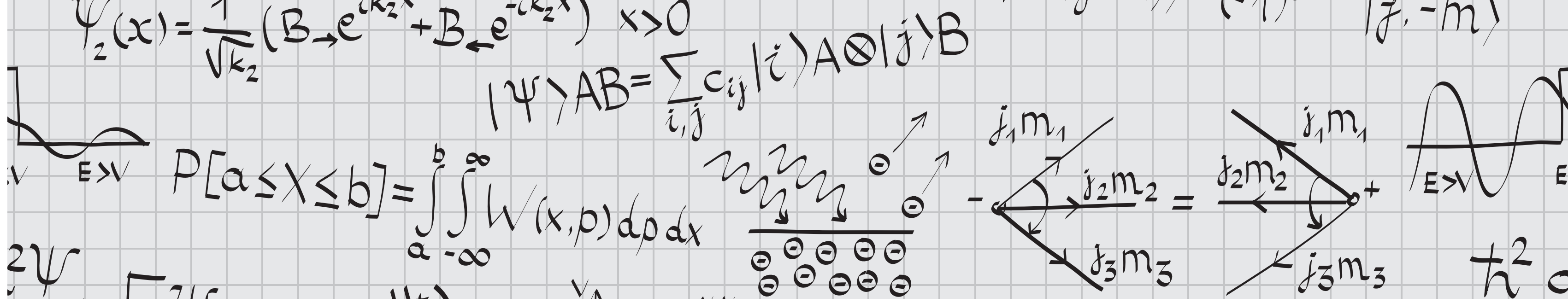

1 Classical Mechanics

Classical mechanics is the cornerstone of the GRE, making up 20% of the exam, and at the same time has the dubious distinction of being the subject that turns so many people *away* from physics. Your first physics class was undoubtedly a mechanics class, at which point you probably wondered what balls, springs, ramps, rods, and merry-go-rounds had to do in the slightest with the physics of the real world. So rather than (a) attempt the impossible task of covering your 1000-page freshman-year textbook in this much shorter reference, or (b) risk turning you away from physics before you've even taken the exam, we'll structure this chapter a little differently than the rest of the book. We're not going to review such things as Newton's laws, force balancing, or the definition of momentum; you should know these things in your sleep, or the rest of the exam will seem impossibly hard. Rather than review basic *topics*, we'll review standard *problem types* you're likely to encounter on the GRE. The more advanced topics will get their own brief treatment as well. After finishing this chapter, you will have reviewed nearly all the material you'll need for the classical mechanics section of the test, but in a format that is much more useful for the way the problems will likely be presented on the test. If you need a more detailed review of any of these topics, just open up any undergraduate physics text.

1.1 Blocks

One of the first things you learned in the first semester of freshman year physics was probably how to balance forces using free-body diagrams. Rather than rehash that discussion, which you can find in absolutely any textbook, we'll review it through a series of example problems that are GRE favorites. They involve objects, usually called "blocks," with certain masses, doing silly things like sitting on ramps, being pushed against springs, and traveling on carts. So here we go.

1.1.1 Blocks on Ramps

Here's a basic scenario (Fig. 1.1): a block of mass m is on a ramp inclined at an angle θ, and suppose we want to know the coefficient of static friction μ required to keep it in place. The usual solution method is to resolve any forces $\mathbf{F}$ into components along the ramp ($F_\parallel$) and perpendicular to the ramp ($F_\perp$). Rather than fuss with trigonometry or similar triangles, we can just do this by considering limiting cases, a theme that we'll return to throughout this book. In this case, we have to resolve the gravitational force $\mathbf{F}_g$. If the ramp is flat ($\theta = 0$), then there is no force in the direction of the ramp, so gravity acts entirely perpendicularly, and $F_{g,\parallel} = 0$. On the other hand, if the ramp is sheer vertical ($\theta = \pi/2$), then gravity acts entirely parallel to the ramp ($F_{g,\perp} = 0$), and the block falls straight down. Knowing that there must be sines and cosines involved, and the magnitude of $\mathbf{F}_g$ is mg, this uniquely fixes

$$F_{g,\parallel} = mg\sin\theta, \qquad F_{g,\perp} = mg\cos\theta.$$

For the block not to accelerate perpendicular to the ramp, we need the perpendicular forces to balance, which fixes the normal force to be $N = mg\cos\theta$. Then the frictional force is $F_f = \mu mg\cos\theta$, which must balance the component of gravity parallel to the ramp, $F_{g,\parallel} = mg\sin\theta$. Setting these equal gives

$$\mu mg\cos\theta = mg\sin\theta \implies \mu = \tan\theta.$$

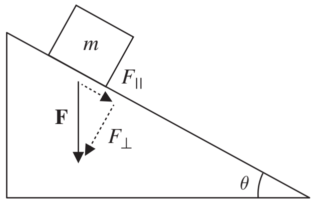

Figure 1.1 Free-body diagram of forces for a block on an inclined ramp.

Again, we can check this by limiting cases. If $\theta = 0$, then we don't need any friction to hold the block in place, and $\mu = 0$. If $\theta = \pi/2$, we need an infinite amount of friction to glue the block to the ramp and keep if from falling vertically, so $\mu = \infty$. Both of these check out.

Standard variants on this problem include applied forces and blocks attached to pulleys which hang over the side of the ramp, but surprisingly, neither the basic problem nor its variants have shown up on recent exams. Perhaps it is considered *too* standard by the GRE, such that most students will have memorized the problem and its variants so completely that it's not worth testing. In any case, consider it a simple review of how to resolve forces into components by using a limiting-cases argument, as this can potentially save you a lot of time on the exam.

1.1.2 Falling and Hanging Blocks

The next step up in complexity is to have two or more blocks interacting – for example, two blocks tied together with a rope, falling under the influence of gravity, or the same blocks hanging from a ceiling. These kinds of questions test your ability to identify precisely which forces are acting on which blocks. A foolproof, though time-consuming, method is to use *free-body diagrams*, where you draw each individual block and *only* the forces acting on it. This avoids the pitfall of double-counting, or applying the same force twice to two different objects, and ensures that you take into careful account the action/reaction balance of Newton's third law. See Example 1.1.

Sometimes, though, simple physical reasoning will suffice, especially in situations where the blocks aren't really distinct objects. For example, consider placing one block on top of another and letting them both fall under the influence of gravity. If we ignore air resistance, there is absolutely no physical distinction between the block–block system, and one larger block with the combined mass of both. In fact, a variant of precisely this argument was used in support of Galileo's discovery that the gravitational acceleration of objects was independent of their mass. We could even put a massless string between the two blocks, and the argument would *still* hold: since the whole system must fall with acceleration g, there can be no tension in the string. (Do the free-body analysis and check this yourself!) When interactions between the blocks become important, for example when they exert forces on one another through friction, then we must usually treat them as independent objects, though, as we'll see in Section 1.1.3, there are cases where the same kind of reasoning works.

EXAMPLE 1.1

A 5 kg block is tied to the bottom of a 20 kg block with a massless string. When an experimenter holds the 20 kg block stationary, the tension in the string is T_1. The experiment is repeated with the 20 kg block hanging under the 5 kg block, and the tension in the string is now T_2. What is T_2/T_1?

Our physical intuition tells us that $T_1/g = 5$ kg and $T_2/g = 20$ kg, since in both cases the function of the string is to support the weight of the lower block. So we expect $T_2/T_1 = 4$. This intuition is confirmed by a limiting-cases analysis: if the mass of the lower block is zero, then no matter the mass of the upper block, the string just dangles below the block with no tension, so the tension must be proportional to the mass of the lower block but independent of the mass of the upper one.

Let's check the intuition by doing a full free-body analysis. In order to treat both cases at once, call the mass of the top block m_1 and that of the bottom block m_2, as in Fig. 1.2. The forces on the two blocks are illustrated in Fig. 1.3. F is the force applied by the experimenter. Notice how the string tension acts up on the bottom block but down on the top block, and that the magnitude of T is the same for both blocks. For the purposes of the GRE, this is the *definition* of a massless string: it carries the same tension at every point. Setting the acceleration of m_2 equal to zero, since it is stationary, let's solve for T: $T - m_2 g = 0$, so indeed, $T = m_2 g$, the weight of the bottom block, and our intuition is correct. In this case it wasn't even necessary to consider the forces on the top block, a convenient time-saver!

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EXAMPLE 1.1 (Cont.)

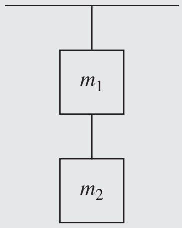


Figure 1.2 Two blocks suspended from one another by a massless string.

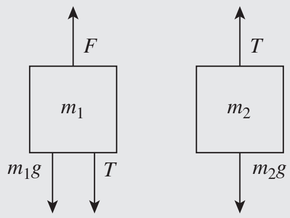


Figure 1.3 Free-body diagram for two blocks on a string.

1.1.3 Blocks in Contact

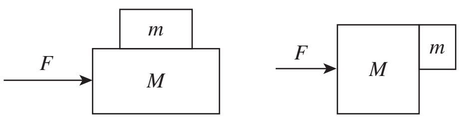
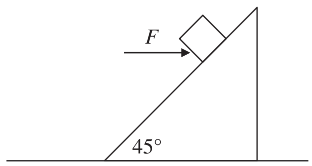


Figure 1.4 Typical setups for blocks moving together with friction.

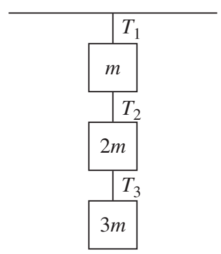
There are two standard setups for these kinds of problems, illustrated in Fig. 1.4. Both get at all the core concepts of force balancing, Newton's second and third laws, and friction. In the second setup, you might be asked, given friction between the two blocks, what the minimum force is such that the mass m does not fall down due to gravity, or, if m is placed on the surface as well, how the force of one block on another changes depending on whether F is applied to M or m. As with the falling and hanging blocks, the key is to remember that the blocks are *independent* objects, so we must consider the forces on each independently. See Example 1.2.

1.1.4 Problems: Blocks

1. A block of mass 5 kg is positioned on an inclined plane at angle 45°. A force of 10 N is applied to the block, parallel to the ground. If the coefficient of kinetic friction is 0.5, which of the following is closest to the acceleration of the block? Assume there is no static friction.



(A) $\sqrt{2}$ m/s^2 up the ramp
(B) $\sqrt{2}$ m/s^2 down the ramp
(C) $5\sqrt{2}$ m/s^2 up the ramp
(D) $5\sqrt{2}$ m/s^2 down the ramp
(E) $25\sqrt{2}$ m/s^2 down the ramp



2. Three blocks of masses m, $2m$, and $3m$ are suspended from the ceiling using ropes, as shown in the diagram. Which of the following correctly describes the tension in the three rope segments, labeled T_1, T_2, and T_3?

(A) $T_1 < T_2 < T_3$
(B) $T_1 < T_2 = T_3$

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EXAMPLE 1.2

Here's an example using the setup shown in Fig. 1.4 (left): A block of mass 2 kg sits on top a block of mass 5 kg, which is placed on a frictionless surface. A force of 10 N is applied horizontally to the 5 kg block. What is the minimum coefficient of static friction between the two blocks such that they move together without slipping?

We could do a full free-body diagram of all the forces in the problem, but simple physical reasoning provides a useful shortcut. Note that, as long as the blocks don't slip, the two blocks are really behaving as one object of mass $M + m$, just like the falling blocks attached by a massless string in Section 1.1.2 above. Thus we expect the final expression for μ to depend on the combination $M + m$, rather than M or m individually, since μ determines whether the two blocks stick together and act as a composite system.

To see this explicitly, let's analyze the motion of the top block first. The forces on the top block are its weight $-mg$, the normal force N_1 provided by the bottom block, and the frictional force $F_f = \mu N_1$. Since the top block is not accelerating vertically, we must have $N_1 = mg$ and the net force forward is $F_f = \mu mg$. Now the top block will begin to slip just as the force F_1 on it is equal to the maximum force that friction can supply; in other words, the slipping condition is $F_1 = F_f = \mu mg$. But by definition we also know that $F_1 = ma$, where a here is the acceleration of the two-block system – since both blocks are stuck together, they experience the same acceleration. The mass of the total system is $M + m$ and the applied force is F, so $F = (M + m)a$. Substituting the values for a and F_1 into $F_1 = ma$, we find

$$\mu mg = m\frac{F}{M+m} \implies \mu = \frac{F}{(M+m)g},$$

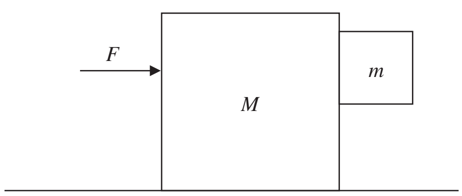
which as expected depends on $M + m$. Notice that we didn't ever have to do a free-body analysis of the second block alone: instead, we applied Newton's second law to the two-block system in the second step.

Of course, we can also do a free-body analysis for the block of mass M. We have the applied force F acting forwards, but there is also a force acting *backwards*, from Newton's third law: the bottom block is providing a frictional force which pushes the top block forwards, so the bottom block feels an equal force backwards. The net horizontal force is then $F - \mu mg$, where the second term is the magnitude of the friction force derived above. The acceleration of the bottom block is $a = \frac{1}{M}(F - \mu mg)$, and we want the frictional force on the top block to provide at least this acceleration, $a = F_f/m$, or the blocks will slip. Thus

$$\frac{1}{M}(F - \mu mg) = \frac{\mu mg}{m} \implies \mu = \frac{F}{(M+m)g},$$

the same answer as before. Plugging in the numbers, we find $\mu \approx 0.14$.

(C) $T_1 = T_2 = T_3$
(D) $T_1 = T_2 > T_3$
(E) $T_1 > T_2 > T_3$



3. Two blocks of masses M and m are oriented as shown in the diagram. The block M moves on a surface with coefficient of kinetic friction μ_1, and the coefficient of static friction between the two blocks is μ_2. What is the minimum force F which must be applied to M such that m remains stationary relative to M?

