THE UNIVERSITY OF ZAMBIA

INSTITUTE OF DISTANCE EDUCATION

MAT 2901 INTRODUCTION TO PROBABILITY

MODULE 1

Acknowledgements

The Institute of Distance Education of the University of Zambia wishes to thank Mr Urban N. Haankuku for writing this module **MAT 2901 introduction to probability**.

About this introduction to probability Module

The introduction to probability has been produced by The University of Zambia, Institute of Distance education.

Contents

How this probability Module 1 is structured

The course overview

The course overview gives you a general introduction to the course. Information contained in the course overview will help you determine:

- If the course is suitable for you.
- What you will already need to know.
- What you can expect from the course.
- How much time you will need to invest to complete the course.

The overview also provides guidance on:

Study skills.

Where to get help.

Course assignments and assessments.

Activity icons.

Units.

We strongly recommend that you read the overview *carefully* before starting your study.

The course content

The course is broken down into units. Each unit comprises:

- An introduction to the unit content.
- Unit outcomes.
- New terminology.
- Core content of the unit with a variety of learning activities.
- A unit summary.
- Exercises and Answers to the exercises, as applicable.

Resources

For those interested in learning more on this subject, we provide you with a list of additional resources at the end of this probability Module; these may be books, calculator, articles or web sites.

Your comments

After completing this module, we would appreciate it if you would take a few moments to give us your feedback on any aspect of this course. Your feedback might include comments on:

- Module content and structure.
- Module reading materials and resources.
- Module Exercises and/or assignments.

- Module Answers and/or assessments.
- Module duration.
- Module support (assigned tutors, technical help, etc.)

Your constructive feedback will help us to improve and enhance this course.

Module 1 overview

Welcome to introduction to probability Module.

This Module is designed to give students background knowledge/ information of probability concepts, knowledge and skills in probability. The module gives a background of techniques of counting which is a bases for understanding probability theory, and then proceeds to basics of probability concepts and its axioms, explain the sample space, events, define probability and Bayes rule. Then concepts of random variables and probability distributions are explained, jointly distributed random variables are discussed and lastly discuss other distributions which are a back ground of probability. The module has a number of exercises at the end of each unit for students to practice, at the end of each exercises answers are provided to give students feedback. Probability is a practical subject so you are advised to try all exercises before you come for residential schools. During the residential schools you are expected to come for consultations on areas you found concepts difficult to understand.

Probability Module 1 —is this course for you?

This course is intended for people who are studying Mathematics for their Bachelor's degree in Mathematics Education.

Pre requisite: In order for you to study this module well you should have done Module MAT 1100 (Foundation Mathematics).

Course outcomes

Upon completion of Probability Module you will be able to:

- Make a distinction between a population and a sample.
- Define an outcome of a sample space
- Identify the different types of data: discrete, continuous, categorical, binary.

- Summarize quantitative data graphically using a histogram.
- Construct histograms for discrete data.
- Construct histograms for continuous data.
- Make a distinction between frequency histograms, relative frequency histograms, and density histograms.
- Solve probability problems.

Timeframe

It is recommended that you take **120** Hours to complete this module

Study skills

As an adult learner your approach to learning will be different from the way you used to learn in your school days: you will choose what you want to study, you will have professional and/or personal motivation for doing so and you will most likely be fitting your study activities around other professional or domestic responsibilities.

Essentially you will be taking control of your learning environment. As a consequence, you will need to consider performance issues related to time management, goal setting, stress management, etc. Perhaps you will also need to reacquaint yourself with areas such as essay planning, coping with exams and using the web as a learning resource.

Your most significant considerations will be *time* and *space* i.e. the time you dedicate to your learning and the environment in which you engage in that learning.

We recommend that you take time now—before starting your self-study—to familiarize yourself with these issues. There are a number of excellent resources on the web. A few suggested links are:

- http://www.how-to-study.com/

  The “How to study” web site is dedicated to study skills resources. You will find links to study preparation (a list of nine essentials for a good study place), taking notes, strategies for reading text books, using reference sources, test anxiety

- http://www.ucc.vt.edu/stdysk/stdyhlp.html

  This is the web site of the Virginia Tech, Division of Student Affairs. You will find links to time scheduling (including a "where does time go?" link), a study skill checklist, basic concentration techniques, control of the study environment, note taking, how to read essays for analysis, and memory skills ("remembering").

- http://www.howtostudy.org/resources.php

  Another "How to study" web site with useful links to time management, efficient reading, questioning/listening/observing skills, getting the most out of doing ("hands-on" learning), memory building, tips for staying motivated and developing a learning plan.

The above links are our suggestions to start you on your way. At the time of writing these web links were active. If you want to look for more go to www.google.com and type "self-study basics", "self-study tips", "self-study skills" or similar search form.

Need help?

When you need help in this module you can contact the Director, UNZA IDE or visit **www.mathstutor.com**

Assessments

1. Continuous Assessment	30%
1.1 Assignments/Quizzes	10%
1.2 Tests	20%
2. Final Examination	70%
Total	100%

Getting around this Probability Module.

Margin icons

While working through this module you will notice the frequent use of margin icons. These icons serve to “signpost” a particular piece of text, a new task or change in activity. They have been included to help you to find your way around.

A complete icon set is shown below. We suggest that you familiarize yourself with the icons and their meaning before starting your study.

Activity	Assessment	Assignment	Case study
Discussion	Group activity	Help	Note it!
Outcomes	Reading	Reflection	Study skills
Summary	Terminology	Time	Tip

UNIT 1 TECHNIQUES OF COUNTING

1.1 Introduction:

Welcome to Unit 1 of this module. Before defining probability, it is worth discussing a few concepts where the probability theory is based on. The topic probability can be understood well if the fundamental principles of counting are introduced first because all experiments that may require probability applications do require counting. In this unit, you begin by discussing techniques of counting where probability will be based. You will be introduced to techniques of counting such as: Permutations, Combinations and Tree diagrams.

1.2 Unit Aim:

Counting various quantities is the foremost human activity in which children engage beginning at a very tender **age**. The main property of counting is so fundamental to our perception of quantity that it is seldom enunciated explicitly. The purpose of counting is to assign a numeric value to a group of objects.

1.3 Unit Objectives:

Upon completion of this unit you will able to:

- Apply the multiplication principle.
- Count objects when the objects are sampled with replacement.
- Count objects when the objects are sampled without replacement.
- Use the permutation formula to count the number of ordered arrangements of n objects taken n at a time.
- Use the permutation formula to count the number of ordered arrangements of n objects taken r at a time.
- Use the combination formula to count the number of unordered subsets of r objects taken from n objects.
- Use the combination formula to count the number of distinguishable permutations of n objects, in which r are of the objects are of one type and $n-r$ are of another type.
- Count the number of distinguishable permutations of n objects, when the objects are of more than two types.

- Apply the techniques learned in the lesson to new counting problems.

Terminology

- n- Sample size
- p_r^n - Permutation of n objects taking r at a time
- C_r^n - Combination of n objects taking r at a time
- ! – factorial

1.4 Time Requirement: To complete this unit you need to speed 10 hours

1.5 Unit Topics:

1.5.1 Permutation

Definition: Any arrangement of a set of n objects taking $r \le n$ r at a time is called an r-permutation.

Fundamental Principles of counting: If some event can occur in n_1 different ways, and if, following this event, a second event can occur in n_2 different ways, then the number of ways the events can occur in the order indicated is $n_1.n_2.n_3......$. Using this principle, you can count using the ordinary way of counting say 1, 2, 3 … or using permutations, or combinations or tree diagram.

Example 1 Suppose a license plate for cars contains three letters and four digits where the first digit, not zero. Find the number of different license plates which can be printed?

Solution. Each letter can be printed in 26 different ways, the first digit in 9 ways and each of the other three digits can be printed in 10 different ways.

Hence 26.26.26.9.10.10.10 **= 158,184,000** different plates can be printed.

Notation. p_r^n The number of permutations taken r at a time from n object is called r-permutations.

n! Reads n factorial, which is defined as:

$$n! = n(n-1)(n-2)......1 \tag{1.1}$$

$0! = 1$

$1! = 1$

$3! = 3.2.1 = 6$

$4! = 4.3.2.1 = 24$

Theorem 1.1 $$p_r^n = \frac{n(n-1)(n-2)....(n-4+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!} \qquad (1.2)$$

Corollary 1.1 $$p_r^n = \frac{n!}{(n-r)!} \qquad (1.3)$$

Using this corollary, it is easy to calculate the number of permutation from n object taking r at a time.

Example 2 How many permutations are there of 6 objects taken three at a time?

Solution Our n = 6 and r = 3 therefore, using the corollary we have

$$p_r^n = \frac{n!}{(n-r)!} = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 6.5.4 = 120 \text{ Permutations} \qquad (1.4)$$

1.5.2 Permutations with repetitions

Usually, we may want to find the number of permutations of objects some of which are alike. The general formula is as follows:

Theorem 1.1 The number of permutations of n objects of which n_1 are alike n_2 is alike, …, n_r are alike is

$$\frac{n!}{n_1!n_2!...n_r!} \qquad (1.5)$$

Example 3 How many seven letter words can be formed using the letters of the word 'BENZENE'?

Solution: we want the number of permutations of 7 objects of which three are alike (the three E's), and two are alike (the two N's). By the theorem above, we have

$$\frac{7!}{3!2!} = \frac{7.6.5.4.3!}{3!2.1} = 7.6.5.2 = 420 \text{ Different words.}$$

Example 4 In how many distinct arrangements can be made of the letters of the word ABRACADABRA?

To start, let us distinguish the identical letters by suffixes thus:

$$A_1\, B_1 R_1\, A_2\, CA_3\, DA_4\, B_2\, R_2\, A_5$$

Solution: In this form, there are 11! Possible permutations. However 5! Have the as in the same positions and are distinguished only by the suffixes on the As. Removing these reduces the number of permutations by a factor of 5! Repeating this reasoning for the Rs and Bs, the number of distinct arrangement is $\frac{11!}{5!2!2!} = 83169$.

1.5.3 Ordered samples

Many problems in probability are concerned with choosing a ball from an urn containing n balls (or cards from a deck, or a person from a population). When we choose one ball after the other from the urn, say r times we call each choice an ordered sample of size r. We consider two scenarios: **Sampling with replacement**. Here the ball is replaced in the urn before the next ball is chosen. There are n different ways to choose each ball, by fundamental principles of counting there are:

$$\overbrace{n.n.n.n....n}^{r\ times} = n^r \qquad (1.6)$$

Different ordered samples with replacement of size r. **sampling without replacement**. Here the ball is not replaced in the urn after it is chosen. That means there is no repetition in the ordered sample. That is an ordered sample of size r without replacement is merely an r- a permutation of the objects in the urn. That is, there are

$$p_r^n = \frac{n!}{(n-r)!} \qquad (1.7)$$

Different ordered samples of size r without replacement from a population of n objects.

Example 5 In how many ways can one choose three cards in succession from a deck of 52 cards (i) with replacement, (ii) without replacement?

Solution. (i) If it is replaced in the deck, we then pick another card, then each card can be selected in 52 different ways. Hence there is a $52.52.52 = 52^3 = 140608$ different ordered Sample of size 3 with replacement.

(ii) If no replacement, the first card can be chosen in 52 different ways, the second card in 51 different ways and the third card can be chosen in 50 different ways. Hence there are $52.51.50 = 132{,}600$ different ordered samples of size three without replacement. This r =3 and n = 52 which can be calculated as:

$$P_r^n = \frac{n!}{(n-r)!} = \frac{52!}{(52-3)!} = \frac{52.51.50.49!}{49!} = 52.51.50 = 132{,}600$$

Some ordered samples of size three without replacement.

1.5.4 Combination

A combination of n objects taken r at a time is any selection of r of the objects where order does not matter. This is denoted by C_r^n which is defined as

$$C_r^n = \frac{n!}{(n-r)!r!} \qquad (1.8)$$

Note that the binomial coefficient was defined as $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ this means $C_r^n = \binom{n}{r}$ is used here interchangeably.

Example 6 A farmer buys three cows, two pigs and four hens from a man who has six cows, five pigs, and eight hens. How many choices does the farmer have?

Solution. The farmer can choose the cows in $\binom{6}{3}$ ways, the pigs in $\binom{5}{2}$ ways, and the hens in $\binom{8}{4}$ ways. So altogether he can choose the animals in $\binom{6}{3}\binom{5}{2}\binom{8}{4} = \frac{6.5.4}{1.2.3}.\frac{5.4}{1.2}.\frac{8.7.6.5}{1.2.3.4} = 20.10.70 = 14000$ ways.

Example 7 A competition consists of selecting three suitable gifts from a list of twenty possible wedding presents. Find the number of ways can this be done?

The number of combinations $= \frac{20!}{(20-3)!3!} = \frac{20!}{17!3!} = 1140$ presents.

1.5.5 Partitions and cross partition

A partition of a set X is a subdivision of X into subsets which are disjoint and whose union is X, that is, such that each $a \in X$ belongs to one and only one of the subsets of X that is a collection $\{A_1, A_2, ..., A_m\}$ of subsets of X is just a partition of X if and only if

(i) $X = A_1 \cup A_2 \cup ... \cup A_m$

(ii) For any A_i, A_j either $A_i = A_j$ or $A_i \cap A_j = \phi$

The subsets of a partition are called cells

Example 8. Consider the following classes of subsets of X = {1, 2, 3… 8, 9}

(i) [{1,3,5}, {2,6}, {4,8,9}]

(ii) [{1,3,5}, {2,4,6,8}, {5,7,9}]

(iii) [{1,3,5}, {2,4,6,8}, {7,9}]

Then (i) is not a partition of X, since $7 \in X$, do not belong to any set in (i)

(ii) Is not a partition of X, since $5 \in X$, belongs to two sets.

(iii) Is a partition of X since it satisfies the condition of partitions.

1.5.6 Ordered partitions

Suppose box A contains seven balls numbered 1 to 7. You want to compute the number of ways you can choose two balls from the box, and then three balls from the box, and lastly two balls from the box. That is, you want to compute the number of ordered partitions $\{A_1, A_2, A_3\}$, of the set of 7 balls into cells A_1, containing two balls, A_2 containing three balls and A_3 containing two balls. These are ordered partitions since you distinguish between [{ 1,2} {3,4,5}, {6,7} & [{6,7}, {3,4,5} {1,2}], each of which determines the same partition of A. You begin with 7 balls in the box, so there are $\binom{7}{2}$ ways of drawing the first 2 balls, i.e., of determining the first cell A_1:

following this there are 5 balls, left in the box and so there are $\binom{5}{3}$ ways of drawing the 3 balls, i.e., in determining the second cell A_2; finally, there are 2 balls left in the box, and so there are $\binom{2}{2}$ ways of determining the last cell A_3.

There are $\binom{7}{2}\binom{5}{3}\binom{2}{2} = \frac{7.6}{1.2}.\frac{5.4.3}{1.2.3}.\frac{2.1}{1.2} = 210$ different ordered partitions of A into cells A_1 containing two balls, A_2 containing three balls and A_3 containing two balls.

Theorem 1.3: Let A contain n elements and let $n_1 + n_2 + ... + n_r = n$, then there exist

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}...\binom{n-n_1-n_2...n_{r-1}}{n_r} \tag{1.9}$$

Different ordered partitions of A of the form $A_1, A_2,..., A_r$ where A_1 contains n_1 elements, A_2 n_2 elements, and A_r contains n_r elements.

It can be observed that.

$$\binom{7}{2}\binom{5}{3}\binom{2}{2} = \frac{7!}{2!3!2!}$$

Theorem 1.4 Let a contain n elements and $n_1, n_2,..., n_r$ be positive integers with $n_1 + n_2 + ... + n_r = n$ then there exist

$$\frac{n!}{n_1!n_2!...n_r!} \tag{1.10}$$

Different ordered partitions of A of the form $A_1, A_2,..., A_r$ where A_1 contains n_1 elements, toys A_2 contains n_2 elements… and A_r contains n_r elements.

Example 9 In how many ways can nine be divided among four children if the youngest child is to receive three toys and each of the others two toys?

Solution. We wish to find the number of partitioning nine toys into four cells containing 3, 2, 2 and two toys respectively by the above theorem; there $\frac{9!}{3!2!2!2!} = 2520$ are such ordered partitions.

1.0 Unit activity

(1) Calculate the number of different three-card hands which can be dealt from a pack of 52 cards.

(2) The judges in a beauty contest have to arrange ten competitors in order of merit. In how many ways can this be done? Two competitors are to be selected to go on to further competition. In how many ways can this selection be made?

(3) (a)Evaluate

(i) p_3^9 (ii) p_4^6 (iii) $\binom{11}{3}$ (iv) $\binom{11}{8}$

(b) Show that $\binom{n}{r} = \binom{n}{n-r}$

(4) From the sixth form of 30 boys and 32 girls, two boys and two girls are to be chosen to represent their school. How many possible selections are there?

(5) Find the number of arrangements there are of the letters of the words

(a) DEAR, (b) DEER?

(6) In how many different ways can a committee of four men and four women be seated in a row on a platform if (a) they can sit in any position, (b) no person is seated next to a person of the same sex?

(7) (a) Four boys and two girls sit in a line on stools in front of a coffee bar. (i) In how many arrangements can they sit among themselves so that the two girls are together? (ii) In how many ways can they sit if the two girls are not together?

(b) Ten people arrange to travel in two cars, a large saloon car and a mini car. If the saloon car has seats for six and the mini has seats for four, find the number of ways the party can travel, assuming that the order of seating in each car does not matter and all the people are drivers.

(8) (a) How many integers are there between 1234 and 4321 which contain each of the digits 3, 2, 5 and six once and once only?

(b) A teacher can take any or all of six boys with him on an expedition. He decides to take at least three boys. Find the number of ways can the party be made up?

(9) (a)Calculate how many different numbers altogether can be formed by taking one, two, three and four digits from the digits 9, 8, 3 and 2, repetitions not being allowed.

(b) Calculate how many of the numbers in part (a) are odd and greater than 800.

(c) If one of the numbers in part (a) is chosen at random, calculate the probability that it will be greater than 300.

(10) (a) Find the number of odd integers that can be formed from the figures 1, 2, 3 and five if repetitions are not allowed?

(b) A diagonal of a polygon is defined as any line joining any two non-adjacent vertices, find the number of diagonals for following polygon of (i) 5 sides5, (ii) 6 sides9 (iii) n sides?

(c) (i) Six different books lie on a table, and a girl is told that she can take away as many as she likes but she must not leave empty-handed. How many different selections can she make?

(ii) One of these books is a Bible. How many of these selections will include this Bible?

(11) Giving a brief explanation of your method, calculate the number of different ways in which the letters of the word TRIANGLES can be arranged if no two vowels may come together.

1.5.7 Tree Diagram

To count many ways an event can occur if the selection is made more than once, the method tree diagram can be used when enumerating some ways an event can occur. A tree diagram is A device which can be used to count all possibilities of each event which can occur in a finite number of ways is called a tree diagram. The construction of a tree diagram is illustrated below:

Example 10 Construct the tree diagram for the number of permutation of $\{a,b,c\}$.

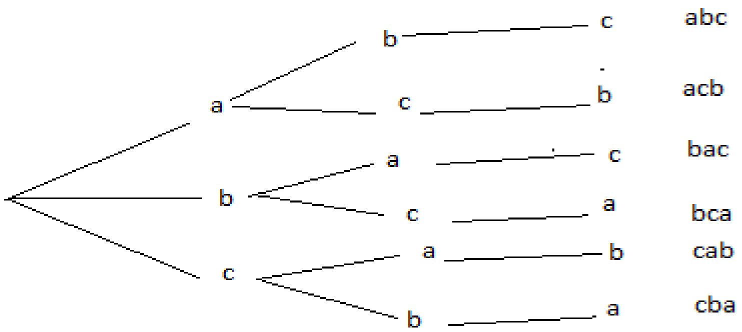


Figure 1 Tree diagram of example 10

There are six permutations. Note that the arrangement of permutation order matters that is why the permutation abc is different from acb (see definition of permutation).

Example 11 Find the product set {1,2,3}X{2,4}X{2,3,4} by constructing the appropriate tree diagram

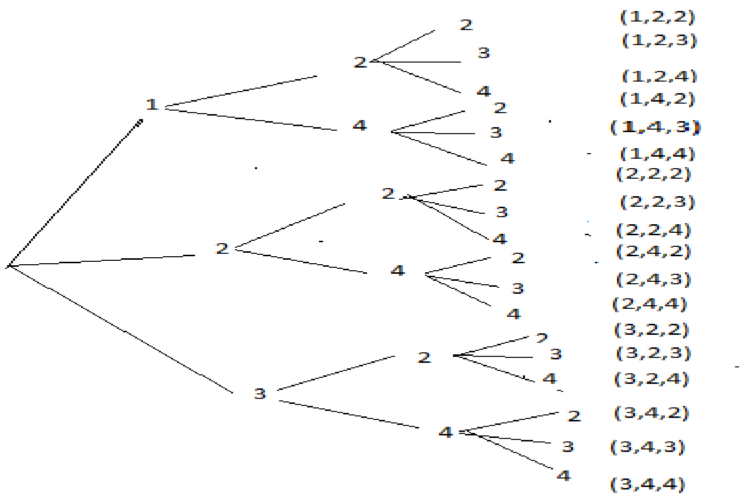


Figure 2 The tree diagram of example 11

There are eighteen elements of a product set as shown on the right of the tree diagram

Example 12 A man is at the origin on the x-axis and take a step either to the left or right. He stops if he reaches 3 or -3 or if he occupies any position, other than the origin, more than once. Find the number of different paths the man can travel.

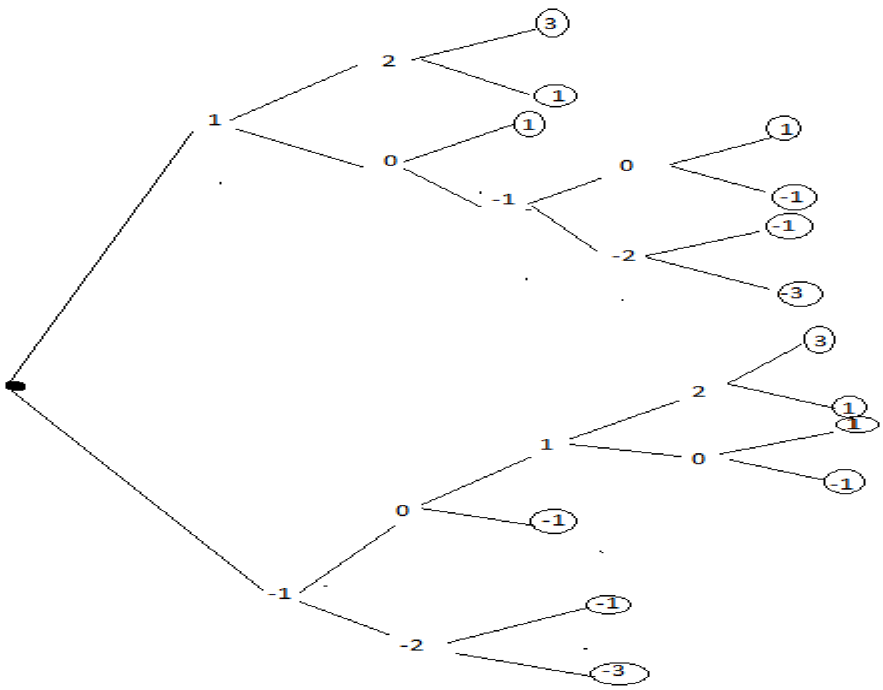


Figure 3 The tree diagram of example 12

There are 14 different paths; each path corresponds to an endpoint of the tree diagram.

1.4 Unit Activities

1. Prove that the number of different ways in which r objects can be selected from a group of n is $\dfrac{n!}{(n-r)!r!}$

2. A shuffled pack of 52 cards is dealt out to four players, each receiving thirteen cards. Show that the probability that a particular player receives the four aces is 0.0026.
   How many deals are necessary so that the probability of a particular player receiving all four aces in at least one game exceeds 0.5?

3 Sixteen soccer players, from many different clubs, form a squad from which the eleven players in the national team are selected. Find the number of different teams which could be selected from the squad irrespective of the positions in which the men play.

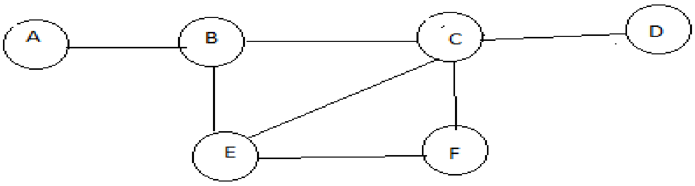
4. If repetition is not permitted.
   (a) In how many three digit numbers can be formed from the six digits: 2, 3, 5, 6, 7 and 9?
   (b) How many are less than 400?
   (c) How many are even?
   (ci) How many are odd?
   (cii) How many are multiples of 5?

5. If repetition is permitted.
   (a) In how many three digit numbers can be formed from the six digits: 2, 3, 5, 6, 7 and 9?
   (b) How many are less than 400?
   (c) How many are even?
   (d) How many are odd?
   (e) How many are multiples of 5?

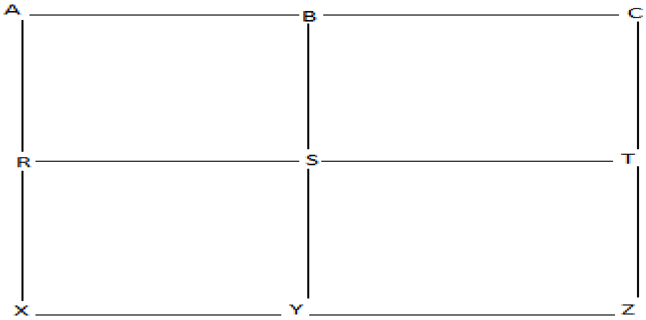
6. Find the total number of positive integers that can be formed from the digits 1, 2, 3, and 4, if repetition on digits in any one integer is not allowed.

7. A student is to answer 8 0out of 10 questions in an exam.
   (i) How many choices has he?
   (ii) How many if he must answer the first three questions?
   (iii) How many if he must answer the at least 4 of the first five questions?

8. In the following diagram let A, B, …, F denotes islands, and the lines connecting them are bridges. A man begins at A and walks from island to island. He stops for lunch when he cannot continue to walk without crossing the same bridge twice. Find the number of ways he can take his walk before eating lunch.



9. Consider the diagram below with nine points A, B,C,R,S,T,X,Y,Z. A man begins at X and is allowed to move horizontally or vertically, one step at a time. He stops when he cannot continue to walk without reaching the same point more than once. Find the number of ways he can take his walk if the first moves from X to R.



10. A bag contains six white items and five black items. Find the number of ways four items can be drawn from the if (i) they are of any colour? (ii) 2 must be white and two black? (iii) they must all be of the same colour?

Unit Summary

Permutations are the **arrangements** of n objects taken r at a time were order matters given by

$$p_r^n = \frac{n!}{(n-r)!}$$

Combinations are the arrangements of n objects taking *r* at a time where the order is not necessary is given by

$$C_r^n = \binom{n}{r} = \frac{n!}{(n-r)r!}$$

UNIT 2 PROBABILITY

2.1 Introduction

Probability is just a study of random or non-deterministic experiments. If a coin is tossed in the air, it is certain that the coin will come down, but it is not certain that say ahead will appear. However, suppose we repeat this experiment of tossing a coin; Let s be the number of times a Head appears (say success) and let n be the number of tosses. Then it has been empirically observed that the ratio $f = \frac{s}{n}$, called the relative frequency, becomes stable in the long run, which is approaching a limit. This stability is the basis of probability theory.

2.2 Unit Aims:

To introduce practical data analysis techniques using the statistical computing packages. To enable students to write a small report summarizing and interpreting an appropriate data set. To introduce the fundamental concepts in elementary probability theory. To introduce and study properties of standard univariate probability distributions. To introduce the basic concepts of statistical inference and assessing significance.

2.3 Unit Objectives: Having successfully completed this module you will be able to:

- Explore data analysis.
- Write a short-report describing a simple statistical data set.
- Solve probability problems and apply probability in every life
- Write the laws of probability and the use of Bayes theorem.
- Write the standard univariate distributions and their properties.
- Apply the Central Limit Theorem and its application.
- Carry out statistical inference.

Terminology

- $p(A)$ - Probability of event A
- A defined as an event
- S = sample space
- n = sample size
- $p = p(A) = \frac{s}{n}$

2.4 Time required: You should spend 10 hours on this unit

2.5 Unit Topics

2.5.1 Sample space and events

The set S of all possible outcome of some given experiment is called the **Sample space**. A particular outcome, that is, an element in S is called a sample point or sample. An event A is a set of outcomes or a **subset** of the sample space.

Definition. If A is an event of some experiment and S is the total possible ways A can occur, then $P(A)$ = the probability of event A is:

(i) $$P(A) = \frac{Number\, of\ elements\, in\ A}{Number\, of\ elements\, in\ S} \tag{2.1}$$

(ii) $$P(A) = \frac{Number\, of\ ways\ A\, can\ occur}{Number\, of\ ways\ S\ can\, occur}$$

In probability theory, we define a mathematical model of the above phenomenon by assigning "probability" (or the limit values of the relative frequencies) to each possible outcome of an experiment. Since the relative frequency of each outcome is non-negative, and the sum of the relative frequencies of all possible outcomes is unity, we require that our assigned "probabilities" also satisfy these two axioms. The reliability of our mathematical model for a given experiment depends upon the closeness of the assigned probabilities to the actual relative frequency. This gives rise to a problem of testing and reliability which is the subject matter of statistics.

Probability began with the study of games such as roulette and cards. The probability p of an event A is defined as follows:

If A can occur in s ways out of a total n equally likely ways, then

$$p = p(A) = \frac{s}{n} \tag{2.2}$$

For example, in tossing a die, an even number can occur in 3 ways out of 6 equally likely ways. Hence, $p = \frac{3}{6} = \frac{1}{2}$ is the probability that an even number occurs.

Theorem 2.1: The probability function p defined on the class of all events in a finite probability space satisfies the following axioms.

(i) For every event A, $0 \le p(A) \le 1$

(ii) $P(S) = 1$

(iii) If A and B events are mutually exclusive, then $p(A \cup B) = p(A) + p(B)$

(iv) $p(\phi) = 0$

(v) $p(A') = 1 - p(A)$

(vi) If A and B are any events $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

Example 1 Toss a die once and find the probability that a prime number appears

Solution: We find the sample space of this experiment. Since a die has six faces, our sample space is 6 and is given by $S = \{1,2,3,4,5,6\}$, S has six elements. The event that a prime number appears is: $A = \{2,3,5\}$ This means there are three numbers in S which are prime.

Using the definition: $P(A) = \dfrac{Number\,of\;elements\,in\;A}{Number\,of\;elements\,in\;S}$ you have $p(A) = \dfrac{3}{6} = \dfrac{1}{2}$ as the probability that a prime number will occur

Example 2 Toss a pair of fair dice. Let S be the sum of two numbers that appear. Find the probability p that:

(i) the sum is less than 5

(ii) the sum is 7

Solution: We first find the sample space of the experiment. This experiment implies we are tossing a pair of dice and the sample space consists of the sum of two dies.

Table 1 The sample space of example 2

S =

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

It can be observed that S has 36 elements as shown in the table above. The event A is shown in table 2.

Table 2 The event of A that the sum of the two numbers is less than 5

S =

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	A 4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$A = \{2,3,3,4,4,4\}$ The event A has six elements, hence,

$$P(A) = \frac{Number\, of\ elements\, in\ A}{Number\, of\ elements\, in\ S} = \frac{6}{36} = \frac{1}{6}$$

(i)The sum is seven is shown in table 3

Table 3 Giving the outcome of the sum is 7

S =

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Let the event B be the sum is seven. Therefore the sample space is: $B = \{7,7,7,7,7,7\}$

$$P(B) = \frac{Number\ of\ elements\ in\ B}{Number\ of\ elements\ in\ S} = \frac{6}{36} = \frac{1}{6}$$

Note that the sample space depends on how it has been defined in the question.

2.0 Unit activity

(1) Give the probability of

(a) Getting one head when a fair coin is tossed,

(b) Picking a red card from a pack of 52 cards,

(c) Picking a king from a pack of cards,

(d) Picking a diamond from a pack of cards,

(e) Tossing a 1 or a 6 with a die.

(2) In a given circle of radius r, we are to construct a chord. The following three methods of construction are suggested:

(a) Let *AB* be a fixed diameter in the circle. Draw the chord from *A* at an angle α to this diameter, where α is a value chosen at random between $\frac{1}{2}$ - $½^n$ and $½^n$.

