1)

$$\ell(\theta) = \frac{1}{2}\sum_{i=1}^{N}\left(y_i - \exp(\theta x_i)\right)^2$$

$$\ell'(\theta) = -\sum_{i=1}^{N}\left(y_i - \exp(\theta x_i)\right) \cdot x_i \exp(\theta x_i)$$

$$\ell''(\theta) = -\sum_{i=1}^{N}\Big[\left(-x_i \exp(\theta x_i)\right) x_i \exp(\theta x_i)$$

$$+ \left(y_i - \exp(\theta x_i)\right) \exp(\theta x_i)$$

$$+ \left(y_i - \exp(\theta x_i)\right) x_i^2 \exp(\theta x_i)\Big]$$

$$= -\sum_{i=1}^{N}\Big[-x_i^2 \exp(\theta x_i)^2 + \left[y_i - \exp(\theta x_i)\right]\exp(\theta x_i)$$

$$+ x_i^2\left(y_i - \exp(\theta x_i)\right)\exp(\theta x_i)\Big]$$

So the algorithm is

$$\theta(t) = \theta(t-1) - \frac{\ell'(\theta)}{\ell''(\theta)}$$

where $\ell'(\theta)$ and ℓ'' are shown above

2) $\hat{\theta} = 0.99025516$

3) We take as model

$$y_i = \exp(\hat{\theta} x_i) + \sigma \varepsilon$$

where σ was estimated with $\sigma^2 = \frac{(\sum y_i - e^{\theta x_i})^2}{N-1}$

$\sigma = 1.005169$ and ε is an error with standard normal distribution. We generate a 100 samples of y_i.

Then we apply Newton-Raphson and obtain a $\hat{\theta}^*$. We repeat this process 100 times. So we obtain 100 $\hat{\theta}^*$. We take the sample variance of these $\hat{\theta}^*$; which will be the approximation for $Var(\hat{\theta})$

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```r


for(kk in 1:100)
{

for (i in 1:100)
  {
  yyy[i]<-exp(t*x[i])+sigma*rnorm(1)
 }


 # initial value of theta
 tt[1]<-1

 k<-2
 difference<-1

 while(k<=1000)
 {

   u=-sum((yyy-exp(tt[k-1]*x))*x*exp(tt[k-1]*x))

   p1= -1*x*x*exp(tt[k-1]*x)*exp(tt[k-1]*x)

   p2=(yyy-exp(tt[k-1]*x))*exp(tt[k-1]*x)
   p3=x*x*(yyy-exp(tt[k-1]*x))*exp(tt[k-1]*x)
   d=-sum(p1+p2+p3)
   #estimation of theta by Newton Raphson
   tt[k]<-tt[k-1]-u/d

   difference<-abs(tt[k]-tt[k-1])
   k<- k+1
 }

 m_x<-mean(x)
 print(m_x)
 b[kk]<-tt[k-1]
 t_x[kk]<-tt[k-1]*m_x

 }
 variance_t<-var(b)
 variance_t_x<-var(t_x)

 print(b)
 print(variance_t)
```

generate samples for y_i

initial value for θ

store 100 θ's

obtain variance for those thetas

4) $Var(\hat{\theta}) \simeq 0.0001007871$

5) $\ln(\hat{y}) = \hat{\theta}\bar{x}$

$$\hat{\theta} \pm C_{\alpha/2}\, SD(\hat{\theta}^* x_1, \dots, \hat{\theta}^* x_{100})$$

$$= -0.08405655 \pm 0.001735513$$

```
variance_t<-var(b)
variance_t_x<-var(t_x)