(A) $\frac{\mu_1}{\mu_2}mg$
(B) $\frac{\mu_1}{\mu_2}\frac{Mm}{M+m}g$
(C) $(\mu_1 M + \mu_2 m)\,g$
(D) $\left(\mu_1 + \frac{1}{\mu_2}\right)(m+M)g$
(E) $\left(\mu_1 M + \frac{m}{\mu_2}\right)g$

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1.2 Kinematics

Kinematics is the first physics that almost everyone learns, so it should be burned into the reader's mind already. For almost all problems it is sufficient to know the equations of motion for a particle undergoing constant acceleration. The primary types of problem worth reviewing are projectile motion problems and problems involving reference frames. To solve projectile motion in two dimensions, you only need the equations of motion for the x- and y-coordinates of the particle,[1]

$$x(t) = v_{0x}t + x_0, \qquad y(t) = -\frac{1}{2}gt^2 + v_{0y}t + y_0, \tag{1.1}$$

where we define coordinates such that gravity acts in the negative y-direction and $g = 10$ m/s^2. Restricting to one dimension, there is another useful formula relating the initial and final velocities of an object, v_i and v_f, its acceleration a, and the change in position between the initial and final states Δx, if the acceleration is constant:

$$v_f^2 - v_i^2 = 2a\Delta x. \tag{1.2}$$

A two-line derivation of this formula uses the work–energy theorem, reviewed in Section 1.3.4.

For problems involving reference frames, just solve the problem in one frame, and then transform to the frame that the problem is asking about. For example, consider the situation in Fig. 1.5: a ball is thrown out of a car moving at constant velocity. Ignoring air resistance, in the frame of the car, the ball moves directly perpendicular to the road. In the frame of an observer at rest, the car is moving forwards, so the motion of the ball is the sum of the two velocities. In other words, the ball moves diagonally, both forward and away from the road. See Example 1.3.

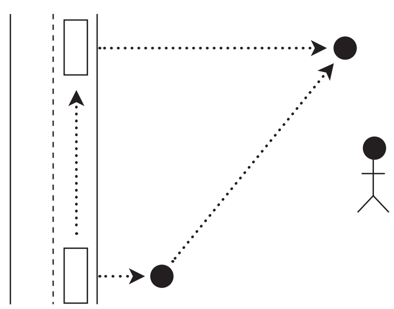

Figure 1.5 A ball thrown out of a moving car, in the frame of a stationary bystander.

From the point of view of solving problems, however, one should avoid kinematics like the plague. It often results in having to solve quadratic equations, and although this is simple in principle, it is usually a huge waste of time. As a rule of thumb, only resort to kinematics if you need to know the explicit time dependence of a system. In nearly all other cases, the basic energy considerations discussed in Section 1.3 will be faster and computationally simpler.

1.2.1 Circular Motion

One kinematic situation that arises often on GRE questions is circular motion. We will consider this in slightly more detail in Section 1.6 when we discuss orbits. For now, consider a particle moving on a circular path. Its acceleration vector can always be decomposed into radial and tangential components. If its tangential acceleration is zero, then its tangential velocity is constant; it is moving in *uniform circular motion* about the center of the circle. But its radial acceleration is nonzero, and has value

$$a = \frac{v^2}{r}, \tag{1.3}$$

where v is the speed of the particle and r is the radius of its orbit. From this, we can immediately also infer that the force needed to keep the particle in its orbit, the *centripetal force*, is

$$F = \frac{mv^2}{r}. \tag{1.4}$$

Indeed, since the tangential acceleration is zero, it must experience some force, directed radially inwards, that keeps it moving in a circular path at a constant speed. Remember that this does not tell you what *kind* of force is acting on the body. It just tells you that if you see a body moving uniformly in a circle of radius r with *constant* speed v, then you can determine what centripetal force must be acting on it.

While uniform circular motion is perhaps the most common example, it is certainly not the most general. There are many cases of nonuniform circular motion: for example, a roller-coaster going around a circular loop-the-loop, or a vertical pendulum attached to a rigid rod with sufficient initial speed to complete a full revolution. In these cases the angle between the gravitational force vector and the velocity vector varies as the object goes around the circle, giving a varying tangential acceleration in addition to the centripetal force, and the above formulas do not apply throughout the whole orbit. However, the uniform circular motion equations *do* apply

[1] In this book, we use the convention of numbering *only equations describing general results worth memorizing for the exam.* We therefore numbered the kinematics formulas here, while we didn't number the equations in the previous section that applied to a specific problem involving blocks. This should help you focus on remembering the equations that actually matter for the exam. We have listed all numbered equations in the equation index at the back of the book, along with page numbers, for your convenience.

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EXAMPLE 1.3

Suppose an astronaut is on a rocket that is moving vertically at constant speed u. When the rocket is at a height h, the astronaut throws a ball horizontally out of the rocket with velocity w, as shown in Fig. 1.6. What is the speed of the ball when it hits the ground?

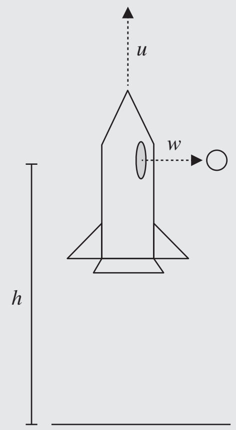

Figure 1.6 A ball is thrown horizontally at velocity w out of a rocket moving vertically upwards at constant velocity u.

In the frame of the rocket, the ball's initial y-velocity is zero, but in the ground frame its initial velocity is u, the relative velocity between the two reference frames. From our kinematic formula (1.2) above, we have for the y-component of the velocity

$$v_y = \sqrt{u^2 + 2gh}.$$

The x-component of the velocity is always the same, $v_x = w$, since no forces act in the x-direction, so we have a total speed

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + w^2 + 2gh}.$$

at two very special places: the top and bottom of the circle, where gravity acts purely vertically, and thus radially, such that the object is instantaneously in uniform circular motion. At all other points in the orbit, other methods (such as energy conservation) must be used to find the velocity.

The centripetal force equation is not so interesting on its own, so a very common class of problems involves combining it with some other type of physics. A typical template might look roughly like this: A particle is moving in a circle. Identify the physics that is causing the centripetal force. Set the expression for this force equal to the centripetal force. Then solve for whatever quantity is requested. See Example 1.4.

1.2.2 Problems: Kinematics

1. A cannonball is fired with a velocity v. At what angle from the ground must the cannonball be fired in order for it to hit an enemy that is at the same elevation, but a distance d away?

   (A) $\arcsin(v/gd)$
   (B) $\arcsin(gd/(2v))$
   (C) $\arcsin(2gd/v)$
   (D) $(1/2)\arcsin(gd/v^2)$
   (E) $\arcsin(gd/v^2)$

2. A satellite (mass m) is in geosynchronous orbit around the Earth (mass M_E), such that its orbit has the same period as the Earth's rotation. If the Earth has angular rotational velocity ω, what is the radius of a geosynchronous orbit?

   (A) $\dfrac{GM_E}{\omega^2}$
   (B) $\dfrac{Gm}{\omega^2}$
   (C) $\left(\dfrac{GM_E}{\omega^2}\right)^{1/3}$
   (D) $\sqrt{\dfrac{GM_E}{\omega^2}}$
   (E) There is no possible geosynchronous orbit.

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EXAMPLE 1.4

An electron (charge e) moves perpendicularly to a uniform magnetic field of magnitude B. If the kinetic energy of the particle doubles, by how much must the magnetic field change for the particle's trajectory to remain unchanged?

We know that the magnetic force on the electron is perpendicular to its motion (see Section 2.2.6 for a review), so it is a *centripetal* force, and the electron moves in a circle. More specifically, the forces are constant, so the electron executes uniform circular motion. Setting the magnetic and centripetal forces equal gives us

$$evB = \frac{mv^2}{r}.$$

Rearranging just a little, we can find the magnetic field

$$B = \frac{mv}{er}.$$

If the kinetic energy of the particle doubles, its velocity increases by $\sqrt{2}$, so B must increase to $\sqrt{2}B$ in order to maintain the same radius. This template occurs very frequently. Though circular motion can involve many different types of physics, identifying the centripetal force and setting it equal to mv^2/r will give you an additional equation to help solve the problem at hand.

1.3 Energy

Conservation of energy can be stated as follows:

> *If an object is acted on only by conservative forces, the sum of its kinetic and potential energies is constant along the object's path.*

Conservative forces are those for which the work done by the force is independent of the path taken between the starting and ending points, but the most useful definition (although it seems tautological) is *a force to which you can associate a (time-independent) potential energy.* The most common examples are gravity, spring forces, and electric forces. The most common example of a force that is *not* conservative, and probably the only such example you'll see on the GRE, is friction: an object traveling from point A to B and back to A will slow down due to friction the whole way through, even though the starting point is the same as the ending point.

A standard subset of GRE classical mechanics problems are most easily solved by straightforward application of conservation of energy. It's important to recognize these problems so you immediately jump to the fastest solution method, rather than fish around for the right kinematics formulas, so we'll state a general principle:

> *If you want to know how fast or how far something goes, use conservation of energy.*
>
> *If you want to know how much time something takes, use kinematics.*

It's baffling that this simple dichotomy isn't introduced in first-year physics courses. It's based on the idea that total energy is a combination of kinetic energies, which depend on velocities, and potential energies, which depend on positions. Setting $E_{\text{initial}} = E_{\text{final}}$ lets you solve for one in terms of the other, but nowhere in the equation does time appear explicitly. On the other hand, kinematics gives you explicit formulas for position and velocity as a function of time t (see equation (1.1)). Of course, some problems will require a combination of both methods, for example using conservation of energy to solve for a velocity which you then plug into a kinematics formula, but, as a very general rule, if time doesn't appear in the problem then you can leave kinematics out of the picture. However, we'll address a common exception to this rule at the end of Section 1.3.2.

1.3.1 Types of Energy

To begin with, you should know the following formulas *cold*:

Translational kinetic energy:	$\frac{1}{2}mv^2$	(1.5)
Rotational kinetic energy:	$\frac{1}{2}I\omega^2$	(1.6)
Gravitational potential energy on Earth:	mgh	(1.7)
Spring potential energy:	$\frac{1}{2}kx^2$	(1.8)

Hopefully the standard notation is familiar to you: v is linear velocity, ω is angular velocity, m is mass, I is the

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moment of inertia, h and x are displacements, g is gravitational acceleration at Earth's surface (which should *always* be approximated to 10 m/s^2 on the GRE when numerical computations are required), and k is the spring constant. There are two important points to remember about potential energy:

- It is only defined up to an additive constant: we are free to choose the zero of potential energy wherever is most convenient, which is usually some physically relevant position such as the bottom of a ramp or the uncompressed length of a spring.
- It is measured from the *center of mass* of an extended object. The usefulness of the center of mass concept (see Section 1.4.4) is that it allows us to treat extended objects like point masses, with all their mass concentrated at the location of the center of mass.

There are other types of potential energy, but all can be summarized by a definition. For any conservative force $\mathbf{F}$, the change in potential energy ΔU between points a and b is

$$\Delta U = -\int_a^b \mathbf{F} \cdot d\mathbf{l}. \tag{1.9}$$

The line integral looks scary but it really isn't, since in all cases of interest the integral will be along the direction of the force vector. Probably the only time you might have to use this formula is if you can't remember the electrostatic or gravitational potential right away, so we'll do that example here. The gravitational force between two masses m_1 and m_2 is

$$\mathbf{F}_{\text{grav}} = \frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}. \tag{1.10}$$

You may have seen this equation in the form

$$\mathbf{F}_{\text{grav, 1 on 2}} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}},$$

stating that the force on mass m_2 from m_1 points along the vector $\hat{\mathbf{r}}$ *from* m_1 *to* m_2, with the minus sign to indicate that the force is attractive. As we'll see, there are minus signs everywhere, so even though it's (deliberately) a bit ambiguous, we find (1.10) a more useful mnemonic for the GRE – just remember that gravity is attractive, and fill in the signs depending on which force (1 on 2 or 2 on 1) you're computing. See Example 1.5.

Alternatively, if you're given the potential, you can compute the force by inverting equation (1.9):

$$\mathbf{F} = -\nabla U. \tag{1.11}$$

Again, watch the minus sign!

1.3.2 Kinetic/Potential Problems

The simplest energy problem involves a mass on a ramp of some complicated shape, asking about its final velocity given that it starts at a certain height, or what initial height it will need to get over a loop-the-loop, or something like that. Because gravity is a conservative force, the shape of the ramp is irrelevant, as long as it's frictionless. If there's friction, then the shape of the ramp *does* matter because the work done by friction depends on the distance traveled – we'll get to that in a bit. First we'll look at a standard example.

EXAMPLE 1.5

Let's find the gravitational potential of a satellite of mass m in the gravitational field of the Earth, of mass M. The most common choice is to set the zero of potential energy at $r = \infty$, so the potential of the satellite at a finite distance r from the center of the Earth is

$$U(r) = -\int_\infty^r -\frac{GmM}{r'^2}\,dr' = -\left.\frac{GmM}{r'}\right|_\infty^r = -\frac{GmM}{r}.$$

Note the signs: the force on the satellite is directed *towards* the Earth, or in the $-\hat{\mathbf{r}}$ direction, but $d\mathbf{l} = +\hat{\mathbf{r}}\,dr'$, so the dot product is negative. The final sign makes sense because gravitational potential decreases (that is, becomes more negative) as the satellite gets closer to the Earth; in other words, it is attracted towards the Earth. Probably the most confusing part of this whole business is the signs, which the GRE *loves* to exploit. Rather than worrying about putting the signs in the right place throughout the whole problem, it may be best to just compute the unsigned quantity, then fill in the sign at the end with physical reasoning.

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EXAMPLE 1.6

A block slides down a frictionless quarter-circle ramp of radius R, as shown in Fig. 1.7. How fast is it traveling when it reaches the bottom?

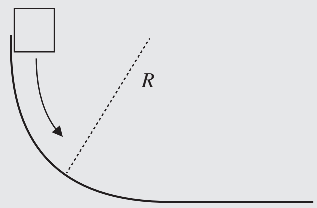

Figure 1.7 Block sliding down a quarter-circle ramp.

The quarter-circle shape is irrelevant except for the fact that it gives us the initial height: the block starts at height R above the bottom. At the top, the block is stationary, so its velocity is zero and there is no kinetic energy; all the energy is potential. Here the obvious choice is to set the zero of gravitational potential energy at the bottom of the ramp, so that the potential at the top is mgR. Wait a minute – the problem didn't tell us the mass of the block! Let's call it m, and see if we can resolve the situation as we finish the problem. At the bottom of the ramp, all the energy is kinetic, because we've defined the potential energy to be zero there. If the block's speed at the bottom is v, then its kinetic energy is $\frac{1}{2}mv^2$. We now apply conservation of energy:

$$0 + mgR = \frac{1}{2}mv^2 + 0$$
$$\implies v = \sqrt{2gR}.$$

Conveniently enough, the mass cancels out since both the kinetic and potential energies are directly proportional to m.

