LECTURES ON QUANTUM FIELD THEORY II

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1. RENORMALISATION

1.A. Some Important Exact Reults

The central object which links perturbation theory with experimental measurements is the Scattering matrix, for short the S-matrix. This matrix is constructed from the N point Green's functions.

Let us consider a real scalar field $\varphi(x)$ with a polynomial interaction. The actions we shall consider will generically have the form,

$$I[\varphi] = \int d^n x \{ \frac{1}{2}\varphi(x)(-\partial^2 - m_0^2)\varphi(x) + \mathcal{L}_{\mathcal{I}}(\varphi)\} \tag{1.1}$$

In the initial part of our lectures the cases of interest for us will be,

$$\mathcal{L}_{\mathcal{I}}(\varphi) = -\frac{\lambda_0}{3!}\varphi^3 \ + g\varphi \tag{1.2}$$

$$\mathcal{L}_{\mathcal{I}}(\varphi) = -\frac{\lambda_0}{4!}\varphi^4 \tag{1.3}$$

The reason for attaching a subscript 0 on the mass parameter m_0^2 and couplings λ_0 will be discussed in detail later. For now it suffices to say that in an interacting quantum theory m_0^2 *will not be the physical mass* of the particle created by the field φ. We shall denote the quantum field operator corresponding to the classical field $\varphi(x)$ by $\hat{\varphi}(x)$. This is a Heisenberg picture field operator satisfying the non-linear field equation,

$$(-\partial^2 - m_0^2)\hat{\varphi}(x) = -\frac{\partial \mathcal{L}_{\mathcal{I}}(\hat{\varphi})}{\partial \hat{\varphi}(x)} \tag{1.4}$$

The single particle physical states in this model are characterised by their mass shell 4-momentum $k^\mu = (k^0, k^i)$ such that $k^0 = \omega(\mathbf{k}^2) = \sqrt{\mathbf{k}^2 + m^2}$. Here m^2 denotes the *physical* mass.

We would like to give a closed formula, known as the LSZ reduction formula, for the scattering matrix elements from an initial state $|\mathbf{k}_1, ...\mathbf{k}_r >$ to a final state of

$|\mathbf{k}'_1, ...\mathbf{k}'_s>$. As you have already seen in QFT1 this is given by,

$$S(\mathbf{k} \to \mathbf{k}') = \int \Pi_i d^4 y_i \int \Pi_j d^4 z_j$$
$$\Pi_i e^{-ik_i y_i} \frac{i}{Z^{\frac{1}{2}}} (\partial^2_{y_i} + m^2) \Pi_j [e^{ik'_j zj} \frac{i}{Z^{\frac{1}{2}}} (\partial^2_{z_j} + m^2) G_N(y_1, ...y_r, z_1, ..., z_s) \quad (1.5)$$

where,

$$G_N(y_1, ...y_r, z_1, ..., z_s) =< \Omega|T\hat{\varphi}(y_1)...\hat{\varphi}(y_r)\hat{\varphi}(z_1)...\hat{\varphi}(z_s))|\Omega > \quad (1.6)$$

where $|\Omega>$ denotes the exact ground state. The LSZ reduction formula, looks somewhat involved. In reality, however, it has a simple structure. It consists of r integrations over $y_i, i = 1, ...r$ and s integrations over $z_j, j = 1, ...s$. In the integrand we have the products of the single free particle wave functions $e^{-ik_i y_i}$ for the incoming particle with the 4-momentum k_i and $\frac{i}{Z^{\frac{1}{2}}}$ times the Klein-Gordon operator $(\partial^2_{y_i} + m^2)$ for the i th incoming particle. We have analogous factors for each one of the outgoing particles except that the sign of the momenta in the free particle wave functions are changed. Finally the most important non-kinematical element is the $N = r + s$ function $G_N(y_1, ...y_r, z_1, ..., z_s)$ which is given by the time ordered product of N Heisenberg field operators $\hat{\varphi}(x)$.

A considerable part of these lectures will be dedicated in learning how to handle these functions N-pint functions G_N. Here we list some facts about these functions without resorting to any approximate method of calculating them.

Firstly in deriving the LSZ formula it was assumed that,

$$< 0|\hat{\varphi}(x)|0 >= 0 \quad (1.7)$$

Indeed if $< 0|\hat{\varphi}(x)|0 >$ is different from zero by Poincar invariance of the theory it must be a constant c. We can then re-define the field by $\varphi'(x) = \varphi(x) - c$. Then

$\varphi'(x)$ will have zero v.e.v. The shifting of the field will introduce new terms in the Lagrangian which will be important as we shall see in the future lectures.

Secondly as you have seen in QFT1 the function $< 0|\hat{\varphi}(x)|\mathbf{k} >$ has a general structure of

$$< 0|\hat{\varphi}(x)|\mathbf{k} >= Z^{\frac{1}{2}} e^{ik.x} \tag{1.8}$$

where Z is the constant appearing in the LSZ formula. It is the value of this matrix element at the origin, $Z^{\frac{1}{2}} =< 0|\hat{\varphi}(0)|\mathbf{k} >$. For the free field operator we have $Z = 1$. In the interacting theory it will be a function of the coupling constants, masses etc. It can not be a function of $\mathbf{k}$ by Lorentz invariance.

Thirdly we also know important facts about the structure of the exact two point function, namely, the Källen-Lehmann spectral decomposition formula which you have learned in QFT1,

$$\tilde{G}_2(p) = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{M_0^2}^{\infty} dM^2 \mu(M^2) \frac{i}{p^2 - M^2 + i\epsilon} \tag{1.9}$$

where $\tilde{G}_2(p)$ is the Fourier transform of,

$$G_2(x, y) =< 0|T\hat{\varphi}(x)\hat{\varphi}(y))|0 >$$

This result shows that the exact 2-point function $\tilde{G}_2$ has a simple pole at $p^2 = m^2$, where m is the *physical* mass of the particle. Furthermore the residue at this pole is iZ. Note that to obtain these results no dynamics of the interacting field were used. They are obtained on the basis of relativistic invariance and definition of the free field propagator.

We can rewrite the expression 1.9 in a neater form by redefining the field and introducing a new field ϕ_R by relation:

$$\varphi(x) = Z^{\frac{1}{2}} \phi_R(x) \tag{1.10}$$

Therefore,

$$< 0|\hat{\phi}_R(x)|\mathbf{k} >= e^{ik.x} \tag{1.11}$$

In terms of this *renormalized* field $\hat{\phi}$ the exact two point function looks like,

$$\tilde{G}_{2R}(p) = \frac{i}{p^2 - m^2 + i\epsilon} + \int_{M_0^2}^{\infty} dM^2 \sigma(M^2) \frac{i}{p^2 - M^2 + i\epsilon} \tag{1.12}$$

where $\tilde{G}_{2R}(p)$ is the Fourier transform of ,

$$G_{2R} =< 0|T\hat{\phi}(x)\hat{\phi}(y))|0 >$$

Here the function $\sigma(M^2)$ is related to the function $\mu(M^2)$ in Eq.(1.9) by $\mu(M^2) = Z\sigma(M^2)$. Note that the part which exhibits the simple pole at the physical mass m^2 has precisely the form of a free particle propagator. The residue at the pole is i. This is an important fact which will be used later to determine the renormalisation constant Z in perturbation theory.

The fourth and final point which is important to know about the exact N-point function G_N is more conveniently expressed in terms of the Fourier transform of this function. Let us denote this transform by $\tilde{G}_N(k_i, ...k_N)$. The LSZ formula, upto constant factors is given by

$$(k_1^2 - m^2)...(k_N^2 - m^2)\tilde{G}_N(k_i, ...k_N) \tag{1.13}$$

Clearly in order to obtain a non zero result the function $\tilde{G}_N(k_i, ...k_N)$ should have precisely a simple pole whenever each one of its arguments goes to the mass shell. The simple pole of the 2-point function exhibited in Eq.(1.12) is a special case of the general result which is valid for any N- point function $\tilde{G}_N(k_i, ...k_N)$. The very specific pole structure when any of the k_i in $\tilde{G}_N(k_i, ...k_N)$ approaches the mass shell can be be proven by applying the methods and arguments analogous to the ones which was used to obtain the Kllen- Lehmann formula.

The LSZ formula can be generalised to include fields of spin 1/2 and 1. Since we are not going to use it we shall not give it here.

To sum up: in order to evaluate the S-Matrix elements for any process we will need to know the explicit form of the N-point functions. In general there is no way of knowing these functions exactly in an interacting theory. Hence we have to resort to approximative methods and perturbation theory. This will be our main concern. However, it is worthwhile to note that if we had a way of calculating the N point functions exactly then we would be able to obtain an exact result for the S matrix elements and there would be no need of perturbation theory, Feynman graphs etc.

Broadly speaking the aim of these lectures is to cover the following topics:

-Renormalisation

-Functional integral formulation

-Non abelian gauge groups

-Effective action

-Goldstone theorem

-Higgs mechanism

-Chiral anomalies

To achieve this aim we shall learn many topics and calculational techniques which are not listed above.

This course will not follow any single book although the material presented can be found in many books. At some stage I will give you a list of books which I have consulted over the years. Since the programme is quite intensive I advise you to first try to digest the material in the lecture notes, work out the exercises spread throughout the notes. Then you will be in a better position to benefit from the advanced books. The background needed to follow these lectures is minimal : Special Relativity, Quantum Mechanics and some basics of Lagrangian field theory (free field quantisation, QED action Feynman graph technics, etc.).

Exercises

Give the complete set of Feynman rules for the model given by the interaction Lagrangian Eq.(1.3) and draw all Feynman graphs contributing to the 2-point function up to the 2-loop order in perturbation theory.

1.B. Perturbation Theory

There is a closed formula which relates the the exact N- point functions $G(x_1, ...x_N)$ to the matrix elements of the free field operator $\hat{\phi}_0(x)$. This operator is the *interaction picture* field operator. The field $\hat{\phi}_0(x)$ satisfies the fee Klein-Gordon equation,

$$(-\partial^2 - m_0^2)\hat{\phi}_0(x) = 0 \tag{1.14}$$

This formula is given by,

$$G_N(x_1, ..., x_N) = \frac{< 0|T\hat{\phi}_0(x_1)...\hat{\phi}_0(x_N)e^{i\int dy\mathcal{L}_{\mathcal{I}}(\hat{\phi}_0(y))}|0 >}{< 0|Te^{i\int dy\mathcal{L}_{\mathcal{I}}(\hat{\phi}_0(y))}|0 >} \tag{1.15}$$

where $\mathcal{L}_{\mathcal{I}}(\hat{\phi}_0(y))$ stands for the interaction part of the Lagrangian. The r.h.s. of this expression can be calculated in perturbation theory by expanding the exponentials in a power series and applying Wick's theorem. As soon as we go beyond the leading order terms we shall encounter integrals which diverge. The first part of this course will be dedicated to understanding the N point functions and how to extract meaningful finite result from divergent expressions.

But first we shall start from simple things.

2. ULTRAVIOLET DIVERGENCES

Our aim is to calculate the r.h.s. of Eq. (1.15) in perturbation theory. The most efficient way to calculate the N point functions, at least in lower orders of perturbation theory, is to convert the right hand side of this equation into a sum of diagrams. The principle is simple. Expand $\hat{S} \equiv T e^{i \int dy \mathcal{L}_{\mathcal{I}}(\hat{\phi}_0(y))}$ in a power series of $\mathcal{L}_{\mathcal{I}}(\hat{\phi}_0(y))$ and apply Wick's theorem to convert the vacuum expectation values into a sum of products of Δ_F's and then integrate over all the internal points y coming from the expansion of $\mathcal{L}_{\mathcal{I}}(\hat{\phi}_0(y))$.

2.A. The Simplest Example

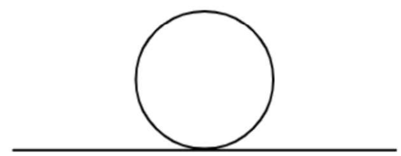

FIG. 1. 1-loop contribution to the 2-point function up to 1-loop including counterterms

Consider the 2-point function in the ϕ^4 model to the 1-loop order. It is easy to see that the only terms which contribute to the r.h.s of Eq.(1.15) are,

$$\begin{aligned} G_2(x_1, x_2) &= \Delta_F(x_1 - x_2) \\ &\quad + (\frac{-i\lambda_0}{2}) \int d^4y \Delta_F(x_1 - y) \Delta_F(y - y) \Delta_F(y - x_2) + ... \end{aligned} \tag{2.1}$$

where

$$\Delta_F(x_1 - x_2) = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} e^{-ik.(x_1 - x_2)} \tag{2.2}$$

is the Feynman propagator built for the free field operator $\hat{\phi}_0$. Going to Fourier space this expression takes the simple form of,

$$\begin{aligned} G(p) &= \Delta_F(p) \\ &\quad + (\frac{-i\lambda_0}{2}) \Delta_F(p) \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} \Delta_F(p) + ... \\ &= \Delta_F(p) + (\frac{-i\lambda_0}{2}) \Delta_F(p) I \Delta_F(p) + ... \end{aligned} \tag{2.3}$$

where,

$$\Delta_F(p) = \frac{i}{k^2 - m_0^2 + i\epsilon} \tag{2.4}$$

is the Fourier space Feynman propagator and I represents the integral:

$$I = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} \tag{2.5}$$

Recall that $k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$. Our first task is to evaluate this integral. As is seen with a little inspection I as it stands is a infinite. We would like to understand it and also a way of making sense out of our calculation notwithstanding this result. To this end it is convenient to rewrite the integral expression for I in a slightly different form. First we consider a contour integral the complex plane of the form

$$\int_C dz \frac{i}{z^2 - \omega^2(\mathbf{k}) + i\epsilon} \tag{2.6}$$

The counter C has been shown in Fig. 2 and $\omega^2(\mathbf{k}) = \mathbf{k}^2 + m_0^2$. The counter C does not include any singularities of the integrand. Hence by Cauchy's theorem the integral equals to zero. We next decompose C into 4 segments as shown in Fig.2 and write,

$$\begin{aligned} 0 &= \int_C dz \frac{i}{z^2 - \omega^2(\mathbf{k}) + i\epsilon} \\ &= \int_{-\infty}^{+\infty} dx \frac{i}{x^2 - \omega^2(\mathbf{k}) + i\epsilon} + \int_{C_+} id\theta Re^{i\theta} \frac{i}{(Re^{i\theta})^2 - \omega^2(\mathbf{k}) + i\epsilon} \\ &+ \int_{+\infty}^{-\infty} d(iy) \frac{i}{(iy)^2 - \omega^2(\mathbf{k}) + i\epsilon} + \int_{C_-} id\theta Re^{i\theta} \frac{i}{(Re^{i\theta})^2 - \omega^2(\mathbf{k}) + i\epsilon} \end{aligned} \tag{2.7}$$

where C_+, C_- are a quarter of a circle with large radius R. As $R \to \infty$ these integrals behave as $\frac{1}{R}$ and they vanish. If we change the integration variable in the first integral from x to k_0 and in the third integral from y to k_4 we shall obtain,

$$\int_{-\infty}^{+\infty} dk_0 \frac{i}{k_0^2 - \omega^2(\mathbf{k}) + i\epsilon} = \int_{-\infty}^{+\infty} dk_4^2 \frac{1}{k_4^2 + \omega^2(\mathbf{k}) - i\epsilon} \tag{2.8}$$

The denominator is now the length squared of a Euclidean 4- vector $k_E^2 = k_4^2 + k_1^2 + k_2^2 + k_3^2$ and our original integral I becomes,

$$I = \int_{-\infty}^{\infty} \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m_0^2} \tag{2.9}$$

where m_0^2 now denotes $m_0^2 - i\epsilon$. The integrand is is rotational invariant in the 4 dimensional Euclidean space. So we can use the polar coordinates and write, $\int d^4 k_E = \int_{S^3} d\Omega \int_0^\infty k^3 dk$ where $d\Omega$ denotes the surface integral on a unit sphere S^3. The integral I becomes,

$$\begin{aligned} I &= \frac{1}{(2\pi)^2} \int_{S^3} d\Omega \int_0^\infty k^3 dk \frac{1}{k^2 + m_0^2} \\ &= \frac{(2\pi^2)}{(2\pi)^4} \int_0^\Lambda k^3 dk \frac{1}{k^2 + m_0^2} \end{aligned} \tag{2.10}$$

where we performed the integral over S^3 and in the integral over k we replaced the upper bound by Λ called a momentum space cutt off. This procedure of cutting the integral at high values of momentum is called an *ultraviolet regularisation*. Obviously as $\Lambda \to \infty$ this integral diverges. In fact it can be easily evaluated and is given by,

$$\int_0^\Lambda k^3 dk \frac{1}{k^2 + m_0^2} = \frac{1}{2}\Lambda^2 - \frac{1}{2} m_0^2 ln(\frac{\Lambda^2}{m_0^2} + 1) \tag{2.11}$$

Thus,

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} \\ &= \frac{1}{16\pi^2}[\Lambda^2 - m_0^2 ln(\frac{\Lambda^2}{m_0^2} + 1)] \end{aligned} \tag{2.12}$$

Having obtained and explicit expression for I as a function of the cut off Now we can go back to Eq.(2.19) and write it as

$$
\begin{aligned}
G(p) &= \Delta_F(p) + (\frac{-i\lambda_0}{2})\Delta_F(p) I \Delta_F(p) + ... \\
&= \Delta_F(p) \frac{1}{1 + \frac{i\lambda_0}{2} I \Delta_F(p)}
\end{aligned}
\tag{2.13}
$$

This expression is assumed only up to the first order term in λ_0. If we substitute for the Feynman propagator its explicit form from Eq.(2.4) we obtain,

$$
G(p) = \frac{i}{k^2 - m_0^2 - \frac{\lambda_0}{2} I + i\epsilon}
\tag{2.14}
$$

This result which is valid only to the first order term in λ_0 can be compared with the exact result for the 2-point function given by Eq.(1.12). The obvious outcome of this comparison is that the mass parameter m_0 introduced in the Lagrangian is not the physical mass. In fact the comparison also gives a relation between the physical mass m and the mass parameter m_0, viz,

$$
\begin{aligned}
m^2 &= m_0^2 + \frac{\lambda_0}{2} I \\
&= m_0^2 + \frac{\lambda_0}{32\pi^2}[\Lambda^2 - m_0^2 ln(\frac{\Lambda^2}{m_0^2} + 1)]
\end{aligned}
\tag{2.15}
$$

Only if the interactions are absent,i.e. if $\lambda_0 = 0$ the two parameters agree. We rewrite this expression expressing m_0^2 as a function of the finite physical mass and the cut off,

$$
m_0^2 = m^2 - \frac{\lambda_0}{32\pi^2}[\Lambda^2 - m_0^2 ln(\frac{\Lambda^2}{m_0^2} + 1)]
\tag{2.16}
$$

As stated before m^2 is a finite quantity determined by the experimental measurement. The difference between m_0^2 and m^2 is of the order λ_0. We can thus replace m_0^2

by m^2 and the error will be of the order of λ_0^2 which, at this order, we are ignoring. Thus,

$$m_0^2 = m^2 - \delta m^2 \tag{2.17}$$

Where $\delta m^2(\lambda_0, m^2; \Lambda) = \frac{\lambda_0}{32\pi^2}[\Lambda^2 - m^2 ln(\frac{\Lambda^2}{m^2} + 1)] + ...$ The ... indicate the terms of λ_0^2 and higher powers of the coupling which can be calculated successively.

The important upshot is that the constant m_0^2 entering the action must indeed depend on the physical mass and on an ultraviolet or short distance cut-off Λ. We have chosen this dependence in such away that the propagator in the interacting theory has a pole at the physical mass m^2. The steps can and must be repeated in higher orders of perturbation theory. Thus in general the function $\delta m^2(\lambda_0, m^2; \Lambda)$ will be a power series expansion in the couplings the coefficients of which will depend on the masses and the cut-off. The terms of this series can be determined order by order in perturbation theory by imposing the two conditions that i) the 2- point function $G_2(p)$ has a simple pole at precisely the physical mass m^2 and ii) the residue at this pole is i. These two conditions are examples of what is called *Renormalisation Conditions*. The same two conditions will also determine the constant Z order by order in perturbation theory.

The next topic will be to formalise the conditions and obtain a systematic way calculating finite physical result at each order of perturbation theory.

Exercise

Go back to the exercise of Lecture 1 and write down the algebraic expression corresponding to each graph both in the momentum as well as the momentum space. For each graph calculate the numerical factor with which it must be multiplied.

As explained at the end of the last subsection we can systematise the calculations outlined above so that the extension to higher orders and less trivial situations becomes transparent. To this end let us substitute m_0^2 by $m_0^2 = m^2 - \delta m^2(\lambda_0, m^2; \Lambda)$ in the Lagrangian given by

$$\begin{aligned} \mathcal{L}(\varphi) &= \frac{1}{2}\varphi(x)(-\partial^2 - m_0^2)\varphi(x) - \frac{\lambda_0}{4!}\varphi^4 \\ &= \frac{1}{2}\varphi(x)(-\partial^2 - m^2)\varphi(x) + \frac{1}{2}\delta m^2\varphi^2(x) - \frac{\lambda_0}{4!}\varphi^4 \end{aligned} \tag{2.18}$$

where δm^2 has a power series expansion in the coupling λ_0. Thus the interaction part of the Lagrangian is,

$$\mathcal{L}_{\mathcal{I}}(\varphi) = \frac{1}{2}\delta m^2\varphi^2(x) - \frac{\lambda_0}{4!}\varphi^4$$

This is the interaction Lagrangian which must be used in Eq. (1.15). Consequently additional terms will appear on the r.h.s. of Eq.(2.19) and therefore also on the r.h.s. of Eq.(2.19)

$$G(p) = \Delta_F(p) + (\frac{-i\lambda_0}{2})\Delta_F(p)I(m^2)\Delta_F(p) + i\delta m\Delta_F(p)\Delta_F(p) \tag{2.19}$$

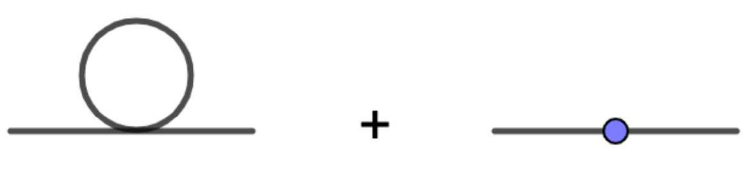

FIG. 2. 1-loop contribution to the 2-point function including counterterms

Since the mass term in the free part of the Lagrangian is m^2 (rather than m_0^2) in the Feynman propagator $\Delta_F(p)$ we must use m^2 rather than than m_0^2.

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \\
&= \frac{1}{16\pi^2}[\Lambda^2 - m^2 ln(\frac{\Lambda^2}{m^2} + 1)]
\end{aligned}
\tag{2.20}
$$

The rest of the calculation will be as before with the substitution of m_0^2 by m^2 and the result can immediately be read from Eq.(2.13)

$$
\begin{aligned}
G(p) &= \Delta_F(p) + (\frac{-i\lambda_0}{2})\Delta_F(p) I \Delta_F(p) + i\delta m \Delta_F(p)\Delta_F(p) + ... \\
&= \Delta_F(p) \frac{1}{1 + \frac{i\lambda_0}{2} I \Delta_F(p) - i\delta m \Delta_F(p)}
\end{aligned}
\tag{2.21}
$$

By simple manipulation which led to Eq. (2.14) we finally obtain,

$$
G(p) = \frac{i}{k^2 - m^2 - \frac{\lambda_0}{2} I + \delta m + i\epsilon} \tag{2.22}
$$

We can now impose the two renormalisation conditions. The first condition demands that $G_2(p)$ has a simple pole at m^2. This is obtained if,

$$
-\frac{\lambda_0}{2} I + \delta m = 0 \qquad \Rightarrow \qquad \delta m = \frac{\lambda_0}{2} I + ... \tag{2.23}
$$

The "..."indicated that this is the value of δm only up to the first order in the coupling constant λ_0. Up to this order we obtain the same result as before for the δm^2.

With is result the $G_2(p)$ becomes,

$$
G(p) = \frac{i}{k^2 - m^2 + i\epsilon} \tag{2.24}
$$

The residue at the pole is obviously i. So up to this order the second renormalisation condition is already fulfilled.

2.C. Terminology and Rewriting of the Lagrangian

The mass m_0^2 is called a bare or undressed mass. Likewise the fields φ and the coupling λ_0 are called bare or undressed field and coupling respectively. On the other hand m^2 is called a renormalised or dressed mass. We can also introduce renormalised field ϕ_R and renormalised coupling λ. This will be done in two steps. First we use Eq. (1.10) to rewrite the Lagrangian in terms of ϕ_R

$$\mathcal{L} = \frac{1}{2} Z \phi_R(x)(-\partial^2 - m^2)\phi_R(x) + \frac{1}{2}\delta m^2 Z \phi_R^2(x) - \frac{\lambda_0}{4!} Z^2 \phi_R^4 \tag{2.25}$$

Next we write,

$$\lambda_0 Z^2 = Z_1 \lambda \tag{2.26}$$

Thus the form of the Lagrangian written in terms of the renormalised objects becomes,

$$\mathcal{L} = \frac{1}{2} Z \phi_R(x)(-\partial^2 - m^2)\phi_R(x) + \frac{1}{2}\delta m^2 Z \phi_R^2(x) - \frac{\lambda}{4!} Z_1 \phi_R^4 \tag{2.27}$$

We have already hinted that $Z = 1$ in the free theory. Similar to δm^2 in the interacting theory Z and Z_1 will have power series expansions in λ with coefficients depending on m^2 and a cut-off λ. To develop perturbation theory we write,

$$Z = 1 + A, \quad Z\delta m^2 = Bm^2, \quad \lambda Z_1 = 1 + C \tag{2.28}$$

The dimensionless constants A, B and C cab be expanded in a power series of λ

$$A = a_1\lambda + a_2\lambda^2 + ... \tag{2.29}$$

$$B = b_1\lambda + b_2\lambda^2 + ... \tag{2.30}$$

$$C = c_1\lambda + c_2\lambda^2 + ... \tag{2.31}$$

The dimensionless constants a_i, b_i, c_i and g_i, i=1, 2, ... are determined order by order in perturbation theory as functions of m^2 and a cut-off Λ by imposing the renormalisatioin constants. For instance we have already determined a_1 and b_1, viz,

$$a_1 = 0 \qquad\qquad b_1 = \frac{1}{2}I \tag{2.32}$$

In this model the renormalisation conditions will be :

1. The two point function $G_{2R}(p^2)$ has a simple pole at $p^2 = m^2$.
2. The residue at this pole is i
3 . The value of the Fourier transform of the 4 point function $G_4(x_1, x_2, x_3, x_4)$ at some specific point in momentum space is λ.
4. $< 0|\hat{\phi}(x)|0 >= 0$

The conditions (1) , (2) and (3) will determine Z, δm^2 and Z_1 and the condition (4) will be automatic in this model due to the Z_2 symmetry of the Lagrangian under $\phi \to -\phi$. These conditions will determine A, B, C in terms of the physical parameters m^2 and λ and the cut-off Λ.

Henceforth we shall drop the subscript R from ϕ_R.

To apply perturbation theory we use Eq.(2.28) to rewrite the Lagrangian Eq. (2.27) as,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

where,

$$\mathcal{L}_{int} = \mathcal{L}_1 + \mathcal{L}_{ct} \tag{2.33}$$

where

$$\mathcal{L}_0 = \frac{1}{2}\phi(x)(-\partial^2 - m^2)\phi(x) \tag{2.34}$$

$$\mathcal{L}_1 = \frac{\lambda}{4!}\phi^4 \tag{2.35}$$

and

$$\mathcal{L}_{ct} = A\frac{1}{2}\phi(x)(-\partial^2 - m^2)\phi(x) + \frac{1}{2}Bm^2\phi(x)^2 - \frac{1}{4!}\lambda C\phi^4 \tag{2.36}$$

The terms $\mathcal{L}_{ct}$ is called the counterterm Lagrangian.

Now in applying Feynman Eq. (1.15)we must use the total $\mathcal{L}_{int} = \mathcal{L}_1 + \mathcal{L}_{ct}$. Thus in addition to the usual 4 point vertex coming from $\mathcal{L}_1 = \frac{\lambda}{4!}\phi^4$ there will be new interaction vertices originating from the counter term Lagrangian $\mathcal{L}_{ct}$.

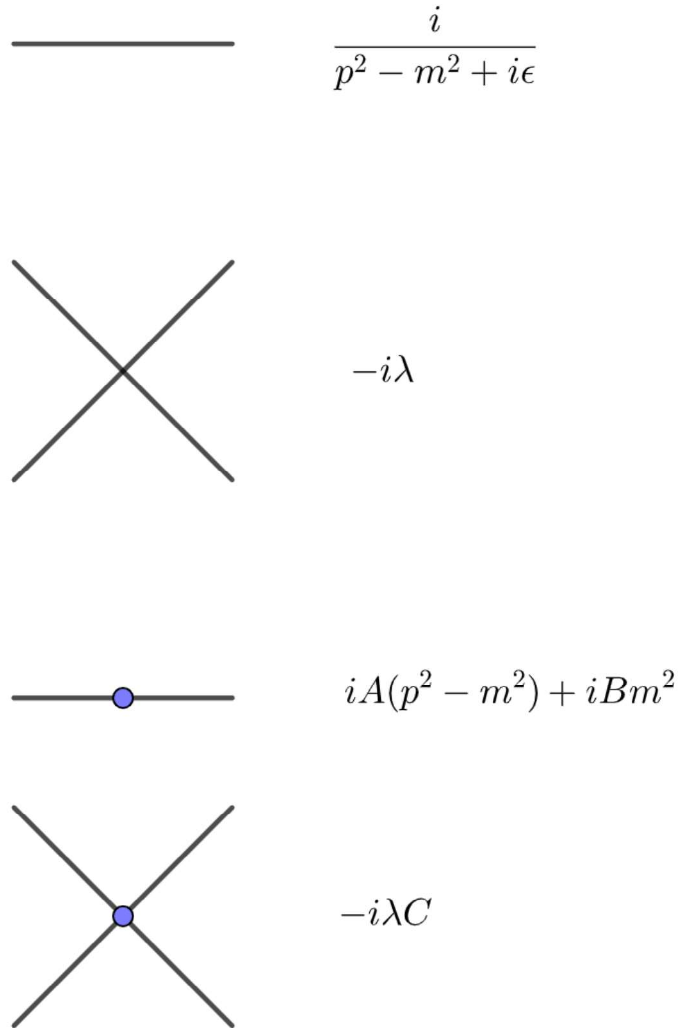

FIG. 3. Feynman rules in ϕ^4 model including counterterms

The same kind of manipulation can be applied to the Lagrangian of a scalar field theory with a cubic interaction term, the so called ϕ^3 model. The result will be quasi identical to the above except that the interaction term will have 3 powers of ϕ rather than 4 powers, viz,

$$\mathcal{L}_1 = \frac{\lambda}{3!}\phi^3 \tag{2.37}$$

Likewise for the counter-term Lagrangian we shall have,

$$\mathcal{L}_{ct} = A\frac{1}{2}\phi(x)(-\partial^2 - m^2)\phi(x) + \frac{1}{2}Bm^2\phi(x)^2 - \frac{\lambda}{3!}C\phi^3 + g\phi \qquad (2.38)$$

In this case there no symmetry to forbid the linear term in ϕ so we added to counter-term Lagrangian.

