Review Susy $\boxed{2022}$

~~Test~~ What ~~things~~ bosons and fermions have in difference and what in common?

Bosons **Fermions**

$[\,,\,] \rightsquigarrow \{\,,\,\}$

Fock Space $\{1,0\}$ ← particle or vacuum state

⋮

Could we organize Bosons and Fermion in only one framework? Answer:

$\boxed{\text{SuSy}}$ (some caveats)

- [,] ← super commutator [,]

which space? $A = A^+ \oplus A^- = A^0 \oplus A^1$

$$\begin{tikzcd} A^+ \arrow[d, equal] & A^- \arrow[d, equal] & A^0 \\ A^0 \arrow[d, equal] & A^1 \arrow[d, equal] & \uparrow \\ \text{even}(A) & \text{odd}(A) & \text{etc.} \end{tikzcd}$$

many names

⤳ A has a splitting
↑ super algebra:

(A super vector space equipped with superproduct)

(CAN I ADD

- super vector space = vector space + a splitting
- super product preserves 'grading'

$A_\pm \times A_\pm \to A_+$

$A_\pm \times A_\mp \to A_-$

$\mathrm{Sym}^*(V)$

$= S^*V \otimes \Lambda^* V$

Ex: $\mathrm{Sym}(\mathbb{R}^{n|n})$

$= \mathbb{R}[x_i, dx^i]$

$\rightsquigarrow$ Super Commutator $[\ ,\ ]_s = ab - (-1)^{|a||b|}ba$

$|a||b| = 0 \cdot 0 = 0$

$\rightsquigarrow [a,b]_s = [a,b]$ if $a,b \in V^+$

$[a,b]_s = \{a,b\}$ is $a,b \in V^-$ $\Big\}$ ✓

$\uparrow$ $|a||b| = 1 \cdot 1 = 1$

...

Super $\rightsquigarrow$ put super in everything abstractly

- Super Lie Algebra
- Super Commutator ~~a [a,b~~
- super trace $\rightsquigarrow$ $M = \begin{array}{c|c|c} & V_+ & V_- \\ \hline V_+ & A\ \boxed{++} & -+ \\ \hline V_- & +- & \boxed{--}\ B \end{array}$

$\rightsquigarrow \operatorname{str} M = \operatorname{tr} A - \operatorname{tr} B.$

- $\operatorname{sdet} = \dfrac{\det A}{\det B}$

Representation theory of this spaces:

Ex: $\mathfrak{gl}(1|1) \leftarrow$ linear transform of $\mathbb{R}^{1|1}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \nearrow$ $\rightsquigarrow$ generators

$\alpha = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right] \leftarrow \in \mathfrak{g}_+$

$\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow \in \mathfrak{g}_-$

$\gamma = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \leftarrow \in \mathfrak{g}_-$

$\delta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow \in \mathfrak{g}_+$

note g_+ → transforms that dont flip character.

↳ commutator relations!

anticommutator

usual commutator

$[\alpha,\alpha] \overset{\downarrow}{=} 0$

$[\alpha,\delta] = 0$

$[\delta,\delta] = 0$

g_+ is commutative

$[\alpha,\beta] = \beta$

$[\delta,\beta] = -\beta$

$[\alpha,\gamma] = -\gamma$

$[\delta,\gamma] = +\gamma$

$[\beta,\beta] \overset{\downarrow}{=} 2\beta^2 = 0$

$[\gamma,\gamma] = 2\beta^2 = 0$

~~[illegible]~~

$[\beta,\gamma] = \underline{\underline{\alpha+\delta}}$

Difference with $gl(2)$?

yes! (almost ~~the~~ same) except for

$\underbrace{[\beta,\gamma] = \alpha+\delta}_{\text{~~super~~ } gl(1|1))} \leftrightarrow \underbrace{[\beta,\gamma] = \alpha-\delta}_{gl(2)}$

↓ $sl(1|1)$? $str(M)=0$ ⇝ $sl(1|1) = \langle \beta,\gamma,\alpha+\delta$

$\neq$

$sl(2) = \langle \beta,\gamma,\alpha-\delta\rangle$

in $sl(1|1))$ what is the $(\alpha-\delta)$?

$\alpha-\delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u \\ -v \end{pmatrix}$ ← $\alpha-\delta$ parity op!

Unitarity? • $g : V \otimes_{\mathbb{R}} V \to \mathbb{C}$

- $\mathbb{C}$ linear on 2^{nd} entry
- $\mathbb{C}$ conjugate linear on 1st
- $g(v,w) = (-1)^{|v||w|}\,\overline{g(w,v)}$

Ex.

$$\begin{bmatrix} u \\ \hline v \end{bmatrix} \cdot \begin{bmatrix} u' \\ \hline v' \end{bmatrix} = \bar{u}u' + i\bar{v}v'$$

is a hermitian inner product

- $$\left(\begin{pmatrix} u \\ v \end{pmatrix} + \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) \cdot \begin{pmatrix} u' \\ \hline v' \end{pmatrix}$$

$$= \bar{u}u' + i\bar{v}v' + \bar{\lambda}\alpha u' + i\bar{\lambda}\bar{\beta}v'$$

$$= \begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix} + \bar{\lambda} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix} \checkmark$$

- $\mathbb{C}$ linear on 2^{nd} entry $\checkmark$

- $$\begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u' \\ w' \end{pmatrix} = \overline{\overline{\bar{u}u' + i\bar{v}v'}}$$

$$= \overline{\bar{u}'u + (-1)\bar{v}'v} = [illegible]$$

$$= \overline{\begin{pmatrix} u' \\ v' \end{pmatrix} \cdot \begin{pmatrix} u \\ -v \end{pmatrix}} = (-1)\overline{\begin{pmatrix} u' \\ v' \end{pmatrix} \cdot \begin{pmatrix} -u \\ v \end{pmatrix}}$$

If ~~$px=0$~~ $\rightsquigarrow v=0 \rightsquigarrow \left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\| = 0 \rightsquigarrow$

~~So~~
$$\begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix} \overset{!}{=} \overline{\begin{pmatrix} u' \\ v' \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}}$$

$$\hookrightarrow \begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix} = \overline{\begin{pmatrix} u' \\ v' \end{pmatrix} \cdot \begin{pmatrix} u \\ \cancel{v}^{0} \end{pmatrix}} = \checkmark$$

If $u=0$

$$\begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix} \overset{!}{=} (-1)\, \overline{\begin{pmatrix} u' \\ v' \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}}$$

$$\hookrightarrow = \overline{\begin{pmatrix} u' \\ v' \end{pmatrix} \cdot \begin{pmatrix} \cancel{u}^{0} \\ -v \end{pmatrix}} = -\overline{\begin{pmatrix} u' \\ v' \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v \end{pmatrix}} \checkmark$$

<u>Ex:</u> $SU(2) \leftarrow$ 2x2 supermatrices . traceless
$+ \cdot A^\dagger = -A \leftarrow$

Cond

$$u^\dagger u = 1 \cancel{\Leftrightarrow} \Rightarrow (1+\varepsilon A)^\dagger (1+\varepsilon A) = 1$$

$$\shortparallel$$

~~$(1+\varepsilon A$~~

$$(1+\varepsilon A)^{-1} = 1 - \varepsilon A + \ldots$$

$$\subset \frac{1}{1-(-\varepsilon A)} = \cancel{1+}\sum_n (-\varepsilon A)^n = 1 - \varepsilon A + \cancel{\mathcal{O}(\varepsilon^2)}^{0}$$

$$\Rightarrow (1+\varepsilon A)^\dagger \overset{!}{=} 1 - \varepsilon A \qquad \overline{\boxed{A^\dagger = -A}}\ ?$$

$$\shortparallel$$

$$1 + \varepsilon A^\dagger = 1 - \varepsilon A \rightsquigarrow \boxed{A^\dagger = -A}$$