(b) Again let *AB* be a fixed diameter in the circle. Choose a point *C* at random on the diameter *AB*. Draw the chord perpendicular to *AB* through *C*.

(c) Choose a point at random within the area of the circle. Make this the centre of the chord. For *each* method of construction evaluate the probability that the length of the chord is more significant than r.

Probability problems often involve the enumeration of possible outcomes, and the following examples illustrate two convenient methods of doing this. Probability problems often involve the enumeration of possible outcomes, and the following examples illustrate two convenient methods of doing this.

Example 3 Find the probability of throwing a total score of 6 with two dice?

Here the point of the sample space are best found by making a table (see Table 4)

Table 4 Sample space for the total score when two dice are thrown

Scores on the first die

S = Score on the second die

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

E

All the points in the sample space are equally likely.

S = {total score on two dice}, $n(S) = 36$

E = {total score on two dice = 6}, $n(E) = 5$

P (score of 6) = $\frac{5}{36}$ 0.1389 5/36 or 13.89%

Example 4 What is the probability of obtaining two heads and one tail when three coins are tossed? Where three (or more) objects are involved in the layout of the previous diagram is not possible. Instead, the possible outcomes can be found by using a tree diagram as in Figures 1. The first ‘branches' of the tree give the possible outcomes of tossing the first coin, the next branches the outcome of tossing the second coin and so on.

Example 5. Toss a coin three-time and observe a sequence of heads (H) and tails (T) occurs. If precisely two heads appear in a row what is its probability?

Solution: This experiment can be given in two ways: either toss a coin three times or toss three coins once. Both give the same sample space and event. We can use a tree diagram approach to enumerate the sample space from which the event can be drawn.

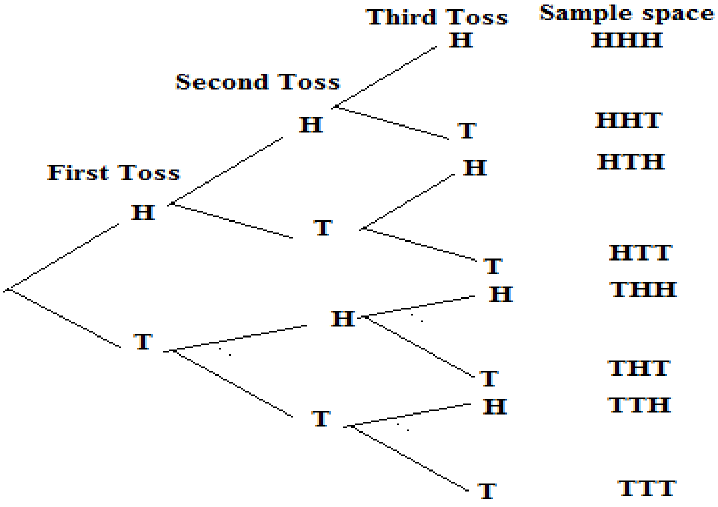


Figure 1 Tree diagram showing the possible outcomes when three coins are tossed

So the sample space is: $S = \{HHH, HHT, HTH, HTH, HTT, THH, THT, TTH, TTT\}$ these can be read from the tree diagram easily. The sample space is eight possible outcomes in this experiment. The event that precisely two heads appear in a row is $A = \{HHT, THH\}$. That is the event has two elements.

Therefore, $P(A) = \dfrac{Number\, of\ elements\, in\ A}{Number\, of\ elements\, in\ S} = \dfrac{2}{8} = \dfrac{1}{4}$

Note that it is possible to write the sample space without using the tree diagram.

There are eight equally likely outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT
Of which three contain two heads. So we have

S = {outcomes of throwing three coins},

$n\{S\}$ = number of ways S can occur

E = {outcomes of throwing three coins in which two are heads}, $n\{E\} = 3$

P (two heads) = $\dfrac{n\{E\}}{n\{S\}} = \dfrac{3}{8}$

2.1 Unity Activity 1

(1) Use Table 1 to find the probabilities of the following outcomes when two dice are thrown.

(a) A double.

(b) A total score higher than 9.

(c) A score of 6 or less.

(2) Two tetrahedral dice, with the faces of each labeled 1, 2, 3 and 4, are thrown. The total score is the *product* of the numbers on the bottom faces. Construct a sample space of the possible outcomes and from it find the probability that the total score is

(a) greater than 8,

(b) 6 or less,

(c) a multiple of 3.

(3) Extend the tree diagram in Figure 1 so that it represents the results of tossing four coins. From your diagram find the probability of an equal number of heads and tails.

(4) A box contains three ‘Scrabble' tiles, one with A on it, one with E and the third with T. A tile is taken at random, the letter noted and the tile replaced. This is repeated once. Construct a tree diagram to find the possible outcomes. What is the probability of getting two vowels? The experiment is modified so that the first tile is not replaced before the second tile is chosen. Modify your tree diagram and find the new value for the probability of getting two vowels.

The addition law. How is the probability that one of several events occurs related to the probabilities of the individual events?

To take a specific example, what is the probability that we throw two or more heads and how is it related to P(two heads) and P(three heads)?

Figure 2 shows the sample space for this problem where E_1 = {outcomes with three heads} and E_2 = {outcomes with two heads}. In this case, E_1 and E_2 are **mutually exclusive**.

This means that they cannot both occur at the same time.

From our definition of probability (equation 2.1)

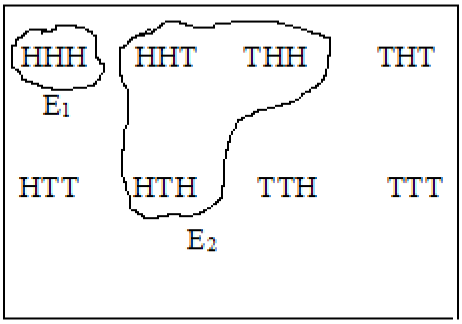


Figure 2 Sample space for the possible outcomes when three coins are tossed

P (two heads or three heads) = $\dfrac{n(E_1 \cup E_2)}{n(S)} = \dfrac{4}{8} = \dfrac{1}{2}$,

Since E_1 and E_2 are mutually exclusive we can use property (iii) of Theorem 1 which states $P(A \cup B) = p(A) + p(B)$ (2.3)

This value can be related to the individual probabilities as follows:

P (two heads or three heads) = $\dfrac{n(A_1 \cup E_2)}{n(S)} = \dfrac{n(E_1) + n(E_2)}{n(S)} = \dfrac{1}{8} + \dfrac{3}{8} = \dfrac{1}{2}$

The probability that E_1 or E_2 occurs is denoted by $p(E_1 \cup E_2)$ so that we have for two

$P(E_1 \cup E_2) = p(E_1) + p(E_2)$.

This can be generalized for *n* mutually exclusive events to

$$P(E_1 \cup E_2 \cup ... \cup E_n) = p(E_1) + p(E_2) + ... + p(E_n) \qquad (2.4)$$

This is the **additional law for mutually exclusive events**. The following example shows how it must be modified for events which are not mutually exclusive.

Example 6 Find the probability p of drawing an ace or a spade from a pack of 52 well-shuffled cards?

Solution: Here E_1 = {card is a spade} and E_2 = {card is an ace}, but the events are not mutually exclusive as a card can be a spade *and* an ace. So we use property (iv) of the Theorem above which states If A and B are any events.

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) \qquad (2.5)$$

P(spade or ace) = $\dfrac{n(E_1 \cup E_2)}{n(S)} = \dfrac{16}{52}$

The probability can be related to individual probabilities as follows:

$$p(E_1 \cup E_2) = P(E_1) + p(E_2) - p(E_1 \cap E_2)$$

$$= \frac{n(E_1 \cup E_2)}{n(S)} = \frac{n(E_1) + n(E_2) - n(E_1 \cap E_2)}{n(S)} = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52}$$

$p(E_1 \cap E_2)$ Denotes the probability that both E_1 and E_2 occur. This probability must be subtracted so that the ace of spades is not counted twice, once as a spade and once as an ace.

In this example, we have established the general addition law for two events.

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) \qquad (2.6)$$

Which is applicable whether or not the events are mutually exclusive? If they are mutually exclusive, then the last term on the right-hand side of the equation is zero.

It is worth pointing out that the expression 'E_1 or E_2 occurs' is somewhat ambiguous in the English language but is taken in probability problems to mean that 'E_1 and E_2 occurs', as is indicated by the way the probability is expressed in set notation.

Set notation for probability can be extended to embrace complements. For example, $p(E_1 \cup E_2')$ is the probability that E_1 occurs and E_2 does not? The following example illustrates how such probabilities may be calculated merely from a Venn diagram.

Example 7. If $p(A) = 0.3$, $p(B) = 0.4$ and $p(A \cap B) = 0.1$, find (a) $p(A \cup B)$, (b) $p(A \cup B)'$ (c) $p(A \cup B')$ (d) $p(A' \cap B)$

(a)The general addition law (equation 2.5) can be used to calculate $P(A$ u $B)$.

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$
$$0.3 + 0.4 - 0.1 = 0.6$$

(b) $p(A \cup B)'$ This is not the probability of neither A nor B occurs. It is the complement of $A \cup B$ so $p(A \cup B)' = 1 - p(A \cup B) = 1 - 0.6 = 0.4$

The Venn diagram in Figure 3 shows the probability associated with each region of the diagram. Each probability represents the fraction of events falling in a region, and so the sum of the probabilities for all the regions must be 1. From this diagram, the remaining probabilities can be found.

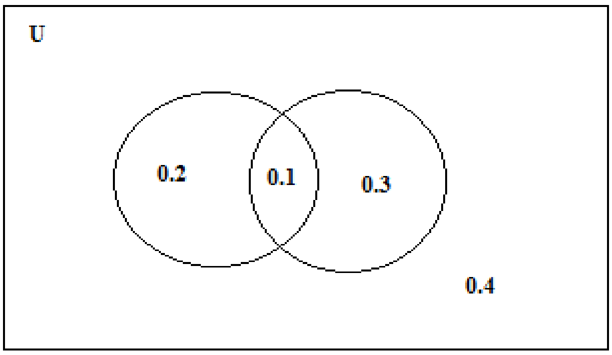


Figure 3 Venn diagram to illustrate Example 7

(c) $p(A \cup B')$ Is the probability that *A* occurs and *B* does not occur? Using the general addition law

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$
$$0.3 + 0.6 - 0.2 = 0.7$$

(d) $p(A' \cap B)$ the probability p that *A* does not occur and *B* does occur?

$$p(A' \cap B) = 0.3$$

Two or more events are said to be **exhaustive** if at least one of them must happen. Expressed in set notation, this condition is met by

$$P(E_1 \cup E_2 \cup ... \cup E_n) = 1$$

By their nature, an even and its complement must be exhaustive.

2.1 Unity Activity 2

(1) State whether the following pairs of events *A* and *B* are (i) mutually exclusive, (ii) exhaustive.

(a) Selected a child at random from a class. *A* = {a child has blue eyes}, *B* = {a child has brown eyes}

(b) A die is tossed. *A* = {a number is a multiple of 3}, *B* = { a number is a multiple of 2}

(c) A card is selected from a park of cards. *A* = {the card is a picture}, *B* = {a king is drawn)

(d) Toss a coin. *A* = {toss gives a head}, *B* = {toss gives a tail}

(2) *C* and *D* are events such that *P*(*C*) = 0.1, *P*(*D*) = 0.2 and *P*(*C* u *D*) = 0.3. Are *C* and *D* mutually exclusive events? Justify your answer. Find the values of (a) *P*(*C*) , (b) *P*(*C* ∩ *D*).

(3) In a group of 50 pupils, 30 study chemistry, and physics. If 20 study chemistry and 15 study physics, what is the probability that a pupil chosen at random studies (a) chemistry, (b) physics, (c) chemistry and physics, (d) Chemistry and Physics?

(4) *If* events *A* and *B* are such that *P*(*A*) = 0.2, *P*(*B*) = 0.9 and *P*(*A* ∩ *B*) = 0.1. By finding *P*(*A* u *B*) show that the events *A* and *B* are exhaustive.

(5) *E* and *F* are events such that *P*(*E*) = .07, *P*(*F*) = 0.6 and *P*(*E* u *F*) = 0.8. Find

(a) *P*(*E* ∩ *F*) (b) *P*(*E*’ ∩ *F*) (c) *P*(*E* or *F* but not both occurs)

(d) *P*(*E*’ ∩ *F*’)

2.5.2 Conditional Probability

Suppose that an event A has occurred. This means everything outside A is not a possible outcome. What this means we only consider outcomes inside the event A. So we have a reduced space $S_r = A$ from a sample space of S. The relationship of A and another event B is the only part of event B which is also part of event A.

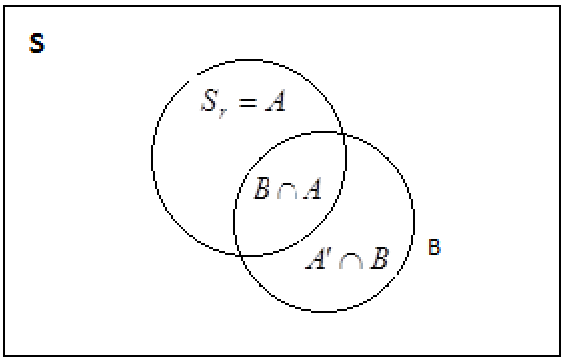


Figure 4

That is $B \cap A$ in the figure above shows that given that event A has occurred, the reduced space is now the event A, and the only relevant part of event B is $B \cap A$.

Given that event A has occurred, the total probabilities in the reduced space must sum up to 1. The probability of B given A is the unconditional probability of that part of B that is also in A, multiplied by the scale factor $\frac{1}{p(A)}$. This now becomes the conditional probability of event B given A.

$$p(B/A) = \frac{p(A \cap B)}{p(A)}$$

It can be observed that the conditional probability $p(B/A)$ is proportional to the joint probability $p(A \cap B)$ but has been rescaled so that the probability of the reduced space equal 1.

If A and B are independent events, then the conditional probability is given by;

$$p(B/A) = \frac{p(A \cap B)}{p(A)}$$

Since $p(\cap B) = p(A)xp(B)$ for independent events and the factor $p(A)$ canceled out.

The knowledge about A does not affect the probability of B occurring when A and B are independent events. If the roles are reversed of the two events A and B, the conditional probability of A given B would be;

$$p(A/B) = \frac{p(A \cap B)}{p(B)}$$ This is called multiplicative rule.

Bayes' Theorem In probability theory, Bayes' theorem can be described as the probability that relates conditional probabilities. If A and B are two events, $p(A/B)$ denotes the conditional probability of A occurring, given that B has occurred.

Using the definition of conditional probability

$p(B/A) = \dfrac{p(A \cap B)}{p(B)}$ The marginal probability of event A is found by summing the probability of its disjoint parts. Since $A = (A \cap B) \cup (A \cap B')$ and $A \cap B$, $A \cap B'$ are disjoint.

So $p(A) = p(A \cap B) + p(A \cap B')$, when substituted in the definition of conditional probability we get;

$$p(B/A) = \frac{p(A \cap B)}{p(A \cap B) + p(A \cap B')}$$

Using the multiplicative rule in finding each of the disjoint probabilities, this gives Bayes' Theorem for a single event:

$$p(B/A) = \frac{p(A/B)xp(B)}{p(A/B)xp(B) + p(A/B')xp(B')}$$

Hence, Bayes' Theorem is a statement of the conditional probability $p(B/A)$ where

(i) The probability of A is merely the sum of the probabilities of its disjoint parts of

$A \cap B$ and $A \cap B'$,

(ii) Each of the disjoint probability is found by the multiplicative rule.

It can be seen that the union of B and B' is the sample space S and that they are disjoint.

A set of events partitioning the sample space.

Suppose the set $B_1, B_2, ..., B_n$ such that

- The $\bigcup_{i=1}^{n} B_i = S$ sample space and
- $B_i \cap B_j = \phi$ For $i = 1,2,...,n$, $j = 1,2,...,n$ and $i \neq j$.

Then the events $B_i, i = 1,2,...,n$, partitioning the sample space. An event A can be partitioned into parts by partitioning

$$A = (A \cap B_1) \cup (A \cap B_2) \cup ... \cup (A \cap B_n).(A \cap B_i)$$

and $A_i \cap B_j$ are disjoint since B_i and B_j are disjoint

Therefore $p(A) = \sum_{i=1}^{n} p(A \cap B_i)$. This is called the law of total probabilities. Using the multiplicative rule on each joint probability gives

$$p(A) = \sum_{j}^{n} p(A / B_j) x p(B_j)$$

The conditional probability $p(B_i / A)$, I =1,2,…,n is found by dividing each probability by the probability of the event A

$$p(B_i / A) = \frac{p(A \cap B_i)}{p(A)}$$

So the joint probability can be found in the numerator together with the law of total probabilities in the denominator given as;

$$p(B_i / A) = \frac{p(A / B_i) x p(B_i)}{\sum_{j=1}^{n} p(A / B_j) x p(B_j)}$$

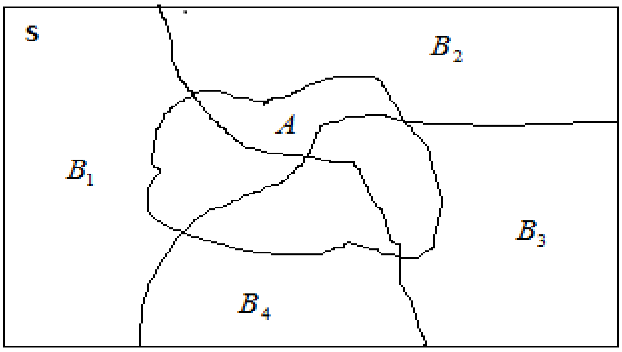


Figure 5 Four events B_i, $i = 1,2,3,4.$ that partitions S along with events

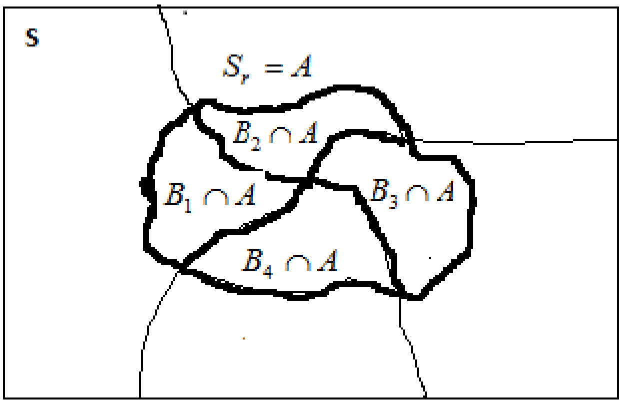


Figure 6 The reduced space given A has occurred together with the four events partitioning the sample space

Example 8 If the events A and B are such that $p(A) = \frac{1}{3}$, $p(B) = \frac{1}{2}$ and $p(A/B) = \frac{1}{4}$P, find

(a) $p(A \cap B)$, (b) $p(A \cup B)$, (c) $p(B/A')$

Solution:

(a) Since $P(A/B) = \dfrac{p(A \cap B)}{p(B)}$

We have $p(A \cap B) = p(A/B).p(B) = \frac{1}{4}.\frac{1}{2} = \frac{1}{8}$

(b) From the addition law

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$
$$= \frac{1}{3} + \frac{1}{2} - \frac{1}{8} = \frac{17}{24}$$

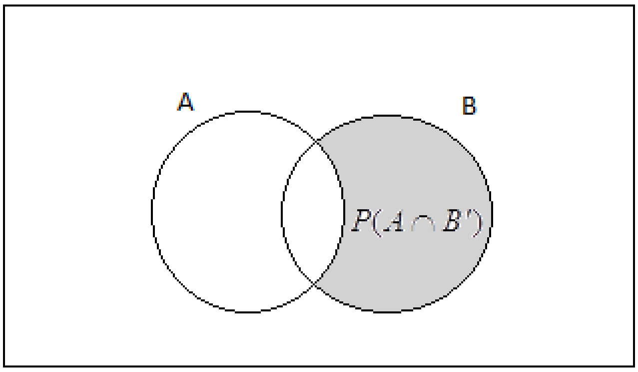


Figure 7 Diagram to illustrate Example 8

(c) The Venn diagram in Figure 2.2.8 shows that

$$p(B \cap A') = p(B) - p)A \cap B) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

and

$$p(A') = 1 - p(A) = 1 - \frac{1}{3} = \frac{2}{3}$$

So $$p(B/A') = \frac{p(B \cap A')}{p(A')} = \frac{3/8}{2/3} = \frac{9}{16}$$

In many cases, conditional probabilities can be calculated without equation (2.1), as in the following example.

Example 9 Two cards are drawn from a well-shuffled pack of 52 without replacement. If the first card is an ace, what is the probability that the second card is also an ace?

The term **without replacement** means that items, once drawn, are not replaced. When the first card is an ace, of the remaining 51 cards, 3 are aces, so that the probability that the second card is also an ace is 3/51.

2.2 Unity Activity

(1) When **A** and **B** are events which are exhaustive, such that $p(A \cap B) = \frac{1}{4}$, $p(A/B) = \frac{1}{3}$

Find (a) $p(B)$, (b) $p(A)$, (c) $p(B/A)$

(2) What is the probability that the second card drawn from a well-shuffled pack is a heart if (a) the first card drawn was a heart? (b) the first card chosen was not a heart?

(3) The probability that a person plays billiards is 1/10. The probability that he smokes if he

(4) plays billiards is 14/15. The probability that he plays billiards if he smokes is 3/10. Find the probability that a person smokes?

(5) In the third year of a particular school, each pupil has to choose one ‘option’. Table 5 shows the results with boys and girls listed separately.

Table 5

Option	Boys	Girls

G = {pupil is a girl}

B = {pupil is a boy}

T = {pupil chooses ‘The theatre’)

H ={pupil chooses ‘Healthy living’)

S = {pupil chooses ‘Sport for all’}

Give the values of (a) P(G), (b) P(B), (c) P(T), (d) P(H), (e) P(S|B), (f) P(G|T), (g) P(H/G), (h) P(T’|G), (i)P(H|G’).

2.5.3 The Product law

Rearranging formula (2.7) and (2.8) we have

$$p(E_1 \cap E_2) = p(E_2)p(E_1 / E_2) = p(E_1)p(E_2 / E_1)$$

This is the **product law for joint events**.

Example 10 Find the probability that if two cards are selected from a pack without replacement. (a) They are both aces, (b) neither is an ace, (c) at least one is an ace?

Solution:

(a) E_1 = {first card an ace}, E_2 = {second card an ace}

$$p(E_1) = \frac{2}{52} = \frac{1}{13} \quad p(E_2 / E_1) = \frac{3}{51} = \frac{1}{17}$$

$$p(E_1 \cap E_2) = p(E_1)p(E_2 / E_1) = \frac{1}{13}.\frac{1}{17} = \frac{1}{221}$$

(b) Similarly $p(E_1' \cap E_2') = p(E_1')p(E_2' / E_1') = \frac{48}{52}.\frac{47}{51} = \frac{188}{221}$

(c) Since the events 'neither is an ace' and at least one is an ace' are exhaustive and mutually exclusive, this probability can be found by subtraction.

$$\text{P(least one ace)} = 1 - \frac{188}{221} = \frac{33}{221}$$

The product law can be extended to multiple events if each probability is calculated assuming the occurrence of previous events.

Example 11 A box contains six red and ten black items. Find the probability that if four items are chosen without looking, they are all black?

$$p(E_1 \cap E_2 \cap E_3 \cap E_4) = p(E_1)p(E_2/E_1)p(E_3/E_2 \cap E_1)p(E_2/E_1)p(E_4/E_3 \cap E_2 \cap E_1)$$
$$= \frac{10}{16}.\frac{9}{15}.\frac{8}{24}.\frac{7}{21} = 0.115$$

Example 12 An absolute rare genetic condition affects 0.01% of the population. A test has been developed which can detect the condition, if it is present, with a probability of 95%. Unfortunately, when the condition is not present, there is a probability of 0.05% that the test will still give a positive result. Calculate the probability that (a) a person has the condition and the test gives a positive result, (b) the test gives a positive result, (c) a person has the condition given that the test gave a positive result.

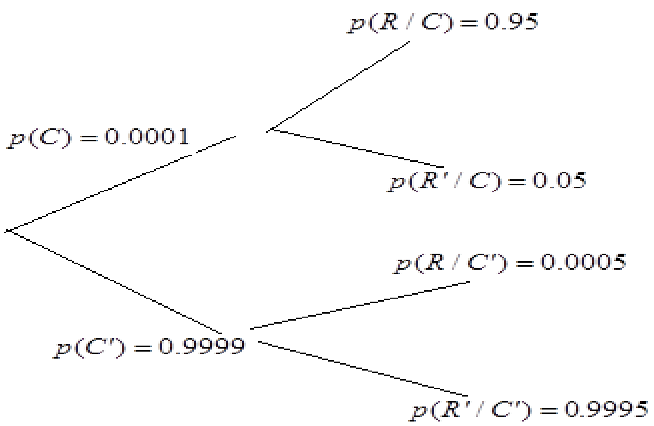


Figure 8 Tree diagram to illustrate Example 12

A convenient way of solving problems of this type is with a tree diagram as shown in

Figure 8. Let C be the event that a person suffers from the condition and R the event that a person shows a positive reaction. The first branches show the possibilities that a person does or does not have the donation followed by the possibilities that they give a positive or

negative reaction to the test. The branches must be in this order because the probability of R depends on whether or not the person has the condition rather than vice versa.

(a) The probability required is

$$p(C \cap R) = p(C).p(R/C) = 0.0001x0.95 = 0.000095$$

(b) We can use the addition law for mutually exclusive events to find the required probability, $p(R)$

$$\begin{aligned} p(R) &= p(C \cap R) + p(C' \cap R) \\ &= p(C)xp(R/C) + p(C')xp(R/C') \\ &= 0.001x0.95 + 0.9999x0.0005 = 0.000894 \end{aligned}$$

(c) This asks for the value of P(C | R) which can be calculated from the answers to parts (a) and (b). $p(C/R) = \dfrac{p(C \cap R)}{p(R)} = \dfrac{0.000095}{0.00059495} = 0.16(2dp)$

Consequently, the probability that a person has not got the condition when an affirmative result is obtained is 0.84, i.e. $(1-0.16)$, 1, which suggests that the test may be of little practical value.

Example 13 A box contains ten green and six white item. An item is chosen at random, its color noted, and it is not replaced. This is repeated once more. What is the probability that the items are of the same colour?

This is another problem which may be conveniently represented by a tree diagram as shown in Figure 9.

P(marbles are the same colour) = P(two greens) + P(two whites)

$$= \frac{10}{16} x \frac{9}{15} + \frac{6}{16} x \frac{5}{15} = \frac{1}{2}$$

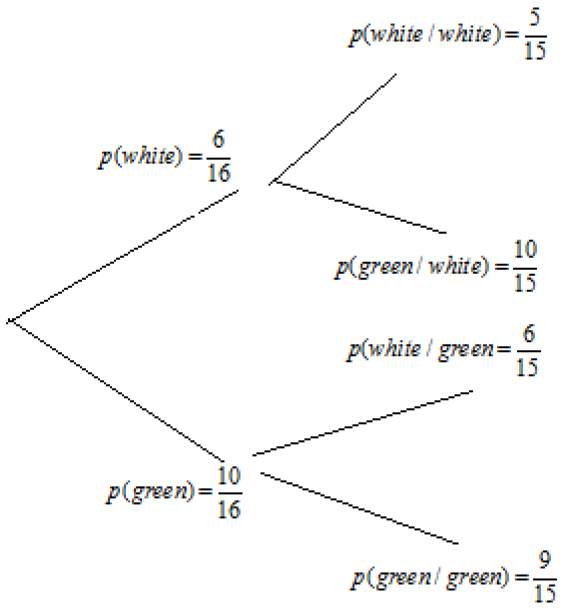


Figure 9 Tree diagram to illustrate Example 13

2.3 Unity Activity

(1) If A and B be are events such that P(A) = ½ , P (B) = 1/3 and P(A u B) = 2/3.

(a) Find (i) P(A ∩ B), (ii) P(A|B), (iii) P(B'|A).

(b) Are the events A and b (i) independent, (ii) mutually exclusive? Justify your answers.

(2) Repeat Example 4.5.4 for the case when the first marble is replaced before the second marble is taken.

(3) For Example 4.5.3, calculate the probability that a person (a) had the condition, (b) Does not have the condition, given that the reaction to the test is adverse.

(4) I try to catch the bus to work each day, but if I miss it, I have to walk. The probability that I catch the bus is 0.7. If I catch the bus, the probability that I arrive on time is 0.98 but if I walk this probability falls to 0.84. What is the probability that (a) I catch the bus and I am on time, (b) I am on time, (c) I caught the bus given that I arrived on time?

(5) (a) Given that, for two events R and S.

P(R) = 0.3, P(R U S) = 0.6. Find P(S) and P(R ∩ S) when R and S are (i) mutually exclusive events, (ii) Independent events.

(b) A company selling wines has two methods of customer sales approach, the current method A and a new method B which is being introduced. Each sales associate uses either method A or method B but not both. Only 25% of sales people are using method A. The probability of success is 2/3 for A and ¾ for B. One sales success is chosen at random from all the Company's sales success. Using a tree diagram method, or otherwise, find the probability it was achieved by method B.

(6) The probability of a person catching a particular disease when exposed to it is o.2 if the person has been inoculated against the disease. This probability rises to 0.9 if the person has not been inoculated. If 70% of the population have been inoculated against the disease, what is the probability that a person who caught the disease when exposed to it had been inoculated against the disease?

2.5.4 Independent events

Events A and B are independent events of each other if the probability the B occurs is not affected by whether A has or has not occurred. That is If the probability of B equals the conditional probability of B given A: $p(B) = p(B/A)$ Now substituting $p(B)$ for $p(B/A)$ in the multiplications theorem $p(A \cap B) = p(A)p(B/A)$, we obtain

$$p(A \cap B) = p(A)p(B) \tag{2.7}$$

Definition: If events A and B are independent $p(A \cap B) = p(A)p(B)$ then; otherwise they are dependent. This is called the product law for independent events. It can be generalized to several events as;

$$p(E_1 \cap E_2 \cap .... \cap E_n) = p(E_1)p(E_2)...p(E_n) \tag{2.8}$$

Example 14 A coin and a die are tossed. What is the probability of a 6 and a tail occurring?

E_1 = {a tail} E_2 = { a 6}

$$p(E_1) = \frac{1}{2} \; p(E_2) = \frac{1}{6} \; p(E_1 \cap E_2) = \frac{1}{2}.\frac{1}{6} = \frac{1}{12}$$

Example 15 A coin is tossed five times. What is the probability that all the throws are heads?

H = {a head} $p(H) = \frac{1}{2}$

$p(H).p(H).p(H).p(H).p(H)$

P(five heads) = $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$

Example 16 die is thrown three times. Find the probability that (a) exactly one 6, (b) at least one 6?

Solutions

(a) P(exactlyone6)= $P(6)xP(6')xP(6') + P(6')xP(6)xP(6') + P(6')xp(6')xp(6)$

$$= \frac{1}{6}x\frac{5}{6}x\frac{5}{6} + \frac{5}{6}x\frac{1}{6}x\frac{5}{6} + \frac{5}{6}x\frac{5}{6}x\frac{1}{6} = \frac{25}{72}$$

(b) Since the events ‘at least one throw results in a 6’and ‘no throws result in a 6’ are exhaustive and mutually exclusive, the required probability can be found by subtraction

$$p(at\, least\, one\;\, 6) = 1 - p(no\, 6s)$$

$$= 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$$

Example 17 Two dice are thrown. What is the probability that a ‘double' (i.e., both dice showing the same score) is thrown?

You want the probability of six mutually exclusive, joint events, i.e. (1 1), (2 2), (3 3), (4 4), (5 5), (6 6). The throws on the dice may be assumed independent so that, using the product law,

$$p(1 \cap 1) = p(1)xp(1) = \frac{1}{6}x\frac{1}{6} = \frac{1}{36}$$

Using the addition law for mutually exclusive events

$$p(double) = p(1 \cap 1) + p(2 \cap 2) + p(3 \cap 3) + p(4 \cap 4) + p(5 \cap 5) + p(6 \cap 6)$$

$$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}$$

$$= \frac{1}{6}$$

Example 18 If three marksmen take part in a shooting contest their chances of hitting a target are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. Find the probability that one and only one bullet will hit the target if all men fire simultaneously.

