... ~~orig~~ at arbitrary point X.

... back to conformal algebra

e.g. $[D, \phi_\alpha(x)] = [D, U\phi_\alpha(0)U^{-1}]$

$$= U[D, \phi_\alpha(0)U^{-1}] + [D, U]\phi_\alpha U^{-1}$$

$$= U\underbrace{[D, \phi_\alpha]}_{i\Delta_\phi \phi_\alpha(0)}U^{-1} + U\phi_\alpha[D, U^{-1}] + [D, U]\phi_\alpha U^{-1}$$

$U \sim e^{ix\cdot P} \Rightarrow [D, U] = x^\nu P_\nu U.$

$$[D, \phi_\alpha(x)] = i(\Delta_\phi + x^\nu \partial_\nu)\phi_\alpha(x)$$

$$[J_{\mu\nu}, \phi_\alpha(x)] = -i(S_{\mu\nu})_\alpha{}^\beta \phi_\beta(x) + i(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi_\alpha$$

$$[K_\mu, \phi_\alpha(x)] = -i(2x_\mu x\cdot\partial - x^2\partial_\mu)\phi_\alpha(x) - 2ix_\mu \Delta_\phi \phi_\alpha(x)$$
$$-2ix^\sigma(S_{\sigma\mu})_\alpha{}^\beta\phi_\beta(x)$$

Similarly we can get for descendants.

ϕ_α is a primary that transforms on $\{\Delta_\phi, (S_{\mu\nu})_\alpha{}^\beta\}$

could be spin $0, \frac{1}{2} \ldots$

Encodes the commutation relations $[K, P] = (\ldots D \ldots)$

Finite dilatations

$x \to \underbrace{(1+\alpha)}_{\lambda}x$ $\quad U_D(\alpha)$ satisfies $U_D(\alpha_1)U_D(\alpha_2) = U_D(\alpha_1+\alpha_2)$

$\hookrightarrow U(\lambda) = [U(\tfrac{\lambda}{n})]^n = (1 - \tfrac{i\lambda D}{n})^n \Rightarrow U_D(\lambda) = e^{-i\lambda D}$

Consider ϕ scalar, $\phi'(x) = U_D\phi(x)U_D^{-1} = \phi(x) - i\alpha[D, \phi(x)] + O(\alpha^2)$

$$= \phi(x) + \alpha\Delta_\phi \phi(x) + \alpha x^\mu\partial_\mu\phi$$

$$= (1+\alpha)^{\Delta_\phi}\phi(x+\alpha x)$$

this corresponds to $\phi'(x) = \lambda^{\Delta_\phi} \phi(\underbrace{\lambda x}_{x'})$

$\phi'(x') = \lambda^{-\Delta_\phi} \phi(x)$ ← scale dimension

Ex/ $S = \int d^d x \, (\partial_\mu \phi(x) \partial^\mu \phi(x)) \xrightarrow{D} \int d^d x' (\partial'_\mu \phi'(x') \partial'^\mu \phi'(x'))$

$= \int d^d x \, \lambda^{d-2\Delta_\phi-2} \partial_\mu \phi \partial^\mu \phi$

Invariant if $\boxed{\Delta = \frac{d-2}{2}}$

Classically conformal if

Quantum is much more difficult.

More generally $x \to x'$ $\quad J^\alpha{}_\mu = \frac{\partial x'^\alpha}{\partial x^\mu}$

$\phi'^{\alpha_1 \dots \alpha_n}(x') = (\det J)^{-\Delta/d} J^{\alpha_1}{}_{\mu_1} \cdots J^{\alpha_n}{}_{\mu_n} \phi^{\mu_1 \dots \mu_n}(x)$

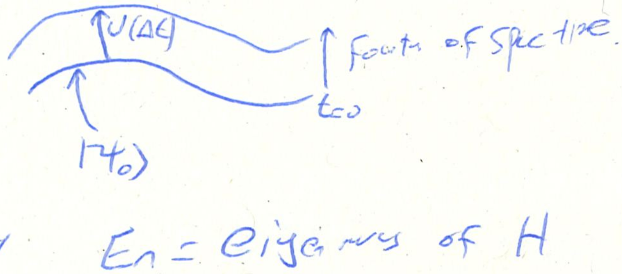
with $\det J = \lambda^d$ ← dilatation.

$\det J = (1 - 2x\cdot b + b^2 x^2)^{-d}$ ← SCT.

using this one can build classically conformally invariant fields.

<u>Radial quantization</u>

Recall



$U(t) = e^{-iHt} = e^{H\tau}$

Usually E_n = eigenvalues of H $\quad |E_n\rangle \xrightarrow{U(t)} e^{iE_n t} |E_n\rangle$

for CFT we label by states of eig(D).

Eigen states of D:

$$|\psi\rangle := \mathcal{O}(0)|0\rangle$$

$\uparrow$ primary $\qquad \uparrow$ vac

$$D|\psi\rangle = D\,\mathcal{O}(0)|0\rangle = [D, \mathcal{O}(0)]|0\rangle = i\Delta|\psi\rangle$$

operator $\longleftrightarrow$ state

$|\psi\rangle$ is eigenstate of D. $\quad U_p = e^{-ix^\mu P_\mu}$ on $|\psi\rangle \rightsquigarrow$

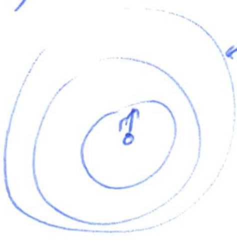
$$|\tilde{\psi}\rangle = U_p|\psi\rangle = \sum_{n=0}^{\infty} \frac{(-ix\cdot P)^n}{n!}|\psi\rangle$$ generates all posible descendants of $|\psi\rangle$.

