Nondeterministic Asynchronous Dataflow in Isabelle/HOL

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Motivation

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Context:

- Stream Processing: programs that compute (possibly) unbounded sequences of data (streams)
- Frameworks: Google Cloud Dataflow, Apache Flink, and Timely Dataflow

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Our long term goal:

Mechanically Verify Timely Dataflow algorithms

A Good Foundation

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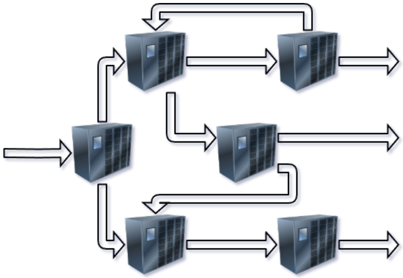

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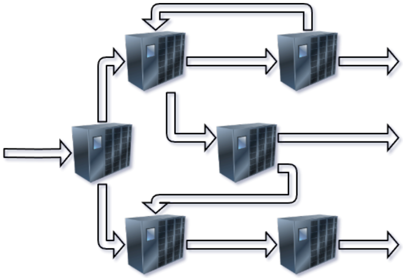

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  - Dataflow: Directed graph of interconnected operators

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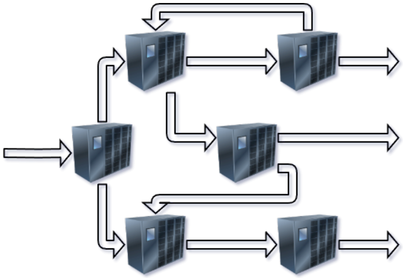

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  - Asynchronous:
    - Operators execute independently: processes without an orchestrator
    - Operators can freely communicate with the network (read/write); do silent computation steps
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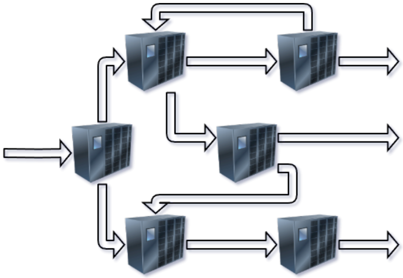

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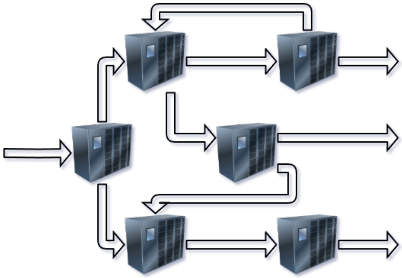

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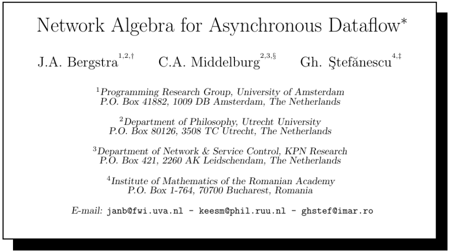
Question

How do we know something is Nondeterministic Asynchronous Dataflow?

The Algebra for Nondeterministic Asynchronous Dataflow

- Bergstra et al. presents an algebra for Nondeterministic Asynchronous Dataflow
- Primitives:
  sequential and parallel composition; feedback loop...
- 52 axioms
- A process calculus instance



Network Algebra for Asynchronous Dataflow*

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Main Contributions

- An Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
    - Operators as a shallow embedding as codatatypes
    - 51 axioms proved

Isabelle/HOL Preliminaries

What is Isabelle/HOL?

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  Simple Typed Lambda Calculus + axiom of choice + axiom of infinity + rank-1 polymorphism

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Why Isabelle/HOL?

