Nondeterministic Asynchronous Dataflow in Isabelle/HOL

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Motivation

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Context:

- Stream Processing: programs that compute (possibly) unbounded sequences of data (streams)
- A common problem in the industry
- Frameworks: Apache Flink, Google Cloud Dataflow, and Timely Dataflow
- Why use frameworks?
  - Highly Parallel
  - Low latency (output as soon as possible)
  - Incremental computing (re-uses previous computations)

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Our goal:

Mechanically Verify Timely Dataflow algorithms

A Good Foundation

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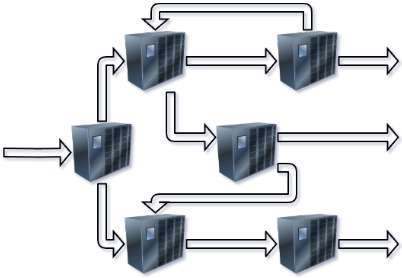

- Nondeterministic Asynchronous Dataflow

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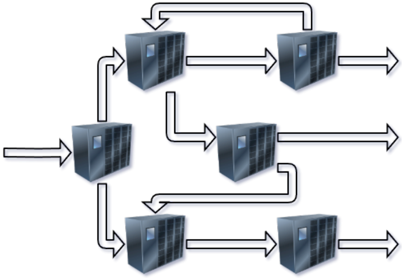

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  - Dataflow: Directed graph of interconnected operators

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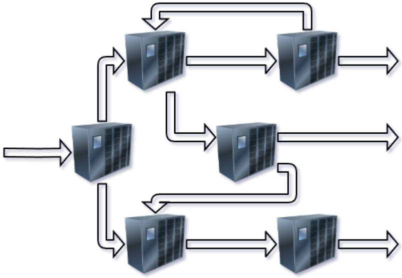

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  - Dataflow: Directed graph of interconnected operators
  - Asynchronous:
    - Operators execute independently: processes without an orchestrator
    - Operators can freely communicate with the network (read/write); do silent computation steps
    - Networks are unbounded FIFO queues

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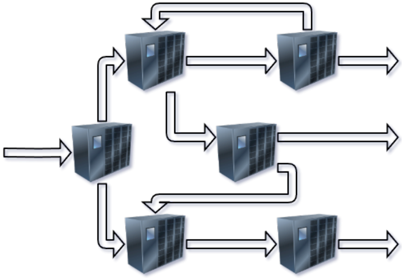

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    - Networks are unbounded FIFO queues
  - Nondeterministic:
    - Operators can make nondeterministic choices

- Bergstra et al. presents an algebra for Nondeterministic Asynchronous Dataflow
- Primitives:
  sequential and parallel composition; feedback loop...
- 52 axioms
- A process calculus instance

Network Algebra for Asynchronous Dataflow*

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Main Contributions

- A Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
  - Operators as a shallow embedding as codatatypes
  - 51 axioms proved
- Executable via code extraction to Haskell

Isabelle/HOL Preliminaries

Isabelle/HOL

- Classical higher-order logic (HOL):
  Simple Typed Lambda Calculus + axiom of choice + axiom of infinity + rank-1 polymorphism

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Why Isabelle/HOL?

- Codatatypes: (possibly) infinite data structures (e.g., lazy lists, streams)
- Corecursion: always eventually produces some codatatype constructor
- Coinductive predicate: infinite number of introduction rule applications
- Coinduction: reason about coinductive predicates

Operators as a Codatatype

Operators

Operators in Isabelle/HOL

$$\textbf{codatatype}\ (\mathsf{inputs}\colon 'i, \mathsf{outputs}\colon 'o, 'd)\ op =$$
$$\quad \mathsf{Read}\ 'i\ ('d \Rightarrow ('i, 'o, 'd)\ op) \mid \mathsf{Write}\ (('i, 'o, 'd)\ op)\ 'o\ 'd$$
$$\quad \mathsf{Silent}\ ('i, 'o, 'd)\ op \mid \mathsf{Choice}\ (('i, 'o, 'd)\ op)\ cset$$

