HOMEWORK 11
415G 001 COMBINATORICS AND GRAPH THEORY

DUE FRIDAY 12/9

Exercises

1. In the early versions of the Enigma machine, used in Germany in the 1930's, the plugboard swapped six pairs of distinct letters of the alphabet. In how many ways can this be done (assuming 26 letters)?

2. For a sequence of numbers $\{a_n\}_{n\geq 0}$ we define its *exponential generating function (e.g.f.)* to be the formal power series

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

Let $B(n)$ be the n-th Bell number and $B(x) = \sum_{n\geq 0} B(n)\frac{x^n}{n!}$.

(A). Use the recursion for the Bell numbers, $B(0) = 1$ and

$$B(n) = \sum_{k\geq 0} \binom{n-1}{k} B(k) \qquad n \geq 1,$$

to prove that

$$\frac{dB(x)}{dx} = e^x B(x).$$

(B). Use the differential equation above to prove that $B(x)$ is given by

$$B(x) = e^{e^x - 1}.$$

(C). Finally expand the formula in part (B) to give the explicit closed formula for the nth Bell numbers

$$B(n) = \frac{1}{e} \sum_{j\geq 0} \frac{j^n}{j!}.$$

Suggested exercises

From the book. 5.1, 5.2, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10