Poisson Equation & Jacobi solver

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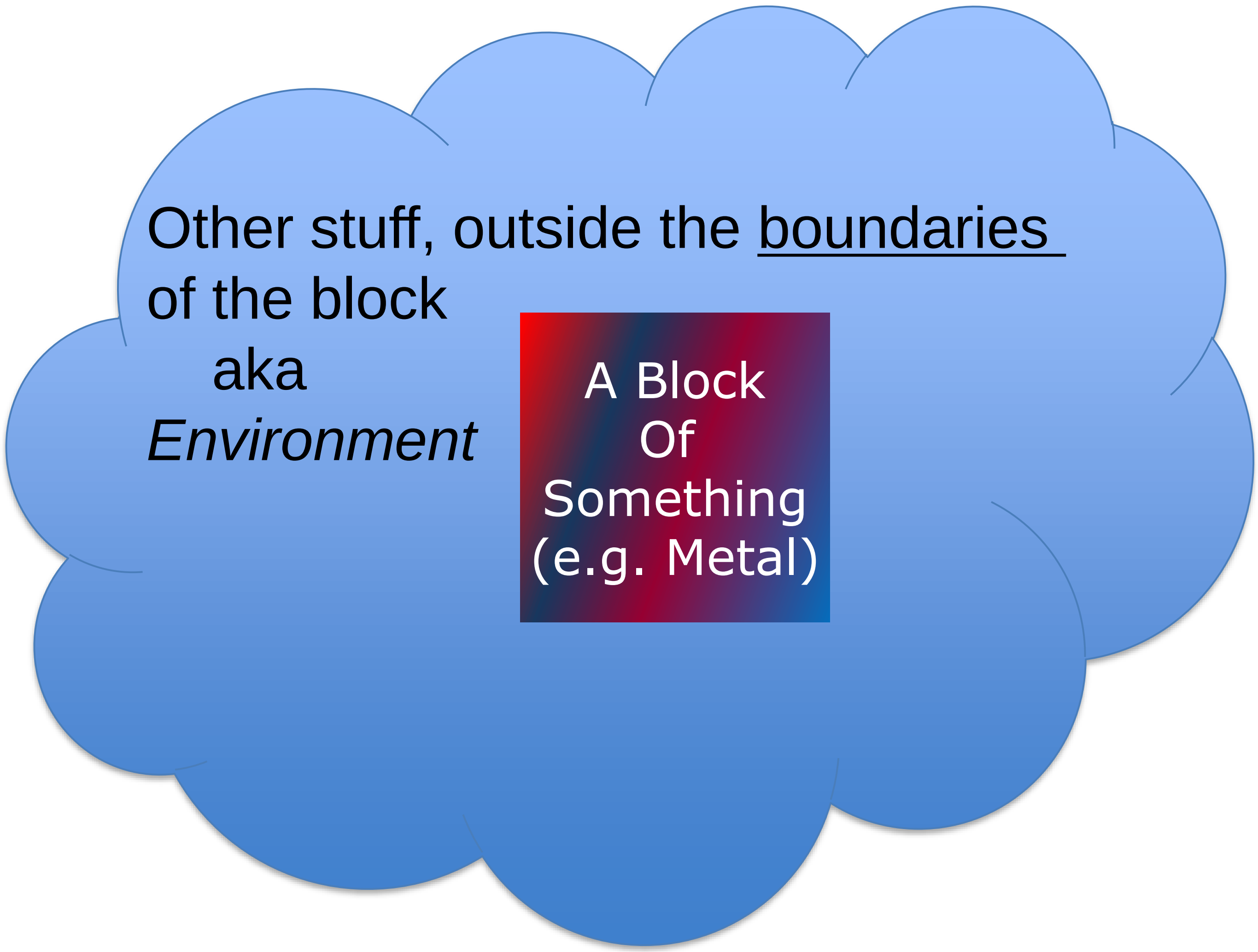
Solving a Heat Transfer Equation

Physics / Math
are NOT part of the exam!

