Verified Time-Aware Stream Processing

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What is this PhD/Status seminar about?

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- Distributed Systems
    - Stream processing frameworks
        - Dataflow models
        - Time-Aware Computations

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- Distributed Systems
  - Stream processing frameworks
    - Dataflow models
      - Time-Aware Computations
- Formal Methods
  - Verification using proof assistants
    - Isabelle proofs
      - Verified + executable + efficient code
- Formalization of Time-Aware Stream Processing

Introduction

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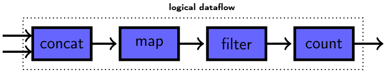
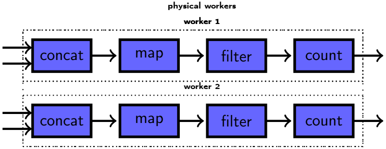
    - Directed graph of interconnected operators that perform event-wise transformations
    - E.g.: Apache Flink, Apache Samza, Apache Spark, Google Cloud Dataflow, and Timely Dataflow

Cloud DataFlow Flink

- Highly Parallel





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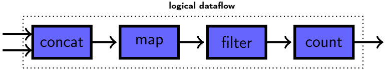

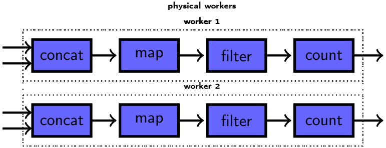

- Time-Aware Computations
  - Timestamps: Metadata associating the data with some data collection
  - Watermarks: Metadata indicating the completion of a data collection

Preliminaries

Isabelle/HOL

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- Isabelle: A generic proof assistant

- Isabelle/HOL: Isabelle's flavor of HOL
- All functions in Isabelle/HOL must be total

Isabelle/HOL: (Co)datatypes

- Datatypes and Codatatypes

  codatatype (lset: $'a$) *llist* = lnull: LNil | LCons (lhd: $'a$) (ltl: $'a$ *llist*)
  for map: lmap **where** ltl LNil = LNil

- Examples:
  - LNil
  - LCons *1* (LCons *2* (LCons *3* LNil))
  - LCons *0* (LCons *0* (LCons *0* (...)))

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- Coinductive principle for lazy list equality:
  - Show that there is a "pair of goggles " (relation) that makes them to look the same:
    - The first lazy list is empty iff second is
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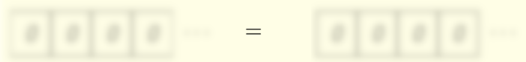
    - Their tail looks the same



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$$\boxed{0}\,\boxed{0}\,\boxed{0}\,\boxed{0}\;\cdots \quad = \quad \boxed{0}\,\boxed{0}\,\boxed{0}\,\boxed{0}\;\cdots$$

Isabelle/HOL: Recursion and While Combinator

- Recursion

$\textbf{fun}$ lshift $:: {}'a\ list \Rightarrow {}'a\ llist \Rightarrow {}'a\ llist$ (*infixr* @@ *65*) $\textbf{where}$
  lshift [] $lxs = lxs$
| lshift $(x \# xs)$ $lxs =$ LCons x (lshift xs lxs)

- While Combinator

$\textbf{definition}$ while_option $:: ({}'a \Rightarrow bool) \Rightarrow ({}'a \Rightarrow {}'a) \Rightarrow {}'a \Rightarrow {}'a\ option$ $\textbf{where}$
while_option $b\ c\ s = \ldots$

- While rule for invariant reasoning (Hoare-style):
  - There is something that holds before a step; that thing still holds after the step

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$$\mathsf{lapp}\ \mathit{lxs}\ \mathit{lys} = \mathtt{case}\ \mathit{lxs}\ \mathtt{of}\ \mathsf{LNil} \Rightarrow \mathit{lys} \mid \mathsf{LCons}\ x\ \mathit{lxs}' \Rightarrow \mathsf{LCons}\ x\ (\mathsf{lapp}\ \mathit{lxs}'\ \mathit{lys})$$