There are a couple things to note about Example 1.6:

- This was the very simplest version of the problem. The block could have had a nonzero speed at the top, in which case it would have had nonzero kinetic energy there. So don't automatically assume that conservation of energy is equivalent to "potential at top equals kinetic at bottom," which is *not* true in general!
- This problem can easily be extended to a kinematics problem by asking how far the block travels after it is launched off the bottom of the ramp, assuming the ramp is some height above the ground.[2] The first step of this problem would still be finding the initial velocity when it leaves the ramp, exactly as we found above.
- The fact that the mass cancels out is actually quite common in problems involving *only* a gravitational potential, since both kinetic and potential energies are proportional to m. So if the problem doesn't give you a mass, don't panic! That's actually a strong clue that the right approach is conservation of energy.

1.3.3 Rolling Without Slipping

A common variant of the above problem is a round object (sphere, cylinder, and so forth) rolling down a ramp. If the object *rolls without slipping*, then its linear velocity v and angular velocity ω are related by

$$v = R\omega, \tag{1.12}$$

where R is the radius. (Dimensional analysis dictates where to put the R so that v comes out with the correct units.) Then in addition to its kinetic energy, $\frac{1}{2}mv^2$, the object also has rotational kinetic energy $\frac{1}{2}I\omega^2$, where I is its moment of inertia. The rolling-without-slipping condition (1.12) lets

[2] Note that this is an exception to our rule about distances being associated with energy rather than kinematics: the block travels with constant horizontal speed once it leaves the ramp, so the only thing dictating how far it goes is the *time* it takes to fall vertically to the ground, which we must get from kinematics. So this is an exception only because it's actually a two-dimensional problem.

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you substitute v for ω and express everything in terms of v, after which you can solve for v exactly as above. Incidentally, it's *friction* that causes rolling without slipping, as friction is responsible for resisting the motion of the point of contact with the object so that it can instantaneously rotate around this pivot. In this situation friction does no work, but instead is responsible for diverting translational energy into rotational energy. Without friction, all objects would simply slide, rather than roll.

Rolling-without-slipping problems almost always boil down to the kinds of cancellations shown in Example 1.7: the kinetic energy is of the form αmv^2, with α some number that accounts for the moment of inertia. Here, α was $3/4$ for the cylinder and $7/10$ for the sphere. Notice that the problem didn't ask which object arrives *first*, only which object had the greater velocity at the bottom: the former is a kinematics question, which by our general principle can't be answered by conservation of energy alone.

EXAMPLE 1.7

A cylinder of mass m and radius r, and a sphere of mass M and radius R, both roll without slipping down an inclined plane from the same initial height h, as shown in Fig. 1.8. The cylinder arrives at the bottom with greater linear velocity than the sphere

(A) if $m > M$
(B) if $r > R$
(C) if $r > \frac{4}{5}R$
(D) never
(E) always

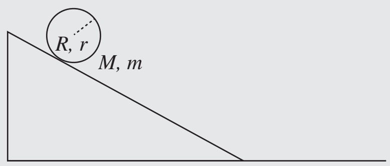

Figure 1.8 Ball or cylinder rolling down an inclined ramp.

You should immediately recognize that the mass is a red herring: since the moment of inertia is proportional to the mass, the same arguments as in Section 1.3.2 go through, and the mass cancels out of the conservation of energy equation for both objects. But let's see how this works explicitly. The moments of inertia are $\frac{1}{2}mr^2$ for the cylinder and $\frac{2}{5}MR^2$ for the sphere (neither of which you should memorize, since they're among the few useful quantities given in the table of information at the start of the test). The energy conservation equations read

$$mgh = \frac{1}{2}mv_{\text{cyl}}^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega_{\text{cyl}}^2 \qquad \text{(cylinder)},$$

$$Mgh = \frac{1}{2}Mv_{\text{sph}}^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\omega_{\text{sph}}^2 \qquad \text{(sphere)}.$$

As promised, we can cancel m from both sides of the first equation, and M from both sides of the second, which lets us equate the two right-hand sides. Now, substituting $\omega_{\text{cyl}} = v_{\text{cyl}}/r$ and $\omega_{\text{sph}} = v_{\text{sph}}/R$, we have

$$\left(\frac{1}{2} + \frac{1}{4}\right)v_{\text{cyl}}^2 = \left(\frac{1}{2} + \frac{1}{5}\right)v_{\text{sph}}^2.$$

The radii also cancel! So we can read off immediately that $v_{\text{cyl}} < v_{\text{sph}}$, and the cylinder *always* arrives slower, choice D.

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1.3.4 Work–Energy Theorem

Since energy is conserved only if the forces acting in the problem are conservative, you might well ask how we can quantify the effects of nonconservative forces such as friction. The answer is simple: we just add a work term to one side of the energy balance equation:

$$E_{\text{initial}} + W_{\text{other}} = E_{\text{final}}, \tag{1.13}$$

where W_{other} is the work due to nonconservative forces. Because work is a signed quantity, the signs can get a little tricky, but you can usually figure them out just by reasoning logically. For example, friction always acts to oppose an object's motion, so the work done by friction is always negative, and this means that $E_{\text{final}} < E_{\text{initial}}$: the object is losing energy due to friction, as it should. You may also be used to seeing this equation in the form

$$W = \Delta \text{KE}. \tag{1.14}$$

Here, the right-hand side is the change in *kinetic* energy, while the left-hand side is the work done by *all* forces, including the conservative ones. The alternate form (1.13) simply absorbs the effect of conservative forces into the definition of the total energy by rewriting the work as a potential energy. Indeed, recall the general definition of work,

$$W = \int \mathbf{F} \cdot d\mathbf{l}, \tag{1.15}$$

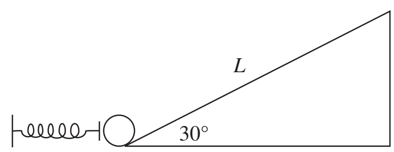
which is of the same form as (1.9) up to a minus sign.

Example 1.8 works through a standard example, which can be tweaked in several ways to make it less straightforward:

- The quarter-circle ramp could have had a coefficient of friction as well. In that case, the frictional force would have varied at different points on the ramp. Note that we could *not* simply apply the formulas for uniform circular motion to determine the normal force, because the block is not moving with constant velocity (see the discussion in Section 1.2.1). This is actually a pretty interesting problem, but it requires solving an ugly differential equation for v, which is far beyond anything you'll see on the GRE.
- Similarly, the frictional surface might not have been flat, in which case the normal force at different points would also have changed.

But the problem we've solved is entirely typical of GRE problems, and illustrates possible shortcuts you should be on the lookout for. Never solve more of a problem than you absolutely must!

1.3.5 Problems: Energy



The following three questions refer to the diagram: a pinball machine launch ramp consisting of a spring of force constant k and a 30° ramp of length L.

EXAMPLE 1.8

Let's revisit the quarter-circle ramp problem (Example 1.6), but this time suppose that, after exiting the ramp, the block slides along a flat surface with coefficient of friction μ. How far down this surface does the block travel before it stops? Now, we could start where we left off, by using the speed $v = \sqrt{2gR}$ at the bottom of the ramp, then computing the kinetic energy, and continuing from there. But that would actually be too much work (no pun intended!). Instead, let's just apply the work–energy theorem directly. The block's initial energy is mgR. The frictional force is μmg along the flat surface (recall that μ is the proportionality constant between the normal force and the frictional force), and so after traveling a distance x, friction does work μmgx. When the block has stopped, its energy is zero. Applying the work–energy theorem, we have

$$mgR - \mu mgx = 0 \implies x = \frac{R}{\mu}.$$

That's it! We never even had to solve for the velocity at the bottom of the ramp. As a sanity check, we can examine the limiting cases $\mu \to 0$ and $\mu \to \infty$: as $\mu \to 0$, there is no friction, so the block never stops, and as $\mu \to \infty$, infinite friction means that the block stops right away. You can do a similar analysis for $R \to 0, \infty$.

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1. You want to launch the pinball (a sphere of mass m and radius r) so that it just barely reaches the top of the ramp without rolling back. What distance should the spring be compressed? You may assume friction is sufficient that the ball begins rolling without slipping immediately after launch.

   (A) $\sqrt{\frac{2mgL^2}{5kr}}$

   (B) $\sqrt{\frac{mgL}{k}}$

   (C) $\sqrt{\frac{2mgL}{k}}$

   (D) $\sqrt{\frac{mgr}{k}}$

   (E) $\sqrt{\frac{2mgr}{k}}$

2. What is the ball's speed immediately after being launched?

   (A) $\sqrt{gL}$

   (B) $\sqrt{\frac{2}{5}gL}$

   (C) $\sqrt{\frac{5}{7}gL}$

   (D) $\sqrt{\frac{7}{10}gL}$

   (E) $\sqrt{\frac{10}{7}gL}$

3. Now suppose the ramp is waxed, so there is no friction. What is the distance the spring should be compressed this time?

   (A) $\sqrt{\frac{2mgL^2}{5kr}}$

   (B) $\sqrt{\frac{mgL}{k}}$

   (C) $\sqrt{\frac{2mgL}{k}}$

   (D) $\sqrt{\frac{mgr}{k}}$

   (E) $\sqrt{\frac{2mgr}{k}}$

1.4 Momentum

If you have gotten this far in physics, then you don't need a refresher on the physics of conservation of momentum. Some of the problems, however, take a bit of practice to learn to solve quickly. In general, you just need to remember that

Momentum is always conserved in a system in the absence of external forces.

This caveat about external forces is sometimes important: for example, if two balls collide in mid-air, the total horizontal momentum is conserved, but not the total vertical momentum because gravity is acting in that direction. In fact, the vertical momentum will continually increase in the downward direction according to Newton's second law $\mathbf{F} = \dot{\mathbf{p}}$. The trick with momentum problems is just to be sure that you are counting all types of momenta – linear and angular – and writing down the correct conservation equations.

1.4.1 Linear Collisions

This class of problem involves point particles that undergo collisions or explosions: for the purposes of the GRE,

If things are colliding, try conservation of momentum first.

Collisional forces can be arbitrarily complicated, but because they are all internal among the colliding particles, the total momentum is conserved as long as there are no additional external forces such as gravity. You just need to set the initial momentum equal to the final momentum and solve for the necessary variables. A special case is when the initial and final energies of the system are the same – this is known as an *elastic* collision, and imposing conservation of energy can give you an additional equation to solve (to find outgoing velocities, for example). Don't assume a collision is elastic unless you are explicitly told so, as this can lead to many trap answers. See Example 1.9.

Solving momentum conservation problems like this one invariably reduces to solving systems of linear equations. This often gets complicated, and if you're like most people, it is easy to make algebraic errors. Don't do it! Exhaust all limiting cases and dimensional analysis arguments before doing algebra. After this, if you think the algebra is easy, then do it. If you think it will be messy, just skip it and come back later. Since you may not even have time to finish the exam, triage is essential.

1.4.2 Rotational Motion and Angular Momentum

Like linear momentum problems, the game here is always to write the angular momentum in the initial state and in the

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EXAMPLE 1.9

A ball of mass M strikes another ball of mass m initially at rest. The ball of mass M scatters at an angle θ relative to its initial direction. Suppose the ball of mass M initially has speed V, and both balls have a final speed v. What is the scattering angle ϕ of the ball of mass m, as defined in Fig. 1.9?

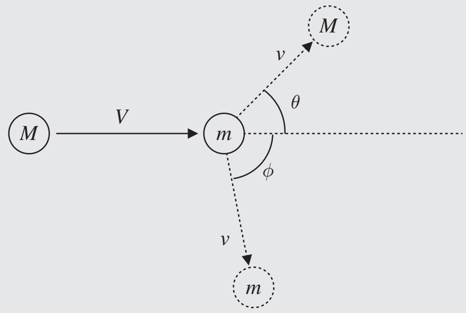

Figure 1.9 Collision of two balls in two dimensions.

Applying conservation of momentum in two dimensions, we get the two equations

$$MV = Mv\cos\theta + mv\cos\phi \qquad \text{(parallel to initial direction)},$$
$$0 = Mv\sin\theta + mv\sin\phi \qquad \text{(perpendicular to initial direction)}.$$

We know M, m, V, v, and θ here and we are solving for ϕ. Thus all we actually need is the second of the two equations, which gives us the result that

$$\phi = \arcsin\left(-\frac{M}{m}\sin\theta\right).$$

The minus sign makes good physical sense: if θ is positive, the mass-M ball goes up, giving a negative ϕ. The ball of mass m goes down, conserving momentum perpendicular to the initial direction. For practice, do a limiting-case analysis for the M and m dependence as well.

final state, then set the two equal. The angular momentum of a point particle of linear momentum $\mathbf{p}$ is defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{1.16}$$

where $\mathbf{r}$ is the vector from a chosen reference point to the particle. Remember that rotational motion is always defined with respect to a reference point or axis. For an extended body we also have

$$\mathbf{L} = I\boldsymbol{\omega}, \tag{1.17}$$

where I is the moment of inertia and $\boldsymbol{\omega}$ is the angular velocity vector. Conceptually, I plays the same role as the mass m in the definition of linear angular momentum $\mathbf{p} = m\mathbf{v}$. Extending the metaphor, the analogue of force $\mathbf{F}$ for rotational motion is the torque

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \tag{1.18}$$

Classic problems include merry-go-rounds and spinning disks. For instance, if a person jumps onto a spinning disk with a known moment of inertia, how does the rotational velocity change? Just equate $\mathbf{L} = I\boldsymbol{\omega}$ in the initial and final states.

We wrote angular momentum and torque in their vector form for completeness above. Note, however, that the vector form is only really needed for the definitions of $\mathbf{L}$ and $\boldsymbol{\tau}$. The analogues of the equations $\mathbf{p} = m\mathbf{v}$ and $\mathbf{F} = d\mathbf{p}/dt$ are only used on the GRE in their scalar forms:

$$L = I\omega, \tag{1.19}$$
$$\tau = \frac{dL}{dt}. \tag{1.20}$$

Problems involving angular momentum can also be conceptual, asking for the configuration of momentum, velocity, and acceleration vectors for a system involving rotational motion. The key point to remember is that the angular

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momentum vector $\mathbf{L}$ is generally *parallel*[3] to the angular velocity vector $\boldsymbol{\omega}$, which points along the axis of rotation, just like an object's linear momentum is parallel to its velocity. The direction of $\mathbf{L}$ is determined by the right-hand rule: curl the fingers of your right hand in the direction of rotation, and your thumb gives the direction of $\mathbf{L}$.

