Any non-finite sigma algebra is not countable

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Let's prove the statement of the title. But first

Definition 1. Let Ω be a non-empty set. Then any collection of subsets of Ω, namely $\mathcal{A}$, is a σ-algebra of Ω if it the satisfies following properties

(i) $A \in \mathcal{A} \implies A \subseteq \Omega$;

(ii) $\emptyset \in \Omega$;

(iii) For any countable subset of $\mathcal{A}$ we have that its union is still an element of $\mathcal{A}$. That is:

$$\forall \mathcal{F} \subseteq \mathcal{A}, \text{ if } \mathcal{F} \text{ is countable } \implies \bigcup \mathcal{F} \in \mathcal{A}.$$

(iv) For any $A \in \mathcal{A}$ we have $A^c \in \mathcal{A}$.

Now, given the definition of a σ-algebra, we will prove the desired.

Proposition 1. *Any non-finite σ-algebra of a set Ω is not countable.*

Demonstração. Let $\mathcal{A}$ be a non-finite σ-algebra of subsets of Ω (Ω non-empty). We start noticing that it must exist a subset $A_0 \in \mathcal{A}$ such that

$$A_0 \neq \emptyset \quad \text{and} \quad A_0 \neq \Omega.$$

With that we can break Ω into two slices A_0 and A_0^c. Here $A_0^c \in \mathcal{A}$ because of the property of a σ-algebra. That is

Now, it comes from the hypothesis that $\mathcal{A}$ is not finite that either

$$\operatorname{card}\{A \in \mathcal{A};\ A \subseteq A_0\} < \infty \quad \text{or} \quad \operatorname{card}\{A \in \mathcal{A};\ A \subseteq A_0^c\} < \infty.$$

This is true due to the fact that for all $A \in \mathcal{A}$ we have $A = (A \cap A_0) \cup (A \cap A_0^c)$ which means

$$\mathcal{A} = (\mathcal{A} \cap A_0) \bigcup (\mathcal{A} \cap A_0^c),$$

where

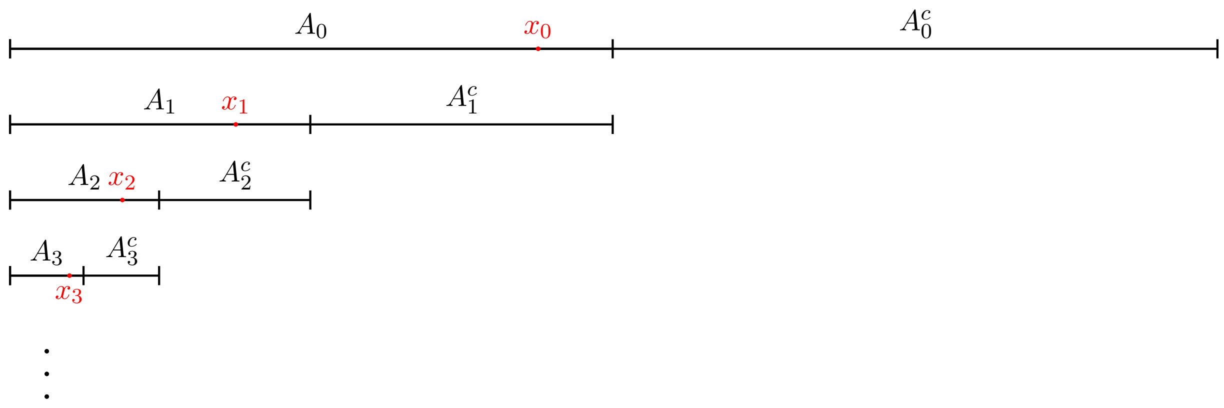
$$\mathcal{A} \cap B := \{A \cap B;\ A \in \mathcal{B}\}$$

and $B \subseteq \Omega$ a set. Then, if both $\mathcal{A} \cap A_0, \mathcal{A} \cap A_0^c$ were finite $\mathcal{A}$ would be also finite, which is impossible by hypothese.

