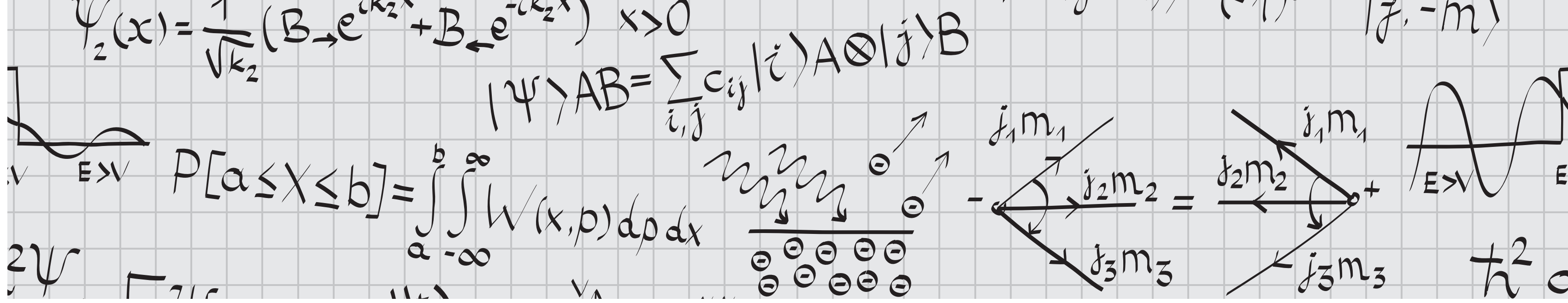


3 Optics and Waves

The Optics and Waves part of the GRE weighs in at 9% of the test, and contains a mix of some very basic material and some rather advanced material. Optics is a part of any standard freshman physics course, while waves appear in all areas of physics, and their treatment can vary greatly in difficulty and sophistication. For the purposes of the GRE, "optics" refers to geometric optics (lenses, mirrors, and so on), while "waves" refers to properties such as interference and diffraction as well as some more advanced topics such as Rayleigh scattering. We'll first discuss general properties of waves, including behavior such as diffraction and interference that can occur with any type of general wave, then we'll go over specific examples involving light waves, finishing with geometric optics. Many of the equations required for solving optics problems arise from fairly technical calculations that are outside the scope of the exam. It is therefore worth memorizing the key equations in this chapter and knowing the situations where they can be applied.

3.1 Properties of Waves

3.1.1 Wave Equation

Roughly speaking, a wave is a disturbance that propagates in time. More precisely, a wave (in one dimension) is any solution to the *wave equation*,

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}. \tag{3.1}$$

It turns out that, for *any* function $f(x)$, the related functions $f(x \pm vt)$ solve the wave equation. We interpret these as disturbances of fixed shape, given by the function $f(x)$, which propagate either to the left or to the right with constant speed v. Now, the crucial property of the wave equation is its linearity in the function f, which leads to the *principle of superposition*: for any two solutions $f(x,t)$ and $g(x,t)$ that solve the wave equation, the function $f+g$ *also* solves the wave equation. This makes analyzing wave behavior quite easy, since we can always break up any complicated wave profile into a sum of simpler pieces.

The waves described above, $f(x \pm vt)$, are known as *traveling waves*: just as $f(x-a)$ represents the graph of $f(x)$ translated to the right by a units, so does $f(x-vt)$ represent the shape $f(x)$ translated to the right by $\Delta x = vt$ units after time t. In other words, the wave travels to the right. If, instead, a solution to the wave equation looks like $f(x,t) = A(x)B(t)$, we say that it represents a *standing wave*. Indeed, there is no longer any translation in time, and the shape at $t = 0$, given by $A(x)$ (up to a constant factor $B(0)$), is modulated in time by the function $B(t)$. See Example 3.1.

3.1.2 Nomenclature and Complex Notation

Often, instead of being given the wave equation (and the associated constant v, which represents the speed of the wave), we are simply given the functional form of the wave: for example,

$$f(x,t) = A\cos(kx - \omega t + \delta). \tag{3.2}$$

It turns out that all solutions to the wave equation can be written as sums of functions of this form, with different values of the constants A, k, ω, and δ; so without loss of generality, whenever we say "wave" we will often mean (3.2). Let's now give the constants names:

A: amplitude

k: wavenumber

https://doi.org/10.1017/9781108296977.006

> **EXAMPLE 3.1**
>
> Consider the standing wave
>
> $$f(x,t) = \cos x \cos vt$$
>
> for various values of t. At $t = 0$, the wave has a shape given by $\cos x$, but at time $t = \pi/2v$, it disappears entirely since the second term $\cos vt$ vanishes. At $t = \pi/v$, it now has the shape $-\cos x$, and so on. But the essential point is that, while the shape oscillates in time, it doesn't *go* anywhere, hence the term "standing wave." However, using the product-to-sum formula for cosines, we can write
>
> $$f(x,t) = \frac{1}{2}\left(\cos(x+vt) + \cos(x-vt)\right),$$
>
> so in fact the standing wave is secretly the sum of a left-moving wave and a right-moving wave, with equal amplitudes. In fact, we can decompose *any* solution of the wave equation into the sum of a left-moving and a right-moving wave, possibly of different shapes and different amplitudes.

ω: angular frequency

δ: phase

The amplitude A is the maximum value of the function $f(x,t)$, and represents the "size" of the wave. It is a general fact that the energy carried by waves is proportional to the *square* of the amplitude, A^2. In the very common case of light waves, the energy (up to some constants) is called the *intensity*.

The wavenumber k is related to the period of $\cos x$ as a function of x: since cosine has period 2π, $\cos kx$ has period $2\pi/k$. In other words, the wave starts repeating after a distance $\lambda = 2\pi/k$, called the *wavelength*. Up to a factor of 2π, the wavenumber counts the number of wave crests contained within a distance x, but the more useful quantity is often the wavelength. Similarly, ω is related to the period T by $T = 2\pi/\omega$, and to the (ordinary, not angular) frequency[1] f by $\omega = 2\pi f$. Summarizing these definitions,

$$\lambda = \frac{2\pi}{k}, \qquad T = \frac{2\pi}{\omega}, \qquad \omega = 2\pi f. \tag{3.3}$$

Finally, the phase represents the "offset" of the shape of the cosine function from the usual one centered at $x = 0$. You've likely seen all these names before, since, with the exception of wavenumber, the notation and nomenclature are identical to those of the classical simple harmonic oscillator. These same concepts also show up in quantum mechanics, with k related to momentum by the de Broglie relation $p = \hbar k$ and frequency related to energy by the Einstein relation $E = \hbar\omega$.

The above information is conveniently represented in terms of complex numbers: we can write

$$f(x,t) = \mathrm{Re}(Ae^{i(kx-\omega t)}),$$

where the amplitude A is now allowed to be complex, $A = |A|e^{i\delta}$. This allows the phase to be absorbed in the amplitude, and by working with complex numbers through the whole calculation and taking the real part at the end, we can avoid using some of the more annoying trig identities.[2]

The generalization to waves traveling in three dimensions is quite simple. We just replace kx with $\mathbf{k}\cdot\mathbf{r}$, and call $\mathbf{k}$ the *wavevector*. Its magnitude $|\mathbf{k}|$ is the wavenumber, and its direction is the direction in which the wave propagates. Such a wave solves the wave equation in three dimensions in Cartesian coordinates, and is called a *plane wave*. This is in contrast to a spherical wave, whose oscillations take the form $e^{i(kr-\omega t)}$, where r is the distance from the origin in spherical coordinates. Note the distinction from the case of plane waves: a spherical wave has a wavevector that is always parallel to the position vector $\mathbf{r}$, which is *not* constant.

[1] In many physics texts, the frequency is denoted by ν, the Greek letter "nu." But this looks too much like v, which we've used for wave velocity, so to avoid any notational confusion we'll use f for frequency wherever possible, which is the standard convention in engineering. Also, be careful not to confuse f and ω! The first quantity f is the frequency in of units of Hz or cycles per second. The second quantity is the rate at which the argument of the sine or cosine functions change, in units of radians per second. When working with waves, it's easy to forget the factors of 2π that connect these quantities.