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- Friendly function
  - Preserves productivity: it may consume at most one constructor to produce one constructor.
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- Coinduction up to congruence: Coinduction for Lazy list equality can be extended to compare an entire finite prefix through a congruence relation

Isabelle/HOL: (Co)inductive Predicates

- Inductive predicate
  - Finite number of introduction rule applications

inductive in_llist $:: {'a} \Rightarrow {'a}\ llist \Rightarrow bool$ **where**
  In_llist: in_llist x (LCons x lxs)
| Next_llist: in_llist x lxs $\Rightarrow$ in_llist x (LCons y lxs)

in_llist 2 (LCons 1 (LCons (2 ($\ldots$))))

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$$\mathsf{in\_llist}\ \mathit{2}\ (\mathsf{LCons}\ \mathit{1}\ (\mathsf{LCons}\ (\mathit{2}\ (\ldots))))$$

- Coinductive predicate
  - Infinite number of introduction rule applications

$$\textbf{coinductive}\ \mathsf{lprefix} :: {'a}\ \mathit{llist} \Rightarrow {'a}\ \mathit{llist} \Rightarrow \mathit{bool}\ \textbf{where}$$
$$\mathsf{LNil\_lprefix}\colon \mathsf{lprefix}\ \mathsf{LNil}\ \mathit{lys}$$
$$\mid \mathsf{LCons\_lprefix}\colon \mathsf{lprefix}\ \mathit{lxs}\ \mathit{lys} \Rightarrow \mathsf{lprefix}\ (\mathsf{LCons}\ x\ \mathit{lxs})\ (\mathsf{LCons}\ x\ \mathit{lys})$$

$$\mathsf{lprefix}\ (\mathsf{LCons}\ \mathit{1}\ (\mathsf{LCons}\ (\mathit{2}\ (\ldots))))\ (\mathsf{LCons}\ \mathit{1}\ (\mathsf{LCons}\ (\mathit{2}\ (\ldots))))$$

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coinductive lprefix $:: 'a\ llist \Rightarrow 'a\ llist \Rightarrow bool$ **where**
  LNil_lprefix: lprefix LNil lys
| LCons_lprefix: lprefix lxs lys $\Rightarrow$ lprefix (LCons x lxs) (LCons x lys)

lprefix (LCons 1 (LCons (2 (...)))) (LCons 1 (LCons (2 (...))))

- Coinduction principle
- But not coinduction up to congruence for free

Lazy Lists Processors

Operator formalization

- Operator as a codatatype
  - Taking $'i$ as the input type, and $'o$ as the output type:
    $\mathbf{codatatype}\ ('o, 'i)\ op = \mathrm{Logic}\ (\mathrm{apply}: ('i \Rightarrow ('o, 'i)\ op \times 'o\ list))$

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  - Infinite trees: applying the selector apply "walks" a branch of the tree

Execution formalization

- Produce function: applies the logic (co)recursively throughout a lazy list

$\textbf{definition}$ $\mathsf{produce}_1$ op lxs $=$ $\mathsf{while\_option}$ $\ldots$

$\textbf{corec}$ $\mathsf{produce}$ $\textbf{where}$
  $\mathsf{produce}$ op lxs $=$ $(\mathtt{case}$ $\mathsf{produce}_1$ op lxs $\mathtt{of}$ $\mathsf{None} \Rightarrow \mathsf{LNil}$
    $\mid \mathsf{Some}\ (op', x, xs, lxs') \Rightarrow \mathsf{LCons}\ x\ (xs\ @@\ \mathsf{produce}\ op'\ lxs'))$

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- produce_1 has an induction principle based on the while invariant rule

