

Nondeterministic Asynchronous Dataflow in Isabelle/HOL

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Motivation

Context:

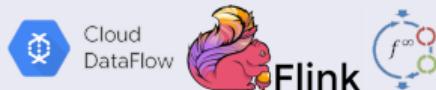
- Stream Processing: programs that compute (possibly) unbounded sequences of data (streams)
- Frameworks: Google Cloud Dataflow, Apache Flink, and Timely Dataflow



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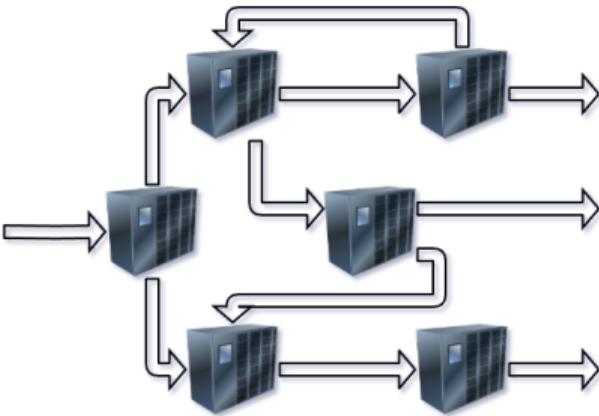
Our long term goal:

Mechanically Verify Timely Dataflow algorithms

A Good Foundation

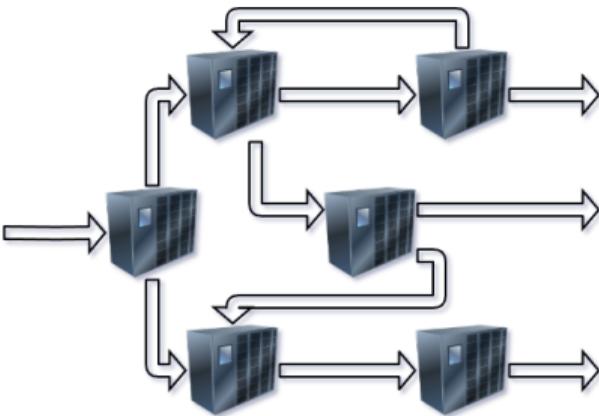
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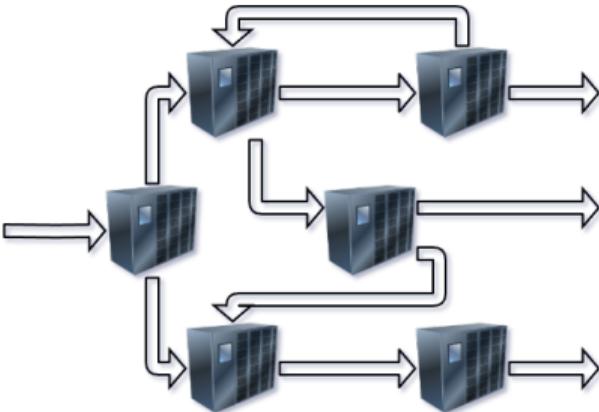
- Nondeterministic Asynchronous Dataflow



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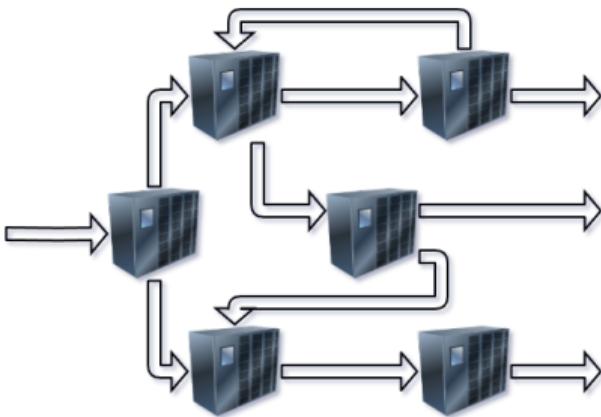
- Nondeterministic Asynchronous Dataflow
 - Dataflow: Directed graph of interconnected operators





- Nondeterministic Asynchronous Dataflow
 - Dataflow: Directed graph of interconnected operators
 - Asynchronous:
 - Operators execute independently: processes without an orchestrator
 - Operators can freely communicate with the network (read/write); do silent computation steps
 - Networks are unbounded FIFO queues

A Good Foundation



- Nondeterministic Asynchronous Dataflow
 - Dataflow: Directed graph of interconnected operators
 - Asynchronous:
 - Operators execute independently: processes without an orchestrator
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 - Networks are unbounded FIFO queues
 - Nondeterministic:
 - Operators can make nondeterministic choices

Question

How do we know something is Nondeterministic Asynchronous Dataflow

The Algebra for Nondeterministic Asynchronous Dataflow

- Bergstra et al. presents an algebra for Nondeterministic Asynchronous Dataflow
- Primitives:
sequential and parallel composition; feedback loop...
- 52 axioms
- A process calculus instance

Network Algebra for Asynchronous Dataflow*

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Main Contributions

- An Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
 - Operators as a shallow embedding as codatatypes
 - 51 axioms proved
- Executable via code extraction to Haskell

Isabelle/HOL Preliminaries

What is Isabelle/HOL?