So $|\tilde{\psi}\rangle$ is not an eigenstate of D.

Ex/ work in $\mathbb{R}^d$ $\quad ds^2 = dx_i dx^i = dr^2 + r^2 d\Omega_{d-1}^2$

$\uparrow$ foliate by spheres.

$\sim S^{d-1}$



we want to have a copy of $\mathcal{H}$ on each S^{d-1} and evolve $S^{d-1} \longrightarrow S^{d-1}$ with Dilations.

$$D = -i x^i \frac{\partial}{\partial x^i} \quad \text{in } dx_i dx_i \left(-i x^\mu \frac{\partial}{\partial x^\mu}\right) \neq dx_i dx_i$$

$$x^i \frac{\partial}{\partial x^i} = r\frac{\partial}{\partial r} = \frac{\partial}{\partial(\log r)}$$, generally $D = -ir\frac{\partial}{\partial r}$ moves

$$S^{d-1}_r \to S^{d-1}_{r+dr} \quad \text{so} \quad U_D = e^{-iD\log r}$$

$$\left\{ \begin{array}{l} \text{Consider } -i\frac{\partial}{\partial t} \longleftrightarrow e^{H t_E} \\ -i\frac{\partial}{\partial \log r} \longleftrightarrow e^{-iD\log r} \end{array} \right\}$$

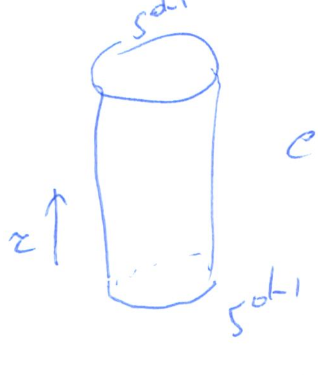
Euclidean $\mathbb{R}^d$ $\longleftrightarrow$ radial "time" evolution $e^{-iD\log r}$

$$U_D = \lim_{n\to\infty}\left(1 - i\frac{dD}{n}\right)^n$$

Plane → cylinder

Identify evolution $\tau = \log r$

$$ds^2 = dr^2 + r^2 d\Omega_{d-1}^2 = e^{2\tau}\underbrace{\left(d\tau^2 + d\Omega_{d-1}^2\right)}_{\text{Cylinder } \mathbb{R}\times S^{d-1}}$$



evolution in τ moving up & down in cylinder.

if $\tau \to -\infty$: $r \to 0$

$\tau \to \infty$: $r \to \infty$

Operators at $r=0$ are infinite ∞ past and $r\to\infty$ is future.

$\boxed{H = -iD}$ ← convention. this has real eigenvalues.

→ well defined CFT and bounded from below.

Unitarity bounds

"Reflection positivity"

· In Lorentzian $\phi_L(t,x) = e^{-iP_0 t - iP_j x^j}\phi_L(0)\, e^{iP_0 t + iP_j x^j}$

$$\left[\phi_L(t,x)\right]^\dagger = e^{-iP_0 t \dots}\phi_L(0)^\dagger e^{iP_0 t \dots} \rightsquigarrow \phi_L(x) \text{ hermitian iff } \phi_L(0).$$

In euclidean $t = -i\tau$

$$\phi(\tau, x) = \phi_L(-i\tau, x)$$

$$\hookrightarrow \left[\phi(\tau,x)\right]^\dagger = \phi(-\tau, x)$$

⇒ Euclidean reflection sends $\tau \to -\tau$ what does it mean in cylinder example?

$\tau \to -\tau \Rightarrow \ln r \to -\ln r \Rightarrow r \to \frac{1}{r} \Rightarrow x^i \to \frac{x^i}{x^2}$

Conjugation (*) is an inversion in radial quantity.

E.g. $P_\mu = -i\frac{\partial}{\partial x^\mu}$, $x'^\mu = \frac{x^\mu}{x^2}$

$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu}$ so $\partial_{\mu'} = x^2\partial_\mu - 2x_\mu(x\cdot\partial)$

so $P_\mu \to P_{\mu'} = -i(x^2\partial_\mu - 2x_\mu(x\cdot\partial)) = K_\mu$

So Conjugation maps $P_\mu \to K_\mu$

$\Rightarrow \boxed{P_\mu^\dagger = K_\mu \;\&\; K_\mu^\dagger = P_\mu}$

In the Cylinder we can easily quantize the theory (radially) and apply conjugation and we see what it means on the plane.
• positivity and so on.

we can show also.

$$\boxed{\begin{array}{l} J_{\mu\nu}^\dagger = J_{\mu\nu} \\ D^\dagger = -D \end{array}}$$

<u>Bounds</u> - look at a scalar primary $|\Delta\rangle$,

• $\langle\Delta|\Delta\rangle > 0$ positivity

• $|\psi\rangle_\mu = P_\mu|\Delta\rangle$

$\||\psi_\mu\rangle\|^2 = \langle\psi_\mu|^\dagger|\psi_\mu\rangle = \langle\Delta|\overset{K_\mu}{\overset{\|}{P_\mu^\dagger}} P_\nu|\Delta\rangle = \langle\Delta|[K_\mu, P_\nu]|\Delta\rangle$

$[K_\mu, P_\nu] = -2i\eta_{\mu\nu}D + J_{\mu\nu}$ $\swarrow$ annihilates Δ.