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- Type parameters:
  inputs/output ports; data
- Operator's actions
- inputs/outputs:
  Sets of used ports
- Possibly infinite trees

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Uncommunicative operators

$$\oslash = \text{Choice}\ \{\}_c$$

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More examples

$$\text{ex1} = \text{Choice}\ \{\text{Write ex1 1 42}, \oslash\}_c$$

$$\text{ex2} = \text{Choice}\ \{\text{Write ex2 1 42}, \text{ex2}\}_c$$

$$\text{ex3} = \text{Choice}\ \{\text{Write ex3 1 42}, \text{Silent ex3}\}_c$$

Operators Equivalences: Motivation

An ideal equivalence relation

- Only equate operators that **behave** the same
- Useful reasoning principle

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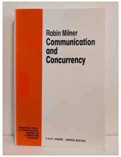
- Milner's approach: Weak Bisimilarity



- Based on labeled transition systems (LTS)
- Labels (actions): Read, Write, Silent (τ)

Operators Equivalences: Label Transition System

Label Transition System

$$\mathbf{datatype}\ ('i,'o,'d)\ IO = \mathrm{Inp}\ 'i\ 'd \mid \mathrm{Out}\ 'o\ 'd \mid \mathrm{Tau}$$

$$\mathbf{inductive}\ \mathrm{step}\ \mathbf{where}$$

$$\mathrm{step}\ (\mathrm{Inp}\ p\ x)\ (\mathrm{Read}\ p\ f)\ (f\ x) \mid \mathrm{step}\ (\mathrm{Out}\ q\ x)\ (\mathrm{Write}\ op\ q\ x)\ op$$

$$\mid \mathrm{step}\ \mathrm{Tau}\ (\mathrm{Silent}\ op)\ op$$

$$\mid op \in_c ops \implies \mathrm{step}\ io\ op\ op' \implies \mathrm{step}\ io\ (\mathrm{Choice}\ ops)\ op'$$

Operators Equivalences: Weak Bisimilarity

Weakly Simulates

$$\text{estep Tau} = (\text{step Tau})^{==}$$

$$\text{estep } io = \text{step } io$$

$$\text{wstep } io = (\text{step Tau})^{**} \text{ OO } (\text{estep } io) \text{ OO } (\text{step Tau})^{**}$$

$$\text{wsim } R\ op_1\ op_2 = (\forall io\ op_1'.\ \text{step } io\ op_1\ op_1' \longrightarrow (\exists op_2'.\ \text{wstep } io\ op_2\ op_2' \wedge R\ op_1'\ op_2'))$$

- A Relation R is a *weak simulation* if $\forall op_1\ op_2.\ R\ op_1\ op_2 \longrightarrow \text{wsim } R\ op_1\ op_2$.

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Weak Bisimilarity

$$\textbf{coinductive}\ \text{wbisim}\ (\textbf{infix} \approx 40)\ \textbf{where}$$

$$\text{wsim } (\approx)\ op_1\ op_2 \implies \text{wsim } (\approx)\ op_2\ op_1 \implies op_1 \approx op_2$$

- R is a *weak bisimulation* when both R and its converse R^{-1} are weak simulations. Weak bisimilarity is the largest weak bisimulation.

Operators Equivalences: Weak Bisimilarity

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Weak Bisimilarity

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- R is a *weak bisimulation* when both R and its converse R^{-1} are weak simulations. Weak bisimilarity is the largest weak bisimulation.
- $\approx$ has a useful coinduction principle
- $\oslash \approx \otimes \approx \odot$ and ex1 $\approx$ ex2 $\approx$ ex3

Asynchronous Dataflow Operators

Buffer Infrastructure

- Buffers: $'a \Rightarrow 'd\ list$

Auxiliary functions for buffers

$$\mathrm{BHD}\ p\ buf = \mathrm{hd}\ (buf\ \ p)$$
$$\mathrm{BTL}\ p\ buf = buf(p := \mathrm{tl}\ (buf\ \ p))$$
$$\mathrm{BENQ}\ p\ x\ buf = buf(p := buf\ \ p\ @\ [x])$$
$$buf_1 \gg buf_2 = (\lambda p.\ buf_2\ p\ @\ buf_1\ p)$$

