Homework 5

Problem 1 Consider the linear model

$$y = X\beta + \sigma\epsilon$$

where y is a $n \times 1$ vector of observations, X is the $n \times p$ design matrix, β is the $p \times 1$ vector of coefficients, σ^2 is the variance and ϵ is a $n \times 1$ vector of independent standard normal random variables.

(1) (8 pts) Using the notations in slides,

$$QX'XQ' = \Lambda^2 \qquad PXQ' = D = \begin{pmatrix} \Lambda \\ 0 \end{pmatrix},$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_p)$ is the diagonal matrix with entries the square roots of the eigenvalues of $X'X$, do a Principal Component Analysis (PCA) using the linear transformation $T = XQ'$. Set $\gamma = Q\beta$, find $X\beta$ in terms of T and γ.

(2) (10 pts) Hence, show that regressing y on T is equivalent to regressing $z = Py$ on D

(3) (12 pts) Use the provided dataset "HW5_data.csv", where $n = 200$ and $p = 5$, to perform a PCA and find the least squares estimator for γ and β. You can use programming software (R, Python, etc.) to do the computations. You do not need to submit the code but show your working, including equations and conclusions.

(4) (10 pts) Find $\hat{\gamma}$ using only 4 principle components.

(5) (10 pts) Now suppose we want to keep m components where m is the smallest integer for which

$$\frac{\sum_{j=1}^{m} \lambda_j^2}{\sum_{j=1}^{p} \lambda_j^2} > 0.9.$$

Find m and the corresponding $\hat{\beta}_{pca}$

Problem 2

Consider the Bayesian linear model

$$y_i \sim N(x_i\beta,\ \sigma^2), \qquad i = 1, \ldots, n,$$

$$\text{where} \quad \sum_{i=1}^{n} x_i = 0, \quad \sum_{i=1}^{n} x_i^2 = n \quad \text{and} \quad \sum_{i=1}^{n} x_i y_i = \gamma.$$

The prior for β and the dummy variable z is given by

$$\pi(\beta|z) = (1-z)\delta_0(\beta) + zN(\beta|0, \tau^2) \quad \text{and} \quad \pi(z) = q^z(1-q)^{1-z}.$$

Suppose σ, τ and q are all known and $\delta_0(\beta)$ is the indicator function which is 1 when $\beta = 0$ and is 0 otherwise. So, $\beta = 0$ if $z = 0$ and $\beta = N(0, \tau^2)$ if $z = 1$.

(1) (8 pts) Find $p(y_1, \ldots, y_n|z = 0)$.

(2) (12 pts) By integrating out β, find $p(y_1, \ldots, y_n|z = 1)$.

(3) (12 pts) Hence, find $P(z = 1|y_1, \ldots, y_n)$.

(4) (8 pts) What is $P(z = 1|y_1, ..., y_n)$ when $\gamma = 0$ and with large n?

(5) (10 pts) Under the condition in (4), give an intuitive explanation why $P(z = 1|y_1, ..., y_n)$ takes this value when $n \to \infty$.