But, it helps you to understand
what you are doing
and why
…

Temperature Changes On a small Block



Temperature Changes On a small Block

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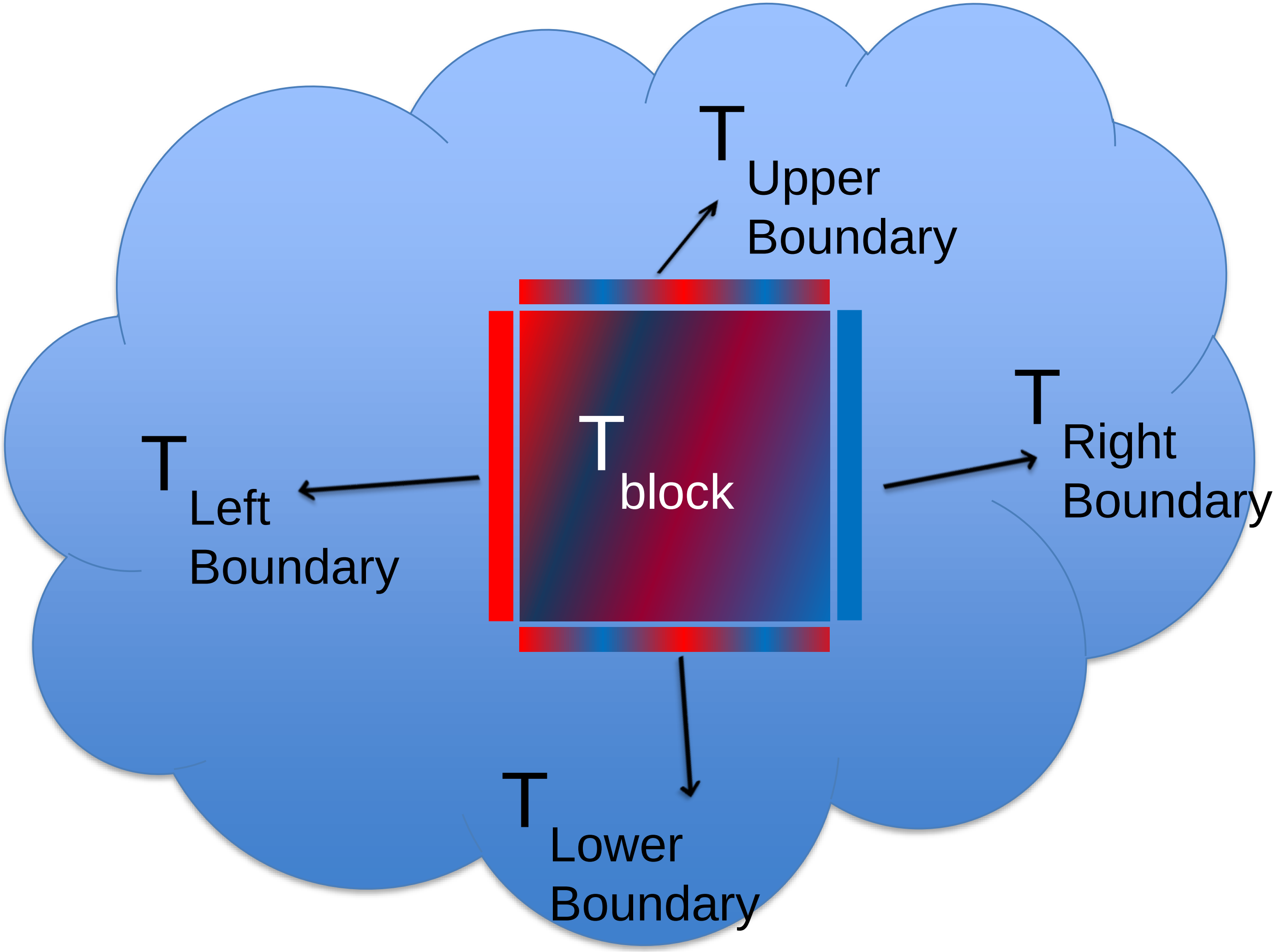
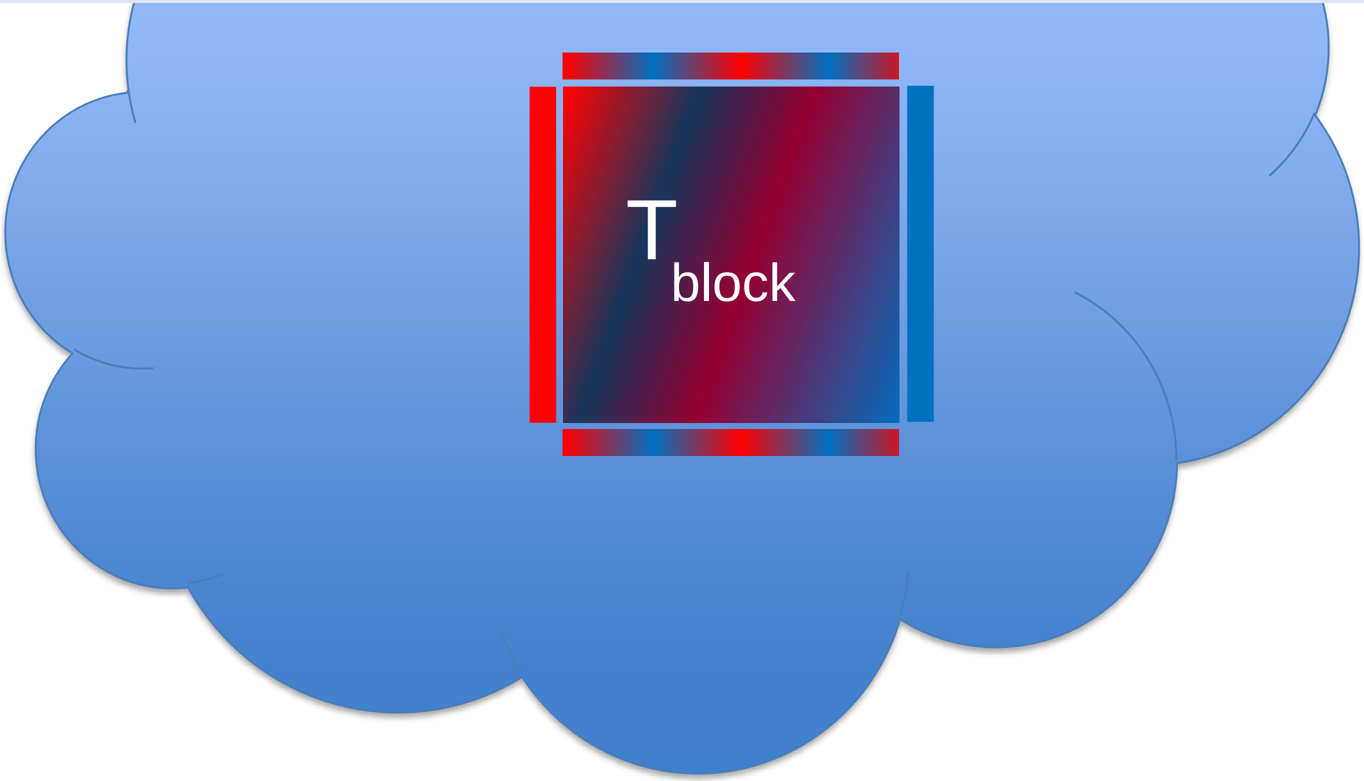


Temperature Changes On a small Block

We are interested in the Temperature changes over Time

Caused by difference in temperature at different location.



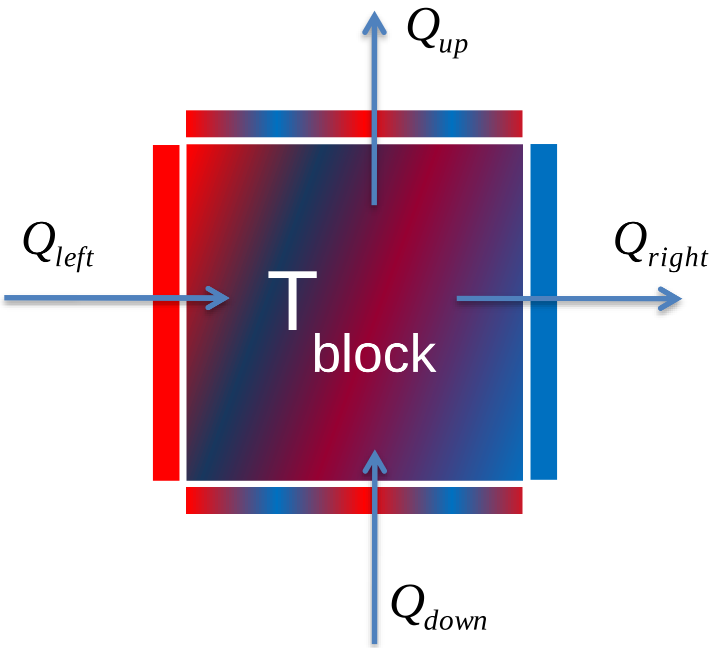
Temperature Changes On a small Block

$$\underset{t=t}{T_{block}} - \underset{t=0}{T_{block}} =$$

$$\frac{Heat\ Entered\ the\ block}{C_p \times V} =$$

$$\frac{(Q_L - Q_R + Q_D - Q_U) \times time}{C_p \times V} =$$

$$\frac{Q_L - Q_R + Q_D - Q_U}{C_p \times V} \times t$$



Reformulate:

$$(C_p \times V) \times \frac{\underset{t=t}{T_{block}} - \underset{t=0}{T_{block}}}{t} = Q_L - Q_R + Q_D - Q_U$$

Where C_p is material property (constant) , and V is block volu me.

