Lecture notes on
Lie Groups, Lie Algebras and representations

November 16, 2024

Abstract

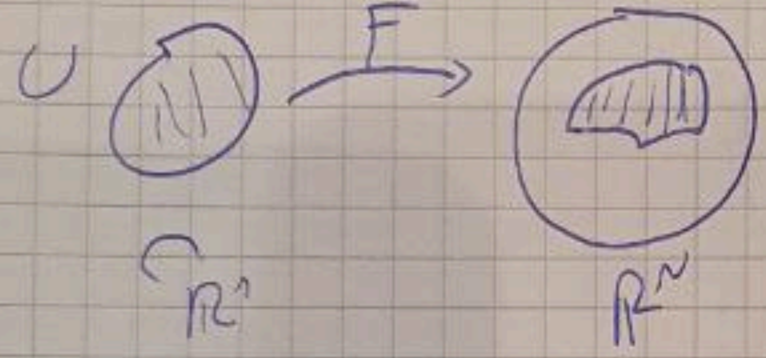
Lie Groups, $SU(3)$, enveloping algebras and so on...

Contents

Lie groups 1) Manifolds $M \subset \mathbb{R}^n$

Def: Let $U \subset \mathbb{R}^n$ be an open set. A C^∞-map $F: U \to \mathbb{R}^N$ is an immersion at $p \in U$ if $(dF)_p : \mathbb{R}^n \to \mathbb{R}^N$ is injective linear map.

A C^∞-map $F: U \to \mathbb{R}^N$ is an immersion if F is an immersion at all $p \in U$ and F is injective.



ex: $f: \mathbb{R}^n \to \mathbb{R}^l \in C^\infty$ $\rightsquigarrow$ $g: \mathbb{R}^n \to \mathbb{R}^{n+l}$ graph of f

$x \mapsto (x, f(x))$

is always an immersion.

$$dg_x = \begin{pmatrix} \overbrace{Id}^{n} \\ df_x \end{pmatrix} \begin{matrix} \}\, n \\ \}\, l \end{matrix}$$

ex/

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \mapsto (x^3+y^3, x^3-y^3)$ not an immersion at 0

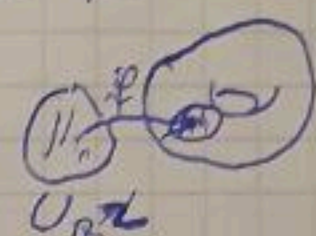
$$df_{(x,y)} = \begin{pmatrix} 3x & 3x \\ 3y & -3y \end{pmatrix}$$ injective iff two columns are indep. they are as long as $y \neq 0$

ex $(0,\pi) \times (0, 2\pi) \to \mathbb{R}^3$

$(\theta, \varphi) \mapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is an immersion def: A submanifold of $\mathbb{R}^N$ of dim n is a subset $M \subset \mathbb{R}^N$ s.t.

$$\forall m \in M \ \exists \begin{cases} V \ni m & \text{open nbhd in } M \\ U & \text{open in } \mathbb{R}^n \\ \varphi : U \to V \subset \mathbb{R}^N \end{cases}$$

s.t φ is an immersion (in $\mathbb{R}^N$)



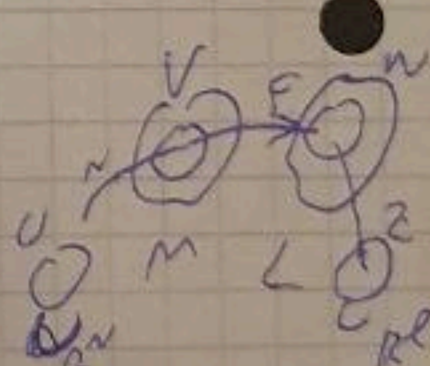
~~Def: A submanifold of $\mathbb{R}^N$ of dim n is a subset $M \subset \mathbb{R}^N$ s.t. $\forall m \in M \ \exists$~~

⇝ Get a covering by opensets, each with its system of local coordinates ("charts")

⇝ $\mathcal{C}^\infty$ funcs on M: funcs $f : M \to \mathbb{R}$ s.t. $\forall$ chart $\varphi : U \to V$ the map $U \xrightarrow{\varphi} V \xrightarrow{f} \mathbb{R}$ is $\mathcal{C}^\infty$.

Let M, L submfds of $\mathbb{R}^N$ & $\mathbb{R}^K$ resp.

$f : M \to L$ is $\mathcal{C}^\infty$ if on charts it is.



is a continuous func $f : M \to L$ s.t. $m \in M$, $\exists$ charts

$$\begin{cases} m \in V \xleftarrow{\varphi} U & , \ U \to \mathbb{R}^N \\ & V = \mathrm{Im}(\varphi) \\ f(m) \in W \xleftarrow{\psi} Z \end{cases}$$

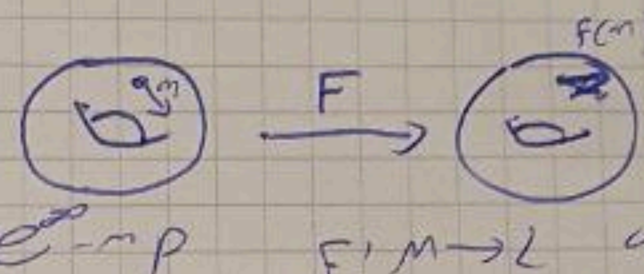
s.t. $f(V) \subset W$ and $U \to V \xrightarrow{F} W \to Z$ is $\mathcal{C}^\infty$.

A map $f: M \to L$ is a C^∞-diffeomorphism if, locally,

$$\begin{tikzcd} U \arrow[r] \arrow[rrr, bend right] & V \arrow[r, "\sim"] & W \arrow[r] & Z \end{tikzcd}$$ is a diffeomorphism

(between open subsets of $\mathbb{R}^n = \mathbb{R}^l$)

~~Tangent space~~: TM_m



Define differential of a C^∞-map $f: M \to L$ as

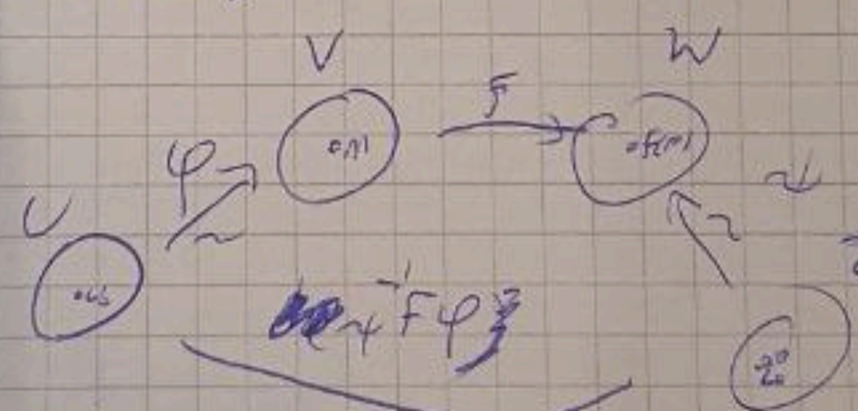
$(df)_m : TM_m \to TL_{f(m)}$ by

$TM_m \subset \mathbb{R}^M$ as $(d\varphi)_{u_0}(\mathbb{R}^n)$ where $\varphi: \underset{\mathbb{R}^n}{\overset{\cap}{U}} \to M \subset \mathbb{R}^M$ is a chart

$u_0 \mapsto m$

(one checks that this is indep of choice of charts)

Then $(d f_m) :=$



$d(\psi^{-1} F \varphi): \underset{\cong \mathbb{R}^n}{TU_{u_0}} \to \underset{\cong \mathbb{R}^l}{TZ_{\psi^{-1}(f(m))}}$

Using identifications $TM_m \simeq TU_{u_0}$, $TL_{f(m)} \simeq TZ_{\psi^{-1} f(m)}$

we get $df: TM_m \to TL_{f(m)}$.

Lie groups A Lie group is a submanifold equipped with C^∞-maps $\mu: G \times G \to G$ and $\iota: G \to G$.

satisfying usual axioms.

Def A Lie subgroup H of a Lie group G

is a <u>closed</u> subset of G which is a subgroup

($\Rightarrow$ submanifold)

not obvious.

Ex/ $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$ are Lie groups

$\Rightarrow$ any closed subgroup of $GL(n,\mathbb{K})$ is also a Lie subgroup.

Thm: The universal covering $\tilde{G}$ of a Lie group G has a (canonical) Lie group structure

Aim of representations study "reps" of Lie Groups

$$\rho : G \longrightarrow GL(V)$$

$\mathbb{K}$-vectorspace.

morphism of Lie groups.

- In particular, if G is compact its "nice"
- In general

$$\rho : G \longrightarrow GL(V)$$

$$d\rho_e : TG_e \rightarrow TGL(V)_e \simeq End(V)$$

Lie algebra

Lie algebra rep

If G is simply connected rep theory of G & of TG_e are "the same". Else, to $\tilde{G}$ & TG_e.

Lecture 3 ~~[Lec 4]~~

Recap:

Manifolds: $M \subset \mathbb{R}^N$, $\dim M = n$,

s.t. $\forall m \in M$, $\exists$

- $V \subset M$ (open) neighborhood of m
- $\varphi : U \xrightarrow{\sim} V \subset \mathbb{R}^N$

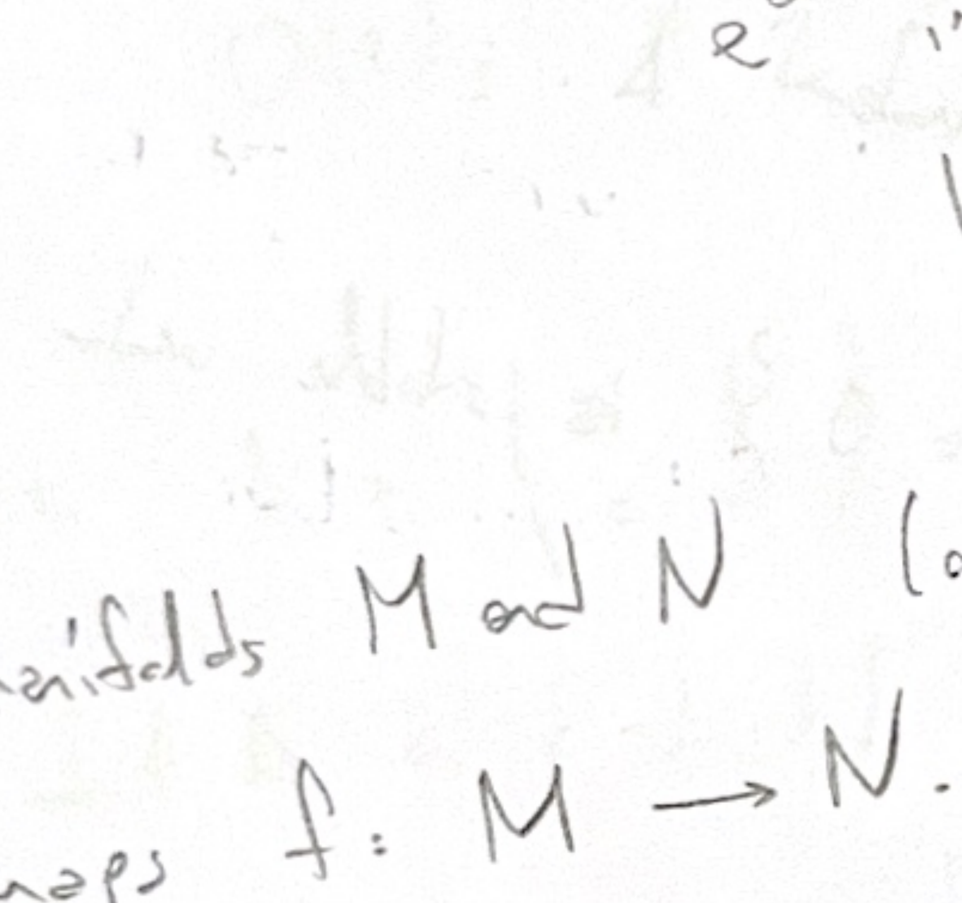
C^∞ immersion where U open in $\mathbb{R}^n$.

- Given manifolds M and N (of dim d_M, d_N), we can talk about C^∞-maps $f: M \to N$.

- Given $f: M \to N$ a C^∞-map, the differential $df_m : TM_m \to TN_{f(m)}$ is a "linear approx." of f around m.

TM_m → tangent space to M at m

ex:



$$S = \{(x,y,z) \mid x^2+y^2+z^2=1\}$$

$$TM = \{(m,\xi) \mid m \in M,\ \xi \in TM_m\}$$

$$\exists\, \pi_M : TM \to M$$
$$(m,\xi) \longmapsto m$$

then,

$$df : TM \longrightarrow TN$$

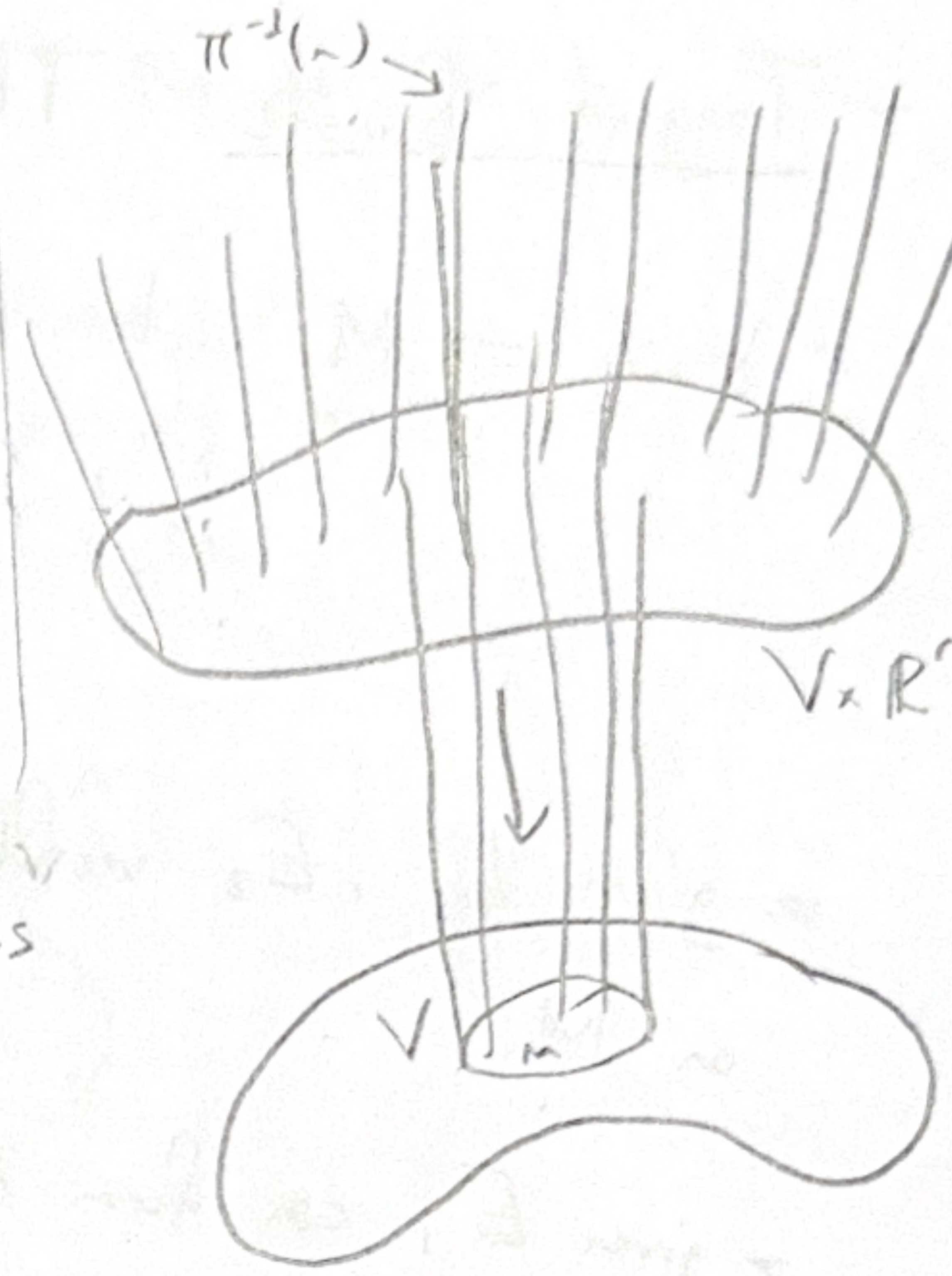
map of manifolds (linear along fibers of π_M, π_N)

$$df(m,\xi) = (f(m), df_m(\xi))$$

i.e.: $\forall$ fixed m,

$$df(m, \cdot) : TM_m \longrightarrow TN_{f(m)}$$

is linear.



Def (Vector Bundle):

A vector bundle of rank r on a manifold M is

→ a manifold E equipped with a map

$$\pi : E \longrightarrow M$$

s.t. $\forall m \in M$, $\pi^{-1}(m)$ has the structure of a vector space of dim r.

→ $\forall m \in M$, $\exists$ open neighborhood $U \ni m$ & "trivialization"

$$S_V : \pi^{-1}(V) \simeq V \times \mathbb{R}^r$$

$$\begin{tikzcd}
E \\
\pi^{-1}(V) \arrow[u, phantom, "\cup" description] \arrow[r, "\sim\, S_V"] \arrow[d, "\pi"] & V \times \mathbb{R}^r \arrow[dl, "pr_1"] \\
V \\
M \arrow[u, phantom, "\cap" description]
\end{tikzcd}$$

ex: Trivial vector bundle

$$\begin{array}{ccc} E & : & M \times \mathbb{R}^r \\ & & (m, v) \\ \downarrow & & \downarrow \\ E & & m \end{array}$$

→ Tangent bundle $TM \to M$

→ $\forall\, E \to M$ vector bundle, one can form E^*, dual vector bundle

$$\pi^{-1}_{E^*}(m) = \left(\pi^{-1}_{E}(m)\right)^*$$

→ given E_1, E_2 vector bundles on M, $E_1 \otimes E_2$ is a vector bundle on M.

→ given E, can form $\Lambda^2 E$

$$\left(\Lambda^2 F = \text{span}\{ f_1 \otimes f_2 - f_2 \otimes f_1 \mid f_1, f_2 \in F\} \subset F^{\otimes 2}\right.$$

On any M, one can form,

$$(TM)^{\otimes l} \otimes (T^*M)^{\otimes n} \otimes \Lambda^l(TM) \otimes \Lambda^l(T^*M) \otimes \dots$$

ex: TS^1;

$\exists$ a compatible identification of any $TS^1_t \simeq \mathbb{R}$ → a point on the circle

for $v_t \in TS^1_t \Rightarrow (t, u v_t) \longleftrightarrow (t, u)$

$$TS^1 \simeq S^1 \times \mathbb{R}$$

TS^2;

non-trivial !

else:

if $\exists\ TS^2 \xrightarrow{\sim} S^2 \times \mathbb{R}^2$

$\Rightarrow \exists$ a map $\quad S^2 \to TS^2$

$$(m) \longmapsto (m, (1,0))$$

Def: A vector field on M is $\mathcal{C}^\infty$-map $\Delta : M \to TM$

s.t. $\pi \circ \Delta = Id_M$ where $\pi : TM \to M$.
projective map

(i.e. we choose in a $\mathcal{C}^\infty$ fashion a tangent vector at any point of M)

In local coordinates:

$\{(u_1, \dots, u_d)\} = U \subset \mathbb{R}^d \xrightarrow{\varphi} V \subset M \subset \mathbb{R}^n$

a function on M (locally in V) is $f(u_1, \dots, u_d)$

a vector field is $\sum_{i=1}^{d} f_i(u_1, \dots, u_d) \frac{d}{du_i}$

Recall: A Lie group is a $\mathcal{C}^\infty$ manifold G equipped with

$\mu : G \times G \to G$ ($\mathcal{C}^\infty$-maps)

$i : G \to G$

satisfying usual axioms. (+ unit $e \in G$)

ex: $GL(n)$, $SL(n)$, $SO(n)$, $SO(p,q)$, $U(n)$, $Sp(2k)$, ...

Representation of G:

$\rho : G \longrightarrow GL(V)$ for V a vector space

(Lie group morphism)

Idea: any (connected) Lie group is generated by any neighborhood of e.

$\to$ any rep $\rho : G \to GL(V)$ is fully determined by its restriction to any neighborhood of e.

$\hookrightarrow$ is ρ fully determined by $d\rho : TG_e \to TGL(V)_I$?

III. Lie Algebra of a Lie Group

Let G be a Lie group.

$$\mathfrak{g} := \mathrm{Lie}(G) = TG_e$$

If G is of dim d, then $\mathfrak{g}$ is a vector space of dim d.

<u>Question:</u> What kind of structure does the product in G induce on $\mathfrak{g}$?

<u>naive:</u> $\mu : G \times G \longrightarrow G$

$(e, e) \longmapsto e$

$$d\mu_{(e,e)} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$(e + hx)\cdot(e + hy) = e + hx + \overset{\text{small}}{h}y + \dots$$

$$\Rightarrow d\mu_{(e,e)}(x, y) = \underbrace{x + y}_{\text{too trivial}}$$

Fix $g \in G$,

$\mathrm{Ad}_g : G \longrightarrow G$

$h \longmapsto g h g^{-1}$

$(e \longmapsto e)$

$$(d\,\mathrm{Ad}_g)_e : \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$g(e + hx)g^{-1} = e\, h g x g^{-1} + \dots$$

<u>ex:</u> For $G = GL(n) \underset{\text{open}}{\subset} \mathrm{End}(\mathbb{K}^n) = \mathfrak{gl}(n)$

$\Rightarrow TG_e = \mathfrak{gl}(n)$

$d(\mathrm{Ad}_g)_e : \mathfrak{gl}(n) \longrightarrow \mathfrak{gl}(n)$

$x \longmapsto g x g^{-1}$

→ viewing this as

$$\mathrm{Ad}: G \to GL(\mathfrak{g})$$
$$g \mapsto d(\mathrm{Ad}_g)_e$$

note: $(e+hy)^{-1} = e - hy + \dots$

$$d\,\mathrm{Ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

$$\mathrm{Ad}_{e+hy}(x) = (e+hy) \times (e-(hy+\dots))$$
$$= x + h(yx - xy) + O(h^2)$$

$$\mathrm{ad}: \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$
$$y \mapsto (x \mapsto yx - xy) \quad (\text{for } GL(n) \text{ still})$$

$$\Rightarrow [\,,\,]: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$(y, x) \longmapsto (d\,\mathrm{Ad})_y(x)$$

* gives the structure of a Lie algebra.

One can also define $[\,,\,]$ in one step:

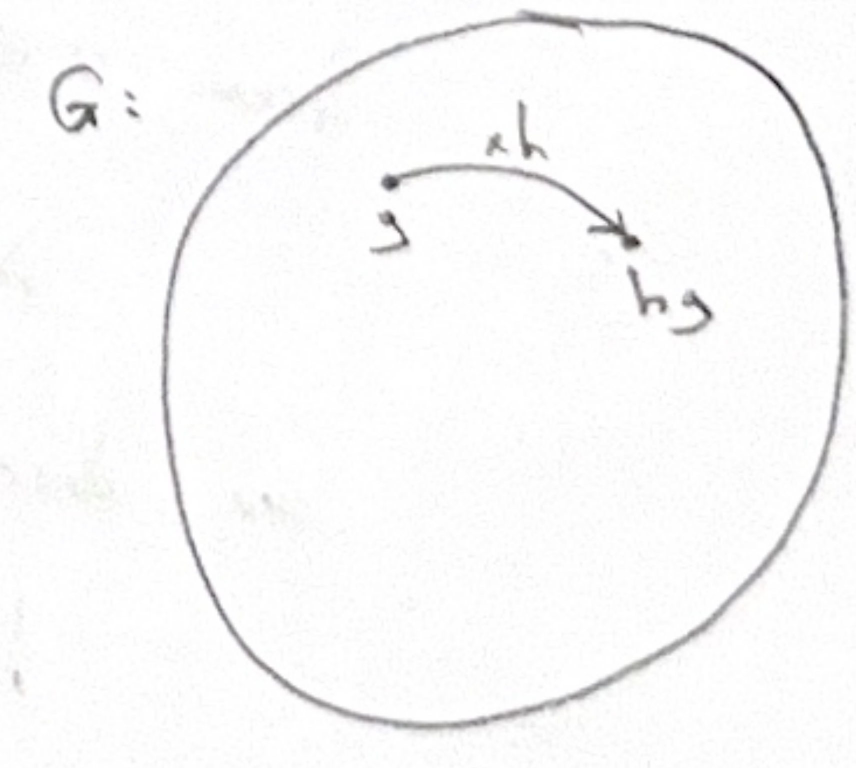
$$G \times G \longrightarrow G$$
$$(g, h) \longmapsto g h g^{-1} h^{-1}$$

$$(e+hx)(e+hy)(e-hx)(e-hy) = e + 0h + h^2(xy - yx - y^2 - x^2)$$

Alternative Construction:

Def. A (left) invariant vector field on G is a vector field $\lambda: G \to TG$ s.t.

