Reps

$$\mathcal{U}(1+\omega, 0) = 1 + \frac{i}{2}\omega_{\rho\sigma} J^{\rho\sigma}$$

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

①

$K^\mu \to K^\mu + \omega^\mu{}_\nu K^\nu$ we want to stabilize K^μ $\to$ $\omega^\mu{}_\nu K^\nu = 0$

so $\boxed{\omega_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} n^\rho K^\sigma}$ satisfies

thus $\omega_{\rho\sigma} J^{\rho\sigma} = J^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma} n^\mu K^\nu = 2 n^\rho W_\rho$

$\boxed{\frac{1}{2} J^{\nu\mu} \cdot K^\sigma \cdot \epsilon_{\rho\nu\mu\sigma}}$

Pauli-Lubanski.

So we want to argue
we find that W_ρ (pauli-lubanski) becomes the generator
and n the parameters. generates the LG.

- $W_0 = \frac{1}{2}\epsilon_{\mu\nu 0\rho} J^{\mu\nu} p^\rho = \frac{1}{2}\epsilon_{ijk} J^{ij} p^k = \underline{J} \cdot \underline{p}$

operators

proj of J onto p.

- $W^i = -J^i p^0 - (\underline{K} \times \underline{p})^i$

helicity operator

Ex/. massive rep $K^\mu = (m, 0, 0, 0)$

$\hookrightarrow$ $W_0 |K^\mu, m^2, \omega^2, \sigma\rangle = \underline{J} \cdot \underline{p} |\ldots\rangle = 0$ $\quad \underline{K} = 0$

$W_i | K^\mu, m^2, \omega^2, \sigma\rangle = -J_i m |K^\mu, m^2, \omega^2, \sigma\rangle$

basically acts as J_i on massive.

So LG is generated by $\{J_i\} \to so(3)$

$p^0 > 0$

Casimir $\boxed{W_\mu W^\mu} = -m^2 J_i^2$

- massless $p^2 = 0$, $K^\mu = (E, 0, 0, E)^\mu$

lightcone coordinates

generates with p and SO.

- $W_0 |K^\mu, \omega^2, \sigma\rangle = \underline{J} \cdot \underline{p} |\ldots\rangle = E J_3 |K^\mu, \omega^2, \sigma\rangle$
- $W_1 |K^\mu, \omega^2, \sigma\rangle = E(K^2 - J^1) |K^\mu, \omega^2, \sigma\rangle$
- $W_2 |K^\mu, \omega^2, \sigma\rangle = E(K^1 + J^2) |K^\mu, \omega^2, \sigma\rangle$

$W_3 |K^\mu, \omega^2, \sigma\rangle = -E J_3 |\ldots\rangle$

so $(J_3, A = K^1 + J^2, B = \ldots$

$[A, B] = 0$, $[J_3, A] = iB$

$[J_3, B] = -iA$

3

$$k^\mu = (m,0,0,0)^\mu$$

Massive ⇝ look at little group

$SO(3)$ ← generators are $\begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{ijk} \end{pmatrix} = J_3$

← Casimir $\begin{pmatrix} 0 & 0 \\ 0 & j(j+1)\delta_{ij} \end{pmatrix}$

↑ spin-1 ⇝ 2.

So polarization vectors

$e_\pm^\mu = \frac{1}{\sqrt{2}}(0, \pm i, 1, 0)^\mu$ ← eigenvectors of J_3 with $\{\pm 1, 0\}$ e.v. — same as groups.

$e_0^\mu = (0,0,0,1)^\mu$

$$V^\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \frac{d^3p}{\sqrt{2p^0}} \left(e_\sigma^\mu(p)\, a(p,\sigma)\, e^{ip\cdot x} + (e_\sigma^\mu)^* a^\dagger(p,\sigma)\, e^{-ip\cdot x} \right)$$

clearly $k^\mu e_{\sigma\mu}(p) = 0$.

Lorentz transform? we now to lift a representation from the rep of the little group!

$$W(\Lambda,p)^\mu{}_\nu\, e^\nu_\sigma = \sum D_{\sigma'\sigma}(\Lambda,p)\, e^\mu_{\sigma'}$$

So $\Lambda^\mu{}_\nu e^\nu{}_\sigma(p) = \left(L(\Lambda p)\, W(\Lambda,p)\, L^{-1}(p)\right)^\mu{}_\nu e^\nu{}_\sigma(p)$

$$= \sum C_{\sigma'\sigma}(\Lambda,p)\, e^\mu_{\sigma'}(\Lambda p)$$

So $U(\Lambda) V^\mu U(\Lambda)^{-1} = \frac{1}{(2\pi)^{3/2}} \sum_{\sigma\sigma'} \int \frac{d^3p}{\sqrt{2p^0}} \left(\underbrace{e_\sigma^\mu(p)\, C^*_{\sigma\sigma'}(\Lambda p)}\, a(\Lambda p,\sigma')\, e^{ipx} + h.c.\right.$

$= \;\text{"}\quad\text{"}\; \left((\Lambda^{-1})^\mu{}_\nu\, e^\nu_{\sigma'}(\Lambda p)\, a(\Lambda p,\sigma')\, e^{ip\cdot x} + h.c.\right.$

$= (\Lambda^{-1})^\mu{}_\nu V^\nu(x)$ ✓

Massless spin-1 case $\quad k^\mu = (k,0,0,k)^\mu$, $LG = ISO(2)$

states $|p^\mu, \overset{\text{helicity}}{\pm 1}\rangle$ $\quad$ we have $\underbrace{A = K_1 + J_2,\ B = K_2 - J_1,\ J_3}$.

$W = \mathbb{I} + i\alpha A + i\beta B + i\theta J_3$ $\quad$ ↙ use reps of K, J.

$$= \mathbb{I} + \begin{pmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\alpha \\ \beta & 0 & 0 & -\beta \\ 0 & \alpha & \beta & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbb{I} + S + J$$

So $e^\mu_k = \{e^\mu_+, e^\mu_-\}$ ↖ Eigen vectors of J_3 (gives helicity states)

$$e_\pm^\mu \xrightarrow{S} S^\mu{}_\nu e^\nu_\pm = e^\mu_\pm + \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \pm i\beta \\ 0 \\ 0 \\ \alpha \pm i\beta \end{pmatrix}^\mu = e^\mu_\pm + \underbrace{\frac{\alpha \pm i\beta}{\sqrt{2}\,k}}_{\text{non trivially!}} k^\mu$$

In a general frame $\quad p^\mu = L^\mu{}_\nu(p)\, k^\nu$, $\quad e^\mu_\pm \to e^\mu_\pm(p)$

$$\Lambda^\mu{}_\nu e^\nu_\pm(p) = \left[L(\Lambda p)\, W(\Lambda, p)\, L^{-1}(p)\right]^\mu{}_\nu e^\nu_\pm(p)$$

$$= L(\Lambda p)^\mu{}_\nu \underbrace{W^\nu{}_\lambda e^\lambda_\pm}_{e^{\pm i\theta}(e^\nu_\pm)'} = \underbrace{e^{\pm i\theta}}_{\text{rot part } e^{i\theta J_3}} \left(e^\nu_\pm(\Lambda p) + \frac{\alpha + i\beta}{\sqrt{2}\,k}(\Lambda p)^\nu\right)$$

massless spin-1 fields

So $\quad |p^\mu, \pm 1\rangle \xrightarrow{U(\Lambda, p)} e^{\pm i\theta} |\Lambda p^\mu, \pm 1\rangle$

$$\boxed{\begin{aligned} U(\Lambda)\, a_k(p)\, U^{-1}(\Lambda) &= e^{-ik\theta} a_k(\Lambda p) \\ U(\Lambda)\, a_k^\dagger(p)\, U^{-1}(\Lambda) &= e^{ik\theta} a_k^\dagger(\Lambda p) \end{aligned}}$$

$D_{kk'} = e^{i\theta J_3} \delta_{kk'}$

Then, $\quad A^\mu_{(x)} = \frac{1}{(2\pi)^{3/2}} \sum \int d^3p \left(e^\mu_k(p)\, a_k(p)\, e^{ip\cdot x} + h.c.\right)$

Recall LT of massless particles. ②

$$U(W) = \underbrace{\exp(i\alpha A + i\beta B)}_{\text{acts trivially on states as otherwise produce continuous rep. (this is why we sometimes take merely } SO(2) \text{ the little group and its OK)}} \exp(i\theta J_3)$$