Temperature Changes On a small Block

$$Rate\ of\ Heat\ Transfer =$$

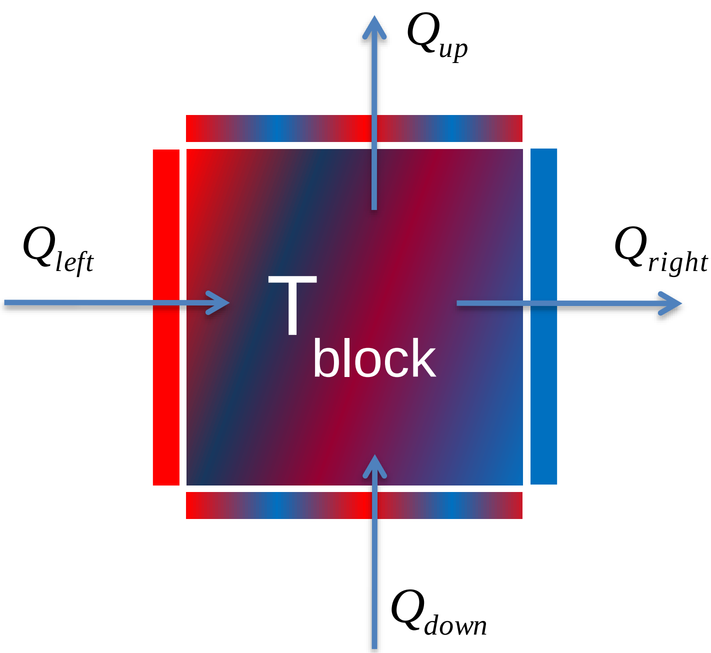
$$Q_{a,b} =$$

$$-K \cdot A \cdot \left.\frac{\partial T}{\partial x}\right|_x \approx$$

$$-K \cdot A \cdot \frac{T_b - T_a}{\mathrm{Distance}}$$

Where K is material property (constant),

and A is heat transfer area.



Temperature Changes On a small Block

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$$(C_p \times V) \times \frac{T_{\substack{block\\t=t}} - T_{\substack{block\\t=0}}}{t} =$$

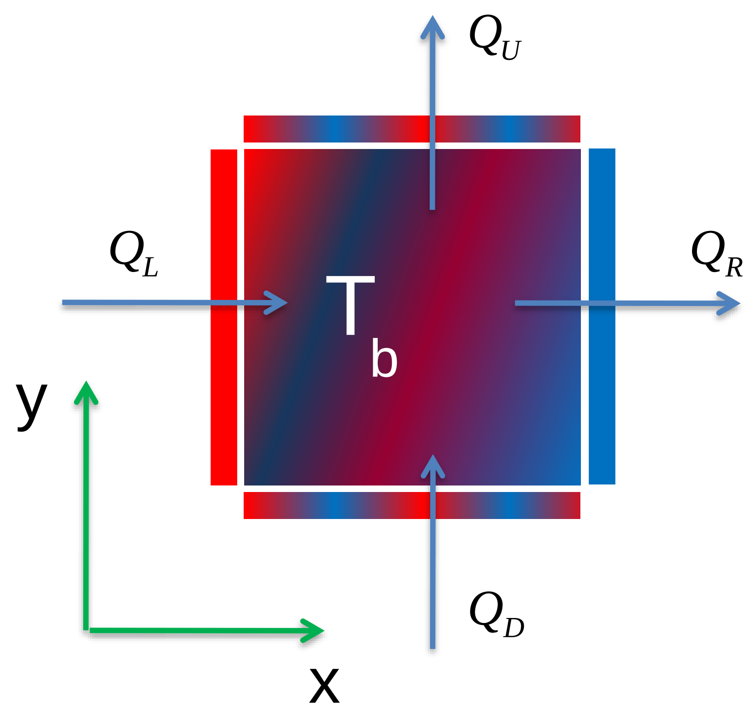
$$Q_L - Q_R + Q_D - Q_U =$$

$$(-AK \frac{T_b - T_L}{\mathrm{X_b} - \mathrm{X_L}})$$

$$-\ (-AK \frac{T_R - T_b}{\mathrm{X_R} - \mathrm{X_b}})$$

$$+\ (-AK \frac{T_b - T_D}{\mathrm{Y_b} - \mathrm{Y_D}})$$

$$-\ (-AK \frac{T_U - T_b}{\mathrm{Y_U} - \mathrm{Y_b}})$$



Temperature Changes On a small Block

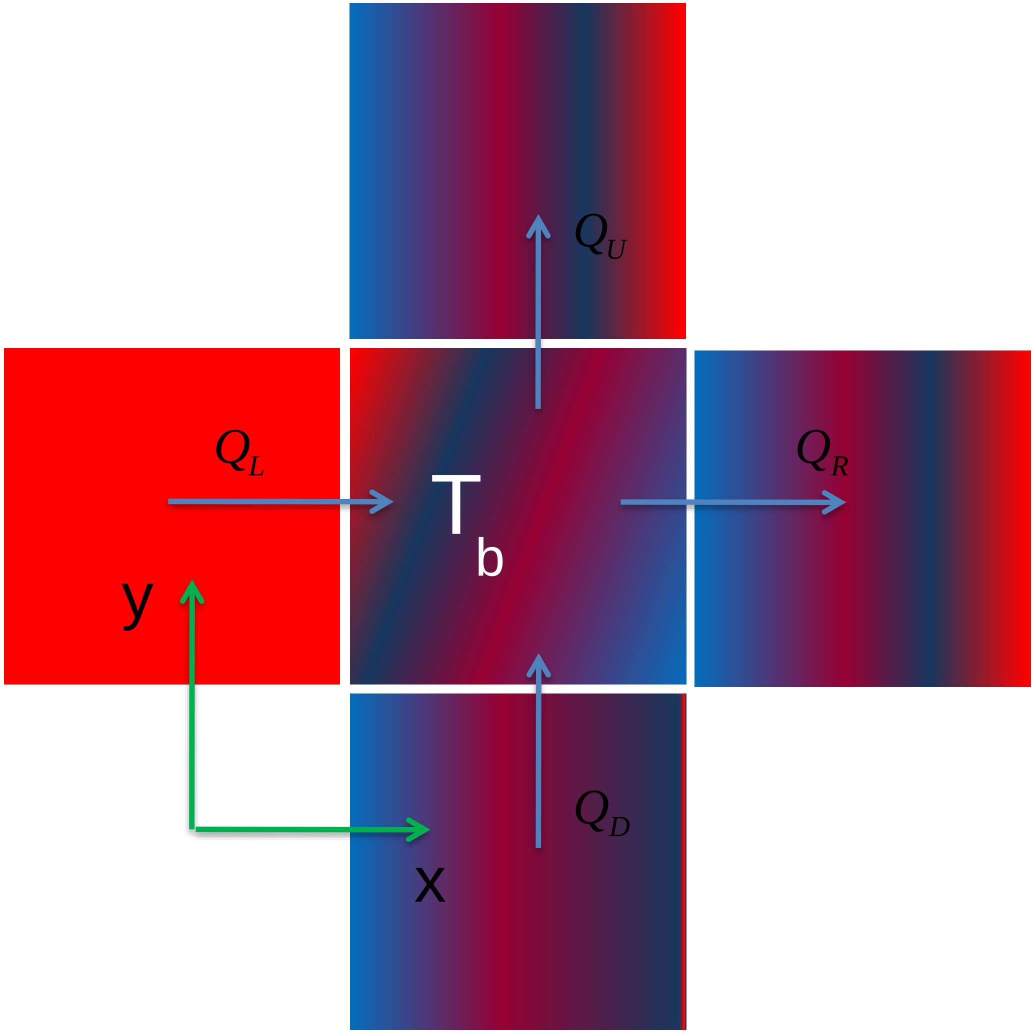
Equidistant grid

$$X_b - X_L = X_R - X_b = \Delta x$$
$$=$$
$$Y_b - Y_D = Y_U - Y_b = \Delta y$$
$$= h$$

