A brief course on integrals

(Indefinite to definite integrals and how to solve them)

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1. Why?

First of all there are two things that must be said: I'm not a leading expert but I'am terrebly curious for the mysterys behind any integral. I'am not pretending that know it all but insted of that I consider that this book is a self-study rather than a exposition of my knowledge. The study of integrals from a measure theory perspective give us a brief view of functions wich neither could be integrable nor described by compostion of elementary functions. As we may say the set of functions wich have a primitive function is of measure zero and further more, the set of functions wich are integrable[1] also is of measure zero. But never the less we could study each of one and conclude incredible things.

It is in our responsability to study (and propose new) the problem-solving of integrals for the seak of knowledge wich may also be of valuable meaning in many fields of applied mathematics or beautiful it self.

Raging from the economical science to pure mathematics the integrals form part of the elements of the day to day on every scientist, therfore, we may sharpen our skills to solve, evaluate and identify some of the most beautiful integrals.

We may see if time and work allow, derivatives, transformation of derivatives ,convergence of functions, convergence of integrals... integrals and sums, sums itself, alternative convergence of functions, ramanujan convergence etc. Of course, we live in the XXI century and with it modern tecnologies are develop, we cover also some integration methods in c++ and integration with other tools such as MATHEMATICA and...

[1] With any definition of integrals

2. About derivatives I

2.1 Derivatives, types and forms (not differential forms)

As many of you could know, the derivatives are the opposite side of the integrals. Nevertheless, the study of integrals is closely related via FTC[1] [Rud76] to the study of derivatives, therefore, in order to accomplish our deep study of integrals we have to became (if already not) as familiar to derivatives as we are familiar with numbers to get an intuition of functions. Of course, the study of integrals are not constrained to be able to find some anti derivative but we may wish deeply this part too. It is important to became familiar to anti derivatives to get an intuition of the meaning (if so) of more general integrals, many of them will not have a primitive function but still be integrable or furthermore, we may wish non analytical functions or in the extreme case we may want to integrate sets with respect some measure.

2.1.1 Functions and definitions: continuous functions

Firs of all we explore some continuous functions

Polynomials:

Each of one defined as

$$\sum_{i=0}^{n} a_i \cdot x^i = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x + a_0$$

The polynomials are a powerful tool but also a very risky type of functions when introduced on a integral. For example, consider:

$$\int_0^{\pi/4} \frac{x}{(\cos(x) + \sin(x))\cos(x)} dx$$

http://masteringolympiadmathematics.blogspot.com/2015/08/evaluate-displaystyleintdfracpi40.html

[1] Fundamental theorem of calculus part 1. Many standard proofs will be skipped but all of them are referenced to a particular book, for example, Rudin in this case.

This integral at first does not seem difficult but it is. In fact, if the interval were not given it is impossible to find a closed form of this. Will be a lot easier to solve and in fact would have a anti derivative if the x will be 1.

Power functions:
Analytic functions of real values may take the form of:

$$f(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$$

In the real case, the series converges within some interval, note that the interval may also be empty.

Exponential functions: $\exp(x) = e^x$
One of the most important functions is the exponential function wich are of the form:

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n} \iff \frac{df}{dx} = f \iff \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{nx}$$

Many definitions of this may come. Note that e is just a special case of functions defined by a^x which may be presented after the definition of log

Trigonometric: $\sin, \cos, \tan, \ldots$
Trigonometric functions give us a perspective of geometry and algebraic theory. Trigonometric functions, specialy in physics simplify several deductions. We define, despite our real treatment of functions sin and cos as follows:

$$\cos x = \mathscr{R}(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \mathscr{I}(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}$$

Where $\mathscr{R}(f(x))$ is the real part of $f(x)$ and $\mathscr{I}(f(x))$ is the imagniary part of $f(x)$ Many readers could argue the use of complex functions to deal with real functions but of course knowing a more general system allow us to use more tools in order to deduce several properties of this functions. Also, it is rather easy to integrate e^x than some power series. We also remember some other definitions,

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{1}{\tan x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Our definition of trigonometric functions allow us to derive very straightforward some of the most important identities for example:

Theorem 2.1.1 $sin^2 x + \cos^2 x = 1$

Proof.

$$\sin^2 x + \cos^2 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 + \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 = \left(-\frac{e^{i2x} + e^{-2ix} - 2}{4} + \frac{e^{i2x} + e^{-2ix} + 2}{4}\right) = \frac{4}{4} = 1$$

■

Theorem 2.1.2 $\sin x + y = \sin x \cos y + \sin y \cos x$

Proof.

$$\sin(x+y) = \left(\frac{e^{i(x+y)} - e^{-i(x+y)}}{2i}\right) = \left(\frac{e^{ix}e^{iy} + e^{ix}e^{-iy} - e^{ix}e^{-iy} - e^{-ix}e^{-iy}}{2i}\right)$$
$$= \left(\frac{e^{ix}(e^{iy} + e^{-iy}) - e^{-iy}(e^{ix} + e^{-ix})}{2i}\right) = -i\cos y e^{ix} + ie^{-iy}\cos x$$
$$= \cos y(\sin x - i\cos x) + \cos x(\sin y + i\cos x) = \sin x \cos y + \sin y \cos x$$

■

One last example of this are the so called "reduction formula" which converts a $\sin^2$ or $\cos^2$ into a linear cos.