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- Isabelle: A generic proof assistant



- Classical higher-order logic (HOL):
Simple Typed Lambda Calculus + axiom of choice + axiom of infinity + rank-1 polymorphism

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Why Isabelle/HOL?

- Codatatypes: (possibly) infinite data structures (e.g., lazy lists, streams)
- Corecursion: always eventually produces some codatatype constructor
- Coinductive predicates: infinite number of introduction rule applications
- Coinduction: reason about coinductive predicates

Operators as a Codatatype

Operators

Operators in Isabelle/HOL

```
codatatype (inputs: 'i, outputs: 'o, 'd) op =  
  Read 'i ('d ⇒ ('i, 'o, 'd) op) | Write (('i, 'o, 'd) op) 'o 'd  
  Silent ('i, 'o, 'd) op | Choice (('i, 'o, 'd) op) cset
```

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- Type parameters:
 inputs/output ports; data
- Operator's actions
- inputs/outputs:
 Sets of used ports
- Possibly infinite trees

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Uncommunicative operators

$$\emptyset = \text{Choice } \{\}^c$$

$$\odot = \text{Silent } \odot$$

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Uncommunicative operators

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More examples

$$\text{ex1} = \text{Choice } \{\text{Write ex1 1 42}, \emptyset\}^c$$

$$\text{ex2} = \text{Choice } \{\text{Write ex2 1 42}, \text{ex2}\}^c$$

$$\text{ex3} = \text{Choice } \{\text{Write ex3 1 42}, \text{Silent ex3}\}^c$$

Operators Equivalences: Weak Bisimilarity

An ideal equivalence relation

- Only equate operators that **behave** the same
- Useful reasoning principle

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- Milner's approach!
The classic chapter on **weak bisimilarity**



- Based on labeled transition systems (LTS)
- Labels (actions): Read, Write, Silent (τ)

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- Based on labeled transition systems (LTS)
- Labels (actions): Read, Write, Silent (τ)
- Weak bisimilarity of operators (\approx): LTS + simulation + greatest bisimulation
- \approx has a useful coinduction principle
- $\emptyset \approx \otimes \approx \odot$ and $\text{ex1} \approx \text{ex2} \approx \text{ex3}$

Asynchronous Dataflow Operators

Auxiliary Definitions

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- Buffers: $'p \Rightarrow 'd \ list$

Buffer functions

BHD $p \ buf = \text{hd} \ (buf \ p)$

BTL $p \ buf = buf(p := \text{tl} \ (buf \ p))$

BENQ $p \ x \ buf = buf(p := buf \ p @ [x])$

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choices function

choices $\otimes = \{\}$

Silent $op' \in_c \text{choices } op \longleftrightarrow \text{step Tau } op \ op'$

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Mapping ports functions

map_op :: $('i_1 \Rightarrow 'i_2) \Rightarrow ('o_1 \Rightarrow 'o_2) \Rightarrow ('i_1, 'o_1, 'd) \ op \Rightarrow ('i_2, 'o_2, 'd) \ op$

$\curvearrowright :: ('a + 'b) + 'c \Rightarrow 'a + 'b + 'c$

$\curvearrowleft :: 'a + 'b + 'c \Rightarrow ('a + 'b) + 'c$

Equation of the identity operator

$\text{id_op } buf = \text{Choice}$

\cup_c

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$((((\lambda p. \text{Read } p (\lambda x. \text{id_op} (\text{BENQ } p \times buf)))) \ `_c \ \mathfrak{U}_c)$
 \cup_c

- \mathfrak{U}_c is the set of usable ports provide by its type

Equation of the identity operator

$\text{id_op } buf = \text{Choice}$

$((\lambda p. \text{Read } p (\lambda x. \text{id_op} (\text{BENQ } p x buf))) \ `_c \ \mathfrak{U}_c)$

\cup_c

$((\lambda p. \text{Write} (\text{id_op} (\text{BTL } p buf)) p (\text{BHD } p buf)) \ `_c \ \{p \in_c \mathfrak{U}_c \mid buf \neq []\})$

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- \mathfrak{U}_c is the set of usable ports provide by its type
- Stream delayer: always can read, and it may eventually output

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- \mathfrak{U}_c is the set of usable ports provide by its type
- Stream delay: always can read, and it may eventually output

Identity operator with an empty buffer

$\mathcal{I} = \text{id_op} (\lambda_. \ [])$

Composition operator type

```
comp_op :: ('o1 ⇒ 'i2 option) ⇒ ('i2 ⇒ 'd list) ⇒ ('i1, 'o1, 'd) op ⇒ ('i2, 'o2, 'd) op ⇒  
      ('i1 + 'i2, 'o1 + 'o2, 'd) op
```

