

Nondeterministic Asynchronous Dataflow in Isabelle/HOL

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Motivation

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Context:

- Stream Processing: programs that compute (possibly) unbounded sequences of data (streams)
- A common problem in the industry
- Frameworks: Apache Flink, Google Cloud Dataflow, and Timely Dataflow



- Why use frameworks?
 - Highly Parallel
 - Low latency (output as soon as possible)
 - Incremental computing (re-uses previous computations)

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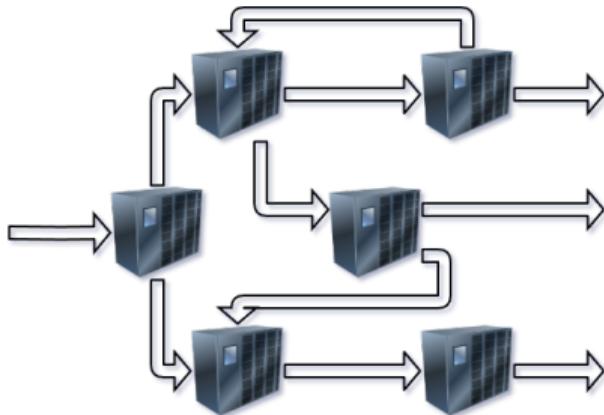
Our goal:

Mechanically Verify Timely Dataflow algorithms

A Good Foundation

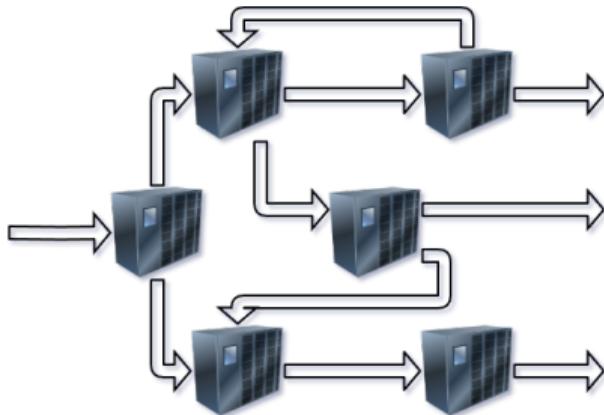
A Good Foundation

- Nondeterministic Asynchronous Dataflow

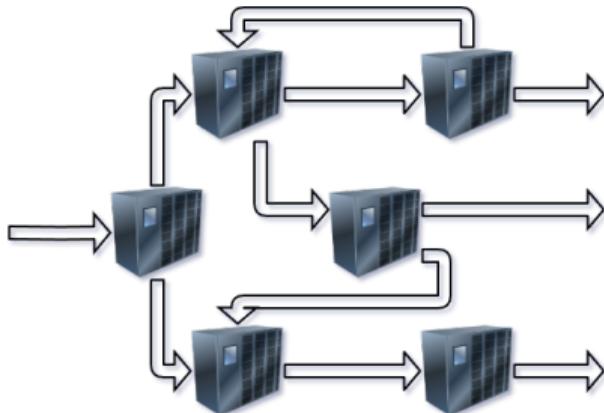


A Good Foundation

- Nondeterministic Asynchronous Dataflow
 - Dataflow: Directed graph of interconnected operators



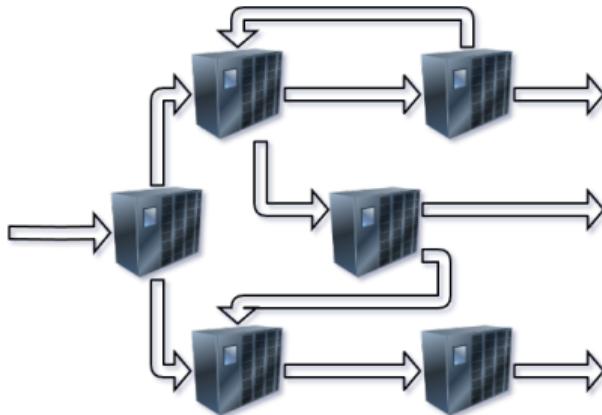
A Good Foundation



- Nondeterministic Asynchronous Dataflow

- Dataflow: Directed graph of interconnected operators
- Asynchronous:
 - Operators execute independently: processes without an orchestrator
 - Operators can freely communicate with the network (read/write); do silent computation steps
 - Networks are unbounded FIFO queues