Let

E_1 = {first man hits the target}

E_2 = {second man hits the target}

E_3 = {third man hits the target}

Then you have

$$p(E_1) = \frac{1}{2} \;\; p(E_1') = \frac{1}{2}$$

$$p(E_2) = \frac{1}{3} \; p(E_2') = \frac{2}{3}$$

$$p(E_3) = \frac{1}{4} \; p(E_4') = \frac{3}{4}$$

You require the probabilities of three joint events. Assuming the events are independent, you have, using the product law.

$$p(E_1 \cap E_2' \cap E_3') = \frac{1}{2} x \frac{2}{3} x \frac{3}{4} = \frac{1}{4}$$

$$p(E_1' \cap E_2 \cap E_3') = \frac{1}{2} x \frac{1}{3} x \frac{3}{4} = \frac{1}{8}$$

$$p(E_1' \cap E_2' \cap E_3) = \frac{1}{2} x \frac{2}{3} x \frac{1}{4} = \frac{1}{12}$$

These three joint events are mutually exclusive, so the probability that one of them will occur is given by the addition law for mutually exclusive events:

P (one and only one bullet strikes target) $= \frac{1}{4} + \frac{1}{8} + \frac{1}{12} = \frac{11}{24}$

Example 19. Two teams A and B play a football match against each other. The probabilities p of a team of scoring 0, 1, 2, three goals are shown in Table 6. Calculate the probability of (a) A winning, (b) a draw, and (c) B winning.

Table 6

	Probability of scoring	
Number of goals	A	B
0	0.3	0.2
1	0.3	0.4
2	0.3	0.3
3	0.1	0.1

The possible results of a match are best shown in tabular form as in table 6, where the probabilities of each joint event have been calculated assuming independent and using the product law. The cells may be thought of a point in a sample space, but they no longer represent equiprobable events.

These events are mutually exclusive so that the total probability of A winning can be found by adding the probability or each joint event in which A wins. These events are surrounded by the massive line.

Table 7 Sample space to illustrate Example 19

Number of goals	0	1	2	3
0	0.3 x 0.2 = 0.06	0.3 x 0.2 = 0.06	0.3 x 0.2 = 0.06	0.1 x 0.2 = 0.02
1	0.3 x 0.4 = 0. 12	0.3 x 0.4 = 0. 12	0.3 x 0.4 = 0. 12	0.1 x 0.4 = 0.04
2	0.3 x 0.3 = 0.09	0.3 x 0.3 = 0.09	0.3 x 0.3 = 0.09	0.1 x 0.3 = 0.03
3	0.3 x 0.1 = 0.03	0.3 x 0.1 = 0.03	0.3 x 0.1 = 0.03	0.3 x 0.1 = 0.01

(a) $p(A\,wins(= 0.06 + 0.06 + 0.02 + 0.12 + 0.04 + 0.03 = 0.33)$

(b) $p(draw) = 0.06 + 0.12 + 0.09 + 0.01 = 0.28$ (values on leading diagonal)

(c) $p(B\,wins) = 0.12 + 0.09 + 0.09 + 0.03 + 0.03 = 0.39$

The three probabilities calculated above are exhaustive as well as mutually exclusive so as a check we make sure that their sum is 1.

Example 20 A bag contains three red, four white and five black balls. If three balls are taken what is the probability that they are all the same colour?

We have three possible joint events

P(all white) = $\frac{4}{12} x \frac{3}{11} x \frac{2}{10} = \frac{24}{1320}$ (using the product law with conditional probabilities)

P(all red) = $\frac{3}{12} x \frac{2}{11} x \frac{1}{10} = \frac{6}{1320}$

P(all black) = $\frac{5}{12} x \frac{4}{11} x \frac{3}{10} = \frac{60}{1320}$

The total probability can found by adding these probabilities since the events are mutually exclusive:

P(all the same colour) = $\frac{24}{1320} + \frac{6}{1320} + \frac{60}{1320} = \frac{3}{44}$

Example 21 Of the bicycles in a school bicycle shed, 60% belong to boys and the rest to girls. 90% of the bicycles belonging to boys are racers as are 70% of the bicycles belonging to girls. Dynamos are fitted to 5% of the non-racing and 1% of the racing bicycles irrespective of whether the bicycle is owned by a boy or a girl. If a bicycle is picked at random, find the probability that it is (a) a racer with a dynamo belonging to a boy, (b) a racer without a dynamo. A bicycle is chosen at random. Find the probability if it is not a racer, that it belongs to a girl.

Figure 10 shows the information in the form of a tree diagram.

(a) $p(boy \cap racer \cap dynamo) =$

$p(boy) x p(racer / boy) x p(dynamo / racer) = 0.6x0.9x0.01 = 0.0054$

(b) Racing bicycles without dynamos can belong to boys or girls.

$$p(girl \cap racer \cap dynamo') = p(girl) x p(racer / girl) x p(dynamo')$$
$$= 0.4x0.7x0.99 = 0.2772$$

Similarly $p(boy \cap racer \cap dynamo') = 0.6x0.9x0.99 = 0.5346$

Using the addition law for mutually exclusive events

$$p(racer \cap dynamo') = 0.2772 + 0.5346 = 0.8118$$

(c) $p(girl / racer') = \dfrac{p(girl \cap racer')}{p(racer'0)}$

$$\begin{aligned} P(racer') &= p(boy \cap racer') + p(girl \cap racer') \\ &= p(boy) x p(racer' / boy) + p(girl) x p(racer' / girl) \\ &= 0.6x0.1 + 0.4x0.3 + = 0.18 \end{aligned}$$

= 0.6 x 0.1 + 0.4 x 0.3 = 0.06 + 0.12 = 0.18

$$p(girl / racer') = \frac{0.12}{0.18} = \frac{2}{3}$$

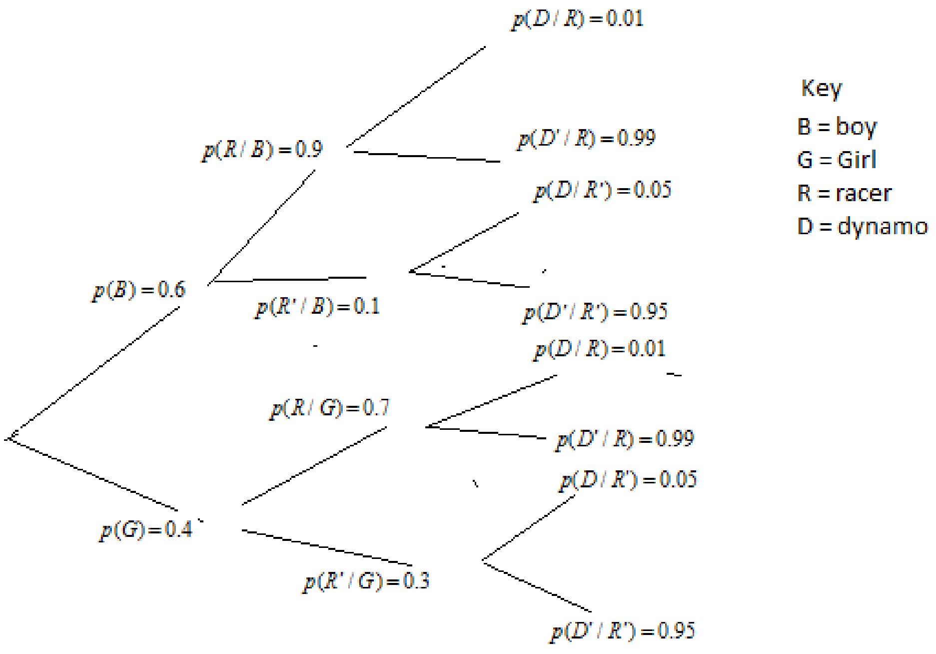


Figure 10 Tree diagram to illustrate Example 21

Example 22 The events A, B and C are such that A and B are independent and A and C are mutually exclusive. Given that $p(A) = 0.4$, $p(B) + 0.2$, $p(C) = 0.3$, $p(B \cap C) = 0.1$, calculate (a) $p(A \cup B)$, (b) $p(C / B)$, (c) $p(B / A \cup C)$. Also calculate the probability that one and only one of the events B, will occur.

Figure 11 shows a Venn diagram of the sample space. A and C are mutually exclusive, and they do not overlap on the Venn diagram.

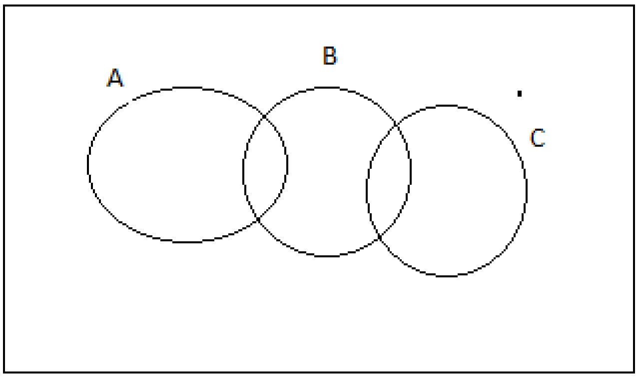


Figure 11 Sample space to illustrate Example 22

(a) $p(A \cup B) = p(A) + p(B) - p(A \cap B)$ A and B are independent.

$$p(A \cap B) = p(A)p(B) = 0.4x0.2 = 0.08$$

and $p(A \cup B) = p(A) + p(B) - p(A \cap B) = 0.4 + 0.2 - 0.08 = 0.52$

(b) $p(C / B) = \dfrac{p(C \cap B}{p(B)} = \dfrac{0.1}{0.2} = 0.5$

(c) $p(B / A \cup C) = \dfrac{p\{B \cap (A \cup C)\}}{p(A \cup C)}$

(d) From Figure 11 you can see that.

$$B \cap (A \cup C) = (B \cap A) \cup (B \cap C)$$

Now $p(B \cap A) = 0.08$ as already calculated and $p(B \cap C) = 0.1$ (given) giving

$p(B \cap (A \cup C) = 0.08 + 0.1 = 0.18$ (Using equation 2.4 for mutually exclusive events)

mutually exclusive events mutually exclusive events

A and C are mutually exclusive $p(A \cup C) = p(A) + p(C) = 0.4 + 0.3 = 0.7$, this

gives $p(B / A \cup C) = \dfrac{0.18}{0.7} = \dfrac{9}{35}$

The probability that B and C occurs corresponds to the area on the diagram inside B and C but not $B \cap C$. Thus this probability is

$$\begin{aligned} & p(B \cup C) - p(B \cap C) \\ &= p(B) + p(C) - 2(B \cap C) \\ &= 0.2 + 0.3 - 2x0.1 \\ &= 0.3 \end{aligned}$$

Example 23 A shopkeeper buys a particular kind of light bulb from three manufacturers A_1, A_2 and A_3. She buys 30% of her stock from A_1, 45% from A_2 and 25% from A_3. This means that if she picks a bulb at random $p(A_1) = 0.3$, $p(A_2) = 0.45$ and $p(A_3) = 0.25$. In the past, she has found that 2% of A_3's bulbs are faulty whereas only 1% of A_1's and A_2's are. Suppose that she chooses a bulb and finds it is faulty. What is the probability that it was on the A_3's bulbs?

The probability will be higher than $p(A_3)$ (= 0.25) since A_3 produces a more significant proportion of faulty bulbs than A_1 and A_2. If F is the event that the bulb is faulty, then $p(A_3 / F)$ is the probability that we require. We have

$p(A_1) = 0.3$, $p(A_2) = 0.45$ $p(A_3) = 0.25$

$p(F / A_1) = 0.01$, $p(F / A_2) = 0.01$, $p(F / A_3) = 0.02$

The information is illustrated as a tree diagram in Figure 12

$p(A_1) = 0.3$, $p(F / A_1) = 0.01$, $p(F' / A_1) = 0.99$

$p(A_2) = 0.45$, $p(F / A_2) = 0.01$, $p(F' / A_2) = 0.99$

$p(A_3) = 0.25$, $p(F / A_3) = 0.02$, $p(F' / A_3) = 0.98$

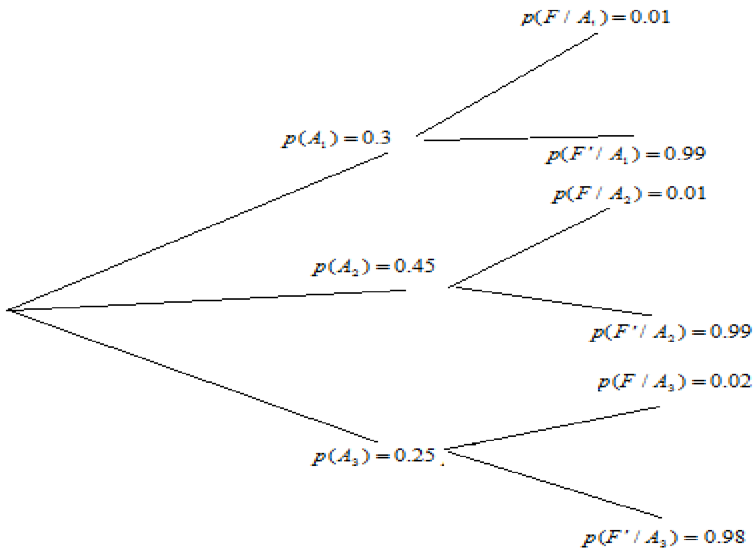


Figure 12 Tree diagram to illustrate the example 23

$$p(F \cap A_3) = p(F/A_3)p(A_3) = p(A_3/F)p(F)$$

Rearranging, the probability we require is

$$p(A_3/F) = \frac{p(F/A_3)pA_3)}{p(F)} \qquad (2.9)$$

$p(F)$, the probability that a bulb is faulty (and comes from A_1 A2 or A_3), is given by the addition law for mutually exclusive events:

$$\begin{aligned} p(F) &= p(F \cap A_1) + p(F \cap A_2) + p(F \cap A_3) \\ &= p(F/A_1)pA_1) + p(F/A_2)p(A_2) + p(F/A_3)p(A_3) \end{aligned}$$

Substituting for P(F) in (4.8.1) gives

$$p(A_3/F) = \frac{p(F/A_3)p(A_3)}{p(F/A_1)pA_1) + p(F/A_2)p(A_2) + p(F/A_3)p(A_3)} \qquad (2.10)$$

$$\begin{aligned} p(A_3/F) &= \frac{p(F/A_3)p(A_3)}{p(F/A_1)pA_1) + p(F/A_2)p(A_2) + p(F/A_3)p(A_3)} \\ &= \frac{0.02x0.25}{0.01x0.3 + 0.01x0.45 + 0.02x0.25} \\ &= 0.4 \end{aligned}$$

Similarly, if we want to find the probability that the faulty bulb was supplied by A_1 we have

$$p(A_1/F)=\frac{p(F/A_1)p(A_1)}{p(F/A_1)pA_1)+p(F/A_2)p(A_2)+p(F/A_3)p(A_3)}$$

$$=\frac{0.01x0.3}{0.01x0.3+0.01x0.45+0.02x0.25}$$

$$=0.24$$

Moreover, since the A_1, A_2 and A_3 are mutually exclusive.

$$p(A_2/F)==1-0.4-0.24=0.36$$

This means that, of the broken bulbs, 24% come from A_1, 36% from A_2 and 40% from A_3.

Equation (2.9) is an example of Bayes' theorem which may be stated as follows:

If $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive events in a sample space S, and B is another event in S then

$$p(A_k/B)=\frac{p(B/A_k)p(A_K)}{\sum_{i=1}^{n}p(B/A_i)p(A_i)} \qquad \text{For } k=1,2,3,4,\ldots,n$$

In practice, the solution to a particular problem is usually made more evident by drawing a tree diagram rather than using equation (2.9).

Example 24 A bag contains five white, six red and seven blue balls. If three items are selected at random, what is the probability that they are (a) all red, (b) all different colours?

The total number of ways, $n\{S\}$, in which three balls can be selected out of the eighteen balls in the bag is $\binom{18}{3}$ the sample space.

(a)The number of ways in which three red balls can be drawn is $\binom{6}{3}$ this is the event that the balls are red. So

$$\text{P(three red balls)} = \frac{Numbet\ of\ ways\ red\ balls\ can\ occur}{Number\ of\ ways\ S\ can\ occur}$$

$$\text{P(three red balls)} = \frac{\binom{6}{3}}{\binom{18}{3}} = \frac{\frac{6!}{(6-3)!!3!}}{\frac{18!}{(18-3)!3!}}$$

$$\text{P(three red balls)} = \frac{6.5.4}{1.2.3}.\frac{1.2.3}{18.17.16} = \frac{6.5.4}{18.17.16} = \frac{5}{3.17.4} = 0.0245$$

This is the probability that if three balls are drawn three are red.

(b) If the balls are all different colours, then one red, one blue and one white must be drawn.

The event that they are of a different colour is: $\binom{5}{1}\binom{6}{1}\binom{7}{1}$

Therefore, p (they are of different colour) = $\dfrac{\binom{5}{1}\binom{6}{1}\binom{7}{1}}{\binom{18}{3}} = 0.257$ this is the probability that if three balls are drawn, they are of any color.

Example 25 Two cards are drawn at random from an ordinary deck of 52 cards. Find the probability p that;

(i) Both are spade

(ii) One is a spade, and one is heart

Solution: The experiment given is a combination in which the sample space is drawing two cards from a deck of 52 cards, and this has

$\binom{52}{2} = \dfrac{52!}{(52-2)!2!} = \dfrac{52.51.50!}{(52-2)!2!} = \dfrac{52.51}{1.2} = 26.51 = 1326$ different ways of selecting two cards.

(i) The event that both are spades is calculated as:

Let A be an event that both cards are spade

(i) Let B be an event that One is spade and one is hear

Example 26 Three light bulbs are chosen at random from 15 bulbs of which 5 are defectives. Find the probability p that:

(i) Non is defective

(ii) Exactly one is defective

(iii) At least one is defective

(iv) At most two are defective

Solution: We first need to know the sample space of this experiment. Let X be a random variable of defective items. So $X = \{0,1,2,3\}$ and we can find the probability of each of these random variables as we use the combination to find S as follows.

$S=\binom{12}{3}=\frac{12!}{(12-3)!3!}=\frac{12!}{9!3!}=\frac{12.11.10.9!}{9!.1.2.3}=4.11.5=220$ Different ways of drawing three items from the box. For each event, we find the number of ways it can occur.

(i) The event A_1 = no defective item can occur in $A_1=\binom{3}{0}=\frac{3!}{0!3!}=1$ that means all items drawn are non-defective, so there are $\binom{9}{3}=\frac{9!}{6!3!}=\frac{9.8.7.6!}{6!.1.2.3}=\frac{9.8.7}{1.2.3}=3.4.7=84$

Therefore, $p(A_1=0)=\frac{\binom{3}{0}\binom{9}{3}}{\binom{12}{3}}=\frac{1.84}{220}=0.382$

(ii) The event A_2 = precisely two defective items are drawn can occur in

$\binom{3}{2}=\frac{3!}{1!2!}=3$ different ways. Therefore the probability that exactly two defective

items are drawn is $p(X=2)=\frac{\binom{3}{2}\binom{9}{1}}{220}=\frac{\frac{3!}{1!2!}\,\frac{9!}{8!1!}}{220}=\frac{3.9}{220}=\frac{27}{220}=0..123$

(iii) The event that at least one defective occurs means our X = {1, 2, 3} so we seek to find $p(1)+p(2)+p(3)$. To get this probability, we find $p(at\ least\ 1 defective\ occurs)=1-p(0)=1-0.382=0.618$ we have used complement law here otherwise we would have calculated the probabilities of $p(1)+p(2)+p(3)$.

(iv) The event A_4 = at most two occurs we seek the probabilities of

X = {0,1,2}, $p=p(0)+p(1)+p(2)=0.382+0.491+0.123=0.996$

Example 27 Let A and B be events with $p(A)=\frac{3}{8}$, $p(B)=\frac{1}{2}$ and $p(A\cap B)=\frac{1}{4}$. Find

(i) $P(A\cup B)$ (ii) $p(A')$ (iii) $p(A'\cap B')$ (iv) $p(A\cap B')$

Solution: Since the probability of the intersection of A and B is non-empty, it means these events are not mutually exclusive.

(i) Hence you use the property of the set theory.

$p(A \cup B) = p(A) + p(B) - p(A \cap B)$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$
$$p(A \cup B) = \frac{3}{8} + \frac{1}{2} - \frac{1}{4}$$
$$p(A \cup B) = \frac{3+4-2}{8} = \frac{5}{8}$$

(ii) To evaluate $p(A')$, we use the property $P(A') = 1 - p(A)$ called the complement law of sets.

$$P(A') = 1 - p(A)$$
$$= 1 - \frac{3}{8} = \frac{5}{8}$$

(iii) To evaluate $p(A' \cap B')$ we use the De Morgan's law of union over intersection given

(iv) as $P(A \cup B)' = p(A' \cap B')$ This means by complement law. $p(A' \cap B') = p(A \cup B)' = 1 - P(A \cup B)$

$$p(A' \cap B') = 1 - \frac{5}{8} = \frac{3}{8}$$

(v) To evaluate $p(A \cap B')$ we use the property. $p(A \cap B') = P(A) - P(A \cap B)$

$$p(A \cap B') = P(A) - P(A \cap B)$$
$$= \frac{3}{8} - \frac{1}{4} = \frac{1}{8}$$

2.4 Unity activity

(1) A box contains 12 eggs of which one is bad. If three eggs are selected at random, what is the probability that one of them will be bad?

(2) In playing a game of bridge, the pack of 52 cards is distributed equally between four players. Find the probability that a particular player is void in one suit (i.e., has no cards of one suit)?

(3) A bag contains twenty chocolates, fifteen toffees, and twelve peppermints. If three sweets are chosen at random what is the probability that they are

(a) all different,

(b) all chocolates,

(c) all the same,

(d) all toffees or peppermints?

(4) A housewife has some cartons of yogurt in her refrigerator. Five is ‘raspberry,' three ‘orange,' four ‘grapefruit' and three ‘strawberry.' When her three children come for dinner, she picks three cartons at random. Calculate the probability that she picks

(a) three ‘orange’ ones,

(b) first a ‘raspberry’ one, then a ‘strawberry’ one, then a ‘grapefruit’ one

(c) one raspberry’, one ‘strawberry’ and one ‘grapefruit’ in any order.

5. A class contains 30 children, 18 girls, and 12 boys. Four complimentary theatre tickets are distributed at random to the children in the class. Find the probability that all four tickets go to girls,

6. A pack of 52 contains four suits each of thirteen cards. If thirteen cards are taken at random from the pack what is the probability that exactly ten of them are spades? (You may take (52) = 6.35×10^{11}.)

7. (a) From an ordinary pack of 52 cards, two are dealt face downwards on a table. What is the probability that (i) the first card dealt is a heart, (ii) the second card dealt is a heart, (iii) both cards are hearts, (iv) at least one card is a heart?

8. Bag A contains three white counters and two black counters whilst bag B contains two white and three black. One counter is removed from bag A and placed in bag B without its color being seen. What is the probability that a counter removed from bag B will be white?

9. A box contains 24 eggs is known to contain four old and twenty new. If three eggs are picked at random, determine the probability that (i) two are new and the other old, (ii) they are all new.

10. Just four players in the squad, none of whom is a goalkeeper, belong to a Liver-pool club. A team is selected at random from the squad of three goalkeepers and thirteen other players. Calculate the probability that all of the four players from the Liverpool club are included in the team.

11. Two cards are chosen at random from 10 cards number 1 to 10. Find the probability p the sum is odd if

(i) the two cards are drawn together?

(ii) the two cards are drawn one after another without replacement

(iii) the two cards are drawn one after another with replacement

12. Six couples are in a room.

(i) Find the probability p if two persons are chosen at random,

(a) they are a couple

(b) there is one male and one female

(ii) If four people are chosen at random, find the probability p that

(a) Two couples are chosen

(b) No couple is among the 4

(c) Precisely one couple is among the 4

(iii) If the 12 people are divided into six pairs, find the probability p that

(a) Each pair is a couple

(b) one is male, and one is female

13. We are given three boxes as follows:

Box I has ten light bulbs of which 4 are defectives

Box II has ten light bulbs of which one are defective

Box III has ten light bulbs of which 3 are defectives

A box is drawn at random and then draw a bulb at random. What is the probability that a bulb is defective?

14.(a) The pages of a book are numbered from 1 to 200. If a page is chosen at random, what is the probability that its number will continue just two digits?

(b) The probability that a January night will be icy is $\frac{1}{4}$. On an icy night, the probability that there will be a car accident at a specific dangerous corner is 1/25. If it is not icy, the probability of an accident is $\frac{1}{100}$. What is the probability that

(i) 13 January will be icy and there will be an accident,

(ii) there will be an accident on 13 January?

15. Two events A and B are such that P(A) = $\frac{1}{3}$ and P(B) = $\frac{1}{2}$. If A' stands for complement of A calculate $p(' \cap B)$ in each of the cases when

(a) $p(' \cap B) = \frac{1}{8}$

(b) A and B are mutually exclusive.

(c) A is a subset of B.

16. An unbiased die F has its faces numbered 1, 2, 3, 4, 5 and 6. Another unbiased die S

has its faces numbered 1, 1, 2, 2, 3 and 3. In a game, a card is selected at random from a pack of 52 playing cards and if a diamond is obtained die F is thrown; otherwise, die S is thrown. Find the probability of scoring 2.

Unit Summary

P(E) denotes the probability that event E occurs. E or E' is called the complement of E and P(E) denotes the probability that E does not occur. $p(E') = 1 - p(E)$

For two events E_1 and E_2

$p(E_1 \cup E_2)$ Denotes the probability that E_1 and E_2 occur.

$p(E_1 \cap E_2)$ Denotes the probability that both E_1 E_2 occurs.

$p(E_1 / E_2)$ Denotes the **conditional probability** that E_1 occurs given that E_2 has occurred.

Additional law: $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Mutually exclusive events are events which cannot both occur. If E_1, E_2 are mutually exclusive then. $p(E_1 \cap E_2) = 0$, $p(E_1 / E_2) = 0$ $p(E_2 / E_1) = 0$ and the additional law becomes $p(E_1 \cup E_2) = p(E_1) + p(E_2)$

Product law: $p(E_1 \cap E_2) = p(E_2) x p(E_2 / E_2) = p(E_1) x p(E_2 / E_1)$

Independent events are events such that the occurrence of one does not affect the probability of the occurrence of the other. If E1 and E2 are independent then.

$p(E_1 / E_2) = p(E_1) p(E_2 / E_1) = p(E_2)$ Moreover, the product law becomes

$p(E_1 \cap E_2) = p(E_1).p(E_2)$

Exclusive events are a group of events such that they include all possible outcomes. If E_1 and E_2 are exhaustive then $p(E_1 \cap E_2) = 1$.

UNIT 3 RANDOM VARIABLE AND ITS PROBABILITY DISTRIBUTION

3.1 Introduction

Welcome to this unit called random variables and its probability distributions. In this unit you will learn what is a random variable is, types of random variables, their properties and their probability distributions.

3.2 Unit Aims:

To explore properties and probabilities of the discrete random variables and general discrete probability distributions.

3.3 Unit Objectives:

At the end of this unit you should be able to:

- Distinguish between discrete and continuous random variables.
- Compute and interpret the expected value, variance, and standard deviation for a discrete random variable.
- Compute probabilities using any given probability distribution.
- Differentiate between how probabilities are computed for discrete and continuous random variables.
- Compute probability values for a continuous uniform probability distribution and be able to compute the expected value and variance for such a distribution.
- use the probability mass function of a discrete or continuous random variable to find probabilities.
- apply the material learned in this lesson to new problems

Terminology

- X – Random variable
- $\sum$ Summation

3.4 Time Required:

You should spend 10 hours on this unit

3.5 Unit Topics:

3.5.1 Random Variables

A variable is something which can change its value. It may vary with different outcomes of an experiment. If the value of a variable depends upon the outcome of a random experiment it is a random variable. A random variable can take up any real value. Mathematically, a random variable is a real-valued function whose domain is a sample space S of a random experiment. A random variable is always denoted by capital letter like X, Y, M etc. The lowercase letters like x, y, z, m etc. represent the value of the random variable. Consider the random experiment of tossing a coin 20 times. You will earn K5 is you get head and will lose K5 if it a tail. You and your friend are all set to see who will win the game by earning more money. Here, you see that the value of getting head for the coin tossed for 20 times is anything from zero to twenty. If you denote the number of a head by X, then X = {0, 1, 2, … , 20}. The probability of getting a head is always 1/2.

Properties of a Random Variable

- It only takes the real value.
- If X is a random variable and C is a constant, then CX is also a random variable.
- If X_1 and X_2 are two random variables, then $X_1 + X_2$ and $X_1 X_2$ are also random.
- For any constants C_1 and C_2, $C_1X_1 + C_2X_2$ is also random variable.
- |X| is a random variable.

Types of Random Variable

A random variable can be categorized into two types. There is discrete random variable and continuous random variables.

3.5.2 Discrete random variable

A discrete random variable is a variable which can only take particular values or a countable number of real values i.e, it is discrete in nature. The value of the random variable depends on chance, because a real-valued function defined on a discrete sample space is a discrete random variable. The number of calls a person gets in a day, the number of items sold by a company, the number of items manufactured, number of accidents, number of gifts received on birthday etc. are some good examples of the discrete random variables.

Probability Distribution

For any event of a random experiment, you can find its corresponding probability. For different values of the random variable, you can find its respective probability. The values of random variables along with the corresponding probabilities are the probability distribution of the random variable. Assume X is a random variable. A function P(X) is the probability distribution of X. Any function f defined for all real x by $f(x) = P(X \leq x)$ is called the distribution function of the random variable X.

Properties of Probability Distribution

- The probability distribution of a random variable X is $P(X = x_i) = p_i$ for $x = x_i$ and

  $P(X = x_i) = 0$ for $x \neq x_i$.

- The range of probability distribution for all possible values of a random variable is from 0 to 1, i.e., $0 \leq p(x) \leq 1$.

3.5.3 Probability Distribution of a Discrete Random Variable

If X is a discrete random variable with discrete values $x_1, x_2, \dots, x_n, \dots$ then the probability function is $p(x) = p_X(x)$. The distribution function is

$F_X(x) = p(X \leq x_i) = \sum_i p(x_i) = p_i$ if $x = x_i$ and is 0 for other values of x. Here, i = 1, 2, …, n, … is called cumulative density function.

Consider an example of tossing of two fair coins. The possible outcomes for this random experiment are S = {HH, HT, TH, TT}. If X is a random variable for the occurrence of the tail, the possible values for X are 0, 1, and 2. The distribution function for X is $F(x) = P(X \leq x)$ is

Table 1

Value of X	0	1	2
$P(X = x) = p(x)$	1⁄4	2⁄4	1⁄4
$F(X) = P(X \leq x) = \sum_i p(x_i)$	1⁄4	3⁄4	4⁄4 = 1

Example 1 Three fair coins are tossed. Let X = the number of heads, Y = the number of head runs. (A 'head run' is a consecutive occurrence of at least two heads.) Find the

probability function of X and Y.

Solution: The possible outcomes of the experiment is S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}. X is the number of heads. It takes up the values 0, 1, 2, and 3.

Table 2

Event	Random variable	Random variable
	X	Y
HHH	3	1
HHT	2	1
HTH	2	0
HTT	1	0
THH	2	1
THT	1	0
TTH	1	0
TTT	0	0

- P(no head) = p(0) = 1⁄8
- P(one head) = p(1) = 3⁄8
- P(two heads) = p(2) = 3⁄8
- P(three heads) = p(3) = 1⁄8

Table 3

Value of X, x	0	1	2	3
p(x)	1⁄8	3⁄8	3⁄8	1⁄8

Y is the number of head runs. It takes up the values 0 and 1. P(Y = 0) = p (0) = 5⁄8, and P(Y = 1) = p(1) = 3⁄8.

Table 4

Value of Y, y	0	1
p(y)	5⁄8	3⁄8

Example 2 Select three fans randomly at a football game in which ZESCO is playing Ndola united. Identify whether the fan is a Zesco(Z) or a Ndola united fan (N). This experiment yields the following sample space: *S* = {ZZZ, ZZN, ZNZ, NZZ, NNZ, NZN, ZNN, NNN}. Let *X* = the number of ZESCO fans selected. The possible values of *X* are, therefore, either 0, 1, 2, or 3. Now, you can find probabilities of individual events, *P*(ZZZ) or *P*(ZZN), for example. Alternatively, you can find *P*(*X* = *x*), the

probability that X takes on a particular value x.