2.D. 2-Point Function in the ϕ^3 model

The relevant graphs contributing up to the 1-loop order have been shown bellow. The last graph is the contribution of the graphs arising from the new vertices which originate from $\mathcal{L}_{ct}$. The sum of the first two graphs denoted by G_2' is given by,

$$\begin{aligned} G_2'(x_1, x_2) &= \Delta_F(x_1 - x_2) \\ &+ \frac{(-i\lambda_0)^2}{2}\int d^n y_1 \int d^n y_2 \Delta_F(x_1 - y_1)\Delta_F(y_1 - y_2)^2\Delta_F(y_2 - x_2) \end{aligned} \qquad (2.39)$$

Going over the momentum space and including the contribution of the counter-terms this becomes,

$$\tilde{G}_2(p) = \Delta_F(p) + \Delta_F(p)[-i\Pi(p)]\Delta_F(p) + \Delta_F(p)\{iA(p^2 - m^2) + iBm^2\}\Delta_F(p) \qquad (2.40)$$

where, $\Pi(p)$ is defined by,

$$-i\Pi(p) \equiv -\frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^n}\frac{i}{[(p-k)^2 - m^2 + i\epsilon]}\frac{i}{[k^2 - m^2 + i\epsilon]} \qquad (2.41)$$

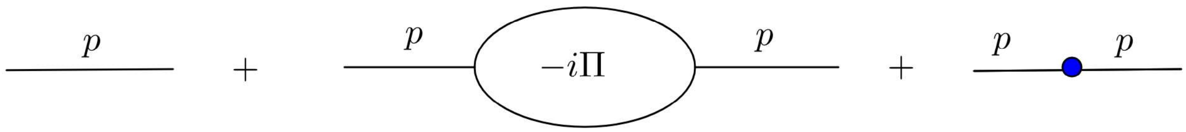

FIG. 4. Renormalised 2-point function up to 1-loop including counterterms

Our first task is to calculated this function.One of the standard ways for the next step is first to combine the two denominators into a single one by using a trick due to Feynman,

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A + (B - A)x]^2} \tag{2.42}$$

Choose $A = (p - k)^2 - m^2 + i\epsilon$ and $B = k^2 - m^2 + i\epsilon$. Then Eq. (2.41) becomes,

$$-i\Pi(p) = \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{[(k - p(1 - x))^2 + p^2x(1 - x) - m^2 + i\epsilon]^2} \tag{2.43}$$

Now we can shift the integration variable $k^\mu \to k^\mu + p^\mu(1 - x)$ and obtain,

$$-i\Pi(p) = \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{[k^2 + p^2x(1 - x) - m^2 + i\epsilon]^2} \tag{2.44}$$

The integration over k^0 has poles both in the lower and upper complex planes. As we did before we can Wick rotate the integration to the Euclidean space and obtain

$$-i\Pi(p) = \frac{\lambda^2}{2} i \int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{[k_E^2 - p^2x(1 - x) + m^2]^2} \tag{2.45}$$

This is effectively replacing k_0 by ik_4 and thus $-k^2 = -k_0^2 + k_1^2 + k_2^2 + k_3^2 \to k_E^2 \equiv k_1^2 + ... + k_4^2$. For the simplicity of writing the $i\epsilon$ has been absorbed in m^2. To proceed we assume that

$$a^2 \equiv m^2 - p^2x(1 - x) \geq 0 \tag{2.46}$$

Then the integrand will have no poles. The integral over k then becomes

$$I_4(p) = \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+a^2]^2} \tag{2.47}$$

Since the integrand is rotational invariant we can use spherical polar coordinates in 4 dimensions

$$d^4k = d\Omega dk k^3 \quad , \quad k_E^2 = k^2$$

where $d\Omega$ is the measure on the angular coordinates. Substitute $\int d\Omega = 2\pi^2$ and obtain

$$I_4(p) = \frac{1}{8\pi^2}\int_0^\Lambda dk \frac{k^3}{(k^2+a^2)^2} \tag{2.48}$$

where Λ is an ultraviolet cut-off. Return to Eq.(2.45) and obtain

$$\Pi(p) = -\frac{\lambda^2}{2}\int_0^1 dx I_4(p) \tag{2.49}$$

The evaluation of the k integral in Eq.(2.48) is straightforward and the result is

$$I_4(p) = \frac{1}{16\pi^2}[ln\frac{(\Lambda^2}{a^2}+1) - \frac{\Lambda^2}{\Lambda^2+a^2}] \tag{2.50}$$

In the limit of large Λ^2 the second term approaches 1. Substitute this result in the penultimate equation to obtained the regularised expression for $\Pi(p)$

$$\Pi(p) = -\frac{\lambda^2}{32\pi^2}\int_0^1 dx[ln\frac{\Lambda^2}{a^2(p^2)} - 1] \tag{2.51}$$

Before substituting this result in Eq.(2.40), to keep the expressions, should we introduce $\Pi_R^*(p^2)$ by

$$\Pi_R^*(p^2) = \Pi_R(p^2) - A(p^2-m^2) - Bm^2 \tag{2.52}$$

and write Eq.(2.40) as

$$\begin{aligned}\tilde{G}_2(p) &= \Delta_F(p) + \Delta_F(p)[-i\Pi^*(p)]\Delta_F(p) \\ &= \Delta_F(p)\frac{1}{1+i\Pi^*(p)\Delta_F(p)} \\ &= \frac{i}{p^2-m^2+i\epsilon+\Pi^*(p)}\end{aligned} \tag{2.53}$$

The fraction is understood to be given by its power series expansion upto the given order of λ^2.

Now we can impose the renormalisation condition. This means studying $G_2(p)$ in the vicinity of $p^2 - m^2 \to 0$. Thus for convenience we can expand $\Pi^*(p^2)$ around this point,

$$\Pi^*_R(p^2) = \Pi^*(p^2=m^2) + (p^2-m^2)\Pi^{'*}(p^2=m^2) + \frac{1}{2}(p^2-m^2)^2\Pi^{'*}(p^2=m^2) + ... \tag{2.54}$$

Substitute this back in Eq.(2.53) and impose the renormalistion conditions. The first condition that G_2 has a simple pole at $p^2 = m^2$ is satisfied provided we have,

$$\Pi^*(p^2=m^2) = 0 \qquad \rightarrow \qquad Bm^2 = \Pi(p^2=m^2) \tag{2.55}$$

The second condition that the residue at the pole is i gives us,

$$\Pi^{'*}(p^2=m^2) = 0 \qquad \rightarrow \qquad A = \Pi^{'}(p^2=m^2) \tag{2.56}$$

Substitute these values of A and B in Eq.(2.52) to obtain,

$$\Pi^*_R(p^2) = \Pi(p^2) - \Pi(p^2=m^2) - (p^2-m^2)\Pi^{'}(p^2=m^2) \tag{2.57}$$

It is not hard to see that this expression is independent of Λ and therefore finite as $\Lambda \to \infty$. The renormalised 2-point function G_2 is thus given by,

$$\tilde{G}_2(p) = \frac{i}{p^2-m^2+i\epsilon+\Pi(p^2)-\Pi(p^2=m^2)-(p^2-m^2)\Pi'(p^2=m^2)} \tag{2.58}$$

Exercise

1.Show that the constant A is in fact Λ independent in this example and that $\Pi^*(p)$ given by Eq.(2.57) is finite as $\Lambda \to \infty$.

2. Assume that the space has 5 instead of 3 dimensions and consider the ϕ^3 model in 6-dimensional space-time . Calculate the renormalisaton constants A and B up the order of λ^2 in this model.

2.E. Coupling Constant Renormalisation in the ϕ^4 Model

Consider the $2 \to 2$ scattering. The graphs which contribute up to the order of λ^2 are shown in Fig.13.

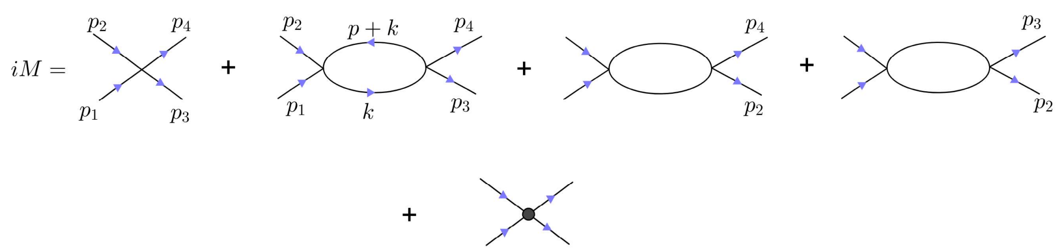

FIG. 5. The amplitude for $2 \to 2$ scattering up to 1-loop. The graph in the second line is the vertex counter-term.

Define $p = p_1 + p_2$ and denote the vertex by $i\Gamma_4$. Then up to the second order in λ we shall have the following contributions to the scattering matrix element

$$i\mathcal{M} = -i\lambda + \{i\Gamma_4(s) + i\Gamma_4(t) + i\Gamma_4(u)\} - i\lambda C \tag{2.59}$$

where s, t and u are the Mandelstam variables, $s = p^2, t = (p_1 - p_3)^2, u = (p_1 - p_4)^2$. Since we are calculating a S matrix element the momenta must be on shell and there is a relation between s,t,u,

$$s + t + u = 4m^2 \tag{2.60}$$

Applying the Feynman rules to the first graph in Fig. 5 gives us the single function $\Gamma_4(s)$, which determines the contributions of all the three 1-loop graphs. This function is given by,

$$i\Gamma_4(s) = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{[(p+k)^2 - m^2 + i\epsilon]}\frac{i}{[k^2 - m^2 + i\epsilon]} \tag{2.61}$$

The r.h.s. of this expression is the same integral we encountered in calculating Π in Eq.(2.41). It is seen that,

$$\begin{aligned}\Gamma_4(p^2) &= -\Pi(p^2) \\ &= \frac{\lambda^2}{2}\int_0^1 dx I_4(p)\end{aligned} \tag{2.62}$$

and thus the result of the integration can be read from Eq.(2.51)

$$\Gamma(p) = +\frac{\lambda^2}{32\pi^2}\int_0^1 dx[ln\frac{\Lambda^2}{a^2(p^2)} - 1] \tag{2.63}$$

where $a^2(p^2) = a^2 \equiv m^2 - p^2x(1-x)$, as in given by Eq. (3.19). Unlike the 2-point function where the renormalisation constants were fixed by imposing renormalisation conditions at the physical mass there is no obvious condition to impose on $i\mathcal{M}(s,t,u)$. We simply declare that at some given values s_0, t_0 and u_0, $\mathcal{M}(s_0,t_0,u_0) = -\lambda$. This will then fix the value of C up to the second order in λ. One $s_0 = 4m^2$ and $t_0 = u_0 = 0$. This condition then gives,

$$\lambda C = \Gamma_4(s = 4m^2) + 2\Gamma_4(0) \tag{2.64}$$

Substituting this value for C in Eq. (2.59) we obtain,

$$\begin{aligned}\mathcal{M}(s,t,u) =& -\lambda \\ &+[\Gamma_4(s) - \Gamma_4(4m^2)] + [\Gamma_4(t) - \Gamma_4(0)] + [\Gamma_4(u) - \Gamma_4(0)]\end{aligned} \tag{2.65}$$

Putting an ultraviolet cut-off is one example of regularising a divergent. But this method of regularisation is inconvenient if we consider multi-loop graphs or in theories with local gauge symmetries. There are other regularisation schemes such as lattice and Pauli- Villars regularisations. But the most commonly used is the dimensional regularisation. To explain this method let us consider as an example the integral in Eq.(2.47) in n-dimensions,

$$I_n(p) = \int \frac{d^n k_E}{(2\pi)^n} \frac{1}{[k_E^2 + a^2]^2} \tag{3.1}$$

Since the integrand is rotational invariant we can use spherical polar coordinates in n dimensions

$$d^n k = d\Omega dk k^{n-1} \quad , \quad k_E^2 = k^2$$

where $d\Omega$ is the measure on the angular coordinates and

$$I_n(p) = \frac{\Omega}{(2\pi)^n} \int_0^\infty dk \frac{k^{n-1}}{(k^2 + a^2)^2}$$

where,

$$\Omega = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \tag{3.2}$$

It is clear that for $n > 3$ the integral over k diverges and the divergence comes from the large values of k in the integral. For this reason it is called *ultravolet divergence*. In principle we can introduce a cut off Λ and evaluate $I_n(p)$ as a function of Λ,

$$I_n(p) = \frac{\Omega}{(2\pi)^n} \int_0^\Lambda dk \frac{k^{n-1}}{(k^2 + a^2)^2} \tag{3.3}$$

Let us return to Eq. (3.1) and convert the n-dimensional integrals over the component of k_E into the product of n Gaussian integrals. First note that,

$$\frac{1}{[k_E^2 + a^2]^2} = \frac{1}{\Gamma(2)} \int_0^\infty ds s e^{-s(k_E^2+a^2)} \tag{3.4}$$

Where the Γ function is defined by

$$\Gamma(z) = \int_0^\infty ds s^{z-1} e^{-s} \quad , \quad Re z > 0 \tag{3.5}$$

With these rewritings $I_n(p)$ of Eq. (2.47) becomes,

$$I_n(p) = \int \frac{d^n k_E}{(2\pi)^n} \int_0^\infty ds s e^{-s(k_E^2+a^2)} = \frac{1}{(2\pi)^n} \int_0^\infty ds s (\frac{\pi}{s})^{\frac{n}{2}} e^{-sa^2} \tag{3.6}$$

Using the definition of Γ function we can rewrite this,

$$I_n(p) = \frac{a^{n-4}}{(4\pi)^{\frac{n}{2}}} \Gamma(2 - \frac{n}{2}) \tag{3.7}$$

We need some basic properties of Γ function. All of these properties can be proven with the help of the basic definition Eq. (3.5). It is easy to see from this equation that,

$$\Gamma(z+1) = z\Gamma(z) \tag{3.8}$$

From this it follows that for an integer z

$$\Gamma(z) = (z-1)! \tag{3.9}$$

By making use of this property one can show that $\Gamma(z)$ has simple poles for non negative integers z. In fact for any non negative integer l and $z = \eta - l$ with small η we have

$$\Gamma(\eta - l) = \frac{(-1)^l}{l!} [\frac{1}{\eta} - \gamma + \Sigma_1^l \frac{1}{k} + O(\eta)] \tag{3.10}$$

where $\gamma = 0.5772...$ is the Euler constant. $O(\eta)$ stands for the terms with positive powers of η which will vanish as $\eta \to 0$. Eq. (3.10) shows that as $\eta \to 0$, i.e. when the argument of the Γ function approaches 0 or a negative integer the function develops a simple pole.

In order to see how this result works in practice let us consider the scattering amplitude given in Eq.(3.11)

$$\begin{aligned}\Gamma_4(p^2) &= \frac{\lambda^2}{2}\int_0^1 dx I_n(p) \\ &= \frac{\lambda^2}{2(4\pi)^{\frac{n}{2}}}\Gamma(2-\frac{n}{2})\int_0^1 dx a^{n-4}\end{aligned} \tag{3.11}$$

amplitude given in Eq.(3.11)

$$\begin{aligned}\Gamma_4(p^2) &= \frac{\lambda^2}{2}\int_0^1 dx I_n(p) \\ &= \frac{\lambda^2}{2(4\pi)^{\frac{n}{2}}}\Gamma(2-\frac{n}{2})\int_0^1 dx a^{n-4}\end{aligned} \tag{3.11}$$

A simple analysis shows that in n dimensional space-time the mass dimension of the coupling constant λ is $4-n$. This is the same as the dimension of the $\mathcal{M}(s,t,u)$. We introduce and arbitrary mass parameter μ and replace the coupling λ by a dimensionless coupling g defined by,

$$\lambda = (\mu^2)^{2-\frac{n}{2}} g$$

Eq. (3.11) then becomes,

$$\Gamma_4(p^2) = \frac{g^2\mu^{4-n}}{2(4\pi)^{\frac{n}{2}}}\Gamma(2-\frac{n}{2})\int_0^1 dx(\frac{a^2}{\mu^2})^{\frac{n}{2}-2} \tag{3.12}$$

Henceforth we shall drop Let us define δ by,

$$\delta = 4-n \tag{3.13}$$

Eventually we are interested in the region of $n \to 4$. In terms of δ introduced in Eq. (3.13) we have $\Gamma(2-\frac{n}{2}) = \Gamma(\frac{\delta}{2})$. As $\delta \to 0$ Eq.(3.10) gives us,

$$\Gamma(\frac{\delta}{2}) = [\frac{2}{\delta} - \gamma + O(\delta)] \tag{3.14}$$

We can thus write,

$$
\begin{aligned}
\Gamma_4(s) =& \frac{g^2\mu^{4-n}}{2}\frac{1}{(4\pi)^{\frac{n}{2}}}\Gamma(2-\frac{n}{2})\int_0^1 dx(\frac{\mu}{a})^{4-n} \\
=& \frac{g^2\mu^{4-n}}{2}\frac{1}{(4\pi)^2}\Gamma(\delta)\int_0^1 dx(\frac{4\pi\mu^2}{a^2})^{\frac{\delta}{2}} \\
=& \frac{g^2\mu^{4-n}}{32\pi^2}[\frac{2}{\delta}-\gamma+O(\delta)]\int_0^1 dx[1+\frac{\delta}{2}ln(\frac{4\pi\mu^2}{a^2})] \\
=& \frac{g^2\mu^{4-n}}{32\pi^2}[\frac{2}{\delta}-\gamma+\int_0^1 dx ln(\frac{4\pi\mu^2}{a(s)^2})]
\end{aligned} \tag{3.15}
$$

In this regularisation scheme the ultraviolet divergence reappears in the limit of $\delta \to 0$. The regularisation parameter is dimensionless number δ. The mass parameter μ is completely arbitrary and thus no physical quantity like scattering cross sections or decay rates should depend on it although (as we shall see) the parameters such as g will depend on it and in principle we could write $g(\mu)$. This will give us a useful notion of an energy dependent or running coupling constant and it will be important to know how the coupling changes with increasing of the energy.

Substitue the above result for Γ_4 in Eq. (2.59) to obtain,

$$
\begin{aligned}
\mathcal{M} =& -g\mu^{4-n} + \{\Gamma_4(s)+\Gamma_4(t)+\Gamma_4(u)\} - g\mu^{4-n}C \\
=& -g\mu^{4-n} + 3\times\frac{g^2\mu^{4-n}}{32\pi^2}[\frac{2}{\delta}-\gamma] + \frac{g^2\mu^{4-n}}{32\pi^2}\int_0^1 dx[ln(\frac{4\pi\mu^2}{a(s)^2})+ln(\frac{4\pi\mu^2}{a(t)^2})+ln(\frac{4\pi\mu^2}{a(u)^2})] \\
& - g\mu^{4-n}C
\end{aligned} \tag{3.16}
$$

3.A. MS and $\overline{MS}$ Subtraction Schemes

The effect of the renormalisation conditions on the 2 point function was to subtract from it the first two terms in its Taylor expansion in powers of $p^2 - m^2$. This operation renders the 2-point function finite. Analogous operation renders also the 4-point function finite as we saw in the previous section with a cut off regularization.

A popular way to implement these subtractions is simply to subtract the pole terms in $\delta = 4 - n$ This scheme is called the minimal subtraction scheme or MS scheme. $\overline{MS}$ scheme is a slight generalisation in which in addition to the pole terms the Euler constants γ and the 4π are also subtracted. Let us examine the scattering amplitude $\mathcal{M}$ in the $\overline{MS}$ scheme. The choice of the renormalisation constant C which implements the $\overline{MS}$ scheme is then,

$$C = \frac{3g}{32\pi^2}(\frac{2}{\delta} - \gamma + 4\pi) \tag{3.17}$$

Substitute this value for C in Eq. (3.16) and letting $n \to 4$ to obtain the vale of the scattering amplitude in the $\overline{MS}$,

$$\mathcal{M}(s,t,u) = -g - \frac{g^2}{32\pi^2}\int_0^1 dx[ln\frac{a^2(s)}{\mu^2} + ln\frac{a^2(t)}{\mu^2} + ln\frac{a^2(u)}{\mu^2}] \tag{3.18}$$

Substitute for a^2 from Eq.(3.19)

$$a^2(p^2) \equiv m^2 - p^2x(1-x) \geq 0 \tag{3.19}$$

to obtain,

$$\begin{aligned}\mathcal{M}(s,t,u) = & -g \\ & - \frac{g^2}{32\pi^2}\int_0^1 dx[ln\frac{m^2 - sx(1-x)}{\mu^2} + ln\frac{m^2 - tx(1-x)}{\mu^2} + ln\frac{m^2 - tx(1-x)}{\mu^2}]\end{aligned} \tag{3.20}$$

Exercise

Consider the 2-point function up to the 1-loop order in ϕ^4 model. Use dimensional regularisation and renormalise it in the three subtraction schemes MS, $\overline{MS}$ and the mass shell. Find the relationship between the renormalisation constants in these schemes. How are the renormalised mass parameters related to each other.

As another example we consider the point function in ϕ^3 model in $n = 6$. Define η by

$$n = 6 - \eta \tag{3.21}$$

Then by making use of Eq. (3.10) we can write,

$$\Gamma(2 - \frac{n}{2}) = \Gamma(\frac{\eta}{2} - 1) = -[\frac{2}{\eta} - \gamma + 1 + O(\eta)] \tag{3.22}$$

Note that in this model in n dimensions the coupling constant λ has a mass dimension of $\mu^{-\frac{n}{2}+3}$. We define a new dimensionless coupling $\tilde{g}$ by

$$\lambda = \mu^{-\frac{n}{2}+3}\tilde{g} \tag{3.23}$$

We substitute this expression in $\Pi(p)$

$$\Pi(p) = -\frac{\mu^{-n+6}\tilde{g}^2}{2}\frac{1}{(4\pi)^{\frac{n}{2}}}\Gamma(2 - \frac{n}{2})\int_0^1 dxa^{n-4} \tag{3.24}$$

to obtain,

$$-\Pi(p) = \mu^\eta\frac{\tilde{g}^2}{2}\frac{1}{(4\pi)^{3-\frac{\eta}{2}}}\{-[\frac{2}{\eta} - \gamma + 1]\}\int_0^1 dxa^2a^{-\eta} \tag{3.25}$$

Since we are interested in the $\eta \to 0$ limit we dropped the positive powers of η, the $O(\eta)$ terms. To identify the finite part in the $\eta \to 0$ limit we must expand the prefactors also in powers of η,

$$\frac{1}{(4\pi)^{\frac{n}{2}}} = \frac{1}{(4\pi)^{3-\frac{\eta}{2}}} = \frac{1}{(4\pi)^3}e^{\frac{\eta}{2}ln4\pi} = \frac{1}{(4\pi)^3}(1 + \frac{\eta}{2}ln4\pi)$$

and

$$\mu^\eta a^{n-4} = a^2\mu^\eta a^{-\eta} = a^2(1 + \eta ln\frac{\mu}{a})$$

Substituting all this in Eq.(3.25) we obtain,

$$\Pi(p) = \frac{\tilde{g}^2}{2}\frac{1}{(4\pi)^3}\int_0^1 dxa^2\{(\frac{2}{\eta}+1) + ln\frac{4\pi e^{-\gamma}\mu^2}{a^2}\} \tag{3.26}$$

We introduce $\tilde{\mu}$ by

$$\tilde{\mu} = \sqrt{4}\pi e^{-\frac{\gamma}{2}}\mu \tag{3.27}$$

Remembering the definition of a^2 in Eq. (3.19) we also have,

$$\int_0^1 dxa^2 = m^2 - \frac{p^2}{6} \tag{3.28}$$

Substitute these expressions in Eq.(3.26) to obtain the final result of this calculation,

$$\begin{aligned}\Pi(p^2) = &\frac{\tilde{g}^2}{(4\pi)^3}\frac{1}{\eta}(m^2 - \frac{1}{6}p^2) + \frac{\tilde{g}^2}{2}\frac{1}{(4\pi)^3}(m^2 - \frac{1}{6}p^2) \\ &+\frac{\tilde{g}^2}{2}\frac{1}{(4\pi)^3}\int_0^1 dx[m^2 - x(1-x)p^2]ln[\frac{\tilde{\mu}^2}{m^2 - x(1-x)p^2}]\end{aligned} \tag{3.29}$$

The singular piece is a *polynomial* in p^2.

Having obtained the function $\Pi(p^2)$ we can impose a subtraction scheme to obtain the renormalisation constants A and B. In the on shell scheme we shall have, as before, $A = \Pi'(p^2 = m^2)$ and $m^2B = \Pi(p^2 = m^2)$. Then $\Pi(p^2) - m^2B - (p^2 - m^2)A$ will be independent of η and hence finite as $\eta \to 0$. In the MS scheme on the other We should choose the counter terms such that only the pole parts cancel. This means,

$$\frac{\tilde{g}^2}{(4\pi)^3}\frac{1}{\eta}(m^2 - \frac{1}{6}p^2) - Bm^2 - (p^2 - m^2)A = 0 \tag{3.30}$$

This gives us,

$$
\begin{aligned}
A &= -\frac{\tilde{g}^2}{(4\pi)^3}\frac{1}{6}\frac{1}{\eta} \\
B &= \frac{\tilde{g}^2}{(4\pi)^3}\frac{5}{6}\frac{1}{\eta}
\end{aligned}
\tag{3.31}
$$

The renormalised propagator in the MS scheme then becomes,

$$
\tilde{G}_2(p^2) = \frac{i}{p^2 - m^2 + i\epsilon - \Pi_R(p^2, m^2, \tilde{g}; \tilde{\mu}^2)} \tag{3.32}
$$

where,

$$
\begin{aligned}
\Pi_R(p^2) = \frac{\tilde{g}^2}{2}\frac{1}{(4\pi)^3}(m^2 - \frac{1}{6}p^2) \\
+\frac{\tilde{g}^2}{2}\frac{1}{(4\pi)^3}\int_0^1 dx[m^2 - x(1-x)p^2]ln[\frac{\tilde{\mu}^2}{m^2 - x(1-x)p^2}]
\end{aligned}
\tag{3.33}
$$

Clearly in the MS scheme m^2 is not the physical mass.

3.C. Running Coupling

As mentioned above the scattering amplitude $\mathcal{M}$ must be independent from the arbitrary parameter μ. This implies that the coupling λ must be a function of μ. If we differentiate both sides of Eq.(3.20) with respect to μ we obtain,

$$
\begin{aligned}
\mu\frac{d\mathcal{M}}{d\mu} = 0 = -\mu\frac{d\lambda}{d\mu} - \frac{2\lambda}{32\pi^2}(\mu\frac{d\lambda}{d\mu})\int_0^1 dxln\frac{(a(s)a(t)a(u))^2}{\mu^6} \\
-\frac{\lambda^2}{32\pi^2}(-2\times 3)
\end{aligned}
\tag{3.34}
$$

We can solve this equation for $\beta(\lambda) \equiv \mu\frac{d\lambda}{d\mu}$ to obtain,

$$
\beta(\lambda) \equiv \mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) \tag{3.35}
$$

3.C. Running Coupling

As mentioned above the scattering amplitude $\mathcal{M}$ must be independent from the arbitrary parameter μ. This implies that the coupling g must be a function of μ. If we differentiate both sides of Eq.(3.20) with respect to μ we obtain,

$$\begin{aligned}\mu\frac{d\mathcal{M}}{d\mu} = 0 = &- \mu\frac{dg}{d\mu} - \frac{2g}{32\pi^2}(\mu\frac{dg}{d\mu})\int_0^1 dxln\frac{(a(s)a(t)a(u))^2}{\mu^6} \\ &- \frac{g^2}{32\pi^2}(-2\times 3)\end{aligned} \tag{3.34}$$

We can solve this equation for $\beta(\lambda) \equiv \mu\frac{d\lambda}{d\mu}$ to obtain,

$$\beta(\lambda) \equiv \mu\frac{dg}{d\mu} = \frac{3g^2}{16\pi^2} + O(g^3) \tag{3.35}$$

The $O(g^3)$ terms will originate from the ln terms in solving Eq. (3.34). This dependence will cancel from higher order graphs which we have not included in our calculation.

Equation (3.35) and in particular the sign of g function has important physical significance. It shows that as μ increases (or at high energies i.e. short distances) the effective coupling $g(\mu)$ increases and the theory becomes strongly coupled. Hence the perturbation theory will not be applicable. We can estimate the value of mass or energy where our approximation becomes unreliable. To do this we solve the differential equation (3.35) for $g(\mu)$ and obtain,

$$g(\mu) = \frac{g_0}{1 - \frac{3g_0}{16\pi^2}ln\frac{\mu}{\mu_0}} \tag{3.36}$$

where g_0 is the value of g at some reference energy μ_0. Eq.(3.35) is an example of a renormalisation group equation. We shall see that similar results hold also in QED. In this case the role of g will be played by the fine structure constant α and, as we shall see later, the one loop β function turns out to be $\beta(\alpha) = \frac{2\alpha^2}{3\pi}$. Thus solving

$$\beta(\alpha) \equiv \mu \frac{d\alpha}{d\mu} = \frac{2\alpha^2}{3\pi} \tag{3.37}$$

gives us,

$$\alpha(\mu) = \frac{\alpha_0}{1 - \frac{2\alpha_0}{2\pi} ln \frac{\mu}{\mu_0}} \tag{3.38}$$

The value of $\frac{1}{137}$ for the fine structure constant corresponds to $\alpha_0 = \alpha(m_e)$ where μ_0 is the electron mass m_e. The value of μ for which the denominator of Eq.(3.38) vanishes is called Landau pole. This happens at an incredibly large energy of,

$$\mu = m_e e^{\frac{3\pi}{2\alpha_0}} \tag{3.39}$$

Consider a real scalar field model with $\mathcal{L} = -\frac{\lambda_l}{l!}\phi^l$ in 4 space time dimensions. We attach to a graph with L loops and I internal lines an index D called superficial degree of divergence which is defined by,

$$D = 4L - 2I \tag{3.40}$$

Each loop has a factor of d^4k, hence $4L$ and each denominator has a k^2, thus $-2I$. Clearly if $D \geq 0$ then the graph will diverge (in scalar field theory). On the other hand $D < 0$ does not necessarily mean that the graph will be finite by itself. We shall see an example below. This is the reason for the D being called *superficial* degree of divergence. We would like to write D in terms of the number of the external lines E and the vertices V.

There is a simple relationship between V, I and E, namely,

$$lV = E + 2I \quad \rightarrow \quad I = \frac{1}{2}(lV - E) \tag{3.41}$$

Each vertex will contribute a δ_4 which will ensure the energy- momentum conservation at that vertex. At the end one δ_4 function will survive to ensure the overall energy momentum conservation. We thus have,

$$L = I - (V - 1) \quad \rightarrow \quad L = \frac{1}{2}(lV - E) - V + 1 \tag{3.42}$$

Using Eq. (3.42) and Eq. (3.41) we can write Eq. (32.18) as,

$$\begin{aligned} D &= 2(lV - E) - 4V + 4 - (lV - E) \\ &= 4 - E + (l - 4)V \end{aligned} \tag{3.43}$$

Note that the mass dimension $[\lambda_l]$ of the coupling constant λ_l is

$$[\lambda_l] = 4 - l \tag{3.44}$$

Thus Eq. (3.45) becomes,

$$D = 4 - E - [\lambda_l]V \tag{3.45}$$

For $l = 3$ we have $[\lambda_l] = 1$ and thus $D = 4-E-V$. Only graphs with $E+V \leq 4$ will have a non-negative D. These are $E = V = 0$ (cosmological constant), $E = V = 1$ (1-point function or the tadpole), $E = V = 2$ (2-point function).

For $l = 4$ we have $[\lambda_l] = 0$ and thus $D = 4 - E$. The only supeficialy divergent graphs will be those of $E = 0, 1, 2, 3, 4$. The graphs with $E =, 1, 3$ will vanish because of $\phi \to -\phi$ symmetry. The graph with $E = 0$ is essentially the sum of the zero point energies and will contribute just a constant (cosmological constant). The graphs with $E = 2$ and $E = 4$ are divergent and need to be renormalised.

What about graphs with $D \leq 0$, are they necessarily convergent? The answer to this question is no. For example consider $l = 4$ and $E = 6$. the graph in (Fig. 14) will have $D = -2$. However, it has a subgraph consisting of a 1-loop 4-point vertex which we saw is divergent. Hence the graph will be divergent. Do we need a term like ϕ^6 in the action in order to absorb this divergence by renormalising its coupling? The answer is no. The reason is that together with this graph we should also include a graph in which the 4 point vertex is replaced by the 1-loop vertex counter term of the (Fig. 15). The coefficient of this counter-term has already been determined at the 1-loop level. Then in the sum we will obviously have no divergence. Thus we do not need any new counter-terms for making the above graph finite.

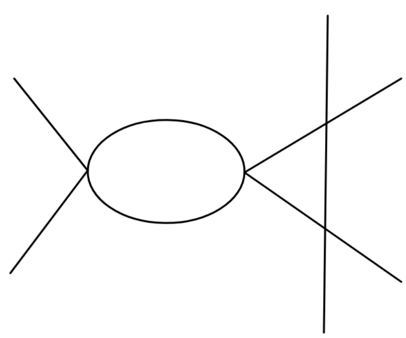

FIG. 6. An example of a superficially divergent graph

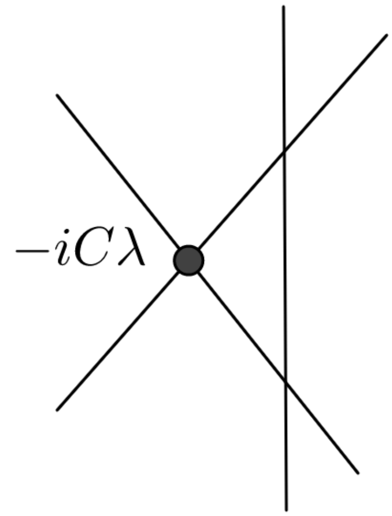

FIG. 7. The 1-loop counter-term which removes the divergence of the graph in Fig.14

The models in which we need only a finite number of counter-terms in order to remove all the divergences by the renormalisation process are called renormalizable theories. This happens only in those models in which there is no coupling constant with negative mass dimensions. If $[\lambda_l] \leq 0$ then for any E, no matter how big, we can simply increase the number of vertices and make D arbitrarily positive. We shall need an infinite number of counter-terms to be able to extract finite results at any order of perturbation theory. Such models can still be useful as effective theories.