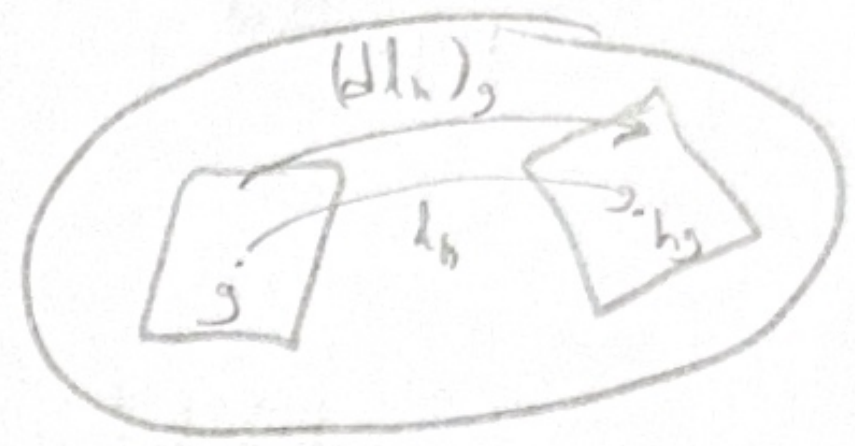
$$l_h: G \xrightarrow{\sim} G$$
$$g \mapsto hg$$



Given a vector field λ and $h \in G$, one can form

$$\lambda_h(\lambda) : G \to TG$$

$$\lambda_h(\lambda)(hg) = (d\ell_h)_g(\lambda(g))$$



A vector field is left invariant if $\forall h \in G$,

$\lambda_h(\lambda) = \lambda$

i.e. $\forall h, \forall g$,

$$(d\ell_h)_g(\lambda(g)) = \lambda(hg)$$

ex: S^1:

value of $\lambda(t)$ = rotation of value of λ at e

Prop: The map

$$\left\{\begin{matrix}\text{left-invariant}\\ \text{vector fields}\end{matrix}\right\} \xrightarrow{\sim} \mathfrak{g}$$

(isomorphism)

$$\lambda \longmapsto \lambda(e)$$

proof:

Injectivity: if

$\lambda(e) = 0$. then $\forall h$

$\lambda(he) = (d\ell_h)(\lambda(e)) = 0$

Surjective:

from $x \in \mathfrak{g} = TG_e$, define

$$\lambda(h) := (d\ell_h)(x)$$

one checks that it is C^∞.

$$\ell_{h_1} \circ \ell_{h_2}(g) = h_1(h_2 g) = \ell_{h_1 h_2}(g)$$

very general construction:

M manifold and ∂_1, ∂_2 vector fields on M.

$\to [\partial_1, \partial_2]$ Lie bracket of vector fields.

locally:
$$\partial_1 = \sum_i a_i(x_1, \dots, x_n) \frac{d}{dx_i}$$
$$\partial_2 = \sum_i b_i(x_1, \dots, x_n) \frac{d}{dx_i}$$

$$[\partial_1, \partial_2] = \sum_{i,j} \left(a_i(\underline{x}) \frac{d}{dx_i} b_j(\underline{x}) \frac{d}{dx_j} - \frac{d}{dx_j} a_i(\underline{x}) b_j(\underline{x}) \frac{d}{dx_i} \right)$$

Fact: If ∂_1 and ∂_2, vector fields on G, are left-invariant, then so is $[\partial_1, \partial_2]$.

Prop A: $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra.

i.e. i) $[\cdot,\cdot]$ is antisymmetric ($[x,y] = -[y,x]$)
($[\cdot,\cdot] : \Lambda^2 \mathfrak{g} \to \mathfrak{g}$).

ii) $[\cdot,\cdot]$ satisfies the Jacobi identity
$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

to remember it: $\partial(fg) = \partial f\, g + f\, \partial g$
$u \mapsto [x,u]$ is a derivation.

Prop B:

i) $Lie(G) = Lie(G^\circ)$
$Lie(\tilde{G}^\circ) = Lie(G^\circ)$

ii) Any Lie group morphism $f: G \to H$ induces a <u>Lie algebra morphism</u>.

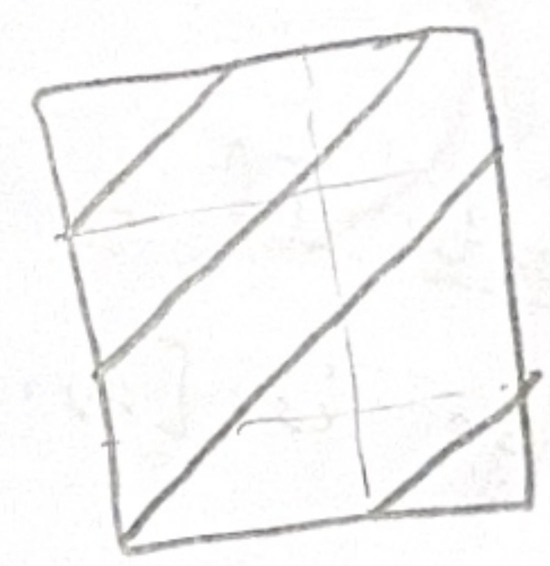
$$(df)_e : Lie(G) \longrightarrow Lie(H)$$

$$(df)_e([x,y]) = [df_e(x), df_e(y)]$$

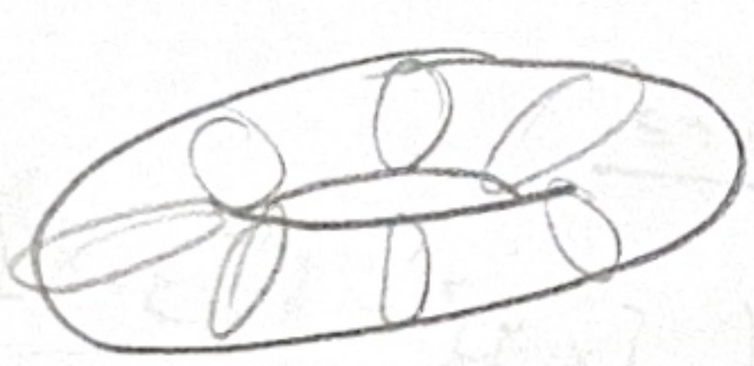
iii) For G, H connected. Let $f: G \to H$ Lie group morphism.

- If df_e is injective, then f is a local immersion.
- If df_e is surjective, then f is surjective (and a submersion).

<u>Useful Corollary</u>: If G is a Lie group, $H \subset G$ a closed subgroup, then $Lie(H) \subset Lie(G)$ is a Lie subalgebra.



<u>ex:</u>

$S^1 \times S^1$:  $\longleftrightarrow S^1$

$$Lie(S^1 \times S^1) = (\mathbb{R}^2, [\cdot,\cdot] = 0) = T_e(S^1 \times S^1) = \mathbb{R}^2$$

$$Lie(S^1) = (\mathbb{R}, [\cdot,\cdot] = 0) = T_e(S^1) = \mathbb{R} \qquad \mathbb{R} \subset \mathbb{R}^2$$

<u>Examples of Lie Algebras:</u>

$Lie(GL(n)) = \mathfrak{gl}(n)$, $[x,y] = xy - yx$

By Cor., the Lie bracket for any $H \subset GL(n)$ is again $[x,y] = xy - yx$.

<u>ex:</u> $O(n) = \{ M \in GL(n) \mid M^tM = Id \}$

$$\mathfrak{o}(n) = \{ x \in \mathfrak{gl}(n) \mid x + {}^tx = 0 \}$$

$$(e + hx)(e + h^tx) = e + h(x + {}^tx) + O(h^2)$$

→ More generally,

$$O_B(n) = \{ M \in GL(n) \mid M^t B M = B \}$$

$$\mathfrak{o}_B(n) = \{ x \in \mathfrak{gl}(n) \mid xB + B^t x = 0 \}$$

IV. Lie Correspondence

For G, H Lie groups,

$$\begin{array}{c} \mathrm{Hom}_{\text{Lie groups}}(G, H) \\ \downarrow \\ \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathfrak{h}) \end{array}$$

where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$

Also: given G,

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{closed subgroups} \\ \text{of } G \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{subalgebras} \\ \text{of } \mathfrak{g} \end{array} \right\} \\ H & \longrightarrow & \mathrm{Lie}(H) \end{array}$$

Def: A representation of a Lie algebra $\mathfrak{g}$ is

$$\rho : \mathfrak{g} \longrightarrow (\mathfrak{gl}(V),\ [x, y] = xy - yx)$$

morphism of Lie algebras.

Thus,

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{representations} \\ \text{of } G \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{representations} \\ \text{of } \mathfrak{g} \end{array} \right\} \\ \rho & \longrightarrow & (d\rho)_e \end{array}$$

Theorem (Lie):

1) Any finite-dim Lie algebra is the Lie algebra of some Lie group.

2) If G_1, G_2 are connected Lie groups, and G_1 is simply connected, then

$$\mathrm{Hom}_{\text{Lie groups}}(G_1, G_2) \xrightarrow{\sim} \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}_1, \mathfrak{g}_2)$$

Corollary: For G_1 simply connected, the representation theory of G_1 is the same as the representation theory of $\mathrm{Lie}(G_1) = \mathfrak{g}_1$.

What to do in order to study $\mathrm{Rep}(G)$ for G a Lie group?

this class → study rep of $\tilde{G}^\circ$ via equivalence with $\mathrm{Rep}(\mathfrak{g})$.

→ determine which reps of $\tilde{G}^\circ$ factor to $\tilde{G}^\circ/K = G^\circ$:

$$\begin{tikzcd} \tilde{G}^\circ \arrow[r, "\rho"] \arrow[d] & GL(V) \\ G^\circ \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}$$

i.e.: find the $\rho: \tilde{G}^\circ \to GL(V)$ s.t. $\rho|_K = \{Id\}$

→ try to extend $\rho: G^\circ \to GL(V)$ to G.

$$1 \to G^\circ \to G \to G/G^\circ \to 1$$

IV.4. 1-param subgroups & exponential map:

one example: 1-dim Lie algebras

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

$\mathfrak{g} = \mathbb{K}x$

$[x, x] = -[x, x] = 0$

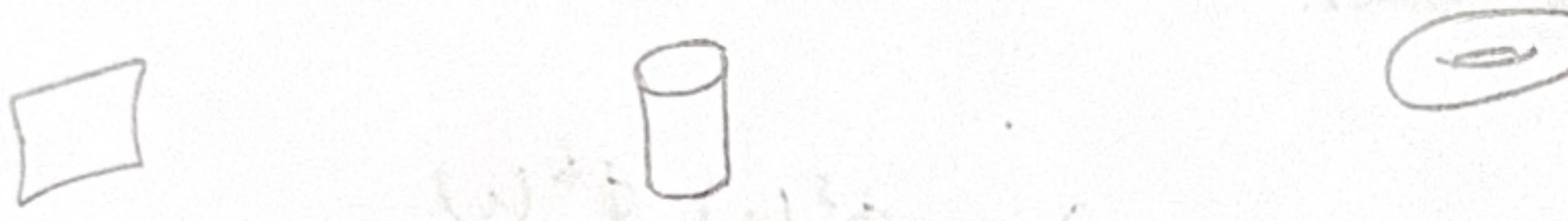
$\mathbb{K} = \mathbb{R}$:

$\mathrm{Lie}(S^1) = \mathbb{R}$, $(\mathbb{R}, +) = (\widetilde{S^1})$, $\mathbb{R} \twoheadrightarrow S^1$ surjective

$\mathrm{Lie}(\mathbb{R}, +) = \mathbb{R}$, $\mathrm{Lie}(S^1) = \mathbb{R} \xrightarrow{\exp} S^1$

$\mathbb{K} = \mathbb{C}$:

$(\mathbb{C}, +) \longrightarrow \mathbb{C}/2i\pi\mathbb{Z} \twoheadrightarrow \mathbb{C}/(\mathbb{Z} \oplus 2i\pi\mathbb{Z})$



$(\mathbb{C}, +) \xrightarrow{\exp} \mathbb{C}^*$

Let G be any Lie group, $x \in \mathfrak{g} = \mathrm{Lie}(G)$.

- get a map $\tau_x : \mathbb{K} \longrightarrow \mathrm{Lie}(G)$

$1 \longmapsto x$

what is the lift of τ_x to a map $\underbrace{(\mathbb{K}, +) \longrightarrow G}_{1\text{-param subgroup}}$?

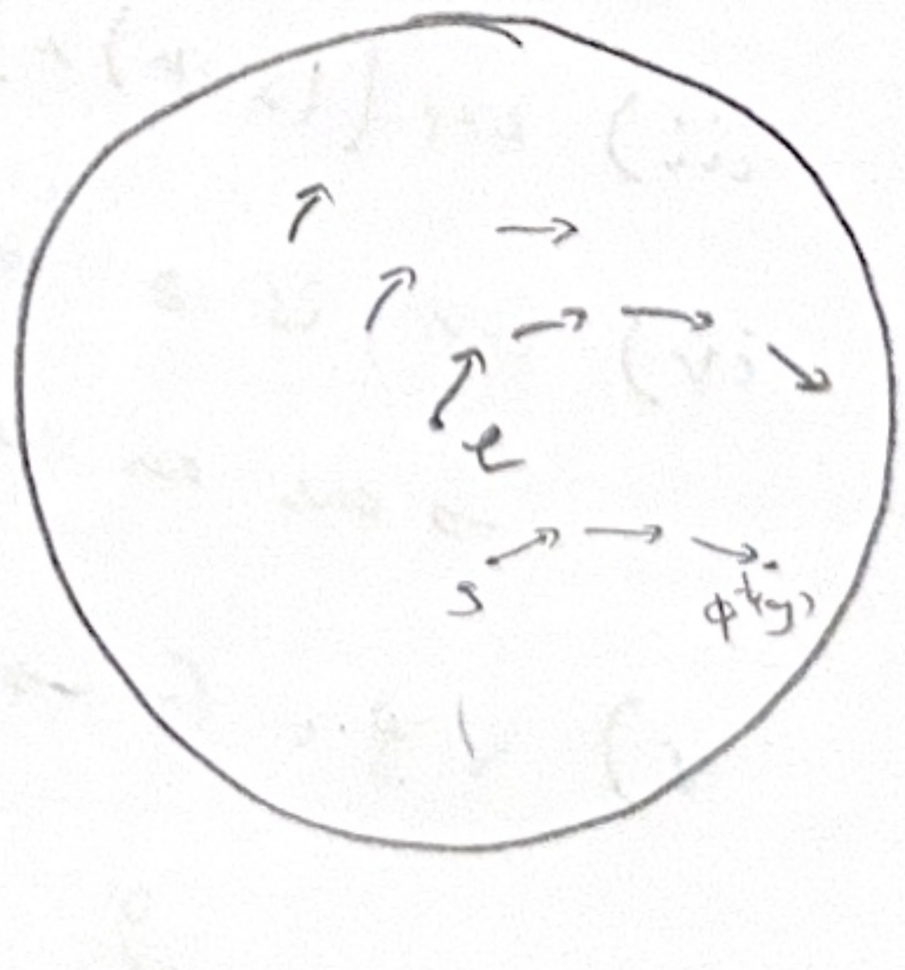
Answer: exponential map.

Prop: $\forall G$, $\forall x \in \mathfrak{g}$, $\exists!$ morphism of Lie groups

$\gamma : (\mathbb{K}, +) \longrightarrow G$

s.t. $(d\gamma)_e(1) = x$

sketch of proof: let θ_x be left-invariant vector field s.t. $\theta_x(e) = x$.



θ_x = left-invariant vector field

By Cauchy-Lipschitz, $\forall g \in G$, $\forall$ small enough ε $\exists$ a (local) flow

$\phi : [0, \varepsilon] \times \mathcal{U} \to G$ ($\mathcal{U}$ a neighborhood of g)

$$\phi(t, g) = \phi^t(g)$$

Claim: $\phi^t(g_1 g) = g_1 \phi^t(g)$ because θ_x is left-invariant.

We deduce that ε can be chosen independently of $g \in G$

$\Rightarrow$ flow is defined for all times.

set $\gamma(t) = \phi^t(e)$

we have $\gamma(s+t) = \phi^t(\phi^s(e) \cdot e) = \phi^s(e)\,\phi^t(e)$

$\Rightarrow \gamma$ morphism of groups.

Define: for any Lie algebra G, the exponential map.

$$\exp : \mathfrak{g} \longrightarrow G$$

$$\exp(x) = \gamma_x(1)$$

where γ is defined previously.

Prop (Standard prop of exp):

i) exp is a $\mathcal{C}^\infty$ map.

ii) $(d\exp)_0 = \mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$

iii) $\exp((\lambda + \mu)x) = \exp(\lambda x) + \exp(\mu x)$

iv) exp is a local diffeo at 0.

$\to$ one can use $\mathfrak{g}$ as a local chart at e.

v) $\forall \varphi : G \to G'$ morphism of Lie group.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\varphi"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\supseteq"] & \mathfrak{g}' \arrow[d, "\exp"] \\ G \arrow[r, "\varphi"] & G' \end{tikzcd}$$

Lecture 3 (look Ömer)

Useful corollary: If G is a Lie group, $H \subset G$ closed subgroup then $\mathrm{Lie}(H) \subset \mathrm{Lie}(G)$ is a subalgebra of Lie.

$S^1 \times S^1$  $\hookleftarrow S^1$ $\quad \mathrm{Lie}(S^1 \times S^1) = T_e(S^1 \times S^1) = \mathbb{R}^2$

$= (\mathbb{R}^2, [\,,\,] = 0)$

$\mathrm{Lie}(S^1) = (\mathbb{R}, [\cdot,\cdot] = 0) = T_e(S^1) = \mathbb{R}.$

Examples of Lie algebras:

$\mathrm{Lie}(GL(n)) = \mathfrak{gl}(n), \quad [X,Y] = XY - YX$

By Corollary, the Lie bracket for any $H \overset{\text{closed}}{\subset} GL(n)$ the bracket is again $[X,Y] = XY - YX$

Ex/ $O(n) = \{ M \in GL(n) \mid M^t M = \mathbb{1} \}$

$\mathfrak{o}(n) = \{ X \in \mathfrak{gl}(n) \mid X + X^t = 0 \}$

$M = (e + hX) \cdots$

More generally, $O_B(n) = \{ M \in GL(n) \mid M B M^t = B \}$

$\mathfrak{o}_B(n) = \{ X \in \mathfrak{gl}(n) \mid XB + BX^t = 0 \}$

II (n+2) Lie correspondence: For G, H Lie groups

$$\mathrm{Hom}_{\text{Lie grps}}(G, H) \longrightarrow \mathrm{Hom}_{\text{Lie alg}}(\underset{\mathrm{Lie}(G)}{\mathfrak{g}}, \underset{\mathrm{L}(H)}{\mathfrak{h}})$$

Also: Given G, $\left\{ \begin{matrix} \text{Closed subgroups} \\ \text{of } G \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Subalgebras} \\ \text{of } \mathfrak{g} \end{matrix} \right\}$

Def: A representation of a Lie algebra $\mathfrak{g}$ is

$$\rho : \mathfrak{g} \longrightarrow (\mathfrak{gl}(V), [X,Y] = XY - YX)$$

morphism of Lie algebras.

Thus: $\{\text{rep}^{\text{ns}} \text{ of } G\} \longrightarrow \{\text{rep}^{\text{ns}} \text{ of } \mathfrak{g}\}$, $\rho \longmapsto d\rho_e$

<u>Thm (Lie)</u>: 1) Any finite-dim Lie algebra is the Lie algebra of some Lie group.

2) If G_1, G_2 connected Lie groups and G_1 simply connected, then

$$\mathrm{Hom}_{\text{Lie gps}}(G_1, G_2) \xrightarrow{\sim} \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}_1, \mathfrak{g}_2)$$

in general this map is not surjective only when $G_1 = \tilde{G}$ universal cover

<u>Cor</u>: For G_1 simply connected, rep. theory of G_1 is the same as rep theory of $\mathrm{Lie}(G_1) = \mathfrak{g}_1$.

What to do to study $\mathrm{Rep}(G)$ for G a Lie group?

* Study rep of $\tilde{G}^0$ via equivalence with $\mathrm{Rep}(\mathfrak{g})$

universal cover — connected component

* Determine ~~them~~ which reps of $\tilde{G}^0$ factor to $\tilde{G}^0/K = G^0$

$$\begin{tikzcd} \tilde{G}^0 \arrow[r, "\rho"] \arrow[d] & GL(V) \\ G^0 \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

i.e. find the $\rho: \tilde{G}^0 \to GL(V)$ s.t. $\rho|_K = \{Id\}$

* Try to extend $\rho: G^0 \to GL(V)$ to G.

$$1 \to G^0 \to G \to G/G^0 \to 1$$

II.4 1-parameter subgroups & exponential map.

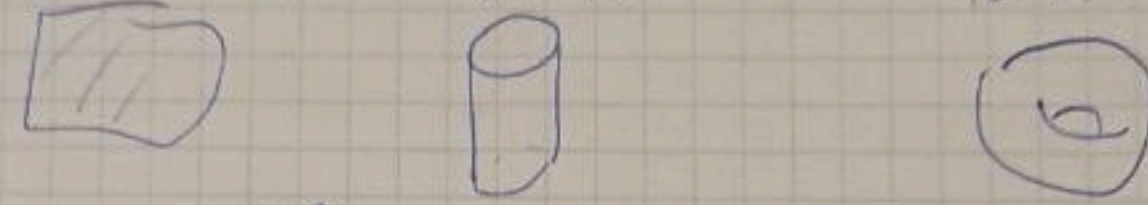
<u>One examples</u>: (1-dim Lie algebra) $K = \mathbb{R}$ or $\mathbb{C}$, $\mathfrak{g} = Kx$, $[x,x] = 0$.

~~Case~~ $\boxed{K = \mathbb{R}}$: $\mathrm{Lie}(S^1) = \mathbb{R}$, $\mathrm{Lie}(\mathbb{R}, +) = \mathbb{R}$.

$(\mathbb{R}, +) \cong \widetilde{(S^1)}$ $\quad \mathbb{R} \xrightarrow{\text{surjective}} S^1$ but we also have

$\mathrm{Lie}(S^1) = \mathbb{R} \xrightarrow{\exp} S^1$

$\boxed{\mathbb{K}=\mathbb{C}}$. $(\mathbb{C},+) \twoheadrightarrow \mathbb{C}/_{2i\pi\mathbb{Z}} \longrightarrow \mathbb{C}/_{(\mathbb{Z}\oplus 2i\pi\mathbb{Z})}$



$(\mathbb{C},+) \xrightarrow{\exp} \mathbb{C}^*$

Let G be any Lie group $v \in \mathfrak{g} = Lie(G)$.

define $\tau_v : \mathbb{K} \longrightarrow Lie(G)$, $1 \longmapsto v$

what is the lift of this map, $\underbrace{(\mathbb{K},+) \to G}_{\text{1-parameter subgroup}}$?

I can use Lie's theorem

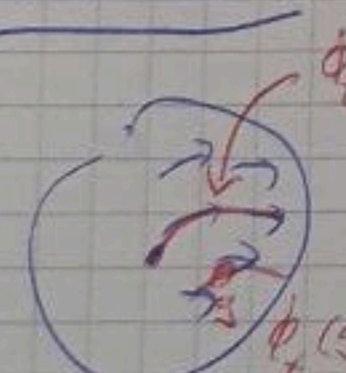
Answer: Exponential map.

Prop: $\forall G$, $\forall v \in \mathfrak{g}$, $\exists!$ morphism of Lie groups

$$\gamma : (\mathbb{K},+) \longrightarrow G, \text{ st } (d\gamma)_e(1) = v.$$

we define $\exp : \mathfrak{g} \longrightarrow G$, $\boxed{\exp(v) = \gamma(v)}$

Proof (sketch)



Θ_v = left-inv vector field s.t. $\Theta_v(e) = v$.

By Cauchy-Lipschitz, $\forall g \in G$, $\forall$ small enough t $\exists$ a local flow $\phi : [0,\varepsilon] \times U \to G$ (U nbhd of g) denoted by $\phi_t(g)$.

Claim: $\phi_t(g_1 g) = g_1 \phi_t(g)$ because Θ_v is left inv. we deduce that ε can be chosen indep. of $g \in G$ $\Rightarrow$ flow defined for all times.

Set $\gamma(t) = \phi_t(e)$ we have $\gamma(s+t) = \phi^t(\phi^s(e) \cdot e)$

$= \phi^s(e)\phi^t(e)$

$\to \gamma$ morphism of groups.

Prop: (Standard prop of exp)

1) exp is a C^∞ map.

2) $(d\exp)_0 = \mathrm{Id} : \mathfrak{g} \longrightarrow \mathfrak{g}$

3) $\exp((\lambda+\mu)X) = \exp(\lambda X)\exp(\mu X)$

4) exp is local diffeo at 0. $\rightsquigarrow$ we can use $\mathfrak{g}$ as local chart etc.

