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# The Unruh Effect and The Measuring Scheme in Relativistic Quantum Field Theory

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*Author:*

Rafael F. CÓRDOBA LÓPEZ

*Supervisor:*

Dr. Rer. Nat. Andrés REYES  
LEGA

Departamento de Física

Facultad de Ciencias

July 14, 2021

## Acknowledgments

*“Forty-two,” said Deep Thought, with  
infinite majesty and calm.*

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## Abstract

Lately, intense discussion regarding measurements in relativistic quantum mechanics has been developed. We present the Fewster-Verch measuring scheme that allows to address problems of locality and causality. An explicit example is given with a particular application to model an Unruh-deWitt type of detector, we compare this results with the traditional approach of the UdW detector, also given in the work. For the sake of completeness, a traditional, as well as an algebraic derivation of the Unruh effect is given. The topics treated in this thesis may attract people from different fields such as QMT, QFT and AQFT. We hope the reader find this work as an useful introduction for those topics and foremost, an interesting review of the Unruh effect and the Fewster and Verch measurement scheme in relativistic quantum mechanics.

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## Introduction

At the beginning of the XX century David Hilbert, Paul Dirac, John von Neumann, and Hermann Weyl among many others, worked on the mathematical foundations of quantum mechanics. The probabilistic nature of Quantum Mechanics (QM) made measurement theory a deeper theory because observables cannot be ascribed definite values. Instead, the pairing between states and observables gives rise to probability distributions which are subject to correlations that cannot be explained by classical probability theory. One of the problems of measurement theory was the proposal of a measurement scheme consistent with a physical experiment and, at the same time, mathematically rigorous to work with it.

In 1932 von Neumann, in his work "Mathematische Begründung der Quantenmechanik" [1] proposed one of the first mathematical schemes for measurements where a system and a probe are coupled quantum systems giving an "experimental setup". Quantum Measure Theory (QMT) is a very robust<sup>1</sup> theory with many applications on non-relativistic quantum systems [2, 3]. Nevertheless, measurement schemes on relativistic systems have not been a great source of attention. The need for the development of a relativistic theory of quantum measurements cannot be overemphasized, as important Quantum Field Theory (QFT) phenomena such as the Unruh effect [4] or the Hawking effect [5] are still primary sources of debate.

One may think that QMT naturally extends to quantum field theories, but it turns out that assumptions of the locality of fields must be addressed. Therefore, if not carefully worked, it may lead to a super-luminal transport of information, as in a measurement protocol proposed many years ago by Sorkin [6].

In chapter 2, Sorkin's protocol is introduced, which suggest that quantum measurement theory is inherently incompatible with relativistic causality. The natural framework of Algebraic Quantum Field Theory (AQFT), also called local quantum physics, allows us to address problems of locality and causality; hence, in chapter 3, some aspects of the AQFT formalism are introduced and developed in order to address the measurement problem.

In chapter 4 we introduce the approach to quantum measurements on quantum fields as developed by Christopher J. Fewster and Reiner Verch [7] call the Fewster-Verch (FV) scheme which is broad enough to allow a general framework of a general Quantum Field Theory on Curved Space-Time (QFT on CST) hence, reconciling (up to other considerations) QFT with QMT.

For the purpose of completeness, chapter 1 is devoted to derive the Unruh effect and the Unruh-deWitt detector outcome. This allows us to gain grasp of the theory that are lately applied (in analogy) to the algebraic approach of the UdW detector.

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<sup>1</sup>Even though the "measurement problem" in QM is far from being solved.

# 1 The Unruh Effect

The Unruh effect (Fulling–Davies–Unruh effect) is perhaps one of the most controversial results that Quantum Field Theory predicts [8, 9, 10, 4, 11]. The Unruh effect states that an observer in accelerated motion at constant acceleration  $a$  will perceive a thermal bath at temperature  $a/2\pi$  and therefore, a source of particles. Despite of the fact that the Unruh effect was first described by Fulling in 1973, Paul Davies in 1975 and W. G. Unruh in 1976 for Minkowski spacetime, many results have been derived in the same spirit. For example, the analog of the Unruh effect in Schwarzschild spacetime was first derived by Hartle and Hawking 1976 [5, 10, 12] in which vacua states (invariant under time translation in the Kruskal extension of Schwarzschild spacetime with mass  $M$ ) are thermal states of temperature  $1/8\pi M$ . This result, in turn, motivates the close relation between the Unruh effect and the Gibbon-Hawking effect (1977), in which a temperature can be associated to each solution of the Einstein field equations that contains a causal horizon<sup>2</sup>. Many other results related to the Unruh effect can be found in [8] and are discussed in great detail.

The theoretical prediction of the Unruh effect is not yet experimentally confirmed even though several concepts of detectors have been designed, for instance the Unruh-DeWitt detector is the most relevant. Nevertheless, the interpretation result of the effect that *different observers "measure" different things* could be a source of misconceptions<sup>3</sup>. Our main focus will not be to present how different results can be made<sup>4</sup>, instead, we formalize the study of how a "measure" must be performed according to Fewster and Verch. In order to understand the effect, we start with a standard derivation, following [12, 10, 8].

## 1.1 Quantum Field Theory on Curved Space Time (QFT on CST)

**Definition 1.1.1.** *A Lorentzian manifold  $M$  is a pseudo-Riemannian manifold equipped with a smooth Lorentzian metric  $g$  of signature  $(+, -, \dots, -)$  and a fixed time orientation.*

**Definition 1.1.2.** *A space-time is a smooth, Lorentzian, Hausdorff, paracompact manifold of dimension  $n$  with at most finitely many connected components.*

In order to construct the Penrose diagram of a given space-time  $M$  we introduce the conformal transformation.

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<sup>2</sup>These results where first observed in 1977 in which Gibbons and Hawking showed that acceptable deSitter-invariant vacuum state of the free scalar field in deSitter spacetime with Hubble constant  $H$  is a thermal state of temperature  $H/2\pi$  of the theory inside the cosmological horizon with the de Sitter boost generator fixing the horizon as the Hamiltonian.

<sup>3</sup>Crispino et al. [8] argues that in fact, different results are different measures of some detector. Thus, inconsistencies arise from the fact that there is no definition of what measurements in QFT should be.

<sup>4</sup>For a further discussion look [8] section III.A.5.



**Definition 1.1.3.** *Given a metric*

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad dx^\mu dx^\nu := \frac{1}{2}(dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu),$$

*a conformal transformation (of the metric) is the metric  $\bar{g}$  described as*

$$\bar{g}_{\mu\nu}(p) = \Omega^2(x) g_{\mu\nu}(x).$$

*We can choose  $\Omega$  such that all the distances on  $M$  have an upper bound and therefore, a diagram of the space-time can be drawn inside a compact region (say, on a A4 size sheet of paper).*

**Definition 1.1.4.** *Let  $M$  be a space-time with line element*

$$ds^2 = N(x)^2 dt^2 - G_{ab}(x) dx^a dx^b.$$

*We say that the function  $N$  is the lapse function whereas  $G$  the metric on the space-like hyper-surface with constant  $t$ . The space-time is static if the lapse function and the metric  $G$  are independent of  $t$ .*

**Example 1.1.1.** *Consider the 1+1-dimensional Minkowski space-time with metric given by the line element*

$$ds^2 = dt^2 - dz^2 = du dv, \tag{1.1}$$

*where  $u, v$  are the null coordinates given by  $u = t - x, v = t + x$ . Clearly the spacetime is static, furthermore, we can bound the metric*

$$g_{\mu\nu} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} (du \otimes dv + dv \otimes du).$$

*Meaning that we can rescale infinities " $\infty$ " into a finite number while preserving the causal structure. In this case, with the help of the transformations  $u' = 2 \arctan u, v' = 2 \arctan v$ . Thus, the line element in the new variables becomes*

$$ds^2 = \frac{1}{4} \sec^2(u'/2) \sec^2(v'/2) du' dv'.$$

*Hence, if we choose  $\Omega^2(x) = (\sec^2(u'/2) \sec^2(v'/2)/4)^{-1}$  then, the conformal transform give us a Minkowski-like metric*

$$\bar{g}_{\mu\nu} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} (du' \otimes dv' + dv' \otimes du'),$$

*however, in this case over a compact region. The resulting figure is shown in figure 1 and, the diagram is known as the Penrose diagram of Minkowski spacetime.*

**Remark 1.1.1.**

The (real) scalar free field  $\varphi$  is the field that satisfies the Klein-Gordon (K-G) equation. On curved space times, the K-G equation can be extended by noticing that the d'Alambertian is the

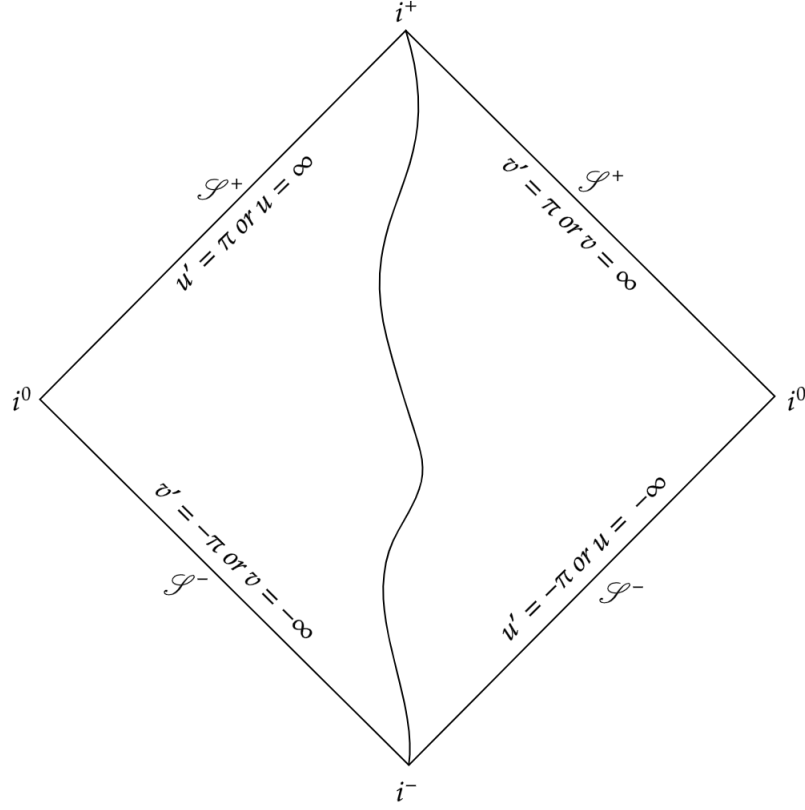


Figure 1: Penrose Diagram: Minkowski space-time:  $\mathcal{S}^\pm$  are the event horizons

Laplace-Beltrami operator on  $M$ ,

$$\square\varphi = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = g^{ij}\left(\frac{\partial^2\varphi}{\partial x^i\partial x^j} - \Gamma_{ij}^k\frac{\partial\varphi}{\partial x^k}\right) = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^j}\left(\sqrt{-g}g^{ij}\frac{\partial\varphi}{\partial x^i}\right).$$

Therefore, the Klein-Gordon equation in curved space times is

$$\left[\square + m^2 + \xi R(x)\right]\varphi(x) = 0, \quad \square\varphi = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi, \quad (1.2)$$

with a Lagrangian (Lagrangian density),

$$\mathcal{L}(x) = \frac{1}{2}[-g(x)]^{1/2} \left\{ g^{\mu\nu}(x)\varphi(x)_\mu\varphi(x)_\nu - [m^2 + \xi R(x)]\varphi^2(x) \right\}. \quad (1.3)$$

The case  $\xi = 0$  or  $\xi(n) = 1/4[(n-2)/(n-1)]$  are the minimally and conformally coupled case respectively. From now on, we work with the minimally coupled case,

$$\left[\square + m^2\right]\varphi(x) = 0, \quad \square\varphi = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi. \quad (1.4)$$

In order to exemplify the quantization, we introduce the notation

$$f \overset{\leftrightarrow}{\partial}_\mu g := f \left( \frac{\partial g}{\partial x^\mu} \right) - \left( \frac{\partial f}{\partial x^\mu} \right) g.$$

that allows us to define an "inner product"<sup>5</sup>. Indeed, let  $\Sigma \subset M$  be a space-like hypersurface and suppose  $\Sigma$  is also a Cauchy surface and let  $d\Sigma^\mu = n^\mu d\Sigma$  where  $n^\mu$  is a future-directed orthonormal vector. We define a linear form by

$$(\varphi_1, \varphi_2) = -i \int_\Sigma \varphi_1(x) \overset{\leftrightarrow}{\partial}_\mu \varphi_2^*(x) [-g_\Sigma(x)]^{\frac{1}{2}} d\Sigma^\mu \quad (1.5)$$

which is often called the inner product between fields  $\varphi_1$  and  $\varphi_2$  when restricted to  $\Sigma$ . With these assumptions one can show (via Gauss divergence theorem) that the scalar product is independent of  $\Sigma$  in the globally hyperbolic space-time  $M$  and thus, well-defined. Fundamental solutions  $\{u_\omega\}_{\omega \in J}$  of (1.4) satisfy

$$(u_\omega, u_{\omega'}) = \delta_{\omega, \omega'}, \quad (u_\omega^*, u_{\omega'}^*) = -\delta_{\omega, \omega'}, \quad (u_\omega, u_{\omega'}^*) = 0$$

and  $\{u_\omega\}_{\omega \in J}$  constitutes a complete orthonormal set of solutions with respect to the inner product. Thus, allowing us to write every solution of (1.4) as

$$\varphi(x) = \oint [a_\omega u_\omega(x) + a_\omega^\dagger u_\omega^*(x)] d\omega.$$

Canonical quantization is carried by adopting

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad (1.6)$$

the construction of the Fock space, vacuum state, etc. is carried in the same way as the scalar field. For a detailed description, we refer to [13].

**Example 1.1.2.** *Let  $M$  be a flat space-time, i.e. Minkowski space-time we consider the real scalar field. In this case, we can choose a Cauchy surface as a slice of  $M$  with  $t = \text{const.}$  i.e.  $\eta^\mu = (1, 0, 0, 0)$ . Hence, the inner product takes the usual form*

$$(\varphi_1, \varphi_2) = i \int d^3x \varphi_1(x) \overset{\leftrightarrow}{\partial}_0 \varphi_2^*(x),$$

where  $\varphi^* = \varphi$ . We have that fields  $\varphi(x)$  are described by

$$\varphi(x) = \int \frac{d^3p}{2E_p} [u_p(x) a(p) + u_p^*(x) a^\dagger(p)], \quad E_p = \sqrt{m^2 + p^2}.$$

The canonical conjugate momentum is  $\pi(x) := \partial^0 \varphi(x)$  and thus, the momentum  $\Pi(x) := \int d^3x \pi(x)$  and position  $\Phi(x) := \int d^3x \varphi(x)$  satisfy the commutation relations

$$\frac{i}{\hbar} [\Phi(x), \Pi(x')]_{x^0=x'^0} = \delta(x - x')$$

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<sup>5</sup>when restricted to an appropriate subspace

with the other commutation relations vanishing. Furthermore, we can find  $a$  and  $a^\dagger$  as

$$a(p) = (u_p^*, \varphi), \quad a^\dagger(p) = (u_p^*, \varphi)^*. \quad (1.7)$$

One can also see that  $a(p)$  is time-independent and the commutation relations

$$[a(p), a^\dagger(q)] = i \int d^3x u_p^*(x) \overleftrightarrow{\partial}_0 u_q(x) = 2E_p \delta(p - q),$$

$$[a(p), a(q)] = 0 = [a^\dagger(p), a^\dagger(q)].$$

holds. The vacuum state  $|0\rangle$  is realized as the state  $|0\rangle \neq 0$  such that

$$a_i |0\rangle = 0 \quad \text{for all } i.$$

Consequently, the annihilation and creation operators  $a$  and  $a^\dagger$  define the usual Fock space and thus, the number operator  $i$  is  $N_i = a_i^\dagger a_i$  with the total number operator given by

$$N = \sum_{i \in J} N_i.$$

The Hamiltonian is finally

$$H = \frac{1}{2} \int \frac{d^3p}{2E_p} E_p [a^\dagger(p) a(p) + a(p) a^\dagger(p)].$$

**Remark 1.1.2.** In this example we started with a dynamics given by the K-G equation, define the observables (the fields) and, a posteriori, we defined the appropriate Hilbert space in which  $a$  and  $a^\dagger$  acts. Then, and only then, we defined the Fock space. This way of construction will be the main focus of the theory (Algebraic Quantum Field Theory) presented in section 3.

**Remark 1.1.3.** In Minkowski space-time there is no ambiguity of how states  $a$  and  $a^\dagger$  are chosen because  $u_p(x)$  are uniquely determined by

$$u_p(x) = \frac{1}{(2\pi)^{3/2}} e^{-ip \cdot x}.$$

Indeed, a vacuum state is chosen so that admits invariance under the action of the Poncairé group. However, no analogous can be described to curved space-times.

The Stone-von Neumann theorem<sup>6</sup> implies that there is only one representation (up to unitary equivalent) of the CCR algebra e.g. the Schrödinger representation. In infinite dimensions inequivalent representations do exist. This fact is closely related to the non-uniqueness of a vacuum state for a free quantum field in a curved spacetime background.

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<sup>6</sup>If  $(V, \sigma)$  are finite dimensional, all irreducible representations of the Weyl algebra are unitarily equivalent

### 1.1.1 Vaccum states

Let  $\varphi(x)$  be a field satisfying the K-G equation on a curved space-time (possibly flat). Consider two different orthonormal complete set of modes  $\{\bar{u}_j(x)\}_{j \in I}$  and  $\{u_j(x)\}_{j \in I}$ . Then, a field  $\varphi(x)$  can be written as

$$\varphi(x) = \oint a_i u_i(x) + a_i^\dagger u_i^*(x) = \oint \bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x).$$

On the other hand, the vacuum states  $|0\rangle$  and  $|\bar{0}\rangle$  are defined as the states  $|\bar{0}\rangle \neq 0 \neq |0\rangle$  such that for all annihilator operators  $\bar{a}_j$  and  $a_j$  satisfies,

$$\bar{a}_j |\bar{0}\rangle = 0, \quad a_j |0\rangle = 0.$$

Naturally, we look how  $|0\rangle$  and  $|\bar{0}\rangle$  are related. Given two orthonormal basis  $\{u_j\}_{j \in I}$  and  $\{\bar{u}_j\}_{j \in I}$ , we can write

$$\bar{u}_j = \sum_{i \in I} (\alpha_{ji} u_i + \beta_{ji} u_i^*), \quad \bar{u}_j^* = \sum_{i \in I} (\beta_{ji}^* u_i + \alpha_{ji}^* u_i^*), \quad (1.8)$$

for some  $\alpha_{ij}, \beta_{kl} \in \mathbb{C}$  naturally, we want  $\alpha$  and  $\beta$  such that the annihilation and creation operators, associated to the two modes, follow the same CCRs or equivalently, that the transformations are linear. If  $I = 1$ , the transformation  $T$  from the modes  $u_i, u_i^*$  to the  $\bar{u}_j$  is

$$T = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha \end{pmatrix} \implies T^{-1} = \frac{1}{|\alpha|^2 - |\beta|^2} \begin{pmatrix} \alpha^* & -\beta \\ -\beta^* & \alpha \end{pmatrix}. \quad (1.9)$$

By analogy, for general  $I$  we can write  $u_i$  as

$$u_i = \sum_{j \in I} (\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*), \quad u_i^* = \sum_{j \in I} (-\beta_{ji}^* \bar{u}_j + \alpha_{ji} \bar{u}_j^*).$$

Hence, using the former equation and (1.8) we have the relations

$$\bar{u}_j = \sum_{i \in I} (\alpha_{ji} u_i + \beta_{ji} u_i^*), \quad u_i = \sum_{j \in I} (\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*). \quad (1.10)$$

Using (1.10) in itself,

$$\begin{aligned} \bar{u}_j &= \sum_{i \in I} \left( \alpha_{ji} \left[ \sum_{k \in I} (\alpha_{ki}^* \bar{u}_k - \beta_{ki} \bar{u}_k^*) \right] + \beta_{ji} \left[ \sum_{k \in I} (-\beta_{ki}^* \bar{u}_k + \alpha_{ki} \bar{u}_k^*) \right] \right) \\ &= \sum_{i, k \in I} (\alpha_{ji} \alpha_{ki}^* \bar{u}_k - \alpha_{ji} \beta_{ki} \bar{u}_k^* - \beta_{ji}^* \bar{u}_k \beta_{ki} + \beta_{ji} \alpha_{ki} \bar{u}_k^*). \end{aligned}$$

Comparing the coefficient's of  $\bar{u}_l$ , the conditions of  $\alpha$  and  $\beta$  are read to be

$$\sum_k (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) = \delta_{ij}, \quad \sum_k (\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}) = 0. \quad (1.11)$$

**Definition 1.1.5.** The transformations (1.10) from the  $\bar{u}_j(x)$  to the  $u_j(x)$  modes with  $\alpha_{ij}$  and  $\beta_{ij}$

satisfying condition (1.11) are called the Bogolyubov transformations whereas the coefficients  $\alpha_{ij}$  and  $\beta_{ij}$ ,  $i, j \in I$  are called the Bogolyubov coefficients.

Using equation (1.10) we can write  $\varphi$  as

$$\begin{aligned} \varphi(x) &= \sum_{j \in I} [a_j u_j + a_j^\dagger u_j^*] = \sum_{j \in I} [\bar{a}_j \bar{u}_j + \bar{a}_j^\dagger \bar{u}_j^*] \\ &= \sum_{j \in I} \left[ a_j \sum_{k \in I} (\alpha_{kj}^* \bar{u}_k - \beta_{kj} \bar{u}_k^*) + a_j^\dagger \sum_{k \in I} (-\beta_{kj} \bar{u}_k + \alpha_{kj} \bar{u}_k^*) \right] \\ &= \sum_{j \in I} \left[ \underbrace{\sum_{k \in I} (a_j \alpha_{kj}^* - a_j^\dagger \beta_{kj})}_{\bar{a}_j} \bar{u}_k + \underbrace{\sum_{k \in I} (a_j^\dagger \alpha_{kj} - a_j \beta_{kj})}_{\bar{a}_j^\dagger} \bar{u}_k^* \right]. \end{aligned}$$

thus, we have

$$\bar{a}_k = \sum_{j \in I} (a_j \alpha_{kj}^* - a_j^\dagger \beta_{kj}) \quad \bar{a}_k^\dagger = \sum_{j \in I} (a_j^\dagger \alpha_{kj} - a_j \beta_{kj}). \quad (1.12)$$

We are now in position to look how the vacua  $|\bar{0}\rangle$  and  $|0\rangle$  are related. Notice that

$$\bar{a}_j |0\rangle = \sum_{k \in I} (a_j \alpha_{kj}^* - a_j^\dagger \beta_{kj}) |0\rangle = - \sum_{j \in I} \beta_{kj} a_j^\dagger |0\rangle = - \sum_{j \in I} \beta_{kj} |1_j\rangle \neq 0,$$

likewise for  $|\bar{0}\rangle$  we have

$$a_i |\bar{0}\rangle = \sum_{j \in I} \beta_{ji}^* |\bar{1}_j\rangle, \quad \bar{a}_j |0\rangle = - \sum_{j \in I} \beta_{kj} |1_j\rangle. \quad (1.13)$$

Hence, the expected value of the particle number operator are

$$\langle \bar{0} | N_i | \bar{0} \rangle = \sum_{j \in I} |\beta_{ij}|^2, \quad \langle 0 | \bar{N}_i | 0 \rangle = \sum_{j \in I} |\beta_{ij}|^2.$$

Therefore, if  $\beta_{kj} \neq 0$  for all  $k, j \in I$  the modes  $u_j(x)$  define a different vacuum state from the one defined by the modes  $\bar{u}_j(x)$ .

**Remark 1.1.4.** If there exist  $i, j \in I$  such that  $\beta_{ij} \neq 0$  then, the vacuum  $|\bar{0}\rangle$  contains particles in the  $u_j$  mode and vice-versa. Furthermore, given a set of modes  $\{u_j\}_{j \in I}$  and if the modes  $\{\bar{u}\}_{j \in I}$  are defined in the positive frequencies, that is

$$\bar{u}_j = \sum_{i \in I} \alpha_{ji} u_i \quad (1.14)$$

then, the vacuum  $|0\rangle$  and  $|\bar{0}\rangle$  coincide. Notice that equation (1.13) tell us that  $\beta_{kj} = 0$  for all  $k, j \in I$  is a sufficient condition for both vacuums  $|\bar{0}\rangle$  and  $|0\rangle$  to be the same. A necessary condition for the vacua to be different is, as we said earlier, the existence of inequivalent representations of the CCR algebra.

### 1.1.2 The Rindler wedges

In order to discuss the Unruh effect, we pay special attention to accelerated observers. The equivalence principle of general relativity allows us to take a uniformly accelerating observer as an inertial observer under a gravitational field. Thus, allowing the study of an accelerated observer into a suitably curved space-time. It turns out that a uniformly accelerated observer on the Minkowski space-time can be seen as an observer called the Rindler space-time; in this section, we introduce the concepts needed. Consider the Minkowski space-time ( $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ ). The Killing

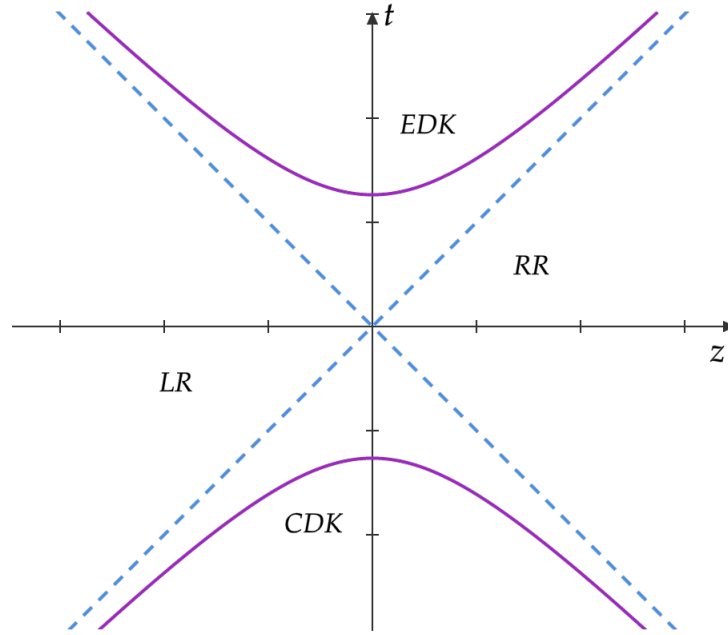


Figure 2: Rindler wedges right and left (RR and LL respectively) and the Kasner degenerate universe (EDK and CDK).

vector  $t\partial_z + z\partial_t$  defines the integral curves,

$$\begin{cases} t = \frac{\partial z(\beta)}{\partial \beta} \\ z = \frac{\partial t(\beta)}{\partial \beta} \end{cases}$$

with

$$z = \rho \cosh \beta, \quad t = \rho \sinh \beta.$$

Therefore, the metric is invariant under transformations of the form

$$\begin{cases} t \rightarrow t \cosh \beta + z \sinh \beta \\ z \rightarrow t \sinh \beta + z \cosh \beta \end{cases}.$$

We can take coordinates along the integral curves, i.e,  $t = \rho \sinh \eta, z = \rho \cosh \eta$  then, the Killing vector in the new coordinates  $(\eta, \rho, x, y)$  will be

$$\begin{aligned} z\partial_t + t\partial_z &= \rho \cosh \eta \partial_t + \rho \sinh \eta \partial_z = \rho \cosh \eta ((\sinh \eta)^{-1} \partial_\rho + (\rho \cosh \eta)^{-1} \partial_\eta) + \\ &\quad \rho \sinh \eta ((\cosh \eta)^{-1} \partial_\rho + (\rho \sinh \eta)^{-1} \partial_\eta) = \rho (\coth \eta + \tanh \eta) (\partial_\rho) + \partial_\eta \\ &= \partial_\eta. \end{aligned}$$

Consequently, world lines with fixed values  $\rho, x, y$  are trajectories (integral curves) of the boost transformations.

**Remark 1.1.5.** Notice that the lines  $t+z=0$  and  $t-z=0$  naturally "separate" the space-time in four regions as shown in figure 2. Observe that  $z^2 - t^2 = \rho^2 \cosh^2 \beta - \rho^2 \sinh^2 \beta = \rho^2 > 0$ , hence, coordinates  $(\eta, \rho, x, y)$  cover the region  $z^2 > t^2$ . This region is known as the Rindler wedges, composed with the Right Rindler wedge (RR) for  $z > |t|$  and the Left Rindler wedge (LR) for  $-z > |t|$ . In other words, the coordinates  $(\eta, \rho, x, y)$  cover the right Rindler and left Rindler wedges. The region  $z^2 < t^2$  is known as the Kasner degenerate space-time (contracting and expanding). The Killing  $\partial_\eta$  is time-like on RR and the LR wedges, space-like in the Kasner degenerate space-time, and null on  $t = \pm z$ . Hence, the lines  $t+z=0$  and  $t-z=0$  indeed separate the space-time into four causally disjoint regions.

The line element (of the RR and LR wedges) is then

$$\begin{aligned} ds^2 &= dt^2 - dx^2 - dy^2 - dz^2 \\ &= \rho(\cosh^2 \beta - \sinh^2 \eta) d\eta^2 \\ &\quad + (\sinh^2 \beta - \cosh^2 \eta) d\rho^2 - dx^2 - dy^2 \\ &= \rho^2 d\eta^2 - d\rho^2 - dx^2 - dy^2. \end{aligned}$$

If we consider only the right (or left) Rindler wedge, we can make a further coordinate transformation,  $\eta = a\tau$ ,  $a > 0$  and  $\rho = a^{-1}e^{a\xi}$ . The overall transformation is

$$t = \rho \sinh \eta = \underbrace{a^{-1}e^{a\xi}}_{\rho} \underbrace{\sinh a\tau}_{\eta}, \quad z = \underbrace{a^{-1}e^{a\xi}}_{\rho} \underbrace{\cosh a\tau}_{\eta}, \quad a > 0.$$

Finally, since  $d\eta^2 = a^2 d\tau^2$ ,  $d\rho^2 = e^{2a\xi} d\xi^2$  we have the line element for the right Rindler wedge

$$ds^2 = \rho^2 d\eta^2 - d\rho^2 - dx^2 - dy^2 = a^{-2} e^{2a\xi} (a^2 d\tau^2) - a^{-1} e^{2a\xi} d\xi^2 - dx^2 - dy^2 \quad (1.15)$$

$$= e^{2a\xi} (d\tau^2 - d\xi^2) - dx^2 - dy^2. \quad (1.16)$$

**Definition 1.1.6.** Let  $M$  be the Minkowski manifold  $(\mathbb{R}^{1,3})$ .



The Right Rindler wedge (RR) is the manifold  $M$  restricted to  $z > |t|$  together with the line element

$$ds^2 = e^{2a\xi}(d\tau^2 - d\xi^2) - dx^2 - dy^2, \quad (1.17)$$

where

$$t = a^{-1}e^{a\bar{\xi}} \sinh a\tau, \quad z = a^{-1}e^{a\bar{\xi}} \cosh a\tau.$$

The Left Rindler wedge (LR) is the manifold  $M$  restricted to  $-z > |t|$  together with the line element

$$ds^2 = e^{2a\bar{\xi}}(d\bar{\tau}^2 - d\bar{\xi}^2) - dx^2 - dy^2, \quad (1.18)$$

where

$$t = a^{-1}e^{a\bar{\xi}} \sinh a\bar{\tau}, \quad z = -a^{-1}e^{a\bar{\xi}} \cosh a\bar{\tau}.$$

We say that a Rindler observer is one whose worldline is parametrized with the Rindler coordinates  $(\eta, \rho, x, y)$ .

**Example 1.1.3.** Consider a trajectory of a Rindler observer on the right Rindler wedge,

$$x^\mu = (a^{-1}e^{-a\xi} \sinh a\tau, a^{-1}e^{-a\xi} \cosh a\tau, 0, 0), \quad \xi, x, y, a \text{ const.}$$

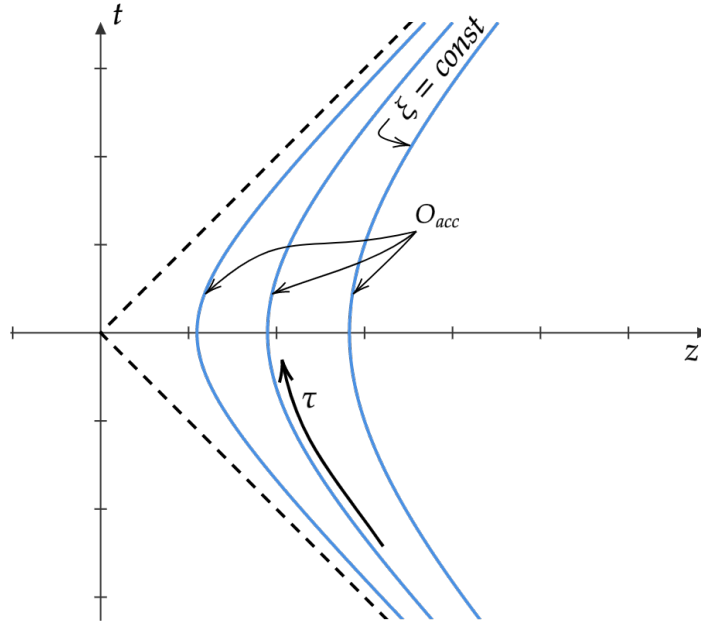


Figure 3: Right Rindler wedge with three different Rindler observers ( $O_{acc}$ ).

The four-velocity vector is

$$\begin{aligned} u^\mu &= \frac{dx^\mu}{d\tau} = (e^{-a\xi} \cosh a\tau, e^{-a\xi} \sinh a\tau, 0, 0) \\ \sqrt{u^\mu u_\mu} &= (g_{00} e^{-2a\xi} \cosh^2 a\tau + g_{11} e^{-2a\xi} \sinh^2 a\tau)^{1/2} \\ &= \sqrt{\cosh^2 a\tau - \sinh^2 a\tau} = 1 \end{aligned}$$

and the four-acceleration  $a^\mu = \nabla_\tau u^\mu = \frac{\partial u^\mu}{\partial \tau} + \Gamma_{\alpha\beta}^\mu u^\alpha u^\beta$ ,

$$\begin{aligned} a^0 &= e^{-a\xi} a \sinh a\tau + 0 \\ a^1 &= e^{-a\xi} a \cosh a\tau + 0 \\ a^i &= 0, \quad i = 1, 2. \\ \sqrt{-a^\mu a_\mu} &= \sqrt{e^{-2a\xi} e^{2a\xi} a^2 \cosh^2 a\tau - e^{-2a\xi} e^{2a\xi} a^2 \sinh^2 a\tau} = \sqrt{a^2} = a = \text{const} \end{aligned}$$

Thus, the proper acceleration of Rindler observers is  $a = \text{const}$  as shown in the figure 3. Therefore, a uniformly accelerated detector with proper acceleration  $a$  will lie on a trajectory with  $\xi = 0$  and  $x, y$  constant, making the Rindler spacetime the framework of the Unruh effect.

## 1.2 The Unruh effect (Part I: A (1+1)D - model)

Having in mind that different vacua states may exist, we will make an explicit computation of this phenomena in Rindler space-time which is essentially the Unruh effect. We present a standard derivation with some similarities to the one derived by William George Unruh (1976) in his work [14].

Since Minkowski and Rindler spacetime are static, let us first see what condition a mode satisfying the K-G equation must follow. Given a line element in a static space-time

$$ds^2 = N(x)^2 dt^2 - G_{ab}(x) dx^a dx^b,$$

notice that  $\sqrt{-g} = \sqrt{-N(x) \prod_i G_{ii}} = \sqrt{N} \sqrt{-G}$  and  $\square\varphi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi)$  thus,

$$\begin{aligned} \square\varphi &= \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi) = \frac{1}{\sqrt{N} \sqrt{G}} \partial_t (\sqrt{N} \sqrt{G} N(x) \partial_t \varphi) \\ &\quad + \frac{1}{\sqrt{N} \sqrt{G}} \partial_a (\sqrt{N} \sqrt{G} (-G^{ab}) g^{ab} \partial_b \varphi). \end{aligned}$$

Hence, the K-G equation (1.4) gives

$$\begin{aligned}
0 &= \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi) + m^2 \varphi \\
&= \frac{1}{\sqrt{-g}} \partial_t (\sqrt{N} \sqrt{G} N(x) \partial_t \varphi) \\
&\quad + \frac{1}{\sqrt{-g}} \partial_a (\sqrt{N} \sqrt{G} (-G^{ab}) g^{ab} \partial_b \varphi) + m^2 \varphi \\
&= \sqrt{N} \partial_t (\sqrt{N} \sqrt{G} N(x) \partial_t \varphi) \\
&\quad + \sqrt{N} \partial_a (\sqrt{N} \sqrt{G} (-G^{ab}) g^{ab} \partial_b \varphi) + N \sqrt{G} m^2 \varphi \\
&= \partial_t (N \sqrt{G} \partial_t \varphi) - \partial_a (N \sqrt{G} G^{ab} g^{ab} \partial_b \varphi) + N \sqrt{G} m^2 \varphi.
\end{aligned}$$

Therefore, a mode satisfying the K-G equation in a static space-time must follow

$$\partial_t (N \sqrt{G} \partial_t \varphi) - \partial_a (N \sqrt{G} G^{ab} g^{ab} \partial_b \varphi) + N \sqrt{G} m^2 \varphi = 0. \quad (1.19)$$

In order to address the Unruh effect, we consider an inertial observer in Minkowski space-time and an observer in accelerated motion. That is, we consider a field  $\varphi$  spanned by the annihilation and creation operators in Minkowski space-time and one spanned by the annihilation and creation operators in Rindler space-time.

