LAB-STA-1. MATHEMATICAL MODELING OF PHYSICAL SYSTEMS. EXAMPLES OF MATHEMATICAL MODELS OF CONTROLLED PROCESSES

A. OBJECTIVES. 1. Application of mathematical modeling principles to processes. 2. Mathematical models (MMs) of different real-world systems: theoretical modeling, simplified linear MMs, simplified MMs obtained through experimental system identification (data-driven identification).

B. THEORETICAL CONSIDERATIONS.

1. Process control (PC) (for controlled processes/plants, CPs) requires knowledge of the mathematical models of the controlled process. These MMs can be:

- detailed MMs obtained analytically (also known as first principles modeling) obtained from the physics laws that govern the controller process. These MMs are often difficult to obtain and the parameter identification is usually laborious;
- simplified MMs, they are often linear or linearized versions of their nonlinear counterparts. These MMs are usually obtained using simplifying assumptions and adequate mathematical transformations of the underlying theoretical MMs;
- simplified MMs obtained through experimental system identification procedures.

The general theory applied to mathematical modeling of different CPs is presented in detail in the literature [1], [2].

2. Different categories of typical MMs used in process control:

(a) Structure-based categorization of MMs (in terms of time-domain representation):

Input-output MMs (IO-MMs)		State-space representation (SS-MMs)	
$y(t)=f\{u(t)\}$	(1.1)	$\dot{\underline{x}}(t)=\underline{f}_1(\underline{x}(t),u(t)),$ $y(t)=\underline{f}_2(\underline{x}(t));$	(1.2)

$f(\ )$ – is a function that describes both the structure and the behavior of the system.

(b) Independent variable domain-based categorization of MMs

- time domain MMs (t-continuous time (CT); $t_k=k\,T_s$- discrete time (DT));
- operational domain (Laplace, Z-domain) MMs (e.g. H(s)|(t=CT), H(z)|(t=DT));
- frequency domain MMs (e.g., $H(j\omega)$).

Please consult the attached 'LaplaceTransform.pdf' and 'Z-Transform.pdf' files for a short introduction to these transforms.

(c) Categorization in terms of MMs properties

- parametric MMs (in different domains);
- nonparametric MMs:
  - time domain response;
  - frequency domain response.

3. Linear (linearized) Single Input-Single Output (SISO) MMs.
Input-output MMs (IO-MMs)

A. Time domain representations

- Continuous-time input-output MMs (CT-IO-MMs):

$$a_n y^{(n)}(t)+\ldots+a_1 y^{(1)}(t)+a_0 y(t)=b_m u^{(m)}(t)+..b_1 u^{(1)}(t)+b_0 u(t)\,, \tag{1.3}$$

$$H(s)\overset{\Delta}{=}\frac{y(s)}{u(s)}=\frac{b_m s^m+..b_1 s+b_0}{a_n s^n+\ldots+a_1 s+a_0},\ m<n; \tag{1.4}$$

- Discrete-time input-output MMs (DT-IO-MMs):

$$a_n y(k+n)+\ldots+a_1 y(k+1)+a_0 y(k)=b_m u(k+m)+..b_1 u(k+1)+b_0 u(k)$$

usually with a_n=1 and $m\le n$ (advance operator form), (1.5)

$$a_n y(k)+\ldots+a_1 y(k-n+1)+a_0 y(k-n)=b_m u(k-n+m)+\ldots+b_0 u(k-n)$$

usually with a_n=1 and $m\le n$ (delay operator form or filter form). (1.6)

The discrete-time recurrent form of MMs in delays (for $m=n$):

$$y(k)=-a_{n-1}y(k-1)-\ldots-a_1 y(k-n+1)+a_0 y(k-n)+b_n u(k)+..b_{n-1}u(k-1)+b_0 u(k-n) \tag{1.7}$$

B. Operational domain representation (CT and DT):

$$H(z)=\frac{y(z)}{u(z)}=\frac{b_m z^m+..b_1 z+b_0}{a_n z^n+\ldots+a_1 z+a_0};\ H(z)=\frac{b_m+\ldots+b_1 z^{-m+1}+b_0 z^{-m}}{a_n+\ldots+a_1 z^{-n+1}+a_0 z^{-n}}z^{-d}, \tag{1.8}$$

with: $n-m=d$, $T_d=d\,T_s$, T_d – process time delay (or dead time), T_s – sampling period.

State-space MMs (SS-MMs)

- Continuous-time state-space MMs (CT-SS-MMs) in zero initial conditions:

$$\begin{aligned}\dot{\underline{x}}(t)&=\underline{A}\,\underline{x}(t)+\underline{b}u(t)\\ y(t)&=\underline{c}^T\,\underline{x}(t)\end{aligned}\ \longleftarrow\!\!\text{-----------}\!\!\longrightarrow\ \begin{aligned}s\underline{x}(s)&=\underline{A}\,\underline{x}(s)+\underline{b}u(s)\\ y(t)&=\underline{c}^T\,\underline{x}(s)\end{aligned} \tag{1.9}$$

with equivalent representations (1.4) and (1.9) (for $m<n$) [3]:

$$H(s)=\underline{c}^T(s\underline{I}-\underline{A})^{-1}\underline{b},\ \underline{A}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-\frac{a_0}{a_n}&-\frac{a_1}{a_n}&-\frac{a_{n-2}}{a_n}&-\frac{a_{n-1}}{a_n}\end{bmatrix},\ \underline{b}=\begin{bmatrix}0\\0\\0\\\frac{1}{a_n}\end{bmatrix}\in\Re^n$$

$$\underline{c}^T=\begin{bmatrix}b_0 & b_1 & \ldots b_m & 0 & \ldots 0\end{bmatrix}\in\Re^n. \tag{1.10}$$