$U(W)|k, j_3\rangle = e^{i\theta j_3}|k, j_3\rangle$

Wigner rotation ⇝ $D_{j_3' j_3} = e^{i\theta j_3}\delta_{j_3' j_3}$

⇝ induces the full rep ⇝ $U(\Lambda, 0)|p, j_3\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}}\, e^{i\theta(\Lambda,p) j_3}|\Lambda p, j_3\rangle$

$\theta(\Lambda, p)$ = defined by relation $W = L^{-1}(\Lambda p)\Lambda L(p)$

Photon $\boxed{j_3 = \pm 1}$ ← CPT conjugate graviton $\boxed{h = \pm 2}$
$h = \pm 1$ $j_3 = \pm 2$

How Lorentz action creation op? $a^\dagger(p,\sigma)|0\rangle = |p,\sigma\rangle$

then $U(\Lambda, a)|p,\sigma\rangle = U(\Lambda,a)\,a^\dagger(p,\sigma)\,U^{-1}(\Lambda,a)\underbrace{U(\Lambda,a)|0\rangle}_{|0\rangle}$

So $\boxed{U(\Lambda,a)\,a^\dagger(p,\sigma)\,U^{-1}(\Lambda,a) = \sum C_{\sigma\sigma'}(\Lambda,a)\,a^\dagger(\Lambda p, \sigma')}$

$U(\Lambda,a)\,a(p,\sigma)\,U^{-1}(\Lambda,a) = \sum_{\sigma'} C^*_{\sigma\sigma'}\,a(\Lambda p, \sigma')$

$$U A^\mu U^{-1} = \frac{1}{(2\pi)^{3/2}} \sum_k \int d\mu \left(\underbrace{e_k^\mu(p)}\, e^{-ik\theta}\, a_k(\Lambda p)\, e^{ipx} + h.c \right)$$

$$\stackrel{\text{II}}{=} \left((\Lambda^{-1})^\mu{}_\nu e_k^\nu(\Lambda p) + \frac{\alpha + i h \beta}{\sqrt{2}\, k} p^\mu\, a_k(\Lambda p)\, e^{ip\cdot x} + h.c \right)$$

$$= (\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x) + \boxed{\partial^\mu \lambda(x)}$$ Not a vector!

Only way out $\rightsquigarrow$ $e^\mu_\pm(p) \underset{\sim}{\approx} e^\mu_\pm + c p^\mu$

$A^\mu \sim A^\mu + \partial^\mu \lambda(x)$

Then, {• Lorentz inv, unitarity, locality and $m=0$, spin 1 we <u>must</u> have gauge symmetry.

Similarly for spin 1/2 $\leftarrow$ diff invariance,

<u>Strength tensor</u> $F_{\mu\nu} \xrightarrow{\Lambda} U(\Lambda) F_{\mu\nu} U^{-1}(\Lambda) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}$

EM $\rightsquigarrow$

Gravity $\rightsquigarrow$ $2R^{\mu\nu\rho\sigma} = \partial^\mu \partial^\sigma h^{\nu\rho} + \partial^\nu \partial^\rho h^{\mu\sigma} - \partial^\mu \partial^\rho h^{\nu\sigma} - \partial^\nu \partial^\sigma h^{\mu\rho}$

$R^{\mu\nu\rho\sigma} \xrightarrow{\Lambda} U R_{\mu\nu\rho\sigma} U^{-1} = \Lambda\Lambda\Lambda\Lambda R\ldots$

gauge inv. and Lorentz covariant. (Linearized Riem. tensor)

So $m=0$ spin-1 $\Rightarrow$ $U(1)$ gauge theory $\to$ Maxwell @ low energies.

$m=0$ spin-2 $\Rightarrow$ diff " " $\to$ Einstein

Soft particles

$$\mathcal{M} = M_\mu \varepsilon^\mu(q)^*$$

has to be function of $M^\mu = p^\mu \underbrace{f(p^2)}_{\text{const}}$ ($p^2 = m^2$)

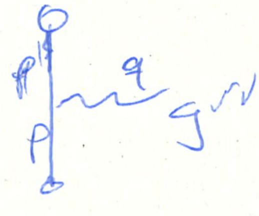
we define charge as

$$\boxed{\mathcal{M} = 2i(2\pi)^4 e\, p^\mu (e^\pm_\mu)^*}$$ soft particle scattering.

Ex/ $\mathcal{L}_{(\text{scalar QED})} \to \mathcal{L}_I = ieA^\mu J_\mu = ieA^\mu(\phi^* \partial_\mu \phi - \partial_\mu \phi^* \phi)$

$$\to \mathcal{M} = ie(2\pi)^4 [p^\mu e^*_\mu(q) + (p^\mu - q^\mu) e^*_\mu(q)]$$

$$\overset{q \to 0}{=} 2i(2\pi)^4 \boxed{e}\, p^\mu \varepsilon^*_\mu(q)$$



$$\mathcal{M} = M^{\mu\nu} \varepsilon_{\mu\nu}(q)^* = 2i(2\pi)^4 g\, p^\mu p^\nu (e^\pm_{\mu\nu})^*$$

(soft) $M^{\mu\nu} = p^\mu p^\nu g(p^2)$ gravitational coupling

Ex/ couple scalars to gravity.

$$S_\phi = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2\right)$$

we want $\phi\phi h$ terms, set $g = \eta + \underset{\sqrt{8\pi G_N}}{\kappa} h$

$$S_\phi = S_\phi[\phi, \eta_{\mu\nu}] + \underbrace{\int d^4x\, \kappa \frac{\delta S_\phi}{\delta h_{\mu\nu}}\bigg|_{h=0} h_{\mu\nu}(x)}_{\int d^4x \frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu}_{\text{flat}}} + \dots$$

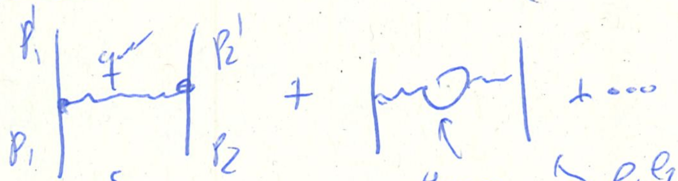
$$T^{\mu\nu}_{\text{flat}} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2}\eta^{\mu\nu}[\partial_\alpha \phi \partial^\alpha \phi + m^2 \dots$$

$$\mathcal{M} = i(2\pi)^4 \frac{\kappa}{2}\left[i p^\mu\left(-i(p'^\nu - q^\nu)\right) + (\mu\leftrightarrow\nu)\right](e^{\pm}_{\mu\nu})^*$$

$$\overset{q\to 0}{=} i(2\pi)^4 \, p^\mu p^\nu \, \frac{2\kappa}{2}$$

So coupling is $g = \frac{\kappa}{2} = \frac{\sqrt{8\pi G}}{2}$ ← as our usual guess from GR.

Two particle potential (2→2) scattering



s

$\frac{e_1 e_2}{q^2}\eta_{\mu\nu}$

4-term correction $\sim \frac{e_1 e_2}{q^2}\eta_{\mu\nu}\Pi$

we can write $i\Pi^{\mu\nu}$ =

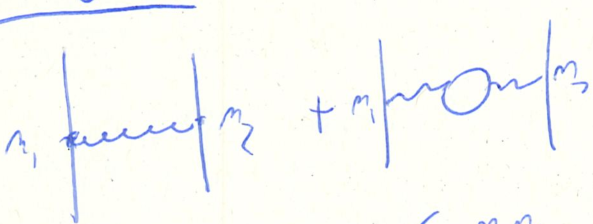
$$= \frac{-i}{q^2 - i\epsilon}\,\frac{\eta_{\mu\nu}}{1-\Pi(q^2)} \approx \frac{-i}{q^2}\left(\eta_{\mu\nu} + \Pi(q^2)\eta_{\mu\nu} + \dots\right)$$

So we see for non relativistic limit $q^0 \ll |q| \rightsquigarrow q^2 = |q|^2$

$$\frac{e_1 e_2}{|q|^2} = \int d^3x\, e^{-iq\cdot x}\, V_0(x) \rightsquigarrow V_0 = \frac{e_1 e_2}{4\pi r} \quad \text{(Coulomb)}$$

$$\frac{e_1 e_2}{|q|^2}\Pi = \int d^3x\, e^{-iqx}\, V_0' \rightsquigarrow V_0' = \underbrace{\frac{e_1^2 e_2^2}{60\pi^2 m^2}\delta(r)}_{\text{quantum corrections}}$$

For gravity



$$V = -\frac{G_N m_1 m_2}{4\pi r} + \underbrace{\frac{N G_N \hbar}{c^3 r^2}\,\frac{G_N m_1 m_2}{4\pi r}}_{\text{quantum corrections}}$$

Soft theorems

S-matrix of γ's and g's are sensitive to charge and mass.