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Identity operator

$$\text{id\_op } \mathit{buf} = \text{Choice}$$

$$\cup_c$$

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- $\mathfrak{U}_c$ is the set of usable ports provide by its type

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&\ ((\lambda p.\ \text{Write } (\text{id\_op } (\text{BTL } p\ \mathit{buf}))\ p\ (\text{BHD } p\ \mathit{buf}))\ `_c\ \{p \in_c \mathfrak{U}_c \mid \mathit{buf}\ \ p \neq []\}))
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- Stream delayer!

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- $\mathfrak{U}_c$ is the set of usable ports provide by its type
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Identity operator with an empty buffer

$$\mathcal{I} = \text{id\_op } (\lambda\_.\ [])$$

Composition

- foo

Asynchronous Dataflow Operators

- foo

Asynchronous Dataflow Properties

$B1\colon op_1 \parallel (op_2 \parallel op_3) \approx \mathsf{map\_op}\ \curvearrowright \curvearrowright\ (op_1 \parallel op_2) \parallel op_3$

$B2\_1\colon op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \mathsf{map\_op}\ \mathsf{Inl}\ \mathsf{Inl}\ op$

$B2\_2\colon (\mathcal{I} :: (0, 0, 'd)\ op) \parallel op \approx \mathsf{map\_op}\ \mathsf{Inr}\ \mathsf{Inr}\ op$

$B3\colon (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$

$B4\_1\colon op\vdash \bullet \mathcal{I} \approx op\vdash$ $\quad$ $B4\_2\colon \mathcal{I} \bullet \dashv op \approx \dashv op$

$B5\colon (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$

$B6\colon \mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$ $\quad$ $B7\colon \mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$

$B8\colon (\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \mathsf{map\_op}\ \mathsf{id}\ (\mathsf{case\_sum}\ \mathsf{Inr}\ \mathsf{Inl})\ \mathcal{I}$

$B9\colon \mathcal{X} \approx \mathsf{map\_op}\ \curvearrowright \curvearrowright\ (\mathcal{X} \parallel \mathcal{I}) \bullet \mathsf{map\_op}\ \mathsf{id}\ \curvearrowleft\ (\mathcal{I} \parallel \mathcal{X})$

$B10\colon (\dashv op_1 \parallel \dashv op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2\vdash \parallel op_1\vdash)$

$F1\colon \mathcal{I}\uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ $\quad$ $F2\colon \mathcal{X}\uparrow \approx \mathcal{I}$

Table 1 Basic network algebra properties

$R1\colon \mathsf{Inr}\ \grave{}\ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}\ \grave{}\ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
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$\mathsf{map\_op}\ \mathsf{Inl}\ \mathsf{Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op)\uparrow) \approx op$

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B9: $\mathcal{X} \approx \mathsf{map\_op} \curvearrowright \curvearrowright (\mathcal{X} \parallel \mathcal{I}) \bullet \mathsf{map\_op}\ \mathsf{id} \curvearrowleft (\mathcal{I} \parallel \mathcal{X})$

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$R2: \mathsf{Inr}\ \text{-`}\ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}\ \text{-`}\ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$(op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$

$R3: \mathsf{Inr}\ \text{-`}\ \mathsf{inputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}\ \text{-`}\ \mathsf{outputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$op_1 \parallel (op_2 \uparrow) \approx (\mathsf{map\_op}\ \curvearrowleft \curvearrowleft\ (op_1 \parallel op_2)) \uparrow$

$R4: \mathsf{Inr}\ \text{-`}\ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}\ \text{-`}\ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$\mathsf{inputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{outputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$(\dashv op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1\vdash) \uparrow$