In Minkowski spacetime, the K-G equation  $(\partial_t^2 - \partial_a \partial^a) \hat{\varphi} = 0$  give us solutions of the form

$$\hat{\varphi}(x) = \int \frac{dk}{\sqrt{4\pi k}} (\hat{a} e^{-ikz} + \hat{a}^\dagger e^{ikz}).$$

If we work only on the Rindler wedges, that is,  $z > |t|$  for the RR and  $-z > |t|$  on the LL then,

$$\hat{\varphi}(x) = \int \frac{dk}{\sqrt{4\pi k}} (\hat{b}_{-k} e^{-ik(t-z)} + \hat{b}_{+k} e^{ik(t+z)} + \hat{b}_{-k}^\dagger e^{-ik(t-z)} + \hat{b}_{+k}^\dagger e^{ik(t+z)})$$

with the usual commutator relations

$$[\hat{b}_{\pm k}, \hat{b}_{\pm k'}^\dagger] = \delta(k - k'), \quad [\hat{b}_{\pm k}, \hat{b}_{\pm k'}] = 0, \quad [\hat{b}_{\pm k}^\dagger, \hat{b}_{\pm k'}^\dagger] = 0, \quad (1.20)$$

with the other commutator relations vanishing because the LR and RR are causally disjoint. Let

$U = t - z$ ,  $V = t + z$  and denote the fields on the RR and the LL by  $\varphi_+$  or  $\varphi_-$  respectively then,

$$\begin{aligned}\hat{\varphi}(x) &= \int d^3k [\hat{b}_{+k} f_k(V) + \hat{b}_{+k}^\dagger f_k^*(V)] + \int d^3k [\hat{b}_{-k} f_k(U) + \hat{b}_{-k}^\dagger f_k^*(U)] \\ &= \underbrace{\int d^3k \frac{1}{(4\pi k)^{3/2}} [\hat{b}_{+k} e^{-ikV} + \hat{b}_{+k}^\dagger e^{ikV}]}_{\hat{\varphi}_+(V)} \\ &\quad + \underbrace{\int d^3k \frac{1}{(4\pi k)^{3/2}} [\hat{b}_{-k} e^{-ikU} + \hat{b}_{-k}^\dagger e^{ikU}]}_{\hat{\varphi}_-(U)} \\ &= \hat{\varphi}_+(V) + \hat{\varphi}_-(U).\end{aligned}$$

where

$$\hat{\varphi}_+(V) = \int dk [\hat{b}_{+k} f_k(V) + \hat{b}_{+k}^\dagger f_k^*(V)], \quad \hat{\varphi}_-(U) = \int dk [\hat{b}_{-k} f_k(U) + \hat{b}_{-k}^\dagger f_k^*(U)] \quad (1.21)$$

with

$$f_k(V) = \frac{1}{\sqrt{4\pi k}} e^{-ikV}. \quad (1.22)$$

Since the RR and LR are causally disjoint, the fields  $\varphi_+$  and  $\varphi_-$  are independent with each other, allowing us to work one at a time. Let us work with the left moving  $\varphi_+(V)$ , the other one is completely analog. As usual, the  $\hat{b}_{+k}$  defines the Minkowski vacuum i.e.  $|0_M\rangle \neq 0$  by  $\hat{b}_{+k}|0_M\rangle = 0$  for all  $k > 0$ .

On the other hand, in the LR wedge with coordinates  $(\tau, \xi, x, y)$ , equation (1.19) give us

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \xi^2} \right) \hat{\varphi} = 0$$

by Lorentz covariance. Let  $v = \tau + \xi$ ,  $u = \tau - \xi$ ,  $\bar{v} = \bar{\tau} - \bar{\xi}$  and  $\bar{u} = \bar{\tau} + \bar{\xi}$ , where  $(\tau, \xi)$  and  $(\bar{\tau}, \bar{\xi})$  are the RR and LR wedge coordinates defined earlier. Notice that the variables  $u, v$  and  $\bar{v}$  and  $U, V$  are related by

$$U = t - z = -a^{-1} e^{-au}, \quad V = t + z = a^{-1} e^{av}, \quad \bar{V} = -a^{-1} e^{-a\bar{v}}.$$

Again, we can separate the field  $\varphi(x)$  into the left and right moving fields, in this case,  $\hat{\varphi}_+(V)$  can be expressed in the LR wedge with the condition  $V < 0 < U$  and in the RR wedge with  $U < 0 < V$ .

Hence, we have

$$\begin{aligned}\hat{\varphi}(x) &= \underbrace{\int d\omega \frac{1}{(4\pi\omega)^{1/2}} [\hat{a}_{+\omega}^R e^{-i\omega v} + \hat{a}_{+\omega}^{R\dagger} e^{i\omega v}]}_{\hat{\varphi}_+(V)} \\ &\quad + \underbrace{\int d\omega \frac{1}{(4\pi\omega)^{1/2}} [\hat{a}_{-\omega}^L e^{-i\omega \bar{v}} + \hat{a}_{-\omega}^{L\dagger} e^{i\omega \bar{v}}]}_{\hat{\varphi}_-(U)} \\ &= \hat{\varphi}_+(V) + \hat{\varphi}_-(U),\end{aligned}$$

where

$$\hat{\phi}_+(V) = \int d\omega [\hat{a}_{+\omega}^R g_\omega(v) + \hat{a}_{+\omega}^{R\dagger} g_\omega^*(v)], \quad \hat{\phi}_-(U) = \int d\omega [\hat{a}_{-\omega}^L g_\omega(\bar{v}) + \hat{a}_{-\omega}^{L\dagger} g_\omega^*(\bar{v})] \quad (1.23)$$

with

$$g_\omega(v) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v}. \quad (1.24)$$

together with the usual commutator relations

$$[\hat{a}_{\pm\omega}, \hat{a}_{\pm\omega'}^\dagger] = \delta(\omega - \omega'), \quad [\hat{a}_{\pm\omega}, \hat{a}_{\pm\omega'}] = 0, \quad [\hat{a}_{\pm\omega}^\dagger, \hat{a}_{\pm\omega'}^\dagger] = 0. \quad (1.25)$$

We define the Rindler Vacuum  $|0_R\rangle$  by  $\hat{a}_{+\omega}^R |0_R\rangle = \hat{a}_{+\omega}^L |0_R\rangle = 0$  for all  $\omega$ .

In summary, we have two different sets of creation operators  $\{\hat{a}_{\pm\omega}\}_{\omega>0}, \{\hat{b}_{\pm k}\}_{k>0}$ , let us see the Bogolyubov coefficients. Let  $\alpha_{\omega k}^R, \beta_{\omega k}^R, \alpha_{\omega k}^L, \beta_{\omega k}^L \in \mathbb{C}$  be the Bogolyubov coefficients of the transformation from  $e^{-ikV}$  to  $e^{-i\omega v}$ . Then, we have the relations

$$\theta(V)g_\omega(v) = \int \frac{d^3k}{(4\pi k)^3} (\alpha_{\omega k}^R e^{-ikV} + \beta_{\omega k}^R e^{ikV}), \quad (1.26)$$

$$\theta(-V)g_\omega(\bar{v}) = \int \frac{d^3k}{(4\pi k)^3} (\alpha_{\omega k}^L e^{-ikV} + \beta_{\omega k}^L e^{ikV}) \quad (1.27)$$

where we use the fact that  $V > 0 > U$  lies on the RR wedge and  $V < 0 < U$  in the left and,  $\theta$  is the Heaviside step function. We use the orthonormal relations of the eigenfunctions  $e^{ikV}/2\pi$  in order to find the Bogolyubov coefficients,

$$\begin{aligned} \int \frac{dV}{2\pi} \theta(V) g_\omega(v) e^{ikV} &= \int \frac{dV}{2\pi} e^{ikV} \int \frac{d^3k}{(4\pi k)^3} (\alpha_{\omega k}^R e^{-ikV} + \beta_{\omega k}^R e^{ikV}) \\ &= \int \int \frac{d^3k}{(4\pi k)^3} \frac{dV}{2\pi} e^{ikV} (\alpha_{\omega k}^R e^{-ikV} + \beta_{\omega k}^R e^{ikV}) \\ &= \alpha_{\omega k}^R \frac{1}{(4\pi k)^{3/2}}. \end{aligned}$$

Hence, using the relation between  $v$  and  $V$ ,  $g_\omega(v) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v} = \frac{1}{\sqrt{4\pi\omega}} (aV)^{-i\omega/a}$  then,

$$\alpha_{\omega k}^R = \sqrt{\frac{k}{\omega}} \int_0^\infty \frac{dV}{2\pi} (aV)^{-i\omega/a} e^{ikV}. \quad (1.28)$$

Notice equation (1.28) has divergence while integrating on  $k$ . For this reason, we introduce the cut-off

$$\delta(x) = \int \frac{dx}{2\pi} e^{ikx - \varepsilon|x|} = \frac{1}{2\pi i} \left[ \frac{1}{k - i\varepsilon} + \frac{1}{k + i\varepsilon} \right].$$

Then, letting  $V \rightarrow V + i\epsilon$  and performing a Wick rotation,  $V = ix/k$  we have

$$\begin{aligned}\alpha_{\omega k}^R &= \frac{ie^{\pi\omega/2a}}{\sqrt{\omega k}} \left(\frac{a}{k}\right)^{-i\omega/a} \int_0^\infty \frac{dx}{2\pi} x^{-i\omega/a} e^{-x} dx \\ &= \frac{ie^{\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{-i\omega/a} \Gamma(1 - i\omega/a),\end{aligned}$$

which gives the first Bogolyubov coefficient. By replacing  $e^{ikV}$  for  $e^{-ikV}$  the coefficients  $\beta_{\omega k}^R$  are also given by

$$\begin{aligned}\beta_{\omega k}^R &= -\frac{ie^{-\pi\omega/2a}}{\sqrt{\omega k}} \left(\frac{a}{k}\right)^{-i\omega/a} \int_0^\infty \frac{dx}{2\pi} x^{-i\omega/a} e^{-x} dx \\ &= -\frac{ie^{-\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{-i\omega/a} \Gamma(1 - i\omega/a).\end{aligned}$$

Similarly, one can find the coefficients  $\alpha_{\omega k}^L$  and  $\beta_{\omega k}^L$ . Therefore, one can show that the Bogolyubov coefficients are given by

$$\alpha_{\omega k}^R = \frac{ie^{\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{-i\omega/a} \Gamma(1 - i\omega/a), \quad (1.29)$$

$$\beta_{\omega k}^R = -\frac{ie^{-\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{-i\omega/a} \Gamma(1 - i\omega/a), \quad (1.30)$$

$$\alpha_{\omega k}^L = -\frac{ie^{\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{i\omega/a} \Gamma(1 + i\omega/a), \quad (1.31)$$

$$\beta_{\omega k}^L = \frac{ie^{-\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{i\omega/a} \Gamma(1 + i\omega/a). \quad (1.32)$$

Furthermore, we immediately identify the relation

$$\beta_{\omega k}^L = -e^{-\pi\omega/a} \alpha_{\omega k}^{R*}, \quad \beta_{\omega k}^R = -e^{-\pi\omega/a} \alpha_{\omega k}^{L*}. \quad (1.33)$$

Now, define a new set of modes on the K-G equation,  $G_\omega$  and  $\bar{G}_\omega$  described by

$$G_\omega(V) := \theta(V)g_\omega(v) + \theta(-V)e^{-\pi\omega a}g_\omega^*(\bar{v}) \quad (1.34)$$

$$\bar{G}_\omega(V) := \theta(V)e^{-\pi\omega a}g_\omega^*(v) + \theta(-V)g_\omega(\bar{v}). \quad (1.35)$$

As we pointed out earlier, a positive frequency solution gives rise to the same vacuum as the one defined by  $g_\omega$  thus, we need only to find that  $G_\omega$  is indeed a positive frequency solution of the K-G equation. A positive frequency solution of the K-G equation is analytic in the lower half plane on the complex  $V$  plane. Thus, solution  $g_\omega(v) = (4\pi\omega)^{-1/2}(V)^{-i\omega/a}$ ,  $V > 0$  should be continued to the negative real line avoiding the singularity  $V = 0$  around a small circle in the lower half plane, leading

to  $(4\pi\omega)^{-1/2}e^{-\pi\omega/a}(-V)^{-i\omega/a}$ ,  $V < 0$ . Equation (1.34) is exactly this analytic continuation of  $g_\omega(V)$  and therefore,  $G_\omega$  must be solutions of positive frequency <sup>7</sup> and therefore, if the field is written with this basis, the operators associated with these frequencies define the same vacuum.

Let us see the associated operators. Using  $G_\omega$  and  $\bar{G}_\omega$ , equation (1.27) can be inverted as

$$\begin{aligned}\theta(V)g_\omega(v) &\propto G_\omega(V) - e^{-\pi\omega/a}\bar{G}_\omega^*(V), \\ \theta(-V)g_\omega(\bar{v}) &\propto \bar{G}_\omega(V) - e^{-\pi\omega/a}G_\omega^*(V).\end{aligned}$$

Next, if we replace back on  $\varphi$ ,

$$\begin{aligned}\hat{\varphi}_+(V) &= \int_0^\infty d\omega \theta(V)[\hat{a}_{+\omega}^R g_\omega(v) + \hat{a}_\omega^{R\dagger} g_\omega^*(v)] + \theta(-V)[\hat{a}_{+\omega}^L g_\omega(\bar{v}) + \hat{a}_\omega^{L\dagger} g_\omega^*(\bar{v})] \\ &= \int_0^\infty d\omega \hat{a}_{+\omega}^R (G_\omega(V) - e^{-\pi\omega/a}\bar{G}_\omega^*(V)) + \hat{a}_\omega^{R\dagger} (G_\omega^*(V) - e^{\pi\omega/a}\bar{G}_\omega(V)) + \\ &\quad \hat{a}_{+\omega}^L (\bar{G}_\omega(V) - e^{-\pi\omega/a}G_\omega^*(V)) + \hat{a}_\omega^{L\dagger} (\bar{G}_\omega^*(V) - e^{\pi\omega/a}G_\omega(V)) \\ &= \int_0^\infty d\omega G_\omega(V)(\hat{a}_{+\omega}^R - e^{\pi\omega/a}\hat{a}_\omega^{L\dagger}) + \bar{G}_\omega^*(V)(-e^{-\pi\omega/a}\hat{a}_{+\omega}^R + \hat{a}_\omega^{L\dagger}) \\ &\quad + \bar{G}_\omega(V)(-e^{\pi\omega/a}\hat{a}_\omega^{R\dagger} + \hat{a}_{+\omega}^L) + G_\omega^*(V)(-e^{-\pi\omega/a}\hat{a}_{+\omega}^L + \hat{a}_\omega^{R\dagger}).\end{aligned}$$

Since  $G_\omega(V)$  and  $\bar{G}_\omega(V)$  are positive frequencies, Minkowski vacuum is defined as the unique state  $|0_M\rangle$  such that for all  $\omega$ ,

$$(\hat{a}_{+\omega}^R - e^{\pi\omega/a}\hat{a}_\omega^{L\dagger})|0_M\rangle = 0, \quad (1.36)$$

$$(\hat{a}_{+\omega}^L - e^{\pi\omega/a}\hat{a}_\omega^{R\dagger})|0_M\rangle = 0. \quad (1.37)$$

Recall that

$$[\hat{a}_{+\omega}^{R/L}, \hat{a}_{+\omega'}^{R/L\dagger}] = \delta(\omega - \omega'), \quad [\hat{a}_{+\omega}^{R/L}, \hat{a}_{+\omega'}^{R/L}] = 0, \quad [\hat{a}_{+\omega}^{R/L\dagger}, \hat{a}_{+\omega'}^{R/L\dagger}] = 0.$$

Finally, we apply  $\hat{a}_{+\omega}^{R\dagger}$  and  $\hat{a}_{+\omega}^{L\dagger}$  into (1.36) and (1.37) respectively. Thus, the number of particles with frequency  $\omega$  on the RR and LR are

$$\langle 0_M | \hat{a}_{+\omega}^{R\dagger} \hat{a}_{+\omega}^R | 0_M \rangle = e^{-2\pi\omega/a} \langle 0_M | \hat{a}_{+\omega}^{L\dagger} \hat{a}_{+\omega}^L | 0_M \rangle + e^{-2\pi\omega/a}. \quad (1.38)$$

The symmetry of the  $R$  and  $L$  operators imply that the number of particles on the right and left wedges must be equal and therefore, using (1.38) the expected value is

$$\langle 0_M | \hat{a}_{+\omega}^{R\dagger} \hat{a}_{+\omega}^R | 0_M \rangle = \langle 0_M | \hat{a}_{+\omega}^{L\dagger} \hat{a}_{+\omega}^L | 0_M \rangle = \frac{1}{e^{2\pi\omega/a} - 1}. \quad (1.39)$$

Hence, the expectation value of the Rindler-particle number is that of a Bose-Einstein particle in a thermal bath of temperature  $T = a/2\pi$ . It can also be expressed without discretization by

$$\hat{a}_{+f}^{R/L} = \int_0^\infty d\omega f(\omega) \hat{a}_{+\omega}^{R/L}$$

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<sup>7</sup>This argument was the original proposed by Unruh [14]

with  $f(\omega)$  being the density of states ( $\int_0^\infty |f(\omega)|^2 = 1$ ). Then, the expected value of the number of particles at a frequency  $f$  is

$$\begin{aligned} \langle 0_M | \hat{a}_{+f}^{R/L\dagger} \hat{a}_{+f}^{R/L} | 0_M \rangle &= \int_0^\infty d\omega f(\omega) f^*(\omega) \langle 0_M | \hat{a}_{+\omega}^{R/L\dagger} \hat{a}_{+\omega}^{R/L} | 0_M \rangle \\ &= \int_0^\infty d\omega \frac{|f(\omega)|^2}{e^{2\pi\omega/a} - 1}. \end{aligned}$$

However, showing that the expectation values are those in a thermal state is not enough to conclude the nature of the state. A sufficient and necessary condition is that the probability of each, right and left Rindler energy eigenstate, corresponds to the grand canonical ensemble if the other Rindler wedge is disregarded. Equations (1.36) and (1.37) imply that  $(\hat{a}_\omega^{R\dagger} \hat{a}_\omega^R - \hat{a}_\omega^{L\dagger} \hat{a}_\omega^L) | 0_M \rangle = 0$  which confirms that number of particles in the right and left sides of Rindles wedges are the same. Allowing us to write

$$| 0_M \rangle = N \prod_i \sum_{n_i=0}^\infty \frac{K_{n_i}}{n_i!} (\hat{a}_{+\omega_i}^{R\dagger} \hat{a}_{+\omega_i}^{L\dagger})^{n_i} | 0_R \rangle, \quad K_{n_i+1} - e^{-\pi\omega_i/a} K_{n_i} = 0,$$

where  $N$  is the normalization constant. The recursive condition  $K_n$  give us  $K_{n_i} = e^{-\pi n_i \omega_i / a} K_0$  and therefore, the Minkowski vacuum can be written as

$$| 0_M \rangle = \prod_i C_i \sum_{n_i=0}^\infty e^{-\pi n_i \omega_i / a} | n_i, R \rangle \otimes | n_i, L \rangle, \quad C_i = \sqrt{1 - \exp(-2\pi\omega_i/a)}. \quad (1.40)$$

Then, the Minkowski vacuum has a density matrix (tracing out the left Rindler states)

$$\hat{\rho}_R = \prod_i \left( C_i^2 \sum_{n_i=0}^\infty e^{-2\pi n_i \omega_i / a} | n_i, R \rangle \langle n_i, R | \right) \quad (1.41)$$

which is exactly the density matrix of the system of free bosons with temperature  $T = a/2\pi$ . Similarly, the LR wedge give the same result. Thus, we can state the Unruh effect as follows:

#### The Unruh effect

The Unruh effect is the effect in which the Minkowski vacuum restricted to the left(right) Rindler wedge is a thermal state with temperature  $T = a/2\pi$  with the boost generator normalized on  $z^2 - t^2 = 1/a^2$  as the Hamiltonian.

### 1.3 The Unruh-DeWitt detector (UDW)

*“Measure what can be measured, and make measurable what cannot be measured.”*

Galileo Galilei

The Unruh-DeWitt (UDW) detector, introduced by DeWitt in 1979 is a detector that tells if the



system has detected a particle (in a thermal state) or not. That is, a two-level point monopole<sup>8</sup>. The Unruh thermal bath is as isotropic as a thermal bath in equilibrium on a general static space-time, which means that Killing observers will see no net energy flux in any spatial direction. However, it is essential to clarify that predictions of the thermal state do not present ambiguity on the measurement but rather the interpretations of the measuring is what historically has been the source of discussion [8]. Audretsch and Müller [15] in 1994 argue that results in principle contradictory are results of different measurements. Let us introduce the UDW detector formally.

We consider a two-level Unruh-DeWitt detector in a Minkowski space-time represented by the Hermitian operator  $\mu_0$  acting on a two dimensional Hilbert space  $\mathcal{H}_D$  whose basis elements are denoted as  $|0\rangle$  and  $|1\rangle$ . Notice that  $\mu_0|0\rangle = |1\rangle$ ,  $\mu_0|1\rangle = |0\rangle$ .

Consider the Hamiltonian of the detector as  $H_D|1\rangle = \frac{E}{2}|1\rangle$ ,  $H_D|0\rangle = -\frac{E}{2}|0\rangle$  thus, the energy gap between the two states is  $E$ . The Hilbert space of the whole system (the system and the probe) is  $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_D$  where  $\mathcal{H}_S$  is the Hilbert space of the field. Associated with the Hilbert space  $\mathcal{H}$  is the Hamiltonian of the system given by

$$H = \underbrace{H_S \otimes 1 + 1 \otimes H_D}_{H_0} + H_{\text{int}}, \quad H_{\text{int}} = \lambda \rho(\tau) \varphi(x(\tau)) \otimes \mu(\tau) \quad (1.42)$$

with  $\mu(\tau) = e^{iH_D\tau} \mu(0) e^{-iH_D\tau}$ , where  $\tau$  is the proper time of the detector,  $x(\tau)$  its world-line,  $\lambda$  the coupling constant and  $\rho$  a switch function compactly supported that controls the coupling region. Assuming the interaction picture then, the evolution operator  $U_{\text{int}}$  with the Dyson series is

$$\begin{aligned} U_{\text{int}} &= \mathcal{T} e^{-i \int d\tau H_{\text{int}}(\tau)} \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d\tau_1 \dots d\tau_n \mathcal{T}(H_{\text{int}}(\tau_1) \dots H_{\text{int}}(\tau_n)). \end{aligned}$$

In order to find the evolution operator, let us first take a look at the order-time  $\mathcal{T}\langle 1|\mu(\tau_1)\dots\mu(\tau_n)|0\rangle$  where  $\mu(\tau)$  is the evolution of the monopole given by the Schrödinger picture:

$$\mu(\tau) = e^{iH_D\tau} \mu(0) e^{-iH_D\tau}, \quad \mu(0) := \mu_0.$$

In this way, the order time is

$$\begin{aligned} \mathcal{T}\langle 1|\mu(\tau_1)\dots\mu(\tau_n)|0\rangle &= \mathcal{T}\langle 1|e^{iH_D\tau_1}\mu(0)e^{-iH_D\tau_1}\dots e^{iH_D\tau_n}\mu(0)e^{-iH_D\tau_n}|0\rangle \\ &= \mathcal{T}\langle 1|e^{iH_D\tau_1}\mu(0)\left(\prod_{i=1}^{n-1} e^{-iH_D(\tau_i-\tau_{i+1})}\mu(0)\right)e^{-iH_D\tau_n}|0\rangle \\ &= \mathcal{T}\langle 1|e^{i\frac{E}{2}\tau_1}\mu(0)\left(\prod_{i=1}^{n-1} e^{-iH_D(\tau_i-\tau_{i+1})}\mu(0)\right)e^{i\frac{E}{2}\tau_n}|0\rangle \\ &= \mathcal{T}e^{i\frac{E}{2}(\tau_1+\tau_n)}\langle 1|\mu(0)\left(\prod_{i=1}^{n-1} e^{-iH_D(\tau_i-\tau_{i+1})}\mu(0)\right)|0\rangle, \end{aligned}$$

---

<sup>8</sup>A monopole (in analogy of electrodynamics) describes the system as coarser as it could be (and finer when increasing the degree of the pole). For example,  $\mu = \sigma^- + \sigma^+$  where  $\sigma^\pm$  are ladder operators determines only two possible outcomes.

where  $\prod_{i=1}^n A_i = A_1 A_2 \dots A_n$ . Notice that for each  $\mu(0)$  there is a permutation interchanging the vectors  $|0\rangle$  with  $|1\rangle$ . Therefore, for  $n$  even, the contribution vanishes (as it give a  $\langle 0|1\rangle$  value) and for  $n$  odd we have

$$\begin{aligned}
&= \mathcal{T} e^{i\frac{E}{2}(\tau_1 + \tau_n)} \langle 1 | \mu(0) \left( \prod_{i=1}^{n-1} e^{-iH_D(\tau_i - \tau_{i+1})} \mu(0) \right) | 0 \rangle \\
&= \mathcal{T} e^{i\frac{E}{2}(\tau_1 + \tau_n)} \langle 0 | \left( \prod_{i=1}^{n-2} e^{-iH_D(\tau_i - \tau_{i+1})} \mu(0) \right) e^{-iH_D(\tau_{n-1} - \tau_n)} \mu(0) | 0 \rangle \\
&= \mathcal{T} e^{i\frac{E}{2}(\tau_1 + \tau_n)} \langle 0 | \left( \prod_{i=1}^{n-2} e^{-iH_D(\tau_i - \tau_{i+1})} \mu(0) \right) e^{-i\frac{E}{2}(\tau_{n-1} - \tau_n)} | 1 \rangle \\
&= \mathcal{T} e^{i\frac{E}{2}(\tau_1 - \tau_{n-1} + 2\tau_n)} \langle 0 | \left( \prod_{i=1}^{n-2} e^{-iH_D(\tau_i - \tau_{i+1})} \mu(0) \right) | 1 \rangle.
\end{aligned}$$

Inductively, we find

$$\mathcal{T} \langle 1 | \mu(\tau_1) \dots \mu(\tau_n) | 0 \rangle = \mathcal{T} e^{i\frac{E}{2}(2\tau_1 - 2\tau_2 + \dots - 2\tau_{n-1} + 2\tau_n)} \langle 1 | \mu_0 | 0 \rangle \quad (1.43)$$

$$= e^{iE(\tau_1 - \tau_2 + \dots - \tau_{n-1} + \tau_n)} \langle 1 | \mu_0 | 0 \rangle. \quad (1.44)$$

Now, we are in position to compute the expected value of detecting a particle given an arbitrary state  $\varphi$ . Write  $|0, 0_M\rangle =: |0\rangle \otimes |0_M\rangle$  then,

$$\begin{aligned}
\mathcal{A}_{0 \rightarrow 1} &= \langle 1, \varphi | U_{int} | 0, 0_M \rangle \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d\tau_1 \dots d\tau_n \left[ \lambda^n \rho(\tau_1) \dots \rho(\tau_n) \langle \varphi | \mathcal{T}(\varphi(x(\tau_1)) \dots \right. \\
&\quad \left. \times \varphi(x(\tau_n))) | 0_M \rangle \times \mathcal{T} \langle 1 | e^{-iH_D(\tau_1 - \tau_2)} \mu \dots e^{-iH_D(\tau_{n-1} - \tau_n)} | 0 \rangle \right] \\
&= \sum_{n \text{ odd}}^{\infty} \lambda^n \frac{(-i)^n}{n!} \int d\tau_1 \dots d\tau_n \rho(\tau_1) \dots \rho(\tau_n) \\
&\quad \times \langle \varphi | \mathcal{T}(\varphi(x(\tau_1)) \dots \varphi(x(\tau_n))) | 0_M \rangle e^{iE(\tau_1 - \tau_2 + \dots - \tau_{n-1} + \tau_n)} \langle 1 | \mu_0 | 0 \rangle.
\end{aligned}$$

Thus, the probability of detecting a state will be

$$\begin{aligned}
P_{0 \rightarrow 1} &= \int d[\varphi] |\langle 1, \varphi | U_{\text{int}} | 0, 0_M \rangle|^2 \\
&= |\langle 1 | \mu(0) | 0 \rangle|^2 \sum_{n, m \text{ odd}}^{\infty} \lambda^{n+m} \frac{(-i)^{n-m}}{n!m!} \int d\sigma_1 \dots d\sigma_m d\tau_1 \dots d\tau_n \\
&\quad \times \left[ \rho(\sigma_1) \dots \rho(\sigma_m) \rho(\tau_1) \dots \rho(\tau_n) \times \langle 0_M | \mathcal{T}(\varphi(x(\sigma_1)) \dots \varphi(x(\sigma_n)))^\dagger \right. \\
&\quad \times \underbrace{\int d\varphi |\varphi\rangle \langle \varphi| \mathcal{T}(\varphi(x(\tau_1)) \dots \varphi(x(\tau_n))) | 0_M \rangle}_1 \\
&\quad \left. \times e^{Ei(\tau_1 - \tau_2 \dots + \tau_n)} e^{-Ei(\sigma_1 - \sigma_2 \dots + \sigma_m)} \right] \\
&= |\langle 1 | \mu(0) | 0 \rangle|^2 \sum_{n, m \text{ odd}}^{\infty} \lambda^{n+m} \frac{(-i)^{n-m}}{n!m!} \int d\sigma_1 \dots d\sigma_m d\tau_1 \dots d\tau_n \\
&\quad \times \left[ \rho(\sigma_1) \dots \rho(\sigma_m) \rho(\tau_1) \dots \rho(\tau_n) \langle 0_M | \mathcal{T}(\varphi(x(\sigma_1)) \dots \varphi(x(\sigma_n)))^\dagger \right. \\
&\quad \times \mathcal{T}(\varphi(x(\tau_1)) \dots \varphi(x(\tau_n))) | 0_M \rangle e^{Ei(\tau_1 - \tau_2 \dots + \tau_n)} e^{-Ei(\sigma_1 - \sigma_2 \dots + \sigma_m)} \left. \right].
\end{aligned}$$

That is,

$$P_{0 \rightarrow 1} = \lambda^2 |\langle 1 | \mu(0) | 0 \rangle|^2 \int d\sigma d\tau \rho(\tau) \rho(\sigma) \underbrace{\langle 0_M | \varphi(x(\sigma)) \varphi(x(\tau)) | 0_M \rangle}_{\mathcal{W}(x(\sigma), x(\tau))} e^{Ei(\tau - \sigma)} + O(\lambda^4). \quad (1.45)$$

**Definition 1.3.1.** Let  $f : M \rightarrow \mathbb{C}$  be a smooth function supported on  $\mathcal{O} \subset M$ .  $\mathcal{O}$  is a bounded open region in the space-time and  $\varphi(x)$  a field.  $f$  is said to be a test function and the smeared field associated is

$$\varphi(f) := \int dx \varphi(x) f(x).$$

**Definition 1.3.2.** Notice that the correlation function of  $n$  smeared fields is a multilinear functional [16]  $(f_1, \dots, f_n) \rightarrow \langle 0_M | \varphi(f_1) \dots \varphi(f_n) | 0_M \rangle$ . Using the Schwarz kernel theorem, the multilinear functional has a kernel  $W$ , such that

$$\langle 0_M | \varphi(f_1) \dots \varphi(f_n) | 0_M \rangle = \int f(x_1) \dots f(x_n) W(x_1, \dots, x_n) dx_1 \dots dx_n.$$

$W$  is called the Wightman function of  $\varphi$ . Observe that the Wightman function is

$$W = \langle 0_M | \phi(x_1) \dots \phi(x_n) | 0_M \rangle.$$

**Remark 1.3.1.**  $\mathcal{W}$  in equation (1.45) is the pull-back of the Wightman function of  $\varphi$  which is exactly the final result on [4],

$$P_{0 \rightarrow 1} = \lambda^2 \int d\tau d\tau' e^{-iE(\tau - \tau')} \rho(\tau) \rho(\tau') \mathcal{W}(\gamma(\tau), \gamma(\tau')) + O(\lambda^4) \quad (1.46)$$

for some smooth function  $\tilde{\rho}$  compactly supported within a compact region  $K \subset M$ . This result will

connect the measurement developed in section 4 and the Unruh detector because in section 4, we work with smeared fields  $\phi(f)$  rather than point-like fields  $\phi(x)$ .

### 1.3.1 An aside: Uniformly accelerated detectors in Minkowski vacuum, inertial observer perspective.

Let us analyze the Unruh detector as a inertial observer perspective as suggested by Crispino et al. [8]. In Cartesian coordinates  $x^\mu = (t, x, y, z)$  of Minkowski space-time with a worldline  $x^\mu(\tau)$  of uniformly accelerated detector along the  $Z$  axis with proper acceleration  $a$  is given by

$$t(\tau) = a^{-1} \sinh a\tau, \quad x(\tau) = x_0, \quad y(\tau) = y_0, \quad z(\tau) = a^{-1} \cosh a\tau$$

$\phi(x) = \int d^3k (u_k a_k^M + u_k^* a_k^{\dagger M})$ , where  $u_k = [2\omega(2\pi)^3]^{-1/2} e^{-ik_\mu x^\mu}$ ,  $k^\mu = (\omega, k)$ ,  $\omega = \sqrt{k^2 + m^2}$  and the CCRs. The proper excitation rate, the excitation probability divided by the total detector proper time  $T$ , associated with the uniformly accelerated detector in the inertial vacuum is given By

$$R_E = \frac{1}{T} \int d^3k |A_{E,k}|^2$$

$$\begin{aligned} A_k &= i \langle E | \otimes \langle k_M | S_I | 0_M \rangle \otimes | E_0 \rangle \\ &= i \int_{-\infty}^{\infty} d\tau \underbrace{\langle E | m(\tau) | E_0 \rangle}_{\langle E | e^{iH\tau} m_0 e^{-iH\tau} | E_0 \rangle} \langle k_M | \phi(x(\tau)) | 0_M \rangle \\ &= i \int_{-\infty}^{\infty} d\tau e^{iE\tau} e^{-iE_0\tau} \underbrace{\langle E | m_0 | E_0 \rangle}_q \int d^3k \frac{1}{\sqrt{2\omega(2\pi)^3}} \langle k_M | (a_k e^{-ik_\mu x^\mu} + a_k^\dagger e^{ik_\mu x^\mu}) | 0_M \rangle \\ &= \frac{qi}{\sqrt{2\omega(2\pi)^3}} \int_{-\infty}^{\infty} d\tau e^{iE\tau} e^{-iE_0\tau} \int d^3k e^{ik_\mu x^\mu} \delta(k - k_M) \\ &= \frac{q}{\sqrt{16\pi^3\omega}} \int d\tau e^{i\Delta E\tau} e^{ik_\mu x^\mu(\tau)} = \frac{q}{\sqrt{16\pi^3\omega}} \int d\tau e^{i\Delta E\tau} \exp[(i\omega/a) \sinh a\tau - (-ik_z/a) \cosh a\tau] \end{aligned}$$

where  $k_x = 0, k_y = 0$ .

$$R = \int d^2k_\perp R^\perp = \frac{1}{T} \int d^3k |A_k|^2$$

$$\begin{aligned} R^\perp &= \frac{|q|^2}{16\pi^3 T} \int_{-\infty}^{\infty} \frac{dk_z}{\omega} \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau'' e^{i\Delta E(\tau' - \tau'')} e^{i\omega(\sinh a\tau' - \sinh a\tau'')/a - ik_z(\cosh a\tau' - \cosh a\tau'')/a} \\ &= \frac{|q|^2}{16\pi^3 T} \int_{-\infty}^{\infty} \frac{dk_z}{\omega} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\sigma e^{i\Delta E\sigma} e^{(2i/a) \sinh a\sigma (\omega \cosh a\tau - k_z \sinh a\tau)} \end{aligned}$$

where  $\tau = (\tau' + \tau'')/2$  and  $\sigma = \tau' - \tau''$ . If the interaction is kept one for a arbitrarily long time, the total time  $T$  diverges, in order to solve this problem, we let  $T = \int_{-\infty}^{\infty} d\tau \quad k_z \rightarrow k'_z = k_z \cosh a\tau - \omega a\tau \implies$

$k'_z \rightarrow k_z = k'_z \cosh a\tau + \omega' \sinh a\tau$  where  $\omega' = [(k'_z)^2 + k_\perp^2 + m^2]^{1/2} = \omega \cosh a\tau - k_z \sinh a\tau$ ,  $k_\perp = \sqrt{k_x^2 + k_y^2}$ . Hence,  $dk'_z/\omega' = dk_z/\omega$ . Thus, we can factor out the  $T$ ,

$$R^\perp = \frac{|q|^2}{16\pi^3} \int_{-\infty}^{\infty} \frac{dk'_z}{\omega'} \int_{-\infty}^{\infty} d\sigma e^{i\Delta E \sigma} e^{(2i\omega'/a) \sinh(a\sigma/2)}.$$

If we make a further change of variables,  $k'_z \rightarrow \zeta = (\omega' + k'_z)/\sqrt{k_\perp^2 + m^2}$ ,  $\sigma \rightarrow \lambda = \exp(a\sigma/2)$  we have,

$$R^\perp = \frac{|q|^2}{8\pi^3 a} \int_1^\infty \frac{d\zeta}{\zeta} \int_0^\infty d\lambda \lambda^{2i\Delta E/a-1} \exp\left[i(k_\perp^2 + m^2)^{1/2}(\lambda - \lambda^{-1})(\zeta + \zeta^{-1})/(2a)\right].$$

Letting  $\kappa = \zeta\lambda$  and  $\varpi = \zeta/\lambda$  then,

$$R^\perp = \frac{|q|^2}{16\pi^3 a} \left| \int_0^\infty \frac{d\kappa}{\kappa^{1-i\Delta E/a}} e^{i(k_\perp^2 + m^2)^{1/2}(\kappa - \kappa^{-1})/(2a)} \right| = \frac{|q|^2 e^{-\pi\Delta E/a}}{4\pi^3 a} \left\{ K_{i\Delta E/a}[(k_\perp^2 + m^2)^{1/2}/a] \right\}^2,$$

where  $K_\nu(x)$  is the modified Bessel function. Remember that  $K_{i\lambda}(x)$  is real if  $x$  and  $\lambda$  are real. Then, the detector response will be

$$R = \frac{|q|^2 a e^{-\pi\Delta E/a}}{2\pi^2} \int_0^\infty d\mu \mu \{ K_{i\Delta E/a}[\sqrt{\mu^2 + (m/a)^2}] \}^2 \quad (1.47)$$

## 2 Quantum Measurement Theory: From von Neumann to Sorkin's protocol

Many measurement schemes are based on the spirit of the von Neumann scheme. The measurement presented in chapter 4 that allows measurements in QFT is no exception. In order to familiarize ourselves with measurement theory, we give a brief introduction. For this purpose, we follow the classical book of Busch [17, 2] and the concise exposition of Montes [18].