Exercise

1. Generalise Eq.(3.45) for (i) a real scalar field theory in n space time dimensions and with the interaction Lagrangian of the form $\mathcal{L} = -\frac{\lambda_l}{l!}\phi^l$, (ii) for the scalar and spinor QED in 4-space-time dimensions. Discuss the renormalisibility of both models.
2. In exercise 1 (i). add a term a term like $(\partial_\mu\phi\partial^\mu\phi)^r$ where r is an integer bigger than 2. Examine the renormalisibility of the resulting model.

Consider the real scalar model with the ϕ^4 interaction and recall from Eq. ([**?**]) the relation between the bare coupling constant λ_0 and the renormalised coupling $\lambda = \mu^\delta g$, viz,

$$Z^2\lambda_0 = Z_1\mu^\delta g \tag{3.46}$$

For the ϕ^4 model $Z = 1$ at the 1-loop order. Furthermore the bare coupling λ_0 does not know about the arbitrary mass parameter μ.

We can then writer,

$$\frac{d}{d\mu}\lambda_0 = 0 = \frac{dZ_1}{d\mu}\mu^\delta g + Z_1\delta\mu^{\delta-1}g + Z_1\mu^\delta\frac{dg}{d\mu} \tag{3.47}$$

Simplify,

$$\frac{dlnZ_1}{d\mu}\mu^\delta g + \delta g + \mu\frac{dg}{d\mu} = 0 \tag{3.48}$$

Since up to the 1-loop order $Z_1 = 1 + gc_1$, where in MS scheme $c_1 = \frac{3}{16\pi^2}\frac{1}{\delta}$ we thus obtain,

$$c_1\beta g + \delta g + \beta = 0 \tag{3.49}$$

where we substituted $\beta = \mu\frac{dg}{d\mu}$. Solve this equation to obatin,

$$(1 + c_1 g)\beta = -\delta g \tag{3.50}$$

Solve this equation to obtain,

$$\beta = -\delta g(1 + c_1 g)^{-1} = -\delta g + \delta c_1 g^2 + ... \tag{3.51}$$

where.... are higher order terms which must be ignored at the 1-loop level. We can now set $\delta = 0$ to obtain,

$$\beta = \frac{3}{16\pi^2} g^2 \tag{3.52}$$

The same as in Eq.(3.35). Note that despite the fact that the starting point, Eq.(3.47)involves the singular functions Z and Z_1 the final result is finite at $\delta \to 0$ limit. This calculation also shows the advantage of the MS and $\overline{MS}$ schemes. In these schemes the renormalisation constants Z and Z_1 do not depend on the mass m. Furthermore the have no explicit μ dependence. They depend on μ through the dependence of the coupling g on μ. The definition of β function is general. However, in other subtraction schemes its derivation can be more complicated.

Exercise

Calculate the 1-loop β function in the scalar field theory in 6-space time dimensions in the $\overline{MS}$ scheme. How does the renormalised coupling constant $\tilde{g}$ of this model behave with the increasing value of μ

4. FUNCTIONAL INTEGRALS

Perturbative field theory is essentially the application of gaussian integrals. First consider a single variable,

$$Z_0[J] = N \int_{-\infty}^{\infty} d\varphi e^{\frac{i}{2}[\varphi(a+i\epsilon)\varphi+2J\varphi]} \tag{4.1}$$

where a and ϵ are both real and $\epsilon > 0$. To shorten the notation let us introduce D by,

$$D = a + i\epsilon$$

The reason for the subscript 0 on Z_0 is that later on we shall consider integrals for which the expression in the exponent of the integrand will be a polynomial of higher degree than 2. The constant N on the r.h.s stands for,

$$N^{-1} = \int_{-\infty}^{\infty} d\varphi e^{\frac{i}{2}\varphi D\varphi} \tag{4.2}$$

Thus,

$$Z_0[J = 0] = 1$$

We are interested in the J dependence of $Z_0[J]$ and to find it we do not need to evaluate any integrals. It is sufficient to complete the square,

$$\varphi D\varphi + 2J\varphi = (\varphi D + J)D^{-1}(D\varphi + J) - JD^{-1}J \tag{4.3}$$

Change the integration variable $\varphi \to \varphi' D \equiv \varphi D + J$ and substitute in Eq. (4.1)

$$\begin{aligned} Z_0[J] &= e^{-\frac{i}{2}JD^{-1}J} N \int_{-\infty}^{\infty} d\varphi' e^{\frac{i}{2}\varphi' D\varphi'} \\ &= e^{-\frac{i}{2}JD^{-1}J} \end{aligned} \tag{4.4}$$

We shall be interested in integrals of the form Eq.(4.7a) when the dimension $n \to \infty$. Consider a quadratic functional of a real scalar field $\varphi(x)$,

$$I_0[\varphi] = \int d^4x \frac{1}{2}\varphi(x)(-\partial^2 - m^2 + i\epsilon)\varphi(x) \tag{4.9}$$

In order to make an identification with our previous notation we introduce a set of basis in which the 4-dimensional position operators $\hat{x}^\mu$ are diagonal, viz,

$$\hat{x}^\mu|x>= x^\mu|x>, \quad , \quad < x'^\mu|x^\nu >= \eta^{\mu\nu}\delta_4(x'-x) \quad , \quad \int_{-\infty}^{\infty} d^4x|x>< x| = 1$$

$$[\hat{x}_\mu, \hat{P}_\nu] = i\eta_{\mu\nu} \quad , \quad \hat{P}_\mu = -i\partial_\mu \tag{4.10}$$

The functions $\varphi(x)$ are then components of a vector $|\varphi>$ relative to this basis,viz,

$$|\varphi>= \int_{-\infty}^{\infty} d^4x|x>< x|\varphi>= \int_{-\infty}^{\infty} d^4x\varphi(x)|x> \tag{4.11}$$

where $\varphi(x) =< x|\varphi>=< \varphi|x>$.

We also introduce the operator D by,

$$\begin{aligned} D &= -\partial^2 - m^2 + i\epsilon \\ &= \hat{P}^2 - m^2 + i\epsilon \end{aligned} \tag{4.12}$$

We then have

$$< x|\hat{P}^2 - m^2 + i\epsilon|x'>= (-\partial^2 - m^2 + i\epsilon)\delta_4(x-x') \tag{4.13}$$

With this notation the action I_0 given in Eq.(4.9)can be written in a compact form as,

$$I_0 = \frac{1}{2} < \varphi|D|\varphi> \tag{4.14}$$

Now consider the integral,

$$Z_0[J] = N_0 \int d[\varphi] e^{\frac{i}{2}[<\varphi|D|\varphi>+<J|\varphi>+<\varphi|J>]} \tag{4.15}$$

where D is given by Eq.(4.12). The result of integration thus is given by the generalization of Eq.(4.8b), viz,

$$Z_0[J] = e^{-\frac{i}{2}<J|D^{-1}|J>} \tag{4.16}$$

The J-independent constant $N_0 = (DetD)^{-\frac{1}{2}}$ is in fact infinite. We shall assume that it has somehow been regularized. The exponent in Eq.(4.16) is given by,

$$< J|D^{-1}|J >= \int d^4x \int d^4x' < J|x >< x|D^{-1}|x' >< x'|J > \tag{4.17}$$

We need to evaluate $< x|D^{-1}|x' >$ as explicit function of x and x'. This is obtained as follows,

$$DD^{-1} = 1 \rightarrow < x|DD^{-1}|x' >= \delta_4(x - x') \tag{4.18}$$

Insert $1 = \int_{-\infty}^{\infty} d^4y|y >< y|$ between D and D^{-1} and make use of Eq.(4.13) to obtain,

$$(-\partial^2 - m^2 + i\epsilon) < x|D^{-1}|x' >= \delta_4(x - x') \tag{4.19}$$

This equation can be solved by Fourier transformation,

$$< x|D^{-1}|x' >= \int \frac{d^4k}{(2\pi)^4} D(k) e^{-ik.(x-x')} \tag{4.20}$$

Then Eq.(4.19) becomes,

$$(k^2 - m^2 + i\epsilon)D(k) = 1 \tag{4.21}$$

Hence,

$$< x|D^{-1}|x' > = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik.(x-x')}$$
$$\equiv -i\Delta_F(x - x') \tag{4.22}$$

Hence in terms of $\Delta_F(x - x')$ Eq.(4.16) becomes,

$$Z_0[J] = N_0 \int d[\varphi] e^{\frac{i}{2}[<\varphi|D|\varphi>+<J|\varphi>+<\varphi|J>]}$$
$$= e^{-\frac{i}{2}<J|D^{-1}|J>}$$
$$= e^{-\frac{1}{2}\int d^4x \int d^4x' J(x)\Delta_F(x-x')J(x')} \tag{4.23}$$

Exercises

Prove the commutation relations given in Eq.(4.10).

Define $Z_0'[J]$ by,

$$Z_0'[J] =< 0|Te^{i\int dyJ(y)\hat{\phi}_0(y)}|0 > \tag{4.24}$$

$\hat{\phi}_0(x)$ is a free scalar field operator satisfying the Klein-Gordon equation,

$$(-\partial^2 - m^2)\hat{\phi}_0(x) = 0 \tag{4.25}$$

We claim that,

$$Z_0'[J] = Z_0[J] = e^{-\frac{1}{2}\int d^4x \int d^4x' J(x)\Delta_F(x-x')J(x')} \tag{4.26}$$

The equality of Z_0 and Z_0' follows from Wick's theorem.

First consider Z_0 and expand it in powers of J.

$$Z_0[J] = \Sigma(\frac{-1}{2})^n \frac{1}{(n)!} \int d^4x_1 ... \int d^4x_{2n} J(x_1)...J(x_{2n})\Delta_F(x_1 - x_2)...\Delta_F(x_{2n-1} - x_{2n}) \tag{4.27}$$

In the integrand the product of J's is symmetric under the exchange of the x's while the product of Δ_F functions is not. So only the symmetric part of the product of the Feynman propagators will contribute to the integral. The symmetrisation is achieved by replacing the product of the propagators as a sum of $(2n)!$ terms obtained from permuting the $(2n)!$ arguments x_i in the Δ_Fs. The contribution of each one of these $(2n)!$ terms will be the same as any other. Hence we must divide by $\frac{1}{(2n)!}$ in order not to obtain $(2n)!$ times the same quantity. Next we note that not all the $(2n)!$ terms will be different. In fact the permutation of , say x_1 and x_2 will leave the corresponding Δ_F invariant. There are 2^n such permutations. Likewise the permutations of, say, x_1 and x_2 with x_3 and x_4 will interchange $\Delta_F(x_1 - x_2)$ with $\Delta_F(x_3 - x_4)$ without producing a new term. There are $n!$ such permutations. To sum up the result of symmetrising the product of the

Δ_Fs in Eq.(4.27) produces $\frac{2^n n!}{(2n)!}\Sigma_{perm}\Pi\Delta_F(x_i - x_j)$ in the integrand where now only the independent permutations are involved in the sum. If we substitute this in Eq. (4.27) and simplify the numerical pre-factor we obtain

$$Z_0[J] = \Sigma_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}\int d^4x_1 ... \int d^4x_{2n} J(x_1)...J(x_{2n})\Sigma_{perm}\Pi\Delta_F(x_i - x_j) \quad (4.28)$$

.

Next expand Z_0' in powers of J. Since the vacuum expectation value of the terms with an odd number of fields is zero we obtain

$$Z_0'[J] = \Sigma_{n=0}^{\infty}\frac{i^{2n}}{(2n)!}\int d^4x_1 ... \int d^4x_{2n} J(x_1)...J(x_{2n}) < 0|T(\hat{\phi}_0(x_1)...\hat{\phi}_0(x_{2n})|0 > \quad (4.29)$$

First of all note that the numerical coefficients in Eq.(4.28) and Eq. (4.29) are the same, namely, $(-1)^n\frac{1}{(2n)!}$. Next applying Wick's theorem to $< 0|T(\hat{\phi}(x_1)...\hat{\phi}(x_{2n})|0 >$ produces precisely $\Sigma_{perm}\Pi\Delta_F(x_i - x_j)$. This completes the proof of $Z_0[J] = Z_0'[J]$. We can the write,

$$Z_0[J] =< 0|Te^{i\int dyJ(y)\hat{\phi}_0(y)}|0 >= e^{-\frac{1}{2}\int d^4x\int d^4x' J(x)\Delta_F(x-x')J(x')} \quad (4.30)$$

The above discussion can be summarised in the following equalities,

$$\begin{aligned} Z_0[J] &=< 0|Te^{i\int dyJ(y)\hat{\phi}_0(y)}|0 >= \\ &= N_0\int d[\varphi]e^{iI_0(\varphi)+i\int J(x)\varphi(x)} \\ &= N_0\int d[\varphi]e^{i\int d^4x\{\frac{1}{2}\varphi(x)(-\partial^2-m^2+i\epsilon)\varphi(x)+J(x)\varphi(x)} \\ &= e^{-\frac{1}{2}\int d^4x\int d^4x' J(x)\Delta_F(x-x')J(x')} \end{aligned} \quad (4.31)$$

where,

$$\Delta_F(x-x') = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} e^{-ik.(x-x')} \tag{4.32}$$

The normalisation factor N_0 has been determined from $Z_0[J=0] =< 0|0 >= 1$,

$$\begin{aligned} N_0^{-1} &= \int d[\varphi] e^{iI_0(\varphi)} \\ &= \int d[\varphi] e^{i\int d^4x \frac{1}{2}\varphi(x)(-\partial^2-m^2+i\epsilon)\varphi(x)} \end{aligned} \tag{4.33}$$

If we expand both sides of Eq. (4.31) in powers of J and compare the same powers of J in both sides we obtain,

$$\begin{aligned} &< 0|T(\hat{\phi}(x_1)...\hat{\phi}(x_{2n})|0 >= \\ &= N_0 \int d[\varphi]\varphi(x_1)....\varphi(x_{2n}) e^{i\int d^4x \frac{1}{2}\varphi(x)(-\partial^2-m^2+i\epsilon)\varphi(x)} \end{aligned} \tag{4.34}$$

Note that the φ appearing under the integral sign on the right hand side is a *commuting* integration variable and thus the ordering is unimportant. The functional integral on the right hand side evaluates the vacuum expectation value of the *Time Ordered* product of operators on the left hand side.

Exercise

Use the canonical quantisation of the field $\hat{\phi}$ to show that $\frac{\delta Z_0'}{i\delta J(x)}$ satisfies the following equation,

$$(\partial^2 + m^2)\frac{\delta Z_0'}{i\delta J(x)} = J(x) Z_0'[J] \tag{4.35}$$

where $Z_0'[J]$ is defined by Eq. (4.24). Show that the appropriate Green function for solving this equation is the Feynman propagator Δ_F and thereby prove the identity Eq. (4.26).

We consider an action $I(\varphi)$ of the form,

$$
\begin{aligned}
I(\varphi) &= I_0(\varphi) + I_I(\varphi) = \\
&= \int d^4x \{\frac{1}{2}\varphi(x)(-\partial^2 - m^2 + i\epsilon)\varphi(x) + \mathcal{L}_{\mathcal{I}}(\varphi)\}
\end{aligned}
\tag{4.36}
$$

where I_0 is given by Eq.(4.9) and $\mathcal{L}_{\mathcal{I}}(\varphi)$ is a polynomial in φ. We define a functional $Z(J)$ by,

$$
\begin{aligned}
Z[J] &= N' \int d[\varphi] e^{iI(\varphi) + i\int J(x)\varphi(x)} \\
&= N' \int d[\varphi] e^{i\int d^4x \{\frac{1}{2}\varphi(x)(-\partial^2 - m^2 + i\epsilon)\varphi(x) + \mathcal{L}_{\mathcal{I}}(\varphi) + J(x)\varphi(x)\}}
\end{aligned}
\tag{4.37}
$$

For a Polynomial $\mathcal{L}_{\mathcal{I}}(\varphi)$ we can write,

$$
\begin{aligned}
e^{i[\mathcal{L}_{\mathcal{I}}(\varphi) + J(x)\varphi(x)]} &= e^{i\mathcal{L}_{\mathcal{I}}(\varphi)} e^{iJ(x)\varphi(x)]} \\
&= e^{i\mathcal{L}_{\mathcal{I}}(\frac{\delta}{i\delta J(x)})} e^{iJ(x)\varphi(x)}
\end{aligned}
$$

Substitute in Eq.(4.37) and take $e^{i\int dx \mathcal{L}_{\mathcal{I}}(\frac{\delta}{i\delta J(x)})}$ out of the integration sign to obtain,

$$
\begin{aligned}
Z[J] &= N' e^{i\int dx \mathcal{L}_{\mathcal{I}}(\frac{\delta}{i\delta J(x)})} \int d[\varphi] e^{i\int d^4x \{\frac{1}{2}\varphi(x)(-\partial^2 - m^2 + i\epsilon)\varphi(x) + J(x)\varphi(x)} \\
&= N e^{i\int dx \mathcal{L}_{\mathcal{I}}(\frac{\delta}{i\delta J(x)})} Z_0[J] \\
&= N < 0|T e^{i\int dx \mathcal{L}_{\mathcal{I}}(\hat{\phi})} e^{i\int dx J(x)\hat{\phi}_0(x)} |0 >
\end{aligned}
\tag{4.38}
$$

where in the second equality we used Eq. (4.31) to substitute for the functional integral in terms of Z_0 and in the third equality we used Eq.(4.26). We also defined $N \equiv \frac{N'}{N_0}$. Note The operator

$$S = Te^{i\int dx \mathcal{L}_{\mathcal{I}}(\hat{\phi})} \tag{4.39}$$

is the S-Matrix.

Using the explicit expression for Z_0 as given by the last equality of Eq.(4.23) we can then write

$$Z[J] = Ne^{i\int dx \mathcal{L}_{\mathcal{I}}(\frac{\delta}{i\delta J(x)})} e^{-\frac{1}{2}\int d^4x \int d^4x' J(x)\Delta_F(x-x')J(x')} \tag{4.40}$$

The normalisation constant N will be fixed bellow. Equation (4.40) gives us the result of functional integration over φ in Eq.(4.37). Usually $\mathcal{L}_{\mathcal{I}}$ has some small parameters in it which can be used as an expansion parameters. We can then expand $e^{i\int dx \mathcal{L}_{\mathcal{I}}(\frac{\delta}{i\delta J(x)})}$ in powers of such parameters and then act on the Gaussian exponential term by term. In practice we need only the n-point functions, defined by,

$$G_n(x_1, ..., x_n) = \frac{\delta}{i\delta J(x_1)} ... \frac{\delta}{i\delta J(x_n)} Z[J] \mid_{J=0} \tag{4.41}$$

The logical procedure is thus to use Eq.(4.40) to evaluate Z up to some order in a coupling constant in $\mathcal{L}_{\mathcal{I}}$ and then insert that Z in Eq. (4.41) to obtain $G_n(x_1, ..., x_n)$ up to that order. Although conceptually simple, in practice this procedure is rather tedious. A more efficient way is to convert the r.h.s of Eq.(4.41) into a sum of Feynman graphs. To this end first substitute for $Z[J]$ from Eq. (4.40) in Eq.(4.41) and commute the functional derivatives so that they act on $Z_0[J]$ to obtain,

$$G_n(x_1, ..., x_n) = Ne^{i\int dx \mathcal{L}_{\mathcal{I}}(\frac{\delta}{i\delta J(x)})} \frac{\delta}{i\delta J(x_1)} ... \frac{\delta}{i\delta J(x_n)} \frac{\delta}{i\delta J(x_1)} ... \frac{\delta}{i\delta J(x_n)} Z_0[J] \mid_{J=0} \tag{4.42}$$

Substitute $Z_0[J] =< 0|Te^{i\int dy J(y)\hat{\phi}_0(y)}|0 >$ and move the derivatives inside the vacuum expectation value,

$$
\begin{aligned}
&G_n(x_1, ..., x_n) = \\
&\quad = N e^{i\int dx \mathcal{L}_{\mathcal{I}}(\frac{\delta}{i\delta J(x)})} \frac{\delta}{i\delta J(x_1)} ... \frac{\delta}{i\delta J(x_n)} \frac{\delta}{i\delta J(x_1)} ... \frac{\delta}{i\delta J(x_n)} < 0|T e^{i\int dy J(y)\hat{\phi}(y)}|0 >|_{J=0} \\
&\quad = N e^{i\int dx \mathcal{L}_{\mathcal{I}}(\frac{\delta}{i\delta J(x)})} < 0|T\hat{\phi}_0(x_1)...\hat{\phi}_0(x_n) e^{i\int dy J(y)\hat{\phi}_0(y)}|0 >|_{J=0} \\
&\quad = N < 0|T\hat{\phi}_0(x_1)...\hat{\phi}_0(x_n) e^{i\int dy \mathcal{L}_{\mathcal{I}}(\hat{\phi}_0(y))}|0 > \qquad (4.43)
\end{aligned}
$$

To fix the normalisation factor N we note that for the 0 point function G_0 is just a constant. We can set it to 1 and obtain,

$$
N^{-1} = < 0|T e^{i\int dy \mathcal{L}_{\mathcal{I}}(\hat{\phi}_0(y))}|0 > \qquad (4.44)
$$

We can then write Eq.(4.43) in the following neat form,

$$
\begin{aligned}
G_n(x_1, ..., x_n) &= \frac{< 0|T\hat{\phi}_0(x_1)...\hat{\phi}_0(x_n) e^{i\int dy \mathcal{L}_{\mathcal{I}}(\hat{\phi}_0(y))}|0 >}{< 0|T e^{i\int dy \mathcal{L}_{\mathcal{I}}(\hat{\phi}_0(y))}|0 >} \\
&= < \Omega|T\hat{\phi}_H(x_1)...\hat{\phi}_H(x_n)|\Omega > \qquad (4.45)
\end{aligned}
$$

In the last equality $\phi_H(x)$ is the fully interacting Heisenberg field operator and the equality is the Gell-Mann Low formula. We conclude that $Z[J]$ is the generating functional of the n-point functions and given by a functional integral,

$$
\begin{aligned}
Z[J] &= N \int d[\varphi] e^{iI(\varphi) + i\int J(x)\varphi(x)} \\
&= N \int d[\varphi] e^{i\int d^4x \{\frac{1}{2}\varphi(x)(-\partial^2 - m^2 + i\epsilon)\varphi(x) + \mathcal{L}_{\mathcal{I}}(\varphi) + J(x)\varphi(x)\}} \\
&= < \Omega|T e^{i\int dy J(y)\hat{\phi}_H(y)}|\Omega > \qquad (4.46)
\end{aligned}
$$

and,

$$N^{-1} = \int d[\varphi] e^{iI(\varphi)} \tag{4.47}$$

1. Use the first equality in Eq.(4.43) to obtain the 1-point function up to the first order in λ for a real scalar field theory with the interaction Lagrangian given by,

$$\mathcal{L}_{\mathcal{I}}(\varphi) = -\frac{\lambda}{3!}\varphi^3$$

Do the same for the 2-point function up to and including the second order in λ.

2. Repeat the same exercise for the interaction

$$\mathcal{L}_{\mathcal{I}}(\varphi) = -\frac{\lambda}{4!}\varphi^4$$

and obtain the 2- point function up to the second order in λ.

3. Represent the results obtained in (1) and (2) by Feynman diagrams technique.

4. (*Optional*) Define $Z'[J]$ by,

$$Z'[J] = < 0_H|Te^{i\int dyJ(y)\hat{\phi}_H(y)}|0_H >$$

Obtain an equation for $\frac{\delta Z'}{i\delta J(x)}$ analogous to the Eq. (4.35). Your result must be the nonlinear generalisation of Eq.(4.35). Show that $Z[J]$ as given by the third equality on the r.h.s. of Eq.(4.38) satisfies the differential equation you have obtained for $Z'[J]$. From here conclude that $Z'[J] = Z[J]$, i.e.

$$\begin{aligned} Z[J] &= < 0_H|Te^{i\int dyJ(y)\hat{\phi}_H(y)}|0_H > \\ &= \frac{< 0|Te^{i\int dx\mathcal{L}_{\mathcal{I}}(\hat{\phi})}e^{i\int dxJ(x)\hat{\phi}(x)}|0 >}{< 0|Te^{i\int dx\mathcal{L}_{\mathcal{I}}(\hat{\phi})}|0 >} \end{aligned}$$

Expanding both sides of this relation in powers of J and identifying the coefficients of the n power on both sides we obtain a derivation of Eq. (4.45).

5. FERMIONS: FUNCTIONAL INTEGRAL FORMULATION

The generalisation of the above scheme to fermions is *not* straightforward. The reason is eventually rooted in the Pauli exclusion principle that the fermions should be quantised by equal time *anti-commutation* rule rather than the equal time commutation rule applied to bosons.

Consider the free Dirac acton,

$$I_D = \int \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{5.1a}$$

The momentum conjugate to ψ^α is $i\psi^{\alpha\dagger}$ and the canonical quantization demands,

$$\{\psi_\alpha(x,t), \psi^\dagger_\beta(\mathbf{x}',t)\} = \hbar\delta(\mathbf{x}-\mathbf{x}')\delta_{\alpha\beta} \tag{5.1b}$$

The formal $\hbar \to 0$ limit of this expression implies that in the "classical" limit we must have complex *anticommuting* functions. Thus we must first develop an algebra and calculus of such numbers and functions. Essentially we need a set of consistent definitions and rules which allow us to do the basic operations of differentiation and integration.

We start from the definition of anticommuting or Grassmann numbers, θ_i, where $i = 1, ...N$ and with the basic property of

$$\theta_i\theta_j = -\theta_j\theta_i \tag{5.2a}$$

for all i and j. We note in particular that

$$\theta_i^2 = 0 \quad , \quad i = 1, ...N \tag{5.2b}$$

We also define *conjugate* Grassmann numbers $\bar{\theta}_i$ with $i = 1, ...N$ with similar properties, viz,

$$\bar{\theta}_i\bar{\theta}_j = -\bar{\theta}_j\bar{\theta}_i \quad , \quad \bar{\theta}_i^2 = 0 \quad , \quad i = 1, ...N \tag{5.2c}$$

Furthermore,

$$\bar{\theta}_i\theta_j = -\theta_j\bar{\theta}_i \quad , \quad i,j = 1,...N \tag{5.2d}$$

Note that a product of an even number of Grassmann numbers will commute with every power of θ's and $\bar{\theta}$'s. Such commuting objects are called Grassmann even, while the anticommuting ones are called Grassmann odd.

Next we extend the set of complex numbers by adding to $\mathbf{C}$ the set of $\theta_1, ...\theta_N, \bar{\theta}_1, ...\bar{\theta}_N$. We have the obvious operations like,

$$(a+b)\theta = a\theta + b\theta, \quad (ab)\theta = a(b\theta) = ab\theta = ba\theta, etc$$

where a and b are arbitrary complex numbers.

The enlarged algebra now has two kinds of elements. The first kind are those which commute with each other. This subset includes all the complex numbers and all the elements formed from the product of an even number of θ's and $\bar{\theta}$'s. Such commuting elements are called Grassmann even. Obviously the product and sum of Grassmann even elements produce Grassmann even elements. The second set are those elements which anticommute with each other. These are called Grassmann odd and they must necessarily consist of monomials with an of odd number of θ's and/or $\bar{\theta}$'s. For this reason the extended algebra is said to be a graded algebra.

Any function of a finite number of θ and $\bar{\theta}$ is a polynomial. For example with $N = 2$ and a a real number we shall have,

$$e^{a\Sigma_i\bar{\theta}_i\theta_i} = 1 + a\Sigma_i\bar{\theta}_i\theta_i + \frac{1}{2}(a\Sigma_i\bar{\theta}_i\theta_i)^2 \tag{5.3}$$

where

$$(\Sigma_i\bar{\theta}_i\theta_i)^2 = 2\bar{\theta}_1\theta_1\bar{\theta}_2\theta_2 = -2\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2$$

The rule for integration is equally simple. We postulate that,

$$\int d\theta_i = 0 = \int d\bar{\theta}_i \tag{5.4}$$

and

$$\int d\theta_i \theta_j = \delta_{ij} = \int d\bar{\theta}_i \bar{\theta}_j \tag{5.5}$$

and obvious anti-commutation rule,

$$d\theta_i \theta_j = -\theta_j d\theta_i \quad , \quad d\bar{\theta}_i \bar{\theta}_j = -\bar{\theta}_j d\bar{\theta}_i \tag{5.6}$$

In other words $d\theta_i$ and $d\bar{\theta}_i$ are Grassmann odd. We also postulate the rule for shifting the integration variable by a constant Grassmann number η,

$$d(\theta + \eta) = d\theta$$

in other words,

$$\int d(\theta + \eta)(\theta + \eta) = \int d\theta\theta = 1$$

With these rules and for $N = 1$ we shall have,

$$\int d\bar{\theta} d\theta e^{-a\bar{\theta}\theta} = a \tag{5.7a}$$

Note the difference with ordinary Gaussian integral. If θ was an ordinary complex variable with $\bar{\theta}$ its complex conjugate the result of integration would give us πa^{-1}. For the Grassmann case instead we obtain a^{+1}.