~~what about SU(1|1)~~ ?

where we used the fact that

$A^\dagger = \overline{A}^t$ ← relation to the unitary?

usual inner product in Q(X)

$$\begin{pmatrix}u\\v\end{pmatrix} \circ \begin{pmatrix}u'\\v'\end{pmatrix} = (\bar{u}\ \bar{v})\begin{pmatrix}u'\\v'\end{pmatrix}$$

$$= \bar{u}u' + \bar{v}v'$$

⇝ $$= \overline{\begin{pmatrix}u\\v\end{pmatrix}}^t \begin{pmatrix}u'\\v'\end{pmatrix}$$ ~~$g(u,v)=$~~

~~SQ(4) (s-sp)~~

~~$\begin{pmatrix}u\\v\end{pmatrix} \circ \begin{pmatrix}u'\\v'\end{pmatrix} = \bar{u}u' + i\bar{v}v'$~~

~~Unitarity in Q(X) ⇝ $g(u, Mv) = g(\ldots)$~~

~~⇝ $g(M^\dagger u, v) = g(Mu, v)$~~

~~$g(u^\dagger M$~~ $g(u,v) = g(M^\dagger M u, v)$

$= g(Mu, Mv)$

$\boxed{\text{unitary}}$ ⇝ $\bar{u}^t v = (\overline{Mu})^t M v$

$= \bar{u}^t \overline{M}^t M v$ $\boxed{M^t \circ F}$

$\mathfrak{sl}(2)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow$ traceless $\rightsquigarrow \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$.

$A^{\dagger}$ ← transpose conjugate

$A^{\dagger} = -A$

$\rightsquigarrow \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & -\bar{a} \end{pmatrix} = -\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$

$\rightsquigarrow \bar{a} = -a \rightsquigarrow a = ir$ ← real

$\bar{b} = -c \rightsquigarrow c = \alpha + i\beta \rightsquigarrow b = -(\alpha - i\beta)$

$\bar{c} = -b$ $\qquad = i\beta - \alpha$

↳ ~~$su(2) = \{$~~ $\begin{pmatrix} i\alpha & i\beta - \gamma \\ \gamma + i\beta & -i\alpha \end{pmatrix}$ ← general element $\in su(2)$

$\rightsquigarrow \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle_{\mathbb{R}} = su(2)$

$= \langle i\sigma_z, i\sigma_x, i\sigma_y \rangle$.

~~what~~ $\rightsquigarrow$ ~~σ_z~~ how we analyze the space? $\{\sigma_z, \sigma_+, \sigma_-\}$ $\sigma_{\pm} = \dfrac{\sigma_x \pm i\sigma_y}{2} \leftarrow \boxed{\tfrac{1}{2}?}$

ladder op.

~~σ_z~~

$$\Rightarrow \begin{pmatrix}1&0\\0&i\end{pmatrix}\overline{A}^{t} = -A\begin{pmatrix}1&0\\0&i\end{pmatrix}$$

$$A=\begin{pmatrix}a&b\\c&d\end{pmatrix},\quad \begin{pmatrix}1&0\\0&i\end{pmatrix}\begin{pmatrix}\overline{a}&\overline{c}\\\overline{b}&\overline{d}\end{pmatrix}=\begin{pmatrix}\overline{a}&\overline{c}\\i\overline{b}&i\overline{d}\end{pmatrix}$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\0&i\end{pmatrix}=\begin{pmatrix}a&ib\\c&id\end{pmatrix}$$

$$\rightsquigarrow \quad \overline{a} = -a$$

$$\overline{a} = -a \rightsquigarrow a = i\gamma$$

$$\left.\begin{matrix}\overline{c} = -ib\\ i\overline{b} = -c\end{matrix}\right\} \rightsquigarrow \overline{c} = -ib$$

$$id = -i\overline{d} \rightsquigarrow d = -\overline{d}$$

$$b = \alpha + i\beta$$
$$c = i(\alpha - i\beta) = i\alpha + \beta$$

- traceless: $\begin{pmatrix}a&b\\c&a\end{pmatrix}=\begin{pmatrix}i\gamma&\alpha+i\beta\\\beta+i\alpha&i\gamma\end{pmatrix}$

$$\rightsquigarrow \left\langle \begin{pmatrix}i&0\\0&i\end{pmatrix}, \begin{pmatrix}0&i\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\i&0\end{pmatrix}\right\rangle_{\mathbb{R}} = su(1,1)$$

Now with $su(1,1)$

- traceless ⇝ $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$ ✓

- unitarity:

$$M\begin{pmatrix} u \\ v \end{pmatrix} \cdot M\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$\overline{\left(M\begin{pmatrix} u \\ v \end{pmatrix} \right)}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M\begin{pmatrix} u' \\ v' \end{pmatrix} =$$

$$= \overline{(u\ v)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$M = 1 + \varepsilon A$$

$$\rightsquigarrow \overline{(u\ v)}\ \underset{(1+\varepsilon \bar{A}^t)}{\underset{\|}{\overline{M}^t}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underset{(1+\varepsilon A)}{\underset{\|}{M}} \begin{pmatrix} u' \\ v' \end{pmatrix} =$$

$$\overline{(u\ v)}\ \underbrace{(1+\varepsilon \bar{A}^t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (1+\varepsilon A)} = \begin{pmatrix} u' \\ v' \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}}$$

$$\cancel{(1+\varepsilon \bar{A}^t} \quad \cancel{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cancel{\varepsilon} A^t + \cancel{\varepsilon} \bar{A}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \cancel{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

1A Susy

$$\begin{pmatrix}u\\v\end{pmatrix}\cdot\begin{pmatrix}u'\\v'\end{pmatrix}=\bar{u}u'+i\bar{v}v'$$

$$\rightsquigarrow \qquad = \overline{(u\ v)}\left(\begin{array}{c|c}1&0\\\hline 0&i\end{array}\right)\begin{pmatrix}u'\\v'\end{pmatrix}$$

$$M\begin{pmatrix}u\\v\end{pmatrix}\cdot M\begin{pmatrix}u'\\v'\end{pmatrix}=\begin{pmatrix}u^*\\v^*\end{pmatrix}\cdot\begin{pmatrix}u'\\v'\end{pmatrix} = \overline{\begin{pmatrix}u\\v\end{pmatrix}}^t\begin{pmatrix}1&0\\0&i\end{pmatrix}\begin{pmatrix}u'\\v'\end{pmatrix}$$

$$=\left(\overline{M\begin{pmatrix}u\\v\end{pmatrix}}\right)^t\begin{pmatrix}1&0\\0&i\end{pmatrix}\underset{\uparrow}{M}\begin{pmatrix}u'\\v'\end{pmatrix}$$

$$=$$

$$\begin{pmatrix}1&0\\0&i\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&b\\ic&id\end{pmatrix}\begin{pmatrix}u'\\v'\end{pmatrix}$$

$$(\bar{u}\ \bar{v})\ \overline{M}^t\begin{pmatrix}1&0\\0&i\end{pmatrix}M\begin{pmatrix}u'\\v'\end{pmatrix}=$$

$$=\overline{\begin{pmatrix}u\\v\end{pmatrix}}^t\begin{pmatrix}1&0\\0&i\end{pmatrix}\begin{pmatrix}u'\\v'\end{pmatrix}$$

$$\rightsquigarrow\ \overline{M}^t\begin{pmatrix}1&0\\0&i\end{pmatrix}M=\begin{pmatrix}1&0\\0&i\end{pmatrix}$$

$$=\ \overline{\begin{pmatrix}a&b\\c&d\end{pmatrix}}^t\begin{pmatrix}a&b\\ic&id\end{pmatrix}=\begin{pmatrix}1&0\\0&i\end{pmatrix}$$

$$=\ \begin{pmatrix}\bar{a}&\bar{c}\\\bar{b}&\bar{d}\end{pmatrix}\begin{pmatrix}a&b\\ic&id\end{pmatrix}=\begin{pmatrix}1&0\\0&i\end{pmatrix}$$

$$=\ \begin{pmatrix}|a|^2+i|c|^2 & \bar{a}b+i\bar{c}d\\ a\bar{b}+i\bar{d}c & |b|^2+i|d|^2\end{pmatrix}=\begin{pmatrix}1&0\\0&i\end{pmatrix}$$