### 2.1 The von Neumann measurement process

Let  $\mathcal{H}_S, \mathcal{H}_D$  be two Hilbert spaces. We want to measure an observable in the quantum system  $S = (\mathcal{H}_S, \mathfrak{B}(\mathcal{H}_S))$  with the help of a quantum system  $D = (\mathcal{H}_D, \mathfrak{B}(\mathcal{H}_D))$  which is said to be the probe. We want to measure an observable  $A_s \in \mathfrak{B}(\mathcal{H})$  with the help of a probe that belongs to system  $D$ . Following [18], the measurement process of an observable  $A_s \in \mathfrak{B}(\mathcal{H})$  can be summarized by the following rules:

1. The interaction takes a finite interaction time  $T$ .
2. The measurement is a quantum process, driven by an interaction Hamiltonian  $H_{int}$  and the system evolves in time with the Schrodinger equation.
3. The measurement updates the state of the probe system.
4. If the system  $S$  is in an eigenstate of the observable  $A_s$ , the measurement does not affect the value of the observable, i.e.

$$\frac{dA_s}{dt} = \frac{i}{\hbar} [H_{int}, A_s] = 0$$

5.  $T$  can be as brief as wanted.

Rules 1-5 are known as the von Neumann measurement scheme. Let us see how the measurement works. Prepare  $P$  on the state  $\hat{\rho}_D$ . The system  $S$  is on a state  $\rho_i$ , possibly unknown. The complete system is described in terms of the Hilbert space  $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_D$ . Thus, before the measurement, the state is  $\hat{\rho} = \hat{\rho}_i \otimes \hat{\rho}_D$ .

We want to infer  $A_s$  from the probe  $D$  and therefore, we want a correlation between  $A_s$  and some property  $B_D$  of the system  $D$ . The initial state  $\hat{\rho}$  is evolved into the final one as usual in the interaction picture,

$$\hat{\rho}' = \hat{U}_{int} \hat{\rho}_i \otimes \hat{\rho}_D \hat{U}_{int}^\dagger, \quad \hat{U}_{int} = e^{-\frac{i}{\hbar} \int_0^T \hat{H}_{int}(t) dt}.$$

The final state of the probe is obtained by tracing out the system  $S$ , i.e., the marginal density

$$\hat{\rho}'_D = \text{Tr}_S(\hat{\rho}').$$

Finally, the measurement of the probe  $B_D$  allows us to identify the result, call it  $b_a$  of the probe with the state  $a$  of the system  $S$  and therefore, a measure of  $A_S$  with result  $a$  has been made. This, of course, is only possible if the interaction leads to appropriate correlations between system and probe.

**Example 2.1.1.** *The probe  $D$  is described by a quantum system of 1 degree of freedom with a canonical pair of observables  $(q_D, p_D)$ , that can be thought of as position and momentum observables. The probe observable is chosen to be  $B_D = p_D$  and the Hamiltonian of interaction is*

$$H_{int}(t) = \rho(t) \hat{A}_s \otimes \hat{p}_D, \quad \rho = \lambda \in \mathbb{C}$$

with  $\rho$  constant with support  $\text{supp } \rho = [0, T]$ . Write  $A_S = \sum_i a_i |a_i\rangle \langle a_i|$  then, the evolution  $\hat{U}$  is given by

$$\hat{U}_{int} = \exp \left\{ -\frac{i}{\hbar} \int_0^T \lambda \sum_i a_i |a_i\rangle \langle a_i| \otimes \hat{p}_D dt \right\} = \sum_i |a_i\rangle \langle a_i| \otimes e^{-\frac{i}{\hbar} \lambda T a_i \hat{p}_D}.$$

If we prepare the probe in a state  $|\psi_0(q)\rangle$  and let  $\rho_i = |\phi\rangle \langle \phi|$ ,  $|\phi\rangle = \sum_i \alpha_i |a_i\rangle$  denote the initial state (possibly unknown) of the system  $S$  then, we have

$$\begin{aligned} \hat{U}_{int}(|\phi\rangle \otimes |\psi(q)\rangle) &= \sum_i \alpha_i |a_i\rangle e^{-\frac{i}{\hbar} \lambda T a_i \hat{p}_D} |\psi(q)\rangle = \sum_i \alpha_i |a_i\rangle \otimes |\psi(q - \lambda T a_i)\rangle \\ &= \sum_i \alpha_i |a_i\rangle \otimes |\psi_i(q)\rangle, \quad \psi_i(q) := \psi_0(q - \lambda T a_i), \end{aligned}$$

where we use the fact that  $e^{-ix_0 \hat{p}}$  is a translation operator, i.e.

$$e^{-ix_0 \hat{p}} |\psi_0(q)\rangle = |\psi_0(q - x_0)\rangle.$$

Consequently, the state after the interaction will be

$$\hat{\rho}' = \sum_{i,j} \alpha_i \bar{\alpha}_j |a_i\rangle \langle a_j| \otimes |\psi_i(q)\rangle \langle \psi_j(q)|, \quad (2.1)$$

$$\hat{\rho}'_D = \text{Tr}_S(\hat{\rho}') = \sum_i |\alpha_i|^2 |\psi_i(q)\rangle \langle \psi_i(q)|.$$

**Remark 2.1.1.** *A (projective) measurement of  $q_D$  will allow us to infer the value of  $A_D$ , provided the uncertainty  $\Delta q$  is small enough ( $\Delta q \ll |\lambda T(a_i - a_j)|$ ), as shown in the figure 4.*

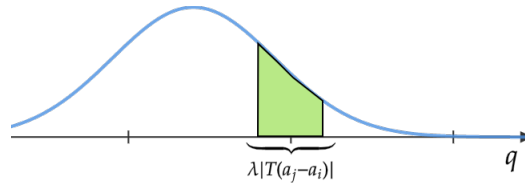


Figure 4: Probability distribution in function of the different values of  $q$ . If the uncertainty  $|\lambda T(a_i - a_j)|$  around  $q_i$  is small enough so that doesn't overlap  $q_{i \pm 1}$  then, one can determine the value of  $q$ .

## 2.2 Measurements in Quantum Field Theory?

When dealing with measurements, the formulation of QFT is unsatisfactory mathematically and physically. The physical problem starts with the measurement of a field. In order to measure a field  $\varphi(x)$ , there is a position on the space-time implicit and thus, due to the Heisenberg uncertainty principle, it would require an infinite amount of energy to measure it [19]. Hence, a better definition of a field is one in which there is no assumption of point-like localization but rather a region of space-time.

On the other hand, the mathematical problem is a subtle one; the problem deals with the localization of the field itself as a whole. The field cannot be supported on the whole space-time but into a (possibly) compact region of the space. Otherwise, causally (open) disjoint regions could interact through the field,  $\varphi(x)$  leading to information pathology's.

One way to handle this situation that solves both, the physical and mathematical problems is the algebraic quantum field theory (AQFT) approach. This theory handles observables locally and also allow us to define thermal state in a satisfactory way. Therefore, AQFT will be introduced in section 3, as a theory that will allow us to define measurements on quantum fields in section 4 and relate them with the Unruh effect. Let us finish this section by explaining the mathematical problem by means of an example.

**Example 2.2.1** (Sorkin's protocol). *Sorkin's protocol is a measurement example on a quantum field that illustrates the problem discussed above. We will follow Sorkin's formulation in his article "Impossible measurements on quantum fields" [6].*

*Consider the following setup, let  $\mathcal{O}_i$ ,  $i = 1, 2, 3$  be three open regions in a space-time  $M$ . Denote  $J_i$  the casual set of the region  $\mathcal{O}_i$  as shown in the figure 5.*

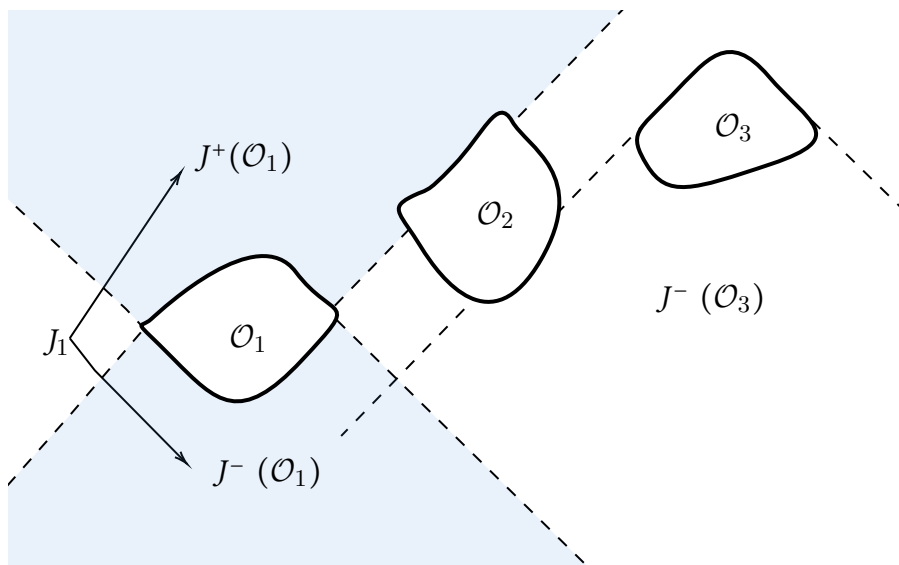


Figure 5: Sorkin's protocol: Regions  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  and  $\mathcal{O}_3$ .



Let  $\mathcal{O}_1$  be causally disjoint of  $\mathcal{O}_3$  whereas  $\mathcal{O}_2$  intersect some causal future/past of both  $\mathcal{O}_1$  and  $\mathcal{O}_3$ , that is,

$$J_1 \cap J_3 = \emptyset, \quad J_1 \cap J_2 \neq \emptyset \neq J_3 \cap J_2.$$

Let  $\phi(x)$  be a free field. In the region  $\mathcal{O}_1$  we perform an operation with the unitary  $U = e^{i\lambda\phi(y)}$  and in the region  $\mathcal{O}_2$  there is a projection  $B = |b\rangle\langle b|$ ,  $|b\rangle = \alpha|0\rangle + \beta|1\rangle$ . Start with the density state  $\rho_0 = |0\rangle\langle 0|$ .

First, in the region  $\mathcal{O}_1$  perform the evolution  $U$ , which gives the state  $\rho$ ,

$$\rho_0 \longrightarrow \rho = U\rho_0U^*. \quad (2.2)$$

Secondly, in the region  $\mathcal{O}_2$  perform a quantum operation with the Kraus operator  $\Lambda_i$ ,  $i = 1, 2$  defined as  $\Lambda_1 = B$ ,  $\Lambda_2 = 1 - B$ ,

$$\rho \longrightarrow \rho' = \Lambda_1\rho\Lambda_1^* + \Lambda_2\rho\Lambda_2^*. \quad (2.3)$$

Finally, in the region  $\mathcal{O}_3$ , we measure  $\phi(x)$ ,

$$\rho' \longrightarrow \langle 0|\phi(x)|0\rangle_{\rho'} = \text{Tr}(\rho'\phi(x)). \quad (2.4)$$

Since  $\mathcal{O}_1$  and  $\mathcal{O}_3$  are causally disjoint, the measurement should be independent of  $\lambda$ . Therefore, the derivative of the measurement should vanish. Let us take a look at this derivative,  $\left. \frac{d}{d\lambda} \right|_{\lambda=0} \text{Tr}(\rho'\phi(x)) =: \mathcal{G}$ .

$$\begin{aligned} \mathcal{G} &= \left. \frac{d}{d\lambda} \right|_{\lambda=0} \langle 0| \underbrace{U^*}_{e^{-i\lambda\phi(y)}} B\phi(x)BU |0\rangle \\ &\quad + \left. \frac{d}{d\lambda} \right|_{\lambda=0} \langle 0| U^* (1-B)\phi(x)(1-B)U |0\rangle \\ &= \left. \frac{d}{d\lambda} \right|_{\lambda=0} \langle 0| [B\phi(x)B - i\lambda[\phi(y), B\phi(x)B] + o(\lambda^2)] |0\rangle + \\ &\quad \left. \frac{d}{d\lambda} \right|_{\lambda=0} \langle 0| [(1-B)\phi(x)(1-B) - i\lambda[\phi(y), (1-B)\phi(x)(1-B)] + o(\lambda^2)] |0\rangle \\ &= -i \langle 0| [\phi(y), B\phi(x)B] |0\rangle - i \langle 0| [\phi(y), (1-B)\phi(x)(1-B)] |0\rangle \\ &= -i \langle 0| [\phi(y), 2B\phi(x)B - \phi(x)B - B\phi(x) + \phi(x)] |0\rangle \\ &= -i \langle 0| [\phi(y), 2B\phi(x)B - \phi(x)B - B\phi(x)] |0\rangle \end{aligned}$$

where we use the Baker-Campbell-Hausdorff (BCH) formula,

$$e^{-i\lambda\phi(y)} A e^{i\lambda\phi(y)} = A - i\lambda[\phi(y), A] + o(\lambda^2)$$

on the second line and that  $[\phi(y), \phi(x)] = 0$  on the fourth line. Notice that  $\overline{\langle 0|AB|0\rangle} = \langle 0|BA|0\rangle$  thus,

$$\langle 0|[A, B]|0\rangle = \langle 0|AB|0\rangle - \overline{\langle 0|AB|0\rangle} = 2i \operatorname{im} \langle 0|AB|0\rangle \text{ consequently,}$$

$$\begin{aligned} \left. \frac{d}{dt} \right|_{\lambda=0} \operatorname{Tr}(\rho' \phi(x)) &= 2 \operatorname{im} \langle 0|\phi(y)(2B\phi(x)B - \phi(x)B - B\phi(x))|0\rangle \\ &= 2 \operatorname{im} [\langle 0|\phi(y)2B\phi(x)B|0\rangle - \langle 0|\phi(y)\phi(x)B|0\rangle \\ &\quad - \langle 0|\phi(y)B\phi(x)|0\rangle]. \end{aligned}$$

Denote  $\psi(x) := \langle 0|\phi(x)|1\rangle$ , since  $B = |b\rangle\langle b|$ ,  $|b\rangle = \alpha|0\rangle + \beta|1\rangle$ , the first term gives us

$$\begin{aligned} 2 \operatorname{im} \langle 0|\phi(y)2B\phi(x)B|0\rangle &= 2 \operatorname{im} \langle 0|\phi(y)2|b\rangle\langle b|\phi(x)|b\rangle\langle b|0\rangle \\ &= 4 \operatorname{im} [\alpha^* (\langle 0|\phi(y)|1\rangle\beta + \langle 0|\phi(y)|0\rangle\alpha) \\ &\quad \times (\langle 0|\phi(x)|0\rangle|\alpha|^2 + \langle 0|\phi(x)|1\rangle\beta\alpha^* \\ &\quad + \langle 1|\phi(x)|0\rangle\alpha\beta^* + \langle 1|\phi(x)|1\rangle|\beta|^2)] \\ &= 4 \operatorname{im} [\alpha^* (\psi(y)\beta + 0)(0 + \psi(x)\beta\alpha^* + \bar{\psi}(x)\alpha\beta^*)] \\ &= 4 \operatorname{im} [(\alpha^*\beta)^2\psi(y)\psi(x) + |\alpha\beta|^2\psi(y)\bar{\psi}(x)]. \end{aligned}$$

Likewise, the third term is

$$\begin{aligned} 2 \operatorname{im} \langle 0|\phi(y)B\phi(x)|0\rangle &= 2 \operatorname{im} \langle 0|\phi(y)|b\rangle\langle b|\phi(x)|0\rangle \\ &= 2 \operatorname{im} |0\rangle\phi(y)\beta|1\rangle\alpha^*\langle 1|\phi(x)|0\rangle = \beta\alpha^*\psi(y)\bar{\psi}(x). \end{aligned}$$

And the second term is

$$\begin{aligned} 2 \operatorname{im} \langle 0|\phi(y)\phi(x)B|0\rangle &= 2 \operatorname{im} \langle 0|\phi(y)\phi(x)(\alpha|0\rangle + \beta|1\rangle)\alpha^* \\ &= 2 \operatorname{im} |\alpha|^2 \langle 0|\phi(y)\phi(x)|0\rangle + \operatorname{im} \alpha^*\beta \underbrace{\langle 0|\phi(y)\phi(x)|1\rangle}_0 \\ &= -i|\alpha|^2 \underbrace{\langle 0|[\phi(y), \phi(x)]|0\rangle}_0 = 0. \end{aligned}$$

Thus, back in  $\left. \frac{d}{dt} \right|_{\lambda=0} \operatorname{Tr}(\rho' \phi(x))$  we have

$$\left. \frac{d}{dt} \right|_{\lambda=0} \operatorname{Tr}(\rho' \phi(x)) = 2 \operatorname{im} [2(\alpha^*\beta)^2\psi(y)\psi(x) + (2|\alpha\beta|^2 - \beta\alpha^*)\psi(y)\bar{\psi}(x)].$$

If we choose  $\alpha = \beta = 1/\sqrt{2}$  then,

$$\left. \frac{d}{dt} \right|_{\lambda=0} \operatorname{Tr}(\rho' \phi(x)) = \operatorname{im} [\psi(y)\psi(x)] \neq 0. \quad (2.5)$$

Given an evolution  $U$  on region  $\mathcal{O}_2$ , the measurement of equation (2.4) must be independent of any action of  $U$  and therefore, of  $\lambda$ . Equation (2.5) tell us that region  $\mathcal{O}_3$  receives information of  $\mathcal{O}_2$ .

*Thus, we arrive at a contradiction, leading to a super-luminal information transport and establishing an apparent paradox which has been the subject of intense discussion recently [20, 21, 7, 22].*

### 3 Algebraic Quantum Field Theory

As we briefly discuss in chapter 1, Algebraic Quantum Field Theory (AQFT) is a satisfactory theory to define thermal states and is also a natural way of dealing with locality and causality, making AQFT a natural starting point for our foundational quest of a QFT measurement scheme and its relation with the Unruh effect. In this chapter we introduce the basic tools and ideas of AQFT as well as its relation with the Unruh effect.

Locality is the principle of field theories and, on the other hand, correlations, for example, entanglement in quantum states, often relies on non-locality principles. Thus, we may want to separate the construction of states from the construction of the observables in order to make a richer theory. Indeed, Algebraic Quantum Field Theory seeks to separate those constructions and, for instance, allows us to have more states than those in the traditional approach (density matrices). Historically, the concepts of algebraic quantum field theory were formally introduced by Wightmann with the axiomatic setting, operator-valued distributions in the early 1950s by taking the idea of fields supported in local regions and later generalized by Haag [23] in 1957 with the so-called Haag-Kastler assumptions in which any idea of fields is forgotten and rather, observables became objects in an abstract algebra. It was until the formulation of the Haag-Ruelle scattering theory that put the eye on these algebraic constructions and later on, it was then made mathematically precise by using the theory of operator algebras, mainly by Araki [24]. Also, AQFT enriched quantum statistical mechanics through Tomita-Takesaki modular theory [25, 26].

In this chapter, we give the basic tools as well as the main theorems in order to understand the formalism of AQFT, KMS states and their relation to the Unruh effect. For this purpose we follow several references, [24, 27, 26, 23, 28, 29, 25].

#### 3.1 Algebraic formalism in a nutshell

**Definition 3.1.1** ( $*$  and  $C^*$  algebras). *An  $*$ -algebra is a complex algebra equipped with an involution  $*$ . An  $C^*$ -algebra is a Banach  $*$ -algebra such that*

$$\|A^*A\| = \|A\|^2, \quad A \in \mathcal{A}.$$

*The 'C' stands for 'closed', first defined by Segal in 1947 as a "uniformly closed, self-adjoint algebra of bounded operators on a Hilbert space" [30] which of course relates with the above definition by means of representation theory.*

**Definition 3.1.2.** *Let  $\mathcal{A}$  be a  $*$ -subalgebra of  $\mathfrak{B}(\mathcal{H})$ . We say that  $\mathcal{A}$  is a von Neumann algebra if  $\mathcal{A}$  is a unital algebra and*

$$(\mathcal{A}')' = \mathcal{A}, \quad \mathcal{A}' = \{B \in \mathfrak{B}(\mathcal{H}) : [B, A] = 0, \forall A \in \mathcal{A}\},$$

where  $\mathcal{A}'$  is called the commutant of  $\mathcal{A}$ . In other words, a von Neumann algebra is a  $*$ -subalgebra of  $\mathfrak{B}(\mathcal{H})$  whose double commutant is itself. In general, a von Neumann algebra need not be unital but will ease computations.

**Definition 3.1.3.** A state on a (unital)  $*$ -algebra  $\mathcal{A}$  is a linear functional  $\omega : \mathcal{A} \rightarrow \mathbb{R}$  such that

$$\omega(AA^*) \geq 0, \quad \forall A \in \mathcal{A} \quad \text{and} \quad \omega(1) = 1.$$

$\omega$  is said to be mixed if it is a convex combination of at least two states  $\omega_1 \neq \omega_2$ . If  $\omega$  is not mixed,  $\omega$  is pure.

States and observables are independent constructions, the Gelfand-Naimark-Segal construction allows us to link these concepts in a suitable Hilbert space.

**Theorem 3.1.1** (Gelfand-Naimark-Segal construction). *Given a  $*$ -representation of a  $C^*$ -algebra on a Hilbert space  $\mathcal{H}$ ,  $\pi : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{H})$  and let  $v \in \mathcal{H}$  be a normalized cyclic vector for  $\pi(\mathcal{A})$  ( $\text{span}\{\pi(\mathcal{A})v\}$  is dense in the uniform topology) then, clearly*

$$\begin{aligned} \omega : \mathcal{A} &\rightarrow \mathbb{R} \\ A &\mapsto \langle \pi(A)v, v \rangle \end{aligned}$$

is a state of  $\mathcal{A}$ . Conversely, if  $\omega$  is a state on a  $C^*$ -algebra  $\mathcal{A}$ , there exists a unique (up to unitary equivalent representations) representation  $\pi_\omega : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{H})$  on some Hilbert space  $\mathcal{H}$  and a unit GNS vector  $\Omega_\omega \in \mathcal{H}$  such that

$$\begin{aligned} \omega(A) &= \langle \Omega_\omega, \pi(A)\Omega_\omega \rangle, \quad \forall A \in \mathcal{A} \\ \pi(\mathcal{A})\Omega_\omega &\text{ is dense in } \mathcal{H}. \end{aligned}$$

*Proof.* Define  $\langle A, B \rangle := \omega(A^*B)$  and  $I_\omega := \{A \in \mathcal{A} : \omega(A^*A) = 0\}$ . Let  $\mathcal{H} = (\overline{\mathcal{A}/I_\omega}, \langle \cdot, \cdot \rangle)$  be the completion of the quotient  $\mathcal{A}/I_\omega$ . Clearly  $\mathcal{H}$  is a complex Hilbert space by the properties of  $\omega$ , the representation  $\pi_\omega$  is such that<sup>9</sup>  $\pi_\omega(A)[B] = [AB]$  on  $\mathcal{A}/I_\omega$  and, the unit vector is  $\Omega_\omega := [1]$ . Clearly, if  $A \in \mathcal{A}$  we have

$$\langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle = \langle \Omega_\omega, [A] \rangle = \omega(A)$$

and  $\pi_\omega(\mathcal{A})\Omega_\omega = \mathcal{A}/I_\omega$  is dense in  $\mathcal{H}$ . Now, suppose  $(\mathcal{H}', \pi')$  is another  $*$ -representation of  $\mathcal{A}$  and  $\Omega'$  is their corresponding cyclic vector in this representation. In this case  $U : \mathcal{H} \rightarrow \mathcal{H}'$  such that  $U\pi_\omega(A)\Omega_\omega = \pi'(A)\Omega'$  is unitary since

$$\begin{aligned} \langle U\pi_\omega(A)\Omega_\omega, U\pi_\omega(B)\Omega_\omega \rangle &= \langle \pi'(A)\Omega', \pi'(B)\Omega' \rangle \\ &= \langle \Omega', \pi'(A^*)\pi'(B)\Omega' \rangle = \omega(A^*B) \\ &= \langle \Omega_\omega, \pi_\omega(A^*)\pi_\omega(B)\Omega_\omega \rangle \\ &= \langle \pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega \rangle. \end{aligned}$$

<sup>9</sup>Given  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  two continuous functions and suppose  $f|_D = g|_D$  where  $D$  is a dense set of  $X$ . Then, is a well known fact that  $f = g$ . Therefore, we sometimes give definitions acting in a dense set rather than the domain.

It follows that  $U^{-1}\pi'(A)U = \pi_\omega(A)$  and  $U\Omega_\omega = \Omega'$ .

Remark:  $I_\omega$  is a (left)ideal. Let  $A \in I_\omega$  and  $B \in \mathcal{A}$ , using the Cauchy-Schwartz inequality

$$\begin{aligned} |\omega((BA)^*BA)|^2 &= |\omega(A^*B^*BA)|^2 = |\omega((B^*BA)^*A)|^2 \\ &\leq \omega((B^*BA)^*(B^*BA))\omega(A^*A) = 0 \end{aligned}$$

therefore,  $BA \in I_\omega$ . ■

The GNS theorem entails that there exists an equivalence between elements and states in the algebra with operators and states on a Hilbert space respectively.

**Definition 3.1.4.** *If an automorphism  $\alpha$  is unitarily implemented*

$$\pi(\alpha(\rho)A) = U(\rho)\pi(A)U(\rho)^{-1}.$$

*on the representation Hilbert space we say that the representation is Poincaré covariant.*

**Theorem 3.1.2.** *Let  $\alpha$  be an automorphism of a unital  $*$ -algebra  $\mathcal{A}$ . If a state  $\omega$  on  $\mathcal{A}$  is invariant under  $\alpha$  then,  $\alpha$  is unitarily implemented in the GNS representation  $\pi_\omega$  of  $\omega$  that leaves the GNS vector  $\Omega_\omega$  invariant. Any group of automorphisms leaving  $\omega$  invariant is unitarily represented in  $\mathcal{H}_\omega$ . The GNS representation of a Poincaré invariant state  $\omega$  is always Poincaré covariant i.e.*

$$\omega(\alpha(\rho)A) = \omega(A), \quad \text{for all } A \in \mathcal{A}(M), \rho \in \mathcal{P}$$

where  $\mathcal{P}$  is the Poincaré group.

*Proof.* Since  $\alpha$  is an automorphism,  $\alpha I_\omega = I_\omega$  where  $I_\omega$  is the GNS (left)ideal defined earlier. Therefore,  $U$  such that in the dense set  $\mathcal{A}/I_\omega$  we have  $U[A] = [\alpha(A)]$  is well-defined. The GNS vector  $\Omega_\omega = [1]$  is invariant and,  $U$  is unitary since

$$\begin{aligned} \langle U[A], U[B] \rangle &= \omega(\alpha(A)^*\alpha(B)) = \omega(\alpha(A^*B)) \\ &= \omega(A^*B) \\ &= \langle [A], [B] \rangle. \end{aligned}$$

Using the representation  $\pi_\omega$  give us

$$\begin{aligned} \pi_\omega(\alpha(A))[B] &= [\alpha(A)B] = [\alpha(A\alpha^{-1}(B))] \\ &= U[A\alpha^{-1}(B)] = U\pi_\omega(A)[\alpha^{-1}B] \\ &= U\pi_\omega(A)U^{-1}[B] \end{aligned}$$

that is,  $\alpha$  is implemented unitarily. Finally, if  $\beta$  is another automorphism leaving  $\omega$  invariant,  $V$  be its unitary implementation. Then

$$UV[A] = [\alpha(\beta(A))] = [(\alpha \circ \beta)(A)]$$

shows that  $UV$  implements  $\alpha \circ \beta$  therefore, any group of automorphisms leaving  $\omega$  invariant is unitarily represented.  $\blacksquare$

An important family of automorphisms are one-parameter group automorphisms because one can implement time evolution operations in the following way.

**Definition 3.1.5.** Let  $\mathcal{A}$  be a  $C^*$ -algebra. If  $\tau_t$  is a one-parameter group of automorphisms such that  $\tau(A) : \mathbb{R} \rightarrow \mathcal{A}$ ,  $\tau_t(A) \in \mathcal{A}$  is continuous for all  $A \in \mathcal{A}$  then, the  $C^*$ -algebra  $\mathcal{A}$  together with the one parameter group  $\tau$  is said to be a  $C^*$ -dynamical system. In this case we write

$$\tau_t(\omega)(A) := \omega(\tau_t(A))$$

and we say that a state  $\omega$  on  $\mathcal{A}$  is invariant under  $\tau$  if

$$\omega(\tau_t(A)) = \omega(A), \quad \forall t \in \mathbb{R}, \forall A \in \mathcal{A}.$$

**Definition 3.1.6.** Let  $\mathcal{B}$  be a von Neumann algebra on a Hilbert space  $H$ . If  $\tau_t$  is a one-parameter group of automorphisms such that  $\tau(B)x : \mathbb{R} \rightarrow \mathcal{H}$  is continuous for all  $B \in \mathcal{B}$  and  $x \in \mathcal{H}$  then, the algebra  $\mathcal{B}$  together with the one parameter group  $\tau$  is said to be a  $W^*$ -dynamical system.

Observe that a  $C^*$ -dynamical system is stronger than the  $W^*$ -dynamical system in the sense that the underlying von Neumann algebra endowed with the  $C^*$ -dynamics inherits the  $W^*$ -dynamics because the topology of the automorphisms on the von Neumann algebra is the weak topology whereas the topology on the  $C^*$  algebra is the strong topology.

**Example 3.1.1.** Given a Hamiltonian  $H$  acting on a Hilbert space  $\mathcal{H}$ , the map

$$\begin{aligned} \tau(A) : \mathbb{R} &\rightarrow \mathfrak{B}(\mathcal{H}) \\ A &\mapsto e^{iHt} A e^{-iHt} \end{aligned}$$

is a one parameter automorphism (inner) automorphisms in the algebra of bounded operators  $\mathfrak{B}(\mathcal{H})$ .

**Theorem 3.1.3.** Let  $\mathcal{A}$  be a  $C^*$ -algebra,  $\tau : \mathcal{A} \rightarrow \text{Aut}(\mathcal{A})$  a one parameter automorphism,  $\omega$  an invariant state under  $\tau$  and  $\pi_\omega : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{H})$  the induced GNS representation. Then, there exists a unique unitary one-parameter group  $U : \mathbb{R} \rightarrow \{A \in \mathfrak{B}(\mathcal{H}) : A^\dagger A = I = A A^\dagger\}$  such that  $U_t \Omega_\omega = \Omega_\omega$  and  $\pi_\omega(\tau_t(A)) = U_t \pi_\omega(A) U_{-t}$ ,  $\forall t \in \mathbb{R}, A \in \mathcal{A}$ .

*Proof.* Look in [26].  $\blacksquare$

**Definition 3.1.7.** Let  $\rho$  be a density operator on a Hilbert space  $\mathcal{H}$ . Notice that  $\omega_\rho : \mathfrak{B}(\mathcal{H}) \rightarrow \mathbb{C}$ ,

$$\begin{aligned} \omega_\rho(A) &= \text{Tr}(A\rho) \\ \omega_\rho(AA^*) &= \text{Tr}(A\rho A^*) \geq 0, \quad \omega_\rho(1) = \text{Tr}(\rho) = 1 \end{aligned}$$

is a state in the sense of definition 3.1.3. If  $\mathcal{A} \subset \mathfrak{B}(\mathcal{H})$  is a  $C^*$ -algebra, the state  $\omega_\rho : \mathcal{A} \rightarrow \mathbb{C}$  is said to be a normal state.

**Remark 3.1.4.** *The algebraic approach takes into account many subtleties that arise in QFT that are hidden in Quantum Mechanics (QM). For instance, in QM the algebra of canonical commutation relations is unique by the Stone-von Neumann theorem and the only states are characterized by density operators, i.e. normal states. However, in quantum field theories there are inequivalent irreducible representations of the canonical commutation relations, allowing the theory to have far more states than QM.*

**Example 3.1.2** (Scalar field with external force). *We follow [23] to depict that different (inequivalent) representations of a quantum field theory may exist. Consider a theory with a Lagrangian*

$$\mathcal{L} = \frac{1}{2} (\nabla_\mu \varphi) \nabla^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \rho \varphi \quad (3.1)$$

where  $\rho$  is a time-independent real valued function. The field equation of this theory is

$$(\square + m^2)\varphi = -\rho(x). \quad (3.2)$$

Using quantum mechanics, one can either use the Schrödinger picture or the interaction picture in order to find the dynamics of the system.

- In the Schrödinger picture one starts with the standard Fock space of the free scalar field with position and momentum satisfying the canonical commutation relations

$$[\varphi(x), \pi(y)] = i\delta(x - y)1,$$

the interaction model is thus described by a Hamiltonian  $H_\rho$  in terms of the canonical variables and the evolution of the states are described by  $e^{-iH_\rho t}$ . Therefore, one can define

$$\varphi_\rho(t, x) = e^{iH_\rho t} \varphi(x) e^{-iH_\rho t}$$

as a state that follows the dynamics (3.2) of the system.

- On the other hand, in the interaction picture the free field  $\varphi_0(t, x)$  acts on the usual Fock space and the evolution is described by the evolution operator  $U_{\text{int}}$  with the Dyson series,

$$\begin{aligned} \hat{U}_{\text{int}} &= \mathcal{T} e^{-i \int d\tau \hat{H}_{\text{int}}(\tau)} \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d\tau_1 \dots d\tau_n \mathcal{T}(\hat{H}_{\text{int}}(\tau_1) \dots \hat{H}_{\text{int}}(\tau_n)). \end{aligned}$$

Either case, ultraviolet or infrared-like divergences may appear and thus, a need of re-normalization theory. On the other hand, one can describe the dynamics in its algebraic nature. We start with equation (3.2) and proceed with the canonical quantization from scratch. In this way, we do not use the Hilbert space of the free scalar field but rather, the one specific of the model. In this model, one



can show that the canonical variables  $\varphi_\rho$  and  $\pi_\rho$  are given by

$$\begin{aligned}\varphi_\rho(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} (a(k) + a(-k)^*) e^{ik \cdot x} \\ \pi_\rho(x) &= -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega}{2}} (a(k) - a(-k)^*) e^{ik \cdot x}\end{aligned}$$

with the free mode frequency  $\omega := \omega(k) = \sqrt{|k|^2 + m^2}$  and impose the Canonical Commutation Relations (CCRs),

$$[a(k), a(k')^*] = (2\pi)^3 \delta(k - k') 1, \quad (3.3)$$

$$[a(k), a(k')] = 0 = [a(k)^*, a(k')^*]. \quad (3.4)$$

The Hamiltonian is then deduced from the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\nabla_\mu \varphi) \nabla^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \rho \varphi$$

as

$$H_\rho = \int \frac{d^3k}{(2\pi)^3} \left( \omega(k) a(k)^* a(k) + \frac{1}{\sqrt{2\omega(k)}} \left( \hat{\rho}(k) a(k) + \hat{\rho}(k) a(k)^* \right) \right),$$

where  $\hat{\rho}(k) = \int d^3x \rho(x) e^{-ik \cdot x}$  and equivalently, if one redefines the annihilator and creation operators,

$$H_\rho = \int \frac{d^3k}{(2\pi)^3} \omega(k) \tilde{a}(k)^* \tilde{a}(k) + E_\rho 1,$$

where  $\tilde{a}$  and the constant  $E_\rho$  are defined by

$$\tilde{a}(k) = a(k) + \frac{\hat{\rho}(k)}{\sqrt{2\omega(k)^3}} 1, \quad E_\rho = - \int \frac{d^3k}{(2\pi)^3} \frac{|\hat{\rho}(k)|^2}{2\omega(k)^2}.$$

The reader may notice that a Hilbert space is not yet specified. However, we want a representation in which  $H_\rho$  is self-adjoint,  $\tilde{a}(k)$  satisfy the CCRs and there is a preferred vacuum state. For this purpose, notice that  $\tilde{a}(k)$  satisfy the same CCRs as  $a(k)$  and thus, we can write a vacuum vector  $\tilde{\Omega}$  annihilated by all  $\tilde{a}(k)$ . This vacuum is the ground state of the Hamiltonian  $H_\rho$  and we may take this energy to be the zero energy state. The Fock space is then realized as usual,

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$$

where  $\tilde{\Omega}$  is the vacuum vector of  $\mathcal{H}$  and  $H_\rho$  is just the standard Hamiltonian of the free field on this space

$$\tilde{H}_\rho = \int \frac{d^3k}{(2\pi)^3} \omega \tilde{a}(k)^* \tilde{a}(k).$$

The redefinition of annihilation and creation operators provide the crucial differentiation between the previous methods (Schrodinger and interaction picture). A calculation shows that the time inde-

pendent Heisenberg picture field is

$$\tilde{\varphi}_\rho(t, x) = e^{i\tilde{H}_0 t} \varphi_\rho(x) e^{-i\tilde{H}_0 t} = \tilde{\varphi}_0(t, x) + \varphi_\rho(x)1, \quad (3.5)$$

where

$$\tilde{\varphi}_\rho(x) = - \int \frac{d^3 k}{(2\pi)^3} \frac{\hat{\rho}(k)}{\omega^2} e^{ik \cdot x} = -(V_Y \star \rho)(x)$$

is a time independent solution of the field equation and

$$\tilde{\varphi}_0(t, x) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} (\tilde{a}(k) e^{-i\omega t + ik \cdot x} + \tilde{a}(k)^* e^{i\omega t - ik \cdot x})$$

is the free scalar field on the Fock space.