Solution. Since the game is a home game, let's suppose that 80% of the fans attending the game are ZESCO fans, while 20% are Ndola United fans. That is, $P(\text{Z}) = 0.8$ and $P(\text{N}) = 0.2$. Then, by independence: $P(X = 0) = P(\text{ZZZ}) = 0.2 \times 0.2 \times 0.2 = 0.008$. And, by independence and mutual exclusivity of NNZ, NZN, and ZNN:

$$P(X = 1) = P(\text{NNZ}) + P(\text{NZN}) + P(\text{ZNN}) = 3 \times 0.2 \times 0.2 \times 0.8 = 0.096$$

Likewise, by independence and mutual exclusivity of ZZN, ZNZ, and NZZ:

$$P(X = 2) = P(\text{ZZN}) + P(\text{ZNZ}) + P(\text{NZZ}) = 3 \times 0.8 \times 0.8 \times 0.2 = 0.384$$

Finally, by independence: $P(X = 3) = P(\text{ZZZ}) = 0.8 \times 0.8 \times 0.8 = 0.512$

There are a few things to note here:

- The results make sense! Given that 80% of the fans in the stands are ZESACO fans, it shouldn't seem surprising that you would be most likely to select 2 or 3 ZESCO fans.
- The probabilities behave well in that (1) the probabilities are all greater than 0, that is, $P(X = x) > 0$ and (2) the probability of the sample space is 1, that is, $P(S) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1$.
- Because the values that it takes on are random, the variable X has a special name. It is called a **random variable**!

Definition. Given a random experiment with sample space S, a **random variable *X*** is a set function that assigns one and only one real number to each element s that belongs in the sample space S. The set of all possible values of the random variable X, denoted x, is called the **support**, or **space**, of X.

Note that the capital letters at the end of the alphabet, such as W, X, Y, and Z typically represent the definition of the random variable. The corresponding lowercase letters, such as w, x, y, and z, represent the random variable's possible outcomes.

Definition. A random variable X is a **discrete random variable** if:

- there are a finite number of possible outcomes of X, or
- There are a countably infinite number of possible outcomes of X.

Recall that a countably infinite number of possible outcomes means that there is a one-to-one correspondence between the outcomes and the set of integers. No such one-to-one correspondence exists for an uncountably infinite number of possible outcomes.

Example 3 Suppose you toss three coins once or one coin three times, and consider a head as a success, then your random variable X is given in the table 1 as:

Table 5 Table of possible outcomes when a coin is tossed three times

Value of X	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
Outcome(No of heads)	3	2	2	2	1	1	1	0

Definition: A random variable (r.v.) is defined as a real number X connected with the outcome of a random experiment E. For this example, if E consists of three coins tossed once, you have a random variable X which denotes the number of heads X = { 0, 1, 2, 3}, possible outcomes.

Thus, to every outcome there corresponds a real number X. Since the points of the sample space corresponds to outcomes, this means that a real number, which is denoted by X, is defined for each $x \in S$. Therefore, X = { 0, 1 , 2, 3 }.Thus, you can define a random variable as a real valued function whose domain is the sample space associated with a random experiment and range is the real line. To find a probability distribution of X, you proceed as follows:

Table 6 Probability distribution of X

X	P(X)
0	1/8
1	3/8
2	3/8
3	1/8

Example 4 A pair of fair dice is tossed. Let X be a random variable which assigns to each point the maximum of the two numbers which occurs. X((a,b)) = max(a,b). Find the probability distribution of X?

Solution.

The sample space S consists of the 36 ordered pairs of numbers from 1 to 6.. Therefore,

S = { (1,1), (1,2), …,(6,6)}

Table 7

		1	2	3	4	5	6
	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
S =	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Since X assigns to each point in S the maximum of the two numbers in the ordered pair, the range space of X consists of the numbers the numbers from 1 to 6: $R_X = \{1,2,3,4,5,6\}$

There is only one point (1,1), whose maximum is 1,; hence p(1) = 1/36. P(2) = 3/36,

P(3) = 5/36, p(4) = 7/36, p(5) = 9/36, p(6) = 11/36

Table 8 The probability distribution of X is:

X	P(x)
1	1/36
2	3/36
3	5/36
4	7/36
5	9/36
6	11/36

3.5.4 Probability Mass Function (Discrete Random Variable)

Suppose X is a one-dimensional discrete random variable taking at most a countable infinite number of values $x_1, x_2, \ldots$ with each possible outcome X; you can associate a number $p_i = p(X = x_i) = p(x_i)$ called the probability of X_i, the numbers P (X_i), i=1, 2, … must satisfy the following conditions:

a) $p_i = P(X=X_i) = P(X_i) \geq 0$ i.e. p_is are all non-negative.

b) $\sum_{i=1}^{\infty} P(X_i) = 1$ i.e. the total probability is one.

c) $0 < p(x_i) < 1$

Example 4 Let ***X*** equal the number of siblings of Penn State students. The support of ***X*** is, of course, 0, 1, 2, 3, ... Because the support contains a countably infinite number of possible values, ***X*** is a discrete random variable with a probability mass function. Find ***f***(***x***) = ***P***(***X*** = ***x***), the probability mass function of ***X***, for all ***x*** in the support.

Table 9

X	P(X=x)
0	0.41
1	0.45
2	0.11
3	0.03

This example illustrated the tabular and graphical forms of a p.m.f. Now let's take a look at an example of a p.m.f. in functional form.

Example 5 Let ***f***(***x***) = ***cx***^{2} for ***x*** = 1, 2, 3. Determine the constant ***c*** so that the function ***f***(***x***) satisfies the conditions of being a probability mass function.

Solution. The key to finding *c* is to use item number b) in the 3.5.4 Probability Mass Function (Discrete Random Variable)

$$f(x) = cx^2 \text{, for X = 1,2,3}$$

$$\begin{aligned}\sum_{x=1}^{3} f(x) &= \sum_{x=1}^{3} cx^2 \\ &= c\left[1^2 + 2^2 + 3^2\right] \\ &= c\left[1+4+9\right] \\ &= c[14] = 1\end{aligned}$$

Therefore, c = 1/14

$$f(x) = \frac{1}{14}x^2, \qquad \text{for X = 1, 2, 3}$$

The support in this example is finite. Let's take a look at an example in which the support is countably infinite.

Example 6 Determine the constant *c* so that the following p.m.f. of the random variable *Y* is a valid probability mass function: $f(y) = c\left(\frac{1}{4}\right)^y$, for *y* = 1, 2, 3, ...

Solution. Again, the key to finding *c* is to use item number b) in the 3.5.4 Probability Mass Function (Discrete Random Variable)

$$f(y) = c\left(\frac{1}{4}\right)^y$$

$$\sum_{y=1}^{\infty} f(y) = \sum_{y=1}^{\infty} c\left(\frac{1}{4}\right)^y = c\left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots\right]$$

$$= c\left[\frac{1/4}{1 - 1/4}\right] = 1$$

Observe that the series given is a geometric series and has sum: $S = \dfrac{a}{1-r}$ where a is the first term and the common ratio is ¼.

c =1/3, to give c = 3

Therefore $f(y) = 3\left(\frac{1}{4}\right)^y$ is the probability mass function

Example 6 Take for example table 10, this satisfies the conditions given above ,hence this is a probability distribution.

Table 10

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

This function p_i = P ($X=X_i$) or p(x) is called the probability function or probability mass function (p.m.f.) of the random variable X and set of all possible ordered pairs {x, p(x)} is called the probability distribution of the random variable X.

Example 7 Two cards are drawn one by one without replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of aces.

Solution: Let X be the random variable, which is the number of aces.

Here X takes values 0, 1, 2

$$P(X=0)=\frac{48}{52}\times\frac{47}{51}=\frac{188}{221}$$

$$P(X=1)=2\left(\frac{4}{52}\times\frac{48}{51}\right)=2\left(\frac{1}{13}\times\frac{16}{17}\right)=2\left(\frac{16}{221}\right)=\frac{32}{221}$$

$$P(X=2)=\frac{4}{52}\times\frac{3}{51}=\frac{1}{13}\times\frac{1}{17}=\frac{1}{221}$$

Hence, the probability distribution is of the random variable X is table 10.

Example 8 Four defective milk pouches are accidently mixed with sixteen good ones and by looking at them it is not possible to differentiate between them. Three milk pouches are drawn at random from the lot. Find the probability distribution of X, the number of defective milk pouches.

Solution: Let X be the random variable, which is the number of defective milk pouches.

Here X takes values 0, 1, 2, 3.

Total number of milk pouches = 4 + 16 = 20

Number of defective milk pouches = 4

$$\therefore P(X=0)=P(\text{No defective milk pouche})=\frac{{}^{16}C_3}{{}^{20}C_3}=\frac{16\times15\times14}{20\times19\times18}=\frac{140}{285}$$

$$P(X=1)=P(\text{one defective milk pouche})=\frac{{}^{4}C_1\times{}^{16}C_2}{{}^{20}C_3}=\frac{4\times16\times15\times6}{2\times20\times19\times18}=\frac{120}{285}$$

$$P(X=2)=P(\text{two defective milk pouches})=\frac{{}^{4}C_2\times{}^{16}C_1}{{}^{20}C_3}=\frac{4\times3\times16\times6}{2\times20\times19\times18}=\frac{24}{285}$$

$$P(X=3)=P(\text{three defective milk pouches})=\frac{{}^{4}C_3}{{}^{20}C_3}=\frac{4\times3\times2}{20\times19\times18}=\frac{1}{285}$$

Hence, the probability distribution is

Table 11

X:	0	1	2	3
P (X):	$\frac{140}{285}$	$\frac{120}{285}$	$\frac{24}{285}$	$\frac{1}{285}$

3.5.5 Mean and Variance of Random Variables

Suppose you want to know or guess about your performance in the five mathematics

tests. The total marks are the same for each of the tests. What can you say about your performance on the basis of the marks scored? What is your overall performance in these tests? You can do so by calculating the average of the marks obtained to get an idea of your overall performance. This average marks will tell you about the marks you are most close to. In probability and statistics, you can find out the average of a random variable. The term average is the mean or the expected value or the expectation in probability and statistics. Once you have calculated the probability distribution for a random variable, you can calculate its expected value. Mean of a random variable shows the location or the central tendency of the random variable.

The expectation or the mean of a discrete random variable is a weighted average of all possible values of the random variable. The weights are the probabilities associated with the corresponding values. It is calculated as,

$$E(X) = \mu = \sum_i x_i p_i \text{ , for i = 1, 2, …, n that is } E(X) = x_1 p_1 + x_2 p_2 + ... + x_n p_n .$$

Let X denotes the random variable which assumes values $x_1, x_2, ..., x_n$ with corresponding probabilities $p_1, p_2, ..., p_n$ .Then the probability distribution be as follows:

Table 12 The probability distribution of a random variable

X:	x_1	x_2	…	x_n
P(X):	p_1	p_2	…	p_n

Then

$$\sum_{i=1}^{n} p_i = p_1 + p_2 + \cdots + p_n = 1$$

The mean (μ) of the above probability distribution is defined as:

$$\mu = \frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{p_1 + p_2 + \cdots + p_n} = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i$$

Properties of Mean of Random Variables

- If X and Y are random variables, then E(X + Y) = E(X) + E(Y).

- If $X_1, X_2, \dots, X_n$ are random variables , then $E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n) = \Sigma_i E(X_i)$.

- For random variables, X and Y, E(XY) = E(X) E(Y). Here, X and Y must be independent.
- If a is any constant and X is a random variable, E[aX] = a E[X] and E[X + a] = E[X] + a.
- For any random variable, X > 0, E(X) > 0.
- $E(Y) \geq E(X)$ if the random variables X and Y are such that $Y \geq X$.

3.5.6 Variance of Random Variables

Suppose you calculated the mean or the average marks in the five tests of mathematics. You can easily see the difference of marks in each of the tests from this average marks. This difference in marks shows the variability of the possible values of the random variable. The random variable being the marks scored in the test.

The variance of a random variable shows the variability or the scatterings of the random variables. It shows the distance of a random variable from its mean. It is calculated as

The variance (σ^2) is defined as:

$$\sigma^2 = \sum (x_i - \mu)^2 p_i = \sum (x_i{}^2 + \mu^2 - 2x_i\mu)p_i = \sum x_i{}^2 p_i + \mu^2 \sum p_i - 2\mu \sum x_i p_i$$

$$= \sum x_i{}^2 p_i + \mu^2(1) - 2\mu(\mu) = \sum x_i{}^2 p_i - \mu^2 = \sum x_i{}^2 p_i - \left(\sum p_i x_i\right)^2$$

Mean of a random variable X is also known as expected value and is denoted by E(X).

$$E(X) = \mu = p_1x_1 + p_2x_2 + \cdots + p_nx_n = \sum p_i x_i$$

$$\text{Variance } (\sigma^2) = E(X^2) - (E(X))^2$$

$Var(X) = \sigma^2 = \sum_i (x_i - \mu)^2 p(x_i) = E(X-\mu)^2$ or, $Var(X) = E(X^2) - [E(X)]^2$.

$E(X^2) = \sum x_i^2 p(x_i)$ and $[E(X)]^2 = [\sum x_i p(_i)]^2 = \mu^2$

If the value of the variance is small, then the values of the random variable are close to the mean.

Properties of Variance of Random Variables

- The variance of any constant is zero i.e, V(a) = 0, where a is any constant.
- If X is a random variable, and a and b are any constants, then $V(aX + b) = a^2 V(X)$.
- For any pair-wise independent random variables, $X_1, X_2, \ldots, X_n$ and for any constants $a_1, a_2, \ldots, a_n$; $V(a_1X_1 + a_2X_2 + \ldots + a_nX_n) = a_1^2 V(X_1) + a_2^2 V(X_2) + \ldots + a_n^2 V(X_n)$.

Example 9 Using table 8 above you can find the expected value of the random variable X. using table 13 below shows the calculation of the expected value and expected variance for the probability distribution.

Table 13 Probability distribution for the number of defective items, X

X	P(X=x)	x P(X=x)	x^2P(X=x)
0	0.382	0	0
1	0.491	0.491	0.491
2	0.123	0.246	0.492
3	0.004	0.012	0.036
Total	1	0.749	1.019

The expected number of defective items in the box is $E(X) = x\sum_{all\,x} xp(X = x)$. Obtained as: $E(X) = \sum_{x=1}^{3} xp(X = x) = 0.382 + 0.491 + 0.123 + 0.004 = 0.749$

Since you do not have decimal items you round it off to the nearest whole number. That is the expected number of defective item in the box is one. The notation can be extended to define the expectation of any function of a random variable. For example the expectation of x^2 is given by $E(x^2) = \sum_{all\,x} x^2 p(X = x)$.

$$E(X^2) = \sum_{x=1}^{3} x^2 p(X = x) = 0 + 0.491 + 0.492 + 0.036 = 1.019$$

Variance

For the probability distribution of the random variable of X can be used to calculate the variance of the random variable X. The formula for calculating variance is

$Var(X) = E(X^2) - [E(X)]^2$, but we know $E(X) = 0.749$ and $E(X^2) = 1.019$

substituting these in equation you get $Var(X) = E(X^2) - [E(X)]^2$

The variance is given as: $\sigma^2 = E(X^2) - \mu^2$

$\sigma^2 = 1.019 - 0.749^2 = 1.019 - 0.5614 = 0.458$. Therefore the variance is 0.458.

Example 10 Find the mean and variance of the number of heads in two tosses of a coin.

Solution: Let X denotes the number of heads obtained in two tosses of a coin. Thus, X takes the values 0, 1, 2. Now p, the probability of getting a head =1/2 and q, the probability of not getting a head = 1- 1/2 = 1/2

$$\therefore \mathrm{P}(\mathrm{X} = 0) = \mathrm{q} \times \mathrm{q} = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$

$$\mathrm{P}(\mathrm{X} = 1) = \mathrm{p} \times \mathrm{q} + \mathrm{q} \times \mathrm{p} = 2\left(\tfrac{1}{2} \times \tfrac{1}{2}\right) = \tfrac{1}{2}$$

$$\mathrm{P}(\mathrm{X} = 2) = \mathrm{p} \times \mathrm{p} = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$

Thus, you have:

Table 14 The probability distribution of X

	p_i	$p_i x_i$	x_i^2	$p_i x_i^2$
0	1/4	0	0	0
1	1/2	1/2	1	1/2
2	1/4	2/4	4	1
Total		1		3/2

$$Hence, the\ mean\ \mu = \sum x_i p_i = 0 + \tfrac{1}{2} + \tfrac{2}{4} = 1\ and\ the\ variance\ \sigma^2 = \sum p_i x_i^2 - \mu^2 = \tfrac{3}{2} - (1)^2 = \tfrac{3}{2} - 1 = \tfrac{1}{2}$$

Example 11 A die is tossed twice. Getting a number greater than 4 is considered a success. Find the variance of the probability distribution of the number of success.

Solution: Here p, probability of a number greater than 4=2/6=1/3 and q, probability of a number not greater than 4 is $1 - \frac{1}{3} = \frac{2}{3}$

$$\mathrm{P}(\mathrm{X} = 0) = \mathrm{q} \times \mathrm{q} = \tfrac{2}{3} \times \tfrac{2}{3} = \tfrac{4}{9}$$

$$P(X=1) = p \times q + q \times p = \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{4}{9}$$

$$P(X=2) = p \times p = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

Thus, you have:

Table 15 Probability distribution of X

x_i	p_i	$p_i x_i$	x_i^2	$p_i x_i^2$
0	4/9	0	0	0
1	4/9	4/9	1	4/9
2	1/9	2/9	4	4/9
Total		6/9		8/9

Hence, the mean $\mu = \sum p_i x_i = \frac{6}{9} = \frac{2}{3}$ and the variance

$$\sigma^2 = \sum p_i x_i^2 - \mu^2 = \frac{8}{9} - \left(\frac{6}{9}\right)^2 = \frac{8}{9} - \frac{36}{81} = \frac{72-36}{81} = \frac{36}{81} = \frac{4}{9}$$

Example 12 If a box has 12 items of which 3 are defectives, the number of defective items is an example of a discrete random variable because they take particular values. The 'number of defective items' is a variable since it can take different numerical values within a given range; it is random because we cannot predict the outcome of counting the number of defective items in a box; it is discrete because it can take only certain values in a given range than all values in that range. In this example of defective items X is defined as a random variable that assigns the number of defective items. So X can take the following values, $X = \{0,1,2,3\}$. If three items are selected from the box, first you examine on the way these are selected so that we determine the technique of counting. If X is 0 it means there is no defective item drawn and it is easy to calculate $p(X=0)$ the probability that there is no defective item among the three items drawn. Then we can answer any probability problem as long the method of counting has been identified. 'The number of yellow flowers in a window box' and 'the number of aces in hand of thirteen cards' are examples of discrete random variables. The term 'variable' is usually used because it can assume any value. The term 'variates' would be used if we have, to say a consignment of packets of balloons and actually counted the number of balloons in each packet.

Example 13 Let a box contain 12 items of which three are defective. Let X be a random variables of defective items in the box. If three items are drawn from the box.

Find the probability p that.

(i) No defective item is drawn

(ii) Two defective items are drawn

(iii) At least one defective item is drawn

(iv) At most two defective items are drawn

Solution: To answer this question we need to know the sample space of this experiment. We let X be a random variable of defective items. So $X = \{0,1,2,3\}$ and we can find the probability of each of these random variables as you use combination to find S as follows. $S = \binom{12}{3} = \frac{12!}{(12-3)!3!} = \frac{12!}{9!3!} = \frac{12.11.10.9!}{9!.1.2.3} = 4.11.5 = 220$

Different ways of drawing three items from the box. For each event we find the number of ways it can occur.

The event A_1 = no defective item can occur in $A_1 = \binom{3}{0} = \frac{3!}{0!3!} = 1$

Therefore, $p(A_1 = 0) = \frac{\binom{3}{0}\binom{9}{3}}{\binom{12}{3}} = \frac{1.\frac{9!}{6!1.2.3}}{220} = \frac{\frac{9.8.7.6!}{1.2.3.6!}}{220} = \frac{3.4.7}{220} = \frac{84}{220} = 0..382$

$$p(1) = \frac{\binom{3}{1}\binom{9}{2}}{220} = \frac{\frac{3!}{(3-1)!1!}.\frac{9!}{(9-2)!2!}}{220} = \frac{3.\frac{9.8.7.!}{7!.1.2}}{220} = \frac{3.9.4}{220} = \frac{108}{220} = 0.491$$

$$p(2) = \frac{\binom{3}{2}\binom{9}{1}}{220} = \frac{3.9}{220} = 0.123$$

$$p(3) = \frac{\binom{3}{3}\binom{9}{0}}{220} = \frac{1}{220} = 0.005$$

So $p(0) = 0.191$, $p(1) = 0.246$, $p(2) = 0.061$ and $p(3) = 0.002$

You can represent these probabilities as a table as follows:

Table 16 The probability distribution of the random variable X

X	P(X=x)
0	0.382
1	0.491
2	0.123
3	0.005
Total	1.000

(i) The event A_1 = no defective item can occur in $A_1 = \binom{3}{0} = \frac{3!}{0!3!} = 1$ that means all items drawn are non-defective, so there are

$$\binom{9}{3} = \frac{9!}{6!3!} = \frac{9.8.7.6!}{6!.1.2.3} = \frac{9.8.7}{1.2.3} = 3.4.7 = 84$$

$$\text{Therefore, } p(A_1 = 0) = \frac{\binom{3}{0}\binom{9}{3}}{\binom{12}{3}} = \frac{1.84}{220} = 0.382$$

(ii) The event A_2 = exactly two defective items are drawn can occur in $\binom{3}{2} = \frac{3!}{1!2!} = 3$ different ways. Therefore the probability that exactly two defective items are drawn is

$$p(X = 2) = \frac{\binom{3}{2}\binom{9}{1}}{220} = \frac{\frac{3!}{1!2!}\frac{9!}{8!1!}}{220} = \frac{3.9}{220} = \frac{27}{220} = 0..123$$

(iii) The event that at least one defective occurs means our X = {1, 2, 3} so we seek p(0)+p(1)+p(3). To get this probability we find $p(at\ \ least\ \ 1 defective\ occurs) = 1 - p(0) = 1 - 0.382 = 0.618$ We have used complement law here otherwise we would have calculated the probabilities of $p(1) + p(2) + p(3)$.

(iv) The event A_4 = at most two occurs we seek the probabilities of X = {0,1,2} $p = p(0) + p(1) + p(2) = 0.382 + 0.491 + 0.123 = 0.996$

Table 1 gives the probabilities of the random variable X = { 0, 1, 2, 3} of defective items in the box.

3.5.7 The Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of the random variable *X* has the following definition: $F_X(t) = p(X \le t)$

The notation $F_X(t)$ means that *F* is the cdf for the random variable *X* but it is a function of *t*.

The cdf of random variable *X* has the following properties:

1. $F_X(t)$ is a non-decreasing function of *t*, for $-\infty <\ t < \infty$.
2. The cdf, $F_X(t)$, ranges from 0 to 1. This makes sense since $F_X(t)$ is a probability.
3. If *X* is a discrete random variable whose minimum value is *a*, then $F_X(a) = p(X \le a) = p(X = a) = f_X(a)$. If *c* is less than *a*, then $F_X(c) = 0$.
4. If the maximum value of *X* is *b*, then $F_X(b) = 1$
5. Also called the *distribution function*.
6. All probabilities concerning *X* can be stated in terms of *F*.

Suppose *X* is a discrete random variable. Let the pmf of *X* be equal to

$$f(x) = \frac{5-x}{10}, \text{ x = 1, 2, 3}$$

Suppose you want to find the cdf of *X*. The cdf is $F_X(t) = p(X \le t)$.

For *t*=1, $p(X \le 1) = p(X = 1) = f(1) = \frac{5-1}{10} = \frac{4}{10}$

For *t*=2, $p(X \le 2) = p(X = 1 or\ X = 2) = p(X = 1) + p(X = 2) = \frac{5-1}{10} + \frac{5-2}{10} = \frac{7}{10}$

For *t*=3, $p(X \le 3) = \frac{5-1}{10} + \frac{5-2}{10} + \frac{5-3}{10} = \frac{9}{10}$

For *t*=4, $p(X \le 4) = \frac{5-1}{10} + \frac{5-2}{10} + \frac{5-3}{10} + \frac{5-4}{10} = \frac{10}{10} = 1$

It is worth noting that $P(X \le 2)$ does not equal $P(X < 2)$; $P(X \le 2) = P(X = 1,2)$ and $P(X < 2) = P(X = 1)$. It is very important for you to carefully read the problems in order to correctly set up the probabilities. You should also look carefully at the notation if a problem provides it.

Consider *X* to be a random variable (a binomial random variable) with the following pmf $f(x) = p(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, for x=0,1,⋯,n.

The cdf of X evaluated at t, denoted $F_X(t)$, is

$F_X(t) = \sum_{x=0}^{t} \binom{n}{x} p^x (1-p)^{n-x}$,for 0 < t < n

When t=0, you have $F_X(0) = \binom{n}{0} p^0 (1-p)^{n-0}$

When t=1, you have $F_X(1) = \binom{n}{0} p^0 (1-p)^{n-0} + \binom{n}{1} p^1 (1-p)^{n-1}$.

When t=2, you ave $F_X(2) = \binom{n}{0} p^0 (1-p)^{n-0} + \binom{n}{1} p^1 (1-p)^{n-1} + \binom{n}{2} p^2 (1-p)^{n-2}$.

And so on and so forth.

Example 14 Suppose you have a family with three children. The sample space for this situation is

$$S = \{BBB, BBG, BGB, GBB, GGG, GGB, GBG, BGG\}$$

Solution Where B = boy and G = girl and suppose the probability of having a boy is the same as the probability of having a girl. Let the random variable X be the number of boys. Then X will have the following pmf:

Table 17

t	0	1	2	3
$P(X = t)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Then, you can use the pmf to find the cdf as follows:

Table 18

t	0	1	2	3
$F_X(t) = p(X < t)$	$\frac{1}{8}$	$\frac{1}{8}+\frac{3}{8}=\frac{4}{8}$	$\frac{4}{8}+\frac{3}{8}=\frac{7}{8}$	$\frac{7}{8}+\frac{1}{8}=1$

Some results on expectation

- E (c) = c , where c is a constant.
- E (cX) =c E(X) , where c is a constant .
- E (aX+b) =a E(X)+b , where a and b are constants.

Addition law of expectation: If X and Y are random variables then E(X+Y)=E(X)+E(Y) i.e. expected value of the sum of two random variables is equal to sum of their expected values.

Multiplication law of expectation: If X and Y are independent random variables then E(X.Y)=E(X).E(Y) i.e. expected value of the product of two random variables is equal to product of their expected values.

Variance in terms of expectations

$$Variance = \sigma^2 = E[X - E(X)]^2 = E[X - \mu^2] = E[X^2] - (E(X))^2$$

3.0 Unity Activities

(1) A random variable R can take the values indicated in Table 19 with the given probabilities.

(a) Calculate the expected value and variance of R.

(b) Calculate the expected value and variance of R^2.

Table 19

r	0	1	2	3	4
P(r)	0.4	0.3	0.1	0.1	0.1

(2) A maths teacher pays her child's pocket money in the following way: she rolls a die and gives the child 10n for each spot on the uppermost face of the die. What is the expected value of the child's pocket money?

(3) A man buys eight tickets from a total of 2000 tickets in a raffle where there is just one prize of K50. The price of a ticket is 10Ngwee. Given that all the tickets are sold, calculate his expected loss.

(4) A bag contains two red and eight black marbles. A sample of four marbles is to be drawn at random from the bag without replacement.

(a) Show that the probability of obtaining exactly two red marbles in the sample is 2/15.

(b) Show that the probability of obtaining exactly one red marble in the sample is 8/15.

(5) Calculate the expected number of red marbles that will be drawn.

Tomorrow I start three days' holiday and I wonder what weather is in store for me. I know that, if it is fine one day, the probability that it will be fine the next day is 4/5, and if it is wet one day, the probability it is we the next day is 3/5. Today it is fine. Draw up the probability distribution for the number of fine days in the next three days. What is the expected number of fine days for my holiday?

Table 20

		B Heads	B Tails
A	Heads	3	x
	Tails	4	6

(6) A and B play a game in which each tosses an unbiased coin. Table 20 shows the amount in pence that A receives from B for each possible outcome of the game. For example, if both players obtain heads, A receives 3 ngwee from B while if both obtain tails A pays B 6 ngwee. Express A's expectation of gain in one game in terms of x and the value of x which makes the game fair to both players. Both players are now given coins which are twice as likely to give heads as tails. How much would A expect to gain in 45 games if x is now 2?

(7) The probability of a man A winning any game against a man B in a match is ½. The first man to win two games in succession or a total of three games wins the match. Calculate the probability that the match takes exactly

(a) Two games, (b) three games, (c) four games, (d) five games. The winner is given K14 if he wins the match in two or three games and K18 if he wins the match in four or five games. If 100 spectators watch the match, how much should each be charged to cover the expected cost of prizes?

(8) In a marble game the challenger has two chances to hit her opponent's marble. If she hits it at the first attempt she wins two marbles. If she misses, she can make another attempt and if this is successful she wins one marble. If she misses at both attempts she pays her opponent two marbles. If p is the probability that the

challenger hits a marble at a single throw, and p is constant, find the number of marbles she can expect to win in ten turns. What value must p take for this expected value to be zero?

(9) A player throws a die whose faces are numbered 1 to 6 inclusive. If the player obtains a 6 he throws the die a second time, and in this case his score is the sum of 6 and the second number: otherwise his score is the number obtained. The player has no more than two throws.

Let X be the random variable denoting the player's score. Write down the probability distribution of X, and determine the mean of X.

10. An event has probability p of success and q(= 1 – p) of failure. Independent trials are carried out until at least one success and one failure have occurred. Find the probability that r trials are necessary (r>2) and show that this probability equals (½y – when p = ½.

A couple decided that they will continue to have children until either they have both a boy and a girls in the family or they have four children. Assuming that boys and girls are equally likely to be born, what will be the expected size of their completed family?

Unit Summary

We introduced the important concept of **random variables**, which are quantitative variables whose value is determined by the outcome of a random experiment.

A random variable is a variable whose values are numerical results of a random experiment. A **discrete random variable** is summarized by its probability distribution and a list of its possible values and their corresponding probabilities. The sum of the probabilities of all possible values must be 1. The probability distribution can be represented by a table, histogram, or sometimes a formula.

$E(X) = x\sum_{all\,x} xp(X = x)$ and

$$Variance = \sigma^2 = E[X - E(X)]^2 = E\left[X - \mu^2\right] = E[X^2] - (E(X))^2$$

The **probability distribution** of a random variable can be supplemented with numerical measures of the center and spread of the random variable.

UNIT 4 MOMENT GENERATING FUNCTIONS

4.1 Introduction

Welcome to this unit called moment generating function. In this unit you will learn a special techniques of calculating the mean and variances of various distributions using moment generating function (mgf). This is a tool which simplifies calculations of parameters which uses methods of differentiation.

4.2 Unit Aims:

Moment-generating functions are special functions which can sometimes make finding the mean and variance of a random variable simpler.

4.2 Objectives

At the end of this unit, you will be able to:

- Define a moment-generating function.
- Find the moment-generating function of any distribution of a random variable.
- Use a moment-generating function to find the mean and variance of a random variable.
- Use a moment-generating function to identify which probability mass function a random variable *X* follows.
- Apply the methods learned in the lesson to new problems.
- Show that the moments of the distribution can be obtained from the derivatives of the MGF at zero

Terminology

- M (t) – Moment generating function

- $\sum$ -summation
- mgf- moment generating function
- pdf – probability density function
- $M'(t)$ - First derivative of M with respect to t
- $M''(t)$ - Second derivative of M with respect

4.4 Time Requirements: You should spend 8 hours on this unit

4.5 Unit Topics:

4.5.1 Moment generating Functions for Discrete Random Variables

The examples you discussed so far shows how mean and standard deviation can be calculated for theoretical probability distributions using the appropriate summations for discrete variables and integration for continuous variables. This calculation can be simplified by using a mathematical device called **Moment generating function**.

Definition. Let *X* be a discrete random variable with probability mass function *f*(*x*) and support *S*. Then: $M(t) = \sum_{all\,x} p(x)e^{xt}$ is the **moment generating function of *X*** as long as the summation is finite for some interval of *t* around 0. That is, *M*(*t*) is the moment generating function ("**m.g.f.**") of *X* if there is a positive number *h* such that the above summation exists and is finite for $-h < t < h$.