Operators: Count

- Example:

$$
\begin{array}{l}
\textbf{corec}\ \mathsf{count\_op}\ \textbf{where}\ \mathsf{count\_op}\ P\ n = \\
\quad \mathsf{Logic}\ (\lambda e.\ \texttt{if}\ P\ e\ \texttt{then}\ (\mathsf{count\_op}\ P\ (n + 1),\ [n{+}1])\ \texttt{else}\ (\mathsf{count\_op}\ P\ n,\ []))
\end{array}
$$

$$stream_1 = \boxed{0}\,\boxed{3}\,\boxed{3}\,\boxed{6}\,\boxed{24}$$

$$\mathsf{produce}\ (\mathsf{count\_op}\ is\_even\ 0)\ \ stream_1 = \boxed{1}\,\boxed{2}\,\boxed{3}$$

Sequential Composition

- Sequential composition: take the output of the first operator and give it as input to the second operator.

definition fproduce $op\ xs$ = fold $(\lambda e\ (op, out).$
  `let` $(op', out') =$ apply $op\ e$ `in` $(op', out\ @\ out'))\ xs\ (op, [])$
corec comp_op **where**
  comp_op $op_1\ op_2$ = Logic $(\lambda ev.$
    `let` $(op_1{}', out) =$ apply $op_1\ ev;\ (op_2{}', out') =$ fproduce $op_2\ out$
    `in` (comp_op $op_1{}'\ op_2{}', out'))$

Sequential Composition: Correctness

- Correctness:

$$\text{produce (comp\_op } op_1 \ op_2) \ lxs = \text{produce } op_2 \ (\text{produce } op_1 \ lxs)$$

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- Proof: coinduction principle for lazy list equality and produce_1 induction principle

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  - Generalization: we must be able to reason about elements in arbitrary positions

$$
\begin{array}{l}
\textbf{corec } \text{skip\_op } \textbf{where} \\
\quad \text{skip\_op } op \ n = \text{Logic } (\lambda ev.\ \texttt{let } (op', out) = \text{apply } op \ ev \ \texttt{in} \\
\qquad \texttt{if } \text{length } out < n \ \texttt{then } (\text{skip\_op } op' \ (n - \text{length } out), []) \\
\qquad \texttt{else } (op', \text{drop } n \ out))
\end{array}
$$

  - Correctness

$$\text{produce } (\text{skip\_op } op \ n) \ lxs = \text{ldropn } n \ (\text{produce } op \ lxs)$$

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Time-Aware Operators

Time-Aware Streams

- Time-Aware lazy lists

$\textbf{datatype}\ ('t{::}order, 'd)\ event = \mathsf{DT}\ (\mathsf{tmp}{:}\ 't)\ (\mathsf{data}{:}\ 'd) \mid \mathsf{WM}\ (\mathsf{wmk}{:}\ 't)$

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- Generalization to partial orders
  - Cycles
  - Operators with multiple inputs

Monotone Time-Aware Streams

- Monotone: watermarks do not go back in time

$$
\begin{array}{l}
\textbf{coinductive}\ \mathsf{monotone} :: ('t{::}order,\ 'd)\ event\ llist \Rightarrow\ 't\ \mathsf{set} \Rightarrow bool\ \textbf{where} \\
\quad \mathsf{LNil}\!:\ \mathsf{monotone}\ \mathsf{LNil}\ W \\
\mid \mathsf{LConsR}\!:\ (\forall wm' \in W.\ \neg\ wm' \geq wm) \longrightarrow \mathsf{monotone}\ lxs\ (\{wm\} \cup W) \longrightarrow \\
\qquad \mathsf{monotone}\ (\mathsf{LCons}\ (\mathsf{WM}\ wm)\ lxs)\ W \\
\mid \mathsf{LConsL}\!:\ (\forall wm \in W.\ \neg\ wm \geq t) \longrightarrow \mathsf{monotone}\ lxs\ W \longrightarrow \\
\qquad \mathsf{monotone}\ (\mathsf{LCons}\ (\mathsf{DT}\ t\ d)\ lxs)\ W
\end{array}
$$

- Up to congruence coinduction principle
- Example:

$stream_2 =$

DT t_4 d	DT t_0 a	DT t_1 c	WM t_1	DT t_2 b
WM t_2	DT t_5 c	DT t_3 a	DT t_5 a	WM t_5 ...