- Codatatypes: (possibly) infinite data structures (e.g., lazy lists, streams)
- Corecursion: always eventually produces some codatatype constructor
- Coinductive predicates: infinite number of introduction rule applications
- Coinduction: reason about coinductive predicates

Operators as a Codatatype

Operators

Operators in Isabelle/HOL

$$
\begin{array}{l}
\textbf{codatatype}\ ('i,'o,'d)\ op = \\
\quad \mathrm{Read}\ 'i\ ('d \Rightarrow ('i,'o,'d)\ op) \mid \mathrm{Write}\ (('i,'o,'d)\ op)\ 'o\ 'd \\
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- Operator's actions
- Possibly infinite trees

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Uncommunicative operators

$$\oslash = \mathsf{Choice}\ \{\}_c$$

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Uncommunicative operators

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More examples

$\mathsf{ex1} = \mathsf{Choice}\ \{\mathsf{Write}\ \mathsf{ex1}\ 1\ 42, \oslash\}_c$

$\mathsf{ex2} = \mathsf{Choice}\ \{\mathsf{Write}\ \mathsf{ex2}\ 1\ 42, \mathsf{ex2}\}_c$

$\mathsf{ex3} = \mathsf{Choice}\ \{\mathsf{Write}\ \mathsf{ex3}\ 1\ 42, \mathsf{Silent}\ \mathsf{ex3}\}_c$

Operators Equivalences: Weak Bisimilarity

An ideal equivalence relation

- Only equate operators that **behave** the same
- Useful reasoning principle

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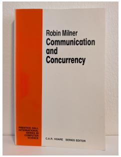
- Milner's approach!
  The classic chapter on **weak bisimilarity**



- Based on labeled transition systems (LTS)
- **Weak**: silent computations are abstracted away

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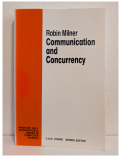

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Weak bisimilarity of operators

- Labels (actions): read, write, silent (τ)
- Weak bisimilarity of operators ($\approx$): LTS + weak simulation + greatest weak bisimulation
- $\approx$ has a useful coinduction principle
- $\oslash \approx \otimes \approx \otimes$ and ex1 $\approx$ ex2 $\approx$ ex3

Asynchronous Dataflow Operators

Auxiliary Definitions

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- Buffers: $'p \Rightarrow 'd\ list$

Buffer functions

$\mathrm{BHD}\ p\ buf = \mathrm{hd}\ (buf\ \ p)$

$\mathrm{BTL}\ p\ buf = buf(p := \mathrm{tl}\ (buf\ \ p))$

$\mathrm{BENQ}\ p\ x\ buf = buf(p := buf\ \ p\ @\ [x])$

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Mapping ports functions

$\mathrm{map\_op} :: ('i_1 \Rightarrow 'i_2) \Rightarrow ('o_1 \Rightarrow 'o_2) \Rightarrow ('i_1, 'o_1, 'd)\ op \Rightarrow ('i_2, 'o_2, 'd)\ op$

$\curvearrowright :: ('a + 'b) + 'c \Rightarrow 'a + 'b + 'c$

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Identity

Equation of the identity operator

$$\text{id\_op}\ buf = \text{Choice}$$

$$\cup_c$$

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- $\mathfrak{U}_c$ is the set of usable ports provide by its type

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$$((\lambda p.\ \text{Write}\ (\text{id\_op}\ (\text{BTL}\ p\ \mathit{buf}))\ p\ (\text{BHD}\ p\ \mathit{buf}))\ \grave{}_c\ \{p \in_c \mathfrak{U}_c \mid \mathit{buf}\ \ p \neq []\}))$$

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Identity operator with an empty buffer

$$\mathcal{I} = \text{id\_op } (\lambda\_.\ [])$$

Composition: Preliminaries

Composition operator type

$$\text{comp\_op} :: ('o_1 \Rightarrow 'i_2\ \mathit{option}) \Rightarrow ('i_2 \Rightarrow 'd\ \mathit{list}) \Rightarrow ('i_1, 'o_1, 'd)\ \mathit{op} \Rightarrow ('i_2, 'o_2, 'd)\ \mathit{op} \Rightarrow ('i_1 + 'i_2, 'o_1 + 'o_2, 'd)\ \mathit{op}$$

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Sequential and parallel composition

$$op_1 \parallel op_2 = \text{comp\_op}\ (\lambda\_.\ \text{None})\ (\lambda\_.\ [])\ op_1\ op_2$$

$$op_1 \bullet op_2 = \text{map\_op projl projr}\ (\text{comp\_op Some}\ (\lambda\_.\ [])\ op_1\ op_2)$$