Consider

LI requires $e^\mu_\pm(q) \simeq e^\mu_\pm(q) + c\, q^\mu$

so $M^\mu(e^\pm_\mu + c\, q_\mu) = M^\mu e^\pm_\mu \rightsquigarrow$ $\boxed{M^\mu(q)\, q_\mu = 0}$ $j=1$

$\boxed{M^{\mu\nu}(q)\, q_\nu = 0}$ $j=2$

Soft photons

$$\mathcal{M}^\pm_{\beta\alpha} \overset{q\to 0}{=} \mathcal{M}_{\beta\alpha} \times \underbrace{\frac{-i}{(2\pi)^4}}_{\text{propagator extra}} \frac{1}{\underbrace{(p_1+q)^2 + m_1^2}_{2p_1\cdot q}} \times 2i(2\pi)^4 e_1 p_1^\mu e^\pm_\mu(q)^*$$

Since $q^\mu \to 0$ the rest of process is the same.

$$\mathcal{M}^\pm_{\beta\alpha} = \mathcal{M}_{\beta\alpha}\, e_1 \frac{p^\mu e^*_\mu(q)}{p_1\cdot q} \qquad (\text{out})$$

$$\mathcal{M}^\pm_{\beta\alpha} = -\mathcal{M}_{\beta\alpha}\, e_1 \frac{p_1^\mu e^*_\mu(q)}{p_1\cdot q} \qquad (\text{in})$$

Emissions from is (in + out)

$$\mathcal{M}^\pm_{\beta\alpha} = \mathcal{M}_{\beta\alpha} \sum_{n=1}^N \underset{\substack{\text{out}=+1\\ \text{in}=-1}}{\eta_n} \frac{e_n p_n^\mu e^*_\mu(q)}{p_n\cdot q}$$

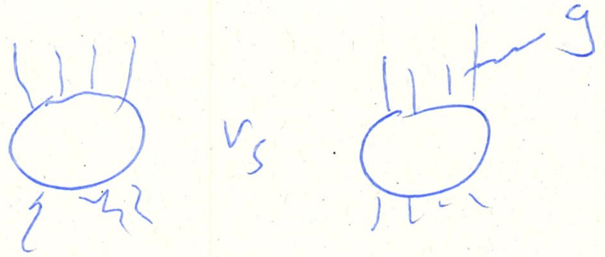
+ Internal emissions of γ but these are irrelevant not have $\frac{1}{q}$

But recall $\mathcal{M}^\pm_{\beta\alpha} = e^*_\mu(q) M^\mu_{\pm\beta\alpha}$ this must vanish

If $e^*_\mu(q) \longmapsto q_\mu$ $\quad 0 = q_\mu M^\mu_{\pm\beta\alpha} = \mathcal{M}_{\beta\alpha} \sum^N \eta_n e_n \frac{p_n\cdot q}{p_n\cdot q}$

$$\Rightarrow\ 0 = \sum_{n=1}^N e_n \eta_n \ \Rightarrow\ \boxed{\sum_{\text{in}} e_n = \sum_{\text{out}} e_n}$$ charge conservation!

Soft gravitons



Similarly
$$\mathcal{M}^{\pm 2}_{\beta\alpha} = \mathcal{M}_{\beta\alpha} \sum_{n=1}^{N} \eta_n g_n \frac{p_n^\mu p_n^\nu e^{\pm}_{\mu\nu}(q)}{p_n \cdot q} + \text{interior terms}$$

$\rightsquigarrow$ since $q_\nu M^{\mu\nu}_{\pm,\beta\alpha} = 0$ we have

$$0 = \mathcal{M}_{\beta\alpha} \sum^{N} \eta_n g_n p_n^\mu \Rightarrow \boxed{0 = \sum^{N} \eta_n g_n p_n^\mu}$$

Since $\sum \eta_n p_n^\mu = 0$ (mom conservation) then the only way to solve this is $\boxed{g_n = g}$ $\rightsquigarrow$ $\boxed{\sum_{in} p_n^\mu = \sum_{out} p_n^\mu}$

Every particle has to couple the same to graviton!

Equivalence principle.

Weinberg-Witten

- A theory containing a Poincaré-covariant J^μ forbids massless charged particles of $j > \frac{1}{2}$ $(\partial_\mu \langle J^\mu \rangle = 0)$
- A theory with a Poinc-cov. conserved $T^{\mu\nu}$ forbids massless "charged" particles of $j > 1$ $\partial_\mu \langle T^{\mu\nu} \rangle = 0$.

  ("charge" is a non-zero $\int T^{0\nu} d^3x$)

Corollary no composite graviton from Poinc-inv

Idea of proof

$$\langle p', \pm h| \; \begin{matrix} J^\mu \\ T^{\mu\nu} \end{matrix} \; |p, \pm h\rangle$$

A) Poincaré cov. and def^n of charges $\neq 0$

B) by Lorentz transf of $|p_{\mu}, \pm h\rangle$ $= 0$ for $\begin{Bmatrix} j > \frac{1}{2} \\ j > 1 \end{Bmatrix}$ $\begin{matrix} \leftarrow J^\mu \\ \leftarrow T^{\mu\nu} \end{matrix}$

so exclude this

A)

$$\langle p'|Q|p\rangle = \overset{\text{conservation of current.}}{\int d^3x} \; \underbrace{\langle p'| J^0(t,\underline{x}) |p\rangle}_{}$$
$$= \int d^3x \langle p' | J^0(t) | p\rangle \, e^{i(p-p')\cdot x} = (2\pi)^3 \delta^3(p-p') \langle p'|J^0|p\rangle$$

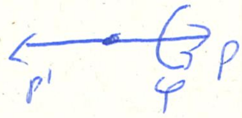
In the limit, $\displaystyle\lim_{p' \to p} \langle p' | J^0 | p\rangle = \frac{q}{(2\pi)^3}$

by Poincaré covariance, $\boxed{\displaystyle\lim_{p' \to p} \langle p' | J^\mu | p\rangle = \frac{q\, p^\mu}{E(2\pi)^3}}$

go to special Lorentz where $(p'+p)^2 = 2p\cdot p' = 2|p||p'|(\cos\phi - 1)$

$p+p'$ is time like except for $\phi = 0$.

Consider $\phi \neq 0$. $\rightsquigarrow$ rest frame $(p'+p)^\mu = (E_{p+p'}, \vec{0})^\mu$



$$U(\Lambda)|p', \pm h\rangle = N e^{\pm i\theta(\Lambda, p)h} |\Lambda p', \pm h\rangle$$

For a rotation along p axis of φ degrees, $\begin{matrix} \theta(\Lambda, p) = \varphi \\ \theta(\Lambda, p') = -\varphi. \end{matrix}$

so $|p', \pm h\rangle \to e^{\mp i\varphi h} |p', \pm h\rangle$

$|p, \pm h\rangle \to e^{\pm i\varphi h} |p, \pm h\rangle$ and $U^{-1} J^\mu U = \Lambda^\mu{}_\nu J^\nu$

So $e^{\pm 2i\varphi h} \langle p'_\varphi, \pm h | J^\mu | p_\varphi, \pm h \rangle$

$= \Lambda(\varphi)^\mu{}_\nu \langle p', \pm h | J^\nu | p, \pm h \rangle$

Simple eigenvalue problem $\rightsquigarrow$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \begin{matrix} \cos & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{matrix} \end{pmatrix} \rightsquigarrow \lambda = \{1, e^{\pm i\varphi}\} \overset{!}{=} e^{\pm 2i\varphi h}$$

Soutn for $h = 0$ or $\frac{1}{2}$ $\rightsquigarrow$ $h > \frac{1}{2}$ not possible. so put (A) ✓

B) use $T^{\mu\nu}$; $\langle p' | P^\mu | p \rangle = p^\mu \delta^3(\vec{p}' - \vec{p})$

$$\| \quad \int d^3x \langle p' | T^{0\mu}(t, x) | p \rangle = (2\pi)^3 \delta^3(p' - p) \langle p' | T^{0\mu} | p \rangle$$

$$\lim_{p' \to p} \langle p' | T^{0\mu} | p \rangle = \frac{p^\mu}{(2\pi)^3} \quad \text{by Lorentz,}$$

$$\langle p' | T^{\mu\nu} | p \rangle = \frac{p^\mu p^\nu}{E (2\pi)^3} \neq 0$$

RHS perspective $p' = (|\vec{p}|, -\vec{p})$, $p = (|\vec{p}|, \vec{p})$

$$e^{\pm 2i\varphi h} \langle p' | T^{\mu\nu} | p \rangle = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \langle p' | T^{\rho\sigma} | p \rangle$$

$$e^{\pm 2i\varphi h} = \{1, e^{\pm i\varphi}, e^{\pm 2i\varphi}\} \rightsquigarrow h = 0, \tfrac{1}{2}, \pm 1$$

So no ~~m≠0~~ $m = 0$ with $h > 1$.