Theorem 2.1.3 $\sin^2 x = \frac{1-\cos(2x)}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$

Proof.

$$\sin^2 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 = -\left(\frac{e^{i2x} + e^{-i2x} - 2}{4}\right) = \left(\frac{-\frac{e^{i2x}+e^{-i2x}}{2} + 1}{2}\right) = \frac{1-\cos(2x)}{2}$$

By 2.1.1 we have:

$$\cos^2 x = 1 - \sin^2 x = \frac{1+\cos(2x)}{2}$$

■

Last but not least, we have the "translation formula", using double angle formula: $\sin(x+\pi/2) = \cos x$.

Exercise 2.1 Prove that $\sin(x+\pi/2) = \cos x$ and use that to prove $\cos(x-\pi/2) = \sin x$ ■

Exercise 2.2 Prove the following equations:

$\cos(x+y) = \cos(x)\cos(y) + \sin(x)\sin(y)$ **(Use the double angle formula and 2.1.)**

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1+\tan(x)\tan(y)}$$
$$\cos(-x) = \cos x$$
$$\sin(-x) = -\sin x$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Hyperbolic functions:
Many students are not familiar with hyperbolic functions I think because of the following reason: Imagine you want to integrate the following family of functions, let a be a real positive constant:

$$\int \frac{1}{\sqrt{x^2+a^2}}dx \tag{2.1}$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx \tag{2.2}$$

Let's consider the case when $a = 1$. For the former, the first idea that many of us will have its to make the substitution $x = \tan u$ because of the identity[2]:

$$1+\tan^2 x = \sec^2 x$$

and for the later we use the substitution $x = \sec u$. And therefore we have the following integral:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+1}}dx &= \int \frac{1}{\sqrt{\sec^2 u}}d(\tan u)\\ &= \int \frac{\sec^2 u}{|\sec u|}du\\ &= \int \sec x\,dx = \log|\sec x+\tan x|+c\\ &= \log|\sqrt{x^2+1}+x|\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-1}}dx &= \int \frac{1}{\sqrt{\tan^2 u}}d(\sec u)\\ &= \int \frac{\sec u\tan u}{|\tan u|}du\\ &= \int \sec x\,dx = \log|\sec x+\tan x|+c\\ &= \log|x+\sqrt{x^2-1}|\end{aligned}$$

This of course under some domain where the function $\sec x$ is positive. Recall that $x/|x| = sgn(x)$ where $sgn(x)$ is the function sign defined by:

$$sgn(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

Define the hyperbolic functions as:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

and in the same way as trigonometric functions define $\tanh x$, $\operatorname{sech} x$ etc.

[2]Divide by $\cos^2 x$ in $\sin^2 x + \cos^2 x = 1$

Exercise 2.3 Prove the following equations:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

$$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$$

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$$

$$\cosh(-x) = \cosh x$$

$$\sinh(-x) = -\sinh x$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech} x^2$$

$$\frac{d}{dx}(\operatorname{sech} x) = \operatorname{sech} x \tanh x$$

■

Inverses:

Inverse of functions pays a crucial role into integration. Note that many substitutions we define some function of our variable defined as $f(x) = u$ where our original variable was x and we want to put all of our expressions in terms of u, we therefore make use of the inverse function of f call it $f^{-1}(u) = x$. Many functions are not bijective and therefore the do not have inverses but all is not lost. The least thing we need is one-to-one functions, we then may find the inverse of functions within some interval of special care. The process is simple, choose a function, choose a interval domain in where the function is one-to-one, and restrict the function over this interval. Then we have a function bijective which is one-to-one(injective) on it's domain and onto(surjective) over it's image. We then have the following identity:

$$\int f(x)dx = \int u\, d(f^{-1}(u))$$

■ **Example 2.1** Find the integral of $f(x) = x^2$ using its inverse. First of all we note that x^2 is not a bijective function over the real line. Over the interval $[0,\infty)$ the inverse function $f^{-1}(x) = \sqrt{x}$ and over the interval $(-\infty, 0]$ the inverse function $f^{-1}(x) = \sqrt{-x}$. i.e:

$$f^{-1}(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ \sqrt{-x} & x < 0 \end{cases}$$

Solving the integral when $x > 0$:

$$\int x^2\, dx = \int u d(\sqrt{u}) = \int \frac{u}{2\sqrt{u}}\, du = \int \frac{1}{2}\sqrt{u}\, du = \frac{1}{3}u^{3/2} + c = \frac{1}{3}x^3 + c$$