$= \langle\Delta|-2iD|\Delta\rangle = 2\Delta\,\||\Delta\rangle\|^2 \geq 0$ so $\Delta \geq 0$ by unitarity "reflection positivity"

Actually a stronger bound exists.

$|\tilde{\psi}\rangle = P_\mu P^\mu|\Delta\rangle \rightsquigarrow 0 \leq \langle\Delta|K_\mu K^\mu P_\nu P^\nu|\Delta\rangle$

so $\boxed{\Delta \geq \frac{d-2}{2}}$ unitarity bound for scalar op..

$\Delta \geq \frac{d-2}{2}$. is the optimal bound

- $\Rightarrow$ free scalar, saturates unitarity bound.

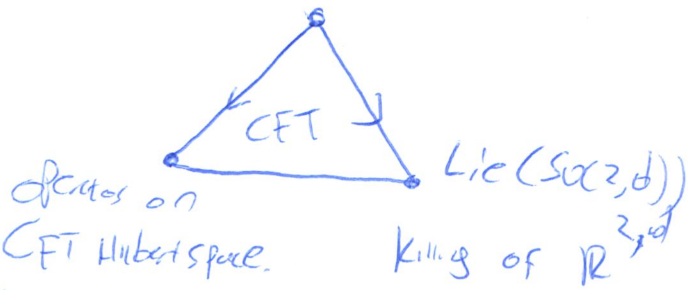
- $\Delta \geq l + d - 2$ non-scalar unitarity bound.

($l=0$ does not recover scalar)
Conserved spin l currents saturate this.

Geometrical representation:

- Recall we started with d-vectors ← conformal Killing vectors generating the ~~isometries~~ symmetry.

we saw how the Killing algebra acts on tensors.
and give precisely $Lie(SO(2,d))$



$\mathbb{R}^{2,d}$ is maximally symmetric?

$\nabla_{(A}\xi_{B)} = 0$

$N^0 \overset{d+2 \text{ is dim.}}{=}$ Number of Killing Vector fields is $\frac{1}{2}(d+2)(d+3)$ while $\dim(SO(2,d)) = \frac{1}{2}(d+2)(d+1)$

then $N^0 - \dim SO(2,d) = d+2$ ← which ones are missing?

- Translations (generated by vectors of $2+d$ components).

so $\underbrace{SO(2,d)}_{CFT_d} \subset \underbrace{ISO(2,d)}_{\text{"Lorentz"} \ltimes \text{translations}}$

Note however $N^0(KVF) \overset{d+1 \text{ dim}}{=} \frac{1}{2}(d+1)(d+2) = \dim(SO(2,d))$
so there is a manifold in $d+1$ dim which realizes $SO(2,d)$ as isometries (and no others)!

Construct $d+1$ dim sub manifold of $\mathbb{R}^{2,d}$

s.t. $SO(2,d)$ is preserved but $P(2,d)$ are broken
translations

$X_A X^A = -L^2$ in $\mathbb{R}^{2,d}$ embedd.

$$ds^2 = \eta_{AB}\, dX^A dX^B = -dX_0^2 + \sum^{d} dX_i^2 - dX_{d+1}^2$$

<u>Coordinates AdS_{d+1}</u>

Let $X_0 = L \dfrac{\cos t}{\cos\rho}$, $X_i = L \tan\rho \, \hat{\Omega}_i$

$X_{d+1} = \dfrac{L \sin(t)}{\cos(\rho)}$

$\{t, \rho, \hat{\Omega}_i\}$ with $\sum_{i=1}^{d} \Omega_i \Omega_i = 1$

$d-1$ angles

Induced metric on AdS_{d+1} restrict $\eta_{AB}\, dX^A dX^B$ to $X_A X^A = -L^2$

e.g. $dX_0 = L\left(-\dfrac{\sin t}{\cos\rho}\, dt + \dfrac{\cos t}{\cos^2\rho} \sin\rho \, d\rho\right)$

$$\vdots$$

$$ds^2 = \frac{L^2}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho \, d\hat{\Omega}_i d\hat{\Omega}_i\right)$$

global coords on AdS_{d+1}

<u>Remark.</u> L_{AB} act as isometries on

$\mathbb{R}^{2,d}$: $L_{AB} = X_A \partial_B - X_B \partial_A$

Use our embedding to work out action on AdS_{d+1}.

$\Rightarrow$ all KVFs of AdS_{d+1} = conformal algebra!

e.g. $\boxed{\text{time translation KVF of } AdS_{d+1} \equiv D \text{ in } CFT_d}$

<u>Conformal compactification of AdS_{d+1}</u>

useful rep. of a spacetime that brings "∞" to finite. and preserves causal structure.

