# HOMEWORK 3

## Problem 1

Given the provided dataset, consider the model regressing $y$ on $x$ and $m$. Conduct a Sobel test to see whether there is a mediation effect. You need to consider the three relevant models,

$$y_i = \beta_{11} + \beta_{12}x_i + \epsilon_i,$$
$$m_i = \beta_{21} + \beta_{22}x_i + \epsilon_i,$$

and

$$y_i = \beta_{31} + \beta_{32}m_i + \beta_{33}x_i + \epsilon_i.$$

Suppose the variance of $\epsilon_i$ is known with $\sigma^2 = 1$ for all the three models. Use the level of significance $\alpha = 0.05$

(1) (4 pts) What is the null hypothesis of the Sobel test?

(2) (8 pts) Regress the three models and find the estimators.

(3) (10 pts) Find the variance of $\hat{\beta}_{22}$ and $\hat{\beta}_{32}$.

(4) (10 pts) Find the $z$ test statistic.

(5) (8 pts) Find the $p$-value and the conclusion of the test.

## Problem 2

Consider the usual linear model,

$$y_i = x_i\beta + \sigma\epsilon_i,$$

where $\sigma$ is unknown and $\sum_{i=1}^{n} x_i^2 = 1$.

(1) (12 pts) Show that the least square estimator $\hat{\beta} = \sum_{i=1}^{n} x_i y_i$, and write this as $z$.

(2) (12 pts) Show that the residual sum of squares is given by

$$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} y_i^2 - z^2.$$

(3) (12 pts) Hence, show that the $F$ statistic for testing $H_0 : \beta = 0$ is given by

$$F = \frac{z^2(n-1)}{\sum_{i=1}^{n} y_i^2 - z^2}.$$

(4) (12 pts) If the null hypothesis is true, what is the distribution of $z$?

(5) (12 pts) Hence, show that the Student–$t$ test statistic for testing the null hypothesis is

$$T = \frac{z}{\hat{\sigma}}, \quad \text{with} \quad \hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

and confirm that the $F$ test and the $T$ test are the same.