Master's Thesis: Orbital Stability of Standing Waves in a Nonlinear Maxwell-Schrödinger System

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March 14, 2023 - June 5, 2023

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Part I

Introduction

0.1. Setting of the problem

The aim of this master's thesis is to follow the path taken in [5] to study the following problem in a three dimensional space, and begin to adapt it for the problem in a two dimensional space.
We consider a nonlinear Schrödinger equation coupled with Maxwell equation, given by

$$i\psi_t + \Delta\psi = e\phi\psi + e^2 |\mathbf{A}|^2 \psi + ie\psi \operatorname{div}\mathbf{A} + 2ie\nabla\psi \cdot \mathbf{A} - |\psi|^{p-1}\psi, \tag{I.0.1}$$

$$\mathbf{A}_{tt} - \Delta\mathbf{A} = e \operatorname{Im}(\overline{\psi}\, \nabla\psi) - e^2|\psi|^2\mathbf{A} - \nabla\phi_t - \nabla \operatorname{div}A, \tag{I.0.2}$$

$$-\Delta\phi = \frac{e}{2}|\psi|^2 + \operatorname{div}\mathbf{A}, \tag{I.0.3}$$

where $\psi : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{C}$, $\mathbf{A} : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\phi : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, $e \in \mathbb{R}$, $1 < p < 3$, and i denotes the unit complex number.
As explained in [5], the system (I.0.1)-(I.0.3) describes the interaction of the Schrödinger wave function ψ with the gauge potential (A, ϕ). The constant e describes the strength of the interaction. When $e = 0$, the equation (I.0.1) reduces to the standard nonlinear Schrödinger equation :

$$i\psi_t + \Delta\psi + |\psi|^{p-1}\psi = 0 \tag{I.0.4}$$

for which the orbital stability of standing waves has been studied for example in [6] and [7].
From here onward, we will impose the Coulomb condition, that is to say we look for a solution $\mathbf{A}$ which satisfies

$$\operatorname{div}\mathbf{A} = 0. \tag{I.0.5}$$

(I.0.3) is then reduced to $-\Delta\phi = \frac{e}{2}|\psi|^2$, and we can express ϕ explicitly using Poisson's fundamental solution for the 2-dimensional space :

$$\phi = \frac{e}{2}(-\Delta)^{-1}|\psi|^2 = -\frac{e}{4\pi} \log|x| * |u|^2. \tag{I.0.6}$$

This is where divergence between the case for dimension $N = 2$ and the case for dimension $N = 3$ begins to appear. Indeed, for $N = 3$, ϕ would be given by $\phi = \frac{e}{8\pi|x|} * |u|^2$ and would always be defined for u in L^2. In the present case however, some conditions on u must be

imposed in order to ensure that the definitions to come are meaningful, see (I.0.9).
We denote by $D^{1,2}(\mathbb{R}^2)$ the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm $\|u\|^2_{D^{1,2}} = \|\nabla u\|^2_{L^2}$. For $\omega > 0$ and $(u, \phi) \in H^1_{\log}(\mathbb{R}^2, \mathbb{C}) \times D^{1,2}(\mathbb{R}^2, \mathbb{R})$, we consider the standing wave for (I.0.1)-(I.0.3) of the form :

$$\psi(x,t) = u(x)e^{i\omega t}, \text{ with } \mathbf{A}(x,t) = \mathbf{0} \text{ and } \phi(x,t) = \phi(x). \tag{I.0.7}$$

This condition turns (I.0.1)-(I.0.3) into the elliptic system

$$\begin{cases} -\Delta u + \omega u + e\phi u - |u|^{p-1}u = 0, \\ -\Delta \phi = \frac{e}{2}|u|^2. \end{cases} \tag{I.0.8}$$

We aim at studying this system from the perspective of the calculus of variations. For this, we define the functional $E : \mathbb{H}^1(\mathbb{R}^2, \mathbb{C}) \times D^{1,2}(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}$ by

$$E_{e,\omega}(u, \phi) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + \omega|u|^2 + e\phi|u|^2 - |\nabla\phi|^2 \, dx - \int_{\mathbb{R}^2} \frac{1}{p+1}|u|^{p+1} \, dx.$$

We then see that if (u, ϕ) is a critical point of E, then it satisfies (I.0.8). Indeed, let $v = (v_1, v_2)^T$ with $v_1 \in C_c^\infty(\mathbb{R}^2, \mathbb{C})$ and $v_2 \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$, and $\varepsilon \in \mathbb{R}$. We have

$$\begin{aligned} |\nabla(u + \varepsilon v_1)|^2 &= |\nabla u|^2 + 2\varepsilon \mathcal{R}e\left(\nabla u \overline{\nabla v_1}\right) + \varepsilon^2|\nabla v_1|^2 \\ |u + \varepsilon v_1|^2 &= |u|^2 + 2\varepsilon \mathcal{R}e\left(u\overline{v_1}\right) + \varepsilon^2|v_1|^2 \\ |\nabla(\phi + \varepsilon v_2)|^2 &= |\nabla\phi|^2 + 2\varepsilon\nabla\phi\nabla v_2 + \varepsilon^2|\nabla v_2|^2 \\ \left(\frac{\partial}{\partial\epsilon}\left(\frac{|u + \varepsilon v_1|^{p+1}}{p+1}\right)\right)_{\varepsilon=0} &= |u|^{p-1}uv_1 \end{aligned}$$

Then we formally compute, in the distribution sense and with an integration by part,

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{E_{e,\omega}\left(\begin{pmatrix} u \\ \phi \end{pmatrix} + \varepsilon\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) - E_{e,\omega}\begin{pmatrix} u \\ \phi \end{pmatrix}}{\varepsilon} \\ &= \int_{\mathbb{R}^2} \mathcal{R}e\left(\nabla u \cdot \overline{\nabla v_1}\right) + \omega\mathcal{R}e\left(u\overline{v_1}\right) + e\phi\mathcal{R}e\left(u\overline{v_1}\right) + \frac{|u|^2}{2}v_2 - \nabla\phi\nabla v_2 - |u|^{p-1}uv_1 \, dx \\ &= \mathcal{R}e\int_{\mathbb{R}^2} -\Delta u\overline{v_1} + \omega u\overline{v_1} + e\phi u\overline{v_1} + \frac{|u|^2}{2}v_2 + \Delta\phi v_2 - |u|^{p-1}uv_1 \, dx \\ &= \mathcal{R}e\int_{\mathbb{R}^2} \begin{pmatrix} -\Delta u + \omega u + e\phi u - |u|^{p-1}u \\ \frac{|u|^2}{2} + \Delta\phi \end{pmatrix} \cdot \begin{pmatrix} \overline{v_1} \\ v_2 \end{pmatrix} \, dx. \end{aligned}$$

We obtain from the fundamental lemma of the calculus of variations what we announced. It is known that such a functional is not bounded from above nor from below, which makes it difficult to handle. However, it is possible to adopt the reduction method from [8] or [9]. Hence, we define

$$S(u) = (-\Delta)^{-1}\left(\frac{1}{2}|u|^2\right) = \frac{-\log|x|}{2\pi} * \left(\frac{1}{2}|u|^2\right) = \frac{1}{4\pi}\int_{\mathbb{R}^2} -\log|x-y||u(y)|^2 \, dy.$$

We get $\phi = eS(u)$ and we define the action I as

$$\begin{aligned}I_{e,\omega}(u) &= E_{e,\omega}(u, eS(u))\\ &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^2} \omega|u|^2\, dx + \frac{e^2}{4}\int_{\mathbb{R}^2} S(u)|u|^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\, dx.\end{aligned}$$

In order to give meaning to this functional, we will have to restrain our study to the subset of functions in $H^1(\mathbb{R}^2)$ for which

$$A(u) := \int_{\mathbb{R}^2} S(u)|u|^2\, dx = -\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y||u(x)|^2|u(y)|^2\, dydx \tag{I.0.9}$$

is finite. We will denote this subset by $H^1_{\log}$, and we will work with $u \in H^1_{\log}$ from now on. We then see that if $u \in H^1_{\log}$ is a critical point of $I(u)$, then $(u, eS(u))$ is a solution of I.0.8, and a sufficient condition for such u is to verify the Euler-Lagrange equation for I given by

$$-\Delta u + \omega u + e^2 S(u)u - |u|^{p-1}u = 0. \tag{I.0.10}$$

In order to work on the orbital stability of (I.0.8), we must reach several intermediary steps. The first step in the present study is to prove that there exists a solution of (I.0.10) $\in H^1_{\log}$. For this, we first proceed as in [5], by defining

$$J_e(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx + \frac{e^2}{4}\int_{\mathbb{R}^2} S(u)|u|^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\, dx \tag{I.0.11}$$

and for $\mu > 0$,

$$B(\mu) = \{u \in H^1_{\log},\ G(u) := \|u\|^2_{L^2} = \mu\}. \tag{I.0.12}$$

We will consider the minimization problem : does a $u \in H^1_{\log}$ exist such that

$$c_e(\mu) = \inf_{v\in B(\mu)} J_e(v) = J_e(u). \tag{I.0.13}$$

Indeed, a critical point u of the functional J_e under the constraint $B(\mu)$ verifies the Lagrange multiplier rule, which states there exists $\tilde{\omega} \in \mathbb{R}$ such that

$$J'_e(u) + \tilde{\omega} G'(u) = 0. \tag{I.0.14}$$

where $J'_e(u)$ and $G'(u)$ denote respectively the Euler-Lagrange operator applied to J_e and G, taken at u.
Therefore, we have built an easier minimization problem than the one for $E_{e,\omega}$ where the frequency $\tilde{\omega}$ appears as a Lagrange multiplier associated to the constraint $\|u\|^2_{L^2} = \mu$. However, the Lagrange multiplier $\tilde{\omega}$ obtained in (I.0.14) is not necessarily equal to ω in (I.0.10), and further work must be done in order to prove that $\omega = \tilde{\omega}$.
To solve the minimization problem (I.0.13), we aim to apply the principle of *concentration-compactness* to $c_e(\mu)$ in order to prove the relative compactness of its minimizing sequences. Once this is done, we therefore have proven that there exists a solution $u_{e,\omega}$ to (I.0.10) and therefore to (I.0.8) for given e and ω. Such a solution is called a *bound state*, and by construction, $\psi_{e,\omega}(t,x) = u_{e,\omega}(x)e^{iwt}$ is a standing wave solution of (I.0.1)-(I.0.3) under the conditions

(I.0.7).
The next step would be to prove that among all bound states of (I.0.10), there exists one that minimizes the energy functional $I_{e,\omega}$. This problem would once again be solved via an $\|u\|_{L^2}$-constrained minimization problem, which will lead to another study on the Lagrange multiplier obtained. Such a solution $u_{e,\omega}$ is called a *ground state*.
Finally, we can start studying the orbital stability of this standing wave $(\phi_{e,\omega}, \mathbf{A}_{\mathbf{e},\omega} = \mathbf{0}, \phi_{\mathbf{e},\omega})$ of (I.0.1)-(I.0.3) in the sense that for an initial value sufficiently close to this standing wave, the solution obtained of (I.0.1)-(I.0.3) remains close to the behaviour of $(\phi_{e,\omega}, \mathbf{A}_{\mathbf{e},\omega} = \mathbf{0}, \phi_{\mathbf{e},\omega})$. At this point, it can be convenient to obtain that the ground state is unique up to phase shift and translation. Indeed, uniqueness of the ground state up to phase shift and translation then rules out the possibility of the solution to orbit around several ground states, which would make the problem of stability harder to tackle.