[2] Caveat: expressing quantities that are *products* of waves in complex notation requires tweaking some definitions, since the real part of a product of complex numbers is not the product of the real parts. See the discussion of the Poynting vector in Section 2.6.1. However, complex notation is entirely straightforward for linear superpositions of waves. You probably won't need complex notation for the GRE, but it's good to be aware of it in case it helps you solve a problem faster.

https://doi.org/10.1017/9781108296977.006

3.1.3 Dispersion Relations

For waves satisfying the wave equation (3.1), the relation

$$\omega = vk \tag{3.4}$$

holds: you can check this by simply plugging (3.2) into (3.1). If we modify the wave equation, for example to incorporate phenomena such as frictional dissipation or changes in the density of the medium, we lose this property, but we can still specify the behavior of the wave by a relation between ω and k, known as a *dispersion relation*. The special case $\omega(k) = vk$ is known as linear dispersion, and is the one obeyed by light in vacuum, with $v = c$ the speed of light. You may have seen it written in the more familiar form $\lambda f = c$. From the dispersion relation, we can calculate two quantities with the dimensions of velocity:

$$\text{Phase velocity: } \frac{\omega}{k}, \tag{3.5}$$

$$\text{Group velocity: } \frac{d\omega}{dk}. \tag{3.6}$$

For linear dispersion, these quantities turn out to be the same, but this is a special case.

In general, phase velocity measures the velocity of an individual wave crest, while group velocity measures the velocity with which a whole bunch of waves (for example, a wave packet of various wavelengths centered around a certain wavelength λ_0) move together. For light waves traveling in a medium (for example, radio waves carrying an FM signal or light signals in a fiberoptic pipe), the group velocity is the speed at which information is transmitted. In fact, there are cases where the phase velocity can be *greater* than the group velocity! If the phase velocity turns out to be greater than the speed of light c, you shouldn't be worried, since information is only transmitted at the group velocity and there is no conflict with special relativity. The fact that the phase velocity can in general be a function of k implies that waves of different wavelengths travel at different speeds; in other words, a wave packet is *dispersed*. This is exactly what happens when white light is passed through a prism.

3.1.4 Examples of Waves

We can see some of these general considerations in action by looking at a few common instances of waves. The amplitude y of waves on a uniform vibrating string, for example, is described by the equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2},$$

where μ is the mass density per unit length and T is the tension in the string. This is exactly the same as equation (3.1), with the replacement

$$v = \sqrt{\frac{T}{\mu}} \tag{3.7}$$

for the velocity of the wave. Memorizing the wave equations for every possible type of wave would obviously be a bit of a nuisance. Thankfully this is unnecessary, since you should usually be able to reconstruct velocities such as equation (3.7) from other physical constants by dimensional analysis. This expression also makes physical sense: increasing the tension in the string makes the restoring force on a displaced segment of string greater, and so the oscillation will occur more rapidly. Similarly, increasing the mass density of the string means a displaced segment of string will accelerate more slowly, reducing the speed of propagation.

Another example is sound waves. By analogy with the string, we might suspect that the parameters that would determine the speed of sound waves in air would be some measure of inertia, such as the density ρ (units of kg/m^3), and some measure of "stiffness," such as the bulk modulus K (units of pressure, Pa or N/m^2). The only combination of these factors with the units of speed is

$$c_s = \sqrt{\frac{K}{\rho}}.$$

Using numbers appropriate for air at STP, this is approximately 340 m/s, as expected.

3.1.5 Index of Refraction

The optical properties of a medium can be roughly summarized by a single number n called the *index of refraction*, which is the factor by which the speed of light is reduced in that medium. In the language of dispersion relations

$$\omega/k = c/n \quad \text{(for light waves).} \tag{3.8}$$

Consider a beam of light with frequency ω and wavenumber k in vacuum, incident on a medium with index of refraction n. Inside the medium, the speed of light changes to c/n, so clearly either ω or k (or both) must change to satisfy (3.8). An analysis of Maxwell's equations at the boundary of the medium shows that the *frequency* of the light in the medium does *not* change. Rather, the wavelength λ is modified to

$$\lambda \to \frac{\lambda}{n} \tag{3.9}$$

https://doi.org/10.1017/9781108296977.006

so that (3.8) holds. By the way, real materials, such as glass, do *not* have a constant index of refraction; rather, $n = n(k)$ is a function of wavenumber, hence dispersion of white light. But in nearly every application on the GRE you can consider n to be a constant that represents the reduction of the speed of light in that medium, or, equivalently, the factor by which the wavelength is reduced.

3.1.6 Polarization

So far we have written waves as real-valued (or sometimes complex-valued) functions. But electromagnetic waves are *vector*-valued because $\mathbf{E}$ and $\mathbf{B}$ are both vectors. The only new ingredient in the mathematical description of the wave is the polarization unit vector $\hat{\mathbf{n}}$. For example, we can write the electric field component of an EM wave as

$$\mathbf{E}(x,t) = E_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)\hat{\mathbf{n}},$$

so while the amplitude is E_0, the vectorial information is contained in $\hat{\mathbf{n}}$. If a source of light emits EM waves with constant $\hat{\mathbf{n}}$, we say the light is *polarized* in the direction $\hat{\mathbf{n}}$, as we have already discussed in Section 2.6.1. If the oscillating vector is polarized perpendicular to the wavevector (as is the case for EM waves in vacuum), the wave is called *transverse*; if the vector is parallel to the wavevector (as can occur for EM waves in waveguides), the wave is *longitudinal*.

There are two very common types of applications of polarization that show up on the GRE:

- **Malus's law.** Suppose we have a device, called a polarization filter or polarizer, for which all light exiting the device is polarized in a certain direction $\hat{\mathbf{n}}_0$ (Fig. 3.1). Then for incident light of intensity I_0, which is polarized at an angle θ with respect to $\hat{\mathbf{n}}_0$ (i.e. $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0 = \cos\theta$), the intensity I of the transmitted light is given by Malus's law:

$$I = I_0 \cos^2\theta. \tag{3.10}$$

This implies some curious properties of polarizers. For example, if we have an arrangement of two polarizers oriented at 90 degrees with respect to one another, the intensity of transmitted light will be *zero*, independent of the initial polarization: light will hit the first filter, emerge polarized with some reduced intensity, then be promptly absorbed by the second filter, since $\cos(90°) = 0$. However, if we place a third filter in between the first two, oriented at 45 degrees with respect to the filters on either side, the emitted intensity of incident light polarized parallel to the first filter is $I_0\cos^4(45°) = I_0/4$, where I_0 is the intensity of light emerging from the first filter. So in this case, placing an extra filter *increases* the intensity of transmitted light, despite the usual intuition that a filter removes light rather than augments it. Another important application is when *unpolarized* light is shined on a polarizer: the transmitted intensity is then the average of $I_0\cos^2\theta$ over all angles θ, which works out to be $I_0/2$. So unpolarized light incident on a polarizer of arbitrary orientation comes out parallel to the polarizer axis, with intensity reduced by half.