For a discrete variable X, the moment generating function (M (t)) is defined by:

$$M(t) = \sum_{all\,x} p(x)e^{xt} \qquad (4.1)$$

Assuming that e^{xt} can be expanded as a series, then equation becomes

$$M(t) = \sum_{all\,x} p(x)\left(1 + xt + \frac{(xt)^2}{2!} + \frac{(xt)^3}{3!} + \ldots\right), \qquad (4.2)$$

on differentiating with respect to t

$$\frac{dM(t)}{dt} = \sum_{all\,x} p(x)\left(1 + xt + \frac{(xt)^2}{2!} + \ldots.\right),$$ and putting t = 0

$$\left(\frac{dM(t)}{dt}\right)_{t=0} = \sum_{all\,x} xp(x) \tag{4.3}$$

The right hand side of this equation is expected value (or mean) of X

Differentiating again, from the equation above

$$\frac{d^2M(t)}{dt^2} = \sum_{all\,x} p(x)(x^2 + x^3t + ... \tag{4.4}$$

Variance of X is given by:

$$\text{Variance (X)} = \sum_{all\,x} x^2 p(x) - \left[\sum_{all\,x} xp(x)\right]^2$$

$$= \left\{\frac{d^2M(t)}{dt^2}\right\}_{t=0} - \left[\left\{\frac{dM(t)}{dt}\right\}_{t=0}\right]^2 \tag{4.5}$$

The power of the moment generating function to calculate means and variances is best illustrated by examples.

4.5.2 Finding Moments

Proposition. If a moment-generating function exists for a random variable *X*, then:

(1) The mean of *X* can be found by evaluating the first derivative of the moment-generating function at $t = 0$. That is:

$$\mu = E(X) = M'(0)$$

(2) The variance of *X* can be found by evaluating the first and second derivatives of the moment-generating function at $t = 0$. That is:

$$\sigma^2 = E(X^2) - [E(X)]^2 = M''(0) - [M'(0)]^2 \tag{4.6}$$

Recall that $E(X)$, $E(X^2)$, ..., and $E(X^r)$ are called **moments about the origin**. It is for this reason, that the function $M(t)$ is called a moment-generating function. That is, $M(t)$ generates moments!. In fact, in general the ***r^{th}* moment about the origin** can be found by evaluating the r^{th} derivative of the moment-generating function at $t = 0$. That is:

$$M^r(0) = E(X^r) \tag{4.7}$$

Proof. We begin the proof by recalling that the moment-generating function is defined as follows:

$$M(t) = E(e^{tX}) = \sum_{x \in s} e^{tx} f(x) \tag{4.8}$$

And, by definition, $M(t)$ is finite on some interval of t around 0. That tells you two things:

1. Derivatives of all orders exist at $t = 0$.
2. It is okay to interchange differentiation and summation.

That said, you can now work on the gory details of the proof:

$$\frac{dM(t)}{dt}\Big|_{t=0} = M'(0) = \sum_{x \in S} xe^{tx} f(x)\Big|_{t=0} = \sum_{x \in S} xf(x) = E(X)$$

$$\frac{d^2M(t)}{dt^2}\Big|_{t=0} = M''(0) = \sum_{t=0} x^2 e^{tx} f(x)\Big|_{t=0} = \sum_{x} x^2 f(x) = E(X^2)$$

Therefore, $\mu = M'(0)$ and $\sigma^2 = M''(0) - [M'(0)]^2$

Example 1 Use the moment-generating function for a binomial random variable ***X***:

$$M(t) = [(1-p) + pe^t]^n$$

to find the mean μ and variance σ^2 of a binomial random variable.

Solution. Keeping in mind that we need to take the first derivative of $M(t)$ with respect to t, we get:

$$M'(t) = n[1 - p + pe^t]^{n-1}(pe^t)$$

And, setting $t = 0$, we get the binomial mean $\mu = np$:

$$\begin{aligned}\mu = M'(0) &= n[1 - p + pe^0]^{n-1} pe^0 \\ &= n[1 - p + p]^{n-1} p \\ &= np\end{aligned}$$

To find the variance, we first need to take the second derivative of $M(t)$ with respect to

t. Doing so, we get:

$$M''(t) = n\left[1 - p + pe^{t}\right](pe^{t}) + (pe^{t})n(n-1)\left[1 - p + pe^{t}\right]^{n-2}(pe^{t})$$

And, setting $t = 0$, and using the formula for the variance, we get the binomial variance $\sigma^2 = np(1 - p)$:

$$\begin{aligned} M''(0) &= n\left[1 - p + pe^{0}\right]^{n-1}(pe^{0}) + (pe^{0})n(n-1)\left[1 - p + pe^{0}\right]^{n-2}(pe^{0}) \\ &= n(1)^{n-1}p + pn(n-1)(1)^{n-2}p(1) \\ &= np + n^2p^2 - np^2 \end{aligned}$$

$$\begin{aligned} \sigma^2 &= M''(0) = -\left[M'(0)\right]^2 = np + n^2p^2 - np^2 - (np)^2 \\ &= np + n^2p^2 - np^2 - n^2p^2 = np - np^2 \\ &= np(1 - p) = npq \end{aligned}$$

Not only can a moment-generating function be used to find moments of a random variable, it can also be used to identify which probability mass function a random variable follows.

4.5.3 Finding distribution

Proposition. A moment-generating function uniquely determines the probability distribution of a random variable.

Proof. If the support S is $\{b_1, b_2, b_3 \ldots\}$, then the moment-generating function:

$M(t) = E(e^{tX}) = \sum_{x \in s} e^{tx} f(x)$ is given by:

$M(t) = e^{tb} \; f(b_1) + e^{tb_2} f(b_2) + \ldots..$

Therefore, the coefficient of:

e^{tb_i}, is the probability: $f(b\ ) = p(X = b_i)$

This implies necessarily that if two random variables have the same moment-generating function, then they must have the same probability distribution.

Example 2 If a random variable X has the following moment-generating function:

$$M(t) = \left(\frac{3}{4} + \frac{1}{4}e^{t}\right)^{20}$$

for all t, then what is the probability mass function of X?

Solution. We previously determined that the moment generating function of a binomial random variable is: $M(t) = \left[(1-p) + pe^t\right]^n$

or $-\infty < t < \infty$. Comparing the given moment generating function with that of a binomial random variable, you can see that X must be a binomial random variable with $n = 20$ and $p = \frac{1}{4}$. Therefore, the p.m.f. of X is:

$$f(x) = \binom{20}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{20-x}, \text{ for } x = 0, 1 \ldots 20.$$

Example 3 If a random variable X has the following moment-generating function:

$$M(t) = \frac{1}{10}e^t + \frac{2}{10.}e^{2t} + \frac{3}{10}e^{3t} + \frac{4}{10}e^{4t}$$

for all t, then what is the p.m.f. of X?

$$M(t) = \frac{1}{10}e^t + \frac{2}{10.}e^{2t} + \frac{3}{10}e^{3t} + \frac{4}{10}e^{4t}$$

$$f(x) = \begin{cases} \frac{1}{10,}, & for\ x = 1 \\ \frac{2}{10}, & for\ x = 2 \\ \frac{3}{10}, & for\ x = 3 \\ \frac{4}{10} & for\ x = 4 \end{cases}$$

4.5.4 Calculation of the mean and variance of the Binomial distribution

The binomial probability distribution is defined by:

$$p(x) = \binom{n}{x}p^x(1-p)^{n-x}, \text{ for } x = 1,2,3,\ldots,n$$

This gives

$$M(t) = \sum \binom{n}{x}(pe^t)^x(1-p)^{n-x}$$

Which is also a binomial expansion, giving

$$M(t) = \left[pe^t + (1-p)\right]^n$$

Differentiating with respect to t

$$\frac{dM(t)}{dt} = pe^t n\left[pe^t + (1-p)\right]^{n-1}$$

And

$$\left\{\frac{dM(t)}{dt}\right\}_{t=0} = np$$

Differentiating again with respect to t

$$\left\{\frac{d^2M(t)}{dt^2}\right\} = pe^t n(n-1)pe^t\left[pe^t + (1-p\right]^{n-2} + pe^t n\left[pe^t + (1-p)\right]^{n-1}$$

Thus

$$\left\{\frac{d^2M(t)}{dt}\right\}_{t=0} = p^2n(n-1) + np$$

You know that,

Mean = np

$$\text{Variance} = \left\{\frac{d^2M(t)}{dt}\right\}_{t=0} - \left(\left\{\frac{dM}{dt}\right\}_{t-0}\right)^2$$

$$= p^2n(n-1) + np - (np)^2$$

$$\begin{aligned} &= n^2p^2 - p^2n + np - n^2p^2 \\ &= np(1-p) \\ &= npq \end{aligned}$$

4.5.5 Mean and variance of a poison distribution

The poison distribution is defined as:

$$p(x) = \frac{\lambda^x e^{-x}}{x!}$$, for x = 1, 2, 3…

This gives $M(t) = \sum_{all\,x}\left(\frac{e^{xt}\lambda^x e^{-\lambda}}{x!}\right)$, taking $e^{-\lambda}$ outside since it is a constant.

$$M(t) = e^{-\lambda}\sum \frac{(\lambda e^t)^x}{x!}$$

By rearranging by putting $e^{\lambda e^t}$ outside the summation and cancelling it by putting $e^{-\lambda e^t}$ inside

$$M(t) = e^{-\lambda} e^{\lambda e^t} \sum_{x=0}^{\infty} \frac{e^{-\lambda e^t}(\lambda e^t)^x}{x!}$$

The effect of this is to make the terms in the expansion those of a poison distribution, mean λe^t . Their sum is 1, giving

$$M(t) = e^{-\lambda} e^{\lambda e^t}$$

$$\frac{dM(t)}{dt} = \lambda e^t e^{-\lambda} e^{\lambda e^t}$$

$$\frac{d^2M(t)}{dt^2} = \lambda e^t e^{-\lambda} e^{\lambda e^t} + \lambda e^t e^{-\lambda} \lambda e^t e^{\lambda e^t}$$

Then

$$\left\{\frac{dM(t)}{dt}\right\}_{t=0} = E(R) = \lambda$$

Then

$$\left\{\frac{d^2M(t)}{dt^2}\right\}_{t=0} = E(R^2) = \lambda + \lambda^2$$

Thus

Mean = $E(R) = \lambda$

Variance = $E(R^2) - [E(R)]^2$

$$= (\lambda + \lambda^2) - \lambda^2$$
$$= \lambda$$

4.5.6 Moment generating function for continuous random variables

The moment generating function of a continuous random variable *X*, if it exists, is:

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \qquad (4.9)$$

for $-h < t < h$.

As before, differentiating the moment generating function provides you with a way of finding the mean:

$$E(X) = M'(0) \qquad (4.10)$$

and the variance:

$$Var(X) = M''(0) - [M'(0)]^2 \qquad (4.11)$$

Example 4 Let *X* be a continuous random variable whose probability density function is: $f(x) = 3x^2$, for $0 < x < 1$. First, note again that $f(x) \neq P(X = x)$. For example, $f(0.9) = 3(0.9)^2 = 2.43$, which is clearly not a probability! In the continuous case, $f(x)$ is instead the height of the curve at $X = x$, so that the total area under the curve is 1. In the continuous case, it is areas under the curve that define the probabilities.

Now, let's first start by verifying that $f(x)$ is a valid probability density function.

What is the probability that *X* falls between ½ and 1? That is, what is $P(\frac{1}{2} < x < 1)$?

What is $P(X = \frac{1}{2})$?

Solution. It is a straightforward integration to see that the probability is 0:

$$\int 3x^2 dx = x^3 \Big|_{1/2}^{1/2} = 0$$

In fact, in general, if *X* is continuous, the probability that *X* takes on any specific value x is 0. That is, when X is continuous, $P(X = x) = 0$ for all x in the support.

An implication of the fact that $P(X = x) = 0$ for all x when X is continuous is that you can be careless about the endpoints of intervals when finding probabilities of continuous random variables. That is:

$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$, for any constants a and b.

Note that you can calculate the expected value and variance of any continuous random variable using the moment generating function very easily.

Example 5 Let X be a continuous random variable whose probability density function is: $f(x) = \dfrac{x^3}{4}$, for an interval $0 < x < c$. What is the value of the constant c that makes $f(x)$ a valid probability density function?

Solution; since the random variable is a pdf, then $\int_{-\infty}^{\infty} f(x)dx = 1$

So $\int_0^c \frac{x^3}{4} dx = 1$

Example 6 Find the moment generating function of the Uniform distribution

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

Find the mean and variance of the distribution.

$$M(t) = \int f(x)e^{xt} dx$$

$$= \int_0^1 1.e^{xt} dx$$

$$= \left[\frac{e^{xt}}{t} \right]_0^1$$

$$= \frac{1}{t}(e^t - 1)$$

Expanding e^t as a series and avoid 0 in the denominator, this gives

$$M(t) = \frac{1}{t}(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots - 1)$$

$$= 1 + \frac{t}{2} + \frac{t^2}{6} + \ldots$$

On differentiating, you get

$$\frac{dM(t)}{dt} = \frac{1}{2} + \frac{t}{3} + \frac{t^2}{8} + \ldots.$$

$$\frac{d^2M(t)}{dt^2} = \frac{t}{3} + \frac{t}{4} + \ldots.$$

Where t = 0

Mean = $\left\{\frac{dM(t)}{dt}\right\}_{t=0} = \frac{1}{2}$

$$\left\{\frac{d^2M(t)}{dt^2}\right\}_{t=0} = \frac{1}{3}$$

Variance = $\left\{\frac{d^2M(t)}{dt^2}\right\}_{t=0} - [\left\{\frac{dM(t)}{dt}\right\}_{t=0}]^2$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

4.0 Unit Activity

1. A random variable X has the probability density function f(x) given by:

$$f(x) = \begin{cases} ce^{-2x} & 0 < x < \infty \\ 0 & elsewhere \end{cases}$$

Find the moment generating function of X and hence, or otherwise, show that the mean is $\frac{1}{2}$ and variance $\frac{1}{4}$. Show also that the median of the distribution is $\frac{1}{2}\ln 2$ and the interquartile range is $\frac{1}{2}\ln 3$.

2. A discrete random variable is such that $p(X = r) = p$, where r = 0, 1, 2, ….. The probability generating function p(t) is defined by $p(t) = \sum_{r=0}^{\infty} p_r t^r$. Show that the moment generating function M(t) of x about the origin is given by $M(t) = \dfrac{e^{nt} - 1}{e^t - 1}$

3. Show that the moment generating function for the random variable X which is Geometrically distributed with parameter p is $M(t) = \dfrac{p}{e^{-t} + p - 1}$ and hence that $E(X) = \dfrac{1}{p}$ and $Var(X) = \dfrac{a - p}{p^2}$

4. Given that $y = \ln x$ and X is a random variable Uniformly distributed in the interval 1 to 10, find the mean value of Y.

5. The random variable X has the density function

$$f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

   If the variable Y is the area of a circle, radius X, find the mean value of Y and its density function.

6. The total surface area, A, of a right circular cone is given by the formula

   $A = \pi r^2 + \pi r l$

   The saint height l is a constant 2a and the base radius r is Uniformly distributed in the interval (0,a). Given that A = n Y,

   (a)Find the mean of Y and hence the mean of A,

   (b) Show that the probability density function f(y) of Y is

$$\frac{1}{2a\sqrt{a^2 + y}}$$

7. The operational lifetime in hundreds of hours of a battery-operated calculator may be regarded as a continuous random variable having probability density function

$$f(x) = \begin{cases} cx(10 - x) & 5 \le x \le 10 \\ 0 & otherwise \end{cases}$$

   (a)Find the value of c and of the expected operational lifetime of such a calculator.

(b) The purchase price of such a calculator is K20 and its running cost (for batteries) amounts to 20ngwee per hundred hours operation. Thus, the overall average cost in ngwee per hundred hours operation of a calculator whose operational lifetime is X hundred hours is given by Y = 20 + (2000/X).

(i) Evaluate E(Y), the expected overall average cost per hundred hours.

(ii) Find the probability that the overall average cost per hundred hours will exceed K2.70.

(8) The maximum length to which a string of natural length a meters can be stretched before it snaps is a(1 + X2) meters, where X is a continuous random variable whose probability density function is

$$f(x) = \begin{cases} 4x & 0.25 \le x \le 0.75 \\ 0 & otherwise \end{cases}$$

(a) Calculate the probability that a string can be stretched to 11/2 times its natural length without snapping.

(b) Find the value of E(X2) and hence find u, the mean maximum stretched length of strings of natural length 1 meter.

(c) Find the probability density functions of Y = 1 + X2, the maximum stretched length of a string of natural length 1 meter, and use it to verify the value of u you obtained in (b).

9. Suppose that Y has the following mgf. $M_Y(t) = \dfrac{e^t}{4-3et}, t < -\ln(0.75)$

(a) Find E(Y) (b) $E(Y^2)$

10. Find the MGF for e^{-x}.

11. Find $E(X^3)$ using the MGF $(1\text{-}2t)^{-10}$.

Unit Summary

Moment generating functions (mgfs) are function of t. You can find the mgfs by using the definition of expectation of function of a random variable. The moment generating function of X

Is $M_X(t) = E(e^{tX}) = E[\exp(tX)]$

Note that exp(X) is another way of writing exp.X. You can see that the moment-generating function uniquely determines the distribution of a random variable. In other words, if the mgf exists, there is one and only one distribution associated with that mgf.". This property of the mgf is sometimes referred to as the *uniqueness property of the mgf*.

Suppose we have the following mgf for a random variable Y

$$M_Y(t) = \frac{e^t}{4 - 3et}, t < -\ln(0.75)$$

Using the information from this unit, you can find the E(Yk) for any k if the expectation exists. Let's find E(Y) and $E(Y^2)$. You can solve these in a couple of ways. You can use the knowledge that $M'(0) = E(Y)$ and. Then you can find variance by using $Var(Y) = E(Y^2) - [E(Y)]^2$.

UNIT 5 THE BINOMIAL DISTRIBUTION

5.1 Introduction

Welcome to this unit called Binomial distribution. In the previous units you discussed the probability of some experiments in which it was possible to find the sample space and the events. If you conduct an experiment in which there are two possible outcomes either “success” or “failure “, but not both. If you define X as a random variable that assigns the number of successes, then this is a Binomial distribution. If you define p as the probability of success and q as a probability of failure, then it is easy to calculate the probability of success.

5.2 Unit Aims:

To give you techniques of how to calculate probabilities of a discrete random variable called the binomial **distribution.** To also give you applications real life problems using binomial distributions

5.3 Unit Objectives:

- Compute probabilities for a Binomial distribution.
- Identify the key properties of the Binomial distribution.
- Compute probabilities for a Binomial distribution.
- Identify differences between the discrete distributions.
- What is a success? What is a failure? Successful Trials: (p) getting the wanted outcome Failure Trials: (1-p=q) getting the unwanted outcome.

Terminology

The following notation is helpful, when we talk about binomial probability.

- *x*: The number of successes that result from the binomial experiment.
- *n*: The number of trials in the binomial experiment.
- *P*: The probability of success on an individual trial.
- *Q*: The probability of failure on an individual trial. (This is equal to 1 - *P*.)
- *n!*: The factorial of n (also known as n factorial).
- b(*x*; *n*, *P*): Binomial probability - the probability that an *n*-trial binomial experiment results in exactly *x* successes, when the probability of success on an individual trial is *P*.

- C_r^n -The number of combinations of *n* things, taken *r* at a time.

5.4 Time Required: You should spend 10 hours on this unit

5.5 Unit Topics:

5.5.1 Binomial Distributions:

In general, if a single trial can have only two possible mutually exclusive and exhaustive results, either 'success' with probability p or 'failure' with probability $q = 1 - p$, then in a series of n trials, the probability of x success is given by

$$p(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad (5.1)$$

Where n is the number of trials, p = the probability of success, q = 1-p the probability of failure and X is the random variable of the number of successes.

Let X be the number of heads that appear when a coin is tossed 20 times and P(X = x) the probability that a head appears x times. Then to find the probability distribution of X we need to calculate the probability of the random variable $X = \{0,1,2,...,20\}$ as. $p(0)$, $p(1),...., p(20)$. The calculation of P(X =15) is more complicated.

Example 1 Toss a fair con six times. Let X be a random variable of the number heads that appears. Calculate the probability distribution of X.

Solution:

The number of heads obtained when six coins are tossed. Here the probability of obtaining a head at a single toss is ½ , and the probability of not obtaining a head is ½ ; we wish to find the probabilities of no, one, two, three, four, five and six heads in a series of six trials. The probability distribution is given in Table 1 Probability distribution for the number of heads obtained when six coins are tossed

Table 1 Probability distribution of X

x		P(X = x)
0	$\binom{6}{0}(\frac{1}{2})^0(\frac{1}{2})^6$	= 1/64 0.0156
1	$\binom{6}{1}(\frac{1}{2})^1(\frac{1}{2})^5$	= 6/64 = 0.0938
2	$\binom{6}{2}(\frac{1}{2})^2(\frac{1}{2})^4$	= 15/64 = 0.2344
3	$\binom{6}{3}(\frac{1}{2})^3(\frac{1}{2})^3$	= 20/64 = 0.3125
4	$\binom{6}{4}(\frac{1}{2})^4(\frac{1}{2})^2$	= 15/64 =0.2344
5	$\binom{6}{5}(\frac{1}{2})^5(\frac{1}{2})^1$	= 6/64 = 0.0938
6	$\binom{6}{6}(\frac{1}{2})^6(\frac{1}{2})^0$	= 1/64 =0.0156

This is called The Binomial distribution of the random variable X of the number of heads that appears, it is applied in many situations and gives the probability distribution of the discrete random variable X where X is the number of successes in n trials when the following criteria apply:

(i) A single trial has only two possible mutually exclusive and exhaustive result, 'success' and 'failure' with probabilities p and 1 - p(=q);

(ii) the value of p and q remain constant throughout the trials;

(iii) the result of each trial is independent of previous trials

(iv) the number of trials, n, is constant.

n and p are the parameters of the distribution, and a convenient short-hand for expressing the fact that X is Binomially distribution with these parameters is to write

X – B(n,p). For example, X – B(5,0.5) means that X is a random variable which is Binomially distributed with n = 5 and P = 0.5.

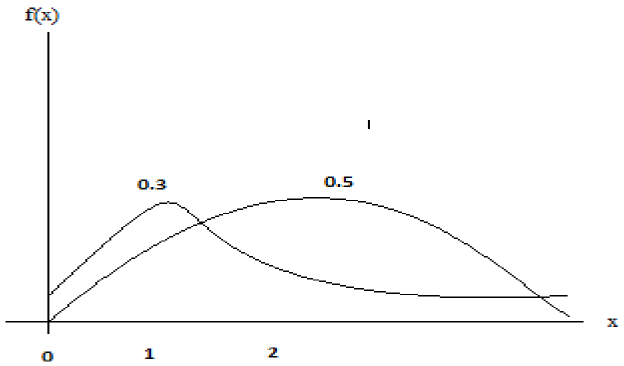


Figure 1 The Binomial distribution for various values of p when n = 8

Figure 1 shows frequency polygons for the Binomial distribution for different values of p when n = 8. Note that the distribution is symmetrical when p = ½, otherwise it is skewed. Figure 1 shows the frequency polygon for n = 50 and p = 0.1. When n is large, as in this case, the distribution is approximately symmetrical even though p = ½. It is important to realize that where three balls were selected from a bag of twenty (three white and seventeen red) cannot be treated by the Binomial distribution because, When one ball has been drawn, the probability of drawing a white (or red) ball has changed. The Binomial distribution can be applied to 'ball in bag' situation provided that either the sample size is small compared with the population from which the sample is drawn so that the probabilities effectively remain constant, or the balls are taken singly and each one is returned to the bag after its colour has been noted.

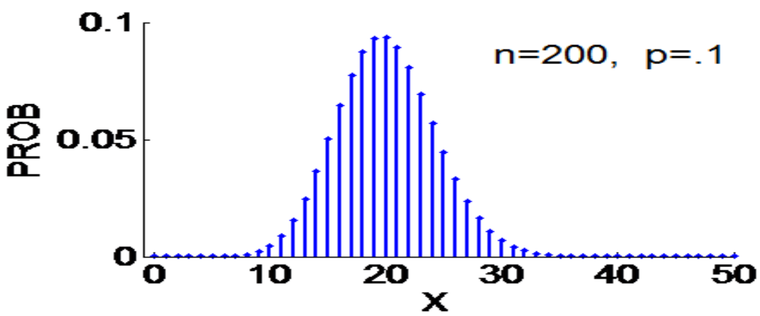


Figure 2 The Binomial distribution for p = 0.1, n = 50

A convenient method of calculating the probabilities for a Binomial distribution is by means of a recurrence formula. This gives P(X = x + 1) in terms of P(X = x). Since

$$p(X = x+1) = \frac{n!}{(x+1)!(n-[x+1])!} p_{x+1}(1-p)^{n-(x+1)} \qquad (5.2)$$

and

$$p(X=x)=\frac{n!}{x!(n-x)}p^{x}(1-p)^{n-x} \qquad (5.3)$$

we have, dividing the first expression by the second,

$$\frac{p(=x+1)}{p(X=x)}=\frac{n!}{(x+1)!(n-x-1)!}.\frac{x!(n-x)!}{n!}.\frac{p^{x+1}}{p^{x}}.\frac{(1-p)^{n-(x+1)}}{(1-p)^{n-x}}$$

$$=\frac{n-x}{x+1}.\frac{p}{(1-p)}$$

and rearranging

$$p(X=x+1)=\frac{(n-x)}{(x+1)}.\frac{p}{(1-p)}.p(X=x) \qquad (5.4)$$

Example 2 Use the moment-generating function for a binomial random variable X:

$$M(t)=\left[(1-p)+pe^{t}\right]^{n}$$

to find the mean μ and variance σ^2 of a binomial random variable.

Solution. Keeping in mind that we need to take the first derivative of $M(t)$ with respect to t, we get:

$$M'(t)=n\left[1-p+pe^{t}\right]^{n-1}(pe^{t})$$

And, setting $t = 0$, we get the binomial mean $\mu = np$:

$$\begin{aligned}\mu=M'(0)&=n\left[1-p+pe^{0}\right]^{n-1}pe^{0}\\&=n[1-p+p]^{n-1}p\\&=np\end{aligned}$$

To find the variance, we first need to take the second derivative of $M(t)$ with respect to t. Doing so, we get:

$$M''(t)=n\left[1-p+pe^{t}\right](pe^{t})+(pe^{t})n(n-1)\left[1-p+pe^{t}\right]^{n-2}(pe^{t})$$

And, setting $t = 0$, and using the formula for the variance, we get the binomial variance

$\sigma^2 = np(1-p)$:

$$M''(0) = n\left[1 - p + pe^0\right]^{n-1}(pe^0) + (pe^0)n(n-1)\left[1 - p + pe^0\right]^{n-2}(pe^0)$$
$$= n(1)^{n-1}p + pn(n-1)(1)^{n-2}p(1)$$
$$= np + n^2p^2 - np^2$$

$$\sigma^2 = M''(0) = -\left[M'(0)\right]^2 = np + n^2p^2 - np^2 - (np)^2$$
$$= np + n^2p^2 - np^2 - n^2p^2 = np - np^2$$
$$= np(1-p) = npq$$

5.0 Unit Activity

(1) (a) Discuss whether the following variables will have a Binomial distribution and, if appropriate, give values for the parameters p and n

(i) The number of heads when a fair coin is tossed five times. .

(ii) The number of tosses of a coin needed to get a head.

(iii)The number of vowels in five-letter words selected at random from a dictionary.

(iv)The number of odd digits in three-digit random numbers generated by a computer.

(v) The number of hearts in a five-card hand dealt from a well-shuffled pack of cards.

(b) Find the probability distribution for the following.

(i) The number of heads when a fair coin is tossed five times.

(ii) The number of 6s when four dice are rolled.

(2) A regular tetrahedron has one white face and red faces. The tetrahedron is allowed to fall on a table in such a way that any face has the same chance of resting on the table. It is thrown four times in succession and the number of times a white face rests on the table is noted.

(a) Find the probability distribution for X where X is the number of times a white face is not contact with the table.

(b) State which value of X is most likely to occur.

(c) Calculate the expected value of X.

(d) Calculate the variance of X.

(3) Six children are born in a maternity ward on one day. If the probability of

having a girl is 0.48 find the probability distribution for the number of girls. Calculate

(a) the most likely number of girls,

(b) the probability that the number of boys is the same as the number of girls,

(c)The probability that there are more boys than girls.

(4) Calculate the probability that in a group of seven people

(a) none has his or her birthday on a Saturday,

(b) two or more have their birthdays on Saturday.

(5) If there is a probability of 0.2 of failure to get through in any attempt to make a telephone call, calculate the most probable number of failures in ten attempts, and also the probability of three or more failures in ten attempts.

(6) A hundred years ago the occupational disease in an industry was such that the work-men had a 20% chance of suffering from it.

(a) If six workmen were selected at random what is the probability that two or less of them contracted the disease?

(b) How many workmen could have been selected at random before the probability that at least one of them contracted the disease became greater than 0.9?

(7) (a)A fair die is cast; then n fair coins are tossed, where n is the number shown on the die. What is the probability of exactly two heads?

(b) A fair die is thrown for as long as necessary for a 6 to turn up. Given that 6 does not turn up at the first throw, what is the probability that more than four throws will be necessary?

(8) The result of each turn at a fruit machine is independent of the results of previous turns and the probability of winning the jackpot at any turn is 0.02.

(a) Find the probability that in a sequence of ten turns there will be

(i) one win of the jackpot,

(ii) more than one win of the jackpot.

(b) Write down the probability that in a sequence of n turns at the machine the jackpot will not be won. Hence find how many turns are necessary for there to be a probability of at least 0.99 of winning the jackpot at least once.

5.5.2 An example from industry

All manufacturing processes inevitably produce some defective items. It is usually too costly to check each item and sometimes impossible since a check would destroy the product (e.g. fireworks). Items are usually produced in batches, and a simply form of check is to test a few items from each batch and reject or accept the whole batch on the evidence of this small sample. This is known as a **single sampling scheme.** The sampling scheme can be improved by taking a second sample from a batch if the first sample showed defective item(s). This is known as a two-stage sampling scheme.

Example 3 A manufacturer produces light bulbs which are tested in the following way. A batch is accepted in either of the following cases:

(i) a first sample of five shows no faulty bulbs;

(ii) a first sample of five shows one or more faulty bulbs but a second sample of five shows no faulty bulbs.

What is the probability that a batch is accepted if

(a) 2%, (b) 10% of the bulbs in it are faulty?

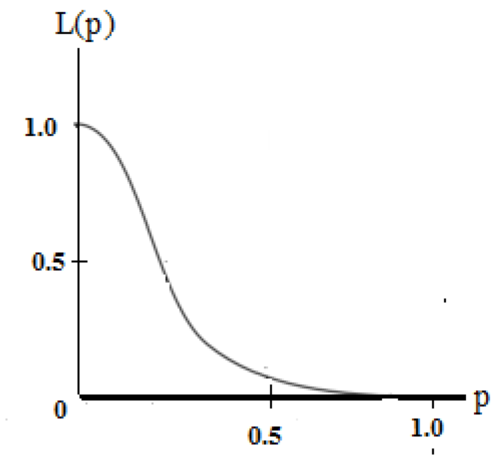


Figure 3 Operating characteristic curve for example 3

We assume that the sample size is sufficiently small compared with the batch size to use the Binomial distribution.

(a) With p = 0.02, q = 0.98, n = 5, we have for (i) $P(X = 0) = (0.98)^5 = 0.904$

This is the probability that the batch is accepted at the first test. The probability that it is not accepted at the first test is 1 – 0.904 = 0.096. The probability that there are no defectives in the second sample is $(0.98)^5 = 0.904$ so that the probability a batch is accepted in case (ii) is 0.096 x 0.904 = 0.087.

The total probability that the batch is accepted for p = 0.02 is 0.904 + 0.087 = 0.991.

(b) Repeating the calculation with p = 0.1, q = 0.9, the probability that a batch is accepted is $(0.9)^5 + (1-0.9^5)\,0.9^5 = 0.832$.