Penrose diagram of flat space

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2$$

$$\left.\begin{array}{c} -\infty < t < \infty \\ 0 < r < \infty \end{array}\right\} \quad \begin{array}{c} 2\tan U = t + r \\ 2\tan V = t - r \end{array} \qquad -\frac{\pi}{2} \le U, V \le \frac{\pi}{2}$$

$$ds^2 = -\frac{4\,dU\,dV}{\cos^2 U \cos^2 V} + (\tan U - \tan V)^2 \, d\Omega_{d-2}^2$$

$\leftarrow$ poles at $U = \frac{\pi}{2}$, $V = \frac{\pi}{2}$

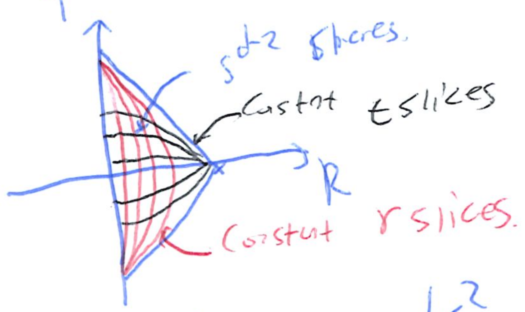
Since we are interested in the causal structure $\rightarrow$ use weyl trnsf to get rid of poles.

$$d\tilde{s}^2 = -4\,dU\,dV + \sin^2(\underbrace{U - V}_{R})\, d\Omega_{d-2}^2$$

not entirely equivalent

$$T = U + V$$

$$d\tilde{s}^2 = -dT^2 + dR^2 + \sin^2 R \, d\Omega_{d-2}^2$$

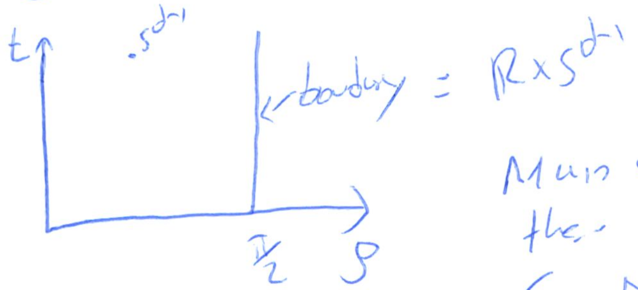


On to AdS_{d+1}: $ds^2 = \frac{L^2}{\cos^2(\rho)} \Big[\underbrace{-dt^2 + d\rho^2 + \sin^2\rho \, d\Omega_{d-1}^2}_{\text{Einstein Static}} \Big]$

already ρ compact.

So just do the weyl rescaling:

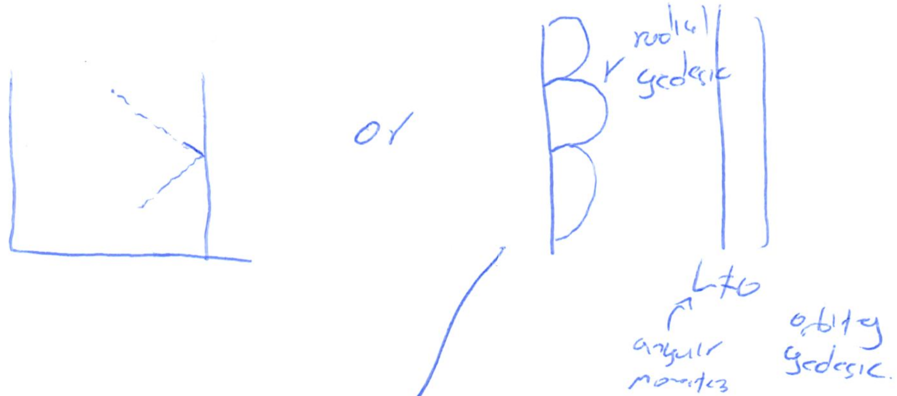
$$d\tilde{s}^2 = -dt^2 + d\rho^2 + \sin^2\rho \, d\Omega_{d-1}^2$$



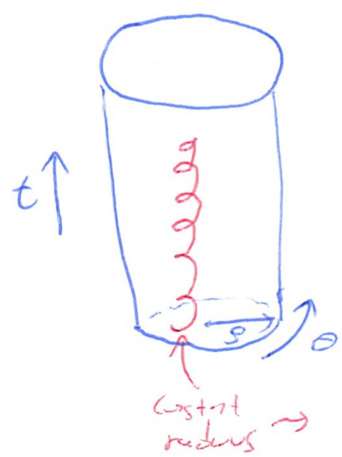
Main difference with Minkowski: the boundary is timelike | (in Minkowski is light like $\diagdown$)

Null rays reach the boundary in finite time!

$\Rightarrow$ AdS_{d+1} <u>not globally hyperbolic.</u>



More generically:



thin Can-picture of AdS_{d+1}

$\rho = $ const. $\leftarrow$ evolves on t but ρ constant.

Boundary of AdS_{d+1} $ds^2 \to \infty$ at $\rho \to \frac{\pi}{2}$

so let $\bar{g} = (\rho - \frac{\pi}{2})^2 g_{AdS}$ then $g_{bound}(t, \Omega_i)$

cancels the pole $\frac{1}{\cos^2(\rho)}$

$= \lim_{\rho \to \frac{\pi}{2}} \bar{g}$ $\leftarrow$ Not unique as we could have put (up to weyl)

$\bar{g}' = f(t, \Omega_i)\,(\rho - \frac{\pi}{2})^2 g_{AdS}$

weyl rescaling.

this gives the $\boxed{\text{"Conformal class"}}$

g'_{bound} & g_{bound} are two representatives in this class.

