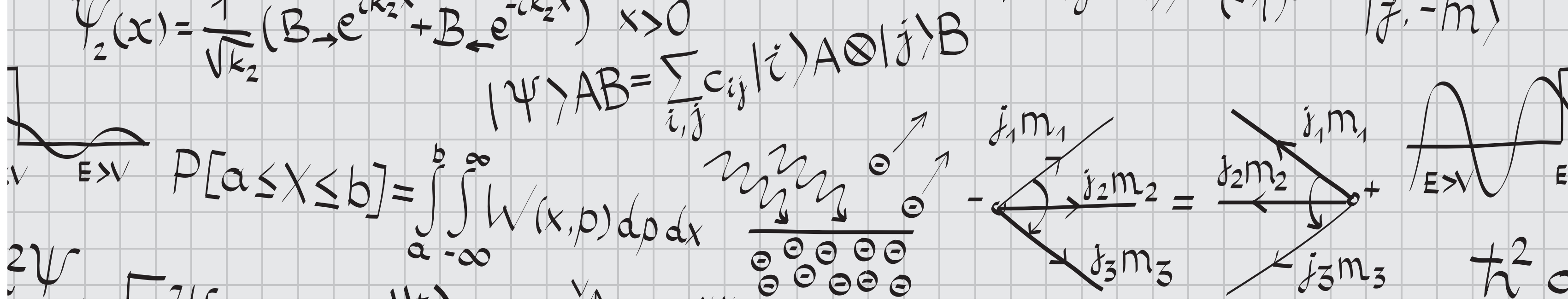

6 Special Relativity

Problems on relativity in the GRE often look simple, but can be tricky as they tend to deal more with conceptual issues than with detailed calculations. Throughout this chapter, keep in mind how important it is to define things precisely: concepts such as time, length, and simultaneity, which seem so obvious in classical mechanics, can actually be quite subtle. Our discussion will necessarily be brief, so by all means consult our recommended references if you want more details.

6.1 Relativity Basics

There are two simple postulates from which all of special relativity follows:

1. The speed of light in vacuum is a constant (denoted c) in all inertial reference frames.
2. The laws of physics are identical in all inertial reference frames.

The crucial idea in both postulates is the *inertial reference frame,* which for the purposes of the GRE just means an observer traveling at constant velocity – that is, in a straight line with constant speed. So another way of stating the second postulate is that the apparent laws of physics do not change as long as the observer is in a frame that is moving at a constant velocity. We should add that only inertial frames with velocity $v < c$ are allowed: this is the familiar statement that no signals propagate faster than light.

These postulates lead directly to the famous Lorentz transformations, which relate the coordinates used to describe two inertial frames. Since this can get a little confusing, let's first consider a simpler example, where the velocities are small compared to c and we can use ordinary coordinate transformations familiar from classical mechanics. Let system S be a person standing on the side of the highway, and system S' be a car traveling on the highway at constant speed v. We'll let (t, x) represent the time and distance along the highway, respectively, for the person in S, and (t', x') represent the analogous quantities for the car in S', where $x' = 0$ corresponds to the front bumper of the car. The first thing to do is synchronize the origin of coordinates in both systems: let $t = t' = 0$ be the instant where the front bumper of the car passes the person on the side of the highway, and let this occur at $x = x' = 0$. The car in S' is moving with constant speed, so its position at time t is $x = vt$. In the frame of the car, then, an object at position x' with respect to the front bumper has traveled an additional distance vt as seen by the person in S: $x = x' + vt$, or $x' = x - vt$. Finally, in classical mechanics, clocks tick at the same rate everywhere, so $t = t'$. Thus we have the *Galilean transformations* relating the coordinates (t, x) to (t', x'):

$$t' = t,$$
$$x' = x - vt.$$

We can check that these make sense: at $t' = t = 0$, we have $x' = x$, corresponding to the synchronizing of clocks described earlier.

However, in special relativity, this simple behavior goes out the window. At velocities near the speed of light, *time and space mix with each other.* The analogous transformations between S and S' are

$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \tag{6.1}$$

$$x' = \gamma\,(x - vt)\,, \tag{6.2}$$

https://doi.org/10.1017/9781108296977.009

where γ is often called the *Lorentz factor*, and is defined by

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{6.3}$$

These are the *Lorentz transformations* for one-dimensional motion along the x-axis. They tell you, given coordinates (t, x) in S, what the analogous coordinates (t', x') are in S'. Notice how the t' equation involves *both* x and t, which is very much unlike the simple statement that $t' = t$ in classical mechanics. The inverse transformations, which express S coordinates in terms of S' coordinates, are

$$t = \gamma \left(t' + \frac{v}{c^2} x' \right), \tag{6.4}$$

$$x = \gamma \left(x' + vt' \right). \tag{6.5}$$

These are extremely easy to remember: just flip the sign of v! This is the concept of "relativity" in action: if S' is moving to the right with respect to S, then from the point of view of an observer in S', it's S that is moving to the left. (Just picture how the scenery seems to fly by backwards as you're driving in a car.)