More advanced classical mechanics texts will discuss rotating reference frames, which are mostly beyond the scope of the GRE. All you need to know is that a reference frame rotating at constant angular velocity $\boldsymbol{\Omega}$ is *not* inertial, but, nonetheless, one can write a formula resembling Newton's second law $F = ma$ at the price of introducing "fictitious" forces, which only appear because of the noninertial choice of coordinates:

$$F_{\text{centrifugal}} = -m\Omega^2 r, \tag{1.21}$$

$$F_{\text{Coriolis}} = -2m\boldsymbol{\Omega} \times \mathbf{v}. \tag{1.22}$$

The centrifugal force (which we emphasize once again is *not a real force*!) is the apparent force on an object in a uniformly rotating frame that pushes it away from the axis of rotation. The Coriolis force vanishes if the object is stationary in the rotating frame, but often appears in the context of the Earth's rotation, which defines a rotating frame. Unless the motion of the object is defined with respect to a rotating frame, in which case you typically need to use the Coriolis force, we recommend sticking with inertial frames to avoid confusion.

1.4.3 Moment of Inertia

As we've seen, an object's moment of inertia is analogous to its mass in the context of rotation, but, unlike mass, it depends on the distance from the center of rotation. Let's start with a point particle of mass m: the moment of inertia scales with the radius as

$$I = mr^2, \tag{1.23}$$

which accounts for the fact that, at fixed rotational frequency, the particle will have a higher linear velocity at higher radii. Thankfully, the moment of inertia of a system of many particles is just the sum of the individual moments of inertia,

[3] Strictly speaking, this is only true if the system is rotating about one of its principal axes. But these correspond to various axes of symmetry, and practically all the rotating objects you'll see on the GRE are symmetric to some extent and rotating about their axes of symmetry, so to the best of our knowledge you can ignore this subtlety for GRE purposes. The exception is in problems involving *precession*, which we haven't yet seen appear on the GRE.

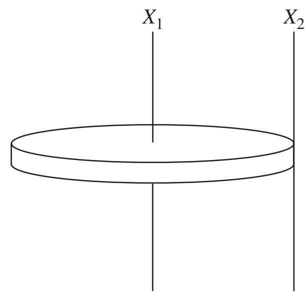

Figure 1.10 A penny is rotated about two axes X_1 and X_2. The moments of inertia for rotation about each axis are related by the parallel axis theorem.

so we can generalize to extended objects with arbitrary mass distributions by integrating:

$$I = \int r^2 \, dm, \tag{1.24}$$

where dm is an infinitesimal mass element (which can depend on position) and the integration is taken over the entire system. Conceptually, objects with more mass further from the axis of rotation are "harder" to rotate and have a larger moment of inertia.

Typically, the integral for moment of inertia is solved by changing integration variables to spatial variables, using the density $dm = \rho dV$. Note that, if you're given the density, for example $\rho = Ar^3$ for a sphere, you actually have to do *two* integrals: one to set the total mass m equal to $\int \rho dV$ to eliminate the constant A, and the other to compute the moment of inertia. The GRE does provide the formulas for the moments of inertia for rods, disks, and spheres on the equations page at the beginning of the test. Typically this is sufficient, though it is useful practice to compute these formulas.

The *parallel axis theorem* is a fast and frequently invaluable tool for computing the moment of inertia of systems built out of smaller pieces whose moments of inertia are known. If we know the moment of inertia I of a system of mass M rotating about an axis through its center of mass (CM), then its moment of inertia about any axis parallel to the CM axis is given by

$$I = I_{\text{CM}} + Mr^2, \tag{1.25}$$

where r is the distance between the CM axis and the parallel axis. For instance, the moment of inertia of a penny rotating about an axis perpendicular to the center of one of its faces is given on the GRE formula page as $I = (1/2)MR^2$. The moment of inertia of the penny when rotating about an axis that passes through the edge of the penny (see Fig. 1.10) would just be $I = (1/2)MR^2 + MR^2 = (3/2)MR^2$.

EXAMPLE 1.10

Consider a rod of mass M and length L whose density varies quadratically, $\rho(x) = Ax^2$, where A is a constant and x is the distance from the left end of the rod, as shown in Fig. 1.11. What is the position of the center of mass of the rod?

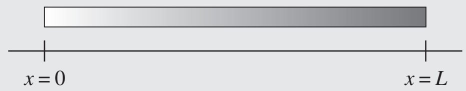

Figure 1.11 A rod of length L with a position-dependent density $\rho(x)$.

The total mass is

$$M = \int_0^L \rho(x)\, dx = \frac{1}{3}AL^3,$$

so $A = 3M/L^3$. The center of mass is then

$$x_{\text{CM}} = \frac{1}{M}\int_0^L x\rho(x)\, dx = \frac{1}{M}\frac{3M}{L^3}\int_0^L x^3\, dx = \frac{3}{L^3}\left(\frac{1}{4}L^4\right) = \frac{3}{4}L,$$

to the right of the center of the rod, which makes sense because the density increases from left to right.

1.4.4 Center of Mass

For reference, the center of mass of an extended object of mass M can be calculated similarly to the moment of inertia using

$$\mathbf{r}_{\text{CM}} = \frac{\int \mathbf{r}\, dm}{M}. \tag{1.26}$$

Instead of weighting the mass by the square of the distance from the axis of rotation, the center of mass weights the mass by the *displacement* from the origin. In principle, equation (1.26) is actually three integrals, one for each coordinate. The corresponding formula for a system of point masses just replaces the integral by a sum

$$\mathbf{r}_{\text{CM}} = \frac{\sum_i \mathbf{r}_i m_i}{M}. \tag{1.27}$$

In particular, for a single mass m at position $\mathbf{r}$, the center of mass is just $\mathbf{r}$, as it should be. See Example 1.10.

1.4.5 Problems: Momentum

1. What is the moment of inertia, about an axis through its center, of a sphere of radius R, mass M, and density varying with radius as $\rho(r) = Ar$?

   (A) $\frac{4}{3}MR^6$

   (B) $\frac{4}{9}MR^2$

   (C) $\frac{2}{5}MR^2$

   (D) $\frac{2\pi}{3}MR^2$

   (E) $\frac{4\pi}{5}MR^2$

2. A block explodes into three pieces of equal mass. Piece A has speed v after the explosion, and pieces B and C have speed $2v$. What is the angle between the directions of piece A and piece B?

   (A) π

   (B) $\pi - \arccos(1/2)$

   (C) $\pi - \arccos(1/3)$

   (D) $\pi - \arccos(1/4)$

   (E) 0

3. A disk of mass M and radius R rotates at angular velocity ω_0. Another disk of mass M and radius r is dropped on top of the rotating disk such that their centers coincide. Both disks now spin at a new angular velocity ω. What is ω?

   (A) $r^2\omega_0/(R^2 + r^2)$

   (B) $R^2\omega_0/(R^2 + r^2)$

   (C) $(R^2 + r^2)\omega_0/r^2$

   (D) $(R^2 + r^2)\omega_0/R^2$

   (E) ω_0

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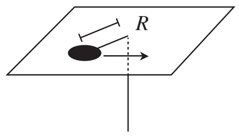

4. A small puck of mass M is attached to a massless string that drops through a hole in a platform, as shown in the diagram above. The puck rotates at radius R when the tension in the string is T. The string is pulled downwards until the radius of rotation is $r < R$. What is the change in energy of the puck when the radius is decreased? You may assume the puck is a point mass.

   (A) $(1/2)TR(R^2/r^2 - 1)$
   (B) $(1/2)TR(r^2/R^2 - 1)$
   (C) $(1/2)Tr(R^2/r^2 - 1)$
   (D) $(1/2)Tr(r^2/R^2 - 1)$
   (E) 0

1.5 Lagrangians and Hamiltonians

As you probably know, Lagrangian and Hamiltonian mechanics provide an elegant way of rewriting the results of classical mechanics, which are most useful for formal results and for dealing with systems with strange constraints, such as a particle confined to the surface of a sphere. Lagrangians and Hamiltonians are fascinating in their own right, and the basis for much of quantum mechanics and quantum field theory, but, as we'll emphasize throughout this book, almost none of this is relevant for the GRE. Instead, Lagrangian and Hamiltonian questions fall into just two categories:

- Write down the Lagrangian or Hamiltonian function.
- Write down the Lagrangian or Hamiltonian equations of motion.

Of course, there are sub-topics within each of these, most importantly conceptual questions dealing with conserved quantities, but these two topics cover all the important bases. Note in particular that *you don't have to solve the equations of motion*. While Lagrangians and Hamiltonians make it easy to write down the equations, they're typically horrible coupled differential equations with no easy solutions. This is sort of the idea of this formulation of mechanics: the actual coordinates of the particle as a function of time aren't so important, but instead one is concerned with properties of the motion (such as energy, momentum, and time dependence) which are easy to see in this framework.

Warning: what follows will be a *drastically simplified* version of Lagrangian and Hamiltonian mechanics. We are leaving out many subtleties and special cases, which are covered in standard treatments, but are not important for the GRE.

1.5.1 Lagrangians

The Lagrangian L of a system is a scalar function described by this absurdly simple formula:

$$L(q, \dot{q}, t) = T - U. \tag{1.28}$$

Here T is the total kinetic energy of the system, U is the potential energy, and q is the collection of all the coordinates describing the degrees of freedom of the system. Note the minus sign! The Lagrangian is *not* the total energy of the system. Note also that the Lagrangian is not only a function of the coordinates q, but also of the *velocities* $\dot{q}$. It is a peculiarity of the Lagrangian formalism that the coordinates and their time derivatives are considered as *independent* variables.

Let's discuss the coordinates in more detail, since that's where almost all of the difficulty of Lagrangians comes in. The power of Lagrangian mechanics lies in being able to choose coordinates to describe *only* the directions in which the system is allowed to move. For example, consider a particle of mass m attached to the end of a massless rod of length ℓ, which is free to rotate in a plane about a pivot (Fig. 1.12). The particle is not allowed to take on any old Cartesian coordinates (x, y), but is forced to move on a circle of radius ℓ. So the most convenient (read "correct") coordinate to use is the angular coordinate θ. But what is the kinetic energy in terms of θ? Here is where things get tricky. We'll now give a recipe for computing the correct expression for T for any question you'll see on the GRE:

- Write down expressions for the *Cartesian coordinates* in terms of your chosen coordinates q.
- Differentiate the Cartesian coordinates (x, y, z) with respect to time to get $(\dot{x}, \dot{y}, \dot{z})$, paying careful attention to the chain rule.
- Form the expression for the Cartesian kinetic energy, $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ for a point particle or $T = \frac{1}{2}I\omega^2$ for an extended object, as appropriate. For the latter, you'll need to express ω in terms of the velocities $\dot{q}$, but this is typically easy because you will have chosen coordinates such that $\dot{q}$ is ω.

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EXAMPLE 1.11

Applying the Lagrangian recipe to Fig. 1.12, we first define our coordinates carefully: choose the origin of Cartesian coordinates (x, y) to be the pivot, so that $\theta = 0$ corresponds to the rod hanging straight down, with the mass at $(0, -\ell)$. We have

$$x = \ell \sin\theta,$$
$$y = -\ell \cos\theta,$$

and taking time derivatives, we get

$$\dot{x} = \ell \cos\theta\, \dot{\theta},$$
$$\dot{y} = \ell \sin\theta\, \dot{\theta}.$$

Notice how the chain rule gets applied to θ. Finally, we form the kinetic energy:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m(\ell^2\cos^2\theta\,\dot{\theta}^2 + \ell^2\sin^2\theta\,\dot{\theta}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2.$$

Not surprisingly, we reproduce exactly the expression for the rotational kinetic energy of a point mass, $\frac{1}{2}I\omega^2$.

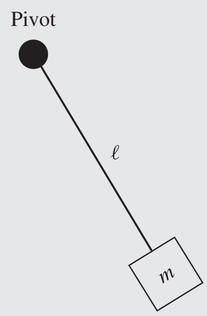

Figure 1.12 A mass m on the end of a rigid rod of length l that rotates about a pivot.

Example 1.11 is probably a simpler example than you'll see on the real exam, so you might have been able to write down the answer right away, but we can't emphasize strongly enough the importance of following this recipe. For example, if the pivot was sliding with velocity v, the total kinetic energy would *not* simply be the sum of the rotational and translational kinetic energies, but would contain an additional cross term $m\ell v\cos\theta\,\dot{\theta}$. You'll see examples of this in our practice tests and in those released by ETS.

1.5.2 Euler–Lagrange Equations

The Lagrangian is a useful quantity because one can derive the equations of motion directly from it. Unlike in Newtonian mechanics, where the equations of motion are the vector differential equations $\mathbf{F} = m\mathbf{a}$, in Lagrangian mechanics the equations of motion are *scalar* equations derived from the scalar quantity L. These equations are known as the Lagrangian equations of motion, or more commonly, the *Euler–Lagrange equations*,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}, \tag{1.29}$$

one equation for each coordinate q.

There are several important things to note about the Euler–Lagrange equations. The first is that signs are *very easy* to mix up, so be careful! A good way to check is to make sure that these equations reduce to $F = ma$ for a particle moving in one dimension x in a potential $U(x)$. In that case, the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - U(x),$$

and we have

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -U'(x),$$

which gives

$$m\ddot{x} = -U'(x) = F,$$

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as expected. (Of course, you still have to remember that $\mathbf{F} = -\nabla U$ with the correct sign, but you should know that already!) The second thing to note is that d/dt is a *total* time derivative, not a partial derivative, which is why it gives $\ddot{x}$ when applied to $\dot{x}$. There are extra terms where the kinetic term happens to have an explicit time dependence – it's unlikely you'll run across something like this on the GRE, but it's good to be careful in any case.

Writing down the correct Lagrangian is typically the hardest part of the problem. Once you have the Lagrangian, the equations of motion are easy, provided you're careful about taking derivatives. Speaking of derivatives, we already know from classical mechanics that quantities whose time derivatives are zero are important – these are *conserved* quantities. Looking at (1.29), we see that if the right-hand side $\partial L/\partial q$ is zero, the quantity $\partial L/\partial \dot{q}$ is conserved! Whether or not this quantity is conserved, it is so important it is given its own name:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}} : \textit{momentum conjugate to } q. \tag{1.30}$$

This name arises because the conjugate momentum is *usually* some kind of momentum (linear or angular), but not always. To reiterate,

> *Iff the Lagrangian is independent of a coordinate q, the corresponding conjugate momentum $\partial L/\partial \dot{q}$ is conserved.*

(Here, "iff" means "if and only if," a shorthand reminder that the converse of the statement is true as well.) In this case, the coordinate q is called *cyclic*. Questions about conserved quantities in Lagrangian mechanics are very common on the GRE, so we've included several representative problems at the end of this section.

