STATEMENT OF RESEARCH INTERESTS

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1. Context

My current research is in Algebraic and Topological Combinatorics. The two areas can be understood as the study of mathematical objects that may have a topological or an algebraic perspective but that are combinatorial in nature. A very good example of this interaction comes from the theory of algebras and operads. An algebra is an object that can be described in a purely algebraic manner by generators and relations. However some algebras are defined by the fact of being the cohomology ring of a topological space. This topological space in turn can have a very combinatorial description. For example, it can be a geometric realization of an abstract simplicial complex whose description can be purely combinatorial. The results coming from this interaction are of interest in Algebra, Topology and Combinatorics.

1.1. **Lie algebras and the topology of the poset of partitions.** I describe now one classical example where we see this interaction. Let $\mathbf{k}$ denote an arbitrary field. A *Lie bracket* on a vector space V is a bilinear binary product $[\cdot,\cdot] : V \times V \to V$ such that for all $x, y, z \in V$,

$$[x, y] + [y, x] = 0 \qquad \text{(Antisymmetry)}, \tag{1.1}$$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \qquad \text{(Jacobi Identity)}. \tag{1.2}$$

The *free Lie algebra on* $[n] = \{1, 2, \ldots, n\}$ is the $\mathbf{k}$-vector space generated by the elements of $[n]$ and all the possible bracketings involving these elements subject only to the relations (1.1) and (1.2). Let $\mathcal{L}ie(n)$ denote the *multilinear* component of the free Lie algebra on $[n]$, ie., the subspace generated by bracketings that contain each element of $[n]$ exactly once. Lets call these bracketings *bracketed permutations*. For example $[[2, 3], 1]$ is a bracketed permutation in $\mathcal{L}ie(3)$, while $[[2, 3], 2]$ is not. The symmetric group $\mathfrak{S}_n$ acts naturally on $\mathcal{L}ie(n)$ making it into an $\mathfrak{S}_n$-module. A permutation $\tau \in \mathfrak{S}_n$ acts on the bracketed permutations by replacing each letter i by $\tau(i)$. For example $(1, 2)\ [[[3, 5], [2, 4]], 1] = [[[3, 5], [1, 4]], 2]$. Since this action respects the relations (1.1) and (1.2), it induces a representation of $\mathfrak{S}_n$ on $\mathcal{L}ie(n)$. It is a classical result that

$$\dim \mathcal{L}ie(n) = (n - 1)!. \tag{1.3}$$

To every partially ordered set (*poset*) P one can associate a simplicial complex $\Delta(P)$ (called the *order complex*) whose faces are the chains of P. Consider now the poset Π_n of set partitions of $[n]$ ordered by refinement. The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n and this action induces isomorphic representations of $\mathfrak{S}_n$ on the unique nonvanishing reduced simplicial homology $\tilde{H}_{n-3}(\overline{\Pi}_n)$ and cohomology $\tilde{H}^{n-3}(\overline{\Pi}_n)$ of the order complex $\Delta(\overline{\Pi}_n)$ of the proper part $\overline{\Pi}_n = \Pi_n \setminus \{\hat{0}, \hat{1}\}$. It is now a classical result observed by Joyal [16] that follows from the computation of the character of $\tilde{H}_{n-3}(\overline{\Pi}_n)$ by Stanley and Hanlon (see [19]) and of an earlier formula of Brandt [7] for the character of $\mathcal{L}ie(n)$ that the following $\mathfrak{S}_n$-module isomorphism holds,

$$\tilde{H}_{n-3}(\overline{\Pi}_n) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(n) \otimes \operatorname{sgn}_n, \tag{1.4}$$

where sgn_n is the sign representation of $\mathfrak{S}_n$.

Joyal [16] gave a proof of the isomorphism using his theory of species. The first purely combinatorial proof was obtained by Barcelo [1] who provided a bijection between known bases for the two $\mathfrak{S}_n$-modules (Björner's NBC basis for $\tilde{H}_{n-3}(\bar{\Pi}_n)$ and the Lyndon basis for $\mathcal{L}ie(n)$). Later Wachs [21] gave a more general combinatorial proof by providing a natural bijection between generating sets of $\tilde{H}^{n-3}(\overline{\Pi}_n)$ and $\mathcal{L}ie(n)$, which revealed the strong connection between the two $\mathfrak{S}_n$-modules.