We're now going to look in detail at the consequences of these transformations. In what follows, think of S as a collection of clocks, one at each point x, all of which are synchronized with each other: a clock at $x = x_1$ and another at $x = x_2$ will all read the same time t. Similarly, S' is a different collection of clocks, all traveling together at velocity v with respect to S, and all synchronized with each other. However, they are *not* all synchronized with the clocks in S! The best we can do is synchronize the origin of coordinates, as we did in our simple example above: we define $t = t' = 0$ to be the instant that the $x' = 0$ clock in S' passes the $x = 0$ clock in S. This is consistent with the Lorentz transformations we wrote down: at $(t, x) = (0, 0)$, we also have $(t', x') = (0, 0)$.

6.1.1 Simultaneity

Suppose two events are *simultaneous* in system S: that is, event A occurs at position x_A, while event B occurs at position x_B, and they both happen at time t. Looking at the t' equation, we find

$$t'_A = \gamma \left(t - \frac{v}{c^2} x_A \right), \quad t'_B = \gamma \left(t - \frac{v}{c^2} x_B \right).$$

These are not the same! In fact, unless $x_A = x_B$ (in which case A and B might as well have been the same event), the times measured in S' are different, and so A and B are *not* simultaneous in S'. Since simultaneity is a concept that is so ingrained in our intuition, and we use it to build other concepts such as length and causality, this is a red flag which we will have to be very careful of in the context of special relativity. This also explains why we can't synchronize all the clocks in S' with all the clocks in S: synchronization is the same as asking for clocks to simultaneously read the same time, and, as we've just seen, that's impossible for two clocks at different positions.

6.1.2 Time Dilation

Suppose we're sitting in S, and we wait for a time interval Δt. How much time elapses in frame S'? Here we have to carefully define what we mean by "time elapsed": since we're thinking of S' as a set of clocks, one at each position, we want to follow a *single* clock, at a *fixed* position x', for a time Δt. So the appropriate equation is (6.4), relating Δt to $\Delta t'$ at fixed x':

$$\Delta t = \gamma \Delta t' \quad (\text{fixed } x'). \tag{6.6}$$

Notice that $\gamma \geq 1$, since $v < c$, so for a given interval of time that elapses in the moving frame S', *more* time elapses in the stationary frame, by a factor of γ. This is known as

Time dilation: moving clocks run slower by a factor of γ.

Here is where things start to get confusing. An easy mistake to make is to use fixed x instead of fixed x': plugging this into the equation for t', we would find $\Delta t' = \gamma \Delta t$, which is precisely the *opposite* of the correct result! The problem is that fixed x corresponds to looking at a whole sequence of clocks in S' as they fly by, one by one, which does *not* correspond to measuring any kind of time elapsed in S' because of the issue of synchronization mentioned earlier. The moral of the story is that, while "time dilation" sounds nice and simple, it's very easy to get mixed up by exactly whose time is being dilated. Luckily, the instances you'll see on the GRE are all pretty standard, and you'll see many examples in the practice problems.

6.1.3 Lorentz Contraction

Suppose we are sitting in S, and want to measure the length of an object in S'. For simplicity, let's put the back end of the object at $x' = 0$ and the front end at $x' = L'$. Now, to measure the length of the object as seen from S, we need to note the positions of the two ends at the *same* S-time t. This brings in simultaneity, but that's OK since the clocks in S are all synchronized with each other, so we can talk about events that are simultaneous in S. Applying the x' equation (6.2) at $x' = 0$

and $x' = L'$, we have

$$0 = \gamma(x_1 - vt),$$
$$L' = \gamma(x_2 - vt).$$

Subtracting the first from the second, and defining $L = x_2 - x_1$ as the length in S, we obtain

$$L' = \gamma L \quad (\text{fixed } t). \tag{6.7}$$

Again, the two definitions of length differ by the same factor γ. But note the direction! Since L, the length as measured in S, is smaller than L', we remember this as

Length contraction: moving objects are shortened by a factor of γ.

Note that we don't say "moving objects *appear* shortened." As far as we can define the concept of length, they *are* shortened.

6.1.4 Velocity Addition

Here's a familiar situation from classical mechanics: you're riding in a car at 100 km/h, and you throw a ball forwards out the window at 5 km/h. Ignoring air resistance, from the point of view of someone standing on the side of the highway, the ball travels at $100 + 5 = 105$ km/h. In our language of inertial reference frames, we would say that the stationary observer defines a reference frame S, the car defines a frame S', and the ball defines a third reference frame S'': we just calculated the velocity of S'' with respect to S, given its velocity with respect to S'.