$g_{bound} = \mathbb{R} \times S^{d-1} = \mathbb{R}^2 \times \mathbb{R}^d$

AdS does not have a metric on the boundary but a "Conformal Class". (good as it allows to study in different foliations)

Relating representations:

building a duality between AdS_{d+1} & CFT_d

$\boxed{A}$ Isometries of AdS → Conf. Isometries of ∂AdS

$\boxed{B}$ QFT in AdS_{d+1}:

$$\underbrace{\hat{\phi}_{AdS}(\rho, x)}_{\text{bulk field}} \longrightarrow \underbrace{\hat{O}(x)}_{\text{bound. CFT operator}}$$

Bulk Isom → Boundary CT.

global: $\{t, \rho, \Theta, \dots\}$

disappear in boundary

$$D = -\frac{\partial}{\partial t}$$

$$P_i = e^{-it}\hat{\Omega}_i\left(-\sin\rho\frac{\partial}{\partial t} + i\cos\rho\frac{\partial}{\partial\rho}\right) + \hat{\Theta}_i$$

$$K_i = P_i^{\dagger}$$

Claim: $\lim_{\rho\to\frac{\pi}{2}} \{D, P_i, K_i, J\}^{AdS}_{iso} = \{D, P_i, K_i, J\}^{\text{Conf. Isom}}$ of $\mathbb{R}^d = \partial AdS_d$

lets just do D:

$$D = -ix^i\frac{\partial}{\partial x^i} = -i\frac{\partial}{\partial\tau} = -\frac{\partial}{\partial t} = \lim_{\rho\to\frac{\pi}{2}} D^{(AdS)}$$

CFT operators $\longleftrightarrow$ AdS fields

Disclaimer: we work partly around the vacuum.

More generally: have qualitatively different challenges.

(how to describe interior of BH using boundary CFT?)

<u>General Idea:</u> $\Phi(\rho, t, \theta, \dots)$ carries a rep of $SO(2,d)$ with dim Δ. $\quad C_2 \Phi = \Delta(\Delta - d)\Phi$

Recall: $C_2 = -D^2 - \frac{1}{2}(K\cdot P + P\cdot K) + \frac{1}{4}J^2$

write in bulk language using $\{D_{\rho t}, K_\mu, P_\mu, J_{\mu\nu}\}$ in the AdS rep.

$$C_2 \Phi = L^2 \Box_{AdS} \Phi = \Delta(\Delta - d)\Phi$$

2nd order diff op. d'Alembert op on AdS.

Φ is a field living on AdS_{d+1} with mass $L^2 m^2 = \Delta(\Delta - d)$

$$\begin{pmatrix}\text{Scalar operator} \\ \mathcal{O} \text{ of dim } \Delta\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\text{bulk field of} \\ \text{mass } \frac{\Delta(\Delta-d)}{L^2}\end{pmatrix}$$

$$\Box_{AdS} = \frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}\, g^{ab}\partial_b\right) \text{ acting on scalar.}$$

Casimir lead to the emergence of field dynamics

we started in d dim conformal algebra leads to a natural object (field) in $d+1$ dim.

- It allows mass negative by Δ bounds. (unitarity)

Conformal family: $|\psi_0\rangle = (|\Delta\rangle)$

$K_\mu |\psi_0\rangle = 0$, the descendants $P_{\mu\cdots}|\psi_0\rangle$ are

$$|\psi_{n\ell j}\rangle = (P_\mu P^\mu)^n \underbrace{P_{\mu_1} \cdots P_{\mu_\ell}}_{\text{traceless symm } SO(d) \text{ irreps.}} |\psi_0\rangle$$

j: spin labels

n: principal quantum number

Ex/ 3 Dimension $|\psi_{n\ell j}\rangle \to |\psi_{n\ell m}\rangle$

ℓ is the principal $SO(3)$ quantum #

$-\ell \leq m \leq \ell$ is spin proj.

Determine Φ. primary Φ_0 satisfies.

1) $K_\mu \Phi_0 = 0$, 2) $D\Phi_0 = i\Delta\Phi_0$, 3) $J_{\mu\nu}\Phi_0 = 0$

primary   Dimension Δ scalar

[Analogy with EM, you satisfy Maxwell wave equation but still need to satisfy linear constraints (Polarization and so on)]

3) $\Rightarrow$ $\Phi_0 = S(\rho, t)$ $\leftarrow$ no θ dependence.

2) $\Rightarrow$ $D\Phi_0 = i\Delta\Phi_0 \rightsquigarrow -i\frac{\partial}{\partial t} S(\rho,t) = i\Delta S(\rho, t)$

$\hookrightarrow$ $S = e^{-i\Delta t} R(\rho)$

1) $\Rightarrow$ $K_\mu \Phi_0 = 0 \Rightarrow R(\rho) = \cos^\Delta(\rho)$

$$\Phi_0(t, \rho, \theta, \ldots) = e^{-i\Delta t} \cos^\Delta(\rho)\, N_{000}$$

you can build $\Phi_{n\ell j}$ by act with various P_μ ....

$$\Phi_{n\ell j} = N_{n\ell j}\, e^{-iE_{n\ell j}t}\, Y_{\ell j}(\theta,\cdots)\, \sin^{\ell}(\rho)\cos^{\Delta}(\rho)\, {}_2F_1\left[-n, \Delta+n+\ell, \ell+\tfrac{d}{2}, \sin^2\rho\right]$$

spheral harmonics

where $E_{n\ell j} = \Delta + 2n + \ell$

we used $(P_\mu P^\mu)^n P_{\mu_1 \cdots \mu_\ell}$ to rise.

this is bulk manifistation of our conformal Family.

build a Hilbert space.

$$\Phi(\rho, t, \cdots) = \sum_{n,\ell,j} \left(\Phi_{n\ell j}\, a_{n\ell j} + \Phi^*_{n\ell j}\, a^\dagger_{n\ell j} \right)$$

modes.