$$
\begin{tikzcd}
C_{t_4}\colon \{\ d\ \} & & & C_{t_5}\colon \{\ a, c\ \} & \\
C_{t_1}\colon \{\ c\ \} \arrow[u] & & C_{t_2}\colon \{\ b\ \} \arrow[ur] & & C_{t_3}\colon \{\ a\ \} \arrow[ul] \\
& & C_{t_0}\colon \{\ a\ \} \arrow[ull] \arrow[u] \arrow[urr] & &
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coinductive productive **where**
  LFinite*:* lfinite *lxs* $\longrightarrow$ productive *lxs*
| EnvWM*:* $\neg$ lfinite *lxs* $\longrightarrow$ $(\exists u \in$ vimage WM (lset *lxs*). $u \geq t)$ $\longrightarrow$
   productive *lxs* $\longrightarrow$ productive (LCons (DT *t d*) *lxs*)
| *SkipWM:* $\neg$ lfinite *lxs* $\longrightarrow$ productive *lxs* $\longrightarrow$
   productive (LCons (WM *t*) *lxs*)

stream$_2$ =

DT t_4 d	DT t_0 a	DT t_1 c	WM t_1	DT t_2 b
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$\cdots$

$$\begin{tikzcd}
C_{t_4}\colon \{\, d \,\} & C_{t_5}\colon \{\, a, c \,\} & \\
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\end{tikzcd}$$

- Up to congruence coinduction principle

Building Blocks: Batch Operator

- Building Blocks: reusable operators
  - Batching and incremental computations

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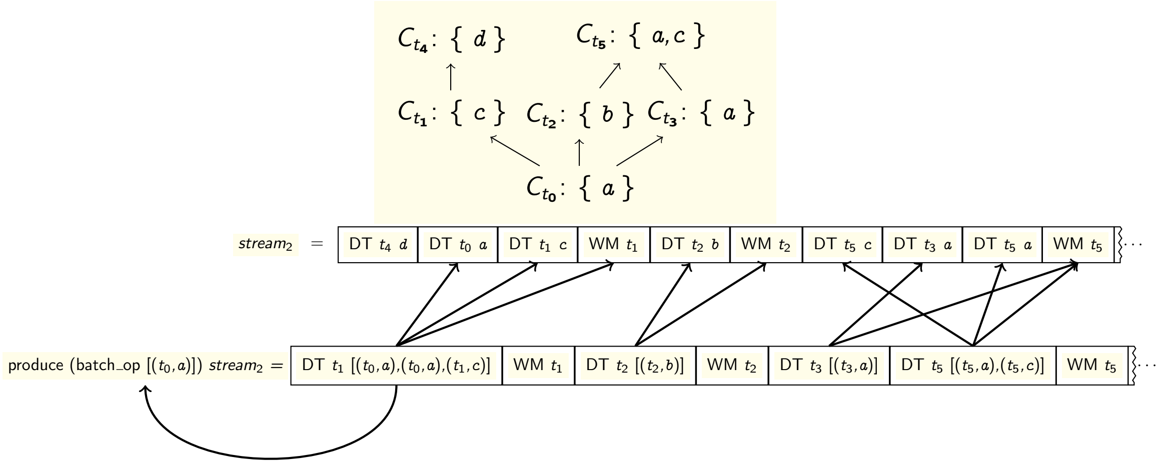
  - Batching and incremental computations
- batch_op : produces batches of accumulated data