$\Rightarrow \quad c=0 \quad |a|^2=1$

$\quad\; b=0 \quad |d|=1$

$\rightsquigarrow \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \leftarrow$ see ~~we~~ seek for

$SU(1|1) \rightsquigarrow$ 0 freedoms $\rightsquigarrow \begin{pmatrix} a & b \\ b\ddagger & a \end{pmatrix}$

~~0 [illegible]~~ ~~$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \; |a|=1$~~

$\hookrightarrow \quad a = e^{i\alpha}$

$\rightsquigarrow \quad \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = ($

$SU(1|1) \simeq$ • $N=2$ Supersymmetric

"SQM"

• origin of other things

as a [illegible].

Resumen $SU(1|1)$ is <u>alot</u> like $SU(2)$

but

• $SU(1|1) = \langle H, Q_+, Q_- \rangle$

$+ \quad - \quad -$

$$\{Q_i, Q_i\} = H$$

$$\{Q_i, Q_{i+1}\} = 0 \qquad [H, Q_i] = 0$$

~~Th Q_i are~~ in QM translations are the <u>only</u> translation symmetries.

In Susy we have "square roots of this"

i.e $\{Q_i, Q_i\} = \frac{\partial}{\partial x}$

$\nearrow$

$N=2$.

For $N=1$ QM would be

$\langle H, Q \rangle = \mathfrak{g}$ $\quad$ $\{Q, Q\} = H$

$[H, Q] = 0$

$\uparrow$ 2d-super Lie algebra

$\rightsquigarrow$ Generalize for N arbitrary.

$\langle H, Q_1, \dots, Q_N \rangle$

$\{Q_i, Q_j\} = \delta_{ij} H_j$

$[H, Q_i] = 0$

- In $d = 3+1$ $\rightsquigarrow$ Four space translations

$\rightsquigarrow$ Q_i^α, $\alpha = 1, 2, \dots, 4$. For $N=1$

$\rightsquigarrow$ $Q^\alpha \leftarrow$ "4 symmetry generators $\rightsquigarrow$ super charges!!

Invariant subalgebras $\langle H \rangle$.

How about its reps? $H|\psi\rangle = \overset{\text{energies?}}{E}|\psi\rangle$

reps of $SU(1|1)$) will fit into one parameter-

Let (W, ψ) be a real vector space.

$\mathfrak{g} = \mathfrak{g}_+ = \mathbb{R}H$, $\mathfrak{g}_- = W \rightsquigarrow \mathfrak{g}$

~~N=d~~ $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}$

$\uparrow$. $\quad W \otimes W \quad \mathbb{R}$

$\mathfrak{g}_+$

$N = \dim(W)$

This algebra has ~~Ro~~ true symetries.

↑ rotations preserving the inner product

Here, R-symmetry is $SO(W)$

For $N=2$ (SQM) $\rightsquigarrow SO(2)$.

rotating Q_1 onto Q_2.

$$\left\{ \begin{array}{l} H, R, Q_1, Q_2 \\ [Q_i, Q_j] = \delta_{ij} H \\ [H, R] = [H, Q_i] = 0 \end{array} \right\} \mathfrak{se}(1|1)$$

$[R, Q_1] = i Q_2 \leftarrow [\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}_+$

$[R, Q_2] = -i Q_1$ inner product

What if I add R-symmetry to the N-add algebra $\overset{SO(N)}{\parallel}$

$$\mathfrak{g}_+ = \mathbb{R}_H \oplus \underbrace{SO(W)}_{\text{R-Symmetry}} \rightsquigarrow \dim(\mathfrak{g}_+) = \frac{N(N-1)}{2 \cancel{+0}} + 1$$

$\mathfrak{g}_- = W, \dim \mathfrak{g}_- = N$

Rep of $SU(2|1) \leftarrow \langle H, Q_1, Q_2 \rangle$

$$\downarrow$$

$Q_\pm \in sl(1|1) = su(1|1) \otimes \mathbb{C}$

⇝ Let $|\psi\rangle$ ~~annihilated by Q_+~~ annihilated by Q_+

1) $Q_-^{2}|\psi\rangle$ ⇝ $\underbrace{Q_-^2 = p^2}_{Q_+^2 = p^2} \Rightarrow Q_\pm^2|\psi\rangle = 0$

$$\downarrow$$

$Q_-|\psi\rangle$ either zero or nonzero.

~~$\{Q_+, Q_-\}|\psi\rangle = 0 \Rightarrow Q_+Q_- +$~~

$\{\overset{Q_1+Q_2/2}{Q_+}, \overset{Q_1-Q_2/2}{Q_-}\} = \frac{2}{2}H = H$ ⇝

$Q_+Q_-|\psi\rangle = H|\psi\rangle - Q_-\overset{0}{\cancel{Q_+}}|\psi\rangle$

⇝ $Q_+Q_-|\psi\rangle = H|\psi\rangle$

∘ Suppose $Q_-|\psi\rangle \neq 0$. → rep has $|\psi\rangle$, $Q_-|\psi\rangle$ ← odd

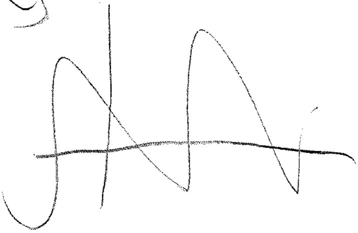
$\lceil Q_+|\psi\rangle = 0,\ Q_-^2 = 0 \ldots \rfloor$ 2-dim.

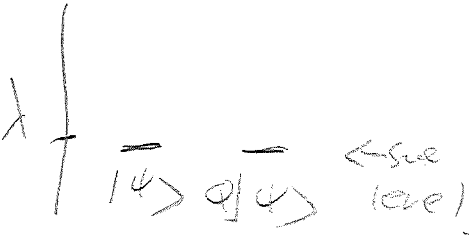
Hence $H|\psi\rangle = \lambda|\psi\rangle$ ← diagonalize at the start.

