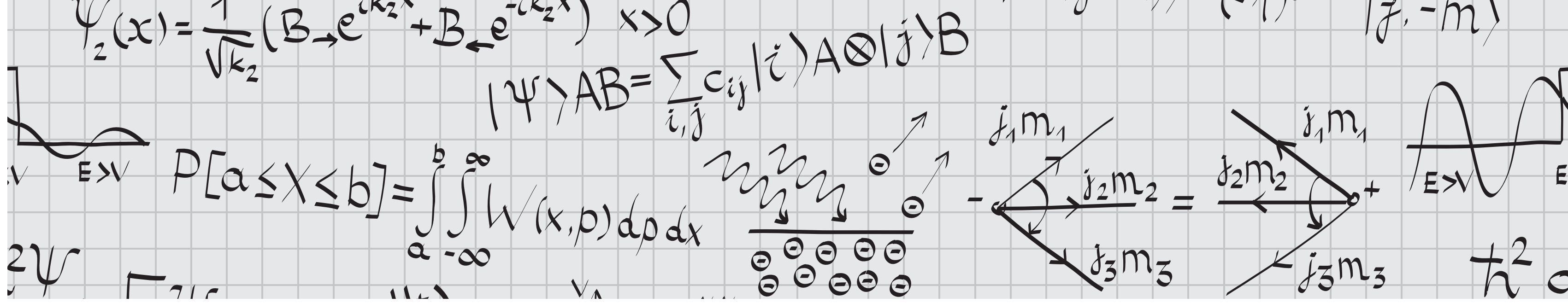

2 Electricity and Magnetism

According to ETS, approximately 18% of the exam covers electromagnetism. Given the format of the exam, the questions tend to emphasize a broad set of topics in electricity and magnetism rather than deep theoretical issues or analysis of complicated charge configurations. You will, for example, find plenty of AC and DC circuits in addition to the more traditional topics of electro- and magnetostatics, induction, Maxwell's equations, and electromagnetic waves.

The general philosophy that you should take away from this chapter is that electromagnetism is conceptually simple. There are only a few key concepts, such as symmetry and boundary conditions, most of which are buried somewhere in Maxwell's equations. The vast majority of the work lies in figuring out how to apply these basic ideas to specific configurations. The key to success on the electromagnetism problems on the GRE is to develop enough intuition with a few specific classes of problems that you can quickly deploy the necessary equations to solve them. We'll try to outline the general ideas concisely and try to illustrate how to choose the fastest methods for the common classes of problems.

2.1 Electrostatics

Electro*statics* refers to problems where charges and fields are not moving or changing in time. If you have a configuration of charges that does not change in time, electrostatics lets you calculate resulting electric fields and forces. We'll generalize this to include time dependence in Section 2.3.

2.1.1 Maxwell's Equations for Electrostatics

The tools needed for analyzing electrostatics problems are very simple. In fact, it just boils down to the first two of Maxwell's equations in free space in the absence of magnetic fields:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{2.1}$$

$$\nabla \times \mathbf{E} = 0 \ \ \text{(electrostatics)}. \tag{2.2}$$

The first equation actually holds true regardless of whether a magnetic field or time dependence is present, but the second is only true in electrostatics. These two equations tell you how to build the electric field $\mathbf{E}$ from a charge distribution ρ. To figure out how a test charge q moves in response to an electric field,[1] you need one more equation,

$$\mathbf{F}_E = q\mathbf{E}. \tag{2.3}$$

That's it! In principle, you can solve for any static electric field with these equations and then compute the motion of a particle in the field, given knowledge of a charge distribution. As you're probably well aware, however, partial differential equations are almost always quite nasty to solve, so we use a few additional tools to calculate the electric field in practice.

2.1.2 Electric Potential

The first tool is the scalar electric potential. Under some fairly general conditions, the fact that $\nabla \times \mathbf{E} = 0$ implies that there is a *scalar potential* field $V(\mathbf{r})$, which we also call the *electric potential* such that

$$\mathbf{E} = -\nabla V. \tag{2.4}$$

[1] This is still electrostatics, since the *field* isn't moving, only the test charge, whose own field we don't care about in this context.

This can be integrated to give V in terms of $\mathbf{E}$,

$$V(b) = -\int_a^b \mathbf{E} \cdot d\mathbf{l}, \tag{2.5}$$

which hopefully looks familiar from potential energy in mechanics. Here the point a has been defined as the zero of potential; as is the case with the potential energy in classical mechanics, electric potential must always be defined relative to some reference location. For electrostatics problems, unless otherwise specified, the reference point is often taken to be infinitely far from the location of interest. As with potential energy in mechanics: *mind the sign!* There is a relative minus sign between the electric field and the electric potential, which is easy to forget. Also note that V is *not* a potential energy; rather, it's a potential energy *per unit charge*, and the real potential energy of a particle of charge q in a region with electric potential V is $U = qV$. Because it is proportional to the true potential energy, the electric potential is also only defined up to a constant, so changing V by a constant value does not change the value of the electric field. Thus it is only *differences* in potential that we measure. Finally, since the potential $V(\mathbf{r})$ is a scalar quantity, not a vector like the electric field, it is sometimes easier to calculate the potential for a particular charge configuration and then convert to the electric field. For practical purposes, both contain equivalent information.

At this point we should mention the fact that both $\mathbf{E}$ and V are *additive*: Maxwell's equations are linear differential equations, so to find $\mathbf{E}$ or V of several charges, you just add up (using vector addition in the first case, and ordinary addition in the second) the corresponding $\mathbf{E}$ and V of the individual charges.

The second tool for solving electrostatics problems is the direct solution of Maxwell's equations. If we plug (2.4) into (2.1), we obtain the single scalar equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}, \tag{2.6}$$

known as *Poisson's equation*. Often we are asked to solve this equation in a region where the charge density ρ is zero, in which case it reduces to *Laplace's equation*,

$$\nabla^2 V = 0 \qquad \text{(empty space)}. \tag{2.7}$$

We can write the general solution to Poisson's equation as an integral over the charge distribution $\rho(\mathbf{r})$:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \tag{2.8}$$

Note the variables carefully: $\mathbf{r}$ labels the point where you're measuring the potential, but $\mathbf{r}'$ labels the position of the charge distribution, and is integrated over so the final answer depends only on $\mathbf{r}$. We should mention that (2.8) is rarely used in practice, since the integral is so nasty, unless the charge density ρ is particularly simple. It is worth remembering mostly as a sanity check for the case of a point charge, where $\rho(\mathbf{r}) \propto \delta^3(\mathbf{r})$. We'll come back to this case below.

2.1.3 Integral Form of Maxwell's Equations

The third useful tool is Maxwell's equations themselves, though in a slightly different form. Using Gauss's theorem and Stokes's theorem, we can rewrite Maxwell's equations in integral form:

$$\oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\epsilon_0}, \tag{2.9}$$

$$\oint_C \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} = 0 \quad \text{(electrostatics)}, \tag{2.10}$$

where Q_{enc} is the charge enclosed by the closed surface S, and C is some closed curve. The first equation is known as *Gauss's law*, and the left-hand side is defined as the *electric flux*. The important implication of Gauss's law is that regions that have no net flux in or out can enclose no net charge. Problems on the GRE that involve actually solving for realistic charge configurations always require the integral forms of Maxwell's equations. The differential forms, if they are ever needed, are mainly involved in conceptual questions or questions that just require you to evaluate the divergence or curl, given some electric field.

There are always problems on the GRE that involve simply solving for the electric field or electric potential of a charge configuration. If a problem has a high degree of symmetry (e.g. spherical, cylindrical, or planar), the fastest route is to use the integral form of Gauss's law (2.9). The recipe to calculate the field is simple and probably familiar from an electromagnetism course.

1. **Figure out the "symmetry" of the problem:** Should the electric field point radially outward from a single point (spherical symmetry), radially outward from a central axis (cylindrical symmetry), or away from a plane (planar symmetry)? This can usually be deduced from the shape of the charge configuration.
2. **Find a Gaussian surface:** Visualize a "Gaussian" surface S such that the electric field $\mathbf{E}$ is always either (a) perpendicular to S with constant magnitude, or (b) parallel to S – this is where the symmetry is important, as you usually *guess* where this surface is, based on the symmetry of the problem.

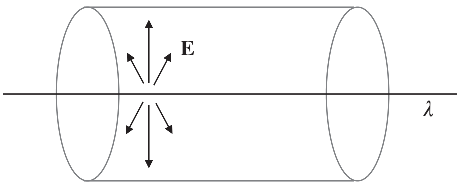

Figure 2.1 Example of a Gaussian surface over which the electric field from the charged line λ is both constant and perpendicular.

3. **Solve for the field:** The dot product $\mathbf{E} \cdot d\mathbf{S}$ then vanishes whenever $\mathbf{E}$ is parallel to S, and is equal to $|\mathbf{E}|dS$ over the rest of S. The constant magnitude $|\mathbf{E}|$ can be pulled out of the integral, and (2.9) reduces to

$$|\mathbf{E}| \oint_{S_\perp} dS = \frac{Q_{\text{enc}}}{\epsilon_0},$$
$$|\mathbf{E}|\, A = \frac{Q_{\text{enc}}}{\epsilon_0},$$

where A is the area of the Gaussian surface $S_\perp$ over which $\mathbf{E}$ is perpendicular to the surface.

We'll treat several more standard applications of Gauss's law in the following Section 2.1.4.

2.1.4 Standard Electrostatics Configurations

As we discussed at the beginning of this chapter, the hard part about electromagnetism is applying the ideas, not understanding them. In this section, we'll therefore summarize the important charge configurations that frequently appear on the exam.

- **Point charges.** The force, electric field, and potential due to a collection of point charges can always be found by summing up the Coulomb terms for each charge in the system. Problems generally fall into two types. In the first type, you are given an arbitrary arrangement of charges and you need to find the force, electric field, or potential due to the charges at some nearby point in space. It is always best to start with symmetry considerations to limit the calculations needed. For example, if the point charges are at the vertices of a regular polygon, then the field and force at the center will be zero because all components will cancel. Only after using as much symmetry as possible should you start writing down terms. If possible, you want to find the potential first and only then calculate the field by taking a derivative, since scalar addition is much simpler than vector addition.

EXAMPLE 2.1

The simplest possible application of all these ideas is to "derive" Coulomb's law. Hopefully you do not need a reminder about Coulomb's law, but this is still a nice simple setting to see how these ideas work without the complications of strange charge configurations. Let's say that we have a point charge q; that is, a charge distribution that is just proportional to a delta function,

$$\rho(\mathbf{r}) = q\delta^3(\mathbf{r}).$$

We know that the electric field due to a point charge is directed radially outward. What surface has a normal vector that points radially outward? A sphere, of course. So we integrate over a sphere of radius r: the enclosed charge is just q since the delta function will integrate to 1 over any region containing the origin, and we get

$$|\mathbf{E}|\,(4\pi r^2) = \frac{q}{\epsilon_0},$$

from which we deduce *Coulomb's law*,

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{2.11}$$

We can deduce the potential from a similar procedure starting with equation (2.8). Or we can simply obtain the potential by integrating the electric field from some reference point, where we set V equal to zero. Here, we pick $r = \infty$ to be our reference point, so we obtain a potential

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r}, \tag{2.12}$$

known as the Coulomb potential.

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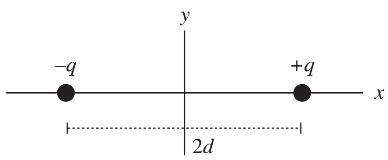

Figure 2.2 Electric dipole.

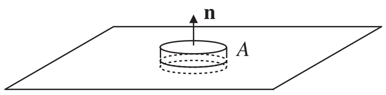

Figure 2.3 Gaussian pillbox for calculating the field of an infinite charged sheet.

In a second common class of problems, you are asked to find the electric field or force of some configuration at a point very far from the origin. A standard example (which we'll treat in a different context in Section 2.4) is two point charges separated by some distance. Suppose we have a charge q and a charge $-q$ separated by a distance $2d$, and choose coordinates such that q is at $(d, 0, 0)$ and $-q$ is at $(-d, 0, 0)$ (Fig. 2.2).

The potential at some point $(x, 0, 0)$ on the x-axis is

$$V(x) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x-d|} - \frac{q}{|x+d|}\right).$$

Taylor expanding this expression for $x \gg d$ tells you approximately how the field behaves far away. Note that the lowest-order terms in the Taylor expansion cancel, but the next nonzero term in the Taylor series does not vanish and gives a much better approximation.

- **Planes.** The classic example here is an infinite flat sheet of charge with surface charge σ. By symmetry, the field must be constant in magnitude: the sheet is infinite, so there is no local measurement you can do to tell you how close you are to it, and it looks the same from every distance. Again by symmetry, the field must point perpendicularly *away* from the sheet, since that is the only preferred direction in the problem. Now we draw a Gaussian "pillbox" surrounding part of the sheet, which cuts out an area A (Fig. 2.3).