**Remark 3.1.5.**

- $H_\rho$  in the new variables is of the same form as the Hamiltonian in the free field defined with the new annihilator/creation operators.
- Given a field equation we found the generators.
- The Fock space was built a posteriori with the help of the vacuum vector once the operators  $\tilde{a}$  and  $\tilde{a}^\dagger$  are defined.
- The collection of fields has both, vector space and ring structure, that is, they form an algebra.

The example 3.1.2 illustrates that the vacuum state lie inside a renormalized Hilbert space  $\mathcal{H}_{\text{renorm}}$  that differs from the Hilbert space  $\mathcal{H}_{\text{free}}$  of the free field. Although an isomorphism is found between  $\mathcal{H}_{\text{renorm}}$  and  $\mathcal{H}_{\text{free}}$ , mainly by the map  $a \mapsto \tilde{a}$ , Haag's theorem<sup>10</sup> implies that no such mapping could deliver unitary equivalent representations of the corresponding canonical commutation relations. In addition, Haag's theorem explains that disjoint states<sup>11</sup> are essential in QFT. In fact, given a group  $G$  acting on the  $C^*$ -algebra of observables, and two  $G$ -invariant states  $\omega_1$  and  $\omega_2$  then either  $\omega_1 = \omega_2$ , or  $\omega_1$  and  $\omega_2$  are disjoint. Since vacuum states must be invariant with respect to the Poincaré group, it follows that vacuum states of both free and an interacting theory must be different. Therefore, by Haag's theorem, they cannot both be represented as density matrices in a single representation. This motivates why the algebraic description of quantum theories is so often used for relativistic quantum mechanics.

### 3.1.1 Thermal states, KMS states and the KMS condition

In the algebraic formalism, a thermal state can be seen as a specific class of states on a  $C^*$ -algebra characterized by Kubo (1957), Martin and Schwinger (1959) called the KMS-states. Let  $\beta \in \mathbb{R}$ ,

<sup>10</sup>Haag's theorem states that if the two representations are unitarily equivalent representations of scalar fields, and both representations contain a unique vacuum state, the two vacuum states are themselves related by the unitary equivalence. Hence neither field Hamiltonian can polarize the other field's vacuum.

<sup>11</sup>states that are normal in one GNS representation and not in the other

define  $\mathfrak{D}_\beta$  as

$$\mathfrak{D}_\beta = \begin{cases} \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \beta\}, & \beta \geq 0 \\ \{z \in \mathbb{C} : \beta < \operatorname{Im} z < 0\}, & \beta < 0 \end{cases},$$

if  $\beta = 0$  then  $\overline{\mathfrak{D}_\beta} = \mathbb{R}$  otherwise  $\overline{\mathfrak{D}_\beta}$  is the closure of  $\mathfrak{D}_\beta$ .

**Definition 3.1.8.** Let  $\mathcal{A}$  together with  $\tau$  a  $C^*$  or  $W^*$  dynamical system,  $\omega$  a state and  $\beta \in \mathbb{R}$ . Suppose there exist a function  $F_{A,B} : \overline{\mathfrak{D}_\beta} \rightarrow \mathbb{C}$ ,  $\forall A, B \in \mathcal{A}$  analytic in  $\mathfrak{D}_\beta$  such that

$$F_{A,B}(t) = \omega(A\tau_t(B)), \quad (3.6)$$

$$F_{A,B}(t + i\beta) = \omega(\tau_t(B)A). \quad (3.7)$$

In that case,  $\omega$  is said to be a  $(\tau, \beta)$ -KMS state or equivalently, a KMS state with respect to the automorphism  $\tau$  at a temperature  $\beta$ . Equations (3.6) and (3.7) are the KMS conditions and  $F$  is called the witness of  $\omega$  being a  $(\tau, \beta)$ -KMS state. A  $(\tau, -1)$ -KMS state is called a  $\tau$ -KMS state.

**Theorem 3.1.6.** Let  $\mathcal{A}$  together with  $\tau$  a  $C^*$  or  $W^*$  dynamical system,  $\omega$  a state and  $\beta \in \mathbb{R}$ . Then,  $\tau$  induce a dynamical system on  $\mathcal{A}$  by  $\alpha : \mathbb{R} \rightarrow \operatorname{Aut}(\mathcal{A})$  where  $\alpha$  is defined so that

$$\begin{aligned} \alpha_t : \mathcal{A} &\rightarrow \mathcal{A} \\ A &\mapsto \alpha_t(A) := \tau_{-\beta t}(A). \end{aligned}$$

Furthermore, if  $\omega$  is a  $(\tau, \beta)$ -KMS state then  $\omega$  is a  $\alpha$ -KMS state. The converse is also true if  $\beta \neq 0$ .

*Proof.* Clearly  $\mathcal{A}$  with  $\alpha$  is a  $C^*$  or  $W^*$ -dynamical system. If  $\omega$  is a  $(\tau, \beta)$ -KMS state, let  $F_{A,B}$  be the witness then,  $G_{A,B}(z) := F_{A,B}(-\beta z)$  is a witness of  $\omega$  being an  $\alpha$ -KMS state. Conversely, if  $\beta \neq 0$  and  $G_{A,B}$  is a witness of  $\omega$  being an  $\alpha$ -KMS state  $F_{A,B}(z) = G_{A,B}(-z/\beta)$  is a witness of  $\omega$  being a  $(\tau, \beta)$ -KMS state. ■

**Theorem 3.1.7.** Let  $\mathcal{A}, \tau$  be a  $C^*$  or  $W^*$ -dynamical system,  $|\beta| > 0$  and,  $\omega$  a  $(\tau, \beta)$ -KMS state. Then, for all  $A \in \mathcal{A}$  and  $t \in \mathbb{R}$  we have  $\omega(\tau_t(A)) = \omega(A)$ .

*Proof.* Let  $\alpha$  be defined as above,  $A \in \mathcal{A}$  and,  $F_{1,A}$  be a witness to  $\omega$  being a  $\alpha$ -KMS state. Notice that,

$$\omega(\tau_t(A)) = \omega\left(\tau_t\left(\frac{A + A^*}{2} + i\frac{A - A^*}{2i}\right)\right) = \omega\left(\tau_t\left(\frac{A + A^*}{2}\right)\right) + i\omega\left(\tau_t\left(\frac{A - A^*}{2i}\right)\right).$$

That is, the condition  $\omega(\tau_t(A)) = \omega(A)$  holds if and only if it holds for all self adjoint elements  $A \in \mathcal{A}$ . Therefore, we might assume that  $A$  is self-adjoint. In this case we have

$$\begin{aligned} \overline{F_{1,A}(t)} &= \overline{\omega(\alpha_t(A))} = \overline{F_{1,A}(t - i)} \\ &= \omega(\alpha_t(A)^*) \\ &= \omega(\alpha_t(A^*)) \\ &= \omega(\alpha_t(A)) = F_{1,A}(t - i) \end{aligned}$$

It follows that  $F_{1,A}(\overline{\mathfrak{D}_{-1}} \setminus \mathfrak{D}_{-1}) \subseteq \mathbb{R}$ . By Liouville theorem <sup>12</sup>  $F_{1,A}$  is constant (which means  $\alpha$  is also constant) and, since  $\tau$  is a reparametrization of  $\alpha$  we have

$$\omega(\tau_t(A)) = \omega(\tau_0(A)) = \omega(A).$$

■

An equivalent (and perhaps more transparent) KMS condition can be given,

**Proposition 3.1.1.** *Let  $\mathcal{A}, \tau$  be a  $C^*$  or  $W^*$ -dynamical system, a state  $\omega$  is a  $(\tau, \beta)$ -KMS state if and only if the condition*

$$\omega(A\tau_{i\beta}(B)) = \omega(BA) \quad (3.8)$$

*holds for all  $A, B \in \mathcal{A}$ .*

*Proof.* Assume  $\omega$  is a KMS state with respect to  $\tau$  at a temperature  $\beta$  and let  $F_{A,B}$  the witness of this. Evaluating (3.15) and (3.16) in  $i\beta$  and 0 respectively gives us

$$\omega(A\tau_{i\beta}(B)) = \omega(BA).$$

Conversely, if the condition (3.8) holds then, define  $F_{A,B}(t) = \omega(A\tau_t(B))$ . It follows that

$$\begin{aligned} F_{A,B}(t + i\beta) &= \omega(A\tau_{t+i\beta}(B)) = \omega(A\tau_{i\beta}(\tau_t(B))) \\ &= \omega(\tau_t(B)A). \end{aligned}$$

■

**Definition 3.1.9.** *Let  $\mathcal{A}$  be a von Neumann algebra and  $\omega$  a faithful and normal state on  $\mathcal{A}$ . Let  $\pi_\omega$  be the GNS representation of  $\mathcal{A}$  associated to  $\omega$  and  $\Delta$  the modular operator (see appendix E.2) associated to  $(\pi_\omega(\mathcal{A}), \Omega_\omega)$ . Then, Tomita's theorem define a one-parameter automorphism<sup>13</sup> group  $\tau_t^\omega \in \text{Aut } \mathcal{A}$  as*

$$\tau_t^\omega(A) = \pi_\omega^{-1}(\Delta^{it}\pi_\omega(A)\Delta^{-it}).$$

*The 1-parameter group is called the modular automorphism group associated with the pair  $(\mathcal{A}, \varphi)$ .*

**Proposition 3.1.2.** *Let  $\mathcal{A}$  be a von Neumann algebra,  $\omega$  a faithful and normal state. Then,  $\omega$  is a KMS state at "temperature"<sup>14</sup>  $\beta = -1$  with respect to the modular automorphism group associated with  $(\mathcal{A}, \omega)$ .*

<sup>12</sup>Every holomorphic function  $f$  for which there exists  $M > 0$  such that  $|f(z)| \leq M$  for all  $z \in \mathbb{C}$  is constant.

<sup>13</sup>Notice this is the converse of theorem 3.1.3.

<sup>14</sup>Some times we refer to the temperature at the value of  $\beta = 1/kT$ .

*Proof.* Consider  $(\mathcal{H}, \pi, \Omega)$  the GNS representation of  $\mathcal{A}$  associated to  $\omega$ .

$$\begin{aligned}
\omega(A\tau_{-i}^\omega(B)) &= \langle \Omega | \pi(A) \pi \left( \pi^{-1} \left( \Delta^{1/2} \Delta^{1/2} \pi(B) \Delta^{-1/2} \Delta^{1/2} \right) \right) \Omega \rangle \\
&= \left\langle \Delta^{1/2} \pi(A^*) \Omega \mid \Delta^{1/2} \pi(B) \underbrace{\Delta^{-1/2} \Delta^{1/2}}_{J\Delta^{1/2}J} \Omega \right\rangle \\
&= \left\langle \Delta^{1/2} \pi(A^*) \Omega \mid \Delta^{1/2} \pi(B) S(\Delta^{1/2} J \Omega) \right\rangle \\
&= \left\langle \Delta^{1/2} \pi(A^*) \Omega \mid \Delta^{1/2} \pi(B) \underbrace{J\Delta^{1/2}}_S \Omega \right\rangle \\
&= \left\langle \Delta^{1/2} \pi(A^*) \Omega \mid \Delta^{1/2} \pi(B) \Omega \right\rangle \\
&= \left\langle J\Delta^{1/2} \pi(B) \Omega \mid J\Delta^{1/2} \pi(A^*) \Omega \right\rangle \\
&= \langle \pi(B^*) \Omega \mid \pi(A) \Omega \rangle = \langle \Omega \mid \pi(BA) \Omega \rangle = \omega(BA).
\end{aligned} \tag{3.9}$$

where we used  $J\Delta^{1/2}J = \Delta^{-1/2} = SJ$  as a result of proposition E.2,  $J$  is the unique anti-linear transformation of  $\mathcal{H}$  such that  $J^2 = 1$  and  $\langle Jf, Jg \rangle = \langle g, f \rangle$  and that  $S(A\Omega) = A^*\Omega$ . Hence, (3.8) holds. ■

**Remark 3.1.8.** *In the former proof, the condition*

$$\left\langle J\Delta^{1/2} \pi(B) \Omega \mid J\Delta^{1/2} \pi(A^*) \Omega \right\rangle = \langle \pi(B^*) \Omega \mid \pi(A) \Omega \rangle$$

*is an equivalent KMS condition. Indeed, the condition uniquely determines the operator  $S$  acting by  $S(\pi(B)\Omega) = \pi(B^*)\Omega$  which, in turn, establishes all the equalities in (3.9) by proposition E.2.*

**Proposition 3.1.3.** *Let  $\mathcal{A}$  be a von Neumann algebra,  $\omega$  a state and let  $(\mathcal{H}, \pi, \Omega)$  be the GNS representation of  $\mathcal{A}$  of  $\omega$ . Then, the KMS condition can be formulated in the following equivalent ways:*

$$1. \left\langle \Delta^{1/2} A \Omega \mid \Delta^{1/2} B \Omega \right\rangle = \langle B^* \Omega \mid A^* \Omega \rangle, \quad \forall A, B \in \mathcal{A}, \tag{3.10}$$

$$2. J\Delta^{1/2} A \Omega = A^* \Omega, \quad \forall A \in \mathcal{A}. \tag{3.11}$$

*Proof.* The first one follows from remark 3.1.8 and the later describes the operator  $S$ . ■

KMS states are mathematically satisfactory states that characterize thermal states in quantum field theory. In order to link the abstract idea with physical applications, let us remember how thermal states are described. Let  $\mathcal{H}$  be a Hilbert space and consider a system with Hamiltonian  $H$  in equilibrium with a heat bath without exchange of particles at inverse temperature  $\beta$ .

**Definition 3.1.10.** *The density  $\rho_\beta$  defined by*

$$\rho_\beta = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}$$

*is said to be a  $\beta$ -Gibbs state and, has the property of being invariant under evolution of the system, that is, the state is in (thermal) equilibrium.*

**Remark 3.1.9.**

- The map  $A \mapsto \text{Tr}(\rho_\beta A)$  defines a state  $(\omega_{\rho_\beta})$  as defined earlier) in the sense of 3.1.3.
- Assume that  $\{|n\rangle : n \in \mathbb{N}\}$  are all the eigenvectors, associated with the eigenvalues  $E_n$ , of  $H$ . Then, the thermal vector is realized as a state

$$|\beta\rangle = \frac{\sum_n e^{-\beta E_n/2} |n\rangle \otimes |n\rangle}{\sqrt{\sum_m e^{-\beta E_m}}} \quad (3.12)$$

in the Hilbert space  $\mathcal{H} \otimes \mathcal{H}$ . In this case,

$$|\beta\rangle \langle \beta| = \frac{\sum_{n,m} e^{-\beta(E_n+E_m)/2} |n\rangle \langle m| \otimes |n\rangle \langle m|}{\sum_m e^{-\beta E_m}} = \rho_\beta.$$

**Theorem 3.1.10.** Let  $(\mathfrak{B}(\mathcal{H}), \tau)$  be the  $W^*$ -dynamical system with Schrödinger evolution,

$$\begin{aligned} \tau : \mathbb{C} &\rightarrow \text{Aut}(\mathfrak{A}(\mathcal{H})) \\ z &\mapsto \tau_z : \mathfrak{B}(\mathcal{H}) \rightarrow \mathfrak{B}(\mathcal{H}) \\ A &\mapsto e^{iHz} A e^{-iHz} \end{aligned}$$

then, a state  $\omega \in \mathfrak{B}(\mathcal{H})$  is a  $\beta$ -Gibbs state if and only if it is a  $(\tau, \beta)$ -KMS state.

*Proof.* Look in [26]. ■

**Corollary 3.1.10.1.** A  $\beta$ -Gibbs state is invariant under evolution of the system.

*Proof.* This is the result of the theorem 3.1.7. ■

### 3.1.2 The KMS condition on the Unruh effect

In section 1, we saw that a Rindler observer identify the Minkowski vacuum (1.40) as a thermal state and, by means of theorem 3.1.10 the state  $\rho$  in (1.41) must be a KMS state in the operator algebra of  $\mathcal{H}$ . Let us observe the explicit form of this result. We consider the discrete level system in (1.38) with a Hamiltonian  $H$  and denote the eigenvectors  $|n, R\rangle$  by  $|n\rangle$  with associated eigenvalues  $E_n$ . The expectation value of the operator  $A$  in a thermal state with temperature  $T = \beta^{-1}$  is

$$\begin{aligned} \langle A \rangle_\beta &= \omega_\rho(A) = \text{Tr}(\rho_\beta A) = \langle \beta | A^{(e)} | \beta \rangle \\ &= \frac{\sum_n e^{-\beta E_n} \langle n | A | n \rangle}{\sum_m e^{-\beta E_m}} \\ &= \frac{\text{Tr}(e^{-\beta \hat{H}} A)}{\text{Tr}(e^{-\beta \hat{H}})} \end{aligned}$$

where  $A^{(e)} = I \otimes A$ . The time evolution operator is taken to be  $\exp(-iH^{(e)}\tau)$ .

**Proposition 3.1.4.** *The time evolution operator is given by*

$$\exp(-iH^{(e)}\tau) = I \otimes \exp(-iH\tau). \quad (3.13)$$

*Proof.* Let  $A, B$  be two operators and denote  $e^A := \exp(A)$ . Then,

$$e^A \otimes e^B = \sum_n \frac{A^n}{n!} \otimes \sum_m \frac{B^m}{m!} = \underbrace{\sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} A^n \otimes B^k}_{L_n}.$$

If  $A$  and  $B$  commutes, the binomial theorem holds in the following form,

$$(A \otimes I + I \otimes B)^n = \sum_{k=0}^n \binom{n}{k} A^n \otimes B^k = L_n.$$

Thus,

$$e^A \otimes e^B = e^{A \otimes I + I \otimes B}.$$

Using the Baker-Campbell-Hasudorff formula, and since  $A$  and  $B$  commute,

$$e^A \otimes e^B = e^{A \otimes I + I \otimes B} = e^{A \otimes I} e^{I \otimes B}.$$

Then,

$$e^{-iH^{(e)}\tau} = e^{-i\tau(I \otimes H + I \otimes 0)} = e^{-i\tau 0} \otimes e^{-i\tau H} = I \otimes \exp(-i\hat{H}\tau).$$

■

Now, define an antiunitary involution  $J^{(e)}$  acting as

$$J^{(e)} \alpha |n\rangle \otimes |m\rangle = \alpha^* |m\rangle \otimes |n\rangle, \quad \alpha \in \mathbb{C}. \quad (3.14)$$

In particular, it is not hard to see that equation (1.40) on section 1.2 implies that the two-dimensional model with only the left movers (or only the right movers) has an involution  $J^{(e)}$  defined by requiring

$$\begin{aligned} J^{(e)} |0_M\rangle &= |0_M\rangle, \\ J^{(e)} \hat{a}_{+\omega}^R J^{(e)} &= \hat{a}_{+\omega}^L, \\ J^{(e)} \hat{a}_{+\omega}^{R\dagger} J^{(e)} &= \hat{a}_{+\omega}^L. \end{aligned}$$

Observe that the operator  $J^{(e)}$  commutes with the time evolution operator as

$$\begin{aligned} J^{(e)} \exp(-iH^{(e)}\tau) \alpha |n\rangle \otimes |m\rangle &= J^{(e)} \alpha |n\rangle \otimes \exp(-iH\tau) |m\rangle \\ &= \alpha^* \exp(-iH\tau) |m\rangle \otimes |n\rangle \\ &= \exp(-iH\tau) \alpha^* |m\rangle \otimes |n\rangle \\ &= \exp(-iH^{(e)}\tau) J^{(e)} \alpha |n\rangle \otimes |m\rangle, \end{aligned}$$

thus,

$$J^{(e)} \exp(-iH^{(e)}\tau) = \exp(-iH^{(e)}\tau) J^{(e)}. \quad (3.15)$$

Therefore,  $J^{(e)}$  satisfies one (and therefore all) conditions of proposition E.2 with respect to the time evolution operator. Furthermore, uniqueness of  $J^{(e)}$  follows from Tomita's theory and the polar decomposition. Once the involution is found, let us take a look at the KMS-condition defined on equation (3.11). Let  $A \in \mathfrak{B}(\mathcal{H})$  be given by a matrix  $A_{mn}$  as  $A|n\rangle = \sum_m A_{mn} |m\rangle$  then,

$$\begin{aligned} \exp(-H^{(e)}\beta/2)A^{(e)}|\beta\rangle &= (1 \otimes \exp(-H\beta/2))1 \otimes A|\beta\rangle \\ &= (1 \otimes \exp(-H\beta/2))1 \otimes Ak \sum_n e^{-\beta E_n/2} |n\rangle \otimes |n\rangle \\ &= k \sum_{n,m} e^{-\beta E_n/2 - \beta E_m/2} A_{mn} |n\rangle \otimes |m\rangle \\ &= k \sum_{n,m} e^{-\beta E_n/2 - \beta E_m/2} J^{(e)} A_{mn}^* |m\rangle \otimes |n\rangle \\ &= k \sum_m e^{-\beta E_m/2} J^{(e)} A^{(e)\dagger} |n\rangle \otimes |n\rangle \\ &= J^{(e)} A^{(e)\dagger} |\beta\rangle, \end{aligned}$$

where  $k$  is the normalization constant of  $|\beta\rangle$ . If we apply  $J$  on both sides, it follows that

$$J^{(e)} \exp(-\hat{H}^{(e)}\beta/2) \hat{A}^{(e)} |\beta\rangle = \hat{A}^{(e)\dagger} |\beta\rangle, \quad (3.16)$$

establishing the KMS-condition (3.11) with respect to the time evolution, making the state  $\rho$  a  $2\pi/a$ -KMS state.

**Remark 3.1.11.** *The former example allows us to define the Unruh effect in the algebraic theory,*

#### *The Unruh effect*

*The Unruh effect is the effect in which the Minkowski vacuum restricted to the right/left Rindler wedge is a  $2\pi/a$ -KMS state.*

**Remark 3.1.12.** *In the two-dimensional model of the Unruh effect the involution  $J^{(e)}$  is in fact the PCT transformation,*

$$\varphi(t, z) \rightarrow \varphi(-t, -z).$$

*This is no coincidence and we shall address this question in section 3.3.*

## 3.2 Algebraic Quantum Field Theory

We motivated the algebraic tools in order to work in quantum field theory now, we present the so-called Algebraic Quantum Field Theory. For this purpose, we define a list of assumptions that a quantum field theory must have. Even though all of them are motivated by physical/structural ideas, we keep the study as broad as possible and thus, an algebraic theory need not fulfill all



the assumptions. However, the ones marked with a  $(\dagger)$  will not be essential in order to define the Fewster-Verch (FV) QMT scheme in section 4. Notations (and some examples) regarding causal/convex geometry in curved spacetimes can be found in the appendix F if not defined in the main body.

### 3.2.1 Assumptions of AQFT in CST (Haag-Kastler)

**Definition 3.2.1.** *Let  $M$  be a space-time. For each open region  $\mathcal{O} \subseteq M$ , an associated unital  $C^*$ -algebra of observables, which can be measured within the spacetime region  $\mathcal{O}$  is denoted by*

$$\mathcal{A}(\mathcal{O}) = \{A \in \mathcal{A} : A \text{ is supported in } \mathcal{O}\}.$$

**Definition 3.2.2.** *We say that*

$$\mathcal{A} = \{\mathcal{A}(\mathcal{O}) : \mathcal{O} \subseteq M \text{ is causally open}\}$$

*is the net of observable algebras.*

**Definition 3.2.3.** *We call  $A$  a quasi-local algebra if it contains observables that can be uniformly approximated by local observables.*

Let  $\mathcal{A}$  be a **non-trivial** net of observables, we require that the following assumptions to hold.

#### A1 Isotony

Let  $\mathcal{O}_1$  and  $\mathcal{O}_2$  be two open regions in the space-time  $M$  such that  $\mathcal{O}_1 \subset \mathcal{O}_2 \subset M$  then,

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2).$$

#### A2 Einstein causality (Microcausality)

Let  $\mathcal{O}_1$  and  $\mathcal{O}_2$  be two causally disjoint regions, that is,  $\mathcal{O}_1 \subseteq \mathcal{O}_2^\perp$  then,

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0.$$

#### A3 Compatibility

Let  $\mathcal{O}$  be a causally convex ( $\mathcal{O} = \text{ch}(\mathcal{O})$ ) open region of space-time, there exist a morphism  $\alpha : \mathcal{A}(M|_{\mathcal{O}}) \rightarrow \mathcal{A}(M)$

$$\alpha_{M;\mathcal{O}}(\mathcal{A}(M|_{\mathcal{O}})) = \mathcal{A}(\mathcal{O})$$

and such that if  $\mathcal{O}_3 \subset \mathcal{O}_2 \subset \mathcal{O}_1$

$$\alpha_{\mathcal{O}_1; \mathcal{O}_2} \circ \alpha_{\mathcal{O}_2; \mathcal{O}_3} = \alpha_{\mathcal{O}_1; \mathcal{O}_3}.$$

$\alpha_{M; \mathcal{O}}$  is an embedding (unital injective  $*$ -homomorphism) from the local algebra  $\mathcal{A}(\mathcal{O})$  onto the local algebra  $\mathcal{A}(M)$ .

#### A4 Time-slice property

If  $M$  is globally hyperbolic and  $N \subseteq M$  contains a Cauchy surface of  $M$  then,  $\mathcal{A}(N) = \mathcal{A}(M)$ .

#### A5 Poincaré covariance

To every transformation  $\rho$  in the identity connected component  $\mathcal{P}_0$  of the Poincaré group, there is an automorphism  $\alpha(\rho)$  of  $\mathcal{A}(M)$  such that

$$\alpha(\rho) : \mathcal{A}(\mathcal{O}) \rightarrow \mathcal{A}(\rho\mathcal{O})$$

such that  $\alpha(id) = id_{\mathcal{A}(M)}$  and naturally  $\alpha(\sigma) \circ \alpha(\rho) = \alpha(\sigma \circ \rho)$  is required for any  $\sigma, \rho \in \mathcal{P}_0$ .

#### A5<sup>†</sup> Haag property

$\mathcal{A}(N^\perp) = \mathcal{A}(N)'$   $x \in \mathcal{A}(M; N)$  means that an observable is localizable in  $N$  if  $x = x^*$  its self adjoint

#### Remark 3.2.1.

- *Isotony assumption carries the idea of any observable measurable in a region  $\mathcal{O}_1$  must be measurable in any region  $\mathcal{O}_2$  containing  $\mathcal{O}_1$ . Also, isotony allow us to relax the construction of the net of observables algebras. Instead of defining a local algebra for each open set, it suffices to associate algebras to diamonds (e.g. in Minkowski space-time double cones) in the space-time and consider the sub algebras within them, a further explanation of these constructions is found in the examples.*
- *The main relativistic assumption of AQFT is contained in the Einstein Causality.*
- *One can restrict ourselves into an open subset of the space-time. In this case, the compatibility requirement states that the algebra can be seen by its own and, as the subalgebra of the space. In both cases, they must coincide.*
- *The existence of a dynamical law is not yet implemented with requirements A1-A4 however, in order to determine the dynamics, it is enough to see the dynamics at a foliation of the space-time. That is, the time-slice property.*

### 3.2.2 Construction of an algebra from generators and relations

The task of specifying local algebras for all open regions by hand is, in general, very restrictive. We may equivalently present a construction of algebras in terms of generators and relations. Start by defining an unital  $\ast$ -algebra  $\mathcal{U}$  spanned by elements  $\varphi(f)$  and  $\varphi(f)^\ast$  labeled by a smooth function  $f \in C_0^\infty(L)$ ,  $L \subseteq M$ . The relations can be implemented by taking the quotient  $\mathcal{U}/I$ , where  $I$  is the  $\ast$ -ideal (an ideal invariant under  $\ast$ -operation) generated by the relations. The resulting algebra  $\mathcal{A}(M) = \mathcal{U}/I$  is defined so that

$$[A][B] = [AB], \quad [A]^\ast = [A^\ast], \quad 1_{\mathcal{A}(M)} = [1_{\mathcal{U}}]$$

holds. Being  $I$  a  $\ast$ -ideal guarantees that these operations are well-defined.

**Example 3.2.1** (The free scalar field, we follow [32]). *Consider the Lagrangian,*

$$\mathcal{L} = \frac{1}{2}(\nabla_\mu \varphi) \nabla^\mu \varphi - \frac{1}{2}m^2 \varphi^2,$$

*which has the field equation*

$$P\varphi = 0, \quad P = \square + m^2.$$

*We want to describe the algebra  $\mathcal{A}(M)$  of this quantum field theory. Let  $G_A$  and  $G_R$  be the Green functions of the operator  $P$  and let  $E_P^+$  and  $E_P^-$  be their corresponding retarded and advanced Green operators. To ensure convergence of these operators, we restrict the domain to  $C_0^\infty(M)$  i.e. smooth functions of compact support. In this case, the operators  $E_P^\pm$  and  $E_P := E_P^- - E_P^+$  satisfy*

$$\text{supp}(E_P^\pm f) \subseteq J^\pm(\text{supp } f), \tag{3.17}$$

$$PE_P^\pm f = f \quad \forall f \in C_0^\infty(\mathbb{R}) \tag{3.18}$$

$$E_P^\pm Pf = f \quad \forall f \in C_0^\infty(\mathbb{R}) \tag{3.19}$$

$$\ker E_P = PC_0^\infty(\mathbb{R}). \tag{3.20}$$

*Therefore,  $\varphi := E_P f := (E_P^- - E_P^+)f$  solves the field equation and, in fact, every solution is of this form (see [33] for details). Define the operator  $E_P : C_0^\infty(M) \times C_0^\infty(M) \rightarrow \mathbb{C}$  by*

$$E_P(f, g) = \int_M f(x) E_P g(x) dv = \int f(x) g(x) E_P(x, y) \tag{3.21}$$

*where  $E_P(x, y)$  is the integral kernel of this operator. Notice that the operator  $E_P$  satisfies:*

$$E_P(x, y)|_{y^0=x^0} = 0 \quad \partial_{y^0} E_P(x, y)|_{y^0=x^0} = \delta(x - y)$$

*that implies that space-like solutions commute.*

**The  $\ast$ -algebra quantization.** Let  $N \subseteq M$ , the net of  $\ast$ -algebras  $\mathcal{A}(N)$  are generated by a unit element together with the generators  $\varphi(f) = E_P f$ ,  $f \in C_0^\infty(N)$  such that:

1.  $f \rightarrow \varphi(f)$
2.  $\varphi(\bar{f}) = \varphi(f)^\ast$ ,  $\forall f \in C_0^\infty(N)$
3.  $\varphi(Pf) = 0$ ,  $\forall f \in C_0^\infty(N)$  where  $P$  is the dynamics,  $P = \square + m^2$ .
4.  $[\varphi(f), \varphi(h)] = iE_P(f, h)\mathbb{1}$ ,  $\forall f \in C_0^\infty(N)$ ,

$$E_P(f, h) := \int_M d\text{vol } f E_P h.$$

Clearly, linear complex combinations of the unit lie in  $\mathbb{C}$  and, finite linear complex combinations of  $\varphi(f)$  lie in  $Q^n$  for some  $n$ . Thus, the underlying vector space of  $\mathcal{A}(M)$  is isomorphic to the symmetric tensor vector space

$$\mathbb{C} \oplus \bigoplus_{n=1}^{\infty} Q^{\otimes n}, \quad Q = C_0^\infty(M)/PC_0^\infty(M), \quad (3.22)$$

where  $Q^n$  is the symmetrized tensor product. In order to see that  $\mathcal{A}(M)$  is non-trivial, we shall look whether the quotient space  $Q$  is non-trivial. In fact, any solution of the field equation can be written as the image under  $E$  of some element of  $C_0^\infty(\mathbb{R})$ , it follows from the isomorphism theorems (as vector spaces),

$$\begin{aligned} Q &:= C_0^\infty(M)/PC_0^\infty(M) = C_0^\infty(M)/\ker E \cong \text{im } E = \ker P \\ &\cong \{\varphi \in C_{sc}^\infty(M) : P\varphi = 0\} =: \text{Sol}(M) \end{aligned} \quad (3.23)$$

where we used equation (3.17) together with the definition of 'spatially compact' support i.e., functions that vanish in the causal complement of the set. Thus,  $Q$  is isomorphic to the space of smooth Klein-Gordon solutions with spatially compact support ( $\text{Sol}(M)$ ) and therefore,  $Q$  is nontrivial. We are in position to verify the assumptions of AQFT.

**A1** If  $\mathcal{O}_1 \subset \mathcal{O}_2$  then, clearly  $f \in C_0^\infty(\mathcal{O}_1) \subset C_0^\infty(\mathcal{O}_2)$ . It follows that  $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$  and thus, isotony is satisfied.

**A2** If  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are causally disjoint regions, let  $f \in C_0^\infty(\mathcal{O}_1)$  and  $g \in C_0^\infty(\mathcal{O}_2)$  then,  $E_P g$  is supported in the union of the causal future and past of  $\text{supp } g$  and thus, the operator defined in (3.21) give  $E_P(f, g) = 0$ . It follows that all generators of  $\mathcal{A}(\mathcal{O}_1)$  commute with all generators of  $\mathcal{A}(\mathcal{O}_2)$  and thus, Einstein causality is satisfied.

**A3** Let  $N \subseteq M$ , notice that the algebra defines a trivial morphism map  $\alpha_{M;N} : \mathcal{A}(N) \rightarrow \mathcal{A}(M)$

$$\alpha_{M;N}\varphi(f) = \varphi(f), \quad f \in C_0^\infty(N) \subseteq C_0^\infty(M)$$

in which every  $\varphi(f)$  can be seen as an element of  $\mathcal{A}(M)$  with generator  $f$  supported on  $N \subseteq M$  satisfying the compatibility requirement.

**A4** Let  $\mathcal{O}_1 \subseteq \mathcal{O}_2$  such that  $\mathcal{O}_1$  contains a Cauchy surface of  $\mathcal{O}_2$ . Choose  $\varphi(f_2) = Ef_2 \in \mathcal{O}_2$  for some  $f_2 \in C_0^\infty(\mathcal{O}_2)$ . We want a  $f_1 \in C_0^\infty(\mathcal{O}_1)$  such that  $\varphi(f_2) = \varphi(f_1)$ . Let  $f_1 := P\chi\phi(f_2)$ , where  $\chi \in C^\infty(\mathcal{O}_2)$  and is defined so that  $\chi$  vanishes on the future of one Cauchy surface of  $\mathcal{O}_2$  contained in  $\mathcal{O}_1$  and is 1 in the past of another (and thus, 1 in the Cauchy surface of  $\mathcal{O}_1$ ). Then,  $f_1$  is supported in  $\mathcal{O}_1$  and  $\varphi(f_2) = \varphi(f_1)$ , which implies that  $\mathcal{A}(\mathcal{O}_2) = \mathcal{A}(\mathcal{O}_1)$  establishing the time-slice property.

**A5** Let  $\rho \in \mathcal{P}_0$ . Then the Poncairé covariance of  $\square + m^2$  and  $E$  can be used to show that the map of generators  $\alpha(\rho)\varphi(f) = \varphi(\rho_*f)$ ,  $(\rho_*f)(x) = f(\rho^{-1}(x))$ , is compatible with the relations and extends to a well defined unit preserving  $*$ -isomorphism

$$\alpha(\rho) : \mathcal{A}(M) \rightarrow \mathcal{A}(M).$$

Clearly  $\alpha(\rho)$  maps each  $\mathcal{A}(\mathcal{O})$  to  $\mathcal{A}(\rho\mathcal{O})$ ; as we also have  $\alpha(\sigma) \circ \alpha(\rho) = \alpha(\sigma \circ \rho)$ , condition 4 holds.

Therefore, the algebra  $\mathcal{A}(M)$  generated by  $\varphi(f)$  and its relations satisfies the AQFT assumptions.

**Remark 3.2.2.** The real scalar field admits a global  $Z_2$  gauge symmetry generated by  $\varphi(f) \rightarrow -\varphi(f)$  and, if we consider a complex field, the algebra admits a further  $U(1)$  gauge symmetry given on the generators by

$$\eta_\alpha(\phi(f)) := e^{-i\alpha\varphi(f)}.$$

In either case, the "physically observable" algebra will be given by a further quotient consisting of all elements of  $\mathcal{A}(\mathcal{O})$  that are invariant under  $U(1)$  or  $U(1) \times Z_2$  gauge actions. Hence, the afore scalar field model a theory of a conserved charge where opposite charge can be physically distinguished.