- **Brewster's angle.** Suppose we have two media with indices of refraction n_1 and n_2, joined at an interface. Some formidable manipulation of Maxwell's equations implies that there exists an angle θ_B at which incident unpolarized light going from n_1 to n_2 will reflect off the interface and emerge *completely polarized* perpendicular to the plane formed by the incident ray and the normal to the surface (the incident plane) (Fig. 3.2). θ_B is known as Brewster's angle, and is given by

$$\theta_B = \arctan\left(\frac{n_2}{n_1}\right). \tag{3.11}$$

By the same reasoning, light polarized *parallel* to the incident plane incident at θ_B will not be reflected at all. The polarization properties of reflected light are what make Polaroid sunglasses useful: even if the incident angle is not exactly θ_B, the reflected light will still be mostly polarized in one direction, so sunglasses whose polarization filters are perpendicular to this direction will block most of the reflected light, reducing glare off a road or off water. Indeed, picturing this scenario helps to make concrete the rather abstract and confusing term "incident plane." If the surface is a flat road, the normal will be a vertical line pointing towards the sky, and if the light beam is coming right

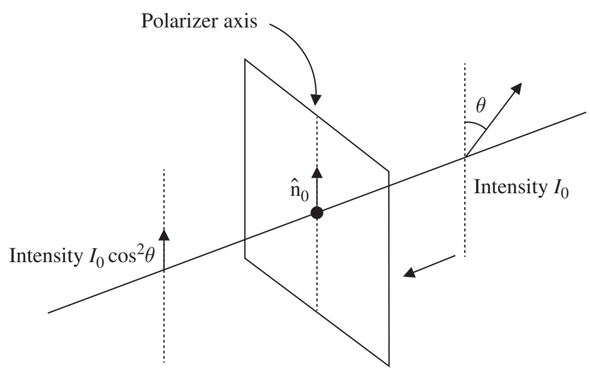


Figure 3.1 Polarized light of intensity I_0 is incident on a polarizer. If the light is polarized at an angle θ with respect to the polarizer axis $\hat{\mathbf{n}}_0$, then the light emerging from the polarizer will have an intensity $I_0\cos^2\theta$ according to Malus's law, and it will be polarized in the direction $\hat{\mathbf{n}}_0$.

https://doi.org/10.1017/9781108296977.006

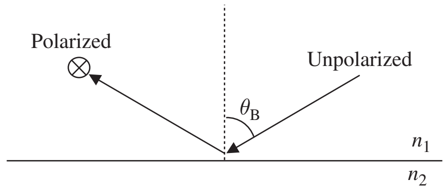


Figure 3.2 Unpolarized light travels in a medium with index n_1 and reflects off an interface with a medium of index n_2. If the angle of incidence (relative to normal incidence) is Brewster's angle θ_B, then the reflected light will emerge linearly polarized in a direction perpendicular to the page.

at you, the polarization of the reflected beam is perfectly horizontal.

3.2 Interference and Diffraction

One of the more striking consequences of the superposition principle for waves is *interference*. Consider two waves:

$$f(x,t) = A\cos(kx - \omega t),$$
$$g(x,t) = A\cos(kx - \omega t + \pi).$$

Both f and g are perfectly acceptable solutions to the wave equation, differing only by their phase δ, but their sum, $f + g$, is identically zero! The phase of g is $\delta = \pi$, so at every (x,t), $g(x,t) = -f(x,t)$, and the two waves cancel identically at every spatial point and time. This is known as *destructive* interference, and occurs whenever the phase difference is an odd multiple of π. Adding 2π to the phase changes nothing because 2π is the period of cosine, so the same destructive interference argument works for π, 3π, etc. Similarly, if the phase difference were $\delta = 0$, the two waves would be identical and would simply add to give a wave of amplitude $2A$; this is known as *constructive* interference. Once again, this argument also holds for phase differences of 2π, 4π, etc. To summarize:

$$\text{Constructive interference} \iff \text{phase difference of } 2m\pi, \tag{3.12}$$

$$\text{Destructive interference} \iff \text{phase difference of } (2m+1)\pi, \tag{3.13}$$

where m can be any integer. The classic examples of interference are when a phase difference arises from a difference in path length, or from traveling between media with different indices of refraction. The former is exemplified by double-slit interference and single-slit diffraction, while the latter arises in situations involving thin films.

3.2.1 Double-Slit Interference

The setup for Young's classic double-slit experiment is a point source of monochromatic light (wavelength λ), shining on a barrier with two narrow slits cut in it a distance d apart, with a screen at a distance L behind the slits (Fig. 3.3).

The waves from the two slits interfere because, at a given point on the screen, light from the two slits will have traveled a different distance; let's call that path difference Δx. This leads to a phase shift

$$\delta = k\Delta x. \tag{3.14}$$

To see why, note that one wave will arrive at the screen with functional form $A\cos(kx - \omega t)$, but the other wave will arrive as $A\cos(k(x+\Delta x) - \omega t) = A\cos(kx - \omega t + k\Delta x)$, where the last term in the cosine can be interpreted as a phase. For constructive interference, we need $\delta = 2m\pi$, so $\Delta x = 2m\pi/k = m\lambda$, and for destructive interference, we need $\delta = (2m+1)\pi$, so $\Delta x = (2m+1)\pi/k = (m+1/2)\lambda$. In words,

Constructive interference occurs when the path difference is an integral multiple of the wavelength, and destructive interference occurs when the path difference is a half-integral multiple of the wavelength.

On the screen, we will see bright bands at the locations of constructive interference, and dark bands at the locations of destructive interference. The bright bands and the dark bands are called interference maxima and minima, respectively.

If the screen distance L is much larger than the distance between the slits d, we can find the positions of the interference maxima and minima using a little geometry. When $L \gg d$, the paths of the two light rays to a fixed point on the screen are very nearly parallel, so the path difference to a point at an angle θ from the center is shown from Fig. 3.4 to be $d\sin\theta$.[3]

The maxima and minima then satisfy:

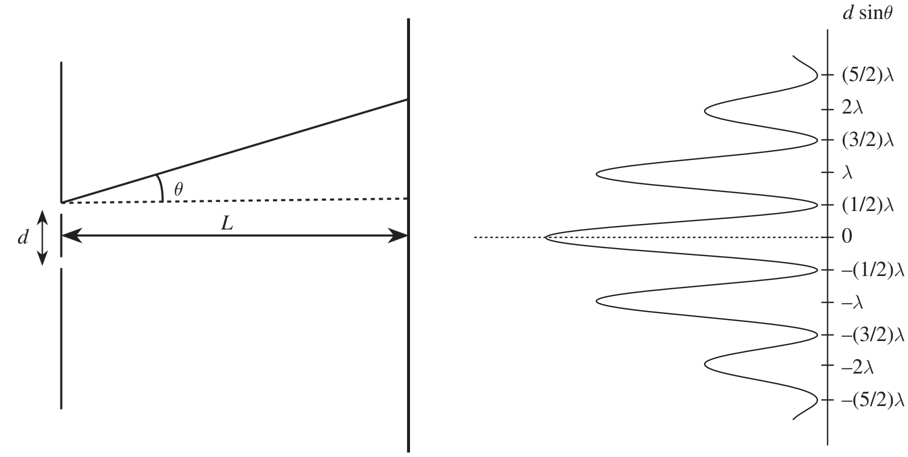
$$\text{Maxima: } d\sin\theta = m\lambda, \tag{3.15}$$

$$\text{Minima: } d\sin\theta = (m+1/2)\lambda. \tag{3.16}$$

Here, m is any integer, positive, negative or zero. Conventions differ, but the problem statement will always be unambiguous with regards to counting maxima and minima. Rather than having to draw the diagram over and over, it's best to just memorize these relations. An easy way to remember which

[3] In Fig. 3.3, θ is measured from the top slit, but in the limit $L \gg d$, θ is the same whether it is measured from the top slit, the bottom slit, or between the two slits. This is *always* the approximation that will be used in double-slit interference, and is known as the Fraunhofer or far-field regime.

https://doi.org/10.1017/9781108296977.006



(a) Geometry of double-slit interference (b) Double-slit interference pattern

Figure 3.3 The interference pattern produced behind a double slit has a maximum when $d \sin\theta$ is an integer multiple of the wavelength λ and a minimum when $d \sin\theta$ is a half-integer multiple of the wavelength.

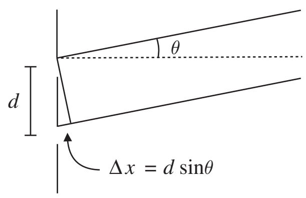


Figure 3.4 Double-slit interference arises because of the difference in path lengths $d \sin\theta$ between light passing through each slit. The path-length differences produce interference maxima and minima as a function of the angle θ relative to the horizontal.

formula applies to maxima is to note that $\theta = 0$ must be a maximum since both waves travel the same distance: this means the right-hand side must be an integer rather than a half-integer, giving (3.15).