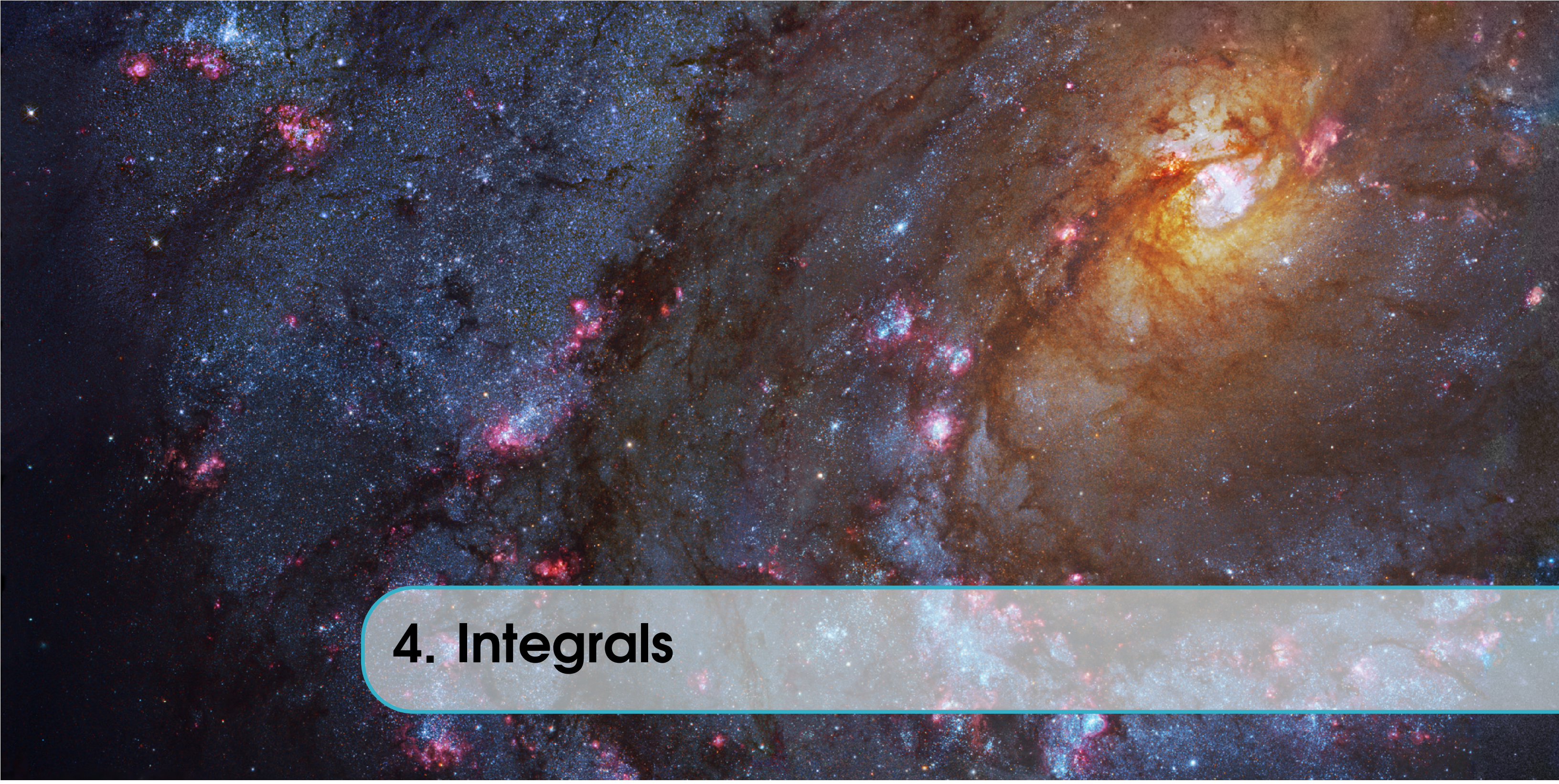
Solving the integral when $x \leq 0$:

$$\int x^2\,dx = \int u d(\sqrt{-u}) = \int \frac{-u}{2\sqrt{-u}}\,du = \int \frac{1}{2}\sqrt{-u}\,du = -\frac{1}{3}(-u)^{3/2} + c = -\frac{1}{3}(-u)^1/2(u) + c = \frac{1}{3}x(-u) = \frac{1}{3}x^3 + c$$

We conclude (because both integrals are the same) that $\int x^2\,dx = x^3/3 + c$ ∎

Logartihms: $\log(x)$ inverses o

3. About derivatives II



4. Integrals

We first develop some standard techniques of solving integrals.

4.1 Integration by substitution

The most important and also the easiest integration technique is by far the integration by substitution or by change of variable. We develop some tips in order to sharp our skills and of course to deeply understand what means to do a substitution.