0.2. Existence of a real valued radially symmetric ground state for $N \geq 3$

This internship started with the reading of [1] and [2]. In these articles, a more general setting than (I.0.10) is considered, but the dimension N is supposed to be atleast 3.
We look for a standing wave to the equation

$$\psi_{tt} - \Delta\psi + e^2\psi - f(\psi) = 0 \tag{I.0.15}$$

where $\psi = \phi(t,x) = u(x)e^{i\omega t}$ is a complex function defined on $\mathbb{R} \times \mathbb{R}^N$ with $u : \mathbb{R}^n \to \mathbb{R}$ and $e \in \mathbb{R}$ is a given constant. $f : \mathbb{R} \to \mathbb{R}$ is assumed to be a real, continuous, odd function verifying $f(0)$=0. This leads to the study of the equation

$$-\Delta u + mu - f(u) = 0 \tag{I.0.16}$$

where $m = e^2 - \omega^2$.
We remark that our problem (I.0.10) fits this description, except for the dimension N. Indeed, since $z \mapsto S(z)$ is even, then $f : z \mapsto |z|^{p-1}z - S(z)z$ is an odd function verifying $f(0) = 0$.
The action corresponding to equation (I.0.16) is wanted to be finite, hence we require $u \in H^1$.
We then express (I.0.16) as the semilinear elliptic problem in $\mathbb{R}^N$

$$\begin{cases} -\Delta u = g(u) \\ u \in H^1(\mathbb{R}^N), \quad u \not\equiv 0 \end{cases} \tag{I.0.17}$$

where $g : \mathbb{R} \to \mathbb{R}$ is an odd continuous function verifying $g(0) = 0$.
The goal of the first article is to seek for solutions to (I.0.17) that are spherically symmetric. Under this restriction, $u = u(r)$, with $r = |x|$, verifies the ordinary differential equation

$$-\frac{d^2u}{dr^2} - \frac{N-1}{r}\frac{du}{dr} = g(u). \tag{I.0.18}$$

We then set

$$G(u) = \int_0^u g(z)\, dz, \qquad V(u) = \int_{\mathbb{R}^N} G(u)\, dx, \qquad T(u) = \int_{\mathbb{R}^N} |\nabla u|^2\, dx. \tag{I.0.19}$$

Pohozaev's identity for (I.0.17) (which we present for our problem in (2.0.8)) is used to show that a solution u to this equation has a positive action, that is, $\frac{1}{2}T(u) - V(u) > 0$. We note

that the action for our problem is given by the functional I.
In order to find a solution to (I.0.17), we consider the constrained minimization problem : find $u \in H^1(\mathbb{R}^N)$ such that

$$\inf\{T(v) \,|v \in H^1(\mathbb{R}^N) \text{ and } V(v) = 1\} = T(u). \tag{I.0.20}$$

With additional conditions on the growth of g near 0 and at infinity, and the condition that there exists a $z > 0$ such that $G(z) > 0$, it is then shown that the solution obtained from this minimization problem happens to be a ground state for (I.0.17). It is also possible to show that a ground state for (I.0.17) is necessarily positive and radially symmetric, therefore we have at our disposal powerful lemmas on spherically symmetric functions for further study of these solutions. Most noticeably, a spherically symmetric function u solution to (I.0.17) is shown to have exponential decay at infinity, as well as its first and second derivatives. No general result at the time of publication of [1] for uniqueness of a ground state is available.
The reasoning shown in this paper is however not valid for the case $N = 2$ because it is not possible using Sobolev embeddings, to obtain from a bound of $\|\nabla u\|_{L^2}$ a bound on $\|u\|_{L^{l+1}}$, with l being the sobolev conjugate exponent $l = \frac{2N}{N-2}$.
In the following article [2], the reasoning in [1] is built upon in order to seek for spherically symmetric solutions of (I.0.17) that are bound states but not ground states. It is shown that (I.0.17) posses infinitely many distinct bound states that are spherically symmetric, and that there exists a sequence $(u_n)_n$ of such bound states for which the action of u_n goes to $+\infty$ as n goes to $+\infty$.
Reading these articles has given more context on our present problem, shown what type of result we can expect, and highlighted a few difficulties we can come across when treating the case $N = 2$.

0.3. The concentration-compactness principle

A portion of the internship has been dedicated to the understanding of the principle of *concentration-compactness* for which a presentation has been given by P.-L. Lions in [3] and [4]. It is a tool he designed to solve minimization problems in unbounded domains which we aim to use for our present problem exposed in the previous section.
This principle cannot be stated as an universal theorem because the setting must be adjusted case-by-case. However, it revolves around a general principle giving an equivalence between the compactness of all minimizing sequences and some conditions of strict sub-additivity of a functional.
The type of problem this tool is designed for is the following : let H be a function space on $\mathbb{R}^n$, and J, G be two functional defined on H taking the form

$$J(u) = \int_{\mathbb{R}^N} j(x, Pu(x))\, dx, \qquad G(u) = \int_{\mathbb{R}^N} g(x, Qu(x))\, dx$$

where $j : \mathbb{R}^N \times R^m \to \mathbb{R}$, $g : \mathbb{R}^N \times R^n \to \mathbb{R}$, g is non-negative and to simplify, we assume $G(0) = 0$. $P : H \to E$, $Q : H \to F$ are potentially nonlinear operators that commute with translations of $\mathbb{R}^N$, for E and F function spaces defined on $\mathbb{R}^N$ with values in $\mathbb{R}^m$ and $\mathbb{R}^n$.
We are interested in the minimization problem

$$c(1) = \inf\, \{J(u) \mid u \in H \text{ and } G(u) = 1\}. \tag{M}$$

which we embed in the family of problems defined for $\mu > 0$.

$$c(\mu) = \inf\{J(u) \mid u \in H \text{ and } G(u) = \mu\}. \tag{M_μ}$$

We note that it is possible to adapt the principle to the minimization problem $c(0)$ instead of $c(1)$. Now, for the idea to work, we must assume that the problem can be also defined «at infinity». It isn't possible to provide a more specific definition to this statement as such setting is to be adapted case-by-case. However, a simple way to describe it can be the possibility to define j^∞, $J^\infty(u) = \int_{\mathbb{R}^N} j^\infty(u(x))\,dx$, g^∞, $G^\infty(u) = \int_{\mathbb{R}^N} g^\infty(u(x))\,dx$ such that for all $p \in \mathbb{R}^m$, $q \in \mathbb{R}^n$,

$$j(x,p) \underset{|x|\to+\infty}{\longrightarrow} j^\infty(p) \quad \text{and} \quad g(x,q) \underset{|x|\to+\infty}{\longrightarrow} g^\infty(q).$$

This leads us to define as well the problem

$$c^\infty(\mu) = \inf\{J^\infty(u) \mid u \in H \text{ and } G^\infty(u) = \mu\}. \tag{M_μ^∞}$$

Finally, some conditions for (M_μ) and (M_μ^∞) will be needed. The precise conditions are established case-by-case. Among these lie notably the condition that for all $\mu > 0$ or for all $\mu \in (0,1]$,

$$\mathcal{G}_\mu = \{u \in H \mid G(u) = \mu\} \neq \varnothing \quad \text{and} \quad c(\mu) > -\infty,$$

and that minimizing sequences for (M_μ) and (M_μ^∞) are bounded in H.

In view of adapting the principle to our current problem, we would set $P, Q = \mathrm{Id}$ and

$$\begin{cases} j(x,z) = \frac{1}{2}|\nabla z|^2 + e^2 S(z)|z|^2 - \frac{1}{p+1}|z|^{p+1} = j^\infty(z), \\ g(x,z) = |z|^2 = g^\infty(z). \end{cases} \tag{I.0.21}$$

As j and g do not depend on x in our case, (M_μ) and (M_μ^∞) are equivalent and we can set aside the problem «at infinity».

The concentration-compactness principle aims at proving a result of the type

Theorem 1.0.1. *For each $\mu > 0$, all minimizing sequences of the problem (M_μ) are relatively compact up to a translation if and only if the following strict subadditivity condition is satisfied:*

$$c(\mu) < c(\alpha) + c(\mu - \alpha), \qquad \forall \alpha \in (0,\mu). \tag{I.0.22}$$

The *strict* inequality is what provides the necessary condition of the assertion. The sufficient condition is brought from a lemma that we will present later.

The sub-additivity

Let us now explain the sub-additivity condition in the context of our current minimization problem. The arguments given here are not rigorous. However, we present them as a general guideline on how to obtain the results we aim for.
First, we show how the loose inequality $c(\mu) \leq c(\alpha) + c(\mu - \alpha)$ is always verified. Indeed, let $\varepsilon > 0$ and u_ε, $v_\varepsilon \in H^1_{\log}$ such that

$$\begin{cases} c(\alpha) \leq J(u_\varepsilon) \leq c(\alpha) + \varepsilon & \text{and} \qquad G(u_\varepsilon) = \alpha, \\ c(\mu - \alpha) \leq J(v_\varepsilon) \leq c(\mu - \alpha) + \varepsilon & \text{and} \qquad G(v_\varepsilon) = \mu - \alpha \end{cases} \tag{I.0.23}$$

is verified for some α such that $0 \leq \alpha < \mu$. As $C_c^\infty(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, we can suppose that u_ε and v_ε have compact support. We denote the translation by $n\chi$ of v_ε for some given $\chi \in \mathbb{S}^1$ and $n \in \mathbb{N}$ by $v_\varepsilon^n = v_\epsilon(\cdot + n\chi)$. Then the distance between $supp(u_\varepsilon)$ and $supp(v_\varepsilon^n)$ goes to $+\infty$ as n goes to $+\infty$, so there exists n_0 sufficiently large such that the two supports are disjoint for all $n \geq n_0$, and we obtain

$$\begin{cases} J(u_\varepsilon + v_\varepsilon^n) \underset{n \to +\infty}{\longrightarrow} (J(u_\varepsilon) + J(v_\varepsilon^n)), \\ G(u_\varepsilon + v_\varepsilon^n) \underset{n \to +\infty}{\longrightarrow} (G(u_\varepsilon) + G(v_\varepsilon^n)). \end{cases} \tag{I.0.24}$$

In addition, j and g do not depend on x, hence J and G are translation-invariant. This gives $\lim\limits_{n \to +\infty} J(v_\varepsilon^n) = J(v_\varepsilon)$ and we deduce

$$\begin{cases} c(\alpha) + c(\mu - \alpha) \leq \lim\limits_{n \to +\infty} (J(u_\varepsilon + v_\varepsilon^n)) = J(u_\varepsilon) + J(v_\varepsilon) \leq c(\alpha) + c(\mu - \alpha) + 2\varepsilon, \\ \lim\limits_{n \to +\infty} (G(u_\varepsilon) + G(v_\varepsilon^n)) = \mu. \end{cases} \tag{I.0.25}$$

Even though we did not need it here because j and g in our problem do not depend on x, the adaptation of the problem « at infinity » we mentioned earlier by defining j^∞ allows for the equality

$$\lim_{n \to +\infty} J(u_\varepsilon + v_\varepsilon^n) = J(u_\varepsilon) + J(v_\varepsilon^n)$$

to hold « at infinity » even when j does depend on x.