- Discrete-time state-space MMs (DT-SS-MMs) in zero initial conditions:

$$\begin{aligned}\underline{x}_{k+1}&=\underline{A}_d\,\underline{x}_k+\underline{b}_d u_k\\ y_k&=\underline{c}^T\,\underline{x}_k\end{aligned}\ \longleftarrow\!\!\text{-----------}\!\!\longrightarrow\ \begin{aligned}z\underline{x}(z)&=\underline{A}_d\,\underline{x}(z)+\underline{b}_d u(z)\\ y(z)&=\underline{c}^T\,\underline{x}(z)\end{aligned} \tag{1.11}$$

with equivalent representations (1.8) and (1.11), similar to the CT-domain case.

- The extended discrete-time MM of the controlled process. The CP is extended with the sampling element (SE) on the output and with the sampling and zero-order hold element (ZOH) on the input (Fig. 1.1). In order to be controlled in discrete time, the CP will have:

- the discrete control signal applied to the ZOH element;
- the controlled output measured at discrete time intervals.

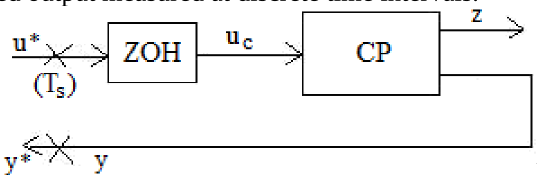


Fig. 1.1.

$$H_{EP}(z) \overset{\Delta}{=} \frac{y(z)}{u(z)} = \frac{z-1}{z} Z\{\ L^{-1}\{\frac{1}{s}H_p(s)\}|\,t_k\ \}, \tag{1.12}$$

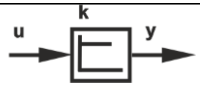
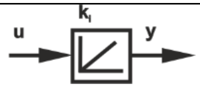
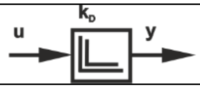
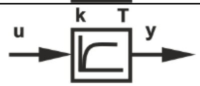
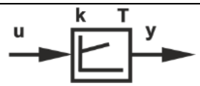
or

$$\underline{A}_d = e^{\underline{A}T_s} = \sum_{i=0}^{\infty}\frac{1}{i!}(\underline{A}T_s),\ \underline{b}_d = \underline{A}^{-1}(\underline{A}_d - \underline{I})\underline{b},\ \underline{c}_d = \underline{c}. \tag{1.13}$$

C. The main transfer elements

The main transfer elements (proportional (P), integral (I), derivative (D), low pass filter / proportional with time (PT1), proportional-integral (PI)) used among with their continuous-time transfer functions (t.f.s) are presented in Table 1.

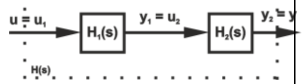
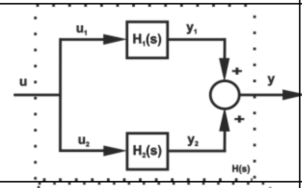
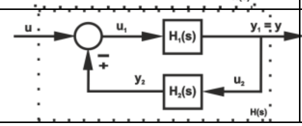
Table 1.

Block type	Block diagram	Continuous-time transfer function
P		$H(s) = \mathrm{k}$
I		$H(s) = \frac{\mathrm{k_i}}{s}$
D		$H(s) = s\mathrm{k_D}$
PT1		$H(s) = \frac{\mathrm{k}}{s\mathrm{T}+1}$
PI		$H(s) = \frac{\mathrm{k}}{s\mathrm{T}}(s\mathrm{T}+1)$

D. The main system connections

The basic systems connections under block diagram representation are the series connection, the parallel connection and the feedback connection illustrated in Table 2.

Table 2.

Connection type	Connection diagram representation	Continuous-time transfer function
series		$H(s) = \frac{\mathrm{y(s)}}{\mathrm{u(s)}} = H_1(s)H_2(s)$
parallel		$H(s) = \frac{\mathrm{y(s)}}{\mathrm{u(s)}} = H_1(s) + H_2(s)$
feedback		$H(s) = \frac{\mathrm{y(s)}}{\mathrm{u(s)}} = \frac{H_1(s)}{1 \pm H_1(s)H_2(s)}$

E. MMs of laboratory applications. Linearization of first principles MMs. MMs and graphical representations through block diagrams.

(1) Single tank system. Control purpose: water level control, that is maintaining constant the level h in the tank by changing the inflow q_a. Inflow change is ensured by variable angular speed of the supply pump (Fig. 1.2). The variable speed of the pump $\boldsymbol{\omega}$ is ensured by the variation of the DC motor input voltage u_a.

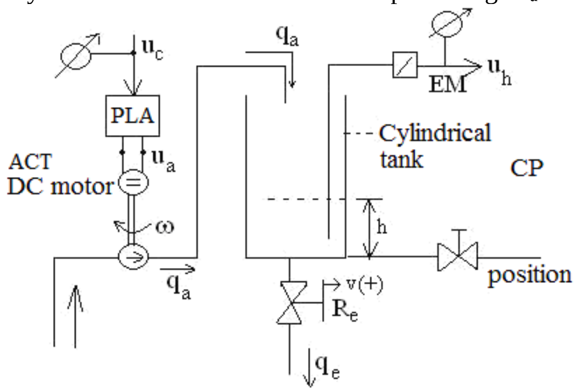


Fig. 1.2.