1.5.3 Hamiltonians and Hamilton's Equations of Motion

There is an alternative formulation of Lagrangian mechanics, called *Hamiltonian mechanics*, which on the surface is nothing more than a change of variables. As the name implies, the formalism depends on a quantity called the Hamiltonian, derived from the Lagrangian as follows:

$$H(p, q) = \sum_i p_i \dot{q}_i - L. \tag{1.31}$$

Here, i runs over all the coordinates q_i, and p_i is the momentum conjugate to q_i, as defined above in equation (1.30). Note that this relation can be inverted, to give $\dot{q}_i$ as a function of p_i and q_i. To construct H, we solve for $\dot{q}_i$ in this way and plug back into both terms on the right-hand side of (1.31), so that the final result is a function of the momenta p_i *rather than the velocities* $\dot{q}_i$.

OK, that was the textbook definition. On the GRE, if you're only asked for the Hamiltonian, you'd prefer not to take the time to first find the Lagrangian, solve for all the momenta, and only then construct H. With two slight restrictions, there is a much simpler definition:

$$H = T + U \quad \text{(if } U \text{ does not depend explicitly on velocities or time).} \tag{1.32}$$

So for all potentials U that only depend on coordinates, the Hamiltonian *is* the total energy, albeit expressed in terms of the funny position and momentum variables. For a simple example, let's consider the particle moving in one dimension again. As we derived above, $p_x = \partial L/\partial \dot{x} = m\dot{x}$, so

$$\dot{x} = \frac{p_x}{m}.$$

Assuming the potential is time independent, we have

$$H = T + U = \frac{1}{2} m \left(\frac{p_x}{m}\right)^2 + U(x) = \frac{p_x^2}{2m} + U(x),$$

an expression we will meet again in quantum mechanics. The tricky part about this formalism is once again the kinetic term, which *usually* takes the form of a momentum squared over twice a mass. In the case of *angular* coordinates, we usually see a moment of inertia in the denominator: you can work out for yourself that the Hamiltonian for a free particle moving in two dimensions in polar coordinates is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2},$$

with the promised moment of inertia mr^2 showing up in the denominator of the angular momentum term. But for complicated examples, you should still go through the first couple of steps of the Lagrangian construction, carefully identifying the kinetic terms. As long as the potential is velocity and time independent, as is true for all ordinary potentials, there is no need to construct the rest of the Lagrangian in order to calculate the momenta.

As with the Lagrangian, the Hamiltonian is a single scalar function encoding the equations of motion. But this time, we get a system of coupled *first*-order differential equations, as opposed to the second-order Euler–Lagrange equations. These are *Hamilton's equations*:

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}. \tag{1.33}$$

Again, the signs are tricky, but again, the same simple example of a particle in a one-dimensional potential will fix them for you. The first equation reduces in that case to

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$$\dot{p}_x = -U'(x),$$

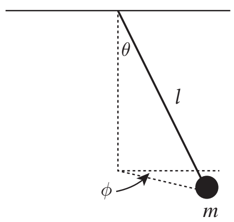
once again reproducing Newton's second law because p_x is precisely the linear momentum of the particle.

Finally, one can derive conservation laws from Hamilton's equations as well. Looking at the first equation, the momentum p is constant if $\partial H/\partial q = 0$. So we have the result:

Iff the Hamiltonian is independent of a coordinate q, the corresponding conjugate momentum p is conserved.

Incidentally, this tells you that if the Lagrangian is independent of q, so is the Hamiltonian, since the conjugate momentum is conserved in both cases.

1.5.4 Problems: Lagrangians and Hamiltonians



The following four questions all refer to a mass m suspended from a rigid massless rod of length l, but free to rotate otherwise (a spherical pendulum). One can take generalized coordinates θ and ϕ as shown in the figure.

1. Which of the following is a possible Lagrangian for the system?
   (A) $\frac{1}{2}ml^2\left(\dot{\phi}^2 + \dot{\theta}^2\right) - mgl\cos\theta$
   (B) $\frac{1}{2}ml^2\left(\dot{\phi}^2 + \sin^2\phi\dot{\theta}^2\right) + mgl\cos\theta$
   (C) $\frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgl\sin\theta$
   (D) $\frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgl\cos\theta$
   (E) $\frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgl\cos\theta$
2. Which of the following is a conserved quantity for the system?
   (A) $ml^2\dot{\phi}$
   (B) $ml^2\sin^2\theta\dot{\phi}$
   (C) $ml^2\dot{\theta}$
   (D) $mgl\cos\theta$
   (E) $mgl\sin\theta$
3. Which of the following is a possible Hamiltonian for the system?
   (A) $\dfrac{p_\theta^2}{2ml^2} + \dfrac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta$
   (B) $\dfrac{p_\theta^2}{2ml^2} + \dfrac{p_\phi^2}{2ml^2} - mgl\cos\theta$
   (C) $\dfrac{p_\theta^2}{2m} + \dfrac{p_\phi^2}{2ml^2} - mgl\cos\theta$
   (D) $\dfrac{p_\theta^2}{2m\sin\phi^2 l^2} + \dfrac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta$
   (E) $\dfrac{p_\theta^2}{2ml^2} + \dfrac{p_\phi^2}{2ml^2\sin^2\theta}$
4. Suppose the pendulum is confined to the plane $\phi = 0$. What is the Euler–Lagrange equation for θ?
   (A) $\dot{\theta} = \frac{g}{l}\sin\theta$
   (B) $\ddot{\theta} = -\frac{g}{l}\theta$
   (C) $\ddot{\theta} = -\frac{g}{l}\sin\theta$
   (D) $\ddot{\theta} = gl\sin\theta$
   (E) $\ddot{\theta} = -gl\cos\theta$

1.6 Orbits

The two fundamental forces of classical mechanics, gravity and electromagnetism, are remarkably similar: they both have $1/r^2$ force laws, but, perhaps more importantly, they are *central forces*, meaning that the force vector points along the line between the two interacting bodies. Without exception, these are the forces that appeared on recent GREs in the context of orbit problems, so we will confine our discussion of orbits to one where all forces are central and spherically symmetric. This means that the force can be derived from a potential function $U(r)$, which only depends on the radial distance between the two bodies; this is known as a *central potential.*

While one can discuss orbits quite straightforwardly using only the language of forces and Newtonian dynamics, our discussion will simplify *immensely* if we throw in a bit of Lagrangian mechanics. As a result, we strongly urge you to study Section 1.5 on Lagrangians and Hamiltonians carefully before reading this section: the material presented there should be more than sufficient to understand our treatment here.

1.6.1 Effective Potential

The fact that our potential has the form $U(r)$ immediately gives us conservation laws which we can put right to use. First, let's write down the Lagrangian for a particle of mass m moving in the potential U: after writing x, y, and z in spherical coordinates, we find

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\dot{\phi}^2 - U(r).$$

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(The polar angle θ shows up in the kinetic energy roughly for the same reason that it shows up in the spherical volume element $r^2 \sin\theta$.) Reverting to Newtonian reasoning for a bit, conservation of angular momentum implies the conservation of a whole *vector* $\mathbf{L}$ (whose magnitude is l), and the fact that the *direction* of this vector is constant means that *the particle is confined to a plane.* By spherical symmetry, we can choose this plane to be at $\theta = \pi/2$; the second term (involving $\dot{\theta}$) vanishes since θ is constant, and $\sin(\pi/2) = 1$ means the third term simplifies as well. We are left with the restricted form:

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - U(r), \tag{1.34}$$

which we will use from now on. Now, since $U(r)$ is independent of the azimuthal angle ϕ, so is the Lagrangian, and that gives us conservation of the conjugate momentum to ϕ, which we identify as the ordinary angular momentum l:

$$l = mr^2\dot{\phi}. \tag{1.35}$$

The radial behavior of the orbit is of course described by the Euler–Lagrange equation for the radial coordinate:

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\phi}^2 - U'(r).$$

Substituting for $\dot{\phi}$ in terms of l using (1.35), we get

$$m\ddot{r} = \frac{l^2}{mr^3} - U'(r).$$

First of all, we have reduced a complex system of partial differential equations in three dimensions to a single ordinary differential equation, which we may have some hope of understanding. And secondly, this looks suspiciously like Newton's second law of motion. We can improve the resemblance by "factoring" the derivative on the right-hand side to find

$$m\ddot{r} = -\frac{d}{dr}\left(\frac{l^2}{2mr^2} + U\right).$$

This is now in exactly the same form as Newton's second law, except that the "potential" now has an additional term depending on l. We call the expression in parentheses the *effective potential*:

$$V(r) = \frac{l^2}{2mr^2} + U(r). \tag{1.36}$$

Now we can draw potential energy graphs just as we would with an ordinary one-dimensional problem, remembering in the back of our minds that we're really dealing with an entire orbit, which also has some angular dependence. The effective potential formalism is most useful for dealing with *shapes* of orbits, $r(\phi)$, rather than time dependences, $r(t)$ and $\phi(t)$. Happily, the GRE only cares about orbit shapes, with one simple exception which we'll discuss when we come to Kepler's laws.

We should briefly mention a subtlety of the most common application of this formalism, namely two bodies orbiting each other under the influence of gravity. In that case, the bodies orbit about their mutual center of mass, but instead of dealing with two separate orbits, one can perform a coordinate transformation to describe the *relative* motion. In doing so, the mass m gets replaced by the *reduced mass*,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{1.37}$$

This is a great one to remember by dimensional analysis and limiting cases. The numerator has to contain the product in order for the whole thing to have dimensions of mass, and in the limit that $m_2 \to \infty$, we have $\mu \approx m_1$, corresponding to a center of mass that is very near the heavy body, m_2, which barely moves at all. So in a two-body problem, replace all instances of m by μ (that is, in *both* the kinetic energy and the effective potential), and you're good to go. The most common situation is the limit $m_2 \gg m_1$ just noted, so $\mu \approx m_1$ is usually a good approximation – but not always!

1.6.2 Classification of Orbits

Let's exploit conservation laws in order to learn something about possible orbits of objects in a central force. Now that we have defined the effective potential, we can define the total energy of the orbit:

$$E = T + V = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + U(r), \tag{1.38}$$

which is conserved if $U(r)$ is time independent. The most interesting part of this formula is the l-dependent term in the effective potential, whose $1/r^2$ dependence acts as a "centrifugal barrier," which imposes an infinite energy cost to get to $r = 0$ if the body has a nonzero angular momentum. To learn more, let's consider a sample shape for $V(r)$, as shown in Fig. 1.13, under the assumptions that $U(r) \to 0$ as $r \to \infty$ and that the centrifugal term dominates as $r \to 0$ (meaning that $U(r)$ must have a smaller power of r in the denominator than $1/r^2$, as is the case for gravity, where $U(r) \sim 1/r$).

Three representative orbit energies are marked, E_1, E_2, and E_3. An orbit with energy $E_1 > 0$ is unbound: the body comes in from infinity, "strikes" the centrifugal barrier, and "reflects" back out to infinity. An orbit with energy E_2 is bound, and has two "turning points," with a minimum distance r_1 and a maximum distance r_2; the body is always stuck between them.

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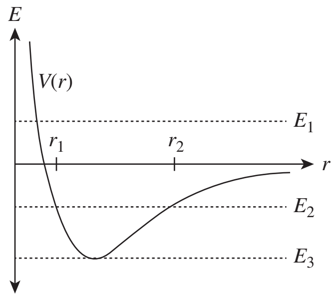


Figure 1.13 Effective potential with some representative orbit energies.

That is *not* to say that the orbit must be periodic: all it means is that, for a general potential, the body's orbit shape is enclosed within a ring-shaped region bounded by the circles of radii r_1 and r_2. The orbit E_3 is a special case, where we sit exactly at the minimum of the effective potential: then, there is not enough energy to change the value of r, so *the minimum of* $V(r)$ *corresponds to circular orbits*. To find the radius of these orbits, just solve $V'(r) = 0$ for r. You should also check for stability by ensuring that $V''(r) > 0$, otherwise we'd be sitting at an unstable *maximum*. Similarly, for more general orbit energies, we can read off the distance of closest approach by solving $E = V(r)$ for r.

We can be much more specific about orbit shapes in the case $U(r) = k/r$, as would be the case for a gravitational potential. Without getting into the details of the derivation, the results are

- $E > 0$: hyperbolic orbit
- $E = 0$: parabolic orbit
- $E < 0$: elliptical orbit

As in the more general discussion, since $U(r)$ falls off at large r, $E < 0$ corresponds to bound orbits, and for a $1/r$ potential these happen to take an elliptical shape. For $E = V_{\text{min}}$, we have the special case of a circular orbit, which has the lowest possible energy.

1.6.3 Kepler's "Laws"

Yes, those scare quotes are there for a reason – Kepler's three "laws" are not really laws at all, in the sense of Newton's laws, which are *always* true (in the context of classical mechanics). Rather, they're three completely logically independent rules of thumb: one of them is almost trivial and totally general, and two of them are far too specific and only approximately true! In any case, here they are, in the traditional order:

I. Planets move in elliptical orbits with one focus at the Sun.
II. Planetary orbits sweep out equal areas in equal times.
III. If T is the period of a planetary orbit, and a is the semi-major axis of the orbit, then $T = ka^{3/2}$, with k the same constant for all planets.

Let's start with the first law. The first part is trivial: planets have bound orbits by definition, and, as we've seen above, a gravitationally bound orbit is either elliptical or circular, where a circular orbit can be considered a limiting case of an elliptical orbit where the two foci coincide. The second part is fairly difficult to derive, and, strictly speaking, it's not even true! Remember that two bodies orbit about their mutual center of mass – the precise statement is that the Sun and planet both undergo elliptical orbits, with a common focus of both ellipses located at their mutual center of mass. The Sun only sits still at the focus under the approximation that it is much heavier than any of the planets, such that the reduced mass is nearly equal to the planetary mass. We can even be a little sloppier, and say that as long as the center of mass of the Sun–planet system lies inside the Sun, the Sun is "at the focus" throughout its motion. Unfortunately, the GRE has been known to ignore this subtlety from time to time, and a question from a 2008 GRE suggested that the statement "the Sun is at one focus" is exactly true.

Kepler's second law is also known as *conservation of areal velocity*, and means that if you drew vectors from an orbiting planet to the Sun, at equal time intervals along the planet's orbit, the orbit would be sliced up into equal-area segments. In Fig. 1.14, if the two shown portions of the orbit are traversed in equal times, the regions marked A and B have equal areas.

