The notion of observable for $*-$algebras

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Motivation I

- Dealing with $*$-algebras algebraic observable and quantum observable do not necessarily agree.
  1. Physical interpretation of those algebraic observables which do not produce quantum observables in some GNS representation
- In the C^*-algebraic setting, an algebraic observable always defines quantum observables in every GNS representation.

  The use of $*$-algebras that are not C^*-algebras is mandatory in some important cases as perturbative QFT.

Motivation II

- POVM provides a notion of generalized observable that can be used in the interpretation of $\pi_\omega(A)$ when it is not essentially selfadjoint for an algebraic observable A.
- POVMs have a nice interplay with the moment problem of the pair (a, ω) and those of the deformations $(a, \omega_b)\,, b \in \mathfrak{A}$.

  1. The moment problem should be tackled for accepting the popular intepretation of $\omega(a)$ as the expectation value of the algebraic observable a.
  2. The moment problem admits a unique (spectral) solution if $\mathfrak{A}$ is a C^*-algebra.
  3. For an algebraic observable a in a $*-$algebra $\mathscr{A}$, if the moment problems of the class of $(a, \omega_b)\,, b \in \mathfrak{A}$, admit unique solutions, then $\pi_\omega(a)$ is essentially selfadjoint (quantum observable) and the POVM of $\pi_\omega(a)$ is a PVM.

Motivation III

4. The class of POVMs decomposing a given GNS representation $\pi_\omega(a)$ of an algebraic observable a are one-to-one with a class of special, physically meaningful, families of solutions of the moment problems for the pairs (a, ω_b).
5. There is only one such family if and only if $\pi_\omega(a)$ admits a unique POVM, i.e., it is maximally symmetric. From this viewpoint, a maximally symmetric $\pi_\omega(a)$ seems to define a good generalization of a quantum observable.

Preliminary definitions I

- A map $\mu : \Sigma \to \mathbb{C}$ is a complex measure if is unconditionally $\sigma-$additive.
- The total variation $||\mu|| := |\mu|(\Sigma)$ is finite.
- $\mathfrak{B}(\mathcal{H})$ denotes the C^*-algebra of bounded operators A in the Hilbert space $\mathcal{H}$.
- $A \subset B$ means $B|_{D(A)} = A$
- A Hermitian operator A is symmetric if $D(A)$ is dense in $\mathcal{H}$

Preliminary definitions II

- a operator on $\mathcal{H}$ is Hermitian if $\langle Ax|y\rangle = \langle x|Ay\rangle$
- A symmetric operator is selfadjoint if $A = A^\dagger$, essentially selfadjoint if it admits a unique selfadjoint extension (equivalently, if $\overline{A}$ is selfadjoint), in this case $\overline{A}$ is the unique selfadjoint extension of A.
- A conjugation is an antilinear isometric map $C : \mathcal{H} \to \mathcal{H}$ such that $CC = I$

Definition

An associative algebra $\mathscr{A}$ is a set equipped (and closed) with

$$+(A,B) := A+B, \quad \lambda(A) := \lambda A \quad \text{and} \quad \cdot(A,B) := AB.$$

An involution is a map $*(A) = A^*$ s.t.
$(\lambda A + B)^* = \lambda^* A^* + B^*, \quad (AB)^* = B^* A^* \quad \text{and} \quad (A^*)^* = A.$

- Complex vector space $V = (\mathscr{A}, +, \{\lambda\}_{\lambda \in \mathbb{C}})$
- Ring $R = (\mathscr{A}, +, \cdot) \to$ unital if R has a unit.

Define a norm $||\cdot|| : \mathscr{A} \to \mathbb{R}^+$ in which the multiplication $\cdot$ is a continuous map i.e.

$$||AB|| \leq ||A|| ||B||$$

If $(V, ||\cdot||)$ is a Banach vector space (complete), the algebra is said to be a Banach algebra.

Definition ($*$ and C^* algebras)

- A $*-$algebra is an algebra equipped with an involution $*$.
- A C^*-algebra is a Banach $*-$algebra such that

$$||A^*A|| = ||A||^2, \quad A \in \mathscr{A}.$$

- A $*-$subalgebra of $\mathfrak{B}(\mathcal{H})$ is a von Neumann algebra if $\mathscr{A}$ is a unital algebra and

$$(\mathscr{A}')' = \mathscr{A}, \quad \mathscr{A}' = \{B \in \mathfrak{B}(\mathcal{H}) : [B, A] = 0, \forall A \in \mathscr{A}\}.$$

$\mathscr{A}'$ is said to be the commutant of $\mathscr{A}$.

Example

Let (M, g) be a G.H S.T, a complex unital $*-$algebra $\mathscr{A}(M, g)$ is (associated to a K-G real scalar field ϕ) generated by finite products of elements $\phi(f)$, where f are real valued smearing functions equipped with the following requirements:

1. $\phi(af + bh) = a\phi(f) + b\phi(h)$
2. $\phi(f)^* = \phi(f)$
3. $\phi(Pf) = 0$
4. $[\phi(f), \phi(g)] = iE(f, g)1$, E is the causal propagator of P.

$\phi(f) = \phi(f') \iff E(f - f') = 0 \iff f - f' = Pg$ for some g.

Definition

A state on a (unital) $*-$algebra $\mathscr{A}$ is a linear functional $\omega : \mathscr{A} \to \mathbb{R}$ such that

$$\omega(AA^*) \geq 0, \quad \forall A \in \mathscr{A} \quad \text{and} \quad \omega(1) = 1.$$

States and observables are independent constructions!