$$
\begin{array}{l}
\textbf{corec}\ \mathsf{batch\_op}\ \textbf{where} \\
\quad \mathsf{batch\_op}\ \mathit{buf} = \mathsf{Logic}\ (\lambda \mathit{ev}.\ \texttt{case}\ \mathit{ev}\ \texttt{of}\ \mathsf{DT}\ t\ d \Rightarrow (\mathsf{batch\_op}\ (\mathit{buf}\ @\ [(t,\ d)]),\ []) \\
\quad \mid \mathsf{WM}\ \mathit{wm} \Rightarrow \texttt{if}\ \exists (t,\ d) \in \mathsf{set}\ \mathit{buf}.\ t \leq \mathit{wm} \\
\qquad \texttt{then}\ \texttt{let}\ \mathit{out} = \mathsf{filter}\ (\lambda (t,\ \_).\ t \leq \mathit{wm})\ \mathit{buf}; \\
\qquad\qquad\qquad \mathit{buf'} = \mathsf{filter}\ (\lambda (t,\ \_).\ \neg\ t \leq \mathit{wm})\ \mathit{buf} \\
\qquad\qquad \texttt{in}\ (\mathsf{batch\_op}\ \mathit{buf'},\ [\mathsf{DT}\ \mathit{wm}\ \mathit{out},\ \mathsf{WM}\ \mathit{wm}]) \\
\qquad \texttt{else}\ (\mathsf{batch\_op}\ \mathit{buf},\ [\mathsf{WM}\ \mathit{wm}]))
\end{array}
$$

Batch Operator: Soundness

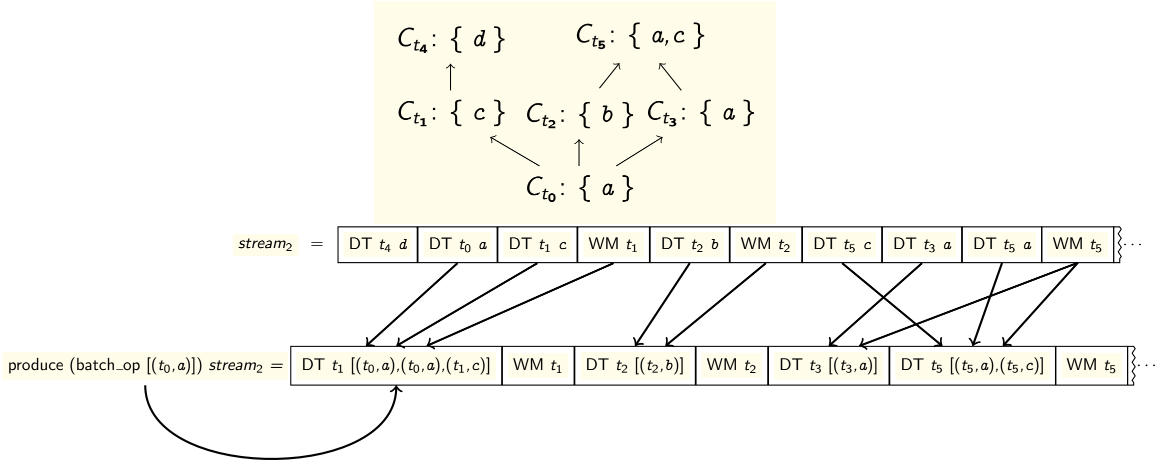
- Given a monotone time-aware stream



- Proof: lset induction, produce$_1$ induction, and generalization with skip_op

Batch Operator: Completeness

- Given a monotone and productive time-aware stream
- if not finite



- Proof: induction over the position (nat) of the element in the input, and soundness of batch_op

Batch Operator: Monotone and productive preservation

- The operators must preserve monotone and productive, so we can compose it with something that needs these properties!

$$\text{monotone } \mathit{lxs}\ W \longrightarrow \text{monotone (produce (batch\_op } \mathit{buf}\text{) } \mathit{lxs}\text{) } W \tag{1}$$

$$\text{productive } \mathit{lxs} \longrightarrow \text{productive (produce (batch\_op } \mathit{buf}\text{) } \mathit{lxs}\text{)} \tag{2}$$