print(b)
print(variance_t)
print(variance_t_x)
print(t*m_x)
print(sqrt(variance_t_x)*abs(qnorm(0.025)))
```

$C_{\alpha/2}\, SD(\hat{\theta}^* x_1, \dots \hat{\theta} x_n)$

Problem 2 $\quad y_i = m(\theta, x_i) + \sigma \varepsilon_i$

$$l(\theta) = \frac{1}{2} \sum_{i=1}^{N} \left(y_i - m(\theta, x_i)\right)^2$$

1) $$l'(\theta) = \frac{\partial l}{\partial \theta} = \cancel{\frac{1}{2}} \sum_{i=1}^{N} \cancel{2} \left(y_i - m(\theta, x_i)\right) \frac{\partial m}{\partial \theta}$$

$$\boxed{l'(\theta) = \sum_{i=1}^{N} \left(y_i - m(\theta, x_i)\right) \frac{\partial m}{\partial \theta}}$$

2) $$E\left(l'(\theta)\right) = E\left(\sum_{i=1}^{N} \left(y_i - m(\theta, x_i)\right) \frac{\partial m}{\partial \theta} \right)$$

Given that there is no randomness in θ and x_i, the expected value only affects the y_i (which are random) $\Rightarrow$

$$E(l(\theta)) = \frac{\partial m}{\partial \theta} \sum_{i=1}^{N} E[y_i - m(\theta, x_i)]$$

$$= \frac{\partial m}{\partial \theta} \sum_{i=1}^{N} \left(E[y_i] - m(\theta, x_i)\right)$$

$$= \frac{\partial m}{\partial \theta} \sum_{i=1}^{N} \left(m(\theta, x_i) - m(\theta, x_i)\right) = 0$$

where we used the linearity of the expectation and that $E[m(\theta, x_i)] = m(\theta, x_i)$

$$E[y_i] = m(\theta, x_i)$$

3) $$\hat{\theta}-\theta \approx \frac{l'(\theta)}{l''(\theta)}$$

$$l'(\theta) = \sum_{i=1}^{N} (y_i - m(\theta, x_i)) \frac{\partial m}{\partial \theta}$$

$$l''(\theta) = \sum_{i=1}^{N} \left[\left(-\frac{\partial m}{\partial \theta}\right)\left(\frac{\partial m}{\partial \theta}\right) + \frac{\partial^2 m}{\partial \theta^2}(y_i - m(\theta, x_i)) \right]$$

$$= \sum_{i=1}^{N} \left[\frac{\partial^2 m}{\partial \theta^2}(y_i - m(\theta, x_i)) - \left(\frac{\partial m}{\partial \theta}\right)^2 \right]$$

$$\Rightarrow \quad \hat{\theta}-\theta \approx \frac{\sum_{i=1}^{N} (y_i - m(\theta, x_i)) \frac{\partial m}{\partial \theta}}{\sum_{i=1}^{N} \left[\frac{\partial^2 m}{\partial \theta^2}(y_i - m(\theta, x_i)) - \left(\frac{\partial m}{\partial \theta}\right)^2 \right]}$$

4) We can get the mean using the approximation

$$E[\hat{\theta}-\theta] = \frac{E[l'(\theta)]}{l''\theta} = 0$$

because $E[l'(\theta)] = 0$. Using linearity of expectation $\Rightarrow$

$$E[\hat{\theta}] - E[\theta] = 0 \Rightarrow$$

$$\boxed{E[\hat{\theta}] = \theta}$$

5) $$Var(\hat{\theta} - \theta) \approx Var\left(\frac{l'(\theta)}{l''(\theta)}\right)$$

using the approximation that there is no randomness in $l''(\theta)$ (we consider it as constant) so

$$Var(\hat{\theta}) - \cancelto{0}{Var(\theta)} \approx \frac{Var(l'(\theta))}{l''(\theta)^2}$$

$$\mathrm{Var}(\hat{\theta}) \approx \frac{\mathrm{Var}(l'(\theta))}{l''(\theta)^2}$$

$$\mathrm{Var}(l'(\theta)) = \mathrm{Var}\left[\sum_{i=1}^{N} (y_i - m(\theta, x_i)) \frac{\partial m}{\partial \theta}\right]$$

$$= \left(\frac{\partial m}{\partial \theta}\right)^2 \underbrace{\mathrm{Var}\left[\sum_{i=1}^{N} (y_i - m(\theta, x_i))\right]}_{\sigma^2}$$

$$= \left(\frac{\partial m}{\partial \theta}\right)^2 \sigma^2$$

$$\Rightarrow \mathrm{Var}(\hat{\theta}) \approx \frac{\sigma^2 \left(\frac{\partial m}{\partial \theta}\right)^2}{l''(\theta)^2}$$

where

$$l''(\theta) = \sum_{i=1}^{N} \left[\frac{\partial^2 m}{\partial \theta^2} (y_i - m(\theta, x_i)) - \left(\frac{\partial m}{\partial \theta}\right)^2\right]$$