The theory follows like this: Suppose you want to integrate some function $f(x)$ we may have to cases: When integrating a function we may develop this two things 1. We know the anti-derivative of the function. 2. We feel that the integral has the "form" of some anti-derivative. We have to take special care in the second rule. Sometimes, we feel that the integral are of one way and we get stuck on solving it because many of them look similar but are different enough to get different approaches. Let's illustrate with an example. Consider the following integral

$$\int \frac{x}{1+x^2}dx \tag{4.1}$$

we immediately (or with a little effort) identify that if we make $u = 1+x^2$ then we have

$$\int \frac{x}{1+x^2}dx = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\log u + c = \frac{1}{2}\log\left(1+x^2\right)+c$$

We may after a few integrals identify the pattern of solving:

$$\boxed{\int \frac{x^{n-1}}{1+x^n}dx = \frac{1}{n}\log\left(1+x^n\right)+c}$$

just by seeing it. Suddenly we are presented a new integral which is:

$$\int \frac{x^{1/2}}{1+x^3}dx$$

First idea may be to try a substitution in the same fashion as in 4.1, i.e. $u = 1+x^3$ then we have:

$$\int \frac{x^{1/2}}{1+x^3}dx = \int \frac{1}{u}\frac{du}{\sqrt[3]{u-1}}$$

We argue, that the substitution that we have made give us no information to solve the integral. Instead of that, our second attempt will be taking $u = x^{3/2}$ and therefore:

$$\int \frac{x^{1/2}}{1+x^3} = \int \frac{2}{3}\frac{1}{1+u^2}du = \frac{2}{3}\arctan u + c = \frac{2}{3}\arctan x^{3/2} + c.$$

Which of course was too easy to solve.

Exercise 4.1 Solve the following integrals:

1. $$\int \frac{1}{x}\frac{1}{(1+x)^n}dx$$

2. $$\int \frac{1}{x+x^n}dx$$

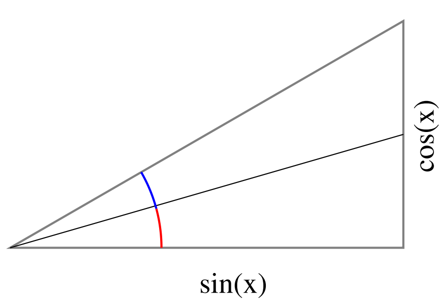
   Hint: Use log

4.2 Weierstraß substitution

Many substitutions have to be chosen having some information about the integral. Weierstraß substitution it's just a standard way of doing difficult trigonometric integrals. We may take care because it is in essence a way of put a trigonometric problem into a pure algebraic integral, that is, a polynomial or radicals.

I emphasize on it's derivation in order to familiarize with trigonometric substitution.

Let $t = \tan(x/2)$ then:



Keeping in mind the graph we have the following derivation:

$$
\begin{aligned}
t &= \tan(x/2) \\
sin(x) &= sin\left(\frac{2x}{2}\right) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\
cos(x) &= cos\left(\frac{2x}{2}\right) = \cos\left(\frac{x}{2}\right)^2 - \sin\left(\frac{x}{2}\right)^2 \\
t &= \tan(x/2) = \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sin(x/2)}{\sqrt{1-\sin(x/2)}} = \frac{\sin(x/2)-1+1}{\sqrt{1-\sin(x/2)}} = -1 + \frac{1}{\sqrt{1-y}} \implies \\
y &= \sin\left(\frac{x}{2}\right) = 1 - \frac{1}{(1+t)^2}
\end{aligned}
$$

Las esenciales (No se incluyen las que todo el tiempo se usan o por lo menos uso):

$f(x)$	$F(x)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\frac{x}{a}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\log\frac{a+x}{a-x}$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\frac{x}{a}$
$\frac{1}{\sqrt{x^2\pm a^2}}$	$\log a+\sqrt{a^2\pm x^2}$
$f(ax)$	$\frac{1}{a}F(ax)$

4.3 Quadratic trinomial

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2 \pm k^2}\,dx$$

$$\frac{c}{a} - \frac{b^2}{4a^2} = \pm k^2$$

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx = \frac{A}{2a}\int \frac{2ax+b}{ax^2+bx+c} + \left(B - \frac{Ab}{2a}\right)\int \frac{1}{ax^2+bx+c}\,dx$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx = \dots$$

$$\int \frac{Ax+b}{ax^2+bx+c}\,dx$$

Let us solve the folowing integral:

$$
\begin{aligned}
\int \sqrt{a^2-x^2}\,dx &= \int \frac{a^2-x^2}{\sqrt{a^2-x^2}\,dx} = a^2\int \frac{1}{\sqrt{a^2-x^2}}\,dx - \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx \\
&= a^2\int \frac{1}{\sqrt{a^2-x^2}}\,dx - \int x\frac{x}{\sqrt{a^2-x^2}}\,dx = a^2\int \frac{1}{\sqrt{a^2-x^2}}\,dx + x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx \implies \\
\int \sqrt{a^2-x^2}\,dx &= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2}
\end{aligned}
$$

$$\int e^{ax}\cos bx + ie^{ax}\sin bx\,dx = \int e^{(ib+a)x}\,dx = \frac{e^{(ib+a)x}}{ib+a} = \frac{e^{(ib+a)x}(a-ib)}{a^{+}b^{2}}$$
$$= \frac{e^{ax}a\cos bx + e^{ax}i\sin bxib + e^{ax}ib\cos bx) + e^{ax}i\sin bxa}{a^2+b^2}$$

Tenemos entonces:

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$
$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(-b\cos bx + a\sin bx)}{a^2+b^2}$$

4.4 Absolute value integrals.

At this point you may be familiar with coninous differentiable functions but of course we may generalize our techniques into either more general or sofisticated functions. The first example of this will ve the integrals over composition of absolute value.