3.2.2 Single-Slit Diffraction

We now consider a modification of the above setup, replacing the two thin slits with a single slit of width a (Fig. 3.5).

Once again, light that passes through different parts of the slit travels different lengths on the way to the screen, but the derivation of the path difference involves a lot of handwaving and is totally useless for the GRE. Instead we will just write down the answer: interference *minima* occur when

$$a \sin\theta = m\lambda, \qquad m = 1, 2, \ldots \tag{3.17}$$

Fortunately this is easy to remember because of the similarity with the double-slit formula, but don't confuse the two equations! The *minima* for single-slit diffraction occur at nearly the same condition as the *maxima* for double-slit diffraction. There is no simple formula for the position of the diffraction maxima for the single slit, although there is clearly a maximum right in the center of the slit; hence we have explicitly specified that $m = 0$ is *not* a minimum. The first minima mark the width of this central maximum. We also see from the diffraction equation why we need $a \sim \lambda$ to see any diffraction effects at all. If $a < \lambda$, then the equation (3.17) has no solution for θ: there are no minima, and the central maximum fills the entire screen. On the other hand, if $a \gg \lambda$, the diffraction minima are so closely spaced that they blur together.

3.2.3 Optical Path Length

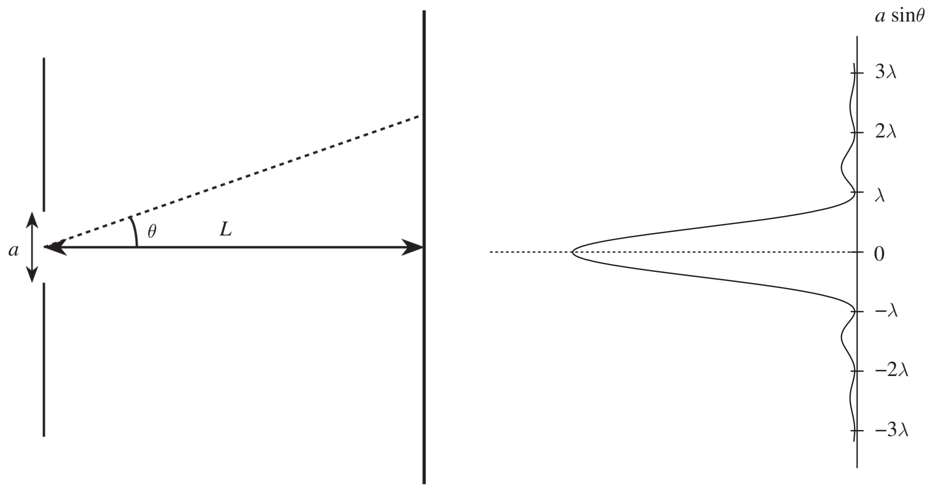
Consider the following situation: a monochromatic beam of light of wavelength λ in a vacuum passes through a medium of index of refraction n, while a nearby beam misses the medium and continues traveling through the vacuum. When the two beams are recombined, will there be interference? At first glance, it seems like there won't be any interference as long as both beams travel the same distance, since there is no path difference like the $k\Delta x$ of the double-slit setup.

But take a look back at equation (3.9): in the medium with index n, the wavelength of light is reduced by a factor of n while its frequency remains the same. If the wave in vacuum travels a distance d, the total phase (i.e. the argument of the sine or cosine) of the wave increases by

$$\delta_1 = kd = \frac{2\pi d}{\lambda},$$

but the total phase of the wave in the medium increases by

$$\delta_2 = \frac{2\pi d}{\lambda/n} = \frac{2\pi nd}{\lambda}.$$



(a) Geometry of single-slit diffraction (b) Single-slit diffraction pattern

Figure 3.5 The diffraction pattern produced behind a single slit has a minimum when $a \sin\theta$ is an integer multiple of the wavelength λ.

Notice that $\delta_1 \neq \delta_2$! This means that, in general, there *will* be interference when the beams are recombined. In fact, the interference pattern will behave precisely as if the wave in the medium has traveled a distance nd while the wave in vacuum traveled a distance d. This suggests that we define the *optical path length* by

$$\Delta x = nd \qquad \text{(optical path length)}, \tag{3.18}$$

where d is the actual distance traveled in the medium. Now that we have this definition, you can feel free to forget all the explanation that came before it: just remember that when you see a situation where a wave passes through a distance d of a medium of index of refraction n, assign it a path length of nd and *not* just d. An easy way of remembering whether the factor of n goes in the numerator or denominator is to consider the limiting case of a medium with $n \to \infty$: there, the wave will be slowed down so much that, as it passes through the medium, it undergoes infinitely many cycles, sending the phase difference to infinity.

3.2.4 Thin Films and Phase Shifts

An additional source of phase shifts is reflection off a boundary between two media. If light is traveling from a medium with index of refraction n_1 toward a medium with index of refraction n_2 and reflects back off the boundary, Maxwell's equations imply

$$n_2 > n_1 : \text{phase shift of } \pi, \tag{3.19}$$

$$n_2 < n_1 : \textit{no} \text{ phase shift.} \tag{3.20}$$

To actually derive these from Maxwell's equations takes an unwieldy amount of algebra, so just memorize them. In a typical thin-film setup, one considers light traveling through air ($n \approx 1$) and striking the boundary of a thin film of soap, oil, or some other material with $n > 1$ and thickness d. The film may be surrounded by air on both sides, or may be placed on another surface with yet another index of refraction. At normal incidence (that is, when the light is shining perpendicular to the surface), light that reflects off the front boundary of the film can interfere with light that passes through the film and reflects off the back boundary. Figure 3.6 shows an example with a film of index of refraction n_2 surrounded by air ($n = 1$) on both sides. Note that the incident wave is supposed to be at normal incidence, and is only shown angled for clarity.

There are now two sources of phase shift: a geometric one, due to the difference in optical path length $2dn_2$ from traversing the thickness of the film twice (the dashed segment in the diagram), and a possible additional phase shift depending on the arrangement of indices of refraction according to rules (3.19) and (3.20). When there are additional phase shifts like

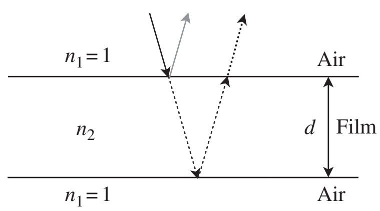


Figure 3.6 Example of reflection off a thin film. If $n_2 > 1$, the light reflecting off the first interface will experience a phase shift of π, but light reflecting off the second interface will have no phase shift. The light reflecting off each interface can interfere either constructively or destructively depending on the thickness d of the film.

this, you can compute the conditions for interference using $\delta = k\Delta x$, tack on an additional phase shift of π, and set this equal to an odd or even multiple of π as appropriate. Or, you can remember that the additional phase shift of π essentially reverses the conditions for constructive and destructive interference. Use whichever method you find conceptually and computationally simplest; the end-of-chapter problems explore the various standard cases.

3.2.5 Miscellaneous Diffraction

A few other types of diffraction scenarios appear on the GRE, mostly because they involve the application of simple formulas. Unfortunately these formulas are rather difficult to derive, and remembering how to do so is frankly a waste of time for the GRE: just memorize them, carefully noting their similarities and differences to the usual diffraction formulas.