Equation 1.4 is a good example of the interaction between a combinatorial object Π_n, a topological object $\Delta(\overline{\Pi}_n)$ and an algebraic object $\mathcal{L}ie(n)$. In my current research I have generalized Equation 1.4 in order to study multibracketed Lie algebras.

2. Doubly bracketed Lie algebra

Two Lie brackets $[\bullet,\bullet]_1$ and $[\bullet,\bullet]_2$ are said to be *compatible* if any linear combination of the brackets is also a Lie bracket, that is, satisfies relations (1.1) and (1.2). Compatibility is equivalent to the *mixed Jacobi* condition: for all $x, y, z \in V$

$$\begin{aligned}(2.1)\qquad & [x,[y,z]_2]_1 + [z,[x,y]_2]_1 + [y,[z,x]_2]_1 + \qquad \text{(Mixed Jacobi)}\\ & [x,[y,z]_1]_2 + [z,[x,y]_1]_2 + [y,[z,x]_1]_2 = 0.\end{aligned}$$

Let $\mathcal{L}ie_2(n)$ be the multilinear component of the free Lie algebra on $[n]$ with 2 compatible brackets. For each i, let $\mathcal{L}ie_2(n,i)$ be the subspace of $\mathcal{L}ie_2(n)$ generated by bracketed permutations with exactly i brackets of the first type and $n-1-i$ brackets of the second type. The symmetric group $\mathfrak{S}_n$ acts naturally on $\mathcal{L}ie_2(n)$ and since this action preserves the number of brackets of each type, we have the following decomposition into $\mathfrak{S}_n$-submodules:

$$(2.2)\qquad \mathcal{L}ie_2(n) = \bigoplus_{i=0}^{n-1} \mathcal{L}ie_2(n,i).$$

Note that $\mathcal{L}ie(n)$ is isomorphic to the submodules $\mathcal{L}ie_2(n,i)$ when $i=0$ or $i=n-1$.

It was conjectured by Feigin and proved independently by Dotsenko-Khoroshkin [8] and Liu [17] that

$$(2.3)\qquad \dim \mathcal{L}ie_2(n) = n^{n-1} \text{ and } \dim \mathcal{L}ie_2(n,i) = |\mathcal{T}_{n,i}|$$

where $\mathcal{T}_{n,i}$ is the set of labeled rooted trees with i descending edges (a parent with a greater label than its child). They also prove that the dimension generating polynomial has a nice factorization:

$$(2.4)\qquad \sum_{i=0}^{n-1} \dim \mathcal{L}ie_2(n,i)t^i = \prod_{j=1}^{n-1}((n-j)+jt).$$

Although Dotsenko and Khoroshkin [8] did not use poset theoretic techniques in their proof of (2.3), they introduced the poset of weighted partitions as a possible approach to establishing Koszulness of the operad associated with $\mathcal{L}ie_2(n)$, a key step in their proof. In a joint paper with Wachs [11] we applied poset theoretic techniques to the poset of weighted partitions to give an alternative proof of (2.3) and to obtain further results on $\mathcal{L}ie_2(n)$. A weighted partition of $[n]$ is a set $\{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ where $\{B_1, B_2, ..., B_t\}$ is a partition of $[n]$ and $v_i \in \{0,1,2,...,|B_i|-1\}$ for all i. The *poset of weighted partitions* Π_n^w is the set of weighted partitions of $[n]$ with order covering relation given by $\{A_1^{w_1}, A_2^{w_2}, ..., A_s^{w_s}\} \lessdot \{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ if the following conditions hold:

- $\{A_1, A_2, \dots, A_s\} \lessdot \{B_1, B_2, \dots, B_t\}$ in Π_n,
- if $B_k = A_i \cup A_j$, where $i \neq j$, then $v_k - (w_i + w_j) \in \{0, 1\}$,
- if $B_k = A_i$, then $v_k = w_i$.

In Figure 1 below, the set brackets and commas have been omitted.