We first analise the most elementary one and extrapolate the others. Consider the following integral:

$$\int_{-4}^{3} |x|\,dx$$
$$\int_{1}^{3} |x|\,dx$$
$$\int_{-4}^{-3} |x|\,dx$$

We could integrate over any interval but of course this three represente all the diferent cases.

Note that the last two integral became trivial thats because the function to be integrates over its interval its just a coninous function and therfore easy to solve. We have then:

$$\int_{1}^{3} |x|\,dx = \int_{1}^{3} x\,dx = \frac{3^2-1^2}{2} = 4$$
$$\int_{-4}^{-3} |x|\,dx = -\int_{-4}^{-3} x\,dx = -\frac{3^2-4^2}{2} = \frac{7}{2}$$

The first case has to be taken more carefully. We will apply a substitution, let $u = |x|$ then:

$$u := \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases} \qquad du := \begin{cases} dx & x > 0 \\ -dx & x < 0 \end{cases}$$

We have then:

$$\int_{-4}^{3} |x|\,dx = -\int_{4}^{0} u\,du + \int_{0}^{3} u\,du = \int_{0}^{4} u\,du + \int_{0}^{3} u\,du = \frac{4^2+3^2}{2} = \frac{25}{2}$$

We may notice after a few integrals that some 'pattern' is presented, lets consider a new notation:

■ **Example 4.1** Solve the folowing integral:

$$\int_{-4}^{8} |x-3|\,dx$$

Solution:

$$\begin{aligned}\int_{-4}^{8} |x-3|\,dx &= \int_{-4}^{0} |x-3|\,dx + \int_{0}^{8} |x-3|\,dx \\ &= -\int_{|-4-3|}^{0} u\,du + \int_{0}^{|8-5|} u\,du = \int_{0}^{7} u\,du + \int_{0}^{5} u\,du = \frac{7^2+5^2}{2} = \frac{74}{2} = 37\end{aligned}$$

Notice that the function integrated is the same in both integrals but over diferent intervals. We will find that in general this will be the case and we may shorten this so we may introduce the following notation:

$$\int_{-4}^{8} |x-3|\,dx = \int_{0}^{|-4-3|} + \int_{0}^{|8-3|} = \frac{7^2+5^2}{2} = \frac{74}{2} = 37$$

■

We practice with the notation:

■ **Example 4.2** Solve the folowing integral:

$$\int_{-3}^{9} |x-2|\,dx$$

Solution:

$$\int_{-3}^{9} |x-2|\,dx = \int_{0}^{|-3-2|} + \int_{0}^{|9-2|} = \frac{7^2+5^2}{2} = \frac{74}{2} = 37$$

■

■ **Example 4.3** Solve the folowing integral:

$$\int_{-3}^{9} |2x-2|\,dx$$

Solution:

$$\int_{-3}^{9} |x-2|\,dx = \frac{1}{2}\int_{0}^{|-3\cdot 2-2|} + \frac{1}{2}\int_{0}^{|9\cdot 2-2|} = \frac{1}{2}\frac{8^2+16^2}{2} = \frac{320}{4} = 80$$

■

4.4.1 The diference between $a|x-c|$ and $|a\cdot x-c|$

You may notice that any of these integrals $a|x-c|$ and $|a\cdot x-c|$ will have the same form of du but of coure it will difer in the intervals. We ilustrate by means of an example.

■ **Example 4.4**

$$\int_{-1}^{7} \left|\frac{x}{3}-2\right|\,dx$$

Solution:

$$\int_{-1}^{7} \left|\frac{x}{3}-2\right|\,dx = 3\int_{0}^{|-\frac{1}{3}-2|} + 3\int_{0}^{|\frac{7}{3}-2|} = 3\frac{(7/3)^2+(1/3)^2}{2} = \frac{25}{3}$$

■

■ **Example 4.5**

$$\int_{-1}^{7} \frac{1}{3}\left|\frac{x}{-}2\right| dx$$

Solution:

$$\int_{-1}^{7} \frac{1}{3}|x-2|\,dx = \frac{1}{3}\int_0^{|-1-2|} + \frac{1}{3}\int_0^{|7-2|} = \frac{3^2+5^2}{2\cdot 3} = 3\frac{34}{2\cdot 3} = \frac{17}{3}$$

■

We are prepared to solve more general questions: What about

$$\int_{-5}^{14} ||x-3|-7|$$

?