$$A = L\ h = h$$

$$V = L\ \Delta x\ \Delta y = h^2$$

$$L = Length\ of\ block = 1$$



Temperature Changes On a small Block

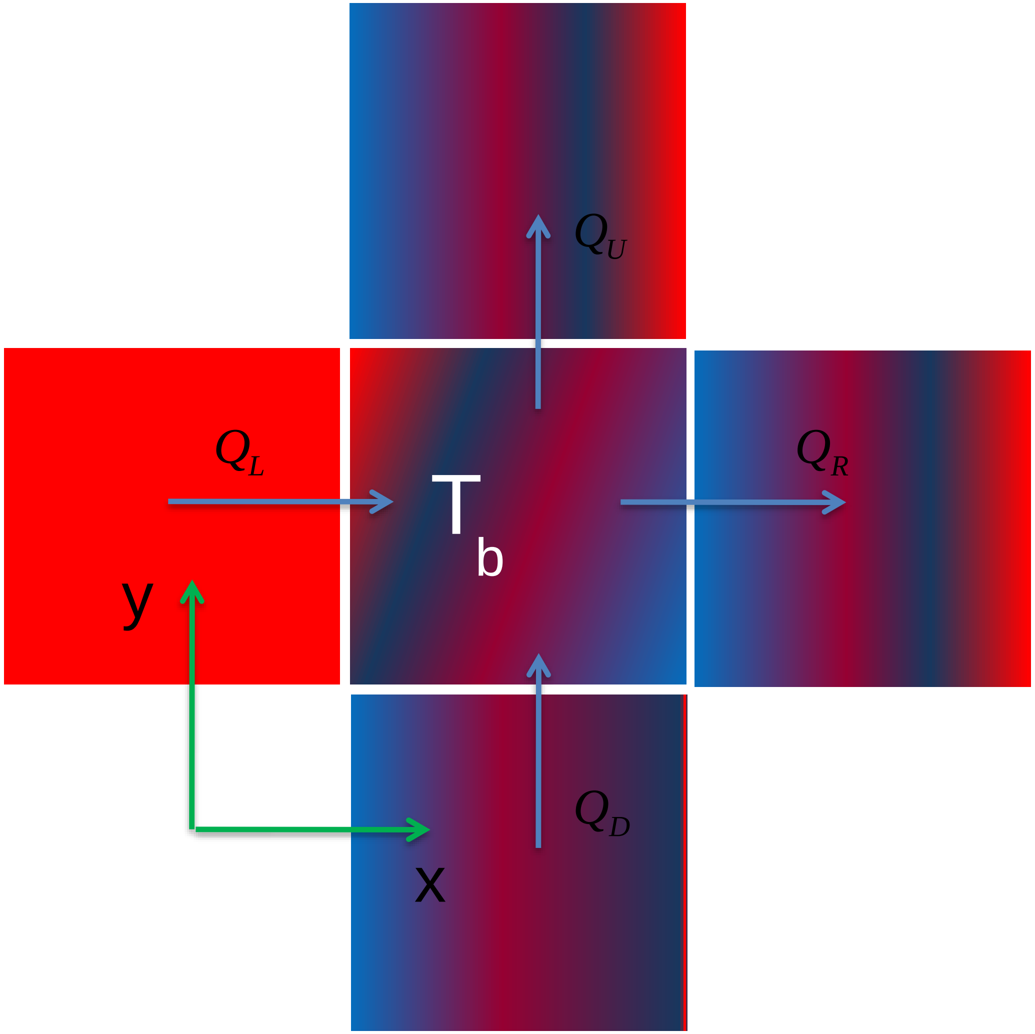
$$-\;\frac{\underset{t=t}{T_{block}} - \underset{t=0}{T_{block}}}{time} \times C_p \times h^2 \;=$$