- **Circular aperture**. A circular hole will also diffract light, but the angular positions of the diffraction minima now require solving some complicated differential equation. The answer is given numerically, as the Rayleigh criterion:

$$\text{First circular diffraction minimum: } D\sin\theta = 1.22\lambda, \tag{3.21}$$

  where D is the diameter of the hole, the circular analogue to the slit width a in single-slit diffraction. The Rayleigh criterion tells us that the angular separation of two point sources observed through a circular aperture (for example, a telescope) must be greater than $\sin^{-1}(1.22\lambda/D)$ for the two sources to be resolved as two different objects, so that the diffraction maxima of the two sources don't overlap and blur together as one big blob. Equation (3.21) is the limiting case where the first minimum of one source lies exactly at the central maximum of the other source.
- **Bragg diffraction**. This technically belongs to Specialized Topics but is conceptually closely related to the other classic diffraction problems. When x-rays[4] shine on a crystal lattice, they bounce off the atoms forming the lattice and interfere by path-length difference just as in double-slit interference. By modeling the crystal as a set of parallel planes a distance d apart, we get

$$\text{Maxima: } d\sin\theta = n\lambda/2, \tag{3.22}$$

  where θ is the angle of the incident x-rays with respect to the plane of the crystal. The only new things to remember in this formula are the factor of 2, which has the same origin as the path-length difference $2dn_2$ in thin-film interference because the light must traverse the distance between adjacent layers twice, and the fact that θ has a different interpretation than in double-slit interference.

[4] Why x-rays, as opposed to some other kind of light? Simply because the plane spacing d for most crystals of interest is comparable to the wavelength of x-rays.

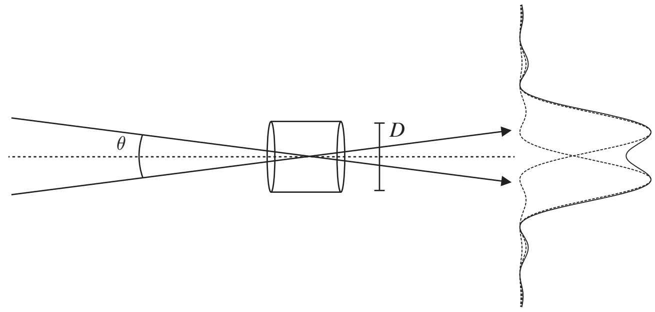


Figure 3.7 Light diffracting through a circular aperture of diameter D will exhibit an interference minimum at $D\sin\theta = 1.22\lambda$. If two objects are separated by an angle θ, this provides a condition, called the Rayleigh criterion, for whether the objects can be resolved through the aperture. At the Rayleigh criterion, the diffraction minimum of the first object coincides with the central maximum for the second object, as shown by the dashed lines on the right. The observed pattern is the sum of the two diffraction patterns, shown by the solid line on the right, which shows that two objects are just barely distinguishable.

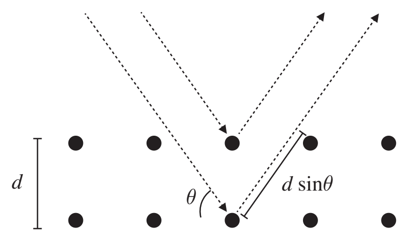


Figure 3.8 X-rays incident on a crystal with interatomic spacing d, at an angle θ relative to the horizontal will experience constructive and destructive interference, known as Bragg diffraction. The x-rays that scatter off the second layer of atoms will travel a total distance $2d\sin\theta$ longer than x-rays that scatter off the first layer. This path difference produces interference in the outgoing x-rays.

3.3 Geometric Optics

Geometric optics is a long-distance approximation to everything discussed in Section 3.2. If the dimensions of the objects in the problem are orders of magnitude larger than the wavelength of light involved, all interference and diffraction effects disappear, and light can be treated as if it travels in straight lines, just like a beam of particles.

3.3.1 Reflection and Refraction

The basic instruments of geometric optics are lenses, which bend (refract) light, and mirrors, which reflect it. The wave

https://doi.org/10.1017/9781108296977.006

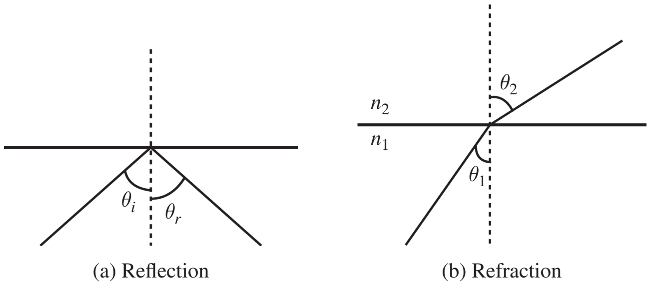


Figure 3.9 Definition of angles used in reflection and refraction problems.

properties of light imply the following laws, which you probably know already:

$$\text{Reflection: } \theta_i = \theta_r \quad \text{(angle of incidence equals angle of reflection)}, \tag{3.23}$$

$$\text{Refraction: } n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad \text{(Snell's law)}. \tag{3.24}$$

As with thin-film phase shifts, the derivations of these equations are long and complicated so the equations must be memorized. Two points are worth mentioning here:

- These laws apply to *any* wave phenomena, not just light: sound, water waves, and so on. In Snell's law, n_1 and n_2 are the indices of refraction of the two media through which the light passes. But by remembering the interpretation of n as the factor by which the speed of light is reduced in the medium, you can easily translate to the relevant equation in terms of the speed of a general wave:
$$c_2 \sin\theta_1 = c_1 \sin\theta_2. \tag{3.25}$$
- Because the equations of electromagnetism are invariant under time reversal, so are geometric optics diagrams: *light rays are reversible*. For a real-world example, if you can see someone's eyes in a mirror, they can see your eyes as well.

3.3.2 Lenses and Mirrors

Suitably interpreted, there is only one equation you need to remember to solve all geometric optics problems:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}. \tag{3.26}$$

The trick lies in remembering the meanings and sign conventions of the various terms in this equation. Starting from the right-hand side, f represents a *focal length*, the distance from the surface of the optical instrument to a special point called the focus (denoted by F in Fig. 3.10). In geometric optics, all mirrors and lenses are idealized so that all incident light rays parallel to the axis (dotted lines in Fig. 3.10) are reflected or refracted through the focus, respectively. Conversely, by reversibility all light rays passing through the focus are either reflected or refracted so that they come out parallel to the axis. The approximation that rays passing through the center of the lens (dashed lines) travel straight through without refraction, combined with some geometry involving similar triangles, leads to (3.26) for lenses.

For the idealized spherical mirrors used in geometric optics,

$$f = R/2, \tag{3.27}$$

where R is the radius of curvature of the mirror. By convention, f is positive if the center of curvature is on the same side of the mirror as the incoming light, which occurs if the mirror is concave. Convex mirrors have the center of curvature on the opposite side, and f is negative. This time, (3.26) comes from the law of reflection, since rays passing through the exact center of the mirror are reflected back at the same angle.

For lenses, f is positive for converging lenses, which have two convex surfaces, and negative for diverging lenses, with two concave surfaces. The lensmaker's equation gives the focal length of a thin lens in terms of the radii of curvature of the two surfaces:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \tag{3.28}$$

The sign conventions are very confusing here because a convex surface viewed from the left becomes a concave surface viewed from the right, so the best way to remember the signs is to imagine the simple case of a converging lens. There, the focal length is positive no matter what the radii of curvature, so R_1 is positive and R_2 is negative.