• Rules out constructions of emergent <u>composite gravitons</u>.

Ex/ if graviton is composite at some E it dissociates and UV theory is LI renormalizable QFT. X LI & conserved $T^{\mu\nu}$

• not good for:

a) E&M: massless spin-1, γ not charged

b) massless Yang-Mills: it doesn't have J^μ conserved and gauge invariant.

c) GR: massless spin 2: no gauge invariant Lorentz covariant

• Does WW kill all <u>emergent gravity</u>?

No! a) Non LI

b) Holography (AdS/CFT)

<u>Gravity as an EFT</u> Main ingredients:

• Gravity = QFT of massless spin-2.

↳ unitary, local, LI, S-mat.

• unitary irreps of poincare group. (spin-2)

1) $h_{\mu\nu} \simeq h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}$

↳ hints — $R_{\mu\nu\rho\sigma} \sim h\partial h$

— $h_{\mu\nu}T^{\mu\nu}$

2) soft gravitons

↳ equivalence principle.

$\boxed{\text{Einstein-Hilbert}}$

Action requirements

Free gravitons. $\rightsquigarrow$ - quadratic in h
- two derivatives in kinetic term
- invariant under $\delta h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}$

$$S \sim M_p^2 \int d^4x \, (h \partial^2 h)$$

$\uparrow$ index contractions will allow the inv. property

<u>Guess:</u>

a s.t. invariance is satisfy.

$$S = \alpha M_p^2 \int d^4x \left(\eta^{\mu\rho} h^{\nu\sigma} + a h \eta^{\mu\rho}\eta^{\nu\sigma}\right) R^{(1)}_{\mu\nu\rho\sigma}$$

want $\delta_\xi \mathcal{L}$ is a total derivative, lets look at the role of a.

$$\delta \mathcal{L} = \left(\eta^{\mu\rho}\partial^\nu \xi^\sigma + a(\partial_\lambda \xi^\lambda)\eta^{\mu\rho}\eta^{\nu\sigma}\right) R^{(1)}_{\mu\nu\rho\sigma}$$

$$= \partial^\nu \xi^\sigma R^{(1)}_{\nu\sigma} + a(\partial_\lambda \xi^\lambda) R^{(1)}$$

$$= \tfrac{1}{2}(\partial^\nu \xi^\sigma)\left(-\Box h_{\nu\sigma} + h^\alpha_{\sigma,\nu\alpha} + h_{\nu\alpha,}{}^\alpha{}_\sigma - h_{,\nu\sigma}\right)$$

$$+ a(\partial_\lambda \xi^\lambda)\left(h^{\mu\alpha}{}_{,\mu\alpha} - \Box h\right)$$

To fix a look at terms involving trace i.e. terms proportional to $\partial_\lambda \xi^\lambda$

$$-\tfrac{1}{2}(\partial^\nu \xi^\sigma)\partial_\nu\partial_\sigma h^\lambda{}_\lambda - a(\partial_\mu \xi^\mu)\partial^\alpha\partial_\alpha h^\lambda{}_\lambda = \partial_\alpha V^\alpha + \left(\tfrac{1}{2}+a\right)(\partial_\sigma\partial_\lambda \xi^\lambda)\partial^\sigma h$$

with $V^\alpha = -\tfrac{1}{2}(\partial^\nu \xi^\alpha)\partial_\nu h + a$ term. $\uparrow$ total derivative. So

set $a = -\tfrac{1}{2}$. other terms are also total derivatives.

so

$$S_h^{(2)} = \alpha \frac{M_p^2}{2} \int d^4x \left(h^{\mu\nu}\Box h_{\mu\nu} - h\Box h + 2h\partial_\mu\partial_\nu h^{\mu\nu} - 2h^{\nu\sigma}\partial_\nu\partial^\mu h_{\mu\sigma}\right)$$

$$= \text{quadratic part of } S_{EH} = \frac{1}{16\pi G}\int d^4x \, R\sqrt{-g}\,\Big|_{g=\eta+h} \;!$$

fixing α to coincide with usual S_{EH}, $\alpha = -\frac{1}{4}$

$$M_p^{-2} = 8\pi G_N$$

Adding interactions

$$S = S_h^{(2)} + \int d^4x \, h_{\mu\nu} T^{\mu\nu}$$

g itself has a stress tensor! Compute from S_h e.g. by coupling to auxiliary metric ~~field~~.

$$\tau^{(2)}_{\mu\nu} = -\frac{1}{4} h_{\alpha\beta} \partial_\mu \partial_\nu h^{\alpha\beta} + \dots$$

$$\sim O(h \partial^2 h) \, !$$

$$S_h^{(3)} = S_h^{(2)} + \underbrace{\int d^4x \, h_{\mu\nu} \left(-\frac{1}{4} h_{\alpha\beta} \partial^\mu \partial^\nu h^{\alpha\beta} \right)}_{O(h^3)} \quad \tau^{\mu\nu} \checkmark$$

$\propto$ 3 graviton vertex

this coincide with EH linearized to order 3!

Construct $\tau^3_{\mu\nu}$ from $S^{(3)}_{\mu\nu}$ $\rightsquigarrow$ $S_h^{(4)} = S_h^{(3)} + \int d^4x \, h_{\mu\nu} \tau^{\mu\nu}_{(3)}$

again EH expansion!  ✓

this iterative process construct E-H full!

[see Deser]

A different perspective

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \equiv G^{(1)}_{\mu\nu} + G^{(\text{higher})}_{\mu\nu} = 8\pi G \, T_{\mu\nu}$$

$$\underset{\text{linear}}{G^{(1)}_{\mu\nu}} = 8\pi G \left[T_{\mu\nu} - \underbrace{\frac{1}{8\pi G} G^{(\text{high})}_{\mu\nu}}_{\tau^{\mu\nu}_{\text{graviton}}} \right]$$

$$-\frac{1}{8\pi G} G^{(\text{higher})}_{\mu\nu} = \tau^{(2)}_{\mu\nu} + \tau^{(3)}_{\mu\nu} + \dots$$

Corresponds to $\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \partial_\mu [T^{\mu\nu} + \tau^{\mu\nu}] = 0$

- This settles (two-derivatives) classical theory.

S-matrix $\longleftrightarrow$ quantum scattering

S_{EH} is non-renormalizable $\Rightarrow$ EFT!

Superficial degree of divergence

$$D = d + V[\Delta_I - d] - \Delta_{ext}^{N}$$

Negative mass dimension coupling $\rightsquigarrow$ non renormalizable

- If $\Delta_I = d$ finite # of divergences at each loop level.
- If $\Delta_I > d$ more divergences $\Rightarrow$ non renormalizable theory (gravity)

as would be $\left(\frac{E}{g}\right)^{a}$ coupling expansion

<u>But</u> we are interested in low cutoff E_{max} so

$$A(|\alpha\rangle \to |\beta\rangle) \Rightarrow A^0 + \sum_{n \geq 1} \underbrace{\left(\frac{E}{E_{max}}\right)^n}_{\text{suppressed}} A^{(n)}(|\alpha\rangle \to |\beta\rangle)$$

So $E \ll E_{max}$ we can get a finite set.

So we don't need the UV completion but we match the EFT allowed operators with experiments. (all operators satisfying symmetries and constraints).

Some loop corrections only depend on lowest order $\mathcal{L}_{EFT}$ and can be clearly isolated $\rightsquigarrow$ predictions of quantum gravity.

$$\mathcal{L} = \sqrt{-g}\left(\Lambda + \frac{M_p^2}{2} R + a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + b_1 R^3 + c_1 \Box R \dots\right)$$

we will see later to isolate this!

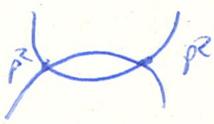
Power Counting Amplitudes

what terms include in $\mathcal{L}_{EFT}$ for specified accuracy?