Composition: Equation

Equation of the composition operator

$$\text{comp\_op}\ \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ \mathit{op}_2 =$$

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$\text{comp\_op}\ \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ \mathit{op}_2 = \text{Choice}$

$\cup_c$

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Equation of the composition operator

comp_op *wire* *buf* op_1 op_2 = Choice

$`_c$ choices op_1) $\cup_c$

Composition: Equation

Equation of the composition operator

comp_op *wire buf* op_1 op_2 = Choice ((($\lambda op.$ <u>case</u> *op* <u>of</u>

Read *p f* $\Rightarrow$ Read (Inl *p*) ($\lambda x.$ comp_op *wire buf* (*f x*) op_2)

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Equation of the composition operator

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\begin{array}{l}
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\quad \mid \text{Some}\ q \Rightarrow \text{Silent}\ (\text{comp\_op}\ \mathit{wire}\ (\text{BENQ}\ q\ x\ \mathit{buf})\ \mathit{op}\ \mathit{op}_2)) \\
\mid \text{Silent}\ \mathit{op} \Rightarrow \text{Silent}\ (\text{comp\_op}\ \mathit{wire}\ \mathit{buf}\ \mathit{op}\ \mathit{op}_2))\ \grave{}_c\ \text{choices}\ \mathit{op}_1)\ \cup_c \\
((\lambda \mathit{op}.\ \underline{\text{case}}\ \mathit{op}\ \underline{\text{of}} \\
\quad \text{Read}\ p\ f \Rightarrow \underline{\text{if}}\ p \in \text{ran}\ \mathit{wire} \\
\qquad \underline{\text{then}}\ \text{Silent}\ (\text{comp\_op}\ \mathit{wire}\ (\text{BTL}\ p\ \mathit{buf})\ \mathit{op}_1\ (f\ (\text{BHD}\ p\ \mathit{buf}))) \\
\qquad \underline{\text{else}}\ \text{Read}\ (\text{Inr}\ p)\ (\lambda x.\ \text{comp\_op}\ \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ (f\ x)) \\
\mid \text{Write}\ \mathit{op}\ p\ x \Rightarrow \text{Write}\ (\text{comp\_op}\ \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ \mathit{op})\ (\text{Inr}\ p)\ x \\
\mid \text{Silent}\ \mathit{op} \Rightarrow \text{Silent}\ (\text{comp\_op}\ \mathit{wire}\ \mathit{buf}\ \mathit{op}_1\ \mathit{op}))\ \grave{}_c\ \text{sound\_reads}\ \mathit{wire}\ \mathit{buf}\ (\text{choices}\ \mathit{op}_2)))
\end{array}
$$

Asynchronous Dataflow Operators

- Identity: $\mathcal{I}$
- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

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$\uparrow :: ('i + 'm, 'o + 'm, 'd)\ op \Rightarrow ('i, 'o, 'd)\ op$

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Transpose operator type

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Transpose operator type

$\mathcal{X}$ :: $('n + 'm, 'm + 'n, 'd)\ op$

Dummy source operator type

i :: $('i, 'o, 'd)\ op$

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$\mathrm{i}\ ::\ ('i, 'o, 'd)\ op$

Sink operator type

$!\ ::\ ('i, 'o, 'd)\ op$

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Transpose operator type

$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd)\ op$

Dummy source operator type

$\mathrm{i} :: ('i, 'o, 'd)\ op$

Sink operator type

$! :: ('i, 'o, 'd)\ op$

Split operator type

$\wedge :: ('n, 'n + 'n, 'd)\ op$

Asynchronous Dataflow Operators

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- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

Feedback operator type ($op \uparrow$)