With this formalism, it's easy to re-evaluate this situation in the context of special relativity, and as expected it differs from the classical result. Suppose S' travels at velocity v, and the ball travels at velocity u with respect to S'. Then a bit of algebra with the Lorentz transformation equations gives the *Einstein velocity addition rule* for the velocity w as seen in frame S:

$$w = \frac{v + u}{1 + vu/c^2}. \tag{6.8}$$

This formula is useful to memorize since it is tricky to derive, but simple to remember. Here are a couple sanity checks that will make memorizing this formula easy. The factor of $1/c^2$ in the denominator (required for dimensional consistency) means that at velocities very small compared to the speed of light, $u, v \ll c$, the formula reduces to the usual addition of velocities, $w = u + v$. That gives you the numerator. On the other hand, plugging in $u = c$ (corresponding to shooting a laser beam out of the car, instead of throwing a ball), we find $w = c$. This recovers the first postulate of relativity, that the velocity of light is the same in all inertial frames: here, we've shown it to be true for a frame moving at velocity v. Finally, note that the signs must always match: if we throw the ball backwards rather than forwards, we should change the sign of u in both the numerator and the denominator. By the way, this velocity addition formula only applies to one-dimensional motion, but that's the only case you'll see on the GRE: the general formula is considerably more complicated and isn't worth memorizing.

6.2 4-Vectors

We're now going to introduce some notation that will make the previous results easy to remember, and generalizes easily to other useful physics situations. Define

$$x^0 = ct, \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z, \tag{6.9}$$

which emphasizes the fact that space and time are treated on a similar footing in relativity: they're both just coordinates. Note that the superscripts are labels, *not* exponents! This notation is totally standard, so, as confusing as it is, we'll stick with it because it will match what you'll see on the exam. We can collect these coordinates into a single object called a *4-vector*:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z). \tag{6.10}$$

In this notation, the superscript μ is again a label that takes the values 0, 1, 2, or 3.

6.2.1 Lorentz Transformation Matrices

Now, we can write the Lorentz transformations as a matrix equation involving the vector x^μ. Defining

$$\beta = v/c \tag{6.11}$$

as the velocity of S' in units of c, we have

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{6.12}$$

The top 2×2 block of the matrix reproduces (6.1) and (6.2), but is much easier to remember because all the annoying factors of c have been absorbed into the various symbols γ, β, x^0, and x^1. The rest of the matrix comes from the fact that a Lorentz transformation along the x-axis does not touch the y or z coordinates. By the way, a Lorentz transformation is often called a *boost*. It is unlikely that you will see a problem on the GRE that will ask you to simply plug in numbers to

https://doi.org/10.1017/9781108296977.009

the Lorentz transformation, but we have encountered GRE questions that required identifying the form of the Lorentz transformation. The generalization to boosts along the other coordinate axes is straightforward, and you'll see an example in the problems at the end of this chapter.

In fact, there are several other quantities whose Lorentz transformation properties use that exact same matrix. In addition to position $x^\mu = (ct, x, y, z)$, which we have already discussed, the other useful 4-vectors are

$$\text{Energy–momentum: } p^\mu = (E/c, \mathbf{p}), \tag{6.13}$$

$$\text{Current density: } j^\mu = (c\rho, \mathbf{J}), \tag{6.14}$$

$$\text{Wavevector: } k^\mu = (\omega/c, \mathbf{k}). \tag{6.15}$$

The first one needs some clarification: when we write $\mathbf{p}$, we mean the *relativistic* momentum, which differs from the usual definition of momentum by a crucial factor of γ:

$$\mathbf{p} = \gamma m\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1 - |\mathbf{v}|^2/c^2}}. \tag{6.16}$$

With this caveat, all of the 4-vectors listed above satisfy the matrix equation (6.12) for a boost along the x-axis, with the components x^0, x^1, etc. replaced by the appropriate components of the 4-vector. Note that not every random collection of four objects satisfies this property, just like not every collection of three objects transform correctly under rotations, as would be true for an ordinary vector. But for the purposes of the GRE, you don't need to know where these 4-vectors come from: in fact, it's probably sufficient just to memorize the energy–momentum 4-vector, as it is by far the most common.

Speaking of the energy–momentum 4-vector, we should mention a couple of key properties you're probably already familiar with, but are very important for the GRE. Consider a particle of mass m. In its rest frame, its velocity is zero, so $p^\mu = (E_0/c, 0, 0, 0)$. It turns out that the *rest energy* E_0 is

$$E_0 = mc^2, \tag{6.17}$$

an equation so famous it barely even needs explaining. In another inertial frame, the zeroth component of p^μ will still contain a contribution from the rest energy, and we define the remainder as the kinetic energy:

$$T = E - mc^2. \tag{6.18}$$

In fact, plugging p^μ into (6.12) we see that the energy in a frame other than the rest frame is given by

$$E = \gamma mc^2, \tag{6.19}$$

so

$$T = (\gamma - 1)mc^2. \tag{6.20}$$

It's an excellent exercise to Taylor expand this last equation for $v \ll c$ and see that we recover the correct nonrelativistic expression for the kinetic energy T.