$\rightsquigarrow$ $H|\psi\rangle = \lambda|\psi\rangle$, $\lambda > 0$ $\left\{\begin{array}{l}\text{one boson}\\ \text{one fermion}\end{array}\right.$

$Q_+(Q_-|\psi\rangle) = H|\psi\rangle = \lambda|\psi\rangle$

↳





• If ~~$|\psi\rangle$~~ $Q|\psi\rangle = 0$ $\rightsquigarrow$ trivial rep

$Q_+|\psi\rangle = 0$ $\rightsquigarrow$ $\langle\psi|H|\psi\rangle = \langle\psi|\{Q_-, Q_+\}|\psi\rangle$

$= \langle\psi|Q_-Q_+|\psi\rangle + \langle\psi|Q_+Q_-|\psi\rangle$

$= \| Q_-|\psi\rangle\|^2 + \|Q_+|\psi\rangle\|^2$

≥ 0 iff $Q_- = Q_+ = 0$ on $|\psi\rangle$

Claim $|\psi\rangle$ is Homogeneous

~~$[Q,\Pi]|\psi\rangle = Q[|\psi_+\rangle - |\psi_-\rangle] + \Pi Q$~~

$\parallel$

$|\psi_+\rangle + |\psi_-\rangle$

$[Q,\Pi] = -Q$

$$\Pi|\psi\rangle = |\psi_+\rangle - |\psi_-\rangle$$

$$\leadsto [Q,\Pi] = Q[|\psi_+\rangle - \cancel{|\psi_-\rangle}] \text{\textcircled{$\pm$}} \Pi Q|\psi\rangle$$

$$= -Q|\psi_+\rangle + \cancel{|\psi_-\rangle}$$

$$-Q|\psi\rangle = Q|\psi_+\rangle \text{\textcircled{$\pm$}} \Pi[\cancel{Q}|\psi_+\rangle \overset{Q}{+} |\psi_-\rangle]$$

$$= Q|\psi_+\rangle \text{\textcircled{$\pm$}} (-(Q|\psi_+\rangle) + Q|\psi_-\rangle)$$

$$-\cancel{Q|\psi_+\rangle} = Q|\psi_+\rangle \mp \cancel{Q|\psi_+\rangle} \pm Q|\psi_\mp\rangle$$

$$Q|\psi_+\rangle = Q|\psi_-\rangle$$ Homogeneous

Ex: $N=2$ Susy. $\leadsto$ Hilbert space $L^2(\mathbb{R}) \ni \psi(x)$

$$H = -\frac{\partial^2}{\partial x^2} = p^2.$$

$$\mathbb{C}_{susy} \leftarrow \mathcal{H} = L^2 \otimes \mathbb{C}^{1|1} \ni \begin{bmatrix}\psi(x)\\ \phi(x)\end{bmatrix}$$

$$\leadsto \ni \psi(x) + \varepsilon\phi(x)$$

$$\leadsto H = -\frac{\partial^2}{\partial x^2} \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix}$$ ← Factor

$$\{Q_i, Q_j\} = \delta_{ij} H = \delta_{ij} P^2$$

Since $\begin{bmatrix}1&0\\0&1\end{bmatrix} = \cancel{\alpha\pm\beta}\left[\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}\right]_S$

$$= \left\{\underset{\beta}{\begin{pmatrix}0&1\\0&0\end{pmatrix}}, \underset{\gamma}{\begin{pmatrix}0&0\\1&0\end{pmatrix}}\right\} =$$

$\rightsquigarrow$ $Q^\dagger = P\otimes\beta$, $\cancel{Q}$ $\beta^\dagger = \gamma$

$Q = P\otimes\gamma$

$\rightsquigarrow$ $\{Q, Q^\dagger\} = \{P\otimes\beta, P\otimes\gamma\}$

~~= [illegible] $\{\beta,\beta\}\otimes$~~

$= P\otimes\beta)(P\otimes\gamma) + (P\otimes\gamma)(P\otimes\beta)$

$= P^2\otimes\beta\gamma + P^2\otimes\gamma\beta$

$= P^2\otimes\{\beta,\gamma\} = \underset{\overset{\shortparallel}{-\frac{\partial^2}{\partial x^2}}}{P^2} \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix}$ ✓

$$\rightsquigarrow \quad Q = \left(\begin{array}{c|c} 0 & 0 \\ \hline -i\frac{\partial}{\partial x} & 0 \end{array}\right)$$

$$Q\begin{pmatrix} \psi(x) \\ \hline \phi(x) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -i\frac{\partial}{\partial x} & 0 \end{pmatrix}\begin{pmatrix} \psi(x) \\ \hline \phi(x) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -i\psi'(x) \end{pmatrix}$$

$$Q^{+} = \begin{pmatrix} 0 & -i\frac{\partial}{\partial x} \\ 0 & 0 \end{pmatrix} \rightsquigarrow Q\begin{pmatrix} \psi \\ \hline \phi \end{pmatrix} = \begin{pmatrix} -i\phi' \\ 0 \end{pmatrix}$$

$$\hookrightarrow \quad Q_1 = \frac{Q + Q^{+}}{2} \qquad Q_2 = \frac{Q_1 - Q^{\perp}}{2i}$$

Works for a $\mathbb{Z}_2$-grad Hilbert spaces!

write $\begin{pmatrix} \psi \\ \phi \end{pmatrix} \rightsquigarrow \psi(x) + \varepsilon\phi(x)$

$\uparrow$ odd nbE

$\psi(x) + dx\,\phi(x) \simeq \Omega^{*}(\mathbb{R})$

$$\left.\begin{array}{l} d_Q(\phi\, d\cdot) = \frac{\partial\psi}{\partial x}dx \\ d_Q(\psi) = 0 \end{array}\right\} Q \text{ is the deRham diff}$$

$Q^\dagger$ is the deRham adjoint, with respect to

$$\langle \alpha, \beta \rangle = \int d\bar{\alpha} \wedge \star\beta$$

This [illegible] [illegible] [illegible] ocFOS $\mathcal{H} = L^2(\mathbb{R})\otimes\mathbb{C}''$

i.e $Q^\dagger = d^\dagger$, $Q = d$

$$\rightsquigarrow \quad d = dx^i \frac{\partial}{\partial x^i} \quad \Bigg| \quad f + g\,dx \rightarrow \frac{\partial f}{\partial x} dx$$

$$\begin{pmatrix} f \\ g \end{pmatrix} \xrightarrow{Q} \begin{pmatrix} 0 \\ \frac{\partial f}{\partial x} \end{pmatrix} \checkmark$$

$$Q^\dagger \rightsquigarrow f + g\,dx \xrightarrow{Q^\dagger} -\frac{\partial g}{\partial x}$$

This works for any mfd M.

$$\mathcal{H} = \Omega^0(M)_{L^2} = \underset{\overline{\bigoplus \Omega^{\text{even}}(M)}}{\underset{\|}{\mathcal{H}_+}} \oplus \underset{\oplus\Omega^{\text{odd}}(M)}{\underset{\|}{\mathcal{H}_-}}$$

$$\langle \alpha, \beta \rangle = \int_M \bar{\alpha} \wedge \star\beta$$

↳ $H = \Delta$ Laplace opp on diff forms.

- The spectrum depends on mass m
but vacuum not.

+ Can I introduce more interactions? w/o
breaking super symmetry.