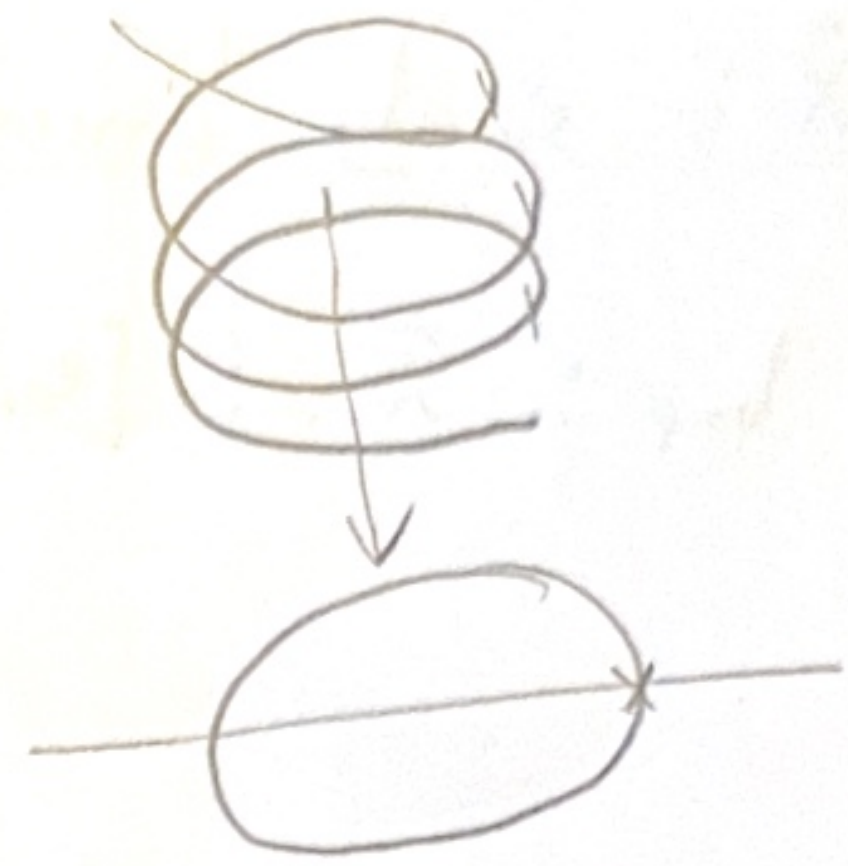
5) $\forall \varphi : G \to G'$ morphism of Lie groups

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\varphi"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{g}' \arrow[d, "\exp"] \\ G \arrow[r, "\varphi"] & G' \end{tikzcd}$$

$\to p : (\mathbb{R}, +) \longrightarrow U(1)$

$x \longmapsto e^{2\pi i x}$

$\to \pi_1(U(1)) = p^{-1}(\{1\}) = \mathbb{Z}$



Remark on the last lecture,

2 Oct

Lecture 4

$$K \to G$$

Lie algebra level $\quad \mathbb{K} \longrightarrow Lie(G)$

$$1 \longmapsto u$$

Lie group level $\quad \gamma : (\mathbb{K}, +) \longrightarrow G$

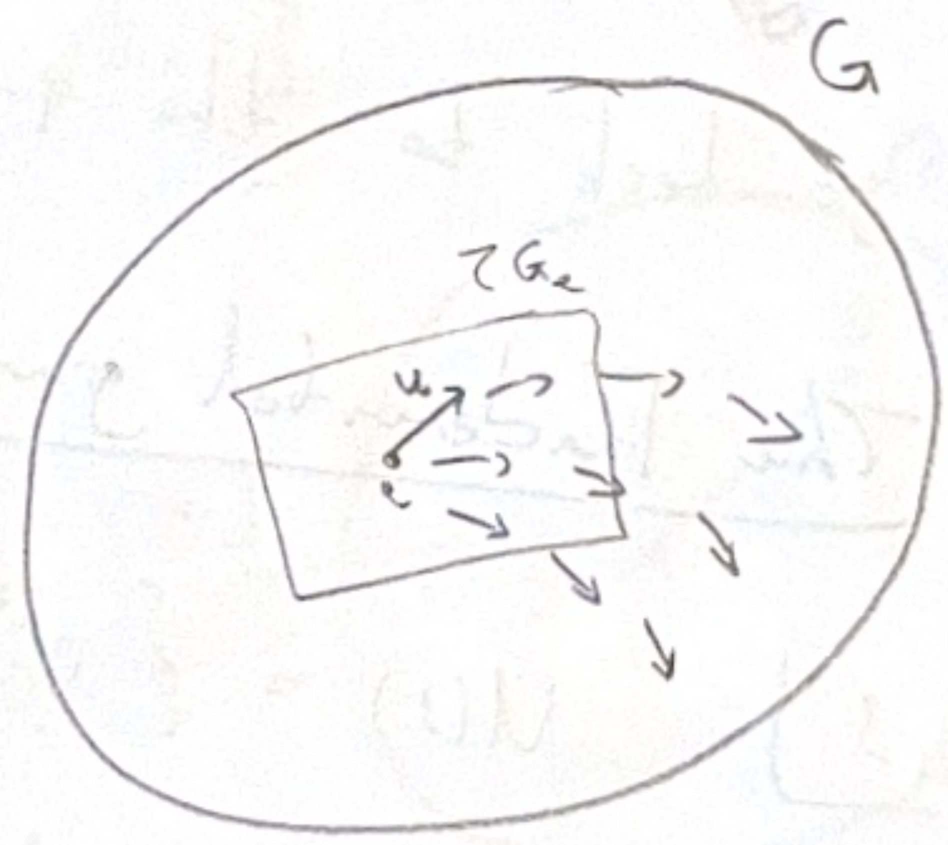
s.t. $d\gamma_0(1) = u$

Construction:

$u \in TG_e$

$\exists !$ left-inv. vector field θ_u

on G s.t. $\theta_u(e) = u$.



By general principles (Cauchy-Lipschitz)

$\exists$ (locally) integral curves for θ_u

i.e. $\gamma : [0, \varepsilon] \longrightarrow G$

s.t. $\gamma'(t) = \theta_u(\gamma(t))$

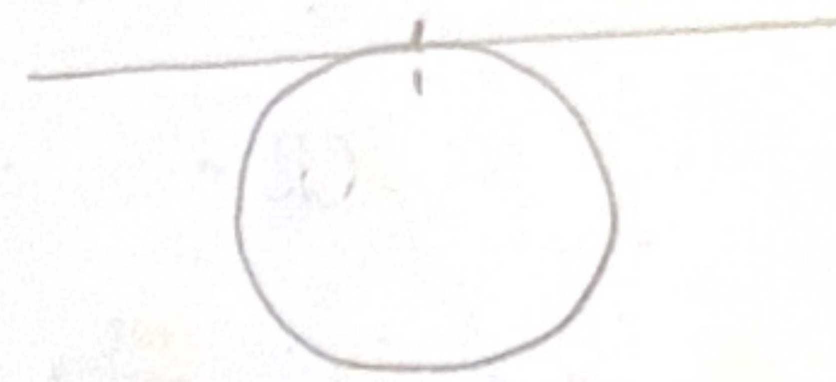
using group structure:

i) γ can be extended at all times.

ii) $\gamma(t_1 + t_2) = \gamma(t_1)\gamma(t_2)$

($\rightsquigarrow$ group morphism)

Recall: $\begin{cases} G & \text{Lie group} \\ \mathfrak{g} := & \text{Lie}(G) \end{cases}$



$$\underset{\text{Lie}(S^1)}{\overset{}{\exp:}}\ \ \begin{array}{c}\text{Lie}(S^1)\\ \parallel \\ \mathbb{R}\end{array} \to S^1$$

$$\theta \mapsto e^{i\theta}$$

Prop:

i) $\exp : \mathfrak{g} \to G$ is smooth map.

ii) $d\exp_0 = Id : \mathfrak{g} \to \mathfrak{g}$

iii) $\exp((\lambda+\mu)x) = \exp(\lambda x)\exp(\mu x)$

iv) Any $\rho : G \to K$ yields,

$$\begin{tikzcd} G \arrow[r, "\rho"] & K \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\rho"] & \mathfrak{k} = \text{Lie}(K) \arrow[u, "\exp"'] \end{tikzcd}$$

Rem: For $x, y \in \mathfrak{g}$ close to 0.

$$\exp^{-1}(\exp(hx)\exp(hy))$$

$$= h(x+y) + h^2 \tfrac{1}{2}[x,y] + O(h^3)$$

Recall: $\forall u \in \mathfrak{g}$, $\exp|\mathbb{K}u \subset G$ 1-param subgroup.

$\underline{\mathbb{K} = \mathbb{R}}$: $\exists$ 2 possibilities:

$\exp(tu) : (\mathbb{R}, +) \longrightarrow G$

1) is injective.

$\rightsquigarrow$ $\exp(u)$ does not belong to a compact 1-par subgroup.

2) factors $(\mathbb{R}, +) \longrightarrow S^1 \longrightarrow G$

$\rightsquigarrow$ $\exp(u)$ belongs to a compact subgroup of G.

$\mathbb{K} = \mathbb{C}$:

1) $\mathbb{C} \longrightarrow G$ is injective.

-OR-

2) $\mathbb{C} \xrightarrow{\exp} \underset{\text{"}\mathbb{R} \times S^1\text{"}}{\underset{\text{s.t. top.}}{\mathbb{C}^*}} \longrightarrow G$

Compact Lie Groups:

Simplest Examples: G finite group.

A rep. of G is a pair (ρ, V),

$$\rho : G \longrightarrow GL(V)$$

Def: (ρ, V) is a simple representation if $\nexists$ proper subrep $W \subset V$.
(same as irrep)

$\rho : G \to GL(V)$ group morphism.

$$g \longmapsto \begin{pmatrix} a_{11}(g) & & \\ & \ddots & \\ & & a_{nn}(g) \end{pmatrix}$$

Subrep: $W \subset V$ s.t.

$$\forall g \in G, \quad \rho(g)(W) \subseteq W$$

in matrix form: in a good basis

$B = B_W \sqcup B_U$ s.t. $B = W \oplus U$

$$\forall g: \quad \rho(g) \in \begin{array}{c c} & \begin{array}{cc} B_W & B_U \end{array} \\ \begin{array}{c} B_W \\ B_U \end{array} & \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array}$$

If W is a subrep of V, then

V/W is also a G-rep (quotient group)

"V is cooked up from W & V/W"

"V an extension of V by W"

V is <u>simple (irreducible) rep</u> if the only subreps are $\{0\}$ and V.

V is <u>semisimple</u> if V is a <u>direct sum</u> of simple reps.

In matrix form:

$$V \simeq S_1 \oplus S_2 \oplus \cdots \oplus S_\ell$$

$$\forall g \in G, \quad \rho(g) \in \begin{array}{c} \begin{matrix} B_{S_1} & B_{S_2} & \cdots & B_{S_\ell} \end{matrix} \\ \begin{pmatrix} \boxed{\times} & & & 0 \\ & \boxed{\times} & & \\ & & \ddots & \\ 0 & & & \boxed{\times} \end{pmatrix} \end{array}$$

<u>Counter example</u>:

$$G = (\mathbb{C}, +), \quad \rho(t) = \begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & 1 & t \\ e_2 & 0 & 1 \end{array} \in GL(\mathbb{C}^2).$$

$\mathbb{C}e_1$ is a subrep.

$\mathbb{C}^2/\mathbb{C}e_1$ is a quotient rep.

$\mathbb{C}^2$ is not $\simeq \mathbb{C}e_1 \oplus \mathbb{C}^2/\mathbb{C}e_2$

Theorem:

simple is also semisimple

1) Any finite-dim rep of G is semisimple.

i.e. $V \simeq S_1^{\oplus m_1} \oplus \cdots \oplus S_\ell^{\oplus m_\ell}$

where $\begin{cases} S_1, \ldots, S_\ell & \text{are simple reps} \\ m_1, \ldots, m_\ell & \in \mathbb{N}_{\geq 0} \end{cases}$

2) The set of irreducible reps is in canonical bijection with the set of conjugacy classes in G.

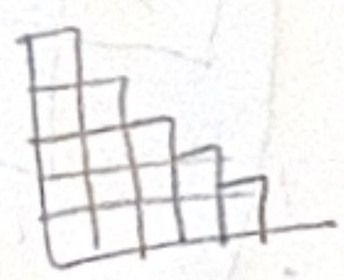
<u>ex:</u> - $\mathbb{Z}/_n\mathbb{Z}$ conj. classes are $\{g\}$ $g \in \mathbb{Z}/_n\mathbb{Z}$

<u>Irred. reps:</u>

$$\rho : \mathbb{Z}/_n\mathbb{Z} \longrightarrow GL(1)$$
$$u \longmapsto e^{\frac{2i\pi}{n} l u}$$

for $l \in \mathbb{Z}/_n\mathbb{Z}$.

- $G = \mathfrak{S}_n$ conj. classes are in bij. with partitions of n.

$\lambda =$ 

→ for finite groups, all the representation theory boils down to understanding <u>simple reps.</u>

<u>Sketch of proof of 1):</u>

Let V be a $\mathbb{C}$-rep of G. Let $(\cdot,\cdot)$ be any Hermitian scalar product.

Define a new Hermitian product,

$$(x,y)' = \frac{1}{|G|} \sum_{\gamma \in G} (\gamma x, \gamma y)$$

Clearly $(\cdot,\cdot)'$ is Hermitian and G-invariant

$$(gx, gy)' = \frac{1}{|G|} \sum_{\gamma \in G} (\gamma g x, \gamma g y) = \frac{1}{|G|} \sum_{\gamma \in G} (\gamma x, \gamma y)$$
$$= (x,y)'$$

Let $S_1 \subset V$ be a smallest dimensional proper subrep (a simple G rep).

We have
$$V \simeq S_1 \oplus S_1^{\perp}$$

and $S_1^{\perp}$ is also a subrep;

If $(S_1, y) = 0$, then
$$(\underbrace{g S_1}_{S_1}, gy) = 0 \quad \Rightarrow \quad gy \in S_1^{\perp}$$

We iterate the process,
$$\rightsquigarrow V = S_1 \oplus S_2 \oplus \dots \oplus S_l$$
$\square$

with S_i simple $\forall i$.

Let G be a compact Lie group.

Def: A (left) Haar measure is a (Borel) measure $d\mu$ on G s.t.
$$d\mu(M) = d\mu(gM) \quad \forall M \subset G$$

In part.,
$\forall f \in L^1(G, \mathbb{C}, d\mu)$
$$\int_G f \, d\mu = \int_G l_g^* f \, d\mu$$

Thm: Let G be compact. $\exists$ a unique left Haar measure on G s.t.
$$\int_G 1 \, d\mu = 1$$

Thm: Let G be a compact Lie group. Then any finite-dim $\mathbb{K}$-rep is semisimple.

proof: Let V be a $\mathbb{K}$-rep of G. Let $(,)$ be a (Hermitian / positive) scalar product.

$$(x,y)' = \int_G (\gamma x, \gamma y)\, d\mu$$

$(,)'$ is a G-invariant scalar product:

$$(gx, gy)' = \int_G (\gamma g x, \gamma g y)\, d\mu$$

$$= \int_G (\gamma x, \gamma y)\, d\mu$$

$\uparrow$ invariance property of $d\mu$

$\square$

$\rightsquigarrow$ the argument is the same as for the finite group case.

Cor: Any compact Lie group $G \subset GL(V)$ is conjugate to a subgroup of $\underbrace{U(V)}_{\mathbb{K}=\mathbb{C}}$ (or $\underbrace{O(V)}_{\mathbb{K}=\mathbb{R}}$)

$$\exists g \in GL(V) \;\Big|\; G \subseteq O\big(\underbrace{g\cdot(,)_{st}}_{(gx,gy)_{st}}\big) \Rightarrow g G g^{-1} \subset O((,)_{st})$$

$$O((,)) = \{ g \in GL(V) \mid (gx, gy) = (x,y) \quad \forall x,y \in V \}$$

Cor: Let V be a finite-dim. $\mathbb{C}$-rep of a compact Lie group G. Then for any $g \in G$, $\rho(g)$ is diagonalizable.

Cor: Let G be a Lie group and let $g \in G$ belong to a compact 1-par subgroup: $\left(\exists \; \begin{array}{c} S^1 \to G \\ t_0 \mapsto g \end{array} \right)$

Then $\forall$ finite-dim $\mathbb{C}$-rep (ρ, V) of G, $\rho(g)$ is diagonalizable.

Thm: If G is compact and connected,

$\exp: \mathfrak{g} \to G$ is surjective.

Examples for $SL(2,\mathbb{R})$, $SU(2)$, $SL(2,\mathbb{C})$:

1) $SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; \begin{array}{c} |a|^2 + |b|^2 = 1 \\ a, b \in \mathbb{C} \end{array} \right\}$

$\simeq S^3 \subset \mathbb{R}^4$

$\to$ compact, connected & simply-connected.

$\text{Lie } SU(2) = \left\{ \begin{pmatrix} ix & b \\ -\bar{b} & -ix \end{pmatrix} \;\middle|\; \begin{array}{c} x \in \mathbb{R} \\ b \in \mathbb{C} \end{array} \right\}$

$Z(SU(2)) = \{ \pm Id \}$

$SU(2) / Z(SU(2)) = SO(3)$

Conjugacy Classes: every $g \in SU(2)$ is diagonalizable,

$g \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $\quad \theta \in [0, 2\pi[\, / \pm 1$

for $\theta \neq 0, \pi$:

$$\mathrm{Stab}_{SU(2)} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \;\middle|\; |a|^2 = 1 \right\}$$

$$SU(2) \Big/ \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \;\middle|\; |a| = 1 \right\} \simeq S^2$$

2) $\underline{SL(2,\mathbb{R})}$: 3-dim Lie group,
$\underline{\text{not}}$ simply-connected

(act on any nonzero element of $\mathbb{R}^2$

$$SL(2,\mathbb{R}) \cdot u = (\mathbb{R}^2) / \{0\}$$

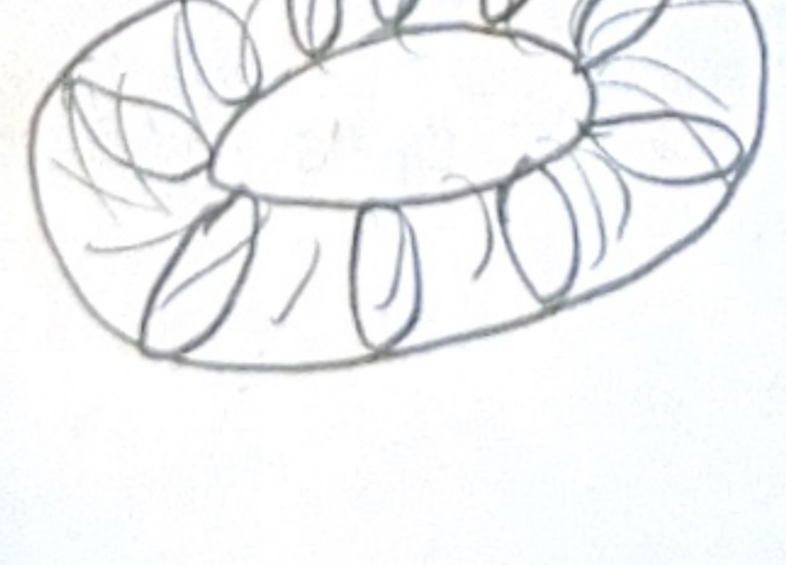
$\rightsquigarrow$ $SL(2,\mathbb{R})$ cannot be simply connected.)

$SL(2,\mathbb{R})$ is conjugate, inside $SL(2,\mathbb{C})$ with

$$\left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \;\middle|\; \begin{array}{c} |a|^2 - |b|^2 = 1 \\ a, b \in \mathbb{C} \end{array} \right\}$$

(via $g = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$)

$$\rightsquigarrow SL(2,\mathbb{R}) \sim S^1 \times D^\circ = \{ z \in \mathbb{C} \mid |z| < 1 \}$$

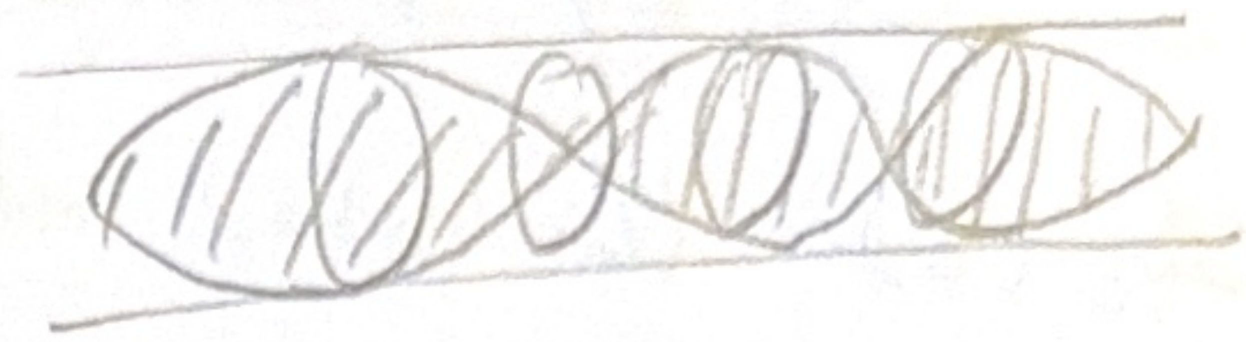


$$\underbrace{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}}_{\to\, 2\times 2} \longmapsto \left(\frac{a}{|a|}, \underbrace{\frac{b}{a}}_{\to\, \text{division}} \right)$$

$\pi_1(SL(2,\mathbb{R})) = \mathbb{Z}$

$\rightsquigarrow \widetilde{SL(2,\mathbb{R})}$ universal covering

$\sim \mathbb{R} \times D^\circ$



3 types of conjugacy classes in $SL(2,\mathbb{R})$:

1) $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $a \in \mathbb{R}^*$ "hyperbolic" non-compact

2) $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ "elliptic" compact

3) $\begin{pmatrix} \pm 1 & u \\ 0 & \pm 1 \end{pmatrix}$ $u \in \mathbb{R}$ "parabolic" non-compact

3) $\underline{SL(2,\mathbb{C})}$: 3-dim complex, connected, and simply-connected.

Conjugacy Classes:

i) diagonalizable $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $a \in \mathbb{C}^*$; if $|a| = 1 \rightsquigarrow$ compact subgroup

ii) not diagonalizable $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in \mathbb{C}$

Crucial Point:

$\operatorname{Lie}(SL(2,\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C} \simeq \operatorname{Lie}(SU(2)) \otimes_{\mathbb{R}} \mathbb{C} \simeq \operatorname{Lie}(SL(2,\mathbb{C}))$

$(\mathbb{R},+)$ $(S^1,\cdot)$

We say that $SL(2,\mathbb{R})$, $SU(2)$ are 2 real forms of $SL(2,\mathbb{C})$.

$\rightsquigarrow$ Rep theory of $SL(2,\mathbb{R})$ and $SU(2)$ are related and related to rep. theory of $SL(2,\mathbb{C})$.

V. Finite-dim reps of $SL(2,\mathbb{C})$

Because $SL(2,\mathbb{C})$ is connected and simply-connected, we can consider reps of $sl_2(\mathbb{C})$ (Lie algebra) instead.

$$sl_2(\mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We have:

$$\begin{cases} [h,e] = 2e \\ [h,f] = -2f \\ [e,f] = h \end{cases}$$ fundamental relations

h acts <u>semisimply</u> on $sl_2(\mathbb{C})$ (by $x \mapsto [h,x]$)
and $\mathbb{C}e, \mathbb{C}f, \mathbb{C}h$ are the eigenspaces.

On the other hand, e acts <u>nilpotently</u> on $sl_2(\mathbb{C})$

$$[e,f] = h, \quad [e,h] = -2e, \quad [e,e] = 0$$

Same for f.

In basis $\{e,h,f\}$, we have:

$$ad(e) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad ad(f) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

$$ad(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$
\begin{tikzcd}
0 & \mathbb{C}f \arrow[l, bend left, "\mathrm{ad}\, f"] \arrow[r, bend left, "\mathrm{ad}\, e"] & \mathbb{C}h \arrow[l, bend left, "\mathrm{ad}\, f"] \arrow[r, bend left, "\mathrm{ad}\, e"] & \mathbb{C}e \arrow[l, bend left, "\mathrm{ad}\, f"] \arrow[r, bend left, "\mathrm{ad}\, e"] & 0
\end{tikzcd}
$$

h eigenvalues: $\quad -2 \quad 0 \quad 2$

Rem: h is special because e^{ih} belongs to a compact subgroup of $SL(2,\mathbb{C})$.

$\Rightarrow \rho(h)$ will be diagonalizable in ANY finite-dim rep. (ρ, V).

h act semisimply on any $\mathfrak{sl}_2(\mathbb{C})$-rep.

Let (ρ, V) be a finite-dim $\mathfrak{sl}_2(\mathbb{C})$-rep.