6.2.2 Relativistic Dot Product

With this covariant notation it is easy to write down an extremely important quantity, the 4-vector product, or *relativistic dot product*. The dot product of two 4-vectors is defined to be[1]

$$a \cdot b \equiv a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3. \tag{6.21}$$

At this point many relativity texts go into enormous detail about covariant versus contravariant indices, the metric tensor, Einstein summation convention, and so forth. Forget about all that: (6.21) is all you ever have to remember for the GRE about the relativistic dot product. Its key property is that it is *invariant under Lorentz transformations*, in exactly the same way that the ordinary 3-vector dot product is invariant under rotations of the coordinate axes. Practically speaking, this means that you can evaluate the dot product in *any inertial frame* you want: you'll get the same answer no matter which frame you use. The individual components of the 4-vectors will change, but the combination $a \cdot b$ remains the same. This often allows us to work in the reference frame with the simplest physics.

There are two important special cases of this formula, both involving dotting a 4-vector with itself. The first will give us a classification of spacetime events based on the sign of the dot product, and the second is a useful formula relating energy and momentum. The power of both of these results is that, because they use the invariant dot product, they are independent of the reference frame, and hold regardless of which coordinate system we choose for the 4-vectors themselves.

- **Invariant interval.** Given two position 4-vectors x_A^μ and x_B^μ, we can define the *displacement 4-vector*

  $$\Delta x^\mu \equiv x_B^\mu - x_A^\mu$$

  that represents the spacetime vector between two events, A and B, occurring at x_A and x_B respectively. The reason it's easier to work with displacement rather than position is that relative positions, rather than absolute positions, are

[1] **Warning!** Some texts define the dot product with an extra overall minus sign, so be careful! This convention is fairly standard, but be prepared to be flexible about sign conventions depending on where you've learned relativity previously.

https://doi.org/10.1017/9781108296977.009

actually meaningful. Now, dotting Δx^μ with itself gives the spacetime "distance" between the two events, also known as the *invariant interval*, $(\Delta x)^2$. (The notation p^2 for the relativistic dot product of a 4-vector p^μ with itself is standard, but don't confuse it with the square of a scalar, or the ordinary dot product!) Crucially, this quantity can be positive, negative, or zero. Each of these cases has a name and a corresponding physical interpretation:

$$\text{Timelike: } (\Delta x)^2 > 0, \tag{6.22}$$

$$\text{Spacelike: } (\Delta x)^2 < 0, \tag{6.23}$$

$$\text{Lightlike or null: } (\Delta x)^2 = 0. \tag{6.24}$$

Two events that are timelike-separated are in causal contact: there exists an inertial frame where both events occur at the same *place*. It's useful to imagine this frame as a spaceship that travels between events A and B: in the frame of the spaceship, both events occur at the same spatial point (namely, the origin of the spaceship coordinate system), but at different times (corresponding to how long it takes the spaceship to travel between them).

For events that are spacelike-separated, there exists an inertial frame where both events occur at the same *time*. This gives a precise condition for simultaneity: simultaneous events must be spacelike-separated (though not all spacelike-separated events are simultaneous, since that depends on the frame). Incidentally, these events are *not* in causal contact: if there is a frame such that event A occurs before event B, there exists another frame such that the order is reversed, and event B occurs before event A! Thus the whole notion of causality doesn't make sense for spacelike-separated events. The intuition is that these kinds of events are so "far away" from each other that there is no inertial frame traveling slower than light that can go between them.

Finally, lightlike-separated events correspond to paths of light rays: A and B are lightlike-separated if and only if they lie on a trajectory traveling at the speed of light. The signs and names here are quite tricky, but the best way to remember them is to consider the simplest of all possible displacement 4-vectors: $\Delta x^\mu = (c\Delta t, 0, 0, 0)$, corresponding to sitting in the same place for a time Δt. This 4-vector clearly has $(\Delta x)^2 > 0$, and it only has a time component (hence "timelike"), and furthermore it represents the displacement in the frame where the two events occur at the same place.

- **Energy–momentum formula.** The second important application of the relativistic dot product is to the energy–momentum 4-vector. Consider a particle of mass m. As we noted above, in its rest frame the energy–momentum 4-vector is $p^\mu = (mc, 0, 0, 0)$, which satisfies $p \cdot p = m^2c^2$. On the other hand, plugging its components into a general frame from (6.13) into (6.21), we find $p \cdot p = E^2/c^2 - \mathbf{p}^2$. Setting these expressions equal and rearranging, we find the very useful formula

$$E^2 = \mathbf{p}^2c^2 + m^2c^4. \tag{6.25}$$

This lets us determine a particle's energy given its momentum without ever having to deal with its velocity, which can save a lot of time in calculations.

6.3 Relativistic Kinematics

One of the main applications of relativity is to kinematics problems: systems of particles that decay and collide, but whose speeds are large and so must be treated relativistically. It's important to remember that whenever relativity is involved, we *must* use the relativistic energy $E = \gamma mc^2$ and relativistic momentum $\mathbf{p} = \gamma m\mathbf{v}$: forgetting the factors of γ will likely lead to trap answers. But apart from that, the setup of the problems should be very familiar from classical mechanics. Despite the importance of these kinds of problems in the physics curriculum, they are fairly under-represented on the GRE, at least based on our experience and the exams that have been released so far. A careful treatment with many examples can be found in both books by Griffiths, but we'll stick to the bare-bones treatment here.