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- Nondeterministic Asynchronous Dataflow
 - Dataflow: Directed graph of interconnected operators
 - Asynchronous:
 - Operators execute independently: processes without an orchestrator
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 - Networks are unbounded FIFO queues
 - Nondeterministic:
 - Operators can make nondeterministic choices

The Algebra for Nondeterministic Asynchronous Dataflow

- Bergstra et al. presents an algebra for Nondeterministic Asynchronous Dataflow
- Primitives:
sequential and parallel composition; feedback loop...
- 52 axioms
- A process calculus instance

Network Algebra for Asynchronous Dataflow*

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Main Contributions

- A Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
 - Operators as a shallow embedding as codatatypes
 - 51 axioms proved
- Executable via code extraction to Haskell

Isabelle/HOL Preliminaries

Isabelle/HOL

- Classical higher-order logic (HOL):
Simple Typed Lambda Calculus + axiom of choice + axiom of infinity + rank-1 polymorphism

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- Isabelle/HOL: Isabelle's flavor of HOL

Why Isabelle/HOL?

- Codatatypes: (possibly) infinite data structures (e.g., lazy lists, streams)
- Corecursion: always eventually produces some codatatype constructor
- Coinductive predicate: infinite number of introduction rule applications
- Coinduction: reason about coinductive predicates