Assume now A_{ij} are elements of a $N \times N$ diagonal matrix. We shall then have (repeated indices are summed over),

$$\int d\bar{\theta}_1 d\theta_1 ... \int d\bar{\theta}_N d\theta_N e^{-\bar{\theta}_i A^{ij} \theta_j} = DetA \tag{5.8}$$

Again compare with the ordinary Gaussian integration which would give us $(DetA)^{-1}$. If A is not diagonal we can transform to the basis of its eigenvalues and obtain the same result as in Eq(5.8). Let us introduce the bra-ket notation and write,

$$\bar{\theta}_i A^{ij} \theta_j =< \bar{\theta}|A|\theta > \tag{5.9}$$

As in the bosonic case we shall be interested in the case when $N \to \infty$ and assume that the indices i, j stand for discrete as well as continuous labels. For example if we consider a Dirac field $\psi(x)$ which is a 4- component spinor field the index i will stand for the discrete spinor index $\alpha = 1, 2, 3, 4$ as well as the space time index x which is a 4- vector under the Lorenz group. By matching the notation to the one in Eq.(5.9) we can write the Dirac action as,

$$I_{Dirac} = \int d^4x \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m + i\epsilon)\psi(x) = i < \bar{\psi}|D|\psi > \tag{5.10}$$

where,

$$D \equiv -i(i\gamma^\mu \partial_\mu - m + i\epsilon) \tag{5.11}$$

The generalisation of Eq.(5.8) will be,

$$\int [d\bar{\psi}][d\psi] e^{i \int d^4x [\bar{\psi}(x)(i\gamma^\mu \partial_\mu - m + i\epsilon)\psi(x)} = Det D \tag{5.12}$$

To obtain a generating functional consider two 4-component spinor fields $\bar{\eta}(x), \eta(x)$. These fields are the analogue of the source $J(x)$ in the bosonic case. Define $Z[\bar{\eta}, \eta]$ by

$$\begin{aligned} Z[\bar{\eta}, \eta] &= N_0 \int [d\bar{\psi}][d\psi] e^{-<\bar{\psi}|D|\psi> + i<\bar{\eta}|\psi> + i<\bar{\psi}|\eta>} \\ &= N_0 \int [d\bar{\psi}][d\psi] e^{i \int d^4x [\bar{\psi}(x)(i\gamma^\mu \partial_\mu - m + i\epsilon)\psi(x) + \bar{\eta}(x)\psi + \bar{\psi}(x)\eta]} \end{aligned} \tag{5.13}$$

As in bosonic case we must complete the square in the exponent of the integrand,

$$\begin{aligned} - < \bar{\psi}|D|\psi > + i < \bar{\eta}|\psi > + i < \bar{\psi}|\eta > = &- < \bar{\psi} - i\bar{\eta}D^{-1}|D|\psi - iD^{-1}|\eta > \\ &- < \bar{\eta}|D^{-1}|\eta > \end{aligned} \tag{5.14}$$

Now the result of the integral can immediately be written, viz,

$$Z[\bar{\eta},\eta] = N_0[DetD]e^{-<\bar{\eta}|D^{-1}|\eta>} \tag{5.15}$$

The normalization factor N_0 is chosen such that $Z[0,0] = 1$. This gives

$$N_0^{-1} = DetD \tag{5.16}$$

and Eq(5.17) becomes,

$$Z[\bar{\eta},\eta] = e^{-<\bar{\eta}|D^{-1}|\eta>} \tag{5.17}$$

To see what D^{-1} is start from

$$DD^{-1} = 1 \quad \Rightarrow \int dy < x|D|y >< y|D^{-1}|x' >= \delta_4(x - x') \tag{5.18}$$

Substitute

$$< x|D|y >= -i(i\gamma^\mu\partial_\mu - m + i\epsilon)\delta_4(x - y)$$

Introduce S by,

$$S = D^{-1} = i(i\gamma^\mu\partial_\mu - m + i\epsilon)^{-1} \tag{5.19}$$

and $S(x - y)$ by

$$S(x - y) \equiv< x|D^{-1}|y > \tag{5.20}$$

Substitute these matrix elements in Eq(5.18) to obtain

$$-i(i\gamma^\mu\partial_\mu - m + i\epsilon)S(x - y) = \delta_4(x - y)$$

or,

$$(i\gamma^\mu\partial_\mu - m + i\epsilon)S(x - y) = i\delta_4(x - y) \tag{5.21}$$

We can then write Eq. (5.17) in the following form

$$Z[\bar{\eta},\eta] = e^{-\int d^4x d^4y \bar{\eta}(x) S(x-y)\eta(y)} \tag{5.22}$$

From canonical quantization we know that for a positive ϵ the solution of Eq. (5.21) is given by the Dirac propagator for a free Dirac spinor,

$$S(x-y) =< 0|T(\hat{\psi}(x)\hat{\bar{\psi}}(y)|0 > \tag{5.23}$$

Combining Eqs.(5.13), (5.16), (5.22) and (5.23) we obtain,

$$\begin{aligned} S(x-y) &=< 0|T(\hat{\psi}(x)\hat{\bar{\psi}}(y)|0 >= \\ &= \frac{\delta}{i\delta\bar{\eta}(x)}\frac{\delta}{i\delta\eta(y)} Z[\bar{\eta},\eta] \mid_{\eta=0,\bar{\eta}=0} \\ &= N_0 \int [d\bar{\psi}][d\psi]\psi(x)\bar{\psi}(y) e^{i\int d^4x \bar{\psi}(i\gamma^\mu\partial_\mu + m + i\epsilon)\psi} \end{aligned} \tag{5.24}$$

5.A. Exercise

1. Use canonical quantization of a free Dirac field to show that

$$< 0|Te^{i\int d^x(\bar{\eta}(x)\hat{\psi}(x)+\hat{\bar{\psi}}(x)\eta(x)}|0 > = e^{-\int d^4x d^4y \bar{\eta}(x)S(x-y)\eta(y)} \tag{5.25}$$

Comparison with Eq.(5.22) then shows that

$$Z[\bar{\eta},\eta] =< 0|Te^{i\int d^x(\bar{\eta}(x)\hat{\psi}(x)+\hat{\bar{\psi}}(x)\eta(x)}|0 >$$

Thus we obtain the same functional of $\bar{\eta}$ and η from the canonical quantization and functional integral methods.

2. Consider the Dirac action in the background of a vector potential $A_\mu(x)$ and

define a functional $W(A)$ by

$$e^{iW(A)} = N \int [d\bar{\psi}][d\psi] e^{i\int d^4x [\bar{\psi}(x)(i\gamma^\mu(\partial_\mu + ieA\mu) - m + i\epsilon)\psi(x)]} = Det D(A) \tag{5.26}$$

where,

$$iD(A) = i\gamma^\mu(\partial_\mu + ieA\mu) - m + i\epsilon) \tag{5.27}$$

Expand $W(A)$ in a power series of A and show that the n-th term in your series expansion is given by a 1-loop fermion graph with n vector field A attached to it. Show that each term in your expansion will have the correct minus sign necessary for a fermion loop in front of it.

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$$\begin{aligned} S(x-y) &=< 0|T(\hat{\psi}(x)\hat{\bar{\psi}}(y)|0 >= \\ &= \frac{\delta}{i\delta\bar{\eta}(x)}\frac{\delta}{i\delta\eta(y)} Z[\bar{\eta},\eta] \mid_{\eta=0,\bar{\eta}=0} \\ &= N_0 \int [d\bar{\psi}][d\psi]\psi(x)\bar{\psi}(y) e^{i\int d^4x \bar{\psi}(i\gamma^\mu\partial_\mu + m + i\epsilon)\psi} \end{aligned} \tag{5.24}$$

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2. Consider the Dirac action in the background of a vector potential $A_\mu(x)$ and

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where,

$$iD(A) = i\gamma^\mu(\partial_\mu + ieA\mu) - m + i\epsilon) \tag{5.27}$$

and N is given by the normalisation condition,

$$e^{iW(A=0)} = 1$$

Expand $W(A)$ in a power series of A and show that the n-th term in your series expansion is given by a 1-loop fermion graph with n vector field A attached to it. Show that each term in your expansion will have the correct minus sign necessary for a fermion loop in front of it.

The QED action is given by,

$$I[A,\psi,\bar{\psi}] \equiv \int d^4x\{\bar{\psi}(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\} \tag{6.1}$$

where,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This action is invariant under the local gauge transformation,

$$\psi(x) \to \psi'(x) = e^{-ie\Lambda(x)}\psi(x) \quad , \quad A_\mu(x) \to A'_\mu(x) = A_\mu + \partial_\mu\Lambda(x)$$

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where $d\mu \equiv [dA_\mu d\bar{\psi}d\psi]$. An example is the following,

$$< 0|T\hat{F}_{\mu_1\nu_1}(x_1),....\hat{F}_{\mu_n\nu_n}(x_n)|0 >= \frac{\int[d\mu]F_{\mu_1\nu_1}(x_1),....F_{\mu_n\nu_n}(x_n)e^{iI[A,\psi,\bar{\psi}]}}{\int[d\mu]e^{iI[A,\psi,\bar{\psi}]}} \tag{6.3}$$

The action is quadratic both in the vector field as well as in the spinor field. Let us write it in the following form,

$$I[A,\psi,\bar{\psi}] = \int d^4x\{\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}A^\mu M_{\mu\nu}A^\nu + A_\mu J^\mu\} \tag{6.4}$$

where $J^\mu(x) \equiv -e\bar{\psi}(x)\gamma^\mu\psi(x)$ and

$$M_{\mu\nu} = \partial^2 \eta_{\mu\nu} - \partial_\mu \partial_\nu \tag{6.5}$$

The integration over the fermion fields can be evaluated and the result will be a functional of A which must be subsequently integrated over A. Structurally the same is also true about the integrals over A. However, the integral over A does not make sense as it stands. The reason is that in order to perform this integral we must evaluate the inverse of the operator $M_{\mu\nu}$. This operator, however, does not have an inverse. To see this it is enough to note that any vector of the form $\partial^\nu \chi(x)$ is an eigenvector of M with eigenvalue zero,

$$M_{\mu\nu} \partial^\nu \chi(x) = 0$$

This problem manifest itself also in evaluating the vacuum expectation values of the gauge invariant operators as in Eq.(6.2) and Eq.(6.3). This is due to gauge invariance.

Another manifestation of gauge invariance is that the integrals on the numerator and denominator of the right hand side are divergent. The reason for this divergence is that the integrals integrate not only over A_μ but also over all its gauge equivalents. This infinity, of course, has nothing to do with the ultraviolet divergence.

Let us understand this geometrically. Denote the space of all vector potentials A_μ with $\mathcal{A}$ as in Fig 8. Each point in this space denotes the entire profile of a vector potential. The gauge groups acts in this space by mapping a point A_μ into anther point A'_μ which is related to A_μ by a gauge transformation. In this way starting from any point A_μ an entire orbit is generated. The field strength is invariant along each orbit. Hence it is an orbit function. The space $\mathcal{A}$ becomes fibered by the non intersecting orbits.

To avoid over counting we must choose a single point from each orbit. Joining these point will produce a section in $\mathcal{A}$ which cuts each orbit at a single point. This is a choice of a gauge.

A section in $\mathcal{A}$ is a surface given by the equation,

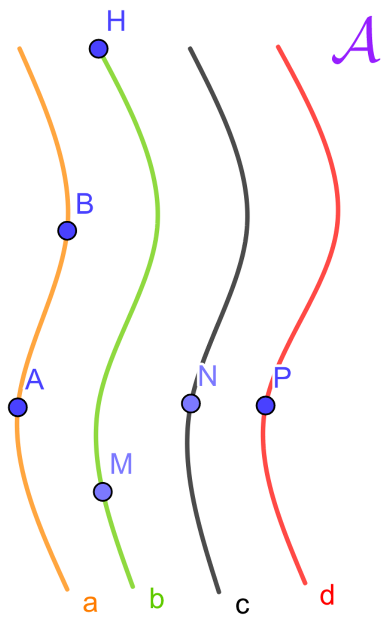

FIG. 8. Each point in the space $\mathcal{A}$ represents the entire profile of a vector potential A_μ. The lines $a, b, c, d, ...$ represent gauge orbits, i.e. all the points belonging to the same line are gauge related and hence have the same $F_{\mu\nu}$. Points A, M, N, P belong to different orbits and hence correspond to different tensor $F_{\mu\nu}$. The tensor $F_{\mu\nu}$ is a function on the space of orbits.

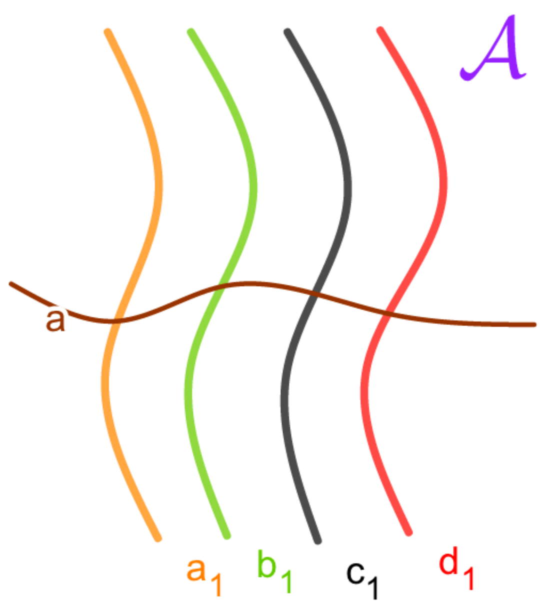

FIG. 9. A section a which cuts each orbit at a single point. Different sections will represent different gauge choices. Physics must be invariant under this arbitrariness.

$$f(A) = 0$$

Examples are,

$$f(A) = A_0 \quad , \quad f(A) = \partial^i A_i \quad , \quad f(A) = \partial_\mu A^\mu$$

Essentially we would like to restrict the domain of integration from $\mathcal{A}$ to the surface $f(A) = 0$. This is going to be done by inserting a δ functional of A in the integrand of Eq.(6.2) and Eq.(6.3). This is the so called Faddeev-Popve method and it works as follows.

Define a functional of $\Delta(A)$ by the following functional integral,

$$\Delta^{-1}(A) = \int d\Lambda \delta(f^\Lambda(A))$$

where Λ denotes a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ and $f^\Lambda(A)$ denotes the gauge transform of f, viz,

$$f^\Lambda(A) \equiv f(A + \partial_\mu \Lambda)$$

Note that for any arbitrary but fixed gauge transformation with the parameter Λ' we have $\Delta(A) = \Delta(A + \partial\Lambda')$. In other words $\Delta(A)$ is a gauge invariant functional of A

We shall insert 1 in the form of

$$1 = \Delta(A) \int d\Lambda \delta(f^\Lambda(A)) \tag{6.6}$$

in the integrand of Eq.(6.2) and change the order of integration over A and Λ. We obtain,

$$< 0|TG(\hat{A}, \hat{\bar{\psi}}, \hat{\psi})|0 >= \frac{\int [d\Lambda] \int [d\mu] G(A, \bar{\psi}, \psi) \Delta(A) \delta(f^\Lambda(A)) e^{iI[A,\psi,\bar{\psi}]}}{\int d\Lambda] \int [d\mu] \Delta(A) \delta(f^\Lambda(A)) e^{iI[A,\psi,\bar{\psi}]}} \tag{6.7}$$

For any given Λ we perform a gauge transformation with a parameter $-\Lambda$. This maps $f^\Lambda(A)$ to $f(A)$. Since every other factor in the integrand is gauge invariant the integrand becomes independent of Λ and the (infinite) integral over Λ cancels from the numerator and denominator. We obtain,

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We must calculate $\Delta(A)$.

The QED action is given by,

$$I[A,\psi,\bar{\psi}] \equiv \int d^4x\{\bar{\psi}(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\} \tag{6.1}$$

where,

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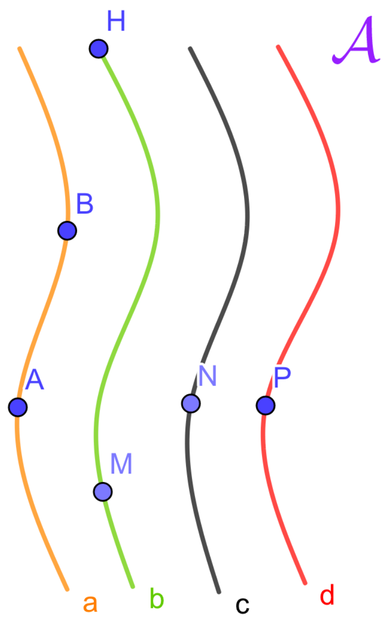

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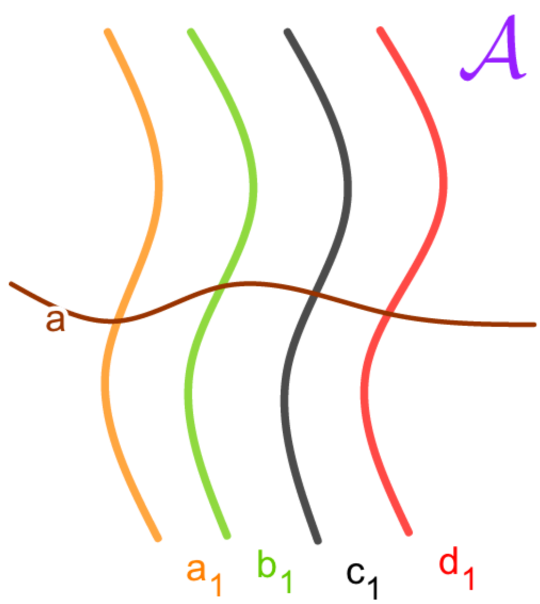

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We must calculate $\Delta(A)$. Since in Eq.(6.8) we need $f(A)$ near zero we can take Λ to be infinitesimal in Eq.(6.6). We can then approximate $f(A + \partial\Lambda)$ with $f(A) + \frac{\partial f}{\partial A_\mu}\partial^\mu\Lambda = M\Lambda$ where,

$$M = \frac{\partial f}{\partial A_\mu}\partial^\mu \tag{6.9}$$

.

Eq. (6.6) then gives us,

$$\begin{aligned} 1 &= \Delta(A) \int d\Lambda \delta(M\Lambda) \\ &= \Delta(A) \frac{1}{detM} \end{aligned}$$

Or,

$$\Delta(A) = detM \tag{6.10}$$

6.A. Gauge Independence and Faddeev-Popov Ghost Fields

Does the matrix element on the left hand side of Eq.(6.8) depend on the choice of the section $f(A) = 0$? Intuitively the answer is clearly no. We started from a gauge invariant expression and inserted 1 under the integration sign. All we have done was to manipulate this insertion. We can verify this by choosing a different section $g(A) = 0$ and repeat the procedure. Attach a subscript f or g to the matrix elements to denote the gauge choice $f = 0$ or $g = 0$, respectively, and in the integrand of Eq. (6.8) insert 1 in the form of

$$1 = \Delta(A) \int d\Lambda \delta(g^\Lambda(A))$$

$$
\begin{aligned}
<0|TG(\hat{A},\hat{\bar{\psi}},\hat{\psi})|0>_f &= \frac{\int[d\mu]G(A,\bar{\psi},\psi)\Delta(A)\delta(f(A))\Delta(A)\int d\Lambda\delta(g^{\Lambda}(A))e^{iI[A,\psi,\bar{\psi}]}}{D} \\
&= \frac{\int d\Lambda\int[d\mu]G(A,\bar{\psi},\psi)\Delta(A)\delta(f(A))\Delta(A)\delta(g^{\Lambda}(A))e^{iI[A,\psi,\bar{\psi}]}}{D} \\
&= \frac{\int d\Lambda\int[d\mu]G(A,\bar{\psi},\psi)\Delta(A)\delta(f^{-\Lambda}(A))\Delta(A)\delta(g(A))e^{iI[A,\psi,\bar{\psi}]}}{D} \\
&= \frac{\int[d\mu]G(A,\bar{\psi},\psi)\Delta(A)\delta(g(A))e^{iI[A,\psi,\bar{\psi}]}}{D} \\
&=<0|TG(\hat{A},\hat{\bar{\psi}},\hat{\psi})|0>_g
\end{aligned}
\tag{6.11}
$$

where D denotes the functional integral in the denominator. We perform the same operations in D as we do on the numerator.

In the second equality of Eq. (6.11) we have interchanged the order of integrations and pulled out $\int[d\Lambda]$. In the third equality we have performed a gauge transformation with the parameter $-\Lambda$. This takes $g^{\Lambda}(A)$ to $g(A)$ and $f(A)$ to $f^{-\Lambda}(A)$. In the fourth equality we have made use of

$$
1=\Delta(A)\int d\Lambda\delta(f^{-\Lambda}(A))
$$

By virtue of this identity $\Delta(A)\delta(f^{-\Lambda}(A))$ has disappeared from the integrands in the numerator and denominator of the right hand side of Eq(6.11). In summary we have proven that for any gauge invariant observable $\hat{G}$ the choice of the gauge (or the section or the integration surface in the space $\mathcal{A}$) is immaterial,

$$
<0|TG(\hat{A},\hat{\bar{\psi}},\hat{\psi})|0>_f=<0|TG(\hat{A},\hat{\bar{\psi}},\hat{\psi})|0>_g \tag{6.12}
$$

This allows us to write the $\delta(f)$ in Eq.(6.8) in a convenient form. We shall clarify this in the context of a class of popular gauge choice, namely,

$$
f(A)=\partial_\mu A^\mu(x)-h(x)
$$

where h is any local function. Since the matrix elements of the gauge invariant operators do not depend on f, and hence neither they depend on h, we can multiply the integrand of the numerator and denominator of Eq.(6.8) by any functional of h and integrate over it. We then use the basic property of the δ functional, namely,

$$\int [dh]\delta(\partial_\mu A^\mu - h)F[h] = F[\partial_\mu A^\mu]$$

For instance the denominator of Eq.(6.8) becomes,

$$\begin{aligned}\int [d\mu]\Delta(A)\delta(f(A))e^{iI[A,\psi,\bar{\psi}]} &= \int [dh]\int [d\mu]\Delta(A)\delta(\partial_\mu A^\mu - h)F[h]e^{iI[A,\psi,\bar{\psi}]} \\ &= \int [d\mu]\Delta(A)F[\partial_\mu A^\mu]e^{iI[A,\psi,\bar{\psi}]}\end{aligned}$$

Choose

$$F[h] = e^{-\frac{i}{2\xi}\int dx h^2(x)} \tag{6.13}$$

.

where ξ is a real number. We then obtain,

$$\int [d\mu]\Delta(A)\delta(f(A))e^{iI[A,\psi,\bar{\psi}]} = \int [d\mu]\Delta(A)e^{iI[A,\psi,\bar{\psi}]-\frac{i}{2\xi}\int dx(\partial_\mu A^\mu(x))^2]}$$

The same steps must be repeated in the denominator of Eq.(6.8) as well.

It is also easy to calculate $\Delta(A) = det(\frac{\delta f}{\delta A_\mu}\partial_\mu)$ for $f(A) = \partial_\mu A^\mu(x) - h(x)$. We have

$$\frac{\delta f(A(y))}{\delta A_\mu(x)}\partial_\mu = \partial_\mu \delta(x-y)\partial^\mu$$

Hence,

$$\Delta(A) = det\partial^2$$

This is a constant and cancels from the numerator and the denominator of Eq.(6.8).

Collecting all the results above we finally obtain write Eq.(6.8) in the following neat form,

$$< 0|TG(\hat{A},\hat{\bar{\psi}},\hat{\psi})|0 >= \frac{\int[d\mu]G(A,\bar{\psi},\psi)e^{iI[A,\psi,\bar{\psi}]-\frac{i}{2\xi}\int dx(\partial_\mu A^\mu(x))^2]}}{\int[d\mu]e^{iI[A,\psi,\bar{\psi}-\frac{i}{2\xi}\int dx(\partial_\mu A^\mu(x))^2]]}} \tag{6.14}$$

We see that, in the covariant gauges in QED the final effect of the Faddeev-Popov method is just to add a gauge fixing term to the action. Due to the presence of this term the difficulty with the integration over A_μ field disappears.

The total action of QED then becomes,

$$I_{total} = I[A,\psi,\bar{\psi}] - \frac{1}{2\xi}\int dx(\partial_\mu A^\mu(x))^2 \tag{6.15}$$

Space-time Symmetries of QED

QED has two independent symmetry groups.

1. The Poincare invariance under which

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu + b^\mu, \quad \Lambda \in O(1,3), \quad b^\mu \in R^4$$

Under this transformation the fermion field ψ and the Maxwell field A_μ transform like a 4 component spinor and a real component vector fields, respectively. Each term in QED action is independently invariant under Poincare transformations.

2. $U(1)$- Gauge Symmetry of QED

This is independent of the Poincare symmetry and acts by a local phase transformation of ψ

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x)$$

The Dirac mass term is invariant under this transformation, but the kinetic term

$$\bar{\psi}'(x)\not{\partial}\psi' = \bar{\psi}(x)\not{\partial}\psi + \bar{\psi}(x)(i\not{\partial}\alpha)\psi$$

The Role of A_μ From geometrical point of view the role of vector field is to restore the invariance of the Dirac action under a local $U(1)$ gauge transformations. Even if Maxwell had not discovered his equations we could discover them by insisting on the local gauge invariance of the Dirac equation!

Covariant Derivative: $\not{\nabla} \equiv \not{\partial} + ieA_\mu(x)$ Replace the $\not{\partial}$ by $\not{\nabla}$ defined by,

$$\not{\nabla}\psi(x) \equiv (\not{\partial} - ie\not{A})\psi(x)$$

Demand that under a gauge transformation $\not{\nabla}$ transforms as,

$$\not{\nabla}' = e^{i\alpha(x)}\not{\nabla}e^{-i\alpha(x)}, \quad \not{\nabla}' = \not{\partial} - ie\not{A}'$$

This implies,

$$\nabla'_\mu = e^{i\alpha(x)} \nabla_\mu e^{-i\alpha(x)}$$

Under this condition we shall have

$$\begin{aligned} \not{\nabla}' \psi' &= [e^{i\alpha(x)} \not{\nabla} e^{-i\alpha(x)}][e^{i\alpha(x)} \psi] \\ &= e^{i\alpha(x)} \not{\nabla} \psi \end{aligned}$$

This means that $\not{\nabla}\psi$ transorms in the same way as ψ itself. This is the reason why it is called a *covariant derivative.*

$$\begin{aligned} \bar{\psi}'(x) \not{\nabla}' \psi' &= [\bar{\psi}(x) e^{-i\alpha(x)}][e^{i\alpha(x)} \not{\nabla} e^{-i\alpha(x)}][e^{i\alpha(x)} \psi] \\ &= \bar{\psi}(x) \not{\nabla} \psi \end{aligned}$$

The transformation rule of A_μ is obtained from $\not{\nabla}' = e^{i\alpha(x)} \not{\nabla} e^{-i\alpha(x)}$,

$$\begin{aligned} \not{\partial} - ie\not{A}' &= e^{i\alpha(x)} \not{\partial} e^{-i\alpha(x)} + \not{A} \\ &= \not{\partial} - i\not{\partial}\alpha - ie\not{A} \qquad \Rightarrow \qquad \not{A}' = \not{A} + \frac{1}{e}\not{\partial}\alpha \end{aligned}$$

Or,

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \alpha$$

Field Strength $F_{\mu\nu}$:

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu] &= [\partial_\mu - ieA_\mu, \partial_\nu - ieA_\nu] \\
&= -ie\partial_\mu A_\nu + ie\partial_\nu A_\mu \\
&= -ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \\
&= -ieF_{\mu\nu}
\end{aligned}
$$

Gauge Invariance of $F_{\mu\nu}$

$$
\begin{aligned}
-ieF'_{\mu\nu} &= [\nabla'_\mu, \nabla'_\nu] \\
&= [e^{i\alpha(x)}\nabla_\mu e^{-i\alpha(x)}, e^{i\alpha(x)}\nabla_\nu e^{-i\alpha(x)}] \\
& e^{i\alpha(x)}\nabla_\mu e^{-i\alpha(x)} e^{i\alpha(x)}\nabla_\nu e^{-i\alpha(x)} - e^{i\alpha(x)}\nabla_\nu e^{-i\alpha(x)} e^{i\alpha(x)}\nabla_\mu e^{-i\alpha(x)} \\
&= e^{i\alpha(x)}\nabla_\mu \nabla_\nu e^{-i\alpha(x)} - e^{i\alpha(x)}\nabla_\nu \nabla_\mu e^{-i\alpha(x)} \\
&= e^{i\alpha(x)}[\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu] e^{-i\alpha(x)} \\
&= e^{i\alpha(x)}[\nabla_\mu, \nabla_\nu] e^{-i\alpha(x)} \\
&= e^{i\alpha(x)}(-ieF_{\mu\nu}) e^{-i\alpha(x)} \\
&= -ieF'_{\mu\nu} \qquad \Rightarrow \qquad F'_{\mu\nu} = F_{\mu\nu} \qquad (7.1)
\end{aligned}
$$

7.A. $SU(2)$ Gauge Theory

Now consider the group $SU(2)$ and a Dirac spinor ψ in its fundamental representation. This means that ψ has two different kind of indices, $\psi_{\alpha i}(x)$, where $\alpha = 1, 2, 3, 4$ is the spinor index w.r.t the Lorentz transformations and $i = 1, 2$ is the $SU(2)$ index. We can consider ψ as a doublet of $SU(2)$. Each one of two components ψ_i of this doublet is a 4 component Dirac spinor. Thus ψ has altogether 8 complex components. Under $u \in SU(2)$ ψ transforms as,

$$
\psi'(x) = u\psi(x), \qquad \Rightarrow \qquad \psi'_i(x) = u_i^j \psi_j(x) \qquad (7.2)
$$

Now consider the Dirac Lagrangian,

$$L = \bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi$$

Clearly if $u \in SU(2)$ is an x-independent 2×2 unitary matrix then each term in L will be invariant. We would like to consider a generalisation in which u is a 2×2 unitary matrix with x dependent elements. Then $\partial_\mu u \neq 0$ and, like the $U(1)$ case the kinetic term will not be invariant. Indeed,

$$\begin{aligned}\bar{\psi}'(x)\partial\!\!\!/\psi'(x) &= \bar{\psi}(x)u^\dagger(x)\partial\!\!\!/(u(x)\psi(x)) \\ &= \bar{\psi}(x)\partial\!\!\!/\psi(x) + \bar{\psi}(x)(u^\dagger(x)\partial\!\!\!/u(x))\psi(x))\end{aligned}$$

The offending term has a 2×2 matrix $u^\dagger(x)\partial\!\!\!/u(x)$ sandwiched between the row matrix $\bar{\psi}$ and the column matrix ψ.

Hence to restore the invariance under $S(2)$ we must introduce a set of vector fields assembled into a 2×2 matrix $(A_\mu)_i^j, i, i = 1, 2$. We shall suppress the matrix indices and introduce the $SU(2)$ covariant derivative,

$$\begin{aligned}\nabla_\mu\psi(x) &= \partial_\mu\psi - ieA_\mu(x)\psi(x) \\ &= (\partial_\mu - ieA_\mu(x))\psi(x)\end{aligned}$$

In component notation this equation means,

$$\nabla_\mu\psi_i(x) = \partial_\mu\psi_i - ie(A_\mu(x))_i^j\psi_j(x)$$

In order for ∇_μ to be a covariant derivative it mast transform like ψ itself,

$$\begin{aligned}\nabla'_\mu\psi'(x) &= u(x)\nabla_\mu\psi(x) \\ &= u(x)\nabla_\mu u^{-1}(x)\psi'(x)\end{aligned}$$

Thus,

$$\nabla'_\mu = u(x)\nabla_\mu u^{-1}(x)$$

Or,

$$\begin{aligned}\partial_\mu - ieA'_\mu(x) &= u(x)(\partial_\mu - ieA_\mu(x))u^{-1}(x)\\ &= \partial_\mu + u(x)\partial_\mu u^{-1}(x) - ieu(x)A_\mu(x)u^{-1}(x)\end{aligned}$$

Thus,

$$A'_\mu(x) = u(x)A_\mu(x)u^{-1}(x) + \frac{i}{e}u(x)\partial_\mu u^{-1}(x) \tag{7.3}$$

It is easy to see that the second term on the r.h.s. is a traceless Hermitian matrix (see Exercises). We shall take the matrix A_μ to be likewise traceless and Hermitian. Then A'_μ will have the same properties.

Note that if in Eq.(10.3) $u(x) = e^{i\alpha(x)}$ with $\alpha(x)$ an arbitrary scalar function, then this relation reduces to the transformation rule of the Maxwell field.

SU(2) Gauge Invariant Dirac Lagrangian:

We have built up the transformation properties of ψ and A_μ in such a way that the Lagrangian,

$$I_{Dirac}(\bar{\psi}, \psi, A) = \int d^4x[\bar{\psi}\nabla\!\!\!/\psi - m\bar{\psi}\psi]$$

is invariant under local $SU(2)$ gauge transformations. This means that,

$$I(\bar{\psi}', \psi', A') = I(\bar{\psi}, \psi, A)$$

where

$$\psi'(x) = u(x)\psi(x)' \qquad and \qquad \bar{\psi}'(x) = \bar{\psi}(x)u^\dagger(x)$$

and where $A'_\mu(x)$ and $A_\mu(x)$ are related by the transformation rule given by Eq. (10.3).

Component Notation

An arbitrary 2×2 unitary matrix can be written as,

$$u(x) = e^{i\frac{\vec{\alpha}(x)}{2}.\vec{\sigma}}$$

Here $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the three Pauli matrices and $\vec{\tau} = (\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_2, \frac{1}{2}\sigma_3)$ the generators of $SU(2)$ in the fundamental representation, i.e.

$$[\tau_a, \tau_b] = i\epsilon_{abc}\tau_c$$

Since A_μ is a 2×2 traceless Hermitian matrix it can be written as a linear combination of Pauli matrices, or the generators τ_a of $SU(2)$,

$$A_\mu = A^a_\mu(x)\tau_a$$

The expansion coefficients $A^a_\mu(x), a = 1, 2, 3$ form a triplet of vector potentials. $SU(2)$. Indeed under a *constant* $u \in SU(2)$ we have,

$$\begin{aligned} A'^a_\mu(x)\tau_a &= u(x)A^b_\mu(x)\tau_b u^{-1}(x) + \frac{i}{e}u(x)\partial_\mu u^{-1}(x) \\ &= A^b_\mu(x)u(x)\tau_b u^{-1}(x) + \frac{i}{e}u(x)\partial_\mu u^{-1}(x) \end{aligned}$$

Group theory tells us that,

$$u(x)\tau_b u^{-1}(x) = D^a_b(u^{-1}(x))\tau_a$$

The 3×3 matrices $D^a_b(u^{-1}(x)), a, b = 1, 2, 3$ represent the group element $u(x) \in SU(2)$ in the adjoint representation of $SU(2)$. Using the identity $tr(\tau_a\tau_b) = \frac{1}{2}\delta_{ab}$ we thus obtain the following transformation rule for the component fields $A^a_\mu(x)$,

$$A'^a_\mu(x) = A^b_\mu(x) D^a_b(u^{-1}(x)) + 2 \times \frac{i}{e} tr[u(x)\partial_\mu u^{-1}(x)\tau^a] \tag{7.4}$$

If it were not for the presence of the second term on the right hand side A_μ would transform homogeneously under gauge transformations.