$\uparrow$ :: $('i + 'm, 'o + 'm, 'd)$ $op \Rightarrow ('i, 'o, 'd)$ op

Transpose operator type

$\mathcal{X}$ :: $('n + 'm, 'm + 'n, 'd)$ op

Dummy source operator type

i :: $('i, 'o, 'd)$ op

Sink operator type

! :: $('i, 'o, 'd)$ op

Split operator type

Λ :: $('n, 'n + 'n, 'd)$ op

Merge operator type

$\mathcal{V}$ :: $('n + 'n, 'n, 'd)$ op

Asynchronous Dataflow Operators

- Identity: $\mathcal{I}$
- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

Feedback operator type ($op \uparrow$)

$\uparrow :: ('i + 'm, 'o + 'm, 'd)\ op \Rightarrow ('i, 'o, 'd)\ op$

Transpose operator type

$\mathcal{X} :: ('n + 'm, 'm + 'n, 'd)\ op$

Dummy source operator type

$\text{¡} :: ('i, 'o, 'd)\ op$

Sink operator type

$! :: ('i, 'o, 'd)\ op$

Split operator type

$\Lambda :: ('n, 'n + 'n, 'd)\ op$

Merge operator type

$\mathcal{V} :: ('n + 'n, 'n, 'd)\ op$

Copy operator type

$\mathcal{C} :: ('n, 'n + 'n, 'd)\ op$

Asynchronous Dataflow Operators

- Identity: $\mathcal{I}$
- Parallel composition: $op_1 \parallel op_2$
- Sequential composition: $op_1 \bullet op_2$

Feedback operator type ($op \uparrow$)

$\uparrow :: ('i + 'm, 'o + 'm, 'd)\ op \Rightarrow ('i, 'o, 'd)\ op$

Transpose operator type

$\mathcal{X}\ :: ('n + 'm, 'm + 'n, 'd)\ op$

Dummy source operator type

$¡\ :: ('i, 'o, 'd)\ op$

Sink operator type

$!\ :: ('i, 'o, 'd)\ op$

Split operator type

$\Lambda\ :: ('n, 'n + 'n, 'd)\ op$

Merge operator type

$\mathcal{V}\ :: ('n + 'n, 'n, 'd)\ op$

Copy operator type

$\mathcal{C}\ :: ('n, 'n + 'n, 'd)\ op$

Equality test operator type

$\mathcal{Q}\ :: ('n + 'n, 'n, 'd\ option)\ op$

Asynchronous Dataflow Properties

Basic Network Algebra Properties

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \text{map\_op} \curvearrowright \curvearrowright (op_1 \parallel op_2) \parallel op_3$

B2_1: $op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \text{map\_op Inl Inl}\ op$

B2_2: $(\mathcal{I} :: (0, 0, 'd)\ op) \parallel op \approx \text{map\_op Inr Inr}\ op$

B3: $(op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$

B4_1: $op \bullet \mathcal{I} \approx op$ B4_2: $\mathcal{I} \bullet op \approx op$

B5: $(op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$

B6: $\mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$ B7: $\mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$

B8: $(\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \text{map\_op id (case\_sum Inr Inl)}\ \mathcal{I}$

B9: $\mathcal{X} \approx \text{map\_op} \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map\_op id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$

B10: $(op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$

F1: $\mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ F2: $\mathcal{X} \uparrow \approx \mathcal{I}$

R1: $op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$

R2: $(op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$

R3: $op_1 \parallel (op_2 \uparrow) \approx (\text{map\_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$

R4: $(op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$

R5: $\text{map\_op Inl Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op) \uparrow) \approx op$

R6: $(op \uparrow) \uparrow \approx (\text{map\_op} \curvearrowright \curvearrowright op) \uparrow$

Basic Network Algebra Properties

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \text{map\_op} \curvearrowright \curvearrowright (op_1 \parallel op_2) \parallel op_3$

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B9: $\mathcal{X} \approx \text{map\_op} \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map\_op id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$

B10: $(op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$

F1: $\mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ F2: $\mathcal{X} \uparrow \approx \mathcal{I}$

R1: $op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$

R2: $(op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$

R3: $op_1 \parallel (op_2 \uparrow) \approx (\text{map\_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$