  The thickness of the pillbox doesn't matter, because, as we argued, $\mathbf{E}$ has constant magnitude. The integral in Gauss's law is $\oint \mathbf{E} \cdot d\mathbf{S} = 2|\mathbf{E}|A$ (one from each of the two opposite faces of the pillbox), the charge enclosed is $Q = \sigma A$, and solving for $\mathbf{E}$ we find

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}, \tag{2.13}$$

  where $\hat{\mathbf{n}}$ is a unit normal pointing away from the plane. This particular result shows up so often that it's best to memorize it, so we've given it an equation number. You'll see below in Section 2.1.5 how precisely this same argument can be used to get the boundary conditions for $\mathbf{E}$ at a surface.

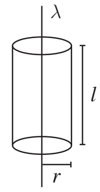

Figure 2.4 Cylindrical Gaussian surface for calculating the field of an infinite line charge.

- **Line charges and cylinders**. Problems involving infinite line charges and cylinders are usually solved with cylindrical Gaussian surfaces. For example, consider the electric field due to an infinite line of charge with charge per unit length λ (Fig. 2.4).

  The field of a point charge points radially outward, but here we have an infinite line charge; by symmetry, the field can't have a component along the line, and by the Maxwell equation $\nabla \times \mathbf{E} = 0$, the only option is for it to point in the $\hat{\mathbf{r}}$ direction in cylindrical coordinates. From Gauss's law applied to a cylinder of height l and radius r surrounding the line, we have

$$|\mathbf{E}|2\pi rl = \frac{\lambda l}{\epsilon_0},$$
$$|\mathbf{E}| = \frac{\lambda}{2\pi\epsilon_0 r},$$
$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}}.$$

  Notice that the height l of the cylinder cancels out when we express the result in terms of the linear charge density λ.
- **Spherical surfaces**. Concentric spherical surfaces are another common geometry in electrostatics problems. Similar to the case of the cylinder, solutions can usually be obtained by using a sphere as the Gaussian surface to compute the electric field at each radius. See Examples 2.1 and 2.2.

2.1.5 Boundary Conditions

In the previous section, we learned everything needed to calculate electric fields and potentials in the bulk of regions – such as inside or outside of a sphere. But what about on the boundary *between* regions? Figuring out how fields behave on boundaries between regions of space is so important that it deserves its own section. Luckily, like everything else, it too follows from Maxwell's equations.

Let's say that we have a continuous surface surrounded by vacuum, and we zoom in so close that the surface appears flat. We now have a plane, which we can use to define $\mathbf{E}^{\parallel}$, the two

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EXAMPLE 2.2

Find the electric field created by a solid sphere of radius a with charge density $\rho(r) = \alpha r$ and a spherical cavity of radius b at its center (Fig. 2.5).

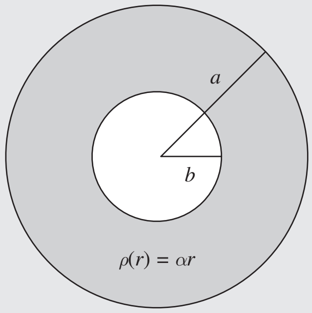

Figure 2.5 A spherically symmetric charge distribution with a cavity at the center.

For $r < b$, there is no electric field because a spherical Gaussian surface lying entirely inside the cavity contains no charge. This is an important result in its own right: *the electric field inside an empty spherically symmetric cavity is always zero*. In the solid region $b < r < a$, we need to use Gauss's law with a sphere as our Gaussian surface, so we can use the usual trick of pulling out $|\mathbf{E}|$ from the surface integral. The electric field is given by

$$\begin{aligned}
\oint_S \mathbf{E} \cdot d\mathbf{S} &= \frac{1}{\epsilon_0} \int_b^r \rho(r') d^3\mathbf{r}', \\
|\mathbf{E}|\,(4\pi r^2) &= \frac{4\pi}{\epsilon_0} \int_b^r \alpha r'^3 dr', \\
|\mathbf{E}|\,(r^2) &= \frac{\alpha}{4\epsilon_0}(r^4 - b^4), \\
\mathbf{E} &= \frac{\alpha}{4\epsilon_0} \frac{r^4 - b^4}{r^2}\hat{\mathbf{r}}.
\end{aligned}$$

In the outermost region $r > a$, the electric field is just given by Coulomb's law for the total charge in the sphere. The total charge in the sphere is just the integral of the charge density:

$$\begin{aligned}
Q &= \int_b^a \rho(r') d^3\mathbf{r}' \\
&= 4\pi \int_b^a \alpha r'^3 dr' \\
&= \pi\alpha(a^4 - b^4),
\end{aligned}$$

so the electric field is given by

$$\mathbf{E} = \frac{\alpha(a^4 - b^4)}{4\epsilon_0 r^2}\hat{\mathbf{r}}.$$

Notice that the electric field is everywhere continuous: at $r = b$ our expression for the field in the region $b < r < a$ gives 0, as did our argument for the field in the region $r < b$. Similarly, at $r = a$, both expressions give $\alpha(a^4 - b^4)/(4\epsilon_0 a^2)$ for the magnitude of the field. As we will see in Section 2.1.5, the continuity of $\mathbf{E}$ is due to the fact that there are no surface charges in this problem, only volume charge densities.

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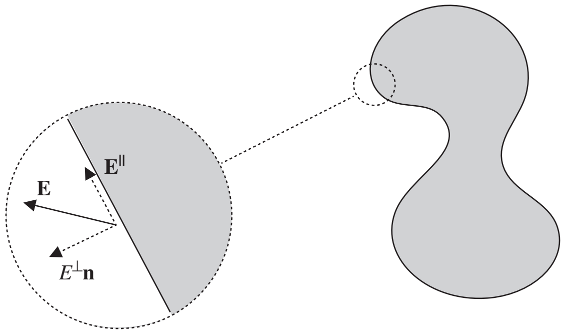

Figure 2.6 Parallel and perpendicular components of **E** at a boundary.

components of the three-component vector **E** that are parallel to this plane, and $E^{\perp}$, the third component perpendicular to the plane. Mind the notation: $\mathbf{E}^{\parallel}$ is bold because it's a (two-component) vector, but $E^{\perp}$ is a one-component scalar. Next, let's integrate over a narrow rectangular path with long edges parallel to the surface and short edges perpendicular to the surface. Using the second of Maxwell's equations in integral form, equation (2.10), the line integral implies that

$$\mathbf{E}^{\parallel}_{\text{out}} - \mathbf{E}^{\parallel}_{\text{in}} = 0, \tag{2.14}$$

where $\mathbf{E}_{\text{out}}$ is the electric field just outside the surface, and $\mathbf{E}_{\text{in}}$ is the field just inside the surface. This is true since the lengths of the paths perpendicular to the surface can be taken to be arbitrarily small, and the other two opposite sides of the rectangle contribute equally and opposite in the integral.

Next, let's consider the same surface, but integrate over the surface of a Gaussian pillbox and use Gauss's law, equation (2.9). If we take the height of the box to be extremely small, then, using the same arguments as in the plane symmetry discussion in the previous section, we end up with the condition that

$$E^{\perp}_{\text{out}} - E^{\perp}_{\text{in}} = \frac{\sigma}{\epsilon_0}, \tag{2.15}$$

where σ is the surface charge density. Equations (2.14) and (2.15) let you patch together electric fields in different regions of space across surfaces, where strange things may be happening. A simple, classic example of why this is useful is the case of conductors, which we discuss below in Section 2.1.6.

Before we do that, however, we should talk about how the potential behaves at boundaries, since working with the potential is almost always easier than working with the electric field. It's not difficult to show that equations (2.14) and (2.15) and the fact that $\mathbf{E} = -\nabla V$ imply that

V is always continuous.

Similarly, the first derivative of V is constrained so that

Derivatives of V are continuous, except at charged surfaces.

These rules for the potential and electric field boundary conditions will allow you to match solutions across any type of boundaries.

2.1.6 Conductors

While the GRE might throw a question or two your way that relies on knowing formal aspects of boundary conditions, you are most likely to use these ideas when solving problems involving conductors. An ideal conductor is a material where charge can flow freely without resistance: usually "metal" and "conductor" are synonymous for the purposes of the GRE. There is only one fact you really need to know about conductors:

V is constant throughout a conductor.

From this fact, you can derive four important corollaries:

- The electric field inside a conductor is zero.
- The net charge density inside a conductor is zero.
- Any net charge on a conductor is confined to the surface.
- The electric field just outside a conductor is perpendicular to the surface.

These properties are a direct consequence of the fact that an ideal conductor has no resistance to the movement of charges. If there *were* any bulk electric fields, free charges would experience a force until they had arranged themselves to cancel out any electric fields.

On the GRE, you will occasionally work with *grounded* conductors. Usually a conductor is said to be grounded if it is connected to the reference for the electric potential. In other words, something is grounded if it is connected to $V = 0$. You can think of this as setting the constant reference scale for the potential. The other important property of the idealized ground in electrostatics problems is that it is an infinite sink and source of charge. So, if you put a charge near a grounded conductor, some charge will be induced on the grounded conductor with no cost in energy. See Example 2.3.

2.1.7 Method of Images

The examples that we have discussed so far have been simplified by totally spherical, cylindrical, or planar geometry which allowed us to use simple arguments with Gauss's law.

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EXAMPLE 2.3

As an example of a problem involving conductors, consider the simple example of a thick, uncharged, conducting shell of inner radius r_1 and outer radius r_2, with a point charge q in the center, shown in Fig. 2.7. Further suppose that the potential at infinity is zero, $V(r = \infty) = 0$. What is the potential everywhere in space?

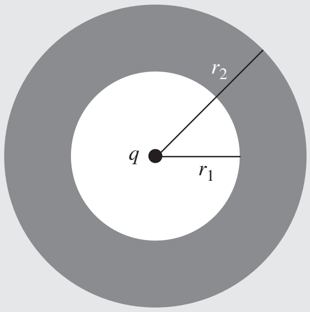

Figure 2.7 Conducting shell with point charge q at center.

From Coulomb's law, the field in the region $0 < r < r_1$ is simply

$$\mathbf{E}(r) = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}.$$

Inside the conductor, the electric field must be zero for $r_1 < r < r_2$:

$$\mathbf{E}(r) = 0.$$

Outside of the conductor ($r > r_2$), we can use Gauss's law to find the electric field. The conductor is uncharged, so the enclosed charge is just q. Thus, the electric field is

$$\mathbf{E}(r) = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}.$$

If we specify that the potential at infinity is 0, then the potential outside the conductor is

$$V(r) = \frac{q}{4\pi\epsilon_0 r}, \quad r > r_2.$$

Inside the conductor, we demand that V be constant. By continuity at $r = r_2$, we must have

$$V(r) = \frac{q}{4\pi\epsilon_0 r_2}, \quad r_1 < r < r_2.$$

Finally, inside the shell ($r < r_1$), the potential is of the form

$$V(r) = \frac{q}{4\pi\epsilon_0 r} + \text{const.}$$

Requiring continuity at $r = r_1$, we determine the value of the constant and find that the potential is

$$V(r) = \frac{q}{4\pi\epsilon_0 r} + \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right), \quad r < r_1.$$

Since the electric field is zero inside the conductor, we conclude from Gauss's law, after integrating over a sphere of radius $r_1 < r < r_2$, that

$$Q_{\text{enc}} = 0.$$

Since there is a charge q at the center, the charge on the inner surface of the conductor must be $-q$. This charge is said to be *induced* by the charge at the center. Since the conductor has zero net charge, there must be a corresponding charge of $+q$ uniformly distributed on the outer surface of the conductor. This gives us exactly the field structure we have shown.

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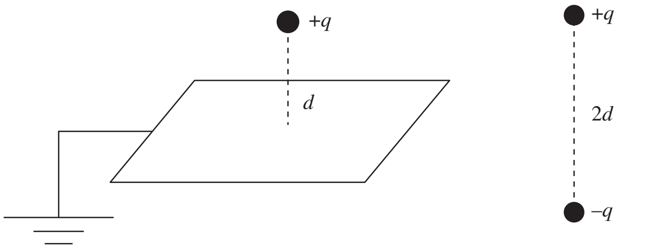

(a) Charge and grounded conducting plate (b) Image charge

Figure 2.8 Setup for the method of images.

As you might imagine, solving for the potential and induced charges of a configuration without this kind of total symmetry can be extremely complicated. In general, this is certainly true, but there are many tricks for systematically dealing with more complex geometries, particularly involving conductors. The only one of these methods that you are likely to need for the GRE is the so-called "method of images," which relies on a property of Laplace's equation (2.7), the *uniqueness of solutions*: if you can guess the potential V and it satisfies the boundary conditions, it must be *the* correct answer.