$R5: \mathsf{Inr}\ \text{-`}\ \mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}\ \text{-`}\ \mathsf{outputs}\ op = \{\} \Longrightarrow$
$\mathsf{map\_op}\ \mathsf{Inl}\ \mathsf{Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op) \uparrow) \approx op$

$R6: \mathsf{Inr}\ \text{-`}\ \mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}\ \text{-`}\ \mathsf{outputs}\ op = \{\} \Longrightarrow$
$\mathsf{Inr}\ \text{-`}\ \mathsf{Inl}\ \text{-`}\ \mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}\ \text{-`}\ \mathsf{Inl}\ \text{-`}\ \mathsf{outputs}\ op = \{\} \Longrightarrow$
$(op \uparrow) \uparrow \approx (\mathsf{map\_op}\ \curvearrowright \curvearrowright\ op) \uparrow$

Basic network algebra properties

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ (op_1 \parallel op_2) \parallel op_3$

B2_1: $op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \mathsf{map\_op}\ \mathsf{Inl}\ \mathsf{Inl}\ op$

B2_2: $(\mathcal{I} :: (0, 0, 'd)\ op) \parallel op \approx \mathsf{map\_op}\ \mathsf{Inr}\ \mathsf{Inr}\ op$

B3: $(op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$

B4_1: $op\vdash \bullet\, \mathcal{I} \approx op\vdash$ B4_2: $\mathcal{I} \bullet \dashv op \approx \dashv op$

B5: $(op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$

B6: $\mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$ B7: $\mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$

B8: $(\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \mathsf{map\_op}\ \mathsf{id}\ (\mathsf{case\_sum}\ \mathsf{Inr}\ \mathsf{Inl})\ \mathcal{I}$

B9: $\mathcal{X} \approx \mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ (\mathcal{X} \parallel \mathcal{I}) \bullet \mathsf{map\_op}\ \mathsf{id}\ \curvearrowleft\ (\mathcal{I} \parallel \mathcal{X})$

B10: $(\dashv op_1 \parallel \dashv op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2\vdash \parallel op_1\vdash)$

F1: $\mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ F2: $\mathcal{X} \uparrow \approx \mathcal{I}$

Table 1 Basic network algebra properties

R1: $\mathsf{Inr}\ `\ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}\ `\ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$

R2: $\mathsf{Inr}\ `\ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}\ `\ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$(op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$

R3: $\mathsf{Inr}\ `\ \mathsf{inputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}\ `\ \mathsf{outputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$op_1 \parallel (op_2 \uparrow) \approx (\mathsf{map\_op}\ \curvearrowleft\ \curvearrowleft\ (op_1 \parallel op_2)) \uparrow$

R4: $\mathsf{Inr}\ `\ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}\ `\ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$\mathsf{inputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{outputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$(\dashv op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1\vdash) \uparrow$

R5: $\mathsf{Inr}\ `\ \mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}\ `\ \mathsf{outputs}\ op = \{\} \Longrightarrow$
$\mathsf{map\_op}\ \mathsf{Inl}\ \mathsf{Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op) \uparrow) \approx op$

R6: $\mathsf{Inr}\ `\ \mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}\ `\ \mathsf{outputs}\ op = \{\} \Longrightarrow$
$\mathsf{Inr}\ `\ \mathsf{Inl}\ `\ \mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}\ `\ \mathsf{Inl}\ `\ \mathsf{outputs}\ op = \{\} \Longrightarrow$
$(op \uparrow) \uparrow \approx (\mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ op) \uparrow$

Properties of equality test, merge, copy, split, source and sink operators

B1: $op_1 \parallel (op_2 \parallel op_3) \approx \mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ (op_1 \parallel op_2) \parallel op_3$

B2_1: $op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \mathsf{map\_op\ Inl\ Inl}\ op$

B2_2: $(\mathcal{I} :: (0, 0, 'd)\ op) \parallel op \approx \mathsf{map\_op\ Inr\ Inr}\ op$

B3: $(op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$

B4_1: $op\vdash \bullet \mathcal{I} \approx op\vdash$ B4_2: $\mathcal{I} \bullet \dashv op \approx \dashv op$