We may therfore:

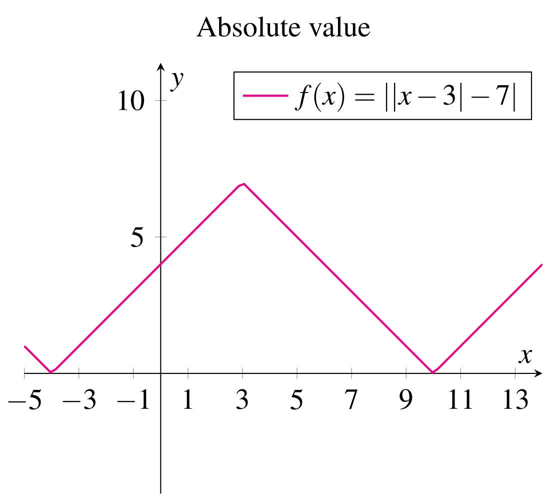
■ **Example 4.6**

$$\int_{-5}^{14} ||x-3|-7| = \int_0^{|-5-3|} + \int_0^{|14-3|} \tag{4.2}$$

$$= \int_0^{|0-7|} + \int_0^{|8-7|} + \int_0^{|0-7|} + \int_0^{11-7} = \frac{7^2+1^2+7^2+4^2}{2} = \frac{115}{2} \tag{4.3}$$

We have to take spacial care with this, our notation is powerfull but we have to have in mind the first line it's been integrated the function $|u-7|$ and in the second the function u. ■

The reader may also been questioning the nature of this.



What will happend if we integrate over diferent domains? That is, for example, consider

$$\int_{-3}^{12} ||x-3|-7|$$

We will be integrating over a region with two local extremal points (insted of three in the previos one). Lets see what will change. As $(-3,12)$ contains a zero of the function $|x-3|$ we will mantain the first step in the same fashion. But the first integral is less than 7 the interval and therfore over it's domain will be just the positive value. That is:

■ **Example 4.7**

$$\int_{-3}^{12} ||x-3|-7| = \int_{0}^{|-3-3|} + \int_{0}^{|12-3|} \tag{4.4}$$

$$= \int_{6-7}^{|0-7|} + \int_{0}^{|9-7|} + \int_{0}^{|0-7|} = \frac{7^2-1^2+2^2+7^2}{2} = \frac{101}{2} \tag{4.5}$$

■

One more example.

■ **Example 4.8**

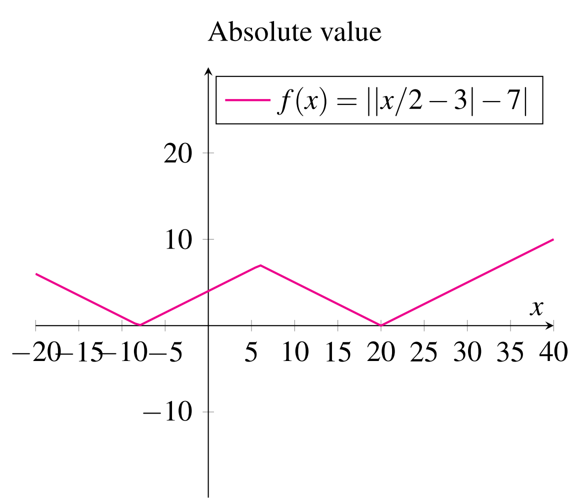
$$\int_{5}^{13} ||x-3|-7| = \int_{|5-3|}^{|10|} \tag{4.6}$$

$$= \int_{0}^{|2-7|} + \int_{0}^{|10-7|} = \frac{5^2+3^2}{2} = 17 \tag{4.7}$$

■

4.4.2 The diference between $|a|x-b|-c|$ and $||a \cdot x-b|-c|$

You may notice that any of these integrals $|a|x-b|-c|$ and $||a \cdot x-b|-c|$ will have the same form of du but of coure it will difer in the intervals. We ilustrate by means of an example.

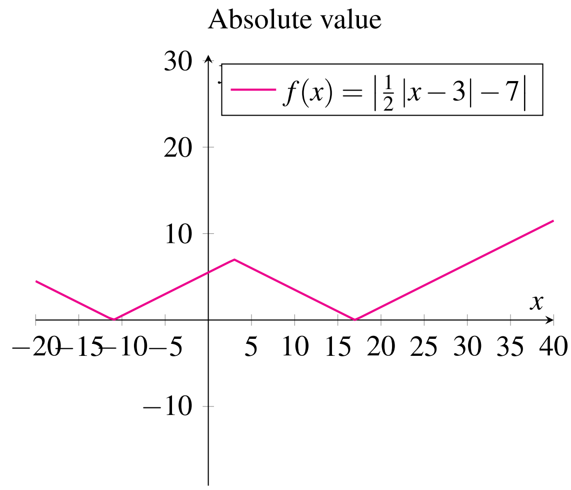


■ **Example 4.9**

$$\int_{-20}^{40} ||x/2-3|-7|\,dx = 2\int_{0}^{|-20/2-3|} + 2\int_{0}^{|40/2-3|} \tag{4.8}$$

$$= 2\int_{0}^{7} + 2\int_{0}^{|13-7|} + 2\int_{0}^{7} + 2\int_{0}^{|17-7|} = 2\frac{7^2+6^2+7^2+10^2}{2} = 234 \tag{4.9}$$