+ Is $(\mathcal{H}, H)$ susy harm osc quantization
of a classical theory?

on flat spaces
$$\begin{tikzcd} L^2(\mathbb{R}^2) \simeq & L^2(\mathbb{R}_x) \arrow[d] \otimes & L^2(\mathbb{R}_y) \arrow[d] \\ & L(\mathbb{R}) \otimes \mathbb{C}[dx] & L^2(\mathbb{R}_y) \end{tikzcd}$$

$\rightsquigarrow$ supersymmetric $L^2(\mathbb{R}^2) \otimes \mathbb{C}[dx, dy]$
$$\mathbb{C}^{2|2}$$

Recall we need:
$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0$$
$$\{Q, Q^\dagger\} = 2H.$$

replace $Q \to e^{\lambda} Q$, $\lambda \in \mathbb{C}$
$$Q^\dagger \to Q^\dagger e^{\lambda}$$

Clearly $\{Q, Q^\dagger\} \Rightarrow \{e^{\lambda} Q, e^{-\lambda} Q^\dagger\}$ ✓

satisfy the same relations.

using $\boxed{d}$ ~> generalize

$d = Q \to \tilde{Q} = e^{-\lambda w} d e^{\lambda w}$, $\lambda \in \mathbb{R}$

$d^\dagger = Q^\dagger \to e^{\lambda w} d^\dagger e^{-\lambda w}$

~~$\{\tilde{Q}, \tilde{Q}\}$~~ $= \tilde{Q}\tilde{Q} = e^{-\lambda w} d e^{\lambda w} e^{-\lambda w} d e^{\lambda w}$

$= e^{-\lambda w} d^2 e^{\lambda w} = 0$

~> ~~$\{\tilde{Q}, \tilde{Q}\} = 0$~~

~~$\{\tilde{Q}^\dagger, \tilde{Q}^\dagger\} = 0$~~

~~$\tilde{Q}\tilde{Q}^\dagger = e^{-\lambda w} d e^{2\lambda w} d^\dagger e^{-\lambda w}$~~

~~$= e^{-\lambda w} d d^\dagger e^{-\lambda w}$~~

Lets see $\{\tilde{Q}, \tilde{Q}^\dagger\} = ?$

$$\tilde{Q} = Q_\lambda$$

$$Q_\lambda(f + g\,dx) = e^{-\lambda w}\, d\, e^{\lambda w}(f + g\,dx)$$

$$= e^{-\lambda w}\left(\frac{\partial e^{\lambda w} f}{\partial x}\, dx\right)$$

$$= \frac{\partial f}{\partial x}\, dx + \lambda w' f\, dx$$

$$= dx\left(\frac{\partial f}{\partial x} + \lambda w' f\right) = df + \lambda\, dw\, f$$

$$\rightsquigarrow Q_\lambda \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\partial f}{\partial x} + \lambda w' f \end{pmatrix}$$

$$\rightsquigarrow Q_\lambda = \left(\begin{array}{c|c} 0 & 0 \\ \hline \frac{\partial}{\partial x} + \lambda w' & 0 \end{array}\right) = \underbrace{\left(\frac{\partial}{\partial x} + \lambda w'\right)}_{P} \otimes \underset{\beta}{\underset{\|}{\left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \end{array}\right.}}$$

$$Q_\lambda^\dagger = \left(\begin{array}{c|c} 0 & -\frac{\partial}{\partial x} + \lambda w' \\ \hline 0 & 0 \end{array}\right) = \left(-\frac{\partial}{\partial x} + \lambda w'\right) \otimes \gamma$$

$$\langle \alpha, Q_\lambda \beta \rangle = \langle Q^\dagger \alpha, \beta \rangle = \int d\alpha \wedge \ast\, d\beta$$

$$\{Q^{\cancel{\dagger}}, Q^{\dagger}\} = \{\frac{\partial}{\partial x} + \lambda w', -\frac{\partial}{\partial x} + \lambda w'\} \otimes \mathbb{1}$$

$$- [\frac{\partial}{\partial x} + \lambda w', -\frac{\partial}{\partial x} + \lambda w'] \otimes \sigma_z$$

$$= \{(\frac{\partial}{\partial x} + \lambda w') \otimes \begin{pmatrix}0&0\\1&0\end{pmatrix}, (-\frac{\partial}{\partial x} + \lambda w') \otimes \begin{pmatrix}0&1\\0&0\end{pmatrix}\}$$

$$\{Q, Q^{\dagger}\} = \{P \otimes \beta, P' \otimes \gamma'\}$$

$$\{Q, Q^{\dagger}\} = \{(\frac{\partial}{\partial x} + \lambda w') \otimes \beta, (-\frac{\partial}{\partial x} + \lambda w') \otimes \gamma'\}$$

$$= (\frac{\partial}{\partial x} + \lambda w')(-\frac{\partial}{\partial x} + \lambda w') \otimes \beta\gamma'^{(\mathbb{1} - \begin{pmatrix}1&0\\0&0\end{pmatrix})}$$

$$+ (-\frac{\partial}{\partial x} + \lambda w') \otimes (\frac{\partial}{\partial x} + \lambda w') \otimes \gamma'\beta^{\begin{pmatrix}1&0\\0&0\end{pmatrix}}$$

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$= ([\frac{\partial}{\partial x} + \lambda w', -\frac{\partial}{\partial x} + \lambda w']$$

$$\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$- (-\frac{\partial}{\partial x} + \lambda w')(\frac{\partial}{\partial x} + \lambda w')) \otimes (\mathbb{1} - \begin{pmatrix}1&0\\0&0\end{pmatrix}) =$$

$$\{\frac{\partial}{\partial x} + \lambda w', -\frac{\partial}{\partial x} + \lambda w'\} \otimes \mathbb{1} - \{\cdots\} \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$- (-\frac{\partial}{\partial x} + \lambda w')(\frac{\partial}{\partial x} - \lambda w') \otimes \mathbb{1} + 2(-\frac{\partial}{\partial x} + \lambda w')(\frac{\partial}{\partial x} \cdots) \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$= \{\ ,\tfrac{\partial}{\partial x}\} \otimes \mathbb{1} - \{\ ,\tfrac{\partial}{\partial x}\} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$- \left(\tfrac{\partial}{\partial x}\right)(\quad) \otimes \mathbb{1} + 2\left(-\tfrac{\partial}{\partial x}\right)(\quad) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \{\ ,-\tfrac{\partial}{\partial x}\} \otimes \mathbb{1} - (\quad)\left(\tfrac{\partial}{\partial x}\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$- \left(-\tfrac{\partial}{\partial x}\right)(\quad) \otimes \mathbb{1} + \left(-\tfrac{\partial}{\partial x}\right)(\quad) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\cancel{+ \left(-\tfrac{\partial}{\partial x}\right)(\quad) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

$$= \left(\sigma_z + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \{\ ,-\tfrac{\partial}{\partial x}\} \otimes \mathbb{1} + \left[-\tfrac{\partial}{\partial x},\ \right] \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$- \left(-\tfrac{\partial}{\partial x}\right)(\quad) \otimes \mathbb{1}$$

$$+ \left[\tfrac{\partial}{\partial x},\ \right] \otimes \sigma_z$$

$$= \{\ ,-\tfrac{\partial}{\partial x}\} \otimes \mathbb{1} + \left[\tfrac{\partial}{\partial x},\ \right] \otimes \sigma_z$$

$$+ \underbrace{\left[-\tfrac{\partial}{\partial x},\ \right] \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \left(-\tfrac{\partial}{\partial x}\right)(\quad) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$- (\quad)\left(-\tfrac{\partial}{\partial x}\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(-\tfrac{\partial}{\partial x}\right)(\quad) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$- \{\ ,-\tfrac{\partial}{\partial x}\} \otimes \mathbb{1}$$

$$\{A \otimes B, A' \otimes B'\} = AA' \otimes BB' + A'A \otimes B'B$$

$$= \tfrac{1}{4}\left(\{A, A'\} + [A, A']\right) \otimes \left(\{B, B'\} + [B, B']\right)$$

$$+ \tfrac{1}{4}\left(\{A', A\} + [A', A]\right) \otimes \left(\{B', B\} + [B',$$