Characteristic variables of the CP:

- Input variables: - u_c, the command to the pump which is adjusted through the power level adaptor (PLA) to u_a (the pump is the actuator); - the disturbance outflow through an output orifice: $q_e(v(t))$;
- The controlled output: h[m], the water level; u_h is the measured output [V].

- The first principles equations of the process are:

For the actuator (ACT in Fig. 1.2, represented by the pump):

$$u_a(t) \approx k_A u_c(t); \qquad \omega(t) \approx k_m u_a; \qquad q_a(t) \approx k_p \omega(t)$$

Sensing element (the sensor or measuring element, ME in Fig. 1.2):

$$u_h(t) = k_M h(t); \tag{1.14}$$

The controlled process CP equations:

$$\dot{h}(t) = \frac{1}{A}(q_a(t) - q_e(t));$$

Bernoulli's principle states that the outflow through a tiny hole with section area S, S<<A is given by $q_e(t) = S\sqrt{2gh(t)}$.

Gathering equations (1.14), the first principles MM of the CP is:

$$\dot{h}(t) = \frac{1}{A}k_E u_c(t) - \frac{S}{A}\sqrt{2gh(t)}, \qquad k_E = k_A k_m k_p \tag{1.15}$$

$$A\dot{h}(t) + S\sqrt{2g}\sqrt{h(t)} = k_E u_c(t).$$

Remark: The first principles MM is nonlinear.

- Linearization of the first principle MM: the nonlinear term refers to the outflow from the tank which stems from the Bernoulli's principle.
- *Important*: if q_e=0, the first principle MM is linear and outlines the integrator behavior of the CP (for constant non-zero input, the water level rises).

The linearization around a steady-state operating point $P(h(t)=\text{const}=h_0, u_c(t)=\text{const})$ allows a first-order Taylor series expansion approximation for the nonlinear term

$$\frac{S}{A}\sqrt{2g}\sqrt{h(t)} \approx \frac{S}{A}\sqrt{2g}\sqrt{h_0} + \frac{S}{A}\frac{\sqrt{2g}}{2\sqrt{h_0}}\Delta h(t) \tag{1.16}$$

Substituting back in (1.15) leads to

$$\Delta\dot{h}(t) = \frac{1}{A}k_E \Delta u_c(t) - \frac{S}{A}\frac{\sqrt{2g}}{2\sqrt{h_0}}\Delta h(t)$$

or

$$\Delta\dot{h}(t) + \frac{S}{A}\frac{\sqrt{2g}}{2\sqrt{h_0}}\Delta h(t) = \frac{1}{A}k_E \Delta u_c(t) \tag{1.17}$$

PT1-like behavior *control signal (input) effect*

Units of measure for various variables and parameters:

<h>:<m>; <u_c>:<V>; <g>:<m/sec^2>;
<A>:<m^2>; <k_E>:<m^3/sec/V>; <k_{Re}>:<m>

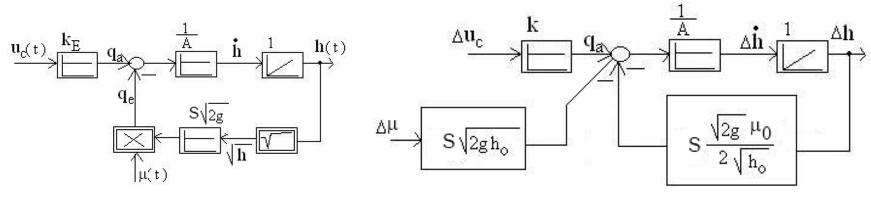
$$T = \frac{A}{S}\frac{2\sqrt{h_0}}{\sqrt{2g}}$$ with <T>:<sec> → time constant of the process.

Remark 1: The time constant value T depends on the steady-state nominal point (h_o,u_{co}).

Remark 2: The variable section of the outflow orifice can be modeled as $\mu(t)\cdot S, 0 \le \mu \le 1$. Then the coefficient $\mu(t)$ can be regarded as a time-varying disturbance.

- The corresponding block diagrams of the single tank system are shown in Fig. 1.3 in the case with varying-section outflow orifice:

(a) nonlinear block diagram,
(b) linearized block diagram.



(a) (b)

Fig. 1.3.

The block diagram representation is a conventional graphical depiction of dynamical systems. Each block has a unique characteristic transfer function whose parameters are given in the close vicinity of the box. The blocks are also called Transfer Elements. The simple-boxed blocks stand for linear systems whereas the double-boxed blocks are nonlinear ones. For a complete characterization of the blocks please consult 'Transfer_Elements_Table.pdf'.

(2) Modeling a DC motor. The structure of the system is presented in Fig. 1.4 (laboratory equipment). The DC motor has constant excitation relative to the magnetic field created by the stator magnet which produces the magnetic flux (Φ_e=const).

- Characteristic variables of the CP:

- inputs: - control signal (input) u_c;
  - disturbance input m_s;
- outputs: - shaft's (rotor) angular speed (rotational velocity)/sensor voltage corresponding to ME-ω: ω and u_ω;
  - DC motor current: i_a.

Remark: Both outputs are also state variables of the systems.

