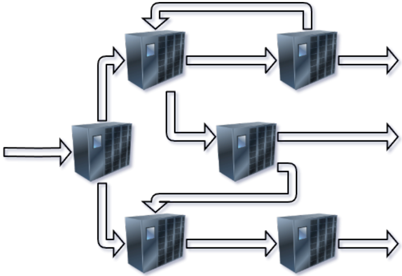

- Nondeterministic Asynchronous Dataflow
  - Dataflow: Directed graph of interconnected operators
  - Asynchronous:
    - Operators execute independently: processes without an orchestrator
    - Operators can freely communicate with the network (read/write); do silent computation steps
    - Networks are unbounded FIFO queues

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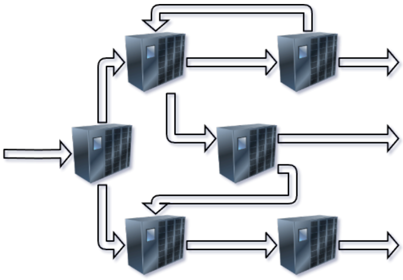

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  - Nondeterministic:
    - Operators can make nondeterministic choices
    - Operators are relations between inputs and outputs sequences

The Algebra for Nondeterministic Asynchronous Dataflow

- Bergstra et al. presents an algebra for Nondeterministic Asynchronous Dataflow
- Primitives:
  sequential and parallel composition; feedback loop...
- The 52 axioms
- An process calculus instance

Network Algebra for Asynchronous Dataflow*

J.A. Bergstra[1,2,†] C.A. Middelburg[2,3,§] Gh. Ştefănescu[4,‡]

[1]Programming Research Group, University of Amsterdam
P.O. Box 41882, 1009 DB Amsterdam, The Netherlands

[2]Department of Philosophy, Utrecht University
P.O. Box 80126, 3508 TC Utrecht, The Netherlands

[3]Department of Network & Service Control, KPN Research
P.O. Box 421, 2260 AK Leidschendam, The Netherlands

[4]Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, 70700 Bucharest, Romania

E-mail: janb@fwi.uva.nl - keesm@phil.ruu.nl - ghstef@imar.ro

Isabelle/HOL Preliminaries

Isabelle/HOL

- Classical higher-order logic (HOL):
  Simple Typed Lambda Calculus + axiom of choice + axiom of infinity + rank-1 polymorphism

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- Isabelle/HOL: Isabelle's flavor of HOL

Why Isabelle/HOL?

- Codatatypes: (possibly) infinite data structures (e.g., lazy lists, streams)
- Corecursion: always eventually produces some codatatype constructor
- Coinductive predicate: infinite number of introduction rule applications
- Coinduction: reason about coinductive predicates

Operators as a Codatatype

Operators

Operators in Isabelle/HOL

$$\textbf{codatatype}\ (\text{inputs}: 'i, \text{outputs}: 'o, 'd)\ op =$$
$$\text{Read}\ 'i\ ('d \Rightarrow ('i, 'o, 'd)\ op) \mid \text{Write}\ (('i, 'o, 'd)\ op)\ 'o\ 'd$$
$$\text{Silent}\ ('i, 'o, 'd)\ op \mid \text{Choice}\ (('i, 'o, 'd)\ op)\ cset$$

- Type parameters: inputs/output ports; data
- Operator's actions
- Possibly infinite trees
- inputs/outputs: Sets of used ports

Examples 1

Uncommunicative operators

abbreviation

$\oslash \equiv \text{Choice } \{\}_c$

corec silent_op ($\odot$) **where**

$\odot = \text{Silent } \odot$

corec spin_op ($\otimes$) **where**

$\otimes = \text{Choice } ((\lambda\_.\ \otimes)`_c\ \{()\}_c)$

lemma spin_op_code:

$\otimes = \text{Choice } \{\otimes\}_c$

- They have the same meaning!
- A small quirk: any corecursive call guarded by the Choice constructor must be applied using the map function on *cset*

Operators Equivalences: Motivation

- foo

Operators Equivalences: Strong Bisimilarity

- foo

Operators Equivalences: Weak Bisimilarity

- foo

Asynchronous Dataflow Operators

Buffer Infrastructure

- foo

Asynchronous Dataflow Properties

Basic network algebra properties

$B1: op_1 \parallel (op_2 \parallel op_3) \approx \mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ (op_1 \parallel op_2) \parallel op_3$

$B2\_1: op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \mathsf{map\_op}\ \mathsf{Inl}\ \mathsf{Inl}\ op$

$B2\_2: (\mathcal{I} :: (0, 0, 'd)\ op) \parallel op \approx \mathsf{map\_op}\ \mathsf{Inr}\ \mathsf{Inr}\ op$

$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$

$B4\_1: op{\vdash} \bullet \mathcal{I} \approx op{\vdash}$ $\qquad$ $B4\_2: \mathcal{I} \bullet {\dashv}op \approx {\dashv}op$

$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$

$B6: \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$ $\qquad$ $B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$

$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \mathsf{map\_op}\ \mathsf{id}\ (\mathsf{case\_sum}\ \mathsf{Inr}\ \mathsf{Inl})\ \mathcal{I}$

$B9: \mathcal{X} \approx \mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ (\mathcal{X} \parallel \mathcal{I}) \bullet \mathsf{map\_op}\ \mathsf{id}\ \curvearrowleft\ (\mathcal{I} \parallel \mathcal{X})$

$B10: ({\dashv}op_1 \parallel {\dashv}op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2{\vdash} \parallel op_1{\vdash})$

$F1: \mathcal{I}\uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ $\qquad$ $F2: \mathcal{X}\uparrow \approx \mathcal{I}$

Table 1 Basic network algebra properties

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Conclusion

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- Isabelle/HOL has a good tool set to formalize and reason about stream processing:
  - Codatatypes, coinductive predicates, corecursion with friends, reasoning up to friends (congruence),
    - Coinduction up to congruence principle is automatically derived for codatatypes (but not for coinductive principles)
- Next step: Feedback loop

Questions, comments and suggestions