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda \quad \text{where} \quad U_\lambda = \{ x \in V \mid \rho(h) x = \lambda x \}$$

We know that $\rho(e), \rho(h), \rho(f)$ satisfy the following commutation rules in $\mathrm{End}(V)$

$$\rho(h)\rho(e) - \rho(e)\rho(h) = 2\rho(e)$$
$$\rho(h)\rho(f) - \rho(f)\rho(h) = -2\rho(f)$$
$$\rho(e)\rho(f) - \rho(f)\rho(e) = 2\rho(h)$$

Simplify notation: drop "ρ".

Lemma:
$$e(V_\lambda) \subseteq V_{\lambda+2}$$
$$f(V_\lambda) \subseteq V_{\lambda-2}$$

$\to$ make sure to understand the next proof!

proof: Let $x_\lambda \in V_\lambda$.

$$h e(x_\lambda) = e \underbrace{h(x_\lambda)}_{\lambda x_\lambda} + 2e(x_\lambda) = (\lambda+2) e(x_\lambda) \qquad \square$$

Same for f.

$$\begin{tikzcd} V_{\lambda-2} \arrow[r, bend left, "e"] & V_{\lambda} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V_{\lambda+2} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V_{\lambda+4} \arrow[l, bend left, "f"] \end{tikzcd}$$

$\lambda-2 \qquad \lambda \qquad \lambda+2 \qquad \lambda+4$

<u>Cor:</u> If V is irreducible finite-dim, then

$\exists\ \lambda_0 \in \mathbb{C}$ and $l \in \mathbb{N}$ s.t.

$V_\lambda \neq \{0\} \iff \lambda \in \{\lambda_0, \lambda_0+2, \dots, \lambda_0+2l\}$

Let V be irreducible

$\rightsquigarrow \exists\ \lambda_{max} \in \{\lambda \text{ st } V_\lambda \neq \{0\}\}$

$\exists\ \lambda_{min} \in \quad$ " $\quad$ "

s.t. $\{\lambda \in V_\lambda \neq 0\} = \{\lambda_{min}, \lambda_{min}+2, \dots, \lambda_{max}\}$

$\Rightarrow \quad e(V_{\lambda_{max}}) = f(V_{\lambda_{min}}) = \{0\}$

$V_{\lambda_{min}} \quad \circ \qquad \circ \qquad \circ \quad \dots \quad \circ \qquad V_{\lambda_{max}}$

$x_{\lambda_{max}}$

$f^l(x_{\lambda_{max}}) \in V_{\lambda_{max}-2l}$

$$\begin{aligned} e f^l(x_{\lambda_{max}}) &= (ef) f^{l-1}(x_{\lambda_{max}}) = (h + fe) f^{l-1}(x_{\lambda_{max}}) \\ &= \left(\left(\lambda_{max} - 2(l-1)\right) + fe\right) f^{l-1}(x_{\lambda_{max}}) \\ &= \left(\left(\lambda_{max} - 2(l-1) - 2(l-2) \dots \right) f^{l-1}(x_{\lambda_{max}}) \\ &= \left(l\lambda_{max} - 2(l-1) - 2(l-2) \dots \right) f^{l-1}(x_{\lambda_{max}}) \\ &= \left(l\lambda_{max} - l(l-1)\right) f^{l-1}(x_{\lambda_{max}}) \end{aligned}$$

iterate e to kill $x_{\lambda_{max}}$

2 conclusions:

1) $\hat{\bigoplus}_{l=0} \mathbb{C} f^l v_{\lambda max}$ is a subrepresentation

$\Rightarrow$ as V is simple,

$$V = \hat{\bigoplus}_{l=0} \mathbb{C} f^l v_{\lambda max}$$

Better normalization:

$$\frac{f^l}{l!} v_{\lambda max} =: u_l$$

$$\Rightarrow \boxed{e\, u_l = (\lambda_{max} - (l-1))\, u_{l-1}}$$

If $u_l = 0$, then

either $u_{l-1} = 0$,

or $\lambda_{max} - (l-1) = 0$

In particular, if $l = n+1 = \frac{\lambda_{max} - \lambda_{min}}{2} + 1$, then

$$u_{l-1} = \frac{f^n}{n!} v_{\lambda max} \neq 0 \quad \text{by definition of } \lambda_{min}$$

$$\Rightarrow \lambda_{max} = l-1 = n$$

$$\Rightarrow \lambda_{max} \in \mathbb{N}$$
$$\lambda_{min} = -\lambda_{max}$$

Prop: Any irreducible finite-dim. $sl_2(\mathbb{C})$-rep has a decomposition.

$$V \simeq \hat{\bigoplus}_{l=0} \mathbb{C}\, u_l$$

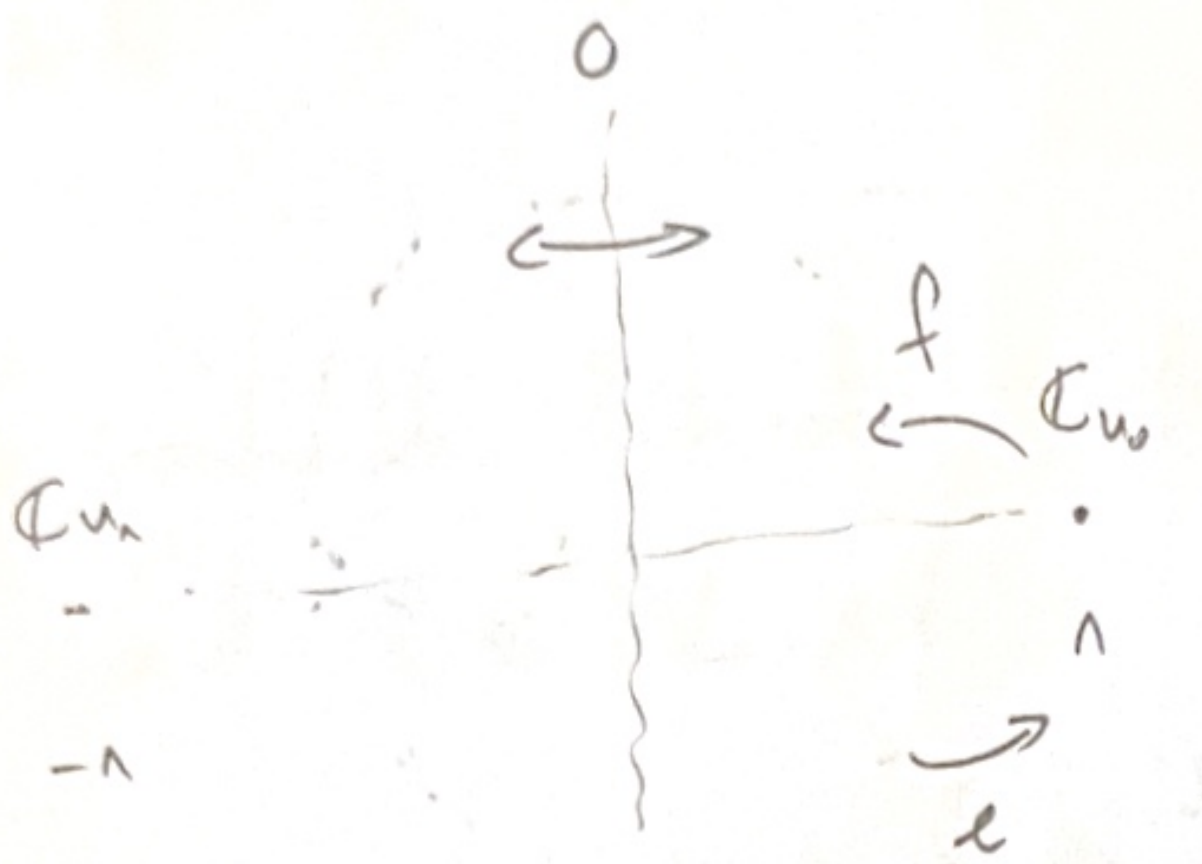
where

$$f(u_l) = (l+1)\, u_{l+1}$$

$$e(u_l) = (n-(l-1))\, u_{l-1}$$

$$h(u_l) = (n-2l)\, u_l$$

Picture:



$\Rightarrow$ similar picture will be on the FINAL !

$n \in \mathbb{N}$

$n=0 \rightsquigarrow \mathbb{C}$

$n=1 \rightsquigarrow \mathbb{C}^2$

$n=2 \rightsquigarrow \mathrm{adj},\ sl_2(\mathbb{C})$

13 Oct

TD 4

7.4 Claim: $SU(2)$ is homeomorphic to S^3.

proof: $\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$, $A^{\dagger} = A^{-1}$ and $\det A = 1$

i.e. $\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad - bc = 1$

This is equivalent to,

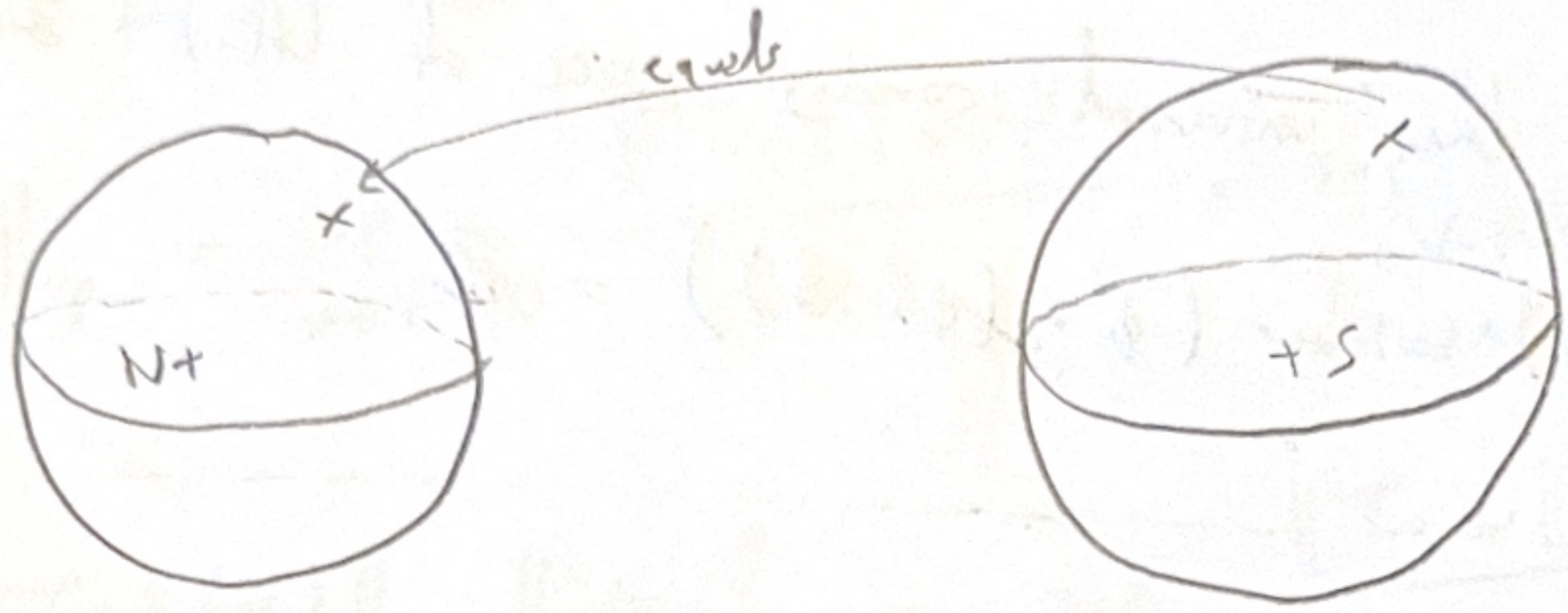
$\begin{cases} \bar{a} = d \\ \bar{c} = -b \end{cases}$ and $|a|^2 + |b|^2 = 1$

Thus, as sets:

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} ; \ a, b \in \mathbb{C}, \ |a|^2 + |b|^2 = 1 \right\}$$

Hence $SU(2)$ is homeomorphic to S^3.

S^3:



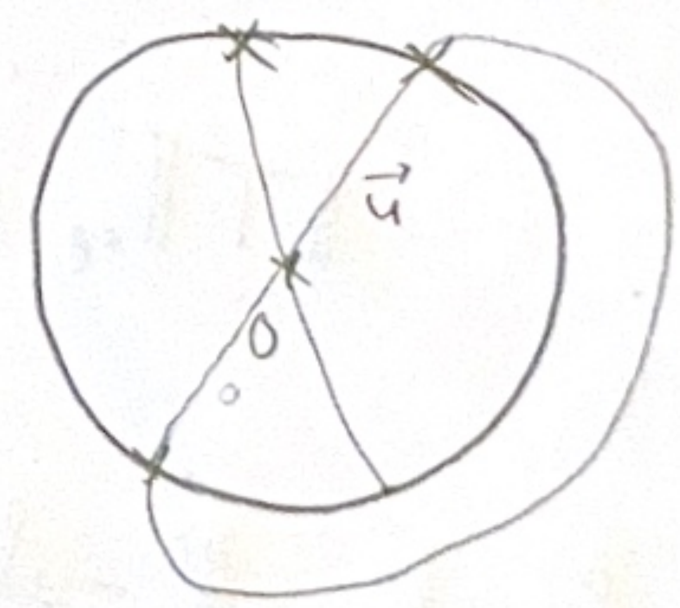
Thus,

$$\pi_1(SU(2)) = \pi_1(S^3) = \{1\}$$

Rem:

$$R_{\vec{u}}(\theta) = \theta \vec{u}$$
$$R_{\vec{u}}(\pi) = R_{\vec{u}}(-\pi)$$

Identity antipodal points



$\boxed{SO(3)}$

$$\phi : SU(2) \to \boxed{SO(3)}$$
$$\|$$
$$\widetilde{SO(3)}$$

Let $n \in \mathbb{N}$, $n \geq 2$.

Rem: $\forall n \geq 2$

$$\pi_1(SU(n+1)) \overset{\Downarrow}{=} \pi_1(SU(n))$$
$$= \pi_1(SU(2)) = \{1\}$$

7.5] 9.11, Ex. 4

$$\mathbb{R} \times SO(n) \xrightarrow{\phi \times id} U(1) \times SU(n) \xrightarrow{\sim} U(n)$$

Now, $\pi_1(\mathbb{R} \times SU(n)) = \pi_1(\mathbb{R}) \times \pi_1(SO(n)) = \{1\}$

Hence $\mathbb{R} \times SU(n)$ is the universal covering space of $U(n)$ and

$$\pi_1(U(n)) = \ker(\phi \circ (\phi \times id)) = \mathbb{Z}$$

Exercise Set #2

1 - The Exponential Map on a Unital Banach Algebra:

1.1] Let V be a Banach $\mathbb{K}$-vector space and let

$$B(V) = \{ T \in End(V) : \exists C \in \mathbb{R}_+, \forall x \in V, \|T(x)\| \leq C\|x\| \}$$

Then, $(B(V), +, \cdot, \circ, \|-\|_{op})$ where

$$\|T\|_{op} = \sup_{x \in V - \{0\}} \frac{\|T(x)\|}{\|x\|},$$

is a Banach $\mathbb{K}$-algebra.

proof: Clearly $(B(V), +, \cdot, \circ)$ is an associative $\mathbb{K}$-algebra.

i) $(B(V), +, \cdot, \|-\|_{op})$ is a Banach space.

ii) $\forall T, S \in B(V)$

$$\|T \circ S\|_{op} \leq \|T\|_{op} \|S\|_{op}$$

i) Clearly, $(\mathcal{B}(V), +, \cdot, \|-\|_{op})$ is a normed $\mathbb{K}$-v.s.

Hence, it suffices to check that it is complete.

Let's prove ii).

Take $T, S \in \mathcal{B}(V)$. Then,

$$\|T \circ S\|_{op} = \sup_{x \in V - \{0\}} \frac{\|T \circ S(x)\|}{\|x\|} = \sup_{\substack{x \in V - \{0\} \\ S(x) \neq 0}} \frac{\|T \circ S(x)\|}{\|x\|}$$

$$= \sup_{\substack{x \in V - \{0\} \\ S(x) \neq 0}} \frac{\|T \circ S(x)\|}{\|S(x)\|} \frac{\|S(x)\|}{\|x\|} \leq \sup_{w \in S(V) - \{0\}} \frac{\|T \circ w\|}{\|w\|} \sup_{x \in V - \{0\}} \frac{\|S(x)\|}{\|x\|}$$

$S(V) - \{0\} \subseteq V - \{0\}$

$$\leq \sup_{w \in V - \{0\}} \frac{\|T(w)\|}{\|w\|} \sup_{x \in V - \{0\}} \frac{\|S(x)\|}{\|x\|}$$

$$= \|T\|_{op} \|S\|_{op}$$

Now, let's prove that $(\mathcal{B}(V), +, \cdot, \|-\|_{op})$ is complete.

In order to do so, let

$(T_n)_{n \in \mathbb{N}} \in \mathcal{B}(V)^{\mathbb{N}}$ be a Cauchy sequence, i.e.

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m, n \geq N$$

$$\|T_m - T_n\|_{op} < \varepsilon \qquad (1)$$

In particular, $\forall x \in V$

$$\|T_m(x) - T_n(x)\| < \varepsilon \|x\|$$

Thus, $(T_n(x))_{n \in \mathbb{N}} \in V^{\mathbb{N}}$ is a Cauchy sequence, but since V is Banach, we must have that

$$T_n(x) \xrightarrow[n \to +\infty]{} t_x \in V.$$

Now, $\forall \alpha, \beta \in \mathbb{K}$; $\forall x, w \in V$; we have $\forall n \in \mathbb{N}$

$$\begin{array}{ccc} T_n(\alpha x + \beta w) & = & \alpha T_n(x) + \beta T_n(w) \\ \downarrow{\scriptstyle n \to +\infty} & & \downarrow{\scriptstyle n \to +\infty} \\ t_{\alpha x + \beta w} & = & \alpha t_x + \beta t_w \end{array}$$

Thus, setting

$$T : x \longmapsto t_x$$

defines a linear operator on V. On the other hand, $\forall n \in \mathbb{N}$,

$$\begin{aligned} \|T(x)\| &= \|T(x) - T_n(x) + T_n(x)\| \\ &\leq \|T(x) - T_n(x)\| + \|T_n(x)\| \\ &\leq \varepsilon \|x\| + C_n \|x\| \end{aligned}$$

$\exists\, C_n \in \mathbb{R}_+$ since $T_n \in B(V)$

Hence $T \in B(V)$.

Finally, taking $n \to +\infty$ in (1) yields

$$\|T_m(x) - T(x)\| < \varepsilon \|x\|$$

i.e. $\|T_m - T\|_{op} \xrightarrow[m \to +\infty]{} 0$ $\square$

1.2 | $\mathcal{G}(A)$ is a group with

- $*$ as multiplication
  (associativity ✓)

- 1_A is unit. ✓

- $x \mapsto x^{-1}$ as inversion. ✓

1.3 | prop: Setting $\forall x \in \mathcal{G}(A)$, $\forall y \in A$,

$$Ad_x(y) = x * y * x^{-1}$$

makes Ad_x a continuous algebra automorphism of A.

proof. Let $x \in \mathcal{G}(A)$ and let $y, z \in A$, let $\alpha, \beta \in \mathbb{K}$. Then

- $Ad_x(\alpha y + \beta z) = x * (\alpha y + \beta z) * x^{-1}$

$$= \alpha x * y * x^{-1} + \beta x * z * x^{-1}$$

$$= \alpha Ad_x(y) + \beta Ad_x(z)$$

- $Ad_x(y * z) = x * y * z * x^{-1}$

$$= x * y * x^{-1} * x * z * x^{-1}$$

$$= Ad_x(y) * Ad_x(z)$$

Hence Ad_x is a $\mathbb{K}$-alg. hom.

- $\forall y \in A$;

$Ad_{x^{-1}} \circ Ad_x(y) = x^{-1} * x * y * x^{-1} * x = y$

$\quad \|$

$Ad_x \circ Ad_{x^{-1}}(y) = \dots$

Hence $Ad_{x^{-1}} \circ Ad_x = Ad_x \circ Ad_{x^{-1}}$ and $Ad_x^{-1} = Ad_{x^{-1}}$

Ad_x is an automorphism.

Finally, $\forall x \in \mathcal{G}(A)$, $\forall y \in A$,

$$\| Ad_x(y) \| = \| x * y * x^{-1} \| \leq \|x\| \, \|x^{-1}\| \, \|y\|$$

$\square$

Thus, Ad_x is continuous.

$\boxed{1.4}$ Let $f : \mathbb{C} \to \mathbb{C}$ be analytic at $z_0 \in \mathbb{K}$, i.e. there exists $R > 0$ and $(a_n)_{n \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ s.t. $\forall z \in \mathbb{C}$

$$|z - z_0| < R$$
$$f(z) = \sum_{n \in \mathbb{N}} a_n (z - z_0)^n$$

Prop: We define

$$f_A : A \longrightarrow A$$

by setting $\forall x \in A$ s.t.

$$\| x - z_0 1_A \| < R$$
$$f_A(x) = \sum_{n \in \mathbb{N}} a_n (x - z_0 1_A)^n$$

proof:

$\forall x \in A$ $\underbrace{\| x - z_0 1_A \|}_{r} < R$

$$\| f_A(x) \| \leq \sum_{n \in \mathbb{N}} |a_n| \, \| x - z_0 1_A \|^n$$

$$= \sum_{n \in \mathbb{N}} |a_n| r^n < \infty$$

$r < R$

$\square$

1.5 exp is an entire function ($R = +\infty$) defined by setting, $\forall z \in \mathbb{C}$

$$\exp(z) = \sum_{n \in \mathbb{N}} \frac{z^n}{n!}$$

Thus, for any unital Banach algebra $(A, +, \cdot, *, \|-\|)$, we can define $\exp_A : A \to A$ by setting

$$\forall x \in A \qquad \exp_A(x) = \sum_{n \in \mathbb{N}} \frac{x^{*n}}{n!}$$

1.6 Prop: $\forall x \in \mathcal{G}(A)$, $\mathrm{Ad}_x \circ \exp_A = \exp_A \circ \mathrm{Ad}_x$

proof: Let $x \in \mathcal{G}(A)$ and let $y \in A$. Then, by definition

$$\mathrm{Ad}_x \circ \exp_A(y) = \mathrm{Ad}_x\left(\sum_{n \in \mathbb{N}} \frac{y^n}{n!}\right) = x * \left(\sum_{n \in \mathbb{N}} \frac{y^n}{n!}\right) * x^{-1}$$

$$= \sum_{n \in \mathbb{N}} \frac{x * y^n * x^{-1}}{n!} = \sum_{n \in \mathbb{N}} \frac{(x * y * x^{-1})^n}{n!}$$

$$= \sum_{n \in \mathbb{N}} \frac{(\mathrm{Ad}_x(y))^n}{n!} = \exp_A(\mathrm{Ad}_x(y)).$$

$\square$

1.7 Prop: $\forall x, y \in A$ s.t.

$$x * y = y * x$$

we have

$$\exp_A(x + y) = \exp_A(x) * \exp_A(y)$$

proof: Let $x, y \in A$ be s.t. $x * y = y * x$. We have by definition,

$$\exp_A(x+y) = \sum_{n\in\mathbb{N}} \frac{(x+y)^{*n}}{n!}$$

$$= \sum_{n\in\mathbb{N}} \sum_{p=0}^{n} \binom{n}{p} \frac{x^{*p} * y^{*n-p}}{n!}$$

$$= \sum_{n\in\mathbb{N}} \sum_{p=0}^{n} \frac{x^{*p}}{p!} * \frac{y^{*n-p}}{(n-p)!}$$

$$= \sum_{\substack{m\in\mathbb{N}\\ p\in\mathbb{N}}} \frac{x^{*p}}{p!} * \frac{y^{*m}}{m!} = \exp_A(x) * \exp_A(y) \quad \square$$

- $\forall x \in A$, let

$$\gamma_x : \mathbb{R} \to g(A)$$
$$t \mapsto \exp_A(tx)$$

1.8 **Prop.** γ_x is a continuous group homomorphism.

proof: By the previous question, $\forall x \in A$, $\forall s,t \in \mathbb{R}$, we have

$$\gamma_x(t+s) = \exp_A((t+s)x)$$
$$= \exp_A(sx+tx) = \exp_A(sx) * \exp_A(tx)$$
$$= \gamma_x(s) * \gamma_x(t).$$

Thus, γ_x is a group homomorphism. Indeed, $\forall t \in \mathbb{R}$, $\forall x \in \mathbb{R}$

$$\gamma_x(t) = \exp_A(tx) \in g(A) \quad \square$$

since $\gamma_x(t)^{-1} = \exp(-tx)$.

$t \mapsto \exp_A(tx)$ is C^0 indeed smooth.