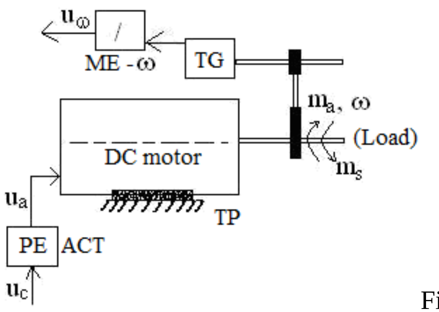


Fig. 1.4.

- First principles equations of the CP:

The actuator (ACT) represented by power electronics (PE):

$$u_a = k_E u_c$$

The process (technical process, TP, it is the DC motor):

$$\frac{L_a}{R_a}\frac{di_a}{dt} + i_a = \frac{1}{R_a}(u_a - e_\omega) \qquad e_\omega = k_e\omega, \qquad T_a = \frac{L_a}{R_a}$$

$$m_a = k_m i_a, \qquad m_f = k_f\omega \qquad (k_f \approx 0) \tag{1.18}$$

$$J\frac{d\omega}{dt} = m_a - m_f - m_s$$

Here m denotes the torques and e_ω is the back emf voltage.

The sensor (measuring element, ME):

$$u_\omega = k_{M\omega}\omega, \qquad u_i = k_{Mi} i_a$$

Remark: The first principles model is linear.

Using the physical laws, typical IO-MMs and SS-MMs are obtained. The block diagram is presented in Fig. 1.5 (CD - the control device).

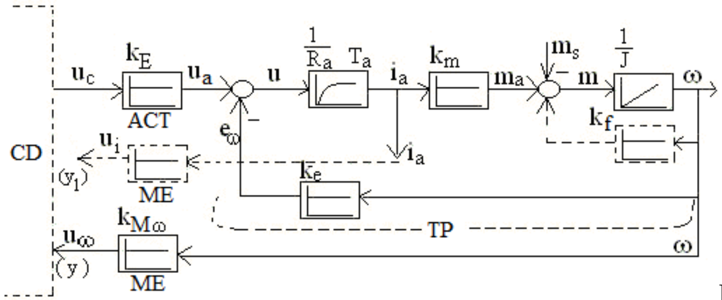


Fig. 1.5.

The typical IO-MMs and SS-MMs are obtained by proper rearrangement of the first principles equations:

- SS-MM:

$$\begin{aligned}
\dot{x}_1 &= -\frac{1}{T_a}x_1 - \frac{k_e}{L_a}x_2 + \frac{k_E}{L_a}u_c \\
\dot{x}_2 &= -\frac{k_m}{J}x_1 - \frac{k_f}{J}x_2 - \frac{1}{J}m_s\,. \\
y_1 &= k_{Mi}i_a \qquad x_1 = i_a \\
y_2 &= k_{M\omega}\omega \qquad x_2 = \omega
\end{aligned} \tag{1.19}$$

Homework: Please write down the matrices <u>A</u>, <u>B</u> ,<u>C</u>, <u>D</u> in explicit form.

- IO-MM: Accepting as a simplifying assumption that k_f=0, the IO-MMs of the physical system (the technical process, TP) illustrated in Fig. 1.6 are obtained relatively easily:

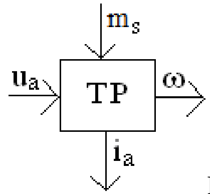


Fig. 1.6.

$$H_{\omega u_a}(s) = \left.\frac{\omega(s)}{u_a(s)}\right| = \frac{1/k_e}{1 + sT_m + s^2T_mT_a}\,(a) \qquad H_{\omega m_s}(s) = \frac{\omega(s)}{m_s(s)} = -\frac{R_a}{k_m}\frac{1+sT_a}{1 + sT_m + s^2T_mT_a}\,(b)$$

$$H_{i_a u_a}(s) = \frac{i_a(s)}{u_a(s)} = \frac{sT_m/R_a}{1 + sT_m + s^2T_mT_a} \quad (c) \qquad H_{i_a m_s}(s) = \frac{i_a(s)}{m_s(s)} = -\frac{R_a}{k_m}\frac{1}{1+sT_m+s^2T_mT_a}\,(d) \tag{1.20}$$

$$T_a = \frac{L_a}{R_a}, \quad and \quad T_m = \frac{JR_a}{k_m k_e}$$

Homework: Calculate the above transfer functions using the MM (1.19) and equation (1.10).

Usually $T_m >> T_a$ and the characteristic polynomial $\Delta_p(s)$ is written as the denominator of a second-order transfer function element:

$$\Delta_p(s) = 1 + sT_m + s^2T_mT_a \approx (1+sT_a)(1+sT_m) \tag{1.21}$$

- Experimental system identification (data-driven experimentation) of the CP. In many cases the theoretical and experimental identification of the MMs parameters is difficult. Therefore, the experimental identification of MMs (1.20) is preferred.
- The experimental identification of the MM (1.20a) via step response method by analyzing the transient response.
- Different techniques can be used for processing the recorded data.