Field Strength $F_{\mu\nu}$

We follow the same steps as we did for the $U(1)$ case and study the commutator of two covariant derivatives, $\partial_\mu - ieA_\mu(x)$

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] &= [\partial_\mu - ieA_\mu, \partial_\nu - ieA_\nu] \\
&= -ie(\partial_\mu A_\nu - \partial_\nu A_\mu) - e^2[A_\mu, A_\nu] \\
&= -ie\{\partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]\}
\end{aligned}$$

Call the right hand side $-ieF_{\mu\nu}$, where

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ie[A_\mu(x), A_\nu(x)] \tag{7.5}$$

We thus have,

$$[\nabla_\mu, \nabla_\nu] = -ieF_{\mu\nu}(x)$$

Substituting $A_\mu(x) = A^a_\mu(x)\tau_a$ we can expend $F_{\mu\nu}(x)$ on the basis of τ_a,

$$\begin{aligned}
F^a_{\mu\nu}(x)\tau_a &= (\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x))\tau_a - ie[A^b_\mu(x)\tau_b, A^c_\nu(x)\tau_c] \\
&= (\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x))\tau_a - ieA^b_\mu(x)A^c_\nu(x)[\tau_b, \tau_c] \\
&= (\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x))\tau_a - ieA^b_\mu(x)A^c_\nu(x) i\epsilon_{bca}\tau_a \\
&= (\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + eA^b_\mu(x)A^c_\nu(x)\epsilon_{bca})\tau_a
\end{aligned}$$

Thus,

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + e\epsilon_{abc}A^b_\mu(x)A^c_\nu(x) \tag{7.6}$$

Unlike the QED the non Abelian version of the field strength tensor has a quadratic term in the definition of $F_{\mu\nu}$. This non-linear term is at the origin of vastly difference physics of the non abelian gauge theories.

How does $F_{\mu\nu}$ transform under gauge transformations?

The most straightforward way would be to substitute the transformation rule of A_μ on the right hand side of Eq. (10.5) or Eq. (10.6) and obtain the transformation rule of $F_{\mu\nu}$. A more elegant and shorter way is to repeat what we did in the case of QED, namely, use Thus, the definition of F as the commutator of two covariant derivatives,

$$\begin{aligned}
-ieF'_{\mu\nu} &= [\nabla'_\mu, \nabla'_\nu] \\
&= [u(x)\nabla_\mu u^{-1}(x), u(x)\nabla_\nu u^{-1}(x)] \\
&= u(x)[\nabla_\mu, \nabla_\nu]u^{-1}(x) \\
&= u(x)(-ieF_{\mu\nu})u^{-1}(x)
\end{aligned}$$

Thus,

$$F'_{\mu\nu} = u(x)F_{\mu\nu})u^{-1}(x) \tag{7.7}$$

This shows that unlike the QED case $F_{\mu\nu}$ is not guage invariant, but also unlike the non Abelian vector potential, it transforms homogeneously. In terms of components, this means,

$$F'^b_{\mu\nu}(x) = F^b_{\mu\nu}(x)D^a_b(u^{-1}(x)) \tag{7.8}$$

We saw above the way we can generalise the Dirac action so that it is invariant under local $SU(2)$ gauge transformations. In this action the gauge field is a background non propagating field. In order to include the dynamics of the gauge field in our action in a gauge invariant way we must employ the field strength tensor $F_{\mu\nu}$ which, unlike the vector potential transforms in a linear representation of $SU(2)$ without an additive inhomogeneous term. The most obvious invariant we can construct from this tensor is the trace of its square,

$$\begin{aligned} I_{YM}(A) &= \int d^4x[-\frac{1}{2}trF_{\mu\nu}(x)F^{\mu\nu}(x)] \\ &= \int d^4x[-\frac{1}{2}F^a_{\mu\nu}(x)F^{b\mu\nu}(x)]tr(\tau^a\tau^b) \end{aligned}$$

where we used $tr(\tau^a\tau^b) = \frac{1}{2}\delta^{ab}$. Hence,

$$I_{YM}(A) = \int d^4x[-\frac{1}{4}F^a_{\mu\nu}(x)F^{a\mu\nu}(x)] \tag{7.9}$$

Note that unlike Maxwell's action the non Abelian version has self interactions between the vector potential A^a_μ with itself. This is due to the fact that F is a sum of a linear and quadratic terms in A. Hence its square will contain, quadratic, cubic and quartic terms in A. To see this in detail let us substitute for F from Eq.(10.6)

$$I_{YM}(A) = -\frac{1}{4}\int d^4x(\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + e\epsilon_{abc}A^b_\mu(x)A^c_\nu(x))^2$$

The quadratic part will be a sum of 3 copies of the Maxwell action, namely,

$$I_{2YM}(A) = \int d^4x[-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})] \tag{7.10}$$

After gauge fixing this part will give us the propagator for the vector potential.

The cubic part will contain one power of the gauge coupling e

$$I_{3YM}(A) = -\frac{e}{2}\int d^4x(\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x))\epsilon_{abc}A^b_\mu(x)A^c_\nu(x)) \tag{7.11}$$

This an interaction term with derivative coupling. Thus the Feynman rules in Fourier space corresponding to this term in the interaction Lagrangian will contain a momentum factor. Finally the quartic term is given by,

$$I_{4YM}(A) = -\frac{e^2}{4}\int d^4x(\epsilon_{abc}A^b_\mu(x)A^c_\nu(x))^2$$

Due to the presence of the cubic and quartic terms in the Yang-Mills action integral the such fields, unlike Maxwell's theory, the Yang-Mills fields are always interacting fields even in the absence of the matter fields. The gauge invariance does not allow the existence of non-interacting non- abelian gauge fields. In this sense the physics of non-abelian gauge theories is more alike that of gravity than electromagnetism.

The full action integral of matter and Yang-Mills fields is obtained by adding the matter and Yang-Mills actions $I(\bar{\psi}, \psi, A) = I_{Dirac}(\bar{\psi}, \psi, A) + I_{YM}(A)$,

$$I(\bar{\psi}, \psi, A) = \int d^4x[-\frac{1}{4}F^a_{\mu\nu}(x)F^{a\mu\nu}(x) + \bar{\psi}\nabla\!\!\!\!/\,\psi - m\bar{\psi}\psi] \tag{7.12}$$

The above construction of a $SU(2)$ non abelian Yang-Mills theory can be generalised in several directions. First of all we can consider an arbitrary compact gauge group G and a r-dimensional irreducible unitary representation of this group. Denote the generators in this representation by $r \times r$ Hermitian matrices t_a, $a = 1, 2, ...d$, where $d \equiv DimG$. This set forms a Lie algebra,

$$[t_a, t_b] = if_{abc}t_c$$

where f_{abc} are fully antisymmetric structure constants of G.

Assume the set of Dirac spinors $\psi_i(x)$, $i = 1, ...r$ transforms in the r dimensional representation of G. This means under any $g \in G$ and assume that this group element is characterised by the set of group parameters $\alpha_1, ...\alpha_d$. A local gauge transformation can be considered to be a map from space-time R^4 to the group manifold G.

$$g : R^4 \to G, \qquad g : x \in R^4 \to g(x) \in G$$

This means that the group parameters α_a become functions of $x \in R^4$

Thus in any representation of G the matrices $u(g)$representing an element $g \in G$ will become functions of the space-time coordinates x. We shall write the transformation law as

$$\psi_i'(x) = u_i^j(x)\psi_j(x)$$

where $u(x) = e^{i\alpha(x).t}$, is a $r \times r$ unitary matrix which represents the group element $g(x) \in G$. We use the abbreviation,

$$\alpha(x).t \equiv \Sigma_1^d \alpha_a(x)t_a = \alpha_a(x)t_a$$

From this point on the definitions of the gauge covariant derivative, etc parallels exactly that of $SU(2)$ with the sole exception that the generators τ_a of $SU(2)$ should be replaced by the matrices t_a. The vector potentials will be written as a matrix

$$A_\mu(x) = A_\mu^a(x) t_a$$

Hence there will always be as many $A_\mu^a(x)$'s as the dimension d of G, regardless to which representation the matter fields belong. The gauge transformation of A_μ^a will be given by an equation exactly similar to Eq. (6.4),

$$A_\mu'^a(x) = A_\mu^b(x) D_b^a(g^{-1}(x)) + 2 \times \frac{i}{e} tr[g(x)\partial_\mu g^{-1}(x) t^a]$$

The$d \times d$ matrices $D_b^a(g^{-1}(x))$ represent the group element $g(x) \in G$ in the adjoint representation. Likewise $F_{\mu\nu}^a$ transform in the adjoint representation.

Example: Quantum Chromodynamics, QCD

The basic constituents for the strong interactions are quarks. Quarks do participate in the weak and electromagnetic interactions. Here we would like to concentrate only on the strong interactions and, for the time being ignore other ones. We shall include them later.

There are 6 observed types of quarks denoted by u, d, c, s, t and b. Each type is called a flavour. As far as as QCD is concerned each quark is a 4-component Dirac spinor.

First we consider one flavour, say u quark. Its strong interactions is governed a $SU(3)$ Yang-Mills theory. Having specified the gauge group the only other element we need to specify the action fully is the $SU(3)$ representation of the quark. It turns out that the right choice is the fundamental 3-dimensional representation of $SU(3)$. We shall denote the spinor field as $\psi_u(x)$. This field thus has 4×3 complex components. The indices will be suppressed. The covariant derivative of $\psi_u(x)$ will thus be,

$$\nabla_\mu \psi_u(x) = \partial_\mu \psi_u(x) - ig_3 G_\mu(x)\psi_u(x)$$
$$= (\partial_\mu - ig_3 G_\mu(x))\psi_u(x)$$

where,

$$G_\mu(x) = G^r_\mu(x)t_r, \qquad r = 1, ...8$$

$t_r, 1, ..., 8$ are the 8 generators of $SU(3)$ in its fundamental. They are 3×3 hermitian matrices acting on the 3 components of ψ_u. The action thus becomes,

$$I(\bar{\psi}, \psi, A) = \int d^4x[-\frac{1}{4}G^r_{\mu\nu}(x)G^{r\mu\nu}(x) + \bar{\psi}_u i \not{\nabla} \psi_u - m\bar{\psi}_u\psi_u]$$

where,

$$G^r_{\mu\nu}(x) = \partial_\mu G^r_\nu(x) - \partial_\nu G^r_\mu(x) + g_3 f_{rst} G^s_\mu(x) G^t_\nu(x)$$

We can generalise this action to include all the quark flavours. We simply need to replace the the subscript u by an index n which runs from 1 to 6 with $\psi_1 = \psi_u$, $\psi_2 = \psi_d$, $\psi_3 = \psi_c$, ...

$$I(\bar{\psi}, \psi, A) = \int d^4x[-\frac{1}{4}G^r_{\mu\nu}(x)G^{r\mu\nu}(x) + \Sigma^6_{n=1}(\bar{\psi}_n i \not{\nabla} \psi_n - m_i\bar{\psi}_n\psi_n)]$$

Inclusion of Scalars

So far the only matter fields we considered are the Dirac spinor fields. Next we consider a set of scalar fields $\phi_m(x)$, $m = 1, ...k$ transforming in a k-dimensional representation of G. Denote the generators in this representation by $k \times k$ matrices, $T_a, a = 1, ...d$. Under the G-action we shall have,

$$\phi'^m(x) = D^m_n(g(x))\phi^n(x) \tag{6.13}$$

We assemble the components ϕ into a column matrix with k elements and define the G-covariant derivative of ϕ by,

$$\nabla_\mu \phi(x) = (\partial_\mu + iA^a_\mu(x)T^a)\phi(x) \tag{6.14}$$

This derivative will transform covariant manner,

$$\nabla'_\mu \phi'(x) = D(g(x))\nabla_\mu \phi(x) \tag{6.15}$$

Thus we can construct gauge invariant action for the ϕ field of the form,

$$I_{Scalar}(\phi^\dagger, \phi, A) = \int d^4x[(\nabla^\mu \phi(x))^\dagger \nabla_\mu \phi(x) - V(\phi^\dagger \phi)]$$

Where V is any function of the gauge invariant quantity $\phi^\dagger(x)\phi(x)$, such as,

$$V(\phi^\dagger \phi) = m^2 \phi^\dagger(x)\phi(x) + \frac{\lambda}{4}(\phi^\dagger(x)\phi(x))^2$$

Thus a more general action including spinors in the r dimensional representation and scalars in the k dimensional represent will be given by,

$$\begin{aligned} I(\bar{\psi}, \psi, A) = \int d^4x[-\frac{1}{4}F^a_{\mu\nu}(x)F^{a\mu\nu}(x) + \bar{\psi} i \not{\nabla} \psi - m\bar{\psi}\psi \\ + (\nabla^\mu \phi(x))^\dagger \nabla_\mu \phi(x) - V(\phi^\dagger \phi) + L_{Yuk}] \end{aligned} \tag{6.16}$$

where the so called Yukawa terms L_{Yuk} contains the sum of gauge invariant monomials which can be constructed from ψ and ϕ. For example, assume $G = SU(2)$. Let us take ψ and ϕ to transform in the doublet and triplet representations of G. Then,

$$L_{Yuk} = f_y \bar{\psi}(x)\tau^a \phi^a(x)\psi(x)$$

is a possible candidate for a Yukawa term. The coupling f_y is called a Yukawa coupling.

Exercises

1. Prove that the covariant derivative satisfies Leibniz rule.
1. Show that $iu(x)\partial_\mu u^{-1}(x)$ is traceless hermitian matrix.
3. Consider a $U(1)$ gauge potential $A_\mu(x)$ and a path P from y to x given by

$$z^\mu = z^\mu(\lambda), \qquad , \tag{6.17}$$

where $0 \leq \lambda \leq 1$ and such that and $z^\mu(\lambda = 0) = x^\mu$ and $z^\mu(\lambda = 1) = y^\mu$. Define the the Parallel transport operator (Wilson loop)

$$W_P(x, y) = exp[i \int_0^1 d\lambda \frac{dz^\mu}{d\lambda} A_\mu(z(\lambda))]$$

(i). Show that under a gauge transformation $A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x)$ we have,

$$W'_P(x, y) = e^{i\alpha(x)} W_P(x, y) e^{-i\alpha(y)}$$

(ii). Now let $\phi(x)$ be a charged scalar field and show that,

$$W'_P(x, y)\phi'(x) - \phi'(x) = e^{i\alpha(x)}(W_P(x, y)\phi(y) - \phi(y))$$

This relation allows us to compare $\phi(x)$ and $\phi(y)$ in a gauge covariant way.

(iii). Now let $y = x + \delta x$ and show that,

$$W_P(x, y)\phi(y) - \phi(y) = i\delta x^\mu \nabla_\mu \phi(x) + O(\delta x^2)$$

where,

$$\nabla_\mu \phi(x) \equiv \partial_\mu \phi(x) - iA_\mu \phi(x)$$

(iv). Consider a closed path (i.e. $z^\mu(\lambda = 0) = z^\mu(\lambda = 1)$). Then $W_P(x = y)$ becomes a gauge invariant functional of A_μ. Hence it must be expressible in terms of $F_{\mu\nu}$. Find this functional.

4. In the example of $SU(3)$ gauge theory let $U(x) \in SU(3)$ be of the form,

$$U(x) = 1 + i\Sigma_{r=1}^{8}\theta^r(x)t^r$$

where the parameters $\theta^r(x), r = 1, ..., 8$ are infinitesimal. Define the infinitesimal variations by,

$$\begin{aligned} &\delta G_\mu^r(x) \equiv G_\mu'^r(x) - G_\mu^r(x) \\ &\delta\psi^i(x) \equiv \psi'^i(x) - \psi^i(x), \qquad i = 1, 2, 3 \end{aligned} \tag{6.18}$$

Evaluate $\delta G_\mu^r(x)$, $\delta G_{\mu\nu}^r(x)$ and $\delta\psi^i$ up to the first order terms in $\theta^r(x)$. Show that the action is invariant up to the first order terms in $\theta^r(x)$

The problems which we faced in applying the straightforward functional integration technics to QED persist also for the Yang-Mills theories. The origin of these difficulties is the same, namely, local gauge invariance which leads to multiple counting of the contributions of the physically equivalent vector potentials. We proceed along the lines which we followed for the simpler case of QED in order to arrive to definition of the vacuum expectation values of gauge invariant operators which avoids multiple counting.

We shall consider vacuum expectation values of gauge invariant operators. One important class of such operators is given by,

An example is the following,

$$Trace(\hat{F}_{\mu_1\nu_1}(x_1), ...\hat{F}_{\mu_n\nu_n})$$

where, $F_{\mu\nu}(x) = F^a_{\mu\nu}(x)t^a$ is the matrix valued field strength tensor, But many others can be considered too.

Let $G(A, \psi, \bar{\psi})$ be any gauge invariant functional of the vector and spinor fields. The naive formula for the vacuum expectation value of $T\hat{G}$ given by the following functional integral,

$$< 0|TG(\hat{A}, \hat{\bar{\psi}}, \hat{\psi})|0 >= \frac{\int[d\mu]G(A, \bar{\psi}, \psi)e^{iI[A,\psi,\bar{\psi}]}}{\int[d\mu]e^{iI[A,\psi,\bar{\psi}]}} \tag{7.1}$$

will be ill defined as it counts the same gauge invariant contributions infinite number of times. As before in this formula $d\mu \equiv [dA_\mu d\bar{\psi}d\psi]$.

Like the QED we need to select a section in the space of the gauge field (or in general in the space of all fields in the model). This is the gauge fixing procedure. We shall proceed as in QED and insert 1 in the form of,

$$1 = \Delta(A)\int[dg]\delta(f^g(A)), \qquad \Rightarrow, \qquad \Delta(A)^{-1} = \int[dg]\delta(f^g(A)) \tag{7.2}$$

where, the integration is over the group manifold and $\delta[f(A)]$ imposes a gauge condition , such as,

$$f^a(A) = \partial^\mu A^a_\mu(x),$$
$$f^a(A) = \partial^i A^a_i(x),$$
$$f^a(A) = A^a_0(x) \tag{7.3}$$

In general, in order to fix the gauge completely, we must impose as many gauge conditions as there are group parameters.

The functional integration in Eq. (7.2)is over the manifold of the gauge group G and $[dg]$ denotes a group invariant measure of integration over this manifold. This means that for any fixed group element $g_0 \in G$ which maps G into itself $g_0 : G \to G$ by $g_0 : g \to g_0 g$ we have,

$$[g_0 dg] = [dg]$$

Because of this condition the functional $\Delta(A)$ in Eq. (7.2) is gauge invariant,

$$\Delta(A^{g_0}) = \Delta(A), \qquad \forall g_0 \in G$$

The steps to follow are the same as in QED, namely, we insert 1 in the form of Eq. (7.2) in Eq. (7.1) then change the order of integration and finally factor out the (infinite !) volume of gauge group. The result will formally be identical to the $U(1)$ case,

$$< 0|TG(\hat{A}, \hat{\bar{\psi}}, \hat{\psi})|0 >= \frac{\int [d\mu] G(A, \bar{\psi}, \psi)\Delta(A)\delta(f(A))e^{iI[A,\psi,\bar{\psi}]}}{\int [d\mu]\Delta(A)\delta(f(A))e^{iI[A,\psi,\bar{\psi}]}} \tag{7.4}$$

To evaluate $\Delta(A)$ it is sufficient to consider infinitesimal gauge transformations. Let us consider the case of $G = SU(2)$. The generalisation to arbitrary compact groups is trivial. Thus we assume that the parameters $\alpha(x)$ are infinitesimal. Then,

$$g(x) = 1 + i\alpha(x).\tau$$

Substituting this in the general gauge transformation rule of A_μ given before we obtain,

$$\begin{aligned}
A'_\mu(x) &= (1 + i\alpha(x).\tau)A_\mu(x)(1 - i\alpha(x).\tau) + \frac{i}{e}(1 + i\alpha(x).\tau)\partial_\mu(-i\alpha(x).\tau) \\
&= A_\mu(x) + i\alpha(x).\tau A_\mu(x) - A_\mu(x)i\alpha(x).\tau + \frac{i}{e}(1)\partial_\mu(-i\alpha(x).\tau \\
&= A_\mu(x) + i[\alpha(x).\tau, A_\mu] + \frac{1}{e}\partial_\mu\alpha(x).\tau
\end{aligned}$$

substitute $\alpha(x).\tau = \alpha^a(x)\tau^a$ to obtain

$$A'_\mu(x) = A_\mu(x) + i\alpha^c(x)[\tau^c, A_\mu] + \frac{1}{e}\partial_\mu\alpha(x)^a.\tau^a$$

Writing $A_\mu(x) = A^a_\mu(x)\tau^a$ we obtain the transformation rule for the component fields $A^a_\mu(x)$,

$$\begin{aligned}
A'_\mu(x) &= A_\mu(x) + i\alpha^c(x)[\tau^c, A^b_\mu\tau^b] + \frac{1}{e}\partial_\mu\alpha(x).\tau \\
&= A_\mu(x) + i\alpha^c(x)i\epsilon^{cba}\tau^a A^b_\mu + \frac{1}{e}\partial_\mu\alpha(x).\tau \\
&= [A^a_\mu(x) + \epsilon^{abc}\alpha^c(x)A^b_\mu + \frac{1}{e}\partial_\mu\alpha^a(x)]\tau^a
\end{aligned}$$

Thus,

$$A'^a_\mu(x) = A^a_\mu(x) + \epsilon^{abc}A^b_\mu\alpha^c(x) + \frac{1}{e}\partial_\mu\alpha^a(x)$$

Define $\delta A^a_\mu(x) \equiv A'^a_\mu(x) - A^a_\mu(x)$, then

$$\delta A_\mu^a(x) = \frac{1}{e}\nabla_\mu \alpha^a \tag{7.5}$$

where,

$$\nabla_\mu \alpha^a \equiv \partial_\mu \alpha^a(x) + e\epsilon^{abc} A_\mu^b \alpha^c(x) \tag{7.6}$$

This is the usual covariant derivative acting on an object which transforms in the adjoint transformation of $SU(2)$, as α^a does.

We need $f^a(A^g)$ for an infinitesimal gauge transformation. To be concrete let us choose the gauge condition,

$$f^a(A) = \partial^\mu A_\mu^a(x) \tag{7.7}$$

Our gauge conditions are $f^(A) = 0$.

We can then write,

$$\begin{aligned} f^a(A^g) &= \partial^\mu (A_\mu^a(x) + \frac{1}{e}\nabla_\mu \alpha^a) \\ &= \partial^\mu A_\mu^a(x) + \frac{1}{e}\partial^\mu \nabla_\mu \alpha^a \\ &= \frac{1}{e}\partial^\mu \nabla_\mu \alpha^a \end{aligned}$$

where in the last equality we used the fact that our integration surface is defined by,

$$\partial^\mu A_\mu^a(x) = 0$$

To determine $\Delta(A)$ we return to Eq. (7.2) and substitute the above expression for $f^a(A^g)$ in it. The result is then,

$$\Delta(A)^{-1} = \int [d\alpha^a]\delta(\frac{1}{e}\partial^\mu\nabla_\mu\alpha^a)$$
$$= [Det(\frac{1}{e}\partial^\mu\nabla_\mu]^{-1} \tag{7.8}$$

$$\Delta(A) = Det(\frac{1}{e}\partial^\mu\nabla_\mu)) \tag{7.9}$$

7.A. Faddeev-Popv Ghost Fields

The positive power of a determinant can always be written as a Gaussian integral over Grassmann variables. Since we have the determinant of a differential operator the Gaussian integral must be a functional integral. We thus introduce to anti-commuting set of fields $c^a(x)$ and $\bar{c}^a(x)$ and write,

$$\Delta(A) = \int [d\bar{c}^a dc^a] e^{i\int d^4x[\bar{c}^a(x)\partial^\mu\nabla_\mu c^a(x)]} \tag{7.10}$$

where we absorbed $\frac{1}{e}$ in the normalization of the fields c^a and $\bar{c}^a$.

Note that these fields have spin zero. They are anti-commuting but they are not spinor fields. Hence they satisfy the wrong spin-statistics. For this reason they are called ghost fields. They can not appear as the external particles. Their role is effectively to circulate in the quantum loops and since they have wrong statistics, they will cancel some contributions of the vector potentials from the loop. These are the contributions of the longitudinal modes of the vector potential.

Now inserting $\Delta(A)$ in Eq.(7.4) we obtain,

$$< 0|TG(\hat{A},\hat{\bar{\psi}},\hat{\psi})|0 >=$$
$$= N\int [d\mu]G(A,\bar{\psi},\psi)\delta(f(A))e^{iI[A,\psi,]+iI_{gh}[\bar{c},c]} \tag{7.11}$$

where, $I_{gh}[A,\bar{c},c]$ is given by,

$$
\begin{aligned}
I_{gh}[\bar{c}, c] &= \int d^4x \partial^\mu \bar{c}^a(x) \nabla_\mu c^a(x) \\
&= \int d^4x \partial^\mu \bar{c}^a(x) [\partial_\mu c^a(x) + e\epsilon^{abc} A^b_\mu c^c(x)]
\end{aligned} \tag{7.12}
$$

The term,

$$
\int d^4x \partial^\mu \bar{c}^a(x) \partial_\mu c^a(x) \tag{7.13}
$$

defines the propagator for the ghost field, while the trilinear term,

$$
e \int d^4x \partial^\mu \bar{c}^a(x) \epsilon^{abc} A^b_\mu c^c(x) \tag{7.14}
$$

defines an interaction vertex between the ghosts and the Y-M. vector potentials. Hence in developing Feynman rules we must include these vertices like any other vertex.

Not that the coupling of the ghost field will depend on the choice of the integration surface in the space of fields. To clarify this let us consider the following integration surface,

$$
f^a(A) = A^a_0(x) \tag{7.15}
$$

To obtain $\Delta(A)$ we proceed as before and evaluate $f(A^g)$ for this choice of the gauge,

$$
\begin{aligned}
f^a(A^g) &= A^a_0(x) + \frac{1}{e} \nabla_0 \alpha^a) \\
&= A^a_0(x) + \frac{1}{e} (\partial_0 \alpha^a + e\epsilon^{abc} A^b_0 \alpha^c \\
&= \partial_0 \alpha^a
\end{aligned}
$$

Thus,

$$\Delta(A)^{-1} = \int [d\alpha^a]\delta(\partial_0 \alpha^a)$$
$$= [Det(\partial_0]^{-1} \tag{7.16}$$

The Faddeev- Popov determinant is hence independent of any fields and it will factor out from the functional integrals in the numerator and denominator of Eq. (7.4). It thus cancels from the numerator and denominator. This shows also that the ghost fields can not be physical.

As we did for the $U(1)$ case we can choose a family of gauges by selecting the gauge surface to be and the discussion around it),

$$f^a(A) = \partial^\mu A^a_\mu(x) - h^a(x)$$

We can the multiply the integrand of Eq. (7.4) with $exp[\frac{i}{2\xi}h^a(x)h^a(x)]$ and integrate over $h^a(x)$. The result of this operation will be,

where

$$< 0|TG(\hat{A},\hat{\bar{\psi}},\hat{\psi})|0 >=$$
$$= N\int [d\mu][dh]G(A,\bar{\psi},\psi)\delta(f(A)-h)e^{iI[A,\bar{\psi},\psi]+iI_{gh}[\bar{c},c]-\frac{i}{2\xi}\int dx(h^a h^a)}$$

Use the δ functional to integrate over h and write the result as,

$$< 0|TG(\hat{A},\hat{\bar{\psi}},\hat{\psi})|0 >=$$
$$= N\int [d\mu][dh]G(A,\bar{\psi},\psi)e^{iI_{tot}[A,\bar{\psi},\psi,\bar{c},]}$$

where,

$$I_{tot}[A,\bar{\psi},\psi,\bar{c},c] = I[A,\bar{\psi},\psi] - \frac{1}{2\xi}\int dx(\partial_\mu A^\mu(x))^2 + \int d^4x\partial^\mu\bar{c}^a(x)\nabla_\mu c^a(x) \tag{7.17}$$

Starting from this action we can derive Feynman rules for a Yang- Mills theory coupled to matter fields.

Exercises

1. Derive the Feynman rules from the action of Eq. (7.17).

2. In model by the action of Eq. (7.17), draw all the Feynman graphs which contribute to the propagator of the vector potential up to and including 1-loop order.

Consider the action for a real scalar field ϕ

$$
\begin{aligned}
I(\varphi) &= I_0(\varphi) + I_I(\varphi) = \\
&= \int d^4x \{ \frac{1}{2}\varphi(x)(-\partial^2 - m^2)\varphi(x) + \mathcal{L}_{\mathcal{I}}(\varphi)\}
\end{aligned}
\tag{8.1}
$$

Recall our definition of the generating functional of the connected Green's functions,

$$
\begin{aligned}
e^{iW(J)} &= Z[J] = \\
&= < 0_H | T e^{i\int dy J(y)\hat{\phi}_H(y)} | 0_H > \\
&= \frac{< 0 | T e^{i\int dx \mathcal{L}_{\mathcal{I}}(\hat{\phi})} e^{i\int dx J(x)\hat{\phi}(x)} | 0 >}{< 0 | T e^{i\int dx \mathcal{L}_{\mathcal{I}}(\hat{\phi})} | 0 >}
\end{aligned}
$$

Derivatives of $Z[J]$ evaluated at $J = 0$ give us the set of all Greens functions, connected and disconnected, while the derivatives of $W(J)$ give us only the connected Greens functions. Consider,

$$
\phi(x; J) \equiv \frac{\delta W(J)}{\delta J(x)}
\tag{8.2}
$$

Using the definition of $W(J)$ in terms of Z we see that,

$$
\begin{aligned}
\phi(x; J) &= \frac{1}{i} \frac{1}{Z[J]} \frac{\delta Z(J)}{\delta J(x)]} \\
&= \frac{1}{Z[J]} < 0_H | T[\hat{\phi}_H(x) e^{i\int dy J(y)\hat{\phi}_H(y)}] | 0_H >
\end{aligned}
\tag{8.3}
$$

Thus at $J = 0$ the function $\phi(x; J)$ will be equal to the v.e.v of the Heisenberg picture field operator $\hat{\phi}_H(x)$,

$$\phi(x;J)\mid_{J=0}=< 0_H|\hat{\phi}(x)|0_H >$$

In general $\phi(x;J)$ will be a local function of x and a functional of J. For example in a free field theory (i.e. when the interaction Lagrangian $\mathcal{L}_{\mathcal{I}}(\hat{\phi})=0$) we have,

$$Z[J]=e^{-\frac{1}{2}\int d^4x\int d^4x' J(x)\Delta_F(x-x')J(x')}$$

and thus,

$$iW(J)=-\frac{1}{2}\int d^4x\int d^4x' J(x)\Delta_F(x-x')J(x') \tag{8.4}$$

Hence,

$$\phi(x;J)=i\int d^4y\Delta_F(x-y)J(y) \tag{8.5}$$

This is a simple linear functional which vanishes at $J=0$ giving us,

$$\begin{aligned}\phi(x;J)\mid_{J=0} &=< 0_H|\hat{\phi}(x)|0_H >= \\ &= 0\end{aligned}$$

as expected for a free field.

In an interacting theory $\phi(x;J)$ will be a more complicated functional of $J(x)$.

If we could invert the relation Eq. (8.2) we could express J as a functional of $\phi(x;J)$. It turns out that this inversion is a rather simple operation and goes through a functional Γ defined by,

$$\Gamma\equiv W(J)-\int dxJ(x)\phi(x;J) \tag{8.6}$$

The first question we must answer is: "what does Γ depend on?" To answer this question we shall look at the total functional derivative of Γ,

$$\begin{aligned}\delta\Gamma &= \delta W(J) - \int dx[\delta J(x)\phi(x;J) + J(x)\delta\phi(x;J)] \\ &= \int dx \frac{\delta W(J)}{\delta J(x)}\delta J(x) - \int dx[\delta J(x)\phi(x;J) + J(x)\delta\phi(x;J)] \\ &= \int dx[\frac{\delta W(J)}{\delta J(x)} - \phi(x;J)]\delta J(x) - \int dx J(x)\delta\phi(x;J)]\end{aligned} \tag{8.7}$$

If we use Eq. (8.2) the integrand of the first integral on the r.h.s. will vanish and we shall obtain,

$$\delta\Gamma = -\int dx J(x)\delta\phi(x;J)]$$

This shows that Γ is a functional of $\phi(x;J)$

$$\frac{\delta\Gamma(\phi(x;J))}{\delta\phi(x;J)} = -J(x)$$

We thus have a pair of equations,

$$\phi(x;J) = \frac{\delta W(J)}{\delta J(x)} \qquad , \qquad \frac{\delta\Gamma(\phi)}{\delta\phi(x;J)} = -J(x) \tag{8.8}$$

If we know $W(J)$ then the first equation expresses $\phi(x)$ explicitly as a functional of J. If we know $\Gamma(\phi)$ then the second equation expresses J as a functional of ϕ. There is a functional relationship between $\phi(x)$ and $J(x)$.