R4: $(op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$

R5: $\text{map\_op Inl Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op) \uparrow) \approx op$

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Basic Network Algebra Properties

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \text{map\_op} \curvearrowright \curvearrowright (op_1 \parallel op_2) \parallel op_3$

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B9: $\mathcal{X} \approx \text{map\_op} \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map\_op id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$

B10: $(op_1 \parallel op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \parallel op_1)$

F1: $\mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ F2: $\mathcal{X} \uparrow \approx \mathcal{I}$

R1: $op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$

R2: $(op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$

R3: $op_1 \parallel (op_2 \uparrow) \approx (\text{map\_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$

R4: $(op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$

R5: $\text{map\_op Inl Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op) \uparrow) \approx op$

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F1: $\mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ F2: $\mathcal{X} \uparrow \approx \mathcal{I}$
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R3: $op_1 \parallel (op_2 \uparrow) \approx (\text{map\_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2)) \uparrow$
R4: $(op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1) \uparrow$
R5: $\text{map\_op Inl Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op) \uparrow) \approx op$
R6: $(op \uparrow) \uparrow \approx (\text{map\_op} \curvearrowright \curvearrowright op) \uparrow$

Properties of Equality Test, Merge, Copy, Split, Source and Sink operators

$A1: (\mathcal{Q} \parallel \mathcal{I}) \bullet \mathcal{Q} \approx \mathsf{map\_op} \curvearrowleft \mathsf{id}\ ((\mathcal{I} \parallel \mathcal{Q}) \bullet \mathcal{Q})$

$A2: \mathcal{X} \bullet \ogreaterthan \approx \ogreaterthan \quad \text{for } \ogreaterthan \in \{\mathcal{Q}, \mathcal{V}\}$ $\qquad$ $A6: \olessthan \bullet \mathcal{X} \approx \olessthan \quad \text{for } \olessthan \in \{\mathcal{C}, \Lambda\}$

$A3_{\mathcal{Q}}: ((\mathsf{i} :: (0, 'a, 'd)\ op) \parallel \mathcal{I}) \bullet \mathcal{Q} \approx (! :: (0 + 'a, 0, 'd)\ op) \bullet \mathsf{i}$

$A3_{\mathcal{V}}: ((\mathsf{i} :: (0, 'a, 'd)\ op) \parallel \mathcal{I}) \bullet \mathcal{V} \approx \mathsf{map\_op\ Inr\ id}\ \mathcal{I}$

$A4: \ogreaterthan \bullet ! \approx ! \parallel ! \quad \text{for } \ogreaterthan \in \{\mathcal{Q}, \mathcal{V}\}$ $\qquad$ $A8: \mathsf{i} \bullet \olessthan \approx \mathsf{i} \parallel \mathsf{i} \quad \text{for } \olessthan \in \{\mathcal{C}, \Lambda\}$

$A5: \mathcal{C} \bullet (\mathcal{C} \parallel \mathcal{I}) \approx \mathsf{map\_op\ id} \curvearrowleft (\mathcal{C} \bullet (\mathcal{I} \parallel \mathcal{C}))$

$A7: \mathcal{C} \bullet (! \parallel \mathcal{I}) \approx \mathsf{map\_op\ id\ Inr}\ \mathcal{I}$ $\qquad$ $A9: \mathsf{i} \bullet ! \approx \mathcal{I}$

$A10: \mathcal{Q} \bullet \mathcal{C} \approx (\mathcal{C} \parallel \mathcal{C}) \bullet (\mathsf{map\_op} \curvearrowright \curvearrowright (\mathsf{map\_op} \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I})) \bullet (\mathcal{Q} \parallel \mathcal{Q})$

$A11: \mathcal{C} \bullet \mathcal{Q} \approx \mathcal{I}$ $\qquad$ $A12: \mathsf{i} \approx (\mathcal{I} :: (0, 0, 'd)\ op)$

$A16: ! \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ $\qquad$ $A13: \mathsf{i} \approx \mathsf{i} \parallel \mathsf{i}$ $\qquad$ $A17: ! \approx ! \parallel !$