$= \frac{1}{\sqrt{4}} ($

$$\{ P \otimes \gamma , P \otimes \beta \}$$

$$= \frac{1}{4} \Big(\{P, P\} \otimes (\mathbb{1} + \sigma_z)$$

$$+ \{P, P\} \otimes (\mathbb{1} - \sigma_z) \Big)$$

$$= \frac{1}{2} \{P, P\} \otimes \mathbb{1} + 0 = P^2 \otimes \mathbb{1}$$

$$\{a_\lambda^{\dagger}, a_\lambda^{\dagger}\} = \{(\frac{\partial}{\partial x} + \lambda\omega') \otimes \gamma , (-\frac{\partial}{\partial x} + \lambda\omega') \otimes \beta\}$$

~~$= \frac{1}{4} (\{-\frac{\partial}{\partial x} + \lambda\omega'\}$~~

~~$= \frac{1}{4} ((\{ -\frac{\partial}{\partial x} + \lambda\omega' \} \frac{\partial}{\partial x} + \lambda\omega' \} \otimes$~~

$$= \frac{1}{4} \Big((\{ -\frac{\partial}{\partial x} + \lambda\omega' , \frac{\partial}{\partial x} + \lambda\omega' \} + [-\frac{\partial}{\partial x} + \lambda\omega' , \frac{\partial}{\partial x} + \lambda\omega'])$$

$$\otimes (\mathbb{1} + \sigma_z)$$

$$+ (\{ -\frac{\partial}{\partial x} - \lambda\omega' , \frac{\partial}{\partial x} + \lambda\omega' \} - [-\frac{\partial}{\partial x} + \lambda\omega' , \frac{\partial}{\partial x} + \lambda\omega'])$$

$$\otimes (\mathbb{1} - \sigma_z) \Big)$$

$$\frac{1}{4} (2 \{ - , \} \otimes \mathbb{1} + 2 [- ,] \otimes \sigma_z) = \frac{1}{2} (\{ - , \} \otimes \mathbb{1} + [- ,] \otimes \sigma_z)$$

$$H = \underbrace{-\frac{\partial^2}{\partial x^2} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{p^2} + \lambda^2 w'^2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$- \lambda w'' \otimes \sigma_z$$

$$\rightsquigarrow \text{potential} \begin{pmatrix} \lambda^2 w'^2 - \lambda w'' & 0 \\ 0 & \lambda^2 w'^2 + \lambda w'' \end{pmatrix}$$

u^{Boson}

u^{Fermion}

Harmc oscillator? potential $= \frac{1}{2} m^2 \omega^2 x^2 +$

$(w = \pm c x^2/2 \rightsquigarrow w' = \pm c x,\ w'' = \pm c)$

$$\rightsquigarrow u_{bos} = \underbrace{\lambda^2 c^2 x^2}_{\text{Harmc osc}} \pm \boxed{\lambda c}$$

Vacuum?

$$\left(-\frac{\partial}{\partial x} + \lambda w'\right) \psi(x) = 0 \rightsquigarrow \frac{\partial \psi}{\partial x} = \lambda w'(x) \psi$$

$$\rightsquigarrow \psi(x) = c\, e^{\pm \int_0^x \lambda w'}$$

$$\left(+\frac{\partial}{\partial x} - \lambda w'\right) \psi = 0 \rightsquigarrow$$

$$\rightsquigarrow \frac{\partial f}{\partial x} = \lambda w' f$$

$$\begin{pmatrix} 0 & 0 \\ -\frac{\partial}{\partial x} + \lambda w' & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\partial f}{\partial x} + \lambda w' f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

reps of $su(1|1) = \langle P, Q_1, Q_2 \rangle$?

$$P = i\begin{pmatrix}1&0\\0&1\end{pmatrix} \quad Q_1 = \begin{pmatrix}0&i\\0&0\end{pmatrix} \quad Q_2 = \begin{pmatrix}0&1\\i&0\end{pmatrix}$$

→ Let $|\psi\rangle$ a vector annihilated by β

($\beta = Q_+$, $\gamma = iQ_+$)

$\hookleftarrow$ ~~easy~~ ladder op.

$Q_+|\psi\rangle$ ∈ zero or non zero and has opp

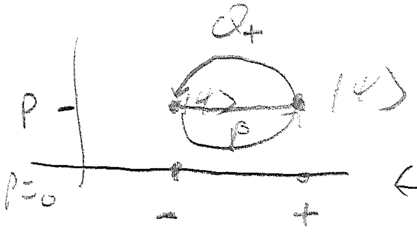
~~then Π ← parity $= \alpha + \beta = \gamma^2 = \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}$~~

⊘ parity of $|\psi\rangle$ $\left(Q_+ \in g_- \right)$

$$\underset{\overset{\|}{0}}{\gamma^2}|\psi\rangle = Q_+^2|\psi\rangle = 0 \quad ✓$$

$$P = \begin{pmatrix}0&0\\0&1\end{pmatrix} + \begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$\beta\gamma|\psi\rangle = Q_-Q_+|\psi\rangle = \{\beta, \gamma\}|\psi\rangle = 2P|\psi\rangle$$

→ 

← 2 reps with $P \neq 0$ (distinguished only by parity)

← 1 rep $|\psi\rangle = 0$

only no fermion

Generators of $sl(2) = \langle P, Q_1, Q_2 \rangle$

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}$$

→ Ladder op:

$$Q_\pm = \frac{Q_1 \pm i Q_2}{2} = \frac{Q_1 \pm}{2}$$

~~$Q_\pm = \begin{pmatrix} 0 & \pm i \\ -1 \mp i & 0 \end{pmatrix} = (\, 0 \; i$~~

~~$Q_\pm \Rightarrow$~~ $Q_\pm = \frac{1}{2}\begin{pmatrix} 0 & 1 \pm 1 \\ 1 \mp 1 & 0 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ -i \end{pmatrix}\right.$

$\rightarrow Q_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$Q_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$Q_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$Q_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \pm i \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \pm \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}$ ✓ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 0 & 1 \mp 1 \\ i \pm i & 0 \end{pmatrix} \rightarrow$ 4 or $\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}$

$\gamma = 2\alpha - \beta$

$\overline{M}^t M = 1$ unitarity

$\|$

$\overline{(1+\varepsilon A)}^t (1+\varepsilon A) = 1$

$\|$

$(1+\varepsilon\overline{A}^t) = (1-\varepsilon A) \rightsquigarrow \overline{A}^t = -A$.

$\hookrightarrow \overline{M}^t \begin{pmatrix}1&0\\0&i\end{pmatrix} M = \begin{pmatrix}1&0\\0&i\end{pmatrix}$ kupmity.