B5: $(op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$

B6: $\mathcal{I} \parallel \mathcal{I} \approx \mathcal{I}$ B7: $\mathcal{X} \bullet \mathcal{X} \approx \mathcal{I}$

B8: $(\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \mathsf{map\_op\ id\ (case\_sum\ Inr\ Inl)}\ \mathcal{I}$

B9: $\mathcal{X} \approx \mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ (\mathcal{X} \parallel \mathcal{I}) \bullet \mathsf{map\_op\ id}\ \curvearrowleft\ (\mathcal{I} \parallel \mathcal{X})$

B10: $(\dashv op_1 \parallel \dashv op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2\vdash \parallel op_1\vdash)$

F1: $\mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ F2: $\mathcal{X} \uparrow \approx \mathcal{I}$

Table 1 Basic network algebra properties

R1: Inr ` $\mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}$ ` $\mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$

R2: Inr ` $\mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}$ ` $\mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$(op_1 \uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I})) \uparrow$

R3: Inr ` $\mathsf{inputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}$ ` $\mathsf{outputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$op_1 \parallel (op_2 \uparrow) \approx (\mathsf{map\_op}\ \curvearrowleft\ \curvearrowleft\ (op_1 \parallel op_2)) \uparrow$

R4: Inr ` $\mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}$ ` $\mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$\mathsf{inputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{outputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$(\dashv op_1 \bullet (\mathcal{I} \parallel op_2)) \uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1\vdash) \uparrow$

R5: Inr ` $\mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}$ ` $\mathsf{outputs}\ op = \{\} \Longrightarrow$
$\mathsf{map\_op\ Inl\ Inl}\ ((op :: ('i + 0, 'o + 0, 'd)\ op) \uparrow) \approx op$

R6: Inr ` $\mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}$ ` $\mathsf{outputs}\ op = \{\} \Longrightarrow$
Inr ` $\mathsf{Inl}$ ` $\mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}$ ` $\mathsf{Inl}$ ` $\mathsf{outputs}\ op = \{\} \Longrightarrow$
$(op \uparrow) \uparrow \approx (\mathsf{map\_op}\ \curvearrowright\ \curvearrowright\ op) \uparrow$

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B2_1: $op \parallel (\mathcal{I} :: (\mathit{0}, \mathit{0}, 'd)\ op) \approx \mathsf{map\_op\ Inl\ Inl}\ op$

B2_2: $(\mathcal{I} :: (\mathit{0}, \mathit{0}, 'd)\ op) \parallel op \approx \mathsf{map\_op\ Inr\ Inr}\ op$

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B10: $(\dashv op_1 \parallel \dashv op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2\vdash \parallel op_1\vdash)$

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Table 1 Basic network algebra properties

R1: Inr -` $\mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}$ -` $\mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$op_2 \bullet (op_1\uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1)\uparrow$

R2: Inr -` $\mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}$ -` $\mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$(op_1\uparrow) \bullet op_2 \approx (op_1 \bullet (op_2 \parallel \mathcal{I}))\uparrow$

R3: Inr -` $\mathsf{inputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}$ -` $\mathsf{outputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$op_1 \parallel (op_2\uparrow) \approx (\mathsf{map\_op} \curvearrowleft \curvearrowleft (op_1 \parallel op_2))\uparrow$

R4: Inr -` $\mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}$ -` $\mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$\mathsf{inputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{outputs}\ op_2 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$(\dashv op_1 \bullet (\mathcal{I} \parallel op_2))\uparrow \approx ((\mathcal{I} \parallel op_2) \bullet op_1\vdash)\uparrow$

R5: Inr -` $\mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}$ -` $\mathsf{outputs}\ op = \{\} \Longrightarrow$
$\mathsf{map\_op\ Inl\ Inl}\ ((op :: ('i + \mathit{0}, 'o + \mathit{0}, 'd)\ op)\uparrow) \approx op$