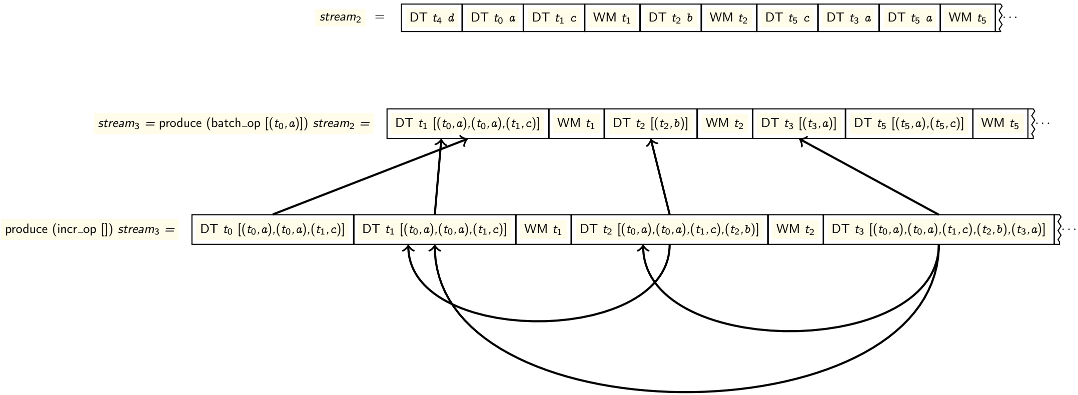
- Proof: coinduction up to congruence

Building Blocks: Incremental Operator

- Incremental computations
- incr_op : produces accumulated batches of accumulated data

$$
\begin{array}{l}
\textbf{corec } \mathsf{incr\_op} \textbf{ where} \\
\quad \mathsf{incr\_op}\ \mathit{buf} = \mathsf{Logic}\ (\lambda\ \mathit{ev}.\ \mathtt{case}\ \mathit{ev}\ \mathtt{of}\ \mathsf{DT}\ \mathit{wm}\ \mathit{batch} \Rightarrow \\
\qquad \mathtt{let}\ \mathit{out} = \mathsf{map}\ (\lambda t.\ \mathsf{DT}\ t\ (\mathit{buf}\ @\ \mathit{batch}))\ (\mathsf{remdups}\ (\mathsf{map}\ \mathsf{fst}\ \mathit{batch})) \\
\qquad \mathtt{in}\ (\mathsf{incr\_op}\ (\mathit{buf}\ @\ \mathit{batch}), \mathit{out}) \\
\quad \mid \mathsf{WM}\ \mathit{wm} \Rightarrow\ (\mathsf{incr\_op}\ \mathit{buf}, [\mathsf{WM}\ \mathit{wm}]))
\end{array}
$$

Incremental Operator: Soundness



- Proof: produce$_1$ induction, and generalization with skip_op

Incremental Operator: Completeness

$stream_2$ = | DT t_4 d | DT t_0 a | DT t_1 c | WM t_1 | DT t_2 b | WM t_2 | DT t_5 c | DT t_3 a | DT t_5 a | WM t_5 | $\cdots$

$stream_3$ = produce (batch_op $[(t_0,a)]$) $stream_2$ = | DT t_1 $[(t_0,a),(t_0,a),(t_1,c)]$ | WM t_1 | DT t_2 $[(t_2,b)]$ | WM t_2 | DT t_3 $[(t_3,a)]$ | DT t_5 $[(t_5,a),(t_5,c)]$ | WM t_5 | $\cdots$

produce (incr_op []) $stream_3$ = | DT t_0 $[(t_0,a),(t_0,a),(t_1,c)]$ | DT t_1 $[(t_0,a),(t_0,a),(t_1,c)]$ | WM t_1 | DT t_2 $[(t_0,a),(t_0,a),(t_1,c),(t_2,b)]$ | WM t_2 | DT t_3 $[(t_0,a),(t_0,a),(t_1,c),(t_2,b),(t_3,a)]$ | $\cdots$

- Proof: induction over the position (nat) of the element in the input

Incremental Operator: Monotone and productive preservation

$$\text{monotone } lxs\ W \longrightarrow \text{monotone (produce (incr\_op []) } lxs)\ W \tag{3}$$

$$\text{productive } lxs \longrightarrow \text{productive (produce (incr\_op []) } lxs) \tag{4}$$

