Exercise 1 - chapter 4. Book: "Introdução a Medida e Integração"

Rafael Polli Carneiro

August 2021

Contents

Here we will prove the following statement

Proposition 0.1. *Let $(\Omega, \mathcal{A}, \lambda)$ be a measurable space where $\lambda : \mathcal{A} \to [0, \infty]$ is the Lebesgue extension of the measure $\lambda : \mathcal{S} \to [0, \infty)$, $\mathcal{S}$ is the semi-ring of the limited intervals on $\mathbb{R}$, and*

$$\lambda(I) = \sup(I) - \inf(I), \forall I \in \mathcal{S}.$$

Now, for any $\alpha \in (0, 1) \subseteq \mathbb{R}$ there is a measurable set A_0 whose interior is empty but its measurable is equal to α:

$$\lambda(A_0) = \alpha. \tag{1}$$

Proof. The idea follows the same steps done to create the ternary set of Cantor. Firstly, notice that, for any $\beta \in \mathbb{R}$ we have

$$\sum_{i=1}^{\infty} \frac{\beta}{2^i} = \beta \sum_{i=1}^{\infty} \frac{1}{2^i} = \beta.$$

This series gives us the steps necessary to produce each interval that should be taken away from the interval $[0, 1]$. We will do it in the following manner. For $i = 1$ we define

$$K_1 = \bigcup_{a \in \gamma_1} \left[a, a + \frac{1 + \alpha}{2^{1+1}} \right],$$

with $\gamma_1 = \{0, \frac{1+\alpha}{2^{1+1}} + \frac{1-\alpha}{2^1}\}$, then, by induction, considering K_n, γ_n well defined for all naturals, we set

$$K_{n+1} = \bigcup_{a \in \gamma_{n+1}} \left[a, a + \frac{1 + \alpha}{2^{(n+1)+1}} \right],$$

with

$$\gamma_{n+1} = \left\{ a + l(\frac{1+\alpha}{2^{2(n+1)}} + \frac{1-\alpha}{2^{2((n+1)-1)+1}});\ l = 0 \text{ or } l = 1 \text{ and } a \in \gamma_n \right\}.$$

This is the same done at the construction of the Cantor's set, differing only at the lenght of the slices being cut at each interaction.

It is worth mentioning that $1/4$ of each interval cut at interaction n is less than the lenght of the intervals left at the same step of the iteraction. That is

$$\frac{1-\alpha}{2^{2(n-1)+1}} 2^{-2} < \frac{1+\alpha}{2^{2(n)}}.$$

Now, as the standard construction, we notice that for all $n \in \mathbb{N}$ K_n are closed sets (finite union of closed sets) and therefore the set

$$K = \bigcap_{n \in \mathbb{N}} K_n$$

is closed as well. Since we are working with the $\sigma-$algebra of the borelian sets. We conclude that K is a measurable set. Its interior is empty. In fact, since we are shrinking down the lenght of each interval within K_n to zero, none interval can be included at K.

Finally, recall that at each step of the induction we remove 2^{n-1} disjoint intervals of length equal to $\frac{1-\alpha}{2^{2(n-1)+1}}$. Thus

$$\lambda(K) = 1 - \sum_{n=1}^{\infty} 2^{n-1} \frac{1-\alpha}{2^{2(n-1)+1}} = 1 - (1-\alpha) \sum_{n=1}^{\infty} 2^{-n} = \alpha$$

$\square$