Moving on to the left-hand side of (3.26), s represents the position of an object, and s' represents the position of the image of that object formed from the single lens or mirror. Typically (3.26) is used to solve for the image position s' in terms of s and f. The sign conventions for s and s' are as follows:

$$\text{Positive distances} \iff \text{same side as light rays (incoming for } s\text{, outgoing for } s'\text{)}, \tag{3.29}$$

$$\text{Negative distances} \iff \text{opposite side as light rays (incoming for } s\text{, outgoing for } s'\text{)}. \tag{3.30}$$

Both s and s' can be positive or negative depending on the situation. The image distance s' is positive if the image is *real* (formed by the intersection of actual light rays), but negative if the image is *virtual* (formed on the opposite side of the optical

https://doi.org/10.1017/9781108296977.006

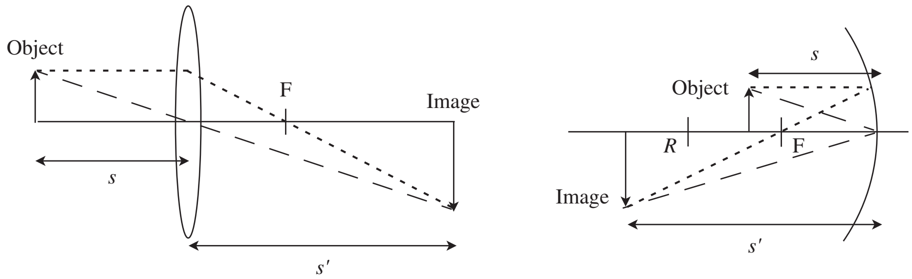


Figure 3.10 Geometry of typical lens and mirror problems, showing raytracing and definitions of common variables. The location of the image of a converging lens in (a) is given by finding the intersection of two lines: (1) draw a line from the top of the object through the center of the lens, and (2) draw a line intersecting the focal point and the point on the lens directly in front the object. The point where these lines intersects is the top of the image. The image of a spherical concave mirror in (b) is given by an analogous procedure, except that the rays reflect off the mirror rather than pass through as they did in the case of the lens. For a spherical concave mirror, the focal length is given by $f = R/2$.

instrument from the outgoing light rays). Typically s is positive, but it can be negative in a configuration of multiple lenses if an image formed by one lens acts as an object for a second lens. If the first image is on the opposite side of the second lens from the *incoming* light rays, its object distance is negative.

Once we have solved for s', we can obtain the magnification $m = -s'/s$. The absolute value $|m|$ is the ratio of the size of the image to the size of the object, and the sign determines the orientation of the image: positive if the image is upright, and negative if inverted. To solve problems involving several lenses and/or mirrors, simply solve (3.26) successively for each instrument. The image formed by the first instrument becomes the object for the second instrument, and so on, with magnifications multiplying at each step. A couple of practice problems are provided at the end of this chapter, but for tons of standard practice problems, just refer to any freshman physics textbook.

3.4 Assorted Extra Topics

3.4.1 Rayleigh Scattering

Why is the sky blue? To answer this, we have to look at how light scatters off small particles. For a given wavelength of light λ there are three regimes depending on the size of the particle a: $\lambda \ll a$, $\lambda \sim a$, or $\lambda \gg a$. The first regime is covered by geometric optics, the second by a theory called Mie scattering, and the last by Rayleigh scattering. We've already discussed geometric optics at length, and Mie scattering is quite complicated because of the coincidence of scales between the wavelength and the scatterer. For the purposes of the GRE you should be familiar with Rayleigh scattering, mostly because the formula takes a rather simple form:

$$I \propto I_0 \, \lambda^{-4} \, a^6. \tag{3.31}$$

As with most of the equations in this section, the derivation of this formula is complicated, the prefactors are irrelevant, and all that matters is the λ^{-4} dependence. (You may be asked about the a^6 dependence of the particle size, but such a question hasn't shown up on any of the recent tests.)

The most common physical application of this formula is the scattering of solar light by air and water molecules in the upper atmosphere. The light hitting the atmosphere approaches from only one direction, but is scattered in all directions, and what we observe on Earth is the scattered light. The λ^{-4} dependence means that shorter wavelengths are scattered much more strongly than longer wavelengths, and hence we observe a blue color (the reason we don't see purple has to do with the photoreceptors in our eyes and not with any underlying physics per se). On the other hand, at sunset when we look directly at the Sun, we are receiving light that has *not* been scattered away, and so we see what's left of the spectrum, which has a red color.

If you look back at Section 2.6.2, you'll notice the λ^{-4} dependence in Rayleigh scattering is the same as the ω^4 dependence from dipole radiation: in fact these are equivalent descriptions of the same process, by which light from the Sun causes the molecules to radiate as dipoles. So we can deduce that the blue light from the Sun is *polarized*, accounting for the effectiveness of Polaroid sunglasses in yet another context.

3.4.2 Doppler Effect

To conclude our review of optics and waves, we shift gears a bit from light waves to sound waves. You're undoubtedly familiar with the fact that an ambulance driving by at high speed with its siren blaring has a characteristic sound, where

https://doi.org/10.1017/9781108296977.006

the frequency appears to drop as the ambulance drives past. This is an example of the Doppler effect, which arises any time a source of waves (sound waves, in our example) is moving relative to the observer. If the source is moving towards the observer, the wave crests get squished together, resulting in an apparent increase in the frequency; if the source is moving away, the wave crests get spread out, resulting in a decrease. The precise relationship between the emitted frequency f_0 and the received frequency f is

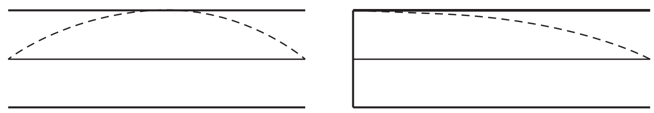
$$f = \left(\frac{v + v_r}{v - v_s}\right) f_0, \tag{3.32}$$

where v is the velocity of the waves in the medium, v_r is the velocity of the receiver relative to the medium in which the waves propagate, and v_s is the velocity of the source with respect to the medium. The sign conventions for the velocities have v_r and v_s positive if the source and receiver are approaching each other, and negative if they are moving away from each other. This equation is rather tricky to derive, and should be memorized. One shortcut to remember the signs: as v_s approaches v, the observed frequency f should blow up to infinity, accounting for the minus sign in the denominator. Also note that f goes negative if $v_s > v$: this equation simply does not apply when the source or receiver is moving faster than the wave velocity (instead, we need a theory of shock waves and sonic booms). To add a further caveat, (3.32) only applies when all velocities are small compared to the speed of light. There is a corresponding formula for the relativistic Doppler shift of light due to the motion of the source; see Section 6.4.1.

Note that (3.32) *only* applies if the receiver and source are moving *directly* toward or away from each other; otherwise, there is an additional factor of $\cos\theta$, where θ is the angle between the source velocity and the receiver's position. In fact, note that for v_r and v_s constant, f is *constant* in (3.32). The falling-frequency sound characteristic of an approaching and receding ambulance is due entirely to the varying factor of $\cos\theta$ as the ambulance drives by, since the ambulance does not directly approach the receiver but rather has some nonzero distance of closest approach.

3.4.3 Standing Sound Waves

Like any other kind of wave, sound waves involve oscillations. Unlike light waves, which involve oscillating electric and magnetic fields, sound waves involve *pressure* oscillations in a gaseous medium (usually air). Additionally, they are *longitudinal* waves, which means that the pressure oscillations take place along the same direction that the wave is traveling. Thus there is no concept of polarization for ordinary sound waves. However, all the other considerations of Sections 3.1.1 and 3.1.2 apply. One of the most common situations involves standing sound waves in an open or half-open pipe. A pipe of length L will support standing sound waves: an example is an empty bottle, which when you blow across the opening creates a definite pitch. But the longest-wavelength oscillation is not simply of length L. Indeed, for an idealized pipe, the open end must always be a pressure *node*, where the difference in average pressure compared to atmospheric pressure is *zero* because the air in the pipe can equilibrate with the air outside.[5] Similarly, a closed end is a pressure *antinode*, where the pressure difference is maximal because air is pushing against the fixed endcap of the pipe. These physical considerations serve as boundary conditions for figuring out the possible wavelengths of fully open and half-open pipes. When solving these problems, it's useful to depict the pressure waves by drawing parts of sine and cosine curves inside the pipe, as in the cartoons in Fig. 3.11, which show the longest allowed wavelengths.



(a) Open pipe (b) Half-open pipe

Figure 3.11 Snapshot at a single point in time of a *pressure* standing wave in a pipe with two ends open (a) or one end open (b). The point at the open end always must have a fixed pressure because it is open to the outside, which has constant pressure. Note that, confusingly, some references will draw similar figures to represent a snapshot of *particle displacement* in the wave. In that case, closed ends look like nodes, because the end of the pipe is fixed, and open ends look like antinodes. In either case, however, the wavelengths of the fundamental and its harmonics are the same.