$\|$

$\cancel{(1+\varepsilon A)}$

$M = 1+\varepsilon A$

$\overline{(1+\varepsilon A)}^t \begin{pmatrix}1&0\\0&i\end{pmatrix} \overset{(1+\varepsilon A)}{\cancel{(1+\varepsilon A)}}$ $\overline{(1+\varepsilon A)}^t \begin{pmatrix}1&0\\0&i\end{pmatrix}(1+\varepsilon A)$

$= 1$

$(1+\varepsilon\overline{A})^t (1+\varepsilon A) = 1$

$1 = 1 + \varepsilon A + \varepsilon\overline{A}^t + \varepsilon^2 \overset{0}{\cancel{\overline{A}^t A}}$

$\Rightarrow A = -\overline{A}^t$ ✓

$$\overline{(1+\varepsilon A)}^t \begin{pmatrix}1&0\\0&i\end{pmatrix}(1+\varepsilon A) = \begin{pmatrix}1&0\\0&i\end{pmatrix}$$

$$\overline{(1+\varepsilon A)}^t \left(\begin{pmatrix}1&0\\0&i\end{pmatrix} + \varepsilon\begin{pmatrix}a&b\\ic&id\end{pmatrix}\right) = \begin{pmatrix}1&0\\0&i\end{pmatrix}$$

$$\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + \varepsilon\begin{pmatrix}\bar{a}&\bar{c}\\\bar{b}&\bar{d}\end{pmatrix}\right)\left(\begin{pmatrix}1&0\\0&i\end{pmatrix} + (\varepsilon)\begin{pmatrix}a&bi\\ic&id\end{pmatrix}\right) = \begin{pmatrix}1&0\\0&i\end{pmatrix}$$

~~$\begin{pmatrix}1&0\\0&i\end{pmatrix} + \varepsilon\begin{pmatrix}a&b\\ic&id\end{pmatrix} + \varepsilon\begin{pmatrix}\bar{a}&\bar{c}\\i\bar{b}&i\bar{d}\end{pmatrix}$~~

$$\cancel{\begin{pmatrix}1&0\\0&i\end{pmatrix}} + \varepsilon\begin{pmatrix}a&ib\\ic&id\end{pmatrix} + \varepsilon\begin{pmatrix}\bar{a}&\bar{c}\\i\bar{b}&i\bar{d}\end{pmatrix} + \cancel{\varepsilon^2}\,0 = \cancel{(1}$$

$$\rightsquigarrow \begin{pmatrix}a&bi\\ic&id\end{pmatrix} = -\begin{pmatrix}\bar{a}&\bar{c}\\i\bar{b}&i\bar{d}\end{pmatrix}$$

$$\rightsquigarrow \left.\begin{array}{l} a = -\bar{a} \\ d = -\bar{d} \\ bi = -\bar{c} \end{array}\right\} \text{~~same as su(2)~~?}$$

$b = i\bar{c}$

$\rightsquigarrow$ tracels $a = d$

$\rightsquigarrow \begin{pmatrix}i\alpha & \beta+i\gamma\\ \gamma+i\beta & i\alpha\end{pmatrix}$

$b = \gamma + i\beta$ ~~$c = -\bar{b}$~~

$b = i(\gamma - i\beta)$

$= \beta + \gamma$

$$\rightsquigarrow su(1,1) = \left\langle \begin{pmatrix}i&0\\0&i\end{pmatrix}, \begin{pmatrix}0&1\\i&0\end{pmatrix}, \begin{pmatrix}0&\\i&\end{pmatrix}\right.$$

$$\rightsquigarrow \bar{a}\, b = \bar{c}\, d$$

$$\shortparallel$$

$$e^{-i\alpha} \cancel{\cos\theta} (\pm \cancel{\sin\theta}) e^{i\gamma} = \cancel{\sin\theta}\, e^{-i\beta} e^{i\alpha} \cancel{\cos\theta}$$

$$\pm e^{i(\gamma-\alpha)} = e^{i(\alpha-\beta)}$$

$$\pm e^{i\gamma} = e^{i(2\alpha-\beta)}$$

$$\overline{\begin{pmatrix} a & b \\ c & -a \end{pmatrix}}^{t} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{b} & -\bar{a} \end{pmatrix}$$

$$\shortparallel \qquad =$$

$$A^{t} = -A$$

Just this

$$\rightsquigarrow$$

$$M = \begin{pmatrix} e^{i\alpha}\cos\theta & \cancel{\pm\sin\theta\, e^{i\gamma}}\; e^{i(2\alpha-\beta)}\sin\theta \\ e^{i\gamma}\sin\theta & -e^{i\alpha}\cos\theta \end{pmatrix}$$

$$\bullet\ \bar{A}^{t} = -A \qquad \circ\ tr(A) = 0 \rightsquigarrow \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

✓

$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & -\bar{a} \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & a \end{pmatrix} \rightsquigarrow$$

$$\bar{a} = -a \rightsquigarrow a = i\gamma$$

$$\bar{c} = -b,\ \cancel{b},\ b = i\gamma + \delta$$

$$\begin{pmatrix} i\alpha & \cancel{\gamma} + i\delta \\ -\cancel{\gamma} + i\delta & -i\alpha \end{pmatrix}$$

$$\rightsquigarrow \rightsquigarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Datum of a super Poinc algebra:

- A quadratic space $(V_{\mathbb{C}}, g)$ ← complexified spacetime

where $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, g is the spacetime metric

- A Lie algebra $\mathfrak{so}((V_{\mathbb{C}}, g))$

(Ex: Euc = $\mathfrak{so}$ is a Euclidean metric)

- A polarization $V_{\mathbb{C}} = \begin{cases} L \oplus L^\vee & \leftarrow \text{even dim} \\ L \oplus L^\vee \oplus \mathbb{C} & \leftarrow \text{odd dim} \end{cases}$

determining the space of Dirac spinors

$S = \wedge^\bullet L^\vee \; (\otimes W \leftarrow)$ in case of extended susy.

← space of Dirac spinors

In case of extended susy, or $\begin{cases} (W, \omega) \text{ symplectic} \\ (W, \delta) \text{ Euclidean} \end{cases}$

As a vector space $\mathfrak{susy}(V_{\mathbb{C}}, g) = V_{\mathbb{C}} \oplus \mathfrak{so}(V_{\mathbb{C}}) \oplus S$

$V_{\mathbb{C}} \times V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$

$(v, v) \longrightarrow 0$ "bracket of translations commute"

$V_{\mathbb{C}} \times S \longrightarrow S$

$(v, \psi) \longrightarrow 0$ " spinor should be translation inv."

$V_{\mathbb{C}} \times \mathfrak{so}(g) \to V_{\mathbb{C}}$ "acts in the standard rep."

$(v, a) \to \rho(a)(v)$ $\rho : \mathfrak{so}(g) \longrightarrow \operatorname{End}(V_{\mathbb{C}})$ standard rep.

$\mathfrak{a}ut(g) \times \mathfrak{a}ut(g) \to \mathfrak{a}ut(g)$

$(a, b) \longrightarrow [a, b]$

$S \times \mathfrak{a}ut(g) \to S$ "spinor rep"

$(\psi, a) \to \tilde{\rho}(a)(\psi)$ $\tilde{\rho} : \mathfrak{a}ut(g) \to End(S)$

$\mathfrak{SPIN}(g) \simeq \{ e_i e_j \in CL(V_g, g) \mid \{e_i\}$ ONB of $V_0 \}$

Lie algebra of $Spin(g)$

"in L acts as contraction and in $L^\vee$ via wedge product"

⇝ $\tilde{\rho}$ acts via Clifford mult.