Here, we have found two functions u_ε and $v_\varepsilon^{n_0}$ verifying $G(u_\varepsilon + v_\varepsilon^{n_0}) = \mu$, so $(u_\varepsilon + v_\varepsilon^{n_0}) \in \mathcal{G}$. By definition of $c(\mu)$, we then have

$$c(\mu) \leq J(u_\varepsilon + v_\varepsilon^{u_0}) = c(\alpha) + c(\mu - \alpha) + 2\varepsilon.$$

This shows that for all $0 \leq \alpha < \mu$, we have

$$c(\mu) \leq c(\alpha) + c(\mu - \alpha). \tag{I.0.26}$$

The necessary condition

The condition (I.0.22) is necessary for the compactness of all minimizing sequences. Indeed, if the *strict* inequality isn't satisfied, then since (I.0.26) is always satisfied, there exists $\alpha \in (0, \mu)$ such that

$$c_e(\mu) = c_e(\alpha) + c_e(\mu - \alpha). \tag{I.0.27}$$

Let $(u_n)_n$ and $(v_n)_n$ be minimizing sequences with compact supports for problems (M_α) and $(M_{\mu-\alpha})$ respectively. Let $(\chi_n)_n \subset \mathbb{R}^2$ such that for $\overline{v_n} = v_n(\cdot + \chi_n)$, we have

$$\text{dist} \left(\text{supp } u_n, \text{ supp } \overline{v_n} \right) \underset{n \to +\infty}{\longrightarrow} +\infty \tag{I.0.28}$$

We then set $w_n = u_n + \overline{v_n}$. We observe that

$$J(\overline{v_n}) = J(v_n) \underset{n\to+\infty}{\longrightarrow} c_e(\mu-\alpha) \quad \text{and} \quad G(\overline{v_n}) \underset{n\to+\infty}{\longrightarrow} \mu-\alpha. \tag{I.0.29}$$

Moreover, for n_0 sufficiently large, we have that for all $n \geq n_0$, u_n and $\overline{v_n}$ have disjoint supports. Since J and G are invariant by translation, we obtain

$$\lim_{n\to+\infty} J(w_n) = \lim_{n\to+\infty} [J(u_n) + J(v_n)] = c_e(\alpha) + c_e(\mu-\alpha) = c_e(\mu) \tag{I.0.30}$$

and

$$G(w_n) \underset{n\to+\infty}{\longrightarrow} \mu. \tag{I.0.31}$$

Hence w_n is a minimizing sequence of (M_μ), but as $(\overline{v_n})_n$ is «slipping to infinity», w_n cannot be relatively compact. Therefore, the *strict* inequality is a necessary condition for the compactness of all minimizing sequences.
We note that the assertion is about *all* minimizing sequences. It does not rule out the possibility for a specific minimizing sequence to be relatively compact even if the strict inequality is not verified.

The sufficient condition

The sufficient condition in (1.0.1) is given by the following lemma.

Lemma 1.0.1. *Let $\mu > 0$ fixed. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \geq 0 \text{ in } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} \rho_n \, dx = \mu. \tag{I.0.32}$$

Then there exists a subsequence $(\rho_{n_k})_{k\in\mathbb{N}}$ satisfying one of the three following possibilities :

i) (compactness) there exists $y_k \in \mathbb{R}^N$ such that $\rho_{n_k}(\cdot + y_k)$ verifies

$$\text{for all } \varepsilon > 0, \text{ there exists } R < +\infty \text{ such that } \int_{y_k+B(0,R)} \rho_{n_k}(x) \, dx \geq \mu - \varepsilon, \tag{I.0.33}$$

ii) (vanishing)

$$\lim_{k\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{y_k+B(0,R)} \rho_{n_k}(x) \, dx = 0, \text{ for all } R < +\infty. \tag{I.0.34}$$

iii) (dichotomy) there exists $\alpha \in (0, \mu)$ such that for all $\varepsilon > 0$, there exist $k_0 \geq 1$ and $\rho_k^1, \rho_k^2 \in L^1_+(\mathbb{R}^N)$ satisfying for all $k \geq k_0$,

$$\begin{cases} \|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L^1} \leq \varepsilon \\ \left| \int_{\mathbb{R}^N} \rho_k^1 \, dx - \alpha \right| \leq \varepsilon \\ \left| \int_{\mathbb{R}^N} \rho_k^2 \, dx - (\mu - \alpha) \right| \leq \varepsilon \\ \mathrm{dist}(\mathrm{Supp}\ \rho_k^1,\ \mathrm{Supp}\ \rho_k^2) \underset{k\to+\infty}{\longrightarrow} +\infty. \end{cases} \tag{I.0.35}$$

This lemma is built on the notion of the concentration function of a measure, expressed by

$$Q_n(t) = \sup_{y\in\mathbb{R}^N} \int_{y+B(0,t)} \rho(x) \, dx. \tag{I.0.36}$$

$(Q_n)_n$ is a sequence of nondecreasing, nonnegative, uniformly bounded functions on $\mathbb{R}^+$. For $\rho_n \in L^1$ such that $\|\rho_n\|_{L^1} = \mu$, we have $\lim\limits_{t\to+\infty} Q_n(t) = \mu$. There also exists a subsequence $(n_k)_k$ and a nondecreasing nonnegative function Q such that $Q_{n_k}(t) \underset{k\to+\infty}{\longrightarrow} Q(t)$ for all $t \geq 0$. Also, there exists $\alpha \in [0, \mu]$ such that $\lim\limits_{t\to+\infty} Q(t) = \alpha$.
Lemma (1.0.1) is then based on the fact that if $\alpha = \mu$, «compactness» occurs. If $\alpha = 0$, «vanishing» occurs and if $\alpha \in (0, \mu)$, it is «dichotomy».
The sufficient condition in (1.0.1) comes from the fact that if the strict inequality (I.0.22) is verified, then it is possible to disqualify the cases «vanishing» and «dichotomy» from occuring, leaving only the option of the case «compactness».

Application of the principle

Let us show how we would apply concentration-compactness principle to our current problem once we would have obtained

$$c(\mu) < 0, \ \ c(\mu) > -\infty, \ \text{for all } \mu > 0 \tag{I.0.37}$$

and

$$c(\mu) < c(\alpha) + c(\mu - \alpha), \ \text{for } \mu > 0 \text{ and for all } \alpha \in (0, \mu). \tag{I.0.38}$$

Suppose that $(u_n)_n \in H^1_{\log}$ is a sequence such that $\|u_n\|^2_{L^2} \underset{n\to+\infty}{\longrightarrow} \mu$ and $J(u_n) \underset{n\to+\infty}{\longrightarrow} c(\mu)$. From the proof of lemma (3.0.9) leading to $c(\mu) > -\infty$, we observe that $(u_n)_n$ is a bounded sequence in $H^1_{\log}$. It is possible to replace u_n by $\frac{\sqrt{\mu}}{\|u_n\|_{L^2}} u_n$ so that $(u_n)_n \subset \mathcal{G}$, and consequently becomes a minimizing sequence of $c(\mu)$.
We now use lemma (1.0.1) with the sequence $\rho_n(x) = |u_n(x)|^2$ to show that our problem lies in the case «compactness». For this, we will rule out the two other cases.

Starting by supposing the «vanishing» case occurs, let us show a contradiction. There exists a subsequence of (ρ_n) that we will still denote by (ρ_n) such that

$$\lim_{k\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{y_k + B(0,R)} \rho_{n_k}(x)\, dx = 0, \ \text{for all } R < +\infty. \tag{I.0.39}$$

From [4, Lemma I.1, p.231], we have $u_n \underset{n\to+\infty}{\longrightarrow} 0$ in $L^q(\mathbb{R}^2)$ for any $2 < q < +\infty$. Notably, in $L^{p+1}(\mathbb{R}^2)$ as long as $p > 1$.
We also know that since (u_n) is a minimizing sequence for $c(\mu)$,

$$c(\mu) + R_n = J(u_n) = \frac{1}{2}\|\nabla u_n\|^2_{L^2} + \frac{e^2}{4}A(u_n) - \frac{1}{p+1}\|u_n\|^{p+1}_{L^{p+1}} \tag{I.0.40}$$

for some $(R_n)_n$ such that $R_n \underset{n\to+\infty}{\longrightarrow} 0$.
We then use the lemma (2.0.5) showing $A(u) \geq 0$ for $u \in H^1_{\log}$ to get

$$\begin{aligned} J(u_n) &= \frac{1}{2}\|\nabla u_n\|^2_{L^2} + \frac{e^2}{4}A(u_n) - \frac{1}{p+1}\|u_n\|^{p+1}_{L^{p+1}} \\ &\geq -\frac{1}{p+1}\|u_n\|^{p+1}_{L^{p+1}}. \end{aligned} \tag{I.0.41}$$

We then pass to the limit as $n \to +\infty$ and obtain

$$0 > c(\mu) \geq 0 \tag{I.0.42}$$

which is absurd, disqualifying the «vanishing» case. We can clearly see in this example how the result $c(\mu) < 0$ is taken advantage of in the application of the principle.