$A14: \mathsf{map\_op\ id\ Inl}\ (\ogreaterthan :: (0 + 0, 0, 'd)\ op) \approx \mathcal{I} \quad \text{for } \ogreaterthan \in \{\mathcal{Q}, \mathcal{V}\}$

$A15: \ogreaterthan \approx \mathsf{map\_op} \curvearrowright \curvearrowright (\mathsf{map\_op} \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \bullet (\ogreaterthan \parallel \ogreaterthan) \quad \text{for } \ogreaterthan \in \{\mathcal{Q}, \mathcal{V}\}$

$A18: \mathsf{map\_op\ Inl\ id}\ (\olessthan :: (0, 0 + 0, 'd)\ op) \approx \mathcal{I} \quad \text{for } \olessthan \in \{\mathcal{C}, \Lambda\}$

$A19: \olessthan \approx (\olessthan \parallel \olessthan) \bullet \mathsf{map\_op} \curvearrowright \curvearrowright (\mathsf{map\_op} \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \quad \text{for } \olessthan \in \{\mathcal{C}, \Lambda\}$

$F3: (\mathsf{map\_op\ id\ Inr}\ \ogreaterthan) \uparrow \approx ! \quad \text{for } \ogreaterthan \in \{\mathcal{Q}, \mathcal{V}\}$

$F4: (\mathsf{map\_op\ Inr\ id}\ \olessthan) \uparrow \approx \mathsf{i} \quad \text{for } \olessthan \in \{\mathcal{C}, \Lambda\}$

$F5: ((\mathcal{I} \parallel \mathcal{C}) \bullet \mathsf{map\_op} \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet (\mathcal{I} \parallel \mathcal{Q})) \uparrow \approx ! \bullet \mathsf{i}$

Properties of Equality Test, Merge, Copy, Split, Source and Sink operators

$A1: (\mathcal{Q} \parallel \mathcal{I}) \bullet \mathcal{Q} \approx \text{map\_op} \curvearrowleft \text{id} ((\mathcal{I} \parallel \mathcal{Q}) \bullet \mathcal{Q})$

$A2: \mathcal{X} \bullet \ogreaterthan \approx \ogreaterthan \quad \text{for } \ogreaterthan \in \{\mathcal{Q}, \mathcal{V}\}$

$A6: \olessthan \bullet \mathcal{X} \approx \olessthan \quad \text{for } \olessthan \in \{\mathcal{C}, \Lambda\}$

$A3_{\mathcal{Q}}: ((\text{i} :: (0, 'a, 'd)\ op) \parallel \mathcal{I}) \bullet \mathcal{Q} \approx (! :: (0 + 'a, 0, 'd)\ op) \bullet \text{i}$

$A3_{\mathcal{V}}: ((\text{i} :: (0, 'a, 'd)\ op) \parallel \mathcal{I}) \bullet \mathcal{V} \approx \text{map\_op Inr id } \mathcal{I}$

$A4: \ogreaterthan \bullet\, ! \approx\, ! \parallel ! \quad \text{for } \ogreaterthan \in \{\mathcal{Q}, \mathcal{V}\}$

$A8: \text{i} \bullet \olessthan \approx \text{i} \parallel \text{i} \quad \text{for } \olessthan \in \{\mathcal{C}, \Lambda\}$

$A5: \mathcal{C} \bullet (\mathcal{C} \parallel \mathcal{I}) \approx \text{map\_op id} \curvearrowleft (\mathcal{C} \bullet (\mathcal{I} \parallel \mathcal{C}))$

$A7: \mathcal{C} \bullet (! \parallel \mathcal{I}) \approx \text{map\_op id Inr } \mathcal{I}$

$A9: \text{i} \bullet\, ! \approx \mathcal{I}$

$A10: \mathcal{Q} \bullet \mathcal{C} \approx (\mathcal{C} \parallel \mathcal{C}) \bullet (\text{map\_op} \curvearrowright \curvearrowright (\text{map\_op} \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I})) \bullet (\mathcal{Q} \parallel \mathcal{Q})$