The vertical axis in these cartoons represents the deviation from average pressure, so both ends of the open pipe are pressure nodes, while the left (closed) end of the half-open pipe is an antinode. We see that the open pipe contains half a wavelength, so the longest-wavelength standing wave has wavelength $2L$, while the half-open pipe contains a quarter wavelength, giving wavelength $4L$ for the lowest mode. The same method will allow you to figure out the other allowed wavelengths, a favorite GRE question.

[5] You can also think of sound waves as displacement waves, where actual chunks of air are moving back and forth, in which case the open end would be a displacement antinode rather than a node since the air is free to slosh back and forth. But in this situation it's often simpler to think in terms of pressure.

3.5 Problems: Optics and Waves

1. Sound waves in air can be described by the equation

$$\frac{\partial^2 \rho}{\partial x^2} = \kappa^2 \frac{\partial^2 \rho}{\partial t^2},$$

where ρ is the deviation from average pressure, and κ is a constant. The speed of sound is

(A) $1/\kappa^2$
(B) $1/\kappa$
(C) $\sqrt{\kappa}$
(D) κ
(E) κ^2

2. Polarized light with polarization vector $\mathbf{n} = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$ is incident on a polarizer oriented at $\mathbf{v} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}}$. The ratio of the intensity of the transmitted light to the initial intensity is

(A) $1/\sqrt{8}$
(B) $\sqrt{8/65}$
(C) $\sqrt{64/65}$
(D) 64/65
(E) 1

3. Let $f(x,t)$ and $g(x,t)$ be two traveling wave solutions to the homogeneous wave equation

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}.$$

Which of the following statements are true?

I. $f + g$ solves the wave equation.
II. fg solves the wave equation.
III. $2f - 3g$ solves the wave equation.

(A) I only
(B) II only
(C) III only
(D) I and III
(E) II and III

4. What is the correct relationship between phase velocity, v_{phase}, and group velocity, v_{group}, for a quantum-mechanical wave packet?

(A) $v_{\text{phase}} = v_{\text{group}}$
(B) $v_{\text{phase}} = 2v_{\text{group}}$
(C) $v_{\text{phase}} = \frac{1}{2}v_{\text{group}}$
(D) $v_{\text{phase}}v_{\text{group}} = c^2$
(E) none of these

5. What is the absolute value of the relative phase between two waves described by $\sin(x - vt + \pi/6)$ and $\cos(x - vt)$?

(A) 0
(B) $\pi/6$
(C) $\pi/3$
(D) $2\pi/3$
(E) π

6. A person standing in the middle of a long straight road sees a truck with its headlights on approaching in the distance. The truck's headlights are 3 m apart. Assuming the headlights are point sources emitting yellow light of wavelength 600 nm and the diameter of the human pupil is 5 mm, approximately how far is the truck from the person when he can first resolve the two headlights as separate sources?

(A) 2 m
(B) 20 m
(C) 200 m
(D) 2 km
(E) 20 km

7. Blue light of wavelength 400 nm and green light of wavelength 500 nm are incident on a slit of width 20 μm, and the light passing through the slit hits a screen 2 m away from the slit. What is the distance on the screen between the first diffraction minimum for blue light and the first minimum for green light?

(A) 1 mm
(B) 5 mm
(C) 1 cm
(D) 4 cm
(E) 5 cm

8. Monochromatic light of wavelength λ is directed at a double-slit arrangement with slit separation d. If the same light is directed at a different double-slit arrangement with slit separation d', the position of the third interference minimum corresponds to the position of the old second interference maximum after the central maximum. What is d' in terms of d and λ?

(A) $4d/5$
(B) $4\lambda^2/5d$
(C) d
(D) $5d/4$
(E) $5d^2/4\lambda$

9. A soap bubble is formed by a thin film of soap (index of refraction 1.5) surrounded on both sides by air. For soap of thickness 1 μm, which of the following wavelengths of light will exhibit constructive interference when reflecting off the bubble at normal incidence?

(A) 400 nm
(B) 500 nm
(C) 600 nm
(D) 800 nm
(E) 900 nm

https://doi.org/10.1017/9781108296977.006

10. A glass window (index of refraction 1.3) is coated with an antireflective film of thickness 2 μm. Which of the following indices of refraction of the film would cause the intensity of reflected light of wavelength 800 nm from a normally incident beam to be suppressed?

    I. 1.1
    II. 1.4
    III. 1.7

    (A) I only
    (B) II only
    (C) III only
    (D) I and II
    (E) I, II, and III

11. A person at the bottom of a swimming pool looking up at the sky observes the Sun at an angle θ from the horizon. What is the true angle of the Sun, in terms of the index of refraction n of the water? (You may assume the index of refraction of air is 1.)

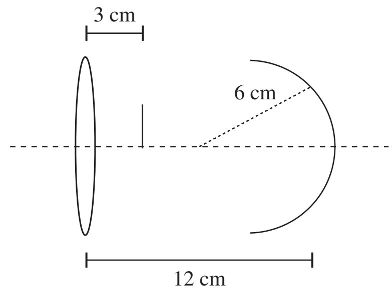
    (A) $\cos^{-1}(n \cos\theta)$
    (B) $\sin^{-1}(n \cos\theta)$
    (C) $\sin^{-1}(n \sin\theta)$
    (D) $\cos^{-1}(\cos\theta/n)$
    (E) $\sin^{-1}(\sin\theta/n)$

12. A sound wave propagates through a region filled with an ideal gas at constant temperature T. It approaches an acoustically permeable but thermally insulating membrane such that the angle between the wave and the plane of the membrane is 30 degrees. On the other side of the membrane is the same gas at a different temperature T'. What is the minimum value of T'/T such that no sound passes across the barrier? (You may find it useful to know that the speed of sound in an ideal gas is proportional to $\sqrt{T}$.)

    (A) 1/2
    (B) 3/4
    (C) 1
    (D) 4/3
    (E) 2

13. Which of the following MUST be true of the image of an object formed by a general configuration of ideal lenses and mirrors, where m denotes the absolute value of the magnification of the object?

    (A) A real image must have $m > 1$.
    (B) A virtual image must have $m > 1$.
    (C) A real image must be inverted.
    (D) A virtual image must be inverted.
    (E) None of the above.



14. A converging lens of focal length 6 cm is placed 12 cm to the left of a concave spherical mirror of radius of curvature 6 cm, as shown in the diagram. An object is placed 3 cm to the right of the lens, in between the mirror and the lens. Which of the following describes the image(s) of the object formed by the lens in this configuration?

    (A) One real image
    (B) One virtual image
    (C) Two real images
    (D) Two virtual images
    (E) One real image and one virtual image

15. A trumpeter on a horse, riding directly towards you, plays a note at 200 Hz. You, standing still, hear a note at 210 Hz. Assuming the speed of sound is 350 m/s, how fast is the horse traveling?

    (A) 8.5 m/s
    (B) 16.7 m/s
    (C) 17.5 m/s
    (D) 333.3 m/s
    (E) 367.5 m/s

3.6 Solutions: Optics and Waves

1. B – This is pure dimensional analysis. κ^2 has units of time^2/length^2, and so $1/\kappa$ has units of velocity.
2. D – Recall that given two vectors $\mathbf{v}$ and $\mathbf{w}$, the cosine of the angle between them is given by
$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|},$$
so using Malus's law,
$$\frac{I}{I_0} = \cos^2\theta = \frac{(\mathbf{n}\cdot\mathbf{v})^2}{(\mathbf{n})^2\,(\mathbf{v})^2} = \frac{(2\cdot 1 + 3\cdot 2)^2}{(2^2+3^2)(1^2+2^2)} = \frac{64}{65},$$
choice D.
3. D – I and III are true by the principle of superposition, but II is not: any linear combinations of f and g are also solutions, but products are not linear combinations.
4. C – You need the Einstein relation $E = \hbar\omega$ and the de Broglie relation $p = \hbar k$, which we discuss further in

https://doi.org/10.1017/9781108296977.006

Chapter 5. Combined with the classical relation for a particle of mass m, $E = p^2/2m$, this gives the dispersion relation $\omega = \frac{\hbar k^2}{2m}$. The phase velocity is $v_{\text{phase}} = \frac{\omega}{k} = \frac{\hbar k}{2m}$, while the group velocity is $v_{\text{group}} = \frac{d\omega}{dk} = \frac{\hbar k}{m}$, so we see that $v_{\text{phase}} = \frac{1}{2} v_{\text{group}}$, choice C.