6.3.1 Conserved vs. Invariant

The reason for introducing all the previous definitions is that the quantities we've defined satisfy certain properties that make calculations easier. As you undoubtedly remember from classical mechanics, momentum is conserved in the absence of external forces. For any relativity question you'll see on the GRE, we can drop that caveat about external forces: *relativistic energy–momentum is conserved*. Note that because the Lorentz transformations mix up space and time components, asking for the relativistic momentum to be conserved *implies* that the whole energy–momentum 4-vector is conserved, which includes the relativistic energy as its first component. We can write the conservation as a 4-vector equation:

$$\sum_i p_i^\mu = \sum_f p_f^\mu, \tag{6.26}$$

where p_i are the incoming 4-vectors and p_f are the outgoing 4-vectors. As with ordinary vector equations, this means that

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each component of the total 4-momentum must match before and after the collision.

We've also introduced the relativistic dot product, which is invariant under Lorentz transformations. The GRE loves to test the subtleties of these two definitions, so let's be totally clear:

Conserved = same before and after. **Invariant** = same in every reference frame.

A very common question will give a list of quantities and ask whether they are conserved, invariant, both, or neither. For example, the total momentum of a system is conserved, but is not invariant, because it can be transformed to zero by going to the center-of-momentum frame. The kinetic energy of a system is neither conserved (since it's not the whole relativistic energy, but only a part of it) nor invariant (because it can be changed by transforming to another frame). An additional example can be found in the problems at the end of the chapter.

6.3.2 Exploiting the Invariant Dot Product

A standard trick in kinematics problems is to exploit two key properties of the relativistic dot product:

- It takes the same value in any reference frame.
- The square of a particle's energy–momentum 4-vector is equal to its mass squared with a factor of c^2: $p^2 = m^2c^2$. (You can put the c's in the right place by remembering that the whole energy–momentum 4-vector has units of momentum.)

This trick, suitably applied, will (almost) always let you calculate energies and momenta without ever having to compute a Lorentz factor γ or a velocity β. The idea is to choose a

EXAMPLE 6.1

Suppose we have a particle of mass M at rest, decaying to two particles of masses m_2 and m_3. What is the energy of m_2?

Let p_1 be the 4-vector of M, and p_2, p_3 be the 4-vectors of m_2 and m_3. By conservation of momentum, $p_1 = p_2 + p_3$, but for reasons we'll see in a moment, we actually want to write this as

$$p_1 - p_2 = p_3.$$

Now we square both sides using the relativistic dot product:

$$p_1^2 + p_2^2 - 2p_1 \cdot p_2 = p_3^2.$$

Note that the usual algebraic rules for squaring a sum apply to the relativistic dot product as well. Now, since M is at rest, $p_1 = (Mc, 0, 0, 0)$. We don't know p_2 yet, but we can always write it as $(E_2/c, \mathbf{p}_2)$. Note that $p_1 \cdot p_2$ exactly isolates E_2:

$$p_1 \cdot p_2 = (Mc)(E_2/c) - 0 = ME_2,$$

which is what we're looking for! We don't know p_3 either, but by the second property of the dot product, $p_3^2 = m_3^2c^2$. Making these replacements, we have

$$M^2c^2 + m_2^2c^2 - 2ME_2 = m_3^2c^2,$$

and solving for E_2 gives

$$E_2 = \frac{(M^2 + m_2^2 - m_3^2)c^2}{2M}.$$

The key step here was to move p_2 to the other side so as to isolate E_2 in the dot product: if we wanted the energy of m_3, we would have done the same with p_3. Just squaring the conservation equation $p_1 = p_2 + p_3$ directly would not have helped, since it would have involved the dot product $p_2 \cdot p_3$ of two 4-vectors we know nothing about. At most, you'll see one of these types of problems on the GRE, but it's still an important trick which can save you precious minutes compared to calculating Lorentz factors directly.

https://doi.org/10.1017/9781108296977.009

EXAMPLE 6.2

Suppose that a galaxy is moving toward us at some substantial fraction of the speed of light, and emitting red light: what wavelength of light do we receive? The motion towards us means that the light will be blueshifted, so the wavelength will decrease, and we should flip the sign of β so that the numerator is smaller than the denominator:

$$\lambda_{\text{rec}} = \sqrt{\frac{1-\beta}{1+\beta}}\lambda_{\text{emit}}.$$

Similarly, from $\lambda f = c$, we can find the change in f by taking the reciprocal:

$$f_{\text{rec}} = \sqrt{\frac{1+\beta}{1-\beta}}f_{\text{emit}}.$$

frame, and a combination of 4-vectors, such that the square has as many zeros as possible and so is easier to calculate. Example 6.1 shows how this technique is used.

6.4 Miscellaneous Relativity Topics

Here are a couple of odd topics not covered by the previous discussion, but which have appeared frequently on the GRE.