Consider the case of a point charge a distance d above an infinite, grounded conducting plane (Fig. 2.8(a)). Naïvely, this looks complicated! The point charge induces some complicated surface density of charge on the conductor. The arrangement does not have spherical or planar symmetry so we cannot rely on Gauss's law easily. But note that the potential on the surface must be constant because the surface is a conductor, and it must be zero because the conductor is grounded. Now switch gears for a moment and imagine a totally different charge distribution, that of the original charge q and a so-called "image" charge $-q$ at $z = -d$ (Fig. 2.8(b)). The potential on the plane $z = 0$ is exactly the same (namely $V = 0$) for these two charges as in the case of the conductor! Furthermore, if we restrict ourselves to the region $z \geq 0$, the sources for Poisson's equation are also the same, namely a single point charge q at $z = d$ in both cases. Therefore, by uniqueness, the potential V produced by the charge and the image charge *must* be the solution to the original problem with the conductor, since it satisfies the same boundary conditions at $z = 0$ with the same sources.

The essence of the method of images is this: whenever we have a conductor, if we can find an arrangement of point charges that exactly reproduces the potential on the surface, then the potential everywhere will be the same as if there were no conductor. But you must beware to place image charges only *below* the conductor, in the region that would be inaccessible in the original setup; if you placed them in the same region as the original charge, you would be adding different sources to Poisson's equation and solving a different problem. The only example you are likely to encounter on the GRE is the case of a grounded, conducting, infinite plane with some charges on one side. In this case, for each charge, just put an opposite mirror charge on the other side of the plane at the same distance away and you will have the potential immediately. While it is good to know this prescription, understanding why it works more generally is important and might come in handy for other problems.

There is one very important subtlety about the method of images. When calculating anything involving work or energy, remember that

There is no energy cost to moving an image charge.

The image charge is just a fake construct, a trick to get the correct potential; in reality, there is no field below the conductor, and so no work can be done on anything below the conductor. This can get confusing because the position of the image charge depends on the position of the real charge. If you move the real charge in towards the conducting plate, the image charge moves with it, but work is only done *on the real charge*. A useful mnemonic that lets you forget about this subtlety is to calculate the electric *field* directly from the image configuration. In other words, do *not* calculate the electric field by taking the derivative of V, because doing so will implicitly count the motion of the image charges as contributing to the energy. A more detailed discussion of this subtlety can be found in Griffiths, Section 3.2.3.

2.1.8 Work and Energy in Electrostatics

As in classical mechanics, using work and energy considerations wherever possible often saves time and energy (no pun intended) when problem solving. The same is true in electrostatics. Let's start with point charges and then generalize to other configurations. The work required to put together n point charges is

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i), \tag{2.16}$$

where q_i is the charge of each point charge and $V(\mathbf{r}_i)$ is the potential due to *all* of the charges, but evaluated at the location of the ith point charge. The intuition for this formula is that each of the $q_i V(\mathbf{r}_i)$ terms gives us the potential energy between one charge and every other charge. When we sum up

EXAMPLE 2.4

Consider a point charge at the center of a thin, grounded, conducting, spherical shell of radius a (Fig. 2.9(a)). The shell is removed and taken to infinity (Fig. 2.9(b)). How much work is done during this process?

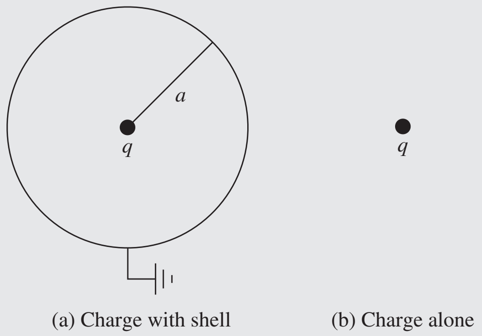

Figure 2.9 Change in energy due to a conducting shell.

We can just compute the change in energy for the two configurations. When the shell is in place, there is no field outside the shell, so the energy is just

$$U_1 = \frac{\epsilon_0}{2}\int_0^a |\mathbf{E}|^2\, 4\pi r^2 dr.$$

After the shell is gone, the energy is

$$U_2 = \frac{\epsilon_0}{2}\int_0^\infty |\mathbf{E}|^2\, 4\pi r^2 dr.$$

The work done must be

$$W = U_2 - U_1 = \frac{\epsilon_0}{2}\int_a^\infty |\mathbf{E}|^2\, 4\pi r^2 dr.$$

Using Coulomb's law, this quantity is easy to evaluate. But beware! If we had actually tried to evaluate U_1 or U_2 explicitly before taking the difference, we would have found them to be infinite! This is an important point: the total energy due to a point charge is infinite. The reasons for this are rather complicated and deep, so we will avoid discussing them. Nevertheless, this example shows that the *difference* between formally infinite quantities still has meaning in electromagnetism.

all of these contributions, we double count the energy so we need to divide by 2 to get the actual work. It should not be too mysterious to see that when we're not dealing with point charges, the work becomes an integral:

$$W = \frac{1}{2}\int \rho(\mathbf{r})V(\mathbf{r})d^3\mathbf{r}. \tag{2.17}$$

We showed that it takes work to move charge around, but there's also energy stored in the *fields* themselves. The energy is just

$$U_E = \frac{\epsilon_0}{2}\int |\mathbf{E}|^2\, d^3\mathbf{r}. \tag{2.18}$$

This simple formula can be extremely useful, as shown in Example 2.4.

2.1.9 Capacitors

Suppose you have two conductors with different net charges – for concreteness, give one $+Q$ and the other $-Q$. This gives rise to an electric field between the conductors, which in turn puts them at different potentials, say 0 and V. (This is a well-defined concept because potential is constant throughout a conductor.) This arrangement is known as a *capacitor*. In many practical cases, Q and V are proportional, and the constant of proportionality is called the *capacitance* C:

$$Q = CV. \tag{2.19}$$

You'll often hear the statement "C only depends on the geometry of the problem." All this means is that the proportionality

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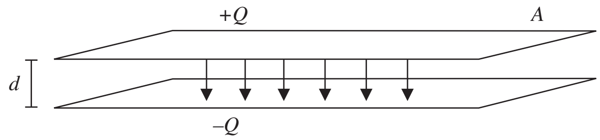


Figure 2.10 Parallel-plate capacitor.

between Q and V holds with the same constant C independent of how the charge got there; it depends only on the shapes and relative orientation of the two conductors.[2]

Finding the capacitance is usually straightforward: given some conductors, imagine putting charges $\pm Q$ on them, find the potential between them, and extract C. The standard example is two parallel plates of area A a distance d apart. We'll assume that $A \gg d^2$ such that the plates are effectively infinite and the field between them is constant. Placing charges $\pm Q$ on the plates gives them surface charge densities $\sigma = \pm Q/A$. A straightforward application of Gauss's law in a plane geometry gives the electric field of each plate as $\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector pointing away from the surface. The fields of the two plates cancel each other outside, but reinforce each other between the plates, such that the total field is $|\mathbf{E}| = \frac{Q}{A\epsilon_0}$ pointing from the positive to the negative plate. Doing the line integral along a straight line between the two plates to get V gives $V = \frac{Qd}{A\epsilon_0}$, or rearranging,

$$Q = V \times \frac{A\epsilon_0}{d}.$$

As promised, Q is proportional to V, so we extract the capacitance:

$$C = \frac{\epsilon_0 A}{d} \quad \text{(parallel-plate capacitor).} \tag{2.20}$$

This result is common enough and important enough that we give it an equation number. Note that it only depends on the geometric constants A and d, along with ϵ_0, which comes along for the ride in any electrostatics problem. By the way, you should reach the point where every step of the derivation is intimately familiar and obvious: the application of Gauss's law to an infinite sheet of charge, integrating $\mathbf{E}$ to find the potential, and checking that the boundary conditions for $\mathbf{E}$ are satisfied at the surfaces of the conductors are all part of classic GRE problems.

The device we have just described is called a *parallel-plate capacitor*. Along with the resistor and the inductor, it is one of the three building blocks of elementary circuits. It has the interesting property that it produces a strong, *uniform* electric field in a limited region of space: the field is $Q/A\epsilon_0$ between the plates, and zero everywhere else. This lets it store electrical energy; indeed, using (2.18), we find that the energy stored in the field of a charged capacitor is

$$U_C = \frac{\epsilon_0}{2}\left(\frac{Q^2}{A^2\epsilon_0^2}\right)(Ad) = \frac{1}{2}\frac{Q^2 d}{A\epsilon_0} = \frac{1}{2}\frac{Q^2}{C}.$$

This can be interpreted as the energy it takes to remove a charge Q from one (initially neutral) plate and put it on the other plate. Using the relation $Q = CV$ this can be expressed in a couple of useful ways:

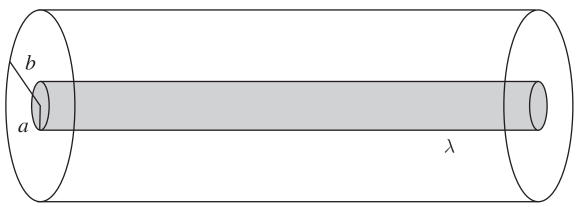
$$U_C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2. \tag{2.21}$$

You should become intimately familiar with capacitors: they are completely defined by (2.19)–(2.21), so learn those equations well. There are other arrangements of conductors that act as capacitors: in such cases, both (2.19) and (2.21) still hold, but you have to derive the analogue of (2.20). One example is two concentric spherical conducting shells, and another example is treated in the problems below.

2.1.10 Problems: Electrostatics

1. A point charge q is brought to a distance d from a grounded conducting plane. What is the magnitude of the force on the plane from the point charge?

   (A) $q^2/(16\pi\epsilon_0 d^2)$
   (B) $q^2/(8\pi\epsilon_0 d^2)$
   (C) $q^2/(4\pi\epsilon_0 d^2)$
   (D) $q^2/(\epsilon_0 d^2)$
   (E) 0



2. A cylindrical wire of charge of radius a and charge per unit length λ is at the center of a thin cylindrical conducting shell of radius b. What is the capacitance per unit length of this configuration?

   (A) ∞
   (B) $2\pi\epsilon_0 ab/(b^2 - a^2)$
   (C) $2\pi\epsilon_0\lambda/\ln(b/a)$
   (D) $2\pi\epsilon_0/\ln(b/a)$
   (E) 0

[2] Note that C is an intrinsically *positive* quantity, so you often don't have to worry too much about sign conventions for V.

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3. A capacitor is formed from two square plates of edge length a and separation d, with $d \ll a$. If all linear dimensions of the capacitor are tripled, by what factor does the capacitance change?

 (A) 1/3
 (B) 1
 (C) 3
 (D) 9
 (E) 27

4. What is the work needed to assemble four point charges q into a regular tetrahedron of side length a?

 (A) $q^2/(4\pi\epsilon_0 a)$
 (B) $q^2/(2\pi\epsilon_0 a)$
 (C) $q^2/(\pi\epsilon_0 a)$
 (D) $3q^2/(2\pi\epsilon_0 a)$
 (E) $3q^2/(\pi\epsilon_0 a)$

5. The electric field inside a sphere of radius R is given by $\mathbf{E} = E_0 z^2 \hat{\mathbf{z}}$. What is the total charge of the sphere?

 (A) $\frac{\pi}{2}\epsilon_0 E_0 R^4$
 (B) $\pi\epsilon_0 E_0 R^3$
 (C) $2\pi\epsilon_0 E_0 R^4$
 (D) $4\pi\epsilon_0 E_0 R^3$
 (E) 0

2.2 Magnetostatics

So far, we have discussed configurations with static electric fields. There is a very similar story for static *magnetic* fields, which are produced by constant currents of charge. This is a slight abuse of terminology, since *moving* charges (not static ones) are what create currents, but we're assuming that there are enough individual charges all moving together that the net current they create is constant in time; this is known as the steady-current approximation.

2.2.1 Basic Tools

The fundamental equations for these problems are the other two of Maxwell's equations, in the absence of changing electric fields:

$$\nabla \cdot \mathbf{B} = 0, \tag{2.22}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{(magnetostatics)}. \tag{2.23}$$

The first equation is simply the statement that there are no magnetic monopoles, which is true in general, not just

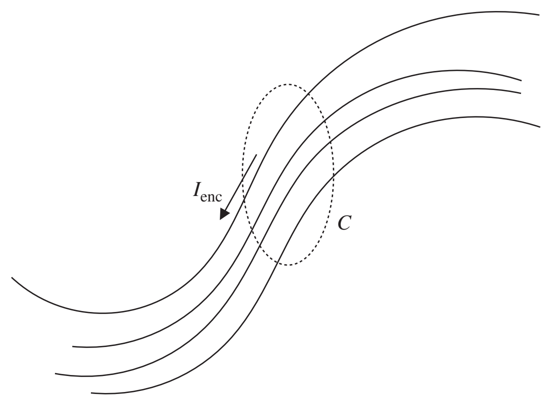

Figure 2.11 Example of a curve enclosing a current as in Ampère's law.

in magnetostatics. The second equation describes how currents act as sources for magnetic fields. As with Maxwell's equations for electrostatics, we can write these equations in integral form:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0, \tag{2.24}$$
$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad \text{(magnetostatics)}, \tag{2.25}$$

where S is a closed surface and C is a closed curve, just as in equations (2.9)–(2.10), and I_{enc} is the current piercing the surface defined by C. Equation (2.24) is sometimes referred to as *Gauss's law for magnetism*, and, just like the electric case, the left-hand side is known as the *magnetic flux*. We should emphasize that S is a *closed* surface, which means that it doesn't have any boundary: just picture a floppy ball. This distinction will be important when we discuss magnetic flux through surfaces that are *not* closed in Section 2.3.2. Equation (2.25) is known as *Ampère's law* and will be discussed in more detail in the following Section 2.2.2.

