Advanced Models

Hw 1.

Part 1.

Question a

A time series to be stationary needs to satisfy

i) The mean function μ_t is constant and does not depend on time

ii) The autocovariance function depends only on times s and t, as the absolute value of the difference.

$$\gamma(s, t) = |t - s|$$

Now, for the time series x_t, we have that

$$\mu_t = E[X_t] = 10 \sin(t/5) + \cancel{E[w_t]}^{\,0}$$

$$= 10 \sin(t/5)$$

$$\gamma(s,t) = \text{cov}(X_s, X_t)$$

$$= E\left[(X_s - \mu_s)(X_t - \mu_t)\right]$$

$$= E\left[\left(\left(\cancel{10\sin(s/5)} + w_s\right) - \cancel{10\sin(s/5)}\right) \cdot \left(\left(\cancel{10\sin(t/5)} + w_t\right) - \cancel{10\sin(t/5)}\right)\right]$$

$$= E[w_s w_t]$$

if $s = t$; $\gamma = E[w_s^2] = \text{Var}(w_s^2) + \cancel{E[w_s]^2} \to 0$

$\therefore$

$$\gamma(s,t) = 1$$

if $s \neq t$

$$\gamma(s,t) = E[w_s w_t] = E[w_s]\, E[w_t] = 0$$

$\hookrightarrow$ (due to independence)

$\therefore \gamma(s,t) = 0$. Hence,

$$\gamma(s,t) = \begin{cases} 1 & \text{if } |s-t| = 0 \\ 0 & \text{if } |s-t| > 0 \end{cases}$$

So, the time series

$X_t = 10\sin(t/5) + w_t$ is NOT

Stationary. This time series mean depends on time, so it violates the first requirement (mean to be constant). This time series does not violate the second requirement (autocovariance function of times s, t as the absolute value of the difference).