Next, we suppose that the case «dichotomy» occurs. We will present here the general idea provided in [3] and made more precise in [4, Section I.2, p.233] to obtain a contradiction

in the context of our problem. A rigorous proof in our context is to be adapted from the presentation in [4].
The goal is to obtain a contradiction to (I.0.22) of the type $c(\mu) \geq c(\alpha) + c(\mu - \alpha)$. First, we show how the splitting of the sequence (ρ_n) in (I.0.35) is obtained. We want to show that there exist $(u_{n,1})_n$, $(u_{n,2})_n \subset H^1_{\log}$, $\alpha \in (0, \mu)$ such that

$$\|u_{n,1}\|^2_{L^2} \underset{n\to+\infty}{\longrightarrow} \alpha, \qquad \|u_{n,2}\|^2_{L^2} \underset{n\to+\infty}{\longrightarrow} \mu - \alpha, \tag{I.0.43}$$

$$\text{supp } (u_{n,1}) \cap \text{supp } (u_{n,2}) = \varnothing \quad \text{ with } \quad \text{dist } (\text{supp } (u_{n,1}) \cap \text{supp } (u_{n,2})) \underset{n\to+\infty}{\longrightarrow} +\infty, \tag{I.0.44}$$

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla u_n|^2 - |\nabla(u_{n,1})|^2 - |\nabla(u_{n,2})|^2 \ dx \geq 0. \tag{I.0.45}$$

We use the concentration function described earlier applied to $|u_n(x)|^2$ defined as

$$Q_n(t) = \sup_{y\in\mathbb{R}^2} \int_{y+B(0,t)} |u_n(x)|^2 \ dx. \tag{I.0.46}$$

We may assume that $Q_n(t) \underset{n\to+\infty}{\longrightarrow} Q(t)$ and that for all $t \geq 0$, $\lim\limits_{t\to+\infty} Q(t) = \alpha$ for some $\alpha \in (0, \mu)$, and that Q is monotonous.
Let $\varepsilon > 0$. We choose $R_0 > 0$ large enough so that $Q(R_0) \geq \alpha - \epsilon$. Then there exists $n_0 \in \mathbb{N}$ and $(y_n)_n \subset \mathbb{R}^2$ such that for all $n \geq n_0$, we have

$$Q_n(R_0) = \int_{y_n+B(0,R_0)} |u_n(x)|^2 \ dx \geq \alpha - 2\varepsilon. \tag{I.0.47}$$

Also, there exists $R_n \underset{n\to+\infty}{\longrightarrow} +\infty$ such that $Q_n(R_n) \leq \alpha + \varepsilon$.
Let $\xi, \varphi \in C_b^\infty$ such that $0 \leq \xi, \varphi \leq 1$, supp $(\xi) \subset B(0, 2)$, $\xi \equiv 1$ on $B(0, 1)$, $\varphi \equiv 0$ on $B(0, 1)$ and $\varphi \equiv 1$ on $\mathbb{R}^2 - B(0, 2)$. We then define

$$\xi_n = \xi\left(\frac{\cdot - y_n}{R_1}\right), \qquad \varphi_n = \varphi\left(\frac{\cdot - y_n}{R_n}\right) \tag{I.0.48}$$

for some $R_1 > R_0$ large enough that we determine from the condition below.
The weak lower semi-continuity of $\|\nabla \cdot\|_{L^2}$ ensures that for an R_1 large enough and n large enough,

$$\begin{cases} \left|\int_{\mathbb{R}^2} |\xi_n \nabla u_n|^2 \ dx - \int_{\mathbb{R}^2} |\nabla(\xi_n u_n)|^2 \ dx\right| \leq \varepsilon \\ \left|\int_{\mathbb{R}^2} |\varphi_n \nabla u_n|^2 \ dx - \int_{\mathbb{R}^2} |\nabla(\varphi_n u_n)|^2 \ dx\right| \leq \varepsilon \end{cases} \tag{I.0.49}$$

Hence, we set $u_{n,1} = \xi_n u_n$ and $u_{n,2} = \phi_n u_n$ and we obtain the two sequences splitting apart described in $(I.0.35)$. Furthermore, replacing respectively $u_{n,1}$ and $u_{n,2}$ by $\frac{\sqrt{\alpha}}{\|u_{n,1}\|^2_{L^2}} u_{n,1}$ and $\frac{\sqrt{\mu-\alpha}}{\|u_{n,2}\|^2_{L^2}} u_{n,2}$, we may assume that $\|u_{n,1}\|^2_{L^2} = \alpha$ and $\|u_{n,2}\|^2_{L^2} = \mu - \alpha$.
We also obtained that for n large enough,

$$\int_{\mathbb{R}^2} |\nabla u_n|^2 \ dx \geq \int_{\mathbb{R}^2} |\nabla(u_{n,1})|^2 \ dx + \int_{\mathbb{R}^2} |\nabla(u_{n,2})|^2 \ dx - 2\varepsilon. \tag{I.0.50}$$

Again from [4, Lemma I.1, p.231], we obtain $\|u_n - u_{n,1} - u_{n,2}\|_{L^{p+1}} \underset{n\to+\infty}{\longrightarrow} 0$. Now, the next step would be to prove a result of the type $A(u_n) - A(u_{n,1}) - A(u_{n,2}) \underset{n\to+\infty}{\longrightarrow} 0$.
Once this is done, we obtain

$$\begin{aligned} c(\mu) = \liminf_{n\to+\infty} J_e(u_n) = &\int_{\mathbb{R}^2} |\nabla u_n|^2 + S(u_n)|u_n|^2 - \frac{|u_n|^{p+1}}{p+1}\,dx \\ \geq &\quad \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla u_{n,1}|^2 + S(u_{n,1})|u_{n,1}|^2 - \frac{|u_{n,1}|^{p+1}}{p+1}\,dx \\ &+ \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla u_{n,2}|^2 + S(u_{n,2})|u_{n,2}|^2 - \frac{|u_{n,2}|^{p+1}}{p+1}\,dx \\ &- 2\varepsilon \\ \geq &\, c(\alpha) + c(\mu-\alpha) - 2\varepsilon \end{aligned} \tag{I.0.51}$$

As $\varepsilon > 0$ is taken arbitrarily, this provides a contradiction to (I.0.22), hence the case «dichotomy» is disqualified.

From there, some additional work is needed from the case « compactness » to prove the existence of a sequence converging to a minimizer $u \in H^1_{\log}$.

Part II

Preliminary results

In order to apply the principle of concentration-compactness, we will need a handful of preliminary results.
We recall that to ensure that $A(u)$ is finite, we will denote by $H^1_{\log}$ the subset

$$H^1_{\log} = \left\{ u \in H^1(\mathbb{R}^2) \mid \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} -\log|x-y||u(y)|^2|u(x)|^2 \, dydx \right| < +\infty \right\}.$$

0.1. Useful norm estimations

To begin with, as a mean to study c_e and establish its boundedness from below and its negativity, we will need to estimate $\|u\|^{p+1}_{L^{p+1}}$.

Lemma 2.0.2. *For any $u \in H^1_{log}$, we have*

$$\|u\|_{L^3} \leq C\|\nabla u\|^{\frac{1}{3}}_{L^2} A(u)^{\frac{1}{6}}.$$

Proof. From the definition of $S(u)$, we have $\Delta S(u) = -\frac{|u|^2}{2}$.
Then integrating by parts, we obtain :

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^2} |\nabla u - a\nabla S(u)|^2 \, dx \\ &= \int_{\mathbb{R}^2} |\nabla u|^2 dx - 2a \int_{\mathbb{R}^2} \nabla u \cdot \nabla S(u) dx + a^2 \int_{\mathbb{R}^2} |\nabla S(u)|^2 dx \\ &= \int_{\mathbb{R}^2} |\nabla u|^2 dx - a \int_{\mathbb{R}^2} |u|^2 u \, dx + a^2 \int_{\mathbb{R}^2} S(u)|u|^2 dx. \end{aligned}$$

Hence, optimizing for a, we obtain :

$$0 \leq \|\nabla u\|^2_{L^2} - \frac{1}{2}\|u\|^6_{L^3} A(u)^{-1}.$$

□

Lemma 2.0.3. *For all $u \in H^1(\mathbb{R}^2)$, $1 \leq p \leq 2$, we have*

$$\|u\|_{L^{p+1}}^{p+1} \leq C\|\nabla u\|_{L^2(\mathbb{R}^2)}^{p-1}\|u\|_{L^2(\mathbb{R}^2)}^2.$$

Proof. We apply the Gagliardo-Nirenberg-Sobolev inequality

$$\|\nabla^j u\|_{L^{p+1}(\mathbb{R}^2)} \leq \|\nabla^m\|_{L^r(\mathbb{R}^2)}^{\theta}\|u\|_{L^q(\mathbb{R}^2)}^{1-\theta} \text{ for } \frac{1}{p+1} = \frac{j}{2} + \theta\left(\frac{1}{r} - \frac{m}{2}\right) + \frac{1-\theta}{q} \text{ and } 0 \leq \theta \leq 1$$

with the values $j = 0$, $m = 1$, $r = 2$, $q = 2$.
This provides the result as long as $0 \leq p - 1 \leq 1$.

□

Lemma 2.0.4. *For all $u \in H^1_{log}$, and for $p \geq \frac{3}{2}$, we have*

$$\|u\|_{L^{p+1}}^{p+1} \leq C\|\nabla u\|_{L^2}^{p-1}A(u)^{\frac{1}{4}}\|u\|_{L^2}.$$

Proof. In one hand, we have from Hölder's inequality for $p \geq \frac{3}{2}$,

$$\begin{aligned}
\|u\|_{L^{p+1}}^{p+1} &= \int_{\mathbb{R}^2} |u|^{p+1-\frac{3}{2}}|u|^{\frac{3}{2}}\,dx \\
&\leq \left(\int_{\mathbb{R}^2} |u|^{2p-1}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2} |u|^3\,dx\right)^{\frac{1}{2}} \\
&\leq \|u\|_{L^{2p-1}}^{\frac{2p-1}{2}}\|u\|_{L^3}^{\frac{3}{2}}.
\end{aligned}$$

On the other hand, we have from the Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^{2p-1}} \leq C\|\nabla u\|_{L^2}^{\frac{2p-3}{2p-1}}\|u\|_{L^2}^{\frac{2}{2p-1}}.$$

Combining these two results, we obtain

$$\|u\|_{L^{p+1}}^{p+1} \leq C\|u\|_{L^3}^{\frac{3}{2}}\|u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{2p-3}{2}}.$$

We then use lemma (2.0.2) to obtain the result.

□

0.2. Results on $S(u)$ and $A(u)$

We aim at proving the negativity of c_e by studying a particular family of functions in $H^1_{\log}$ that happens to point out such property. This particular family of functions would be given by the dilation of a function in $H^1_{\log}$ following $u_\lambda(x) = \lambda^a u(\lambda^b x)$, for a, $b \in \mathbb{R}$ and $\lambda \in R_+^*$.