Similarly to the case of electrostatics, we can construct a potential for $\mathbf{B}$. However, since $\nabla \times \mathbf{B} \neq 0$, we cannot use a scalar potential. We are instead forced to use a *vector* potential, which has the defining property that

$$\nabla \times \mathbf{A} = \mathbf{B}. \tag{2.26}$$

Since the vector potential has three components (just like $\mathbf{B}$), it isn't as useful as the scalar potential V for calculations. The vector potential shows up so rarely on the GRE that it's not even worth discussing further apart from its defining equation.

In addition to Maxwell's equations, we can completely determine the effects of fields on test charges with the Lorentz force law. This gives the force on a test charge q due to a magnetic field:

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}. \tag{2.27}$$

Generalizing this to the force on a wire carrying current I, we have

$$d\mathbf{F}_B = I d\mathbf{l} \times \mathbf{B}. \tag{2.28}$$

The direction of the force is often more important than the magnitude on the GRE, so now is a good time to mention the famed *right-hand rule*: to evaluate the cross product, put the fingers of your right hand in the direction of the wire, curl them around toward the direction of $\mathbf{B}$, and your thumb will point in the direction of $\mathbf{F}_B$. Alternatively, you can use the rules appropriate to right-handed coordinate systems: $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$ in cylindrical coordinates, $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}}$ in spherical coordinates, and so on.

Problems in magnetostatics almost always fall into three general classes:

- Finding the field due to a configuration of currents
- Finding forces on wires or charges due to fields
- Finding energies of fields

The first class can generally be tackled with Ampère's law or the Biot–Savart law (discussed below). The second class can usually be solved with some variant of the Lorentz force law. The final class can be solved by integrating to find the energy in a field configuration. Obviously, these are just general guidelines, but they help to put these topics in perspective.

2.2.2 Ampère's Law and the Biot–Savart Law

Suppose that we have some collection of wires carrying currents. What is the magnetic field produced by the wire configuration? This question can be answered by Ampère's law and the Biot–Savart law.

Ampère's law is generally only useful for configurations that possess a high degree of symmetry, and can be thought of as analogous to Gauss's law for electrostatics. Referring back to equation (2.25), we want to pick the closed curve C such that the magnetic field is parallel to the path and constant. This allows us deduce the magnetic field by

$$|\mathbf{B}| \oint_C dl = \mu_0 I_{\text{enc}},$$
$$|\mathbf{B}| = \frac{\mu_0 I_{\text{enc}}}{L},$$

where L is the length of the curve C (for example, $2\pi r$ for an Amperian loop at a distance r from a current-carrying wire). This method relies crucially on being able to choose a path along which the magnetic field is constant, which is why it only works in highly symmetric problems. When we can use it, however, it dramatically simplifies our work.

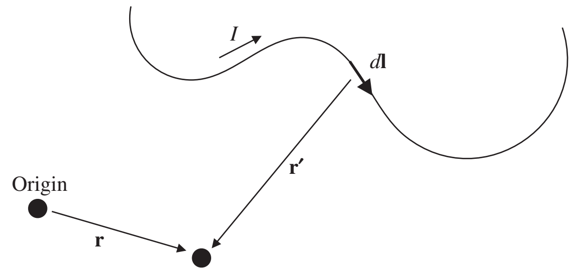

Figure 2.12 Calculating magnetic field from the Biot–Savart law.

In cases that are not so symmetric, we can use the Biot–Savart law instead, integrating over all of the current elements in a configuration. The Biot–Savart law reads[3]

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \hat{\mathbf{r}}'}{r'^2}. \tag{2.29}$$

The notation in this expression can be a bit cryptic and deserves some explanation. Referring to Fig. 2.12, $\mathbf{r}$ is the point where the field is evaluated, the integration is over the entire wire producing the magnetic field, $d\mathbf{l}$ is the line element along the wire, $\mathbf{r}'$ is the vector pointing from the line element to $\mathbf{r}$, and I is the current carried in the wire. This is similar to (2.8), which gives the potential sourced by a charge distribution; there is an analogous expression for the electric field as an integral over a charge configuration, but it's rarely used since we have the advantage of the scalar potential in the electrostatics case. The Biot–Savart law is clearly a good deal more complicated than Ampère's law, so avoid it unless you have no choice. Chances are good that it will arise at least once in a simple form on the GRE; if it does, the line element $d\mathbf{l}$ will likely take some simple form such as a square or a circle. You'll see an example in the problems at the end of this section.

2.2.3 Standard Magnetostatics Configurations

As in the case of electrostatics, there are a few magnetostatic configurations that seem to come up over and over again. This tends to occur because there just are not very many configurations that one can solve analytically in a reasonable amount of time. This is lucky for you! If you can master the following configurations, you should have a good general intuition for most problems that the GRE will throw at you. Our discussion will be extremely brief, because you probably covered these examples at least twice: once in your freshman physics course,

[3] This form of the equation assumes that the current is confined to a wire and has constant magnitude I, which can be pulled outside the integral. As far as we know this will always be the case for GRE problems.

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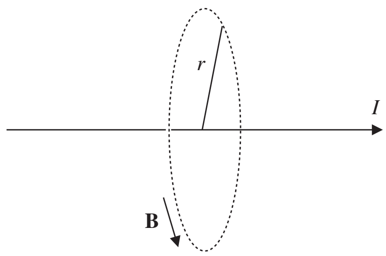

Figure 2.13 Circular Amperian loop for calculating the magnetic field of an infinite wire.

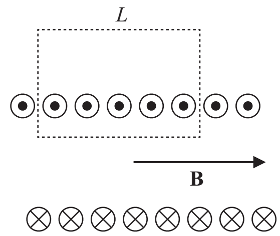

Figure 2.14 Rectangular Amperian loop for calculating the magnetic field of a solenoid.

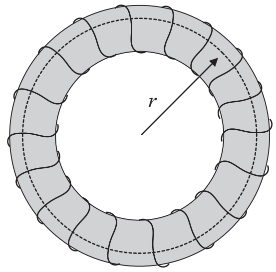

Figure 2.15 Example of a toroid. The dashed circle inside the toroid is a typical Amperian curve which can be used to solve for the magnetic field inside the toroid.

and then again in an advanced electromagnetism course. We recommend going back and reviewing this material in detail, just to convince yourself that the symmetry arguments make sense; but once you do this, you can forget them and just remember the problem-solving techniques for the GRE.

- **Wires.** Since there are no point charges in magnetostatics, the wire is the simplest example. Consider an infinite wire along the z-axis carrying a current I in the positive $\hat{\mathbf{z}}$ direction (Fig. 2.13). Draw an Amperian loop of radius r around the wire, so the current enclosed is I. By (cylindrical) symmetry, the **B**-field must only be a function of the distance r from the wire. Since we know **B** has a curl, we can guess that it "curls" around the wire in a direction dictated by the right-hand rule; in this case, the $\hat{\boldsymbol{\phi}}$ direction.

  This means that it is always parallel to the Amperian loop, so we can use the trick described above and pull out **B** from the integral in Ampère's law:

$$|\mathbf{B}|(2\pi r) = \mu_0 I \implies \mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}}. \tag{2.30}$$

  This result shows up very often and should be memorized.

  A common generalization is a thick wire with a volume current density that changes with radius, such that the current enclosed changes with radius inside the wire. Note that the wire has to be *infinite* for these symmetry arguments to work: to find the field of a finite wire, you have to go all the way back to the Biot–Savart law.
- **Solenoids.** The previous example had a straight line of current and a circular Amperian loop surrounding it. Now imagine the opposite scenario: take a bunch of tightly wound circular coils ("turns") of wire carrying current in the $\hat{\boldsymbol{\phi}}$ direction, stack the coils in a cylinder, and draw a rectangular Amperian loop with one vertical side of length L inside the cylinder and the other outside (Fig. 2.14).

  Various symmetry arguments, which you can find in Griffiths, tell you that the field must point along the axis of the cylinder and be constant inside. If there are n turns per unit length, then the only nonzero term in Ampère's law is from the one inside the cylinder, which gives you BL. The current enclosed is InL, so setting these equal gives

$$B = \mu_0 n I \quad \text{(solenoid)}. \tag{2.31}$$

  The direction of the field can be found, as usual, with the right-hand rule: curl your fingers in the direction of the current going around the coils, and your thumb points in the direction of **B**. This device, with the coils stacked in a cylinder, is called a *solenoid*, and we will discuss it in much more detail in Section 2.3.3. By the way, applying the same arguments to an Amperian loop *outside* the cylinder tells you that the field is identically zero outside, so, just like a capacitor, the solenoid confines a strong uniform field to a limited volume.
- **Toroids.** If you bend the solenoid cylinder around into a circle like a donut, you get a *toroid* (Fig. 2.15). Once again, symmetry arguments tell you that the field still points along the axis of a cylinder, which has now been bent around into the $\hat{\boldsymbol{\phi}}$ direction. Drawing a circular Amperian loop in the plane of the tube, just as in the wire example above, gives you

$$B = \frac{\mu_0 N I}{2\pi r} \quad \text{(toroid)}, \tag{2.32}$$

  where N is the total number of turns. Note that the field is *no longer constant*, but depends on r, the distance from the

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center of the toroid to the point inside where we are measuring the field. Notice also that this result is independent of the cross-sectional shape of the toroid, so long as it is constant. Just as with a solenoid, the field vanishes outside the volume enclosed by the loops of wire.

2.2.4 Boundary Conditions

We can also study the boundary conditions for magnetostatics, in complete analogy to the boundary conditions for electrostatics. They are slightly more complicated, but luckily they're not needed as frequently. As in Section 2.1.5, let's zoom into the surface of a boundary so that it can be taken to be flat, and define $\mathbf{B}^{\parallel}$ as the two-component vector parallel to the surface, and $B^{\perp}$ as the component normal to the surface (Fig. 2.16). If we put a cylinder across the surface and evaluate using equation (2.24), we find that the normal component of the magnetic field must be continuous:

$$B^{\perp}_{\text{out}} - B^{\perp}_{\text{in}} = 0. \tag{2.33}$$

Using equation (2.25), we can integrate over a narrow rectangle around the surface. To be completely general, let's suppose that there is some current density $\mathbf{K}$ on the surface, and let's orient the plane of our loop to be perpendicular to this surface current. If we take the sides of the rectangle perpendicular to the surface to be arbitrarily small, then we can neglect its contribution to the integral, and we are left with the boundary condition for the parallel component of the magnetic field:

$$\mathbf{B}^{\parallel}_{\text{out}} - \mathbf{B}^{\parallel}_{\text{in}} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}, \tag{2.34}$$

where $\mathbf{K}$ is the surface current density and $\hat{\mathbf{n}}$ is a unit vector pointing perpendicular to the surface, from "in" to "out."

A good mnemonic to remember these equations is to notice that they're sort of the reverse of the analogous electrostatic boundary conditions (2.14) and (2.15): normal $\mathbf{B}$ becomes parallel $\mathbf{E}$ and vice versa, and ϵ_0 goes in the denominator while μ_0 goes in the numerator. Also, note that σ (the surface charge density) is a *scalar*, while $\mathbf{K}$ (the surface current density) is a *vector*; this means that σ must be related to the scalar $E^{\perp}$, and $\mathbf{K}$ is related to the vector $\mathbf{B}^{\parallel}$.

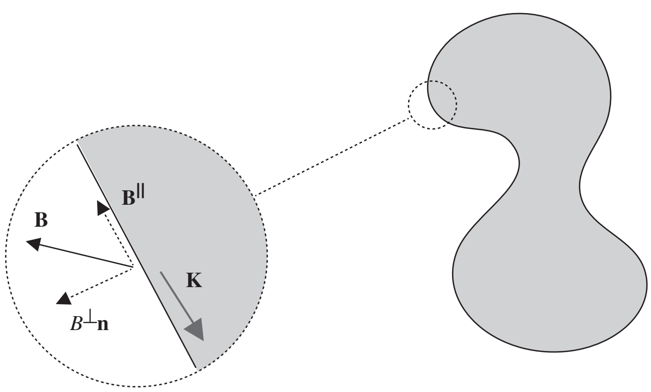

Figure 2.16 Parallel and perpendicular components of $\mathbf{B}$ at a boundary where surface current $\mathbf{K}$ flows along the boundary.

2.2.5 Work and Energy in Magnetostatics

Unlike electric fields, *magnetic fields do no work*. This can be seen immediately from the Lorentz force law (2.27): the cross product $\mathbf{v} \times \mathbf{B}$ means that the force is always perpendicular to the velocity, the speed of a particle cannot increase and $\mathbf{B}$ does no work. This is the cause of much conceptual confusion, as there are many standard problems where work is being done by *something* (usually, an external source or gravity), but it's not the magnetic field. However, magnetic fields store energy just like electric fields:

$$U_B = \frac{1}{2\mu_0} \int |\mathbf{B}|^2 \, d^3\mathbf{r}. \tag{2.35}$$

This is identical to the analogous expression (2.18) for electric fields except for the prefactor.