Composition: Preliminaries

Composition operator type

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```

Sequential and parallel composition

$$op_1 \parallel op_2 = \text{comp_op } (\lambda_. \text{None}) (\lambda_. []) op_1 op_2$$
$$op_1 \bullet op_2 = \text{map_op projl projr } (\text{comp_op Some } (\lambda_. [])) op_1 op_2$$

Composition: Equation

Equation of the composition operator

`comp_op wire buf op1 op2 =`

Composition: Equation

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$\text{comp_op } \text{wire } \text{buf } \text{op}_1 \text{ op}_2 = \text{Choice } (((\lambda \text{op}. \underline{\text{case}} \text{ op } \underline{\text{of}} \text{ Read } p f \Rightarrow \text{Read} (\text{Inl } p) (\lambda x. \text{comp_op } \text{wire } \text{buf } (f x) \text{ op}_2)$

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Composition: Equation

Equation of the composition operator

$$\begin{aligned} \text{comp_op wire buf } op_1 \text{ } op_2 &= \text{Choice } (((\lambda op. \underline{\text{case}} \text{ } op \text{ } \underline{\text{of}}}) \\ &\quad \text{Read } p \text{ } f \Rightarrow \text{Read } (\text{Inl } p) (\lambda x. \text{comp_op wire buf } (f x) \text{ } op_2) \\ | \text{ Write } op \text{ } p \text{ } x &\Rightarrow (\underline{\text{case}} \text{ } \text{wire } p \text{ } \underline{\text{of}}) \\ &\quad \text{None} \Rightarrow \text{Write } (\text{comp_op wire buf } op \text{ } op_2) (\text{Inl } p) \text{ } x \\ | \text{ Some } q &\Rightarrow \text{Silent } (\text{comp_op wire } (\text{BENQ } q \text{ } x \text{ } \text{buf}) \text{ } op \text{ } op_2)) \\ &\quad \backslash_c \text{ choices } op_1) \cup_c \end{aligned}$$

Composition: Equation

Equation of the composition operator

$\text{comp_op wire buf } op_1 \ op_2 = \text{Choice (((}\lambda op. \underline{\text{case}} \ op \ \underline{\text{of}} \ \text{Read } p \ f \Rightarrow \text{Read} (\text{Inl } p) (\lambda x. \text{comp_op wire buf } (f \ x) \ op_2)$

| Write $op \ p \ x \Rightarrow (\underline{\text{case}} \ \text{wire } p \ \underline{\text{of}}$

 None $\Rightarrow \text{Write} (\text{comp_op wire buf } op \ op_2) (\text{Inl } p) \ x$

 | Some $q \Rightarrow \text{Silent} (\text{comp_op wire } (\text{BENQ } q \ x \ \text{buf}) \ op \ op_2))$

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(choices op_2)))

Composition: Equation

Equation of the composition operator

$$\begin{aligned} \text{comp_op wire buf } op_1 \text{ } op_2 &= \text{Choice (((}\lambda \text{op. } \underline{\text{case}} \text{ op } \underline{\text{of}} \\ &\quad \text{Read } p \text{ } f \Rightarrow \text{Read (Inl } p) (\lambda x. \text{comp_op wire buf } (f \text{ } x) \text{ } op_2) \\ | \text{ Write } op \text{ } p \text{ } x \Rightarrow (\underline{\text{case}} \text{ wire } p \text{ } \underline{\text{of}} \\ &\quad \text{None} \Rightarrow \text{Write (comp_op wire buf } op \text{ } op_2) (\text{Inl } p) \text{ } x \\ | \text{ Some } q \Rightarrow \text{Silent (comp_op wire (BENQ } q \text{ } x \text{ } \text{buf) } op \text{ } op_2)) \\ | \text{ Silent } op \Rightarrow \text{Silent (comp_op wire buf } op \text{ } op_2)) \text{ } \backslash_c \text{ choices } op_1 \text{) } \cup_c \end{aligned}$$

$\backslash_c \text{ sound_reads wire buf (choices } op_2\text{))}$

Composition: Equation

Equation of the composition operator

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$\text{Read } p \ f \Rightarrow \text{Read} (\text{Inl } p) (\lambda x. \text{comp_op wire buf } (f \ x) \ op_2)$

| Write $op \ p \ x \Rightarrow (\underline{\text{case}} \ \text{wire } p \ \underline{\text{of}}$

None $\Rightarrow \text{Write} (\text{comp_op wire buf } op \ op_2) (\text{Inl } p) \ x$

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Read $p \ f \Rightarrow \underline{\text{if }} p \in \text{ran wire}$

then $\text{Silent} (\text{comp_op wire } (\text{BTL } p \ \text{buf}) \ op_1 (f (\text{BHD } p \ \text{buf})))$

