Quantum Key Search for Ternary LWE

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Introduction

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- Classical algorithm solves LWE in $\mathcal{S}^{0.24}$.
- Quantum algorithm solves LWE in $\mathcal{S}^{0.19}$.
- Different approach than the current best attacks.
- NTRU still quantum secure.

LWE

- Public $A \in \mathbb{Z}_q^{n \times n}, b \in \mathbb{Z}_q^n$.
- Solve $As = b + e \mod q$ where $s \in \mathbb{Z}_q^n, e \in \mathbb{Z}_q^n$ are the secret key.
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- NTRU: 127 1s, 127 -1s, 255 0s.
- Number of possible s: $\mathcal{S} = \binom{n}{w/2}\binom{n-w/2}{w/2}$.
- NTRU: $\mathcal{S} \approx 2^{754}$.

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- Time & space complexity for NTRU with $n = 509, q = 2048, w = 254$:

$$2^{377} = \mathcal{S}^{\frac{1}{2}}.$$

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- For our NTRU example: $2^{252} = \mathcal{S}^{1/3}$.

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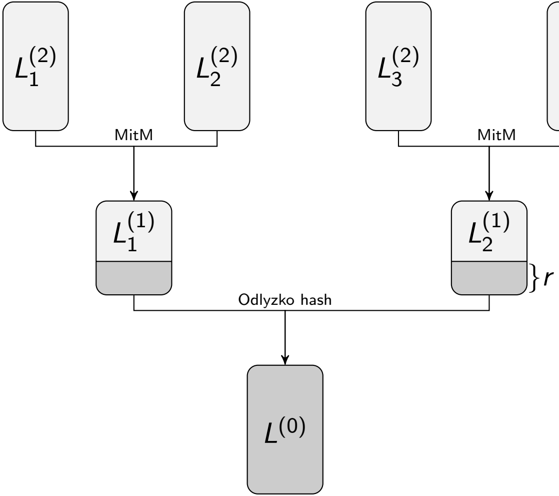
- At highest level do MitM.
- At lowest level check solution using LSH.

Figure



$s_i^{(2)} \in \mathbb{Z}_q^{n/2}$, weight $\frac{w}{4}$

$s_1^{(1)}, s_2^{(1)} \in \mathbb{Z}_q^{n}$, weight $\frac{w}{2}$

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- Final runtime: $2^{282+36} = 2^{318} < 2^{377}$.

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- Further improvement: $1 - 1 = 0$.
- Classically now 2^{267}.
- Example: 4 levels optimal, $2^{155+33} = 2^{188}$.

Results

	(n, q, w)	Meet-LWE	QMeet-LWE	cSVP
NTRU-Enc	$(509, 2048, 254)$	$267 = 193 + 74$	$188 = 155 + 33$	98
	$(677, 2048, 254)$	$313 = 235 + 78$	$223 = 191 + 32$	137
	$(821, 4096, 510)$	$449 = 336 + 113$	$320 = 268 + 52$	164
	$(701, 8192, 468)$	$387 = 295 + 92$	$278 = 235 + 43$	126
NTRU-Prime	$(653, 4621, 288)$	$309 = 236 + 73$	$225 = 190 + 35$	119
	$(761, 4591, 286)$	$344 = 265 + 79$	$245 = 206 + 39$	143
	$(857, 5167, 322)$	$383 = 294 + 89$	$274 = 236 + 38$	163
BLISS I+II	$(512, 12289, 154)$	$206 = 168 + 38$	$149 = 133 + 16$	77
GLP I	$(512, 8383489, 342)$	$250 = 210 + 40$	$193 = 175 + 18$	34

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- γ for time-memory trade-off.