$\{ \mathcal{O}_{i,d} \}$, d derivatives w/ i powers of fundamental field.

$$\mathcal{L}_{EFT} = \underset{\text{Overall Scale to get } [\mathcal{L}]=4}{f^4} \sum_{i,d} \frac{c_{i,d}}{M^d} \mathcal{O}_{i,d} = f^4 \sum \frac{\overset{\text{"naturalness" } \mathcal{O}(1)}{c_n}}{M^{d_n}} \mathcal{O}_n\left(\frac{\overset{\text{fund field}}{\phi}}{\underset{\text{Some scale to leave } \frac{\phi}{v} \text{ massless}}{v}}\right)$$

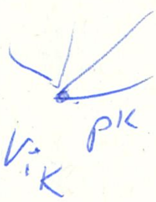
e.g. $\mathcal{L}_{EFT} \subset f^4 \frac{1}{M^2} \frac{1}{v^4} \phi\phi\Box\phi\phi$



$k=2$
$i=4$
$d_n=2$

$\mathcal{A}_E(E) =$ , $\mathcal{L}_{EFT}^{int}$ generates diagrams with

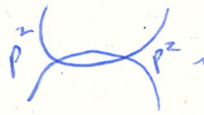
$\underset{\underset{\#\text{ of Extrn} = E}{N_\alpha + N_\beta}}{\mathcal{A}_E}$



- i lines
- k powers of p

so $i = \#$ of fields in $\mathcal{L}_{EFT}^{in}$
$k = \#$ of derivatives

lets consider a diagram $(I, L, V_{i,k})$

e.g.  $\sim (2, 1, V_{4,2} = 2)$

For a given diagram

$$\begin{pmatrix}\text{total vertex} \\ \text{contribution}\end{pmatrix} = \prod_{ik} \left[i(2\pi)^4 \underbrace{\delta^4(p_v)}_{\text{momentum conserv}} \left(\frac{p}{M}\right)^k \frac{f^4}{v^i} \right]^{V_{ik}}$$

$\prod_{ik}$ ← Product of all types of vertex.

$$\begin{pmatrix}\text{each internal} \\ \text{line}\end{pmatrix} = \left[-i \int \frac{d^4p}{(2\pi)^4} \frac{M^2 v^2}{f^4} \frac{1}{p^2+m^2} \right]$$

Integrate over $\delta^4(\sum p_{vertex})$ of all vertex there remains

$L = I - \sum_{ik} V_{ik} + 1$ loop momenta to be integrated.

$$\mathcal{A}_E(E) = i(2\pi)^4 \delta^4\left(\sum_{ext} p\right) \tilde{\mathcal{A}}_E(E)$$

↑ reduced amplitude.

use dim. reg.

$$\tilde{\mathcal{A}}_E(E) = \left(\int \frac{d^4p}{(2\pi)^4}\right)^L \frac{p^X}{(p^2+Q^2)^I} \sim \left(\frac{1}{4\pi}\right)^{2L} Q^Y$$

- $X = \sum_{ik} k V_{ik}$
- $Y = 4L - 2I + \sum_{ik} k V_{ik}$

Note • $2I + E = \sum_{ik} i V_{ik}$

and • $L = 1 + I - \sum_{ik} V_{ik}$

• Q largest scale.

$$\boxed{\tilde{A}_{\xi}(E) \sim f^4 \left(\frac{1}{2}\right)^{\xi} \left(\frac{M^2}{4\pi f^2}\right)^{2L} \left(\frac{Q}{M}\right)^{P}}$$

$$P = 2 + 2L + \sum_{i,k} (k-2) V_{ik}$$

<u>Implications</u>

Fix a certain degree of accuracy $Q \ll M$, and fix $\left(\frac{Q}{M}\right)^P$, we get.

- $L=0$, s.t. $\sum_{ik} (k-2) V_{ik} = P-2$ ← this constrains how many k derivatives you have, as $(k-2)$ actually is positive and V_{ik} is also positive the series terminates.

$L=1$, s.t. $\sum_{ik} (k-2) V_{ik} = P-4$

Further, once you put all operators contributing to $L=0$ there could be <u>no</u> higher order derivatives as decreases.

So for fixed P you can make sense of the <u>theory to all loops!</u>

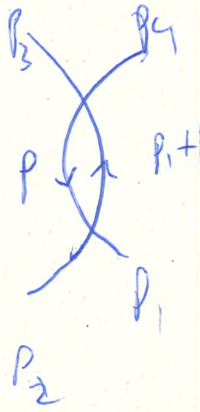
you can work with finite ~~order~~ number of operators in a p expansion contribute!

→ finite counterterms.

<u>EFT is predictive!</u>

loops ~~fact~~ increase our precision of measurements and therefore require less derivatives to get into that order.

Ex/ $\frac{f^4}{v^4}\frac{\lambda}{4!}\phi^4$



$$\sim \frac{M^4}{v^4}\frac{(-i\lambda)^2}{2}\int\frac{d^4p}{(2\pi)^4}\left(\frac{i}{p^2-m^2}\,\frac{i}{(p_1+p_2+p)^2-m^2}\right)$$

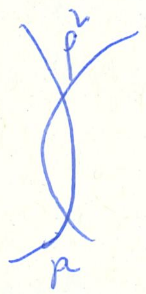
$$= \frac{M^4}{v^4}(-i\lambda)^2\, i\left(-\frac{1}{2}\int_0^1 dx\int\frac{d^dp_E}{(2\pi)^d}\frac{1}{\left[p_E^2-x(1-x)\underbrace{(p_1+p_2)^2}_{Q\sim\text{scale.}}+m^2\right]^2}\right)$$

$$\parallel$$

$$V(p^0) \propto \left(\frac{Q}{M}\right)^p f^4\left(\frac{1}{v}\right)^E\left(\frac{M^2}{f^2}\right)^{2L}$$

$$= Q^0\left(\frac{M^4}{v^4}\right), \quad p=0,\ L=1$$

$\frac{f^4}{v^4}\frac{1}{M^2}\frac{\lambda}{4!}\phi^2\Box\phi^2$



now

$$V(p^2)\sim\left(\frac{Q}{M}\right)^p f^4\left(\frac{1}{v}\right)^E\left(\frac{M^2}{f^2}\right)^{2L}$$

$p=4$

$(L=1)$

Power Counting for gravity (low E quantum gravity)

- low E d.o.f. $g_{\mu\nu}\to\eta_{\mu\nu}+h_{\mu\nu}$
- symmetry: locality + LI $\Rightarrow$ diffeos.
  - $\hookrightarrow$ only ops in energy derivative expansion

$$\mathcal{L}_{EFT}=\sqrt{-g}\left(\Lambda+\frac{M_p^2}{2}R+a_1R^2+a_2R_{\mu\nu}R^{\mu\nu}\right.$$
$$\left.+\,a_3R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}+\frac{b_1}{m^2}R^3+\cdots\right)$$
$$a\,\Box R$$

m defines low energy. eg $m=M_p$, $m=M_{\text{electron}}$

the cosmological constant.

$\Lambda \sim O(1)$ by EFT logic.

nonzero $\mathcal{L}_{EFT} = \sqrt{-g}\,(R - 2\tilde{\Lambda} + \cdots)\frac{M_p^2}{2}$

so since R has 2 derivatives $\boxed{\tilde{\Lambda}_{EFT} \sim O(M_p^2) \sim 6\times 10^{54}(eV)^2}$

$\tilde{\Lambda}_{exp} \sim 10^{-66}(eV)^2$ a mismatch of <u>120 orders</u>

set it equal to zero. "The cosmological const problem"

$$\mathcal{L}_{EFT} = \sqrt{-g}\Big(\frac{M_p^2}{2}R + a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + c_1 \Box R$$
$$+ \frac{b_1}{m^2}R^3 + \cdots\Big)$$

Redundant Interactions & 1 loop ~~coun~~ counter terms:

- $\chi(M) = \frac{1}{32\pi^2}\int_M \sqrt{g}\,\underbrace{(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + R^2 - 4R_{\mu\nu}R^{\mu\nu})}_{\text{Euler density = total derivative.}}$

topological so doesn't respond to metric fluctuations $\Leftrightarrow$ gravitational amplitudes.