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- Identity: \mathcal{I}
- Parallel composition: $op_1 \parallel op_2$
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Transpose operator type

$$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd) \ op$$

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Dummy source operator type

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Sink operator type

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Split operator type

$$\Lambda :: ('n, 'n + 'n, 'd) \ op$$

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Split operator type

$$\Lambda :: ('n, 'n + 'n, 'd) \text{ op}$$

Merge operator type

$$\mathcal{V} :: ('n + 'n, 'n, 'd) \text{ op}$$

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$$\uparrow :: ('i + 'm, 'o + 'm, 'd) \text{ op} \Rightarrow ('i, 'o, 'd) \text{ op}$$

Transpose operator type

$$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd) \text{ op}$$

Dummy source operator type

$$! :: ('i, 'o, 'd) \text{ op}$$

Sink operator type

$$! :: ('i, 'o, 'd) \text{ op}$$

Split operator type

$$\Lambda :: ('n, 'n + 'n, 'd) \text{ op}$$

Merge operator type

$$\mathcal{V} :: ('n + 'n, 'n, 'd) \text{ op}$$

Copy operator type

$$\mathcal{C} :: ('n, 'n + 'n, 'd) \text{ op}$$

Asynchronous Dataflow Operators

- Identity: \mathcal{I}
- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

Feedback operator type ($op \uparrow$)

$$\uparrow :: ('i + 'm, 'o + 'm, 'd) \ op \Rightarrow ('i, 'o, 'd) \ op$$

Transpose operator type

$$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd) \ op$$

Dummy source operator type

$$! :: ('i, 'o, 'd) \ op$$

Sink operator type

$$! :: ('i, 'o, 'd) \ op$$

Split operator type

$$\Lambda :: ('n, 'n + 'n, 'd) \ op$$

Merge operator type

$$\mathcal{V} :: ('n + 'n, 'n, 'd) \ op$$

Copy operator type

$$\mathcal{C} :: ('n, 'n + 'n, 'd) \ op$$

Equality test operator type

$$\mathcal{Q} :: ('n + 'n, 'n, 'd \ option) \ op$$

Asynchronous Dataflow Properties

Basic Network Algebra Properties

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx \text{map_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx \text{map_op} \text{Inl} \text{Inl} op$$

$$B2_2: (\mathcal{I} :: (0, 0, 'd) op) \parallel op \approx \text{map_op} \text{Inr} \text{Inr} op$$

$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

$$B4_1: op \bullet \mathcal{I} \approx op$$

$$B4_2: \mathcal{I} \bullet op \approx op$$

$$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$$

$$B6: \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$$

$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx \text{map_op} \text{id} (\text{case_sum} \text{Inr} \text{Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx \text{map_op} \curvearrowleft \curvearrowleft (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map_op} \text{id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$F2: \mathcal{X} \uparrow \approx \mathcal{I}$$

$$R1: op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$$

$$R2: (op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$$

$$R3: op_1 \parallel (op_2 \uparrow) \approx (\text{map_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$$

$$R4: (op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$$

$$R5: \text{map_op} \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

$$R6: (op \uparrow) \uparrow \approx (\text{map_op} \curvearrowleft \curvearrowleft op) \uparrow$$

Basic Network Algebra Properties

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx map_op \curvearrowleft \curvearrowleft (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx map_op \text{Inl} \text{Inl} op$$

$$B2_2: (\mathcal{I} :: (0, 0, 'd) op) \parallel op \approx map_op \text{Inr} \text{Inr} op$$

$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

$$B4_1: op \bullet \mathcal{I} \approx op$$

$$B4_2: \mathcal{I} \bullet op \approx op$$

$$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$$

$$B6: \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$$

$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx map_op \text{id} (\text{case_sum Inr Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx map_op \curvearrowleft \curvearrowleft (\mathcal{X} \parallel \mathcal{I}) \bullet map_op \text{id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$F2: \mathcal{X} \uparrow \approx \mathcal{I}$$

$$R1: op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$$

$$R2: (op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$$

$$R3: op_1 \parallel (op_2 \uparrow) \approx (map_op \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$$

$$R4: (op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$$

$$R5: map_op \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

$$R6: (op \uparrow) \uparrow \approx (map_op \curvearrowleft \curvearrowleft op) \uparrow$$

Basic Network Algebra Properties

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx map_op \curvearrowleft \curvearrowleft (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx map_op \text{Inl} \text{Inl} op$$

$$B2_2: (\mathcal{I} :: (0, 0, 'd) op) \parallel op \approx map_op \text{Inr} \text{Inr} op$$