We first remark the non-negativity of A, which will be helpful to establish the boundedness from below of c_e.

Lemma 2.0.5. *For any $u \in H^1_{log}$, we have*

$$A(u) \geq 0$$

Proof. We have $u \in H^1$, so $|u|^2 \in W^{1,1}(\mathbb{R}^2)$. From Sobolev's embeddings, we then have $|u|^2 \in W^{0,2}(\mathbb{R}^2)$. As $-\Delta S(u) = \frac{|u|^2}{2}$, elliptic regularity for Poisson's equation provides $S(u) \in W^{2,2}(\mathbb{R}^2) = H^2(\mathbb{R}^2)$. For $R > 0$, an integration by parts on $B_R := B(0, R)$ gives

$$\int_{B_R} S(u)|u|^2\, dx = -2\int_{B_R} S(u)\Delta S(u)\, dx = 2\|\nabla S(u)\|^2_{L^2(B_R)} - 2\int_{\partial B_R} S(u)\ \nabla S(u) \cdot \vec{n}\ d\sigma,$$

where $\vec{n}$ is the unit normal vector to $\partial B(0, R)$. For $\partial B(0, R)$, we have $\vec{n} = \frac{x}{|x|} = \frac{x}{R}$, and $|x \cdot \nabla S(u)| \leq R|\nabla S(u)|$. We then proceed as in the proof of lemma (2.0.8), and show that

$$\int_{\partial B(0,R)} S(u)\ \nabla S(u) \cdot \vec{n}\ d\sigma \underset{R\to+\infty}{\longrightarrow} 0.$$

Hence, we obtain

$$0 \leq \|\nabla S(u)\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{2}\int_{\mathbb{R}^2} S(u)|u|^2\, dx = \frac{1}{2}A(u).$$

□

Such previously mentioned dilation translates to the following formulas.

Lemma 2.0.6. *(Dilation)*

For any $u \in H^1_{log}$, $a, b \in \mathbb{R}$, $\lambda > 0$, we have under the condition $\lambda^b \geq 1$ and for $u_\lambda(x) = \lambda^a u(\lambda^b x)$,

$$S(u_\lambda)(x) = \lambda^{2a-2b}\left[S(u)(\lambda^b x) + b\log|\lambda|\frac{\|u\|^2_{L^2}}{4\pi}\right],$$

$$A(u_\lambda) = \lambda^{4a-4b}\left[A(u) + b\log|\lambda|\frac{\|u\|^4_{L^2}}{4\pi}\right].$$

And by further calculations,

$$J(u_\lambda) = \frac{\lambda^{2a}}{2}\|\nabla u\|^2_{L^2} + \lambda^{4a-4b}\left[A(u) + b\log|\lambda|\frac{\|u\|^4_{L^2}}{4\pi}\right] - \frac{\lambda^{(p+1)a-2b}}{p+1}\|u\|^{p+1}_{L^{p+1}}$$

Proof. We have

$$\begin{aligned} S(u)(x) &= -\frac{1}{2\pi}\log|x| * \frac{1}{2}|u_\lambda|^2 \\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^2}\log|x-y||\lambda^a u(\lambda^b y)|^2\, dy = -\frac{\lambda^{2a}}{4\pi}\int_{\mathbb{R}^2}\log|x-y||u(\lambda^b y)|^2\, dy. \end{aligned}$$

By denoting $Y = \lambda^b y$, we get $dY = \lambda^{Nb} dy = \lambda^{2b} dy$, so that

$$S(u)(x) = -\frac{\lambda^{2a-2b}}{4\pi} \int_{\mathbb{R}^2} \log \left| x - \lambda^{-b} Y \right| |u(Y)|^2 \, dY \tag{II.0.1}$$

Then,

$$\begin{aligned} S(u)(x) &= -\frac{\lambda^{2a-2b}}{4\pi} \int_{\mathbb{R}^2} \log \left| \lambda^{-b}(\lambda^b x - Y) \right| |u(Y)|^2 \, dY \\ &= -\frac{\lambda^{2a-2b}}{4\pi} \left[\int_{\mathbb{R}^2} \log \left| \lambda^{-b} \right| |u(Y)|^2 \, dY + \int_{\mathbb{R}^2} \log \left| (\lambda^b x - Y) \right| |u(Y)|^2 \, dY \right] \\ &= \lambda^{2a-2b} \left[b \log |\lambda| \frac{\|u\|_{L^2}^2}{4\pi} + S(u)(\lambda^b x) \right]. \end{aligned}$$

We initially thought this result to be valid for $\lambda > 0$. Yet this would lead to an error that has been revealed by the development of two contradictory lemmas, which we expose in Part III. Indeed, as opposed to the problem for N=3, we have had to restrict our results on $H^1_{\log}$ in order to ensure the integrability of $\log|x-y||u(y)|^2$ for y over $\mathbb{R}^2$. However, this is a priori not the case of the integrand in (II.0.1) when λ^{-b} is large, that is to say when λ is small. This can also be seen with the help of lemma (2.0.5) which contradicts our result with $A(u_\lambda)$ for some λ sufficiently small. To show this explicitly, we will reformulate the integral as follows :

$$\begin{aligned} -4\pi S(u)(x) &= \int_{\mathbb{R}^2} \log|x-y| \, |u(y)|^2 \, dy \\ &= \int_{|x-y|\leq 1} \log|x-y| \, |u(y)|^2 \, dy + \int_{|x-y|\geq 1} \log|(x-y)| \, |u(y)|^2 \, dy \\ &= \int_{y \in B(x,1)} \log|x-y| \, |u(y)|^2 \, dy + \int_{y \in \mathbb{R}^2 - B(x,1)} \log|(x-y)| \, |u(y)|^2 \, dy \\ &= \int_{z \in B(0,1)} \log(1-|z|) \, |u(x+z)|^2 \, dz + \int_{z \in \mathbb{R}^2 - B(0,1)} \log|z| \, |u(x+z)|^2 \, dz. \end{aligned}$$

We can then express the dilation as

$$
\begin{aligned}
-4\pi S(u_\lambda)(x)& \\
&= \int_{z\in B(0,1)} \log(1-|z|)\,|u_\lambda(x+z)|^2\,dz + \int_{z\in\mathbb{R}^2-B(0,1)} \log|z|\,|u_\lambda(x+z)|^2\,dz \\
&= \int_{z\in B(0,1)} \log(1-|z|)\,|\lambda^a u(\lambda^b(x+z))|^2\,dz \\
&\quad + \int_{z\in\mathbb{R}^2-B(0,1)} \log|z|\,|\lambda^a u(\lambda^b(x+z))|^2\,dz \\
&= \lambda^{2a}\int_{z\in B(0,1)} \log(1-|z|)\,|u(\lambda^b x+\lambda^b z)|^2\,dz \\
&\quad + \lambda^{2a}\int_{z\in\mathbb{R}^2-B(0,1)} \log|z|\,|u(\lambda^b x+\lambda^b z))|^2\,dz \\
&= \lambda^{2a-2b}\int_{z\in B(0,\lambda^b)} \log\left(1-\left|\frac{Z}{\lambda^b}\right|\right)|u(\lambda^b x+Z)|^2\,dZ \\
&\quad + \lambda^{2a-2b}\int_{z\in\mathbb{R}^2-B(0,\lambda^b)} \log\left|\frac{Z}{\lambda^b}\right||u(\lambda^b x+Z))|^2\,dZ.
\end{aligned}
$$

For $\lambda<1$, we have $\left|\frac{Z}{\lambda^b}\right|>|Z|$. The first term of the right member in the last equation is integrable as the integral is over a bounded domain. However, the second term shows clearly the error, as we have

$$
\int_{z\in\mathbb{R}^2-B(0,\lambda^b)} \log\left|\frac{Z}{\lambda^b}\right||u(\lambda^b x+Z))|^2\,dZ \geq \int_{z\in\mathbb{R}^2-B(0,\lambda^b)} \log|Z|\,|u(\lambda^b x+Z))|^2\,dZ.
$$

We have imposed $u\in H^1_{\log}$, hence only the right member of this inequation has been sought to be finite. This shows the result cannot be obtained for $\lambda^b<1$, but is valid for $\lambda^b\geq 1$.

Next, we have

$$
\begin{aligned}
A(u_\lambda) &= \int_{\mathbb{R}^2} S(u_\lambda)(x)|u_\lambda(x)|^2\,dx \\
&= \int_{\mathbb{R}^2} \lambda^{2a-2b}\left[S(u)(\lambda^b x)+b\log|\lambda|\|u\|_{L^2}^2\right]|\lambda^a u(\lambda^b x)|^2\,dx \\
&= \lambda^{4a-2b}\left[\int_{\mathbb{R}^2} S(u)(\lambda^b x)|u(\lambda^b x)|^2\,dx + \int_{\mathbb{R}^2} b\log|\lambda|\frac{\|u\|_{L^2}^2}{4\pi}|u(\lambda^b x)|^2\,dx\right] \\
&= \lambda^{4a-4b}\left[\int_{\mathbb{R}^2} S(u)(x)|u(x)|^2\,dx + b\log|\lambda|\frac{\|u\|_{L^2}^4}{4\pi}\right].
\end{aligned}
$$

□

These show two important differences to notice with the case $N=3$, where we would obtain

$$
S(u_\lambda)(x)=\lambda^{2a-2b}S(u)(\lambda^b x),\quad A(u_\lambda)=\lambda^{4a-5b}A(u).
$$

The first difference is the equality of the coefficients in front of a and b in $A(u_\lambda)$, which will lead to a difficulty later when considering a constraint of a constant $\|u_\lambda\|_{L^2}^2$. The second difference is the coming of an additional term in $\log(\lambda)$. These difference will play a major role in the next part of this study.

The following existence result provides an useful tool for the study of the functional J_e.