[Fierol with modes out of quantum fields in the bulk.
This is analoguos as in Poincare.
we solved casimir equation there $\rightsquigarrow$ K-G equation.
then constans give spin $(0, \frac{1}{2}, 1, \cdots)$
then we built creation and anihilation op (and polarizations)
Here the same <u>but</u> they give rise to field
operators in the d+1 dimensional space.]

$$[a_{n\ell j}, a^\dagger_{n'\ell' j'}] = \delta_{nn'}\delta_{\ell\ell'}\delta_{jj'}$$

$\hookrightarrow$ Fock space. $\left\{ (a^\dagger_{n,\ell j})^{\cdots} |0\rangle \right\}$ $\rightsquigarrow$ canonical quantization.

of free scalar in AdS.

$$S = \int_{AdS} d^{d+1} \sqrt{-g} \left(\frac{1}{2} (\nabla\Phi)^2 - \frac{m^2}{2} \Phi^2 \right)$$

$$m^2 L^2 = \Delta(\Delta - d) \quad , \quad H = \sum (\Delta + 2n + l)\, a^{\dagger}_{nlj} a_{nlj}$$

free multiparticle QFT in AdS_{d+1}

"Generalize free field theory"

what happen with diffrent primaries?

we saw scalar, what about others?

How to go for fields on the bulk and operators on boundary

Operator $\longleftrightarrow$ bulk field.

<u>Def</u>

$$\mathcal{O}(t, \theta, \cdots) = \lim_{\varepsilon \to 0} \frac{\Phi(\overset{\text{radial}}{\rho}, t, \theta, \cdots)}{\underset{\text{regulator.}}{\varepsilon^{\Delta}}} \Bigg|_{\rho = \frac{\pi}{2} - \varepsilon e^{-it}}$$

we do this as we have not a metric but conformal class.

two representatives are related by a weyl transf.

then $\rho - \frac{\pi}{2} = -\varepsilon \underline{e^{-it}} \Rightarrow$ boundary is $\mathbb{R}^d \cdot \mathbb{R}$

$f(t)$ is weyl factor $\uparrow$ $(i e^{-it})^{2} \ldots = \ldots \mathbb{R}^2$

we could have used another frame

transforms of $\mathcal{O}(x)$

we want to see how conformal symmetries act on operators $\mathcal{O}$.

under D: $\delta\Phi = [D,\Phi] = -\partial_t \Phi$ $\quad \lim \phi_I = \lim_{\varepsilon\to 0} \left.\frac{\phi_I}{\varepsilon^\Delta}\right|_{\rho=\frac{\pi}{2}-\varepsilon e^{-it}}$

always means

$$\partial_t \mathcal{O} = \partial_t \sum_{\substack{I \\ \text{index}}} \left(a_I \lim \Phi_I + h.c. \right)$$

$$\lim \Phi_I = N_I \lim \Big(e^{-iE_{n\ell j} t} Y_{\ell j}(\dots) \sin^\ell(\rho) \cos^\Delta(\rho) \times$$

$$\times {}_2F_1[-n, E_{n\ell j}, \ell+\tfrac{d}{2}, \sin^2\rho] \Big)$$

$\sin^\ell(\rho)\to 1 \quad \cos^\Delta(\rho) \to \rho^\Delta \to \varepsilon^\Delta e^{-i\Delta t}$

$$= N_I \lim \left(e^{-iE_{n\ell j}t} Y_{\ell j}(\dots) \varepsilon^\Delta e^{-i\Delta t} \times 1 \right)$$

taking t derivative:

$$\partial_t \lim \Phi_I = -i(E_{n\ell j}+\Delta) \lim \Phi_I$$

$$\therefore \partial_t \mathcal{O} = -i\Delta\mathcal{O} - i\sum_I \left(E_I \lim \Phi_I a_I + h.c. \right)$$

$$= -i\Delta\mathcal{O} + \lim \partial_t \Phi_I$$

$$\to \boxed{\lim \partial_t \Phi = (i\Delta + \partial_t)\mathcal{O}}$$

$\partial_t = x^\nu \partial_\nu$

$t = -i\tau \to \tau = \log r, \quad r\frac{\partial}{\partial r} = x^i \frac{\partial}{\partial x^i} = x^\mu \frac{\partial}{\partial x^\mu}$

so $\boxed{\delta\mathcal{O} = i(\Delta + x^\mu \partial_\mu)\mathcal{O} = [D^{(\mathbb{R}^d)}, \mathcal{O}]}$

Caction of ~~conformal~~ dilatation on operator $\mathcal{O}$.

So from Φ we found how D acts! (if we started with $\mathcal{O}$ then consistency check).

$\mathcal{O}$ transforms as a primary of a CFT_d under D.

- Can show correct properties under all other generators of $SO(2,d)$

$$\Phi_{bulk} \longleftrightarrow \mathcal{O}_{bound}$$

↳ radial quantization!