Example:

What is $\Gamma(\phi)$ in free field theory?

First using Eq.(8.5) we obtain J as a functional of ϕ. This we do by acting with $-\partial^2 - m^2$ on both sides of Eq. (8.5)

$$
\begin{aligned}
(-\partial^2 - m^2)\phi(x;J) &= i\int d^4y(-\partial^2 - m^2)\Delta_F(x-y)J(y) \\
&= i\int d^4y i\delta_4(x-y)J(y) \\
&= -J(x)
\end{aligned}
$$

Now use $\frac{\delta\Gamma(\phi)}{\delta\phi(x))} = -J(x)$,

$$\frac{\delta\Gamma(\phi)}{\delta\phi(x))} = (-\partial^2 - m^2)\phi(x)$$

Integrate w.r.t ϕ to obtain,

$$\Gamma(\phi) = \int dx\frac{1}{2}\phi(-\partial^2 - m^2)\phi(x) \tag{8.9}$$

Hence in the free field theory $\Gamma(\phi)$ coincides with the free field action. We shall see that this result is valid in the interacting theory at the limit of $\hbar \to 0$, namely, in this limit $\Gamma(\phi)$ will be given by the interacting classical action.

9. LOOP EXPANSION OF $\Gamma(\phi)$

Notations :

1. To simplify the notation we shall drop J from $\phi(x;J)$ and write it simply as $\phi(x)$.

2. We shall often consider multicomponent fields such as scalar fields belonging to some representation of a symmetry group G which we shall denote by $\phi^i(x)$. To simplify the notation we shall assume i denotes all the indices including the space-

time points x and denote the field by ϕ^i. In this notation sums and integration shall be denoted by repeated indices. For example,

$$
\begin{aligned}
\phi^i\phi^i &= \int d^4x \phi^i(x)\phi^i(x) \\
J_i\phi^i &= \int d^4x J_i(x)\phi^i(x)
\end{aligned}
\tag{9.1}
$$

Recall that,

$$
\begin{aligned}
e^{\frac{i}{\hbar}W(J)} &= Z[J] = \\
&= N\int [d\varphi] e^{\frac{i}{\hbar}[I(\varphi)+J_i\varphi^i\}]}
\end{aligned}
\tag{9.2}
$$

$W(J)$ can be evaluated as a power series in $\hbar$. This expansion is the same as an expansion in the number of loops. To see this we note from Eq. (9.2) that each propagator will have a factor of $\hbar$ in it while each vertex will be accompanied with a factor of $\frac{1}{\hbar}$. Thus in a graph with I propagators and V vertices the power of $\hbar$ will be $I - V$. Recall from Eq. (3.42) that the number of loops is related to I and V by $L = I - V + 1$. Hence the power of $\hbar$ in a graph with L loops in the expansion of $\frac{i}{\hbar}W(J)$ will be

$$
\hbar^{I-V} = \hbar^{L-1}
$$

Likewise we consider the expansion of $\Gamma(\phi)$ in powers of $\hbar$

$$
\Gamma(\phi) = \Gamma_0(\phi) + \frac{\hbar}{i}\Gamma_1(\phi) + (\frac{\hbar}{i})^2\Gamma_2(\phi) + ... \tag{9.3}
$$

We shall evaluate Γ_0 and Γ_1 for an arbitrary action.

Substitue $W(J) = \Gamma(\phi) + J_i\phi^i$ in Eq. (9.2)

$$
e^{\frac{i}{\hbar}[\Gamma(\phi)+J_i\phi^i]} = N\int [d\varphi] e^{\frac{i}{\hbar}[I(\varphi)+J_i\varphi^i\}]}
$$

Or

$$e^{\frac{i}{\hbar}\Gamma(\phi)} = N\int[d\varphi]e^{\frac{i}{\hbar}[I(\varphi)+J_i(\varphi^i-\phi^i\}]} \tag{9.4}$$

In this equation J_i must be understand to be given by Eq. (8.8). In any case it is independent from the integration variable φ^i.

Define a new integration variable χ^i by,

$$\hbar^{\frac{1}{2}}\chi^i = \varphi^i - \phi^i$$

Since this is a constant shift of the integration variable φ we have $[d\varphi] = [\hbar^{\frac{1}{2}}d\chi]$. We also have,

$$\begin{aligned} I(\varphi) &= I(\phi + \hbar^{\frac{1}{2}}\chi) \\ &= I(\phi) + \hbar^{\frac{1}{2}}\chi^i I_{,i}(\phi) + \frac{1}{2!}\hbar\chi^i\chi^j I_{,ij}(\phi) + \frac{1}{3!}\hbar^{\frac{3}{2}}\chi^i\chi^j\chi^k I_{,ijk}(\phi) + ... \end{aligned}$$

Thus,

$$\begin{aligned} \frac{i}{\hbar}[I(\varphi) + J_i(\varphi^i - \phi^i\}) &= \frac{i}{\hbar}I(\phi) \\ &+ \frac{i}{\hbar}[\hbar^{\frac{1}{2}}\chi^i(I_{,i} + J_i)] \\ &+ \frac{i}{\hbar}[\frac{1}{2!}\hbar\chi^i\chi^j I_{,ij}(\phi) + \frac{1}{3!}\hbar^{\frac{3}{2}}\chi^i\chi^j\chi^k I_{,ijk}(\phi) + ...] \\ &= \frac{i}{\hbar}I(\phi) \\ &+ \frac{i}{\hbar^{\frac{1}{2}}}[\chi^i(I_{,i} + J_i)] \\ &+ \frac{i}{2!}\chi^i\chi^j I_{,ij}(\phi) \\ &+ i\frac{1}{3!}\hbar^{\frac{1}{2}}\chi^i\chi^j\chi^k I_{,ijk}(\phi) + ...] \end{aligned}$$

Or,

$$\begin{aligned}\frac{i}{\hbar}[I(\varphi)+J_i(\varphi^i-\phi^i\}) &= \frac{i}{\hbar}I(\phi)+\frac{i}{2!}\chi^i\chi^j I_{,ij}(\phi)\\ &+\frac{i}{\hbar^{\frac{1}{2}}}[\chi^i(I_{,i}+J_i)]\\ &+i\Sigma_{n=3}\frac{1}{n!}\hbar^{\frac{n}{2}-1}\chi_1^i....\chi_n^i I_{,i_1\ldots i_n}(\phi)\end{aligned}$$

We shall write,

$$\frac{i}{\hbar}[I(\varphi)+J_i(\varphi^i-\phi^i\}) = \frac{i}{\hbar}I(\phi)+\frac{i}{2!}\chi^i\chi^j I_{,ij}(\phi)+I_{int}(\phi;\chi) \tag{9.5}$$

where, (substitute $J_i = -\Gamma_,i$)

$$I_{int}(\phi;\chi) = \frac{i}{\hbar^{\frac{1}{2}}}\chi^i(I_{,i}-\Gamma_{,i})+i\Sigma_{n=3}\frac{1}{n!}\hbar^{\frac{n}{2}-1}\chi_1^i....\chi_n^i I_{,i_1\ldots i_n}(\phi) \tag{9.6}$$

Now we return to Eq. (9.4) and use Eq. (9.5) to substitute for the integrand to obtain,

$$\begin{aligned}e^{\frac{i}{\hbar}\Gamma(\phi)} &= Ne^{\frac{i}{\hbar}I(\phi)}\int[d\chi]e^{\frac{i}{2!}\chi^i\chi^j I_{,ij}(\phi)+iI_{int}(\phi;\chi)}\\ &= Ne^{\frac{i}{\hbar}I(\phi)}\int[d\chi]e^{\frac{i}{2!}\chi^i\chi^j I_{,ij}(\phi)}[1+\Sigma_{k=1}\frac{1}{k!}I_{int}^k(\phi;\chi)]\end{aligned} \tag{9.7}$$

The integrals can be evaluated term by term. The terms will be,

$$\begin{aligned}e^{\frac{i}{\hbar}\Gamma(\phi)} &= Ne^{\frac{i}{\hbar}I(\phi)}\int[d\chi]e^{\frac{i}{2!}\chi^i\chi^j I_{,ij}(\phi)}\\ &= Ne^{\frac{i}{\hbar}I(\phi)}[DetI_{,ij}(\phi)]^{-\frac{1}{2}}+....\end{aligned} \tag{9.8}$$

Assuming $\Gamma(\phi=0)=0$ we obtain,

$$N = [DetI_{,ij}(\phi = 0)]^{\frac{1}{2}}$$

We thus obtain,

$$\Gamma(\phi) = I(\phi) - \frac{1}{2}\frac{\hbar}{i} lnDet[M_0^{-1}M(\phi)] + (\frac{\hbar}{i})^2[....] \tag{9.9}$$

where $M(\phi)$ is the operator with the matrix elements,

$$M_{ij}(\phi) \equiv I_{,ij}(\phi) \tag{9.10}$$

Eq. (9.9) shows that the classical action is the tree level effective action. The one loop effects are given by the determinant of the operator defined by quadratic part. The higher loop graphs are give by the interaction vertices in I_{int}. and $M_0 \equiv M(\phi = 0)$

In Eq. (9.9) the terms indicate by .... represent the contributions of the subleading terms which will originate from the interaction terms $\Sigma_{k=1} I^k_{int}(\phi; \chi)$ in Eq. (9.7). In principle it is straightforward to calculate the contribution of these terms in each order of $\hbar^{\frac{1}{2}}$. It can be shown that only the integer powers of $\hbar$ will contribute.

Eq. (9.9) shows that the classical action is the tree level approximation to the quantum effective action. The 1-loop effects are given by the determinant of the operator defined by the quadratic part of the action. The higher loop effects are given by the vertices in I_{int}.

Differentiate the first of Eqs.(8.8) w.r.t to ϕ,

$$\delta^i_j = \frac{\delta J_k}{\delta \phi^j} \frac{\delta^2 W(J)}{\delta J_k \delta J_i} \tag{9.11}$$

where we used the compact notation and the chain rule of differentiation. The first factor can be worked out from the the second equation in Eq.(8.8). We thus obtain,

$$\delta^i_j = -\frac{\delta^2 \Gamma}{\delta \phi^j \delta \phi^k} \frac{\delta^2 W(J)}{\delta J_k \delta J_i} \tag{9.12}$$

The two point function is given by,

$$iG^{ki} = \frac{\delta^2 W(J)}{\delta J_k \delta J_i} \tag{9.13}$$

Substitute in Eq.(9.14)

$$i\delta^i_j = \Gamma_{,jk} G^{ki} \tag{9.14}$$

Thus the matrices $\Gamma_{,jk}$ and $-iG^{ki}$ are inverses of each other. The poles of $-iG^{ki}$ will be the zeros of $\Gamma_{,jk}$.

We must set $J_i = 0$. Then the effective quantum equation becomes,

$$\frac{\delta \Gamma(\phi)}{\delta \phi^i} = 0 \tag{10.1}$$

This is an equation which determines ϕ^i. Recall that when $J_i = 0$ we have,

$$\phi^i =< 0_H | \hat{\phi}^i | 0_H >$$

Thus our equation essentially determines the vacuum expectation value of the Heisenberg field operator $\hat{\phi}^i$. We shall examine Eq. (10.1) first at the tree level for the $O(N)$ model, for which the action is given by,

$$I(\varphi) = \int d^n x \{ \frac{1}{2} \varphi^a(x)(-\partial^2 - m^2)\varphi^a(x) - \frac{\lambda}{4}(\vec{\varphi}^2(x))^2 \} \tag{10.2}$$

where $\vec{\varphi}^2(x) \equiv \varphi^a(x)\varphi^a(x)$.

Remark :Note that we put $\frac{1}{4}$ rather than $\frac{1}{4!}$ in front of the coupling λ.

At the tree level we have $\Gamma_0(\phi) = I(\phi)$ and thus Eq. (10.1) reduces to,

$$\frac{\delta \Gamma_0(\phi)}{\delta \phi^i} = \frac{\delta I(\phi)}{\delta \phi^i} = 0$$

For our model this becomes,

$$(-\partial^2 - m^2)\phi^a(x) - \lambda \vec{\phi}^2(x)\phi^a(x) = 0$$

For a Poincare invariant ground state we shall have,

$$\phi^i(x) = < 0_H | \hat{\phi}^i | 0_H >$$
$$\equiv v^a = constant$$

Hence the effective equations become

$$-m^2 v^a - \lambda \vec{v}^2 v^a = 0 \tag{10.3}$$

We assume $\lambda > 0$ but allow for m^2 to have either sign. The possible solutions of this equation are,

$$\begin{array}{lll} 1. & v^a = 0 & \\ 2. & \lambda \vec{v}^2 + m^2 = 0, \quad \Rightarrow & \vec{v}^2 = -\dfrac{m^2}{\lambda} \end{array} \tag{10.4}$$

The first solution is just a single point in the space ϕ^a fields and for a real ϕ^a it exists both for positive as well as for negative m^2. The second equation has a real solution only if $m^2 < 0$. In this case it defines a S^{N-1} in the N-dimensional space of the ϕ fields. Any point on this sphere is an acceptable v.e.v of $\hat{\phi}$ at the tree level.

To understand what kind of physics corresponds to each case we must look at the poles of the propagator and at the interaction vertices. The exact propagator is given by i times the inverse of the matrix $\Gamma_{,ij}(\phi)$. The poles of the propagator will correspond to the zero eigenvalues of the matrix $\Gamma_{,ij}(\phi)$ evaluated at the solutions of Eq. (10.1),

For our model and at the tree level we have,

$$\Gamma_{0,ab} = \delta(x - x')[(-\partial^2 - m^2 - \lambda \vec{v}^2)\delta_{ab} - 2\lambda v_b v_a]$$

In Fourier space this becomes,

$$\Gamma_{0,ab}(p) = (p^2 - m^2 - \lambda \vec{v}^2)\delta_{ab} - 2\lambda v_b v_a$$

For each p we have an $N \times N$ matrix and thus N eigenvalues. The matrix corresponding to the case 1. of Eq. (10.4) (i.e. $v = 0$) becomes,

$$\Gamma_{0,ab}(p) = (p^2 - m^2)\delta_{ab}$$

There are N equal eigenvalues $p^2 - m^2$ and they vanish at,

$$p^2 = m^2$$

They correspond to N simple poles of the propagators. There are thus N spin zero massive particles with equal masses of m^2. Clearly if m^2 is negative these particles will all be tachyons. That is why the solution $v = 0$ is physically acceptable only if m^2 is non negative.

The case 2. of Eq. (10.4) requires, $m^2 < 0$ and gives us the matrix,

$$\Gamma_{0,ab}(p) = p^2\delta_{ab} - 2\lambda v_b v_a$$

To propagator in this case will be given by i times the inverse of the matrix $\Gamma_{0,ab}(p)$ and, as always, the masses will be given by the simple poles of the propagator with the residue i.

It is easiest to use the rotational invariance of our construction and rotate v to have the form, $v = (0, ..., v)$, where only the N-th component is non-zero. In this case $v_a v_b$ will be a square $N \times N$ matrix with only one non zero component at the position N-th row and N-th column. We shall thus have,

$$\begin{aligned} \Gamma_{0,ij}(p) &= p^2\delta_{ij} \qquad i, j = 1, ...N - 1 \\ \Gamma_{0,NN}(p) &= p^2 - 2\lambda v^2 \end{aligned} \tag{10.5}$$

The propagator will be i times the inverse of this diagonal matrix.

$$\Delta_{F,ij}(p) = \frac{i}{p^2}\delta_{ij} \qquad i,j = 1,...N-1$$
$$\Delta_{F,NN}(p) = \frac{i}{p^2 - 2\lambda v^2} \tag{10.6}$$

From here we see that there are $N-1$ massless spin zero particles in the model and one massive spin zero particle with the mass given by $\mu^2 = 2\lambda v^2 = -2m^2$.

What we have just obtained goes under the name of Goldstone theorem.

Note that in the case of $m^2 > 0$ the mass spectrum reflects the full $O(N)$ invariance whereas in the case of $m^2 < 0$ only $N-1$ masses are equal and the spectrum shows the symmetry under $O(N-1)$. We also note that $dimO(N) - dimO(N-1) = N-1$.

Thus the number of zero mass particles is precisely equal to $dimO(N) - dimO(N-1)$.

These facts are related to the invariance group of the ground state. In the case of $m^2 > 0$ the ground state is invariant under the full $O(N)$ while when $m^2 < 0$ the invariance group of the ground state is $O(N-1)$. The symmetry is said to be spontaneously broken to $O(N-1)$. Any element of $O(N)$ which lies in this $O(N-1)$ subgroup will leave the ground state invariant. This is easily seen by choosing the axis in the space of the ϕ fields such that the Nth axis lies in the direction of $\vec{v}$. The elements of $O(N)$ outside this unbroken subgroup are generated by $N-1$ generators of $O(N)$. These are called broken generators. We conclude that the number of massless particles equals to the number of broken generators. What we have seen at the tree level approximation is called the Goldstone theorem. This theorem is valid to all orders of perturbation theory. It states:

If a continuous global symmetry group G breaks spontaneously to a subgroup H, there will be $dimG - dimH$ massless spin zero particles in the spectrum.

We continue with the $O(N)$ example of the previous section. As we saw in our analysis of that section, when $m^2 > 0$ the vacuum expectation value of $\hat{\phi}_H(x)$ is zero and there are N particles with equal masses in the spectrum. On the other hand when $m^2 < 0$ the spectrum consists of $N-1$ massless particles and one massive particle of $mass^2 = -2m^2$. How can we see this at the level of the action?

Recall that in our discussion of the LSZ reduction formula we argued that if $<0|\hat{\phi}^a(x)|0>= v^a \neq 0$ then we must shift the field operator by the constant $\vec{v}$. Hence we define a new field $\hat{\chi}^a(x) \equiv \hat{\phi}^a(x) - v^a$. We shall then have $<0|\hat{\chi}^a(x)|0>= 0$. In terms of this new field the action given by Eq. (10.2) then becomes,

$$I(\varphi) = \int d^n x \{ \frac{1}{2}\chi^a(x)(-\partial^2)\chi^a(x) - \frac{1}{2}m^2(\chi^a(x)+v^a)^2 - \frac{\lambda}{4}[(\vec{\chi}(x)+\vec{v})^2]^2 \} \quad (10.7)$$

To simplify this expression we denote $\chi^a(x)$ for $a =, 1, ..N-1$ by $\sigma^a(x)$ and $\chi^N(x)$ by $\rho(x)$,

$$\vec{\chi}(x) + \vec{v} = \begin{pmatrix} \sigma_1(x) \\ . \\ . \\ . \\ \sigma_{N-1}(x) \\ \rho(x) + v \end{pmatrix}$$

or in short,

$$\vec{\chi}(x) + \vec{v} = \begin{pmatrix} \vec{\sigma}(x) \\ \rho(x) + v \end{pmatrix}$$

where $\vec{\sigma}(x)$ has only $N-1$ components. Writing the action in terms of $\vec{\sigma}(x)$ and

$\rho(x)$ we obtain,

$$I(\varphi) = \int d^n x \{ \frac{1}{2}\vec{\sigma}(x)(-\partial^2)\vec{\sigma}(x) + \frac{1}{2}\rho(x)(-\partial^2)\rho(x) - \frac{1}{2}(-2m^2)\rho(x)^2 - V_{int}(\vec{\sigma},\rho)\} \tag{10.8}$$

where,

$$V_{int}(\vec{\sigma},\rho) = \frac{\lambda}{4}[(\vec{\sigma}^2(x))^2 + \rho^4(x) + 2\vec{\sigma}^2(x)\rho(x)^2 + 4v\vec{\sigma}^2(x)\rho + 4v\rho^3(x) - v^4] \tag{10.9}$$

We see from the action given in Eq. (10.8) that we have $N-1$ massless fields $\sigma_1(x), ...\sigma_{N-1}(x)$ and one massive spin zero field with the squared $mass^2 = -2m^2 = 2\lambda v^2$. Furthermore the action is manifestly invariant under the symmetry global group $O(N-1)$. Thus the original symmetry group of $O(N)$ is broken to a subgroup $O(N-1)$. Note that the number of massless fields is equal to the $dimO(N) - dimO(N-1) = N-1$. The group $O(N-1)$ is the largest subgroup of $O(N)$ which leaves the $\vec{v}$ invariant. In other words any element of $O(N)$ which is outside of this subgroup will change $\vec{v}$ to some other vector. These elements are said to be spontaneously broken while the subgroup $O(N-1)$ forms the unbroken subgroup of $O(N)$.

We can express this situation in terms of the generators of $O(N)$. Let us denote them by $\{T^A, A = 1, ...d\}$ where $d = \frac{1}{2}N(N-1)$. We divide this set into two subsets: the first subset denoted by $\{T^\alpha, \alpha = 1, ...d'\}$ are generators of the subgroup $O(N-1)$ and thus $d' = \frac{1}{2}(N-1)(N-2)$. The second subset denoted by $T^i, i = 1, ..., N-1$ are all the remaining generators. An element of the subgroup $O(N-1)$ of the form $e^{i\theta^\alpha T^\alpha}$ will leave $\vec{v}$ invariant,

$$e^{i\theta^\alpha T^\alpha}\vec{v} = \vec{v}$$

This implies that,

$$T^\alpha \vec{v} = 0, \quad \alpha = 1, ..., d'$$

On the other hand for any element of $O(N)$ of the form $e^{i\theta^i T^i}$ we shall have

$$e^{i\theta^i T^i}\vec{v} \neq \vec{v}$$

Thus,

$$T^i\vec{v} \neq 0, \quad i = 1, ..., N-1$$

Example: G=O(3)

Let's consider an example of $G = O(3)$. the generators are given by,

$$T_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}$$

$$T_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix}$$

$$T_3 = i\begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is then seen that,

$$T_1\vec{v} \neq 0, \qquad T_2\vec{v} \neq 0, \qquad T_3\vec{v} = 0, \tag{10.10}$$

Thus the unbroken group is generated by T_3 which is the set of all rotations around third axis (in the 12 plane). There are two broken generators (symmertries) and hence there will be two massless spin zero particles and one massive spin zero particles.

Quantum mechanically to each generator T^A there will correspond a quantum operator Q^A constructed through Noether theorem. They are given by,

$$Q^A = \int d^3x j_0^A(x)$$

where, $j_\mu^A(x)$ are the Noether currents given by,

$$j_\mu^A(x) = i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a(x))}(T^A)^{ab}\varphi^b(x)$$

Noether theorem states that if the Euler-Lagrange equations are satisfied then the currents are conserved,

$$\partial^\mu j_\mu^A(x) = 0$$

This in turn implies that the charges Q^A are time independent. It is a useful exercise to construct them for the $O(N)$ model and verify that they satisfy the Lie algebra of the symmetry grou,viz,

$$[Q^A, Q^B] = if^{ABC}Q^C \tag{10.11}$$

The symmetry is implement in the Hilbert space of states by,

$$e^{-i\theta^A Q^A}\hat{\phi}_H^a(x)e^{i\theta Q^A} = [e^{i\theta^A T^A}]^{ab}\hat{\phi}^b(x) \tag{10.12}$$

Now let us make the same division of the generators into two subsets of $\{Q^\alpha, \alpha = 1,...d'\}$ are generators of the subgroup $O(N-1)$ and thus $d' = \frac{1}{2}(N-1)(N-2)$. The second subset denoted by $Q^i, i = 1,..., N-1$ which are all the remaining generators.

We then see from Eq. (10.12) that

$$\begin{aligned} < 0_H|e^{-i\theta^\alpha Q^\alpha}\hat{\phi}^a_H(x)e^{i\theta^\alpha Q^\alpha}|0_H > &= [e^{i\theta^\alpha T^\alpha}]^{ab}v^b \\ &= v^a \\ &=< 0_H|\hat{\phi}^a_H(x)|0_H > \end{aligned} \tag{10.13}$$

This will be valid if,

$$Q^\alpha|0 >= 0, \quad \alpha = 1, ..., d'$$

An analogues argument will give us,

$$Q^i|0 >\neq 0, \quad \alpha = 1, ..., N-1$$

If a generator of the invariance group of the Lagrangian does not annihilate the ground state we say that the symmetry operation generated by that operator is *Spontaneously Broken*. As we observed above in our example of the $O(N)$ model the number $N-1$ of massless particles is precisely equal to the number of the so called *Broken Generators*, i.e. those for which $Q^{\bar{\alpha}}|0_H >\neq |0_H >$.

If we compare this situation with the case of $m^2 > 0$ and thus $\vec{v} = 0$ we have $e^{i\theta.Q}|0_H >= |0_H >$ and hence $Q_A|0_H >= 0, \forall A = 1, ..., \frac{1}{2}N(N-1)$. Hence there are no broken generators.

The observation above is generally valid and it goes under the name of Goldstone's theorem which states that for any generator $Q_{\bar{\alpha}}$ of a continuous symmetry which does not annihilate the ground state there corresponds a massless particle. The quantum numbers of these particles are the same as those of the broken generators. These particles are called Goldstone bosons. In our example the Goldstone bosons are massless spin zero particles transforming in the fundamental representation of the unbroken subgroup $SO(N-1)$.

Note that the symmetry argument given above is exact valid to all orders of perturbation theory, while the spectrum we discovered in the previous section was only valid at the tree approximation.

Remark: To make everything well defined the system must be put in a finite volume. Spontaneous symmetry breaking happens only if the volume goes to infinity. In this limit the charges Q_A become ill defined and the symmetry is not unitarily realised. The Noether currents continue to be conserved and in practice this is what matters. The proof of the Goldstone's theorem given in the next section does not make use of the unitary implementation of the $O(N)$ symmetry.

Exercises

1. Use Euler Lagrange equations to show $\partial^\mu j_\mu^A = 0$.
2. Prove that the operators $Q^A = \int d^3x j_0^A(x)$ satisfy the Lie algebra of $O(N)$.

The discussion which follows is quite general but in order to be concrete we shall consider the example of the renormalizable N-component scalar field theory with global $O(N)$ invariance. The classical (or the tree level) action is given by,

$$I(\varphi) = \int d^n x \{ \frac{1}{2}\varphi^a(x)(-\partial^2 - m^2)\varphi^a(x) - \frac{\lambda}{4}(\vec{\varphi}^2(x))^2 \}$$

which is invariant under,

$$\vec{\varphi}'(x) = \vec{\varphi}(x) + iT.\theta\vec{\varphi}(x), \qquad \Rightarrow \qquad \delta\vec{\varphi}(x) = iT.\theta\vec{\varphi}(x)$$

We can regard this transformation as a change of variable in the functional integral. Note that the measure of integration is invarain,

$$[d\varphi] = [d\varphi']$$

Hence,

$$\begin{aligned}
e^{iW(J)} &= Z[J] = \\
&= N\int [d\varphi'] e^{i[I(\varphi') + \int d^4x J_a(x)\varphi'^a(x)]} \\
&= N\int [d\varphi] e^{i[I(\varphi) + \int d^4x J_a(x)\{\varphi^a(x) + i(T.\theta\vec{\varphi}(x))^a\}} \\
&= N\int [d\varphi] e^{i[I(\varphi) + \int d^4x J_a(x)\varphi^a(x)]}[1 - \int d^4x J_a(x)(T.\theta\vec{\varphi}(x))^a + ...] \\
&= e^{iW(J)} - N\int [d\varphi] e^{i[I(\varphi) + \int d^4x J_a(x)\varphi^a(x)]}\int d^4x J_a(x)(T.\theta\vec{\varphi}(x))^a
\end{aligned}$$

Henece,

$$0 = N \int [d\varphi] e^{i[I(\varphi)+\int d^4x J_a(x)\varphi^a(x)]} \int d^4x J_a(x)(T.\theta\vec{\varphi}(x))^a$$

Recall that

$$\begin{aligned}\phi(x)^a &\equiv \phi(x;J)^a \\ &= e^{-iW(J)} N \int [d\varphi] e^{i[I(\varphi)+\int d^4x J_a(x)\varphi^a(x)]} \varphi(x)^a\end{aligned}$$

Using this we then obtain,

$$\int d^4x J_a(x)(T.\theta\vec{\phi}(x))^a = 0$$

Now substitute for J from

$$-J_a(x) = \frac{\delta\Gamma(\phi_J)}{\delta\phi^a(x;J)} \equiv \Gamma_{,a}(x;\phi_J))$$

to obtain,

$$0 = \int d^4x \Gamma_{,a}(x;\phi_J)(T.\theta\vec{\phi}(x;J))^a$$

Differentiate this equation once more relative to $\phi^b(x;J)$,

$$0 = \int d^4x \{\Gamma_{,ba}(x,x';\phi_J)(T.\theta\vec{\phi}(x;J))^a + \Gamma_{,a}(x;\phi_J)(T.\theta)^a_b \delta_4(x-x')\}$$

If we set $J_a(x) = 0 = \Gamma_{,a}(x;\phi_J)$ then $\phi^a(x;J) =< 0_H|\hat{\phi}^a_H(x)|0_H >= v^a$ and we obtain,

$$0 = \int d^4x \Gamma_{,ba}(x,x';v)(T.\theta\vec{v})^a \tag{11.1}$$

By translational invariance we can write,

$$\Gamma_{,ab}(x,x';v) = \int \frac{d^4p}{(2\pi)^4}\tilde{\Gamma}_{,ab}(p;v)e^{ip.(x-x')} \tag{11.2}$$

Eq. (11.1) then yield,

$$\tilde{\Gamma}_{,ba}(p,v)\mid_{p=0}(T.\theta\vec{v})^a = 0$$

Since θ are arbitrary we obtain,

$$\tilde{\Gamma}_{,ba}(p,v)\mid_{p=0}(T^A)^a_c v^c = 0 \tag{11.3}$$

The matrix $\tilde{\Gamma}_{,ab}(p;v)\mid_{p=0}$ is -i times the inverse propagator evaluated at $p=0$. This is what we call the mass matrix. Thus Eq. (11.3) is an eigenvalues equation for the mass matrix with the eigenvalues equal to zero. This equation shows that the mass matrix will have as many zero eigenvalues as there are linearly independent non-zero vectors $T_A\vec{v}$, $A=1,...,dimO(N)$. In turn this is equal to the number of the generators T_A of $O(N)$ for which

$$T_A\vec{v} \neq 0 \tag{11.4}$$

We need to investigate the solutions of Eq. (11.3). First of all the vacuum expectation value v^a of the field operator $\hat{\phi}^a_H(x)$ is determined by solving the effective equations,

$$\Gamma_{,a}(v) = 0$$

If these equations give $v^a=0$ then Eq. (11.3) gives us no useful information on mass spectrum. In this case it follows that

$$Q_A|0_H>=0, \qquad \forall A=1,...,\frac{1}{2}N(N-1)$$

Hence for $\vec{v}=0$ the symmetry is not spontaneously broken.

On the other hand if $\vec{v}\neq 0$ any T_A which generates a rotation leaving $\vec{v}$ invariant will kill $\vec{v}$. These form a closed sub-algebra of $O(N)$ namely the algebra of all rotations in the isotropy subgroup of $\vec{v}$. This subalgebra is isomorphic to the Lie algebra of $O(N-1)$. Any generator outside this algebra will map $\vec{v}$ to some other non zero vector. The number of such generators is $N-1$. Hence there will be $N-1$ zero eigenvalues of the mass matrix $\tilde{\Gamma}_{,ab}(p,v)\mid_{p=0}$. Note that our argument does not say anything about mass of the Nth eigenvalue of the mass matrix. It must be calculated from the explicit expression of the matrix $\tilde{\Gamma}_{,ab}(p,v)\mid_{p=0}$ as we did in the leading order approximation.

Exercises

1. Assume that $\varphi^a(x)$, $a=1,...,n$ transforms in some n -dimensional irreducible representation of a compact group G. Define $\vec{v}=<0_H|\hat{\vec{\phi}}(x)|0_H>$. Show that the set of vectors $T_{\bar{\alpha}}\vec{v}$, which are non-zero are linearly independent.