1.9 Prop: Let $x \in A$. Then, γ_x is the only solution of the initial value problem:

$$\begin{cases} \gamma' = x * \gamma \\ \gamma(0) = 1_A \end{cases}$$

proof: clearly, $\forall t \in \mathbb{R}$

$$\gamma_x'(t) = x * \gamma_x(t)$$

Moreover,

$$\gamma_x(0) = 1_A + \sum_{n \in \mathbb{N}^*} \frac{0^n}{n!}$$

→ Uniqueness follows from Cauchy-Lipschitz or assume η is another solution and prove that $t \longmapsto \exp(-tx) * \eta(t)$ is constant and equal to 1_A. □

→ $\forall x \in A$, $ad_x : A \to A$

$$y \longmapsto Ad_x(y) = x * y - y * x = [x, y]_*$$

1.10 Prop: $\forall x \in A$, $ad_x \in \mathcal{B}(A)$

proof: Let $x \in A$, then

$\forall \alpha, \beta \in \mathbb{K}$; $\forall y, z \in A$; we have

$$ad_x(\alpha y + \beta z) = [x, \alpha y + \beta z]_* = \alpha [x, y]_* + \beta [x, z]_*$$

$$= \alpha \, ad_x(y) + \beta \, ad_x(z)$$

and $ad_x \in End(A)$.

Moreover $\forall y \in A$,

$$\|ad_x(y)\| = \|[x,y]_*\|$$
$$\leq \|x*y\| + \|y*x\|$$
$$\leq 2\|x\|\,\|y\|$$

$\square$

Thus, $ad_x \in \mathcal{B}(A)$.

1.11 Prop: $\forall x \in A$

$$Ad_{\exp_A(x)} = \exp_{\mathcal{B}(A)}(ad_x)$$

proof: Let $x \in A$, then, $\forall y \in A$, we have by definition

$$\exp_{\mathcal{B}(A)}(ad_x)(y) = \sum_{n\in\mathbb{N}} \frac{ad_x^{\,n}}{n!}(y)$$

$$= \sum_{n\in\mathbb{N}} \frac{1}{n!} \underbrace{[x,[x,\dots,[x,y]_*\dots]_*}_{n \text{ times}}$$

$$= \sum_{n\in\mathbb{N}} \sum_{p=0}^{n} \frac{(-1)^p}{n!} \underbrace{\binom{n}{p}}_{\frac{n!}{p!(n-p)!}} x^{*n-p} * y * x^{*p}$$

$$= \sum_{n\in\mathbb{N}} \sum_{p=0}^{n} \frac{x^{*n-p}}{(n-p)!} * y * \frac{(-x)^{*p}}{p!}$$

$$= \sum_{n,m\in\mathbb{N}} \frac{x^{*n}}{m!} * y * \frac{(-x)^{*p}}{p!}$$

$$= \exp_A(x) * y * \underbrace{\exp_A(-x)}_{\exp_A(x)^{-1}} = Ad_{\exp_A(x)}(y).$$

$\square$

1.12 For every $x_0 \in A$, $\forall r > 0$,

$$B_A(x_0, r) = \{ x \in A \,;\, \|x - x_0\| < r \}$$

Prop: $B_A(1_A, 1) \subseteq \mathcal{G}(A)$

proof: Let $x \in B_A(1_A, 1)$. It suffices to show that x i) invertible, for that $x \in \mathcal{G}(A)$

Now, $\forall x \in B_A(1_A, 1)$, we have $\|x - 1_A\| < 1$

$$\frac{1}{x} = \frac{1}{1_A + x - 1_A} = \sum_{n \in \mathbb{N}} (-1)^n (x - 1_A)^n. \quad \square$$

1.13 **Prop:** $\mathcal{G}(A)$ is a topological group.

proof: $\mathcal{G}(A)$ is a group - see q.3.

→ $*$ is obviously continuous since $\forall x, y \in \mathcal{G}(A) \subset A$

$$\|x * y\| \leq \|x\| \cdot \|y\|$$

Let $x_0 \in \mathcal{G}(A)$. Then, there exists $R > 0$ s.t.

$$B_A(x_0, R) \subseteq \mathcal{G}(A)$$

Let $x \in B_A(x_0, R)$. Then $\|x - x_0\| < R$. Thus

$$\frac{1}{x} = \frac{1}{x_0 + x - x_0} = x_0^{-1} * \frac{1}{1 + x_0^{-1} * (x - x_0)}$$

$$= x_0^{-1} * \sum_{n \in \mathbb{N}} (-1)^n \left(x_0^{-1} * (x - x_0) \right)^n$$

$$\|x_0^{-1} * (x - x_0)\| \leq \|x_0^{-1}\| \, \|x - x_0\| < 1$$

↳ $R < r = \frac{1}{\|x_0^{-1}\|}$

Now,

$$\| x^{-1} - x_0^{-1} \| = \| x_0^{-1} * \sum_{n \in \mathbb{N}} (-1)^n \left(x_0^{-1} * (x - x_0) \right)^n \|$$

$$\leq \| x_0^{-1} \| \sum_{n \in \mathbb{N}^*} \| x_0^{-1} \| \; \| x - x_0 \|^n$$

$$\xrightarrow{x \to x_0} 0 \quad \text{Hence } x \mapsto x^{-1} \text{ is } C^0 \text{ over } \mathcal{G}(A).$$

$\square$

V Finite dim reps of $SL(2,\mathbb{C})$

Because $SL(2,\mathbb{C})$ is connected & simply connected we can consider $\mathfrak{sl}(2,\mathbb{C})$ (Lie Alg.) instead.

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

ad(h): $X \mapsto [h,X]$ is diagonalizable element of $End(\mathfrak{sl}_2(\mathbb{C}))$

we have
$$\begin{cases} [h,e] = 2e \\ [h,f] = -2f \\ [e,f] = h \end{cases}$$
fundamental relations.

<u>Remark</u> h acts <u>semisimple</u> on $\mathfrak{sl}_2(\mathbb{C})$ (by ad: $X \mapsto [h,X]$)

& $\mathbb{C}e$, $\mathbb{C}f$, $\mathbb{C}h$ are the eigenspaces.

e acts <u>nilpotently</u> on $\mathfrak{sl}_2(\mathbb{C})$

$[e,f]=h$, $[e,h]=-2e$, $[e,e]=0$

$[f,f]=0$, ...

In basis $\{e,h,f\}$

$$ad(e) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$ad(f) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$ad(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\begin{tikzcd} 0 & \mathbb{C}f \arrow[l, bend left, "ad\, f"] \arrow[r, bend left, "ad\, e"] & \mathbb{C}h \arrow[l, bend left, "ad\, f"] \arrow[r, bend left, "ad\, e"] & \mathbb{C}e \arrow[l, bend left, "ad\, f"] \arrow[r, bend left, "ad\, e"] & 0 \end{tikzcd}$$

h-eigenvalues $\{-2 \quad 0 \quad 2\}$

<u>Rem.</u> h is special because e^{ih} belongs to a compact subgroup of $SL(2,\mathbb{C})$ $\Rightarrow$ $\rho(h)$ will be diagonalizable in <u>any</u> f.d. rep (ρ, V).

<u>Claim</u> h acts semisimply on any $\mathfrak{sl}_2(\mathbb{C})$-rep. Let (ρ, V) be f.d. $\mathfrak{sl}_2(\mathbb{C})$-rep. $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$

$V_\lambda = \{ v \in V \mid \rho(h) v = \lambda v \}$

we know $\rho(e), \rho(h), \rho(f)$ satisfy the following

$$[\rho(h), \rho(e)] \cancel{\rho(e)\rho(h)} = +2\rho(e)$$

$$[\rho(h), \rho(f)] = -2\rho(f)$$

$$[\rho(e), \rho(f)] = \rho(h)$$

<u>Lemma:</u> $e(V_\lambda) \subseteq V_{\lambda+2}$ & $f(V_\lambda) \subseteq V_{\lambda-2}$

<u>Proof:</u> Let $v_\lambda \in V_\lambda$, we drop "ρ" $he v_\lambda = e\underbrace{h(v_\lambda)}_{\lambda v_\lambda} + 2e(v_\lambda)$

$= (\lambda+2)\, e(v_\lambda)$

similarly for f.

$$\begin{tikzcd} \cdots V_{\lambda-2} \arrow[r, bend left, "e"] & V_\lambda \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V_{\lambda+2} \cdots \arrow[l, bend left, "f"] \end{tikzcd}$$

<u>Cor:</u> If V is irreducible f.d. then $\exists\ \lambda_0 \in \mathbb{C}$ & $l \in \mathbb{N}$ s.t.

$V_\lambda \neq \{0\} \iff \{\lambda_0, \lambda_0+2, \dots, \lambda_0+2l\}$

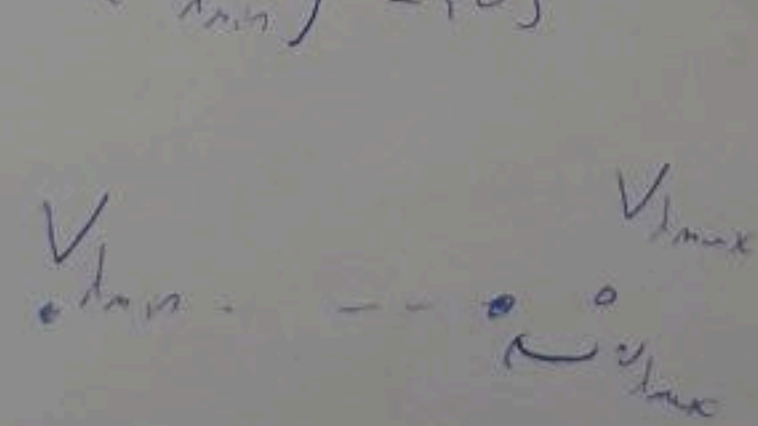
Let V be irred. $\rightsquigarrow \exists\ \lambda_{max} \in \{\lambda \text{ s.t. } V_\lambda \neq \{0\}\}$

$\lambda_{min} \in$ ______

s.t. $\{\lambda \in V_\lambda \neq 0\} = \{\lambda_{min}, \lambda_{min}+2, \dots, \lambda_{max}\}$

$\lambda_{min} = \lambda_{max} - 2l$

$\rightarrow e(V_{\lambda_{max}}) = f(V_{\lambda_{min}}) = \{0\}$



$f^{\ell}(v_{\lambda_{max}}) \in V_{\lambda_{max}-2\ell}$, $\quad e f^{\ell}(v_{\lambda_{max}}) = \underbrace{(ef)}_{h+fe} f^{\ell-1}(v_{\lambda_{max}})$

$= \left((\lambda_{max} - 2(\ell-1)) + fef^{\ell-1}\right)(v_{\lambda_{max}}) = (\ell\lambda_{max} - 2(\ell-1) - 2(\ell-2) - \cdots) f^{\ell-1}(v_{\lambda_{max}})$

$= (\ell\lambda_{max} - \ell(\ell-1)) f^{\ell-1}(v_{\lambda_{max}})$

2 conclusions 1) $\bigoplus_{\ell=0}^{n} \mathbb{C} f^{\ell} v_{\lambda_{max}}$ is a subrep. $\Rightarrow$ as V is simple (irreducible)

$V = \bigoplus_{\ell=0}^{n} \mathbb{C} f^{\ell} v_{\lambda_{max}}$

Choose a normalization $\frac{f^{\ell}}{\ell!} v_{\lambda_{max}} =: u_{\ell}$.

$\rightarrow \quad e u_{\ell} = (\lambda_{max} - (\ell-1))\, u_{\ell-1}$.

If $u_{\ell} = 0$ then either $\begin{cases} u_{\ell-1} = 0 & \text{or} \\ \lambda_{max} - (\ell-1) = 0 \end{cases}$

In particular, if $\ell = n+1 = \frac{\lambda_{max} - \lambda_{min}}{2} + 1$

then $u_{\ell-1} = \frac{f^{n}}{n!} v_{\lambda_{max}} \neq 0$ by def of $\lambda_{min} \Rightarrow \lambda_{max} = \ell - 1 = n \in \mathbb{N}$

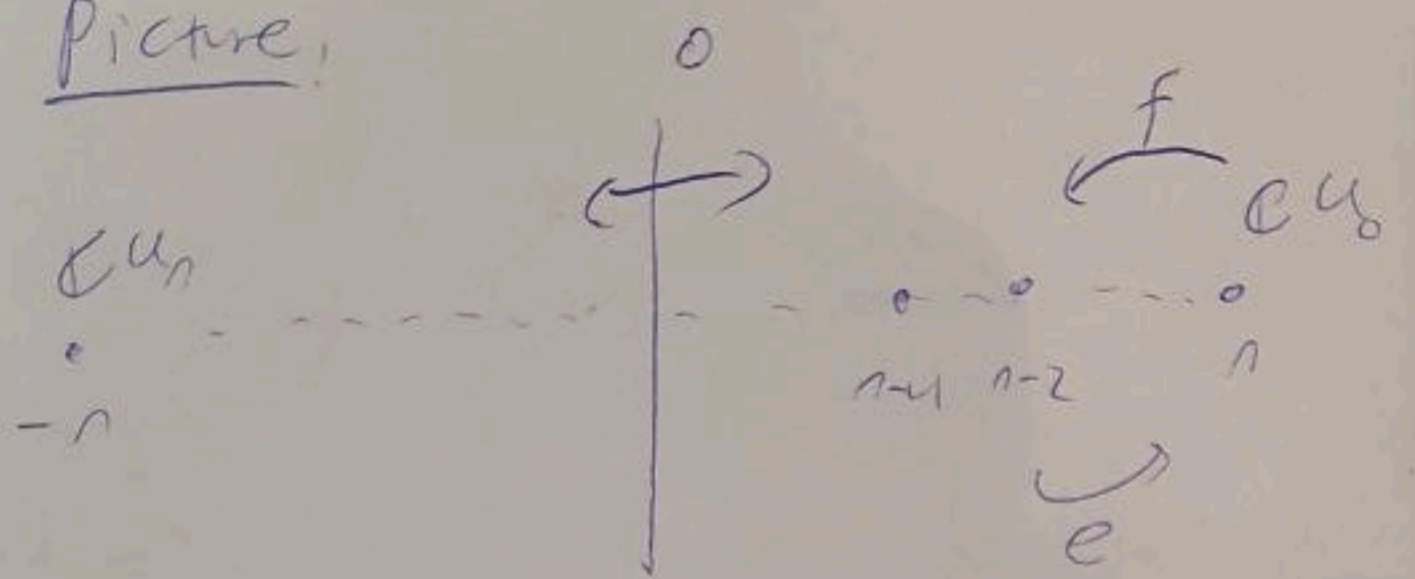
$\Rightarrow \lambda_{min} = -\lambda_{max}$

<u>Prop</u> Any irreducible finite dim $sl_2(\mathbb{C})$-rep has decomp.

$$V \simeq \bigoplus_{\ell=0}^{n} \mathbb{C} u_{\ell}$$

where: $\begin{cases} f(u_{\ell}) = (\ell+1)\, u_{\ell+1} \\ e(u_{\ell}) = (n-(\ell-1))\, u_{\ell-1} \\ h(u_{\ell}) = (n-2\ell)\, u_{\ell} \end{cases}$

Picture:



$n \in \mathbb{N}$

$n = 0 \rightsquigarrow \mathbb{C}$

$n = 1 \rightsquigarrow \mathbb{C}^2$

$n = 2 \rightsquigarrow \mathrm{adj}\ \mathfrak{sl}_2(\mathbb{C})$

LIE GROUPS $\mathfrak{g} = sl_2(\mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$

with sl commutation relations $\begin{cases} [h,e] = -e2 \\ [h,f] = f2 \\ [e,f] = h \end{cases}$

Lemma: In any finite dim $sl_2(\mathbb{C})$-rep h is semisimple (diagonalizable)

Proof: 1) $\exp : sl_2(\mathbb{C}) \to SL_2(\mathbb{C})$

$\exp(ih) \in SU(2,\mathbb{C})$ compact.

2) by Lie's thms, any finite dim $sl_2(\mathbb{C})$ rep (ρ, V) arises as the diff. of a f.d rep of $SL_2(\mathbb{C})$ (P, V) $(\rho = dP)$.

Since $S^1 = \exp(i\mathbb{R}h)$ is compact, $V \simeq \bigoplus_j V_j$ ← Simple S^1-module's $\Rightarrow V_j$ 1-dim.

$\Rightarrow$ by differently V_j is a $i\mathbb{R}h$-module.

$\Rightarrow$ ih acts in a diagonalizable way on V (as $\mathbb{R}$-v.space but also as $\mathbb{C}$-vector space)

$\Rightarrow$ h acts in a diagonalizable way on V as $\mathbb{C}$ vector space.

$$\left(\begin{array}{lccc} \text{Lie Groups} & S^1 \subset & SU(2) \subset & SL(2,\mathbb{C}) \\ & \downarrow \text{Lie} & \downarrow \text{Lie} & \downarrow \text{Lie} \\ \text{Lie algebras} & i\mathbb{R}h \subset & su(2) \subset & sl_2(\mathbb{C}) \end{array}\right)$$

☺

Let V be any simple f.d. rep. of $sl_2(\mathbb{C})$.

$\Rightarrow \exists! \, n, \; V = \bigoplus_{i=0}^{n} V_{n-2i}$ where $\begin{cases} h|_{V_{n-2i}} = (n-2i)\,\mathrm{Id} \\ e : V_k \mapsto V_{k+2} \\ f : V_k \mapsto V_{k-2} \end{cases}$ and $\dim V_1 = 1$

acts by $F, F^2, \dots F^{n_{max}}$

we get $V^1 := \bigoplus_{i=1}^{d} L_{n_{max}} \subset V$. where L_n = simple rep. with h.w. n.

If $V_{n_{max}-1} \subset V^1$ we pick a supplementary subspace in

$V_{n_{max}-1} \cap \ker(E)$.

This generates a subrep V^2 s.t. $V^2 \cap V^1 = \{0\}$.

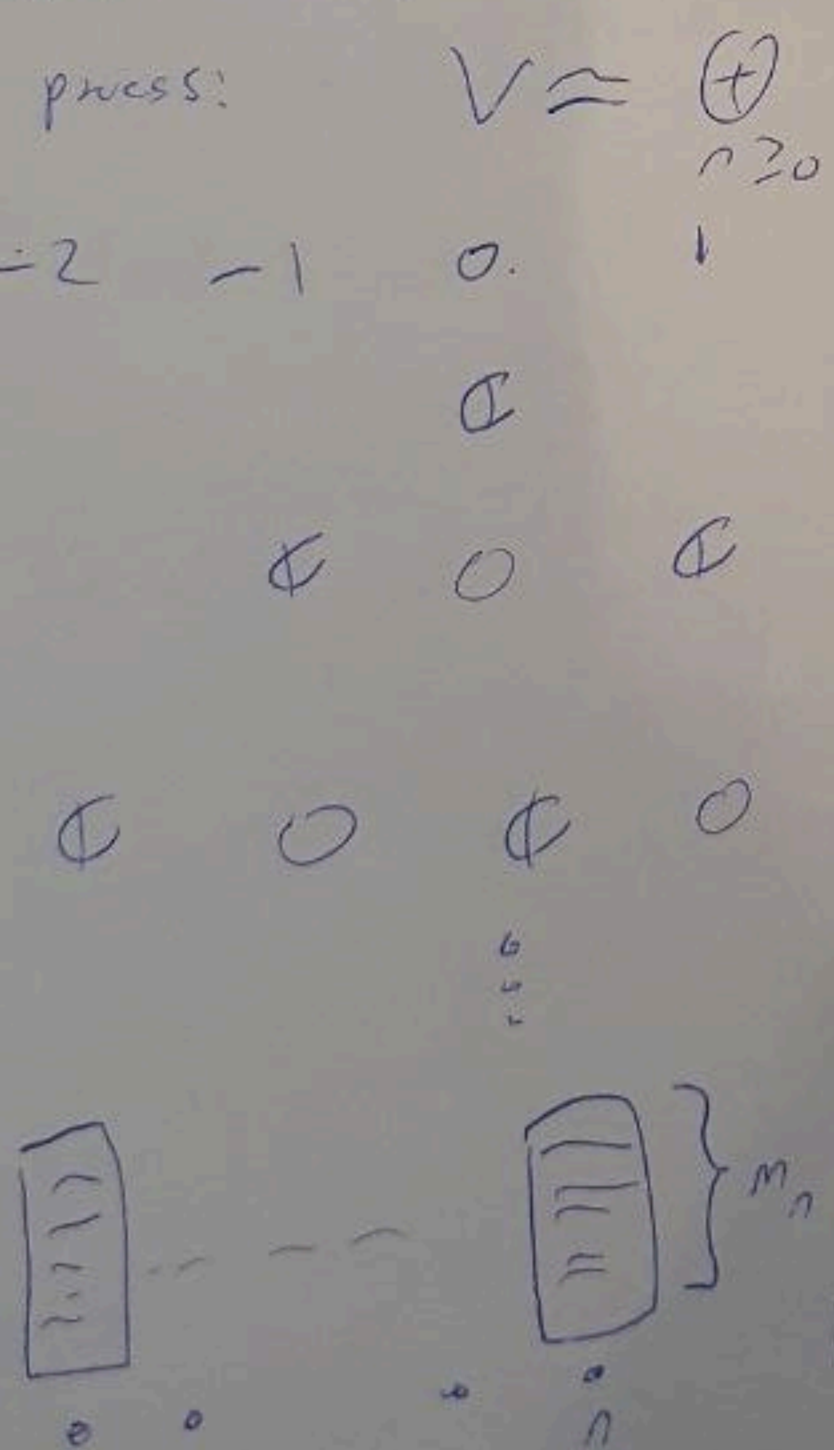
$\rightsquigarrow V^1 \oplus V^2 \subset V$.

We iterate process: $V \simeq \bigoplus_{n \geq 0} L_n^{\oplus m_n}$ which is:

	-2	-1	0	1	2
L_0			$\mathbb{C}$		
L_1		$\mathbb{C}$	0	$\mathbb{C}$	
L_2	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$

$\vdots$

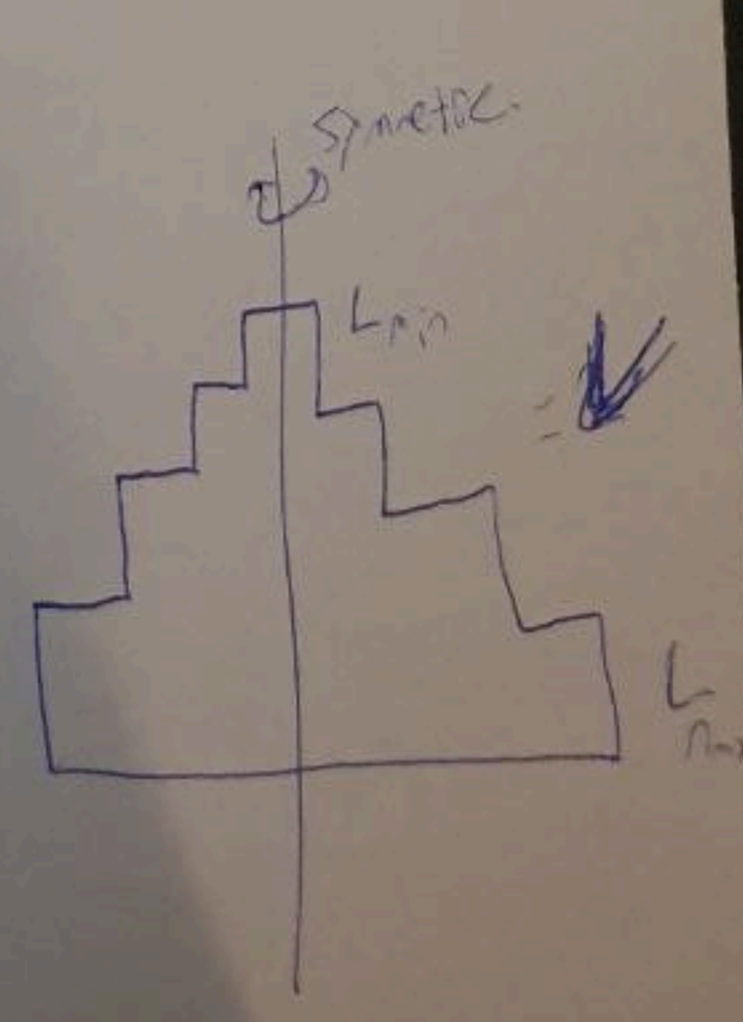
$L_n^{\oplus m_n}$



$\vdots$

$L_{n-i}^{\oplus m_{n-i}}$

i.e.



Cor: $\forall$ f.d. $sl_2(\mathbb{C})$-rep V

1) $\dim V_\ell = \dim V_{-\ell}$ $\forall \ell$.