5. C – Note that the answer is *not* choice B because there is an inherent phase difference of $\pi/2$ between sine and cosine. To figure out whether the answer should be $\pi/2 + \pi/6$ or $\pi/2 - \pi/6$, ignore the phase shift of $\pi/6$ for now and remember that cosine is shifted to the *left* from sine by $\pi/2$. So $\cos x = \sin(x + \pi/2)$, and the phase difference is $\pi/2 - \pi/6 = \pi/3$, choice C.
6. E – Although probably a little too involved for a real GRE problem, this is a classic example of the Rayleigh criterion. Let d be the distance of the truck from the person in meters. The angular separation θ of the two sources is given by $\tan(\theta/2) = 1.5/d$; since θ is likely small, we can approximate $\tan\theta/2 \approx \theta/2$ to get $\theta \approx 3/d$. Now, the distance D appearing in the Rayleigh criterion is the aperture diameter of 5 mm, so we get (again using the small-angle approximation)

$$(5\text{ mm})(3/d) = 1.22(600\text{ nm}) \implies d = \frac{15\text{ mm}}{1.22(600 \times 10^{-6}\text{ mm})}\text{ m} \approx 2 \times 10^4\text{ m} = 20\text{ km}.$$

This is choice E, and also seems physically reasonable – on a clear night and a straight road, we ordinarily have no trouble distinguishing the headlights of an approaching vehicle.
7. C – From the single-slit formula, $a \sin\theta = \lambda$ for the first minimum, so $\sin\theta_{\text{blue}} = 2 \times 10^{-2}$ and $\sin\theta_{\text{green}} = 2.5 \times 10^{-2}$. These are small enough that we are justified in using the small-angle approximation, $\sin\theta \approx \tan\theta \approx \theta$. Since $\theta = x/L$, where x is the distance on the screen and L is the distance to the screen (2 m in this case), $x_{\text{blue}} = 4$ cm and $x_{\text{green}} = 5$ cm, and the distance between them is 1 cm, which is C.
8. D – The position of the old second maximum (after the central maximum) is given by $d \sin\theta = 2\lambda$, and the new third minimum is $d' \sin\theta = 5\lambda/2$. Setting the two expressions for $\sin\theta$ equal, we get

$$\frac{2\lambda}{d} = \frac{5\lambda}{2d'} \implies d' = \frac{5}{4}d.$$

Note that this is independent of λ!
9. A – Let $n = 1.5$ be the index of refraction of the soap. At the front boundary, we have $n > n_{\text{air}}$ so there is a phase shift of π. At the back boundary, $n_{\text{air}} < n$ so there is no phase shift. The optical path length is $2nd$, where $d = 1\ \mu$m. Thus the total phase shift is $2ndk + \pi$, where k is the wavenumber, and the condition for constructive interference is that the total phase be a multiple of 2π:

$$2dn(2\pi/\lambda) + \pi = 2m\pi \implies \lambda = \frac{4dn}{2m-1}.$$

For $d = 1\ \mu$m, we can take $m = 8$ to get 400 nm, which is A. None of the other choices correspond to possible values of m.
10. D – Now we are looking for destructive interference. There is already a phase shift of π at the boundary between the air and the coating because all the given indices of refraction are greater than 1, so the only question is whether there is an additional phase shift of π at the coating–glass boundary. If the coating has index of refraction $n < 1.3$, there is an additional phase shift, so the condition for destructive interference is

$$2dn = (m - 1/2)\lambda \implies \lambda = \frac{4dn}{2m-1}.$$

If $n > 1.3$, there is no additional phase shift, so we get instead

$$2dn = m\lambda \implies \lambda = \frac{2dn}{m}.$$

Choice I must satisfy the first condition, which it does when $m = 6$. Choice II must satisfy the second condition, which it does when $m = 7$. Choice III fails the second condition (it satisfies the first condition, but that does not apply since $n > 1.3$), so the correct options are I and II, choice D.
11. A – Referring to Fig. 3.12, let $\theta = \theta_{\text{obs}}$, the observed angle of the Sun, and θ_{true} be the true angle. Since we're given angles with respect to the horizon, *not* the normal, we have to be careful applying Snell's law. If α is the angle to the horizontal, and $\beta = \pi/2 - \alpha$ is the angle to the normal, then $\sin\beta = \cos\alpha$, so we can forget about the

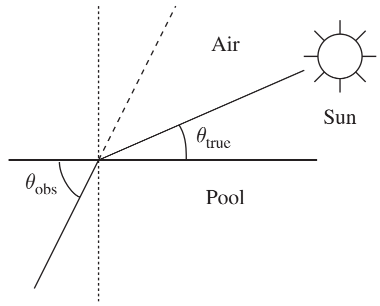


Figure 3.12 Solution for problem 11.

normal and just use cosines rather than sines. Since the index of refraction of air is 1, we have

$$\cos\theta_{\text{true}} = n\cos\theta_{\text{obs}} \implies \theta_{\text{true}} = \cos^{-1}(n\cos\theta_{\text{obs}}).$$

This matches choice A. For a quick limiting-cases analysis, notice that if $n = 1$ we should have $\theta_{\text{true}} = \theta_{\text{obs}}$, which gets rid of choice B.

12. D – This is the phenomenon of *total internal reflection*, applied to the unfamiliar context of sound waves. Total internal reflection occurs when Snell's law has no solution for θ_2, so that there is no refracted wave at all: this occurs when $\sin\theta_2 > 1$, or $n_1\sin\theta_1/n_2 > 1$. Here, θ_1 (which, remember, is the angle to the *normal*) is 60°, so $\sin\theta_1 = \sqrt{3}/2$. We'll see in the following chapter on thermodynamics that the speed of sound in an ideal gas is proportional to $\sqrt{T}$, and the "index of refraction" is proportional to the reciprocal of the wave speed, so we have the condition

$$\frac{\sqrt{T'}(\sqrt{3}/2)}{\sqrt{T}} > 1 \implies \frac{T'}{T} > \frac{4}{3},$$

choice D. Choice E is the classic mistake of taking θ_1 as the angle to the boundary, rather than the angle to the normal, while choice B results from forgetting that n_1 is the reciprocal of the speed.

13. E – This is a little tricky. If there is only *one* lens or mirror, then by $m = -s'/s$ and the sign conventions for s and s', a real image always has $m < 0$ and hence is inverted, leading us to suspect choice C. However, in a more general configuration a real image can serve as the object for another lens or mirror, and a second successive real image causes another inversion and the resulting object is upright. Counterexamples for the remaining choices are easy. For A, place the object beyond the center of curvature of a single concave mirror; for B and D, place the object anywhere outside a single convex mirror.

14. E – There are two possibilities for light rays coming from the object: they can either go to the left and pass straight through the lens, or they can go to the right, hit the mirror first, then pass through the lens. Each of these paths can potentially give rise to an image. For the first possibility, note that the object is inside the focal length of the lens, so we will get a virtual image; you can also see this from $1/s + 1/s' = 1/f$, which shows s' must be negative. This eliminates A and C. As for the second possibility, the mirror has focal length $R/2 = 3$ cm. The object is 9 cm to the left of the mirror, so the lens equation gives $s' = 4.5$ cm, a real image. But this is now outside the focal length of the lens, which will give a real image somewhere far to the left of the lens. Thus choice E is correct.

15. B – Straightforward application of the Doppler effect. Here $v_r = 0$, and we are solving for v_s:

$$\frac{f}{f_0} = \frac{210}{200} = \frac{350}{350 - v_s}$$
$$\implies v_s = 16.7 \text{ m/s}.$$