$$h\,K\,\frac{T_b - T_L}{h}$$

$$-\;h\,K\,\frac{T_R - T_b}{h}$$

$$+\;h\,K\,\frac{T_b - T_D}{h}$$

$$-\;h\,K\,\frac{T_U - T_b}{h}$$

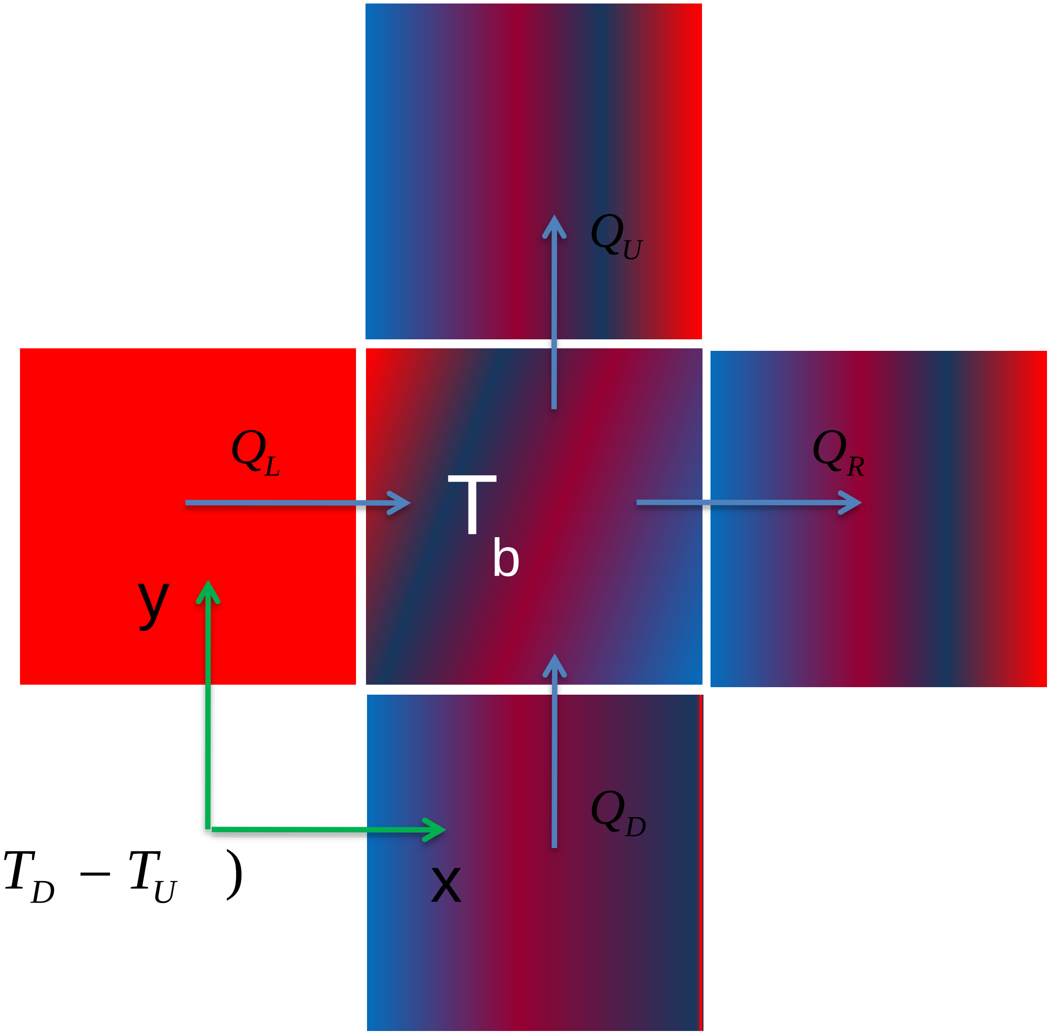


Temperature Changes On a small Block

$$\frac{T_{\substack{block\\t=t}} - T_{\substack{block\\t=0}}}{t} \times C_p =$$

$$\frac{-K}{h^2}\Big(T_b - T_L - (T_R - T_b) + T_b - T_D - (T_U - T_b) \Big) =$$

$$\frac{-K}{h^2}\Big(4T_b - T_L - T_R - T_D - T_U \Big)$$



Temperature Changes On a small Block

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$$\underset{t=t}{T_{block}} - \underset{t=0}{T_{block}} =$$

$$\frac{-t.K}{h^2 \times C_p} \left(4T_b - T_L - T_R - T_D - T_U \right) =$$

$$\frac{t.K}{h^2 \times C_p} \left([T_L - 2T_b + T_R] + [T_D - 2T_b + T_U] \right) =$$

$$\cong t.\alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\underset{t=t}{T_{block}} - \underset{t=0}{T_{block}}}{t} \cong \frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{T(x+\Delta x, y) - 2T(x,y) + T(x-\Delta x, y)}{(\Delta x)^2}$$

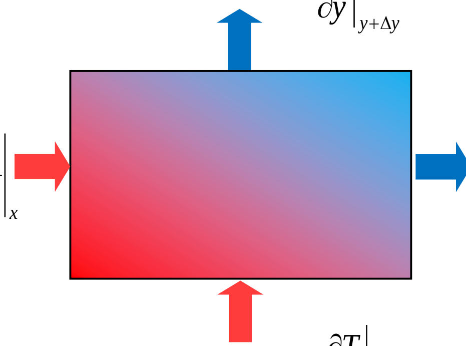
$$\rightarrow \left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \cong \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}$$

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \nabla.\nabla T = \Delta T$$

Heat Equation

$$Q_4 = -K \cdot \Delta x \cdot \left.\frac{\partial T}{\partial y}\right|_{y+\Delta y}$$

$$Q_1 = -K \cdot \Delta y \cdot \left.\frac{\partial T}{\partial x}\right|_{x}$$



$$Q_2 = -K \cdot \Delta y \cdot \left.\frac{\partial T}{\partial x}\right|_{x+\Delta x}$$

$$Q_3 = -K \cdot \Delta x \cdot \left.\frac{\partial T}{\partial y}\right|_{y}$$

$T(x, y, t)$ Temperatur e function

$\frac{\partial T}{\partial x}$ *and* $\frac{\partial T}{\partial y}$ variation of temperatur e

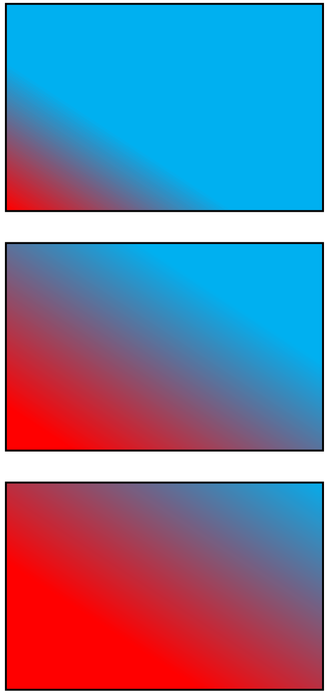
Total Heat gain by an element per unit time

$$\frac{\partial T}{\partial t} = Q_1 - Q_2 + Q_3 - Q_4 = K \cdot \Delta y \left(-\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x} + \left.\frac{\partial T}{\partial x}\right|_{x} \right) + K \cdot \Delta x \left(-\left.\frac{\partial T}{\partial y}\right|_{y+\Delta y} - \left.\frac{\partial T}{\partial y}\right|_{y} \right)$$

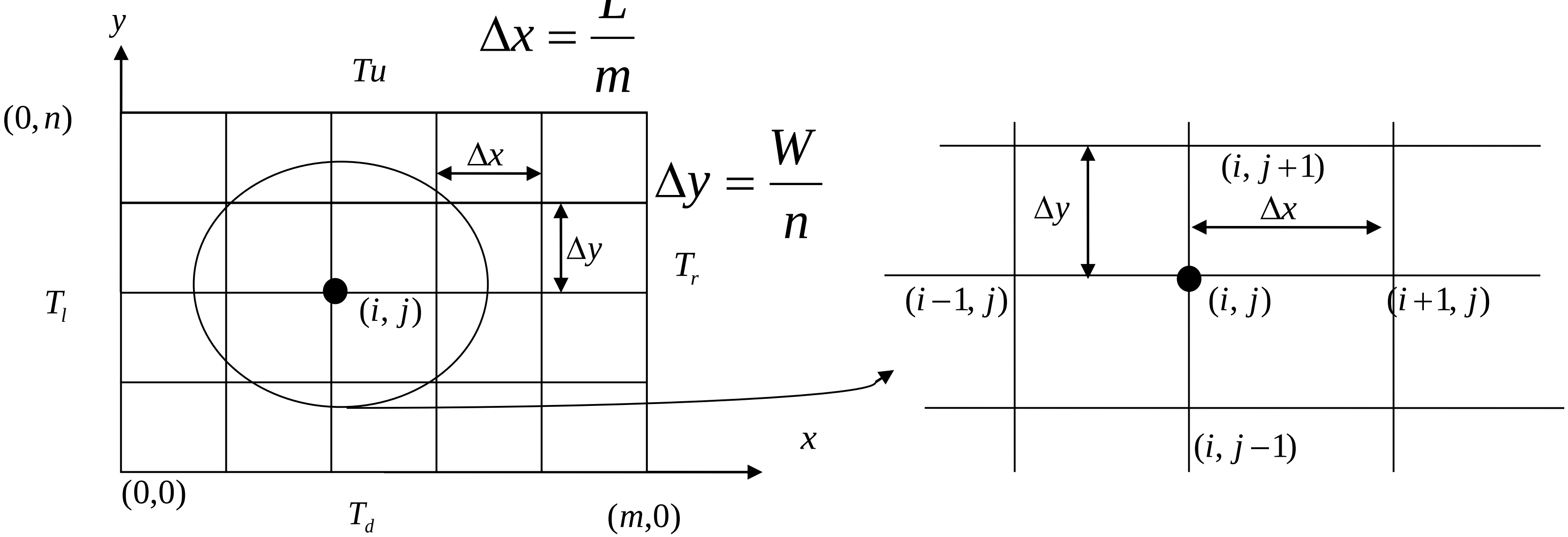
$$\frac{\partial T}{\partial t} = K \cdot \Delta x . \Delta y \left[\frac{\left(-\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x} + \left.\frac{\partial T}{\partial x}\right|_{x} \right)}{\Delta x} + \frac{\left(-\left.\frac{\partial T}{\partial y}\right|_{y+\Delta y} - \left.\frac{\partial T}{\partial y}\right|_{y} \right)}{\Delta y} \right].$$

$$\frac{\partial T}{\partial t} = f(x, y) \quad \text{and} \quad \Delta x, \Delta y \to 0$$

$$f(x, y) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \quad \textit{or} \quad \Delta u = b \quad \text{Poisson Equation}$$