Question b

$$SNR = \frac{\text{Amplitude}}{\sigma_w}; \quad \sigma_w = 1, \; A = 10$$

$$\therefore \; SNR = \frac{10}{1} = \underline{\underline{10}}$$

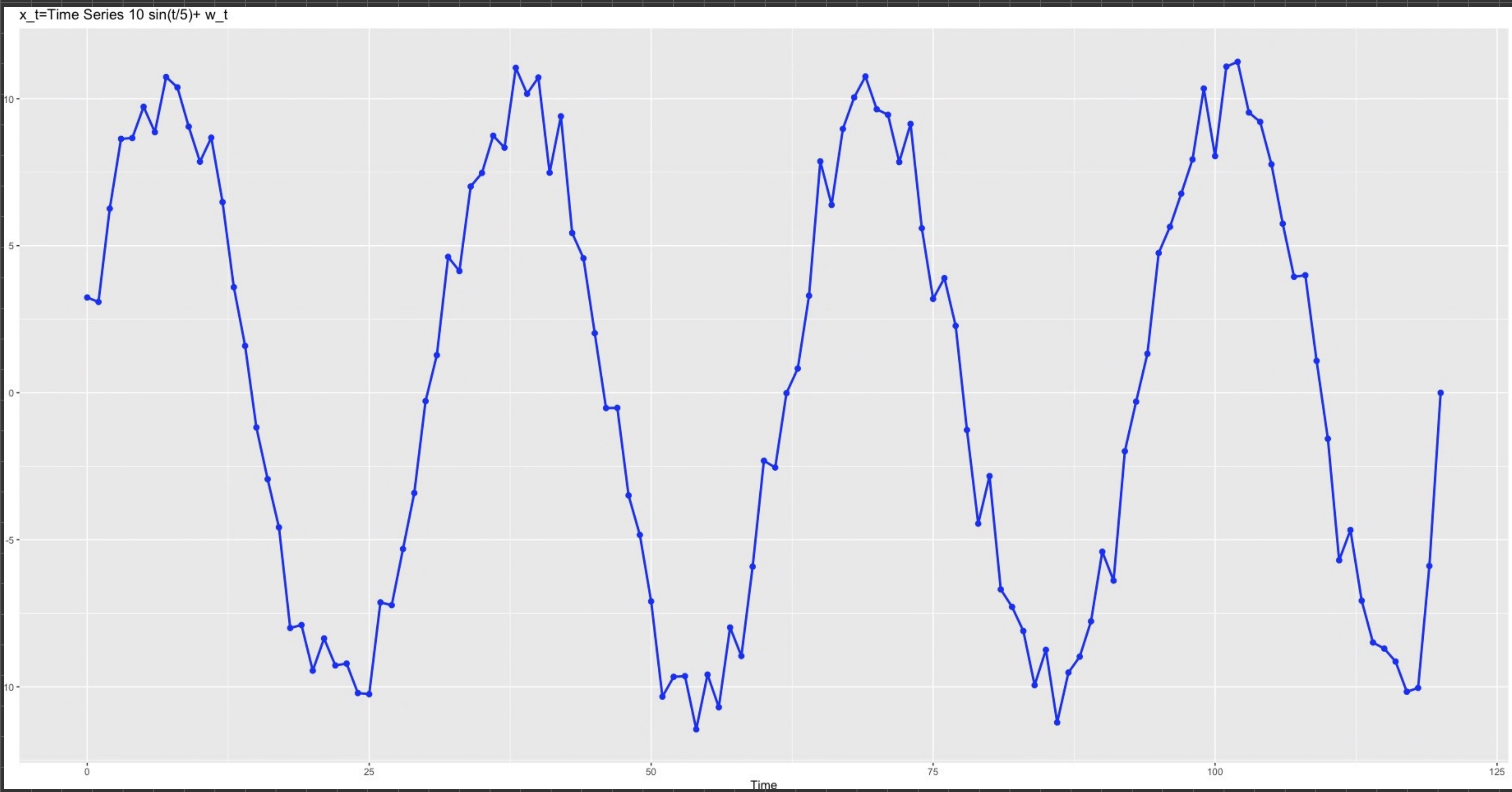
Question C

```r
#The time series mean is 0 and variance 1
mu<-0.0
sigma2<- 1.0
#We create 121 white noise elements taken from the normal distribution N(0,1)
w<-rnorm(121, mu, sqrt(sigma2))
#Create a vector of 121 elements
x<- rep(0, 121)

#Create the time series
for (t in 0:120)
{
  x[t]<-10*sin(t/5)+w[t]
}
# We make the time series as a data frame
sim_tim_series <- data.frame("Time" = 0:120,"x" =x)
# construct a plot of the time series
gg_time_series <- ggplot(sim_tim_series,
                         aes(x = Time, y = x)) +
  geom_line(size = 1, color = "blue") +
  geom_point(shape = 19, size = 2, fill = "white", color = "blue")  +
  ggtitle("x_t=Time Series 10 sin(t/5)+ w_t")

#plot the time series
gg_time_series
```