- So drop $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$.
- $\sqrt{-g}\,\Box R = \partial_\mu(\sqrt{-g}\,\nabla^\mu R)$ total derivative.
- Field redefinitions $g_{\mu\nu} \to g_{\mu\nu} + Y_{\mu\nu}$

$$\delta\Big[\frac{M_p^2}{2}\int d^4x\sqrt{-g}\,R\Big] = -\frac{M_p^2}{2}\int d^4x\sqrt{-g}\,\Big(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\Big)Y_{\mu\nu}$$

If $M_p^2 Y_{\mu\nu} = 2a_2 R_{\mu\nu} - (a_2 + 2a_1)Rg_{\mu\nu}$ we eliminate R^2 and $R_{\mu\nu}R^{\mu\nu}$

and we can always eliminate terms that vanish on lowest order EOM. $g_{\mu\nu} \to g_{\mu\nu} + \frac{R_{\mu\nu}}{M_p^2}$ higher order. $R^2_{\mu\nu}$

$\mathcal{L}_{EFT}$ contains no $(\text{curv})^2$ interactions in pure gravity

First correction is R^3 $\Rightarrow$ no 1-loop counter term

$\to$ ('t Hooft & Veltman): pure gravity is finite @ 1-loop

Power counting:

$A_E(E) =$ 

$E = N_A + N_B$

external gravitons

$$\mathcal{L}_{EFT} = \sqrt{-g}\left(\frac{M_p^2}{2}R + a_1 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \dots + \frac{b_1}{m^2}R^3 + \dots\right)$$

$$h\Box h + h^2\Box h + \dots$$

$$\sim \frac{1}{m^2}\Big(M_p^2\, h\,\underbrace{\Box^2 h + M_p^2 \Box^2}_{4\ \text{derivatives}} h^3$$

Interactions with arbitrary number of particles and higher ∂ powers

$$\eta_{\mu\nu} + h_{\mu\nu} \to \eta_{\mu\nu} + \frac{1}{M_p}h_{\mu\nu}$$

$$\mathcal{L}_{EFT} = \left(h\Box h + \frac{1}{M_p}h^2\Box h\right) + \frac{1}{m^2}\left(h\Box^2 h + \frac{1}{M_p}\Box^2 h^3 + \dots\right)$$

$$\sim m^2 M_p^2\left(\frac{h}{M_p}\left(\frac{p}{m}\right)^2\frac{h}{M_p} + \left(\frac{h}{M_p}\right)^2\left(\frac{p}{m}\right)^2\left(\frac{h}{M_p}\right) + \dots\right)$$

$$\mathcal{L}^{\text{int}}_{EFT} = m^2 M_p^2 \sum_n \frac{c_n}{m^{d_n}}\,\mathcal{O}_n\left(\frac{h_{\mu\nu}}{M_p^2}\right)$$

n: # of derivatives

M_p: # of graviton fields

$m^2 M_p^2 \sim f^4$: overall scale.

$$\mathcal{A}_{\mathcal{E}}(E) \simeq m^2 M_p^2 \left(\frac{1}{M_p}\right)^{\mathcal{E}} \left(\frac{m}{4\pi M_p}\right)^{2L} \left(\frac{E}{m}\right)^{\rho}$$

$$\rho = 2 + 2L + \sum_i{}' (d-2) V_{id}$$

Einstein-Hilbert kinetic term

vertex with i lines of $h_{\mu\nu}$ & d factors of p.

<u>Note:</u>

1. $\left(\frac{E}{m}\right)^{2+2L+\sum' (d-2) V_{id}}$ $\Rightarrow$ only the power of $\left(\frac{E}{m}\right)$

2. Case of two scales M_p & m is important in string theory & AdS/CFT

$M_p \sim$ Planck mass

$m \sim \frac{1}{\ell_s} \sim$ string length (tension)

3. Gravity at low E $\quad E \ll M_p$, $E \ll m$ is weakly coupled!

4. Some low order contributions:

- $L = 0$, $\rho = 2$ $\leftarrow$ tree level two derivatives diagrams.
  - $\sim$ full exp of $\sqrt{-g}\,R$
  - $\sim$ classical GR

- First corrections:

  either $L = 1$ $\quad V_{id} = 0 \quad \forall\, d > 2$

  (i.e. 1-loop E-Hilbert)

  or $L = 0$, $\sum V_{i4} = 1$, $V_{i2} = $ arbitrary, all other $V_{id} = 0$

  (i.e. $(\text{Curvature})^2$ interaction + only # of EH interaction)

how to compute coefficient of V_{i4}? Local $L=0$

Long range $L=1$

5. with matter couplings

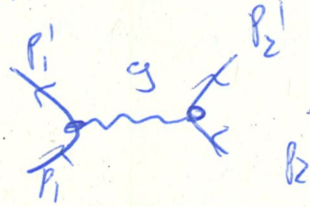
- 1-loop physical divergences
- we're interested in case of external non-relativistic matter (stars & planets)

$\rightsquigarrow$ external scale dominated by mass no momentum.

Predictions of gravity EFT

External source $\rho(\underline{r}) = \sum M \delta^3(\underline{r}-\underline{r}_i)$

Definition of potential:

$\langle f | T | i \rangle =$  $\sim (2\pi)^4 \delta^4(p-p')\, \mathcal{M}(q)$

EFT matrix element

$$=: \underbrace{-(2\pi)\delta(E-E')\langle f | \hat{V}(q) | i \rangle}_{\text{Born approx in QM}}$$

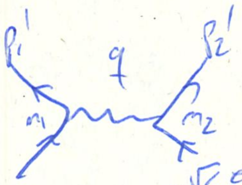
$$\rightsquigarrow \quad V(r) = \frac{1}{2m_1 2m_2}\int \frac{d^3q}{(2\pi)^3} e^{i q\cdot r}\, \mathcal{M}(q)$$

Fourier transform of potential.

Schematically (Donoghue)

$$\mathcal{M}(q) = \frac{K_2}{q^2} + \underbrace{\frac{K_1}{\sqrt{q^2}} + K_0 \log(q^2)}_{\text{non analytic}} \rightsquigarrow \text{long range } \frac{1}{r^\#}$$

$$+ \underbrace{\mathcal{M}_{analytic}(q^2)}_{\text{short range}} \propto \delta^3(r)$$

i) tree: 

$\sqrt{G}$ each vertex

$$i\mathcal{M}_{tree} = \tau_1^{\mu\nu}(p_1, p_1'; m_1)\; i\frac{P_{\mu\nu\alpha\beta}}{q^2}\; \tau_2^{\alpha\beta}(p_2, p_2'; m_2)$$

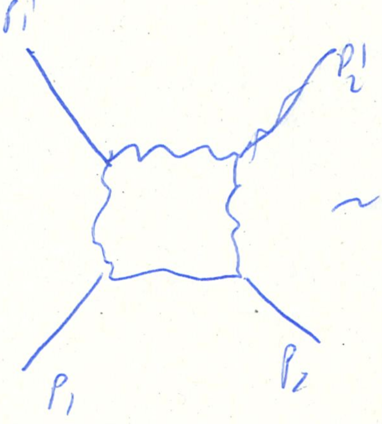
propagator w/ 4 indices

current of gravitons

$$\longrightarrow -4\pi G_N \frac{m_1 m_2}{q^2} \Rightarrow \boxed{V_{tree} = -G_N \frac{m_1 m_2}{r}}$$

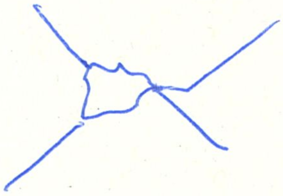
classical newtonian correction.

ii) 1-loop corrections

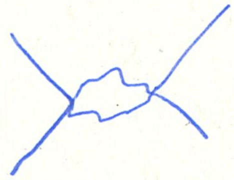


$\sim \# \, G_N^2 m_1 m_2 \log(q^2)$

$$V_{\square}^{loop} = -\frac{47}{3} \frac{m_1 m_2}{\pi} \frac{G^2}{r^3}$$



$\sim G^2 m_1 m_2 \left(\# \log q^2 + \# \frac{m_1}{|q|} \right)$



$\sim \# \, G^2 m_1 m_2 \log(q^2)$

$\vdots$

$$V^{1\text{-loop}}_{total}(r) = -\frac{G m_1 m_2}{r} \Big[\underbrace{\frac{3G(m_1+m_2)}{r}}_{\text{relativistic correction}} + \underbrace{\frac{41}{10\pi} \frac{G \hbar}{r^2}}_{\text{quantum correction}} \Big]$$

iii) tree diagrams from R^2 interactions

$\Rightarrow V(r) \sim G m_1 m_2 (m_1+m_2) \, \delta(\underline{r})$

extremely short range!