**Example 3.2.2** (Real scalar field with external source, [32]). Let  $\rho \in D'(M)$  be a distribution such that  $\overline{\rho(f)} = \rho(\bar{f})$ .

$$(\square + m^2)\Phi_\rho = -\rho. \tag{3.24}$$

**The  $\ast$ -algebra quantization.** Let  $N \subseteq M$ , the net of  $\ast$ -algebras  $\mathcal{A}(N)$  are generated by a unit element together with the generators

$$\varphi_\rho(f) = \varphi(f) + \phi_\rho(f)1, \quad f \in C_0^\infty(N)$$

where  $\phi_\rho \in D'(M)$  is any weak solution to  $(\square + m^2)\phi_\rho = -\rho$  and  $\varphi(f)$  are the generators of the free field. Notice that the generators satisfy the relations

1.  $f \rightarrow \phi_\rho(f)$  is complex linear
2.  $\phi_\rho(f)^\ast = \phi_\rho(\bar{f})$
3.  $\phi_\rho((\square + m^2)f) + \rho(f)1 = 0$
4.  $[\phi_\rho(f), \phi_\rho(g)] = iE(f, g)1$

**Remark 3.2.3.** The algebras in the examples closely relate to each other, this is a general feature of AQFT which allow us to have a standard construction and, by hand, introduce further restrictions. However, the important construction is how the elements can be labeled by functions, in this case with the help of the Green operators.

### 3.2.3 Quasifree states for the free scalar field

Given the relations and generators on an  $\ast$ -algebra  $\mathcal{A}(M)$ , an element  $A \in \mathcal{A}(M)$  can be written as a polynomial

$$A = c_0 1 + \sum_{i_1} c_1^{i_1} \phi(f_{i_1}^1) + \cdots + \sum_{i_1, \dots, i_n} c_n^{i_1, \dots, i_n} \phi(f_{i_1}^n) \dots \phi(f_{i_n}^n) = c_i^{i_1, \dots, i_k} \phi(f_{i_1}^n \dots f_{i_k}^n)$$

on the generators of  $\mathcal{A}(M)$ . Consequently, given a state  $\omega : \mathcal{A}(M) \rightarrow \mathbb{C}$ , its action on  $A$  is completely determined once the action of the  $n$ -point functions of  $\omega$ ,

$$\omega_n(f_1, \dots, f_n) := \omega(\phi(f_1) \dots \phi(f_n))$$

are known. Since  $\mathcal{A}(M)$  is an  $\ast$ -algebra, we have not yet endowed  $\mathcal{A}(M)$  with a topology and questions about continuity and convergence of the functionals  $\omega_n(f_1, \dots, f_n)$  may be discussed in the final topology of the  $\omega_n$ 's that is, where  $\omega_n$  is continuous in the usual test function topology on  $C_0^\infty(M)$ . By the Schwartz kernel theorem, we can write the  $n$ -point function in terms of its distributional kernel:

$$\omega_n(f_1, \dots, f_n) = \int_M \omega_n(x_1, \dots, x_n) f_1(x_1) \dots f_n(x_n) dv. \quad (3.25)$$

Hence,  $\omega$  being normalized can be verified equivalent in the  $n$ -point functions by setting

$$1 = c_0 + \sum_{i_1} c_1^{i_1} \omega_1(f_{i_1}^1) + \cdots + \sum_{i_1, \dots, i_n} c_n^{i_1, \dots, i_n} \omega(f_{i_1}^n, \dots, f_{i_n}^n).$$

Despite this characterization, the positive requirement  $\omega(A^*A) \geq 0$  is still unknown and need to be verified for each observable  $A$ . Therefore, the construction of states and their corresponding GNS representations are nontrivial problems. For instance, given a "state"  $\omega$ , the positive requirement need to compute (and verify)  $\omega(AA^*) \geq 0$  for all  $A \in \mathcal{A}$  and, in the other hand, the GNS representation need to be specified by hand. Nevertheless, this task is greatly simplified if the state is one of the so-called quasifree states. Quasifree states (also called Gaussian states) became useful because they are completely determined by their two-point functions and their underlying Hilbert space is canonically build so that imitates the Fock space of the Minkowski vacuum which allows us to retrieve many results familiar from QFT.

**Definition 3.2.4.** Let  $\omega : \mathcal{A} \rightarrow \mathbb{C}$  be a state of the algebra  $\mathcal{A}$ .  $\omega$  is said to be quasifree if its  $n$ -point functions satisfy

$$\omega_n(f_1, \dots, f_n) = \begin{cases} 0 & \text{for } n \text{ odd} \\ \sum_{\mathcal{G}_{2n}} \omega_2(x_{i_1}, x_{i_2}) \dots \omega_2(x_{i_{n-1}}, x_{i_n}) & \text{for } n \text{ even} \end{cases}$$

where the sum is taken to all partitions of  $\{1, \dots, n\}$  into  $n/2$  pairwise disjoint subsets of elements  $\{i_1, i_2\}, \{i_3, i_4\}, \dots, \{i_{n-1}, i_n\}$  with  $i_{2k-1} < i_{2k}$  for  $k = 1, 2, \dots, n/2$ .

**Remark 3.2.4.** Since quasifree states are also characterized with the 2-point functions, we may write the two point function  $\omega_2(f, g)$  for quasifree states as

$$W(f, g) := \omega_2(f, g).$$

Therefore, a quasifree state can be written (that's why  $\omega$  is also called Gaussian) as

$$\omega(e^{i\varphi(f)}) = e^{-W(f, f)/2}, \quad f \in C_0^\infty(M, R). \quad (3.26)$$

The importance of quasifree states relies on their representations. For this class of states the  $*$ -algebras and Weyl  $C^*$ -algebras give equivalent representations and therefore, as we saw earlier (or the appendix), many constructions are given explicitly.

**Example 3.2.3.** The Minkowski vacuum state  $\omega_0$  is a quasifree state. Indeed, given a free field  $\varphi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{\omega}} (a_k e^{-ik \cdot x} + a_k^\dagger e^{ik \cdot x})$ , the smeared field  $\varphi(f)$  is given by

$$\varphi(f) = \int d^4x f(x) \varphi(x).$$

Therefore, the two-point function is given by

$$\begin{aligned}
W(f, g) &= \langle 0_M | \varphi(f) \varphi(g) | 0_M \rangle \\
&= \langle 0_M | \int d^4 y \frac{d^3 k'}{(2\pi)^3} \frac{1}{\sqrt{2\omega'}} f(y) \left( a_{k'} e^{-ik' \cdot y} + a_{k'}^\dagger e^{ik' \cdot y} \right) \\
&\quad \times \int d^4 x \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} f(x) \left( a_k e^{-ik \cdot x} + a_k^\dagger e^{ik \cdot x} \right) | 0_M \rangle \\
&= \int d^4 y \frac{d^3 k'}{(2\pi)^3} d^4 x \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega} f(y) g(x) \langle 0_M | a_{k'} e^{-ik' \cdot y} a_k^\dagger e^{ik \cdot x} | 0_M \rangle \\
&= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega} \overline{f(\hat{k})} \hat{g}(k)
\end{aligned}$$

where  $\omega = \sqrt{|k|^2 + m^2}$  and  $\hat{g}(k) = \int d^4 x e^{ik \cdot x} g(x)$ .

One can define a one-particle Hilbert space in the following way<sup>15</sup>: Let  $\mathcal{H}$  be the square integrable functions in the hyperboloid  $H_m^+$   $k \cdot k = m^2$ ,  $k^0 > 0$  in  $\mathbb{R}^4$  with respect to the measure

$$d\mu(S) = \frac{d^3 k}{(2\pi)^3} \frac{\chi_S(k)}{2\omega} \quad \chi_S(k) = \begin{cases} 1 & k \in S \\ 0 & k \notin S \end{cases}.$$

Notice that the set of functions  $\hat{f}$  supported in the hyperboloid is dense in  $\mathcal{H}$  thus, we can define an inner product such that

$$\langle K\varphi(f), K\varphi(g) \rangle_{\mathcal{H}} = W(f, g)$$

where the map  $K : \text{Sol}_{\mathbb{R}}(M) \rightarrow \mathcal{H}$  is given by  $KEg = \hat{g}|_{H_m^+}$ . In this case the Hilbert space is constituted as  $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ . Then, a representation of  $\varphi(g)$  is given by

$$\pi_0(\varphi(g)) = a(\hat{g}|_{H_m^+}) + a^*(\hat{g}|_{H_m^+}), \quad g \in C_0^\infty(M, \mathbb{R}),$$

where

$$a(\hat{g}|_{H_m^+}) = \underbrace{\int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} \hat{g}(\mathbf{k}) a(\mathbf{k})}_{\int d\mu \hat{g}(\mathbf{k}) a(\mathbf{k})}, \quad a^*(\hat{g}|_{H_m^+}) = \underbrace{\int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} \hat{g}(\mathbf{k}) a^*(\mathbf{k})}_{\int d\mu \hat{g}(\mathbf{k}) a^*(\mathbf{k})}. \quad (3.27)$$

If one chooses  $g$  such that  $KEg = e^{ik \cdot r}$  we retrieve our usual free field

$$\pi_0(\varphi(g)) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} \left( a(\mathbf{k}) e^{-ik \cdot r} + a^*(\mathbf{k}) e^{ik \cdot r} \right). \quad (3.28)$$

**Remark 3.2.5.** The choice of  $g$

The above example gives an idea of how to build a (canonical) Hilbert space. In general, let  $W$  be the two point function defined on  $C_0^\infty(M)$ .  $W$  obeys  $W(f, g) - W(g, f) = iE(f, g)$  and therefore, let

<sup>15</sup>For further details such as existence and general features of quasifree states we refer to [34], particularly, Sec. 5.2.4



$w$  be a skew-linear form on  $\text{Sol}(M)$  (defined in (3.23)) by the formula

$$w(Ef, Eg) = W(\bar{f}, g).$$

If the Cauchy-Schwartz inequality in  $w$  is saturated then, one can define a (complex) Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$  where  $\mathcal{H}$  is the completion of  $\text{Sol}_{\mathbb{R}}(M)$  in the norm  $\|\phi\|_w := w(\phi, \phi)^{1/2}$  and  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is defined by

$$\begin{aligned} \langle \phi_1 | \phi_2 \rangle_{\mathcal{H}} &= \langle \phi_1 | \phi_2 \rangle_w + i \langle \phi_1 | J\phi_2 \rangle_w, \\ &= \text{Re } w(\phi_1, \phi_2) + i \text{Im } w(\phi_1, \phi_2), \quad \phi_i \in \mathcal{H} \end{aligned}$$

where we used that multiplication by  $-i$  is implemented by  $J$  because  $w(\phi_1, \phi_2) = iw(\phi_1, J\phi_2)$ . In this Hilbert space the inclusion of  $\text{Sol}_{\mathbb{R}}(M)$  into  $\mathcal{H}$ ,  $K : \text{Sol}_{\mathbb{R}}(M) \rightarrow \mathcal{H}$  has dense image (being  $\mathcal{H}$  the completion of  $\text{Sol}_{\mathbb{R}}(M)$ ) and therefore, the inner product is defined so that

$$\langle KEf | KEg \rangle_{\mathcal{H}} = W(f, g).$$

Accordingly, given a quasifree state  $\omega$ , the representation  $\pi_{\omega} : \mathcal{A}(M) \rightarrow \mathcal{H}$  is defined by the formula

$$\pi_{\omega}(\varphi(f)) = a(KEf) + a^*(KEf)$$

where  $a(\psi)$  and  $a^*(\psi)$  are the annihilation and creation operators on the Fock space which obey the CCRs

$$[a(\phi), a^*(\psi)] = \langle \phi | \psi \rangle_{\mathcal{H}} 1$$

and, we also require that  $\pi_{\omega}$  is  $C^*$ -linear,

$$\pi_{\omega}(\varphi(f)) := \pi_{\omega}(\varphi(\text{Re } f)) + i\pi_{\omega}(\varphi(\text{Im } f))a(KEf) + a^*(KEf)$$

for complex-valued  $f \in C_0^{\infty}(M)$ . Now, we check that the map  $\pi$  correctly represents the algebra.

- The representation obeys the field equation in the sense  $\pi_{\omega}(\varphi(Pf)) = 0$
- The representation is hermitian by the quantization of the  $*$ -algebra  $\mathcal{A}(M)$ .
- For the CCRs, we compute

$$\begin{aligned} [\pi_{\omega}(\varphi(f)), \pi_{\omega}(\varphi(g))] &= [a(KEf), a^*(KEg)] + [a^*(KEf), a(KEg)] \\ &= \langle KEf | KEg \rangle_{\mathcal{H}} 1 - \langle KEg | KEf \rangle_{\mathcal{H}} 1 \\ &= (W(f, g) - W(g, f)) 1 \\ &= iE(f, g) 1. \end{aligned}$$

- The Fock vacuum vector  $\Omega_{\omega}$  satisfies

$$\langle \Omega_{\omega} | \pi_{\omega}(\varphi(f_1)) \cdots \pi_{\omega}(\varphi(f_n)) \Omega_{\omega} \rangle = \omega(\varphi(f_1) \cdots \varphi(f_n)). \quad (3.29)$$

Indeed, using  $\pi_0(\varphi(g)) = a(\hat{g}|_{H_m^+}) + a^*(\hat{g}|_{H_m^+})$ , given an odd number  $n$  the right hand side vanishes and, by the definition of the quasifree state the  $n$ -point function also vanishes. Now, if  $n$  is even, clearly for  $n = 0, 2$  the equality holds and for  $n > 3$ , using the CCRs in the inductive step it is not hard to see that the equality holds.

Consequently,  $\omega$  is seen to be a vector state on  $\mathcal{A}(M)$  and  $(\mathcal{F}(\mathcal{H}), \pi_\omega, \Omega_\omega)$  is its GNS representation whose dense domain  $\mathcal{D}_\omega$  consisting of finite linear combinations of finite products of operators  $a^*(KEf)$  acting on  $\Omega_\omega$ .

**Remark 3.2.6.** Notice that one-particle states (elements of  $\mathcal{H}$ ) can be identified with (complex linear combinations of) vectors generated from  $\Omega_\omega$  by a single application of the field as shown in the example. Due to remark 1.1.4, these vectors are identified as wavepackets of positive frequency modes relative to the choice of the quasifree state  $\omega$ .

### 3.2.4 Vacuum states and the detector problem

Let  $G$  be the symmetry group of  $M$  e.g.  $M$  is Minkowski and  $G$  the Poincaré group. A vacuum state in a algebra  $\mathcal{A}$  must be at least translation invariant under the action of  $G$  but also it may coincide with the definition of being defined by the positive frequency modes that is, a vacuum state must have positive energy in every Lorentz frame. Neither of the above conditions are trivial nor satisfied with the above assumptions; we first formalize the concept vacuum states and afterwards we introduce a further assumptions in order to define a vacuum in an AQFT. We follow [28, 4].

Let  $\omega$  be a translationally invariant state. As we saw in the GNS construction, the translation group can be implemented unitarily by a strongly continuous function  $U(x)$  which also can be written as  $U(x) = e^{iP_\mu x^\mu}$  where  $P^\mu$  are self-adjoint operators by Stone's theorem. In this case, the projection operator  $E(\Delta)$  corresponding to the test of whether the result of 4-momentum measurement  $P^\mu$  would be found to lie in a measurable set  $\Delta$ . The assignment  $\Delta \mapsto E(\Delta)$  is then a PVM and, one can write

$$U(x) = \int e^{ip_\mu x^\mu} dE(p).$$

**Definition 3.2.5.** The state  $\omega$  is said to satisfy the spectrum condition if the support of  $E$  lies in the closed forward cone  $\overline{V^+} = \{p \in \mathbb{R}^4 : p^\mu p_\mu \geq 0, p^0 \geq 0\}$ <sup>16</sup>.

**Definition 3.2.6.** A vacuum state is a translationally invariant state obeying the spectrum condition, whose GNS vector is the unique translationally invariant vector (up to scalar multiples) in the GNS Hilbert space. The corresponding GNS representation is called the vacuum representation.

**A6<sup>†</sup> Weak additivity** For any causally convex open region  $\mathcal{O}$  the  $*$ -algebra is generated by the algebras  $\pi(\mathcal{A}(\mathcal{O} + x))$  as  $x$  runs over  $\mathbb{R}^4$ .

**Remark 3.2.7.**

<sup>16</sup>This implies that the spectrum of  $P^\mu$  lies in  $\overline{V^+}$ .

- The algebra generated by translation is the algebra whose elements are of the form

$$U(x_1)Q_1U^{-1}(x_1)\dots U(x_n)Q_nU^{-1}(x_n) = U(x_1)Q_1U(x_2-x_1)Q_2U(x_3-x_2)\dots U(x_n-x_{n-1})Q_n$$

for  $n \in \mathbb{N}$  and some  $Q_1, \dots, Q_n \in \pi(\mathcal{A}(\mathcal{O} + x))$ ,  $x \in \mathbb{R}^4$ . Weak additivity not also asserts that that this algebra is dense in  $\mathcal{A}(M)$  but also that arbitrary observables can be built as limits of algebraic combinations of translates of observables in any given region  $\mathcal{O}$  (as one would expect in a quantum field theory).

- The translation group is Abelian and thus, its mean  $\mu$  is a translation invariant, positive linear functional of bounded functions on the group  $G$  which is measurable with respect to the Haar measure. As usual, we can define an averaged state  $\rho$  which is of course invariant. Indeed, given a state  $\omega$  of  $A$  define  $\rho$  as

$$\rho(A) := \int \omega(\alpha_x A) d\mu(x).$$

Then, if  $\alpha$  is an action of the translation group on  $\mathcal{A}$ , translation invariant states of  $\mathcal{A}$  exist. In general, the group symmetry is not Abelian for example, the Lorentz group does not admit an invariant mean [28] and thus, the existence of invariant states is not guaranteed.

**Theorem 3.2.8** (Reeh-Schlieder). *Let  $\mathcal{O}$  be any causally convex bounded open region.*

- (a) *The set  $\{A\Omega : A \in \pi(\mathcal{A}(\mathcal{O}))\}$  is dense in  $\mathcal{H}$  (the GNS vector is cyclic)*
- (b) *If  $A \in \pi(\mathcal{A}(\mathcal{O}))$  is such that  $A\Omega = 0$  then,  $A = 0$  (the GNS vector is separating).*

The existence of a cyclic and separating vector on a (von Neumann) algebra is the starting point of Tomita–Takesaki theory.

*Proof.*

- (a) If  $\Omega$  is not cyclic in  $\pi(\mathcal{A}(\mathcal{O}))$  there exist a vector  $\Psi$  such that  $\langle \Psi | A\Omega \rangle = 0$  for all  $A \in \pi(\mathcal{A}(\mathcal{O}))$ .

Let  $\mathcal{O}_1$  be a causally open region such that<sup>17</sup>  $\overline{\mathcal{O}_1} \subset \mathcal{O}$  and, for any  $Q_1, \dots, Q_n \in \pi(\mathcal{A}(\mathcal{O}_1))$  there exist an  $\varepsilon > 0$  such that for all  $\|x\| < \varepsilon$  we have  $U(x)Q_iU(x)^{-1} \in \pi(\mathcal{A}(\mathcal{O}))$  and

$$\langle \Psi | U(x_1)Q_1U(x_2-x_1)Q_2U(x_3-x_2)\dots U(x_n-x_{n-1})Q_n\Omega \rangle = 0 \quad (3.30)$$

for  $\|x_1\| + \dots + \|x_n\| < \varepsilon$ . The spectrum condition implies that  $U(x)$  can be extended to complex vectors. In this case,  $U(x)$  is strongly continuous on  $\mathbb{R}^4 + i\overline{V}^+$  and holomorphic on  $\mathbb{R}^4 + i\text{int } \overline{V}^+$ . It follows that the function

$$F : (\zeta_1, \dots, \zeta_n) \mapsto \langle \Psi | U(\zeta_1)Q_1U(\zeta_2)Q_2\dots U(\zeta_n)Q_n\Omega \rangle$$

<sup>17</sup>This  $\mathcal{O}_1$  always exist because the space is Hausdorff and regular.

is continuous in  $(\mathbb{R}^4 + i \operatorname{int} \overline{V^+})^n \subset (\mathbb{C}^4)^n$  whose boundary value on  $(\mathbb{R}^4)^n$  vanishes in  $(\{|x| < \varepsilon/n : x \in \mathbb{R}^4\})^n$ . The 'edge of the wedge' theorem<sup>18</sup> [16] implies that  $F$  vanishes identically in  $(\mathbb{R}^4 + i \operatorname{int} \overline{V^+})^n$  i.e. condition (3.30) holds for all  $x_i$ ,  $i = 1, \dots, n$ . By weak additivity, the function  $A \rightarrow \langle \psi | A \Omega \rangle$  vanishes in the dense set of translations and therefore vanishes in  $\pi(\mathcal{A}(M))$  i.e.  $\langle \Psi | \pi(\mathcal{A}(\mathbb{M})) \Omega \rangle = 0$  but  $\Omega_\omega$  is cyclic in  $\mathcal{H}$  which means that  $\Psi = 0$ .

- (b) Suppose that  $A \in \pi(\mathcal{A}(\mathcal{O}_1))$  annihilates  $\Omega$  and choose  $\mathcal{O}_2 \subset \mathcal{O}_1^\perp$ . For each  $\Psi \in \mathcal{H}$  and all  $B \in \pi(\mathcal{A}(\mathcal{O}_2))$  we have

$$\langle A^* \Psi | B \Omega \rangle = \langle \Psi | A B \Omega \rangle = \langle \Psi | B A \Omega \rangle = 0$$

using Einstein causality. The left most side implies that  $A^* \Psi$  vanishes ( $A^* \Psi$  is orthogonal to a dense) and, as  $\Psi \in \mathcal{H}$  is arbitrary it follows  $A = 0$ .

■

**Remark 3.2.9.**

- The Reeh-Schlieder theorem tell us that any operation performed in  $\mathcal{O}$  may be approximated as close as we want by realizing operations in a region elsewhere. Furthermore, the vacuum is separating for each local algebra and thus, every local event has a nonzero probability of occurring in the vacuum state, giving no possibility of existing a local number operators. This in turn, tell us how deeply entangled states in relativistic QFT typically are, stranger than the non-locality in non-relativistic QM.
- The use of the spectrum condition may indicate that the R-S theorem is a purely relativistic result in which many authors [28, 4] argue that has no analog in non-relativistic QM or QFT. Indeed, we will see that the Unruh/Hawking effects are consequence of the Reeh-Schlieder theorem.

As we pointed out in the first chapter, one may require a notion of a particle in which a detector registers an outcome of an experiment. The problem of defining a detector lies in the R-S theorem. A detector is well localized (in a bounded region  $K$ ) observable  $C$  that registers 0 in the vacuum state and 1 otherwise. It follows, by the Reeh-Schlieder theorem, that  $C$  must be identically zero, therefore the detector cannot be defined as an observable! Nonetheless, the solution to this problem also lies in the R-S theorem, mainly we approximate the localization of  $C$  with observables in  $\mathcal{A}$  in the sense that

$$C = \int f(x) \alpha_x(A) dx$$

where  $f$  is a smooth function whose Fourier transform has support in the complement of the causal future of  $\mathcal{O}$  and  $A \in \mathcal{A}(\mathcal{O})$  with  $0 \leq A \leq I$ . Then  $C \Omega = 0$  and the function  $f$  is chosen so that  $C$  is "close" in the norm topology to an operator in  $\mathcal{A}(\mathcal{O})$ . This notion of approximately localized detectors is employed extensively in Haag- Ruelle scattering theory and, we will use this fact in the measuring scheme of an Unruh-deWitt detector.

<sup>18</sup>Holomorphic functions on two "wedges" with an "edge" in common are analytic continuations of each other provided they both give the same continuous function on the edge.

### 3.3 The Unruh effect (Part II: An algebraic approach)

Section 1 was devoted to derive the Unruh effect in a two-dimensional space. In this section, we derive a rigorous characterization of the Unruh effect using the full power of algebraic quantum field theory, which is essentially the result of the Bisognano-Wichmann theorem. Let  $\mathcal{A}$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$ ,  $\omega$  a faithful, normal state and let  $\Omega$  the representation of the state  $\omega$ . Recall that the state  $\omega$  is a KMS state if  $\Omega$  satisfy the KMS condition

$$J \exp\{(-\beta K/2)\} A \Omega = A^* \Omega, \quad \text{for all } A \in \mathcal{A}, \quad (3.31)$$

where  $J$  is the unique anti-linear transformation of  $\mathcal{H}$  such that

$$J^2 = 1 \text{ and } (Jf, Jg) = (g, f).$$

Notice that once the KMS state is verified, equation (3.31) determines  $J$  and  $\beta$  uniquely therefore, we shall verify condition (3.31). Let  $M$  be Minkowski and denote by  $M^\pm$  the right and left Rindler wedges. If  $x = (x^0 = ct, \dots, x^3) \in M$  then  $T$  and  $L$  defined by

$$\begin{aligned} T(t)x^\mu &= (x^0 + ct, x^1, x^2, x^3), \\ L(\tau)x^\mu &= (x^0 \cosh \tau + x^1 \sinh \tau, x^1 \cosh \tau + x^0 \sinh \tau, x^2, x^3), \end{aligned}$$

are the subgroups of the Poncarè group related to time-translation and Lorentz boosts respectively. Let us see how  $L$  acts on  $M^+$ . Given a point  $x^\mu = (\xi, \tau', x^2, x^3) \in M^+$  then, we have

$$\begin{aligned} L(\tau')x^\mu &= (a^{-1}e^{a\xi} \sinh \tau \cosh \tau' + a^{-1}e^{a\xi} \cosh \tau \sinh \tau', a^{-1}e^{a\xi} \cosh \tau' \cosh \tau + a^{-1}e^{a\xi} \sinh \tau \sinh \tau', x^2, x^3) \\ &= (a^{-1}e^{a\xi} \sinh(\tau + \tau'), a^{-1}e^{a\xi} \cosh(\tau + \tau'), x^2, x^3) \end{aligned}$$

where we use the fact that  $t = a^{-1}e^{a\xi} \sinh \tau'$  and  $z = a^{-1}e^{a\xi} \cosh \tau'$ . It follows that Lorentz boosts in the right Rindler wedge act by the equation

$$L(\tau)(\xi, \tau', x^2, x^3) = (\xi, \tau' + \tau, x^2, x^3) \in M^+. \quad (3.32)$$

Similarly, one can show the same for  $M^-$ . Thus,  $M^\pm$  are stable under  $L$  and  $L$  is the time translation group of the Rindler wedges. In order to formulate the Unruh effect, let us state a few theorems related with the algebraic theory of the scalar field satisfying the assumptions A1-A7.

**Theorem 3.3.1** (PCT theorem). *There is a unique  $J_0$  of  $\mathcal{H}$  such that*

$$J_0 \varphi(f_1) \dots \varphi(f_k) \psi = \varphi(f_k^\dagger) \dots \varphi(f_1^\dagger) \psi, \quad \forall f_1, \dots, f_k \in \mathcal{S}(X)$$

where  $f^\dagger(x) := f^*(-x) = \overline{f}(-x)$ .

*Proof.* Look in [16]. ■

**Theorem 3.3.2** (Reeh-Schlieder theorem). *Let  $A$  be an arbitrary open subset of  $M$ ,  $\mathcal{A}$  the subalgebra of  $\mathcal{A}$  given by the polynomials in  $\{\varphi(f); f \in \mathcal{S}(X), \text{supp } f \subset A\}$ . Then,  $\mathcal{A}(A)\psi$  is dense in  $\mathcal{H}$ .*

**Theorem 3.3.3** (Bisognano-Wichmann). *Let  $iK$  be the infinitesimal generator of  $\hat{L} = U(L)$ . Then,*

$$J_0 \hat{\rho} e^{-\pi K} A \psi = A^* \psi, \quad \forall A \in \mathcal{A}(M^+),$$

where  $J_0$  is the PCT conjugation,  $\hat{\rho} = U(\rho)$  and  $\rho$  is the partial inversion  $(x^2, x^3) \mapsto (-x^2, -x^3)$ .

*Proof.* Look in [27], for curved space times look in [35]. ■

The Bisognano-Wichmann (BW) theorem tells that the vacuum state  $\psi$  satisfies the KMS condition at a specified temperature with respect to the Lorentz boosts

$$\begin{cases} t \rightarrow t \cosh \beta + z \sinh \beta \\ z \rightarrow t \sinh \beta + z \cosh \beta \end{cases}$$

if Lorentz boost were the time translation group which in turn, generates the Hamiltonian. Therefore, if  $\psi$  is restricted to the right Rindler wedge then, by means of (3.32) Lorentz boosts correspond to time-translations  $\tau \rightarrow \tau + \tau'$  for uniformly accelerated observers which generates the dynamics of dynamical system of the algebra  $\mathcal{A}(M)$ . It follows that the B-W theorem states the Unruh effect provided that the observer is restricted to the Rindler wedges.

**Proposition 3.3.1.** *The restriction of the state  $\psi$  to  $\mathcal{A}(M^+)$  is a thermal one satisfying the following KMS condition with respect the time-translation group  $\hat{L}^+ := U(L)$  for  $O_{acc}$ .*

$$J e^{-\pi K^+} A \psi = A^* \psi, \quad \forall A \in \mathcal{A}(M^+), \quad (3.33)$$

where  $iK^+$  is the generator of the restriction of  $L$  in  $M^+$ ,  $J = J_0 \hat{\rho}$ ,  $J_0$  is the PCT operator and  $\hat{\rho}$  the unitary representative of the partial inversion  $(x^2, x^3) \mapsto (-x^2, -x^3)$ .

**Remark 3.3.4.**

- In this derivation it becomes clear that the conjugation operator  $J$  is the PCT operator  $J_0$  corrected by the partial inversion  $\hat{\rho}$ . Thus,  $J$  is the PCT conjugation associated with the restricted inversion  $(x^0, x^1) \mapsto (-x^0, -x^1)$ . In the  $1+1$ -dimensional case, the partial inversion  $\rho$  is trivial and thus,  $J$  is the PCT operator as we saw in remark 3.1.12.
- Comparing equations (3.33) and (3.31) one find that the temperature of the thermal state satisfies

$$\frac{K}{2kT} = \pi K.$$

This implies that the temperature of the thermal state is

$$T = \frac{1}{2\pi k}.$$

However, this temperature is based on the time  $\tau$  instead of the proper time  $\tau_\alpha = \xi\tau/c = c\tau/a$ . Therefore, the temperature observed by  $O_{acc}$  will correspond to a re-scaling of the time  $\tau$  to  $\tau_\alpha$ , i.e.,

$$T_\alpha = \frac{a}{2\pi kc} = \frac{a}{2\pi}$$

where we used natural units. Thus, establishing a rigorous generalization of Unruh's result for fields in flat space-time.

- The Unruh effect arises from the B-W theorems which, in turn, derives from the PCT and R-S theorem. Therefore, as long these theorems can be generalized to arbitrary quantum fields in spacetime then, the Unruh effect will still hold. This is, for example, the case of the Hawking effect and many other results.

## 4 Quantum Measurements on Relativistic Quantum Field Theory

Having introduced the power of AQFT, we are now in position to discuss the Fewster and Verch measurement scheme. This scheme behaves in the spirit of the von Neumann measurement scheme with particular emphasis in taking account the locality of the theory. The measurement takes the following steps,

1. Prepare the system,
2. Make a measurement,
3. Measure the probe elsewhere.

In this chapter we introduce the relevant concepts in order to formalize the steps above and determine the correlation between the probe and the system. We strongly follow [7] for the general measure scheme and, [21] for the Sorkins protocol section. Notations (and some examples) regarding causal/convex geometry in curved spacetimes can be found in the appendix F if not defined in the main body.

### 4.1 A General Measurement Scheme on AQFT

#### 4.1.1 The framework

Let  $M$  be a space-time. We take the probe and the system coupled in a compact region  $K \subset M$ . The probe and the system are equipped with an (algebraic) quantum field theory  $\mathcal{B}$  and  $\mathcal{A}$  respectively, e.g., both being free fields. Given the compact region  $K$ , define  $M^\pm = M \setminus J^\mp(K)$ .

We start with the uncoupled algebra of observables  $\mathcal{U}(M) = \mathcal{A}(M) \otimes \mathcal{B}(M)$  and the coupled algebra  $\mathcal{C}(M)$  equipped with the compatibility map  $\alpha_{M;N} \otimes \beta_{M;N}$  and  $\gamma_{M;N}$  respectively. If there is no risk of confusion, we drop the sub-indices for ease of read.

The interaction takes place in region  $K$  and therefore, we shall assume that  $\mathcal{C}$  reduces to  $\mathcal{A} \otimes \mathcal{B}$  outside  $K$ . That is, for every  $L \subset M \setminus K$  there exist an isomorphism  $\chi_L : \mathcal{A}(L) \otimes \mathcal{B}(L) \rightarrow \mathcal{C}$  such that the following diagram commutes:

i.e.  $\chi$  is compatible with the coupled and uncoupled structures.

**Remark 4.1.1.** *These regions are determined covariantly once the interaction region is specified and without reference to any observer's clock or time coordinate thus, making the following construction covariant.*



$$\begin{array}{ccc}
\mathcal{A}(L') \otimes \mathcal{B}(L') & \xrightarrow{\alpha_{L,L'} \otimes \beta_{L,L'}} & \mathcal{A}(L) \otimes \mathcal{B}(L) \\
\chi_{L'} \downarrow & & \downarrow \chi_L \\
\mathcal{C}(L') & \xrightarrow{\gamma_{L,L'}} & \mathcal{C}(L)
\end{array}$$

Figure 6: Compatibility of  $\chi$ .

Assume the regions  $M^\pm$  are big enough to have Cauchy surfaces then, by the time-slice property the morphisms  $\alpha, \beta$  and  $\gamma$  are also isomorphism. We define

$$\kappa^\pm = \gamma^\pm \circ \chi^\pm : \mathcal{A}(M^\pm) \otimes \mathcal{B}(M^\pm) \rightarrow \mathcal{C}(M).$$

**Definition 4.1.1.** Let  $M, \mathcal{U}, \mathcal{C}, \alpha, \beta, \gamma$  and  $\chi$  as above. Define  $\tau^\pm : \mathcal{U}(M) \rightarrow \mathcal{C}(M)$  by

$$\tau^\pm := \kappa^\pm \circ (\alpha^\pm \otimes \beta^\pm)^{-1} = \gamma^\pm \circ \chi^\pm \circ (\alpha^\pm \otimes \beta^\pm)^{-1}.$$

$\tau^\pm$  defines an isomorphism that identifies the uncoupled system with the coupled at early  $(-)$  or late  $(+)$  times;  $\tau^\pm$  is said to be the retarded  $(+)$  and advanced  $(-)$  response maps.

**Definition 4.1.2.** We define the scattering morphism  $\Theta : \mathcal{U}(M) \rightarrow \mathcal{U}(M)$  as

$$\Theta = (\tau^-)^{-1} \circ \tau^+.$$

The scattering map determines the evolution after an interaction in the region  $K$  has been made.

**Proposition 4.1.1.** Let  $\Theta$  be the scattering morphism of the interaction taking place in  $K$  then, the following statements holds:

- (a)  $K \subset \hat{K}$  with  $\hat{K}$  compact then, the scattering morphism  $\hat{\Theta}$  constructed in  $\hat{K}$  is the same as  $\Theta$ .
- (b)  $L \subset K^\perp$  then  $\Theta$  acts trivially on  $\mathcal{U}(M; L)$
- (c)  $L^\pm \subset M^\pm$  open causally convex and  $L^+ \subset D(L^-)$ . Then  $\Theta(\mathcal{U}(M; L^+)) \subset \mathcal{U}(M; L^-)$

*Proof.* All of them are a consequence of  $\chi$  being a isomorphism and the definition of the Cauchy development.

- (a) Notice that  $\hat{M}^\pm = M \setminus J^\mp(\hat{K}) \subset M^\pm$ . Thus, we can define  $(\alpha^\pm)^{-1} \circ \hat{\alpha}^\pm = \alpha_{M^\pm, \hat{M}^\pm}$  by the ryle for consecutive inclusions and the time-slice property.

Observe that

$$\begin{aligned}
\kappa^\pm \circ (\alpha^\pm \otimes \beta^\pm)^{-1} \circ (\hat{\alpha}^\pm \otimes \hat{\beta}^\pm) &= \gamma^\pm \circ \chi^\pm \circ (\alpha_{M^\pm, \hat{M}^\pm} \otimes \beta_{M^\pm, \hat{M}^\pm}) = \gamma^\pm \circ \gamma_{M^\pm, \hat{M}^\pm} \circ \hat{\chi}^\pm = \hat{\gamma}^\pm \circ \hat{\chi}^\pm \\
&= \hat{\kappa}^\pm
\end{aligned}$$

it follows that  $\hat{\tau}^\pm = \tau^\pm$  and hence  $\hat{\Theta} = \Theta$ .