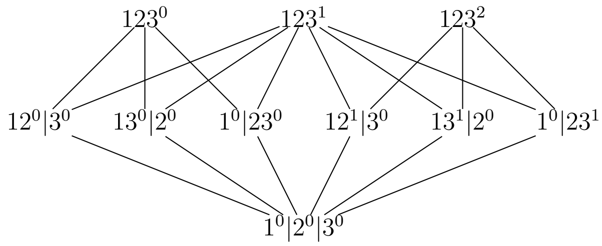


Figure 1. Weighted partition poset for $n = 3$

The poset Π_n^w has a minimum element $\hat{0} := \{\{1\}^0, \{2\}^0, \dots, \{n\}^0\}$ and n maximal elements $\{[n]^0\}$, $\{[n]^1\}, \dots, \{[n]^{n-1}\}$. For each i, the maximal intervals $[\hat{0}, [n]^i]$ and $[\hat{0}, [n]^{n-1-i}]$ are isomorphic to each other, and the two maximal intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$ are isomorphic to Π_n. In [11] we found a nice EL-labeling that generalized a classical EL-labeling of Π_n due to Björner and Stanley (see [2]). An EL-labeling of a poset is a labeling of the edges of the Hasse diagram of the poset that satisfies certain requirements. Such a labeling has important topological and algebraic consequences, such as the determination of homotopy type of each open interval of the poset. The so called ascent-free chains give a basis for cohomology of the open intervals. A poset that admits an EL-labeling is said to be EL-shellable. See [2] and [22] for further information.

Theorem 1 (González D'León and Wachs [11]). *The poset $\widehat{\Pi_n^w} := \Pi_n^w \cup \{\hat{1}\}$ is EL-shellable and hence Cohen-Macaulay. Consequently, for each $i = 0, \dots, n-1$, the order complex $\Delta((\hat{0}, [n]^i))$ has the homotopy type of a wedge of spheres.*

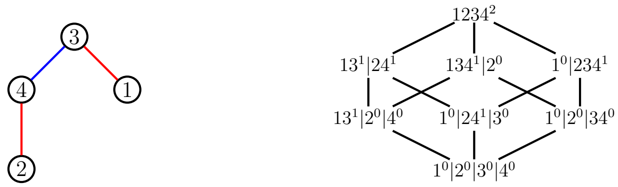
It follows from this and an operad theoretic result of Vallette [20] that the following $\mathfrak{S}_n$-module isomorphism holds:

$$\tilde{H}_{n-3}((\hat{0}, [n]^i)) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n, i) \otimes \mathrm{sgn}_n\,. \tag{2.5}$$

For $i = 0$ or $i = n-1$, Equation 2.5 reduces to Equation 1.4. In [11] we gave an explicit $\mathfrak{S}_n$-module isomorphism using Wachs' technique in [21]. We also constructed bases for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ and $\mathcal{L}ie_2(n, i)$ that generalize the classical Lyndon tree basis and the comb basis for $\tilde{H}^{n-3}(\overline{\Pi_n})$ and $\mathcal{L}ie(n)$. In particular, our general Lyndon basis is obtained from the ascent-free chains of the EL-labeling of Theorem 1. We also define a basis for $\tilde{H}_{n-3}(\overline{\Pi_n})$ in terms of labeled rooted trees that generalizes the Björner NBC basis for homology of $\bar{\Pi}_n$ (see [3, Proposition 2.2]). Indeed, we can associate a weighted partition $\alpha(F)$ with a labeled binary forest $F = \{T_1, \dots, T_k\}$ on node set $[n]$, by letting

$$\alpha(F) = \{A_1^{w_1}, \dots, A_k^{w_k}\},$$

where A_i is the node set of T_i and w_i is the number of descents of T_i. Let T be a rooted tree on node set $[n]$. For each subset E of the edge set $E(T)$ of T, let T_E be the subgraph of T with node set $[n]$ and edge set E. Clearly T_E is a forest on $[n]$. We define Π_T to be the induced subposet of Π_n^w on the set $\{\alpha(T_E) : E \in E(T)\}$. See Figure 2 for an example of Π_T. The poset Π_T is clearly isomorphic to the boolean algebra $\mathcal{B}_{n-1}$. Hence $\Delta(\overline{\Pi_T})$ is the barycentric subdivision of the boundary of the $(n-2)$-simplex. We let ρ_T denote a fundamental cycle of the spherical complex $\Delta(\overline{\Pi_T})$. The set $\{\rho_T : T \in \mathcal{T}_{n,0}\}$ is precisely the interpretation of the Björner NBC basis for homology of $\bar{\Pi}_n$ given in [21, Proposition 2.2].