Result



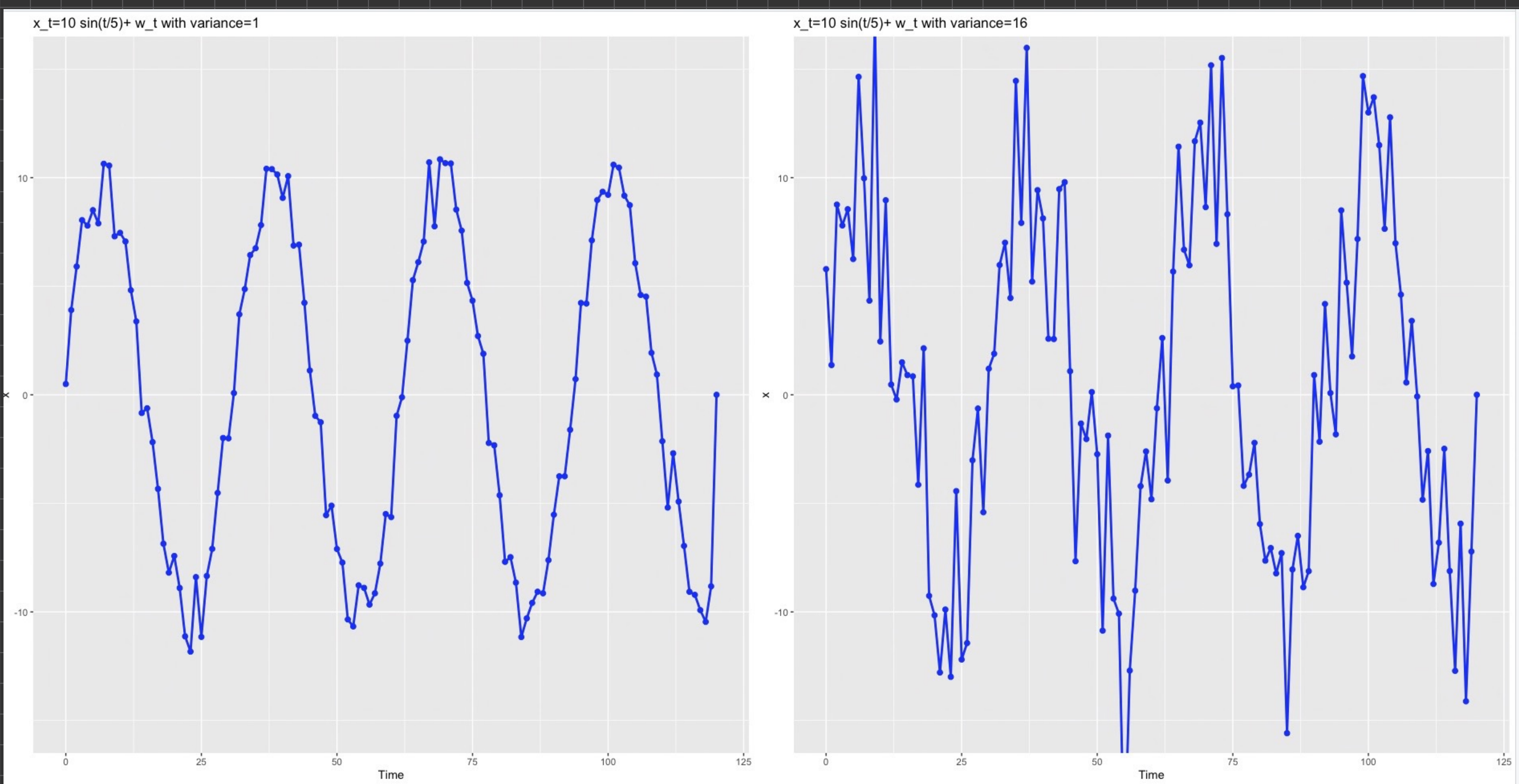
Question d

```
mu<-0.0
sigma2<- 1.0
w<-rnorm(121, mu, sqrt(sigma2))
x<- rep(0, 121)
for (t in 0:120)
{
  x[t]<-10*sin(t/5)+w[t]
}

sim_tim_series <- data.frame("Time" = 0:120,"x" =x)
# construct a plot of the independent time series
gg_time_series <- ggplot(sim_tim_series,
                         aes(x = Time, y = x)) +
  geom_line(size = 1, color = "blue") +
  geom_point(shape = 19, size = 2, fill = "white", color = "blue")+
  coord_cartesian(ylim = c(-15,15))+
  ggtitle("x_t=10 sin(t/5)+ w_t with variance=1")
#The new white noise has variance=16.
mu_2<-0.0
sigma2_2<- 16.0
#We create 121 random elements from the distribution N(0,16), because the time series starts from t=0
#up to t=120
w_2<-rnorm(121, mu_2, sqrt(sigma2_2))
# Create an array full of 0`s
x_2<- rep(0, 121)
#We get the values of the time series starting from t=0 up to t=120 with the new white noise
for (t in 0:120)
{
  x_2[t]<-10*sin(t/5)+w_2[t]
}
# Make the created time series as dataframe
sim_tim_series_2 <- data.frame("Time" = 0:120,"x" =x_2)
# construct a plot of the new time series
gg_time_series_2 <- ggplot(sim_tim_series_2,
                         aes(x = Time, y = x)) +
  geom_line(size = 1, color = "blue") +
  geom_point(shape = 19, size = 2, fill = "white", color = "blue")+
  coord_cartesian(ylim = c(-15,15)) +
  ggtitle("x_t=10 sin(t/5)+ w_t with variance=16")
#Plot both time series, first time series in the left and the new times series plotted to the right
grid.arrange(arrangeGrob(gg_time_series,gg_time_series_2,ncol=2))
```