■



Example 4.10

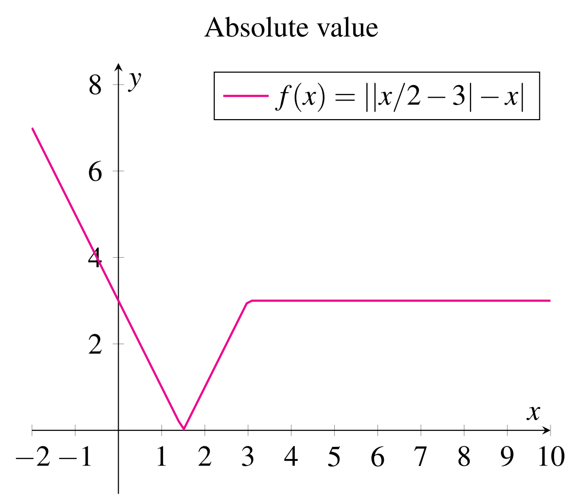
$$\int_{-20}^{40}\left|\frac{1}{2}|x-3|-7\right|dx = 2\int_0^{|-20-3|/2} + 2\int_0^{|40-3|/2} \tag{4.10}$$

$$= 2\int_0^{|0-7|} + 2\int_0^{|23/2-7|} + 2\int_0^{|0-7|} + 2\int_0^{|37/2-7|} \tag{4.11}$$

$$= 2\frac{7^2+9^2/2^2+7^2+23^2/2^2}{2} = \frac{501}{2} \tag{4.12}$$

■

Once we master this all nested integrals will be easy to solve as a generalization of this. The final las examples is, suppose we want to find the following integral:



Example 4.11

$$\int_{-2}^{10}||x-3|-x|\,dx = \int_0^{|10-3|}|-3|+|\int_0^{|-2-3|}|2u-3| \tag{4.13}$$

$$= \frac{1}{2}\int_0^{|-3|} + \frac{1}{2}\int_0^{|2\cdot5-3|} + 3\cdot 7 = \frac{3^2/2+7^2/2+21\cdot 2}{2} = \frac{71}{2} \tag{4.14}$$

■

4.5 The Cauchy formula

One of the main theorems in complex analysis is named after Augustine Louis Cauchy, we refere him as Cauchy (To be pronounced Cau as in **co**op and chy as the word she). Louis Augustine Cauchy was a french mathematician and perhaps the most influencial mathematician in modern analysis. My admiration began at a young age. I was on a familiy trip in Paris, France and my father took me to the Eiffel tower. Besides of the marvelous architecture of the tower I was more perplexed by the names of the people that were written on the arcs of the tower. I was reading the names; Cauchy, Fourier, Lagrange, Laplace, Legendre, etc. the most influencial mathematicians were there. Cauchy not only wrote on pure mathematics but also work on applied mathematics such as, wave mechanics, elasticity, mechanics etc. Also, in mathematics he develop a great contribution for number theory as well.

His main works are colected in "Analyse Algébrique" (1821), "Le Calcul infinitésimal" (1823), "Leçons sur les applications de calcul infinitésimal; La géométrie" (1826–1828). but of course it has many other remarkable works.

One of the most remarkable theorems named after him is the so called "Cauchy formula" [**cauchy**].

Theorem 4.5.1 Let U be an open subset of the complex plane $\mathbb{C}$, and suppose the closed disk D defined as $D = \left\{z : |z - z_0| \leq r\right\}$ is completely contained in U. Let $f : U \to C$ be a holomorphic function, and let γ be the circle, oriented counterclockwise, forming the boundary of D. Then for every a in the interior of D,

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - a} dz.$$

5. Miscellaneous

Many of the following integrals will look familiar, that is of course because many of them were presented before but as the good old say "From failure you learn. From succes not so much."

5.1 Definite integrals:

1.

$$\int_0^{\pi/4} \frac{x}{\cos(x)+\sin(x)} \frac{dx}{\cos(x)}$$

2. (From a math meme)