(A) T (B) Π_T

FIGURE 2. Example of a tree T with two descent edges (red edges) and the corresponding poset Π_T

Theorem 2 (González D'León and Wachs [11]). *The set $\{\rho_T : T \in \mathcal{T}_{n,i}\}$ is a basis for $\tilde{H}_{n-3}((\hat{0}, [n]^i))$*

3. Multibracketed Lie algebras

One of my main results answers the following question of Liu.

Question 1 (Liu [17], Question 11.7). *Is it possible to define $\mathcal{L}ie_k(n)$ for any $k \geq 1$ so that it has nice rank formulas like those for $\mathcal{L}ie(n)$ and $\mathcal{L}ie_2(n)$? What are the right combinatorial objects for $\mathcal{L}ie_k(n)$, if it can be defined?*

We say that a set B of Lie brackets is *compatible* if any linear combination of the brackets in B is a Lie bracket. We now consider compatible Lie brackets $[\cdot,\cdot]_j$ indexed by positive integers $j \in \mathbb{P}$. A *weak composition* μ of n is a sequence of nonnegative integers $(\mu(1), \mu(2), \dots)$ such that $\sum_{i\geq 1} \mu(i) = n$. Let wcomp_n be the set of weak compositions of n. For $\mu \in \mathrm{wcomp}_n$, define $\mathcal{L}ie(\mu)$ to be the multilinear component of the multibracketed free Lie algebra on $[n]$ generated by bracketed permutations with $\mu(j)$ brackets of type j for each j. For example $\mathcal{L}ie(0,1,2,0,1)$ is generated by bracketed permutations that contain one bracket of type 2, two brackets of type 3, one bracket of type 5 and no brackets of any other type.

For $\mu \in \mathrm{wcomp}_n$ define its *support* $\mathrm{supp}(\mu) = \{j \in \mathbb{P} \mid \mu(j) \neq 0\}$ and for a subset $S \subseteq \mathbb{P}$ let

$$\mathcal{L}ie_S(n) := \bigoplus_{\substack{\mu\in \mathrm{wcomp}_{n-1} \\ \mathrm{supp}(\mu)\subseteq S}} \mathcal{L}ie(\mu). \tag{3.1}$$

Note that $\mathcal{L}ie_k(n) := \mathcal{L}ie_{[k]}(n)$ generalizes $\mathcal{L}ie(n) = \mathcal{L}ie_1(n)$ and $\mathcal{L}ie_2(n)$.

The isomorphisms (1.4) and (2.5) provide a way to study the algebraic objects $\mathcal{L}ie(n)$ and $\mathcal{L}ie_2(n)$ by applying poset topology techniques to Π_n and Π_n^w. In particular the dimensions of the modules can be read from the structure of the posets and the bases for the cohomology of the posets can be directly translated into bases of $\mathcal{L}ie(n)$ and $\mathcal{L}ie_2(n)$.

Question 2. *Is there a poset that allow us to extend the isomorphisms (1.4) and (2.5) to $\mathcal{L}ie(\mu)$?*

I introduce in [11] a more general poset of weighted partitions Π_n^k where the weights are given by weak compositions supported in $[k]$. A (k-)weighted partition of $[n]$ is a set $\{B_1^{\mu_1}, B_2^{\mu_2}, ..., B_t^{\mu_t}\}$ where $\{B_1, B_2, ..., B_t\}$ is a set partition of $[n]$ and $\mu_i \in \text{wcomp}_{|B_i|-1}$ with $\text{supp}(\mu_i) \subseteq [k]$. The *poset of k-weighted partitions* Π_n^k is the set of k-weighted partitions of $[n]$ with covering relations defined similarly to that of Π_n^w, but now we need an ordering of the weight set wcomp_n. For $\nu, \mu \in \text{wcomp}_n$ we say that $\mu \leq \nu$ if $\mu(i) \leq \nu(i)$ for every i. The poset Π_n^k has a minimum element $\hat{0} := \{\{1\}^{(0,\ldots,0)}, \{2\}^{(0,\ldots,0)}, \ldots, \{n\}^{(0,\ldots,0)}\}$ and maximal elements $\{[n]^\mu\}$ given by weak compositions $\mu \in \text{wcomp}_{n-1}$ supported in $[k]$. Note that $\Pi_n^1 \simeq \Pi_n$ and $\Pi_n^2 \simeq \Pi_n^w$. Using Wachs' technique I give an explicit isomorphism that proves the following theorem.