To analyses a sampling plan in more detail an **operating characteristic (OC) curve** can be plotted. This gives the probability, L(p), of a batch being accepted for different values of p, the proportion of defectives. If all the items are defective, i.e. p = 1, then the batch is bound to be rejected and L(1)= 0. If all the items are perfect, i.e. p = 0, the batch is bound to be accepted and L(0) = 1. Figure 3 shows the OC curve for the sampling scheme described above for which the reader may verify that:

$$L(p) = 2(1-p)^5 - (1-p)^{10}$$

In practice the manufacturer and customer have to obtain a balance between two conflicting aims: the manufacturer's desire that he will not reject too many perfect bulbs, and the customer's wish that he is not supplied with too many faulty bulbs.

5.5.3 Mean and variance of a Binomial distribution

Five dice are tossed together *N* times (where *N* is a large number) and the number of 3s counted at each throw. What would we expect the mean number of 3s at each throw to be? We can consider the experiment in a different light as 5*N* single throws of a die. Since the probability of getting a 3 when a die is thrown is ⅙, we would expect to get 5*N* x ⅙3s altogether. Since these are shared over *N* throws of five dice, the average number of 3s at each throw of five will be

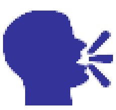
$$\frac{5N * \frac{1}{6}}{N} = 5 * \frac{1}{6}$$

i.e. the value of N is immaterial. This argument can be generalized to give the mean of a Binomial distribution as *np*:

$$\mu = np \qquad (5.5)$$

where p is the probability of success at a single trial and n is the number of trials.
It is also possible to show mathematically that the variance of a Binomial distribution is *npq*.

$$\text{Variance} = \sigma^2 = npq = np(1-p) \qquad (5.6)$$



Example 4

Calculate the probability distribution for X, the number of 6s obtained when three dice are thrown. Calculate the mean and variance and check that the values obtained agree with the formulae above.

Table 1 Calculation for Example 4

x	P(X = x)	xP(X = x)	x^2P(X = x)
0	$\binom{3}{0}(\frac{5}{6})^3 = \frac{125}{216}$	0	0
1	$\binom{3}{1}(\frac{5}{6})^2 \frac{1}{6} = \frac{75}{216}$	$\frac{75}{216}$	$\frac{75}{216}$
2	$\binom{3}{2}(\frac{5}{6})\ (\frac{1}{6})^2 = \frac{15}{216}$	$\frac{30}{216}$	$\frac{60}{216}$
3	$\binom{3}{3}(\frac{1}{6})^3 = \frac{1}{216}$	$\frac{3}{216}$	$\frac{9}{216}$

$$\mu = E(X) = \sum_{all\,x} xp(X = x) = \frac{108}{216} = \frac{1}{2}$$

$$E(X^2) = \sum_{all\,x} x^2 p(X = x) \frac{144}{216} = \frac{2}{3}$$

Variance = $E(X^2) - [E(X)]^2$

$$= \frac{2}{3} - [\frac{1}{2}]^2 = \frac{5}{12}$$

Using the formula: from equation (5.1) mean = $\mu = np = \frac{1}{2}$,

from equation (5.2) variance = $\sigma^2 = npq = 3x\frac{1}{6}x\frac{5}{6} = \frac{5}{12}$

5.5.4 Derivation of the mean and variance of a Binomial distribution

You know that the mean, μ, is

$$E(X) = \sum_{all\,x} xp(X = x)$$

$$= \sum x.\frac{n!}{(n-x)!\,x!} p^x q^{n-x}$$

Writing out the series term by term

$$\mu = 0xq^n + 1xnpq^{n-1} + 2\frac{n(n-1)}{2x1} p^2 q^{n-2} + 3x\frac{n(n-1)(n-2)}{3x2x1} p^3 q^{n-3} + ... + nxp^n$$

$$= npq^{n-1} + n(n-1)p^2q^{n-2} + \frac{n(n-1)(n-2)}{2x1}p^3q^{n-3} + ... + nxp^n$$

Taking out the factor np gives

$$\mu = np\left[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)}{2x1}p^2q^{n-3} + ... + p^{n-1}\right]$$

The square bracket contains the binomial expansion of $(p + q)^{n-1}$ and, since p + q = 1, this is also 1. Hence $\mu = np$ and the variance is variance = $E(X^2) - [E(X)]^{2.}$

In this case you have already found $E(X)$ so it remains to find $E(X^2)$. This is most easily done using the identity $E[X(X - 1)] = E(X^2) - E(X)$ which can be proved as follow:

$$E[X(X-1)] = \sum_{all\,x} x(x-1)p(X+x)$$

$$= \sum_{all\,x}(x^2 - x)p(X = x)$$

$$= \sum_{all\,x} x^2 p(X = x) - \sum_{all\,x} xp(X = x)$$

$$= E(X_2) - E(X)$$

For the Binomial distribution

$$E[X(X-1)] = \sum_{all\,x} x(x-1)\frac{n!}{(n-x)!\,x!}p^x(1-p)^{n-x}$$

Writing out the series term by term, omitting the first two terms since they are both zero,

$$E[X(X-1)] = 2x1x\frac{n(n-1)}{2x1}p^2q^{n-2} + 3x2x\frac{n(n-1)(n-2)}{3x2x1}p^3q^{n-3} + 4x3\frac{n(n-1(n-2)(n-3)}{4x3x2x1}p^4q^{n-4} + ...+ n(n-1)p^n$$

Taking out the factor $n(n-1)p^2$ gives

$$E[X(X-1)] = n(n-1)p^2\left[q^{n-2} + (n-2)pq^{n-3} + \frac{(n-2)(n-3)}{2x1}p^2q^{n-4} + ... + q^{n-4}\right]$$

Again the sum of the terms in the square bracket is 1, this time because they are the expansion of $(q + p)^{n-2}$. This gives $E[X(X-1)] = E(X^2) - E(X) = n(n-1)p_2$

Rearranging $E(X^2) = n(n-1)p^2 + E(X)$

Since we have shown E(X) = np, we have

$E(X^2) = n(n-1)p^2 + np$ and variance = $E(X^2) - [E(X)]^2$

$$= n(n-1)p^2 np - (np)^2$$
$$= np - np^2$$
$$= np(1-p)$$
$$= npq$$

5.3 Unit Activities

(1) The probability that a person chosen at random is left-handed is 0.1. What is the probability that in a group of ten people there is one, and only one, who is left-handed? What is the most likely number of left-handed people in a group of ten?

(2) (a) In a certain manufacturing process, it is known that approximately 10% of the items produced are defective. A quality control scheme is set up, by selecting twenty items out of a large batch, and rejecting the whole batch if three or more are defective. Find the probability that the batch is rejected.

(b) Two boys, John and David, play a game with a die. The die will be thrown four times.

David will give John Kx if there is an odd number of 6n; otherwise John will give David K1. 0.32

If the game is to be a fair one to both John and David, find the value of x.

(3) The probability of a success in a single trial is p. Show that the probability of r success in n independent trials is $\dfrac{n!\,p^r(1-p)^{n-r}}{r!(n-r)!}$

One-third of the inhabitants of Mongu have blood group P. find the probability (as a fraction) that at least two people will have blood group P in a random sample of four Mongu's. Would your answer be correct for a randomly chosen Mongu family of four people? State briefly the reasons for your answer.

(4) Items produced by a certain industrial process are checked by examining samples of ten. It is done 30 times. The data are given in Table 2

Estimate what proportion of defective items there is in the complete consignment, on the basis of the Binomial distribution law.

Table 2

Number defective in sample	0	1	2	3	4	5	6 or more
Number of times in 30 samples	13	11	4	1	1	0	0

(5) In a packet of flower seeds one-third are known to be pink flowering and the remainder are yellow flowering. The seeds are well-mixed and sown in 162 rows with 4 seeds in each row. Assuming that all the seed germinates,

(a) Calculate the expected mean and standard deviation of the number of pink flowering plants per row;

(b) Copy and complete Table 3

What is the most likely number of pink flowering plants in a row?

Table 3

Number of pink flowering plants per row	0	1	2	3	4
Expected number of rows					

(6) A shopkeeper found that 5% of the eggs received from a central distributing agency were stale on delivery, and reduced his prices by 5%. A housewife requiring ten fresh eggs was advised to purchase a dozen, the shopkeeper claiming that it was more likely than not that all the eggs in dozen would be fresh, and furthermore that there was only a one in ten chance of two eggs in the dozen being stale.

Are the shopkeeper's claims valid?

What is the probability that, of the dozen eggs purchased,

(a) one egg is stale,

(b) not more than two eggs are stale?

If each customer accepted the shopkeeper's claims, and increased his or her egg order in the same ratio as the housewife, determine the net percentage change in the shopkeeper's daily receipts from egg sales.

(7) In an experiment a certain number of dice are thrown and the number of 6s obtained is recorded. The dice are all biased and the probability of obtaining a 6 with each individual die is *p*. In all there were 60 experiments and the results are shown in Table 4.

Calculate the mean and the standard deviation of these data.

By comparing these answers with those expected for a Binomial distribution, estimate

(a) the number of dice thrown in each experiment,

(b) the value of *p*.

Table 4

==

Number of 6s obtained in an experiment	0	1	2	3	4	More than 4
Frequency	19	26	12	2	1	0

(8) Give a clear and concise definition of the Binomial distribution suitable for readers who understand probability but have never heard of the Binomial distribution.

In some families the probability that a child will have red hair is ¼. If one of these families contains five children in all,

(a) find the probability that it will contain at least two children with red hair

(9) Assuming boys and girls are equally probable, find (i) the probability that there is no red-headed boy, (ii) the probability that there is no red-headed girl and (iii) the probability that there is no red-headed child. Hence, or otherwise, find the probability of at least one red-headed boy and at least one red-headed girl. Define the Binomial distribution, stating the conditions under which it will arise, and find its mean.

A manufacturer of glass marbles produces equal large numbers of red and blue marbles. These are thoroughly mixed together and then packed in packets of six marbles which are random sample from mixture. Find the probability distribution of the number of red marbles in a packet purchased by a child.

Two boys, Fred and Tom, each buy a packet of marbles. Fred prefers the red ones and Tom prefers the blue ones, so they agree to exchange the marbles as far as possible, in order that at least one of them will have six of the colour he prefers. Find the probabilities that after exchange (a) they will both have the colour they prefer. (b)

Fred will have three or more blue ones.

(10) Consider a young man waiting for this young lady who is late. To amuse himself while waiting, he decides to take a walk under the following set of rules. He tosses a fair coin. If the coin falls heads he walks 10 metres north; if the coin falls tails he walks 10 metres south. He repeats this process every 10 metres and thus executes what is called a 'random walk'. What is the probability that after walking 100 metres he will be

(a) back at his starting point,

(b) no more than 10 metres from his starting point,

(c) exactly 20 metres away from his starting point?

(11) A mother has found that 20% of the children who accept invitations to her children's birthday parties do not come. For a particular party she invites twelve children but has available only ten party hats. What is the probability that there is not a hat for every child to the party?

The mother knows that there is a probability of 0.1 that a child who comes to a party will refuse to wear a hat. If this is taken into account, what is the probability that the number of hats will not be adequate?

(12) Derive the mean and variance of the Binomial distribution.

Mass production of miniature hearing aids is a particularly difficult process and so the quality of these products is monitored carefully. Samples of size sic are selected regularly and tested for correct operation. The number of defective in each sample is recorded. During one particular week 140 samples are taken and the distribution of the number of defective per sample is given in Table 6

Table 5

Number of defectives per sample (x)	0	1	2	3	4	5	6
Number of samples with x defectives (f)	27	36	39	22	10	4	2

Find the frequencies of the number of defectives per sample given by a Binomial distribution having the same mean and total as the observed distribution.

Unit Summary

The Binomial probability distribution is a suitable model for the discrete random variable X, where X is the number of 'successes' in a series of trials, provided that.

(i) each trial has only two exhaustive and mutually exclusive outcomes, 'success' and 'failure';

(ii) the probability, p, of success is constant;

(iii) the trials are independent;

(iv) n, the number of trials, is constant.

$$p(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Mean $\mu = E(X) = np$, Variance = $= \sigma^2 = npq$, where q = 1 –p

Recurrence formula $p(X = x+1) = \left(\frac{n-x}{x+1}\right)\left(\frac{p}{1-p}\right) p(X = x)$.

UNIT 6 THE POISSON DISTRIBUTION

6.1 Introduction

Welcome to this unit called Poisson distribution. This is a discrete distribution which is employed when occurrences (usually rare events such as accidents) are randomly distributed in space and time. It is discrete and that the mean number of time in a given interval is a constant. That is the mean will remain constant throughout the experiment.

6.2 Unit Aims:

Understand in which circumstances Poisson distribution will occur, and learn the meaning of the parameter. In addition, know how to compute the probability using the PMF, and know the E[X] and Var[X] for Poisson. Also realize the fact that the sum of independent Poisson is still Poisson

6.3 Unit Objectives: At the end of this unit you will be able to:

- Identify the circumstances where Poisson distribution will occur.
- Calculate the parameter of the poisson distribution
- Compute the probability using the PMF, the E[X] and Var[X] for Poisson distribution.

Terminology

The following notation is helpful, when we talk about the Poisson distribution.

- *e*: A constant equal to approximately 2.71828. (Actually, *e* is the base of the natural logarithm system.)
- μ: The mean number of successes that occur in a specified region.
- *x*: The actual number of successes that occur in a specified region.
- P(*x*; μ): The **Poisson probability** that exactly *x* successes occur in a Poisson experiment, when the mean number of successes is μ.

6.4 Time Required: You need 8 hours to complete this unit

6.5 Unit Topics:

6.5.1 Poisson Distributions

What kind of distribution would you expect in this situation? You might be tempted to think that the Binomial distribution is applicable since the number of calls is a discrete variable. Although you know the number of calls in each time interval, how many 'non-calls' are there? In other words, what number corresponds to the Binomial parameter n and what meaning can you give to p, the probability of a 'success', in this case the occurrence of a telephone call? Leaving aside these problems for the moment, there is none statistic which you can calculate and this is the mean number of telephone calls, I, for a five-minute time interval. From Table 1

$$\lambda = \frac{\sum_i f_i x_i}{\sum_i f_i} = \frac{37}{100} = 0.37$$

You can apply the Binomial distribution to the problem if you imagine the five-minute time interval divided into n small time intervals of δt, where n is so large that there can be no more than one telephone call arriving in δt. It is snow possible to give a meaning to the probability p of a call in time δt, corresponding to the probability of a 'success' in the Binomial distribution:

Probability of a call in $\delta t = \dfrac{mean\ number\ of\ calls\ in\ five\ \min utes}{Number\ of\ \delta t's\ in\ five\ \min utes} = \dfrac{\lambda}{n}$

This mean that q, the probability of a 'non-call' or 'failure', is 1 – p = q and the number of trials is n, the number of δt s in five minutes. Using the Binomial distribution, the probability P(X = x) of x calls in five minutes is given by

$$p(X = x) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

The value of δt, and consequently n, has still to be specified: the criterion for δt in effect you require $\delta t \to 0$.

Table 1 shows the frequency of 0, 1, 2 etc telephone calls arriving at a small telephone exchange in 100 consecutive time intervals of five minutes. (The significance of the last column will become apparent later.)

Table 1 Frequency distribution of phone calls arriving at an exchange

Number of girls, x_i	Number of time intervals, f_i	$f_i x_i$
0	71	0
1	21	23
2	4	8
3	2	6
4 or more	0	0
	100	37

and hence $n \to \infty$, while the mean number of calls, λ , in a five-minute interval remains, constant.

$$p(X = x) = \frac{\lambda e^{-\lambda}}{x!}$$

Using this expression we can calculate the probability distribution for the number of calls, taking equal to the observed mean (i.e. 0.37).

$$p(X = 0) = e^{-0.37} = 0.691$$

$$p(X = 1) = 0.37e^{-0.37} = 0.256$$

$$p(X = 2) = \frac{0.37^2 e^{-0.37}}{2!} = 0.047$$

$$p(X = 3) = \frac{0.37^3 e^{-0.37}}{3!} = 0.006$$

(These values are given to three decimal places.)

$P(X \geq 4)$ is found by adding these probabilities and subtracting the total from 1, giving $P(X \geq 4) = 0.000$ (to three decimal places). The expected frequencies are found by multiplying these probabilities by the total frequency, i.e. 100. Table 2 compares the expected and observed frequencies and it will be seen that reasonable agreement is obtained.

Table 2 Comparison of observed and expected Poisson frequencies for the data in Table 1

Number of calls	Observed frequency	Expected frequency
0	71	69.1
1	23	25.6
2	4	4.7
3	2	0.6
≥ 4	0	0

The discrete distribution which we have obtained is called the Poisson distribution. It gives the probability of x events occurring in a given interval as

$$p(X = x) = \frac{\lambda e^{-\lambda}}{x!} \qquad (6.1)$$

Where the mean number of events in the same interval is λ . Besides being applicable to events randomly distribution in time such as telephone calls, it is also applicable to events randomly distributed in space, for example the number of flaws in a given length of rope. For it is to be applicable the following conditions must be satisfied:

(i) two or more events cannot occur simultaneously;
(ii) the events are independent;
(iii) The mean number of events in a given interval is constant.

λ Is the parameter of the distribution? A convenient short-hard for denoting that the random variable X is Poisson distributed with mean λ is X is equivalent to $p_o(\lambda)$

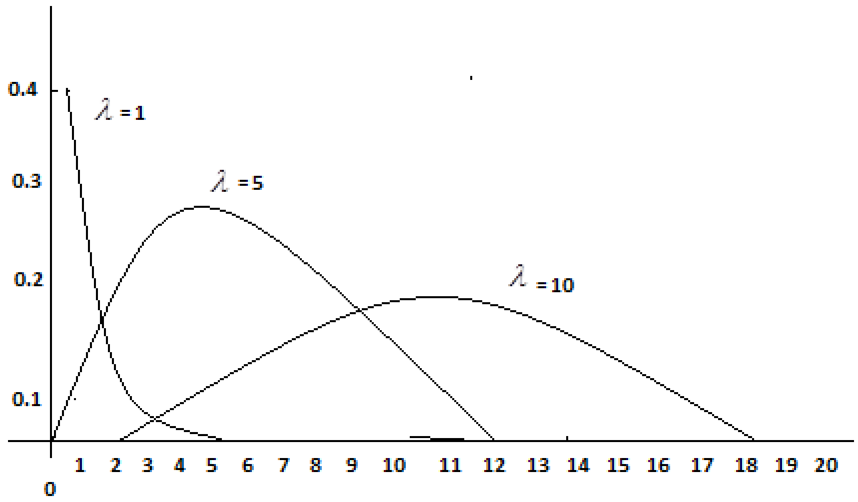


Figure 1 The Poisson distribution for various values of λ

Figure 1 shows the Poisson distribution for different values of λ . Note that as λ increases the distribution becomes more symmetrical.

Example 1 The number of cars per minute passing a certain point on a road is Poisson distribution with mean 4. Find

(a) the probability that 4 cars pass in a minute,

(b) the probability that 8 cars pass in two minutes.

Solution

(a) The mean number of cars in a minute is 4 so

$$p(x=4)=\frac{4^4 e^{-4}}{4!}=0.195$$

(b) The mean number of cars in a minute is 8 so

$$p(x=8)=\frac{4^8 e^{-8}}{8!}=0.140$$

As for the Binomial distribution a recurrence formula can be used to calculate P(X = x + 1) from P(X = x) since $\dfrac{p(X=x+1)}{p(X=x)}=\dfrac{\lambda^{x+1}e^{-\lambda}/(x+1)!}{\lambda^x e^{-\lambda}/x!}$

which on cancelling and rearranging gives

$$p(X=x+1)=\frac{\lambda}{(x+1)}\,p(X=x)$$

6.0 Unit activities

(1) The mean number of particles emitted per second by a radioactive source is 3. Calculate the probabilities of 0, 1, 2, 3, 4, 5 and 6 emissions per second. What is the probability that at least one particle is emitted in a second?

(2) The number of demands for taxis to a taxi firm is Poisson distributed with a mean of four demands in 30 minutes. Find the probabilities of

(a) no call in 30 minutes,

(b) one call in 30 minutes,

(c) one call in 1 hour

(d) two calls in 1 hour

Table 3

Number of vehicles in the period	0	1	2	3	4	5	6	7
frequency	252	306	235	137	42	24	3	1

(3) The average number of faults in a metre of cloth produced by a particular machine is 0.1. What is the probability that a length of 4 m is free from faults? How long would a piece have to be before the probability that it contains no flaws is less than 0.957?

(4) In checking the proofs of a book for publication it is found that the probability that there are no errors on a page is 0.05. Estimate the mean number of errors per page to the nearest whole number.

If the book contains 300 pages, estimate the number of pages on which you would expect to find fewer than four errors.

(5) In a traffic survey a count was made of the number of vehicles passing the survey point in each period of 15 seconds. The survey continued for 1000 such periods, and the results were as in Table 3.

Calculate the mean number of vehicles per period and the theoretical Poisson frequencies with this mean.

(6) Gnat larvae are distributed at random in pond water so that the number of larvae contained in a random sample of 10 cm3 of pond water may be regarded as a

random variable having a Poisson distribution with mean 0.2. Ten independent random samples, each of 10 cm3, of pond water, are taken by a zoologist. Determine (correct to three significant figures)

(a) The probability that none of the samples contain larvae,
(b) The probability that one sample contains a single larva and the remainder contain no larvae,
(c) The probability that one sample contains two or more larvae and the remainder contain no larvae,
(d) The expectation of the total number of larvae contained in the ten samples,
(e) The expectation of the number of samples containing no larvae.

6.5.2 The mean and variance of a Poisson distribution

These can be deduced from the mean and variance of the Binomial distribution in the limit when $n \to \infty$, $p \to 0$ but $np \to \lambda$ remains constant. For a Binomial distribution $\mu = np$. Therefore for a Poisson distribution

$$\mu = \lambda \qquad (6.2)$$

for a Binomial distribution $\sigma^2 = npq$ o2 . As $n \to \infty$, $p \to 0$ and q –1. Writing $np = \lambda$ and q = 1, gives

$$\sigma^2 = \lambda \qquad (6.3)$$

rigorous proof of these results is given in Section 6.4) we thus find an important

property of the Poisson distribution which is that is mean is equal to its variance. This gives a simple test of whether or not a distribution is approximately Poissonian.

Table 3 extends Table 1 and shows the calculation of the mean and variance for the number of calls in a five-minute time interval. In this case the mean is approximately equal to the variance thus indicating that the distribution is approximately Poissonian.

Table 4 Calculation of mean and variance for data in

Number of calls xi	Frequency f_i	$f_i x_i$	$f_i x_i^2$
0	71	0	0
1	23	23	23
2	4	8	16
3	2	6	18
≥4	0	0	0
	100	37	57

Mean $= \dfrac{\sum_i f_i x_i}{\sum_i f_i} = \dfrac{37}{100} = 0.37$

Variance $\dfrac{\sum_i f_i x_i}{\sum_i f_i} - \left(\dfrac{\sum_i f_i x_i}{\sum_i f_i} \right)^2 = \dfrac{57}{100} - 0.37^2 = 0.43$

Unit activity

Calculate P(X = 0), P(X = 1), P(X = 2), P(X = 3), P(X = 4), P(X =5), P(X > 6) for a Poisson variable with mean 1.2. Using this probability distribution calculate the mean and variance and confirm that they are equal to each other within 0.01. (There is a slight discrepancy since values of P(X = x) for x 6 have grouped together.)

Proof that the mean and variance of a Poisson distribution are λ .

We have, for a Poisson distribution,

$$p(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Mean = E(X)

$$= \sum_{x=0}^{\infty} x p(X = x)$$

$$= \sum_{x=0}^{\infty} \frac{x \lambda^x e^{-\lambda}}{x!}$$

$$= \frac{0x\lambda^0 e^{-\lambda}}{0!} + \frac{1x\lambda e^{-\lambda}}{1!} + \frac{2x\lambda^2 e^{-\lambda}}{2!} + \frac{3x\lambda^3 e^{-\lambda}}{3!} + \ldots + \frac{(x+1)\lambda^{x+1} e^{-\lambda}}{(x+1)!} + \ldots$$

$$= 0 + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{1!} + \frac{\lambda^3 e^{-\lambda}}{2!} + \ldots + \frac{\lambda^{x+1} e^{-\lambda}}{x!} + \ldots$$

$$= \lambda\left[e^{-\lambda} + \frac{\lambda e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \ldots + \frac{\lambda^x e^{-\lambda}}{x!} + \ldots.\right]$$

The terms in the bracket are those of the Poisson distribution. Their sum is therefore 1, since they are the probabilities of mutually exclusive and exhaustive events. Therefore,

$$\text{Mean} \quad = \lambda \qquad (6.4)$$

Turning to the variance we require $E\left\{(X-\mu)^2\right\}$.

$$E\left\{(X-\mu)^2\right\} = \sum_{x=0}^{\infty}(x-\lambda)^2\,\frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty}[x(x-1) + x(1-2\lambda) + \lambda^2]\,\frac{\lambda^x e^{-\lambda}}{x!}$$

(The term in the square bracket has been written in this way so that the first term cancels with the first two terms of x!)

$$\text{Variance} = E\left\{(X-\mu)^2\right\} \qquad (6.5)$$

The second summation is E(X) which we have already found to be λ and the third sum is $\sum_{x=0}^{\infty}$ P(X =x) which is 1. The first two terms of the first sum are both zero, so expanding we have:

$$\sum_{x=0}^{\infty} x(x-1)\frac{\lambda^x e^{-\lambda}}{x!} = \frac{2x1x\lambda^2 e^{-\lambda}}{2!} + \frac{3x2\lambda^2 e^{-\lambda}}{3!} + \frac{4x3\lambda e^{-\lambda}}{4!} + \ldots. + \frac{(x+2)(x+1)\lambda^{x+2} e^{-\lambda}}{(x+2)!} + \ldots.$$

$$= \lambda^2 e^{-\lambda} + \lambda^3 e^{-\lambda} + \frac{\lambda^4 e^{-\lambda}}{2!} + \ldots + \frac{\lambda^{x+2} e^{-\lambda}}{x!} + \ldots.$$

$$= \lambda^2\left[e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2!} + \ldots + \frac{\lambda^x e^{-\lambda}}{x!} + \ldots.\right]$$

$$= \lambda^2$$

Since again the terms in the bracket have the sum 1. Substituting for the three summations in equation (6.5) you have

$$\text{Variance} = E\left\{(X-\mu)^2\right\}$$

$$= \lambda^2 + (1-2\lambda)\lambda + \lambda^2$$
$$= \lambda^2 + \lambda - 2\lambda^2 + \lambda^2$$
$$= \lambda$$

6.5.3 Application of the Poisson distribution

The Poisson distribution is used in biological research for counting the number of cells of a particular type in a dilute solution. The well-shaken solution is placed on a slide which is divided into squares and viewed through a microscope. The number of cells in each of the squares is counted, and from the mean, the number of cells per unit volume can be estimated. Agreement of the observed frequencies with a Poisson distribution with the same means is used to rest that the cells are distributed randomly through the solution.

Alternatively, if it is known that the cells are randomly distributed, a quick method is available for estimating the total number of cells present as is shown in the following example.

Example 2 An ambulance station receives on average one emergency call every hour. If there are three ambulances available and the average time for which an ambulance is out to a call is half an hour, what is the probability that the ambulance station cannot cover the emergency calls?

Solution If the ambulance station receives three or less emergency calls in half an hour, then it can deal with incoming calls. The mean number of calls in half an hour is ½. The probability that the ambulance station can cope with incoming calls is equal to

$$p(X=0) + p(X=1) + p(X=2) + P(X=3)$$
$$= e^{-0.5} + 0.5e^{-0.5} + \frac{(0.5)^2 e^{-0.5}}{2!} + \frac{(0.5)^3 e^{-0.5}}{3!} = 0.998$$

The probability that the ambulance station cannot cope is equal to 1 – 0.998 = 0.002. (This problem oversimplifies what happens in practice since the mean number of emergency calls in a given time interval will vary during the day.)

Example 3 A shop sells five pieces of shirt every day, then what is the probability of selling three shirts today?

Solution: Mean value for 1 day, m=5

Probability of selling 3 shirts, P(3;5) = e−5 533! = 0.006737947×1255! = 0.007

Hence, the probability of selling three shirts is 0.007 when at the average 5 shirts are being sold each day.

Example 4 If three persons, on an average, come to ABC company for job interview, then find the probability that less than three people have come for interview on a given day.

Solution: The mean for Poisson random variable, m=3

$P(x<3;3)=P(0;3)+P(1;3)+P(2;3)$

$P(0;3)$ = e−3 300! = 0.04978706837

$P(1;3)$ = e−3 311! = 0.1493612051

$P(2;3)$ = e−3 322! = 0.22404180766

Hence, $P(x<3;3) = P(0;3)+P(1;3)+P(2;3)$

= 0.04978706837+0.1493612051+0.22404180766

= 0.423

The probability of less than three persons coming for interview on a certain day is 0.423.

Example 5 Number of calls coming to the customer care center of a mobile company per minute is a Poisson random variable with mean 5. Find the probability that no call comes in a certain minute.

Solution: The mean value, m=5

you need to find the probability of getting zero calls when 5 calls are known to come every minute.

$P(0;5)$ = e−5 500! = 0.006737947

Hence, the probability of getting zero calls in a minute is 0.006737947.

Example 6 There are five students in a class and the number of students who will participate in annual day every year is a Poisson random variable with mean 3. What will be the probability of more than 3 students participating in annual day this year?

Solution:

Mean for Poisson random variable, m = 3

$P(x>3;3)=P(4;3)+P(5;3)$

$P(4;3)$ = e−3 344! = 0.16803135574

$P(5;3)$ = e−3 355! = 0.10081881344

Hence, $P(x>3;3) = P(4;3)+P(5;3)$=0.268850169

The probability of getting more than three students participating is 0.268850169.

Example 7 The deals cracked by an agent per day is a random Poisson variable with mean 2. Given that each day is independent of other day, find the probability of getting 2 deals cracked on first day and 1 deal to be cracked the next day.

Solution: The probability of getting 2 deals in a day is $P(2;2)$ and the probability of getting 1 deal is $P(1;2)$.

The probability of getting 2 deals on first day and one deal on second day = $P(2;2) \times P(1;2)$

$P = e{-}2\ 222! \times e{-}2\ 211!$

$= 0.27067056647+0.27067056647=0.54134113295$

The probability the first day two deals are cracked and the second day one deal is cracked is 0.54.

Example 8 The average number of homes sold by the Acme Realty Company is 2 homes per day. What is the probability that exactly 3 homes will be sold tomorrow?

Solution: This is a Poisson experiment in which you know the following:

- $\mu = 2$; since 2 homes are sold per day, on average.
- $x = 3$; since we want to find the likelihood that 3 homes will be sold tomorrow.
- $e = 2.71828$; since *e* is a constant equal to approximately 2.71828.

You plug these values into the Poisson formula as follows:

$$p(x) = \frac{\mu^x e^{-\mu}}{x!}$$

$$p(3) = \frac{2^3 e^{-2}}{3!}$$
$$= 0.180$$

Thus, the probability of selling 3 homes tomorrow is 0.180.

6.2 Unit activities

(1) The probability that a brand of light bulb is faulty is 0.01. the light bulbs are packed in boxes of 100. What is the probability that a box chosen at random contains (a) no faulty light bulbs, (b) two faulty light bulbs, (c) at least four faulty light bulbs?

A buyer accepts a consignment of 50 boxes provided that when two boxes are chosen at random they contain at most two faulty light bulbs altogether. What is the probability that a consignment is accepted?

(2) A large number of screwdrivers from a trial production run is inspected. It is found that the cellulose acetate handles are defective on 1% and that the chrome steel blades are defective on 1 ½ of the screwdrivers, the defects occurring independently.

(a) What is the probability that a sample of 80 contains no screwdrivers with defective handles?

(b) What is the probability that a sample of 80 contains more than two defectives screwdrivers?

(c) What is the probability that a sample of 80 contains at least one screwdriver with both a defective handle and a defective blade?

(3) X is a random variable having a Poisson distribution given by

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2 \ldots$$

Prove that the mean of X is m and state the variance of X.

(4) The number of telephone calls received per minute at the switchboard of a certain office was logged during the period 10 a.m. to noon on a working day. The results were as in Table 5 f is the number of minutes with x calls per minute.

By consideration of the mean and variance of this distribution show that a possible model is a Poisson distribution.

Using the calculated mean and on the assumption of a Poisson distribution calculate

(a) the probability that two or more calls were received during any one minute,

(b) the probability that no calls were received during any two consecutive minutes.