$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

$$B4_1: op \bullet \mathcal{I} \approx op$$

$$B4_2: \mathcal{I} \bullet op \approx op$$

$$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$$

$$B6: \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$$

$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx map_op \text{id} (\text{case_sum Inr Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx map_op \curvearrowleft \curvearrowleft (\mathcal{X} \parallel \mathcal{I}) \bullet map_op \text{id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$F2: \mathcal{X} \uparrow \approx \mathcal{I}$$

$$R1: op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$$

$$R2: (op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$$

$$R3: op_1 \parallel (op_2 \uparrow) \approx (map_op \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$$

$$R4: (op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$$

$$R5: map_op \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

$$R6: (op \uparrow) \uparrow \approx (map_op \curvearrowleft \curvearrowleft op) \uparrow$$

Basic Network Algebra Properties

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx \text{map_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx \text{map_op} \text{Inl} \text{Inl} op$$

$$B2_2: (\mathcal{I} :: (0, 0, 'd) op) \parallel op \approx \text{map_op} \text{Inr} \text{Inr} op$$

$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

$$B4_1: op \bullet \mathcal{I} \approx op$$

$$B4_2: \mathcal{I} \bullet op \approx op$$

$$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$$

$$B6: \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$$

$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx \text{map_op} \text{id} (\text{case_sum} \text{Inr} \text{Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx \text{map_op} \curvearrowleft \curvearrowleft (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map_op} \text{id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$F2: \mathcal{X} \uparrow \approx \mathcal{I}$$

$$R1: op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$$

$$R2: (op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$$

$$R3: op_1 \parallel (op_2 \uparrow) \approx (\text{map_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$$

$$R4: (op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$$

$$R5: \text{map_op} \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

$$R6: (op \uparrow) \uparrow \approx (\text{map_op} \curvearrowleft \curvearrowleft op) \uparrow$$

Basic Network Algebra Properties

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx \text{map_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx \text{map_op} \text{Inl} \text{Inl} op$$

$$B2_2: (\mathcal{I} :: (0, 0, 'd) op) \parallel op \approx \text{map_op} \text{Inr} \text{Inr} op$$

$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

$$B4_1: op \bullet \mathcal{I} \approx op$$

$$B4_2: \mathcal{I} \bullet op \approx op$$

$$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$$

$$B6: \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$$

$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx \text{map_op} \text{id} (\text{case_sum} \text{Inr} \text{Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx \text{map_op} \curvearrowleft \curvearrowleft (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map_op} \text{id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$F2: \mathcal{X} \uparrow \approx \mathcal{I}$$

$$R1: op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$$

$$R2: (op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$$

$$R3: op_1 \parallel (op_2 \uparrow) \approx (\text{map_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$$

$$R4: (op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$$

$$R5: \text{map_op} \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

$$R6: (op \uparrow) \uparrow \approx (\text{map_op} \curvearrowleft \curvearrowleft op) \uparrow$$

Basic Network Algebra Properties

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx \text{map_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx \text{map_op} \text{Inl} \text{Inl} op$$

$$B2_2: (\mathcal{I} :: (0, 0, 'd) op) \parallel op \approx \text{map_op} \text{Inr} \text{Inr} op$$

$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

$$B4_1: op \bullet \mathcal{I} \approx op$$

$$B4_2: \mathcal{I} \bullet op \approx op$$

$$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$$

$$B6: \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$$

$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx \text{map_op} \text{id} (\text{case_sum} \text{Inr} \text{Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx \text{map_op} \curvearrowleft \curvearrowleft (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map_op} \text{id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$F2: \mathcal{X} \uparrow \approx \mathcal{I}$$

$$R1: op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$$

$$R2: (op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$$

$$R3: op_1 \parallel (op_2 \uparrow) \approx (\text{map_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$$

$$R4: (op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$$

$$R5: \text{map_op} \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

$$R6: (op \uparrow) \uparrow \approx (\text{map_op} \curvearrowleft \curvearrowleft op) \uparrow$$

Properties of Equality Test, Merge, Copy, Split, Source and Sink operators

$$A1: (\mathcal{Q} \parallel \mathcal{I}) \bullet \mathcal{Q} \approx \text{map_op } \curvearrowleft \text{id } ((\mathcal{I} \parallel \mathcal{Q}) \bullet \mathcal{Q})$$

$$A2: \mathcal{X} \bullet \otimes \approx \otimes \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\} \qquad \qquad A6: \otimes \bullet \mathcal{X} \approx \otimes \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A3_{\mathcal{Q}}: ((i :: (0, 'a, 'd) op) \parallel \mathcal{I}) \bullet \mathcal{Q} \approx (! :: (0 + 'a, 0, 'd) op) \bullet i$$

$$A3_{\mathcal{V}}: ((i :: (0, 'a, 'd) op) \parallel \mathcal{I}) \bullet \mathcal{V} \approx \text{map_op Inr id } \mathcal{I}$$

$$A4: \otimes \bullet ! \approx ! \parallel ! \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\} \qquad \qquad A8: i \bullet \otimes \approx i \parallel i \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A5: \mathcal{C} \bullet (\mathcal{C} \parallel \mathcal{I}) \approx \text{map_op id } \curvearrowleft (\mathcal{C} \bullet (\mathcal{I} \parallel \mathcal{C}))$$