Theorem 3 (González D'León [12])**.**

$$\tilde{H}_{n-3}((\hat{0}, [n]^\mu)) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(\mu) \otimes \text{sgn}_n, \tag{3.2}$$

I also construct an EL-labeling that generalizes that of Π_n and $\widehat{\Pi_n^w}$.

Theorem 4 (González D'León [12])**.** *The poset $\widehat{\Pi_n^k}$ is EL-shellable and hence Cohen-Macaulay. Consequently, for each $\mu \in \text{wcomp}_{n-1}$, the order complex $\Delta((\hat{0}, [n]^\mu))$ has the homotopy type of a wedge of spheres.*

The ascent-free chains give a colored Lyndon basis for $\tilde{H}^{n-3}((\hat{0}, [n]^\mu))$. I also define a colored comb basis for $\tilde{H}^{n-3}((\hat{0}, [n]^\mu))$. By the isomorphism (3.2) these bases are also bases of $\mathcal{L}ie(\mu)$.

Open problem 1. *Is there a way to generalize the tree basis for $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ to a basis for $\tilde{H}_{n-3}((\hat{0}, [n]^\mu))$?*

It follows from (2.4) that the polynomial $\sum_{i=0}^{n-1} \dim \mathcal{L}ie_2(n, i)\, t^i$ has a property called γ-positivity, i.e, when written in the basis $t^i(1+t)^{n-1-2i}$ it has positive coefficients. Although I don't know of a simple closed formula like that of (2.4) for the symmetric function

$$\sum_{\mu \in \text{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^\mu, \tag{3.3}$$

where $\mathbf{x}^\mu = x_1^{\mu_1} x_2^{\mu_2} \ldots$, in [12] I generalize the γ-positivity property by showing that this symmetric function is e-positive, i.e., it has positive coefficients in the elementary symmetric function basis. Moreover, I have nice combinatorial descriptions of the coefficients involving labeled binary trees on one hand and the Stirling permutations introduced by Gessel and Stanley in [10] on the other hand.

A *Stirling permutation* is a permutation of the multiset $\{1, 1, 2, 2, \cdots, n, n\}$ such that all numbers between the two occurrences of any number m are larger than m. This family of multipermutations is denoted $\mathcal{Q}_n$. To this family of permutations belongs for example 12234431 but not 11322344 since 2 is less than 3 and 2 is between the two occurrences of 3. The Stirling permutations have been also studied by Bóna, Park, Janson, Kuba, Panholzer and others (see [4, 18, 15, 14]).

For a Stirling permutation $\sigma \in \mathcal{Q}_n$ we call (a, b) an *ascending adjacent pair* if $a < b$ and in σ the second occurrence of a is the immediate predecessor of the first occurrence of b. An *ascending adjacent sequence* of σ of length k is a sequence $a_1 < a_2 < \cdots < a_k$ such that (a_j, a_{j+1}) is an ascending adjacent pair for $j = 1, \ldots, k-1$. We can associate a *type* to a Stirling permutation $\sigma \in \mathcal{Q}_n$ by letting $\lambda(\sigma)$ be the partition whose parts are the lengths of maximal ascending adjacent sequences. For example if $\sigma = 158851244667723399$ then the maximal ascending adjacent sequences are 1239, 467, 5 and 8 and so $\lambda(\sigma) = (4, 3, 1, 1)$ which is a partition of $n = 9$. Define the nth *Stirling-Eulerian symmetric function* S_n^2 by

$$S_n^2(\mathbf{x}) = \sum_{\sigma \in \mathcal{Q}_n} e_{\lambda(\sigma)}(\mathbf{x}) \tag{3.4}$$

where e_λ is the elementary symmetric function associated with the partition λ. Note that this symmetric function reduces to the classical second-order Eulerian polynomials of Gessel and Stanley by the simple specialization $e_i \mapsto x$. In [12] I give a bijection between the ascent-free chains in the EL-labeling of Theorem 4 and colored Stirling permutations which enables me to prove the following theorem.