6.4.1 Relativistic Doppler Shift

We've already covered the Doppler shift in Chapter 3, but the relevant formulas change slightly when we include the effects of relativity. Recall that the formula for the Doppler shift depended on both the velocity of the source and the velocity of the emitter. But according to special relativity, if we're asking about the Doppler shift of *light*, there are no privileged reference frames, this distinction is meaningless, and the shift can only depend on the *relative* velocity between the source and the observer. If this relative velocity is $v = \beta c$, then the change in wavelength of the emitted light is

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}}. \tag{6.27}$$

This simple-looking equation is actually quite tricky to derive, so we recommend simply memorizing it. The signs, as well as the corresponding formula for the frequency shift, can be deduced from context and some physical intuition. See Example 6.2.

As with velocity addition, the Doppler shift formula only applies to collinear motion, where the source and emitter move along the same line. There also exists a transverse Doppler effect, but we are not aware of it ever having shown up on the GRE thus far.

6.4.2 Pythagorean Triples

You'll likely be expected to do some number-crunching in the relativity questions, either by calculating length contractions, Doppler shifts as above, or energies of particles in collisions. These all involve the ubiquitous factor $\gamma = 1/\sqrt{1-\beta^2}$, and taking square roots by hand is annoying. Luckily, since the GRE is made to minimize calculations, the presence of the square root *tells* you exactly which numbers to expect: Pythagorean triples! Only certain values of β make the square root easy to compute, and they're the ones for which 1 and β form two parts of a Pythagorean triple where 1 is the hypotenuse. An extremely common example is $\beta = 0.6$, which belongs to the triple $(0.6, 0.8, 1)$, better known as $(3, 4, 5)$. Since things will usually be given in terms of decimals rather than fractions, we're looking for triples with nice denominators, so $(0.28, 0.96, 1)$ (derived from $(7, 24, 25)$) is probably more common than one derived from $(5, 12, 13)$. In any case, it may be helpful to spend just a few minutes reminding yourself of the small Pythagorean triples, just to save a few minutes on arithmetic. For the most common triple, $(0.6, 0.8, 1)$, we have

$$\beta = 0.6 \implies \gamma = 1.25, \tag{6.28}$$
$$\beta = 0.8 \implies \gamma = 5/3. \tag{6.29}$$

6.5 Relativity: What to Memorize

There was a good deal of information presented in this chapter, but since relativity only makes up 6% of the exam, it's important not to go overboard memorizing equations. We recommend memorizing *only* the following, which are simple to state but too time-consuming to derive on the spot in the exam:

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- Definitions:

$$\beta = v/c \tag{6.11}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}} \tag{6.3}$$

$$x^\mu = (ct, x, y, z) \tag{6.10}$$

$$p^\mu = (E/c, \mathbf{p}) \tag{6.13}$$

$$\mathbf{p} = \gamma m \mathbf{v} \tag{6.16}$$

- Lorentz transformation matrix for boost along the x-axis:

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{6.12}$$

- Addition of velocities:

$$w = \frac{v+u}{1+vu/c^2} \tag{6.8}$$

- Relativistic Doppler shift:

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} \tag{6.27}$$

- 4-vector dot product:

$$a \cdot b \equiv a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3 \tag{6.21}$$

- Rest energy of a particle of mass m:

$$E_0 = mc^2 \tag{6.17}$$

Everything else can be derived very quickly from these. In particular, time dilation and length contraction can be derived from (6.12) after specializing to the position 4-vector, the invariant interval and the energy–momentum relationship $E^2 = \mathbf{p}^2c^2 + m^2c^4$ can be derived from (6.21), and so forth. If you feel comfortable with it, an excellent additional simplification is just to set $c = 1$ in all the formulas in this chapter. These are the units typical for particle physics, and you can always restore the factors of c by dimensional analysis.

6.6 Problems: Special Relativity

1. System $\bar{S}$ travels with constant velocity $v \neq 0$ in the $\hat{\mathbf{x}}$-direction with respect to system S. If two events, separated by a distance $x \neq 0$, occur simultaneously at time t in S, do they occur simultaneously in $\bar{S}$?

   (A) Yes, always
   (B) No, never
   (C) Only if $x < vt$
   (D) Only if $x > vt$
   (E) Only if $x < ct$

2. System B travels with respect to system A at constant velocity $\mathbf{v} = \beta c\hat{\mathbf{z}}$. Assuming the origins of both coordinate systems coincide, which of the following represents the Lorentz transformation matrix from the coordinates (ct', x', y', z') of system B to the coordinates (ct, x, y, z) of system A? ($\gamma = 1/\sqrt{1-\beta^2}$.)

   (A) $\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (B) $\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (C) $\begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$

   (D) $\begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$

   (E) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & \gamma\beta \\ 0 & 0 & 1 & 0 \\ 0 & \gamma\beta & 0 & \gamma \end{pmatrix}$

3. A particle of mass M and energy E decays into three identical particles of equal energy. What is the magnitude of the momentum of one of the decay products of mass m?