Revisit operator/state map.

$$\lim \Phi_{n\ell j} = \lim \{ N_{n\ell j}\, e^{-iE_{n\ell j} t}\, Y_{\ell j}(\theta \cdots) \cos^{\Delta}(\rho) \sin^{\ell}(\rho)\, {}_2F_1 [\cdots$$

$$= e^{-i(E_{n\ell j}+\Delta)t}\, G_{n\ell j}$$

isolate the phase.

← boundary limit.

analogously

$$\lim \Phi^*_{n\ell j} = N^*_{n\ell j}\, e^{i(E_{n\ell j}-\Delta)t}\, Y^*_{n\ell}(\cdot\cdot)$$

$$= e^{i(E_{n\ell j}-\Delta)t}\, G^*_{n\ell j}$$

we first take conjugate then limit so remains the same.

this allows some relations! we can write

$\mathcal{O}$ as

$$\mathcal{O} = \sum e^{-i(\underbrace{2\Delta + 2n+\ell}_{E_{n\ell j}+\Delta})t}\, G_{n\ell j}\, a_{n\ell j} + e^{i(2n+\ell)t}\, G^*_{n\ell j}\, a^{\dagger}_{n\ell j}$$

remaining phase. $e^{-i2\Delta}$

Recall $\tau = \log r = it$

$$= \sum \left(r^{-(2\Delta + 2n + \ell)} \sigma_{n\ell j}\, a_{n\ell j} + r^{2n+\ell} \sigma^*_{n\ell j}\, a^\dagger_{n\ell j} \right)$$

act on AdS_{d+1} vacuum $(a_{n\ell j}|0\rangle = 0)$

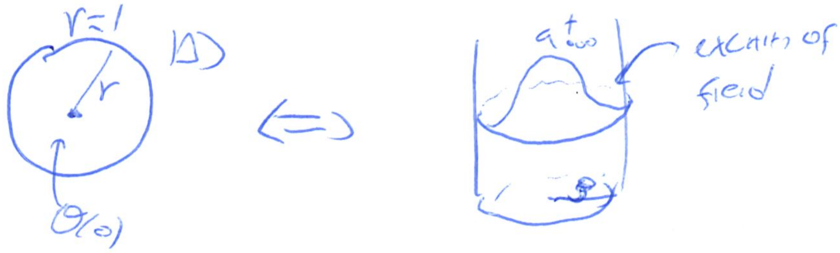
$$\mathcal{O}(r=0)|0\rangle = \sum_{n\ell j} r^{2n+\ell} \sigma^*_{n\ell j}\, a^\dagger_{n\ell j} |0\rangle \Big|_{r=0}$$

$\leftarrow$ only $n=0=\ell$ survives

$$= \sigma^*_{000}\, a^\dagger_{000} |0\rangle \leftarrow \text{only } 0 \text{ mode}$$

corresponds to the primary state in radial quantization.

$a^\dagger_{000}$ acting on AdS vacuum $\Longleftrightarrow$ primary state (in rad. quantization)



Correlation functions simplest case: $\langle \mathcal{O}(1)\, \mathcal{O}(0) \rangle$

$$r_i = \underbrace{e^\tau}_{r} \hat{x}_i = e^\tau \Omega_i$$

$$\boxed{\begin{aligned} \langle \mathcal{O}(r)\, \mathcal{O}(0) \rangle &= \lim \langle \Phi(\rho, r)\, \Phi(\rho, 0) \rangle \\ &= \langle a_{000}\, a^\dagger_{000} \rangle e^{-2i\Delta t} \underset{\text{set } \sigma\sigma^* = 1}{=} \frac{1}{r^{2\Delta}} = \frac{1}{|x-y|^{2\Delta}} \end{aligned}}$$ ✓

CFT_d 2pt func. for primary of dim Δ!

- can be extended to "multi-traces" $\mathcal{O}(0)^k \leftrightarrow (a^\dagger_{000})^k$
- descendents $\leftrightarrow a^\dagger_{n\ell j}$
- non-scalar primaries $J_\mu \leftrightarrow A_\mu$ (bulk gauge)

  $T_{\mu\nu} \leftrightarrow g_{\mu\nu}$ (bulk graviton)

Generalized free theories & large N

higher n-pt fns:

- 3pt function:

$$\langle \mathcal{O}(r_1)\mathcal{O}(r_2)\mathcal{O}(r_3) \sim \langle a_{000}\, a^\dagger_{000}\, a^\dagger_{000}\rangle + \dots$$
$$= 0.$$

- 4-pt functions

pairwise wick contractions.

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle\langle \mathcal{O}(x_3)\mathcal{O}(x_4)\rangle + \langle \mathcal{O}(x_1)\mathcal{O}(x_3)\rangle\langle \mathcal{O}(x_2)\mathcal{O}(x_4)\rangle + \dots$$

$$= \frac{1}{x_{12}^{2\Delta_1}}\frac{1}{x_{34}^{2\Delta_3}} + \text{permutations}$$

where $\langle \mathcal{O}_1(r)\,\mathcal{O}_2(0)\rangle = \begin{cases} 0 & \text{if } \Delta_1 \neq \Delta_2 \\ \frac{1}{r^{2\Delta}} & \text{if } \Delta_1 = \Delta_2 \end{cases}$

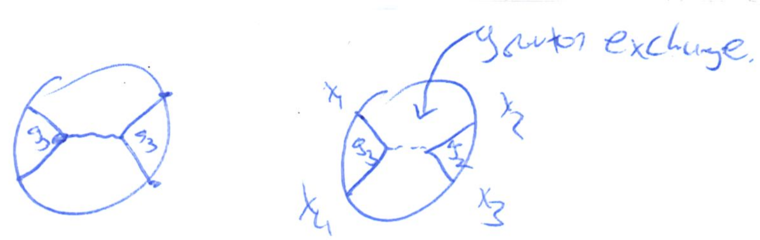
- odd n-point give 0
- even are wick factorized.