2. Consider the $O(N)$ invariant scalar field theory.

a. Show that there is a one to one map between the manifold $O(N)/O(N-1)$ and the S^{N-1} defined by $\vec{v}.\vec{v}=const.$

b. Assume that the vector $\vec{v}$ takes the form given by

$$\vec{v}=\begin{pmatrix} v_1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

Write an arbitrary $\vec{\varphi}(x)$ as

$$\vec{\varphi}(x) = R(x)\vec{v}', \qquad R(x) \in O(N) \tag{11.5}$$

where $\vec{v}'$ is defined by,

$$\vec{v}' = \begin{pmatrix} v_1 + \rho(x) \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

Substitute this $\vec{\varphi}(x)$ in the action and find the mass of the field $\rho(x)$.

c. The notation $R(x)$ indicates that the parameters θ_A of the matrix $R(x)$ depend on x. Show that only $N-1$ of these parameters are significant in the sense that the action becomes a functional of these parameters. These parameters describe the low energy dynamics of the Goldstone bosons. Low energy here refers to energies small compared to the mass of the $\rho(x)$ field. In this situation the dynamics of the $\rho(x)$ field will be frozen and the low energy physics will be that of the Goldstone bosons.

d. (Optional) Show that the low energy action can be written in the form of

$$I(\theta_{\bar{1}}, ...\theta_{\bar{d}}) = \int d^4x g_{\bar{\alpha}\bar{\beta}}(\theta)\partial_\mu\theta^{\bar{\alpha}}(x)\partial^\mu\theta^{\bar{\beta}}(x) \tag{11.6}$$

where $g_{\bar{\alpha}\bar{\beta}}(\theta)$ is the metric on S^{N-1}. The action Eq. (11.6) is the $O(N)$ invariant sigma model. It is a non-linear action for the fields $\theta_{\bar{\theta}}$. Show that this action is invariant under the global $O(N)$ transformations

What does happen to Goldstone's theorem in theories with local gauge symmetry? To answer this question let us consider a model with the gauge group $G = SO(3)$ and scalar fields $\phi^a(x), a = 1, 2, 3$ in the adjoint representation of this group. This immediately fixes the gauge covariant derivative of $\phi^a(x)$,

$$\nabla_\mu \phi^a(x) = \partial_\mu \phi^a(x) + e\varepsilon^{abc} A^b_\mu \phi^c(x)$$

The action will be,

$$I(\phi, A) = \int d^4x[-\frac{1}{4}F^a_{\mu\nu}(x)F^{a\mu\nu}(x) + \frac{1}{2}(\nabla^\mu \phi(x))^a \nabla_\mu \phi^a(x) - V(\phi^a\phi^a)] \qquad (12.1)$$

We shall choose $V(\phi^a\phi^a)$ to be given by,

$$V(\phi^a\phi^a) = \frac{m^2}{2}\phi^a\phi^a + \frac{\lambda}{4}(\phi^a\phi^a)^2$$

For $m^2 < 0$ there will be spontaneous symmetry breaking. The local gauge symmetry allows us to rotate the axis in the 3-dimensional space of the ϕ fields such that this field points always in the direction of the 3-axis. With this choice $< 0|\hat{\phi}(x)|0 >$ will also be in the third direction. We can thus the shifted field as,

$$< 0|\hat{\vec{\phi}}_H(x)|0 >= \vec{\phi}(x) = \vec{\chi}(x) + \vec{v} = \begin{pmatrix} 0 \\ 0 \\ \rho(x) + v \end{pmatrix}$$

The covariant derivative written in terms of the shifted field becomes,

$$\nabla_\mu \vec{\phi}(x) = \begin{pmatrix} eA_{2\mu}(v + \rho(x)) \\ -eA_{1\mu}(v + \rho(x)) \\ \partial_\mu \rho \end{pmatrix}$$

We then obtain,

$$
\begin{aligned}
\nabla_\mu \vec{\phi}.(x) \nabla_\mu \vec{\phi}(x) &= \partial_\mu \rho \partial^\mu \rho + e^2 (v+\rho)^2 (A_{1\mu}^2 + A_{2\mu}^2) \\
&= \partial_\mu \rho \partial^\mu \rho + e^2 v^2 (A_{1\mu}^2 + A_{2\mu}^2) + 2ev\rho(x)(A_{1\mu}^2 + A_{2\mu}^2) + e^2 \rho(x)^2 (A_{1\mu}^2 + A_{2\mu}^2)
\end{aligned} \tag{12.2}
$$

After substituting into the action the second term on the r.h.s will indicate that the $A_{1\mu}$ and $A_{2\mu}$ vector fields have become massive and their masses is given by ev.

It is convenient to combine the vector fields A_μ^1 and A_μ^2 into a complex vector field W_μ^+ and its complex conjugate field W_μ^- by,

$$
\begin{aligned}
A_{1\mu} &= \frac{1}{\sqrt{2}} (W_\mu^+ + W_\mu^-) \\
A_{2\mu} &= \frac{i}{\sqrt{2}} (W_\mu^+ - W_\mu^-)
\end{aligned} \tag{12.3}
$$

Substituting this in the formula above we obtain,

$$
\begin{aligned}
\nabla_\mu \vec{\phi}.(x) \nabla_\mu \vec{\phi}(x) &= \partial^\mu \rho \partial_\mu \rho + 2e^2 (v+\rho)^2 W_\mu^+ W_\mu^- \\
&= \partial^\mu \rho \partial_\mu \rho + 2e^2 v^2 W_\mu^+ W_\mu^- + 4ev\rho(x) W_\mu^+ W_\mu^- + 2e^2 \rho(x)^2 W_\mu^+ W_\mu^-
\end{aligned}
$$

The action thus becomes,

$$
\begin{aligned}
I(\phi, A) = \int d^4x [& -\frac{1}{4} F_{\mu\nu}^a (x) F^{a\mu\nu}(x) + \frac{1}{2} \partial^\mu \rho \partial_\mu \rho \\
& e^2 v^2 W_\mu^+ W_\mu^- + 2ev\rho(x) W_\mu^+ W_\mu^- + e^2 \rho(x)^2 W_\mu^+ W_\mu^- \\
& - \frac{m^2}{2} (v+\rho)^2 - \frac{\lambda}{4} (v+\rho)^4]
\end{aligned} \tag{12.4}
$$

What remains is to express the the $-\frac{1}{4} F_{\mu\nu}^a (x) F^{a\mu\nu}(x)$ in terms of W_μ^+, W_μ^- and $A_{3\mu}$. We shall first construct the components of $F_{a\mu\nu}$. A straightforward calculation gives us, vector field,

$$F_{1\mu\nu}(x) = \frac{1}{\sqrt{2}}\{\nabla_\mu(W_\nu^+ + W_\nu^-) - \nabla_\nu(W_\mu^+ + W_\mu^-)\}$$
$$F_{2\mu\nu}(x) = \frac{i}{\sqrt{2}}\{\nabla_\mu(W_\nu^+ - W_\nu^-) - \nabla_\nu(W_\mu^+ - W_\mu^-)$$
$$F_{3\mu\nu}(x) = \{\partial_\mu A_\nu - \partial_\nu A_\mu$$
$$\frac{i}{2}e[(W_\mu^+ + W_\mu^-)(W_\nu^+ - W_\nu^-) - (W_\nu^+ + W_\nu^-)(W_\mu^+ - W_\mu^-))\}$$

Introduce the definition:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

Using these expressions we obtain,

$$-\frac{1}{4}F_{\mu\nu}^a(x)F^{a\mu\nu}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}(\nabla_\mu W_\nu^+ - \nabla_\nu W_\mu^+)(\nabla_\mu W_\nu^+ - \nabla_\nu W_\mu^+)$$
$$+ \frac{1}{2}ieF_{\mu\nu}(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - e^2(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-)^2$$

This expression should be substituted in Eq. (12.4) to obtain the full action,

$$\begin{aligned} I(\phi, A) = \int d^4x[&- \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}(\nabla_\mu W_\nu^+ - \nabla_\nu W_\mu^+)(\nabla_\mu W_\nu^+ - \nabla_\nu W_\mu^+) \\ &+ e^2v^2 W_\mu^+ W_\mu^- + \frac{1}{2}ieF_{\mu\nu}(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - e^2(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-)^2 \\ &+ \frac{1}{2}\partial^\mu\rho\partial_\mu\rho - \frac{1}{2}(-2m^2)\rho^2 - \lambda v\rho^3 - \frac{\lambda}{4}\rho^4 \\ &+ 2ev\rho(x)W_\mu^+ W_\mu^- + e^2\rho(x)^2 W_\mu^+ W_\mu^-] \end{aligned} \tag{12.5}$$

The action shows manifest local $U(1)$ gauge symmetry under which the field W_μ^+ has charge $-e$. The propagating physical degrees of freedom are thus a massless vector potential A_μ a massive charged vector particle W_μ^+ and its antiparticle W_μ^- and a neutral scalar field ρ. This makes a total of nine physical propagating degrees of freedom (2+ 2×3+1=9). This is the same number as in the original action (3× 2 + 3=9).

Exercise

1. Repeat the calculation of the Higgs mechanism for the group $SO(2)$

2.Fill in the claculational details to obtain Eq.(12.5) from Eq. (12.4)

In order to show that the rules for fermion functional integral are correct we shall consider the QED action and integrate over the fermion field. The QED is given by

$$I[A,\psi,\bar{\psi}] = \int d^4x\{\bar{\psi}(i\gamma^\mu\partial_\mu - m - e_0\gamma^\mu A_\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\}$$

where we have denoted the bare coupling constant by e_0. In fact by a simple rescaling of the A field we can put e_0 in front of the F^2 term, viz,

$$A_\mu \to \frac{1}{e_0}A_\mu \tag{13.1}$$

The action then becomes,

$$I[A,\psi,\bar{\psi}] = \int d^4x\{\bar{\psi}(i\gamma^\mu\partial_\mu - m - \gamma^\mu A_\mu)\psi - \frac{1}{4e_0^2}F_{\mu\nu}F^{\mu\nu}\} \tag{13.2}$$

In this section we shall use this form of the action. We shall postpone the integration over the vector field A but perform the functional integrations over $\bar{\psi}$ and ψ in the background of an A field. In this way we obtain a functional $W(A)$ of the A field,

$$\begin{aligned} e^{iW(A)} &= N\int[d\bar{\psi}d\psi]e^{i\int d^4x\{\bar{\psi}(i\gamma^\mu\partial_\mu - m - \gamma^\mu A_\mu)\psi - \frac{1}{4e_0^2}F_{\mu\nu}F^{\mu\nu}\}} \\ &= Ne^{-\frac{i}{4e_0^2}\int dxF_{\mu\nu}F^{\mu\nu}}Det[-i(i\gamma^\mu\partial_\mu - m - \gamma^\mu A_\mu)] \end{aligned} \tag{13.3}$$

The normalisation factor N is determined from the condition that $W(0) = 0$,

$$N^{-1} = Det[-i(i\gamma^\mu\partial_\mu - m + i\epsilon)]$$

This gives us

$$e^{+\frac{i}{4e_0^2}\int dx F_{\mu\nu}F^{\mu\nu}} e^{iW(A)} = [Det(i\gamma^\mu \partial_\mu - m + i\epsilon)^{-1}(i\gamma^\mu \partial_\mu - m - \gamma^\mu A_\mu)]$$
$$= Det[(1 - (i\gamma^\mu \partial_\mu - m + i\epsilon)^{-1}\gamma^\mu A_\mu]$$
$$= Det[(1 + S(i\gamma^\mu A_\mu)] \tag{13.4}$$

where we used Eq. (5.19) to substitute

$$(i\gamma^\mu \partial_\mu - m + i\epsilon)^{-1} = -iS \tag{13.5}$$

Thus,

$$iW(A) = lnDet[(1 + S(i\gamma^\mu A_\mu)] - \frac{i}{4e_0^2}\int dx F_{\mu\nu}F^{\mu\nu} \tag{13.6}$$

Use the formula,

$$lnDetA = TrlnA$$

to obtain,

$$iW(A) = Trln[(1 + S(i\gamma^\mu A_\mu)] - \frac{i}{4e_0^2}\int dx F_{\mu\nu}F^{\mu\nu} \tag{13.7}$$

The ln can be expanded in a power series using the formula

$$ln(1 - L) = -\Sigma_1^\infty \frac{1}{n} L^n$$

For us L is given by the following,

$$L = -iS\gamma^\mu A_\mu \tag{13.8}$$

Using these formulae Eq. (13.7) becomes,

$$iW(A) = -\Sigma_1^\infty \frac{1}{n} TrL^n - \frac{i}{4e_0^2}\int dx F_{\mu\nu}F^{\mu\nu} \tag{13.9}$$

We need to calculate

$$TrL^n = Tr(-iS\gamma^\mu A_\mu)^n$$

The trace is both over the space time basis $|x>$ as well as over the spinor indices. We shall denote the the trace over the spinor indices by tr. We can thus write,

$$\begin{aligned} TrL^n &= \int d^4x < x|trL^n|x > \\ &= \int d^4x \int d^4y_1 tr < x|L|y_1 >< y_1|....... \int d^4y_{n-1}|y_{n-1} >< y_{n-1}|L|x > \end{aligned} \tag{13.10}$$

Using the definition of L in Eq.(13.8) we obtain,

$$\begin{aligned} < y_1|L|y_2 > &= (-i) \int d^4z < y_1|S|z >< z|\gamma^\mu A_\mu|y_2 > \\ &= (-i) \int d^4z S(y_1 - z)\gamma^\mu A_\mu(z) < z|y_2 > \\ &= (-i) \int d^4z S(y_1 - z)\gamma^\mu A_\mu(z)\delta_4(z - y_2) \\ &= S(y_1 - y_2)(-i\gamma^\mu)A_\mu(y_2) \end{aligned}$$

Substitute this result for the matrix elements of L on the right hand side of Eq(13.10)

$$\begin{aligned} TrL^n = \int d^4y_1 ... \int d^4y_n tr[S(y_1 - y_2)(-i\gamma^{\mu_2})A_{\mu_2}(y_2)... \\ ...S(y_n - y_1)(-i\gamma^{\mu_1})A_{\mu_1}(y_1)] \end{aligned} \tag{13.11}$$

where we renamed the integration variable x in Eq(13.11) and called it y_1, then made a renaming $y_1 \to y_2 \to y_3 \to, ..., \to y_{n-1} \to y_n$. Substitute Eq.(13.11) in Eq.(13.9) to obtain,

$$iW(A) = -\frac{i}{4e_0^2}F_{\mu\nu}F^{\mu\nu} + \Sigma_1^\infty \frac{1}{n}\int d^4y_1 ... \int d^4y_n A^{\mu_1}(y_1)...A^{\mu_n}(y_n)$$
$$[-tr(S(y_1-y_2)[(-i\gamma_{\mu_2})......S(y_n-y_1)(-i\gamma_{\mu_1})] \tag{13.12}$$

The trace is over the spinor indices. In each term under the summation sign the quantity under the trace can be represented by a closed fermion loop with n vertices. To each vertex there are attached a vertex factor of $(-i\gamma_\mu)$ and an external photon line A^μ. Had we not used the rescaling of A_μ given in Eq. (13.1) the vertex factor would be $(-ie_0\gamma_\mu)$. Note the overall minus sign, appropriate for a closed fermion loop. This is represented graphically as in the diagram below.

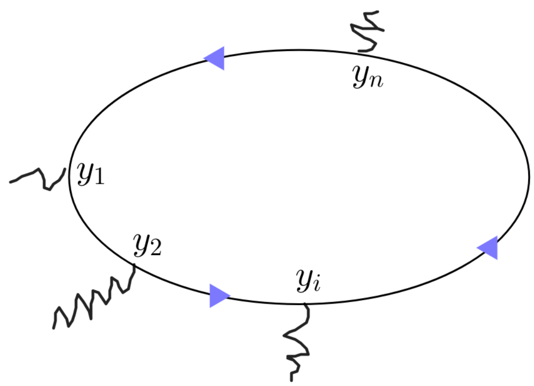

FIG. 8. The fermion loop representing the n-th order term in Eq.(13.12). The external lines represent the external vector field A_μ.

Exercise 1

Start from the QED Feynman rules and apply them to loop graphs given above with arbitrary n and sum them to obtain Eq.(13.12).

Exercise 2

Consider the functional $W(\phi)$ defined by,

$$e^{iW(\phi)} = \int [d\bar{\chi}d\chi]e^{iI(\bar{\chi},\chi,\phi)}$$

where

$$I(\bar{\chi}, \chi, \phi) = \int dx[\partial_\mu \bar{\chi}(x) \partial^\mu \chi(x) - m^2 \bar{\chi}(x)\chi(x) - \phi(x)\bar{\chi}(x)\chi(x)]$$

Here $\chi(x)$ and $\phi(x)$ are, respectively, complex and real scalar fields. Expand $W(\phi)$ in a power series of ϕ and point out the salient differences with the result which you would obtain if χ was a fermion field. Evaluate the coefficients of the first 2-terms (up to and including quadratic terms in ϕ) in the expansion of $W(\phi)$. Use your calculation to argue that in the absence of extra terms the model would not be renormalisable and identify those extra terms which would make the model renormalisable. Use Feynman graphical method to provide a qualitative reason for any extra term needed for the renormalisabiity.

By charge conjugation invariance of QED any graph with an odd number of external photon line is zero (Furry's theorem). Hence the leading non zero term in Eq.(13.12) will have two external photon fields. This term is given by,

$$\begin{aligned} iW(A) &= -\frac{i}{4e_0^2}\int d^4x F_{\mu\nu}F^{\mu\nu} \\ &- \frac{(-i)^2}{2}\int d^4y_1 \int d^4y_2 A^{\mu}(y_1)A^{\nu}(y_2) tr(S(y_1-y_2)\gamma_\mu S(y_2-y_1)\gamma_\nu) \end{aligned} \quad (14.1)$$

Define $\Pi(y_1 - y_2)$ by,

$$-i\Pi_{\mu\nu}(y_1-y_2) \equiv -(-i)^2 tr(S(y_1-y_2)\gamma_\mu S(y_2-y_1)\gamma_\nu) \quad (14.2)$$

Then Eq. (13.12) becomes,

$$iW(A) = -\frac{i}{4e_0^2}\int d^4x F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\int d^4y_1 \int d^4y_2 A^{\mu}(y_1)\Pi_{\mu\nu}(y_1-y_2)A^{\nu}(y_2) \quad (14.3)$$

To evaluate the loop contribution we go to Fourier space and substitute,

$$S(y_1-y_2) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{\not p - m + i\epsilon}e^{-ip(y_1-y_2)}$$

Then Eq.(13.12) can be brought to the following form,

$$W(A) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}^{\mu}(-k)[\frac{1}{e_0^2}(-k^2\eta_{\mu\nu} + k_\mu k_\nu) - \Pi_{\mu\nu}(k)]\tilde{A}^{\nu}(k) \quad (14.4)$$

The loop part then can be written as

$$-i\Pi_{\mu\nu}(k) \equiv -(-i)^2\int \frac{d^np}{(2\pi)^n} tr[\gamma_\mu \frac{i(\not p + m)}{p^2 - m^2 + i\epsilon}\gamma_\nu \frac{i(\not p - \not k + m)}{(p-k)^2 - m^2 + i\epsilon}] \quad (14.5)$$

where anticipating dimensional regularization we have written the integral in n

dimensions.

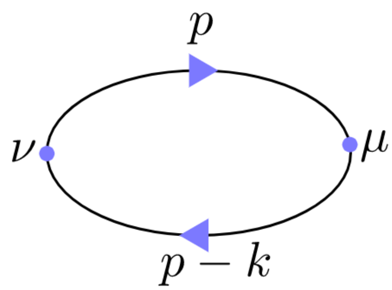

FIG. 9. The 1-lppo graph contributing to the vacuum polarization.

Since the trace of a product of an odd number of γ matrices is zero the numerator of Eq.(14.5) simplify and we obtain

$$
\begin{aligned}
i\Pi_{\mu\nu}(k) &= \int \frac{d^n p}{(2\pi)^n} \frac{tr[\gamma_\mu(\not p + m)\gamma_\nu(\not p - \not k) + m)]}{(p^2 - m^2 + i\epsilon)(p-k)^2 - m^2 + i\epsilon} \\
&= \int \frac{d^4 p}{(2\pi)^4} \frac{tr[\gamma_\mu \not p \gamma_\nu(\not p - \not k) + m^2\gamma_\mu\gamma_\nu]}{(p^2 - m^2 + i\epsilon)[(p-k)^2 - m^2 + i\epsilon]}
\end{aligned}
\tag{14.6}
$$

Use the Feynman trick Eq. (2.42),

$$
\frac{1}{AB} = \int_0^1 dx \frac{1}{[A + (B-A)x]^2}
$$

Choose $A = (p-k)^2 - m^2 + i\epsilon$ and $B = p^2 - m^2 + i\epsilon$. Then Eq. (2.41) becomes,

$$
i\Pi(k)_{\mu\nu} = \int \frac{d^n p}{(2\pi)^n} \int_0^1 dx \frac{tr[\gamma_\mu \not p \gamma_\nu(\not p - \not k) + m^2\gamma_\mu\gamma_\nu]}{[(p - k(1-x))^2 + k^2 x(1-x) - m^2 + i\epsilon]^2}
$$

Complete the square in the expression inside the square bracket in the denominator and shift the integration variable $p \to p + (1-x)k$. The last expression then becomes,

$$
i\Pi(k)_{\mu\nu} = \int \frac{d^n p}{(2\pi)^n} \int_0^1 dx \frac{tr[\gamma_\mu(\not p + (1-x)\not k)\gamma_\nu(\not p + (1-x)\not k - \not k) + m^2\gamma_\mu\gamma_\nu]}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2}
$$

The numerator can be further simplified by noting that the terms with a product of an odd number of p's will integrate to zero

$$i\Pi(k)_{\mu\nu} = \int \frac{d^n p}{(2\pi)^n} \int_0^1 dx \frac{tr[\gamma_\mu \not{p} \gamma_\nu \not{p} + x(x-1)\gamma_\mu \not{k} \gamma_\nu \not{k} + m^2 \gamma_\mu \gamma_\nu]}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2} \tag{14.7}$$
$$= \int \frac{d^n p}{(2\pi)^n} \int_0^1 dx \frac{4[(2p_\mu p_\nu - \eta_{\mu\nu} p^2) + x(x-1)(2k_\mu k_\nu - \eta_{\mu\nu} k^2) + m^2 \eta_{\mu\nu}]}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2}$$

By Lorentz invariance we can write,

$$\int \frac{d^n p}{(2\pi)^n} \frac{p_\mu p_\nu}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2} = \frac{1}{n} \int \frac{d^n p}{(2\pi)^n} \frac{p^2 \eta_{\mu\nu}}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2}$$

Using this the last expression for $i\Pi_{\mu\nu}$ becomes,

$$i\Pi(k)_{\mu\nu} = 4(\frac{2}{n} - 1)\eta_{\mu\nu} \int \frac{d^n p}{(2\pi)^n} \int_0^1 dx \frac{p^2}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2}$$
$$+ 4[x(x-1)(2k_\mu k_\nu - \eta_{\mu\nu} k^2) + m^2 \eta_{\mu\nu}] \int \frac{d^n p}{(2\pi)^n} \int_0^1 dx \frac{1}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2} \tag{14.8}$$

At this stage we perform the Wick rotation and evaluate the p integrals,

$$\int \frac{d^n p}{(2\pi)^n} \frac{p^2}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2} = -\frac{i}{(4\pi)^{\frac{n}{2}}} \frac{\frac{n}{2}\Gamma(1 - \frac{n}{2})}{[m^2 - k^2 x(1-x)]^{1-\frac{n}{2}}} \tag{14.9}$$

and

$$\int \frac{d^n p}{(2\pi)^n} \frac{1}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2} = \frac{i}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(2 - \frac{n}{2})}{[m^2 - k^2 x(1-x)]^{2-\frac{n}{2}}} \tag{14.10}$$

These expressions must be substituted in Eq(14.8).

$$i\Pi(k)_{\mu\nu} = \int \frac{d^np}{(2\pi)^n} \int_0^1 dx \frac{tr[\gamma_\mu \not{p} \gamma_\nu \not{p} + x(x-1)\gamma_\mu \not{k} \gamma_\nu \not{k} + m^2 \gamma_\mu \gamma_\nu]}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2} \tag{14.7}$$
$$= \int \frac{d^np}{(2\pi)^n} \int_0^1 dx \frac{4[(2p_\mu p_\nu - \eta_{\mu\nu} p^2) + x(x-1)(2k_\mu k_\nu - \eta_{\mu\nu} k^2) + m^2 \eta_{\mu\nu}]}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2}$$

By Lorentz invariance we can write,

$$\int \frac{d^np}{(2\pi)^n} \frac{p_\mu p_\nu}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2} = \frac{1}{n} \int \frac{d^np}{(2\pi)^n} \frac{p^2 \eta_{\mu\nu}}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2}$$

Using this the last expression for $i\Pi_{\mu\nu}$ becomes,

$$i\Pi(k)_{\mu\nu} = 4(\frac{2}{n} - 1)\eta_{\mu\nu} \int \frac{d^np}{(2\pi)^n} \int_0^1 dx \frac{p^2}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2}$$
$$+ 4[x(x-1)(2k_\mu k_\nu - \eta_{\mu\nu} k^2) + m^2 \eta_{\mu\nu}] \int \frac{d^np}{(2\pi)^n} \int_0^1 dx \frac{1}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2} \tag{14.8}$$

At this stage we perform the Wick rotation and evaluate the p integrals,

$$\int \frac{d^np}{(2\pi)^n} \frac{p^2}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2} = -\frac{i}{(4\pi)^{\frac{n}{2}}} \frac{\frac{n}{2}\Gamma(1 - \frac{n}{2})}{[m^2 - k^2 x(1-x)]^{1-\frac{n}{2}}} \tag{14.9}$$

and

$$\int \frac{d^np}{(2\pi)^n} \frac{1}{[p^2 + k^2 x(1-x) - m^2 + i\epsilon]^2} = \frac{i}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(2 - \frac{n}{2})}{[m^2 - k^2 x(1-x)]^{2-\frac{n}{2}}} \tag{14.10}$$

These expressions must be substituted in Eq(15.17). The result is,

$$i\Pi(k)_{\mu\nu} = \frac{i}{(4\pi)^{\frac{n}{2}}}\int_0^1 dx[-\frac{\eta_{\mu\nu}4(\frac{2}{n}-1)\frac{n}{2}\Gamma(1-\frac{n}{2})}{[m^2-k^2x(1-x)]^{1-\frac{n}{2}}}$$
$$+ 4[x(x-1)(2k_\mu k_\nu - \eta_{\mu\nu}k^2) + m^2\eta_{\mu\nu}]\frac{\Gamma(2-\frac{n}{2})}{[m^2-k^2x(1-x)]^{2-\frac{n}{2}}} \tag{14.11}$$

In the first term on the right hand side we can write,

$$(\frac{2}{n}-1)\frac{n}{2}\Gamma(1-\frac{n}{2}) = (1-\frac{2}{n})\Gamma(1-\frac{n}{2}) =$$
$$= \Gamma(2-\frac{n}{2}) \tag{14.12}$$

where we used the property

$$(1-\frac{n}{2})\Gamma(1-\frac{n}{2}) = \Gamma(2-\frac{n}{2})$$

We also write,

$$\frac{1}{[m^2-k^2x(1-x)]^{1-\frac{n}{2}}} = \frac{m^2-k^2x(1-x)}{[m^2-k^2x(1-x)]^{2-\frac{n}{2}}} \tag{14.13}$$

Inserting from Eq. (14.12) and Eq. (14.13) on the right hand side of Eq.(14.14) we obtain,

$$i\Pi(k)_{\mu\nu} = \frac{4i}{(4\pi)^{\frac{n}{2}}}\int_0^1 dx\{-\frac{\eta_{\mu\nu}[m^2-k^2x(1-x)]\Gamma(2-\frac{n}{2})}{[m^2-k^2x(1-x)]^{2-\frac{n}{2}}}$$
$$+ [x(x-1)(2k_\mu k_\nu - \eta_{\mu\nu}k^2) + m^2\eta_{\mu\nu}]\frac{\Gamma(2-\frac{n}{2})}{[m^2-k^2x(1-x)]^{2-\frac{n}{2}}}\}$$
$$= \frac{4i}{(4\pi)^{\frac{n}{2}}}\int_0^1 dx$$
$$[-\eta_{\mu\nu}(m^2-k^2x(1-x)) + x(x-1)(2k_\mu k_\nu - \eta_{\mu\nu}k^2) + m^2\eta_{\mu\nu}]\frac{\Gamma(2-\frac{n}{2})}{[m^2-k^2x(1-x)]^{2-\frac{n}{2}}} \tag{14.14}$$

The $\eta_{\mu\nu}m^2$ terms in the numerator cancel and we finally obtain,

$$
\begin{aligned}
i\Pi(k)_{\mu\nu} &= \frac{4i}{(4\pi)^{\frac{n}{2}}}\int_0^1 dx[2x(x-1)(k_\mu k_\nu - \eta_{\mu\nu}k^2)\frac{\Gamma(2-\frac{n}{2})}{[m^2-k^2x(1-x)]^{2-\frac{n}{2}}}] \\
&= \frac{4i}{(4\pi)^{\frac{n}{2}}}\int_0^1 dx[2x(x-1)\frac{(k_\mu k_\nu - \eta_{\mu\nu}k^2)}{m^{4-n}}\frac{\Gamma(2-\frac{n}{2})}{[1-x(1-x)\frac{k^2}{m^2})]^{2-\frac{n}{2}}}] \qquad (14.15)
\end{aligned}
$$

Now we must study the $n \to 4$ limit. In terms $\delta = 4-n$ introduced in Eq. (3.13) we have $\Gamma(2-\frac{n}{2}) = \Gamma(\frac{\delta}{2})$. As $\delta \to 0$ Eq.(3.10) gives us,

$$
\Gamma(\frac{\delta}{2}) = [\frac{2}{\delta} - \gamma + O(\delta)] \qquad (14.16)
$$

Likewise,

$$
[1-x(1-x)\frac{k^2}{m^2}]^{2-\frac{n}{2}} = 1 + \frac{\delta}{2} ln[1-x(1-x)\frac{k^2}{m^2}]
$$

Thus,

$$
\begin{aligned}
\frac{\Gamma(2-\frac{n}{2})}{[1-x(1-x)\frac{k^2}{m^2}]^{2-\frac{n}{2}}} &= [\frac{2}{\delta} - \gamma + O(\delta)][1 - \frac{\delta}{2} ln[1-x(1-x)\frac{k^2}{m^2}]] \\
&= \frac{2}{\delta} - \gamma - ln[1-x(1-x)\frac{k^2}{m^2}] + O(\delta) \qquad (14.17)
\end{aligned}
$$

Substitute this expression in Eq. (14.15) to obtain

$$
\begin{aligned}
\Pi(k)_{\mu\nu} &= \frac{8}{(4\pi)^{\frac{n}{2}}}\int_0^1 dx[x(x-1)\frac{(k_\mu k_\nu - \eta_{\mu\nu}k^2)}{m^{4-n}}\frac{\Gamma(2-\frac{n}{2})}{[(1-x(1-x)\frac{k^2}{m^2})]^{2-\frac{n}{2}}}] \\
&= \frac{8}{(4\pi)^{\frac{n}{2}}}(\frac{2}{\delta} - \gamma)\frac{(k_\mu k_\nu - \eta_{\mu\nu}k^2)}{m^{4-n}}\int_0^1 dx x(x-1) \\
&+ \frac{8}{(4\pi)^{\frac{n}{2}}}\frac{(k_\mu k_\nu - \eta_{\mu\nu}k^2)}{m^{4-n}}\int_0^1 dx x(1-x) ln[1-x(1-x)\frac{k^2}{m^2}]
\end{aligned}
$$

In the first term on the right hand side the integral over x is elementary. We thus obtain,

$$\Pi(k)_{\mu\nu} = -\frac{1}{6}\frac{8}{(4\pi)^{\frac{n}{2}}}(\frac{2}{\delta}-\gamma)\frac{(k_\mu k_\nu - \eta_{\mu\nu}k^2)}{m^{4-n}} + \frac{8}{(4\pi)^{\frac{n}{2}}}\frac{(k_\mu k_\nu - \eta_{\mu\nu}k^2)}{m^{4-n}}\int_0^1 dxx(1-x)ln[1-x(1-x)\frac{k^2}{m^2}]$$

We can now set $n = 4$ in the δ independent terms,

$$\Pi(k)_{\mu\nu} = -\frac{1}{12\pi^2}(\frac{2}{\delta}-\gamma)(k_\mu k_\nu - \eta_{\mu\nu}k^2) + \frac{1}{2\pi^2}(k_\mu k_\nu - \eta_{\mu\nu}k^2)\int_0^1 dxx(1-x)ln[1-x(1-x)\frac{k^2}{m^2}] \quad (14.18)$$

Note that $\Pi_{\mu\nu}(k)$ is transverse,

$$k^\mu \Pi_{\mu\nu}(k) = 0 \quad (14.19)$$

We must substitute $\Pi_{\mu\nu}$ from Eq. (14.18) into Eq. (14.4) to obtain $W(A)$

$$W(A) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{A}^\mu(-k)(-k^2\eta_{\mu\nu} + k_\mu k_\nu)\{\frac{1}{e_0^2} + \frac{1}{12\pi^2}(\frac{2}{\delta}-\gamma) - \frac{1}{2\pi^2}\int_0^1 dxx(1-x)ln[1-x(1-x)\frac{k^2}{m^2}]\}\tilde{A}^\nu(k) \quad (14.20)$$

As a consequence of transversality of $\Pi_{\mu\nu}$ (c.f. Eq (14.19) $W(A)$ remains invariant under a gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu\chi(x)$ which in Fourier space becomes $\tilde{A}_\mu(k) \to \tilde{A}_\mu(k) + ik_\mu\tilde{\chi}(k)$.