Suppose  $L \subset M^\pm$  is an open causally convex set then,

$$\kappa^\pm \circ (\alpha_{M^\pm;L} \otimes \beta_{M^\pm;L}) = \gamma^\pm \circ \chi_{M^\pm} \circ (\alpha_{M^\pm;L} \otimes \beta_{M^\pm;L}) = \gamma^\pm \circ \gamma_{M^\pm;L} \circ \chi_L = \gamma_{M;L} \circ \chi_L. \quad (4.1)$$

(b) Assume  $L \subset M^+ \cap M^-$ , using (4.1) we have

$$\tau^+ \circ (\alpha_{M;L} \otimes \beta_{M;L}) = \tau^- \circ (\alpha_{M;L} \otimes \beta_{M;L}),$$

and hence

$$\Theta \circ (\alpha_{M;L} \otimes \beta_{M;L}) = (\alpha_{M;L} \otimes \beta_{M;L}).$$

(c) We apply equation (4.1) to  $L^+ \subset M^+$  and  $L^- \subset M^-$ , giving

$$\tau^+ \circ (\alpha_{M;L^+} \otimes \beta_{M;L^+}) = \gamma_{M;L^+} \circ \chi_{L^+}, \quad \tau^- \circ (\alpha_{M;L^-} \otimes \beta_{M;L^-}) = \gamma_{M;L^-} \circ \chi_{L^-}$$

the first of which asserts that  $\tau^+ \circ (\alpha_{M;L^+} \otimes \beta_{M;L^+})$  factors through  $\gamma_{M;L^+}$ , while the second implies that  $(\tau^-)^{-1} \circ \gamma_{M;L^-}$  factors through  $\alpha_{M;L^-} \otimes \beta_{M;L^-}$ . As  $\gamma_{M;L^+}$  factors via  $\gamma_{M;L^-}$  due to the assumption  $L^+ \subset D(L^-)$  and the timeslice property, the two observations combine to show that  $\Theta \circ (\alpha_{M;L^+} \otimes \beta_{M;L^+})$  factors through  $\alpha_{M;L^-} \otimes \beta_{M;L^-}$ , as required. ■

**Remark 4.1.2.** *Part (a) shows that the scattering morphism is canonically associated with the theories  $\mathcal{U}$  and  $\mathcal{C}$  and the identifications between them, while parts (b) and (c) are locality properties. In particular, part (b) shows that, as one would expect, the coupling has no effect on observables localized in  $L \subset K^\perp$ .*

*The first property captures the idea that the coupling has no effect in spacelike separated regions, whereas the second property indicates how  $\Theta$  changes the localisation of observables.*

### 4.1.2 The Measurement process

Given the dynamics of the system, we want to describe the measurements of the coupled  $\mathcal{C}(M)$  in terms of the uncoupled observables. Suppose the system to be measured is in a state  $\omega$  of  $\mathcal{A}$  and prepare the probe at a state  $\sigma$  of  $\mathcal{B}$ . In terms of the response maps, we can identify this initial state as

$$\omega_\sigma = ((\tau^-)^{-1})^*(\omega \otimes \sigma), \quad \omega_\sigma(X) = (\omega \otimes \sigma)((\tau^-)^{-1}X)$$

which is uncorrelated at early times. Let  $B \in \mathcal{B}(M)$  be an observable of the probe, in the late times we can identify  $B$  with the observable

$$\tilde{B} = \tau^+(1 \otimes B) \in \mathcal{C}(M).$$

Therefore, the measurement in the probe after the interaction has been made (expectation value) is

$$\omega_\sigma(\tilde{B}) = (\omega \otimes \sigma)(\Theta(1 \otimes B)).$$

We want to identify observables in the system  $\mathcal{A}$  as observables in  $\mathcal{B}$  such that the measurement of the probe is interpreted as measurements on the system (provided that  $\sigma$  was prepared). That is, we want an identification  $A \leftrightarrow B$  such that

$$\omega(A) = \omega_\sigma(\tilde{B}), \quad \text{for all states } \omega \text{ in } \mathcal{A}(M). \quad (4.2)$$

In order to determine  $A$ , consider the linear and continuous map  $\eta_\sigma : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$  such that

$$\begin{aligned} \eta_\sigma : \mathcal{A} \otimes \mathcal{B} &\rightarrow \mathcal{A} \\ A \otimes B &\mapsto \eta_\sigma(A \otimes B) = \sigma(B)A. \end{aligned}$$

Then, define  $A$  as

$$A = \varepsilon_\sigma(B) := \eta_\sigma(\Theta(1 \otimes B)). \quad (4.3)$$

Taking expectation value of  $A$ ,

$$\omega(A) = \omega(\eta_\sigma(\Theta(1 \otimes B))) = (\omega \otimes \sigma)(\Theta(1 \otimes B)) = \widetilde{\omega_\sigma}(\tilde{B}) \quad (4.4)$$

we see that  $A$  is what we have been looking for. Therefore, given an observable  $B$ , we can deduce the observable  $A := \varepsilon_\sigma(B)$  of the system.

**Definition 4.1.3.** *Let  $B$  and  $\sigma$  be an observable and a state on the probe, respectively. The observable  $A := \varepsilon_\sigma(B) \in \mathcal{A}(M)$  is said to be the induced system observable.*

**Remark 4.1.3.** *Let  $C = D \otimes E \in \mathcal{A}(M) \otimes \mathcal{B}(M)$  and  $A \in \mathcal{A}(M)$  then,*

$$\begin{aligned} A\eta_\sigma(C) &= A\sigma(E)D = \eta_\sigma(AD \otimes E) \\ &= \eta_\sigma((A \otimes 1)C), \\ \eta_\sigma(C)A &= \sigma(E)DA = \eta_\sigma(DA \otimes E) \\ &= \eta_\sigma(C(A \otimes 1)). \end{aligned}$$

*The map  $\eta_\sigma$  is completely positive: its tensor products with any finite-dimensional matrix identity map preserve positivity*

**Proposition 4.1.2.** *Prepare the system  $\mathcal{B}$  at a state  $\sigma$ , let  $B \in \mathcal{B}$  and  $A = \varepsilon_\sigma(B)$  the induced observable. If the states (of  $\mathcal{A}$ ) separate<sup>19</sup> observables on  $\mathcal{A}$  then,  $A$  is the unique solution to (4.2). Furthermore,  $\varepsilon_\sigma$  is a completely positive linear map and the following statements holds,*

(a)  $\varepsilon_\sigma(1) = 1$

(b)  $\varepsilon_\sigma(B^*) = \varepsilon_\sigma(B)^*$  and if  $B$  is self adjoint so is the induced observable.

---

<sup>19</sup> $\omega$  is said to separate the observables  $A, B \in \mathcal{A}(M)$  if  $\omega(A) \neq \omega(B)$

- (c)  $\varepsilon_\sigma(B^*B) - \varepsilon_\sigma(B)^* \varepsilon_\sigma(B)$  is a positive operator
- (d) The measurement  $\omega_\sigma(\tilde{B})$  is less sharp than the hypothetical measurement of the induced observable in the system state.
- (e) For fixed  $B$ , the map  $\sigma \mapsto \varepsilon_\sigma(B)$  is weak  $*$ -continuous.

*Proof.* If  $A, E \in \mathcal{A}(M)$  are two different solutions of the relation (4.2) then,

$$\omega(A) = \omega(E)$$

for all states  $\omega$  in  $\mathcal{A}(M)$ , a contradiction. Therefore, the solution of (4.2) is unique.

- (a)  $\varepsilon_\sigma(1) = \eta_\sigma(\Theta(1 \otimes 1)) = \eta_\sigma(1 \otimes 1) = 1$ .
- (b) Note that  $\eta_\sigma((A \otimes B)^*) = \eta_\sigma(A^* \otimes B^*) = \sigma(B^*)A^* = (\eta_\sigma(A \otimes B))^*$ . It follows that  $\varepsilon_\sigma(B^*) = \varepsilon_\sigma(B)^*$ . Clearly self adjoint observables induce self adjoint observables.
- (c) We need to show that  $\eta_\sigma$  is completely positive.

The positive operators of a  $*$ -algebra are finite convex combinations of elements  $A^*A$ . Let  $N > 0$  and consider an element  $C \in M_N(\mathbb{C}) \otimes \mathcal{A}(M) \otimes \mathcal{B}(M)$ ,

$$C = \sum_r M_r \otimes A_r \otimes B_r.$$

We shall see that  $X = (Id_N \otimes \eta_\sigma)(C^*C)$  is positive in  $M_N \otimes \mathcal{A}(M)$ .

$$\begin{aligned} X &= \sum_{r,s} (Id_N \otimes \eta_\sigma)(M_r^* M_s \otimes A_r^* A_s \otimes B_r^* B_s) \\ &= \sum_{r,s} \sigma(B_r^* B_s) M_r^* M_s \otimes A_r^* A_s. \end{aligned}$$

$\sigma$  is a state and thus,  $\sigma(B_r^* B_s)$  is a positive matrix which can be decomposed as

$$\sigma(B_r^* B_s) = \sum_i \overline{v_s^{(i)}} v_s^{(i)}$$

for mutually orthogonal vectors  $v^{(i)}$ . Then,

$$X = \sum_i \left( \sum_s v_s^{(i)} M_s \otimes A_s \right)^* \left( \sum_s v_s^{(i)} M_s \otimes A_s \right).$$

Therefore,  $\eta_\sigma$  is completely positive.

$$C = \sum_r M_r \otimes B_r \in M_N(\mathbb{C}) \otimes \mathcal{B}(M),$$

$$(Id_N \otimes \varepsilon_\sigma)(C^*C) = (Id_N \otimes \eta_\sigma)(D^*D) \geq 0$$

where  $D = \sum_r M_r \otimes \Theta(1 \otimes B_r)$ .

Finally, let  $B \in \mathcal{B}(M)$  and define  $C = \Theta(1 \otimes B) \in \mathcal{A}(M) \otimes \mathcal{B}(M)$ , which can be decomposed as

$$C = \sum_r A_r \otimes B_r.$$

Then,

$$\varepsilon_\sigma(B^*B) - \varepsilon_\sigma(B)^* \varepsilon_\sigma(B) = \sum_{r,s} \left( \sigma(B_r^* B_s) - \overline{\sigma(B_r)} \sigma(B_s) \right) A_r^* A_s,$$

the factors in the parenthesis determine a positive matrix due to the Cauchy-Schwarz inequality then the operator is positive by analogous argument to that used above.

(d) We shall see the variance,

$$\begin{aligned} \text{Var}(\tilde{B}; \omega_\sigma) &= \tilde{\omega}_\sigma(\tilde{B}^2) - \tilde{\omega}_\sigma(\tilde{B})^2 = \omega(\varepsilon_\sigma(B^2)) - \omega(\varepsilon_\sigma(B))^2 \\ &\geq \omega(A^2) - \omega(A)^2 = \text{Var}(A; \omega), \end{aligned}$$

where we used the fact that  $\tilde{B}^2 = \tilde{B}^2$ . Thus, the actual measure is less sharp than the hypothetical measure.

(e) Since  $\eta_\sigma$  is  $*$ -continuous then so is  $\varepsilon_\sigma$ .

■

**Remark 4.1.4.** *The separable assumption may be relaxed by requiring only the induced observable to be separable. In this way, the condition of separability naturally arises, for example, in remark 2.1.1 when we require the instrument of measurement to be able to distinguish small enough outcomes.*

**Proposition 4.1.3.** *The induced observable need to be localized at least partially within the interaction region, that is,  $\varepsilon_\sigma(A) \in \mathcal{A}(M; L), K \subset L$ . Indeed, for each probe observable  $B \in \mathcal{B}(M)$ , the induced system observable  $\varepsilon_\sigma(B)$  is localized in any connected open causally convex set containing  $K$ . Furthermore, if  $B$  is localized in  $K^\perp$  then, the induced observable is trivial i.e.,  $\varepsilon_\sigma(B) = \sigma(B)1$ .*

*Proof.* Let  $L \subset K^\perp$ ,  $B \in \mathcal{B}(M; L)$  a probe observable localized in  $L$ . Since  $L \subset M^+ \cup M^-$  then  $\Theta$  acts trivial and therefore,  $\varepsilon_\sigma(B) = \eta_\sigma(\Theta(1 \otimes B)) = \sigma(B)1$ . Thus, the induced observable is trivial. On the other hand, let  $B \in \mathcal{B}(M)$ ,  $A \in \mathcal{A}(M; L)$ . Notice that  $\eta_\sigma(A \otimes 1) = 1$  then,

$$\begin{aligned} [\varepsilon_\sigma(B), A] &= [\eta_\sigma(\Theta(1 \otimes B)), A] \\ &= \eta_\sigma([\Theta(1 \otimes B), A \otimes 1]) \\ &= \eta_\sigma(\Theta[1 \otimes B, A \otimes 1]) = 0 \end{aligned}$$

as  $\eta_\sigma$  and  $\Theta$  are linear and  $\Theta$  acts trivially on  $A \otimes 1$ . Therefore, all induced observables must be localizable in any connected open causally convex set containing the coupling region  $K$ . ■

### 4.1.3 QMT on QFT

**Definition 4.1.4.** Let  $\Sigma$  be a  $\sigma$ -algebra and  $E : \Sigma \rightarrow \mathcal{A}$  be an EVM. Then, one interprets the elements of  $\Sigma$  as potential outcomes of the measurement and, for each  $X \in \Sigma$ ,  $\omega(E(X))$  is the probability that a value lying in  $\Sigma$  is measured in state  $\omega$ . We can naturally induce an EVM

$$\begin{aligned} \varepsilon_\sigma \circ E : \Sigma &\rightarrow \varepsilon_\sigma(E(\Sigma)) \subset \mathcal{A}(M) \\ X &\mapsto \varepsilon_\sigma(E(X)), \end{aligned}$$

taking values in the effects of the system  $\mathcal{A}$  because  $\varepsilon_\sigma \circ E$  inherits the measure properties of  $E$  by the linearity of  $\varepsilon_\sigma$ . Moreover, the statistical properties of  $\varepsilon_\sigma$  tell us that the induced EVM is in general unsharp even though  $E$  could be sharp, one only knows that

$$\varepsilon_\sigma(E(X))^2 \leq \varepsilon_\sigma(E(X)^2),$$

by proposition 4.1.2.

**Remark 4.1.5.** Consider two experimenters in causally disjoint spacetime regions and suppose that each experimenter can measure a local observable of the probe. Einstein causality implies that these observables commute and therefore are compatible. However, generally speaking, disjoint observables are not always compatible; in order to fix this problem we use a joint EVM  $E$  and consequently a induced (by linearity) EVM  $\varepsilon_\sigma \circ E$ , in which each experimenter EVM are the marginals of an EVM  $E : X_1 \otimes X_2 \rightarrow \mathcal{A}$ . In this case, even if the  $E_i$  are commuting EVMs, the induced system may not commute because  $\varepsilon_\sigma$  is not (generally) a homomorphism. Moreover, if the EVMs  $E_i$  are sharp (PVM),  $\varepsilon_\sigma \circ E_i$  could be sharp by proposition 4.1.2. Thus, incompatible system observables cannot be measured jointly and sharply.

### 4.1.4 Instruments

Let  $A$  and  $B$  be observables of the system and probe respectively, consider a joint measurement of  $A$  and  $B$  at late times. By the same reasoning as used above, the probability of the joint effect being observed is

$$\text{Prob}_\sigma(A \cap B, \omega) = (\omega \otimes \sigma)(\Theta(A \otimes B))$$

and the conditional probability that  $A$  is observed given that  $B$  is observed is

$$\text{Prob}_\sigma(A|B, \omega) = \frac{\text{Prob}_\sigma(A \cap B, \omega)}{\text{Prob}_\sigma(B, \omega)} = \frac{\mathcal{I}_\sigma(B)(\omega)(A)}{\mathcal{I}_\sigma(B)(\omega)(1)}$$

where  $\mathcal{I}_\sigma$  is defined for all  $A \in \mathcal{A}(M)$  as

$$\mathcal{I}_\sigma(B)(\omega)(A) := (\omega \otimes \sigma)(\Theta(A \otimes B)) = \Theta^*(\omega \otimes \sigma)(A \otimes B).$$

In particular, using (4.4), the normalization factor is

$$\mathcal{I}_\sigma(B)(\omega)(1) = \omega(A).$$

**Definition 4.1.5.** We call the map  $\mathcal{I}_\sigma(B) : \mathcal{A}(M)^* \rightarrow \mathcal{A}(M)^*$  a pre-instrument<sup>20</sup>. If defined, the normalized post-selected state conditioned on  $B$  is given by

$$\omega' = \frac{\mathcal{I}_\sigma(B)(\omega)}{\mathcal{I}_\sigma(B)(\omega)(1)}.$$

**Remark 4.1.6.** Suppose a probe-effect  $B$  is tested when the system is in a state  $\omega$ . The post-selected system state, conditioned on the effect observed, should correctly predict the probability of any system effect being observed given that  $B$  was observed.

**Proposition 4.1.4.** The post-selected state  $\omega'$  is a state in the system.

*Proof.* Clearly  $\omega$  is normalized and, since the effect  $B$  is positive, we can write  $B = \sum_i C_i^* C_i$  for some finite set of elements  $C_i \in \mathcal{B}(M)$ . Then

$$\mathcal{I}_\sigma(B)(\omega)(A^*A) = \sum_i \Theta^*(\omega \otimes \sigma)((A \otimes C_i)^*(A \otimes C_i)) \geq 0$$

it follows that  $\omega(A^*A) \geq 0$  for all  $A \in \mathcal{A}(M)$ . ■

In a non-selective probe measurement, there is no filtering conditional on the measurement outcome. Explicitly, the updated state resulting from the non-selective measurement is

$$\omega'_{\text{ns}}(A) = \mathcal{I}_\sigma(1)(\omega)(A) = (\Theta^*(\omega \otimes \sigma))(A \otimes 1) \quad (4.5)$$

In other words,  $\omega'_{\text{ns}}$  is the partial trace of the state  $\Theta^*(\omega \otimes \sigma)$  over the probe.

**Remark 4.1.7.** The non-selective state depends only in the dynamics of the coupling, if  $A \in \mathcal{A}(M)$  is localised in the causal complement of the coupling region then  $\omega_{\text{ns}}(A) = \omega(A)$  because  $\Theta$  acts trivially. A non-selective measurement cannot influence the results of other experiments in causally disjoint regions.

**Example 4.1.1.** Let  $A$  be a system observable localized in  $K^\perp$  and  $B$  be a probe effect. Using that  $\Theta(A \otimes 1) = A \otimes 1$  notice that

$$A\varepsilon_\sigma(B) = A\eta_\sigma\Theta(1 \otimes B) = \eta_\sigma((A \otimes 1)\Theta(1 \otimes B)) = \eta_\sigma(\Theta(A \otimes B)).$$

The pre-instrument is then given by

$$\mathcal{I}_\sigma(B)(\omega)(A) = \omega(\eta_\sigma(\Theta(A \otimes B))) = \omega(A\varepsilon_\sigma(B)).$$

---

<sup>20</sup>Given an EVM  $E : \mathcal{X} \rightarrow \mathcal{A}(M)$ , the composition of the pre-instrument with  $E$  is a instrument in the sense of [2] i.e., a measure  $X \rightarrow \mathcal{I}_\sigma(E(X))$  on the  $\sigma$ -algebra of measurement outcomes.

Therefore, the normalized post-selected state, conditioned on the effect being observed, for a localizable in  $K^\perp$  is

$$\omega'(A) = \frac{\omega(A\varepsilon_\sigma(B))}{\omega(\varepsilon_\sigma(B))}.$$

It follows that  $\omega'(A) = \omega(A)$  if and only if  $A$  is uncorrelated with  $\varepsilon_\sigma(B)$  in  $\omega$ .

Up until now we have not assumed any rule that  $\omega$  changes  $\omega'$  across a surface in  $M$  nor have we found any indication that such a rule is desirable.

**Proposition 4.1.5.** *If  $\omega$  has a Reeh-Schluder property (e.g. Minkowski vacuum state) then*

$$\omega'(A) = \omega(A) \iff \varepsilon_\sigma(B) = \omega(\varepsilon_\sigma(B))1$$

for observables localizable in  $K^\perp$ .

*Proof.*

$$\begin{aligned} \omega'(A_1^*A_2) - \omega(A_1^*A_2) &= \frac{\omega(A_1^*A_2\varepsilon_\sigma(B))}{\omega(\varepsilon_\sigma(B))} - \omega(A_1^*A_2) \\ &= \omega\left(\frac{A_1^*A_2\varepsilon_\sigma(B)}{\omega(\varepsilon_\sigma(B))} - A_1^*A_2\right) = \omega\left(A_1^*\left(\frac{\varepsilon_\sigma(B)}{\omega(\varepsilon_\sigma(B))} - 1\right)A_2\right) \end{aligned}$$

Hence, if  $\omega' = \omega$  the right hand side vanishes and, by the Reeh-Schluder property, it follows that  $\varepsilon_\sigma(B) = \omega(\varepsilon_\sigma(B))1$  because  $A_1$  and  $A_2$  are arbitrary. Conversely, if  $\varepsilon_\sigma(B) = \omega(\varepsilon_\sigma(B))1$  the left hand side vanishes i.e.  $\omega' = \omega$ .  $\blacksquare$

**Remark 4.1.8.** *Any nontrivial probe measurement necessarily alters the state in the causal complement of the coupling region in this highly correlated case. Fewster and Verch suggests that this property allows one to probe strong form of quantum entanglement between causally disjoint regions e.g. to detect EPR-like correlations.*

### 4.1.5 Successive measurements

Let  $K_1, K_2 \subset M$  be two compact regions and consider a measurement scheme composed with a system and two probes whose algebras are  $\mathcal{A}$ ,  $\mathcal{B}_1$  and  $\mathcal{B}_2$  respectively. Denote the uncoupled system as  $\mathcal{U}(M) := \mathcal{A} \otimes \mathcal{B}_1 \otimes \mathcal{B}_2$  and the coupling system as  $\mathcal{C}(M)$ , the scattering morphisms of each coupling  $i = 1, 2$  with the system are  $\Theta_i : \mathcal{A}(M) \otimes \mathcal{B}_i(M) \rightarrow \mathcal{A}(M) \otimes \mathcal{B}_i(M)$ . In this way, one can extend  $\Theta_i$  into  $\hat{\Theta}_i : \mathcal{U}(M) \rightarrow \mathcal{U}(M)$  by choosing  $\hat{\Theta}_1 := \Theta_1 \otimes 1_{\mathcal{B}_2}$  and  $\hat{\Theta}_2 := \Theta_2 \otimes 1_{\mathcal{B}_1}$ , where  $\otimes_i$  indicate the slot into which the second factor is inserted.

In this scenario, the probe system consist of  $\mathcal{B}_1 \otimes \mathcal{B}_2$  with coupling region  $K_1 \cup K_2$ . We assume further that the scattering morphism  $\Theta : \mathcal{U} \rightarrow \mathcal{U}$  obeys the causal factorization,

$$\Theta = (\Theta_1 \otimes 1) \circ (\Theta_2 \otimes 1).$$



**Theorem 4.1.9.** *Consider two probes as described above with  $K_2 \cap J^-(K_1) = \emptyset$ . For all probe preparations  $\sigma_i$  of  $\mathcal{B}_i(M)$  and all probe observables  $B_i \in \mathcal{B}_i(M)$ . If one can order  $K_1$  and  $K_2$  so that  $K_2$  is later according to some observer then*

$$\mathcal{J}_{\sigma_2}(B_2) \circ \mathcal{J}_{\sigma_1}(B_1) = \mathcal{J}_{\sigma_1 \otimes \sigma_2}(B_1 \otimes B_2).$$

Furthermore, if there is no causal order between  $K_1$  and  $K_2$ ,

$$\mathcal{J}_{\sigma_2}(B_2) \circ \mathcal{J}_{\sigma_1}(B_1) = \mathcal{J}_{\sigma_1}(B_1) \circ \mathcal{J}_{\sigma_2}(B_2).$$

**Remark 4.1.10.** *This result implies that measurements act in a coherent fashion allowing us to analyzing experiments in their own causal structure, for example, the Sorkin's protocol in 2.2.1.*

*Proof.* Let  $\omega$  a state and  $A$  an observable of  $\mathcal{A}$ , we have

$$\begin{aligned} \mathcal{J}_{\sigma_2}(B_2)(\mathcal{J}_{\sigma_1}(B_1)(\omega))(A) &= (\mathcal{J}_{\sigma_1}(B_1)(\omega) \otimes \sigma_2)(\Theta_2(A \otimes B_2)) \\ &= \mathcal{J}_{\sigma_1}(B_1)(\omega)(\eta_{\sigma_2}(\Theta_2(A \otimes B_2))) \\ &= (\Theta_1^*(\omega \otimes \sigma_1))(\eta_{\sigma_2}(\Theta_2(A \otimes B_2)) \otimes B_1) \\ &= (\Theta_1^*(\omega \otimes \sigma_1) \otimes_2 \sigma_2)(\Theta_2(A \otimes B_2)) \otimes_3 B_1 \\ &= (\Theta_1^*(\omega \otimes \sigma_1) \otimes_3 \sigma_2)(\Theta_2(A \otimes B_2)) \otimes_2 B_1 \\ &= (\omega \otimes \sigma_1 \otimes \sigma_2)((\Theta_1 \otimes_3 1)(\Theta_2(A \otimes B_2) \otimes_2 B_1)) \\ &= (\omega \otimes \sigma_1 \otimes \sigma_2)((\hat{\Theta}_1 \circ \hat{\Theta}_2)(A \otimes B_1 \otimes B_2)) \\ &= (\omega \otimes \sigma_1 \otimes \sigma_2)(\hat{\Theta}(A \otimes B_1 \otimes B_2)) \\ &= \mathcal{J}_{\sigma_1 \otimes \sigma_2}(B_1 \otimes B_2)(\omega)(A) \end{aligned}$$

where we permuted the second and third tensor factor of both, elements and functionals, in line 5. ■

**Corollary 4.1.10.1.** *Consider two probes as described above,  $B_i \in \mathcal{B}_i(M)$  and probe preparation states  $\sigma_i$  ( $i = 1, 2$ ). Suppose  $B_1$  has nonzero probability of being observed in system state  $\omega$  and that  $B_2$  has nonzero probability of being observed in system state  $\omega_1$ , the post-selected system state conditioned on  $B_1$  being observed in state  $\omega$ . Then the post-selected state  $\omega_{12}$  conditioned on  $B_2$  being observed in state  $\omega_1$  coincides with the post-selected state  $\omega_{12}$  conditioned on  $B_1 \otimes B_2$  being observed in state  $\omega$ .*

*Proof.* We shall compute

$$\omega'_1 = \frac{\mathcal{J}_{\sigma_1}(B_1)(\omega)}{\mathcal{J}_{\sigma_1}(B_1)(\omega)(1)} \quad \text{and} \quad \omega''_{12} = \frac{\mathcal{J}_{\sigma_2}(B_2)(\omega'_1)}{\mathcal{J}_{\sigma_2}(B_2)(\omega'_1)(1)}$$

conditioned on both effects being observed, post-selecting on the  $B_1$  measurement at the intermediate step. Obviously, the normalization factors applied to  $\omega_1$  cancel in the formula for  $\omega$ .

$$\omega''_{12} = \frac{\mathcal{I}_{\sigma_2}(B_2)(\mathcal{I}_{\sigma_1}(B_1)(\omega))}{\mathcal{I}_{\sigma_2}(B_2)(\mathcal{I}_{\sigma_1}(B_1)(\omega))(1)} = \frac{\mathcal{I}_{\sigma_1 \otimes \sigma_2}(B_1 \otimes B_2)(\omega)}{\mathcal{I}_{\sigma_1 \otimes \sigma_2}(B_1 \otimes B_2)(\omega)(1)} = \omega'_{12}$$

and so we also have where the denominators are equal by setting  $A = 1$ . ■

**Remark 4.1.11.** *If the  $K_i$  are causally disjoint then, the post-selection may be made in either order with the same result. Causal factorization formula must be verified for concrete models of system-probe interactions. Exactly as one would hope, we have shown that experiments conducted in causally disjoint regions [that is, for which the coupling regions are causally disjoint] may be conducted in ignorance of one another, or combined as a single overall experiment by coordinating their results.*

### 4.1.6 Example: Sorkin's protocol part III

Recall the arrangement of the Sorkin's protocol on example 2.2.1, Alice, Bob and Charlie each perform actions in the regions  $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$  that satisfy

1.  $\mathcal{O}_2 \cap J^-(\mathcal{O}_1) = \emptyset$
2.  $\mathcal{O}_3 \cap J^-(\mathcal{O}_2) = \emptyset$ ;
3.  $\mathcal{O}_3$  is spacelike separated from  $\mathcal{O}_1$ ;
4.  $\mathcal{O}_3$  has compact closure  $\overline{\mathcal{O}_3}$ .

Suppose Alice chooses to make a non-selective measurement, Bob makes a non selective measurement, can Charly determine whether Alice performed the measurement?

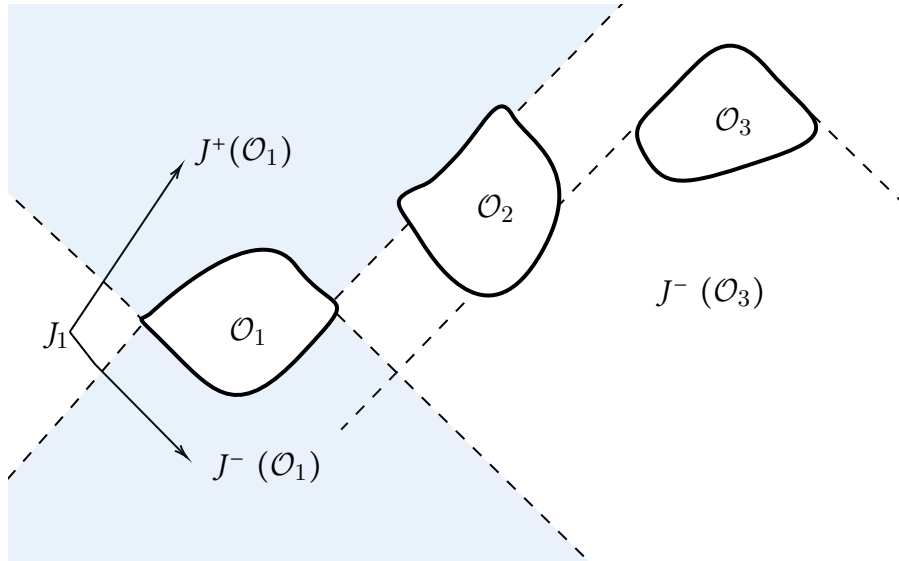


Figure 7: Sorkin's protocol: Alice is in region  $\mathcal{O}_1$ , Bob in region  $\mathcal{O}_2$  and Charly on  $\mathcal{O}_3$ .

In order to use the formalism described above set the system observables  $\mathcal{A}$  to Charlie and the two probe theories for Alice and Bob. The coupling of the probes are localized in the regions  $K_1 \subset \mathcal{O}_1$  and  $K_2 \subset \mathcal{O}_2$  which defines the in-and-out regions  $M_1^\mp, M_2^\pm$  respectively, suppose  $\omega$  is the prepared

state in the system and  $\sigma = \sigma_1 \otimes \sigma_2$  the joint probe state where  $\sigma_1$  and  $\sigma_2$  are the states of Alice and Bob and, let  $C$  be a system observable by Charlie.

Once the measurement is made, the updated (non-selective) state will be

$$\omega'(C) = (\omega \otimes \sigma)(\Theta(C \otimes 1)).$$

In this case, if Alice and Bob each perform a measurement, the expectation value for Charlie's measurement is therefore given by<sup>21</sup>

$$\omega_{AB}(C) := (\omega \otimes \sigma_1 \otimes \sigma_2) \left( (\hat{\Theta}_1 \circ \hat{\Theta}_2)(C \otimes 1 \otimes 1) \right). \quad (4.6)$$

However, since Alice is causally disjoint of Charlie, the measurement should be as if Alice does not perform the experiment i.e.

$$\omega_B(C) = (\omega \otimes \sigma_2)(\Theta_2(C \otimes 1)).$$

The problem that arises in example 2.2.1 is that  $\omega_{AB} \neq \omega_B$ . The following theorem shows that the presented pathologies illusterd before in the traditional approach are non present within the algebraic formalism. In fact, it shows the equality  $\omega_B(C) = \omega_{AB}(C)$  even if  $C$  is not an induced observable.

**Theorem 4.1.12.** *In the notation above, suppose the following assumptions hold:*

1.  $K_2 \cap J^-(K_1) = \emptyset$ ;
2.  $\mathcal{O}_3$  is a region with compact closure;
3.  $\mathcal{O}_3 \cap J^-(K_2) = \emptyset$ ;
4.  $\overline{\mathcal{O}_3}$  is spacelike separated from  $K_1$ .

Then, for all  $C \in \mathcal{S}(\mathcal{O}_3)$  we have

$$(\hat{\Theta}_1 \circ \hat{\Theta}_2)(C \otimes 1 \otimes 1) = \hat{\Theta}_2(C \otimes 1 \otimes 1). \quad (4.7)$$

**Corollary 4.1.12.1.** *Charlie's measurement is independent of Alice's experiment.*

*Proof.*

$$\begin{aligned} \omega_{AB}(C) &= (\omega \otimes \sigma_1 \otimes \sigma_2) \left( (\hat{\Theta}_1 \circ \hat{\Theta}_2)(C \otimes 1 \otimes 1) \right) \\ &= (\omega \otimes \sigma_2) (\Theta_2(C \otimes 1)) \\ &= \omega_B(C). \end{aligned} \quad (4.8)$$

■

**Lemma 4.1.13** (Geometric lemma). *Let  $K_1, K_2, L$  be compact subsets of  $M$  such that  $K_2 \cap J^-(K_1) = \emptyset$  and  $L \cap J^-(K_1) = \emptyset$ . Then there exists a Cauchy surface  $\Sigma$  of  $M_1^+$  such that*

$$\Sigma \subseteq M \setminus (J^-(K_1) \cup J^+(K_2) \cup J^+(L)). \quad (4.9)$$

---

<sup>21</sup>Assuming causal factorization that Alice realize the experiment after Bob.

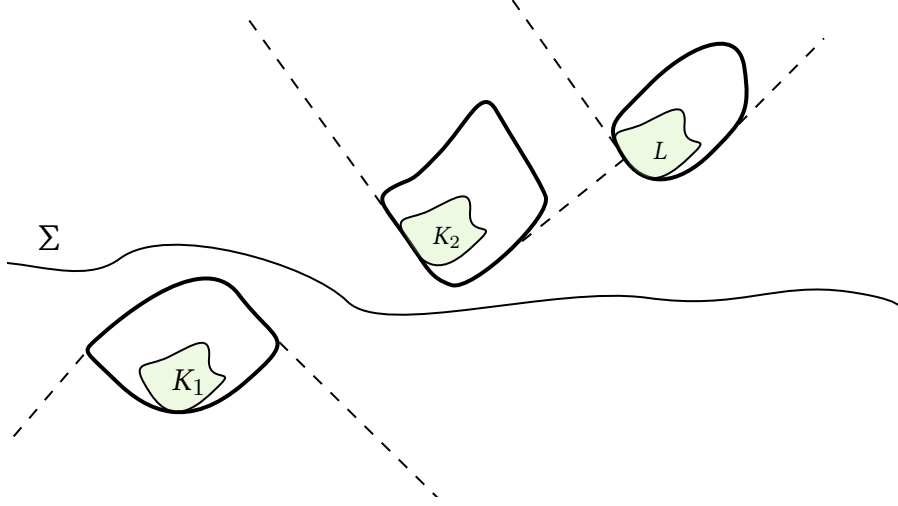


Figure 8: Diagram of the relative causal position of the compact sets  $K_1$ ,  $K_2$  and  $L := \overline{O_3}$  and the Cauchy surface  $\Sigma$ .

*Proof.* Look in [21]. ■

Taking  $K_1$  and  $K_2$  the coupling regions of the probe and  $L = \overline{O_3}$ , the former lemma gives

**Lemma 4.1.14.** *Let  $K_1, K_2$  be compact subsets of  $M$  such that  $K_2 \cap J^-(K_1) = \emptyset$ . Then for every region  $O_3$  with compact closure such that  $O_3 \cap J^-(K_2) = \emptyset$  and  $\overline{O_3} \subseteq K_1^\perp$  it holds that  $O_3 \subseteq D(K_1^\perp \cap M_2^-)$ .*

*Proof.* Using lemma 4.1.13 there exist a Cauchy surface for  $M_1^+$  that satisfy the condition (4.9). Notice that  $\Sigma$  is disjoint from  $J^-(K_1)$  and  $J^-(\overline{O_3})$  is disjoint from  $J^+(K_1)$  as  $\overline{O_3} \subseteq K_1^\perp$  by assumption. In this case we have  $T := J^-(\overline{O_3}) \cap \Sigma \subseteq K_1^\perp \cap M_2^-$ . Now  $O_3 \subseteq D(T) \subseteq D(K_1^\perp \cap M_2^-)$ . ■

*Proof of Theorem 4.1.12.* Using the lemma,  $O_3 \subseteq D(K_1^\perp \cap M_2^-)$ . Since  $C \otimes 1 \otimes 1$  must be localized in  $O_3$  then,  $\hat{\Theta}_2(C \otimes 1 \otimes 1)$  must lie in  $K_1^\perp \cap M_2^-$ . On the other hand, since  $K_1^\perp \cap M_2^- \subseteq K_1^\perp$  then, by proposition 4.1.1  $\hat{\Theta}_1$  acts trivially i.e.

$$\hat{\Theta}_1 \circ \hat{\Theta}_2(C \otimes 1 \otimes 1) = \hat{\Theta}_2(C \otimes 1 \otimes 1).$$

■

**Remark 4.1.15.** *The skeptical reader might noticed that no assumption on the algebra (or the dynamics) has been said. This is due to the spirit of the result i.e., a purely geometrical result derived from the principles of causality and locality of the theory. The only assumption of the dynamics used was causal factorization and, in practice, one need to check the factorization for each specific algebra. For the purpose of completeness, we verify this condition for a class of algebras, mainly those generated by Green-hyperbolic functions (thus, applying these results to the algebras illustrated earlier and the system introduced in the next section). In this case, one can show in*

Sorkin's protocol (example 2.2.1), that pathologies are still present when used a field  $\varphi(x)$  instead of projector  $|b\rangle\langle b|$ . Then, one can model the algebras as generated by Green hyperbolic functions and thus, the above result (together with the following propositions) implies that the algebraic approach fix those problems of Sorkin.

