$$\mathcal{H}_0 = \sum_{i\mathbf{k}\sigma} \varepsilon_{i\mathbf{k}} c^\dagger_{i\mathbf{k}\sigma} c_{i\mathbf{k}\sigma} + \sum_{i\mathbf{k}} \left(\Delta_i c^\dagger_{i-\mathbf{k}\uparrow} c^\dagger_{i\mathbf{k}\downarrow} \right) , \tag{1}$$

where $\varepsilon_{i\mathbf{k}} = s_i \left(\mathbf{k}^2/2m_i - \varepsilon_{F_i} \right)$ and the superconducting order parameter is self-consistently determined by $\Delta_i = \sum_{j\mathbf{k}} U_{ij} \langle c_{j-\mathbf{k}\downarrow} c_{j\mathbf{k}\uparrow} \rangle$.

$$\mathcal{H}_1 = -\sum_{i\mathbf{k}\mathbf{k}'\sigma} \mathbf{J}_{i\mathbf{k}\mathbf{k}'} \cdot \mathbf{A}\, c^\dagger_{i\mathbf{k}\sigma} c_{i\mathbf{k}'\sigma} + \sum_{i\mathbf{k}\sigma} \frac{s_i e^2}{2m_i} \mathbf{A}^2\, c^\dagger_{i\mathbf{k}\sigma} c_{i\mathbf{k}\sigma}$$

$$\begin{aligned} \langle |\mathbf{e} \cdot \mathbf{J}_{i\mathbf{k}\mathbf{k}'}|^2 \rangle_{\mathrm{Av}} &= \int \frac{d\Omega_{\mathbf{k}}}{4\pi} \frac{d\Omega'_{\mathbf{k}}}{4\pi} |\mathbf{e} \cdot \mathbf{J}_{i\mathbf{k}\mathbf{k}'}|^2 \\ &\approx \frac{(ev_{F_i})^2}{3\pi N_i(0)} \frac{\gamma_i}{(\varepsilon - \varepsilon')^2 + \gamma_i^2} \end{aligned}$$

Discussion of A, A^2.

The full Hamiltonian is given by $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$.

$$\mathbf{j} = -\left\langle \frac{\delta \mathcal{H}}{\delta \mathbf{A}} \right\rangle = \mathbf{j}_P + \mathbf{j}_D$$

$$\mathbf{j}_p = \sum_{i\mathbf{k}\mathbf{k}'\sigma} \mathbf{J}_{i\mathbf{k}\mathbf{k}'} \cdot \mathbf{A}\, \langle c^\dagger_{i\mathbf{k}\sigma} c_{i\mathbf{k}'\sigma} \rangle$$

$$\mathbf{j}_D = -\sum_{i\mathbf{k}\sigma} \frac{s_i e^2}{2m_i} \mathbf{A}\, \langle c^\dagger_{i\mathbf{k}\sigma} c_{i\mathbf{k}\sigma} \rangle$$

Next we construct the density matrix

$$\rho = |\psi_0\rangle \langle \psi_0| = \begin{pmatrix} \rho^{11}_{i\mathbf{k}\mathbf{k}'} & \rho^{12}_{i\mathbf{k}\mathbf{k}'} \\ \rho^{21}_{i\mathbf{k}\mathbf{k}'} & \rho^{22}_{i\mathbf{k}\mathbf{k}'} \end{pmatrix} = \begin{pmatrix} \langle c^\dagger_{i\mathbf{k}\uparrow} c_{i\mathbf{k}'\uparrow} \rangle & \langle c^\dagger_{i\mathbf{k}\uparrow} c^\dagger_{i-\mathbf{k}'\downarrow} \rangle \\ \langle c_{i-\mathbf{k}\downarrow} c_{i\mathbf{k}'\uparrow} \rangle & \langle c_{i-\mathbf{k}\downarrow} c^\dagger_{i-\mathbf{k}'\downarrow} \rangle \end{pmatrix}$$