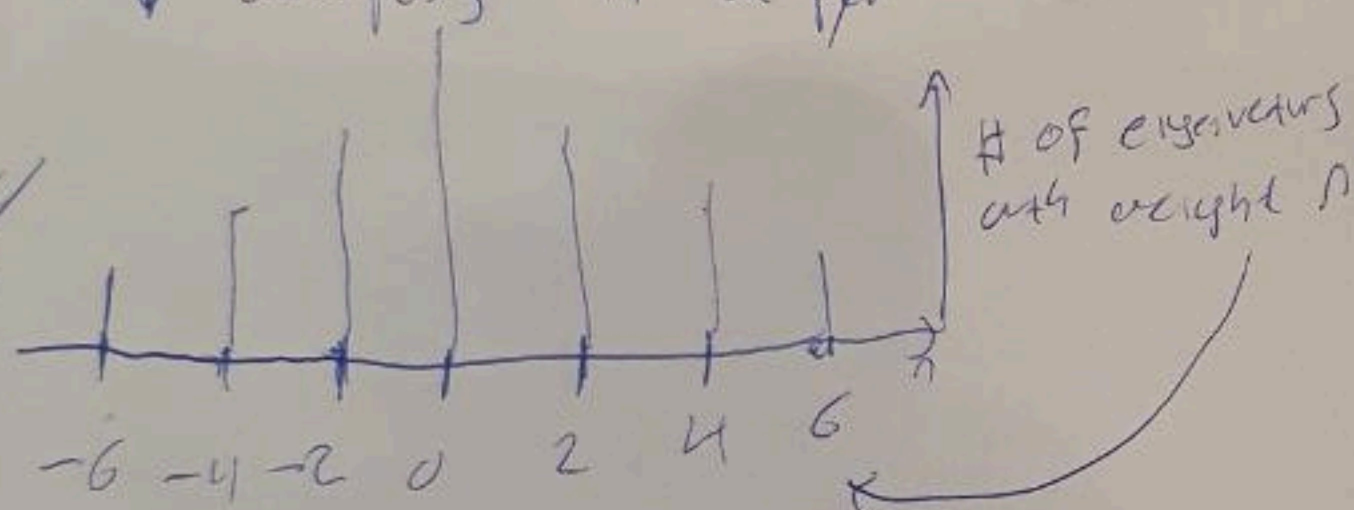
2) Spec^{ct} spectrum even an spectrum odd. (every the spectrum jumps by 2 with c.f.)

$\hookrightarrow$ $V = V^{even} \oplus V^{odd}$ as sl_2-reps.

$\dim V^{even}_\sigma \leq \dim V^{even}_{\sigma'}$ if $0 \leq \sigma' \leq \sigma$ (similarly for odd)

"V decomposes in a pyramid".

ex/ Spec^{even}



Def: character $ch(V) := \sum_n \dim V_n e^n$ (formal sum)

Cor: $ch(V)$ determines V up to isomorphism.

$$ch(L_n) = e^{-n} + e^{-n+2} + \cdots + e^{n-2} + e^{n} = \frac{e^{n+1} - e^{-n-1}}{e - e^{-1}}$$

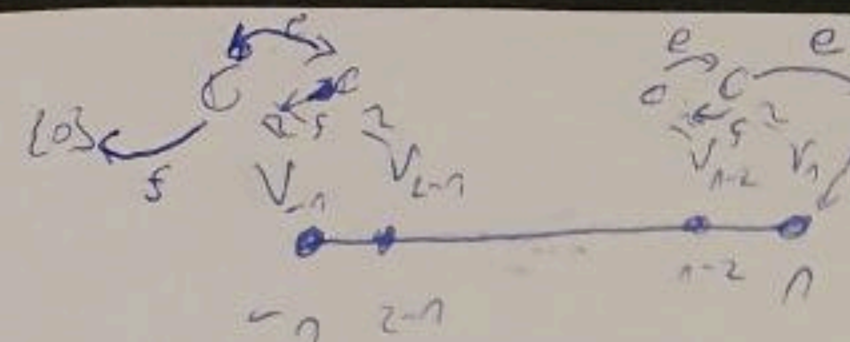
Thm: (Classification of sl_2-reps)

1) any f.d. sl_2-rep is semisimple.

2) The simple f.d. reps are in bij with $\mathbb{N}_{\geq 0}$

$n \longleftrightarrow L_n$ where L_n is the unique f.d. rep. gen. by highest weight vector $v \in (L_n)_n$

3) $ch(L_n) = \dfrac{e^{n+1} - e^{-n-1}}{e^{1} - e^{-1}}$ 4) The assignement $V \to ch(V)$ is a bijection between $\{\text{f.d. } sl_2\text{-rep}\}/\sim$



weight n (weight = eigenvalue of h)

Terminology
- n = highest-weight of V
- any $v \in V_n$ = highest weight vector.
- $-n$ = lowest weight ~~vector~~ of V & any $v \in V_{-n}$ = lowest vector of V.

Rem: $\forall v \in V_{-n} \setminus \{0\}$, $\{v, ev, e^2 v, \dots, e^n v\}$ is a basis of V.

Rem: weights are symmetric wrt $\{0\}$. $\dim V_\ell = \dim V_{-\ell}$.

Prop: Any f.d. rep of $sl_2(\mathbb{C})$ is semisimple. i.e.
$V \cong S_1 \oplus \cdots \oplus S_\ell$ for (not necessarily distinct) simple reps $S_1, \dots, S_\ell$.

Proof: Let V be a f.d. rep. Apriori, $\exists$ smallest subrep $S' \subset V$ which is simple. replace V by V/S'

iterate: this gives us a filtration

$$\underset{\substack{\parallel \\ S_1}}{F_1} \subset F_2 \subset \cdots \subset F_t = V$$

s.t. F_i / F_{i-1} is simple sl_2-rep.

Also, h acts semisimply on V, i.e. h has eigenvalues in $\mathbb{Z}$ for all F_i/F_{i-1} $\rightsquigarrow$ $\mathrm{spec}(h, V) = \bigcup_i \mathrm{spectrum}(h, F_i/F_{i-1})$

$$\begin{pmatrix} \boxed{F_1} & * & * \\ 0 & \boxed{F_2/F_1} & * \\ 0 & 0 & \boxed{F_3/F_2} \end{pmatrix}$$

~~nu~~ Construct the decomposition as follows:
$n_{max} = \max\{\underbrace{\mathrm{spectrum}(h, V)}_{wt_h(V)}\}$. Let $\{v_1, \dots, v_j\}$ be a basis of $V_{n_{max}}$

$\overset{\sim}{\longleftrightarrow} \left\{ \sum_n d_n e^n \;\middle|\; \begin{array}{l} d_n \geq 0 \ \forall n,\ d_{-n} = d_n \ \forall n \text{ if } 0 \leq l \leq n \\ \text{and } l \equiv n \pmod 2 \text{ then } d_l \geq d_n \end{array} \right\}$

Algorithm:

To determine the structure of a f.d. sl_2-rep: V.

1) Compute $ch(V)$ (i.e. diagonalize h)
2) express $ch(V) = \sum m_n \, ch(L_n)$

$$\Rightarrow V \cong \bigoplus L_n^{\oplus m_n}$$

Tensor Products

V_1, V_2 sl_2-reps $\rightsquigarrow V_1 \otimes V_2$ is also rep.

$\forall x \in sl_2 \qquad x \cdot (v_1 \otimes v_2) = (x \cdot v_1) \otimes v_2 + v_1 \otimes (x \cdot v_2)$

Sanity check: in sl_2 $[e,f] = h$.

$$ef(v_1 \otimes v_2) = ef\, v_1 \otimes v_2 + e v_1 \otimes f v_2 + f v_1 \otimes e v_2 + v_1 \otimes ef v_2$$

$fe(v_1 \otimes v_2) = \cdots$

$$\Rightarrow [e,f](v_1 \otimes v_2) = ([e,f]v_1) \otimes v_2 + v_1 \otimes [e,f] v_2 \quad = h(v_1 \otimes v_2) \checkmark$$
$$= h v_1 \otimes v_2 + v_1 \otimes h v_2$$

This comes from: V_1, V_2 G-reps. $\rightsquigarrow V_1 \otimes V_2$ G-rep

$g \cdot (v_1 \otimes v_2) = g v_1 \otimes g v_2 \xrightarrow{\text{expand infinitesimally}}$

Lemma If V, W are two f.d. sl_2-rep,

$$ch(V \otimes W) = ch(V)\, ch(W)$$

Moreover in case 2) $Z_\ell \simeq M_{-\ell-2}$.

It's "easy" to compute $ch(M_\lambda) = e^\lambda + e^{\lambda-2} + e^{\lambda-4} + \cdots$

$$= \frac{e^{\lambda+1}}{e-e^{-1}}$$

$$ch(L_n) = ch(M_n) - ch(M_{-n-2})$$

$$= \frac{e^{n+1} - e^{-n-1}}{e-e^{-1}}$$

IV General theory of Lie algebras.

Let $\mathfrak{g}$ be a f.d. Lie alg/$\mathbb{C}$.

A subalgebra is a subspace $\mathfrak{h} \subset \mathfrak{g}$ s.t. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$

An ideal is a ——— $I \subset \mathfrak{g}$ s.t. $[I, \mathfrak{g}] \subseteq I$.

- If $I \subset \mathfrak{g}$ is an ideal, the quotient $\mathfrak{g}/I$ has a Lie alg. structure

- $Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid \forall y, [x, y] = 0\}$ is the center

Def. A ~~simple~~ Lie algebra $\mathfrak{g}$ is simple if it has $\boxed{no}$ non-trivial ideals.

If $\mathfrak{g}$ is not simple, $\exists$ a short exact sequence (s.e.s)

$$0 \longrightarrow I \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/I \longrightarrow 0$$

of Lie algebras.

In general, $\exists$ a filtration

$$F_1 \subset F_2 \cdots \subset F_t = \mathfrak{g}$$

s.t. F_i/F_{i-1} is simple Lie algebra.

Proof: let $v \in V_n$, $w \in W_m$

$$h \cdot (v \otimes w) = (n+m)\, v \otimes w$$

$$\rightsquigarrow \operatorname{ch}(V \otimes W) = \sum_{n,m} \dim(V_n \otimes W_m) e^{n+m}$$

$$= \sum_{n,m} \dim(V_n)\dim(W_m)\, e^n e^m$$

$$= \operatorname{ch}(V)\operatorname{ch}(W)$$

Cor: (Clebsh-Gordon) $n \geq m$ $L_n \otimes L_m = L_{n+m} \oplus L_{n+m-2}$ $+ \dots + L_{n-m}$

Proof check that

$$\left(\frac{e^n - e^{-n}}{e - e^{-1}}\right)\left(\frac{e^m - e^{-m}}{e - e^{-1}}\right) = \sum_{l=0}^{m} \frac{e^{n+m-2l+1} - e^{-(\dots)}}{e - e^{-1}}$$

ex/

$$L_1^{\otimes n} = L_n \oplus \dots$$

$$L_1 \otimes L_1 = L_2 \oplus L_0$$

$$L_1 \otimes L_1 \otimes L_1 = (L_2 \oplus L_0) \otimes L_1 = L_3 \oplus L_1 \oplus L_1 \oplus L_1$$

$$= L_3 \oplus 2L_1$$

Nice Cor L_1 "generates" all Sl_2-reps

i.e. $\forall V$ $\exists a_0, \dots a_l \in \mathbb{Z}$ s.t.

$$\operatorname{ch}(V) = \sum_l a_l \operatorname{ch}(L_1^{\otimes l})$$

ex. $L_2 = L_1^{\otimes 2} \ominus L_1^{\otimes 0}$ i.e. $L_2 \oplus L_1^{\otimes 0} \simeq L_1^{\otimes 2}$

A ~~se~~ Lie algebra $\mathfrak{g}$ is <u>semisimple</u> if it's isomorphic to $\bigoplus_{i=1}^{s} \mathfrak{g}_i$ with each $\mathfrak{g}_i$ simple & different from $\mathbb{C}$

<u>Rem:</u> A Lie algebra of the form $\bigoplus_{i=1}^{s} \mathfrak{g}_i \oplus \mathbb{C}^r$ with each $\mathfrak{g}_i$ simple, of $\dim > 1$, $\mathbb{C}^r$ abelian Lie algebra, is called <u>reductive</u> ($\Rightarrow Z(\mathfrak{g}) = \mathbb{C}^r$)

Ex/ - $\mathfrak{sl}_r$ is simple, $\mathfrak{gl}_r$ is reductive ($Z(\mathfrak{gl}_r) = \mathbb{C}\,Id$)

$$\mathfrak{gl}_r \cong \mathfrak{sl}_r \oplus \mathbb{C}$$

- $\mathfrak{b}_r := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset \mathfrak{gl}_r$

(Lie) subalgebra

$\mathfrak{h}_r := \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \subset \mathfrak{b}_r$

subalgebra but not ideal.

$\mathfrak{n}_r := \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\} \subset \mathfrak{b}_r$

subalgebra

not ideal in $\mathfrak{gl}_r$ but ideal in $\mathfrak{b}_r$.

In fact $[\mathfrak{b}_r, \mathfrak{b}_r] \subseteq \mathfrak{n}_r$

$\mathfrak{b}_r / \mathfrak{n}_r \simeq \mathbb{C}^r$ (abelian Lie algebra)

$$\begin{tikzcd} \mathfrak{h}_r \arrow[r] \arrow[rr, bend right, "\text{can}", "\sim"'] & \mathfrak{b}_r \arrow[r] & \mathfrak{b}_r / \mathfrak{n}_r \end{tikzcd}$$

One way to construct simple reps of $sl_2(\mathbb{C})$ are <u>Verma modules</u>

Start with $\lambda \in \mathbb{C}$ "highest weight" & v_λ h.w.v

s.t. $\begin{cases} h v_\lambda = \lambda v_\lambda \\ e v_\lambda = 0 \end{cases}$

$\exists!$ sl_2-rep M_λ generated by v_λ i.e. $M_\lambda = \mathrm{Span}\{e^a h^b f^c v_\lambda\}$ $\forall a,b,c$

and no extra relations except for those of $sl_2(\mathbb{C})$,

$$v_\lambda \to \begin{cases} \{f v_\lambda\} \rightsquigarrow \begin{cases} e(f v_\lambda) = f \overset{0}{\cancel{e v_\lambda}} + \underbrace{[e,f]}_{h} v_\lambda = h v_\lambda \\ h f v_\lambda = (\lambda - 2) f v_\lambda \end{cases} \\ f^2 v_\lambda \rightsquigarrow \begin{cases} e f^2 v_\lambda = \cancel{\quad} \doteq (2\lambda - 2) f v_\lambda \\ \vdots \end{cases} \\ \vdots \end{cases}$$

$$\rightsquigarrow M_\lambda = \bigoplus_{l \geq 0} \mathbb{C} f^l v_\lambda$$

$\cdots \bullet \bullet \quad \bullet \; -- \; f^2 v_\lambda \; f v_\lambda \; v_\lambda$

By previous sl_2 - study of f.d reps

$$e f^n v_\lambda = 0 \quad \text{iff} \quad \lambda \in \mathbb{N}, \; n = \lambda + 1.$$

else $\quad e : \mathbb{C} f^n v_\lambda \xrightarrow{\sim} \mathbb{C} f^{n-1} v_\lambda$

<u>Structure of verma modules</u>

1) If $\lambda \notin \mathbb{N}_{\geq 0}$, M_λ is simple (∞-dim)

2) If $\lambda = k \in \mathbb{N}_{\geq 0}$, <u>$\exists!$ submodule</u> $Z_k \subset L_k$, $Z_k = \bigoplus_{K \geq n+1} \mathbb{C} f^K v_\lambda$

In that case $M_n / Z_k \simeq L_k$ "universal construct of simple f.d. modules"

2. Nilpotent Lie algebras

<u>Def</u> A Lie algebra $\mathfrak{g}$ is nilpotent if the sequence of ideals
$$\begin{cases} \mathcal{C}_0 = \mathfrak{g} \\ \mathcal{C}_i = [\mathfrak{g}, \mathcal{C}_{i-1}] \end{cases}$$
vanishes for i finite.

$$\mathcal{C}_0 = \mathfrak{g}, \quad \mathcal{C}_1 = [\mathfrak{g},\mathfrak{g}], \quad \mathcal{C}_2 = [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]].$$

- An element $x \in \mathfrak{g}$ is nilpotent if $ad_x : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x,y]$ is a nilpotent operator. i.e. $\forall y \in \mathfrak{g}$ $ad_x^n(y) = [x,[x,[\cdots,[x,y]]]] = 0$.

Ex/ $\mathfrak{n}_r = \left\{ \begin{pmatrix} 0 & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$ nilpotent

$\mathfrak{b}_r = \left\{ \begin{pmatrix} * & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$ not nilpotent.

"solvable" Lie algebra.

<u>Exercise</u> Let $\mathfrak{g}$ be nilpotent Lie algebra

1) $\forall \mathfrak{h} \subset \mathfrak{g}$, subalgebra, $\mathfrak{h}$ is nilpotent.
2) $\forall I \subset \mathfrak{g}$ ideal, $\mathfrak{g}/I$ is nilpotent.
3) Let $\mathfrak{g}$ be any Lie algebra, $I \subset \mathfrak{g}$ an ideal. If I & $\mathfrak{g}/I$ nilpotent then $\mathfrak{g}$ is nilpotent.

GENERAL THEORY OF LIE ALG (f.d / $\mathbb{C}$)

→ simple, ½ semisimple Lie Alg.

Reductive = ½ simple ⊕ center (semisimple)

→ nilpotent Lie Algebras:

$$\mathfrak{e}_i = [\mathfrak{g}, \mathfrak{e}_{i-1}], \quad \mathfrak{e}_0 = \mathfrak{g} \quad \text{then } \mathfrak{g} \text{ nilpotent iff } \mathfrak{e}_i = 0$$

for $i \gg 0$ i.e. $[x_1, [x_2, [x_3, \dots [x_{n-1}, x_n]]]] = 0$

ex
$$\begin{pmatrix} 0 & * & * \\ & \ddots & * \\ 0 & & 0 \end{pmatrix}.$$

$\mathfrak{g}$ nilp. ⇒ any subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is nilpotent

& any ideal $I \subset \mathfrak{g}$, $\mathfrak{g}/I$ is nilpotent

$$\big([x+I, y+I] = [x,y] + \underbrace{[I,y] + [x,I] + [I,I]}_{\subset I}$$

<u>ex</u> $\mathfrak{b} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \mathfrak{gl}(2)$

$I = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ nilpotent. → $\mathfrak{b}/I$ abelian ⇒ nilpotent.

but $\mathfrak{b}$ is <u>not</u> nilpotent,

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] = 2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathrm{ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^N \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 2^N \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$$

<u>Question</u>: $\mathfrak{g}$ nilp ⇒ $\forall x \in \mathfrak{g}$, ad_x is nilpotent ✓

$\boxed{\overset{?}{\Leftarrow}}$ yes!

Thm. (Engel) Let V be a f.d. $\mathbb{C}$-v.s. & $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra s.t. $\forall x \in \mathfrak{g}$, x is nilpotent (as an op. on V). Then, ∃ a flag of subspaces

$L_1 \subset L_2 \subset \cdots \subset L_d = V$ s.t. $\begin{cases} 1)\ \dim L_i = i \\ 2)\ g(L_i) \subseteq L_{i-1} \end{cases}$

Cor: (reformulation) up to change of basis,

$$g \subset \left\{ \begin{pmatrix} 0 & * & * & \cdots & * \\ & 0 & * & & \vdots \\ \vdots & & 0 & \ddots & \\ & & & \ddots & * \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \right\}$$

Proof: If $\dim g = 1 \rightsquigarrow$ Jordan-Form.

(Thm: Let V be a f.d. $\mathbb{C}$-vectorspace, $x \in End(V)$ nilpotent. Then, $\exists$ a basis $(e_1, \ldots, e_d)$ of V s.t.

$$x \in \begin{pmatrix} 0 & * & * \\ \vdots & \ddots & * \\ & 0 & \ddots \\ 0 & 0 & 0 \end{pmatrix}$$

By induction on $\dim g$, $\dim V$, if $\dim V = 1$, $g = 0$ is obvious.
Let's fix (g, V) as above & assume thm valid for any (g', V') with $\dim g' \leq \dim g$ & $\dim V' < \dim V$.

Lemma: g contains a codim 1 ideal I.

Proof: We'll show for any subalg. $h \subset g$, $\exists$ another subalgebra $\tilde{h} \subseteq g$ s.t. $h \subset \tilde{h}$, $\dim \tilde{h}/h = 1$ & $[\tilde{h}, h] \subseteq h$.

There's an adjoint action of H on g/h. This satisfies hypothesis of thm. Indeed, if $x \in gl(V)$ is nilpotent, then $ad_x \in End(gl(V))$ is nilpotent.

$ad(x)^N(y) = \sum_{i=0}^{N} c_i x^{N-i} y x^i$ For $N > 2$(index of nilpotency of x) $\Rightarrow x^{N-i}$ or x^i is $0 \Rightarrow ad(x)^N(y) = 0$

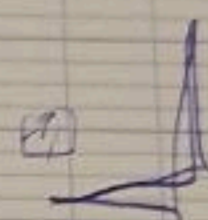
by induction hypothesis, since $\dim(\mathfrak{h}) < \dim(\mathfrak{g})$,

$\exists\, L_1 \subset \mathfrak{g}/\mathfrak{h}$ s.t. $\mathfrak{h}(L_1) = 0$ (in $\mathfrak{g}/\mathfrak{h}$).

Then $L_1 + \mathfrak{h} =: \tilde{\mathfrak{h}}$. By construction $[\mathfrak{h}, \tilde{\mathfrak{h}}] \subset \mathfrak{h}$

$\Rightarrow [\tilde{\mathfrak{h}}, \tilde{\mathfrak{h}}] \subseteq \tilde{\mathfrak{h}}$. We start with any $\mathfrak{h}_0 = \mathbb{C}a$ for $a \in \mathfrak{g}$,

and iterate the process $\mathfrak{h}_0 \subset \underset{\tilde{\mathfrak{h}}_0}{\overset{\parallel}{\mathfrak{h}_1}} \subset \underset{\tilde{\mathfrak{h}}_1}{\overset{\parallel}{\mathfrak{h}_2}} \subset \cdots \subset I \subset \underset{I \oplus}{\overset{\parallel}{\underset{\approx}{\mathfrak{g}}}}$



By induction hypothesis, applied to (I, V), $\exists\, L_1 \subset V$ s.t.

$I(L_1) = \{0\}$. Set $\underset{L_1}{\overset{}{\underset{\cup\!\mid}{U}}} := \{v \in V \mid I(v) = 0\}$

Claim: $\mathfrak{g}(U) \subseteq U$. Let $x \in \mathfrak{g}$, $u \in U$, $i \in I$

$$\Rightarrow ix(u) = \underbrace{\underbrace{[i,x]}_{\in I}(u)}_{0} + x\underbrace{i(u)}_{0} = 0.$$

Let us write $\mathfrak{g} = I \oplus \mathbb{C}x$ $\Rightarrow$ x acts nilpotently

on U $\Rightarrow \exists\, L_1' \subset U$ s.t. $x(L_1') = 0$.

Then $\begin{cases} I(L_1') = 0 & \text{as } L_1' \subset U \\ \mathfrak{g}(L_1') = 0 \end{cases}$

$\Rightarrow$ we set $\mathbb{L}_1 := L_1'$. Then $\mathfrak{g}$ acts on $V/\mathbb{L}_1$

and satisfies hypothesis of thm. By induction hyp.

$\exists\, \bar{\mathbb{L}}_2 \subset \cdots \subset \bar{\mathbb{L}}_d \subset V/\mathbb{L}_1$ s.t. $\mathfrak{g}(\bar{\mathbb{L}}_i) \subseteq \bar{\mathbb{L}}_{i-1}$

Set $\mathbb{L}_i := \mathbb{L}_1 + \bar{\mathbb{L}}_{i-1}$. $\square$

General strategy. $\mathfrak{g} \subset \mathfrak{gl}(V)$ want to find

$L_1 \subset \cdots \subset V$ s.t. $\mathfrak{g}(L_i) \subseteq L_{i-1}$.

Step 1 show that $\{ v \in V \mid \mathfrak{g}(v) = 0 \} \neq \{0\}$

$$\begin{pmatrix} 0 & * \cdots * \\ \vdots & \boxed{\; * \;} \\ 0 & \end{pmatrix}$$

Cor: A Lie Alg. $\mathfrak{g}$ is nilpotent iff ad_X is nilpotent for any $x \in \mathfrak{g}$.

Proof: $\mathfrak{g}$ nilp $\Rightarrow$ ad_x nilp $\forall x$.

$\Leftarrow$: consider $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by hyp.

$Im(ad)$ is a Lie subalgebra $\left([ad_x, ad_y] = ad_{[x,y]} \right)$

- $\forall x \in \mathfrak{g}$, $ad_x \in \mathfrak{gl}(\mathfrak{g})$ is a nilpotent op. by hypothesis

By Engel's thm $\exists$ $L_1 \subset L_2 \subset \cdots \subset L_{\dim \mathfrak{g}} = \mathfrak{g}$

s.t. $\forall x \in \mathfrak{g}$,

$$\underbrace{ad_x(L_i)}_{[x, L_i]} \subseteq L_{i-1}$$

which says that $\mathfrak{g}$ is nilpotent.

Reformulation of Engel's thm: If $\mathfrak{g}$ is a Lie algebra acting on V, via nilpotent op., then V is a successive extension of the trivial rep $\mathbb{C}$ of $\mathfrak{g}$.