Discretizing the PDE



$$\Delta x = \frac{L}{m}$$

$$\Delta y = \frac{W}{n}$$

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{T(x+\Delta x, y) - 2T(x,y) + T(x-\Delta x, y)}{(\Delta x)^2} \rightarrow \left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \cong \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 T}{\partial y^2} \cong \frac{T(x, y+\Delta y) - 2T(x,y) + T(x, y-\Delta y)}{(\Delta y)^2} \rightarrow \left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j} \cong \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

Discretizing the PDE

Substituti ng approximat ion to Poission Equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=F(x,y) \quad \Rightarrow \quad \frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{(\Delta x)^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{(\Delta y)^2}=F(x,y)$$

if, $\quad \Delta x=\Delta y.$

Poission' s Equation can be written as

$$T_{i+1,j}+T_{i-1,j}+T_{i,j+1}+T_{i,j-1}-4T_{i,j}=F(x,y)$$

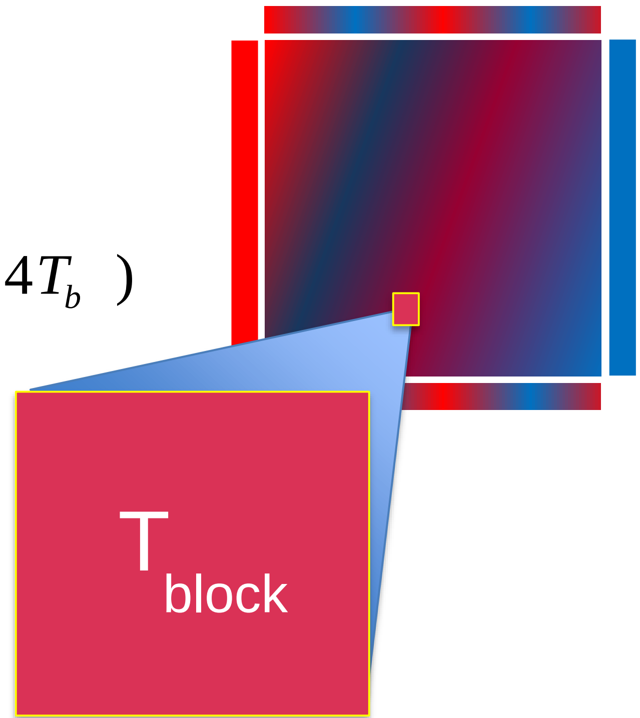
Note : In an stable condition, the value of temperatu re at each point of the domain wou ld not change in time.

$$\underset{t=t}{T_{i,j}} = \underset{t=0}{T_{i,j}} \Rightarrow \underset{t=t}{T_{block}} - \underset{t=0}{T_{block}} = F(x,y)=0$$

Temperature Changes On a Bigger Block

Inside the domain:

$$\underset{t=t}{T_{block}} - \underset{t=0}{T_{block}} = \alpha \left(T_L + T_R + T_D + T_U - 4T_b \right)$$



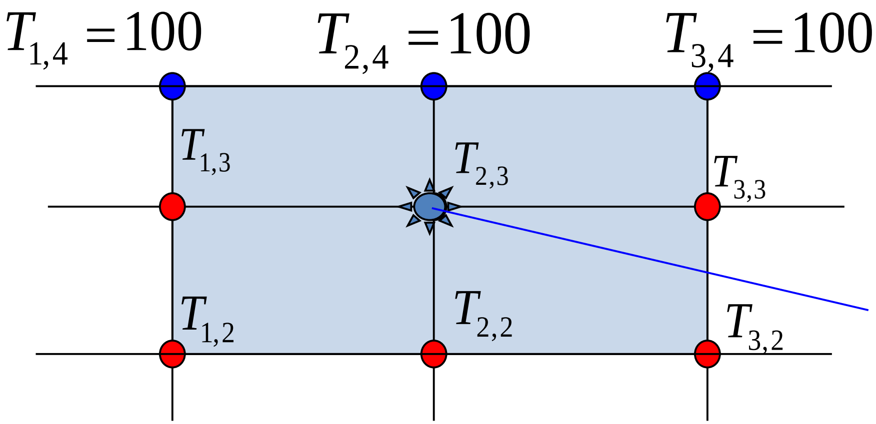
On the boundaries,
two possible Boundary Conditions (B.C):

- We have the value of Temperature (Dirichlet B.C)
- We have the value for derivative of Temperature (Neumann B.C)

Dirichlet B.C on the Upper side:

stable condition

$$T_{2,4} = 100\ °C$$



$$T_{1,3} + T_{2,2} + T_{3,3} + T_{2,4} - 4\ T_{2,3} = 0$$

$$T_{2,4} = 100$$

$$T_{1,3} + T_{2,2} + T_{3,3} + 100 - 4\ T_{2,3} = 0$$

$$T_{1,3} + T_{2,2} + T_{3,3} \qquad - 4\ T_{2,3} = -100$$

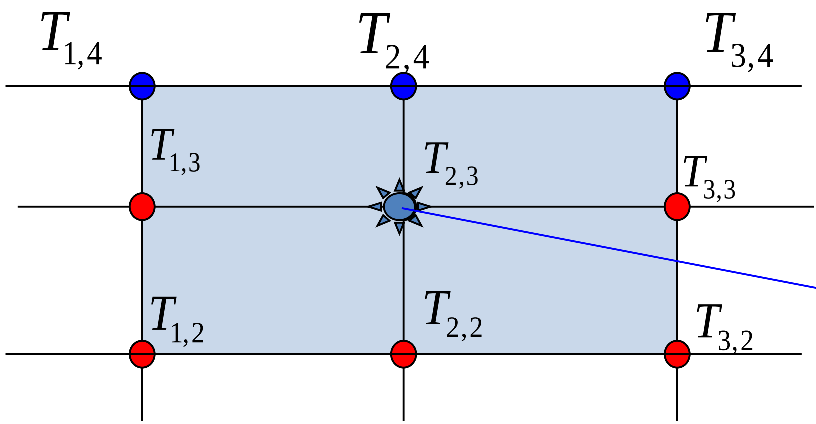
- Known
- Unknown

Neumann B.C

Neumann B.C on the Upper side:

$$\left.\frac{\partial T}{\partial y}\right|_{2,3} = 0$$

stable condition



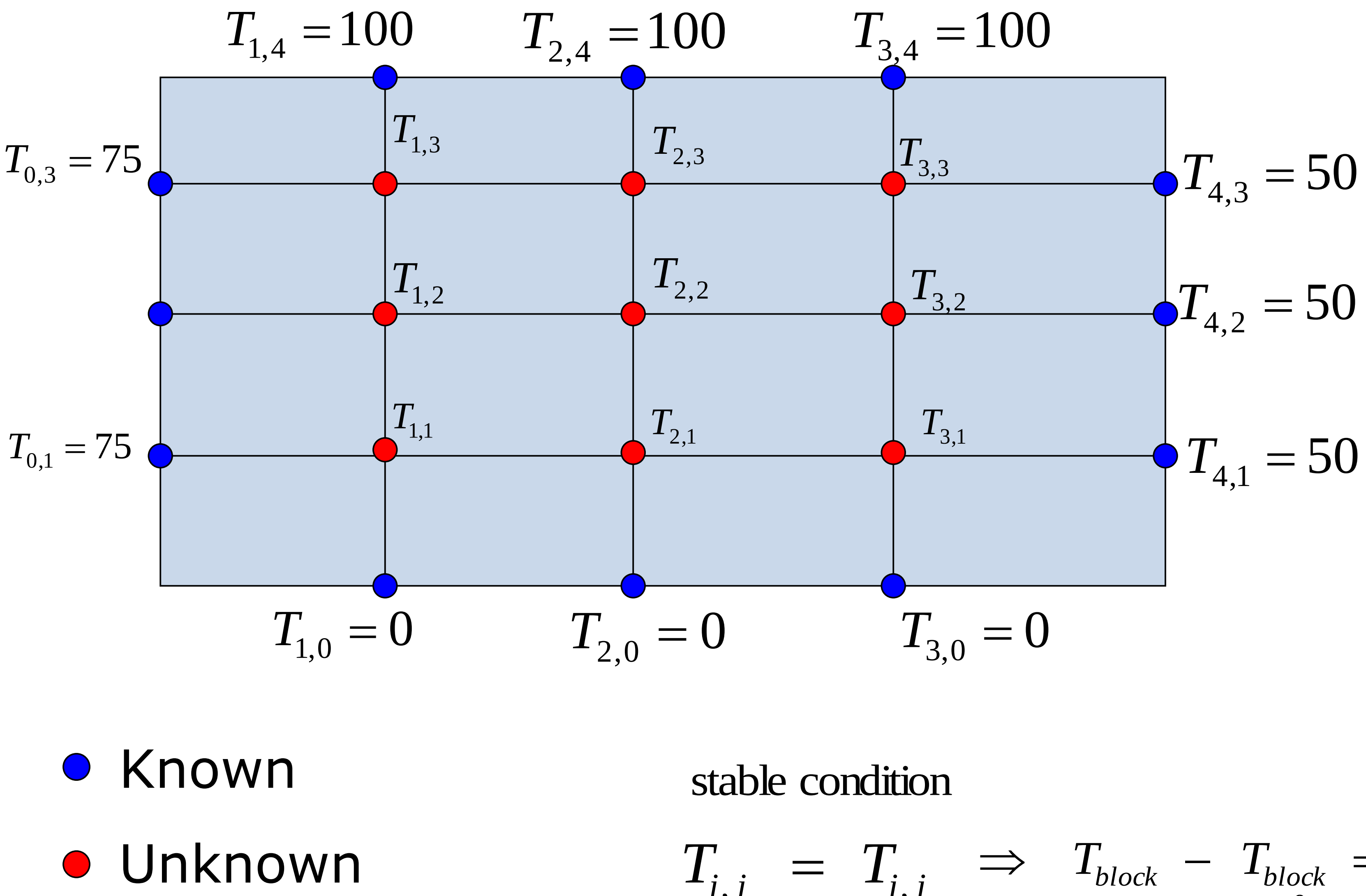
$$T_{1,3} + T_{2,2} + T_{3,3} + T_{2,4} - 4\,T_{2,3} = 0$$

$$\left.\frac{\partial T}{\partial y}\right|_{2,3} = 0 \quad T_{2,4} = T_{2,2}$$

$$T_{1,3} + 2\,T_{2,2} + T_{3,3} \qquad - 4\,T_{2,3} = 0$$

- Known
- Unknown

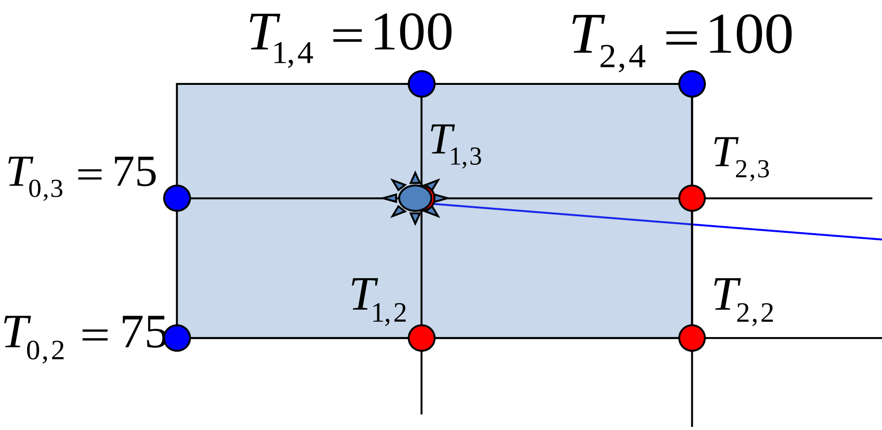
Example



stable condition

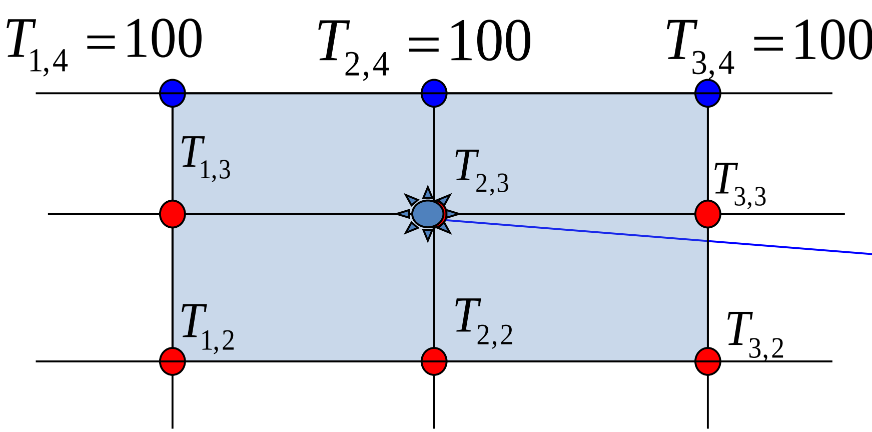
$$\underset{t=t}{T_{i,j}} = \underset{t=0}{T_{i,j}} \Rightarrow \underset{t=t}{T_{block}} - \underset{t=0}{T_{block}} = 0$$