- Proof: coinduction up to congruence

Compositional Reasoning

- batch_op and incr_op can be composed

  definition incr_batch_op *buf1 buf2* = comp_op (batch_op *buf1*) (incr_op *buf2*)

- Soundness, completeness, and monotone and productive preservation

Case Study

Histogram

- A histogram count the elements of a collection
- Incremental histogram: timestamps smaller or equal
- $H_{t_5} = C_{t_0} + C_{t_0} + C_{t_1} + C_{t_2} + C_{t_5}$
- paths to t_5: $\{t_0, t_2\}$ and $\{t_0, t_3\}$

$$
\begin{tikzcd}
C_{t_4}\colon \{\, d \,\} & & & C_{t_5}\colon \{\, a, c \,\} & \\
C_{t_1}\colon \{\, c \,\} \arrow[u] & & C_{t_2}\colon \{\, b \,\} \arrow[ur] & & C_{t_3}\colon \{\, a \,\} \arrow[ul] \\
& & C_{t_0}\colon \{\, a \,\} \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
H_{t_4}\colon \{\, a, c, d \,\} & & & H_{t_5}\colon \{\, a, a, a, a, b, c \,\} & \\
H_{t_1}\colon \{\, c,\ a \,\} \arrow[u] & & H_{t_2}\colon \{\, b,\ a \,\} \arrow[ur] & & H_{t_3}\colon \{\, a,\ a \,\} \arrow[ul] \\
& & H_{t_0}\colon \{\, a \,\} \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
$$

Histogram Operator

$\textbf{corec}$ map_op $\textbf{where}$ map_op f = Logic (λ ev. case ev of
    WM wm $\Rightarrow$ (map_op f, [WM wm]) | DT t d $\Rightarrow$ (map_op f, [DT t (f t d)]))

$\textbf{definition}$ incr_coll t xs = mset ...

$\textbf{definition}$ incr_hist_op $buf1$ $buf2$ =
  comp_op (incr_batch_op $buf1$ $buf2$) (map_op incr_coll)

Histogram Operator: Correctness

- Correctness: (1) Soundness + (2) Completeness + (3) Monotone Preservation + (4) Productive Preservation
- Given a monotone and productive time-aware stream

$$stream_2 = \boxed{\text{DT } t_4\ d} \boxed{\text{DT } t_0\ a} \boxed{\text{DT } t_1\ c} \boxed{\text{WM } t_1} \boxed{\text{DT } t_2\ b} \boxed{\text{WM } t_2} \boxed{\text{DT } t_5\ c} \boxed{\text{DT } t_3\ a} \boxed{\text{DT } t_5\ a} \boxed{\text{WM } t_5} \cdots$$

$$\text{produce (incr\_hist\_op [] []) } stream_2 = \boxed{\text{DT } t_0 \text{ (mset } [a])} \boxed{\text{DT } t_1 \text{ (mset } [a, c])} \boxed{\text{WM } t_1} \boxed{\text{DT } t_2 \text{ (mset } [a, b])} \boxed{\text{WM } t_2} \boxed{\text{DT } t_3 \text{ (mset } [a, a])} \boxed{\text{DT } t_5 \text{ (mset } [a, a, a, b, c])} \boxed{\text{WM } t_5} \cdots$$

- Proof: soundness, completeness, monotone and productive preservation of incr_batch_op

Efficient Histogram Operator

- Efficient histogram operator incr_hist_op′ for timestamp in total order
  - State of the operator: last computed histogram, and buffer of newly accumulated data
- Equivalent incr_hist_op only for monotone time-aware stream (equivalence relation)

Join Operator

- Relation Join
- Use the `sum` type in the timestamps to represent two stream as one
- Partial order for the `sum`: lefts and rights are incomparable
- Defined using `incr_batch_op`
- Soundness, Completeness, Monotone
  - WIP: Productive

Next Steps

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- Feedback loop
- Exit argument
- Connect to the Isabelle-LLVM refinement framework

Questions, comments and suggestions