Rem: $\mathfrak{g} \subset \mathfrak{gl}(V)$ consist of nilpotent op.

$\cancel{\Downarrow}$

$\mathfrak{g}$ is a nilpotent Lie alg.

Def: A Lie Alg. is solvable if the sequence of Lie subalgebras

$$\begin{cases} D^0(g) = g \\ D^{i+1}(g) = [D^i(g), D^i(g)] \end{cases}$$

vanishes for $i \gg 0$. (i.e. $\exists i$ s.t. $D^i(g) = \{0\}$)

Ex/ • nilp $\Rightarrow$ solvable.

• g satisfies $D(g) = [g, g]$ is nilpotent $\Rightarrow$ g is solvable.

Ex/ 1) g solvable $\Rightarrow$

• $\forall h$ subalgebra of g, h is solvable

• $\forall I$ ideal of g, g/I is solvable.

2) If $I \subset g$ ideal s.t. I is solvable and g/I too then g is solvable Lie alg.

Alternative way of thinking about solvable Lie Alg.

$$0 \to D(g) \to g \to \underbrace{g/D(g)}_{\text{abelian}} \to 0$$

$[g, g]$

i.e. A Lie Algebra is solvable iff it is a successive extension of <u>abelian</u> Lie algebras.

Thm (Lie) Let g be a solvable Lie alg, V a f.d. g-rep. Then $\exists\ L_1 \subset \cdots \subset L_d = V$ s.t.

$$\begin{cases} g(L_i) \subset L_i \\ \dim L_i = i \end{cases}$$

Proof: (Riche)

Cor: Any f.d. rep of a solvable Lie alg. is a successive extension of 1-dim reps.
(any simple rep. of g is 1-dim)

Useful fact: (exercise) $D(g)$ is nilpotent $\Leftrightarrow$ g is solvable

$$\begin{pmatrix} * & & \\ & \ddots & * \\ 0 & & * \end{pmatrix}$$

ex/ $\mathbb{C}\,\mathrm{Id}$ is a 1-dim Lie alg. $\Rightarrow$ nilpotent
but Id is not nilpotent operator on V.

<u>Thm</u> Let $\mathfrak{g}$ be nilpotent Lie alg.
V a (f.d) $\mathfrak{g}$-~~Lie alg~~ rep. There $\exists\ L_1 \subset \cdots \subset L_n = V$
st $\mathfrak{g}(L_i) \subseteq L_{i-1}$

i.e. up to change of basis,

$$\mathfrak{g} \subset \left\{ \begin{pmatrix} x & & x \\ & \ddots & \\ 0 & & x \end{pmatrix} \right\}$$

<u>Cor</u> Any f.d. rep of a nilpotent Lie Alg. is a successive extension of 1-dim reps. i.e. any simple rep of a nilpotent Lie alg. is 1-dim.

<u>Exercise</u> Given a Lie alg. $\mathfrak{g}$, construct <u>all</u> 1-dim reps of $\mathfrak{g}$.

$$\rho([\mathfrak{g},\mathfrak{g}]) = 0$$

Any linear form $\rho : \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \longrightarrow \mathbb{C} = \mathfrak{gl}(1)$

is a 1-dim rep

$$\rightsquigarrow \boxed{\left(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \right)^*}$$ ~~answer~~ are all 1-dim reps.

~~$\rho : \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \longrightarrow \mathfrak{gl}(1) \simeq \mathbb{C}$~~

Solvable Lie Alg

<u>Heuristics</u>: prototypical ex. of nilpotent Lie Alg is $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\} \subset \mathfrak{gl}(V)$. Prot. ... of solvable $\mathfrak{b} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset \mathfrak{gl}(V)$

Nilpotent & solvable radicals

Idea: Break down any f.d. Lie Alg. as

$$0 \to \mathfrak{r} \to \mathfrak{g} \to \mathfrak{g}^{ss} \to 0$$

$\mathfrak{r}$: solvable (radical = ideal); $\mathfrak{g}^{ss}$: semisimple

Prototip ex:

$$\mathfrak{g} = \left\{ \begin{pmatrix} \boxed{x} & x & x \\ & \boxed{x} & x \\ 0 & & \boxed{x} \end{pmatrix} \right\} \subset \mathfrak{gl}(V)$$

$$\mathfrak{r} = \left\{ \begin{pmatrix} \boxed{\lambda_1 1_{d_1}} & x & x \\ & \boxed{\lambda_2 1_{d_2}} & x \\ & & \boxed{\lambda_3 1_{d_3}} \end{pmatrix} \right\}, \quad \mathfrak{g}^{ss} = \left\{ \begin{pmatrix} \boxed{\mathfrak{sl}_{d_1}} & & 0 \\ & \boxed{\mathfrak{sl}_{d_2}} & \\ 0 & & \boxed{\mathfrak{sl}_{d_3}} \end{pmatrix} \right\} = \bigoplus_{i=1}^{3} \mathfrak{sl}_{d_i}$$

Prop: Let $\mathfrak{g}$ be a f.d. Lie Alg. $\exists!$ maximal ideal solvable $\mathfrak{r} \subset \mathfrak{g}$.

Proof: Let $\mathfrak{r}_1, \mathfrak{r}_2$ be solvable ideals. Then $\mathfrak{r}_1 + \mathfrak{r}_2$ is an ideal solvable ($0 \to \underbrace{\mathfrak{r}_1}_{\text{solvable}} \to \underbrace{\mathfrak{r}_1 + \mathfrak{r}_2}_{\text{extension}} \to \underbrace{\mathfrak{r}_2 / \mathfrak{r}_1 \cap \mathfrak{r}_2}_{\text{solvable}} \to 0$ $\rightsquigarrow$ solvable) extend inductively $\rightsquigarrow \mathfrak{r}$.

Def: $\mathrm{rad}(\mathfrak{g})$ = unique maximal ideal solvable.

Exercise: 1) Let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal, $\mathfrak{h} \subset \mathrm{rad}(\mathfrak{g})$. Then, $\mathrm{rad}(\mathfrak{g}/\mathfrak{h}) = \mathrm{rad}(\mathfrak{g})/\mathfrak{h}$

2) $\mathrm{rad}(\mathfrak{g}/\mathrm{rad}(\mathfrak{g})) = \{0\}$

(because $0 \to \mathfrak{r} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{r} \to 0$ ← radical of this is $\mathfrak{r}'$.

$0 \subseteq \mathfrak{r} \subseteq \mathfrak{r}' \subseteq \mathfrak{g}$ $\quad \mathfrak{r}'/\mathfrak{r}$ solvable)

SES. $0 \to V_1 \xrightarrow{a} V \xrightarrow{b} V_2 \to 0$ for vector spaces

means:
- a is injective (V_1 subspace of V)
- b is surjective / V_2 is a quotient of V)
- $\mathrm{Im}(a) = \mathrm{Ker}(b)$ ($V_2 \cong V/V_1$)

For algebras ~> same but a, b maps of Lie algebras $\Rightarrow V_1$ is an ideal (Ker is ideal)

For this you cannot find s.p. space inside V to fit V_2 (in general)

In matrices $\begin{pmatrix} \boxed{V_1} & * \\ & \boxed{V_2} \end{pmatrix}$ $\leftarrow$ $0 \to V_1 \to V \to V_2 \to 0$

on vector spaces the sequence splits

but in general not

"Def": A f.d. Lie alg. $\mathfrak{g}$ is semisimple if $\mathrm{rad}(\mathfrak{g}) = 0$

Thm: (Weyl) The two def. agree. $\mathrm{rad}(\mathfrak{g}) = \{0\} \Leftrightarrow \mathfrak{g} \simeq \mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_\ell$, $\mathfrak{s}_i$ simple

Less important: $S := \bigcap_{(\rho, V)\text{-simple } \mathfrak{g}\text{-rep}} \mathrm{Ker}(\rho) = \{x \in \mathfrak{g} \mid x \text{ acts by zero in any f.d. simple rep}\}$

- S is an ideal (nothing)
- S is nilpotent

Proof of S nilpotent: S acts on $\mathfrak{g}$ by adjoint action.

Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots \subseteq \mathfrak{a}_\ell = \mathfrak{g}$ be a Jordan-Hölder filtration of $\mathfrak{g}$ as $\mathfrak{g}$-rep, i.e. $\mathfrak{a}_i / \mathfrak{a}_{i-1}$ is simple $\mathfrak{g}$-rep $\forall i$.

By construction, $S(\mathfrak{a}_i / \mathfrak{a}_{i-1}) = \{0\}$ i.e. $S(\mathfrak{a}_i) \subseteq \mathfrak{a}_{i-1}$

$\Rightarrow S$ acts nilpotently on $\mathfrak{g}$ (by adjoint action)

$\Rightarrow$ " " " " " S i.e. S is nilpotent.

Def S = nilpotent radical of $\mathfrak{g}$. $S(\mathfrak{g}) \subseteq \mathrm{rad}(\mathfrak{g}) \subseteq \mathfrak{g}$.

Thm: $\forall$ f.d. Lie alg $\mathfrak{g}/\mathbb{C}$ then we have the sequence

$$\begin{array}{l|c|l}
\mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \text{ semisimple} & \mathfrak{g} & \mathfrak{g}/S(\mathfrak{g}) \text{ is reductive } \& \ \mathfrak{g}/S(\mathfrak{g}) \simeq \mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{g})/S(\mathfrak{g}) \\
 & \mathrm{rad}(\mathfrak{g}) & \\
\mathrm{rad}(\mathfrak{g})/S(\mathfrak{g}) \text{ abelian (central part)} & & \mathrm{rad}(\mathfrak{g}) \text{ solvable} \\
 & S(\mathfrak{g}) & \\
S(\mathfrak{g}) \text{ nilpotent} & & \\
 & \{0\} &
\end{array}$$

7

LIE GROUPS (6)

Recap $\mathfrak{g}$ f.d. Lie Alg. $/\mathbb{C}$ $\rightsquigarrow$ $\mathrm{rad}(\mathfrak{g}) :=$ largest solvable ideal in $\mathfrak{g}$

$\rightsquigarrow$ $S(\mathfrak{g}) := \bigcap\limits_{\substack{\text{simple} \\ \text{repr. } (V,\rho)}} \ker \rho$ (nilp.)

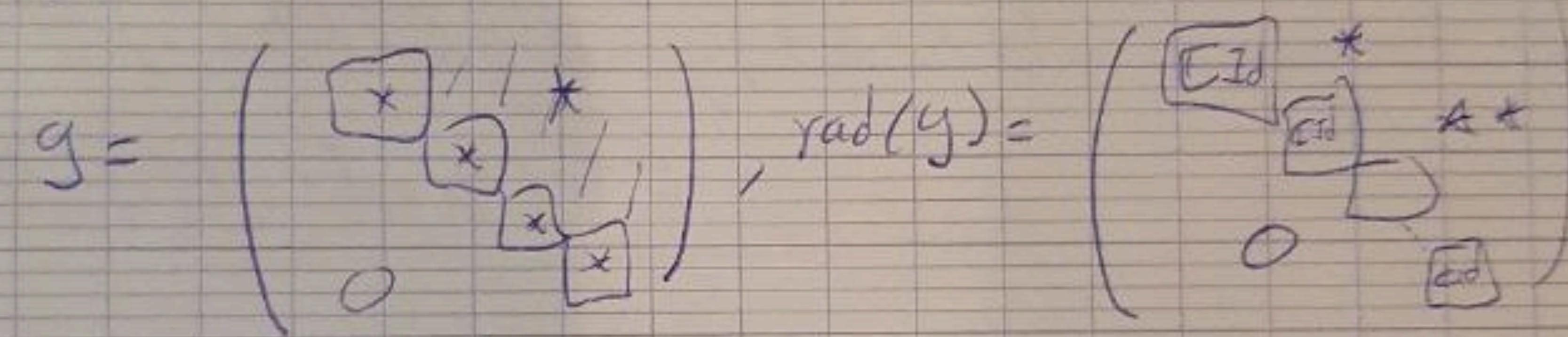
$\rightsquigarrow$ $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ ← semisimple, $\mathfrak{g}/S(\mathfrak{g})$ ← reductive

$$\underbrace{\underbrace{\mathfrak{g} \text{ --- }}_{\text{semisimple}}\overbrace{\mathrm{rad}(\mathfrak{g}) \text{ --- } S(\mathfrak{g})}^{}}_{} \text{ --- } \{0\}$$

$\mathfrak{g}$ — $\mathrm{rad}(\mathfrak{g})$ — $S(\mathfrak{g})$ — $\{0\}$: reductive = semisimple + center (over $\mathfrak{g}$ — $S(\mathfrak{g})$); semisimple (over $\mathfrak{g}$ — $\mathrm{rad}(\mathfrak{g})$); solvable (over $\mathrm{rad}(\mathfrak{g})$ — $\{0\}$)

- Most important part is semi-simple (for rep. theory)

Ex/



$$\mathfrak{g}/\mathrm{rad}(\mathfrak{g}) = \bigoplus_i \mathfrak{sl}(n_i) \quad \text{semisimple.}$$

Criteria of Lie algebra reductive / semisimple

Def: A bilinear form $(\ ,\ ): \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is invariant if $([x,y],z) = (x,[y,z])$.

Notice $(\ ,\ ) \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ so $\mathfrak{g}$ acts on $\mathfrak{g}^*$, $\mathfrak{g}^* \otimes \mathfrak{g}^*$

explicitly: $(x \cdot_{\mathfrak{g}^*} \lambda)(y) := -\lambda([x,y])$ ← ($g\lambda(x) = \lambda(g^{-1}x)$ in groups)

$$x \cdot ((\ ,\ ))(y,z) = -\big(([x,y],z) + (y,[x,z])\big)$$

$\rightsquigarrow$ $\mathfrak{g}$ acts inv. on $\mathfrak{g}^* \otimes \mathfrak{g}^*$ if $x \cdot (\ ,\ ) = 0 \ \forall x \in \mathfrak{g}$

we write $(\ ,\ ) \in (\mathfrak{g}^* \otimes \mathfrak{g}^*)^{\mathfrak{g}}$ ← invariant under $\mathfrak{g}$.

Some steps of proof:

Def: A derivation of a Lie alg $\mathfrak{g}$ is an endomorphism $\partial \in \mathrm{End}(\mathfrak{g})$ s.t.

$$\partial([x,y]) = [\partial x, y] + [x, \partial y]$$

Ex/ $\partial = \mathrm{ad}(z)$ for some z.

Question: $\mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathfrak{g}) := \{\text{derivations of } \mathfrak{g}\}$ No! but in semi simple yes. ($\underset{\uparrow \text{semisimple}}{\mathfrak{g}} \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathfrak{g})$ is surjective)

Lemma: Let $u \in \mathrm{Der}(\mathfrak{g})$ then u_{ss}, u_n are also in $\mathrm{Der}(\mathfrak{g})$.

Proof: Since $u_n = u - u_{ss}$, it's enough to show u_{ss} is a derivation.

Decompose $\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_\lambda$, $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid (u - \lambda \mathrm{Id})^N(x) = 0 \text{ for } N \gg 0\}$

On $\mathfrak{g}_\lambda$, u_{ss} acts as $\lambda \mathrm{Id}$.

Claim: $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\mu+\lambda}$.

Proof: $x \in \mathfrak{g}_\lambda$, $y \in \mathfrak{g}_\mu$

$$(u - (\lambda+\mu)\mathrm{Id})([x,y]) = [(u - \lambda \mathrm{Id})x, y] + [x, (u - \mu \mathrm{Id})y]$$

$$(u - (\lambda+\mu)\mathrm{Id})^N([x,y]) = \sum_{k=0}^{N} \binom{N}{k} \left([(u-\lambda \mathrm{Id})^k x, (u - \mu \mathrm{Id})^{N-k} y] \right)$$

For $N \gg 0$, any term

$$[(u - \lambda \mathrm{Id})^k x, (u - \mu \mathrm{Id})^{N-k} y] = 0$$

$\boxed{}$

$$\forall \lambda, \mu \quad u_{ss}([\mathfrak{g}_\lambda, \mathfrak{g}_\mu]) = (\lambda+\mu)[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = [u_{ss}(\mathfrak{g}_\lambda), \mathfrak{g}_\mu] + [\mathfrak{g}_\lambda, u_{ss}(\mathfrak{g}_\mu)]$$

$\Rightarrow u_{ss}$ is a derivation $\boxed{}$

Ex/ Let $\mathfrak{g}$ be solvable. $\forall$ $\mathfrak{g}$-rep V, show $D_\mathfrak{g} \subset \ker(,)_V$

$\boxed{\text{use Lie's theorem}}$ $\rightsquigarrow$ upper triangular matrix (all) $\to$ multiply both $\rightsquigarrow$ upper triangular (upper & strictly upper; $(\mathfrak{g})$, $(D_\mathfrak{g})$) $\rightsquigarrow$ traceless

$D_\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$

~~and~~ i.e. $D_\mathfrak{g} \neq 0 \Rightarrow (,)_V$ is degenerate.

Thm (Cartan) Let $\mathfrak{g}$ be f.d. Lie alg and let $K_\mathfrak{g}$ it's Killing form.

1) $\mathfrak{g}$ nilpotent $\Rightarrow K_\mathfrak{g} = 0$
2) $\mathfrak{g}$ is solvable iff $K_\mathfrak{g}(\mathfrak{g}, D_\mathfrak{g}) = 0$
3) $\mathfrak{g}$ semisimple* iff $K_\mathfrak{g}$ is non-degenerate
4) $\mathfrak{g}$ reductive $\Leftrightarrow \exists V$ s.t. $(,)_V$ is non-degenerate

semisimple* = $\mathrm{rad}\, \mathfrak{g} = \{0\}$

5 Weyl's semisimplicity of thm

Thm. A f.d. Lie alg $\mathfrak{g}/_\mathbb{C}$ is a direct sum of simple Lie alg iff $\mathrm{rad}(\mathfrak{g}) = \{0\}$

Proof: "$\Rightarrow$" $\mathfrak{g} = \bigoplus_i \mathfrak{a}_i$, $\mathfrak{a}_i$ simple $\Rightarrow \mathrm{rad}(\mathfrak{g}) = \bigoplus_i \mathrm{rad}(\mathfrak{a}_i) = \{0\}$

"$\Leftarrow$" Assume $\mathrm{rad}(\mathfrak{g}) = \{0\}$. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ we will show $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ (as Lie alg.)
w.r.t Killing form $K_\mathfrak{g}$.

Lemma: Let $\mathfrak{g}$ be any Lie alg, $(,)$ an inv. form.

1) If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal then so is $\mathfrak{a}^\perp = \{x \in \mathfrak{g} \mid (x, \mathfrak{a}) = 0\}$
2) If $(,)$ is non-degenerate and $\mathfrak{b}$ is an isotropic ideal (i.e $(\mathfrak{b}, \mathfrak{b}) = 0$) then $\mathfrak{b}$ is abelian.

Proof: Let $x, y \in \mathfrak{g}$, $x \in \mathfrak{a}^\perp$

$$([x, y], \mathfrak{a}) = \underbrace{(x, [y, \mathfrak{a}])}_{= \{0\}} \subset (x, \mathfrak{a})$$

b) By assumption, $\mathfrak{b} \subset \mathfrak{b}^\perp$ and by 1), $\mathfrak{b}^\perp$ is ~~abelian~~ is an ideal.

$$([\mathfrak{b}, \mathfrak{b}], y) = (\mathfrak{b}, [\mathfrak{b}, y]) \subseteq (\mathfrak{b}, \mathfrak{b}) = \{0\}$$

true for any $y \rightsquigarrow$ since non-degenerate, $[\mathfrak{b}, \mathfrak{b}] = \{0\}$.

End of proof: Let $\mathfrak{a}$ be an ideal. Since $rad(\mathfrak{g}) = \{0\}$ $K_\mathfrak{g}$ is non-deg. by Cartan's criterion. We have

- $\mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal • $(\mathfrak{a} \cap \mathfrak{a}^\perp, \mathfrak{a} \cap \mathfrak{a}^\perp) = \{0\}$

$\overset{\text{Lemma}}{\Longrightarrow}$ $\mathfrak{a} \cap \mathfrak{a}^\perp$ is $\underbrace{\text{abelian}}_{\text{abelian} \Rightarrow \text{solvable}}$ ideal $\Rightarrow \mathfrak{a} \cap \mathfrak{a}^\perp \subset rad(\mathfrak{g}) = \{0\}$

$\Rightarrow \mathfrak{a} \cap \mathfrak{a}^\perp = \{0\}$ $\Rightarrow [\mathfrak{a}, \mathfrak{a}^\perp] = \{0\}$ as $[\mathfrak{a}, \mathfrak{a}^\perp] \subset \mathfrak{a} \cap \mathfrak{a}^\perp$

We deduce $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ (direct sum of Lie alg.)

We have $\begin{cases} rad(\mathfrak{a}) \subset rad(\mathfrak{g}) = \{0\} \\ rad(\mathfrak{a}^\perp) \subset \quad " \quad\quad " \end{cases}$

We continue by induction on $\dim \mathfrak{g} \rightsquigarrow \mathfrak{g}$ ~~[illegible]~~ is sum of simple

Cor: The adjoint rep. of a semi-simple Lie alg. is semisimple.

Thm: (Weyl's semisimple) Any finite dim rep of a semisimple Lie alg is semisimple.

Secret reason: Any semisimple Lie alg. is the complexification of the Lie alg. of a compact Lie group
semisimple reps

Exercises: Let $\mathfrak{g}_1, \mathfrak{g}_2$ semisimple Lie alg., $\mathfrak{g}$ Lie alg. s.t.

$$\begin{cases} \mathfrak{g}_1 \subset \mathfrak{g} \text{ is an ideal} \\ \mathfrak{g}/\mathfrak{g}_1 \simeq \mathfrak{g}_2 \end{cases} \Rightarrow \mathfrak{g} \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad \text{as Lie alg.}$$

2) If V_1, V_2 are simple reps of a semisimple Lie alg.

Then
$$\underset{\substack{\| \\ \{f : V_1 \to V_2 \mid x \cdot f(v_1) = f(x \cdot v_1) \ \forall v_1 \in V_1, x \in \mathfrak{g}\}}}{\mathrm{Hom}_{\mathfrak{g}}(V_1, V_2)} = \begin{cases} \mathbb{C} & \text{if } V_1 \simeq V_2 \\ 0 & \text{if } V_1 \not\simeq V_2 \end{cases}$$

(Schur's lemma)

6. JORDAN DECOMP IN Semi-simple Lie alg (or reductive)

<u>Thm</u> (Jordan - Kronecker)

We begin with $\mathfrak{gl}(V)$, $k = \mathbb{C}$. There exists a <u>unique</u> pair $(X_{ss}, X_n) \in \mathfrak{gl}(V)^2$ s.t. $X = X_{ss} + X_n$

1) X_{ss} diagonalizable
2) X_n is nilpotent
3) $[X_n, X_{ss}] = 0$

(s.t. $X_n, X_{ss} \in \mathbb{C}[X]$)

What about Lie alg? What does <u>an element</u> to be semi-simple ss or nilpotent mean?

Notice that $\forall \mathfrak{g}$ f.d. Lie alg$/\mathbb{C}$, $X \in \mathfrak{g}$

$$\underset{\|}{(\mathrm{ad}(X))_{ss}} + \underset{\|}{(\mathrm{ad}(X))_n} = \mathrm{ad}(X) \quad \exists!\ X_{ss}, X_n \in \mathfrak{g}$$

s.t. $\mathrm{ad}(X_{ss}) + \mathrm{ad}(X_n) = \mathrm{ad}(X)$?

<u>Thm</u> (Jordan decomp) Let $\mathfrak{g}$ be semisimple. $\forall X \in \mathfrak{g}$

$\exists!\ X_{ss}, X_n \in \mathfrak{g}$ s.t.
$$\begin{cases} \mathrm{ad}(X_{ss}) = \mathrm{ad}(X)_{ss} & \text{as an element of } \mathfrak{gl}(\mathfrak{g}) \\ \mathrm{ad}(X_n) = \mathrm{ad}(X)_n \\ X_{ss} + X_n = X \\ [X_{ss}, X_n] = 0 \end{cases}$$

Exercise: 1) $\Rightarrow$ 2) & 3) (ad is faithful)

Moreover, $\forall$ f.d. rep (ρ, V)

$$\begin{cases} \rho(X)_{ss} = \rho(X_{ss}) \\ \rho(X)_n = \rho(X_n) \end{cases}$$

How do we construct (symmetric) inv. bilinear forms?