Writing the density matrix in vector form,

$$\rho = \begin{pmatrix} \rho^{11}_{i\mathbf{k}\mathbf{k}'} \\ \rho^{12}_{i\mathbf{k}\mathbf{k}'} \\ \rho^{21}_{i\mathbf{k}\mathbf{k}'} \\ \rho^{22}_{i\mathbf{k}\mathbf{k}'} \end{pmatrix} ,$$

the Heisenberg equation of motion can be calculated as follows

$$i \frac{d\rho_{i\mathbf{k}\mathbf{k}'}}{dt} = \sum_{\mathbf{q}} \left[H^{(1)}_{i\mathbf{k}'\mathbf{q}} \rho_{i\mathbf{k}\mathbf{q}} - H^{(2)}_{i\mathbf{q}\mathbf{k}} \rho_{i\mathbf{q}\mathbf{k}'} \right]$$

where we have defined the following two matrices:

$$H^{(1)}_{i\mathbf{k}\mathbf{k}'} = \begin{pmatrix} h^{11}_{i\mathbf{k}\mathbf{k}'} & h^{12}_{i\mathbf{k}\mathbf{k}'} & & \\ h^{21}_{i\mathbf{k}\mathbf{k}'} & h^{22}_{i\mathbf{k}\mathbf{k}'} & & \\ & & h^{11}_{i\mathbf{k}\mathbf{k}'} & h^{12}_{i\mathbf{k}\mathbf{k}'} \\ & & h^{21}_{i\mathbf{k}\mathbf{k}'} & h^{22}_{i\mathbf{k}\mathbf{k}'} \end{pmatrix}$$

$$
H^{(2)}_{i\mathbf{k}\mathbf{k}'} = \begin{pmatrix} h^{11}_{i\mathbf{k}\mathbf{k}'} & & h^{21}_{i\mathbf{k}\mathbf{k}'} & \\ & h^{11}_{i\mathbf{k}\mathbf{k}'} & & h^{21}_{i\mathbf{k}\mathbf{k}'} \\ h^{12}_{i\mathbf{k}\mathbf{k}'} & & h^{22}_{i\mathbf{k}\mathbf{k}'} & \\ & h^{12}_{i\mathbf{k}\mathbf{k}'} & & h^{22}_{i\mathbf{k}\mathbf{k}'} \end{pmatrix}
$$

Let us multiply this out:

$$
i\frac{d}{dt}\begin{pmatrix} \rho^{11}_{i\mathbf{k}\mathbf{k}'} & \rho^{12}_{i\mathbf{k}\mathbf{k}'} \\ \rho^{21}_{i\mathbf{k}\mathbf{k}'} & \rho^{22}_{i\mathbf{k}\mathbf{k}'} \end{pmatrix} = \sum_{\mathbf{q}} \begin{pmatrix} h^{11}_{i\mathbf{k}'\mathbf{q}}\rho^{11}_{i\mathbf{k}\mathbf{q}} + h^{12}_{i\mathbf{k}'\mathbf{q}}\rho^{12}_{i\mathbf{k}\mathbf{q}} - h^{11}_{i\mathbf{q}\mathbf{k}}\rho^{11}_{i\mathbf{q}\mathbf{k}'} - h^{21}_{i\mathbf{q}\mathbf{k}}\rho^{21}_{i\mathbf{q}\mathbf{k}'} & h^{21}_{i\mathbf{k}'\mathbf{q}}\rho^{11}_{i\mathbf{k}\mathbf{q}} + h^{22}_{i\mathbf{k}'\mathbf{q}}\rho^{12}_{i\mathbf{k}\mathbf{q}} - h^{11}_{i\mathbf{q}\mathbf{k}}\rho^{12}_{i\mathbf{q}\mathbf{k}'} - h^{21}_{i\mathbf{q}\mathbf{k}}\rho^{22}_{i\mathbf{q}\mathbf{k}'} \\ h^{11}_{i\mathbf{k}'\mathbf{q}}\rho^{21}_{i\mathbf{k}\mathbf{q}} + h^{12}_{i\mathbf{k}'\mathbf{q}}\rho^{22}_{i\mathbf{k}\mathbf{q}} - h^{12}_{i\mathbf{q}\mathbf{k}}\rho^{11}_{i\mathbf{q}\mathbf{k}'} - h^{22}_{i\mathbf{q}\mathbf{k}}\rho^{21}_{i\mathbf{q}\mathbf{k}'} & h^{21}_{i\mathbf{k}'\mathbf{q}}\rho^{21}_{i\mathbf{k}\mathbf{q}} + h^{22}_{i\mathbf{k}'\mathbf{q}}\rho^{22}_{i\mathbf{k}\mathbf{q}} - h^{12}_{i\mathbf{q}\mathbf{k}}\rho^{12}_{i\mathbf{q}\mathbf{k}'} - h^{22}_{i\mathbf{q}\mathbf{k}}\rho^{22}_{i\mathbf{q}\mathbf{k}'} \end{pmatrix}
$$