Theorem 5 (González D'León [12])**.**

$$\sum_{\mu \in \mathrm{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^\mu = S_{n-1}^2(\mathbf{x}). \tag{3.5}$$

In particular,

$$\dim \mathcal{L}ie_k(n) = S_{n-1}^2(\overbrace{1, \dots, 1}^{k \text{ times}}, 0, 0, \dots). \tag{3.6}$$

The characters of the representation of $\mathfrak{S}_n$ on $\mathcal{L}ie(n)$ and $\mathcal{L}ie_2(n)$ were computed in ([7, 19] and [8]). Let $\mathrm{ch}\,\mathcal{L}ie(\mu)$ denote the Frobenius characteristic in variables $\mathbf{y} = (y_1, y_2, \dots)$ of the representation $\mathcal{L}ie(\mu)$ and Λ_R be the ring of symmetric functions in $\mathbf{y}$ with coefficients in the ring of symmetric functions $R = \Lambda_\mathbb{Q}$ in $\mathbf{x}$.

Theorem 6 (González D'León [12])**.**

$$\sum_{n \geq 1} \sum_{\mu \in \mathrm{wcomp}_{n-1}} \mathrm{ch}\,\mathcal{L}ie(\mu)\, \mathbf{x}^\mu = -\Big(-\sum_{n \geq 1} h_{n-1}(\mathbf{x}) h_n(\mathbf{y})\Big)^{<-1>} \tag{3.7}$$

where h_n is the complete homogeneous symmetric function and $(\cdot)^{<-1>}$ denotes the plethystic inverse in Λ_R.

Open problem 2. *Can we find explicit character formulas for $\mathcal{L}ie(\mu)$? What are the multiplicities of the irreducibles?*

Since we are dealing with symmetric functions in both the $\mathbf{x}$ and the $\mathbf{y}$-variables we can write

$$\sum_{\mu \in \mathrm{wcomp}_{n-1}} \mathrm{ch}\,\mathcal{L}ie(\mu)\, \mathbf{x}^\mu = \sum_{\lambda \vdash n-1} C_\lambda(\mathbf{y}) e_\lambda(\mathbf{x}). \tag{3.8}$$

Conjecture 7. *The coefficients $C_\lambda(\mathbf{y})$ are Schur positive.*

The conjecture basically asserts that $C_\lambda(\mathbf{y})$ is the Frobenius characteristic of a representation of dimension $\left\langle\!\left\langle {n \atop \lambda} \right\rangle\!\right\rangle$, the number of Stirling permutations in $\mathcal{Q}_{n-1}$ of type λ. An approach to proving the conjecture is to find such a representation.

In [13] I study the Stirling permutations and the Stirling-Eulerian symmetric functions further. I provide bijections which establish refinements of work of Bona [4] on equidistribution of different statistics on Stirling permutations. I also relate them to the work of Drake [9] on computing compositional inverses of exponential generating functions. Moreover, I consider a generalization to r-Stirling permutations and define the more general r-Stirling-Eulerian symmetric function $S_n^r(\mathbf{x})$. Let $f \in \mathbb{R}[x_1, ..., x_k]$ be a multivariate complex polynomial with real coefficients. The polynomial f is said to be *stable* if f is non-vanishing whenever all the variables x_i are taken to be in the open upper half-plane $H = \{a + bi | \, b > 0\}$ (see [5, 6]). Stability reduces to real-rootedness in the one variable case and hence this property is related to γ-positivity, which can be viewed as a special case of e-positivity. We know from the definition that $S_n^r(\mathbf{x})$ is e-positive.

Open problem 3. *What is the relation between stability of homogeneous symmetric polynomials and e-positivity? and Schur-positivity?*

Open problem 4. *Are the polynomials $S_n^r(x_1, \ldots, x_k)$ stable?*

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