Exercise 7 - chapter 4. Book: "Introdução a Medida e Integração"

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Definition 0.1. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space. Then, we say that any measurable set $A \in \mathcal{A}$ is an atom if

$$\forall B \in \mathcal{A}\, (B \subseteq A \implies \mu(B) = 0 \text{ or } \mu(B) = \mu(A).)$$

Thus we say that a masure μ is non-atomic if it doesn't admit an atomic measurable space as element of its $\sigma-$algebra. In other words, μ is non-atomic if

$$\forall A \in \mathcal{A}, \exists B \in \mathcal{A} \setminus \{\emptyset\}\, (B \subseteq A \text{ and } 0 < \mu(B) < \mu(A))$$

Now that we know the meaning of a measure being non-atomic we will proove the following

Proposition 0.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space and μ a non-atomic measure. Then for any measurable set $A \in \mathcal{A}$ such that*

$$\mu(A) < \infty$$

we have guaranteed that for any real number $\alpha \in \mathbb{R}$ satisfying

$$0 < \alpha < \mu(A)$$

there is $Bin\mathcal{A}$ where the equality

$$\mu(B) = \alpha$$

holds.

Proof. The proof for this proposition is quite trick. I will make use of the Zorn's Lemma to proceed with the proof (check the section below **??**). □

1 Zorn's Lemma

We say that a non empty set X is endowed with a partial order $\leq$ if

(i) $\leq$ is *reflexive*, which means

$$\forall x \in X(x \leq x)$$

(ii) $\leq$ is *anti-simmetric*:

$$\forall x, y \in X(x \leq y, y \leq x \implies x = y)$$

(iii) $\leq$ is *transitive*:

$$\forall x, y, z \in X(x \leq y, y \leq z \implies x \leq z).$$

Every set X in posse of a partial order is called as a partially ordered set.

Before we adress the Zorn's Lemma we must introduction some notions. Consider X a partially ordered set and $Y \subseteq X$ a subset. We say that

1. $x_0 \in X$ is an *upper bound* of Y if $\forall y \in Y, y \leq x_0$;
2. $x_0 \in X$ is a *maximal element* of X if $\forall x \in X, x \leq x_0$;
3. Y is a *chain* if $\forall x, y \in Y, x \leq y$ or $y \leq x$.

Now we are able of stating the Zorn's Lemma, which is a proposition equivalent to the axiom of choice

Proposition 1.1. *Let $X, \leq$ be a partially ordered set. If any chain $Y \subseteq X$ admits an upper bound on X, that is*

$$\forall Y \subseteq X, \forall y \in Y, \exists y_0 \in X \left(Y \text{ non empty and it is a chain } \implies y \leq y_0\right),$$

then X does have a maximal element.

2 Inducing a partial order into a measurable space

As usual, consider $(\Omega, \mathcal{A}, \mu)$ a measurable space. (Ω non empty). We say that two measurable sets $X, Y \in \mathcal{A}$ are almost equal if

$$\mu(X \setminus Y \cup Y \setminus X) = 0.$$

From this definition we state the following relation

$$\forall X, Y \in \mathcal{A} X \sim Y \iff \mu(X \setminus Y \cup Y \setminus X) = 0.$$

Notice that

(i) $\forall X \in \mathcal{A} \implies X \sim X$;

(ii) $\forall X, Y \in \mathcal{A}$, clearly $X \sim Y$ and $Y \sim X$;

(iii) $\forall X, Y, Z \in \mathcal{A}$ such that

$$X \sim Y \quad \text{and} \quad Y \sim Z$$

then

$$\begin{aligned}
\mu(Z \setminus X \cup X \setminus Z) &= \mu([(Z \setminus X) \cup (X \setminus Z)] \cap (Y \cup Y^c)) \\
&= \mu([(Z \setminus X) \cap (Y \cup Y^c)] \cup [(X \setminus Z) \cap (Y \cup Y^c)]) \\
&= \mu([(Z \cap X^c \cap Y) \cup (Z \cap X^c \cap Y^c)] \cup [(X \cap Z^c \cap Y) \cup (X \cap Z^c \cap Y^c)]) \\
&\le \mu(Z \cap X^c \cap Y) + \mu(Z \cap X^c \cap Y^c) + \mu(X \cap Z^c \cap Y) + \mu(X \cap Z^c \cap Y^c) \\
&\le \mu(X^c \cap Y) + \mu(Z \cap Y^c) + \mu(Z^c \cap Y) + \mu(X \cap Y^c) \\
&= 0.
\end{aligned}$$