R6: Inr -` $\mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}$ -` $\mathsf{outputs}\ op = \{\} \Longrightarrow$
Inr -` $\mathsf{Inl}$ -` $\mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}$ -` $\mathsf{Inl}$ -` $\mathsf{outputs}\ op = \{\} \Longrightarrow$
$(op\uparrow)\uparrow \approx (\mathsf{map\_op} \curvearrowright \curvearrowright op)\uparrow$

Well-Behaved Operators

$B1: op_1 \parallel (op_2 \parallel op_3) \approx \mathsf{map\_op}\ \curvearrowright \curvearrowright\ (op_1 \parallel op_2) \parallel op_3$

$B2\_1: op \parallel (\mathcal{I} :: (0, 0, 'd)\ op) \approx \mathsf{map\_op}\ \mathsf{Inl}\ \mathsf{Inl}\ op$

$B2\_2: (\mathcal{I} :: (0, 0, 'd)\ op) \parallel op \approx \mathsf{map\_op}\ \mathsf{Inr}\ \mathsf{Inr}\ op$

$B3: (op_1 \bullet op_2) \bullet op_3 \approx op_1 \bullet (op_2 \bullet op_3)$

$B4\_1: op\vdash \bullet \mathcal{I} \approx op\vdash$ $\quad$ $B4\_2: \mathcal{I} \bullet \dashv op \approx \dashv op$

$B5: (op_1 \parallel op_2) \bullet (op_3 \parallel op_4) \approx (op_1 \bullet op_3) \parallel (op_2 \bullet op_4)$

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$B8: (\mathcal{X} :: ('i + 0, 0 + 'i, 'd)\ op) \approx \mathsf{map\_op}\ \mathsf{id}\ (\mathsf{case\_sum}\ \mathsf{Inr}\ \mathsf{Inl})\ \mathcal{I}$

$B9: \mathcal{X} \approx \mathsf{map\_op}\ \curvearrowright \curvearrowright\ (\mathcal{X} \parallel \mathcal{I}) \bullet \mathsf{map\_op}\ \mathsf{id}\ \curvearrowleft\ (\mathcal{I} \parallel \mathcal{X})$

$B10: (\dashv op_1 \parallel \dashv op_2) \bullet \mathcal{X} \approx \mathcal{X} \bullet (op_2\vdash \parallel op_1\vdash)$

$F1: \mathcal{I} \uparrow \approx (\mathcal{I} :: (0, 0, 'd)\ op)$ $\quad$ $F2: \mathcal{X} \uparrow \approx \mathcal{I}$

Table 1 Basic network algebra properties

$R1: \mathsf{Inr}\ \text{-`}\ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}\ \text{-`}\ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
$op_2 \bullet (op_1 \uparrow) \approx ((op_2 \parallel \mathcal{I}) \bullet op_1) \uparrow$

$R2: \mathsf{Inr}\ \text{-`}\ \mathsf{inputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow \mathsf{Inr}\ \text{-`}\ \mathsf{outputs}\ op_1 \cap \mathsf{defaults} = \{\} \Longrightarrow$
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$\mathsf{Inr}\ \text{-`}\ \mathsf{Inl}\ \text{-`}\ \mathsf{inputs}\ op = \{\} \Longrightarrow \mathsf{Inr}\ \text{-`}\ \mathsf{Inl}\ \text{-`}\ \mathsf{outputs}\ op = \{\} \Longrightarrow$
$(op \uparrow) \uparrow \approx (\mathsf{map\_op}\ \curvearrowright \curvearrowright\ op) \uparrow$

Related Work

Related Work

- A Isabelle/HOL instance of Nondeterministic Asynchronous Dataflow
    - Operators as a shallow embedding as codatatypes
    - 51 axioms proved
- Executable via code extraction to Haskell

Conclusion

Conclusion

- Isabelle/HOL has a good tool set to formalize and reason about stream processing:
  - Codatatypes, coinductive predicates, corecursion with friends, reasoning up to friends (congruence),
    - Coinduction up to congruence principle is automatically derived for codatatypes (but not for coinductive principles)
- Next step: Feedback loop

Questions, comments and suggestions