In fact, this law is completely general, for *any central potential*, not just gravitational force laws, since it follows immediately from conservation of angular momentum. Recall the definition of l:

$$l = mr^2 \frac{d\phi}{dt}$$
$$\implies \frac{l}{m} dt = r^2 d\phi.$$

The expression on the right-hand side is precisely the area element in polar coordinates (up to a factor of 2), and l/m is constant, so integrating both sides gives us the second law.

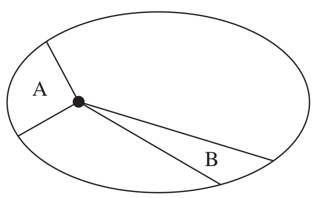


Figure 1.14 Areal sections illustrating Kepler's second law.

The third law is rather tricky to derive, so best to just memorize it. The proportionality between T and $a^{3/2}$ is exactly true for a pure $1/r$ potential. You probably won't need it, but the proportionality constant k is

$$k = \frac{2\pi}{\sqrt{G(m_{\text{planet}} + m_{\text{Sun}})}}.$$

Like the first law, the statement that k is the same for all planets is only true in the approximation that the Sun is infinitely massive; otherwise, the m_{planet} term in the denominator of k matters, and k then varies from planet to planet.

1.6.4 Problems: Orbits

1. A particle of mass m is attached to one end of a spring with spring constant k in a zero-gravity environment; the other end of the spring is attached to a fixed pivot. The spring's equilibrium position is fully compressed. The spring is stretched to a length r and the particle is given an initial angular momentum l. Which of the following is a possible value of r so that the spring stays at a constant extension r throughout the entire motion of the particle?

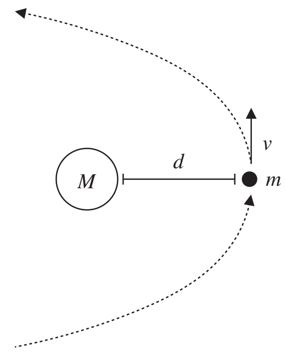
   (A) $(l^2/mk)^{1/4}$
   (B) $(l^2/mk)^{1/2}$
   (C) $\sqrt{mkl^2}$
   (D) ml^2/k
   (E) $(mk/l)^{1/3}$

2. Suppose a new planet were discovered orbiting the Sun, whose orbital period was exactly twice that of Mars. Assuming the new planet's mass is much smaller than the Sun's mass, which of the following must be true?

   I. The new planet's mass is exactly twice that of Mars.
   II. The major axis of the new planet's orbit is smaller than that of Mars.
   III. The major axis of the new planet's orbit is bigger than that of Mars.

   (A) I only
   (B) II only
   (C) III only
   (D) I and II
   (E) I and III

3. An asteroid of mass m orbits the Sun (mass M) on a parabolic trajectory. Which of the following relates its distance of closest approach d to its orbital velocity v at the point of closest approach? You may assume m is negligibly small compared to M.



   (A) $d = \dfrac{GM}{v^2}$
   (B) $d = \dfrac{2GM}{v^2}$
   (C) $d = \dfrac{Gm}{v^2}$
   (D) $d = \dfrac{2Gm}{v^2}$
   (E) $d = \dfrac{GM}{2v^2}$

1.7 Springs and Harmonic Oscillators

Spring problems appear in many different forms, though they tend to use only a few basic facts. Obviously Hooke's law is the starting point for most problems:

$$F = m\ddot{x} = -kx, \tag{1.39}$$

where x is the displacement of the spring from equilibrium, and k is the spring constant. This is an ordinary differential equation describing a *harmonic oscillator* whose solutions are of the form $x(t) = A\cos(\omega t + \phi)$. The angular frequency is given by

$$\omega = \sqrt{\frac{k}{m}}. \tag{1.40}$$

Note that the amplitude A is *not* determined by Hooke's law, but is instead a constant of integration fixed by the initial conditions. The phase ϕ is the second constant of integration. Since complex exponentials

$$x(t) = Ae^{i\omega t}, \tag{1.41}$$

with A allowed to be complex, also satisfy Hooke's law, it is often easier to write the solutions in terms of complex numbers and take the real part at the end of the problem. This convenient shorthand will be elaborated in the chapter on waves, Section 3.1.2.

The potential energy of a spring of spring constant k at displacement x is given by

$$U = \frac{1}{2}kx^2. \qquad \text{(p.7)} \tag{1.8}$$

You can prove this to yourself just by integrating the force from Hooke's law over the displacement of the spring to obtain the potential energy.

In fact, this is all that you really need to know for solving spring problems! Like much of the rest of mechanics, spring problems – at worst – just reduce to solving second-order ordinary differential equations. Simple as it sounds, this can sometimes be a time-consuming task, so you should use the potential energy considerations whenever possible. To summarize, the order of operations for spring problems should be:

1. Try limiting cases, dimensional arguments, and symmetry.
2. Try conservation of energy.
3. Try writing down a differential equation and solving it.

Even this last method should not be too bad for problems on the GRE, but consider it a last resort.

1.7.1 Normal Modes

Now suppose that, instead of having one body attached to a spring, we have two bodies, and we want to solve for the motion of both. This is complicated because we have a set of two coupled differential equations with a large family of solutions. It turns out that all of the solutions, however, are just superpositions of two basic solutions, called *normal modes*, which have the usual sinusoidal form of a harmonic oscillator. One very common GRE question is to ask for the normal modes of a system.

The mathematical setup is as follows. Consider the general case of n masses attached to springs, whose displacements[4] are q_k. Then the equations of motion for the entire system can be written

$$\sum_k (A_{jk}q_k + m_{jk}\ddot{q}_k) = 0.$$

This is just the most general linear combination of coordinates that we can form out of acceleration terms and forces from Hooke's law.

With great foresight, use the ansatz

$$q_k(t) = a_k e^{i\omega t}, \tag{1.42}$$

whose form is inspired by (1.41). For this guess, $\ddot{q}_k = -\omega^2 q_k$, so the equations of motion reduce to

$$\sum_k (A_{jk} - \omega^2 m_{jk})a_k = 0.$$

This is just a matrix equation for the coefficients a_k, and linear algebra teaches us that in order for it to have a nontrivial solution the determinant must vanish:

$$\det(A_{jk} - \omega^2 m_{jk}) = 0. \tag{1.43}$$

This *secular equation* defines a polynomial whose solutions give n frequencies ω_i, which are called the *normal frequencies* of the system. The normal modes alluded to above are just the solutions at which the *entire* system oscillates collectively at a fixed normal frequency, with the motion in time given by equation (1.42). A linear combination of solutions to an ordinary differential equation is still a solution, so we can build up more complicated motion by taking linear combinations of normal mode solutions with various coefficients.

Often, it's important to determine what kind of motion normal frequencies correspond to. For Example 1.12, we can immediately tell that the normal mode with frequency $\sqrt{k/m}$ will correspond to the two blocks moving in sync with fixed separation. In this case, the middle spring has no effect on the system, and the motion reduces to a single block–spring system with total mass $2m$ and total spring constant $2k$, thus giving $\omega = \sqrt{2k/2m} = \sqrt{k/m}$. (This should remind you of the discussion of blocks stuck together in Section 1.1.3.) We might guess that the other normal mode corresponds to motion of the blocks in exactly opposite directions. The frequency of this motion should be higher than $\sqrt{k/m}$ because the middle spring is now exerting an additional restoring force on the two blocks. Depending on answer choices given in the problem, these observations may be sufficient to pick the correct answer.

Solving for the normal modes of a system is a very common problem, in both quantitative and qualitative contexts. For a quantitative solution, the recipe is exactly as above:

- Write down the equations of motion for the system.
- Determine the matrices A_{jk} and m_{jk} and write down the secular equation (1.43).
- Solve to find the normal frequencies ω_i.

In many cases, this machinery is overkill, especially as most GRE problems are designed to be solved in about one minute. For most problems, you should ask yourself:

- What are the simplest ways that the system can oscillate at a fixed frequency? Drawing a picture is often helpful.

[4] In this section, by "displacement" we will always mean "displacement from equilibrium."

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EXAMPLE 1.12

Two blocks of mass m are coupled to each other and to two walls by springs of spring constant k, as shown in Fig. 1.15. What are the normal modes of the system?

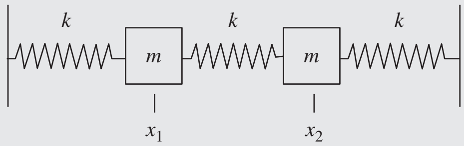

Figure 1.15 Two blocks and three springs.

We start by writing the equations of motion. Let x_1 and x_2 denote the displacements of the left and right blocks, respectively. The force on the left block is the force due to each of the adjacent springs, giving an equation of motion

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2).$$

Similarly, for the right block, we have

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1).$$

Using our guess (1.42) for x_1 and x_2, we find that the equations of motion become

$$m\omega^2 a_1 = 2ka_1 - ka_2,$$
$$m\omega^2 a_2 = 2ka_2 - ka_1.$$

Writing this explicitly in matrix form we have

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0.$$

Notice that we didn't actually have to plug in the exponential guess solution to find this matrix equation; with practice, you can determine the matrices A_{jk} and m_{jk} just by staring at the equations of motion for x_1 and x_2, and jump straight to the secular equation (1.43). Taking the determinant of the matrix and setting to zero gives

$$(2k - m\omega^2)^2 - k^2 = 0.$$

Luckily, we don't even need the quadratic equation to solve this: rearranging and taking square roots gives

$$2k - m\omega^2 = \pm k,$$

handing us immediately the two solutions

$$\omega = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}.$$

- What is the normal frequency corresponding to the entire system oscillating together? A general rule of thumb is that *symmetric* motion will have lower frequency than asymmetric motion, with the most symmetric mode (corresponding to collective oscillations, as in the case above of the blocks moving in sync) having the lowest frequency. This is often $\sqrt{K/M}$, where K and M are the effective spring constant and mass of the entire system ($2k$ and $2m$ in the example above).
- Can this information be used to eliminate incorrect answers?

1.7.2 Damping, Driving, and Resonance

What happens when there is an oscillatory system with a damping force? One example would be a block–spring system placed underwater, where drag forces act to oppose the block's motion. As always in classical mechanics, our strategy is to

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write down the equations of motion of the system and find solutions. Damping terms such as drag or air resistance usually appear as a force proportional to the *velocity* of a particle, $F_{\text{damp}} = -b\dot{x}$, so the general equation of motion for an oscillator with damping is

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{1.44}$$

There are three qualitative types of solutions: underdamped, critically damped, and overdamped. These are most conveniently expressed in terms of a damping parameter $\beta = b/2m$ and the natural frequency $\omega_0 = \sqrt{k/m}$. The underdamped solution, for $\beta^2 < \omega_0^2$, corresponds to motion in which the oscillations follow an exponentially decaying envelope. The equation of motion takes the convenient form

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta),$$

where $\omega_1^2 = \omega_0^2 - \beta^2$. It is not critical to remember all of the constants in this expression, but it is valuable to remember that underdamped oscillations are a sinusoid with an exponentially decaying envelope. This should make intuitive sense: a small amount of damping will produce the same behavior as free oscillation, but with gradually decreasing energy (and thus amplitude) because of the damping force.

The overdamped solution, for $\beta^2 > \omega_0^2$, corresponds to motion in which the damping is so strong that no oscillation occurs. While the solutions to the underdamped case are sinusoidal functions (i.e. *complex* exponentials), the solutions to the overdamped oscillator are *real* exponentials, with the oscillator returning exponentially to its equilibrium position. The intermediate case between these two is called critical damping, and it corresponds to $\beta^2 = \omega_0^2$.

A harmonic system can also be driven by some external sinusoidal force. In this case, the equation of motion picks up an additional term due to the driving; in the most general case including possible damping,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t.$$

The solutions to this equation of motion can be found using elementary methods for solving inhomogeneous ordinary differential equations. Without going into details, the important point is that, unlike the case of the oscillator with no driving force, the amplitude of the steady-state solution (that is, after a long time has elapsed) is not a free parameter but instead depends on the coefficient A and the frequency ω of the driving force. For a given A, the amplitude is maximized when the driving frequency equals the so-called *resonant frequency*:

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2}. \tag{1.45}$$

It is also possible to calculate the amplitude of oscillation when the driving frequency is different from resonance. It's extremely unlikely that you would ever need the exact expression for the GRE, but it could be worth remembering the scaling relation in the absence of damping. The amplitude D of an undamped oscillator of natural frequency ω_0 subject to a driving force at frequency ω is proportional as follows:

$$D \propto \frac{1}{|\omega_0^2 - \omega^2|}.$$

Note that this expression diverges at a driving frequency $\omega = \omega_0$! Of course, we don't see infinite amplitude in real-life oscillators because of small damping forces such as friction. But this proportionality should hold well near resonance in the weak damping limit.

To summarize, we have three different types of motion with three different characteristic frequencies, in order of increasing generality:

- Free oscillation: $\omega^2 = k/m$
- Damped oscillation:
  - overdamping
  - critical damping
  - underdamping, with characteristic frequency $\omega_1^2 = \omega_0^2 - \beta^2$
- Driven oscillation: $\omega_R^2 = \omega_0^2 - 2\beta^2$

1.7.3 Further Examples

There are many additional examples of the methods presented in this section. We show a few common examples. The basic unifying feature of all the examples in this section is that the behavior of the system can be described with a system of second-order linear ordinary differential equations with constant coefficients.

- **Pendulums.** In the limit of small displacements, the equation of motion of a pendulum of length L is described by simple harmonic motion. The full equation of motion for a pendulum with angular displacement θ is (see problem 4 in Section 1.5.4)

  $$mL\ddot{\theta} = -mg\sin\theta.$$

  For small displacements, this becomes

  $$m\ddot{x} = -mgx/L, \tag{1.46}$$

  which describes simple harmonic motion of angular frequency

  $$\omega = \sqrt{\frac{g}{L}}. \tag{1.47}$$

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A similar equation holds for an extended object of mass m swinging on a massless rod of length R, which has moment of inertia I about the pivot. More precisely, R is the distance between the center of mass of the object and the pivot. The oscillation frequency, easily derived using Lagrangian mechanics, is now

$$\omega = \sqrt{\frac{mgR}{I}}, \tag{1.48}$$

which is easy to remember using dimensional analysis once you know equation (1.47).