"Generalized free theories" (GFT)

- n-pt functions factorize.
- can have any conformal dim.

Not actually a free theory on CFT_d but on AdS_{d+1}

NB: for $g_3 \Phi^3$



Boundary interpretation:

adding interactions - gives non-trivial connected parts
- modifies anomalous dimensions
- shifts OPE coefficients λ_{ijk}

In a CFT you have $\{\{\mathcal{O}_\Delta, \Delta\}, \lambda_{ijk}\}$ ← rule

$$\mathcal{O}_i(x)\,\mathcal{O}_j(0) \sim \lambda_{ijk}\,\mathcal{O}_k$$ OPE coefficents.

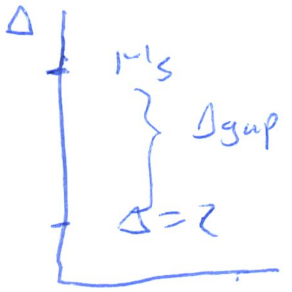
does any CFT have a nice gravity dual?

NO. we require: - all CFT n-point functions to be interpreted as coming from local bulk interactions

- there is a GFT limit (large N)
- at low-E, bulk reduces to AdS EFT:

$$S = \int h\Box^2 h + \frac{1}{M_p} h^2 \Box h + \frac{1}{M_s}\left(\partial^2 h\, \partial^2 h + \frac{1}{M_p} h \partial^2 h \partial^2 h\right)$$

...tis b/you have to have two parameters / where ...lled)



"Large-gap Conjecture:
all single trace primaries of $\Delta > 2$ have parametric large dim.

Ex/ $AdS_5 \times S^5 \longleftrightarrow N=4$ SYM

$$\frac{1}{\Delta_{gap}} \sim \frac{1}{\lambda_{t'hooft}} \to \infty$$ in semi classical gravity limit

~~the basic field is a matrix of O(N),~~

~~$O(x) \sim (\phi_a \phi^a)^k$ factorize.~~

- SYK$_{d+1}$ / tensor models $\rightsquigarrow$ AdS_2 gravity + "higher spins"

Questions:

what CFT with GFT limit have "reasonable" bulk gravity dual?

CFT is well defined in the UV. $\rightsquigarrow$ we define fully UV complete quantum gravity in AdS_{d+1} by CFT_d

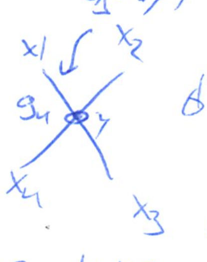
Bulk interactions:

$$S[\Phi] = S_{free} + \int d^{d+1}\sqrt{g}\left(\frac{g_4}{4!}\Phi^4\right)$$

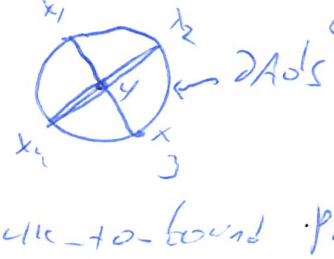
$g_4 \sim \frac{1}{N}$

bulk prop $(\Box_{AdS} - m^2)\, G_\Delta(x,y) = \delta_{AdS}(x-y)$

4-pt:

$$\langle \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle = \int$$  $$d^{d+1}y$$

$\downarrow$ push $(x_1,\dots,x_4)$ to ∂AdS. i.e. take "lim"

$$\langle O(x_1) \cdots O(x_4)\rangle =$$  $\longleftarrow$ "Witten diagram"

push $G_\Delta \rightsquigarrow G_\partial$ bulk-to-bound propagator.

what kind of theories behave like this?

Large N factorization gives GFT

Ex/ - matrix large N ($\mathcal{N}=4$ SYM $SU(N)$)

$$\{ \underset{\uparrow\,\text{gauge}}{A^{\mu}_{ab}},\ \underbrace{\overset{\text{fermion}}{\psi_{ab}},\ \overset{\text{boson}}{\Phi_{ab}}}_{\text{adjoint rep of } SU(N)} \}$$

operators $\mathcal{O} \sim tr(\Phi(x)^2) \leftarrow \mathcal{O}^K \sim \underbrace{tr(\ldots)\,tr\ldots(\ldots)}_{\text{multi traces}}$

- $\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle \sim \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \cdots \langle \mathcal{O}_n \rangle + \frac{1}{N^{\alpha}}$ connected

- $\Delta(tr(\Phi^2))$ pick up anomalous dimensions ($\exists$ non-trivial coupling "'t hooft" coupling $\lambda = g^2_{YM} N$)

duality $\mathcal{N}=4$ SYM $\Longleftrightarrow$ (local) theory of gravity on AdS_5

First duality.

- vector like large N ($O(N)$ vector model)

$\mathcal{O}(x) \sim (\phi_a \phi^a)^K$ factorize.

duality $O(N)$ 3D theory theory $\longleftrightarrow$ Vasiliev higher spins in AdS_4

non local theory