$A11: \mathcal{C} \bullet \mathcal{Q} \approx \mathcal{I}$

$A12: \text{i} \approx (\mathcal{I} :: (0, 0, 'd)\ op)$

$A16: ! \approx (\mathcal{I} :: (0, 0, 'd)\ op)$

$A13: \text{i} \approx \text{i} \parallel \text{i}$

$A17: ! \approx\, ! \parallel !$

$A14: \text{map\_op id Inl } (\ogreaterthan :: (0 + 0, 0, 'd)\ op) \approx \mathcal{I} \quad \text{for } \ogreaterthan \in \{\mathcal{Q}, \mathcal{V}\}$

$A15: \ogreaterthan \approx \text{map\_op} \curvearrowright \curvearrowright (\text{map\_op} \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \bullet (\ogreaterthan \parallel \ogreaterthan) \quad \text{for } \ogreaterthan \in \{\mathcal{Q}, \mathcal{V}\}$

$A18: \text{map\_op Inl id } (\olessthan :: (0, 0 + 0, 'd)\ op) \approx \mathcal{I} \quad \text{for } \olessthan \in \{\mathcal{C}, \Lambda\}$

$A19: \olessthan \approx (\olessthan \parallel \olessthan) \bullet \text{map\_op} \curvearrowright \curvearrowright (\text{map\_op} \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \quad \text{for } \olessthan \in \{\mathcal{C}, \Lambda\}$

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$F5: ((\mathcal{I} \parallel \mathcal{C}) \bullet \text{map\_op} \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet (\mathcal{I} \parallel \mathcal{Q})) \uparrow \approx\, ! \bullet \text{i}$

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$A18: \text{map\_op Inl id } (\textcircled{<} :: (0, 0 + 0, 'd)\ op) \approx \mathcal{I} \quad \text{for } \textcircled{<} \in \{\mathcal{C}, \Lambda\}$

$A19: \textcircled{<} \approx (\textcircled{<} \parallel \textcircled{<}) \bullet \text{map\_op} \curvearrowright \curvearrowright (\text{map\_op} \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \quad \text{for } \textcircled{<} \in \{\mathcal{C}, \Lambda\}$

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A3$_{\mathcal{V}}$: $((\text{i} :: (0, 'a, 'd)\ op) \parallel \mathcal{I}) \bullet \mathcal{V} \approx \text{map\_op Inr id } \mathcal{I}$

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$\mathsf{A2}: \mathcal{X} \bullet \circledgt \approx \circledgt \quad \text{for } \circledgt \in \{\mathcal{Q}, \mathcal{V}\}$ $\qquad$ $\mathsf{A6}: \circledless \bullet \mathcal{X} \approx \circledless \quad \text{for } \circledless \in \{\mathcal{C}, \Lambda\}$

$\mathsf{A3}_{\mathcal{Q}}: ((\mathsf{i} :: (0, 'a, 'd)\ op) \parallel \mathcal{I}) \bullet \mathcal{Q} \approx (! :: (0 + 'a, 0, 'd)\ op) \bullet \mathsf{i}$

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$\mathsf{A4}: \circledgt \bullet ! \approx ! \parallel ! \quad \text{for } \circledgt \in \{\mathcal{Q}, \mathcal{V}\}$ $\qquad$ $\mathsf{A8}: \mathsf{i} \bullet \circledless \approx \mathsf{i} \parallel \mathsf{i} \quad \text{for } \circledless \in \{\mathcal{C}, \Lambda\}$

$\mathsf{A5}: \mathcal{C} \bullet (\mathcal{C} \parallel \mathcal{I}) \approx \mathsf{map\_op\ id} \curvearrowleft (\mathcal{C} \bullet (\mathcal{I} \parallel \mathcal{C}))$

$\mathsf{A7}: \mathcal{C} \bullet (! \parallel \mathcal{I}) \approx \mathsf{map\_op\ id\ Inr}\ \mathcal{I}$ $\qquad$ $\mathsf{A9}: \mathsf{i} \bullet ! \approx \mathcal{I}$