↑ we can forget in the IR corrections and just see

long-range effects entirely from non-analytic (loop) effects in EM

In IR only filter non-analytic that are predictions of our EFT so only those that do not filter must be used as to low order we already know this is the theory

How are this corrections?

e.g. $r = R_\odot$ then $\frac{GM_\odot}{rc^2} \sim 10^{-6}$, $\frac{G\hbar}{r^2c^3} \sim 10^{-88}$
small

but at planck scale $\frac{G\hbar}{r^2c^3} \sim O(1)$, $r = \sqrt{\frac{c^3}{G\hbar}} = l_p$

So quantum gravity corrections are important

(naive EFT)
Quantum gravity becomes non perturbative @ l_p.
how to construct a consistent S-matrix of spin-2
($m=0$) object at planck scale?

(a) String theory ✓ — holography ✓

The AdS/CFT correspondence:

Flat space: poincare group is the isometry

$w^{\mu\nu}$ (Lorentz) (6) generators
a^μ (translations) (4) generators
10 isometries e.g. $\mathbb{R}^{1,3}$.

maximally symmetric space.
exactly 10 isometries and not more.

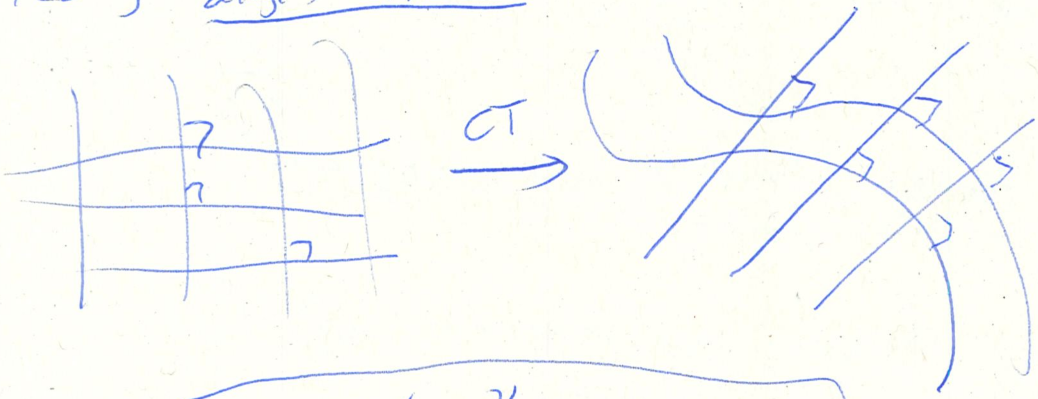
Gravity vs Einstein EFT. great at low Energy, but problematic at UV.

To fix this: 1) Gravity is emergent
2) replace Poincare with conformal

Conformal transformations — they are not isometries of $\mathbb{R}^{1,d-1}$ ← already exhausted by poincaré!

Def A CT map, i.e. $T: x^\mu \to (x')^\mu(x)$

leaving angles invariant



i.e. $$\boxed{\eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} = \Omega^2(x)\,\eta_{\alpha\beta}}$$ PDE constraining $x^\mu \to (x')^\mu$

- is a conformal isometry $\supset$ isometry (with $\Omega = 1$)
- not all CT can be reached by CT

Remark: weyl symmetry allows all type of $\Omega(x)$

CT symmetry not

solve for linearized CT, $(x')^\mu = x^\mu + \varepsilon^\mu(x)$

$\Omega = 1 + \delta\Omega(x)$

$$g_{\mu\nu}(\underset{x+\varepsilon}{x'}) \frac{\partial \overset{x+\varepsilon}{x'^\mu}}{\partial x^\alpha} \frac{\partial x'}{\partial x^\beta} = \Omega^2(x)\, g_{\alpha\beta}(x) = (1 + 2\delta\Omega)\, g_{\alpha\beta}$$

$$\Leftrightarrow \quad \nabla_{(\mu}\varepsilon_{\nu)} = (\delta\Omega)\, g_{\mu\nu} \Rightarrow \frac{1}{d}\nabla\cdot\varepsilon = \delta\Omega$$

so $$\boxed{\nabla_{(\mu}\varepsilon_{\nu)} = \frac{1}{d}\, g_{\mu\nu}\, \nabla\cdot\varepsilon}$$ Conformal Killing equation

conformal killing vector

Solutions for $g_{\mu\nu} = \eta_{\mu\nu}$

$$\partial_{(\mu}\epsilon_{\nu)} = \frac{1}{d}(\partial_\lambda \epsilon^\lambda)\eta_{\mu\nu}$$

1) act with ∂_σ and (3 perms)

$$\partial_\mu\partial_\sigma\epsilon_\nu = \frac{1}{d}\left(\eta_{\mu\nu}\partial_\sigma(\partial\cdot\epsilon) + \eta_{\sigma\nu}\partial_\mu(\partial\cdot\epsilon) - \eta_{\sigma\mu}\partial_\nu(\partial\cdot\epsilon)\right)$$

2) trace (w/ $\eta^{\mu\nu}$)

$$\Box\epsilon_\nu = \frac{1}{d}(-d+2)\partial_\nu(\partial\cdot\epsilon) = \left(\frac{2}{d}-1\right)\partial_\nu(\partial\cdot\epsilon)$$

3) apply ∂_μ and symmetrize

$$\Box\partial_{(\mu}\epsilon_{\nu)} = \left(\frac{2}{d}-1\right)\partial_\mu\partial_\nu(\partial\cdot\epsilon) = \frac{1}{d}\eta_{\mu\nu}\Box(\partial\cdot\epsilon)$$

$\hookrightarrow$ $\boxed{(d-1)\Box(\partial\cdot\epsilon)=0}$ $\quad \mathbb{R}^{1,d-1}$

<u>Comments:</u>
- $d=1$ every $\epsilon^\mu(\tau)$ is a conf transformation.
- $d=2$ special (later)
- $d>2$: $(\partial\cdot\epsilon) = a + b_\mu x^\mu$

$$\partial_\mu\partial_\sigma\epsilon_\nu = c_{\mu\sigma\nu} \quad \text{(constant)}$$

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\sigma}x^\nu x^\sigma$$

plug in in conformal killing equation and determine $a_\mu, b_{\mu\nu}, c_{\mu\nu\sigma}$

$O(x^0)$: $\quad \epsilon_\mu = a_\mu$ ✓

$O(x)$: $\quad \epsilon_\mu = b_{\mu\nu}x^\nu \Rightarrow b_{(\mu\nu)} = \frac{1}{d}b_\lambda{}^\lambda\eta_{\mu\nu}$

antisymmetric

so $b = \omega_{\mu\nu} + \alpha\eta_{\mu\nu}$

$$\varepsilon_\mu = \underbrace{\omega_{\mu\nu} x^\nu}_{\text{Lorentz}} + \underbrace{\alpha x_\mu}_{\text{Scale transf.}}$$

$O(x^2)$: $\varepsilon_\mu = c_{\mu\nu\sigma} x^\nu x^\sigma$

$$\Rightarrow \varepsilon_\mu = \underbrace{2(b\cdot x)x_\mu - b_\mu x^2}_{\text{special conf transf.}}$$

$$x^\mu \to (x')^\mu = \begin{cases} x^\mu + a^\mu & \text{translations} \\ \Lambda^\mu{}_\nu x^\nu & \text{lorentz} \\ \lambda x^\mu & \text{scale/dilation} \\ \dfrac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2} & \text{special conf transformation} \end{cases}$$

Conformal = {poincare} ∪ {Dilation, special conf}

<u>Aside</u>

$$x^\mu \xrightarrow{\text{Inversion}} \frac{x^\mu}{x^2} \xrightarrow{\text{translation}} \frac{x^\mu}{x^2} - b^\mu \xrightarrow{I} \frac{\frac{x^\mu}{x^2} - b^\mu}{\left(\frac{x^\mu}{x^2} - b^\mu\right)^2}$$

translate the point at infinity

Invert again.

$$= \frac{x^\mu - x^2 b^\mu}{1 - 2b\cdot x + b^2 x^2} \quad (= \underset{\text{special}}{S}\,\underset{\text{conformal}}{C}\,\underset{\text{transformation}}{T})$$

essentially the generated is like a ~~conform~~ momentum acting on the inversion (end of the world).

In 2D $(d=2)$ $\quad ds^2 = dx^2 + dy^2 = \cancel{d+dz}\, dz\, d\bar{z}$

$z = x+iy$
$\bar{z} = x-iy$ $\quad$ any $\quad z \to f(z)$, $\bar{z} \to \bar{f}(\bar{z})$ $\quad$ is a C.t.

In 2D conf group is infinite (much bigger than for $d>2$) "Virasoro 'group'"

$d>2$ case: global conf. group.