Operators as a Codatatype

Operators

Operators in Isabelle/HOL

```
codatatype (inputs: 'i, outputs: 'o, 'd) op =  
  Read 'i ('d ⇒ ('i, 'o, 'd) op) | Write (('i, 'o, 'd) op) 'o 'd  
  Silent ('i, 'o, 'd) op | Choice (('i, 'o, 'd) op) cset
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- Type parameters:
inputs/output ports; data
- Operator's actions
- inputs/outputs:
Sets of used ports
- Possibly infinite trees

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Uncommunicative operators

$$\emptyset = \text{Choice } \{\}^c$$

$$\odot = \text{Silent } \odot$$

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More examples

$$\text{ex1} = \text{Choice } \{\text{Write ex1 1 42}, \emptyset\}^c$$

$$\text{ex2} = \text{Choice } \{\text{Write ex2 1 42}, \text{ex2}\}^c$$

$$\text{ex3} = \text{Choice } \{\text{Write ex3 1 42}, \text{Silent ex3}\}^c$$

An ideal equivalence relation

- Only equate operators that **behave** the same
- Useful reasoning principle

Operators Equivalences: Motivation

An ideal equivalence relation

- Only equate operators that **behave** the same
- Useful reasoning principle

- Milner's approach: Weak Bisimilarity



- Based on labeled transition systems (LTS)
- Labels (actions): Read, Write, Silent (τ)

Label Transition System

datatype ('i,'o,'d) IO = Inp 'i 'd | Out 'o 'd | Tau

inductive step where

step (Inp $p\ x$) (Read $p\ f$) ($f\ x$) | step (Out $q\ x$) (Write $op\ q\ x$) op
| step Tau (Silent op) op
| $op \in_c ops \implies$ step $io\ op\ op'$ \implies step $io\ (Choice\ ops)\ op'$

Operators Equivalences: Weak Bisimilarity

Weakly Simulates

$$\text{estep } \text{Tau} = (\text{step } \text{Tau})^{==}$$

$$\text{estep } io = \text{step } io$$

$$\text{wstep } io = (\text{step } \text{Tau})^{**} \text{ OO } (\text{estep } io) \text{ OO } (\text{step } \text{Tau})^{**}$$

$$\text{wsim } R \text{ } op_1 \text{ } op_2 = (\forall io \text{ } op'_1. \text{ step } io \text{ } op_1 \text{ } op'_1 \longrightarrow (\exists op'_2. \text{ wstep } io \text{ } op_2 \text{ } op'_2 \wedge R \text{ } op'_1 \text{ } op'_2))$$

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Weak Bisimilarity

coinductive wbisim (infix \approx 40) where

$\text{wsim } (\approx) \text{ } op_1 \text{ } op_2 \implies \text{wsim } (\approx) \text{ } op_2 \text{ } op_1 \implies op_1 \approx op_2$

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- \approx has a useful coinduction principle
- $\oslash \approx \otimes \approx \odot$
- ex1 \approx ex2 \approx ex3

Asynchronous Dataflow Operators

- Buffers $buf :: 'a \Rightarrow 'd list$

Buffer auxiliary functions

BHD $p\ buf = \text{hd}\ (buf\ p)$

BTL $p\ buf = buf(p := \text{tl}\ (buf\ p))$

BENQ $p\ x\ buf = buf(p := buf\ p @ [x])$

$buf_1 \gg buf_2 = (\lambda p. buf_2\ p @ buf_1\ p)$

Identity operator

$\text{id_op } buf = \text{Choice}$

\cup_c

- Stream delayer!
- $\mathcal{I} = \text{id_op } (\lambda_. \text{[]})$

Identity operator

$$\begin{aligned} \text{id_op } buf &= \text{Choice} \\ (((\lambda p. \text{Read } p (\lambda x. \text{id_op } (\text{BENQ } p x buf)))) \ `_c \ \mathfrak{U}_c) \\ &\cup_c \end{aligned}$$

- \mathfrak{U}_c is the set of usable ports provide by its type
- Stream delay!
- $\mathcal{I} = \text{id_op } (\lambda_. \ [])$

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$((\lambda p. \text{Write} (\text{id_op} (\text{BTL } p buf)) p (\text{BHD } p buf)) \ `_c \ \{p \in_c \mathfrak{U}_c \mid buf \neq []\}))$

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Composition

- foo

Asynchronous Dataflow Operators

- foo

Asynchronous Dataflow Properties

Basic network algebra properties

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx \text{map_op} \curvearrowright \curvearrowright (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx \text{map_op} \text{Inl} \text{Inl} op$$

$$B2_2: (\mathcal{I} :: (0, 0, 'd) op) \parallel op \approx \text{map_op} \text{Inr} \text{Inr} op$$

$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

$$B4_1: op \text{Inl} \bullet \mathcal{I} \approx op \text{Inl}$$

$$B4_2: \mathcal{I} \bullet \text{Inp} op \approx \text{Inp} op$$

$$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$$

$$B6: \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$$

$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx \text{map_op} \text{id} (\text{case_sum} \text{Inr} \text{Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx \text{map_op} \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map_op} \text{id} \curvearrowright (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (\text{Inp} op_1 \parallel \text{Inp} op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \text{Inl} \parallel op_1 \text{Inl})$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