$\mathsf{A10}: \mathcal{Q} \bullet \mathcal{C} \approx (\mathcal{C} \parallel \mathcal{C}) \bullet (\mathsf{map\_op} \curvearrowright \curvearrowright (\mathsf{map\_op} \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I})) \bullet (\mathcal{Q} \parallel \mathcal{Q})$

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$\mathsf{A15}: \circledgt \approx \mathsf{map\_op} \curvearrowright \curvearrowright (\mathsf{map\_op} \curvearrowleft \curvearrowleft (\mathcal{I} \parallel \mathcal{X}) \parallel \mathcal{I}) \bullet (\circledgt \parallel \circledgt) \quad \text{for } \circledgt \in \{\mathcal{Q}, \mathcal{V}\}$

$\mathsf{A18}: \mathsf{map\_op\ Inl\ id}\ (\circledless :: (0, 0 + 0, 'd)\ op) \approx \mathcal{I} \quad \text{for } \circledless \in \{\mathcal{C}, \Lambda\}$

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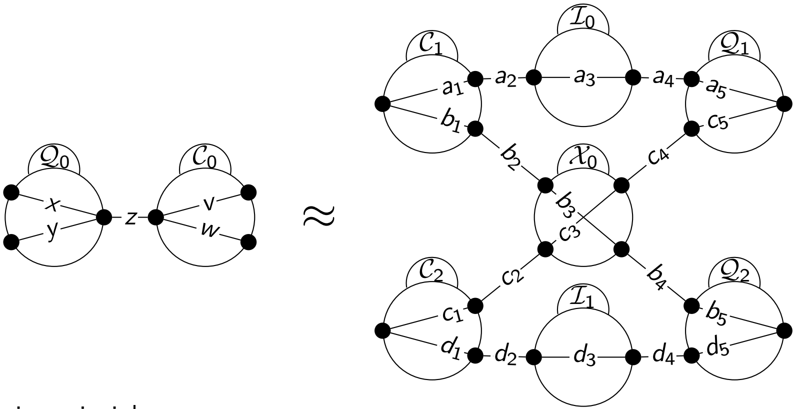
A10



- $\approx$ coinduction principle
- Buffers generalization
- Buffers invariant

Related Work

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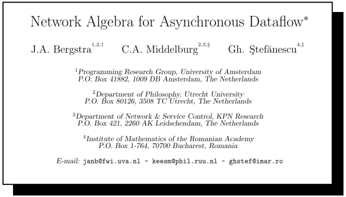
- Instance in Algebra of Communicating Processes (ACP)
- Our operators closely follow theirs


Network Algebra for Asynchronous Dataflow*

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- Our operators can be seen as a domain specific variant of choice trees

Interaction Trees

Representing Recursive and Impure Programs in Coq

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- **Synchronous** dataflow languages have some proof assistant mechanization

A Formally Verified Compiler for Lustre

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- Flo: a framework for representing streaming computations originating from different systems like Flink and DBSP

Flo: a Semantic Foundation for Progressive Stream Processing

SHADAJ LADDAD, UC Berkeley, USA
ALVIN CHEUNG, UC Berkeley, USA
JOSEPH M. HELLERSTEIN, UC Berkeley, USA
MAE MILANO, Princeton University, USA

Conclusion

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  - Axiom A1 for the merge operator does not hold in our instance when equivalence is $\approx$. We conjecture it also does not hold in the process algebra from Bergstra et al.

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- Around 28 000 lines of definitions and proofs

Conclusion

- An Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
- Isabelle/HOL has a good tool set:
  - Codatatypes, corecursion, coinductive predicates
- 51/52 axioms proved
  - Axiom A1 for the merge operator does not hold in our instance when equivalence is $\approx$. We conjecture it also does not hold in the process algebra from Bergstra et al.
- Jonsson's trace model, trace equivalence
- Around 28 000 lines of definitions and proofs
- Future work:
  - Brock–Andersen Time anomaly
  - Formalize Timely Dataflow infrastructure
  - Verify Timely Dataflow algorithms

Questions, comments and suggestions