HOMEWORK 4
415G 001 COMBINATORICS AND GRAPH THEORY

DUE FRIDAY 10/16

Exercises

1. Find the ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$ satisfying the recurrence $a_{n+2} = 2a_{n+1} - a_n$ for $n \geq 0$ with initial conditions $a_0 = 0$ and $a_1 = 1$.

2. Find the ordinary generating function for the Fibonacci sequence defined by the recursion,

$$F_n = F_{n-1} + F_{n-2} \quad n \geq 2 \quad F_0 = 1, F_1 = 1$$

and use it to find a closed formula for F_n.

3. Find the following coefficients:

(a) $[x^n]\dfrac{1}{(1-ax)(1-bx)}, \quad a \neq b.$

(b) $[x^9](1+x^2+x^4)(1+x)^m$

(c) $[x^{17}]\left(\dfrac{1-x^{10}}{1-x}\right)^6$

4. Build a generating function for the number a_n of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = n,\ 2 \leq x_i \leq 8, x_1 \text{ odd, } x_2 \text{ even.}$$

5. How many positive integers less than $1,000,000$ have the sum of their digits equal to 17? (Hint: Use generating functions)

6. (Bonus) A *plane tree* is a rooted tree (it has one special node called the *root*) defined recursively as follows:

- A single vertex $\bullet$ (the root) is a plane tree.
- If we attach a new root connecting to all the roots of an ordered sequence $(P_1, P_2, \cdots, P_k)$ of plane trees we obtain a plane tree.

Figure 1 shows all plane trees with 4 vertices. Prove that the number P_n of plane trees with n vertices is C_{n-1} the $(n-1)$-th Catalan number by exhibiting a bijection with a family of known Catalan objects. (Hint: Find a bijection with Dyck paths or with Ballot sequences).

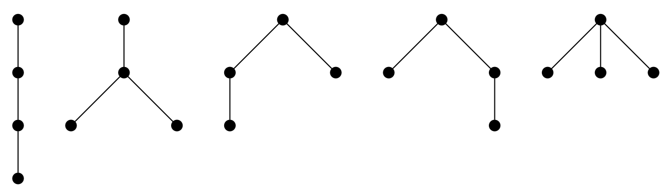

FIGURE 1. Plane trees with 4 vertices

Suggested exercises

From the book. 2.4, 2.16, 2.17, 2.18, 2.20