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$$R1: \text{Inr} \text{- inputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \text{Inr} \text{- outputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \\ op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$$

$$R2: \text{Inr} \text{- inputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \text{Inr} \text{- outputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \\ (op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$$

$$R3: \text{Inr} \text{- inputs } op_2 \cap \text{defaults} = \{\} \Rightarrow \text{Inr} \text{- outputs } op_2 \cap \text{defaults} = \{\} \Rightarrow \\ op_1 \parallel (op_2 \uparrow) \approx (\text{map_op} \curvearrowright \curvearrowright (op_1 \parallel op_2)) \uparrow$$

$$R4: \text{Inr} \text{- inputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \text{Inr} \text{- outputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \\ \text{inputs } op_2 \cap \text{defaults} = \{\} \Rightarrow \text{outputs } op_2 \cap \text{defaults} = \{\} \Rightarrow \\ (\text{Inp} op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet \text{Inp} op_1) \uparrow$$

$$R5: \text{Inr} \text{- inputs } op = \{\} \Rightarrow \text{Inr} \text{- outputs } op = \{\} \Rightarrow \\ \text{map_op} \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

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$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx \text{map_op} \text{id} (\text{case_sum} \text{Inr} \text{Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx \text{map_op} \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map_op} \text{id} \curvearrowright (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (\sqsupseteq op_1 \parallel \sqsupseteq op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \sqsubseteq \parallel op_1 \sqsubseteq)$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$F2: \mathcal{X} \uparrow \approx \mathcal{I}$$

Table 1 Basic network algebra properties

$$R1: \text{Inr} \dashv \text{inputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \text{Inr} \dashv \text{outputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \\ op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$$

$$R2: \text{Inr} \dashv \text{inputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \text{Inr} \dashv \text{outputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \\ (op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$$

$$R3: \text{Inr} \dashv \text{inputs } op_2 \cap \text{defaults} = \{\} \Rightarrow \text{Inr} \dashv \text{outputs } op_2 \cap \text{defaults} = \{\} \Rightarrow \\ op_1 \parallel (op_2 \uparrow) \approx (\text{map_op} \curvearrowright \curvearrowright (op_1 \parallel op_2)) \uparrow$$

$$R4: \text{Inr} \dashv \text{inputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \text{Inr} \dashv \text{outputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \\ \text{inputs } op_2 \cap \text{defaults} = \{\} \Rightarrow \text{outputs } op_2 \cap \text{defaults} = \{\} \Rightarrow \\ (\sqsupseteq op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1 \sqsubseteq) \uparrow$$

$$R5: \text{Inr} \dashv \text{inputs } op = \{\} \Rightarrow \text{Inr} \dashv \text{outputs } op = \{\} \Rightarrow \\ \text{map_op} \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

$$R6: \text{Inr} \dashv \text{inputs } op = \{\} \Rightarrow \text{Inr} \dashv \text{outputs } op = \{\} \Rightarrow \\ \text{Inr} \dashv \text{Inl} \dashv \text{inputs } op = \{\} \Rightarrow \text{Inr} \dashv \text{Inl} \dashv \text{outputs } op = \{\} \Rightarrow \\ (op \uparrow) \approx (\text{map_op} \curvearrowright \curvearrowright op) \uparrow$$

Properties of equality test, merge, copy, split, source and sink operators

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx \text{map_op} \curvearrowright \curvearrowright (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx \text{map_op} \text{Inl} \text{Inl} op$$

$$B2_2: (\mathcal{I} :: (0, 0, 'd) op) \parallel op \approx \text{map_op} \text{Inr} \text{Inr} op$$

$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

$$B4_1: op \sqsubseteq \bullet \mathcal{I} \approx op \sqsubseteq$$

$$B4_2: \mathcal{I} \bullet \sqsupseteq op \approx \sqsupseteq op$$

$$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$$

$$B6: \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$$

$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx \text{map_op} \text{id} (\text{case_sum} \text{Inr} \text{Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx \text{map_op} \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map_op} \text{id} \curvearrowright (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (\sqsupseteq op_1 \parallel \sqsupseteq op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \sqsubseteq \parallel op_1 \sqsubseteq)$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