Lemma 2.0.7. *For every $u \in H^1_{log}$ and for $p \geq \frac{3}{2}$, we define $C^* = C^*(p) > 0$ as the quantity*

$$C^* = \sup_{u \in H^1_{log}, u \neq 0} \quad \frac{\|u\|_{L^{p+1}}^{p+1}}{A(u)^{\frac{1}{4}} \|\nabla u\|_{L^2}^{p-1} \|u\|_{L^2}}.$$

Then $C^* < +\infty$ and for any $0 < K < C^*$ and $\mu > 0$, there exists $v \in H^1_{log}$ such that $\|v\|_{L^2}^2 = \mu$ and

$$\|v\|_{L^{p+1}}^{p+1} > K\sqrt{\mu} A(v)^{\frac{1}{4}} \|\nabla v\|_{L^2}^{p-1}.$$

Proof. From lemma (2.0.4), C^* is well defined. For any $0 < K < C^*$, there exist $u_0 \in H^1_{\log}$ such that

$$\|u_0\|_{L^{p+1}}^{p+1} > A(u_0)^{\frac{1}{4}} \|\nabla u_0\|_{L^2}^{p-1} \|u_0\|_{L^2}.$$

Setting $\lambda = \frac{\sqrt{\mu}}{\|u_0\|_{L^2}}$ and renormalizing such that $v = \lambda u_0$, we get from lemma (2.0.6) :

$$\begin{aligned}
\|v\|_{L^{p+1}}^{p+1} &= \int_{\mathbb{R}^2} |\lambda u_0|^{p+1}\, dx = \lambda^{p+1} \|u_0\|_{L^{p+1}}^{p+1} \\
\|v\|_{L^2} &= \lambda \|u_0\|_{L^2} = \sqrt{\mu} \\
\|\nabla v\|_{L^2}^{p-1} &= \lambda^{p-1} \left(\int_{\mathbb{R}^2} |\nabla u_0|^2\, dx \right)^{\frac{p-1}{2}} = \lambda^{p-1} \|\nabla u_0\|_{L^2}^{p-1} \\
A(v)^{\frac{1}{4}} &= \lambda A(u_0)^{\frac{1}{4}}.
\end{aligned}$$

Hence, combining these calculations, we obtain that $\|v\|_{L^2}^2 = \mu$ and v verifies

$$\|v\|_{L^{p+1}}^{p+1} > K A(v)^{\frac{1}{4}} \|\nabla v\|_{L^2}^{p-1} \|v\|_{L^2}.$$

$\square$

0.3. Rework of J_e and $I_{e,\omega}$

It can appear useful further in the study to express J_e and different terms constituting J_e in a different way.
For this purpose, we dispose of the two following well-known identities, namely Nehari's and Pohozaev's identities. Although they did not come in useful during this internship, we will express them in the setting of our present problem.

Lemma 2.0.8. *Nehari's and Pohozaev's identities*

For all $u \in H^1_{log}$ solution of (I.0.10), it stands

$$i)\ N_{e,\omega}(u) = \int_{\mathbb{R}^2} |\nabla u|^2\,dx + \int_{\mathbb{R}^2} \omega|u|^2\,dx + e^2\int_{\mathbb{R}^2} S(u)|u|^2\,dx - \int_{\mathbb{R}^2} |u|^{p+1}\,dx = 0$$
$$ii)\ P_{e,\omega}(u) = \int_{\mathbb{R}^2} \omega|u|^2\,dx + e^2\int_{\mathbb{R}^2} S(u)|u|^2\,dx - \frac{2}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\,dx = 0$$

Proof. i) Starting with Nehari's identity, we multiply (I.0.10) by u and integrate over $B_R := B(0,R)$, we then get

$$\int_{B_R} -\Delta u u + \omega u^2 + e^2 S(u)u^2 - |u|^{p-1}u^2\,dx = 0.$$

We proceed as in [1]. Integrating by parts, we get

$$\int_{B_R} |\nabla u|^2 + \omega u^2 + e^2 S(u)u^2 - |u|^{p-1}u^2\,dx = \int_{\partial B_R} u\nabla u\cdot\vec{n}\,d\sigma, \tag{II.0.2}$$

where $\vec{n}$ is the normal unit vector to ∂B_R. In this case, $\vec{n} = \frac{x}{|x|} = \frac{x}{R}$. We want to show that the left-hand side of the equation vanishes as $R \to +\infty$. We notice that $|x\cdot\nabla u| \leq R\,|\nabla u|$ on ∂B_R. Hence, we have

$$\int_{\partial B_R} u\nabla u\cdot\vec{n}\,d\sigma \leq \int_{\partial B_R} |u|\,|\nabla u|\ d\sigma.$$

As $u \in H^1(\mathbb{R}^2)$, then $u\nabla u \in L^1(\mathbb{R}^2)$ and subsequently,

$$\int_{\mathbb{R}^2} |u|\,|\nabla u|\ dx = \int_0^{+\infty}\left(\int_{\partial B_R} |u|\,|\nabla u|\ d\sigma\right)dR < +\infty.$$

Hence, there exists at least one sequence $R_n \underset{n\to+\infty}{\longrightarrow} +\infty$ such that

$$R_n\int_{\partial R_n} |u|\,|\nabla u|\,d\sigma \underset{n\to+\infty}{\longrightarrow} 0.$$

Indeed, by contradiction, let us suppose that

$$\liminf_{R\to+\infty} R\int_{\partial B_R} |u|\,|\nabla u|\,d\sigma = \alpha > 0.$$

Then $\int_{\partial B_R} |u|\,|\nabla u|\,d\sigma = O(\frac{1}{R})$ and $\int_{\partial B_R} |u|\,|\nabla u|\,d\sigma \notin L^1(0,+\infty)$, which contradicts $u\nabla u \in L^1(\mathbb{R}^2)$. This show that the right-hand side of (II.0.2) vanishes as $R \to +\infty$ and we obtain $N_{e,\omega}(u) = 0$.

ii) To obtain Pohozaev's identity, we multiply again (I.0.10) by $x\cdot\nabla u$ and integrate over $B(0,R)$ and obtain

$$\int_{B_R}\left(\underbrace{\sum_{1\leq i,j\leq 2}-u_{ii}u_jx_j}_{=:T_1}+\underbrace{\sum_{1\leq j\leq 2}\omega uu_jx_j}_{=:T_2}+\underbrace{\sum_{1\leq j\leq 2}e^2S(u)uu_jx_j}_{=:T_3}-\underbrace{\sum_{1\leq j\leq 2}|u|^{p-1}uu_jx_j}_{=:T_4}\right)dx.$$

Starting with T_2, we have

$$\begin{aligned}\int_{B_R}T_2\,dx&=\int_{B_R}\sum_{1\leq j\leq 2}\omega uu_jx_j\,dx\\&=\int_{\partial B_R}\sum_{1\leq j\leq 2}\omega|u|^2x_j\frac{x_j}{R}\,d\sigma-\int_{B_R}\sum_{1\leq j\leq 2}\omega|u|^2\,dx\\&=R\int_{\partial B_R}\omega|u|^2\,d\sigma-2\int_{B_R}\omega|u|^2\,dx.\end{aligned}$$

Using the same type of arguments as before with $|u|^2\in L^1(\mathbb{R}^2)$, we establish

$$R\int_{\partial B_R}\omega|u|^2\,d\sigma\underset{R\to+\infty}{\longrightarrow}0\qquad\text{and}\qquad\int_{B_R}T_2\,dx\underset{R\to+\infty}{\longrightarrow}-2\int_{\mathbb{R}^2}\omega|u|^2\,dx.$$

Now with T_1, we have

$$\begin{aligned}\int_{B_R}T_1\,dx&=\int_{B_R}\sum_{1\leq i,j\leq 2}-u_{ii}u_jx_j\,dx\\&=\int_{\partial B_R}\sum_{1\leq i,j\leq 2}-u_iu_jx_j\frac{x_i}{|x|}\,d\sigma+\int_{B_R}\sum_{1\leq i,j\leq 2}\left(u_iu_{ij}x_j+u_iu_j\delta_{ij}\right)\,dx\\&=\int_{\partial B_R}\sum_{1\leq i,j\leq 2}-u_iu_jx_j\frac{x_i}{R}\,d\sigma+\int_{B_R}\sum_{1\leq i,j\leq 2}\left(\frac{(|u_i|^2)_j}{2}x_j\right)+\sum_{1\leq j\leq 2}|u_j|^2\,dx\\&=\int_{\partial B_R}\sum_{1\leq i,j\leq 2}-u_iu_jx_j\frac{x_i}{R}+|u_i|^2\frac{x_j^2}{2R}\,d\sigma+\int_{B_R}\sum_{1\leq i,j\leq 2}\left(\frac{-|u_i|^2}{2}\right)+|\nabla u|^2\,dx\\&=\int_{\partial B_R}-|x\cdot\nabla u|^2\frac{1}{R}+|\nabla u|^2\frac{R}{2}\,d\sigma+\int_{B_R}\left(-|\nabla u|^2+|\nabla u|^2\right)\,dx.\end{aligned}$$

Once again, the integral over ∂B_R vanishes as $R\longrightarrow+\infty$ and we obtain

$$\int_{B_R}T_1\,dx\underset{R\to+\infty}{\longrightarrow}0.$$

With T_3, we recall that $-\Delta S(u)=\frac{1}{2}|u|^2$. As in proof of lemma (2.0.5), we have $S(u)\in$

$H^2(\mathbb{R}^2)$ and we will temporarily write $S = S(u)$ for the sake of simplicity. This gives

$$\begin{aligned}
\int_{B_R} T_3 \, dx &= \int_{B_R} \sum_{1\leq j\leq 2} e^2 S u u_j x_j \, dx \\
&= \int_{B_R} \sum_{1\leq j\leq 2} e^2 S \frac{(|u|^2)_j}{2} x_j \, dx \\
&= \int_{\partial B_R} \sum_{1\leq j\leq 2} e^2 S |u|^2 \frac{x_j^2}{2R} \, d\sigma - e^2 \int_{B_R} \sum_{1\leq j\leq 2} \left(S_j \frac{|u|^2}{2} x_j + S \frac{|u|^2}{2} \right) \, dx \\
&= \frac{R}{2} \int_{\partial B_R} e^2 S(u) |u|^2 \, d\sigma - e^2 \int_{B_R} \left((\nabla S \cdot x) \frac{|u|^2}{2} + S|u|^2 \right) \, dx.
\end{aligned}$$

We also have

$$\begin{aligned}
-e^2 \int_{B_R} (\nabla S \cdot x) \frac{|u|^2}{2} \, dx &= e^2 \int_{B_R} (\nabla S \cdot x) \Delta S \, dx \\
&= e^2 \int_{B_R} \sum_{1\leq i,j\leq 2} S_j x_j S_{ii} \, dx \\
&= -e^2 \int_{B_R} \sum_{1\leq i,j\leq 2} S_j S_i \delta_{ij} + S_i S_{ij} x_j \, dx + e^2 \int_{\partial B_R} \sum_{1\leq i,j\leq 2} S_j S_i x_i \frac{x_j}{R} \, d\sigma \\
&= -e^2 \int_{B_R} |\nabla S|^2 + \sum_{1\leq i,j\leq 2} \frac{(|S_i|^2)_j}{2} x_j \, dx + e^2 \int_{\partial B_R} \frac{|x \cdot \nabla S|^2}{R} \, d\sigma \\
&= -e^2 \int_{B_R} |\nabla S|^2 - |\nabla S|^2 \, dx + e^2 \int_{\partial B_R} \frac{|x \cdot \nabla S|^2}{R} - \sum_{1\leq i,j\leq 2} |S_i|^2 \frac{x_j^2}{R} \, d\sigma \\
&= e^2 \int_{\partial B_R} \frac{|x \cdot \nabla S|^2}{R} - R|\nabla S|^2 \, d\sigma.
\end{aligned}$$

Finally we obtain

$$\int_{B_R} T_3 \, dx = e^2 \int_{\partial B_R} \frac{R}{2} S(u)|u|^2 + \frac{|x \cdot \nabla S(u)|^2}{R} - R|\nabla S(u)|^2 \, d\sigma - e^2 \int_{B_R} S(u)|u|^2 \, dx.$$

Since $S(u) \in W^{2,1}(\mathbb{R}^2)$, we deduce from Sobolev embeddings that $S(u) \in W^{1,2}(\mathbb{R}^2)$, which gives once again

$$\int_{B_R} T_3 \, dx \underset{R\to+\infty}{\longrightarrow} -e^2 \int_{B_R} S(u)|u|^2 \, dx.$$

T_4 is treated similarly, and summing all four results, we obtain the identity $P_{e,\omega}(u) = 0$.