If (ρ, V) is a $\mathfrak{g}$-rep, $(x,y)_V := \mathrm{Tr}|_V(\rho(x)\rho(y))$

Ex/ $\mathfrak{g} = \mathfrak{gl}_n$, $V = \mathbb{C}^n$.

$$(E_{ij}, E_{kl})_{\mathbb{C}^n} = \mathrm{Tr}(\delta_{jk} E_{il}) = \delta_{jk}\delta_{il}$$

- non-degenerate. - it pairs non-trivially elements sym along diagonal.

$$\begin{pmatrix} \cdot & \cdot & \cdot\, E_{23} \\ \cdot & \cdot & \cdot \\ \cdot\, E_{32} & & \cdot \\ \cdot & \cdot & \cdot \; \cdot \end{pmatrix}$$

Ex/ $(\,,\,)_V$ is inv.

$$([x,y],z)_V = \mathrm{Tr}(\rho([x,y])\rho(z))$$
$$\overset{\text{cyclicity}}{=} \mathrm{Tr}(\rho(x)[\rho(y)\rho(z) - \rho(z)\rho(y)])$$
$$= (x,[y,z])_V$$

Taking $V = \mathfrak{g}$ (adjoint rep),

$$\underset{\text{"Killing" form}}{K_{\mathfrak{g}}} := \mathrm{Tr}_{\mathfrak{g}}(ad(x)\, ad(y))$$

Ex/ $\mathfrak{g} = \mathfrak{sl}_2$, $ad(e) = \begin{array}{c|c} & \begin{matrix} e & h & f \end{matrix} \\ \begin{matrix} e \\ h \\ f \end{matrix} & \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$, $ad(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

$ad(f) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

$K_{\mathfrak{g}}(e,e) = K_{\mathfrak{g}}(f,f) = 0$, $K_{\mathfrak{g}}(h,h) = 8$, $K_{\mathfrak{g}}(e,f) = 4$
$= K_{\mathfrak{g}}(e,h) = K_{\mathfrak{g}}(f,h)$

K is nondegenerate as "e is upper triangular components whilst f is lower triangular"
is semisimple not solvable.

$$\begin{array}{c|c} & \begin{matrix} e & f & h \end{matrix} \\ \begin{matrix} e \\ f \\ h \end{matrix} & \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \end{array}$$

Def: $\mathfrak{g}$ semisimple Lie algebra: $\begin{cases} \cdot\ \mathfrak{g}^{ss} = \{x \in \mathfrak{g} \mid x = x_{ss}\} & \text{"Semisimple"} \\ \cdot\ \mathcal{N} = \{x \in \mathfrak{g} \mid x = x_n\} & \text{"Nilpotent"} \end{cases}$

Typically $\mathfrak{g} = \mathfrak{sl}_2$. $h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{g}^{ss}$. e, f nilpotent.

Facts: 1) $\mathfrak{g}^{ss}$ is a dense subset of $\mathfrak{g}$

2) $\mathcal{N} \subset \mathfrak{g}$ is closed, conical

$\exists$ open $\mathfrak{g}^{reg,ss} \subset \mathfrak{g}^{ss}$

open & dense in $\mathfrak{g}$

Ex/ $\mathfrak{sl}_n$, $\mathfrak{g}^{reg\,ss} = \left\{ x \in \mathfrak{sl}_n \mid \begin{matrix}\text{all eigenvalues} \\ \text{disjoint}\end{matrix} \right\}$ ~~Eagle~~

Geometric picture

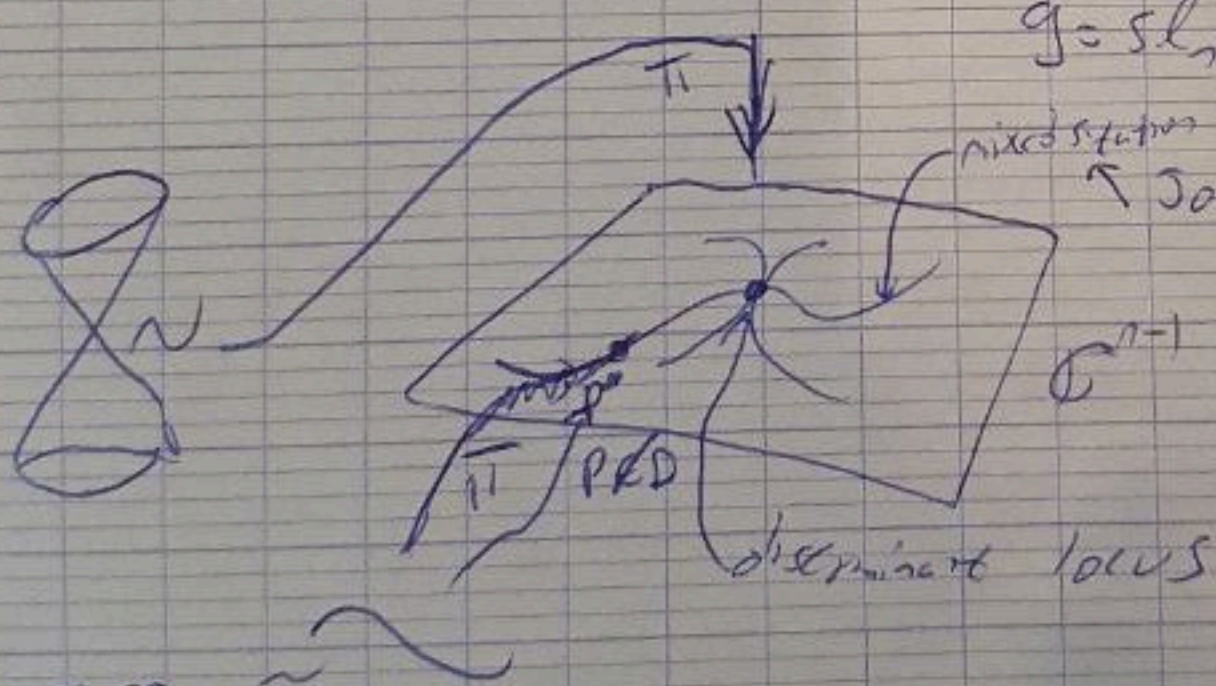
$$\pi : \mathfrak{sl}_n \longrightarrow \mathbb{C}^{n-1} \subseteq \mathbb{C}[\lambda]_n$$

$$x \longmapsto P_x = \det(x - \lambda \mathrm{Id}) \leq (-1)^n (\lambda^n + \overset{0}{c_{n-1}\lambda^{n-1}} + c_{n-2}\lambda^{n-2} + \dots + c_0) \qquad \underbrace{}_{n-1 \text{ coeff}}$$

$$\mathcal{N} = \pi^{-1}(\{0\}), \quad \mathfrak{g}^{reg,ss} = \pi^{-1}\Big(\mathbb{C}^{n-1} \setminus \underbrace{\text{discriminant locus}}\Big)$$

$\mathfrak{g} = \mathfrak{sl}_n$ polynomial multiple roots

P_x with



$\simeq$ smooth

SL_n / T $\cong$

$\nearrow$ any two elements are conjugates.

$\{SL_n \text{orbit}\}$ ~~...~~ i.e in this point I have only one orbit.

$\cong$

SL_n / stab

Ex/

1 Introduction

2 Structure of semisimple Lie Algebras ($k = \mathbb{C}$)

2.1 sl_2

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

With

$$\begin{cases} [h,e] = 2e \\ [h,f] = -2f \\ [e,f] = h \end{cases}$$

h is semisimple element. $\mathrm{ad}(h)$ is diagonalizable in $End\mathfrak{sl}_2$

$$\begin{array}{cccccc} \mathfrak{sl}_2 \equiv & E_{-2} & \oplus & E_0 & \oplus & E_2 \\ & \| & & \| & & \| \\ & \mathbb{C}f & & \mathbb{C}h & & \mathbb{C}e \end{array}$$

$$\begin{array}{ccccccc} & \mathbb{C}f & & \mathbb{C}h & & \mathbb{C}e & \\ \rule{4em}{0.4pt} & * & \rule{4em}{0.4pt} & * & \rule{4em}{0.4pt} & * & \rule{4em}{0.4pt} \\ & -2 & & 0 & & 2 & \end{array}$$

2.2 $\mathfrak{sl}_n$

First idea: Pick a semisimple element and decompose $\mathfrak{sl}_n$ accordingly. Better ideas is to pick a maximal subspace in $\mathfrak{sl}_n$ of commuting, semisimple element. One example is

$$\mathfrak{h} = \{ \begin{pmatrix} * & & .. \\ & * & \\ ... & & * \end{pmatrix} \}$$

We can decompose $\mathfrak{sl}_n$ into simultaneous $\mathrm{ad}(h)$ eigenspaces for all $h \in \mathfrak{h}$.

$$\mathfrak{g} = \mathfrak{sl}_n = \bigoplus_{\lambda \in \mathfrak{h}^*} g_\lambda$$

where $g_\lambda = \{x \in g | \forall h \in \mathfrak{h},\ [h,x] = \lambda(h)x\}$

$$\mathfrak{h} = \{diag(\varepsilon_1, ..., \varepsilon_n) | \sum \varepsilon_i = 0\}$$

$$\mathfrak{h}^* = \bigoplus \mathbb{C}\epsilon_i / \left(\sum \epsilon_i\right)$$

where $\epsilon_i(\varepsilon_j) = \delta_{ij}$. Set of $\lambda \in \mathfrak{h}^*$ s.t. $g_\lambda \neq \{0\}$ and g_λ.

$$\left[\sum \varepsilon_i E_i i, \sum a_{ijE_{ij}}\right] = \sum_{ij}(\varepsilon_i - \varepsilon_j)a_{ij}E_{ij} =_? \lambda(\varepsilon_i, \varepsilon_j)\sum_{ij} a_{ij}E_{ij}$$

i.e. $\varepsilon_i - \varepsilon_j = \lambda(\varepsilon_i, \dots, \varepsilon_n)$ s.t. $a_{ij} \neq 0$.

hence, either $a_{ij} = 0$ for all but one pair (i,j) with $i \neq j$ or $a_{ij} = 0$ for all $i \neq j$. This implies

$$g_0 = \mathfrak{h}$$

and

$$g_{\epsilon_i - \epsilon_j} = \mathbb{C}E_{ij}, \quad i \neq j$$

We have a nice piramid "Skeleton of $\mathfrak{sl}_n$"

$$\begin{array}{r}
\epsilon_1 - \epsilon_n \\
\vdots \\
\epsilon_1 - \epsilon_3\, \epsilon_2 - \epsilon_4\, \dots \epsilon_{n-2} - \epsilon_n \\
\epsilon_1 - \epsilon_2\, \epsilon_2 - \epsilon_3\, \dots\dots \epsilon_{n-1} - \epsilon_n \\
\hline \\
\epsilon_2 - \epsilon_1\, \epsilon_3 - \epsilon_2\, \dots\dots \epsilon_n - \epsilon_{n-1} \\
\vdots \\
\vdots \\
\epsilon_n - \epsilon_1
\end{array}$$

What happens when we consider $[g_\lambda, g_\mu]$? Let $x_\lambda \in g_\lambda$, $x_\mu \in g_\mu$, $h \in \mathfrak{h}$.

$$[h, [x_\lambda, x_\mu]] = [[h, x_\lambda], x_\mu] + [x_\lambda, [h, x_\mu]] = [\lambda(h)x_\lambda, x_\mu] + [x_\lambda, \mu(h)x_\mu] = (\lambda + \mu)(h)[x_\lambda, x_\mu]$$

i.e. $[g_\lambda, g_\mu] \subset g_{\lambda+\mu}$. Since $\dim g_\sigma = 1$ if $\sigma \neq 0$ then if $\lambda, \mu, \lambda + \mu \neq 0$ this describes completely $[g_\lambda, g_\mu]$ up to scalar.

Notation:

$$\Delta := \{\lambda \in \mathfrak{h}^* | g_\lambda \neq \{0\}\} \backslash \{0\}$$

is called the root system of g with respect to $\mathfrak{h}$ and an element in Δ is a root. In the piramid, $\Delta^+ = -\Delta^-$ being upper and lower triangles in the graph. Indeed,

Choosing

$$\Delta^+ := \{\epsilon_i - \epsilon_j | i < j\}$$

$$\Delta^- := \{\epsilon_i - \epsilon_j | i > j\}$$

we get $\Delta^+ = -\Delta^-$ and $(\Delta^\pm + \Delta^\pm) \cap \Delta \subset \Delta^\pm$ which implies

$$\bigoplus_{\lambda \in \Delta^+} g_\lambda$$

is a Lie subalgebra. but one would also get some splitting by choosing

$$\Delta^+ := \{\epsilon_i - \epsilon_j | \sigma(i) < \sigma(j)\}$$

, $\Delta^+ = -\Delta^-$ for any fixed $\sigma \in S_n$

Example 2.1. *$n = 3$ and $\sigma = (12)(3)$. So $\Delta^+ = \{\epsilon_2 - \epsilon_1, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3\}$*

$$\begin{pmatrix} 0 & 0 & * \\ * & 0 & * \\ & & 0 \end{pmatrix}$$

example: first row $*$ is $\epsilon_1 - \epsilon_3$.

2.3 Better picture of the root system

Recall that $\mathfrak{sl}_n$ has a Killing form $\mathcal{K}_g$ which is invariant and nondegenerate.

Lemma 2.2. *$\mathcal{K}_g(g_\lambda, g_\mu) = 0$ if $\lambda + \mu \neq 0$.*

Proof. Pick $x_\lambda \in g_\lambda$, $x_\mu \in g_\mu$, $h \in \mathfrak{h}$.

$$\mathcal{K}_g([h, x_\lambda], x_\mu) + \mathcal{K}_g(x_\lambda, \underbrace{[h, x_\mu]}_{\mu(h)x_\mu}) = 0 = (\lambda + \mu)(h)\mathcal{K}_g(x_\lambda, x_\mu)$$

□

Corollary 2.3. *1. $\mathcal{K}_g|_{\mathfrak{h}}$ is nondegenerate.*

2. $g_\lambda^* = g_{-\lambda}$ and in particular they are one dim.

We can use the isomorphisme $\mathfrak{h} \sim \mathfrak{h}^*$ coming from 1. to transport $\mathcal{K}_g$ to nondegenerate form on h^*.

Digression 2.4. *V a vector space. $(,) : V \otimes V \to \mathbb{C}$ non degenerate. $\forall v \in V$, define $\lambda_v : u \mapsto (v, u)$ so the map $v \mapsto \lambda_v$*

$$V \to V^*$$

is an isomorphism.

This implies $g_\lambda \subset g \sim g^* - - - - >> g^*_{-\lambda}$

Lemma 2.5.

- *$span_{\mathbb{C}} = \mathfrak{h}^*$*
- *$span_{\mathbb{R}} = \mathfrak{h}^*_{\mathbb{R}}$ is a real form of h^* i.e. $\mathfrak{h}^*_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = h^*$*
- *$(h^*_{\mathbb{R}}, (,))$ is a euclidean vector space.*

Example 2.6. *Up to normalization*

$$\mathcal{K}_g(E_{ij}, E_{ji}) = 1$$

$$\mathcal{K}_g(E_{ii}, E_{jj}) = \delta_{ij}$$

all others are 0. $\mathfrak{h}^* = \bigoplus_{i=1}^{n-1} \mathbb{C}\alpha_i$ *and* $\alpha_i := \epsilon_i - \epsilon_{i+1}$, $\mathfrak{h}^*_{\mathbb{R}} = \bigoplus_{i=1}^{n-1} \mathbb{R}\alpha_i$ *so*

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\alpha_i, \alpha_j) = \begin{cases} 0 & |i-j| > 1 \\ -1 & |i-j| = 2 \\ 2 & i = j \end{cases}$$

Matrix of $(,)$ *in basis* $\{\alpha_i\}$ *is*

$$\begin{pmatrix} 2 & -1 & .. \\ -1 & ... & 0 \\ ... & & -1 \\ ... & -1 & 0 \\ & & 2 \end{pmatrix}$$

This is positive definite. Check.

Example 2.7. *For* $\mathfrak{sl}_3$ $\mathfrak{h}^* = \mathbb{C}\alpha_1 \oplus \mathbb{C}\alpha_2$ *so* $(\alpha_i, \alpha_i) = 2$, $|\alpha_1| = \sqrt{2} = |\alpha_2|$, $(\alpha_1, \alpha_2) = -1$ *so* $\cos(angle(\alpha_1, \alpha_2)) = -1/2$ *so* $angle(\alpha_1, \alpha_2) = 2\pi/3$.

$$\Delta = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$$

$$\begin{tikzcd}
 & \alpha_2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \arrow[rr] \arrow[dl] & & \alpha_2 + \alpha_1 \sim \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \arrow[dr] & \\
-\alpha_1 \sim \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & & h & & \alpha_1 \sim \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 & -\alpha_1 - \alpha_2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \arrow[rr] \arrow[ul] & & -\alpha_2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix} \arrow[ur] &
\end{tikzcd}$$

Possible choices for Δ^+*:*

$$\begin{cases} \sigma = (1)(2)(3) & \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \\ \sigma = (13)(2) & -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2 \\ \sigma = (12)(3) & -\alpha_1, \alpha_1 + \alpha_2, \alpha_2 \\ \sigma = (23)(1) & -\alpha_1, , \alpha_1 + \alpha_2, \alpha_2 \\ \vdots \end{cases}$$

$\alpha_1 = \epsilon_1 - \epsilon_2 \in \mathfrak{h}^*$, $g_{\alpha_1} = E_{12}$,

$$[diag(\varepsilon_1, \varepsilon_2, \varepsilon_3), E_{12}] = (\varepsilon_1 - \varepsilon_2)E_{12}$$

Group of symmetries of the root system is D_3 *which is isomorphic to* S_3*.*

We may use the same picture to understand finite dimensional representations. Let V be a finite dimensional $\mathfrak{sl}_n$-representation. $\mathfrak{h}$ acts by diagonalizable operators commuting between them. Therefore,

$$V = \bigoplus_{\lambda \in h^*} V_\lambda$$

where $V_\lambda = \{v \in V | \forall v \in \mathfrak{h},\, h \cdot v = \lambda(h)v\}$ $wt(V) := \{\lambda \in \mathfrak{h}^* | V_\lambda \neq \{0\}\}$.

Lemma 2.8. $g_\lambda \cdot V_\mu \subset V_{\lambda+\mu}$

$g \otimes V \to V$ then $g_\lambda \otimes V_\mu \to V_{\lambda+\mu}$

Proof.

$$\begin{aligned} h \cdot (x_\lambda v_\mu) &= [h, x_\lambda] \cdot v_\mu + x_\lambda \cdot (h \cdot v_\mu) \\ &= \lambda(h) x_\lambda \cdot v_\mu + \mu(h) x_\lambda \cdot v_\mu \end{aligned}$$

□

Example 2.9. $\mathfrak{sl}_3$*.* $V = \mathbb{C}$ *with usual action.* $wt(V) = \{\epsilon_1, \epsilon_2, -\epsilon_1 - \epsilon_2\}$ *Indeed,* $h = diag \epsilon_1, \epsilon_2, \epsilon_3$*,* $v^T = (v_1, v_2, v_3)$*. We want to solve* $h \cdot v = \lambda(h)v$

$$\begin{cases} \lambda v_{\epsilon_1} = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \\ \lambda v_{\epsilon_2} = \begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix} \\ \lambda v_{-\epsilon_1 - \epsilon_2} = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix} \end{cases}$$

On the picture,
$\epsilon_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$, $\epsilon_2 = -\frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$ *and* $\epsilon_3 = -\frac{2}{3}\alpha_1 - \frac{1}{3}\alpha_2$.

Weights form a triangle pointing down inside the hexagon. This shows you the action of the rep.

Finally, the dual $wt(V^*) = -wt(V)$, *the weights form a pointing up triangle.*

2.4 General Case (statements and definitions)

Let g be a semisimple Lie Algebra over $\mathbb{C}$.

Definition 2.10. *A Cartan subalgebra is a Lie subalgebra* $\mathfrak{h} \subset g$ *s.t.*

1. $\mathfrak{h}$ *is abelian*
2. *any* $h \in \mathfrak{h}$ *is semisimple*
3. $\mathfrak{h}$ *is maximal with respect to 1. and 2.*

Theorem 2.11. *Chavalley Any two Cartan subalgebras are* $Aut(g)-$*conjugate.*

Definition 2.12. $rank(g) = \dim \mathfrak{h}$ *(for any Cartan subalgebra* $\mathfrak{h}$*)*

Example 2.13. $rank(\mathfrak{sl}_n) = n - 1$

How two construct Cartan subalgebras?

Definition 2.14. *An element* $x \in g$ *is regular if* $g^x := \{y \in g | [x, y] = 0\}$ *is of minimal dimension among all* x.

Fact: The function $x \mapsto \dim g^x$ is upper semicontinous. i.e. $\forall n$, $f^{-1}([0, n])$ is open. In particular, $g^{reg} = \{x | x \text{ is reg}\}$ is open in g.

Example 2.15. $g = \mathfrak{sl}_n$

$$x = \begin{pmatrix} Id_n 1 a_1 & & .. \\ & Id_2 a_2 & \\ ... & & Id_n a_n \end{pmatrix}$$

$a_i \neq a_j$ *for* $i \neq j$. *Then*

$$g^x = \begin{pmatrix} \boxed{*} & & .. \\ & \boxed{*} & \\ ... & & \boxed{*} \end{pmatrix}$$

$\dim g^x$ *is minimal if* $n_1 = ... = n_k = 1$. *When it happens, it is equal to* $n - 1$.

Fact: For $\mathfrak{sl}_n$, $n - 1$ is the minimal dimension for g^x.

$$\{diag(a_1, ..., a_n) | a_i \neq a_j \text{ for } i \neq j\} \subset g^{reg}$$

Jordan block are in g^{reg}. Exercise: x regular $\iff$ $\exists$ at most one Jordan block for every eigenvalue λ.

Proposition 2.16. *$g^{reg} \cap g^{ss}$ is open and dense in g. Exercise: For $\mathfrak{sl}_n$, $g^{reg} \cap g^{ss}$ is the set of matrices with characteristic polynomial with simple roots.*

Theorem 2.17.

1. *$\forall x \in g^{reg} \cap g^{ss}$, g^x is cartan subalgebra*
2. *Any cartan subalgebra is of this form.*

Example 2.18. *$\mathfrak{sl}_n$, let $x = diag(a_1, ..., a_n)$ $a_i \neq a_j$. then*

$$g^x = \{ \begin{pmatrix} * & & .. \\ & * & \\ ... & & * \end{pmatrix} \}$$

2.4.1 Root Space Decomposition

Fix a Cartan subalgebra $\mathfrak{h}$. For $\alpha \in h^*$,

$$g_\alpha := \{x \in g | \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$$

$$g = \bigoplus_{\lambda \in \mathfrak{h}^*} g_\lambda$$

Definition 2.19.

$$\Delta := \{\alpha \in \mathfrak{h}^* \{0\} | g_\alpha \neq 0\}$$

is the root system of g with respect to $\mathfrak{h}$.

Theorem 2.20.

1. *$g_0 = \mathfrak{h}$ ($\implies \dim g_0 = rh(g)$)*
2. *$\forall \alpha \in \Delta, \dim g_\alpha = 1$*
3. *$[g_\lambda, g_\mu] \subset g_{\lambda+\mu}$ $\forall \lambda, \mu$.*

2.4.2 Metric structure on $\mathfrak{h}^*$

Exercise:$span_{\mathbb{C}} = \mathfrak{h}^*$ (hint: $Z(g) = \{0\}$).
Let $\mathcal{K}_g$ be the Killing form on g.

Lemma 2.21.

1. *$\mathcal{K}_g(g_\lambda, g_\mu) = 0$ if $\lambda + \mu \neq 0$*
2. *$\mathcal{K}_g|_{\mathfrak{h}}$ is non-degenerate.*

Using $\mathfrak{h} \tilde{\to} \mathfrak{h}^*$ we can transport $\mathcal{K}_g$ to a non-degenerate form on $\mathfrak{h}^*$ Notation: $t_\alpha \in \mathfrak{h}$ corresponds to $\alpha \in \mathfrak{h}^*$ $\mathcal{K}_g(t_\alpha, h) = \alpha(h)$ $\forall h \in \mathfrak{h}$.

Lemma 2.22.

$$[g_\alpha, g_{-\alpha}] = \mathbb{C} t_\alpha$$

and $\mathcal{K}_g(t_\alpha, t_\alpha) \neq 0$

Proposition 2.23. *$\mathfrak{h}^*_{\mathbb{R}} := span_{\mathbb{R}}$ is a Euclidean space and a real form of $\mathfrak{h}^*$.*

Example 2.24. *There exists a Lie Algebras called g_2 of rank 2 whose root system is:*

$$
\begin{tikzcd}
 & & & 2\alpha_2+3\alpha_1 & & & \\
\alpha_2 \arrow[urrr] \arrow[rr] \arrow[ddd] & & \alpha_2+\alpha_1 \arrow[rr] \arrow[dl] & & \alpha_2+2\alpha_1 \arrow[rr] \arrow[dr] & & \alpha_2+3\alpha_1 \arrow[ulll] \arrow[dd, dash] \\
 & -\alpha_1 & & 2 \arrow[ulll] \arrow[rr] & & \alpha_1 & \\
 & & & & & & \bullet \arrow[d] \\
\bullet \arrow[drrr] & & \bullet \arrow[uul] \arrow[rr] & & \bullet \arrow[uur] & & \bullet \\
 & & & \bullet \arrow[urrr] & & &
\end{tikzcd}
$$

$\dim g_2 = 14$

Test