Let's consider, without loss of generalization, that $\mathcal{A} \cap A_0$ is not finite. What we will observe is that we can continue this argument recursively. This is so beacuse $\mathcal{A} \cap A_0$ is also a *sigma*-algebra, which as showed before is not finite. Therefore, we can build by induction a sequence of σ-algebras

$$\mathcal{A} \cap A_0 \supseteq \mathcal{A} \cap A_1 \supseteq \mathcal{A} \cap A_2 \supseteq \cdots \supseteq \mathcal{A} \cap A_n \supseteq \mathcal{A} \cap A_{n+1} \cdots$$

where for each index $i \in \mathbb{N} \cup \{0\}$ there is a point x_i such that $x_i \notin A_{i+1}$. In other words, we have the following



Notice that this property is merely a consequence of what was showed at the beggining and used at the induction.

Now comes the interesting part. Firstly, we must look at the sequence of non-empty measurable sets

$$B_0 = A_0 \setminus A_1 \in \mathcal{A}, B_1 = A_1 \setminus A_2 \in \mathcal{A}, \ldots B_i = A_i \setminus A_{i+1} \in \mathcal{A}, \ldots$$

which satisfies $B_i \cap B_j = \emptyset$, for indexes $i \neq j$. Now, for convinience we will define the collections

$$\mathcal{F}_i := \{B_{2i-1}, B_{2i-2}\}, \forall i \in \mathbb{N}$$

which will look like

$$\mathcal{F}_1 = \{B_1, B_0\}, \mathcal{F}_2 = \{B_3, B_2\}, \mathcal{F}_3 = \{B_5, B_4\}, \ldots$$

Then, remember that the set

$$\mathcal{C} := \{f : \mathbb{N} \to \{0, 1\};\ f \text{ a function}\}$$

is not countable (check the Cantor's diagonal argument to prove this fact [1]). and define the sequence of functions

$$\begin{aligned} \xi_i : \{0,1\} &\to \mathcal{F}_i = \{B_{2i-1}, B_{2i-2}\} \\ j &\mapsto \begin{cases} B_{2i-2} & \text{if } j = 0; \\ B_{2i-1} & \text{if } j = 1, \end{cases} \end{aligned}$$

with $i \in \mathbb{N}$. Notice that the set $\mathcal{C}$ is the key to prove the proposition succesfully.

Finally, we just need to prove that the following function

$$\begin{aligned} \phi : \mathcal{C} &\to \mathcal{A} \\ f &\mapsto \bigcup_{i \in \mathbb{N}} \xi_i(f(i)). \end{aligned}$$

is injective. Whenever the injection is proved we conclude that

$$\operatorname{card}(\mathcal{C}) = \operatorname{card}(\operatorname{Im}(\phi)) \leq \operatorname{card}(\mathcal{A})$$

Let $f, g \in \mathcal{C}$ be two different functions. That is, we can find a natural number $j \in \mathbb{N}$ such that $f(j) \neq g(j)$. Without loss of generalization, lets assume that $f(j) = 0$ and $g(j) = 1$. This gives us

$$\xi_j(f(j)) = B_{2j-2}, \quad \text{and} \quad , \xi_j(g(j)) = B_{2j-1}.$$

[1] https://en.wikipedia.org/wiki/Cantor's_diagonal_argument

Thus, from the fact that

$$\forall i \neq j,\ B_i \cap B_j = \emptyset$$

we obtain

$$B_{2j-2} \cap \bigcup_{i \in \mathbb{N}} \xi_i(g(i)) = \bigcup_{i \in \mathbb{N}} (\xi_i(g(i)) \cap B_{2j-2}) = (B_{2j-1} \cap B_{2j-2}) \cup \bigcup_{i \in \mathbb{N} \setminus \{j\}} (\xi_i(g(i)) \cap B_{2j-2}) = \emptyset.$$

Consequently, $\xi(f) \neq \xi(g)$ and ξ is a injective function. Hence, the σ-algebra is not countable!!! $\square$