Now we expand the equation of motion in orders of $\mathbf{A}$. The zeroth-order components are:

$$
\rho_{i\mathbf{k}\mathbf{k}'}\big|_0 = \delta_{kk'} \begin{pmatrix} \frac{1}{2}\left(1 - \frac{\varepsilon_{i\mathbf{k}}}{E_{i\mathbf{k}}}\right) + \frac{\varepsilon_{i\mathbf{k}}}{E_{i\mathbf{k}}} f_{i\mathbf{k}} \\ -\frac{\Delta_i^{eq}}{2}(1 - 2f_{i\mathbf{k}}) \\ -\frac{\Delta_i^{eq}}{2}(1 - 2f_{i\mathbf{k}}) \\ \frac{1}{2}\left(1 + \frac{\varepsilon_{i\mathbf{k}}}{E_{i\mathbf{k}}}\right) - \frac{\varepsilon_{i\mathbf{k}}}{E_{i\mathbf{k}}} f_{i\mathbf{k}} \end{pmatrix}
$$

$$
H^{(1)}_{i\mathbf{k}\mathbf{k}'}\big|_0 = \delta_{\mathbf{k}\mathbf{k}'} \begin{pmatrix} \varepsilon_{i\mathbf{k}} & \Delta_i^{eq} & & \\ \Delta_i^{eq} & -\varepsilon_{i\mathbf{k}} & & \\ & & \varepsilon_{i\mathbf{k}} & \Delta_i^{eq} \\ & & \Delta_i^{eq} & -\varepsilon_{i\mathbf{k}} \end{pmatrix}
$$

$$
H^{(2)}_{i\mathbf{k}\mathbf{k}'}\big|_0 = \delta_{\mathbf{k}\mathbf{k}'} \begin{pmatrix} \varepsilon_{i\mathbf{k}} & & \Delta_i^{eq} & \\ & \varepsilon_{i\mathbf{k}} & & \Delta_i^{eq} \\ \Delta_i^{eq} & & -\varepsilon_{i\mathbf{k}} & \\ & \Delta_i^{eq} & & -\varepsilon_{i\mathbf{k}} \end{pmatrix}
$$

Now we proceed with the first order.

$$
i\frac{d}{dt}\rho_{i\mathbf{k}\mathbf{k}'}\big|_1 = \left(H^{(1)}_{i\mathbf{k}'\mathbf{k}'} - H^{(2)}_{i\mathbf{k}\mathbf{k}}\right)\bigg|_0 \rho_{i\mathbf{k}\mathbf{k}'}\big|_1 + \left(H^{(1)}_{i\mathbf{k}'\mathbf{k}}\big|_1 \rho_{i\mathbf{k}\mathbf{k}}\big|_0 - H^{(2)}_{i\mathbf{k}'\mathbf{k}}\big|_1 \rho_{i\mathbf{k}'\mathbf{k}'}\big|_0\right)
$$

$$
H^{(1)}_{i\mathbf{k}\mathbf{k}'}\big|_0 = -\mathbf{J}_{i\mathbf{k}\mathbf{k}'} \cdot \mathbf{A} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}
$$

$$
H^{(2)}_{i\mathbf{k}\mathbf{k}'}\big|_0 = -\mathbf{J}_{i\mathbf{k}\mathbf{k}'} \cdot \mathbf{A} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
$$