(a) For first-order systems with delay (e.g., Fig. 1.7):

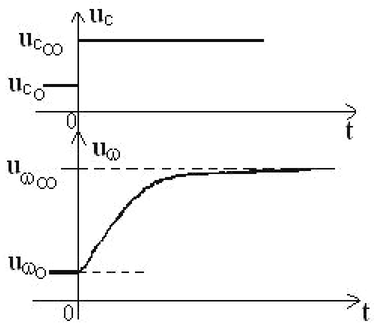


Fig. 1.7.

$$\tilde{H}_p(s) \overset{\Delta}{=} \frac{u_\omega(s)}{u_c(s)} = \frac{k_p}{(1+sT)} e^{-sT_m}, \quad k_p = \frac{u_{\omega\infty} - u_{\omega 0}}{u_{c\infty} - u_{c0}},$$

T *and* T_m *graphically.*

(b) For second-order MMs (with or without delay):

$$\tilde{H}_p(s) \overset{\Delta}{=} \frac{u_\omega(s)}{u_c(s)} = \frac{k_p}{(1+sT_1)(1+sT_\Sigma)} e^{-sT_m} \qquad (1.22)$$

$$k_p = \frac{u_{\omega\infty} \quad u_{\omega 0}}{u_{c\infty} \quad u_{c0}}$$

T_1 , T_2 , T_m are determined through various graphic-analytical methods (e.g., Fig. 1.8).

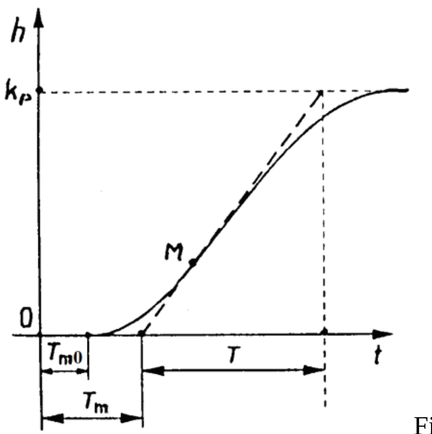


Fig. 1.8.

For validating the MM, the behavior of the identified model is simulated by applying a step input and compared to the real-world recorded data.

(3) Electrically heated room temperature control, Fig. 1.9 (a). Purpose: ensuring the thermic comfort in the room by keeping the temperature constant, θ_c=const, independent of the outside changing temperature θ_e. This is a typical regulating control problem. The main variables of the CP are:

- u_c, control input (u), input to the thyristor bridge TB;
- p_e, electrical dissipated power;
- θ_c, inside temperature (room temperature, z in general notation), is the controlled variable;
- θ_e, external temperature, regarded as a disturbance, generally noted as v.

Parameters: C_p, C_c are the thermal capacities of the heating element and of the room respectively.

The first principles equations of the CP are:

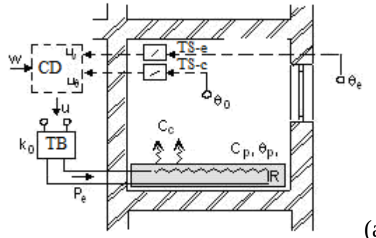
The actuator (ACT): $p_e = k_E u_c(t)$; k_E-determined experimentally

The plant (TP):
$$\begin{aligned} C_p\dot{\theta}_p &= p_e - k_p(\theta_p - \theta_c) \\ C_c\dot{\theta}_c &= k_p(\theta_p - \theta_c) - k_c(\theta_c - \theta_e) \\ z &= \theta_c \end{aligned} \qquad (1.23)$$

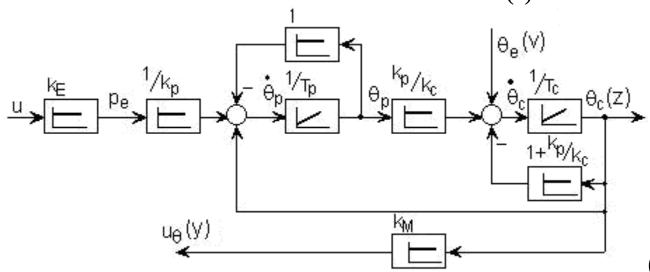
The sensor (ME, TS indicates temperature sensor):

$$u_\theta = k_M \theta_c \qquad (1.24)$$

Time constants notations: $T_p = \dfrac{C_P}{k_p}$, $T_c = \dfrac{C_c}{k_c}$.



(a)



(b)

Fig. 1.9.

Using the first principles equations, the following MMs are obtained:

- SS-MM: $x_1 = \theta_p$, $x_2 = \theta_c$:

$$\begin{bmatrix}\dot{\theta}_p\\ \dot{\theta}_c\end{bmatrix}=\begin{bmatrix}-\dfrac{1}{T_p} & \dfrac{1}{T_p}\\ \dfrac{k_p}{k_cT_c} & -\dfrac{1+k_p/k_c}{T_c}\end{bmatrix}\begin{bmatrix}\theta_p\\ \theta_c\end{bmatrix}+\begin{bmatrix}\dfrac{k_E}{k_pT_p} & 0\\ 0 & \dfrac{1}{T_c}\end{bmatrix}\begin{bmatrix}u_c\\ \theta_e\end{bmatrix} \tag{1.25}$$

$$\theta_e=\theta_c; \qquad u_\theta=k_M\theta_c$$

where the process constant parameters are: k_E = 1000, T_p = 60 (s), k_p = 500 (W/degrees), k_c = 125, T_c = 300 (s). The open-loop CP conditions of simulation (operation) are: the simulation time 12000 (s), u_c = 2.5 (V) and θ_e = -10 (ºC) after 5000 (s).