- **Circuits.** The structure of the differential equations describing damped and driven oscillations is identical to the differential equations describing electrical circuits under the replacements in Table 1.1. All differential equations remain perfectly valid for electrical systems after these substitutions, with the technical caveat that the quantities in the table refer to the effective quantities for the *whole* circuit or system, not individual circuit elements or springs. This shouldn't be too surprising since, while one can add resistors in series or in parallel, it doesn't make much sense to "add damping resistance in parallel." In particular, while it is true that $1/k_{\rm eq} = C_{\rm eq}$, one *cannot* just replace all individual capacitors by springs, though there is a simple rule for equivalent spring constants for springs in series and parallel that mirrors the rule for capacitors. See below for more details.
- **Parallel and series springs.** Pursuing the electrical analogy, we can consider springs attached to a block of mass m in both parallel and series configurations; see Example 1.13.

EXAMPLE 1.13

When the springs are in parallel (Fig. 1.16, left), the equation of motion is

$$m\ddot{x} = -kx - kx = -2kx,$$

so the effective spring constant is the sum of the two spring constants $k + k = 2k$. When the springs are in series (Fig. 1.16, right), the situation is slightly more complicated. Call the displacement of the (massless) joint between the springs x_1, and call the displacement of the block x_2. To determine the equations of motion, we can use a trick: pretend a very small mass is attached to the joint between the two springs. In this case, we just have a two-mass/two-spring system, very similar to the one we already solved in Section 1.7.1! Simply copying down the first equation of motion and sending the small mass to zero gives

$$0 = -kx_1 - k(x_1 - x_2).$$

Notice that this is no longer a differential equation, but an algebraic constraint equation, $x_1 = x_2/2$. This enforces the condition that there must be zero force acting on the joint between springs; otherwise, since the joint is massless, the acceleration would be infinite. The block just experiences a force due to the spring touching it, so its equation of motion is

$$m\ddot{x}_2 = -k(x_2 - x_1).$$

Plugging in the constraint, we get

$$m\ddot{x}_2 = -\frac{k}{2}x_2,$$

giving an effective spring constant of $k/2$.

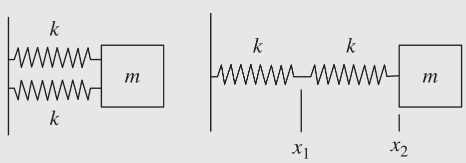

Figure 1.16 Springs in series and parallel.

Table 1.1 Correspondence of quantities in the analogy between electrical and mechanical systems.

Mechanical system		Circuit system	
x	displacement	q	charge
$\dot{x}$	velocity	I	current
m	mass	L	inductance
b	damping resistance	R	electrical resistance
$1/k$	spring stiffness	C	capacitance
F	amplitude of driving force	V	amplitude of driving voltage

If you go through the derivation in Example 1.13 with two different spring constants k_1 and k_2, you can prove to yourself that the rule for computing equivalent spring constants is $1/k_{\text{eq}} = 1/k_1 + 1/k_2$, the same as the rule for computing equivalent capacitances for capacitors in series, $1/C_{\text{eq}} = 1/C_1 + 1/C_2$.

This leads to a subtle pitfall: the electrical analogue of our series spring system above is two capacitors in series. If we were to replace each spring k by a capacitor C, the rule of Table 1.1 gives $1/k \to C$. But the equivalent capacitance of this system is $C/2$, and the equivalent spring constant is $k/2$, so the rule $1/k_{\text{eq}} \to C_{\text{eq}}$ becomes $2/k \to C/2$, or $1/k \to C/4$! So the electrical–mechanical analogue is not as simple as just replacing $1/k$ by C everywhere in the circuit. To keep things straight, it's best to remember the two rules as logically distinct:

– The rule for adding springs k in series and parallel is the same as adding capacitors C in series and parallel.
– The electrical analogue to a mechanical system with equivalent spring stiffness $1/k_{\text{eq}}$ is an equivalent capacitor C_{eq}.

1.7.4 Problems: Springs

1. A block of mass M is attached to two springs, both with spring constants k, in series. Another block of mass m is attached to three springs, each of spring constant k, in parallel. What is the ratio of the oscillation frequency of the block of mass M to the frequency of the block of mass m?

   (A) $\sqrt{3m/(2M)}$
   (B) $\sqrt{2M/m}$
   (C) $\sqrt{3M/m}$
   (D) $\sqrt{m/(6M)}$
   (E) $\sqrt{6m/M}$

2. A ball of mass m is launched at 45° from the horizontal by a spring of spring constant k which is depressed by a displacement d. What horizontal distance x does the ball travel before returning to its height at launch?

   (A) $2mgd/k$
   (B) mgd/k
   (C) $kd^2/(gm)$
   (D) $kd^2/(2gm)$
   (E) $2kd^2/(gm)$

3. Suppose a motor drives a block on a spring at a frequency ω, and the natural frequency of the spring–block system is ω_0. If damping is negligible, by what factor does the amplitude of oscillation change when the driving frequency is increased from $\omega = 2\omega_0$ to $\omega = 3\omega_0$?

   (A) 4/9
   (B) 3/8
   (C) 2/3
   (D) 9/64
   (E) 9/4

1.8 Fluid Mechanics

In general physics courses, fluid dynamics problems typically appear only in two simple forms. The first is in applications of Bernoulli's principle – essentially a reformulation of conservation of energy. The second uses the concept of buoyant forces.

1.8.1 Bernoulli's Principle

Consider a fluid that is traveling through some pipe. The pipe may go up and down, and it may change diameter. Regardless, the following quantity is constant along a streamline of the fluid moving through the pipe:

$$\frac{v^2}{2} + gz + \frac{p}{\rho} = \text{constant}, \tag{1.49}$$

where v is the velocity of the fluid, g is gravitational acceleration, z is the height of a point along the streamline, p is the

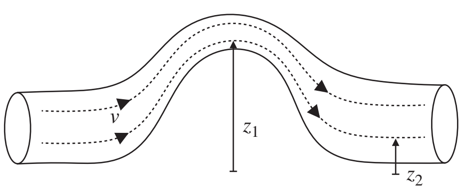

Figure 1.17 General setup for the Bernoulli equation, describing a fluid traveling through a pipe of variable size and height. Dashed lines represent streamlines.

EXAMPLE 1.14

To see how Bernoulli's principle works in context, consider a horizontal square pipe with two sections: one of side length a, and another of side length b (Fig. 1.18). The cross-sectional areas of the two pipes are a^2 and b^2, respectively. The amount of fluid flowing past a point in the first section in a time Δt must equal the amount of fluid flowing past a point in the second section, since the fluid can't appear or disappear. Mathematically, we require the following *fluid conservation equation* to hold:

$$\rho v_1 A_1 \Delta t = \rho v_2 A_2 \Delta t,$$

or just

$$v_1 A_1 = v_2 A_2, \tag{1.50}$$

where A_1 and A_2 are the cross-sectional areas of the two sections of pipe. Note that the density entering into both sides of the above equation is the same because of our assumption that the fluid is incompressible. It doesn't matter what the pressure is – an incompressible fluid will always have the same density. In our specific case, we have

$$v_1 a^2 = v_2 b^2.$$

From Bernoulli's principle, we know that we must have

$$\frac{v_1^2}{2} + g z_1 + \frac{p_1}{\rho} = \frac{v_2^2}{2} + g z_2 + \frac{p_2}{\rho}. \tag{1.51}$$

Since the pipe is horizontal (except in the region between the two pipe sections), streamlines are horizontal and $z_1 = z_2$. Substituting equation (1.50) into equation (1.51), we find that

$$\frac{v_1^2}{2} + \frac{p_1}{\rho} = \frac{v_1^2 a^4}{2b^4} + \frac{p_2}{\rho}.$$

If we know the pressure and velocity in the first part of the pipe, we can now calculate the pressure in the second part of the pipe:

$$p_2 = \frac{\rho v_1^2}{2}\left(1 - \frac{a^4}{b^4}\right) + p_1.$$

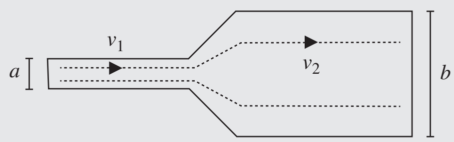

Figure 1.18 Fluid flows through two horizontal segments of square pipe, with side lengths a and b.

pressure, and ρ is the density of the fluid. Nearly all common fluids are *incompressible*, in the sense that ρ is constant for any reasonable range of pressures p. You can remember this equation by noting its relation to conservation of energy. The first term is a form of kinetic energy, the second term is a form of gravitational potential energy, and the third term is an energy associated with pressure. Since the first two terms are just the usual kinetic and potential energy terms with the mass divided out, the units of the constant on the right-hand side of (1.49) must be energy per unit mass, a fact that can help you remember the form of the third term. Problems involving Bernoulli's principle typically just require applying this relation to two different points along a streamline, often in conjunction with a fluid conservation equation.

From Example 1.14, we can abstract the general strategy for problems involving Bernoulli's principle:

- Write down the Bernoulli equation at all relevant points for the system.
- Write down the equation for fluid conservation.
- Solve these equations for the desired variables.

Remember that not all problems need both the fluid conservation and the Bernoulli equation. If you need to know something about pressure, for example, you'll certainly need Bernoulli, but maybe not the conservation equation. If you need to know something about the velocity of the fluid, the reverse could be true.

1.8.2 Buoyant Forces

Consider a block that floats in water (Fig. 1.19). Obviously the force of gravity pushes down on it, yet it does not sink. This is, of course, because of the buoyant force of the water that the block displaces. If the block displaces a volume of V of water when it is floating, then the buoyant force pushing back up on it will be

$$F = \rho V g, \tag{1.52}$$

where ρ is the density of water (or whatever fluid the block is floating in). You can think of this as the weight of the displaced water pushing up on the submerged object. The mass of the displaced water is ρV, and so its weight is $\rho V g$. Problems involving floating and submerged blocks can usually be solved by simply assigning all forces as usual, and then adding in the buoyant force which pushes upward on the object.

For example, suppose you blow all the air out of your lungs and sink to the bottom of a pool. How much do you weigh underwater? Suppose you weigh 60 kg and your volume is 50 L. You are displacing 50 kg of water, so there is about 500 N of buoyant force pushing up on you. But your weight is 600 N, so your net weight is only 100 N. Quite an effective weight-loss program!

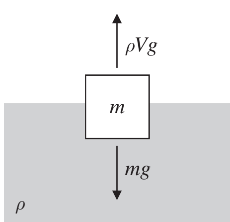

Figure 1.19 Example of buoyant forces. A block floating in water displaces a mass ρV of water, whose gravitational force pushes up against the weight of the block.

1.8.3 Problems: Fluid Mechanics

1. A diver in water picks up a lead cube of side length 10 cm. How much force is needed to lift the cube? The density of lead is approximately 11 g/cm^3.

   (A) 10 N
   (B) 11 N
   (C) 50 N
   (D) 100 N
   (E) 110 N

2. A vertical cylindrical tube has radius 1 cm. The tube is plugged with a stopper at the bottom end and filled with water so that the top of the water is 1 m above the bottom stopper. What is the frictional force required to hold the stopper in the tube?

   (A) 3.1 N
   (B) 2.4 N
   (C) 1.0 N
   (D) 0.62 N
   (E) 0 N

3. An aqueduct consisting of a pipe filled completely with water passes up a hill that is 10 m high. At the bottom of the hill, a flowmeter measures the speed of the fluid to be 2.0 m/s. At the top of the hill, a flowmeter measures the speed of the fluid to be 1.0 m/s. Which is closest to the difference in the fluid pressure between the bottom and top of the hill?

   (A) 1.50×10^3 Pa
   (B) 9.85×10^4 Pa
   (C) 1.00×10^5 Pa
   (D) 1.02×10^5 Pa
   (E) 9.85×10^7 Pa

1.9 Solutions: Classical Mechanics

Blocks

1. B – The happy fact that the plane is at a 45° angle, and $\sin 45° = \cos 45°$, means that we don't have to be especially careful about decomposing forces since the sines and cosines will always be equal. We know that the applied force contributes a normal force of $5\sqrt{2}$ N and a force *up* the ramp of $5\sqrt{2}$ N, and, similarly, gravity (approximating $g \approx 10$ m/s^2) contributes a normal force of $25\sqrt{2}$ N and a force *down* the ramp of $25\sqrt{2}$ N. So before considering

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friction, the net force is $20\sqrt{2}$ N *down* the ramp. Now, friction contributes a force $(0.5)(30\sqrt{2}) = 15\sqrt{2}$ N opposing the block's motion, which means up the ramp in this case. The total net force is then $5\sqrt{2}$ N down the ramp. Dividing by the mass to find the acceleration, we have $\sqrt{2}$ m/s^2 down the ramp, choice B.

2. E – As explained in the text, we could either do separate free-body diagrams for each of the three blocks and solve for the tensions one by one, or we could use some physical intuition and realize that each rope segment must support the full weight of all the blocks below it. So, as long as the blocks have nonzero masses, $T_1 > T_2 > T_3$, *always*.

3. D – As always, we begin by separating the two blocks and drawing free-body diagrams as above. Here, F_{bb} is the block-on-block force, which is equal in magnitude for the two blocks by Newton's third law. Note that the normal force, which provides the friction keeping the mass m stationary, is the block-on-block force, *not* its weight! The plan is to solve for F_{bb} using the second block, then plug that in and solve for F using the first block. For the vertical forces to balance, we need $\mu_2 F_{bb} = mg$, so $F_{bb} = mg/\mu_2$. The acceleration of the second block is $a = F_{bb}/m = g/\mu_2$. This must equal the acceleration of the first block for the two to move together. Now, the normal force on the first block is $Mg + \mu_2 F_{bb} = (M + m)g$, where the $\mu_2 F_{bb}$ term is from the action–reaction pair of the frictional force on m. The net force on the first block is then $F - F_{bb} - \mu_1(m + M)g = F - mg/\mu_2 - \mu_1(m + M)g$, so we set its acceleration equal to g/μ_2 and solve:

$$\frac{F - mg/\mu_2 - \mu_1(m+M)g}{M} = \frac{g}{\mu_2}$$
$$\implies F = \left(\mu_1 + \frac{1}{\mu_2}\right)(m+M)g.$$

The limiting cases check out: if $\mu_2 \to 0$, $F \to \infty$ since a vanishing frictional force between the two blocks means that they slip no matter what, and if $\mu_1 \to \infty$, $F \to \infty$

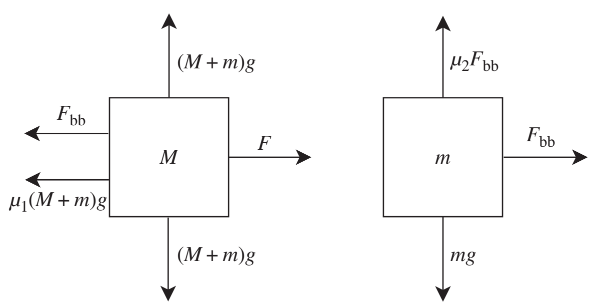

Figure 1.20 Solution for block problem 3.

since the strong kinetic friction prevents block M from accelerating enough and providing a large enough F_{bb}.