**Proposition 4.1.6.** *Let  $P$  be a partial differential operator and suppose that Green hyperbolic operators  $Q_i$  ( $i = 1, 2$ ) agree with  $P$  outside compact sets  $K_i$  ( $i = 1, 2$ ) with  $J^-(K_1) \cap J^+(K_2) = \emptyset$ . Let  $K = K_1 \cup K_2$  and suppose further that  $Q = Q_1 + Q_2 - P$  is also Green-hyperbolic. Then the Green functions  $E_i^\pm$  of  $Q_i$  and  $E^\pm$  of  $Q$  are related by*

$$(1 - (Q - P)E^-)f = (1 - (Q_1 - P)E_1^-)(1 - (Q_2 - P)E_2^-)f, \quad f \in C_0^\infty(M^+)$$

$$(1 - (Q - P)E^+)f = (1 - (Q_2 - P)E_2^+)(1 - (Q_1 - P)E_1^+)f, \quad f \in C_0^\infty(M^-)$$

*Proof.* Notice that  $Q$  satisfies

$$Q = \begin{cases} Q_1, & \text{Outside of } K_2, \\ Q_2, & \text{Outside of } K_1, \\ P, & \text{Outside of } K. \end{cases}$$

If  $f \in C_0^\infty(M^+)$ , the equation  $Q\varphi = f$  has unique solution on a future-compact support for  $\varphi$ .  $\varphi = E^-f$  and thus,  $Q_i\varphi = f - Q\varphi + Q_i\varphi$ . Then,

$$\varphi = E^-f = E_i^-(Q_i\varphi) = E_i^-f - E_i^-(Q - Q_i)\varphi$$

and

$$(Q - Q_1)E^-f = (Q_2 - P)E^-f = (Q_2 - P)E_2^-f = (Q - Q_1)E_2^-f$$

It follows that

$$\begin{aligned} E^-f &= E_1^-f - E_1^-(Q - Q_1)E^-f = E_1^-f - E_1^-(Q_2 - P)E_2^-f \\ &= E_1^-(1 - (Q_2 - P)E_2^-)f. \end{aligned}$$

Then,

$$(Q_1 - P)E^-f = (Q_1 - P)E_1^-(1 - (Q_2 - P)E_2^-)f,$$

consequently,

$$\begin{aligned} (1 - (Q_1 - P)E_1^-)(1 - (Q_2 - P)E_2^-)f &= f - (Q_1 - P)E_1^-(1 - (Q_2 - P)E_2^-)f - (Q_2 - P)E_2^-f \\ &= f - (Q_1 - P)E^-f - (Q_2 - P)E^-f \\ &= (1 - (Q - P)E^-)f. \end{aligned}$$

The case  $f \in C_0^\infty(M^-)$  is similar considering the advanced (+) Green operators and

$$(Q - Q_2)E^+f = (Q_1 - P)E^+f = (Q_1 - P)E_1^+f = (Q - Q_2)E_1^+f.$$

■

**Corollary 4.1.15.1** (Causal factorization). *The scattering morphism  $\Theta$  obeys the causal factorization,*

$$\Theta = \Theta_1 \circ \Theta_2.$$

*Proof.* Suppose that there are two probe fields  $\psi_1$  and  $\psi_2$ , coupled to  $\varphi$  (but not each other) with compactly supported functions  $\rho_i$  in regions  $K_i$  with  $J^-(K_1) \cap J^+(K_2) = \emptyset$ . Writing the corresponding free field equations by  $Q_1$  and  $Q_2$ , the two probes together are described by  $Q = Q_1 \oplus Q_2$ , which is also Green-hyperbolic, while the coupling  $R$  is now  $RF = \rho_1 F_1 + \rho_2 F_2$ . The causal factorization formula gives

$$\hat{\Theta} = \hat{\Theta}_1 \circ \hat{\Theta}_2,$$

where  $\hat{\Theta}_i$  is the scattering morphism for the dynamics given by  $\hat{T}_i = P \oplus (Q_1 \oplus Q_2) + \tilde{R}_i$ . Identifying the overall quantized theory with  $\mathcal{A}(M) \otimes \mathcal{B}_1 \otimes \mathcal{B}_2$ , we have

$$\hat{\Theta}_1 = \Theta_1 \otimes 1_{\mathcal{B}_2}, \quad \hat{\Theta}_2 = \Theta_2 \otimes 1_{\mathcal{B}_1}$$

where the  $\Theta_i$  are the scattering morphisms for the quantized dynamics of  $T_i = P \oplus Q_i + \tilde{R}_i$  relative to  $P \oplus Q_i$ . This establishes the causal factorization formula for system-probe models of this type. ■

## 4.2 Two Scalar Fields Coupled: An explicit example

The  $\ast$ -algebra quantization allows us to address the measurement process of an Unruh-DeWitt type detector. Indeed, by analyzing the system and probe modeled as two scalar fields we can address the detector's measurement (a free field and a two level system) in section 4.3. Consider two scalar fields coupled linearly with a Lagrangian of the form

$$\mathcal{L} = \underbrace{\frac{1}{2}[\Box\varphi - m_1\varphi^2 + \Box\psi - m_2\psi^2]}_{\mathcal{L}_0} - \underbrace{\rho\varphi\psi}_{\mathcal{L}_{\text{int}}} \quad (4.10)$$

i.e. the fields  $\psi$  and  $\varphi$  satisfy the following equations,

$$\underbrace{[\Box + m_1^2]}_P \varphi = 0, \quad \underbrace{[\Box + m_2^2]}_Q \psi = 0, \quad \rho(\varphi + \psi) = 0, \quad (4.11)$$

where we define the operators  $P$  and  $Q$  as the Klein-Gordon operators. Studies of the K-G equation ensures that that Green operators on a globally hyperbolic Lorentzian manifold behave as well as they are in Minkowski space-time in the sense that there exist Green functions  $E_{P/Q}^\pm$  (retarded and advanced Green operators) related with the operators  $P$  and  $Q$  that are unique [36] and supported on the causal future or past of  $K$  ( $J^+$  and  $J^-$  respectively)<sup>22</sup>. For the purpose of simplicity, combine

<sup>22</sup>This condition was essential to prove that the algebra of the scalar field in the examples is non-trivial

both fields as a function  $\Xi : M \rightarrow \mathbb{C}^2$  given by

$$\underbrace{\begin{pmatrix} P & R \\ R & Q \end{pmatrix}}_T \underbrace{\begin{pmatrix} \varphi \\ \psi \end{pmatrix}}_\Xi = 0, \quad T = (P \oplus Q) + \tilde{R}.$$

In this way, one can show that the operator of the coupled  $(T)$  and, the uncoupled  $(P \oplus Q)$  system are also Green-hyperbolic operators [36]. Therefore, as we saw in the examples of the  $\ast$ -algebra quantization, the algebras of the system  $\mathcal{A}$ , the probe  $\mathcal{B}$ , the uncoupled  $\mathcal{U}$  and the coupled  $\mathcal{C}$  theory are generated by elements  $\varphi(f) \in \mathcal{A}$ ,  $\psi(g) \in \mathcal{B}$ ,  $\Xi_0(f \oplus h) \in \mathcal{U}$  and  $\Xi(F) \in \mathcal{C}$  respectively, where  $\Xi_0(f \oplus h) \in \mathcal{U}$  and  $\Xi(F) \in \mathcal{C}$  are defined as

$$\Xi_0(f \oplus h) = \varphi(f) \otimes \mathbb{1}_{\mathcal{B}(M)} + \mathbb{1}_{\mathcal{A}(M)} \otimes \psi(h), \quad \Xi(F), F \in C_0^\infty(M; \mathbb{C}^2),$$

since  $C_0^\infty(M; \mathbb{C}^2) \cong C_0^\infty(M; \mathbb{C}) \oplus C_0^\infty(M; \mathbb{C})$ . The quantization is carried with the relations,

**The  $\ast$ -algebra quantization.** Let  $N \subseteq M$ , the net of  $\ast$ -algebras  $\mathcal{A}(N)$  are generated by a unit element together with the generators  $\varphi(f)$ ,  $f \in C_0^\infty(N)$  such that:

1.  $f \rightarrow \varphi(f)$
2.  $\varphi(\bar{f}) = \varphi(f)^\ast, \quad \forall f \in C_0^\infty(N)$
3.  $\varphi(Pf) = 0, \forall f \in C_0^\infty(N)$  where  $P$  give the dynamics of the theory (e.g.  $P, Q, P \oplus Q$  and  $T$ ).
4.  $[\varphi(f), \varphi(h)] = iE_P(f, h)\mathbb{1}, \quad \forall f \in C_0^\infty(N),$

$$E_P(f, h) := \int_M d\text{vol} f E_P h.$$

Notice that the compatibility diagram 6 commutes because both operators  $E_T$  and  $E_{P \oplus Q}$  agree outside the support of  $\rho$  and therefore, similarly as the example 3.2.1, one can establish the existence of all algebras needed. In order to see the scattering morphism first note that  $\Theta$  can be decomposed in the following diagram,

$$\begin{array}{ccccccc} & & \Theta = (\tau^-)^{-1} \circ \tau^+, & & & & \\ \tau^+ : \mathcal{U}(M) & \xrightarrow{\text{T-S}} & \mathcal{U}(M^+) & \xrightarrow{\gamma} & \mathcal{C}(M^+) & \xrightarrow{\text{T-S}} & \mathcal{C}(M) \\ (\tau^-)^{-1} : \mathcal{C}(M) & \xrightarrow{\text{T-S}} & \mathcal{C}(M^-) & \xrightarrow{\gamma} & \mathcal{U}(M^+) & \xrightarrow{\text{T-S}} & \mathcal{U}(M) \end{array}$$

where every map is an isomorphism either by the time-slice property (T-S) or the natural morphism  $\gamma : \mathcal{A}(M) \rightarrow \mathcal{B}(m)$ ,  $\gamma\varphi(f) := \psi(f)$ . Hence, it is enough to see how  $\Theta$  acts on a generator  $\Xi(F)$  where  $F$  is compactly supported in  $M^+$ .

**Proposition 4.2.1.** *The scattering morphism  $\Theta : \mathcal{U}(M) \rightarrow \mathcal{U}(M)$  acts as*

$$\Theta \Xi_0(F) = \Xi_0(F - (T - P \oplus Q)E_T F) = \Xi_0(F - \tilde{R}E_T^- F), \quad \tilde{R} = \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix} = T - P \oplus Q. \quad (4.12)$$

*Proof.* Choose  $f$  smooth supported on  $M^+$ .  $\tau^+$  acts by the chain of mappings,

$$\Xi_0(f) \rightarrow \Xi_0(f) \rightarrow \Xi(f) \rightarrow \Xi(f) \quad (4.13)$$

and, the map  $(\tau^{-1})^{-1}$  acts as

$$\Xi(f) \rightarrow \Xi(g) \rightarrow \Xi_0(g) \rightarrow \Xi_0(g). \quad (4.14)$$

We want to find  $g$  supported in  $M^-$  such that  $\Xi(f) = \Xi(g)$  because from (4.13) and (4.14), it follows that

$$\Theta \Xi_0(f) = \Xi_0(g).$$

Let us find  $g$ . For simplicity, we write  $S := P \oplus Q$ . We can write  $E_T f$  as a sum of a part compactly supported on  $M^+$  and another on  $M^-$ ,

$$E_T f = \phi^+ + \phi^-, \quad \phi^\pm \in C_{sc,pc/fc}^\infty(M), \text{ supp } \phi^- \subset M^-.$$

Since  $f$  is compactly supported on  $M^+$  then,  $TE_T f = 0 = T\phi^+ + T\phi^-$  which implies

$$T\phi^- = -T\phi^+ \quad (4.15)$$

which is compactly supported because it has support on a spatially future and past compact. By the uniqueness of the Green functions,  $E_T^- T\phi^- = \phi^-$  and, using the above equation,

$$E_T^+ T\phi^- = E_T^+ (-T\phi^+) = -\phi^+.$$

It follows that

$$E_T f = \phi^- + \phi^+ = \phi^- - E_T^+ T\phi^- = \underbrace{(E_T^- - E_T^+)}_{E_T} T\phi^-,$$

which implies

$$\Xi(f) = \Xi(T\phi^-)$$

again, by uniqueness and the quantization of  $\Xi$ . In this way, we have

$$\Theta \Xi_0(f) = \Xi_0(T\phi^-). \quad (4.16)$$

Notice also that there are no further conditions on  $\phi^-$  and therefore, we can choose  $\phi^-$  such that  $E_T^- f = \phi^- + \phi^0$  and  $\phi^0$  is a smooth compactly supported function and thus,

$$E_T f = \phi^- + \phi^0 - E_T^+ f.$$



Consequently,

$$\begin{aligned}
T\phi^- &= f - T\phi^0 \\
&= f - (T - S)\phi^0 - S\phi^0 = f - (T - S)\phi^0 - \underbrace{(T - S)\phi^-}_{0} - S\phi^0 \\
&= f - (T - S)(\phi^0 + \phi^-) - S\phi^0 \\
&= f - (T - S)(E_T^- f) - S\phi^0 - \underbrace{(T - S)(E_T^+ f)}_{T=S} \\
&= f - (T - S)(E_T^- - E_T^+)f - S\phi^0 = f - (T - S)E_T f - S\phi^0,
\end{aligned}$$

where the second line is because  $f$  is supported in  $M^+$  and the third line because  $T = S$  in the support of  $\phi^-$ . Using (4.16), we finally have

$$\begin{aligned}
\Theta \Xi_0(f) &= \Xi_0(T\phi^-) = \Xi_0(f - (T - S)E_T f - S\phi^0) \\
&= \Xi_0(f - (T - S)E_T f) - \underbrace{\Xi_0(S\phi^0)}_0 \\
&= \Xi_0(f - (T - S)E_T f) \\
&= \Xi_0(f - (T - S)E_T^- f).
\end{aligned}$$

■

Now, let  $\psi(h) \in \mathcal{B}(M)$ ,  $h \in C_0^\infty(M)$  be an observable on the probe. In order to find the induced observable, let us take a look at  $\Theta(\mathbb{1} \otimes \psi(h))$  and, we shall assume  $h$  is supported on  $M^+$ . According to the morphism scattering equation (4.12),

$$\Theta \Xi_0(f \oplus h) = \Xi_0(f \oplus h - \tilde{R}E_T^- f \oplus h) \quad (4.17)$$

$$= \Xi_0(f \oplus h) - \Xi_0(\tilde{R}E_T^-(f \oplus h)) \quad (4.18)$$

$$= \Phi(f^-) \otimes \mathbb{1} + \mathbb{1} \otimes \Psi(h^-) \quad (4.19)$$

where  $f^-$  and  $h^-$  are defined as

$$\begin{pmatrix} f^- \\ h^- \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} - \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix} E_T^- \begin{pmatrix} f \\ h \end{pmatrix}. \quad (4.20)$$

In particular, if one choose  $f = 0$ ,

$$\Theta(\mathbb{1} \otimes \psi(h)) = \Theta \Xi_0(0 \oplus h) = \varphi(f^-) \otimes \mathbb{1} + \mathbb{1} \otimes \psi(h^-), \quad (4.21)$$

with  $f^- = -RF_2$  and  $h^- = h - RF_1$  where  $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = E_T^- \begin{pmatrix} 0 \\ h \end{pmatrix}$ . Thus,  $f^-$  is supported within  $\text{supp } \rho$  and  $h^-$

is compactly supported within  $M^+ \cup \text{supp } \rho$ . Finally, the induced observables are given by

$$\varepsilon_\sigma(\psi(h)) = \eta_\sigma(\Theta(\mathbb{1} \otimes \psi(h))) = \eta_\sigma(\varphi(f^-) \otimes \mathbb{1} + \mathbb{1} \otimes \psi(h^-)) = \varphi(f^-)\sigma(\mathbb{1}) + \sigma(\psi(h^-))\mathbb{1} \quad (4.22)$$

$$= \varphi(f^-) + \sigma(\psi(h^-))\mathbb{1}. \quad (4.23)$$

In general, one can also find the variance and the moment-generating function. First note that

$$\begin{aligned} \varepsilon_\sigma(\psi(h)^n) &= \eta_\sigma((\varphi(f^-) \otimes \mathbb{1} + \mathbb{1} \otimes \psi(h^-))^n) = \eta_\sigma\left(\sum_{k=0}^n \binom{n}{k} \varphi(f^-)^{n-k} \otimes \psi(h^-)^k\right) \\ &= \sum_{k=0}^n \binom{n}{k} \varphi(f^-)^{n-k} \sigma(\psi(h^-)^k). \end{aligned}$$

Therefore, the induced observable of  $e^{i\psi(h)}$  is

$$\begin{aligned} \varepsilon_\sigma(e^{i\psi(h)}) &= \varepsilon_\sigma\left(\sum_{n=0}^{\infty} \frac{i^n \psi(h)^n}{n!}\right) = \eta_\sigma\left(\sum_{n=0}^{\infty} \frac{i^n}{n!} (\varphi(f^-) \otimes \mathbb{1} + \mathbb{1} \otimes \psi(h^-))^n\right) \\ &= \eta_\sigma\left(e^{i\varphi(f^-)} \otimes e^{i\psi(h^-)}\right) = \sigma\left(e^{i\psi(h^-)}\right) e^{i\varphi(f^-)}. \end{aligned}$$

This result allow us to taking the variance,

$$\begin{aligned} \text{Var}\left(\widetilde{\psi(h)}; \omega_\sigma\right) &= \omega_\sigma\left(\widetilde{\psi(h)}^2\right) - \omega_\sigma(\widetilde{\psi(h)})^2 \\ &= \omega\left(\varepsilon_\sigma\left(\psi(h)^2\right)\right) - \omega\left(\varepsilon_\sigma(\psi(h))\right)^2 \\ &= \omega\left(\varphi(f^-)^2\right) + 2\omega\left(\varphi(f^-)\right)\sigma\left(\psi(h^-)\right) + \sigma\left(\psi(h^-)\right)^2 \\ &\quad - \left(\omega\left(\varphi(f^-)\right)^2 + 2\omega\left(\varphi(f^-)\right)\sigma\left(\psi(h^-)\right) + \sigma\left(\psi(h^-)\right)^2\right) \\ &= \text{Var}\left(\varphi(f^-); \omega\right) + \text{Var}\left(\psi(h^-); \sigma\right) \end{aligned}$$

and, on the other hand, the moment-generating function is

$$\omega_\sigma\left(e^{i\widetilde{\psi(h)}}\right) = \omega_\sigma\left(\widetilde{e^{i\psi(h)}}\right) = \omega\left(\varepsilon_\sigma\left(e^{i\psi(h)}\right)\right) = \sigma\left(e^{i\psi(h^-)}\right)\omega\left(e^{i\varphi(f^-)}\right).$$

which determines all the  $n$ -point functions of the induced observable  $\psi(h)$  as we remarked earlier.

#### 4.2.1 Perturbative treatment of the detector response

Once our scheme is established, we shall make explicit computations with the aim to compare with the explicit computations of the UdW detector in 1.3; we might find  $f^-$  and  $h^-$  by using a perturbative analysis of (4.20). First, observe that equation  $T\Xi = F$  can be rewritten to

$$(P \oplus Q)\Xi = F - \tilde{R}\Xi \quad (4.24)$$

and, one also has, solving for  $\Xi$ ,  $E_T^- F = \Xi$ . Applying  $T_S^-$  to equation (4.24),  $E_T^- F$  can be written as

$$\begin{aligned} E_T^- F &= E_S^- F - E_S^- \tilde{R} E_T^- F \\ &= E_S^- F - E_S^- \tilde{R} E_S^- F + E_S^- \tilde{R} E_S^- \tilde{R} E_T^- F, \end{aligned}$$

where we use the first line in itself in the second line. Then, replacing  $\rho$  by  $\lambda\rho$  ( $R$  remains the same) we obtain a Born-like series for the operator  $E_T^- F$ ,

$$E_T^- F = E_{P \oplus Q}^- F - \lambda E_{P \oplus Q}^- \tilde{R} E_{P \oplus Q}^- F + \lambda^2 E_{P \oplus Q}^- \tilde{R} E_{P \oplus Q}^- \tilde{R} E_{P \oplus Q}^- F + O(\lambda^3).$$

Therefore, using the Born series with  $F = 0 \oplus h$ ,  $h \in C_0^\infty(M)$  and using (4.20) then,  $f^-$  and  $h^-$  can be found to be

$$\begin{aligned} \begin{pmatrix} f^- \\ h^- \end{pmatrix} &= \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda \tilde{R} E_T^- \begin{pmatrix} 0 \\ h \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda \tilde{R} E_{P \oplus Q}^- \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda E_{P \oplus Q}^- \tilde{R} E_{P \oplus Q}^- \begin{pmatrix} 0 \\ h \end{pmatrix} \\ &\quad + \lambda^2 E_{P \oplus Q}^- \tilde{R} E_{P \oplus Q}^- \tilde{R} E_{P \oplus Q}^- \begin{pmatrix} 0 \\ h \end{pmatrix} + O(\lambda^3) \\ &= \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda \begin{pmatrix} 0 & RE_Q^- \\ RE_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda \begin{pmatrix} 0 & E_Q^- \\ E_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 & RE_Q^- \\ RE_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} \\ &\quad + \lambda^2 \begin{pmatrix} 0 & E_Q^- \\ E_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 & RE_Q^- \\ RE_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 & RE_Q^- \\ RE_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} + O(\lambda^3) \\ &= \begin{pmatrix} -\lambda \rho E_Q^- h + O(\lambda^3) \\ h + \lambda^2 \rho E_P^- \rho E_Q^- h + O(\lambda^4) \end{pmatrix}. \end{aligned}$$

Assuming  $\sigma$  has vanishing one point function<sup>23</sup> and  $h \in C_0^\infty(M^+)$ , the induced observable of  $\psi(h)^* \psi(h)$  is

$$\begin{aligned} \varepsilon_\sigma(\psi(h)^* \psi(h)) &= \eta_\sigma(\Theta(\mathbb{1} \otimes \psi(h)^* \psi(h))) \\ &= \varphi(f^-)^* \varphi(f^-) + \sigma(\psi(h^-)^* \psi(h^-)) \mathbb{1} \end{aligned}$$

where  $f^-$  and  $h^-$  are given by the above equation. Define  $h_1 = \rho E_Q^- h$ ,  $h_2 = \rho E_P^- \rho E_Q^- h$  then, we have an explicit form of the induced observable,

$$\begin{aligned} \varepsilon_\sigma(\psi(h)^* \psi(h)) &= \varphi(-\lambda h_1)^* \varphi(-\lambda h_1) + \sigma(\psi(h)^* \psi(h)) \mathbb{1} + \sigma(\psi(h)^* \psi(\lambda^2 h_2)) \mathbb{1} \\ &\quad + \sigma(\psi(\lambda^2 h_2)^* \psi(h)) \mathbb{1} + \sigma(\psi(\lambda^2 h_2)^* \psi(\lambda h_2)) \mathbb{1}. \end{aligned}$$

<sup>23</sup>This assumption is justified that the state  $\sigma$  will be a quasifree state. Using the GNS representation  $(\pi_\sigma, \Omega_\sigma)$ , one get  $\sigma(\varphi(h)) = \langle 0_M | \varphi(h) | 0_M \rangle = 0$ .

Taking expectation value, the measurement of the induced observable finally is

$$\begin{aligned}
\omega(\varepsilon_\sigma(\psi(h)^*\psi(h))) &= \lambda^2 \omega(\varphi(h_1)^*\varphi(h_1)) + \sigma(\psi(h)^*\psi(h)) \\
&\quad + \lambda^2 \sigma(\psi(h)^*\psi(h_2)) + \lambda^2 \sigma(\psi(h_2)^*\psi(h)) + \lambda^4 \sigma(\psi(h_2)^*\psi(h_2)) + O(\lambda^6) \\
&= \sigma(\psi(h)^*\psi(h)) + \lambda^2 [\omega(\varphi(h_1)^*\varphi(h_1)) + 2 \operatorname{Re} \sigma(\psi(h)^*\psi(h_2))] + O(\lambda^4) \\
&= S(\bar{h}, h) + \lambda^2 [W(\bar{h}_1, h_1) + 2 \operatorname{Re} S(\bar{h}, h_2)] + O(\lambda^4),
\end{aligned}$$

where  $S$  and  $W$  are the so-called two-point functions of  $\sigma$  and  $\omega$  respectively and  $h_1, h_2$  are compactly supported within support  $\rho$ . Notice that in the absence of a coupling ( $\rho = 0$ ), the expected value describes the spontaneous excitation of the probe and can be regarded as background noise.

### 4.3 The Unruh effect (Part III: The UdW detector in the FV scheme)

Having introduced the FV scheme as well as some elementary computations, we are in position to analyze the Unruh–DeWitt detector. As we derived earlier, the Unruh effect is the phenomenon in which a field undergoing accelerated object detects the vacuum state as a thermal state. The detector is then an object moving in accelerated motion (in Minkowski space-time) that measures if a (thermal) particle is detected or not. Within the FV scheme, let  $\varphi(f)$  be a generic element of the algebra  $\mathcal{A}(M)$  of the system,  $\omega$  the vacuum state of  $\mathcal{A}(M)$ ,  $\gamma$  the uniformly accelerated trajectory of the field  $\varphi$  and,  $\psi(g)$  be a generic element of the algebra  $\mathcal{B}(M)$  of observables in the probe.

The interaction between probe and system only takes place in the compact region  $K$  with a linear coupling  $\rho$  supported on  $K$ . We prepare the probe in a state  $\sigma$  and, recall that the measurement in the probe describes the measure in the induced observable  $A := \varepsilon_\sigma(B)$  by the equation

$$\omega(A) = \omega(\tilde{B}).$$

As we saw in section 3, we can assume that the GNS representation of the probe induced by  $\sigma$  is a Fock representation and, that  $h$  is chosen so that  $\psi(h) = a(h)$  or at least  $\psi(h)$  closely approximates (in the strong topology) to an annihilation operator in this representation. Accordingly, the operator  $\psi(h)^*\psi(h) = N_f$  is the operator for the mode annihilated by  $\psi(h)$  and, using our perturbative analysis,

$$\omega(\varepsilon_\sigma(N_f)) = S(\bar{h}, h) + \lambda^2 [W(\bar{h}_1, h_1) + 2 \operatorname{Re} S(\bar{h}, h_2)] + O(\lambda^4) \quad (4.25)$$

is the measure (in the probe) of the induced number of particles in the system, where  $h_1 = \rho E_Q^- h$  and  $h_2 = \rho E_P^- \rho E_Q^- h$ .

We claim (as Fewster-Verch [7] did/suggest) that, under a judicious choice of  $h$ , this is in fact a (indirect) measurement of the Unruh-deWitt detector (carefully worked out in 1.3). Essentially, the correspondence of the measure  $\omega(\tilde{B})$  with the induced observable implies that we are measuring the presence (or absence) of a particle at a state  $\omega$  which is of course the quantity observed in the

UdW detector. Moreover, since the two terms involving  $S$  can be regarded as background noise, the information of the system (at order 2) is stored in the two-point function  $W(\bar{h}_1, h_1)$ . In order to completely justify the claim i.e. that the measure (4.25) contains the information of the traditional analysis of the detector, we require a little bit more computational work. For this purpose it suffices to show in the  $1+1$  dimensional case, however the computations in general cases are completely analogous.

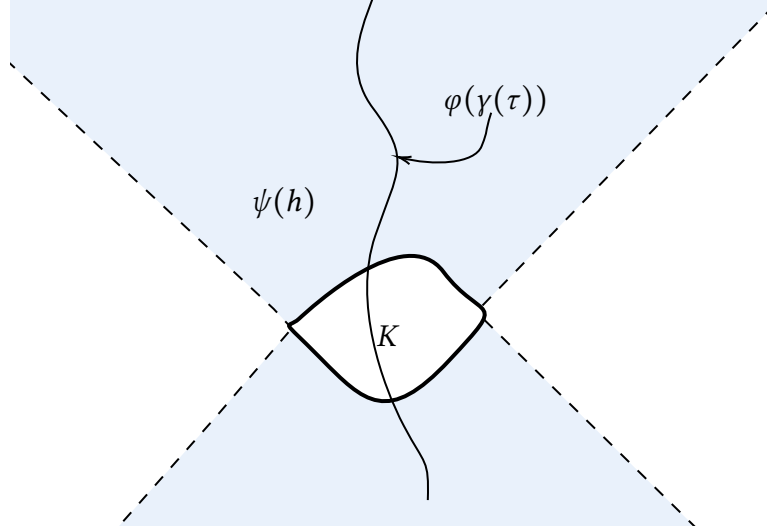


Figure 9: Schematic figure of the UdW detector: The field  $\varphi$  follows a time-like worldline  $\gamma(\tau)$ , the coupling takes place in the region  $K$  and the probe observables are described by  $\psi(h)$ .

Let  $(\pi_\omega, \Omega_\omega)$  be the GNS representations of  $\mathcal{A}$  with respect to the state  $\omega$ . Notice that  $\omega$ , since is a quasifree state (Minkowski vacuum), has the Minkowski vacuum  $|0_M\rangle$  as the GNS vector. It follows that the two point function is given by the equation

$$\begin{aligned} W(\bar{h}_1, h_1) &= \omega(\varphi(\bar{h}_1)\varphi(h_1)) \\ &= \langle \Omega_\omega | \pi_\omega(\varphi(\bar{h}_1))\pi_\omega(\varphi(h_1)) | \Omega_\omega \rangle \end{aligned}$$

where, due to the example 3.2.3, the quasifree state  $\omega$  allows the representation of the field  $\varphi(h_1)$  to be given by

$$\pi_\omega(\varphi(h_1)) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} \overline{\hat{h}_1(k)} a(\mathbf{k}) + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} \hat{h}_1(k) a^*(\mathbf{k}).$$

Observe that the model, described above by the equation (4.10), takes an interaction the form  $\rho\varphi\psi$  for some  $\rho$  real valued. Nonetheless, the interaction of a Unruh-deWitt detector takes place along the uniformly accelerated trajectory  $\gamma$  given by  $\rho(\tau)\varphi(\gamma(\tau))\psi(x)$ , as shown on equation (1.42) and figure 9. In this sense, hereafter we take the UdW detector interaction  $f \mapsto \rho(f)$  acting as a distribution. Accordingly, using the Schwartz kernel theorem, we can write  $\rho(f)$  by the equation

$$\rho(f) = \int d\tau f(\gamma(\tau)) \tilde{\rho}(\tau), \quad (4.26)$$

for some compactly supported function  $\tilde{\rho}$ . Then,  $\hat{h}_1$  (using equation (4.26)) is given by

$$\begin{aligned}\hat{h}_1 &= \int d^4x h_1(x) e^{ik \cdot x} = \int d^4x d\tau \tilde{\rho}(\tau) \gamma^* E_Q^- h e^{ik \cdot x} \\ &= \int d\tau \tilde{\rho}(\tau) \gamma^* \left( \int d^4x E_Q^- h e^{ik \cdot x} \right),\end{aligned}\quad (4.27)$$

where  $\gamma^*$  is just the pullback (precomposition) with  $\gamma$ . In order to find an explicit form of  $h_1$  let us take a look at  $E_Q^- h$ . Recall first that, in the 1 + 1-d model, the Green retarded function is given by (see appendix G for details):

$$G_R(z, t) = i\Theta(t) \int \frac{dk}{2\pi} e^{ikz} \frac{1}{2E} (e^{-iEt} - e^{iEt}). \quad (4.28)$$

Therefore, taking  $h$  concentrated in a point of space<sup>24</sup> i.e.  $h = \delta(x)$ ,

$$\begin{aligned}E_Q^- h &= \int dz' dt' G_R(z - z', t - t') h(z', t') \\ &= \int dz' dt' i\Theta(t - t') \int \frac{dk}{2\pi} e^{ik(z - z')} \frac{1}{2E} (e^{-iE(t - t')} - e^{iE(t - t')}) \delta(x') \\ &= i\Theta(t) \int \frac{dk}{2\pi} e^{ikz} \frac{1}{2E} (e^{-iEt} - e^{iEt}) = i\Theta(t) \delta(z) \frac{1}{2E} (e^{-iEt} - e^{iEt}).\end{aligned}$$

Consequently, the unknown factor on (4.27) is

$$\int d^2x E_Q^- h e^{ik \cdot x} = \frac{i}{2E} e^{-iEt} e^{ik \cdot x}$$

where we only use the negative sign in  $e^{iEt}$  to ensure convergence. It follows that  $\hat{h}_1$ , according to (4.27), is

$$\hat{h}_1 = \frac{i}{2E} \int d\tau \tilde{\rho}(\tau) \gamma^* (e^{-iEt} e^{ik \cdot x}) = \frac{i}{2E} \int d\tau \tilde{\rho}(\tau) e^{-iE\tau} e^{ik \cdot \gamma(\tau)}.$$

Back on the representation  $\pi_\omega(\varphi(h_1))$ , we finally have

$$\begin{aligned}\pi_\omega(\varphi(h_1)) &= \int d\tau \tilde{\rho}(\tau) e^{-iE\tau} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} (a(\mathbf{k}) e^{-ik \cdot \gamma(\tau)} + a^*(\mathbf{k}) e^{ik \cdot \gamma(\tau)}) \\ &= \frac{i}{2E} \int d\tau \tilde{\rho}(\tau) e^{-iE\tau} \varphi(\gamma(\tau))\end{aligned}\quad (4.29)$$

and, putting all back together in the two point-function  $W$ ,

$$\begin{aligned}W(\overline{h_1}, h_1) &= \langle 0_M | \pi_0(\overline{\varphi(h_1)}) \pi_0(\varphi(h_1)) | 0_M \rangle \\ &= \langle 0_M | \int d\sigma d\tau \frac{i}{2E} \rho(\tau) e^{-iE\tau} \varphi(\gamma(\tau)) \frac{i}{2E} \rho(\sigma) e^{-iE\sigma} \varphi(\gamma(\sigma)) | 0_M \rangle \\ &= \frac{1}{4E^2} \int d\sigma d\tau \rho(\tau) \rho(\sigma) e^{-iE(\sigma - \tau)} \underbrace{\langle 0_M | \varphi(\gamma(\tau)) \varphi(\gamma(\sigma)) | 0_M \rangle}_{W(\gamma(\sigma), \gamma(\tau))}\end{aligned}$$

where the function  $W(\gamma(\tau), \gamma(\tau'))$  is the same as described in (1.45). Consequently, if one disregard

<sup>24</sup>This assumption is justified by the fact that  $\pi_\omega(\varphi(\delta(x))) = \varphi(x)$  as we require.

the background noise (the two functions involving  $S$  vanish) in the measure (4.25) we obtain

$$\omega(\varepsilon_\sigma(N_h)) = \lambda^2 \int d\tau d\tau' e^{-iE(\tau-\tau')} \rho(\tau) \rho(\tau') W(\gamma(\tau), \gamma(\tau')) + O(\lambda^4) \quad (4.30)$$

which is exactly the result on section 1.2. That is, the measure that the Unruh-DeWitt detector should give in the absence of any background noise.

**Remark 4.3.1.** *The functions considered in  $h_1$  turns out to not be given in the Hilbert space (it is not square integrable) as expected, because, by the virtue of the Reeh-Schluder theorem, one cannot model a detector as an observable in the probe algebra; one only find an approximation. Particularly, by approximating the Dirac delta function  $\delta$  by square integrable functions.*

## Summary and Outlook

It is important to notice that the algebraic formalism is a powerful tool that has many triumphs, ranging from defining thermal states (that are mathematically satisfactory) to deriving a consistent AQFT. At the same time, the algebraic formalism allows us to have a deeper understanding of relativistic quantum mechanics while allowing us to be able to identify many structures that are hidden in QFT.

Regarding to the measurement problem, it seems rather strange that (traditional) measurement schemes in QMT do not involve the notions of causality. The FV scheme is a novel approach in the sense that takes into account causality and locality of the structures. Thereby, proposing a natural (besides the algebraic formulations) scheme that is very much based in the spirit of QMT. The FV has many advantages, the explanation of the Sorkin protocol and its application to the UdW detector are some triumphs that are coined to the algebraic approach. Of course the FV scheme has also disadvantages, the principal operational one is that the question of finding a probe observable corresponding to a given system observable is not addressed, one only knows the converse. An important result, and perhaps the most important of this work, is the connection between the UdW type detector and the FV scheme realized in the last section. We remark that the results agree under suitable approximations and, connects our discussion in a coherent way.

Having said that, we summarized that this work mainly focuses in three parts, the Unruh effect, the algebraic formalism and the FV measurement scheme. We addressed the main point in each chapter (with particular detail in the FV scheme) in order to understand the theory. However we remark that many important features, that do not relate too much with our approach, are not covered in this work. We list some of the aspects that the reader may want to further consult.

1. The standard derivation of the Unruh effect along with its UdW detector,
  - (a) There are many types of Unruh-DeWitt detector that, as we said before, differ in their outcomes. Detectors in inertial, Rindler, static or circularly moving observers can be found in [8] for further reading.  
 The discussion of each motion in the FV scheme, according to Fewster and Verch [7], “[...] should, in particular, shed light on the localization properties of induced observables of measurements conducted with Unruh-deWitt detectors in arbitrary motion, and lead to a structural understanding of the relation of the coupling and the spacetime structure to the state of the detector (the probe) after measurement.” Therefore, we regard the computations of this clame as as possible future work.
  - (b) Hawking effect and some generalizations of the Unruh effect can be found in [35, 5, 8, 27].
2. The algebraic formalism and the AQFT, We showed the general features of AQFT however, physical theories require more specific tools we list a few concepts that were not used in this work , nevertheless they are important.



- (a) Applications of KMS theory and Tomita-Takesaki [26, 25].
- (b) Scattering theory in AQFT [34] chapter 1.3
- (c) Superslection rules in AQFT [34, 28].

# Appendices

## A Functional-Definitions

**Definition A.1.** A complete normed space is said to be a Banach space

**Definition A.2.** A map  $h : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  is said to be a skew-linear form if it is conjugate linear in the first argument and linear in the second; if  $h$  is positive definite and symmetric  $(f, g) = (g, f)^*$  it is called a inner product

**Definition A.3.** If  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  is complete it is called a Hilbert space

**Definition A.4.** A linear map  $A$  between normed spaces  $X$  and  $Y$  is called a operator with domain  $D(A)$ ,  $D(A)$  is usually required to be dense on  $X$ .