$$A7: \mathcal{C} \bullet (! \parallel \mathcal{I}) \approx \text{map_op id Inr } \mathcal{I} \qquad \qquad A9: i \bullet ! \approx \mathcal{I}$$

$$A10: \mathcal{Q} \bullet \mathcal{C} \approx (\mathcal{C} \parallel \mathcal{C}) \bullet (\text{map_op } \curvearrowleft \curvearrowleft \curvearrowleft (\text{map_op } \curvearrowleft \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I})) \bullet (\mathcal{Q} \parallel \mathcal{Q})$$

$$A11: \mathcal{C} \bullet \mathcal{Q} \approx \mathcal{I} \qquad \qquad \qquad A12: i \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$A16: ! \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$A13: i \approx i \parallel i$$

$$A17: ! \approx ! \parallel !$$

$$A14: \text{map_op id Inl } (\otimes :: (0 + 0, 0, 'd) op) \approx \mathcal{I} \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$A15: \otimes \approx \text{map_op } \curvearrowleft \curvearrowleft \curvearrowleft (\text{map_op } \curvearrowleft \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \bullet (\otimes \parallel \otimes) \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$A18: \text{map_op Inl id } (\otimes :: (0, 0 + 0, 'd) op) \approx \mathcal{I} \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A19: \otimes \approx (\otimes \parallel \otimes) \bullet \text{map_op } \curvearrowleft \curvearrowleft \curvearrowleft (\text{map_op } \curvearrowleft \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$F3: (\text{map_op id Inr } \otimes) \uparrow \approx ! \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$F4: (\text{map_op Inr id } \otimes) \uparrow \approx i \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$F5: ((\mathcal{I} \parallel \mathcal{C}) \bullet \text{map_op } \curvearrowleft \curvearrowleft \curvearrowleft (\mathcal{X} \parallel \mathcal{I}) \bullet (\mathcal{I} \parallel \mathcal{Q})) \uparrow \approx ! \bullet i$$

Properties of Equality Test, Merge, Copy, Split, Source and Sink operators

$$A1: (\mathcal{Q} \parallel \mathcal{I}) \bullet \mathcal{Q} \approx \text{map_op } \curvearrowright \text{id } ((\mathcal{I} \parallel \mathcal{Q}) \bullet \mathcal{Q})$$

$$A2: \mathcal{X} \bullet \otimes \approx \otimes \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\} \qquad \qquad A6: \otimes \bullet \mathcal{X} \approx \otimes \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A3_{\mathcal{Q}}: ((i :: (0, 'a, 'd) op) \parallel \mathcal{I}) \bullet \mathcal{Q} \approx (! :: (0 + 'a, 0, 'd) op) \bullet i$$

$$A3_{\mathcal{V}}: ((i :: (0, 'a, 'd) op) \parallel \mathcal{I}) \bullet \mathcal{V} \approx \text{map_op Inr id } \mathcal{I}$$

$$A4: \otimes \bullet ! \approx ! \parallel ! \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\} \qquad \qquad A8: i \bullet \otimes \approx i \parallel i \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A5: \mathcal{C} \bullet (\mathcal{C} \parallel \mathcal{I}) \approx \text{map_op id } \curvearrowright (\mathcal{C} \bullet (\mathcal{I} \parallel \mathcal{C}))$$

$$A7: \mathcal{C} \bullet (! \parallel \mathcal{I}) \approx \text{map_op id Inr } \mathcal{I} \qquad \qquad A9: i \bullet ! \approx \mathcal{I}$$

$$A10: \mathcal{Q} \bullet \mathcal{C} \approx (\mathcal{C} \parallel \mathcal{C}) \bullet (\text{map_op } \curvearrowright \curvearrowright (\text{map_op } \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I})) \bullet (\mathcal{Q} \parallel \mathcal{Q})$$

$$A11: \mathcal{C} \bullet \mathcal{Q} \approx \mathcal{I} \qquad \qquad \qquad A12: i \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$A16: ! \approx (\mathcal{I} :: (0, 0, 'd) op) \qquad \qquad \qquad A13: i \approx i \parallel i \qquad \qquad \qquad A17: ! \approx ! \parallel !$$

$$A14: \text{map_op id Inl } (\otimes :: (0 + 0, 0, 'd) op) \approx \mathcal{I} \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$A15: \otimes \approx \text{map_op } \curvearrowright \curvearrowright (\text{map_op } \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \bullet (\otimes \parallel \otimes) \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$A18: \text{map_op Inl id } (\otimes :: (0, 0 + 0, 'd) op) \approx \mathcal{I} \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A19: \otimes \approx (\otimes \parallel \otimes) \bullet \text{map_op } \curvearrowright \curvearrowright (\text{map_op } \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$F3: (\text{map_op id Inr } \otimes) \uparrow \approx ! \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$F4: (\text{map_op Inr id } \otimes) \uparrow \approx i \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$F5: ((\mathcal{I} \parallel \mathcal{C}) \bullet \text{map_op } \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet (\mathcal{I} \parallel \mathcal{Q})) \uparrow \approx ! \bullet i$$