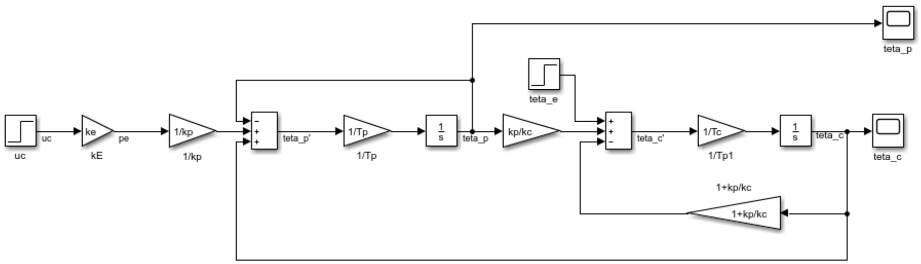
Three different implementations of the electrically heated room temperature process are depicted in Fig. 1.10 (a) – the process implementation using the informational block diagram, Fig. 1.10 (b) – the process implementation using transfer functions, Fig. 1.10 (c) – the process implementation using the SS-MM.

- ***Homework***: The t.f.s of the CP (determined using the block diagram given in Fig. 1.9 (b) and using the block diagram algebra):

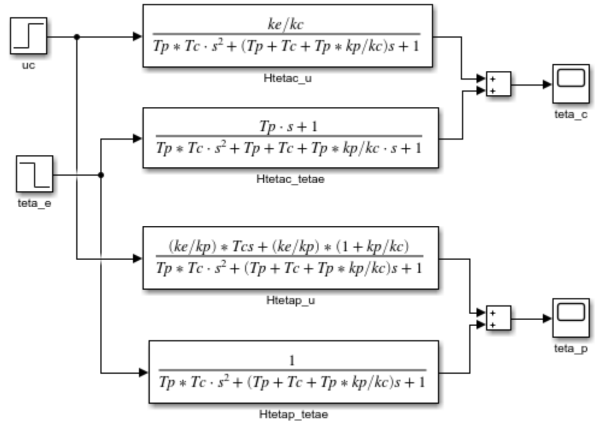
$$H_{\theta_c u_c}(s)\overset{\Delta}{=}\frac{\theta_c(s)}{u_c(s)}=\ldots\ldots\ldots\ldots\ldots\ldots \quad , \quad H_{\theta_c\theta e}(s)\overset{\Delta}{=}\frac{\theta_c(s)}{\theta_e(s)}=\ldots\ldots\ldots\ldots\ldots\ldots \tag{1.26}$$

- Static characteristics can be built using the MMs. The characteristics can then be used to define various control problems.

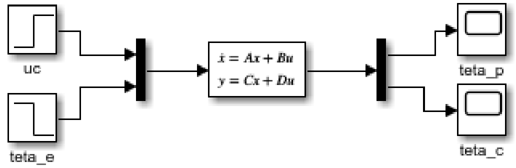
Homework: Using the SS-MM (1.25) calculate the t.f.s (1.26).



(a)



(b)



(c)

Fig. 1.10.

(4). The HIV-virus dynamic mathematical model [8]

The dynamic of the HIV virus infection can be modeled as the following state-space continuous-time nonlinear system:

$$\begin{aligned}\dot{x}(t)&=\lambda-dx(t)-\beta x(t)v(t)\\ \dot{y}(t)&=\beta x(t)v(t)-ay(t)\\ \dot{v}(t)&=ky(t)-uv(t)\end{aligned} \tag{1.27}$$

The model describes the interaction between the virus and its target cells. x denotes the number of uninfected cells, y the number of infected cells and v the number of free virus particles. Uninfected cells are produced at the rate λ and die at the rate dx. Free virus infects uninfected cells to produce infected cells at the rate βxv. The terms βxv reflects the interaction between the two cell populations. Infected cells die at the rate ay. New virus is produced from the infected cells at the rate ky and the virus dies at rate uv.

Nonzero initial conditions have to be assumed either for $y(0)$ or for $v(0)$ in order to be able to simulate the system properly.

This nonlinear model is used to study the dynamics of the virus under drug treatment and helps to understand the virus's resistance to drugs.

(5). An example of discrete-time state-space MM: The predator-prey model [6]

The predator-prey problem refers to an ecological system in which we have two species, one of which feeds on the other. Fig. 1.11 shows a historical record taken over 90 years for a population of lynxes versus a population of hares. As can been seen from the graph, the annual records of the populations of each species are oscillatory in nature.

A model is built using a discrete-time model by keeping track of the rate of births and deaths of each species. H represents the population of hares and L represent the population of lynxes. We can describe the state in terms of the populations at discrete periods of time as shown in Fig. 1.11, where the populations of hares and lynxes was recorded between 1845 and 1935 in Canada.

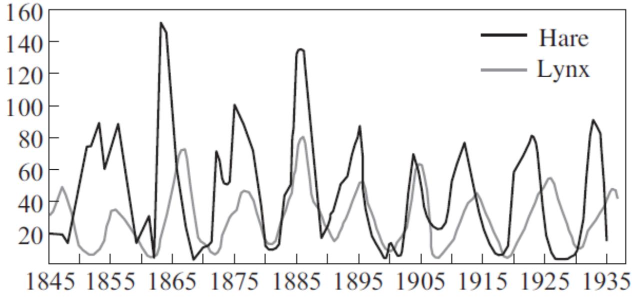


Fig. 1.11 [6].