$S \otimes S \xrightarrow{\Gamma} V_0$ is defined by

$$g(\Gamma(\varphi, \psi), v) = \langle \varphi, v\psi \rangle$$

$\varphi, \psi \in S = \Lambda L^\vee$

$$= \int dvol \, \pi(\varphi^t) v \cdot \psi$$

Ex: $e_i \in L$, $e^j \in L^\vee$ ⇝ $\tilde{\rho}(e_i e^j)\psi = e_i \lrcorner e^j \wedge \psi$

$\Gamma : S \otimes S \to V$

⇝ $S \to V \otimes S^\vee$

$V^\vee \otimes S \to S^\vee$

$\gamma : V \to CL(V)$

$\{e_i\}$ basis of V

ρ rep of $CL(V)$, $\rho : CL(V) \to E$

$\rho(\gamma(e_i)) : S \to S$

$\downarrow$

$\rho(\gamma(-)) : V \otimes S \to S$

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = -2\,\sigma^{\mu}_{\alpha\dot\beta} P_\mu$$

$$\downarrow$$

$$\eta_{\mu\nu}(\sigma^{\mu\,\alpha\dot\beta} \quad P^{\nu})$$

$$\gamma^{\mu}(\Gamma(\varphi\,\psi), \partial)$$

$$[M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta$$

$$\{Q_\alpha \rightsquigarrow -\frac{1}{2} \quad \rightsquigarrow \frac{1}{2}$$

$$Q_\alpha \to -\frac{1}{2}$$

$$\Rightarrow \quad \bar{D}_\pm = -\frac{\partial}{\partial\bar\theta^\pm} + i\,\theta^\pm \frac{\partial}{\partial x^\pm}$$

$$\bar{D}_\pm X = 0$$

$$\Rightarrow \quad X = X(\theta^\pm, \underbrace{x^\pm - i\theta^\pm\bar\theta^\pm}_{y^\pm =}) \quad , \quad x^\pm = x^0 \pm x^1$$

Since $\bar{D}_\pm \theta^\pm = \bar{D}_\pm y^\pm = 0$

$$X = \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm)$$

$$= \phi(x^\pm) - i\theta^+\bar\theta^-\partial_+\phi(x^\pm) - i\theta^-\bar\theta^-\partial_-\phi(x^\pm) + \theta^+\theta^-\bar\theta^+\bar\theta^-\partial\,\phi(x^\pm)$$

$$+ \theta^+\psi_+(x^\pm) - \theta^+\theta^-\bar\theta^-\partial_-\psi_+(x^\pm)$$

$$+ \theta^-\psi_-(x^\pm) + i\theta^-\theta^+\bar\theta^+\partial_+\psi_-(x^\pm)$$

$$+ \theta^+\theta^- F(x^\pm)$$

$$V = \theta^- \bar{\theta}^- \cdot (v_0 - v_1) + \theta^+ \bar{\theta}^+ (v_0 + v_1)$$
$$- \theta^- \bar{\theta}^+ \sigma - \theta^+ \bar{\theta}^- \bar{\sigma} + i \theta^- \theta^+ (\bar{\theta}^- \bar{\lambda}_-$$
$$+ \bar{\theta}^+ \bar{\lambda}_+) + i \bar{\theta}^+ \bar{\theta}^- (\theta^- \lambda_- + \theta^+ \lambda_+)$$
$$+ \theta^- \theta^+ \bar{\theta}^+ \bar{\theta}^- D$$

(similar to HW 9)

(3) supergauge transf: $V \to V - i(A - \bar{A})$

For sure chiral A, $\Sigma := \bar{D}_+ D_- V$

$$\bar{D}_+ D_- (A - \bar{A}) = \bar{D}_+ D_- A = \{\bar{D}_+, D_-\} A$$
$$= 0$$

Σ is not chiral, but twisted chiral i.e.
$$\bar{D}_+ \Sigma = D_- \Sigma = 0$$

If Σ where chiral it would be const. since it is real (HW 7 ex. (h))

$$\rightsquigarrow \Sigma = \sigma(y) + i \theta^+ \bar{\lambda}_+(y) - i \bar{\theta}^- \lambda_-(y)$$
$$+ \theta^+ \bar{\theta}^- (D(y) - i v_{01}(y)),$$ (similar to HW 9)

where $v_{01} = \partial_0 v_1 - \partial_1 v_0$

(n)

$$\mathcal{L} = \int d^4\theta \left(\sum_i \bar{X}_i e^{q_i V} X_i - \frac{1}{2} \overset{\ni \frac{1}{2}D^2}{\bar{\Sigma}\Sigma} \right.$$

$$+ \int d^2\theta\, W(X_i) - \frac{t}{2} \int d^2\theta \underset{\overset{\cup}{tD}}{\Sigma} + h.c$$

$t = r + i\vartheta$

$$\leadsto D\, q\, |\phi_i|^2 - \frac{1}{2} D^2 - tD \subset \mathcal{L}$$

EOM: $q_i |\phi_i|^2 - D - t = 0 \Rightarrow D = q_i |\phi_i|^2 - t$

Substitute in $\mathcal{L} \Rightarrow$

$$\mathcal{L} \supset 2(q_i |\phi_i|^2 - t)^2$$

from (f)

$$X_i = \dots = -i\theta^+ \bar{\theta}^+ \partial_+ \phi(x^\pm) - i\theta^- \bar{\theta}^- \partial_- \phi(x^\mp)$$

$+ \dots$

$$\leadsto \int d^2\theta\, W(X_i) \supset \left|\frac{\partial W}{\partial \phi_i}\right|^2$$ contains only lowest comp fields i.e. scalars

$$U = \sum_{i=0}^{N} |q_i \sigma \phi_i|^2 + \frac{1}{2}\left(\sum_{i=0}^{N} q_i |\phi_i|^2 - r\right)^2$$

$$+ \sum_{i=0}^{N} \left(\left|\frac{\partial W}{\partial \phi_i}\right|^2\right.$$

i) r-term shifts D-field
θ-term topological term

$\rightsquigarrow$ $t = r + i\theta$

ii) $W = X_0\, G(X_1, \dots, X_N)$ assume $\frac{\partial G}{\partial X_i} = 0$

$$\begin{array}{lcccc} & X_0 & X_1 & \dots & X_N \\ U(1) & -N & 1 & & 1 \end{array}$$

only for $X_1, \dots, X_N = 0$

$$U = |\sigma|^2\left(\sum_{i=1}^{N} |\phi_i|^2 + |N\phi_0|^2\right) + \frac{e^2}{2}\left(\sum_{i=1}^{N} |\phi_i|^2 - N|\phi_0|^2 - r\right)^2 + \sum_{i=1}^{N} |\phi_0|^2 \left|\frac{\partial G}{\partial \phi_i}\right|^2 + |G(\phi_1, \dots, \phi_N)|^2$$

Consider $r \gg 0$:

~~$U = |\sigma|^2\left(\sum_{i=1}^{N} |\phi_i|^2 + |N\phi_0|^2\right)$~~

$\sum_{i=1}^{N} |\phi_i|^2 = r + N|\phi_0|^2$ $\rightsquigarrow$ at least one $|\phi_i|$ has to be non vanishing $\rightsquigarrow$ σ acquires mass and must be zero in vacuum.

Since $\frac{\partial G}{\partial X_i} = 0$ only must at $X_i = 0$ $\forall i$ $\rightsquigarrow$ $|\phi_0|$ has to be zero

$\Rightarrow$ $U = 0 \rightarrow \begin{cases} \sum^{N} |X_i|^2 = r \\ |G(X_i)|^2 = 0 \end{cases} \Rightarrow$

$(\sigma = 0,\ \phi_0 = 0)$

Vacuum manifold

$$\left\{(\phi_1,\dots,\phi_N)\in\mathbb{C}^N : \sum|\phi_i|^2=r,\ \sigma(\phi_i)=0\right\}\Big/U(1)$$

$$\simeq \mathbb{C}P^{N-1} = \mathbb{C}P^{N-1}\Big|_{\sigma=0}$$

$$\cong (\mathbb{C}^N\setminus\{0\})\Big/\mathbb{C}^*\Big|_{\sigma=0}$$

if $r \ll 0$

$|\phi_0|\neq 0 \Rightarrow |\sigma|=0,\ |\phi_i|=0$

in $\sum^N |\phi_i|^2 = r + N|\phi_0|^2 \rightsquigarrow$

vacuum manifold $|\phi_0| = \sqrt{\frac{|r|}{d}}$