Table 5

Calls per minute (x)	0	1	2	3	4	5	6	7	8
f	7	18	27	28	20	11	5	3	1

==

(4) (a) The number of accidents notified in a factory per day over a period of 200 days gave rise to Table 6

(i) Calculate the mean number of accidents per day.

(ii) Assuming that this situation can be represented by a suitable Poisson distribution, calculate the corresponding frequencies.

(b) Of items produced by a machine, approximately 3% are defective, and these occur at random. What is the probability that, in a sample of 144 items, there will be at least two which are defective?

Table 6

==

Number of accidents	0	1	2	3	4	5
Number of days	127	54	14	3	1	1

(5) National records for the past 100 years were examined to find the number of deaths in each year due to lighting. The most deaths in any year were four which was recorded once. In 35 years, no death was observed and in 38 years only one death. The mean number of deaths per year was 1.00. Draw up a frequency table of the number of deaths per year, and estimate the corresponding expected frequencies for a Poisson distribution having the same mean. Illustrate both frequency distributions graphically. [1/e = 0.3679.]

(6) The number of emergency admissions each day to a hospital is found to have a Poisson distribution with mean 2.

(a) Evaluate the probability that on a particular day there will be no emergency admission.

(b) At the beginning of one day the hospital has five beds available for emergencies. Calculate the probability that this will be an insufficient number for the day.

(c)Calculate the probability that there will be exactly three admissions altogether on two consecutive days.

(7) (a) Table 7shows the number of phone calls I received over a period of 150 days .

(i) Find the average number of calls per day.

(ii) Calculate the frequencies of the comparable Poisson distribution.

(b) A firm selling electrical components packs them in boxes of 60. One average, 2% of the components are faulty. What is the chance of getting more than two defective components in a box? (use the Poisson distribution.)

Table 7

Number of calls	0	1	2	3	4
Number of days	51	54	36	6	3

(8) A small garage has three cars available for daily hire. The daily demand for these cars may be assumed to have a Poisson distribution with mean 2.

(a)Prove that demands for one and two cars on any day are equally probable. $2e^{-2}$

(b) Find, as accurately as your tables permit, the probabilities that on a given day exactly no, one, two and three car(s) will be hired.

Hence, or otherwise, calculate the mean number of cars hired per day.

(c)The garage owner charges K6 per day for the hire of a car and his total outgoings per car, irrespective of whether or not it is hired, amount to K1 per day. Calculate, to the nearest ngwee, the garage owner's expected daily profit from the hiring of these cars.

(9) (a) The number of organic particles suspended in a volume V cm^3 of a certain liquid follows a

Poisson distribution with mean 0. 1. Volume.

Find the probabilities that a sample of 1 cm3 of the liquid will contain

(i) at least one organic particle.

(ii) exactly one organic particle.

(b) The liquid is sold in vials, each vial containing 10 cm^3 of the liquid. The vials are dispatched for sale in boxes, each box containing 100 vials. Find the probability that a vial will contain at least one organic particle. Hence find the mean and the standard deviation of the number of vials per box of 100 vials that contain at least one organic particle.

(10) Specify the conditions under which a Binomial distribution reduces to a Poisson distribution and derive the expression for the Poisson distribution form the expression for the Binomial distribution. Hence derive expressions for the mean and variance of the Poisson distribution.

A clerk in the ticket issuing office of a suburban railway station noted that the number of tickets issued for journeys away from London was equal to the number of first-class tickets issued for all journeys, and the number of tickets issued in each category was in the proportion of one in five hundred of all tickets issued.

Determine the probability that, of the 2000 travelers departing from that station between 8 a.m. and 9 a.m.,

(a) exactly three required first-class seats,

(b) more than two travelled away from London.

Unit Summary

The Poisson probability distribution is a suitable mode for the random variable X, where X is the number of events in particular interval of time or space, provided that

(i) the events occur independently and at random;

(ii) two or more events cannot occur simultaneously;

(iii) the mean number of events, λ, in the specified interval is constant.

Mean = λ variance = λ

The Poisson distribution with mean np can be used as an approximation to the Binomial distribution with parameters n and p provided that n is large and p is small (in practice n $\geq$50, p < 0.1).

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ x} = 0,1,2,\ldots$$

UNIT 7 HYPER-GEOMETRIC DISTRIBUTION

7.1 Introduction

Welcome to this unit called hyper-geometric distribution. This is a discrete probability distribution that describes the probability of k successes (random draws for which the object drawn has a specified feature) in N draws, without replacement, from a finite population of size N that contains exactly k objects with that feature, wherein each draw is either a success or a failure.

7.2 Unit Aims:

Get the ideas about which situations you can see as a hyper geometric Distribution, understand what its three parameters are, and be able to distinguish this from Binomial distribution. Also, learn the PMF and how to calculate the probability and statistical parameters.

7.3 Unit Objectives: At the end of this unit you will be able to:

- Define hyper geometric distribution correctly;
- Compute the mean, variance and standard deviation of hyper geometric distribution precisely;
- Solve problems involving hyper geometric distribution accurately

Terminology

The following notation is helpful, when you talk about hyper geometric distributions and hyper geometric probability.

- *N*: The number of items in the population.
- *k*: The number of items in the population that are classified as successes.
- *n*: The number of items in the sample.
- *x*: The number of items in the sample that are classified as successes.
- C_x^k : The number of combinations of *k* things, taken *x* at a time.
- h(*x*; *N*, *n*, *k*): **hyper geometric probability** - the probability that an *n*-trial hyper geometric experiment results in exactly *x* successes, when the population consists of *N* items, *k* of which are classified as successes.
- pmf- probability marginal function

7.4 Time Required: You should spend 6 hours on this unit

7.5 Unit Topics:

7.5.1 Hyper- Geometric distribution

The probability distribution of a hyper geometric random variable is called a **hyper geometric distribution.** In this unit the following are described: what a hyper geometric random variables is, hyper geometric experiments, hyper geometric probability, and the hyper geometric distribution and how they are all related.

7.5.2 Hyper geometric Experiments

A **hyper geometric experiment** is a statistical experiment that has the following properties:

- A sample of size n is randomly selected without replacement from a population of N items.
- In the population, k items can be classified as successes, and $N - k$ items can be classified as failures.

Consider the following statistical experiment. You have a box of 10 marbles - 5 red and 5 green. You randomly select 2 marbles without replacement and count the number of red marbles you have selected. This experiment follows a hyper geometric distribution.

A **hyper geometric random variable** is the number of successes that result from a hyper geometric experiment. The probability distribution of a hyper geometric random variable is called a **hyper geometric distribution.** Note that it would not be a binomial experiment. A binomial experiment requires that the probability of success be constant on every trial. With this experiment, the probability of a success changes on every trial. In the beginning, the probability of selecting a red marble was 5/10. If you select a red marble on the first trial, the probability of selecting a red marble on the second trial is 4/9. And if you select a green marble on the first trial, the probability of selecting a red marble on the second trial is 5/9.

Note further that if you selected the marbles with replacement, the probability of success do not change. It would be 5/10 on every trial. Then, this would be a binomial experiment.

Given x, N, n, and k, you can compute the hyper geometric probability based on the following formula:

Hyper geometric Formula.. Suppose a population consists of *N* items, *k* of which are successes. And a random sample drawn from that population consists of *n* items, *x* of which are successes. Then the hyper geometric probability is:

$$h(x; N, nk) = p(X = x) = f(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \qquad (7.1)$$

The hyper geometric distribution has the following properties:

- The mean of the distribution is equal to $\frac{n*k}{N}$.
- The variance is $\frac{n*k*(n-k)*(N-n)}{N^2*(N-1)}$

Example 1 Suppose we randomly select 5 cards without replacement from an ordinary deck of playing cards. What is the probability of getting exactly 2 red cards (i.e., hearts or diamonds)?

Solution: This is a hyper geometric experiment in which we know the following:

- N = 52; since there are 52 cards in a deck.
- k = 26; since there are 26 red cards in a deck.
- n = 5; since we randomly select 5 cards from the deck.
- x = 2; since 2 of the cards we select are red.

You plug these values into the hyper geometric formula as follows:

$$h(x; N, nk) = p(X = x) = f(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

h(*2*; *52*, *5*, *26*) = [${}_{26}C_2$] [${}_{26}C_3$] / [${}_{52}C_5$]

h(*2*; *52*, *5*, *26*) = [325] [2600] / [2,598,960]

h(*2*; *52*, *5*, *26*) = 0.32513

Thus, the probability of randomly selecting 2 red cards is 0.32513.

Example 2 Let the random variable *X* denote the number of aces in a five-card hand dealt from a standard 52-card deck. Find a formula for the probability mass function of *X*.

Solution. The random variable *X* here also follows the hyper geometric distribution. Here, there are N = 52 total cards, n = 5 cards sampled, and m = 4 aces. Therefore, the p.m.f. of *X* is:

$$f(x) = \frac{\binom{4}{x}\binom{48}{5-x}}{\binom{52}{5}}$$, for the support x = 0, 1, 2, 3, and 4.

E(X)= $$= \sum_{x \in S} x \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$$

Example 3 There are 10 black marbles and 10 white marbles out of which 5 marbles are being chosen. Find the probability that there are 2 white marbles in them.

Solution: Total population, N=20, Sample size, n=5, Number of successes in N, K=10 Number of successes in sample, k=2, Probability of getting 2 white marbles, $P(X=2)$ =

$$\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$P(X=2) = \frac{\binom{10}{2}\binom{10}{3}}{\binom{20}{5}} = 0.08707430341$$

Example 4 Out of 100 students qualifying an exam, 10 were drawn randomly. If 35 out of 100 qualified students are female, then find the probability that 6 out of 10 chosen are females.

Solution:

Total population, N=100, Sample size, n=10, Number of successes in N, K=35

Number of successes in sample, k=6, Probability of getting 2 white marbles, $P(X$=6) =

$$\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$P(X=6)=\frac{\binom{35}{6}\binom{65}{4}}{\binom{100}{10}} = 0.06348495671$$

Example 5 Consider a shipment of 1000 items into a factory. Suppose the factory can tolerate about 5% defective items. Let X be the number of defective items in a sample without replacement of size n = 10. Suppose the factory returns the shipment if $X \geq 2$.

(a) Obtain the probability that the factory returns a shipment items which has 5% defective items.

(b) Suppose the shipment has 10% defective items. Obtain the probability that the factory returns such a shipment.

(c) Obtain approximations to the probabilities in part (a) and (b) using appropriate binomial distribution.

Solution. This distribution follows a hyper geometric distribution with pdf as:

$$p(x)=\frac{\binom{N-D}{n-x}\binom{D}{x}}{\binom{N}{n}}, \text{ x = 0, 1, …, n}$$

Hence, let X be the number of defective items.

(a) From the given information, you have N = 1000, n = 10, D = 1000*5% = 50

Since X has a hyper geometric distribution, you have

$$p(X \geq 2) = 1 - p(X = 0) - p(X = 1)$$

$$= 1 - \frac{\binom{1000-50}{10-0}\binom{50}{0}}{\binom{1000}{10}} - \frac{\binom{1000-50}{10-1}\binom{50}{1}}{\binom{1000}{10}}$$

$$= 0.0853$$

(b) Since D = 1000*10%=100 you have

$$p(X \geq 2) = 1 - p(X = 0) - p(X = 1)$$
$$= 1 - \frac{\binom{1000-100}{10-0}\binom{100}{0}}{\binom{1000}{10}} - \frac{\binom{1000-100}{10-1}\binom{100}{1}}{\binom{1000}{10}}$$
$$= 0.2637$$

(c) From the given information, when n = 10, p = 0.05, using the binomial distribution, you have

$$p(X \geq 2) = 1 - p(X = 0) - p(X = 1)$$
$$= 1 - \binom{10}{0}0.05^0(1-0.05)^{10-0} - \binom{10}{1}0.05^1(1-0.05)^{10-1}$$
$$= 0.0861$$

When n = 10, p = 0.1 , using the binomial distribution, you have

$$p(X \geq 2) = 1 - p(X = 0) - p(X = 1)$$
$$= 1 - \binom{10}{0}0.1^0(1-0.1)^{10-0} - \binom{10}{1}0.1^1(1-0.1)^{10-1}$$
$$= 0.2639$$

7.5.3 Cumulative Hyper geometric Probability

A **cumulative hyper geometric probability** refers to the probability that the hyper geometric random variable is greater than or equal to some specified lower limit and less than or equal to some specified upper limit.

For example, suppose you randomly selected five cards from an ordinary deck of playing cards. You might be interested in the cumulative hyper geometric probability of obtaining 2 or fewer hearts. This would be the probability of obtaining 0 hearts plus the probability of obtaining 1 heart plus the probability of obtaining 2 hearts, as shown in the example below.

Example 6 Suppose we select 5 cards from an ordinary deck of playing cards. What is the probability of obtaining 2 or fewer hearts?

Solution: This is a hyper geometric experiment in which we know the following:

- N = 52; since there are 52 cards in a deck.
- k = 13; since there are 13 hearts in a deck.
- n = 5; since we randomly select 5 cards from the deck.
- x = 0 to 2; since our selection includes 0, 1, or 2 hearts.

You plug these values into the hyper geometric formula as follows:

$h(x \le x; N, n, k) = h(x \le 2; 52, 5, 13)$

$h(x \le 2; 52, 5, 13) = h(x = 0; 52, 5, 13) + h(x = 1; 52, 5, 13) + h(x = 2; 52, 5, 13)$

$h(x \le 2; 52, 5, 13) = [\ ({}_{13}C_0)\ ({}_{39}C_5) / ({}_{52}C_5)\ ] + [\ ({}_{13}C_1)\ ({}_{39}C_4) / ({}_{52}C_5)\ ] + [\ ({}_{13}C_2)\ ({}_{39}C_3) / ({}_{52}C_5)\ ]$

$h(x \le 2; 52, 5, 13) = [\ (1)(575{,}757)/(2{,}598{,}960)\ ] + [\ (13)(82{,}251)/(2{,}598{,}960)\ ] + [\ (78)(9139)/(2{,}598{,}960)\ ]$

$h(x \le 2; 52, 5, 13) = [\ 0.2215\ ] + [\ 0.4114\ ] + [\ 0.2743\ ]$

$h(x \le 2; 52, 5, 13) = 0.9072$

Thus, the probability of randomly selecting at most 2 hearts is 0.9072.

Theorem: An easier way to calculate the **variance** of a random variable X is:

$$\mathrm{var}(X) = E(X^2) - [E(X)]^2$$

7.0 Unit Activities

1. There are 6 bulbs in a house out which 3 are defective. If 2 bulbs are picked randomly, find the probability that at least one is defective.
2. If 6 cards are drawn from a deck of 52 cards, find the probability of getting all kings in the draw.
3. There are 40 cards in a deck of cards. Out of these 12 are red cards and 28 are black cards. We draw 10 cards randomly from this pile. We do not replace the card drawn. What is the probability that exactly 8 out of the chosen cards are red?
4. A school has 105 female students and 90 male students. If a delegation of 10 students is to be sent to a cultural event, then what is the probability that 7 out of those 10 would be females?
5. Out of 100 Bulbs produced by a manufacturing company, 35 are white light bulbs and the rest are yellow light bulbs. If 10 bulbs are randomly drawn without replacement, find the probability that 6 out of these 10 would be white light bulbs.
6. Five cards are chosen randomly from a standard shuffled deck of 52 cards. What is

the probability that at least 3 out of the 5 cards are diamonds?

7. A consignment of 20 microprocessors has arrived. 4 out of the 20 in the consignment are actually defective. To check the consignment the buyer randomly checks 3 microprocessors. Find the probability that the buyer find two or more defective processors in the check he conducts.
8. 10 coins are tossed simultaneously where the probability of getting head for each coin is 0.6. Find the probability of getting 4 heads.
9. In an exam, 10 multiple choice questions are asked where only one out of four answers is correct. Find the probability of getting 5 out of 10 questions correct in an answer sheet.

Unit Summary

Hyper-Geometric distribution has two properties:

- A sample size is randomly selected without replacement from a population of N items
- The k items in the population may be classified as successes, and N-k items may be classified as failure.

A random variable x follows a hyper-Geometric distribution, denoted C~ Hyp(n,N,k), if the pmf is given by:

$$p(x)=\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

Where N = Population size

n = Sample size

k = number of success in the population

x = number of success in the sample

UNIT 8 THE GEOMETRIC DISTRIBUTION

8.1 Introduction

Welcome to this unit called geometric distribution. This is a distribution which arises when you have a series of independent trials (such as those considered in connection with Binomial distribution) where each trial can have only two possible mutually exclusive outcomes with constant probabilities. The variable, however, is not the number of successful trials but the number of trials required to achieve a success. Thus the number of trials is not constant but is itself the variable. The **geometric distribution** is a special case of the negative binomial distribution. It deals with the number of trials required for a single success. Thus, the geometric distribution is negative binomial distribution where the number of successes (r) is equal to 1. An example of a geometric distribution would be tossing a coin until it lands on heads. You might ask: What is the probability that the first head occurs on the third flip? That probability is referred to as a **geometric probability** and is denoted by g(x; P).

8.2 Unit Aims:

Geometric distribution is to give you the number of trials needed until the first success, and the parameter is the probability of success. You will learn the PMF and how to calculate the probability for Geometric. In addition, know the E[X] and Var[X] for Geometric distribution.

8.3 Unit Objectives: At the end of this unit you should be able to:

- Derive of the formula for the geometric probability mass function.
- Explore the key properties, such as the mean and variance, of a geometric random variable.
- Calculate probabilities for a geometric random variable.

Terminology

- $\sum$ - Summation
- g' **-** First derivative
- g'' **- Second** derivatives
- p – Probability of success

8.4 Time Required: You should spend 6 hours on this unit.

8.5 Unit Topics

8.5.1 Geometric distribution

This is a distribution which arises when you have a series of independent trials (such as those considered in connection with the Binomial distribution) which each trial can have only two possible mutually exclusive and exhaustive outcomes with constant probabilities. The variable, however, is not the number of successful trials but the number of trials required to achieve a success. Thus the number of trials is not constant but is itself the variable. Suppose you throw a die until a 6 is obtained and define the random variable X as the number of throws required to get a 6. Then, since the probability of a 6 is 1/6, we have

$$P(X = 1) = \frac{1}{6}$$

$$P(X = 2) = (\frac{5}{6})(\frac{1}{6}) = \frac{5}{36}$$

$$P(X = 3) = (\frac{5}{6})^2(\frac{1}{6}) = \frac{25}{216}$$

and so on. A representative from the National Football League's Marketing Division randomly selects people on a random street in Lusaka, until a person who attended the last home football game is found. Let *p*, the probability that he succeeds in finding such a person, equal 0.20. And, let *X* denote the number of people selected until the first success is found. What is the probability mass function of *X*?

P=p(success) = 0.2

1-p=P(failure) = 0.8

Let X = number of people selected until first success

$$\underbrace{F\ F\ F\ F\ F......F}_{x-1\ \text{failures}}\ \underbrace{S}_{1\ \text{success}}$$

Definition. Assume Bernoulli trials that is, (1) there are two possible outcomes, (2) the trials are independent, and (3) *p*, the probability of success, remains the same from trial to trial.

Let X denote the number of trials until the first success. Then, the probability mass function of X is: $f(x) = p(X = x) = (1-p)^{x-1}p$, for x = 1, 2, ... In this case, you can say that X follows a **geometric distribution**.

Properties of a Geometric Random Variable

You state and then prove four properties of a geometric random variable. In order to prove the properties, you need to recall the sum of the geometric series.

Recall. (1) The sum of a geometric series is:

$$g(r) = \sum_{k=0}^{\infty} a + ar + ar^2 + ... = \frac{1}{1-r} = a(1-r)^{-1}$$

(2) Then, taking the derivatives of both sides, the first derivative with respect to r must be: $g'(r) = \sum_{k=1}^{\infty} akr^k = 0 + a + 2ar + 3ar^2 + ... = \frac{a}{(1-r)^2} = a(1-r)^{-2}$

(3) And, taking the derivatives of both sides again, the second derivative with respect to r must be: $g''(r) = \sum_{k=2}^{\infty} ak(k-1)r^{k-2} = 0 + 0 + 2a + 6a + ... = \frac{2a}{(1-r)^3} = 2a(1-r)^{-3}$

Theorem 8.1 The probability mass function:

$f(x) = p(X = x) = (1-p)^{x-1}p$, $0 < p < 1$, x = 1, 2 ... for a geometric random variable X is a valid p.m.f.

(1) For 0<p<1

$$\sum_{x=1}^{\infty}(1-p)^{x-1}p = p[1 + (1-p) + (1-p)^2 + ...]$$

$$= p\left[\frac{1}{1-(1-p}\right] = 1$$

(2) **Theorem 8.2** The mean of a geometric random variable X is:

$$\mu = E(X) = \frac{1}{p}$$

(3) **Theorem 8.3** The variance of a geometric random variable *X* is:

$$\text{Var}(X) = \frac{1-p}{p^2}$$

The probability function, p(x), is

$$P(x) = p(X = x) = (\frac{5}{6})^{x-1}(\frac{1}{6})$$

and in general, when the probability of success is p, we have

$$p(x) = p(X = x) = (1-p)^{x-1}p \qquad (x > 0)$$

The mean of the geometric distribution is given by $E(X) = \sum_{all\,x} x(1-p)^{x-1}p$

$$\begin{aligned} &= 1p + 2(1-p)p + 3(1-p)^2 p + \ldots \\ &= p([1 + 2(1-p) + 3((1-p)^2 + \ldots] \end{aligned} \qquad (8.1)$$

From binomial expansion of this can be verified;

$$(1 - x)^{-2} = 1 + 2x + 3x^2 \ldots$$

so the summation in the bracket of (equation 8.1) is $[1-(1-p)]^{-2} = p^{-2}$, giving

$$E(X) = p * p^{-2} = \frac{1}{p}$$

The cumulative distribution function, F(xo) can be found as follows. Since X < x_o and X> x_o are mutually exclusive and exhaustive events we have

$$F(x_o) = P(X \le x_o) = 1 - P(X > x_o)$$

Now P(X > xo) is the probability that there is no success in the first xo trials giving

$$F(x_o) = 1 - (1 - p)^{x_o}$$

Example 1 A discrete distribution is defined by

$$p(X = r) = \lambda r \qquad (r = 1, 2, \ldots, n)$$

Find the value of – and the mean of the distribution. Show that he variance is

$$\frac{1}{18}(n-1)(n+2)$$

The sum of the probabilities over all r must be 1, giving

$$\sum_{x=1}^{n} \text{P(X = r)} = \sum_{1}^{n} \lambda r = 1$$

Therefore $\lambda \sum_{1}^{n} r = \frac{1}{2} \lambda n(n+1) = 1$ giving $\lambda = \frac{2}{n(n+10}$ and $P(x=r) = \frac{2r}{n(n+1)}$

mean = $\sum_{r=1}^{n} rp(X = r)$

$$= \sum_{R=1}^{N} r \frac{2r}{n(n+1)}$$

$$= \frac{2}{n(n+1)} \sum_{r=1}^{n} r^2$$

$$= \frac{2}{n(n+1)} x \frac{1}{6} n(n+1)(2n+1)$$

$$= \frac{1}{3}(2n+1)$$

For the variance you need to evaluate $\sum_{r=1}^{n} r^2 p(X = x)$.

$$\sum_{r=1}^{n} r^2 p(X = r) = \sum_{r=1}^{n} r^2 \frac{2r}{n(n+1)}$$

$$= \frac{2}{n(n+1)} \sum_{r=1}^{n} r^3$$

$$= \frac{2}{n(n+1)} x \frac{1}{4} n^2 (n+1)$$

$$= \frac{1}{2} n(n+1)$$

$$\text{Variance} = \sum_{r=1}^{n} r^2 p(X = r) - \left[\sum_{r=1}^{n} rp(X = r) \right]^2$$

$$= \tfrac{1}{2} n(n+1) - \frac{1}{9}(2n+1)^2$$

Which after some manipulation gives variance = $\frac{1}{18}(n-1)(n+1)$

When n = 12, λ = 1/78. If x_M is the median then

$$p(X \le x_M) \ge \frac{1}{2} \le p(X \ge x_M$$

This gives

$$\sum_{r=1}^{k} \frac{1}{78} r \geq \frac{1}{2} \leq \sum_{r=k}^{12} \frac{1}{78} r$$

$$\frac{1}{78} x \frac{1}{2} k(1+k) \geq \frac{1}{2} \leq \frac{1}{78} x \frac{1}{2}(12-k+1)(k+12)$$

$$k(1+k) \geq 78 \leq (13-k)(k+12)$$

Giving, by inspection, the solution k = 9.

8.0 Unit Activities

(1) Find the mean and variance for the following random variables

(a) the score when a fair die is thrown,

(b) the result of picking a digit at random from the digits 0 to 9.

(2) I decide to make a telephone call from a call-box one evening and to keep trying different call-boxes until I find one which is empty. If the probability that a call-box is empty is constant and equal to 4/5, what is the probability that I have to try (a) three call-boxes, (b) more than three call-boxes?

What is the expected value for the number of call-boxes tried and the most likely number of call-boxes tried?

(3) When a marksman shoots at a target the probability that he will hit it is 9/10. Assuming that this probability is constant and that the trials are independent, calculate the mean number of shots needed to hit the target.

If the marksman has already had s shots and failed to hit the target, show that the probability that he will fail to hit the target in the nest t shots is independent of the value of s, i.e. $p(X > s+1 | X > s) = p(X > t)$, where X is the number of shots needed to hit the target. Comment on this result.

(4) The discrete variable X has only integer values and takes the value x with probability P(x). Define the mean of X and the variance of X.

Given that P(x) = k|3 – x|, where k is a constant, for x = 1, 2, 3, . . ., 6, and P(x) = 0 for all other x, determine the value of k.

Calculate

(a) the mean of X,

(b) the standard deviation of X,

(c) the mean of X^2.

(5) The number of times that a certain item of electronic equipment operates before having to be discarded is a random variable X with distribution.

$$f(x) = p(X = x) = \begin{cases} k\left(\frac{1}{3}\right)^x, & x = 0,1,.... \\ 0 & elsewhere \end{cases}$$

(a) Show that k = 2/3.

(b) What is the probability that the number of times that the equipment will operate before having to be discarded is

(i) greater than 5, ,

(ii) an even number (regarding 0 as even)?

(c) For integers m(> 0) and n(> 0), find P(X > n), ,

(6) Show that P(X > (m + n)| X > m) = P(X > n) and comment upon this result.

(7) Each of a set of n identical fair dice has faces numbered 1 to 6. Obtain the probability that, when every die is rolled once, all the scores are less than or equal to 5.

Obtain the probability Pr that the highest score in a single throw of all n dice has value r for r = 1, 2, . ., 6.

Determine the mean of r and show that for n = 3 its value is 4(23/24).

(8) Buses arrive punctually at the local bus-stop every fifteen minutes. If I leave my house and walk to the bus-stop without bothering to see whether a bus is due what are the mean and variance of the time which I have to wait for a bus?

Unit Summary

$p(X = x) = ((1-p)^{x-1}p$, for x > 0

$$E(X) = \frac{1}{p}$$

UNIT 9 BERNOULLI DISTRIBUTION

9.1 Introduction

Welcome to this unit called Bernoulli distribution. Bernoulli distribution deals with

Experiments whose outcome which can be classified in one of two mutually exclusive and exhaustive ways, say 'Success' or Failure (e.g Head or Tail, life or death' defective or non-defective). A sequence of Bernoulli trials occurs when a Bernoulli experiment is performed several independent times so that the probability of success, say p remains the same from trial to trial. That is, in such a sequence you let p denote the probability of Success on each trial. Also you use $q = 1 - P$ denotes the probability of failure: that is you can use $q = 1 - P$ Inter changeably.

9.2 Unit Aims:

A Bernoulli Experiment involves repeated independent trials of an experiment with 2 outcomes usually called "success" and "failure".

9.3 Unit Objectives:

- Calculate the expected values of a random variable,
- Write the properties of E[X], which is a linear operator. Variance, Var[X]:
- compute the expected values for the functions of random variables
- Calculate the Probability Mass Function (PMF) for Bernoulli, and its expected value and variance.

Terminology

- $\sum$ -Summation
- p – Probability
- x – Random variable
- N-population size
- n – Sample size

9.4 Time Required: You should spend 6 hours on this unit.

9.5 Unit Topics

9.5.1 Bernoulli Distribution

Suppose you perform an experiment with two possible outcomes: either success or failure. Success happens with probability p , while failure happens with probability

1-p. A random variable that takes value 1 in case of success and in case of failure 0 is called a Bernoulli random variable (alternatively, it is said to have a Bernoulli distribution).

Definition Bernoulli random variables are characterized as follows.

Definition Let X be a discrete random variable. Let its support be

$R_X = \{0,1\}$

Let $p \in (0,1)$. You can say that has a **Bernoulli distribution** with parameter if its probability mass function is

$$p_X(x) = \begin{cases} p \quad if \;\; x = 1 \\ 1 - p \;\; if \;\; x = 0 \\ 0 \;\; otherwise \end{cases}$$

A random variable having a Bernoulli distribution is also called a Bernoulli random variable.

Note that, by the above definition, any indicator function is a Bernoulli random variable.

Let X be a random variable associated with a Bernoulli trial by defining it as follow:

X (success) = 1 and X (failure) = 0

The two outcomes, s and f are denoted by 1 and 0 respectively. The pdf of X is written as

$$f(x) = p^x (1-p)^{1-x}, \text{ x = 0 , 1} \qquad (9.1)$$

You can say X has a Bernoulli distribution

The following is a proof that $p_X(x)$ is a legitimate probability mass function.

Proof. Non-negativity is obvious. We need to prove that the sum of over its support equals 1. This is proved as follows:

$$\sum_{x \in R_X} p_X(x) = p_X(1) + p_X(0)$$
$$= p + (1-p)$$
$$= 1$$

Expected value

The expected value of X is

$$\mu = E(X) = \sum_{x=0}^{1} x p^x (1-p)^{1-x} \qquad (9.2)$$

$$= (0)(1-p) + (1)p = p$$

The expected value of a Bernoulli random variable X is: $E(X) = p$

Proof It can be derived as follows:

$$E(X) = \sum_{all\, x} x p_X(x)$$
$$= 1.p_X(1) + 0.p_X(0)$$
$$= 1.p + .0.(1-p)$$
$$= p$$

Variance

The variance of a Bernoulli random variable X is $Var(X) = p(1-p)$

And the variance of X is

$$\sigma^2 = \mathrm{var}(X) = \sum_{x=0}^{1} (1-p)^2 p^x (1-p)^{1-x} \qquad (9.3)$$

$$= p^2(1-p) + (1-p)^2 p = p(1-p)$$

Proof. It can be derived thanks to the usual variance formula $Var(X) = E[X^2] = -[E(X)]^2$;

$$\begin{aligned} E\left[X^2\right] &= \sum_{x \in R_X} x^2 p_X(x) \\ &= 1^2 \cdot p_X(1) + 0^2 \cdot p_X(0) \\ &= 1 \cdot p + 0 \cdot (1-p) \\ &= p \end{aligned}$$

$$E[X]^2 = p^2$$

$$\operatorname{Var}[X] = E\left[X^2\right] - E[X]^2 = p - p^2 = p(1-p)$$

9.5.2 Moment generating function

The mgf of X is $M(t) = E(e^{tX}) = \sum e^{tx} p^x (1-p)^{1-x}$ (9.4)

$$= (1)(1-p) + pe^t$$

In a sequence of n Bernoulli trials, you let X ; denotes the Bernoulli random variable associated with the ith trial. An observed sequence of n Bernoulli trials has an n – tuple of Zeros and ones.

The moment generating function of a Bernoulli random variable X is defined for any $t \in R$:

$$M_X(t) = 1 - p + p\exp(t)$$

Proof using the definition of moment generating function, you get;

$$\begin{aligned} M_X(t) &= E[\exp(tX)] \\ &= \sum_{x \in R_X} \exp(tx) p_X(x) \\ &= \exp(t \cdot 1) \cdot p_X(1) + \exp(t \cdot 0) \cdot p_X(0) \\ &= \exp(t) \cdot p + 1 \cdot (1-p) \\ &= 1 - p + p\exp(t) \end{aligned}$$

Obviously, the above expected value exists for any $t \in R$.

Distribution function

The distribution function of a Bernoulli random variable X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Proof

Remember the definition of distribution function:

$F_X(x) = p(X \le x)$, and the fact that X can take either value 0 or value 1 . If $x < 0$, then $p(X \le x) = 0$ because X cannot take values strictly smaller than 0. If $0 \le x < 1$, then $p(X \le x) = 1 - p$ because 0 is the only value strictly smaller than 1 that X can take. Finally, if $x \ge 1$, then $p(X \le x) = 1$ because all values X can take are smaller than or equal to 1 .