Letting k be the discrete-time index (e.g., the day or month number), the following equations are expressed:

$$\begin{aligned} H(k+1) &= H(k) + b_r(u)H(k) - aL(k)H(k) \\ L(k+1) &= L(k) + cL(k)H(k) - d_f L(k) \end{aligned} \quad . \qquad (1.28)$$

where $b_r(u)$ is the hare birth rate per unit period and as a function of the food supply u, d_f is the lynx mortality rate and a and c are the interaction coefficients. The interaction term $aL(k)H(k)$ models the rate of predation, which is assumed to be proportional to the rate at which predators and prey meet and is hence given by the product of the population sizes. The interaction term $cL(k)H(k)$ in the lynx dynamics has a similar form and represents the rate of growth of the lynx population (as in the HIV model). This model makes many simplifying assumptions—such as the fact that hares decrease in number only through predation by lynxes (not disease or natural death)—but it often is sufficient to answer basic questions about the system. Also note that the modeled system is nonlinear.

For an initial population of hares and lynxes given as initial conditions for the dynamic system (1.28) ($x(0)$=($H(0)$,$L(0)$)) simulations can be performed in order to observe the dynamics of the two populations over time. Using the parameters a=c=0.014, $b_r(u)$ = 0.6 and d = 0.7 in equation (1.28) with daily updates, the period and magnitude of the lynx and hare population cycles shown in Fig. 1.12 approximately match the data in Fig. 1.11.

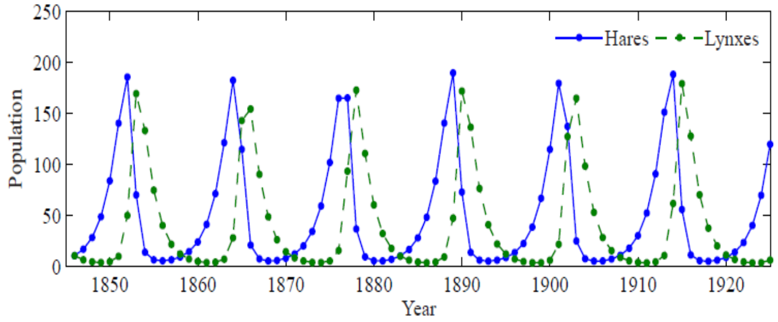


Fig. 1.12 [6].

(6). An email server modeled as a discrete-time system [6]

The IBM Lotus server is an collaborative software system that administers users' e-mail, documents and notes. Client machines interact with end users to provide access to data and applications. The server also handles other administrative tasks. In the early development of the system it was observed that the performance was poor when the central processing unit (CPU) was overloaded because of too many service requests, and mechanisms to control the load were therefore introduced.

The interaction between the client and the server is in the form of remote procedure calls (RPCs). The server maintains a log of statistics of completed requests. The total number of requests being served, called RIS (RPCs in server), is also measured. The load on the server is controlled by a parameter called MaxUsers, which sets the total number of client connections to the server. This parameter is controlled by the system administrator. The server can be regarded as a dynamical system with MaxUsers as the input and RIS as the output. The relationship between input and output was first investigated by exploring the steady-state performance and was found to be linear.

A dynamic model in the form of a first-order difference equation is used to capture the dynamic behavior of this system. Using system identification techniques, the following discrete-time linear input output model can be used:

$$y(k+1) = ay(k) + bu(k) . \qquad (1.29)$$

where $u = MaxUsers - \overline{MaxUsers}$, and $y = RIS - \overline{RIS}$. The parameters a=0.43 and b=0.47 are parameters that describe the dynamics of the system around the operating point, and $\overline{MaxUsers} = 165$ and $\overline{RIS} = 135$ represent the nominal operating point of the system. The number of requests was averaged over a sampling period of 60 s.

Applying the Z-transform to equation (1.29) assuming zero initial conditions, the discrete transfer function from the input u to the output y is $H(z) = Y(z)/U(z) = b/(z-a)$.

The email server model is interesting also for the fact that it has variables that can take only integer values.

(7). A discrete-time model associated to a bank account [7]

A bank account can also be modeled as a discrete-time linear system. Let the system dynamics be written as

$$\begin{aligned} &x(k+1) = (1+\rho)x(k) + u(k), \\ &x(0) = x_0 \end{aligned} \quad (1.30)$$

where k is the year counter, ρ is the interest rate, $x(k)$ is the wealth at the beginning of year k, $u(k)$ is the money saved at the end of the year k, x_0 is the initial money in the bank account. For x_0=10.000€, $u(k)$=5000€ and ρ=10%, the simulation of the bank account evolution over time shows up as:

$$x(k) = (1.1)^k \cdot 10 + (1-1.1^k)/(1-1.1)\cdot 5 = 60\cdot(1.1)^k - 50\,. \quad (1.31)$$

For this simple system, using (1.10) and considering the single state as the output y(k)=x(k), the discrete transfer function can be calculated for the matrices A=1.1, B=1, C=1 to give H(z)=1/(z–1.1). The single root of the denominator is 1.1 and this shows that the system is unstable as predicted by (1.31), where for a constant input the output will grow after each year.

(8). Student dynamics modeled as a discrete-time system [7]

Given a 3 years undergraduate course, the simplifying assumption that the percentages of students promoted, repeaters, and dropouts are roughly constant, the direct enrollment in the 2nd and 3rd years is not allowed and students can not enroll for more than 3 years, a 3rd order linear discrete-time dynamic model can be expressed as the following state-space model (SS-MM):

$$\begin{aligned} &x_1(k+1) = \beta_1 x_1(k) + u(k) \\ &x_2(k+1) = \alpha_1 x_1(k) + \beta_2 x_2(k) \\ &x_3(k+1) = \alpha_2 x_2(k) + \beta_3 x_3(k) \\ &y(k) = \alpha_3 x_3(k) \end{aligned} \quad (1.32)$$

where k is the year, $x_i(k)$ is the number of students enrolled in year i at year k, $u(k)$ is the number of freshmen at year k, $y(k)$ is the number of graduates at year k, α_i is the promotion rate during year i , β_i is the failure rate during year i. Here also zero initial conditions are assumed!