To interpret the singular part of $W(A)$ we note that $W(A)$ is a 1-loop corrected Maxwell action for the classical field A_μ. Observationally we can only measure the combination $\frac{1}{e_0^2} + \frac{1}{12\pi^2}(\frac{2}{\delta}-\gamma)$. Hence we define e^2 by,

$$\frac{1}{e^2} \equiv \frac{1}{e_0^2} + \frac{1}{12\pi^2}(\frac{2}{\delta}-\gamma) \quad \Rightarrow \quad e_0^2 = e^2[1 + \frac{e^2}{12\pi^2}(\frac{2}{\delta}-\gamma)] \quad (14.21)$$

and express $W(A)$ in terms e^2. For later use we shall write the relation between the bare coupling e_0 and the renormalised coupling e in the following form,

$$e_0 = Z_e e \tag{14.22}$$

where the renormalisation constant Z_e is given by,

$$Z_e = 1 + \frac{e^2}{24\pi^2}(\frac{2}{\delta} - \gamma) + O(e^4) \tag{14.23}$$

With this definition Eq.(14.20) becomes,

$$W(A) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}$$
$$\tilde{A}^\mu(-k)(-k^2\eta_{\mu\nu} + k_\mu k_\nu)\{\frac{1}{e^2} - \frac{1}{2\pi^2}\int_0^1 dxx(1-x)ln[1 - x(1-x)\frac{k^2}{m^2}]\}\tilde{A}^\nu(k) \tag{14.24}$$

If we rescale A by $A_\mu \to eA_\mu$ the above becomes,

$$W(A) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}$$
$$\tilde{A}^\mu(-k)(-k^2\eta_{\mu\nu} + k_\mu k_\nu)\{1 - \frac{e^2}{2\pi^2}\int_0^1 dxx(1-x)ln[1 - x(1-x)\frac{k^2}{m^2}]\}\tilde{A}^\nu(k) \tag{14.25}$$

In order to write $W(A)$ in a compact form we introduce the function $\pi(k^2)$ by,

$$\pi(k^2) = -\frac{e^2}{2\pi^2}\int_0^1 dxx(1-x)ln[1 - x(1-x)\frac{k^2}{m^2}] \tag{14.26}$$

and write $W(A)$ as,

$$W(A) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}^\mu(-k)(-k^2\eta_{\mu\nu} + k_\mu k_\nu)[1 + \pi(k^2)]\tilde{A}^\nu(k) \tag{14.27}$$

We can rescale the field A as $A \to e_0 A$ and rewrite Eq.(14.20) as,

$$W(A) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}$$
$$\tilde{A}^\mu(-k)(-k^2\eta_{\mu\nu} + k_\mu k_\nu)\{1 + \frac{e_0^2}{12\pi^2}(\frac{2}{\delta} - \gamma) - \frac{e_0^2}{2\pi^2}\int_0^1 dxx(1-x)ln[1 - x(1-x)\frac{k^2}{m^2}]\}\tilde{A}^\nu(k) \tag{14.28}$$

Recall that the field A is a bare field. As usual we can introduce a renormalised field A_R by,

$$A^\mu(x) = Z_3^{\frac{1}{2}} A_R^\mu(x) \tag{14.29}$$

where Z_3 is the vector field renormalisation constant. We choose Z_3 such that the singular term in Eq. (14.28) disappears up to the order of $e_)^2$, viz,

$$Z_3[1 + \frac{e_0^2}{12\pi^2}(\frac{2}{\delta} - \gamma)] = 1 \qquad \to \qquad Z_3 = [1 - \frac{e_0^2}{12\pi^2}(\frac{2}{\delta} - \gamma)] + ... \tag{14.30}$$

where .... indicate terms of order e_0^4 and higher which we ignore at this order. Up to e_0^2 terms then we obtain,

$$W(A) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}$$
$$\tilde{A}_R^\mu(-k)(-k^2\eta_{\mu\nu} + k_\mu k_\nu)\{1 - \frac{e_0^2}{2\pi^2}\int_0^1 dxx(1-x)ln[1 - x(1-x)\frac{k^2}{m^2}]\}\tilde{A}_R^\nu(k) \tag{14.31}$$

If we compare the 1-loop expression for the charge renormalisation constant Z_e as given by Eq.(14.23) and Z_3 given by Eq.(14.30) we obtain,

$$Z_e = Z_3^{-\frac{1}{2}} \tag{14.32}$$

The remarkable fact is that, on the basis of the Ward identities, this 1-loop result continues to be valid to all orders in perturbation theory. This relation shows that in QED the renormalisation constants are not independent from each other. The important physical consequence of this fact will be discussed later.

Exercises

1. Consider the integral,

$$I(p) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - k.p - m^2 + i\epsilon)^\alpha}$$

Evaluate this integral. You can use your result to obtain integrals of the form,

$$I_{\mu_1 \cdots \mu_r}(p) = \int \frac{d^n k}{(2\pi)^n} \frac{p_{\mu_1} \ldots p_{\mu_r}}{(k^2 - k.p - m^2 + i\epsilon)^\alpha}$$

2. Evaluate $\Pi_{\mu\nu}(k)$ in 2-space-time dimensions by using the dimensional regularisation as well as a momentum space cut off Λ. Comment on the result you obtain.

15. RENORMALIZED QED

15.A. Photon Propagator

With the inclusion of the gauge fixing term the total action of QED is given by,

$$I_{total}[A,\psi,\bar{\psi}] =$$
$$= \int d^4x\{\bar{\psi}(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu)\psi + \frac{1}{2}A^\mu(\partial^2\eta_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu + \frac{1}{2\xi}(A^\mu\partial_\mu\partial_\nu A^\nu(x))\} \tag{15.1}$$

The operator sandwiched between two A's is

$$M_{\mu\nu} = \partial^2\eta_{\mu\nu} - \partial_\mu\partial_\nu + \frac{1}{\xi}\partial_\mu\partial_\nu \tag{15.2}$$

In matrix notation the Feynman propagator for photon is defined by,

$$M^{\mu\lambda}\Delta_{\lambda\nu}(x-y) = i\delta^\mu_\nu\delta(x-y) \tag{15.3}$$

We shall write this equation in Fourier space. Firstly in Fourier space $M_{\mu\nu}$ becomes,

$$\begin{aligned}\tilde{M}_{\mu\nu} &= -k^2\eta_{\mu\nu} + k_\mu k_\nu - \frac{1}{\xi}k_\mu k_\nu \\ &= k^2(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}) - \frac{1}{\xi}\frac{k_\mu k_\nu}{k^2}\end{aligned} \tag{15.4}$$

Eq. (15.3) then gives us

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{i}{k^2+i\epsilon}[-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}] - \frac{i\xi}{k^2+i\epsilon}\frac{k_\mu k_\nu}{k^2} \tag{15.5}$$

For $\xi = 1$ we obtain,

$$\tilde{\Delta}_{\mu\nu}(k) = -\frac{i}{k^2+i\epsilon}\eta_{\mu\nu} \tag{15.6}$$

This is called Feynman gauge. Another popular gauge is the Landau gauge which is obtained for $\xi = 0$,

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{i}{k^2 + i\epsilon}[-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}] \tag{15.7}$$

We introduce the tensors $P_{T\mu\nu}$ and $P_{L\mu\nu}$ by,

$$\begin{aligned} P_{T\mu\nu} &\equiv -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \\ P_{L\mu\nu} &\equiv \frac{k_\mu k_\nu}{k^2} \end{aligned} \tag{15.8}$$

Note that,

$$\begin{aligned} P_{T\mu\nu} P_L^{\nu\lambda} &= 0 \\ P_{T\mu\nu} P_T^{\nu\lambda} &= -P_{T\mu}^{\lambda} \end{aligned} \tag{15.9}$$

Using these notations the photon propagator is written in a compact form,

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{i}{k^2 + i\epsilon} P_{T\mu\nu} - \frac{i\xi}{k^2 + i\epsilon} P_{L\mu\nu} \tag{15.10}$$

15.B. Renormalisation

The QED action, written in terms of bare fields A_0, ψ_0 , bare m_0 ,bare e_0 and bare gauge parameter ξ_0 is given by, is given by,

$$\begin{aligned} I[A_0,&\psi_0, \bar{\psi}_0] = \\ &= \int d^4x \{\bar{\psi}_0(i\gamma^\mu\partial_\mu - m_0)\psi_0 - e_0\bar{\psi}_0\gamma^\mu A_{0\mu}\psi_0 - \frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A_0^\mu(x))^2\} \end{aligned} \tag{15.11a}$$

In order to apply the renormalisation programme we must write the action in terms of the renormalised objects plus counter-terms. We denote the renormalised

15. RENORMALIZED QED

15.A. Photon Propagator

With the inclusion of the gauge fixing term the total action of QED is given by,

$$
\begin{aligned}
I_{total}[A,\psi,\bar{\psi}] = \\
= \int d^4x\{\bar{\psi}(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu)\psi + \frac{1}{2}A^\mu(\partial^2\eta_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu + \frac{1}{2\xi}(A^\mu\partial_\mu\partial_\nu A^\nu(x))\}
\end{aligned}
\tag{15.1}
$$

The operator sandwiched between two A's is

$$
M_{\mu\nu} = \partial^2\eta_{\mu\nu} - \partial_\mu\partial_\nu + \frac{1}{\xi}\partial_\mu\partial_\nu \tag{15.2}
$$

In matrix notation the Feynman propagator for photon is defined by,

$$
M^{\mu\lambda}\Delta_{\lambda\nu}(x-y) = i\delta^\mu_\nu\delta(x-y) \tag{15.3}
$$

We shall write this equation in Fourier space. Firstly in Fourier space $M_{\mu\nu}$ becomes,

$$
\begin{aligned}
\tilde{M}_{\mu\nu} &= -k^2\eta_{\mu\nu} + k_\mu k_\nu - \frac{1}{\xi}k_\mu k_\nu \\
&= k^2(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}) - \frac{1}{\xi}\frac{k_\mu k_\nu}{k^2}
\end{aligned}
\tag{15.4}
$$

Eq. (15.3) then gives us

$$
\tilde{\Delta}_{\mu\nu}(k) = \frac{i}{k^2+i\epsilon}[-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}] - \frac{i\xi}{k^2+i\epsilon}\frac{k_\mu k_\nu}{k^2} \tag{15.5}
$$

For $\xi = 1$ we obtain,

$$
\tilde{\Delta}_{\mu\nu}(k) = -\frac{i}{k^2+i\epsilon}\eta_{\mu\nu} \tag{15.6}
$$

This is called Feynman gauge. Another popular gauge is the Landau gauge which is obtained for $\xi = 0$,

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{i}{k^2 + i\epsilon}[-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}] \tag{15.7}$$

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Using these notations the photon propagator is written in a compact form,

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{i}{k^2 + i\epsilon} P_{T\mu\nu} - \frac{i\xi}{k^2 + i\epsilon} P_{L\mu\nu} \tag{15.10}$$

15.B. Renormalisation

The QED action, written in terms of bare fields A_0, ψ_0 , bare m_0 ,bare e_0 and bare gauge parameter ξ_0 is given by, is given by,

$$\begin{aligned} I[A_0,\psi_0,\bar{\psi}_0] = \\ = \int d^4x \{ \bar{\psi}_0 (i\gamma^\mu \partial_\mu - m_0)\psi_0 - e_0 \bar{\psi}_0 \gamma^\mu A_{0\mu} \psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A_0^\mu(x))^2 \} \end{aligned} \tag{15.11a}$$

In order to apply the renormalisation programme we must write the action in terms of the renormalised objects plus counter-terms. We denote the renormalised coupling, mass and fields without the subscript zero and relate them to their bare counterparts by,

$$A_0^\mu = \sqrt{Z}_3 A^\mu \quad , \qquad \psi_0 = \sqrt{Z}_2 \psi$$
$$e_0 = Z_e e \quad , \qquad m_0 = Z_m m \tag{15.11b}$$

To simplify the writing we introduce Z_1 by,

$$Z_1 = Z_e Z_2 \sqrt{Z}_3 \tag{15.11c}$$

Written in terms of the renormalised objects the QED action then becomes,

$$I = \int d^4x \{ Z_2 \bar{\psi}(i\gamma^\mu \partial_\mu - Z_m m)\psi - Z_1 e\bar{\psi}\gamma^\mu A_\mu \psi - \frac{1}{4} Z_3 F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu(x))^2 \} \tag{15.11d}$$

To set up renormalised perturbation theory we define δ_1, δ_2, δ_3 and δ_m by

$$Z_1 = 1 + \delta_1 \quad Z_2 = 1 + \delta_2 \quad Z_3 = 1 + \delta_3, \quad Z_2 Z_m = 1 + \delta_2 + \delta_m \tag{15.11e}$$

Now we can write the action as,

$$I = I_0 + I_1 + I_{ct}$$

where I_0, I_1 and I_{ct} are defined by

$$I_0 = \int d^4x \{ \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu(x))^2 \} \tag{15.12}$$

$$I_1 = \int d^4x \{ -e\bar{\psi}\gamma^\mu A_\mu \psi \} \tag{15.13}$$

and

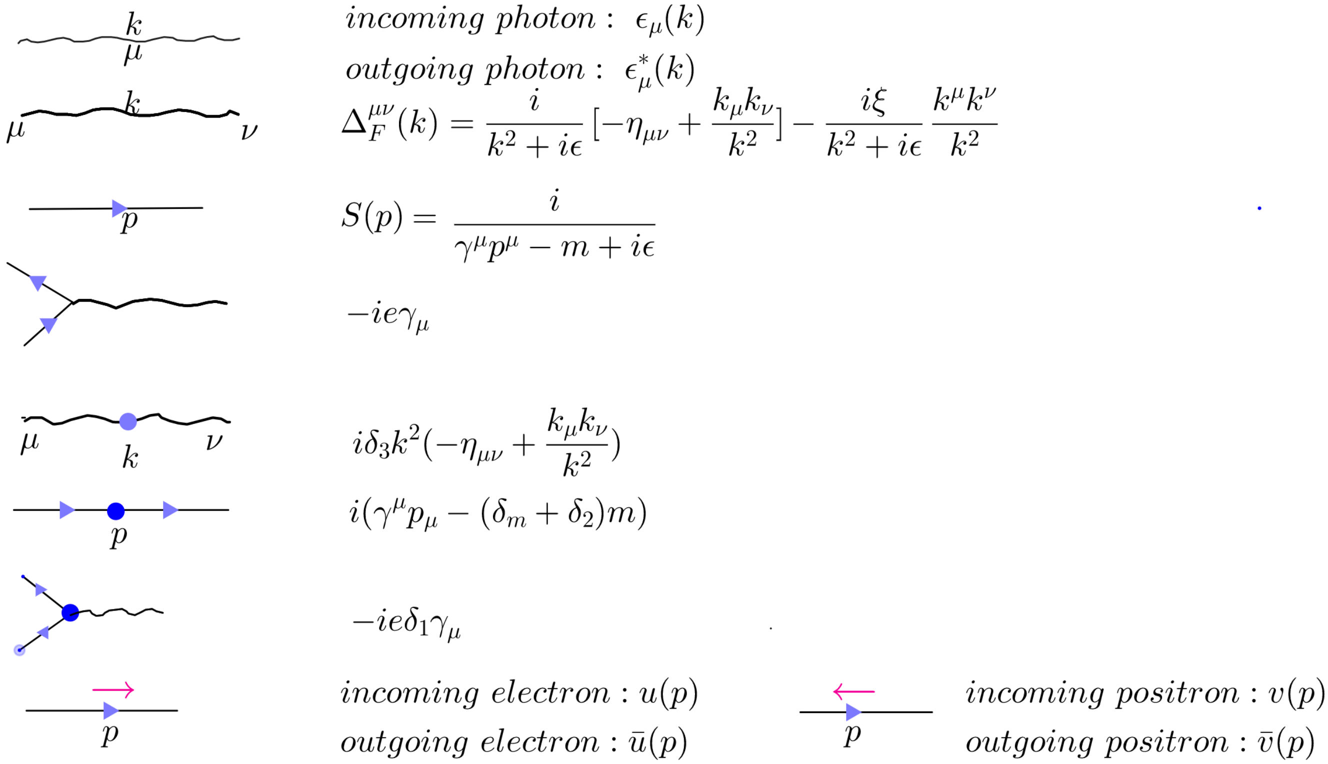

FIG. 10. Complete set of Feynman rules for QED including counterterms.

$$I_{ct} = \int d^4x\{\bar{\psi}[i\delta_2\gamma^\mu\partial_\mu \ - (\delta_2+\delta_m)m]\psi - \frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu(x))^2 - \delta_1 e\bar{\psi}\gamma^\mu A_\mu\psi\} \tag{15.14a}$$

I_0 defines the free part of the action while $I_{int} \equiv I_1 + I_{ct}$ gives the interaction part of the action. The renormalization constants δ_i will be determined by imposing the renormalization conditions on the 2 and the 3 point functions. The Feynman rules including the counter-terms are given in Fig.20. Note that the counter term involving δ_3 takes the compact form of ,

$$i\delta_3 k^2 P_{T\mu\nu} \tag{15.15}$$

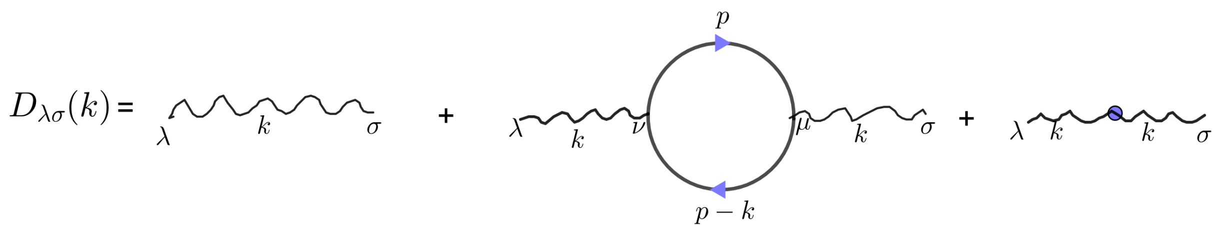

FIG. 11. The fermion loop representing the n-th order term in Eq.(14.3). The external lines represent the external vector field A_μ.

15.C. Vacuum Polarization

The graphs contributing to the photon 2-point function up to the 1-loop order are given in Fig. 21.

The last term on the right hand side is the contribution of the counter-terms. Use the Feynman rules to substitute for each term,

$$D_{\lambda\sigma}(k) = \tilde{\Delta}_{\lambda\sigma}(k) + \tilde{\Delta}_{\lambda\mu}(k)(-i\Pi^{\mu\nu}(k)\tilde{\Delta}_{\nu\sigma}(k) + \tilde{\Delta}_{\lambda\mu}(k)i\delta_3 k^2 P_T^{\mu\nu}\tilde{\Delta}_{\nu\sigma}(k) \qquad (15.16)$$

The $\Pi_{\mu\nu}$ term is essentially the same which we have calculated in Eq.(14.18) which we rewrite it using the tensor $P_{T\mu\nu}$ and adopting to our present action Eq. (15.11d) where the coupling constant is associated to the photon-fermion vertex rather than the Maxwell kinetic energy term in the action,

$$\begin{aligned}\Pi(k)^{\mu\nu} &= -\frac{e^2}{12\pi^2}(\frac{2}{\delta} - \gamma)k^2 P_T^{\mu\nu} \\ &\qquad + \frac{e^2}{2\pi^2}k^2 P_T^{\mu\nu}\int_0^1 dxx(1-x)ln[(1-x(1-x)\frac{k^2}{m^2})]\end{aligned} \qquad (15.17)$$

Introduce A and $B(k^2)$ by,

$$\begin{aligned}A &\equiv -\frac{e^2}{12\pi^2}(\frac{2}{\delta} - \gamma) \\ \pi(k^2) &\equiv \frac{e^2}{2\pi^2}\int_0^1 dxx(1-x)ln[(1-x(1-x)\frac{k^2}{m^2})]\end{aligned} \qquad (15.18)$$

Then we can write $\Pi^{\mu\nu}(k)$ in a more compact form,

$$\Pi(k)^{\mu\nu} = [A + \pi(k^2)]k^2 P_T^{\mu\nu} \tag{15.19}$$

Substitute this expresion in Eq.(15.16) and use the identity,

$$\tilde{\Delta}_{\lambda\sigma}(k)k^2 P_T^{\sigma\nu} = -iP^{\nu}_{T\lambda}$$

to obtain,

$$\begin{aligned} D_{\lambda\sigma}(k) &= \tilde{\Delta}_{\lambda\sigma}(k) \\ &\quad + \frac{i}{k^2}[A + \pi(k^2)]P_{T\lambda\sigma} - \frac{i\delta_3}{k^2}P_{T\lambda\sigma} \end{aligned} \tag{15.20}$$

Now we can impose the renormalisation condition and find the 1-loop contribution to δ_3. This condition is as follows,

$$D_{\lambda\sigma}(k) \mid_{k^2\to 0} \rightarrow \tilde{\Delta}_{\lambda\sigma}(k) \tag{15.21}$$

Since

$$\pi(k^2) \mid_{k^2\to 0} \rightarrow 0$$

we obtain,

$$\delta_3 = A = -\frac{e^2}{12\pi^2}(\frac{2}{\delta} - \gamma) \tag{15.22}$$

Substitute this result in Eq.(15.16) to obtain the final answer,

$$\begin{aligned} D_{\lambda\sigma}(k) &= \frac{i}{k^2 + i\epsilon}\{1 + \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x) ln[(1 - x(1-x)\frac{k^2}{m^2})]\}P_{T\lambda\sigma} \\ &\quad + \frac{i\xi}{k^2 + i\epsilon}P_{L\lambda\sigma} \end{aligned} \tag{15.23}$$

This result indicates an important fact which is valid to all orders in perturbation theory, namely, the loop corrections change only the transverse part of the photon propagator and leave the longititudinal part unaffected. In other words there is no loop correction of the gauge fixing parameter ξ.

15.D. 1-loop Ward Identity

Recall from Eq. (15.11e) that $Z_3 = 1 + \delta_3$. Use the 1-loop value Eq. (15.22) for δ_3 to obtain,

$$Z_3 = 1 - \frac{e^2}{12\pi^2}(\frac{2}{\delta} - \gamma) \tag{15.24}$$

Comparing this with the 1-loop value of the charge renormalization constant Z_e obtained in Eq. (14.23) we observe that,

$$Z_3^{-\frac{1}{2}} = Z_e \tag{15.25}$$

We have proven this identity up to the order of e^2. However, there is a deeper reason for this equality of the two renormalisation constants rooted in the gauge invariance of QED which makes the identity to be valid to all orders of perturbation theory.

The identity Eq. (15.25) has important physical consequences. It implies that,

$$\begin{aligned} e_0 F_{0\mu\nu} &= Z_e e Z_3^{\frac{1}{2}} F_{\mu\nu} \\ &= e F_{\mu\nu} \end{aligned} \tag{15.26}$$

Or,

$$F_{0\mu\nu} = \frac{e}{e_0} F_{\mu\nu}$$

Now let us assume that there are two charged particles with bare charges e_0 and e_0' which couple to the same vector potential. Denote their renormalised charges by e and e', respectively. Then applying the above equation to both of these particles we obtain,

$$\begin{aligned} F_{0\mu\nu} &= \frac{e'}{e_0'} F_{\mu\nu} \\ &= \frac{e}{e_0} F_{\mu\nu} \end{aligned}$$

This implies that

$$\frac{e'}{e_0'} = \frac{e}{e_0}, \quad \Rightarrow \quad \frac{e_0}{e_0'} = \frac{e}{e'}$$

In other words the equality of the bare charges implies the equality of the renormalised charges. This is independent of the internal structure of the charged particles. For example one of the particles can be an electron and the other can be a quark or a proton which is a bound state of 3 quarks (2 up quarks and one down quark). The quarks interact not only electromagnetically but also via strong interactions. Our result shows that, provided that the bare charges are equal, no quantum corrections will violate the equality of $e_p = -e$, where e is taken to be the electric charge of the electron! The fundamental reason for this very interesting and important result is gauge invariance. Another way to see the same effect is to look back to Eq. (15.11c)

$$Z_1 = Z_e Z_2 \sqrt{Z_3}$$

If we use the result $1 = Z_e Z_3^{\frac{1}{2}}$ we obtain,

$$Z_1 = Z_2$$

Now return to Eq. (15.11d) which we copy here ,

$$I = \int d^4x \{ Z_2 \bar{\psi}(i\gamma^\mu \partial_\mu - Z_m m)\psi - Z_1 e\bar{\psi}\gamma^\mu A_\mu \psi - \frac{1}{4} Z_3 F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu(x))^2 \}$$

Setting

$$Z_1 = Z_2$$

makes the interaction term to combine with the term having the derivative of ψ to form a covariant derivative,

$$I = \int d^4x \{ Z_2 \bar{\psi} i\gamma^\mu (\partial_\mu + ieA_\mu)\psi - Z_m Z_2 m\bar{\psi}\psi - \frac{1}{4} Z_3 F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu(x))^2 \}$$

We can sum up all this by saying that the charge renormalisation in QED is a property of the gauge field and not the charged particle!

Exercises

1. Define the superficial degree of divergence

$$D = 4L - 2I_\gamma - I_e$$

where L is the number of loops, I_γ is the number of photon propagator and I_e the number of internal electron lines. We also Introduce E_γ and E_e, the number of the external photon and electron lines respectively. Show that,

$$D = 4 - E_\gamma - \frac{3}{2}E_e$$

List all the values of E_γ and E_e for which D is non negative. Which 1 loop graphs have non negative D?

2. a.) What is the imaginary part of $ln(-a - i\epsilon)$, where a is positive and $\epsilon \to 0^+$?

b.) Use your result to evaluate the imaginary part of $W(A)$ defined by Eq. (14.27), when $k^2 \geq 4m^2$. [Perform the integration over the Feynman parameter x explicitly!]

c.) Use the result you obtained in (b) and the relationship between $W(A)$ and the matrix elements of the scattering operator $\hat{S}$ to obtain the probability of e^+e^- pair creation in a background electromagnetic field.

d.) Express your result in terms of the electric field $\vec{E}$ and the magnetic field $\vec{B}$ and from this answer the following question: can a pure magnetic field create an electron-positron pair?

The relationship between bare and running coupling constants in n- dimensional QED is given by,

$$e_0 = Z_e e \mu^{\frac{\delta}{2}} \tag{15.27}$$

where $\delta = 4 - n$. The coupling e is dimensionless. We calculated the 1-loop $Z_e = Z_3^{-\frac{1}{2}}$. Thus from Eq.(15.24)

$$Z_e = 1 + \frac{e^2}{24\pi^2}(\frac{2}{\delta} - \gamma) \tag{15.28}$$

Using these relations we can evaluate the renormalisation group β function. To this end we start from the fact e_0 is independent from μ. Thus,

$$0 = \frac{d}{d\mu} Z_e e \mu^{\frac{\delta}{2}} + Z_e \beta(e) \mu^{\frac{\delta}{2}} + \frac{\delta}{2} Z_e e \mu^{\frac{\delta}{2}}$$

Hence

$$0 = \frac{d}{dln\mu} ln Z_e e + \beta(e) + \frac{\delta}{2} e$$

Up to the order of e^2 we have, $lnZ_e = \frac{e^2}{24\pi^2}(\frac{2}{\delta} - \gamma)$. Thus,

Hence

$$\frac{d}{dln\mu} ln Z_e = \frac{e}{12\pi^2} \beta(e) (\frac{2}{\delta} - \gamma)$$

Thus,

$$0 = \beta(e)[1 + \frac{e^2}{12\pi^2}(\frac{2}{\delta} - \gamma)] + \frac{\delta}{2} e$$

$$\beta(e) = -\frac{\delta}{2}e[1 + \frac{e^2}{12\pi^2}(\frac{2}{\delta} - \gamma)]^{-1}$$
$$= \frac{e^3}{12\pi^2}$$

We can rewrite this in terms of $\alpha = \frac{e^2}{4\pi}$. A little calculation then gives,

$$\beta(\alpha) = \mu\frac{d\alpha}{d\mu} = \frac{2\alpha^2}{3\pi}$$

We can integrate this equation to obtain,

$$\alpha(\mu^2) \equiv \frac{\alpha}{1 - \frac{\alpha}{3\pi}ln\frac{\mu^2}{m^2}} \tag{15.29}$$

where,

$$\alpha = \alpha(m^2)$$

Here m is a reference mass. This shows that with increasing energy (or at short distances) QED becomes strongly coupled. The point where the running coupling $\alpha(\mu^2)$ becomes very large is called the Landau pole. It happens at

$$\mu^2 = m^2 e^{\frac{3\pi}{\alpha}} \tag{15.30}$$

Qualitatively this is the same short distance behaviour which we had for a scalar field model with a ϕ^4 interaction. an important model with the opposite behaviour, namely, with a negative β function is QCD in which the running coupling decreases as we go to high energies. This means that the quarks become weakly interacting particles in very short distances and they become strongly interacting as the distance between them increases. For that reason it is called an asymptotically free model.

One may ask if the result in Eq.(15.29) can be given a meaning beyond order the order of α. To answer this question we calculate the quantum loop corrected

Coulomb potential. This can be obtained directly from the loop corrected Feynman propagator or from Eq.(14.31) by coupling the vector potential to a source of the form,

$$J_\mu = (J_0 = e\delta^3(\vec{x}), \quad \vec{J} = \vec{0})$$

Then the Fourier space solution of the loop corrected Maxwell's equations give,

$$e\tilde{V}(\vec{k}^2) = \frac{e^2}{\vec{k}^2[1 - e^2\pi(-\vec{k}^2)]}$$

where $\pi(q^2)$ is defined in Eq.(14.26). If $\frac{\vec{k}^2}{m^2} >> 1$ the function $\pi(-\vec{k}^2)$ becomes,

$$\pi(k^2) = \frac{e^2}{12\pi^2}[ln\frac{k^2}{m^2} + const.]$$

where the constant is given by,

$$const. = 6\int_0^1 dxx(1-x)lnx(1-x)$$

Thus we can write the potential in the form,

$$e\tilde{V}(\vec{k}^2) = \frac{e^2}{\vec{k}^2[1 - \frac{e^2}{12\pi^2}[ln\frac{ck^2}{m^2}]} \tag{15.31}$$

Define an effective charge by,

$$e_{eff}(\vec{k}^2) = \frac{e^2}{[1 - \frac{e^2}{12\pi^2}[ln\frac{ck^2}{m^2}]} \tag{15.32}$$

The potential then becomes,

$$e\tilde{V}(\vec{k}^2) = \frac{e^2_{eff}(\vec{k}^2)}{\vec{k}^2}$$

Comparing Eq. (15.32) with Eq.(15.29) we see that with an appropriate choice of

μ^2 the effective fine structure constant $\alpha_{eff}(\vec{k}^2) = \frac{e^2_{eff}(\vec{k}^2)}{4\pi^2}$ is identical with what we obtain from the β function equation.

On the other hand if we consider an infinite chain of 1-loop graphs of the form,

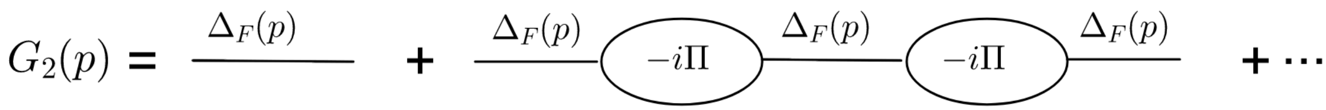

FIG. 12. $G_2(p)$ is the sum of infinite number of graphs. $-i\Pi$ represents the sum of all 1PI irreducible vertices with 2-external points.

the result for the Coulomb potential in the large log limit obtained from the propagator would be precisely the one given in Eq.(15.31). This shows that the $1-loop$ β function gives us the same effective charge which is obtained from the sum of an infinite subset of 1-loop graphs shown above in the large log limit! Thus the β function leads to running fine structure constant which has more than 1-loop information in it.