Setting up equation



$$T_{0,3} + T_{1,4} + T_{1,2} + T_{2,3} - 4T_{1,3} = 0$$

$$75 + 100 + T_{1,2} + T_{2,3} - 4T_{1,3} = 0$$



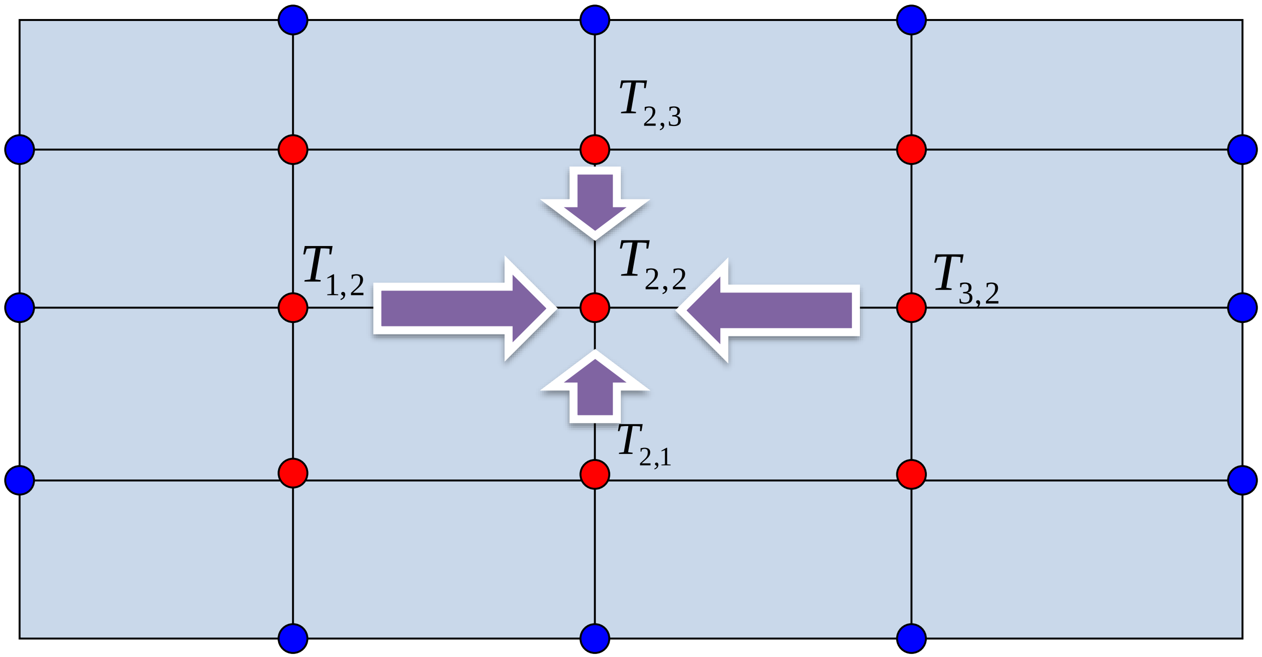
$$T_{1,3} + T_{2,4} + T_{3,3} + T_{2,2} - 4T_{2,3} = 0$$

$$T_{1,3} + 100 + T_{3,3} + T_{2,2} - 4T_{2,3} = 0$$

- Known
- Unknown

Stencil

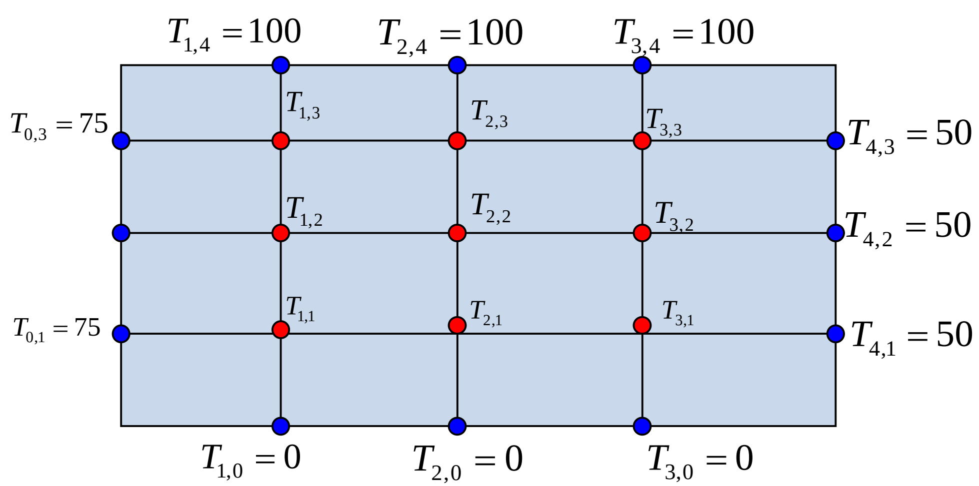
Stencil is only an information. A geometric arrangement of nodes.
It shows how to **update** array elements according to some **fixed pattern**.



Here, to calculate the value of $T_{2,2}$ we use the information from its Four Neighboring Nodes: *Five Point Stencil.*

Equation in matrix form

$$
\begin{pmatrix}
4 & -1 & 0 & -1 & & & & & \\
-1 & 4 & -1 & 0 & -1 & & & & \\
0 & -1 & 4 & 0 & 0 & -1 & & & \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & & \\
 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & \\
 & & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
 & & & -1 & 0 & 0 & 4 & -1 & 0 \\
 & & & & -1 & 0 & -1 & 4 & -1 \\
 & & & & & -1 & 0 & -1 & 4
\end{pmatrix}
\begin{pmatrix}
T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{1,2} \\ T_{2,2} \\ T_{3,2} \\ T_{1,3} \\ T_{2,3} \\ T_{3,3}
\end{pmatrix}
=
\begin{pmatrix}
75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150
\end{pmatrix}
$$



Jacobi method

$A\,x = b$ system of equation

Convert the system to the form $x = Cx + d$

$x^{(1)}, x^{(2)}, \ldots\ldots\ldots, x^{(n)}$ $\qquad x^{(k)} = Cx^{(k-1)} + d$

So, if we write A=L+D+U

for example,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

$$(L + D + U)x = b \;\Rightarrow Dx = b - (L + U)x$$

$$x^{k+1} = D^{-1}[b - (L + U)x^{k}]$$

Compute for new x, till

$\left\| x^{k} - x^{k-1} \right\| <$ error threshold

Jacobi method

Do you notice anything special about the matrix?

Should we store the entire matrix?

How can we do the matrix * matrix and matrix * vector faster?

$$
\begin{bmatrix}
4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4
\end{bmatrix}
\cdot
\begin{bmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16}
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ \vdots \\ b_n
\end{bmatrix}
$$

Thank you!

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