$\square$

Thanks to those two identities, we are able to combine them and obtain various new expressions of J_e and $I_{e,\omega}$ which could appear useful once the existence of a solution for (I.0.10) is proven. For example, in [1], it is shown that the solution found in its context has positive action using Nehari's identity.

Part III

Towards the application of concentration-compactness to c_e

As presented in the setting of (M_μ), let us set $\mu > 0$ and consider the minimizing problem : find $v \in H^1_{\log}$ such that

$$c_e(\mu) = \inf_{u \in \mathcal{B}} \{J_e(u)\} = J_e(v), \tag{III.0.1}$$

where

$$J_e(u) = \frac{\|\nabla u\|_{L^2}^2}{2} + \frac{e^2}{4} A(u) - \frac{\|u\|_{L^{p+1}}^{p+1}}{p+1} \tag{III.0.2}$$

and

$$\mathcal{B} = \{u \in H^1_{\log} \mid \frac{\|u\|_{L^2}^2}{\sqrt{4\pi}} = \mu\}. \tag{III.0.3}$$

In order to apply the principle of concentration-compactness to our minimization problem $c_e(\mu)$, we first need to ensure that $c_e(\mu) \neq -\infty$.

Lemma 3.0.9. *For all $u \in H^1$, $\mu > 0$, $1 < p \leq 2$, we have*

$$c_e(\mu) > -\infty.$$

Proof. Let C be the optimal constant obtained in lemma (2.0.3). From Young's inequality with

$$\varepsilon = \left(\frac{1}{2C}\right)^{\frac{1}{\alpha}}, \qquad \alpha = \frac{2}{p-1}, \qquad \beta = \frac{2}{3-p},$$

we obtain under the restrictions that in one hand $1 < p < 3$ in order to ensure $\alpha, \beta > 1$, and in the other hand for $1 \leq p \leq 2$ so that lemma (2.0.3) is valid,

$$\begin{aligned} \|u\|_{L^2}^{p-1} \|u\|_{L^2}^2 &= \varepsilon \|u\|_{L^2}^{p-1} \frac{1}{\varepsilon} \|u\|_{L^2}^2 \\ &\leq \frac{\varepsilon^\alpha}{\alpha} \left(\|u\|_{L^2}^{p-1}\right)^\alpha \frac{1}{\varepsilon^\beta \beta} \left(\|u\|_{L^2}^2\right)^\beta. \end{aligned}$$

This gives, injecting in (III.0.2)

$$\begin{aligned}
J_e(u) &\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 + [A(u)] \\
&\qquad\qquad - \frac{C}{p+1}\left(\frac{\varepsilon^\alpha}{\alpha}\left(\|u\|_{L^2}^{p-1}\right)^\alpha + \frac{1}{\varepsilon^\beta \beta}\left(\|u\|_{L^2}^2\right)^\beta\right) \\
&= \frac{1}{4}\|\nabla u\|_{L^2}^2 + [A(u)] \\
&\qquad\qquad - \underbrace{\frac{C}{p+1}\frac{1}{\varepsilon^\beta \beta}}_{>0}\left(\|u\|_{L^2}^2\right)^\beta .
\end{aligned}$$

As $A(u) \geq 0$ and $\|u\|_{L^2}^2 = \mu\sqrt{4\pi}$ is fixed, we obtain the result.

□

The next step is to show that $c_e(\mu) < 0$.
As discussed in the previous part, we thought the dilation lemma (2.0.6) to be valid for $\lambda > 0$. This had led to the following result which is a contradiction to lemma (3.0.9) presented above, and thus initiated the search for the error explained in the proof of lemma (2.0.6).
We present nonetheless the incorrect result to make explicit the need to take a different approach.

Incorrect result

Lemma 3.0.10. *For any* $u \in H^1$, $1 < p < 3$, *we have*

$$c_e(\mu) = -\infty.$$

Proof. We fix $u \in H^1$ such that $\frac{\|u\|_{L^2}^2}{\sqrt{4\pi}} = \mu$. Using the Lemma (2.0.6), we have for $u_\lambda(x) = \lambda^a u(\lambda^b x)$,

$$J_e(u_\lambda) = \frac{\lambda^{2a}}{2}\|\nabla u\|_{L^2}^2 + \lambda^{4a-4b}\left[A(u) + b\log(\lambda)\mu^2\right] - \frac{\lambda^{(p+1)a-2b}}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

As we are looking for functions v such that $\frac{\|v\|_{L^2}^2}{\sqrt{4\pi}} = \mu$, we must impose for all $\lambda > 0$ that

$$\mu = \frac{\|u_\lambda\|_{L^2}^2}{\sqrt{4\pi}} = \lambda^{2a-2b}\frac{\|u\|_{L^2}^2}{\sqrt{4\pi}} = \lambda^{2a-2b}\mu. \tag{III.0.4}$$

Hence, we are going to restrain a and b such that $a = b$. Thus, we get

$$J_e(u_\lambda) = \frac{\lambda^{2a}}{2}\|\nabla u\|_{L^2}^2 + \left[A(u) + a\log(\lambda)\mu^2\right] - \frac{\lambda^{(p-1)a}}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{III.0.5}$$

We remark that

$$c_e(\mu) \leq \limsup_{\lambda \to 0^+} J_e(u_\lambda) = -\infty.$$

However, this last step is not valid as we would be using lemma (2.0.6) for $\lambda^b < 1$. □

Once the error was found, we made several attempts at showing a result of the type $c_e(\mu) < 0$. (III.0.5) does not provide any useful information when $\lambda \to 1^+$, and since $1 < p < 3 \iff 0 < (p-1)a < 2a$, it does not for $\lambda \to +\infty$. On a side note, we remark that if $p > 3$, then

$$c_e(\mu) \leq \limsup_{\lambda \to +\infty} J_e(u_\lambda) = -\infty$$

is a valid computation and we can disqualify the cases $p > 3$ from this study right away. Hence, the main idea was to construct two functions a and b which will provide a scaling of the form

$$u_\lambda(x) = \lambda^{a(\lambda)} u(\lambda^{b(\lambda)} x). \tag{III.0.6}$$

However, as shown by (III.0.4), we must impose $a(\lambda) = b(\lambda)$. Consequently, such a dilated function can only possibly lead to

$$J_e(u_\lambda) = \frac{\lambda^{2a(\lambda)}}{2}\|\nabla u\|_{L^2}^2 + \left[A(u) + a(\lambda)\log(\lambda)\mu^2\right] - \frac{\lambda^{(p-1)a(\lambda)}}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{III.0.7}$$

As we are only looking for a single function u_λ, we are looking for some specific $\lambda > 0$ and $a(\lambda)$ such that

$$\begin{cases} J_e(u_\lambda) < 0 \\ \lambda^{b(\lambda)} = \lambda^{a(\lambda)} = e^{\log(\lambda)a(\lambda)} \geq 1, \end{cases} \tag{III.0.8}$$

where the second condition is the one imposed by the proof of lemma (2.0.6). We remark that (III.0.8) is equivalent to

$$\begin{cases} \frac{e^{2\log(\lambda)a(\lambda)}}{2}\|\nabla u\|_{L^2}^2 + \left[A(u) + a(\lambda)\log(\lambda)\mu^2\right] - \frac{e^{(p-1)\log(\lambda)a(\lambda)}}{p+1}\|u\|_{L^{p+1}}^{p+1} < 0, \\ a(\lambda) \geq 0 \text{ if } \lambda > 1, \\ a(\lambda) \leq 0 \text{ if } 0 < \lambda < 1. \end{cases} \tag{III.0.9}$$

Differentiating with respect to λ, we have

$$\frac{\partial J_e(u_\lambda)}{\partial \lambda} = \frac{1}{\lambda}\left(\lambda^{2a}\|\nabla u\|_{L^2}^2 + \mu^2 - \frac{p-1}{p+1}\lambda^{(p-1)a}\|u\|_{L^{p+1}}^{p+1}\right), \tag{III.0.10}$$

and we then see that $J_e(u_\lambda)$ admits a local extrema only if

$$\lambda^{2a}\|\nabla u\|_{L^2}^2 + \mu^2 - \frac{p-1}{p+1}\lambda^{(p-1)a}\|u\|_{L^{p+1}}^{p+1} = 0 \tag{III.0.11}$$

However, we found no success in localizing such potential local minima of this exponential polynomial for all $\mu > 0$.

Part IV

Searching for a different approach

Facing trouble in finding a result of the type $c_e(\mu) < 0$, a different approach was considered towards proving the existence of a ground state.
Let $\eta > 0$. We consider the minimizing problem : find $v \in H^1_{\log}$ such that

$$c_\omega(\eta) = \inf_{u \in \mathcal{B}} \{J_\omega(u)\} = J_\omega(v), \tag{IV.0.1}$$

where

$$J_\omega(u) = \frac{\|\nabla u\|^2_{L^2}}{2} + \omega \frac{\|u\|^2_{L^2}}{2} - \frac{\|u\|^{p+1}_{L^{p+1}}}{p+1} \tag{IV.0.2}$$

and

$$\mathcal{B} = \{u \in H^1_{\log} \mid A(u) = \eta\} \tag{IV.0.3}$$

If there exists $v \in H^1_{\log}$ solution of this minimization problem, then from the Lagrange multiplier rule, there would exist $k \in \mathbb{R}$ such that

$$J'_\omega(v) + kA'(v) = -\Delta(v) + \omega v + kS(v)v - |v|^{p-1}v = 0,$$

where $J'_\omega(v)$ and $A'(v)$ denote respectively the Euler-Lagrange operator applied to J_ω and A taken at v. We would obtain the existence of a ground state for a given frequency $\omega > 0$ and an electromagnetic strength $e^2 = k$ as a Lagrange multiplier which value is to be determined from the problem.