   (A) $\frac{E}{3c}$
   (B) $\frac{1}{3}Mc$
   (C) $\sqrt{\frac{E^2}{9c^2} + m^2c^2}$
   (D) $\sqrt{\frac{E^2}{9c^2} + M^2c^2}$
   (E) $\sqrt{\frac{E^2}{9c^2} - m^2c^2}$

4. An explosion occurs at the spacetime point $(ct, \mathbf{x})$ in one frame, and at $(ct', \mathbf{x}')$ in another frame related by a Lorentz transformation. If $(ct)^2 > |\mathbf{x}|^2$, we can conclude:

   (A) There exists a frame where $t' = 0$.
   (B) There exists a frame where $\mathbf{x}' = 0$.
   (C) There exists a frame where $(ct')^2 = |\mathbf{x}'|^2$.

https://doi.org/10.1017/9781108296977.009

(D) There exists a frame where $(ct')^2 < |\mathbf{x}'|^2$.
(E) None of the above.

5. The classical cyclotron frequency of an electron in a uniform magnetic field is ω_0. What is the cyclotron frequency of an electron of velocity v, as measured by a stationary observer?

(A) ω_0
(B) $\omega_0\sqrt{1 - v^2/c^2}$
(C) $\omega_0(1 - v^2/c^2)$
(D) $\dfrac{\omega_0}{\sqrt{1 - v^2/c^2}}$
(E) $\dfrac{\omega_0}{1 - v^2/c^2}$

6. A massive particle has energy E and relativistic momentum $\mathbf{p}$. Which of the following is true of the quantity $E^2 - \mathbf{p}^2c^2$?

I. It is conserved in elastic collisions.
II. It is invariant under Lorentz transformations.
III. It is equal to zero.

(A) I only
(B) II only
(C) I and II
(D) II and III
(E) I, II, and III

7. The USS *Enterprise*, moving at speed $0.5c$ with respect to a nearby planet, fires a photon torpedo of speed c at a Romulan warship, initially 6000 km away, which is retreating away from the *Enterprise* at constant velocity. According to the *Enterprise's* clock, the torpedo made contact with the warship 0.1 seconds after firing. How fast was the warship traveling, in the frame of the planet?

(A) $\frac{13}{28}c$
(B) $\frac{13}{16}c$
(C) $\frac{13}{14}c$
(D) c
(E) $\frac{13}{10}c$

8. A space-car speeds towards an intergalactic traffic light. The traffic light is red, emitting light of wavelength 750 nm, but the driver sees it as green, at wavelength 500 nm. How fast was the car traveling?

(A) $\dfrac{1}{5}c$
(B) $\dfrac{2}{7}c$
(C) $\dfrac{5}{13}c$
(D) c
(E) The given wavelengths are not consistent with any speed.

9. Spaceship 1, carrying a meter stick, flies past Spaceship 2, carrying a 1 liter container. The occupants of Spaceship 2 measure the meter stick on Spaceship 1 to be 0.5 m long. What volume do the occupants of Spaceship 1 measure for the container on Spaceship 2? Both spaceships travel along parallel trajectories and all dimensions should be measured parallel to the axis of their trajectories.

(A) 0.125 L
(B) 0.25 L
(C) 0.5 L
(D) 1 L
(E) 2 L

10. An 8 kg mass is traveling at 30 m/s. What is the approximate difference between its classical kinetic energy and its relativistic kinetic energy?

(A) 27 pJ
(B) 27 nJ
(C) 27 μJ
(D) 27 mJ
(E) 27 J

6.7 Solutions: Special Relativity

1. B – Following the same method as in the discussion of simultaneity, we arrive at the same equation using the Lorentz transformations:
$$t'_A = \gamma\left(t - \frac{x_A v}{c^2}\right), \quad t'_B = \gamma\left(t - \frac{x_B v}{c^2}\right).$$
These are only equal if $v = 0$ or if $x_B = x_A$, both of which are excluded by the problem statement.
2. D – Since the boost is along the z-axis, we want components of the matrix that mix up the x^0 and x^3 components, and those are the corners of the matrix, as in choices C and D. We're asked for the transformation *from B to A*, so we want the *inverse* Lorentz transformations, which don't have the minus signs. Choice D has the correct signs for the inverse transformations.
3. E – Each final-state particle has the same energy, $E/3$. We now apply the energy–momentum relation (6.25):
$$(E/3)^2 = \mathbf{p}^2c^2 + m^2c^4$$
$$\implies |\mathbf{p}| = \sqrt{\frac{E^2}{9c^2} - m^2c^2},$$
which is choice E. We could also have arrived at this answer purely by logical reasoning: for a given energy E,

https://doi.org/10.1017/9781108296977.009

as the mass m of the decay products increases, eventually there will not be enough available energy to produce them. Choice E is the only one that displays this behavior: indeed, $|\mathbf{p}|$ goes imaginary when $E < 3mc^2$.