2.2.6 Cyclotron Motion

Even though magnetic fields do no work, they can still change the *direction* of a charged particle's motion. The most common situation is the motion of a charge q with mass m in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. Suppose the particle moves with velocity $\mathbf{v} = v\hat{\mathbf{y}}$. Then evaluating the cross product,

$$\mathbf{F} = qvB(\hat{\mathbf{y}} \times \hat{\mathbf{z}}) = qvB\hat{\mathbf{x}}.$$

The force is constant and in the $\hat{\mathbf{x}}$-direction, which is perpendicular to the particle's motion. As the particle begins to accelerate in the $\hat{\mathbf{x}}$-direction, the force will remain perpendicular thanks to the cross product, so what we have is *uniform circular motion*: a charged particle in a constant magnetic field will move in a circle confined to the plane with normal parallel to the $\mathbf{B}$ field. Actually, if the particle has some initial velocity *parallel* to the magnetic field, this velocity component will be unchanged because of the cross product. Thus the most general motion in a constant magnetic field is a *helix*, with the velocity perpendicular to the magnetic field $\mathbf{v}^{\perp}$ playing the role of v above.

By using the uniform circular motion formula from Section 1.2.1, you can easily work out the radius of the circle:

$$R = \frac{mv}{qB}, \tag{2.36}$$

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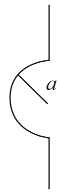
and the angular frequency, known as the *cyclotron frequency*:

$$\omega = \frac{qB}{m}. \tag{2.37}$$

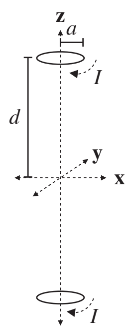
Amusingly, the formula for the cyclotron radius remains the same even at relativistic velocities, provided we replace the numerator mv with the relativistic momentum p (see Chapter 6).

2.2.7 Problems: Magnetostatics



1. A wire consists of a half circle whose ends extend perpendicular to the circle as shown above. If current I flows downward through the wire, what is the magnitude of the magnetic field at the center of the circle?

   (A) $\mu_0 I/(4a^2)$
   (B) $\mu_0 I/(4a)$
   (C) $\mu_0 I/a$
   (D) 0
   (E) $\mu_0 I/(4\pi a)$

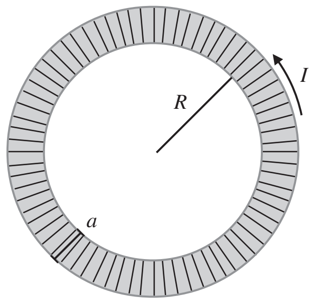


2. Two circular loops of wire, both with radius a, are oriented parallel to the xy-plane with their centers at $(0, 0, -d)$ and $(0, 0, d)$, where $d \gg a$. If both wires carry clockwise currents, which best describes the direction of the force from the loop at $z = d$ on electrons in the loop at $z = -d$?

   (A) Radially inward
   (B) Radially outward
   (C) In the $+\hat{\mathbf{z}}$ direction
   (D) In the $-\hat{\mathbf{z}}$ direction
   (E) There is no force.

3. What is the magnetic force per unit length between two parallel wires, separated by a distance d, each carrying current I in the same direction?

   (A) $\mu_0 I/(2\pi d)$, attractive
   (B) $\mu_0 I/(2\pi d)$, repulsive
   (C) $\mu_0 I^2/(2\pi d)$, attractive
   (D) $\mu_0 I^2/(2\pi d)$, repulsive
   (E) $\mu_0 I^2/(2\pi d^2)$, attractive



4. What is the magnetic energy stored in a toroid of wire with a square cross section of side length a, N total winds, inner radius R, and current I?

   (A) $\dfrac{\mu_0 N^2 I^2 a}{4\pi} \ln\left(\dfrac{R+a}{R}\right)$
   (B) $\dfrac{\mu_0 N^2 I^2 R}{4\pi} \ln\left(\dfrac{R+a}{R}\right)$
   (C) $\dfrac{\mu_0 N^2 I^2 a}{2\pi}$
   (D) $\dfrac{\mu_0 N^2 I^2 a}{4\pi}$
   (E) $\dfrac{\mu_0 N^2 I^2 R}{4\pi}$

2.3 Electrodynamics

So far we have given a treatment of how static electric fields and magnetic fields behave. The story becomes more complicated as we fill in the final piece of the puzzle and ask what happens when charges and currents – the sources of electromagnetic fields – move and change *in time*.

2.3.1 Maxwell's Equations

This story is summarized by the complete Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{2.38}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.39}$$

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$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.40}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{2.41}$$

The only changes from the static case are the time derivative terms in equations (2.40) and (2.41) which describe how a changing magnetic field produces an electric field, and how a changing electric field produces a magnetic field. The former is called inductance, and the latter is called the displacement current, also known as Maxwell's correction to Ampère's law. These are the main new effects that we must account for when we deal with electro*dynamics*. As it turns out, the displacement current is almost always too small an effect to be measured, so we will focus our attention on induction, which is much more practically important.

2.3.2 Faraday's Law

Integrating both sides of equation (2.40) over a surface S and using Stokes's theorem, we find the integral form of the equation:

$$\int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\mathbf{\Phi}_B}{dt},$$

where $\mathbf{\Phi}_B = \int_S \mathbf{B} \cdot d\mathbf{S}$ is the magnetic flux through the (not closed!) surface S with boundary C (a *closed* curve). In most cases of interest, the curve C is a loop of current such as a wire. The surface S can be any surface with C as the boundary; see Fig. 2.17 for an example. The middle term of this expression is just the electric potential around the loop (up to a sign), and the right-hand side is the change in magnetic flux. This expression is telling us that a changing magnetic flux through a loop of wire sets up a potential (and therefore a current) through the wire, much like a battery would. The electric potential in this context is often called by the unfortunate name *electromotive force* (emf) and denoted by $\mathcal{E}$. Most

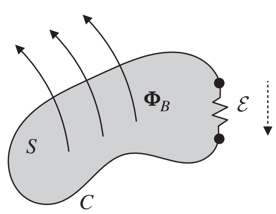

Figure 2.17 Example geometry described by Faraday's law. A wire along a curve C encloses a surface S, through which a changing magnetic flux Φ_B penetrates. A segment of the wire contains a resistor, across which there is a voltage or emf $\mathcal{E}$ in response to a changing magnetic flux. Note that the sign of the voltage must obey Lenz's law: the current induced by the emf must generate a magnetic field that opposes the external change in flux.

often, it is only the emf $\mathcal{E}$ that matters, so you can stick to the easier-to-remember form,

$$\mathcal{E} = -\frac{d\mathbf{\Phi}_B}{dt}, \tag{2.42}$$

provided you remember that (despite the notation) $\mathcal{E}$ is the potential, *not* the electric field.

Equations (2.40) and (2.42) are collectively referred to as Faraday's law. There is, however, one critical additional twist concerning the minus sign on the right-hand side of equation (2.42). This minus sign is often referred to as Lenz's law. The subtlety arises from the following problem. Imagine a wire loop with an increasing magnetic flux from some external source; Faraday's law implies there is a current induced in this wire. But the induced current in the wire also sets up a magnetic field itself. If both the external magnetic field and the magnetic field from the wire point in the same direction then there will be a runaway increase in magnetic field and current, violating conservation of energy! Clearly, energy conservation requires that these two fields point in opposite directions. This is the origin of the minus sign and the essential content of Lenz's law:

Induced currents always oppose changes in magnetic flux.

If you calculate everything correctly and use a minus sign in equations (2.40) and (2.42), then you should obtain the correct answer. But *always* check to make sure that the direction of current in your final solution does not end up violating conservation of energy.

2.3.3 Inductors

When two current loops are positioned close to each other, a changing current in one produces a time-varying magnetic field that can influence the other and vice versa. The flux Φ_{21} through loop 2 is proportional to the current I_1 in loop 1 via

$$\Phi_{21} = M_{12} I_1, \tag{2.43}$$

where M_{12} is a constant entirely dependent on geometry and known as the mutual inductance. It turns out that $M_{12} = M_{21}$, so this relationship is symmetric: $\Phi_{12} = M_{21} I_2 = M_{12} I_2$.

While mutual inductance rarely appears on the GRE, it is related to a much more common quantity that almost certainly will appear in some form on every exam: *self-inductance*. The self-inductance (or simply *inductance*) is generally defined to be the constant L in the expression

$$\Phi_B = LI, \tag{2.44}$$

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in which the magnetic flux through an arrangement of wires, where the field is produced *by the wires themselves*, is proportional to the current carried by the wires. This immediately implies that

$$\mathcal{E} = -L\frac{dI}{dt}, \tag{2.45}$$

where I is the current. The self-inductance of an arrangement of wires is the magnetic analogy of the capacitance of an arrangement of conductors, and, as with capacitance, the self-inductance is purely determined by the geometry of the arrangement. If you ever need to calculate it directly, there is a set of simple guidelines:

- Calculate the magnetic field through the current loop and integrate it to obtain the magnetic flux. This can be done with either the Biot–Savart law or Ampère's law, but obviously Ampère's law is always preferable when the geometry permits.
- Plug the result into Faraday's law, and identify the coefficient L.

In the same way that "capacitor" usually refers to the particular parallel-plate model, "inductor" usually means a solenoid with a large number of turns carrying a current. Just like a capacitor, it produces a strong uniform field in a limited spatial region, and therefore is able to store energy. The inductance of this solenoid is

$$L = \frac{\mu_0 N^2 A}{l} \quad \text{(solenoid)}, \tag{2.46}$$

where N is the total number of turns, l is the length, and A is the cross-sectional area. Rather than derive this for you, we will leave the derivation as an exercise. The factor of N^2 is peculiar, so we recommend memorizing this expression so you don't get tripped up on the exam.

Solenoids store magnetic field energy just as capacitors store electric field energy. From equations (2.32) and (2.35), we can compute the total stored energy in terms of the inductance (2.46) and the current, which works out to

$$U_L = \frac{1}{2}LI^2. \tag{2.47}$$

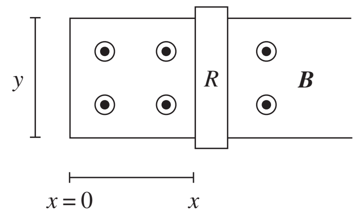
2.3.4 Problems: Electrodynamics

1. A circular loop of wire of radius a and resistance R is oriented in the xy-plane. A uniform magnetic field of magnitude B points in the $+\hat{\mathbf{z}}$-direction. If the loop of wire is rotated about the x-axis with an angular frequency ω, what is the average power dissipated by Joule heating in the loop?

   (A) $\pi^2 a^4 B^2 \omega^2/2R$
   (B) $\pi^2 a^4 B^2 \omega^2/R$
   (C) $\pi^2 a^4 B^2/2R$
   (D) 0
   (E) $\pi a B\omega/R$

2. What is the inductance of a toroid with N winds, circular cross section of radius a, and overall radius R (from the center of the torus to the center of the circular cross section)? You may assume that $a \ll R$.

   (A) $\mu_0 a^2 N/(2R)$
   (B) $\mu_0 a N^2/2$
   (C) $\mu_0 a^2 N^2/(2R)$
   (D) $\mu_0 a N/2$
   (E) $\mu_0 R^2 N/(8a)$



3. A rod of mass m and resistance R is attached to frictionless rails in the presence of a magnetic field of magnitude B pointing out of the page, as shown in the diagram above. The rod and rails form a closed electrical circuit. If the rod is launched from $x = 0$ with velocity v_0 to the right, at what time t is the velocity of the rod v_0/e? Assume that the rails have negligible resistance and neglect the self-inductance of the circuit.

   (A) $my/(R^2B^2)$
   (B) $my/(2R^2B^2)$
   (C) $mR/(y^2B^2)$
   (D) $mR/(2y^2B^2)$
   (E) The rod never reaches this speed because it travels at constant velocity v_0.

4. A capacitor made from two circular parallel plates of area A and separation d is connected in series to a voltage supply which maintains a constant current I in the circuit, charging the capacitor. What is the magnitude of the magnetic field between the parallel plates, as a function of r, the distance from the central axis of the plates? Assume r is smaller than the radius of the plates.

   (A) $\mu_0 r I/(2A)$
   (B) $\mu_0 I/(2\pi r)$

(C) $\mu_0 I/(2d)$
(D) $\mu_0 Id/(2\pi r^2)$
(E) 0

2.4 Dipoles

There is one final parallel between electricity and magnetism that is worth exploring. There are many cases when we are confronted with a situation in which two opposite charges are located very close to each other and we want to find the field far away. (If you remember your chemistry from high school, then you should know that a salt is a good example of a dipole.) Since there are no magnetic monopoles, most configurations of magnetic fields are dipoles. Although the mathematical formalism is identical in both cases, we'll examine them one by one.