setting y limits to the plot

setting y limits to the plot



Signal to Noise Ratio (SNR) $= \frac{A}{\sigma} = \frac{10}{4} = 2.5$

Question e

i) Period: The periods of both time series are the same, because the frequency of oscillation ω is the same.

ii) Signal to Noise Ratio:

When using variance = 1

$SNR_1 = 10.$

While the SNR_2 with variance = 16

$$SNR_2 = \frac{A}{\sigma} = 2.5.$$

Here the $SNR_1 > SNR_2$

SNR_1: signal to noise ratio of part a, b time series.

SNR_2: signal to noise ratio of part d time series.

The signal of second time series is smaller due to the higher variance.

Also, we can notice that the SNR should be smaller for the second time series by looking at the plots of question d). When the signal can be more easily seen, the SNR (left plot) and when

SNR is smaller, the signal looks with more noise (right plot).

iii) Observed value of the generated signal at $t=45$.

Expected value of the new time series

$E[X_t] = 10\sin(t/5) + \cancel{E[X_t]}^{0}$

$= \underline{10 \cdot \sin(t/5)}$. Same result for the time series in a)

```
#We get the time series value at time t=45 of th series x_t= A sin(t/5)+w_t, where
# where w_t= N(0,1)
print(x[45])
#We get the time series value at time t=45 of th series x_t= A sin(t/5)+w_t
# where w_t= N(0,16)
print(x_2[45])
#The expected value for both time series is 10*sin(t/5)
print(10*sin(45/5))
```

time series with $w_t \sim N(0,1)$ at $t=45$

time series with $w_t \sim N(0,16)$ at $t=45$

Expected value for both time series.

```
> #We get the time series value at time t=45 of th series x_t= A sin(t/5)+w_t, where
> # where w_t= N(0,1)
> print(x[45])
[1] 4.236915
> #We get the time series value at time t=45 of th series x_t= A sin(t/5)+w_t
> # where w_t= N(0,16)
> print(x_2[45])
[1] 9.791993
> #The expected value for both time series is 10*sin(t/5)
> print(10*sin(45/5))
[1] 4.121185
```

Result for time series with $w_t \sim N(0,1)$ at $t=45$

Result for time series with $w_t \sim N(0,16)$ at $t=45$

Result of expected value for both series.

Answer:

The observed values of both time series only differ by the white noise. So, the observed values at time $t=45$ are different as expected because the white noise variance of the second time series is much bigger than the variance of the first one.

The expected value is the same for both time series.

Expected value for both series

$$4.121185$$

time series with $w_t \sim N(0,1)$

$X_{45} = 4.236915$

$|E[X_{45}] - X_{45}| = 0.11573$

time series with $w_t \sim N(0,16)$

$X_{45} = 9.791993$

$E[X_{45}] - X_{45} = 5.670808$

As expected for the second series the difference to the expected value is greater than for the first series.

This is due to the bigger variance in the white noise.