Alternatively, by the reasoning of Section 1.1.3, the fact that the two blocks don't slip means that they move as a composite system with mass $m + M$, and so the total force required to make them move together should depend on the combination $m + M$, which singles out choice D. The GRE may not always be this kind to you, but if you see a major simplification that allows you to solve a problem without drawing free-body diagrams, take it! If you have time at the end of the test, of course, come back to the problem and do it the long way to make sure that you didn't oversimplify.

Kinematics

1. D – The projectile will land on the ground at a time given by the solution to

$$0 = -\frac{1}{2}gt^2 + vt\sin\theta.$$

The solution is

$$t = \frac{2v\sin\theta}{g}.$$

The distance away at time t is given by

$$vt\cos\theta = d.$$

Substituting the result above, we find that

$$\cos\theta\,\sin\theta = \frac{gd}{2v^2}.$$

Using the double angle formula, we conclude that

$$\theta = \frac{1}{2}\arcsin\frac{gd}{v^2}.$$

2. C – In a geosynchronous orbit, the satellite will orbit with the same angular velocity as the Earth, which is constant, so its orbital speed is constant and we can use the uniform circular motion formulas. Just equate the centripetal force to the gravitational force to obtain

$$\frac{GM_E m}{r^2} = \frac{mv^2}{r}.$$

The velocity of the satellite at radius r is $v = r\omega$. Thus, we have

$$\frac{GM_E m}{r^2} = mr\omega^2.$$

Solving for r, we find that

$$r = \left(\frac{GM_E}{\omega^2}\right)^{1/3}.$$

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Energy

1. B – From the $30 - 60 - 90$ triangle, the ramp has height $L/2$. Setting the zero of gravitational potential at the bottom of the ramp, the initial energy is all potential energy of the spring, $\frac{1}{2}kx^2$. At the top of the ramp, we have potential $mgh = mgL/2$ and *that's it*: if it just barely reaches the top of the ramp, its kinetic energy (translational *and* rotational) must be zero there. So we have simply

$$\frac{1}{2}kx^2 = \frac{mgL}{2} \implies x = \sqrt{\frac{mgL}{k}}.$$

2. C – Now there is no change in potential energy immediately after being launched, so the ball's energy is all kinetic. Because it rolls without slipping, we have to take into account translational and rotational energies, using the appropriate moment of inertia for a sphere:

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{7}{10}mv^2.$$

(See the calculation at the end of Section 1.3.3 for more details.) We may be tempted to set this equal to the potential energy of the spring, but that's more work than necessary: since energy is conserved everywhere, we can feel free to use the simpler expression $mgL/2$ for the energy at the top. This gives

$$\frac{7}{10}mv^2 = \frac{mgL}{2} \implies v = \sqrt{\frac{5}{7}gL}.$$

3. B – Without doing any work, we know the answer here must be the same as the answer to problem 1. While it's true that, during the trip up the ramp, the ball's kinetic energy will no longer be shared between rotational and translational, this is irrelevant once it gets to the top of the ramp, when all the energy is potential. So the spring is compressed by precisely the same distance. Once again, the *time* it takes the ball to get up the ramp will be different than in the case with friction, but the problem doesn't ask about that, so we need not worry about it.

Momentum

1. B – It is important to keep the notation straight in this problem. The r in the mass density $\rho(r)$ refers to the distance between the origin and a point in the sphere, but when we compute the moment of inertia, the argument of the integral is the square distance from *a point in the sphere to the axis of rotation* – in this case, we can choose coordinates to make it the z-axis. If we use $s = r\sin\theta$ to denote this distance, then the moment of inertia is

$$\begin{aligned} I &= \int s^2\,dm \\ &= \int r^2\sin^2\theta\,\rho(r)\,dV \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} Ar^3\sin^2\theta\, r^2\sin\theta\,dr\,d\theta\,d\phi \\ &= \frac{1}{6}AR^6\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\,d\theta\,d\phi \\ &= \frac{\pi}{3}AR^6\int_0^{\pi}\sin^3\theta\,d\theta \\ &= \frac{4\pi}{9}AR^6. \end{aligned}$$

As usual, we get rid of A by expressing it in terms of the total mass of the sphere. Since the density depends only on radius, we can integrate spherical shells of thickness dr and mass $4\pi r^2\,dr\rho(r)$, so the only integral is the radial one:

$$M = 4\pi\int_0^R Ar(r^2\,dr) = \pi AR^4 \implies A = \frac{M}{\pi R^4}.$$

Plugging this in, we get $I = \frac{4}{9}MR^2$, which has the correct units for a moment of inertia.

2. D – Since the momentum must sum to zero, and since all the masses of the pieces are equal, the velocity vectors must also sum to zero. This means that the velocity vectors can be arranged as the sides of an isosceles triangle. Solving the rest of the problem is just basic geometry. The angle between the long and short sides of the isosceles triangle is given by $\cos\theta = (v/2)/(2v) = 1/4$, so $\theta = \arccos(1/4)$. The angle between the outward-going velocity vectors of the exploding fragments is then $\pi - \arccos(1/4)$.
3. B – Straightforward conservation of angular momentum. The initial angular momentum is $L_i = \frac{1}{2}MR^2\omega_0$. The final angular momentum is $L_f = \frac{1}{2}M(R^2 + r^2)\omega$. Solving for ω, we find choice B.
4. A – Call the initial angular velocity ω_0 and the final angular velocity ω. Angular momentum is conserved because tension acts radially and hence provides no torque, so we have the constraint

$$MR^2\omega_0 = Mr^2\omega,$$

so

$$\omega = \frac{R^2}{r^2}\omega_0.$$

The initial tension must be equal to the centripetal force, so

$$T = MR\omega_0^2.$$

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The change in energy is just

$$\Delta E = \frac{1}{2}M\left(r^2\omega^2 - R^2\omega_0^2\right)$$
$$= \frac{1}{2}M\left(\frac{TR^3}{Mr^2} - \frac{TR}{M}\right)$$
$$= \frac{1}{2}TR\left(\frac{R^2}{r^2} - 1\right).$$

Note that this is a case in which energy is not conserved, but angular momentum is. Work is required to pull the string downwards and change the radius of the rotation, hence the energy increases.

Lagrangians and Hamiltonians

1. D – Following the recipe outlined in Section 1.5.1, we define the origin of coordinates to be at the pivot, giving

$$x = l\sin\theta\cos\phi,$$
$$y = l\sin\theta\sin\phi,$$
$$z = -l\cos\theta.$$

Note the minus sign in front of the z-coordinate! These are not quite spherical coordinates because of the way the angle θ was defined by the problem. This is conventional for pendulums though, so it's good to get used to it. Now, taking time derivatives, we have

$$\dot{x} = l\cos\theta\cos\phi\,\dot{\theta} - l\sin\theta\sin\phi\,\dot{\phi},$$
$$\dot{y} = l\cos\theta\sin\phi\,\dot{\theta} + l\sin\theta\cos\phi\,\dot{\phi},$$
$$\dot{z} = l\sin\theta\,\dot{\theta}.$$

We now form the kinetic energy:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right).$$

The algebra really isn't all that bad once you realize that the cross term cancels in $\dot{x}^2 + \dot{y}^2$, and all the rest collapse using the trig identity $\sin^2\alpha + \cos^2\alpha = 1$. The potential energy is all gravitational,

$$U = mgz = -mgl\cos\theta,$$

so we have

$$L = T - U = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgl\cos\theta.$$

Note that this is not the unique Lagrangian, since we can always add a constant to U, and hence to L, without changing the physics. But it matches one of the answer choices, so we can move on.

2. B – We notice that the Lagrangian is independent of the coordinate ϕ, which means the conjugate momentum p_ϕ is conserved:

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = ml^2\sin^2\theta\dot{\phi}.$$

Happily, this matches answer choice B. When looking for conserved quantities in problems like this, it's a *much* better idea to find them yourself and check them against the answer choices, rather than trying to ascertain whether each of the answer choices individually is conserved. Incidentally, the only other conserved quantity in this problem is the total energy, since L is independent of time.

3. A – Here we can apply the trick mentioned in Section 1.5.3: rather than compute the Hamiltonian using the Legendre transform, just use the fact that $H = T + U$ since there is no time dependence. We've already computed one canonical momentum, so we need the other:

$$p_\theta = ml^2\dot{\theta}.$$

This gives us

$$\dot{\theta} = \frac{p_\theta}{ml^2},$$
$$\dot{\phi} = \frac{p_\phi}{ml^2\sin^2\theta},$$

and plugging into T gives

$$T = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2\sin^2\theta}.$$

Adding, rather than subtracting, the potential energy this time gives us the Hamiltonian:

$$H = T + U = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta.$$

4. C – Restricting to a constant value of ϕ means that we can drop the $\dot{\phi}$ term from the Lagrangian:

$$L_{\phi=0} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta.$$

Now we compute the two quantities we need for the Euler–Lagrange equations:

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta},$$
$$\frac{\partial L}{\partial\theta} = -mgl\sin\theta.$$

Applying the Euler–Lagrange equation (1.29), we have

$$ml^2\ddot{\theta} = -mgl\sin\theta,$$

or canceling ml from both sides and rearranging,

$$\ddot{\theta} = -\frac{g}{l}\sin\theta.$$

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Expanding $\sin\theta \approx \theta$ for small θ, you should recognize this as the simple harmonic oscillator equation. Indeed, this is just an ordinary pendulum, with the correct angular frequency $\sqrt{g/l}$.

Orbits

1. A – Don't let the complicated-seeming problem statement fool you: this is just an application of the orbit formalism to the central potential $U(r) = \frac{1}{2}kr^2$ for a spring. We're looking for the radius of a circular orbit, so we solve $V'(r) = 0$:
$$\frac{-l^2}{mr^3} + kr = 0 \implies r = (l^2/mk)^{1/4}.$$
We can check that $V''(r) = k + 3l^2/mr^4 > 0$, so this is indeed a stable circular orbit. Incidentally, since all of the answer choices have different units, this would have been a perfect chance to practice using dimensional analysis.
2. C – This is a straightforward application of Kepler's third law. In the limit where planetary masses are small compared to that of the Sun, we have
$$\frac{T_{\text{new}}}{T_{\text{Mars}}} = \left(\frac{a_{\text{new}}}{a_{\text{Mars}}}\right)^{3/2} \implies a_{\text{new}} = a_{\text{Mars}} \times 2^{2/3}.$$
$2^{2/3} > 1$, so only III is true. In this limit the planet masses don't show up explicitly in Kepler's third law, so I is not necessarily true.
3. B – A parabolic orbit means that the total energy of the orbit is zero. Because m is negligibly small compared to M, we can use m instead of the reduced mass μ in the formulas for angular momentum and effective potential. Right at the point of closest approach, $\dot{r} = 0$, so all the motion is tangential and we can write $l = mvd$. Substituting into the expression for orbital energy (1.38), we have
$$E = \frac{(mvd)^2}{2md^2} - \frac{GMm}{d} = \frac{1}{2}mv^2 - \frac{GMm}{d}.$$
Setting this equal to zero and solving for d, we find choice B.

Springs and Harmonic Oscillators

1. D – Computing equivalent spring constants is discussed in Section 1.7.3. The equivalent spring constant for the block of mass M attached to the series springs is
$$k_{\text{eq}} = \left(\frac{1}{k} + \frac{1}{k}\right)^{-1} = \frac{k}{2}.$$
The equivalent spring constant for the block of mass m attached to the three parallel springs is
$$k_{\text{eq}} = k + k + k = 3k.$$
The ratio of the frequencies is therefore
$$\frac{\sqrt{k/(2M)}}{\sqrt{3k/m}} = \sqrt{\frac{m}{6M}}.$$
2. C – The kinetic energy of the ball at launch is equal to the potential energy stored in the spring, so
$$\frac{1}{2}mv^2 = \frac{1}{2}kd^2,$$
and
$$v = \sqrt{\frac{k}{m}}d.$$
The velocity in both the vertical and horizontal directions is therefore
$$v_0 = \frac{\sqrt{2}}{2}\sqrt{\frac{k}{m}}d.$$
The horizontal displacement is $x = v_0 t$, and the vertical displacement is $y = -(1/2)gt^2 + v_0 t$. Solving the latter equation for t at $y = 0$, we find
$$t = \frac{2v_0}{g}.$$
Substituting this into the equation for the horizontal displacement, we find
$$x = \frac{2v_0^2}{g} = \frac{kd^2}{gm}.$$
3. B – Recall that the amplitude of oscillation of an undamped driven oscillator scales as
$$A \sim \frac{1}{|\omega_0^2 - \omega^2|}.$$
Thus, as we increase ω from $2\omega_0$ to $3\omega_0$, the amplitude is multiplied by a factor of 3/8.

Fluid Mechanics

1. D – The mass of the lead cube is 11 kg, so the weight of the cube is 110 N. The volume of the cube is 10^3 cm^3, so the mass of the displaced water is 1 kg, since the density of water is 1 g/cm^3. The force required to lift the cube is just the weight minus the buoyant force from the displaced water: $F_{\text{lift}} = 110\text{ N} - 10\text{ N} = 100\text{ N}$.
2. A – By the Bernoulli equation, the pressure exerted by the fluid on the bottom stopper must be $p = \rho g y$, where y

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is the level of the water line. (Atmospheric pressure acts equally at the top and at the bottom, so it cancels out.) The frictional force is therefore $F = pA = \pi\rho g y r^2$. Plugging numbers we have

$$F = \pi(10^3 \text{ kg/m}^3)(10 \text{ m/s}^2)(1 \text{ m})(10^{-2} \text{ m})^2 \approx 3.1 \text{ N}.$$

3. B – By Bernoulli's principle, we have

$$\frac{1}{2}\rho v_t^2 + \rho g y_t + p_t = \frac{1}{2}\rho v_b^2 + \rho g y_b + p_b,$$

where the left-hand side contains quantities measured at the top of the hill, and the right-hand side contains quantities measured at the bottom of the hill. We can rearrange and solve for the difference between the pressures at the top and bottom:

$$p_b - p_t = \rho g(y_t - y_b) + \frac{1}{2}\rho(v_t^2 - v_b^2).$$

Plugging in $y_t - y_b = 10$ m, $\rho = 1000$ kg/m^3, $g = 10$ m/s^2, $v_t = 1$ m/s, and $v_b = 2$ m/s, we find the result 9.85×10^4 Pa.