2D CFT is ~~very~~ powerful.

$$f(x) \xrightarrow{CT} f(\underset{x+\epsilon}{x'}) = f(x) + \epsilon^\mu \partial_\mu f$$

$\boxed{\text{Conformal algebra}}$ consider $f(x)$,

$$\delta_\epsilon f = \epsilon^\mu \partial_\mu f, \quad \delta_\epsilon f = \mathcal{L}_\epsilon f$$

for a general tensor T. $\quad \delta_\epsilon T = \mathcal{L}_\epsilon T$

$$\epsilon^\mu \partial_\mu = i\left(a^\mu P_\mu + \frac{1}{2}\omega^{\mu\nu} J_{\mu\nu} + \alpha D + b^\mu K_\mu\right)$$

defines $\{P, J, D, K\}$

- transl $\epsilon^\mu = a^\mu$
- SCT $\epsilon^\mu = 2(b\cdot x)x^\mu - b^\mu x^2$

SCT = Special conformal transf.

Here
$$\begin{cases} P_\mu = -i\partial_\mu, & J_{\mu\nu} = -2i\, x_{[\mu}\partial_{\nu]} \\ D = -i x^\mu \partial_\mu, & K_\mu = -i\left(x^2 \partial_\mu - 2x_\mu\, x\cdot\partial\right) \end{cases}$$

we use the cuts Lie bracket to get the Lie Algebra.

$$\left.\begin{aligned} &[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\rho} J_{\mu\sigma} + \dots) \quad [P_\mu, P_\nu] = 0 \\ &[J_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \end{aligned}\right\} \text{Poincaré}$$

Dilator $\left\{ \begin{array}{l} [D, P_\mu] = iP_\mu \\ [D, J_{\mu\nu}] = 0 \\ [D, K_\mu] = -iK_\mu \end{array} \right.$

SCT $\left\{ \begin{array}{l} [J_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu) \\ [P_\mu, K_\nu] = 2i(\eta_{\mu\nu} D - J_{\mu\nu}) \end{array} \right.$

Comments

$\boxed{\text{Poincare + Dilatns are closed}}$ subalgebra

so could live without SCT but it seems that

Poincare + D $\Rightarrow$ SCT. So full conf symmetry.

- Casimirs: $C_1 = P^2$, $C_2 = W^2$ of Poinc no longer Casimirs of Conformal group.

Instead $C_2 = -D^2 - \frac{1}{2}(K \cdot P + P \cdot K) + \frac{1}{2} J_{\mu\nu} J^{\mu\nu}$

- how many elements?

$$\left.\begin{array}{l} a^\mu \rightsquigarrow d \\ \omega^{\mu\nu} \rightsquigarrow \frac{d(d-1)}{2} \\ D \rightarrow 1 \\ b^\mu \rightarrow d \end{array}\right\} \begin{array}{l} \frac{1}{2}(d+1)(d+2) = \dim(SO(2,d)) \\ \hookrightarrow \text{no coincidence!} \end{array}$$

Define $SO(2,d)$ generators L_{AB} $\quad A, B = 0, 1, \dots, d+1$

$\eta_{\mu\nu} = (-1, 1, \dots, 1, -1)$

$[L_{AB}, L_{CD}] = i(\eta_{AC} L_{BD} - \eta_{BC} L_{AD} + \dots)$

$$\left.\begin{array}{ll} \cdot\ L_{\mu\nu} = J_{\mu\nu} & \cdot\ L_{d+1\,\mu} = \frac{1}{2}(P_\mu + K_\mu) \\ \cdot\ L_{d\mu} = \frac{1}{2}(P_\mu - K_\mu) & \cdot\ L_{d,d+1} = D \end{array}\right\} \begin{array}{l} \text{Lie}(SO(2,d)) \\ \simeq \text{Lie}\begin{pmatrix}\text{conf group} \\ \text{in } d \text{ dim}\end{pmatrix} \end{array}$$

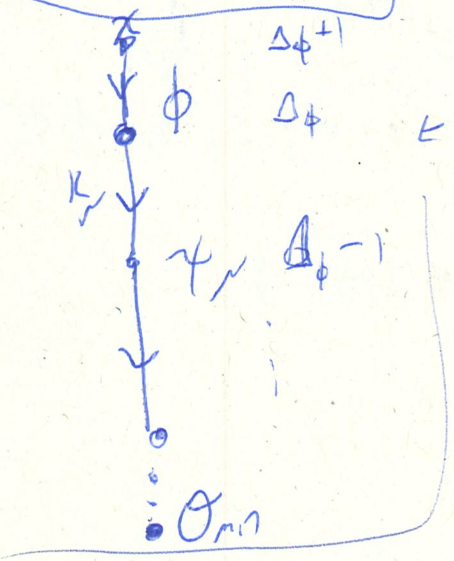
use Jacobi

$$\overset{\checkmark}{=} -[K_\nu,[\phi_\alpha, D]] - [\phi_\alpha,[D,K_\nu]]$$

$$= i\Delta_\phi [K_\nu, \phi_\alpha] + i[\phi_\alpha, K_\nu]$$

$$= i(\Delta_\phi - 1)[K_\nu,\phi_\alpha] = i(\Delta_\phi - 1)\psi_{\alpha\nu}$$

$\boxed{\Delta_\psi = \Delta_\phi - 1}$ similarly $P_\mu : \Delta_\phi \to \Delta_\phi + 1$



← ladder K_μ, P_μ.

$\psi_{\alpha\mu}$ is called a "descendant" operator of ϕ_α.

we can show Δ is bounded from below.

~ (Quantization in which $D \sim$ Hamiltonian, $\Delta \sim$ energy levels)

$\Rightarrow \exists\, \Delta_{min}$ $[K_\mu, \mathcal{O}_{min}(0)] = 0$.

such $\mathcal{O}_{min}$ is called a "primary operator"

but there is ∞-ly many ops by creating with P_μ

$\underbrace{[P_{\mu_1}, [\cdots [P_{\mu_n}}_{n \text{ times}}, \mathcal{O}(x)]\cdots]$ "descendants"

they define a conformal family.

Descendants ~ derivatives

$[P_\mu, \mathcal{O}] \sim \partial_\mu \mathcal{O}$ etc.

tentative: $\mathrm{CFT} = \{ \mathcal{O}_I(x)$, other data (interaction) $\}$

consistent (conformal bootstrap)

primaries of dim Δ_I

Unitary reps

$$\phi'_\alpha(x) = U(\Lambda, a)\, \phi_\alpha(x)\, U^{-1}(\Lambda, a)$$

$$= (\Lambda^{-1})_\alpha{}^\beta\, \phi_\beta(\Lambda x + a) \quad ,$$

$$U(\mathbb{1}, a) = e^{-i a_\mu P^\mu}$$

$$\phi_\alpha(x) = U(x)\, \phi_\alpha(0)\, U^{-1}(x)$$

- $\partial_\mu \phi_\alpha(x) = \partial\left(e^{-ix^\sigma P_\sigma} \phi_\alpha(0)\, e^{ix^\sigma P_\sigma} \right) = e^{-ix^\sigma P_\sigma}\left(-iP_\mu \phi_\alpha(0) + i\,\phi_\alpha(0) P_\mu \right) e^{ix^\sigma P_\sigma}$

  $= -i\,[P_\mu, \phi_\alpha(x)]$

So act on $\phi_\alpha(0)$ and translate to x with P.

- $[J_{\mu\nu}, \phi_\alpha(0)] = -i\,(S_{\mu\nu})_\alpha{}^\beta\, \phi_\beta(0)$

  defines "spin" e.g.
  - ϕ_α is a scalar $\Rightarrow S_{\mu\nu} = 0$
  - ϕ_α is spin $\frac{1}{2}$ $\Rightarrow (S_{\mu\nu})_\alpha{}^\beta = \frac{i}{4}[\gamma_\mu, \gamma_\nu]_\alpha{}^\beta$

  $\vdots$

- $[D, J_{\mu\nu}] = 0$, can choose to work on a basis diagonalize D.

  so $[D, \phi_\alpha(0)] = i\,\Delta_\phi\, \phi_\alpha(0)$

  dilation eigenvalue of ϕ_α

  "scaling dimension", "conformal dimension"

- $[D, P_\mu] = iP_\mu$; $[D, K_\mu] = -iK_\mu$

  ⇝ raising and lowering op of D

Consider $\psi_{\alpha\mu}(x) := [K_\mu, \phi_\alpha(x)]$

⇝ $i\Delta\, \psi_{\alpha\mu}(x) = [D, [K_\mu, \phi_\alpha(x)]]$