2.4.1 Electric Dipoles

If we have two opposite charges q and $-q$ located at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively (Fig. 2.18), then we can define the dipole moment as

$$\mathbf{p} = q\mathbf{r}_1 - q\mathbf{r}_2 = q\mathbf{d}, \tag{2.48}$$

where $\mathbf{d}$ is the vector *from* the negative *to* the positive charge.[4] The dipole moment is a vector, so the dipole moment for several dipoles is the vector sum of the individual dipole moments. This leads to a trick for finding the dipole moments of funny charge configurations: pair them up into dipoles, and add up the vectors. More generally, for an arbitrary collection of point charges we have

$$\mathbf{p} = \sum_i q_i \mathbf{d}_i.$$

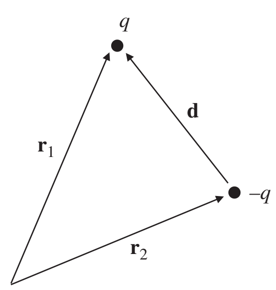


Figure 2.18 Two charges of opposite sign have a dipole moment proportional to their charge and the displacement $\mathbf{d}$ between them. The direction of the dipole moment points from the negative charge to the positive charge.

[4] Watch this sign convention! This is the opposite of the more familiar case where electric fields point from positive to negative charges.

Or, most generally, for a charge density $\rho(\mathbf{r})$, we have a dipole moment

$$\mathbf{p} = \int \mathbf{r}\rho(\mathbf{r})d^3\mathbf{r}. \tag{2.49}$$

You can recover the expression for the dipole moment of two opposite point charges $\pm q$ by just substituting delta functions for the charge density (try it yourself!). Dipoles have a nonzero electric field, which we will not dwell on here because knowing the precise form doesn't seem to be important for actually doing GRE problems. The potential is much easier to work with, and is simply

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}. \tag{2.50}$$

Notice that the potential of a dipole goes as $1/r^2$, which is *not* the same as the $1/r$ potential of a point charge! This comes from the Taylor expansion mentioned in Section 2.1.4, but it's an important enough fact that it's worth remembering on its own. Since $\mathbf{E}$ is related to V by a derivative, this means that the electric field of a dipole goes as $1/r^3$, rather than $1/r^2$.

Electric dipoles tend to align themselves with electric fields because the negative part gets pulled in one direction and the positive part gets pulled in the opposite direction. But since pure dipoles have net charge zero, they are not accelerated by electric fields – they just experience a torque. This torque is given by

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}. \tag{2.51}$$

Since there is a torque, there is also a potential energy for a dipole in an electric field:

$$U = -\mathbf{p}\cdot\mathbf{E}. \tag{2.52}$$

2.4.2 Magnetic Dipoles

Magnetic dipoles behave quite similarly to electric ones, with one key conceptual difference. Magnetic dipoles are *not* composed of two opposite charges because there are no magnetic monopoles. Magnetic dipoles are pure and irreducible: if you chop a magnetic dipole in half, you only get two magnetic dipoles, not two monopoles.

Since we cannot build magnetic dipoles from monopoles like the electric case, we typically build them out of current loops. The magnetic dipole moment associated with a current loop carrying current I is given by

$$\mathbf{m} = I\mathbf{A}, \tag{2.53}$$

where $\mathbf{A}$ is a vector pointing normal to the surface subtended by the current loop, with magnitude equal to the area of the

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surface. As always in this business, the direction of the normal is fixed by the right-hand rule, the direction your right thumb points when your fingers curl around the direction of the current. The torques and potential fields due to a dipole in a magnetic field are analogous to the electric case:

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}, \tag{2.54}$$

$$U = -\mathbf{m} \cdot \mathbf{B}. \tag{2.55}$$

And, just as with electric dipoles, the magnetic field of a magnetic dipole falls off as $1/r^3$.

2.4.3 Multipole Expansion

The idea of a dipole can be generalized with a tool known as the *multipole expansion*, a series that gives a quantitative measure to how "lumpy" a charge distribution is. If it has a net charge, the first term in the series is nonzero; if it is neutral but has a separation of charge within it, the second term is nonzero; and so on. While it is unlikely that you will have to compute anything with a multipole expansion, understanding it will help you guess correct answers. The potential due to an arbitrary charge configuration is given by equation (2.8). By expanding the fraction inside the integral in terms of Legendre polynomials, we arrive at a general series expression:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta')\rho(\mathbf{r}')d^3\mathbf{r}'.$$

This looks complicated, but the idea is that the first few Legendre polynomials are quite simple, so it will be easy to evaluate the first few terms of this series. Since the first few terms dominate, this will often be a good enough approximation for the problem we wish to solve. In addition, *all* the dependence on $\mathbf{r}$ is now contained in the power series $1/r^{n+1}$, so the integral that remains is "easy" in the sense that it only depends on the coordinates $\mathbf{r}'$ of the charge distribution.

The first two terms are very simple, in fact. The first term is the monopole term, which is just given by the total charge of the configuration:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}.$$

As promised, the second term is the dipole term:

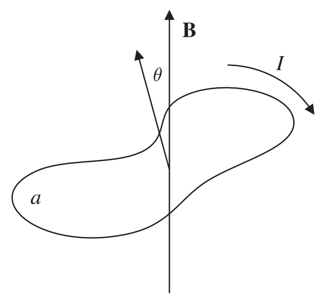
$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2},$$

where $\mathbf{p}$ is defined by (2.49). While this discussion only dealt with the scalar potential V, you could play the same game with the vector potential $\mathbf{A}$; in that case, by Gauss's law for magnetism, you would find that the monopole moment of a current distribution automatically vanishes, and the first term would be the dipole term (2.53).

2.4.4 Problems: Dipoles

1. Consider a pure electric dipole of moment $\mathbf{p} = p\hat{\mathbf{z}}$. A small test particle is located at $(0, 0, z)$ and experiences an electric field of magnitude E. What is the magnitude of the electric field experienced by a test particle at $(0, 0, 2z)$ from a dipole of moment $2\mathbf{p}$?

   (A) $E/4$
   (B) $E/2$
   (C) E
   (D) $2E$
   (E) $4E$



2. Suppose that a current loop of area a carrying current i, with moment of inertia I is placed in a uniform magnetic field of magnitude B. The normal to the loop is initially misaligned from the direction of the magnetic field by a small angle θ. When the loop is released, what is the period of oscillation?

   (A) $2\pi\sqrt{I/(iaB)}$
   (B) $2\pi\sqrt{i/(IaB)}$
   (C) $\sqrt{IaB}$
   (D) $2\pi/(IaB)$
   (E) $1/(IaB)$

2.5 Matter Effects

In everything that we have done so far with electric and magnetic fields, we have implicitly assumed that we are working in vacuum. This is rarely the case in the real world. Matter, for example, is often full of microscopic dipoles that can align themselves to slightly cancel out electric fields. Similar effects occur with magnetic fields in many materials. The behavior of electric and magnetic fields in matter used to be *much* more important on the GRE ten or twenty years ago, but these questions have gradually fallen out of fashion, paralleling a similar development in undergraduate physics curricula.

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You are unlikely to see more than one question on your exam related to matter effects, so our treatment here will be even briefer than usual. In fact, the three most recently released exams from ETS had only one question about matter effects each, both times about capacitors and/or dielectrics.

2.5.1 Polarization

The primary effect of electric fields in matter is the dielectric effect: small dipoles in a material become slightly polarized by the presence of an external electric field. This can be described by a quantity called the *polarization* $\mathbf{P}$, which is the *electric dipole moment per unit volume*. On the GRE, the most you will be asked to do is calculate the electric field, given the polarization. This is straightforward: $\mathbf{P}$ gives rise to effective surface and volume charge densities, known as *bound* charges:

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}, \tag{2.56}$$

$$\rho_b = -\nabla \cdot \mathbf{P}. \tag{2.57}$$

Here, $\hat{\mathbf{n}}$ is the outward-pointing normal to the surface of the polarized object. To calculate the electric field, just apply Gauss's law as usual to the bound charges σ_b and ρ_b, exploiting whatever symmetry is appropriate to the problem.

The situation becomes considerably more complicated in the presence of external electric fields, which will back-react on the polarization. There are also analogous bound currents for magnetized materials. These scenarios are treated in any advanced electrodynamics textbook, but are likely too advanced for the GRE.

2.5.2 Dielectrics

Dielectrics are materials, such as insulators, that can be polarized in an applied field, and thus slightly *cancel* the applied electric field. This effect can be parameterized by making the substitution

$$\epsilon_0 \mapsto \epsilon = \kappa\epsilon_0 \tag{2.58}$$

in all formulas (potential, electric field, etc.), where κ is known as the *dielectric constant*. The most common situation is when a dielectric is placed between the two plates of a parallel-plate capacitor, and the capacitance becomes

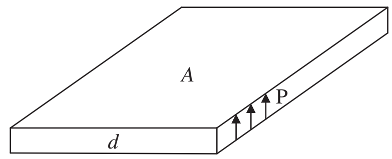
$$C = \frac{\epsilon A}{d} = \kappa\frac{\epsilon_0 A}{d}. \tag{2.59}$$

This is a common way to increase the capacitance of a capacitor.

2.5.3 Problems: Matter Effects

1. What is the work needed to insert a dielectric with dielectric constant $\kappa = 2$ into a parallel-plate capacitor with capacitance C that is maintained at a constant voltage V?

   (A) $\frac{1}{2}CV^2$
   (B) CV^2
   (C) $2CV^2$
   (D) $\frac{1}{2}\frac{C}{V^2}$
   (E) $\frac{C}{V^2}$



2. A thin slab of material of area A and thickness d carries uniform polarization $\mathbf{P}$, as shown in the diagram. What is the magnitude of the electric field just above the slab, assuming $d^2 \ll A$?

   (A) $|\mathbf{P}|/(2\epsilon_0)$
   (B) $|\mathbf{P}|/\epsilon_0$
   (C) $2|\mathbf{P}|\epsilon_0$
   (D) $|\mathbf{P}|\epsilon_0$
   (E) 0

2.6 Electromagnetic Waves

Back in the early twentieth century, there was a famous conflict in physics between experiments such as diffraction, which indicated that light was a wave, and other experiments such as the photoelectric effect, which indicated that light was a particle. Now, of course, we know that light is both a particle and a wave in some sense. Here we will see the argument for light being a wave by showing how classical electrodynamics implies the existence of waves that travel at the speed of light.

2.6.1 Wave Equation and Poynting Vector

Start by taking the curl of equation (2.40). Using the identity from vector calculus that $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$, and assuming that we are in vacuum where $\rho = 0$ and $\mathbf{J} = 0$ we have

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}),$$
$$\nabla^2\mathbf{E} = \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

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By a similar argument,

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

As we will see in Chapter 3, these are just wave equations with velocity

$$c = 1/\sqrt{\epsilon_0 \mu_0}. \tag{2.60}$$

If you plug in the numbers, you really do find that $1/\sqrt{\epsilon_0 \mu_0}$ gives the correct speed of light. So, in vacuum, the electric and magnetic fields have wavelike solutions that travel at the speed of light. The wave solutions have the explicit form

$$\tilde{\mathbf{E}}(\mathbf{r}) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}}, \tag{2.61}$$

$$\tilde{\mathbf{B}}(\mathbf{r}) = \frac{1}{c}\tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}} \times \hat{\mathbf{n}}). \tag{2.62}$$

Here, $\hat{\mathbf{k}}$ is the propagation vector, describing the direction the wave travels in, and $\hat{\mathbf{n}}$ is the polarization vector. Note that when discussing electromagnetic waves, the polarization refers to the direction of the *electric* field only; the magnetic field is polarized in a perpendicular direction, as you can see from the cross product in equation (2.62). The three vectors $\hat{\mathbf{k}}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{k}} \times \hat{\mathbf{n}}$ form a right-handed coordinate system; the fact that the electric and magnetic field vectors are both perpendicular to the propagation vector means that electromagnetic waves in vacuum are *transverse*.

We are using some extremely convenient, but potentially confusing, notation here, where we are representing the electric and magnetic fields as complex exponentials. The magnitude $\tilde{E}_0$ may also be complex. There is nothing imaginary about the fields, but it simplifies the algebra considerably. The *physical* part of the fields is just the real part: $\mathbf{E} = \mathrm{Re}(\tilde{\mathbf{E}})$, and similarly for $\mathbf{B}$. So the rule for working in this notation is to calculate everything in the complex formalism as normal, and then take the real part at the end. This is straightforward when dealing with superpositions of waves, since the real part of a sum is the sum of the real parts, but for products it requires some care in definitions, as we'll see below. For a slightly more in-depth treatment of this material, see the following chapter, Section 3.1.