$$\int_0^{\infty} x \frac{\sinh x}{\cosh 3x}\, dx$$

5.2 Indefinite integrals:

1. $$\int \operatorname{sen}(-3x)dx$$
2. $$\int (4x-5)dx$$
3. $$\int x\sqrt{1+x^2}dx$$
4. $$\int (e^x)^2\,dx$$
5. $$\int \frac{dx}{x+8}$$
6. $$\int \operatorname{sen}^2(\operatorname{sen}x)\cos x dx$$
7. $$\int xe^{x^2}\sqrt{e^{x^2}}dx$$
8. $$\int \frac{(3x+2)^3}{x}dx$$
9. $$\int \frac{dx}{3\cos x - 4\operatorname{sen}x}$$
10. $$\int e^{\sqrt{x}}dx$$
11. $$\int \left(\frac{\ln(x+1)}{x}+\frac{\ln x}{x+1}\right)dx$$
12. $$\int \frac{2x^3+x^2-2}{x^4+x^2}dx$$
13. $$\int x\sqrt{x+4}dx$$
14. $$\int \frac{\ln(xe^x+e^x)}{x+1}dx$$
15. $$\int \frac{\ln(x-3)}{x^2-6x+9}dx$$
16. $$\int \operatorname{sen}(5x)\cos(2x)dx$$
17. $$\int \frac{dx}{(x^2+4)^2}$$
18. $$\int \cos(\ln x)dx$$
19. $$\int (\operatorname{sen}x+\cos x)^2dx$$
20. $$\int \frac{x^2+x+1}{x^3+4x^2+5x+2}dx$$
21. $$\int 4x\left(x^2+5\right)^3dx$$
22. $$\int \left(\ln(\ln x)+\frac{1}{\ln x}\right)dx$$
23. $$\int \tan^2 x\sec^4 x dx$$
24. $$\int \sqrt{9-x^2}dx$$
25. $$\int \frac{x}{1+x^4}dx$$
26. $$\int \operatorname{sen}^4(2x)dx$$
27. $$\int \frac{2\tan x}{1-\tan^2 x}dx$$
28. $$\int \frac{x^2}{(x+1)^4}dx$$
29. $$\int \frac{\ln\left(x^5\right)}{x}dx$$
30. $$\int \cos^2(4x)dx$$
31. $$\int \frac{dx}{1+\operatorname{sen}^2 x}$$
32. $$\int \left(x^5+4\right)\sqrt{1-x^2}dx$$
33. $$\int \frac{1}{1+\left(\frac{1}{x}\right)^2}dx$$
34. $$\int \frac{dx}{x\sqrt{1+x}}$$
35. $$\int \frac{\operatorname{sen}(1/x)}{x^3}dx$$
36. $$\int \frac{dx}{x(\ln x)^3}$$
37. $$\int \frac{dx}{\sqrt{x}+x^{3/2}}$$
38. $$\int \frac{4x^3+5x^2+8x+12}{x^4+x^3+4x^2+4x}dx$$
39. $$\int \cos^5 x dx$$
40. $$\int \frac{dx}{\sqrt{x+4}-\sqrt{x+2}}$$
41. $$\int \frac{8x^2+6x+4}{x+1}dx$$
42. $$\int \frac{dx}{\sqrt{4x-x^2}}$$

43. $$\int \frac{e^x}{e^{2x}+6e^x+9}dx$$

44. $$\int \ln(\cos x)\tan x\,dx$$

45. $$\int (\ln x)^3 dx$$

46. $$\int \sec^3 x dx$$

47. $$\int \frac{dx}{1+e^x}$$

48. $$\int \frac{dx}{\sqrt{\sqrt{x}+1}}$$

49. $$\int e^{1+x+e^x}dx$$

50. $$\int \frac{\ln(\ln x)}{x\ln x}dx$$

51. $$\int \frac{dx}{1+\sqrt{1-x}}$$

52. $$\int \frac{2x}{(x^2+x+1)^2}dx$$

53. $$\int \frac{\tan x}{\sec x}dx$$

54. $$\int \frac{dx}{x^4+4}$$

55. $$\int arcsenx\,dx$$

56. $$\int \ln\sqrt{1+x^2}dx$$

57. $$\int \frac{\cos x}{1+\operatorname{sen} x}dx$$

58. $$\int x^2 \arctan x dx$$

59. $$\int \frac{\sqrt{x^3-2}}{x}dx$$

60. $$\int \sqrt{e^x}dx$$

61. $$\int \sqrt{1-\operatorname{sen} x}dx$$

62. $$\int \sqrt{\cos x}\operatorname{sen}^3 x dx$$

63. $$\int \frac{\sec^2(\ln x)}{5x}dx$$

64. $$\int x^2 \ln x dx$$

65. $$\int \frac{\arccos x}{x^2}dx$$

66. $$\int (x+1)e^{-\left(x+\frac{x^2}{2}\right)}dx$$

67. $$\int e^x\left(x^2+4x+4\right)dx$$

68. $$\int \frac{\sqrt{x^2-1}}{x}dx$$

69. $$\int \frac{4x}{\sqrt{5x^2-7}}dx$$

70. $$\int \sqrt{x}\ln x dx$$

71. $$\int \frac{dx}{(x^2-2x+5)^2}$$

72. $$\int \frac{\cos x}{2+3\operatorname{sen} x}dx$$

73. $$\int \operatorname{sen} x \ln(e^x)\,dx$$

74. $$\int \frac{5x^2+3}{x^4+2x^2}dx$$

75. $$\int \frac{x^5}{\sqrt{x^2+9}}dx$$

76. $$\int e^{3x}\cos(5x)dx$$

77. $$\int dx$$

78. $$\int \frac{3}{x^2+3x}dx$$

79. $$\int \frac{dx}{(x^2-4)^2}dx$$

80. $$\int (\sec x\tan x)^3 dx$$

81. $$\int \frac{(5x+3)(2x-4)}{x}dx$$

82. $$\int \frac{x^2}{x+5}dx$$

83. $$\int 3^{1-x}dx$$

84. $$\int x^3 e^{x^2}dx$$

85. $$\int \frac{dx}{4x-4x^2}$$

86. $$\int \frac{\cos x}{1+\operatorname{sen}^2 x}dx$$

87. $$\int \frac{x^2}{25+x^2}dx$$

88. $$\int \frac{xdx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$$

Bibliography

[Rud76] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill, 1976 (cited on page 7).