4. B – The given information is equivalent to saying that the displacement vector between $(ct, \mathbf{x})$ and the origin $(0, \mathbf{0})$ is timelike. Thus, by our discussion of the invariant interval, there is a frame where the two events occur at the same place. Since the origins coincide for systems related by Lorentz transformations, this place is $\mathbf{x} = 0$. A is characteristic of a spacelike event, and C and D contradict the given information because the invariant interval never changes sign. Notice how the phrasing of this question doesn't commit itself to a particular choice of sign convention for the invariant interval: this is typical of GRE questions on this topic.
5. B – There are at least two valid solution methods. The first is to apply time dilation: in the electron frame, an interval of time $\Delta t'$ is related to frequency by $\omega_o \propto 1/\Delta t'$. The electron's clock runs slow when it is moving, so the interval Δt measured by a stationary observer is longer by a factor of γ. Again, using the inverse relation of frequency and time, we have
$$\Delta t = \gamma \Delta t',$$
$$\frac{1}{\Delta t} = \frac{1}{\gamma}\frac{1}{\Delta t'}$$
$$\implies \omega = \sqrt{1 - v^2/c^2}\,\omega_0.$$
Stated more simply, the stationary observer's time is dilated by γ, so the frequency observed is reduced by γ. Another method is to recall that the formula for cyclotron motion, $p = qBR$, holds relativistically as long as p is interpreted as the relativistic momentum γmv. The classical cyclotron frequency is $\omega_0 = qB/m$, and doing the algebra shows that the factor of γ ends up in the same place, $\omega = qB/(m\gamma)$.
6. C – By the energy–momentum relation (6.25), the given quantity is equal to m^2c^4, which is conserved in elastic collisions where the outgoing particles are the same as the ingoing particles (since the particle's mass doesn't change). It is also invariant, either because the mass of a particle doesn't depend on its reference frame, or because it is equal to p^2, the square of the energy–momentum 4-vector. It is never identically zero unless the particle is massless, but this case is excluded by the problem statement.
7. C – This problem involves the addition of velocities formula with a small twist. For our setting, the addition of velocities formula is
$$s = \frac{u+v}{1+\frac{uv}{c^2}},$$
where u is the speed of the warship in the *Enterprise* frame, v is the speed of the *Enterprise* in the planet frame, and s is the speed of the warship in the planet frame. We are given v in the problem, and we are solving for s. To determine u, we divide the distance Δx traveled by the warship in the *Enterprise* frame while the photon torpedo is in transit by the time Δt taken for the photon torpedo to contact the warship in the *Enterprise* frame. This gives
$$u = \frac{\Delta x}{\Delta t}.$$
On the other hand, we know that, since the torpedo travels at c, we must have
$$\Delta t = \frac{\Delta x + x_0}{c},$$
where x_0 is the distance between the *Enterprise* and the warship when the torpedo is fired. This implies that
$$\Delta x = c\Delta t - x_0,$$
and therefore that
$$u = \frac{c\Delta t - x_0}{\Delta t} = c - \frac{6\times 10^6\ \text{m}}{0.1\ \text{s}} = 2.4\times 10^8\ \text{m/s} = 0.8c,$$
with $c = 3\times 10^8$ m/s. Plugging this result into our expression for s above, we find
$$s = \frac{0.5c + 0.8c}{1 + (0.8c)(0.5c)/c^2} = \frac{1.3}{1.4}c = \frac{13}{14}c,$$
which is C.
8. C – A straightforward application of the relativistic Doppler shift formula gives
$$\frac{750\ \text{nm}}{500\ \text{nm}} = \sqrt{\frac{1+\beta}{1-\beta}}$$
$$\implies \beta = \frac{5}{13},$$
so the car's speed is $\frac{5}{13}c$. Note that the wavelength decreases (the light is blueshifted) because you are traveling towards the source, but the problem was kind enough to give you this fact for free.
9. C – The Lorentz contraction factor between Spaceship 2 and Spaceship 1 is $\gamma = 2$, and so each ship will measure the other's length *in the direction of motion* by a factor of $1/\gamma = 0.5$. But as we can see from the Lorentz transformation equations, there is no change to the coordinates in the perpendicular directions, so volumes are only contracted by a factor γ, from the single length contraction in the direction of motion.

https://doi.org/10.1017/9781108296977.009

10. A – Recall that the total relativistic energy of a particle is given by $E = \gamma mc^2$, so the relativistic kinetic part is $T = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$. If we Taylor expand the $(\gamma - 1)$ factor, the leading term is the classical kinetic energy, and the subsequent terms are the higher relativistic corrections:

$$\begin{aligned}
T &= \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right) mc^2 \\
&= \left(1 + \frac{v^2}{2c^2} + \frac{(-1/2)(-3/2)}{2!}\frac{v^4}{c^4} + \cdots - 1\right) mc^2 \\
&= \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2} + \cdots .
\end{aligned}$$

Plugging in the numbers,

$$\frac{3}{8}\frac{mv^4}{c^2} = 27\ \text{pJ},$$

which is choice A. Even without doing an exact Taylor expansion, we could have reasoned as follows: since the velocity is small, the difference is likely to be *extremely* small, which means it is suppressed by powers of c. The only quantities with units of energy are mv^3/c and mv^4/c^2, which correspond roughly to choices C and A, respectively. Odd powers of c are rare in quantities involving energy, so we might make an educated guess towards choice A.