Properties of Equality Test, Merge, Copy, Split, Source and Sink operators

$$A1: (\mathcal{Q} \parallel \mathcal{I}) \bullet \mathcal{Q} \approx \text{map_op } \curvearrowright \text{id } ((\mathcal{I} \parallel \mathcal{Q}) \bullet \mathcal{Q})$$

$$A2: \mathcal{X} \bullet \otimes \approx \otimes \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$A6: \otimes \bullet \mathcal{X} \approx \otimes \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A3_{\mathcal{Q}}: ((i :: (0, 'a, 'd) op) \parallel \mathcal{I}) \bullet \mathcal{Q} \approx (! :: (0 + 'a, 0, 'd) op) \bullet i$$

$$A3_{\mathcal{V}}: ((i :: (0, 'a, 'd) op) \parallel \mathcal{I}) \bullet \mathcal{V} \approx \text{map_op Inr id } \mathcal{I}$$

$$A4: \otimes \bullet ! \approx ! \parallel ! \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$A8: i \bullet \otimes \approx i \parallel i \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

$$A5: \mathcal{C} \bullet (\mathcal{C} \parallel \mathcal{I}) \approx \text{map_op id } \curvearrowright (\mathcal{C} \bullet (\mathcal{I} \parallel \mathcal{C}))$$

$$A7: \mathcal{C} \bullet (! \parallel \mathcal{I}) \approx \text{map_op id Inr } \mathcal{I}$$

$$A9: i \bullet ! \approx \mathcal{I}$$

$$A10: \mathcal{Q} \bullet \mathcal{C} \approx (\mathcal{C} \parallel \mathcal{C}) \bullet (\text{map_op } \curvearrowright \curvearrowright (\text{map_op } \curvearrowright \curvearrowright (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I})) \bullet (\mathcal{Q} \parallel \mathcal{Q})$$

$$A11: \mathcal{C} \bullet \mathcal{Q} \approx \mathcal{I}$$

$$A12: i \approx (\mathcal{I} :: (0, 0, 'd) op)$$

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$$A14: \text{map_op id Inl } (\otimes :: (0 + 0, 0, 'd) op) \approx \mathcal{I} \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$A15: \otimes \approx \text{map_op } \curvearrowright \curvearrowright (\text{map_op } \curvearrowright \curvearrowright (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \bullet (\otimes \parallel \otimes) \quad \text{for } \otimes \in \{\mathcal{Q}, \mathcal{V}\}$$

$$A18: \text{map_op Inl id } (\otimes :: (0, 0 + 0, 'd) op) \approx \mathcal{I} \quad \text{for } \otimes \in \{\mathcal{C}, \Lambda\}$$

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$$F5: ((\mathcal{I} \parallel \mathcal{C}) \bullet \text{map_op } \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet (\mathcal{I} \parallel \mathcal{Q})) \uparrow \approx ! \bullet i$$

Properties of Equality Test, Merge, Copy, Split, Source and Sink operators

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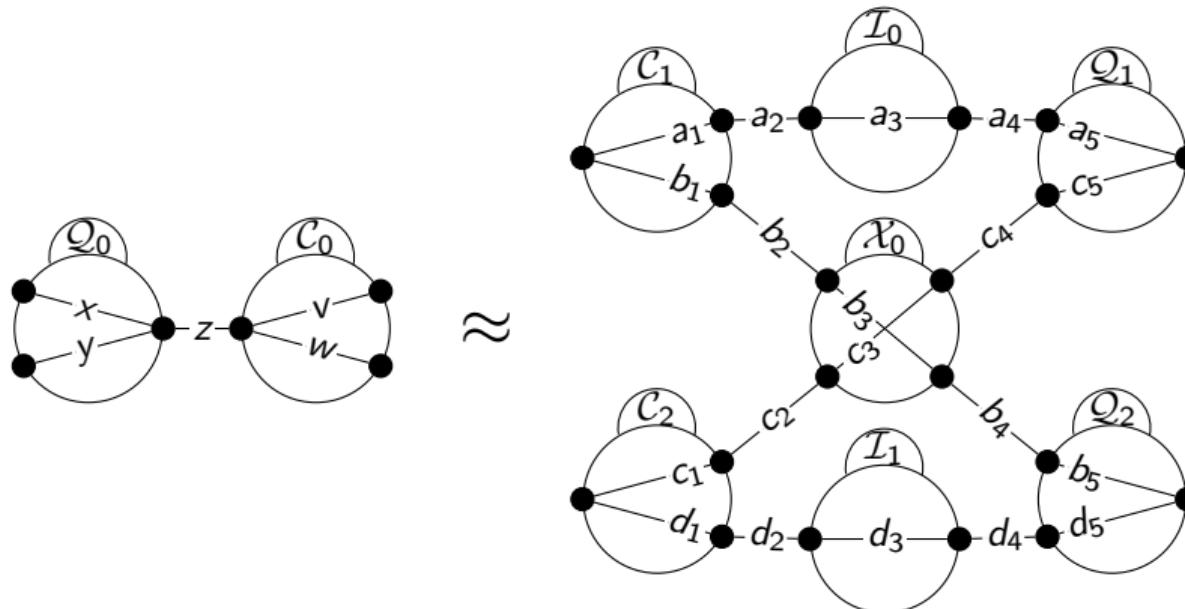
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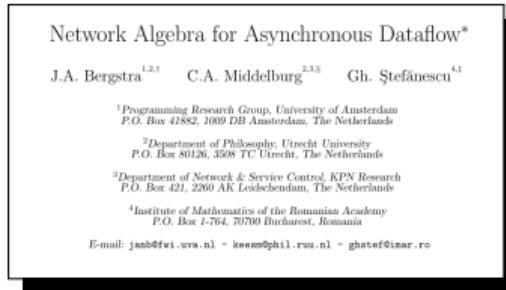
- \approx coinduction principle
- Buffers generalization
- Buffers invariant