**Definition A.5.** The kernel and range of an operator  $A$  on a Hilbert space are defined by

$$\begin{aligned} \text{Ker}(A) &= \{f \in D(A) \mid Af = 0\} \\ \text{Ran}(A) &= \{Af \mid f \in D(A)\} = \text{AD}(A) \end{aligned}$$

**Definition A.6.**  $A$  is Bounded if the norm of  $A$  is finite,

$$\|A\| = \sup_{\|f\|_X=1} \|Af\|_Y < \infty$$

The set of bounded operators from  $X$  to  $Y$  is denoted by  $\mathfrak{B}(X, Y)$ .

A linear operator  $A$  is bounded if and only if it is continuous. Let  $A \in \mathfrak{B}(X, Y)$  and let  $Y$  be a Banach space. If  $D(A)$  is dense, there is a unique (continuous) extension of  $A$  to  $X$  which has the same norm. Indeed, a bounded operator maps Cauchy sequences to Cauchy sequences, thus  $Af = \lim Af_n$  as  $f_n \in D(A)$ ,  $f \in X$ . This definition is independent of the sequence.

**Definition A.7.** Let  $\mathcal{H}$  be a separable Hilbert space and  $\{e_j\}_{j \in J}$  an orthonormal basis of  $\mathcal{H}$ . A bounded linear operator  $A \in \mathfrak{B}(\mathcal{H})$  is of trace class if

$$\text{Tr}(|A|) := \sum_{j \in J} \langle e_j, |A| e_j \rangle < \infty.$$

We denote the family of trace class operators as  $\mathfrak{L}(\mathcal{H})$ .

**Definition A.8.** Let  $\mathcal{H}$  a Hilbert space. An operator  $A : D(A) \rightarrow \mathcal{H}$  is closed if for any sequence  $\{x_n\}_{n \in \mathbb{N}} \subset D(A)$  such that  $x_n \xrightarrow{n \rightarrow \infty} x$  and  $Ax_n \xrightarrow{n \rightarrow \infty} y$  then,  $x \in D(A)$  and  $Ax = y$ .

**Definition A.9.** A operator  $A : D(A) \rightarrow \mathcal{H}$  is closable if it has a closed extension  $B : D(B) \rightarrow \mathcal{H}$  where  $D(A) \subset D(B)$ .

## B Definitions on QMT

**Definition B.1.** Let  $\Omega$  be a nonempty set and let  $\mathcal{F}$  be a  $\sigma$ -algebra of subsets of  $\Omega$ . A countably additive mapping  $E : \mathcal{F} \rightarrow \mathfrak{B}(\mathcal{H})$  is called a POVM or semispectral measure if  $0 \leq E(\Delta) \leq \mathbb{1}$  for all  $\Delta \in \mathcal{F}$  (i.e. each  $E(\Delta)$  is an effect) and  $E(\Omega) = \mathbb{1}$ .

**Definition B.2.** A POVM  $E : \mathcal{F} \rightarrow \mathfrak{B}(\mathcal{H})$  is called a projection-valued measure (PVM) or spectral measure if in addition  $E(\Delta)^2 = E(\Delta)$  for all  $\Delta \in \mathcal{F}$  or (equivalently)  $E(\Delta)E(\Delta) = 0$  whenever  $\Delta \cap \Delta = \emptyset$ .

**Remark B.1.** A POVM is also referred to as an unsharp observable while a PVMs are sharp observables.

**Remark B.2.** The spectral theorem allow us to write every Hermitian operators on  $\mathcal{H}$  as the integral of the identity with respect to the induced measure,

$$\hat{A} = \int_{\mathbb{R}} x d\hat{\mu}_A.$$

**Example B.1.** Partícula con momento  $L = \sqrt{l(l+1)}\hbar$  Está en un espacio de Hilbert  $\mathbb{C}^{2l+1}$ . En la base  $|l, m\rangle$   $m = -l, -(l-1), \dots, l-1, l$  Es egienc vector de el operador  $L_z$  es decit,  $L_z |l, m\rangle = \hbar m |l, m\rangle$  La completez da

$$\hat{1} = \sum |l, m\rangle \langle l, m|$$

SE define  $\hat{\mu}_L$  por

1.  $\hat{\mu}_{L_z}(\emptyset) = 0$  2.  $E \subset \mathbb{R}$  no contiene ningun valor propio de  $L_z$  entonces  $\hat{\mu}_{L_z}(E) = 0$  3. El unico valor prio posible de  $L_z$  contiene  $E \subset \mathbb{R}$  es  $\hbar k$  entonces  $\hat{\mu}_{L_z}(E) = |l, k\rangle \langle l, k|$  4. En general  $E$  contiene valores proios entonces

$$\hat{\mu}_{L_z}(E) = \sum |l, m_i\rangle \langle l, m_i|$$

$$\hat{\mu}_{L_z}(E) = \int_E d\hat{\mu}_{L_z} = \int_E \sum |l, m_i\rangle \langle l, m_i| \delta(x - m\hbar) dx$$

Asi,  $\sum |l, m_i\rangle \langle l, m_i| \delta(x - m\hbar) dx = \hat{\mu}_{L_z}$

$$Y \text{ por tanto, } \hat{L}_z = \int_{\mathbb{R}} x d\hat{\mu}_{L_z} = \int_{\mathbb{R}} \sum |l, m_i\rangle \langle l, m_i| x \delta(x - m\hbar) dx = \sum m\hbar |l, m_i\rangle \langle l, m_i|$$

Si definimos  $E_i = |a_i\rangle \langle a_i|$ ,  $E_i = E_i^\dagger$ ,  $E_i \geq 0$ ,  $\sum E_i = \hat{1}$ ,  $E_i E_j = \delta_{i,j} E_i$  asi,  $\{E_i\}$  define una PVM

La medida de probabilidad se define por

$$p(E_i|\psi) = \begin{cases} |\langle \psi | a_i \rangle|^2 \\ \langle E_i \rangle_\psi \\ \|E_i |\psi\rangle\|^2 \\ \text{Tr}(|\psi\rangle\langle\psi| E_i) \end{cases}$$

$$p(a_i|\hat{\rho}) = \text{Tr}(\hat{\rho} E_i)$$

La caracterizacion de las medidas de probabilidad para cada  $E_i$  es completo y se llama el teorema de Gleason

$$p(\hat{0}) = 0, \quad p(\hat{1}) = \hat{1}, \quad p(E + E') = p(E) + p(E')$$

si  $EE' = 0$  existe matriz de densidad  $\rho$  tal que  $p(E) = \text{Tr}(\hat{\rho}E)$  Es unica si la dim es mayor que 2.

**Theorem B.3** (Naimark). *Every POVM  $\hat{M}$  defined on a Hilbert space  $\mathcal{H}$  can be extended to a PVM in the sense that there exist a PVM  $\hat{\mu}$  defined on  $\mathcal{H}' \supset \mathcal{H}$  such that*

$$\hat{M}(E) = P_{\mathcal{H}} \hat{\mu}(E) \Big|_{\mathcal{H}}$$

where  $P_{\mathcal{H}}$  is the projection into  $\mathcal{H}$ .

**Example B.2.** *The observables  $A_S$  and  $B_M$ . Assume that  $\hat{B}_M = \sum_i b_i |b_i\rangle\langle b_i|$ ,  $b_i$  is the spectrum of  $\hat{B}_M$ . Assume the correlation between  $A_S$  and  $B_M$  is  $a_i \iff b_i$ . Then, the projections  $\{|b_i\rangle\langle b_i|\}_{i \in \mathbb{N}}$  define a PVM mainly by  $b_i \mapsto |b_i\rangle\langle b_i|$ .*

*Assume the initial system is  $\hat{\rho}_i \otimes \hat{\rho}_M$ , where  $\hat{\rho}_M$  is the prepare state  $|\psi\rangle\langle\psi|$ . Then,*

$$\begin{aligned} \Pi(b_i) &= \text{Tr}_M(\hat{U}^\dagger |b_i\rangle\langle b_i| \hat{U} |\psi\rangle\langle\psi|) \\ &= \underbrace{\langle \hat{U} \psi | b_i \rangle \langle b_i | \hat{U} | \psi \rangle}_{\hat{E}_i^\dagger \hat{E}_i} = \hat{E}_i^\dagger \hat{E}_i. \end{aligned}$$

$\Pi(b_i)$  is Hermitian and positive since  $\langle \Pi(b_i) \varphi | \varphi \rangle = \|E_i \varphi\|^2$ . The operators  $E_i$  define a Kraus opertro since

$$\sum_i \hat{E}_i^\dagger \hat{E}_i = \text{Tr}_M(\hat{U}^\dagger \hat{U} |\psi\rangle\langle\psi|) = \text{Id}_{\mathcal{H}_S}.$$

Therefore, the map  $b_i \mapsto \Pi(b_i)$  defines a POVM for the measurement of  $A_S$  and  $\text{Prob}(a_i|\rho_i) = \text{Tr}_S(\hat{E}_i^\dagger \hat{E}_i \hat{\rho}_i)$ .

The measure may change the system's state as

$$\hat{\rho}'_S = \frac{\hat{E}_i \hat{\rho}_i \hat{E}_i^\dagger}{\text{Tr}(\hat{E}_i^\dagger \hat{E}_i \hat{\rho}_i)}.$$

**Definition B.3.** A quantum operator is a linear map  $\Phi : \mathfrak{T}(\mathcal{H}) \rightarrow \mathfrak{T}(\mathcal{G})$  between trace-class operators of a Hilbert space  $\mathcal{H}$  into a trace-class operators of a Hilbert space  $\mathcal{G}$  satisfying that  $\text{Tr}_{\mathcal{G}}(\Lambda(S)) \leq 1$  and  $\Lambda$  is completely positive.

**Theorem B.4** (Kraus). Let  $\mathcal{H}$  and  $\mathcal{G}$  be two Hilbert spaces of dimension  $n$  and  $m$  respectively and  $\Phi$  be a quantum operation between  $\mathcal{H}$  and  $\mathcal{G}$ . Then, there are matrices  $\{B_i\}_{i \in I}$  such that,

$$\Phi(\rho) = \sum_{i \in I} B_i \rho B_i^* \quad (\text{B.1})$$

Conversely, any map

$$\rho \mapsto \sum_{i \in J} B_i \rho B_i^*, \quad \sum_i B_i^* B_i = 1 \quad (\text{B.2})$$

is a quantum operation.

**Definition B.4.** The matrices  $\{B_i\}_{i \in J}$  defined on equation (B.2) are called Kraus operators.

**Definition B.5.** Let  $\mathcal{A}$  be a  $*$ -algebra. An element  $A \in \mathcal{A}$  is an effect if  $A$  and  $1 - A$  are both positive.

**Definition B.6.** Let  $(X, \Sigma)$  be a measurable space and  $\mathcal{A}$  a  $*$ -algebra. Any map  $E : \Sigma \rightarrow \mathcal{A}$  is called an Effect-Valued Measure (EVM) if  $E$  has the properties of a measure, that is,

1.  $E(X)$  is positive for all  $X \in \Sigma$ .
2.  $E(\emptyset) = 0, E(X) = 1$
3.  $\hat{\mu}$  is additive.

and takes its values in the effects of  $\mathcal{A}$ .

Then one interprets the elements of  $X$  as potential outcomes of the measurement and for each  $X \in \Sigma$ ,  $\omega(E(X))$  is the probability that a value lying in  $\Sigma$  is measured in the state  $\omega$ .

## C Algebraic quantum mechanics

In Quantum Mechanics (QM), (pure) states are rays on a (separable) Hilbert space  $\mathcal{H}$  and observables are self-adjoint operators in  $\mathcal{H}$ . Given a self-adjoint operator  $A$  acting on  $\mathcal{H}$  and a normalized vector  $\varphi \in \mathcal{H}$ , the probabilistic nature of QM allows us to assign a probability measure  $\mu$  such that the probability of finding the observable  $A$  on one of the values of  $\Delta$  is

$$\text{Prob}(A \in \Delta; \varphi) = \int_{\Delta} \mu(t) = \langle \varphi | P_A(\Delta) \varphi \rangle,$$

where  $P_A$  is the spectral projection of the operator  $A$  associated with the Borel set  $\Delta \subset \mathbb{R}$ . Notice that the map  $\Delta \rightarrow P_A(\Delta)$  determines a PVM.

## D Canonical Quantization

The quantum system, associated to a dynamical system, is carried by adopting

1. There exists a Hilbert space  $\mathcal{H}$  for a quantum system and the state of the system is required to be described by a vector (or state)  $|\varphi\rangle$  and, two states  $|\varphi\rangle$  and  $c|\varphi\rangle$  describe the same state. The state can also be described as a ray representation of  $\mathcal{H}$ .
2. A physical quantity  $A$  in classical mechanics is replaced by a Hermitian operator  $\hat{A}$  acting on  $\mathcal{H}$ . The operator  $\hat{A}$  is often called an observable. The result obtained when  $A$  is measured is one of the eigenvalues of  $\hat{A}$ .
3. The Poisson bracket in classical mechanics is replaced by the **commutator**

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

multiplied by  $-i/\hbar$ . The unit in which  $\hbar = 1$  will be employed here. The fundamental commutation relations are

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \quad [\hat{q}_i, \hat{p}_j] = i\delta_{ij}$$

Hamilton's equations became

$$\frac{d\hat{q}_i}{dt} = \frac{1}{i}[\hat{q}_i, \hat{H}] \quad \frac{d\hat{p}_i}{dt} = \frac{1}{i}[\hat{p}_i, \hat{H}]$$

When a classical quantity  $A$  is independent of  $t$  explicitly,  $A$  satisfies the same equation as Hamilton's equation. By analogy, for  $\hat{A}$  which does not depend on  $t$  explicitly, one has Heisenberg's equation of motion:

$$\frac{d\hat{A}}{dt} = \frac{1}{i}[\hat{A}, \hat{H}]$$

4. Let  $|\varphi\rangle \in \mathcal{H}$  be an arbitrary state. Suppose one prepares many systems, each of which is in this state. Then, observation of  $A$  in these systems at time  $t$  yields random results in general. Then the expectation value of the results is given by

$$\langle A \rangle_t = \frac{\langle \varphi | \hat{A}(t) | \varphi \rangle}{\langle \varphi | \varphi \rangle}$$

5. For any physical state  $|\varphi\rangle \in \mathcal{H}$ , there exists an operator for which  $|\varphi\rangle$  is one of the eigenstates.

## E The $*$ , $C^*$ and von Neumann algebras

We state some concepts and well known facts about mathematical algebras in operator theory. We follow [28, 29].

**Definition E.1.** *An associative algebra  $\mathcal{A}$  is a set equipped with the maps addition  $+: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ , scalar multiplication  $\lambda \in \mathbb{C} \lambda: \mathcal{A} \rightarrow \mathcal{A}$  and multiplication  $\cdot: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$*

$$+(A, B) := A + B, \quad \lambda(A) := \lambda A \quad \text{and} \quad \cdot(A, B) := AB.$$

The algebra is both, a complex vector space and a ring,  $V = (\mathcal{A}, +, \{\lambda\}_{\lambda \in \mathbb{C}})$  and  $R = (\mathcal{A}, +, \cdot)$  respectively. An algebra is said to be unital if the ring structure has a unit. An involution on an algebra  $\mathcal{A}$  is a map  $*$ :  $\mathcal{A} \rightarrow \mathcal{A}$   $*$ ( $A$ ) :=  $A^*$  such that

$$(\lambda A + B)^* = \lambda^* A^* + B^*, \quad (AB)^* = B^* A^* \quad \text{and} \quad (A^*)^* = A,$$

where  $\lambda^*$  is the complex conjugate of  $\lambda$ . Also, we define a norm  $\|\cdot\|: \mathcal{A} \rightarrow \mathbb{R}^+$  in which the multiplication  $\cdot$  is a continuous map. That is,

$$\|AB\| \leq \|A\| \|B\|.$$

If  $(V, \|\cdot\|)$  is a Banach vector space the algebra is said to be a Banach algebra.

**Definition E.2** ( $*$  and  $C^*$  algebras). *An  $*$ -algebra is an algebra equipped with an involution  $*$ . An  $C^*$ -algebra is a Banach  $*$ -algebra such that*

$$\|A^* A\| = \|A\|^2, \quad A \in \mathcal{A}.$$

The ' $C$ ' stands for 'closed', first defined by Segal in 1947 as a "uniformly closed, self-adjoint algebra of bounded operators on a Hilbert space" [30] which of course relates with the above definition by means of representation theory.

**Definition E.3.** *Let  $\mathcal{A}$  be a  $*$ -subalgebra of  $\mathfrak{B}(\mathcal{H})$ . We say that  $\mathcal{A}$  is a von Neumann algebra if  $\mathcal{A}$  is a unital algebra and*

$$(\mathcal{A}')' = \mathcal{A}, \quad \mathcal{A}' = \{B \in \mathfrak{B}(\mathcal{H}) : [B, A] = 0, \forall A \in \mathcal{A}\}.$$

$\mathcal{A}'$  is called the commutant of  $\mathcal{A}$  and thus, a von Neumann algebra is a  $*$ -subalgebra of  $\mathfrak{B}(\mathcal{H})$  whose double commutant is itself. In general, a von Neumann algebra need not be unital but will ease computations.

Observe that a von Neumann endows the topology on  $\mathfrak{B}(\mathcal{H})$  and thus, we introduce some of them.

**Definition E.4.** *Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be a Hilbert space. The uniform topology on  $\mathfrak{B}(\mathcal{H})$  is defined by the*

norm  $\|\cdot\| : \mathfrak{B}(\mathcal{H}) \rightarrow \mathbb{R}^+$ ,

$$\|A\| = \sup\{ \underbrace{\|Av\|}_{\langle Av, Av \rangle^{1/2}} : v \in \mathcal{H}, \|v\| \leq 1 \}, \quad A \in \mathfrak{B}(\mathcal{H}), \quad (\text{E.1})$$

The sequence  $\{A_i\}_{i \in \omega}$  converges (uniformly) to  $A \in \mathfrak{B}(\mathcal{H})$  if and only if

$$\lim_{i \rightarrow \infty} \|A_i - A\| = 0.$$

**Definition E.5.** The weak topology on  $\mathfrak{B}(\mathcal{H})$  is defined by the family of seminorms  $\{f_{u,v} : u, v \in \mathcal{H}\}$  where

$$f_{u,v}(A) = \langle u, Av \rangle.$$

In other words, the weak topology is the initial topology of the family  $\{f_{u,v} : u, v \in \mathcal{H}\}$ .

The sequence  $\{A_i\}_{i \in \omega}$  converges (weakly) to  $A \in \mathfrak{B}(\mathcal{H})$  if and only if

$$\lim_{i \rightarrow \infty} f_{u,v}(A_i) = f_{u,v}(A), \quad \forall u, v \in \mathcal{H}.$$

**Definition E.6.** The strong topology on  $\mathfrak{B}(\mathcal{H})$  is defined by the family of norms  $\{f_v : v \in \mathcal{H}\}$  with

$$f_v(A) = \|Av\|.$$

In other words, the strong topology is the initial topology with respect to the family  $\{f_v : v \in \mathcal{H}\}$ . The sequence  $\{A_i\}_{i \in \omega}$  converges (strongly) to  $A \in \mathfrak{B}(\mathcal{H})$  if and only if

$$\lim_{i \rightarrow \infty} f_u(A_i) = f_u(A), \quad \forall u \in \mathcal{H}.$$

**Definition E.7.** The ultra weak topology on  $\mathfrak{B}(\mathcal{H})$  is defined in terms of the family  $\{f_v : v \in \mathcal{T}(\mathcal{H})\}$ , where  $\mathcal{T}(\mathcal{H})$  is the set of positive operators with trace 1 on  $\mathcal{H}$  and

$$f_v(A) = \text{Tr}(vA).$$

The sequence  $\{A_i\}_{i \in \omega}$  converges (ultra-weak) to  $A \in \mathfrak{B}(\mathcal{H})$  if and only if

$$\lim_{i \rightarrow \infty} f_u(A_i) = f_u(A), \quad \forall u \in \mathcal{H}.$$

If  $\mathcal{H}$  is finite dimensional the four topologies coincide. Indeed, it is enough to take basis elements. In general, the ultraweak topology is not comparable with the strong topology, the norm topology is the finest of them all and the weak topology is coarser than both, the ultraweak topology and the strong topology.

**Theorem E.1** (von Neumann double commutant theorem).  $\mathcal{A}$  is weakly closed if and only if the double commutant of  $\mathcal{A}$  is  $\mathcal{A}$ .



## E.1 States

**Definition E.8.** A state on a (unital)  $\ast$ -algebra  $\mathcal{A}$  is a linear functional  $\omega : \mathcal{A} \rightarrow \mathbb{R}$  such that

$$\omega(AA^\ast) \geq 0, \quad \forall A \in \mathcal{A} \quad \text{and} \quad \omega(1) = 1.$$

$\omega$  is said to be mixed if it's sum of at least two states  $\omega_1 \neq \omega_2$ . If  $\omega$  is not mixed,  $\omega$  is pure.

**Definition E.9.** Let  $\mathcal{A}$  be an  $C^\ast$ -algebra. A representation (on a Hilbert space  $\mathcal{H}$ ) of  $\mathcal{A}$  is  $\ast$ -homomorphism  $\pi : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{H})$ . The representation  $\pi$  is reducible if  $\pi(\mathcal{A})(\mathcal{H})$  has non-trivial invariant subspaces and,  $\pi$  is faithful if  $\pi$  is an isomorphism.

**Definition E.10.** Let  $\pi : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{H}_1), \phi : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{H}_2)$  be two representations of  $\mathcal{A}$  on some Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$ .

1.  $\pi$  and  $\phi$  are equivalent if there is  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  unitary such that  $U\pi(A) = \phi(A)U, \forall A \in \mathcal{A}$ .  $\phi$  is a subrepresentation of  $\pi$  if there is an isometry  $V : \mathcal{H}_2 \rightarrow \mathcal{H}_1$  such that  $\pi(A)V = V\phi(A), \forall A \in \mathcal{A}$ .
2.  $\pi$  and  $\phi$  are quasiequivalent if the von Neumann algebras  $\pi(\mathcal{A})'', \phi(\mathcal{A})''$  are  $\ast$ -isomorphic.
3.  $\pi$  and  $\phi$  are disjoint if they are not quasiequivalent.

**Definition E.11.** Let  $\mathcal{H}$  be a Hilbert space,  $S \subset \mathfrak{B}(\mathcal{H})$  and  $G \subset \mathcal{H}$ . Then  $G$  is said to be cyclic for  $S$  if  $\text{span}\{SG\}$  is dense with respect to the uniform topology.  $G$  is separating for  $S$  if  $\forall A \in S, AG = 0$  implies  $A = 0$ .

**Definition E.12.** Two projections  $P_1$  and  $P_2$  in a von Neumann algebra  $\mathcal{A}$  are said to be equivalent if there is a  $V \in \mathcal{A}$  such that  $V^\ast V = E, VV^\ast = F$ , in that case we write  $E \sim F$ .

**Definition E.13.** Given two von Neumann algebras  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , the generated von Neumann algebra  $\mathcal{A}$  is the algebra

$$\mathcal{A} = \bigcap_{\mathcal{B} \in \mathfrak{B}} \mathcal{B}, \quad \mathfrak{B} = \{\mathcal{B} : \mathcal{B} \text{ is a von Neumann algebra and } \mathcal{A}_1 \subseteq \mathcal{B}, \mathcal{A}_2 \subseteq \mathcal{B}\}$$

**Definition E.14.** Let  $\mathcal{A}$  a von Neumann algebra. In general the algebra is not closed under commutant, thus, the center of the algebra is the elements that commute with the algebra that lies on the algebra, that is,  $Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$ . Clearly (in a unital von Neumann algebra)  $\mathbb{C}I \subset \mathcal{A}$ , thus,  $Z(\mathcal{A}) \neq \emptyset$ . If  $Z(\mathcal{A}) = \mathbb{C}I$  the algebra is called a factor, the algebra generated by  $\mathcal{A}$  and its commutant is  $\mathfrak{B}(\mathcal{H})$ . A projection that lies on the center of the algebra is called a central projection.

**Definition E.15.** A faithful normaised trace on a von Neumann algebra  $\mathcal{A}$  is a state  $\rho$  on  $\mathcal{A}$  such that

$$\rho(AB) = \rho(BA), \quad \forall A, B \in \mathcal{A}. \quad (\text{E.2})$$

$$\rho(A^\ast A) = 0 \iff A = 0, \quad A \in \mathcal{A}. \quad (\text{E.3})$$

## E.2 Tomita-Takesaki theory

**Theorem E.2.** *Let  $\varphi$  be a state on a von Neumann algebra  $\mathcal{A}$  acting on a Hilbert space  $\mathcal{H}$ . Then  $\varphi$  is normal if and only if there exist a density (positive trace-class) operator  $\rho$ , on  $\mathcal{H}$  with  $\text{Tr } \rho = 1$ , such that*

$$\varphi(A) = \text{Tr}(\rho A) \quad \forall A \in \mathcal{A},$$

where  $\text{Tr}$  is the trace operator.

**Definition E.16.** *Let  $\mathcal{A}$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$ ,  $\Omega \in \mathcal{H}$  a separating and cyclic vector. The operators  $S_0, F_0$  are defined such that*

$$S_0(m\Omega) = A^\ast \Omega, \quad F_0(m\Omega) = A'^\ast \Omega \quad \forall A \in \mathcal{A}, A' \in \mathcal{A}'.$$

$F_0, S_0$  are neither bounded nor defined on the whole Hilbert space. However, they are defined on a dense subset.

**Proposition E.1.** *The operators  $S_0$  and  $F_0$  are closable. Denote  $S = \text{Cl}(S_0)$  and  $F = \text{Cl}(F_0)$  their closed extensions.*

**Theorem E.3** (Polar decomposition).

**Corollary E.3.1.** *There exist a unique, positive, self adjoint operator  $\Delta$ , and a unique anti-unitary operator  $J$  such that*

$$S = J\Delta^{1/2}.$$

**Definition E.17.** *We call  $\Delta$  the modular operator associated with  $\mathcal{A}, \Omega$  and  $J$  its correspondent modular conjugation.*

**Proposition E.2.** *The following relations hold,*

$$\begin{array}{llll} 1. \Delta = FS & 3. \Delta^{-1} = SF & 5. \Delta^{-1/2} = J\Delta^{1/2}J & 7. J^2 = 1. \\ 2. F = J\Delta^{-1/2} & 4. J = J^\ast & 6. S = J\Delta^{1/2} & \end{array}$$

**Theorem E.4** (Tomita-Takesaki theorem). *Let  $\mathcal{A}$  be a von Neumann algebra with cyclic and separating vector  $\Omega$ , let  $\Delta$  the modular operator and  $J$  the associated modular conjugation. Then,*

$$J\mathcal{A}J = \mathcal{A}'$$

holds. Moreover,

$$\Delta^{it}\mathcal{A}\Delta^{it} = \mathcal{A}.$$

**Definition E.18.** *A state  $\phi$  on a von Neumann algebra  $\mathcal{A}$  is faithful if  $\phi(A) > 0$  for all non-zero  $A \in \mathcal{A}$ .*

Tomita-Takesaki theory has meaningful applications in both classical and quantum physics, for a detailed explanation we refer to [26].

**Lemma E.5.** *Let  $\mathcal{A}$  be a von Neumann algebra, and  $\varphi$  a faithful, normal state on  $\mathcal{A}$ . Consider  $\pi_\varphi$ , the GNS representation of  $\mathcal{A}$  associated to  $\varphi$ . Then, the cyclic vector  $\Omega_\varphi$  is separating for  $\pi_\varphi(\mathcal{A})$ .*

*Proof.* Fix  $\pi_\varphi(A), A \in \mathcal{A}$ . Suppose  $\pi_\varphi(A)\Omega_\varphi = 0$ . Then,

$$\begin{aligned} 0 &= \|\pi_\varphi(A)\Omega_\varphi\|^2 = \langle \pi_\varphi(A)\Omega_\varphi | \pi_\varphi(A)\Omega_\varphi \rangle \\ &= \langle \Omega_\varphi | \pi_\varphi(A)^* \pi_\varphi(A) \Omega_\varphi \rangle \\ &= \langle \Omega_\varphi | \pi_\varphi(A^* A) \Omega_\varphi \rangle \\ &= \varphi(A^* A). \end{aligned}$$

Since  $\varphi$  is faithful,  $A^* A = 0$  and hence  $A = 0$ . ■

## F Convex geometry in CST

Given a fixed time orientation in the manifold (space-time), time-like or light-like vectors can be classified as future or past pointing. We define some concepts of causal convex geometry. We follow [36, 7]. Let  $M$  be space-time,

**Definition F.1.** *A subset  $\Sigma \subset M$  is called a Cauchy hypersurface if every inextendible timelike curve in  $M$  meets  $\Sigma$  exactly once.*

**Remark F.1.** *Any Cauchy hypersurface of  $M$  is a submanifold of dimension 1.*

**Definition F.2.** *If  $M$  possesses a Cauchy hypersurface then  $M$  is called a globally hyperbolic space-time.*

**Example F.1.** *Minkowski, Friedmann, Schwarzschild and de Sitter spacetimes are all global hyperbolic.*

**Definition F.3.** *Let  $x \in M$ , the causal future/past  $J^\pm(x)$  of  $x$  is the set of all points that can be reached from  $x$  by smooth future/past directed causal curves (including  $x$  itself).*

*Let  $S \subset M$ , we write  $J^{+/-}(S) := \bigcup_{x \in S} J^{+/-}(x)$  and,  $J(S) = J^+(S) \cup J^-(S)$ . If  $S$  is closed, then so is  $J^\pm(S)$  (connectedness)*

**Definition F.4.** *Let  $S \subset M$ ,  $S$  is called strictly past compact (pc) if  $S$  is closed and there is a compact  $K \subset M$  such that  $A \subset J^+(K)$ .*

*If  $A$  is a strictly past compact then, there exist  $K$  compact such that  $J^+(A) \subset J^+(J^+(K)) = J^+(K)$ . Thus,  $J^+(A)$  is strictly past compact.*

**Definition F.5.** *Let  $A \subset M$ ,  $A$  is called a future compact (fc) if  $A \cap J^+(x)$  is compact for all  $x \in M$ .*

**Example F.2.** *If  $\Sigma$  is a Cauchy hypersurface, then  $J^-(\Sigma)$  is future compact.*

**Definition F.6.** *If  $A \subset M$  is past compact and future compact then we call  $A$  temporally compact.*

**Example F.3.** Let  $M$  be Minkowski space and let  $C \subset M$  be an open cone with tip  $0$  containing the closed cone  $J^-(0) \setminus \{0\}$ . Then  $A = M \setminus C$  is past compact but not strictly past compact. Indeed, for each  $x \in M$ , the set  $J^-(x) \cap A = J^-(x) \setminus C$  is compact. But  $A$  is not strictly past compact because the intersection of  $A$  and space-like hyperplanes is not compact.

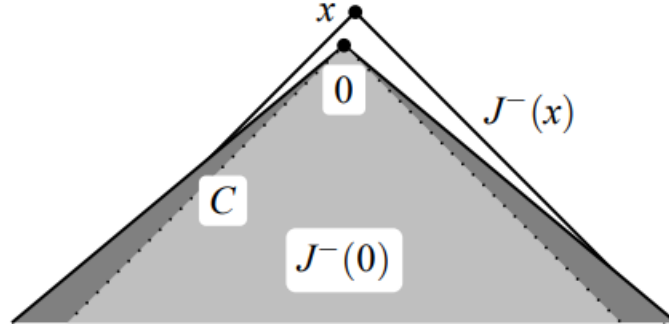


Figure 10: Cones  $J^-(0)$  and the open cone  $C$ . Figure retrieved from [36].

**Definition F.7.** The causal hull of  $S$  is the intersection  $J^+(S) \cap J^-(S)$ . Set of all points that lie on causal curves with both endpoints in  $S$ .

$A \subset M$  is causally convex if it is equal to its causal hull. (contains every causal curve that begins and ends in it) The causal hull of  $S$  is the intersection of all causal convex sets containing  $S$

**Definition F.8.** The space-time is globally hyperbolic if and only if it is devoid of closed causal curves and the causally hull of ant compact is compact.

**Definition F.9.** A Cauchy surface is a set intersected exactly once by every in-extendable smooth time-like curve. Every Lorentizan space-time possessing a Cauchy surface is globally hyperbolic and every globally hyperbolic space-time may be foliated into Cauchy surfaces that are, additionally, smooth space-like hypersurfaces.

Any open causally convex subset of a globally hyperbolic space-time is itself globally hyperbolic when equipped with the induced metric and time-orientation.

**Definition F.10.** The causal complement of a set  $S$  is defined as  $S^\perp = M \setminus J(S)$ .

Sets  $S$  and  $T \subset S^\perp$  are causally disjoint if there is no causal curve joining  $S$  and  $T$ .

In a globally hyperbolic space-time, the causal future and past of an open set are open, while those of a compact set are closed.  $K$  compact,  $K^\perp$  open,  $K \subseteq K^{\perp\perp}$  is closed possibly not compact and in general  $K^{\perp\perp}$  is not the causal hull of  $K$ . (In Minkowski space-time, if  $K$  is a time-like curve segment they coincide but if  $K$  is a subset of a constant time hyper-surface they differ)

Let  $S, T \subset \mathbf{M}$ , if  $J^+(S) \cup J^-(T) = \emptyset$  or  $J^+(T) \cup J^-(S) = \emptyset$  then, there is a Cauchy surface of  $\mathbf{M}$  lying to the future of  $T$  and the past of  $S$  thus, establishing a causal ordering in which  $S$  is later than  $T$ . Also, if  $S$  and  $T$  are causally disjoint it is possible to order  $S$  both later and earlier than  $T$ .

**Definition F.11.** If  $J^+(S) \cup J^-(T) = \emptyset$  or  $J^+(T) \cup J^-(S) = \emptyset$  we say that  $S$  and  $T$  are causally orderable.

**Definition F.12.** The future/past Cauchy development  $D^{+/-}(S)$  is the set of points  $p$  so that every past/future-inextendable pice-wise smooth casual curve through  $p$  meets  $S$

**Definition F.13.** The spaces of smooth functions with supports that are future, past, or spatially compact are designated by subscripts  $fc$ ,  $pc$ ,  $sc$ . For example,  $C_{sc,fc}^\infty(M)$  consists of smooth functions on  $M$  whose supports are both spatially compact and future compact.

## G Green retarded function for the K-G equation in 1 + 1-d

The Klein-Gordon equation in 1 + 1-D is given by the equation:

$$(\partial_t^2 - \partial_z^2 + m^2)\phi = f(z, t). \quad (G.1)$$

The Green's function satisfy

$$(\partial_t^2 - \partial_z^2 + m^2)G(z, t) = \delta(z)\delta(t). \quad (G.2)$$

**Proposition G.1.** The Green retarded function of the K-G equation in 1 + 1-d is given by

$$G(z, t) = i\Theta(t) \int \frac{dk}{2\pi} e^{ikz} \frac{1}{2\sqrt{k^2 + m^2}} \left( e^{-i\sqrt{k^2 + m^2}t} - e^{i\sqrt{k^2 + m^2}t} \right) \quad (G.3)$$

**Lemma G.1.**

$$\lim_{\epsilon \rightarrow 0^+} F_\epsilon(t) := \lim_{\epsilon \rightarrow 0^+} \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{\omega - \omega_0 + i\epsilon} = -ie^{-i\omega_0 t}, \quad \omega_0 \in \mathbb{R} \quad (G.4)$$

*Proof.* Let  $\omega = Re^{i\phi}$  with  $\phi \in (\pi, 2\pi)$ . Notice that  $(t < 0) |e^{-i\omega t}| = |e^{-iRe^{i\phi}t}| = |e^{R\sin\phi t}| \xrightarrow{R \rightarrow \infty} 0$ . Therefore the integral (G.4) coincides (in the limit  $R \rightarrow \infty$ ) with the closed integral in the lower half circle (for  $t < 0$ ). Using the residue theorem,

$$F(t) = \oint \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{\omega - \omega_0 + i\epsilon} = -i\text{Res}\left(\frac{e^{-i\omega t}}{\omega - \omega_0 + i\epsilon}, \omega_0 - i\epsilon\right) = -ie^{-i\omega_0 t} e^{-\epsilon t} \quad (G.5)$$

where we put a minus as we integrate in anticlockwise orientation. Taking limits gives the desired result. ■

*Proof of proposition H.1.* In order to find  $G_R$ , the equation (G.2) in the Fourier space is given by

$$\hat{G}(k, \omega) = \frac{1}{(-\omega^2 + k^2 + m^2)} = -\frac{1}{2\sqrt{k^2 + m^2}} \left( \frac{1}{\omega - \sqrt{k^2 + m^2}} - \frac{1}{\omega + \sqrt{k^2 + m^2}} \right). \quad (G.6)$$

which gives the function

$$G(z, t) = - \int \frac{dz}{2\pi} e^{ikz} \frac{1}{2\sqrt{k^2 + m^2}} \int \frac{d\omega}{2\pi} e^{-i\omega t} \left( \frac{1}{\omega - \sqrt{k^2 + m^2}} - \frac{1}{\omega + \sqrt{k^2 + m^2}} \right). \quad (G.7)$$

The Green retarded function is given by shifting both poles into the lower half plane and, using the lemma,

$$G(z, t) = i\Theta(t) \int \frac{dk}{2\pi} e^{ikz} \frac{1}{2\sqrt{k^2 + m^2}} \left( e^{-i\sqrt{k^2 + m^2}t} - e^{i\sqrt{k^2 + m^2}t} \right) \quad (\text{G.8})$$

where  $\Theta(t)$  is introduced due to the convergence of the integrals, for  $t > 0$ , in the lemma. ■

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