Since electromagnetic fields have energy, it should be perfectly natural to expect the wave solutions to transport energy. This is described by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}), \tag{2.63}$$

which gives the flux of energy of the wave (energy per unit area per unit time, or power per unit area). This expression

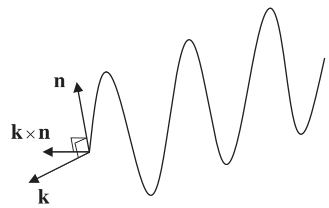

Figure 2.19 Vectors describing propagation and component fields for an electromagnetic wave. The wave propagates in the direction $\mathbf{k}$, the electric field $\mathbf{E}$ is proportional to the vector $\mathbf{n}$, and the magnetic field $\mathbf{B}$ is proportional to the vector $\mathbf{k} \times \mathbf{n}$.

is in terms of the physical (real) fields: if you prefer to use complex notation, use the definition

$$\mathbf{S} = \frac{1}{2\mu_0}\mathrm{Re}(\tilde{\mathbf{E}} \times \tilde{\mathbf{B}}^*) \tag{2.64}$$

instead. Since electromagnetic waves have extremely high frequencies, it's often more useful to average the magnitude of the Poynting vector over one complete cycle in time, which gives the *intensity* $\langle S \rangle$, the *average* power per unit area. Using the helpful fact that the average of $\sin^2$ or $\cos^2$ over one cycle is $1/2$, we obtain

$$I = \langle S \rangle = \frac{1}{2} c \epsilon_0 E_0^2. \tag{2.65}$$

When we are analyzing electromagnetic waves, we often want to know what happens when we pass across an interface of different materials. This is the case in optics, for example, where it is possible to use electrodynamics to derive all of the formulas in geometric optics. We won't do this, but if you want a little practice, it's a nice way to check whether you truly understand all of this material. Snell's law is a good place to begin. The crucial trick when analyzing electromagnetic waves at boundaries is to use the boundary conditions that we described in detail earlier in this chapter. The schematic approach is:

- Figure out the generic form of the field on either side of the interface.
- Write the boundary conditions for the fields at the interface.
- Match the fields using the boundary conditions.

Conductors are good example of this process. The fields inside a perfect conductor must be zero. We also have the boundary condition that $\mathbf{E}^{\parallel}_{\text{out}} = \mathbf{E}^{\parallel}_{\text{in}} = 0$. So, if an electromagnetic wave is normally incident on a conductor, the electric component of the reflected wave points in the opposite direction to the incident wave so as to cancel off the parallel electric field just outside the conductor. Working through the cross

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products, the magnetic fields of the incident and reflected waves therefore point in the same direction.

2.6.2 Radiation

Since moving electric charges can create electromagnetic waves and these electromagnetic waves carry energy, it is possible to produce electromagnetic radiation far away from the charges. Indeed, an accelerating point charge radiates a total power

$$P = \frac{q^2 a^2}{6\pi \epsilon_0 c^3} = \frac{\mu_0 q^2 a^2}{6\pi c}, \tag{2.66}$$

where q is the charge of the oscillating particle, and a is its acceleration. The two forms are related to each other by equation (2.60). This formula, known as the Larmor formula, only holds when the charge is moving at small velocities, $v \ll c$. The prefactors are not so important, what really matters is the q^2 and a^2 dependence.

An oscillating *dipole* will also radiate, and in fact this situation is more common because most molecules in nature are dipoles rather than free charges. Let the dipole have dipole moment $\mathbf{p}(t) = p_0 \cos(\omega t)\hat{\mathbf{z}}$. There are two useful formulas here. The first is the intensity:

$$\langle S \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \frac{\sin^2\theta}{r^2}, \tag{2.67}$$

where θ has its usual meaning in spherical coordinates. As usual, don't worry about the numerical factors, but what is important is the p_0^2 dependence (the same as the q^2 dependence in the Larmor formula), the frequency dependence ω^4, the fact that $\langle S \rangle$ falls off as $1/r^2$, and the $\sin^2\theta$ term which means that *no radiation occurs along the dipole axis*. Integrating (2.67) over a sphere of radius r gives the total power

$$\langle P \rangle_E = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}, \tag{2.68}$$

which has the same p_0^2 and ω^4 dependence. By the way, the reason we deal with dipole radiation rather than monopole radiation (for example, a sphere of charge with an oscillating radius) is the curious fact that *monopoles do not radiate*. This actually follows from Gauss's law, which says the field outside a spherically symmetric charge distribution is independent of the size of the sphere.

The above formulas applied to electric dipoles only. There is an analogous formula for *magnetic* dipole radiation:

$$\langle P \rangle_B = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}, \tag{2.69}$$

where m_0 is the average magnetic dipole moment. The only other change is the additional factor of $1/c^2$, which represents an *enormous* suppression compared to the electric case because c is so large. This means that electric dipole radiation will dominate unless the system is contrived to eliminate an electric dipole moment: for example, when current is driven around a wire loop.

2.6.3 Problems: Electromagnetic Waves

1. An AC current is driven around a loop of wire. Suppose the amplitude and frequency of the current are both doubled. By what factor does the power radiated by the antenna increase?

   (A) 4
   (B) 8
   (C) 16
   (D) 32
   (E) 64

2. A perfectly conductive plate is placed in the yz-plane. An electromagnetic wave with electric field $\mathbf{E} = E_0 \cos(kx - \omega t)\hat{\mathbf{y}}$ is incident on the conductor. If the wave strikes the plate at $t = 0$, what are the directions of the electric and magnetic fields of the reflected wave immediately after reflection?

   (A) $\mathbf{E} \propto -\hat{\mathbf{x}}, \mathbf{B} \propto -\hat{\mathbf{y}}$
   (B) $\mathbf{E} \propto \hat{\mathbf{y}}, \mathbf{B} \propto \hat{\mathbf{z}}$
   (C) $\mathbf{E} \propto -\hat{\mathbf{y}}, \mathbf{B} \propto -\hat{\mathbf{z}}$
   (D) $\mathbf{E} \propto -\hat{\mathbf{y}}, \mathbf{B} \propto \hat{\mathbf{z}}$
   (E) There is no reflected wave.

3. What is the speed of light in a medium with a permeability of $2\mu_0$ and a permittivity of $3\epsilon_0$?

   (A) c
   (B) $c/\sqrt{3}$
   (C) $c/2$
   (D) $c/\sqrt{6}$
   (E) $c/6$

2.7 Circuits

Depending on the flavor of your undergraduate education, your knowledge of circuits might be a little rusty. One of us actually never learned circuits in an undergraduate course, possibly because they were deemed too practical and not of fundamental importance! Apparently the GRE does not share this opinion, so it behooves you learn this material

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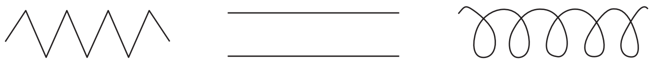

(a) Resistor (b) Capacitor (c) Inductor

Figure 2.20 Symbols for three fundamental circuit elements.

well. Thankfully it is not too difficult at the level the GRE tests it.

2.7.1 Basic Elements

There are three fundamental circuit elements: resistors, capacitors, and inductors. Their icons in circuit diagrams are shown in Fig. 2.20. The voltages across each element are given by

$$V_R = IR, \tag{2.70}$$

$$V_C = \frac{Q}{C}, \tag{2.71}$$

$$V_L = L\frac{dI}{dt}, \tag{2.72}$$

where R is the resistance (units of ohms), C is the capacitance (units of farads), and L is the inductance (units of henries). When confronted with a circuit to analyze, one typically wants to find the current or voltage across some element. For simple circuits, this is done by subdividing the circuit into "equivalent" blocks and computing the total resistance, capacitance, or inductance of each section. The practical formulas for adding elements in series are

$$R_{\text{eq}} = \sum_i R_i \quad \text{(series)}, \tag{2.73}$$

$$\frac{1}{C_{\text{eq}}} = \sum_i \frac{1}{C_i} \quad \text{(series)}, \tag{2.74}$$

$$L_{\text{eq}} = \sum_i L_i \quad \text{(series)}. \tag{2.75}$$

In parallel, we have the rules

$$\frac{1}{R_{\text{eq}}} = \sum_i \frac{1}{R_i} \quad \text{(parallel)}, \tag{2.76}$$

$$C_{\text{eq}} = \sum_i C_i \quad \text{(parallel)}, \tag{2.77}$$

$$\frac{1}{L_{\text{eq}}} = \sum_i \frac{1}{L_i} \quad \text{(parallel)}. \tag{2.78}$$

These rules come from the fact that circuit elements in series see the same current, while circuit elements in parallel see the same voltage.

While capacitors and inductors are created by very particular geometries of conductors and wires, resistance is a property of all materials. More precisely, we define a material-dependent, intrinsic property called resistivity. The resistance is the impedance to electrical current for a particular geometrical configuration of that material. The general relation between the resistivity ρ of a material and the resistance of a geometry is given by

$$R = \frac{\rho \ell}{A}, \tag{2.79}$$

where A is the cross-sectional area of the resistor, and ℓ is the length.

2.7.2 Kirchhoff's Rules

A more general method for solving for currents and voltages in a circuit is to use Kirchhoff's rules. This method is more systematic for larger circuits, but it leads to a system of linear equations which can be cumbersome to solve quickly on the GRE. There are two rules, which are consequences of conservation of charge and energy, respectively:

1. The sum of currents flowing into every node must be zero:

$$\sum_k I_k = 0. \tag{2.80}$$

2. The sum of the voltages across elements around any closed loop must be zero:

$$\sum_k V_k = 0. \tag{2.81}$$

The strategy is then to write an equation for every node and loop in a circuit, and then solve them all simultaneously for the desired current or voltage. Since they are so systematic, Kirchhoff's rules are good to use when you are uncertain how to solve a problem using the series and parallel circuit rules above.

2.7.3 Energy in Circuits

An important distinction between the three circuit elements is between dissipative elements and elements that conserve energy. Resistors *dissipate* energy according to the famous rule

$$P = IV = \frac{V^2}{R} = I^2 R, \tag{2.82}$$

and this power usually shows up as heat, which raises the temperature of circuit elements (which is why the back of your computer gets hot). In contrast, capacitors and inductors do not dissipate energy, but *store* it in electric and magnetic fields, as we have previously discussed. We'll repeat the formulas here for convenience:

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$$U_C = \frac{1}{2}CV^2, \qquad \text{(p. 44)} \tag{2.21}$$

$$U_L = \frac{1}{2}LI^2. \qquad \text{(p. 51)} \tag{2.47}$$

2.7.4 Standard Circuit Types

Circuits generally have two kinds of behavior: transient, which describes charges and currents that die off quickly with time, and steady-state, which describes the state of the circuit after a sufficiently long time has passed. When analyzing circuits that contain a combination of different elements, the transient (as opposed to steady-state) behavior of the circuit is very important. The general approach for analyzing such circuits is to write down the voltage around some section of the loop (e.g. using Kirchhoff's rules), and then use the relations in equations (2.70)–(2.72) to produce a differential equation in I or Q. You can then solve the ODE with exponentials to extract the time dependence. For a more qualitative, but very useful, discussion of the time behavior of circuit elements, see Section 7.3.1.

- **RL circuits.** Consider the example of a circuit with a resistor and inductor in series with a voltage source. The circuit satisfies the equation

$$V = IR + L\frac{dI}{dt}.$$

If the voltage source is suddenly switched on, the current is

$$I = \frac{V}{R}\left(1 - e^{-t/\tau_{RL}}\right).$$

The time constant

$$\tau_{RL} = L/R \tag{2.83}$$

is the characteristic response time of an RL circuit. All transient behavior of RL circuits with DC voltages is exponential with this characteristic time.

