HW 4

1)

a) we need that

$$\sum_{i=1}^{3} Pr(X=i) = 1$$

$$Pr(X=1) + Pr(X=2) + Pr(X=3) = 1$$

$$\theta_1 + 2\theta_1 + \theta_2 = 1$$

$$3\theta_1 + \theta_2 = 1 \Rightarrow \boxed{\theta_2 = 1 - 3\theta_1}$$

So, $\boxed{\theta = [\theta_1, 1 - 3\theta_1]}$

b)

$$Pr(D|\theta) = \prod_{i=1}^{N} Pr(x^{(i)}|\theta)$$

$$\log(Pr(D|\theta)) = \sum_{i=1}^{N} \log(Pr(x^{(i)}|\theta))$$

$$= \sum_{i=1}^{S_1} \log\left(\Pr\left(X^{(i)} = 1 \mid \theta\right)\right)$$

$$+ \sum_{i=1}^{S_2} \log\left(\Pr\left(X^{(i)} = 2 \mid \theta\right)\right)$$

$$+ \sum_{i=1}^{S_3} \log\left(\Pr\left(X^{(i)} = 3 \mid \theta\right)\right)$$

$$= \sum_{i=1}^{S_1} \log(\theta_1) + 2\sum_{i=1}^{S_2} \log \theta_1 + \sum_{i=1}^{S_3} \log(\underbrace{\theta_2}_{1-3\theta_1})$$

$$= S_1 \log(\theta_1) + 2 S_2 \log(\theta_1) + S_3 \log(1 - 3\theta_1)$$

$$= (S_1 + 2S_2)\log(\theta_1) + S_3 \log(1 - 3\theta_1)$$

c) $\frac{\partial}{\partial \theta_1} \log\left(P_1(D|\theta)\right)$

$$\frac{\partial}{\partial \theta_1}\left[(S_1 + 2S_2)\log(\theta_1) + S_3 \log(1-3\theta_1)\right]$$

$$= \frac{S_1+2S_2}{\theta_1} - \frac{3S_3}{1-3\theta_1} = 0$$

$$\Rightarrow \frac{S_1+2S_2}{\theta_1} = \frac{3S_3}{1-3\theta_1}$$

$$\Rightarrow \frac{1-3\theta_1}{\theta_1} = \frac{3S_3}{S_1+2S_2}$$

$$\frac{1}{\theta_1} - 3 = \frac{3S_3}{S_1+2S_2} \Rightarrow$$

$$\frac{1}{\theta_1} = \frac{3S_3}{S_1+2S_2} + 3 \Rightarrow$$

$$\boxed{\hat{\theta}_1 = \frac{1}{\frac{3S_3}{S_1+2S_2} + 3}}$$

And $\hat{\theta}_2 = 1 - 3\hat{\theta}_1$

So, $\boxed{\hat{\theta} = \left[\frac{1}{\frac{3S_3}{S_1+2S_2}+3} , 1 - \frac{1}{\frac{S_3}{S_1+2S_2}+1} \right]}$

2)

$$\Pr(D|\beta) = \prod_{i=1}^{N} f(x^{(i)}|\beta)$$

$$= \frac{1}{\beta^N} \exp\left(-\frac{1}{\beta}\sum_{i=1}^{N} x_i\right)$$

$\Rightarrow$

$$\log\left(\Pr(D|\beta)\right) = -N\log(\beta) - \frac{1}{\beta}\sum x_i$$

$$\frac{\partial}{\partial\beta}\left(\log\left(\Pr(D|\beta)\right)\right) =$$

$$-\frac{N}{\beta} + \frac{1}{\beta^2}\sum x_i = 0 \Rightarrow$$

$$\frac{1}{\beta^2}\sum x_i = \frac{N}{\beta} \Rightarrow$$

$$\boxed{\hat{\beta} = \frac{1}{N}\sum x_i}$$

3)

$$p(\theta|D) = \frac{p(D|\theta)\, p(\theta)}{p(D)}$$

where $p(\theta) = \begin{cases} \frac{1}{1-0}, & \text{if } 0 < \theta \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$$p(x^{(i)}|\theta) = \begin{cases} \theta & \text{if } x = \text{defective} \\ 1-\theta & \text{if } x = \text{non defective} \end{cases}$$

$$p(D|\theta) = \prod_{i=1}^{30} p(x^{(i)}|\theta)$$

$$= (\theta)^5 (1-\theta)^{25}$$

So,

$$p(\theta|D) = \frac{(\theta)^5 (1-\theta)^{25} \cdot 1}{p(D)}$$

$$p(\theta|D) = \frac{\theta^5 (1-\theta)^{25}}{p(D)}$$

proportional

$$\boxed{p(\theta|D) \propto \theta^5 (1-\theta)^{25} \quad \text{Eq 1.}}$$

where the proportionality constant is

$$p(D) = \int_0^1 \theta^5 (1-\theta)^{25} \, d\theta \quad \text{Eq 2.}$$

The probability density function of a beta distribution is given by

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1} \, du} \quad \text{Eq 3.}$$

So, comparing what we obtained in Eq 1, Eq 2 and 3; we notice we obtained a beta distribution

$$\boxed{p(\theta \mid D) = \text{Beta}(6, 26)}$$

b)

$$\Pr(x^{(i)} \mid \mu, 1) = N(\mu, 1)$$

$$\Rightarrow$$

$$\Pr(\{x^{(1)}, \ldots, x^{(n)}\} \mid \mu, \sigma^2) = \prod_{i=1}^{n} N(\mu, 1)$$

$$\Pr(\mu) = \mathcal{N}(0,1)$$

$$\Pr(\mu \mid \{x^{(i)}\}) = \frac{\Pr(x^{(i)} \mid \mu, 1)\Pr(\mu)}{\Pr(\{x^{(i)}\})}$$

$$= \frac{\left(\prod_{i=1}^{n} \mathcal{N}(\mu,1)\right)\mathcal{N}(0,1)}{\Pr(\{x^{(i)}\})}$$

$$\propto \prod_{i=1}^{n} \exp\left(-\frac{(x_i-\mu)^2}{2}\right)\exp\left(-\frac{\mu^2}{2}\right)$$

$$\propto \exp\left(-\frac{1}{2}\sum(x_i-\mu)^2 - \frac{\mu^2}{2}\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\sum(x_i^2 - 2x_i\mu + \mu^2) + \mu^2\right]\right)$$

$$\propto \exp\left(-\frac{1}{2}\left(\sum x_i^2 - 2\mu\sum x_i + (n+1)\mu^2\right)\right)$$

$$\propto \exp\left(-\frac{1}{2}\left(\underbrace{(n+1)}_{A}\mu^2 - 2\underbrace{\sum x_i}_{B}\mu + \underbrace{\sum x_i^2}_{C}\right)\right)$$

$$\propto \exp\left(-\frac{1}{2}\left(A\mu^2 - 2B\mu + C\right)\right)$$

$$\propto \exp\left(-\frac{1}{2}A\left(\mu^2 - 2\frac{B}{A}\mu + \frac{C}{A}\right)\right)$$

$$\propto \exp\left(-\frac{1}{2}A\left(\mu - \frac{B}{A}\right)^2 + \text{const}\right)$$

$$\sim N\left(\frac{B}{A}, \frac{1}{A}\right)$$

where $\frac{B}{A} = \frac{\sum x_i}{n+1}$, $\frac{1}{A} = \frac{1}{n+1}$

$$\boxed{\therefore\ p(\mu \mid \text{Data}) = N\left(\frac{\sum x_i}{n+1}, \frac{1}{n+1}\right)}$$