consequently, we conclude that $\sim$ is an equivalence relation and therefore we can reduce $\mathcal{A}$ into its equivalence classes

$$\mathcal{A}/\sim = \{[A];\ \forall A \in \mathcal{A}\}$$

where

$$\forall A \in \mathcal{A}, [A] = \{B \in \mathcal{A};\ B \sim A\}.$$

Now we need set operators into the set $\mathcal{A}/\sim$. Define, for all $[A], [B] \in \mathcal{A}/\sim$:

$$[A] \sqcup [B] = [A \cup B], \quad [A] \sqcap [B] = [A \cap B]$$

and

$$[A] \setminus [B] = [A \setminus B]$$

Before we carry on, notice that:

- $\mathcal{A}/\sim$ is still a $\sigma-$algebra;
- and μ can be extended to such $\sigma-$algebra.

2.1 $\mathcal{A}/\sim$ is still a $\sigma-$algebra

Whenever operating with equivalence relations, one must show that the operations extended from its base are well definied.

We start by the union. Let A_0, A_1, B_0, B_1 be measurable sets satisfying

$$A_0 \sim A_1 \quad \text{and} \quad B_0 \sim B_1.$$

Let's check what is the value of

$$[A_1] \sqcup [B_1] = [A_1 \cup B_1].$$

We just need to inspect that

$$A_1 \cup B_1 \sim A_0 \cup B_0.$$

Let's do it

$$\begin{aligned}
&\mu\Big(((A_1 \cup B_1) \setminus (A_0 \cup B_0)) \cup ((A_0 \cup B_0) \setminus (A_1 \cup B_1))\Big)\\
&\quad = \mu(((A_1 \cup B_1) \cap (A_0^c \cap B_0^c)) \cup ((A_0 \cup B_0) \cap (A_1^c \cap B_1^c))\\
&\quad = \mu(((A_1 \cap A_0^c \cap B_0^c) \cup (B_1 \cap A_0^c \cap B_0^c)) \cup ((A_0 \cap A_1^c \cap B_1^c) \cup (B_0 \cap A_1^c \cap B_1^c))\\
&\quad \leq \mu(A_1 \cap A_0^c)\mu(B_1 \cap B_0^c) + \mu(A_0 \cap A_1^c)\mu(B_0 \cap B_1^c))\\
&\quad = 0.
\end{aligned}$$

As a consequence

$$A_1 \cup B_1 \sim A_0 \cup B_0 \implies [A_1 \cup B_1] = [A_0 \cup B_0]$$

and we conclude that the unio is well defined.

For the operator $\sqcap$ we will have the same arguments above and hence I won't do it. It only rest to check the complementary operation.

To show that taking the complementar is still a well defined operation we nee to show that

$$[A_1] \setminus [B_1] = [A_1 \setminus B_1]$$

is equal to the same above but changing the indexes to 0. Again, we need to show that $A_1 \setminus B_1 \sim A_0 \setminus B_0$. This is done below

$$\begin{aligned}
&\mu((A_1 \setminus B_1) \setminus (A_0 \setminus B_0) \cup (A_0 \setminus B_0) \setminus (A_1 \setminus B_1))\\
&\quad = \mu(((A_1 \cap B_1^c) \cap (A_0 \cap B_0^c)^c) \cup ((A_0 \cap B_0^c) \cap (A_1 \cap B_1^c)^c))\\
&\quad = \mu(((A_1 \cap B_1^c) \cap (A_0^c \cap B_0)) \cup ((A_0 \cap B_0^c) \cap (A_1^c \cap B_1)))\\
&\quad \leq 0.
\end{aligned}$$

As a result we have shown that the operations ove classes are well defined. Now we need to prove that $\mathcal{A}/\sim$