Related Work

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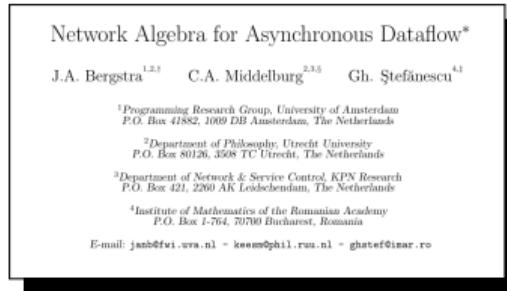
Related Work

- Instance in Algebra of Communicating Processes (ACP)
- Our operators closely follow their ACP counterparts



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- Our operators can be seen as a domain specific variant of choice trees
- Monadic composition

Interaction Trees

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PAUL HE, University of Pennsylvania, USA
CHUNG-KIL HUR, Seoul National University, Republic of Korea
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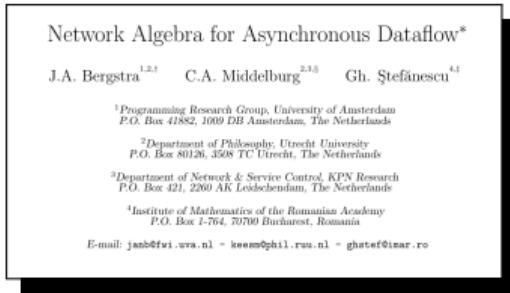
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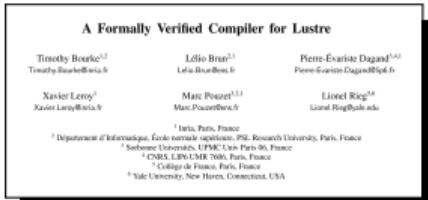
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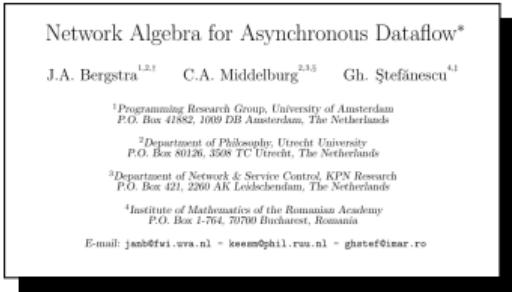
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- **Synchronous** dataflow languages have some proof assistant mechanization

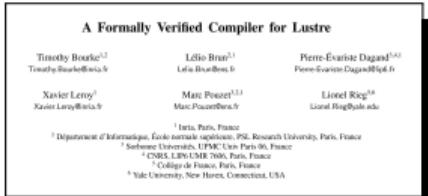


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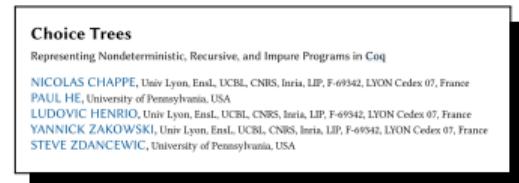
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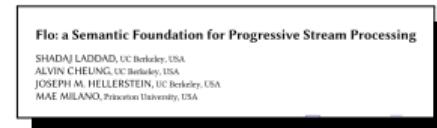
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- Flo: a framework for representing streaming computations originating from different systems like Flink and DBSP



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- Future work:
 - Brock–Andersen Time anomaly
 - Formalize Timely Dataflow infrastructure
 - Verify Timely Dataflow algorithms

Questions, comments and suggestions