HOMEWORK 1
415G 001 COMBINATORICS AND GRAPH THEORY

DUE FRIDAY 9/04

Exercises

1. Prove by induction that $1^3+2^3+...+n^3 = \dfrac{n^2(n+1)^2}{4}$. Then show that $(1+2+\cdots+n)^2 = 1^3+2^3+\cdots+n^3$.

2. Prove by induction that for $a \neq 1$, $1+a+a^2+\cdots+a^n = \dfrac{1-a^{n+1}}{1-a}$.

3. Prove that for positive integers $a > b$ there exist **unique** non negative integers q and r such that $a = qb + r$ with $0 \leq r < b$.

4. Use strong induction to show that any positive integer $n \geq 1$ can be expressed as a sum $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r}$ of powers of 2, where $a_1 > a_2 > \cdots > a_r \geq 0$ (all a_i's are distinct).

5. Prove that for every integer x, if x is odd then there exists an integer k such that $x^2 = 8k + 1$.

6. Let F_n be the number of binary sequences (sequences of ones and zeros) of length n (for example 1101 is a binary sequence of length 4). There is a unique sequence of length 0, the empty sequence. Also having a binary sequence of length n we can obtain one binary sequence of length $n+1$ by adding either a trailing 0 or 1, so there is always twice as much binary sequences of length $n+1$ than binary sequences of length n, i.e., $F_{n+1} = 2F_n$. Guess a closed formula for F_n and prove that the formula is correct using induction.

7. Find a bijection (and prove that your map is indeed a bijection) between the power set $\mathcal{P}([n]) := \{A \mid A \subseteq [n]\}$ of the set $[n] := \{1, 2, \ldots, n\}$ and the set $\mathcal{B}_n$ of binary sequences of length n. Use the previous exercise to conclude a formula for the cardinality of $\mathcal{P}([n])$.