Example 1 Let X and Y be two independent Bernoulli random variables with parameter p . Derive the probability mass function of their sum Z = X + Y

Solution The probability mass function of X is

$$p_X(x) = \begin{cases} p & if \;\; x = 1 \\ 1-p & if \;\; x = 0 \\ 0 & otherwise \end{cases}$$

The probability mass function of y is

$$p_Y(y) = \begin{cases} p & if \;\; y = 1 \\ 1-p & if \;\; y = 0 \\ 0 & otherwise \end{cases}$$

The support of Z (the set of values Z can take) is $R_Y = \{0,1,2\}$

The convolution formula for the probability mass function of a sum of two independent variables is $p_Z(z) = \sum_{y \in R_X} p_X(z-y) p_Y(y)$

Where R_Y is the support of Y. When z = 0, the formula gives

$$
\begin{aligned}
p_Z(0) &= \sum_{y \in R_Y} p_X(-y)\,p_Y(y) \\
&= p_X(-0)\,p_Y(0) + p_X(-1)\,p_Y(1) \\
&= (1-p)(1-p) + 0.p = (1-p)^2
\end{aligned}
$$

When z = 1, the formula gives

$$
\begin{aligned}
p_Z(1) &= \sum_{y \in R_Y} p_X(1-y)\,p_Y(y) \\
&= p_X(1-0)\,p_Y(0) + p_X(1-1)\,p_Y(1) \\
&= p \cdot (1-p) + (1-p) \cdot p = 2p(1-p)
\end{aligned}
$$

When z = 2, the formula gives

$$
\begin{aligned}
p_Z(2) &= \sum_{y \in R_Y} p_X(2-y)\,p_Y(y) \\
&= p_X(2-0)\,p_Y(0) + p_X(2-1)\,p_Y(1) \\
&= 0 \cdot (1-p) + p \cdot p = p^2
\end{aligned}
$$

Therefore, the probability mass function Z of is

$$
p_Z(z) = \begin{cases} (1-p)^2 & \text{if } z = 0 \\ 2p(1-p) & \text{if } z = 1 \\ p^2 & \text{if } z = 2 \\ 0 & \text{otherwise} \end{cases}
$$

Example 2 You flip a coin 10 independent times with heads being considered a success on each trial. This constitute a sequence of 10 Bernoulli trials with p = $\frac{1}{2}$

Example 3 A Box contains 10 red and 20 white balls. Draw five balls at random from the box, one at a time with replacement. Let the draw of a red ball be a success. If the trials are independent, you have five Bernoulli trials with P = $\frac{1}{3}$

Example 4 A fair die is cast four independent times. Let a Six be a success, otherwise the outcome is a failure. A possible observed sequence is (0, 0, 1, 0), in which case a six would have been rolled on the third trial and a non-six on each of the other three trials.

Since the trials are independent, the probability of the outcome are $(\frac{5}{6})(\frac{5}{6})(\frac{1}{6})(\frac{5}{6})$

In this experiment the interest is in the total number of success and not in the order of their occurrence. Let the random variable be Y equal the number of observe success is a Bernoulli trials, the possible values of Y are 0, 1, 2 …n. If y success occur, when y = 0, 1, 2 …n, then n – y failure occur.

The number of ways of selecting y positi0n for the y success in the n trials is

$$\binom{n}{y} = \frac{n!}{y!(n-y)!}$$

Since the trials are in depend and since the probability of success and f on each trial are respectively p and q = 1 - p , The probability of each of these ways is $p^x 1-p)^{n-y}$. The pdf of y, say fly) is the sum of the pro of these $\binom{n}{y}$ mutually exclusive events, that is:

$$f(y) = \binom{n}{y} p^y (1-p)^{n-y}, \text{ for y = 0,1,2, …,n}$$

These probabilities are called Binomial distribution of random variables Y is said to have a binomial distribution

Example 5 The probability of rolling two sixes and three non-sixes in five independent tosses of a die is

$$f(2) = \binom{5}{2}\left(\frac{1}{2}\right)^3\left(\frac{5}{6}\right)^3$$

The random variable of Y that denotes the number of 6s is the n = 5 tosses is $B(5, \frac{1}{6})$

Example 6 Toss a fair die 10 independent time. The probability of $observing y sixes and 10 – y non-sixes is.

$$f(y) = \binom{10}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{10-y}, \text{ y = 0, 1, …,n}$$

Which is the pdf of the number of Y of sixes in n = 10 independent trials of this die.

Example 7 The probability of observing exactly six heads when an unbiased coin is tossed 10 independent trials is

$$\binom{10}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4 = p(x \le 6) - p(x \le 5)$$

$$= 0.828 - 0.6230$$
$$= 0.2051$$

The probability of observing at least six heads

$$\sum_{1}^{10}\binom{10}{y}\left(\frac{1}{2}\right)^y\left(\frac{1}{2}\right)^{10-y} = 1 - p(X \le 5)$$

$$= 1 - 0.6230$$
$$= 0.3770$$

Example 8 Let y b (B15, 0.7). Though the probability P(Y = 11, 12, 13) is clearly equal to P(Z = 2, 3, 4) where Z is equal to the number of failure in n= 15 trials and hence b(15, 0.3). Hence

P(y = 11, 12, 13) $= p(z = 2,3,4)$

$$= p(z \le 4) - p(z \le 1)$$
$$= 0.5153 - 0.0353$$
$$= 0.4802$$

To find the mean and variance of Y, Using moment generating function you have

$$M(t) = E(e^{tY}) = \sum_{y=0}^{n} e^{ty}\binom{n}{y}p^y(1-p)^{n-y}$$

$$= \sum_{y=0}^{n}\binom{n}{y}(pe^t)(1-p)^{n-y}$$

Using Binomial Expansion with b = pe^t and q = 1 – p you get

$$M(t) = [(1-p) + pe^t]^n, \quad M'(t) = n[(1-p) + pe^t]^{n-1}(pe^t)$$

$$M''(t) = n(n-1)\left[(1-p)+pe^t\right]^{n-2}(pe^t)^2 + n\left[(1-p)+pe^t+pe^t\right]^{n-1}\left(pe^t\right)$$

Therefore, $\mu = M'(0) = np$

and $\sigma^2 = M''(0) - \mu^2 = n(n-1)p^2 + np - (np)^2$

$$= np(1-p)$$
$$= npq$$

9.0 Unit activities

1. Akira is completing a quiz that consists of 8 multiple choice questions and that has a pass mark of 5. Each question has 6 possible answers, only one of which is correct. Akira by nature is too lazy to study for her quiz and randomly guesses the answers to the questions. With reference to the information given, answer the following parts:

a) Find out the probability of achieving full marks by Akira

b) Find out the probability that Akira gets all her answers wrong

c) Find out the probability that Akira passes the quiz

2. Lubasi is playing a game which is like tossing a bunch of dice into the air. It is considered a win if exactly 4 of the dice land on four. Referring to this information answer the following parts:

a) Find the probability of Lubasi's win if he tosses *n* number of dice. $p(X = k) = C_k^n p^k (1-p)^{n-k}$.

b) Find the fewest number of dice he must toss to ensure that the probability that he wins is more than 15%

3. The probability of a team winning a football match is 0.75. It played three numbers of matches. What is the probability of winning zero matches? Also, find out the probability that the team shall win at least one of the three matches played?

4. In an exam, Paul had to attempt 10 multiple choice questions. The number of choices given per question is 4. One out of the four options was correct. Find out the probability that Paul gets 5 out of 10 questions correct in an answer sheet

5. A coin is flipped eight times. Find out the probability of getting head exactly four times

6. A committee consists of 9 members. What is the probability of having more

female members than male members provided the probability of having a male or a female member is equal

Unit Summary

The Bernoulli distribution is a discrete distribution having two possible outcomes labelled by $n = 0$and $n = 1$in which $n = 1$("success") occurs with probability $p$and $n = 0$ ("failure") occurs with probability $q \equiv 1 - p$, where $0 < p < 1$. It therefore has probability density function

$$P(n) = \begin{cases} 1 - p & \text{for } n = 0 \\ p & \text{for } n = 1, \end{cases}$$ Which can also be written $P(n) = p^n (1 - p)^{1-n}$.

The corresponding distribution function is

$$D(n) = \begin{cases} 1 - p & \text{for } n = 0 \\ 1 & \text{for } n = 1. \end{cases}$$

The characteristic function is

$\phi(t) = 1 + p\left(e^{it} - 1\right)$, and the moment-generating function is $M(t) = (e^{tn})$

$$= \sum e^{tn} p^n (1 - p)^{1-n}$$

$$= e^0 (1 - p) + e^t p$$

So

$$M(t) = (1 - p) + e^t p$$
$$M'(t) = pe^t$$
$$M''(t) = pe^t$$
$$M^{(n)} = pe^t$$

The mean and variance are then $\mu = p$ and $\sigma^2 = p(1 - p)$

UNIT 10 NEGATIVE BINOMIAL DISTRIBUTION

10.1 Introduction

Welcome to this unit called negative binomial distribution. The negative distribution is a discrete random variable in which the number *X* of repeated trials to produce *r* successes in a negative binomial experiment. The probability distribution of a negative binomial random variable is called a **negative binomial distribution**. The negative binomial distribution is also known as the **Pascal distribution**. Suppose you flip a coin repeatedly and count the number of heads (successes). If you continue flipping the coin until it has landed 2 times on heads, you are conducting a negative binomial experiment. The negative binomial random variable is the number of coin flips required to achieve 2 heads. In this example, the number of coin flips is a random variable that can take on any integer value between 2 and plus infinity. The **negative binomial probability** refers to the probability that a negative binomial experiment results in r - 1 successes after trial x - 1 and r successes after trial x. Given x, r, and P, you can compute the negative binomial probability.

10.2 Unit Aims:

Understand the relationship between Geometric and Negative Binomial, and the difference between Negative Binomial and Binomial. Learn the PMF and how to calculate the probability for Negative Binomial

10.3 Unit Objectives:

- explore the key properties, such as the moment-generating function, mean and variance, of a negative binomial random variable
- Calculate probabilities for a negative binomial random variable.
- Apply the methods learned in the lesson to new problems.

Terminology

The following notation is helpful, when we talk about negative binomial probability.

- x: The number of trials required to produce r successes in a negative binomial experiment.
- r: The number of successes in the negative binomial experiment.
- P: The probability of success on an individual trial.
- Q: The probability of failure on an individual trial. (This is equal to 1 - P.)
- b*(x; r, P): Negative binomial probability - the probability that an x-trial negative binomial experiment results in the *rth* success on the *xth* trial, when the probability of success on an individual trial is P.

- ${}_{n}C_{r}$: The number of combinations of *n* things, taken *r* at a time.
- mgf- moment generating function
- pdf – probability density function

10.4 Time Required: You should spend 6 hours on this unit.

10.5 Unit Topics

10.5.1 Negative Binomial distribution

A **negative binomial random variable** is the number *X* of repeated trials to produce *r* successes in a negative binomial experiment. The probability distribution of a negative binomial random variable is called a **negative binomial distribution**. The negative binomial distribution is also known as the **Pascal distribution**.

Suppose we flip a coin repeatedly and count the number of heads (successes). If we continue flipping the coin until it has landed 2 times on heads, we are conducting a negative binomial experiment. The negative binomial random variable is the number of coin flips required to achieve 2 heads. In this example, the number of coin flips is a random variable that can take on any integer value between 2 and plus infinity. The negative binomial probability distribution for this example is presented below.

Number of coin flips	Probability
2	0.25
3	0.25
4	0.1875
5	0.125
6	0.078125
7 or more	0.109375

Negative Binomial Probability

The **negative binomial probability** refers to the probability that a negative binomial experiment results in *r* - 1 successes after trial *x* - 1 and *r* successes after trial *x*. For example, in the above table, we see that the negative binomial probability of getting the second head on the sixth flip of the coin is 0.078125.

Given *x*, *r*, and *P*, we can compute the negative binomial probability based on the following formula:

$$f(x) = b^*(x; r, p) = C_{r-1}^{x-1} p^r (1-p)^{x-r}$$

Negative Binomial Formula. Suppose a negative binomial experiment consists of *x* trials and results in *r* successes. If the probability of success on an individual trial is *P*, then the negative binomial probability is:

$$f(x) = b^*(x; r, p) = C_{r-1}^{x-1} p^r (1-p)^{x-r}$$

10.5.2 The Mean of the Negative Binomial Distribution

If you define the mean of the negative binomial distribution as the average number of trials required to produce r successes, then the mean is equal to:

$\mu = r / P$ where μ is the mean number of trials, r is the number of successes, and P is the probability of a success on any given trial.

Suppose that the sequences of Bernoulli trials is observed until exactly r successes occur, where r is a fixed positive integer. Let the random variable Y denotes the number of failures before the occurrence of that r^{th} success. The y+r denotes the number of trials required to produce exactly r successes and y failures with the rth success occurring at the $(y+r)^{th}$ trials. By the multiplication rule of probability, the probability density function of y, say g(y), equals the product of the probability

$$\binom{y+r-1}{r-1} p^{r-1} (1-p)^y$$

Of obtaining exactly r-1 successes in the first y+r-1 trials and the probability of a success on the (y+r)th trial. Thus

$$g(y) = \binom{y+r-1}{r-1} p^r (1-p)^y, \text{ y} = 0,1,2,\ldots.$$

Then Y is said to have a negative binomial distribution.

10.5.3 Negative Binomial Experiment

A negative binomial experiment is a statistical experiment that has the following properties:

- The experiment consists of *x* repeated trials.
- Each trial can result in just two possible outcomes. We call one of these outcomes a success and the other, a failure.
- The probability of success, denoted by *P*, is the same on every trial.
- The trials are independent; that is, the outcome on one trial does not affect the outcome on other trials.
- The experiment continues until *r* successes are observed, where *r* is specified in advance.

Consider the following statistical experiment. You flip a coin repeatedly and count the number of times the coin lands on heads. You continue flipping the coin until it has landed 5 times on heads. This is a negative binomial experiment because:

- The experiment consists of repeated trials. We flip a coin repeatedly until it has landed 5 times on heads.
- Each trial can result in just two possible outcomes - heads or tails.
- The probability of success is constant - 0.5 on every trial.
- The trials are independent; that is, getting heads on one trial does not affect whether we get heads on other trials.
- The experiment continues until a fixed number of successes have occurred; in this case, 5 heads.

Example 1 A fair die is tossed on successive independent trials until the second six is observed. The probability of observing exactly ten non-sixes before the second six is tossed is

$$\binom{10+2-1}{1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{10} = 0.049$$

Example 2 Bob is a high school basketball player. He is a 70% free throw shooter. That means his probability of making a free throw is 0.70. During the season, what is the probability that Bob makes his third free throw on his fifth shot?

Solution*:* This is an example of a negative binomial experiment. The probability of success (*P*) is 0.70, the number of trials (*x*) is 5, and the number of successes (*r*) is 3.

To solve this problem, we enter these values into the negative binomial formula.

$f(x) = b*(x;r,p) = C_{r-1}^{x-1} p^r (1-p)^{x-r}$
b*(*5*; *3*, *0.7*) = ${}_4C_2 * 0.7^3 * 0.3^2$
b*(*5*; *3*, *0.7*) = 6 * 0.343 * 0.09 = 0.18522

Thus, the probability that Bob will make his third successful free throw on his fifth shot is 0.18522.

Example 3 Let's reconsider the above problem from Example 2. This time, you will ask a slightly different question: What is the probability that Bob makes his first free throw on his fifth shot?

Solution: This is an example of a geometric distribution, which is a special case of a negative binomial distribution. Therefore, this problem can be solved using the negative binomial formula or the geometric formula. We demonstrate each approach below, beginning with the negative binomial formula.

The probability of success (*P*) is 0.70, the number of trials (*x*) is 5, and the number of successes (*r*) is 1. We enter these values into the negative binomial formula.

b*(*x*; *r*, *P*) = ${}_{x-1}C_{r-1} * P^r * Q^{x-r}$
b*(*5*; *1*, *0.7*) = ${}_4C_0 * 0.7^1 * 0.3^4$
b*(*5*; *3*, *0.7*) = 0.00567

It can be shown that the sum of all probability for the negative binomial distribution is equal to 1.

$$\sum_{y=1}^{\infty} g(y) = 1$$

Using the McLaurin series, you have

$$(1-a)^{-n} = \sum_{k=0}^{\infty} (-1) \frac{(-n)(-n-1)......(-n-k+1)}{k!} a^k \qquad (10.1)$$

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} a^k \ , \ |a| < 1 \qquad (10.2)$$

Using (10.2) you have

$$\sum_{y=0}^{\infty} g(y) = \sum_{y=0}^{\infty} \binom{y+r-1}{r-1} p^r (1-p)^y$$

$$= p^r \sum_{y=0}^{\infty} \binom{r+y-1}{r-1} (1-p)^y$$

$$= p^r (1-[1-p])^{-r} = 1$$

The mgf of Y is given by:

$$M(t) = \sum_{y=0}^{\infty} e^{ty} \binom{y+r-1}{r-1} p^r (1-p)^y$$

$$= p^r \sum_{y=0}^{\infty} \binom{y+r-1}{y} \left[(1-p)e^t\right]^r$$

Expression (10.2) with $a = (1-p)e^t$ gives

$M(t) = p^r \left[1-(1-p)e^t\right]^{-r}$, for $t < \ln(1-p)$

Hence, you can find the expected value and variance as:

$$\mu = M'(0) = \frac{r(1-p)}{p} = \frac{rq}{p}$$

10.0 Unity Activities

1. Robert is a football player. His success rate of goal hitting is 70%. What is the probability that Robert hits his third goal on his fifth attempt?
2. You draw cards from a deck (with replacement) until you get four aces. What is the chance that you will draw exactly 20 times?
3. Each year the Akron Aardvarks have a 10%chance of winning the trophy in chinchilla grooming. Their trophy case has space for five trophies. Let Y be the number of years until their case is full. Find the mean and standard deviation of Y
4. You roll a die until you get four sixes (not necessarily consecutive). What is the mean and standard deviation of the number of rolls you will make?
5. 10% of applicants for a job possess the right skills. A company has three positions

to fill, and they interview applicants one at a time until they fill all three positions. A. What is the probability that they will interview exactly ten applicants? What is the probability that they will interview at least ten applicants?

A. Exactly ten applicants?

B. At least ten applicants?

What is the probability that they will find two or fewer out of the first nine? Use the binomial distribution

6. The company from exercise 5 takes three hours to interview an unqualified applicant and five hours to interview a qualified applicant. Calculate he mean and standard deviation of the time to conduct all the interviews.

Unit Summary

The **negative binomial probability** refers to the probability that a negative binomial experiment results in *r* - 1 successes after trial *x* - 1 and *r* successes after trial *x*. For example, in the above table, we see that the negative binomial probability of getting the second head on the sixth flip of the coin is 0.078125.

Given *x*, *r*, and *P*, we can compute the negative binomial probability based on the following formula:

Negative Binomial Formula. Suppose a negative binomial experiment consists of *x* trials and results in *r* successes. If the probability of success on an individual trial is *P*, then the negative binomial probability is:

$$g(y) = \binom{y+r-1}{r-1} p^r (1-p)^y$$

The Mean of the Negative Binomial Distribution If you define the mean of the negative binomial distribution as the average number of trials required to produce r successes, then the mean is equal to: μ = r / P, where μ is the mean number of trials, r is the number of successes, and P is the probability of a success on any given trial.

Module 1 Summary

In this module you discussed the following topics:

- Techniques of counting which covered methods of counting such as: permutations, combinations and tree diagrams. These techniques are very important in the study of probability theory and probability depends on counting techniques.
- Probability theory covers ordinary definition of probability, defines discrete random variables, probability distributions of distributions of discrete distributions such as: binomial distribution, poisson distributions, geometric distributions, hyper geometric distributions, Bernoulli distributions and negative binomial distributions. Calculations of the expected values and variances is discussed in the module.
- A special device called moment generating function which simplifies calculation of expected value and variance using derivatives is discussed. Each unit has an activity which students are expected to do and answers at the end of the module are given.

PRESCRIBED READINGS

Ross S., (2005) A First Course in Probability –7th Edition. Prentice Hall. ISBN: 0 137 46314 6

Recommended Readings

1. Bain L.J., and Engelhardt M., (2000) Introduction to Probability and Mathematical Statistics. Duxbury Press. ISBN: 0 534 98563 7
2. Jane Miller, (1992). Statistics for advanced level 2nd edition. Cambridge University press.

Appendix 1 Answers to Unit activities

1.0 Unit activities

(1) 22100
(2) 3.6288x106, 45
(3) (a)(i) 504 (ii)360 (iii) 165 (iv)165

(4) 215760
(5) (a) 24, (b) 12
(6) (a) 40320(b) 1152
(7) (a). (i)?240 (ii) 480
(b) 210
(8) (a) 22 (b) 42
(9) (a).64 (b) 18 (c)21/32
(10) (a) 48 (b) (i) 5, (ii) 9 (iii) 1/2n(n-3) (c) (i) 63 (ii) 32
(11) 151200

1.4 Unit activities

2. 263

3 4368,

4.(a) 120 (b) 40

(ciii) 40 c(i) 80), c(ii) 20

5. (a) 216 (b) 72 (c) 72 (d) 144 (e) 36

6. 64

7. (i) 45 (ii) 21 (iii) 35

8. 11

9. 10

10. (i) 330 (ii) 150 (iii) 20

11. 11

12. 10

13. (i) 330 (ii) 150 (iii) t 20

12. 11

13. 10

2.0 Unit activities

(1) (a) ½ (b) ½ (c) 1/13 (d) ¼ (e) 1/3

(2) (a) 2/3, (b) $\sqrt{\frac{3}{2}}$, (c) ¾

2.1 Unit activities 1

1. (a) 1/6 (b) 1/6 (c) 7/16

2. (a) ¼ (b0 5/8 (c) 7/16

3. 3/8

4. 4/9, 1/3

2.1 Unit activities 2

1. (a) Yes , no (b) no, no. (c) no, no. (d) yes, yes.

2. Yes, since C and D are mutually exclusive. (a) 0.9 (b) 0.2

3. (a) 2/5, (b) 3/10(c) 3/5, (d) 1/10

5. (a) 0.5 (b) 0.1 (c) 0.3 (d) 0.2

2.2 Unit activities

1. (a) 3/4 , (b)1/2 (c) 1/2

2. (a) 4/17 (b) 13/51

3. 14/45

4. (a) 62/131 (b) 69/131 (c) 42/131 (d) 43/131 (e) 29/69 (f) 23/42 (g) 22/62 (h) 39/62 (i) 48/69

2.3 Unit activities

1. (a) (i) 1/6 (ii) 1/2 (iii) 2/3. (b) (i) yes. (ii) no

2. 17/32

3. (a) 0.000005 (b) 0.999995

4. (a) 0.686 (b) 0.938 (c) 0.731

5. (a) (i) 0.3, 0.3 .(ii) 3/7, 0.3 (b) 3/11

6. 0.341

2.3 Unit activities

1. 1/4

2. 0.051

3 (a) 0.222 (b) 0.0703 (c) 0.112 (d) 0.180

4 (a) 1/455, (b) 2/91, (c) 12/91

5. 0.112

6. 4.1x10^-6

7. (a) (i)1/4, (ii) ,1/4 (iii) 1/17 (iv) 15/34

8. 13/30

9. (i) 0.375(ii) 0.563

10. 0.294

11. (i) $(\frac{5}{9})$ (ii) $(\frac{5}{9})$ (iii)(1/2)

12. (i) (a) 1/4 (b) 1/4 (ii) (a) 1/33 (b) 16/33 (c) 16/33 (iii) (a) 1/10395 (b) 16/31

13. 45/173

14.(a) 9/20 (b) (i) 1/100, (ii) 7/400

15. (a) 3/8 (b) 1/2 (c) 1/6

16. 7/24

3.0 Unit activities

1. (a) 1.2, 1.76, (b) 3.2, 25.4

2. 35n

3. 60n

1. 4/5

2. 2.208

3. $\frac{1}{4}(x+1)$ x = -1 ; 50n

   (a) 1/2 (b) 1/4 (c) 1/8 (d) 1/8 , 15n

4. $10(p-2-3p^2)$, $\frac{1}{3}$

5. E(X) =49/12

6. 2.75

4.0 Unit activities

1. $M(t)=\frac{c}{2-t}$ where c = **2**

2.

3. $E(X)=\frac{1}{p}$ and $Var(X)=\frac{a-p}{p^2}$

4. 1.56

5. $[\frac{1}{5}\pi x^5]_0^1 = \frac{1}{5}\pi$, $[g(y)=\frac{1}{2\pi}\sqrt{\left(\frac{y}{\pi}\right)}]$

6. (a) $\frac{4}{3a^2}$

7. (a) 3/250; 687.5Hrs (b) (i) 320n (ii) 0.792

8. (a) 1/8 (b) 5/16 , 21/16 (c) g(y)=2 [17/16,25/16]; g(y)=0 elsewhere

9. (a): E(Y)=4, (b) $E(Y^2) = 28$

10. $\int_0^{\infty} e^{tx} \; e^{-x} dx$

12. $E(X^3) = 10{,}560$.

5.0 Unit activities

(1) (a) (i) Yes , fixed number 5 of independent trials , p 1/2 (ii) Number of trials not fixed (iii) No , trials not independent (iv) No, fixes number (3) of effectively independent trial, p=1/2 (v) No, p is not constant.

(b)

(i)

x(number of heads)	0	1	2	3	4	5
p(X=x)	1/32	5/32	10/32	10/32	5/32	1/32

(ii)

x	0	1	2	3	4
p(X=x)	625/1296	590/1296	150/1296	20/1296	1/1296

(2) (a)

X	0	1	2	3	4
p(X=x)	81/256	108/256	54/256	12/256	1/256

(b) 1 (c) 1, (d) 3/4

(3)

R(number of girls)	0	1	2	3	4	5	6
P(r)	0.020	0.109	0.253	0.311	0.215	0.079	0.012

(d) 3 (b) 0.311 (c) 0.382

(4) (a) 0.34 (b) 0.264

(5) 2 ; 0.3222

(6) (a) 0.901 (b) 11

(7) (a) 0.258 (b) 0.579

(8)(a) (i) 0.167 (ii) 0.0162 (b) 0.98^n, 228

5.3 Unit activities

1. P(1) = 0.387 ; 1
2. (a) 0.323 (b) 0.32; 1.49
3. 11/27, no since trial not independent

4. 0.087

5. (a) 4/3 , $2\sqrt{\frac{2}{3}}$ (b) 32, 64, 48, 16, 2; 1 ,

(a) 0.341, 0.54,0.099, second claim valid (b) 0.98; 14%

6. 1; 0.98 (a) 5 (b) 0.2 .

7. (a) 47/128 (b) (i) $(\frac{7}{8})^5$ (ii) $(\frac{7}{8})^5$ (iii) $(\frac{3}{4})^5$; 0.2115

8. (a) 0.226 (b) 0.0730 Prob distribution is binomial p = 1/2 , n = 6

9. (a) 0.246 (b) 0.246 (c) 0.410

10. 0.275; 0.11

11. 16.5, 42.4, 45.4, 25.9, 8.3, 1.4, 0.1

6.0 Unit activities

1. 0.0498, 0.1494, 0.2240, 0.2240, 0.1680, 0.1008, 0.0504, 0.9502

2. (a) 0.0183 (b) 0.0733 (c) 0.00268 (d) 0.01073

3. 0.67; 0.513m

4. Mean 3 errors; 194 pages

5. Mean = 1.5 ; 223.1, 334.7, 251.0 , 125.5, 47.1, 14.1, 3.5, 1.0

6. (a) 0.135 (b) 0.271 (c) 0.0290 (d) 2 € 8.19

6.2 Unit activities

1. (a) 0.368 (b) 0.184 (c) 0.0190 ; 0.677

2. (a) 0.449 (b) 0.32 (c) 0.0119

3. Mean = 2.917, Var= 2.860; (a) 0.788, (b) 0.00293

4. (a) (i) 0.5 (ii) 121.3, 60.7, 15.2, 0.32, 0.003 (b) 0.629

5.

Observ	35	38	20	6	1
Expected	36.8	36.8	18.4	6.1	1.9

6 (a) 0.135 (b) 0.017 (c) 0.095

7 (a) (i) 1.04 (ii) 53.0, 55.1, 28.7, 9.9, 2.6 (b) 0.121

8. (a) $2e^{-2}$ (b) 0.1353, 0.2707, 0.2707, 0.3233; 1.782 (c)7.69

9. (a) (i) 1.04 (ii) 53.0, 55.1, 28.7, 9.9, 2.6 (b) 0.121

10 (a) $2e^{-2}$ A (b) 0.1353, 0.2707, 0.2707, 0.3233; 1.782 (c) 7.69

11. (a) (i) 0.0952 (ii) 0.0905 (b) 0.632; 6302 , 4.82

12. (a) 0.195 (b) 0.762

7.0 Unit activities

1. 0.8
2. 0.000055406780
3. 0.0002207
4. 0.15433; 15.433%.
5. 6.348%.
6. 0.0935.
7. 0.0877.
8. 0.111476736.
9. 0.05839920044

8.0 Unit activities

1. (a) 3.5, 35/12 (b) 4.5, 8.25
2. 1/125, 1.25, 1
3. 10/9. The probability of succeeding does not depend on how many failures there have already been.
4. (a) k = 1/9 (b) 4, (c) 20
5. (b) (i) $\left(\frac{1}{3}\right)^6$ (ii) 3/4 , (c) (i) $\left(\frac{1}{3}\right)^n$ (ii) The probability of a component lasting for n more times or more does not depend on the number of times it has already been used.
6. 23/24; $\left(\frac{5}{6}\right)^n$; $\left(\frac{1}{6}\right)^n$, $\left(\frac{r}{6}\right)^n - \left(\frac{r-1}{6}\right)^n$, for r = 2,3,…,6; $\left(1-\sum_{1}^{6}\left(\frac{r}{6}\right)^n\right)$
7. 7.5 min; 18.5 min

9.0 Unit activities

1. (a) 1/679616 (b) 390625/1679616 (c) $P(X\geq5)$ = 8*C*5× (1/6)5 × (5/6)3 + 8*C*6 × (1/6)6 × (5/6)2 + 8*C*7 × (1/6)7 × (5/6)1 + 8*C*8 × (1/6)8 × (5/6)0
2. (a) *nC*4×(16)4 × (56)*n*−4 (b) *nC*4 ×(1/6)4 × (5/6)*n*−4 > 0.15, n = 10
3. 0.00659; 0.99341
4. 0.0583992
5. 0.2734
6. 0.5

10.0 Unit activities

1. 0.18522

2. $\binom{19}{3}\left(\frac{12^{16}}{13^{20}}\right)$

3. Mean 50years; Var = 450 years

4. Mean 24 roll; Var = 120

5. $36\left(\frac{9^{7}}{10^{10}}\right)$; 97.7%

6. Mean = 96 hours; Var = 2430

Appendix 2 MAT 2901 Introduction to probability Syllabus

Rationale

Probability reasoning at the bottom is only common sense reduced to calculations. The basic probability concepts are important for better understanding of higher probability and statistical concepts. This course empowers students with the necessary skills and techniques in probability.

Objectives

At the end of the course, students should be able to: -

1. Find probabilities involving different kinds of events.
2. Describe different types of random variables and their distributions.
3. Find expectations of functions or random variables for both discrete and continuous random variables.
4. Find marginal and conditional distributions given joint probability functions.

Pre-requisites: MAT 1100 – Foundation Mathematics or MAT 1110 – Foundation Mathematics and Statistics for Social Science or A' Level Mathematics

Course Content

1. Introduction

Definitions and axioms of probability. Sample space and events. Independent events, conditional probability and Bayes theorem. Counting techniques, permutations and combinations.

2. Random variables and probability distributions

Definition of random variables and probability distribution, discrete and continuous. Expectations of random variables, moment generating function. Discrete random variables; Bernoulli, Binomial, Poisson, Geometric, Negative Binomial and Hyper geometric.

Mode of delivery: 3 lectures and 1 tutorial per week.

Prescribed Readings

1. Ross S., (2005) A First Course in Probability –7th Edition. Prentice Hall. ISBN: 0 137 46314 6

Recommended Readings

1. Bain L.J., and Engelhard M., (2000) Introduction to Probability and Mathematical Statistics. Duxbury Press. ISBN: 0 534 98563 7