The second order is

$$
i\frac{d\rho_{i\mathbf{k}\mathbf{k}}}{dt} = \left(H^{(1)}_{i\mathbf{k}\mathbf{k}} - H^{(2)}_{i\mathbf{k}\mathbf{k}}\right)\bigg|_0 \rho_{i\mathbf{k}\mathbf{k}}\big|_2 + \sum_{\mathbf{q}} \left(H^{(1)}_{i\mathbf{k}\mathbf{q}}\big|_1 \rho_{i\mathbf{k}\mathbf{q}}\big|_1 - H^{(2)}_{i\mathbf{q}\mathbf{k}}\big|_1 \rho_{i\mathbf{q}\mathbf{k}}\big|_1\right) + \left(H^{(1)}_{i\mathbf{k}\mathbf{k}} - H^{(2)}_{i\mathbf{k}\mathbf{k}}\right)\bigg|_2 \rho_{i\mathbf{k}\mathbf{k}}\big|_0
$$

$$H^{(1,2)}_{i\mathbf{k}\mathbf{k}}\big|_2 = H^{(1,2)}_{i\mathbf{k}\mathbf{k}}\big|_{2,D} + H^{(1,2)}_{i\mathbf{k}\mathbf{k}}\big|_{2,H} + H^{(1,2)}_{i\mathbf{k}\mathbf{k}}\big|_{2,L}$$

Diamagnetic quasiparticle current contribution

$$H^{(1)}_{i\mathbf{k}\mathbf{k}'}\big|_{0,D} = \delta_{\mathbf{k}\mathbf{k}'}\frac{s_i e^2}{2m_i}\mathbf{A}^2 \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$H^{(2)}_{i\mathbf{k}\mathbf{k}'}\big|_{0,D} = \delta_{\mathbf{k}\mathbf{k}'}\frac{s_i e^2}{2m_i}\mathbf{A}^2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Higgs contribution

$$H^{(1)}_{i\mathbf{k}\mathbf{k}'}\big|_{0,D} = \delta_{\mathbf{k}\mathbf{k}'}\delta\Delta'\big|_2 \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$$

$$H^{(2)}_{i\mathbf{k}\mathbf{k}'}\big|_{0,D} = \delta_{\mathbf{k}\mathbf{k}'}\delta\Delta'\big|_2 \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

Leggett contribution

$$H^{(1)}_{i\mathbf{k}\mathbf{k}'}\big|_{0,D} = \delta_{\mathbf{k}\mathbf{k}'}\delta\Delta''\big|_2 \begin{pmatrix} & i & & \\ -i & & & \\ & & & i \\ & & -i & \end{pmatrix}$$

$$H^{(2)}_{i\mathbf{k}\mathbf{k}'}\big|_{0,D} = \delta_{\mathbf{k}\mathbf{k}'}\delta\Delta''\big|_2 \begin{pmatrix} & & -i & \\ & & & -i \\ i & & & \\ & i & & \end{pmatrix}$$

And third order

$$i\frac{d}{dt}\rho_{i\mathbf{k}\mathbf{k}'}\big|_3 = \left(H^{(1)}_{i\mathbf{k}'\mathbf{k}'} - H^{(2)}_{i\mathbf{k}\mathbf{k}}\right)\Big|_0 \rho_{i\mathbf{k}\mathbf{k}'}\big|_3 + \sum_{\mathbf{q}}\left(H^{(1)}_{i\mathbf{k}'\mathbf{q}}\big|_1 \rho_{i\mathbf{k}\mathbf{q}}\big|_2 - H^{(2)}_{i\mathbf{q}\mathbf{k}}\big|_1 \rho_{i\mathbf{q}\mathbf{k}'}\big|_2\right) + \left(H^{(1)}_{i\mathbf{k}'\mathbf{k}'} - H^{(2)}_{i\mathbf{k}\mathbf{k}}\right)\Big|_2 \rho_{i\mathbf{k}\mathbf{k}'}\big|_1$$