Homework:

The **prediction** problem: For α_1=0.6, α_2=0.8, α_3=0.9, β_1=0.2, β_2=0.15, β_3=0.08, $u(k)$=50 for all k, starting with k=2014, simulate the system until year 2025 and observe the evolution of the number of graduates.

The **control** problem: Suppose that we want 115 graduate students in years 2017, 2018, 2019,… . How should the sequence of the number of freshmen look like?

Given the state-space model (1.32) calculate the discrete transfer function from the input $u(k)$ to the output $y(k)$, using (1.11).

For the system (1.33), a state space representation given the parameters values from the Homework is (in matrix notation):

$$\begin{aligned} &\mathbf{x}(k+1) = \begin{pmatrix} 0.2 & 0 & 0 \\ 0.6 & 0.15 & 0 \\ 0 & 0.8 & 0.08 \end{pmatrix}\mathbf{x}(k) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}u(k) \\ &y(k) = \begin{pmatrix} 0 & 0 & 0.9 \end{pmatrix}\mathbf{x}(k) \end{aligned} \quad (1.33)$$

Using (1.10), the discrete transfer function from the input $u(k)$ to the output is calculated as H(z)=0.432/(z^3-0.43z^2+0.058z-0.0024). The delayed operator form implies the recursive computation of the output as

$$y(k) = 0.43y(k-1) - 0.058y(k-2) + 0.0024y(k-3) + 0.432u(k-3)\,. \quad (1.34)$$

(9). Modeling a supply chain [7]

S purchases the quantity $u(k)$ of raw material each month k as illustrated in Fig. 1.12. A fraction δ_1 of the raw material is discarded and another fraction α_1 is shipped to producer P. P sells a fraction α_2 to retailer R, another fraction δ_2 being discarded. The retailer sells a fraction γ_3 to customers and returns a fraction β_3 of defective products to the producer.

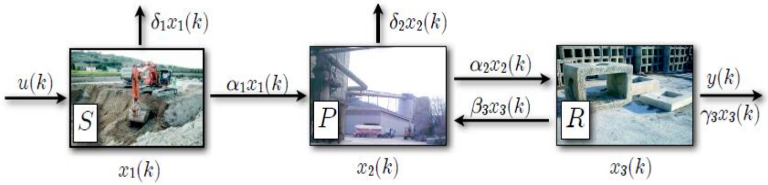


Fig. 1.12 [7].

The discrete-time linear state-space MM can be formulated as:

$$\begin{aligned} &x_1(k+1) = (1-\alpha_1-\delta_1)x_1(k) + u(k) \\ &x_2(k+1) = (1-\alpha_2-\delta_2)x_2(k) + \alpha_1 x_1(k) + \beta_3 x_3(k), \mathbf{x}(0) = \mathbf{0} \\ &x_3(k+1) = (1-\beta_3-\gamma_3)x_3(k) + \alpha_2 x_2(k) \\ &y(k) = \gamma_3 x_3(k) \end{aligned} \quad (1.35)$$

In this example zero initial conditions are also assumed. A state-space representation can be written as:

$$\begin{aligned} &\mathbf{x}(k+1) = \begin{pmatrix} 1-\alpha_1-\delta_1 & 0 & 0 \\ \alpha_1 & 1-\alpha_2-\delta_2 & \beta_3 \\ 0 & \alpha_2 & 1-\beta_3-\gamma_3 \end{pmatrix}\mathbf{x}(k) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}u(k) \\ &y(k) = \begin{pmatrix} 0 & 0 & \gamma_3 \end{pmatrix}\mathbf{x}(k) \end{aligned} \quad (1.36)$$

Homework: Using (1.10), a discrete transfer function can be obtained from the input $u(k)$ to the output $y(k)$. Calculate this transfer function.

Additional homework:

1) Calculate the Laplace transforms of the following functions (explicitly 'by hand') using the definition:

a) $2t^2$

b) $3\sin(2t)$

c) $3e^{-t}$

d) te^{-t}

2) Find the inverse Laplace transform of the following functions:

a) $1/(s^2+2s+1)$

b) $(2s+4)/[(s+1)(s^2-2)]$

Hint: Use partial fraction expansion (decomposition) and the cover-up method of finding the coefficients.

3) Use the Laplace transform to solve the following differential equations:

a) $y'+y=2t^2-1$, $y(0)=-1$.

b) $y''+4y=\sin(2t)$, $y(0)=1$, $y'(0)=0$.

4) Calculate the Z-transform of the following functions:

a) $x(k)=(1/2)^k$

b) $x(k)=k$;

c) $x(k)=e^{ak}$

d) $x(k)=\begin{cases} 2, k \geq 3 \\ 0, k < 3 \end{cases}$

5) Use the Z-transform and tables to find the solution to the exponential growth model $y(k+1)=(1+r)y(k)$ with $r=0.1$ and $y(0)=100$.

6) Using the Z-transform, solve the difference equation $y(k+1)-3y(k)=4^k$ with the initial condition $y(0)=2$.

7) Use the Z-transform to solve the system of difference equations:

$x(k+1)-y(k)=0$,

$y(k+1)+x(k)=0$, with $x(0)=1$ and $y(0)=0$.

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