- **RC circuits.** The situation with RC circuits is very similar to RL circuits. An RC circuit is a circuit containing a resistor and a capacitor in series, and possibly a voltage source as well. The equation of current of an RC circuit with no voltage source is

$$0 = R\frac{dQ}{dt} + \frac{1}{C}Q.$$

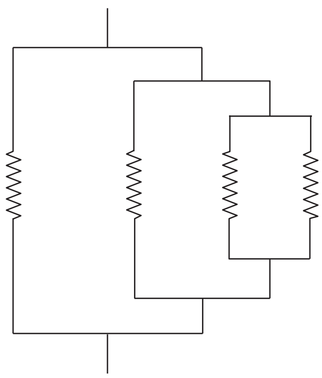
The discharging of a capacitor in such a circuit (or the charging of a capacitor in a circuit with a voltage supply) is again exponential with characteristic time constant

$$\tau_{RC} = RC. \tag{2.84}$$

- **RLC circuits.** Combining all three basic circuit elements gives the most interesting elementary circuit, the RLC circuit. Happily, the mathematics of this situation are *identical* to a problem we've already treated, namely masses and springs with friction. Just refer to Table 1.1 in Section 1.7.3 and make the appropriate substitutions. Note that an LC circuit can be considered a special case where the resistance R goes to zero. In that case, the resonant frequency of the circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{2.85}$$

2.7.5 Problems: Circuits



1. What is the equivalent resistance of the network above, if all resistors have resistance R?
   (A) R
   (B) $R/4$
   (C) $R/2$
   (D) $3R/4$
   (E) $4R/3$

2. A capacitor C is in series with a resistor R. The capacitor is initially charged, and a switch is closed at time $t = 0$ to complete the circuit. After what time t has the resistor dissipated half of the energy originally stored in the capacitor?
   (A) RC
   (B) $(RC \ln 2)$
   (C) $(RC \ln 2)/2$
   (D) $(RC)/2$
   (E) $(RC \ln 2)/4$

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2.8 Solutions: Electricity and Magnetism

Electrostatics

1. A – By the method of images, the configuration is equivalent to two point charges, each a distance d from the putative conducting plane. The force on the image charge, and thus the magnitude of the force on the conducting plane is just given by the Coulomb force law:
$$F = \frac{q^2}{16\pi\epsilon_0 d^2}.$$
2. D – As we have calculated previously, the electric field due to a line of charge is given by
$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}}.$$
The potential between the line and the cylindrical shell is given by
$$\begin{aligned} V &= \int_a^b \frac{\lambda}{2\pi\epsilon_0 r}dr \\ &= \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}. \end{aligned}$$
The capacitance per unit length is consequently
$$\begin{aligned} \frac{C}{\ell} &= \frac{\lambda}{V} \\ &= \frac{2\pi\epsilon_0}{\ln(b/a)}. \end{aligned}$$
3. C – The factor of 3 just comes by scaling the equation for the capacitance of a parallel-plate capacitor:
$$C = \frac{\epsilon_0 A}{d} = \frac{\epsilon_0 a^2}{d}.$$
4. D – The work needed to assemble the configuration is just given by the potential between each pairwise combination of vertices of the tetrahedron, i.e. the number of edges of a tetrahedron times the potential between two point charges. Each point charge is separated by a distance a and there are six edges on a tetrahedron, so the work needed is
$$W = \frac{3q^2}{2\pi\epsilon_0 a}.$$
5. E – To find the total charge enclosed, we can use the first of Maxwell's equations to find the charge density and then integrate it over the sphere. From the first Maxwell equation, we have
$$\begin{aligned} \rho &= \epsilon_0 \nabla\cdot\mathbf{E} \\ &= \epsilon_0\frac{\partial}{\partial z}\left(E_0 z^2\right) \\ &= 2\epsilon_0 E_0 z. \end{aligned}$$
The total enclosed charge is therefore (recalling $z = r\cos\theta$ in spherical coordinates)
$$\begin{aligned} Q &= \int \rho(r)d^3\mathbf{r} \\ &= 2\epsilon_0 E_0 \int_0^R\int_0^\pi\int_0^{2\pi}(r\cos\theta)r^2\sin\theta\, dr\, d\theta\, d\phi \\ &= 0. \end{aligned}$$
In hindsight, this makes sense: the electric field always points in the same direction throughout the sphere, so, thinking in terms of field lines, there is no net charge for the field lines to start or end on. This is equally apparent from the charge distribution $\rho = 2\epsilon_0 E_0 z$, which is positive for $z > 0$ and negative for $z < 0$, with the same magnitude on both sides of the plane $z = 0$.

Magnetostatics

1. B – We can use the Biot–Savart law to find the field. Note that there is no force from either straight portion of the wire, since $d\mathbf{l}$ and $\hat{\mathbf{r}}'$ are parallel and the cross product vanishes. Applying Biot–Savart to the semicircle, we have
$$\begin{aligned} |\mathbf{B}| &= \frac{\mu_0}{4\pi}\int_0^\pi \frac{Iad\phi\left|\hat{\boldsymbol{\phi}}\times(-\hat{\mathbf{r}})\right|}{a^2} \\ &= \frac{\mu_0 I}{4a}. \end{aligned}$$
2. B – The clockwise current of the upper loop produces a magnetic field in the $-\hat{\mathbf{z}}$-direction. The electrons moving in the lower wire are moving in the $+\hat{\boldsymbol{\phi}}$-direction, and therefore feel a force
$$\mathbf{F} = q\mathbf{v}\times\mathbf{B} \propto -e\hat{\boldsymbol{\phi}}\times(-\hat{\mathbf{z}}) \propto \hat{\mathbf{r}},$$
in the outward radial direction in cylindrical coordinates. Beware: by convention current is the flow of positive charges, so negative charges move in the direction opposite to the "direction of current." So a clockwise current is a counterclockwise flow of electrons!
3. C – This is a classic problem and well worth remembering. The field from one wire is $\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}}$. Letting the z-axis run along that wire, the force on a length dl of the other wire is then
$$d\mathbf{F} = (Idl\,\hat{\mathbf{z}})\times\frac{\mu_0 I}{2\pi d}\hat{\boldsymbol{\phi}} = \frac{\mu_0 I^2}{2\pi d}(-\hat{\mathbf{r}})dl.$$
Thus, the force per unit length $d\mathbf{F}/dl$ is $\mu_0 I^2/2\pi d$, towards the first wire. This can easily be remembered as "like currents attract."

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4. A – Recall that the energy stored in a magnetic field $\mathbf{B}$ is given by equation (2.35),

$$U_B = \frac{1}{2\mu_0}\int |\mathbf{B}|^2 d^3\mathbf{r}.$$

Using the expression for the magnetic field of a toroid, we have (using cylindrical coordinates)

$$\begin{aligned} U &= \frac{1}{2\mu_0}\int_0^a \int_0^{2\pi}\int_R^{R+a} \frac{\mu_0^2 N^2 I^2}{4\pi^2 r^2} r\,dr\,d\phi\,dz \\ &= \frac{\mu_0 N^2 I^2 a}{4\pi}\int_R^{R+a}\frac{dr}{r} \\ &= \frac{\mu_0 N^2 I^2 a}{4\pi}\ln\frac{R+a}{R}. \end{aligned}$$

Electrodynamics

1. A – The magnetic flux through the loop of wire is given by

$$\Phi_B = \pi a^2 B\cos\omega t.$$

By Faraday's law, we have

$$\mathcal{E} = \pi a^2 B\omega\sin\omega t.$$

The power dissipated by Joule heating is therefore

$$P = \frac{\mathcal{E}^2}{R} = \frac{\pi^2 a^4 B^2\omega^2\sin^2\omega t}{R},$$

or on average (using the fact that $\langle\sin^2\rangle = 1/2$),

$$\langle P\rangle = \frac{\pi^2 a^4 B^2 \omega^2}{2R}.$$

2. C – To calculate the inductance we just calculate the magnetic field and flux through the toroid and then identify the coefficient of the current term. From Ampère's law, the magnetic field is simply

$$\mathbf{B} = \frac{\mu_0 NI}{2\pi r}\hat{\boldsymbol{\phi}}.$$

Since the field is approximately constant inside the toroid (because $a \ll R$), we can just multiply by the cross-sectional area and the number of turns to obtain the magnetic flux:

$$\begin{aligned} \Phi_B &= N\pi a^2\frac{\mu_0 NI}{2\pi R} \\ &= \frac{\mu_0 a^2 N^2 I}{2R}. \end{aligned}$$

In this problem it is easy to miss the factor of N that must multiply the magnetic flux through one of the turns. Remember that, at a fixed magnetic field in the toroid, as the number of turns grows, the total flux must increase. The self-inductance is then just the coefficient of the current term:

$$L = \frac{\mu_0 a^2 N^2}{2R}.$$

3. C – The magnetic flux through the loop formed by the rod and rails is $\Phi_B(t) = x(t)yB$. The emf is given by

$$\mathcal{E} = -\dot{x}(t)yB.$$

Note that we have neglected the "back emf" from the magnetic field of the induced current. The power dissipated is

$$P = \frac{\mathcal{E}^2}{R}.$$

Using the work–energy theorem, the work performed on the rod, as a function of the velocity, is given by

$$W = \frac{1}{2}mv_0^2 - \frac{1}{2}m\dot{x}^2.$$

Setting power equal to the time derivative of the work, we find that

$$\frac{dv}{dt} = -\frac{vy^2B^2}{mR}.$$

The solution to this differential equation is just

$$v(t) = v_0\exp\left(-\frac{y^2B^2t}{mR}\right),$$

So the time needed to reach a velocity of v_0/e is given by

$$t_0 = \frac{mR}{y^2B^2}.$$

If this problem seems rather involved, note that E can be eliminated by common sense because the problem clearly involves induction, and A and B can be eliminated by dimensional analysis. It's a tough call between C and D, but one could plausibly guess that a factor of 2 should not enter into the solution.

4. A – While 0 is a tempting answer because there is no current flowing across the capacitor, it is not correct. In fact, a charging capacitor is about the only time you'll have to invoke the concept of displacement current. To find the answer carefully, note that the electric field between the two plates is approximately constant with magnitude

$$E = \frac{Q}{\epsilon_0 A},$$

where Q is the charge on the plates. The charge on the plates is equal to the constant current times the time since the beginning of charging, so

$$E = \frac{It}{\epsilon_0 A}.$$

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From the integral form of Ampère's law with Maxwell's correction, we have

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}.$$

Pick a circular loop between the parallel plates, centered at the axis of the plates, of radius r. The electric flux through this loop is

$$\Phi_E = \frac{\pi r^2 It}{\epsilon_0 A}.$$

The magnitude of the magnetic field is therefore

$$B = \frac{\mu_0 r I}{2A},$$

which you can think of as the field sourced by the fictitious "displacement current" between the capacitor plates. Note that without Maxwell's correction to Ampère's law, the answer *would* be zero!

Dipoles

1. A – Recall that the potential of a pure dipole scales as $V \sim \hat{\mathbf{r}} \cdot \mathbf{p}/r^2$. The electric field along the z-direction is simply the z-derivative of the potential V, so the electric field will scale as p/z^3 and will be reduced by a factor of 4 when p and z are both doubled.
2. A – The torque on the dipole is given by

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}.$$

For small oscillations, we have (using Newton's second law in the form $N = I\ddot{\theta}$)

$$\ddot{\theta} = -\frac{iaB}{I}\theta.$$

The angular frequency is therefore

$$\omega = \sqrt{\frac{iaB}{I}},$$

and the period is

$$T = 2\pi\sqrt{\frac{I}{iaB}}.$$

Matter Effects

1. A – The potential energy stored in the capacitor before the dielectric is

$$U_i = \frac{1}{2}CV^2.$$

After the dielectric is introduced, the effective capacitance doubles, since $C' = \kappa C$. So the new energy stored is

$$U_f = CV^2,$$

and the total work done must be

$$W = \frac{1}{2}CV^2.$$

2. E – Since the polarization is uniform, the bound volume charge vanishes. The polarization is normal to the area A, so the bound surface charge is $\sigma_b = \pm|\mathbf{P}|$ on the upper and lower surfaces, respectively, and zero on the sides. Just above the surface, then, the slab looks like an infinite parallel-plate capacitor, where the fields from the two plates reinforce each other between the plates (here, inside the slab), but cancel outside. Thus, the field just above the slab is zero. This picture makes sense if we think of the polarized slab as a bunch of dipoles collected in a rectangular area, which will create an effective sheet of positive charge at the top and negative charge at the bottom.

Electromagnetic Waves

1. E - The total power radiated is proportional to $\omega^4 m_0^2$. Doubling the amplitude of the current doubles m_0, and doubling the frequency doubles ω, so overall the power changes by a factor of $2^6 = 64$.
2. D – Since the plate is a perfect conductor the electric field inside must be identically zero:

$$\mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{refl}} = \mathbf{E}_{\text{transmitted}} = 0.$$

Since $\mathbf{E}^{\parallel}_{\text{out}} = \mathbf{E}^{\parallel}_{\text{in}}$ by the EM boundary conditions, the parallel component of the incident and reflected waves must cancel. At $t = 0$ the electric field vector of the incident wave is $\mathbf{E} = E_0\hat{\mathbf{y}}$, so the reflected wave must be polarized in the $-\hat{\mathbf{y}}$-direction. Since the propagation vector of the reflected wave is $-\hat{\mathbf{x}}$, the magnetic field of the reflected wave is in the $(-\hat{\mathbf{x}}) \times (-\hat{\mathbf{y}}) = \hat{\mathbf{z}}$-direction.
3. D – The speed of an electromagnetic wave in vacuum is

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

Making the substitutions $\mu_0 \mapsto 2\mu_0$ and $\epsilon_0 \mapsto 3\epsilon_0$, we find answer D.

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Circuits

1. B – The rightmost parallel section has a resistance of $R/2$. Adding the next branch to the left we find a resistance $(1/R + 2/R)^{-1} = R/3$. Finally, adding the final section, we get a total resistance of $R/4$.
2. C – After the switch is closed, the charge on the capacitor decreases as

$$Q(t) = Q_0 \exp\left(-t/(RC)\right).$$

The energy initially stored in the capacitor is

$$U = \frac{Q_0^2}{2C},$$

so we want to solve for the time such that

$$\begin{aligned} \frac{Q(t)^2}{2C} &= \frac{1}{2}\frac{Q_0^2}{2C}, \\ \exp\left(-\frac{2t}{RC}\right) &= \frac{1}{2}, \\ t &= \frac{RC\ln 2}{2}. \end{aligned}$$

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