$$F2: \mathcal{X} \uparrow \approx \mathcal{I}$$

$$R1: \text{Inr} \dashv \text{inputs } op_1 \cap \text{defaults} = \{\} \implies \text{Inr} \dashv \text{outputs } op_1 \cap \text{defaults} = \{\} \implies \\ op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$$

$$R2: \text{Inr} \dashv \text{inputs } op_1 \cap \text{defaults} = \{\} \implies \text{Inr} \dashv \text{outputs } op_1 \cap \text{defaults} = \{\} \implies \\ (op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$$

$$R3: \text{Inr} \dashv \text{inputs } op_2 \cap \text{defaults} = \{\} \implies \text{Inr} \dashv \text{outputs } op_2 \cap \text{defaults} = \{\} \implies \\ op_1 \parallel (op_2 \uparrow) \approx (\text{map_op} \curvearrowright \curvearrowright (op_1 \parallel op_2)) \uparrow$$

$$R4: \text{Inr} \dashv \text{inputs } op_1 \cap \text{defaults} = \{\} \implies \text{Inr} \dashv \text{outputs } op_1 \cap \text{defaults} = \{\} \implies \\ \text{inputs } op_2 \cap \text{defaults} = \{\} \implies \text{outputs } op_2 \cap \text{defaults} = \{\} \implies \\ (\sqsupseteq op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1 \sqsubseteq) \uparrow$$

$$R5: \text{Inr} \dashv \text{inputs } op = \{\} \implies \text{Inr} \dashv \text{outputs } op = \{\} \implies \\ \text{map_op} \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

$$R6: \text{Inr} \dashv \text{inputs } op = \{\} \implies \text{Inr} \dashv \text{outputs } op = \{\} \implies \\ \text{Inr} \dashv \text{Inl} \dashv \text{inputs } op = \{\} \implies \text{Inr} \dashv \text{Inl} \dashv \text{outputs } op = \{\} \implies \\ (op \uparrow) \approx (\text{map_op} \curvearrowright \curvearrowright op) \uparrow$$

Table 1 Basic network algebra properties

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx \text{map_op} \curvearrowright \curvearrowright (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx \text{map_op} \text{Inl} \text{Inl} op$$

$$B2_2: (\mathcal{I} :: (0, 0, 'd) op) \parallel op \approx \text{map_op} \text{Inr} \text{Inr} op$$

$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

$$B4_1: op \sqsubseteq \bullet \mathcal{I} \approx op \sqsubseteq$$

$$B4_2: \mathcal{I} \bullet \sqsupseteq op \approx \sqsupseteq op$$

$$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$$

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$$B7: \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$$

$$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd) op) \approx \text{map_op} \text{id} (\text{case_sum} \text{Inr} \text{Inl}) \mathcal{I}$$

$$B9: \mathcal{X} \approx \text{map_op} \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet \text{map_op} \text{id} \curvearrowright (\mathcal{I} \parallel \mathcal{X})$$

$$B10: (\sqsupseteq op_1 \parallel \sqsupseteq op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2 \sqsubseteq \parallel op_1 \sqsubseteq)$$

$$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd) op)$$

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$$R4: \text{Inr} \dashv \text{inputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \text{Inr} \dashv \text{outputs } op_1 \cap \text{defaults} = \{\} \Rightarrow \\ \text{inputs } op_2 \cap \text{defaults} = \{\} \Rightarrow \text{outputs } op_2 \cap \text{defaults} = \{\} \Rightarrow \\ (\sqsupseteq op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1 \sqsubseteq) \uparrow$$

$$R5: \text{Inr} \dashv \text{inputs } op = \{\} \Rightarrow \text{Inr} \dashv \text{outputs } op = \{\} \Rightarrow \\ \text{map_op} \text{Inl} \text{Inl} ((op :: ('i + 0, 'o + 0, 'd) op) \uparrow) \approx op$$

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Well-Behaved Operators

$$B1: op_1 \parallel (op_2 \parallel op_3) \approx \text{map_op} \curvearrowright \curvearrowright (op_1 \parallel op_2) \parallel op_3$$

$$B2_1: op \parallel (\mathcal{I} :: (0, 0, 'd) op) \approx \text{map_op} \text{Inl} \text{Inl} op$$

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$$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$$

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Table 1 Basic network algebra properties

Related Work

Related Work

- A Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
 - Operators as a shallow embedding as codatatypes
 - 51 axioms proved
- Executable via code extraction to Haskell

Conclusion

Conclusion

- Isabelle/HOL has a good tool set to formalize and reason about stream processing:
 - Codatatypes, coinductive predicates, corecursion with friends, reasoning up to friends (congruence),
 - Coinduction up to congruence principle is automatically derived for codatatypes (but not for coinductive principles)
 - Next step: Feedback loop

Questions, comments and suggestions