To prove the existence of such a v, we try once again to apply the principle of concentration-compactness. First, we start by verifying that $c_\omega(\eta) > -\infty$.

Lemma 4.0.11. *For all $\eta > 0$, $\frac{3}{2} \leq p < 3$, $c_\omega(\eta) > -\infty$.*

Proof. From the lemma (2.0.4), we have

$$J_\omega(u) \geq \frac{\|\nabla u\|^2_{L^2}}{2} + \omega \frac{\|u\|^2_{L^2}}{2} - \frac{C\|\nabla u\|^{p-1}_{L^2} A(u)^{\frac{1}{4}} \|u\|_{L^2}}{p+1}.$$

From young's inequality with $\alpha = \frac{2}{p-1}$, $\beta = \frac{2}{3-p}$, $\varepsilon = \left(\frac{\alpha}{4C^\alpha}\right)^{\frac{1}{\alpha}}$, we obtain

$$J_\omega(u) \geq \frac{\|\nabla u\|_{L^2}^2}{4} + \omega\frac{\|u\|_{L^2}^2}{2} - \frac{C\varepsilon^\beta \eta^{\frac{1}{2(3-p)}}}{\beta(p+1)}\|u\|_{L^2}^{\frac{2}{3-p}}.$$

As we must take $1 < p < 3$ to verify Young's inequality and since lemma (2.0.4) imposes $p \geq \frac{3}{2}$, then the result is obtained for $\frac{3}{2} \leq p < 3$. Moreover, η is fixed. Therefore, for $\frac{3}{2} \leq p < 3$, $J_\omega(u)$ is bounded from below.

□

The next step is to prove that the manifold on which we constrain the functional is not empty. For the purpose of showing later a result of the type $c_\omega(\eta) < 0$, we consider $a, b : \mathbb{R} \longrightarrow \mathbb{R}^+$ and a scaling $u_\lambda(x) = \lambda^{a(\lambda)}u(\lambda^{b(\lambda)}x)$, which once again will be valid under the condition

$$\lambda^{b(\lambda)} = e^{\log(\lambda)b(\lambda)} \geq 1. \tag{IV.0.4}$$

Lemma 4.0.12. *For any $\eta > 0$, $\mathcal{B}(\eta) \neq \varnothing$.*

Proof. We first look for the condition on a and b so that $A(u_\lambda) = \eta$ stands atleast locally. That is to say, that there exists $\lambda_0 > 0$ and $\varepsilon > 0$ such that for every $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, it stands that $A(u_\lambda) = \eta$.
From Lemma (2.0.6), such a and b must verify

$$\lambda^{4(a(\lambda)-b(\lambda))}\left[\eta + b(\lambda)\log(\lambda)\frac{\|u\|_{L^2}^4}{4\pi}\right] = \eta. \tag{IV.0.5}$$

To simplify, we define $\mu := \frac{\|u\|_{L^2}^4}{4\pi}$.
As the condition (IV.0.4) imposes $\log(\lambda)b(\lambda) \geq 0$, we are then looking for a and b such that $4\log(\lambda)(a(\lambda) - b(\lambda)) \leq 0$, that is, $a(\lambda) \leq b(\lambda)$.
We proceed by setting a as a constant function $a(\lambda) = a > 0$ for all $\lambda > 0$ and defining $y = b - a$. We then have $y \geq 0$ and $b = y + a$. With this, (IV.0.5) becomes

$$F(y,\lambda) := e^{-4\log(\lambda)y(\lambda)}\left[\eta + (y(\lambda) + a)\log(\lambda)\mu\right] - \eta = 0. \tag{IV.0.6}$$

To determine such a function y, we differentiate F with respect to λ and get as a necessary condition the differential equation

$$\begin{aligned}\frac{\partial F(y,\lambda)}{\partial\lambda} &= y'(\lambda)\log(\lambda)\left[\mu - 4\left(\eta + (y(\lambda)+a)\log(\lambda)\right)\right] \\ &\quad + y(\lambda)\frac{1}{\lambda}\left[\mu - 4\left(\eta + (y(\lambda)+a)\log(\lambda)\mu\right)\right] \\ &\quad + \frac{a\mu}{\lambda} \qquad = 0.\end{aligned} \tag{IV.0.7}$$

Rearanging the equation, we obtain

$$y'(\lambda) = -\frac{y(\lambda)\frac{1}{\lambda}\left[\mu - 4\left(\eta + (y(\lambda)+a)\log(\lambda)\mu\right)\right] + a\mu\frac{1}{\lambda}}{\log(\lambda)\left[\mu - 4\left(\eta + (y(\lambda)+a)\log(\lambda)\right)\right]} =: \Xi(y,\lambda). \tag{IV.0.8}$$

Ξ is defined for $\lambda \in ((0,1) \cup (1,+\infty))$. Let $\lambda_0 \in (1,+\infty)$. Then Ξ is defined for $y(\lambda_0)$ such that

$$\mu - 4\left(\eta + (y(\lambda_0) + a)\log(\lambda_0)\right) \neq 0. \tag{IV.0.9}$$

Hence we define

$$y_{\text{crit}}(\lambda) = \frac{\mu - 4\eta}{4\log(\lambda)} - a. \tag{IV.0.10}$$

And then Ξ is atleast defined for $y(\lambda_0) \in (y_{\text{crit}}(\lambda_0), +\infty)$. As $a > 0$, there exists $\overline{\lambda_0} > 0$ large enough such that for any $\gamma > \overline{\lambda_0}$, we have $y_{\text{crit}}(\gamma) < 0$. For such a γ, we have $0 \in (y_{\text{crit}}(\gamma), +\infty)$. Therefore, we found that the set $\mathcal{S} := (\overline{\lambda_0}, +\infty) \times (y_{\text{crit}}(\overline{\lambda_0}), +\infty)$ is a convex subset of $\mathbb{R}^2$ on which Ξ is defined. Furthermore, Ξ is locally lipschitz on $\mathcal{S}$.
Let $(\gamma, Y) \in \mathcal{S}$. Then there exists $\varepsilon > 0$ and $y : (\gamma - \varepsilon, \gamma + \varepsilon) \longrightarrow \mathbb{R}$ such that

$$\begin{cases} \frac{\partial F(y,\lambda)}{\partial \lambda} = 0 \text{ for } \lambda \in (\gamma - \varepsilon, \gamma + \varepsilon) \\ y(\gamma) = Y. \end{cases} \tag{IV.0.11}$$

Therefore for such a y, we have for all $(\gamma - \varepsilon, \gamma + \varepsilon)$,

$$F(y(\lambda), \lambda) = F(\ y(\gamma) = Y\ ,\ \gamma) = e^{-4\log(\gamma)Y}\left[\eta + (Y + a)\log(\gamma)\mu\right] - \eta \tag{IV.0.12}$$

Keeping in mind that Y is required to be such that $y(\gamma) = Y > 0$ but otherwise is still to be chosen freely, we have for any $\lambda \in (\gamma - \varepsilon, \gamma + \varepsilon)$

$$F(y(\lambda), \lambda) \underset{Y \to +\infty}{\longrightarrow} -\eta < 0 \quad \text{and} \quad F(Y, \lambda) \underset{Y \to 0^+}{\longrightarrow} a\log(\lambda)\mu > 0 \tag{IV.0.13}$$

We deduce that there exists y and $Y_0 > 0$ such that for any $\lambda \in (\gamma - \varepsilon, \gamma + \varepsilon)$

$$F(y(\lambda), \lambda) = F(\ y(\gamma) = Y\ ,\ \gamma\ ) = 0 \tag{IV.0.14}$$

Hence, there exists $a, b, \gamma, \varepsilon$ such that for any $\lambda \in (\gamma - \varepsilon, \gamma + \varepsilon)$, $u_\lambda \in \mathcal{B}(\eta)$ and we conclude that

$$\mathcal{B}(\eta) \neq \varnothing \tag{IV.0.15}$$

□

The next step towards the application of the concentration-compactness principle would be to obtain a lemma of the type

$$\forall\, \eta > 0,\ c_\omega(\eta) < 0.$$

Using an adaptation of lemma (2.0.4) for an u such that $A(u) = \eta$ and injecting in the equation (IV.0.2), there would exist $u \in H^1_{\log}$ such that for any $0 < K < C$,

$$J_\omega(u) \leq \frac{\|\nabla u\|^2_{L^2}}{2} + \omega\frac{\|u\|^2_{L^2}}{2} - \frac{K\|\nabla u\|^{p-1}_{L^2}\eta^{\frac{1}{4}}\|u\|_{L^2}}{p+1} \tag{IV.0.16}$$

An idea to go forwards would be to make use of the scaling $u_\lambda(x) = \lambda^a u(\lambda^{b(\lambda)}x)$ found in (4.0.12) for one specific $\lambda > 0$ to compute in (IV.0.16),

$$J_\omega(u_\lambda) \leq \lambda^{2a}\frac{\|\nabla u\|^2_{L^2}}{2} + \lambda^{2a-2b(\lambda)}\frac{\omega\|u\|^2_{L^2}}{2} - \lambda^{pa-b(\lambda)}\frac{K\|\nabla u\|^{p-1}_{L^2}\eta^{\frac{1}{4}}\|u\|_{L^2}}{p+1}. \tag{IV.0.17}$$

As we set $a > 0$ freely in the proof of lemma (4.0.12), we could consider a here as a variable and study the behaviour of $J_\omega(u_\lambda)$ as a varies.

Part V

References

[1] H. Berestycki, P.-L. Lions, *Nonlinear Scalar Field Equations, I. Existence of a Ground State*, Arch. Rational Mech. Anal. 82, p. 313–345 (1983)

[2] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations, II existence of infinitely many solutions*, Arch. Rational Mech. Anal. 82, p. 347–375 (1983)

[3] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1*, Annales de l'I.H.P. Analyse non linéaire, Volume 1 (1984) no. 2, p. 109-145.

[4] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 2*, Annales de l'I.H.P. Analyse non linéaire, Volume 1 (1984) no. 4, p. 223-283.

[5] M. Colin, T. Watanabe, *Standing waves for the nonlinear Schrödinger equation coupled with the Maxwell equation*, Nonlinearity, Volume 30, Number 5, (2017).

[6] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics 10 (2003), American Mathematical Society.

[7] T. Cazenave, P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Commun. Math. Phys. 85 (1982), 549–561.

[8] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a non linear local term*, J. Funct. Anal. 237 (2006), 655–674.

[9] A. Azzollini, A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. 345 (2008), 90–108.