Given a $*$-representation $\pi : \mathscr{A} \to \mathfrak{B}(\mathcal{H})$ and letting $v \in \mathcal{H}$ be a normalized cyclic vector for $\pi(\mathscr{A})$ (span$\{\pi(\mathscr{A})v\}$ is dense in the uniform topology) then,

$$\begin{aligned} \omega : &\mathscr{A} \to \mathbb{R} \\ &A \mapsto \langle \pi(A)v, v \rangle \end{aligned}$$

is a state of $\mathscr{A}$.

Theorem (Gelfand-Naimark-Segal construction)

Given a state ω on a C^*-algebra $\mathscr{A}$, there exists a unique representation $\pi_\omega : \mathscr{A} \to \mathfrak{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$ and a unit GNS vector $\Omega_\omega \in \mathcal{H}$ s.t.

$$\omega(A) = \langle \Omega_\omega, \pi(A)\Omega_\omega \rangle \quad \textit{and} \quad \pi(\mathscr{A})\Omega_\omega \quad \textit{is dense in } \mathcal{H}.$$

Proof sketch.

Define $\langle A, B \rangle := \omega(A^* B)$ and $I_\omega := \{A \in \mathscr{A} : \omega(A^* A) = 0\}$.
$\mathcal{H} = (\overline{\mathscr{A}/I_\omega}, \langle \cdot, \cdot \rangle)$ is a complex Hilbert space, π_ω is defined so that $\pi_\omega(A)[B] = [AB]$ on $\mathscr{A}/I_\omega$ and $\Omega_\omega := [1]$.

$$\langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle = \langle \Omega_\omega, [A] \rangle = \omega(A)$$

and $\pi_\omega(\mathscr{A})\Omega_\omega = \mathscr{A}/I_\omega$ is dense in $\mathcal{H}$.

$\square$

1. Hilbert-space $\rightarrow$ physical observables are (essentially) selfadjoint operators
2. $*-$algebra $A \rightarrow$ algebraic observables are $a = a^*$.

$\boxed{\text{Does physical and algebraic observables coincide?}}$

In view of the GNS construction, for an algebraic observable $A = A^*$, the physical interpretation of $\omega(A)$ is the expectation value of A in the state ω

$\boxed{\text{In } *-\text{algebras the physical interpretation of } \omega(A) \text{ is still valid?}}$

- For C^*-algebras, it holds $\pi_\omega(A^*) = \pi_\omega(A)^\dagger$

  $\implies$ Hermitian elements of $\mathscr{A}$ are always represented by selfadjoint operators

  $\implies$ the two notions of observable always agree.

- If $\mathscr{A}$ is only a $*-$algebra and $\pi_\omega(A)$ is essentially selfadjoint for every ω

  $\implies$ it admits a one-to-one correspondence between algebraic observables and quantum observables.

Could π_ω have non-unique extensions? Yes!

Could π_ω admit no selfadjoint extension? Yes!

Example

Concider the space $\mathscr{I}$ of complex-valued smooth functions with domain $[0,1]$ and vanishing at the boundary along with all of their derivatives. Concider $\mathscr{A}$ as the $*-$algebra of diff. op. acting on $\mathscr{I}$ generated by $P := -i\frac{d}{dx}$, $Q := f\cdot$ ($f \in \mathscr{I}$ real valued) and 1. Take $A^* := A^\dagger$ where $\dagger$ is the adjoint in $L^2([0,1],dx) \supset \mathscr{I}$ (e.g. $P^* = P$). Concider

$$\omega(A) := \int_0^1 \psi(x)(A\psi)(x)dx$$

with $\psi \in \mathscr{I}$ fixed, non-negative function vanishing only at the boundary.

- GNS $\implies \mathcal{H}_\omega = L^2([0,1],dx)$, $\pi_\omega(A) = A|_{D_\omega}$, $\psi_\omega := \psi$.

Define the deficiency subspaces of A by $K_+ = Ran(\pi_\omega(P)+i)^\perp$ and $K_- = Ran(\pi_\omega(P)-i)^\perp$, in this case $K_\pm = \{ce^\mp, c \in \mathbb{C}\}$.

The operator is not essentially selfadjoint on its GNS domain but it admits a one-parameter class of different selfadjoint extensions according to von Neumann's extension theorem.

Similarly, one can show for $\mathscr{I}$ with domain $[0,\infty)$ that $\dim K_- \neq \dim K_+ \implies$ does not admit a selfadjoint extension.

The interpretation of $\omega(A)$ as expectation value for $a = a^*$ is defined so that

$$\omega(A) = \int_{\mathbb{R}} \lambda d\mu(\lambda)$$

Assumptions:

- μ is defined on the Borel $\sigma-$algebra since it is the case for measures arising from the spectral theory.
- We interpret A^n as the observable whose values are λ^n if λ is a value attained by A. We shall then assume

$$\omega(A^n) = \int_{\mathbb{R}} \lambda^n d\mu(\lambda), \quad \forall n \in \mathbb{N}.$$

With this assumptions, $\omega(A^n)$ is interpreted as the n-th moment, find μ is named the **Hamburger moment problem**.

Connection between interpretations of $\omega(A)$ and $\pi_\omega(A)$

If $\pi_\omega(A)$ is essentially selfadjoint, the GNS constructions gives us a measure μ, it is constructed out of the Projection-Valued Measure (PVM) $P^{\overline{\pi_\omega(A)}}$: Borel-$\mathbb{R} \to \mathfrak{B}(\mathcal{H})$ given by

$$\mu(E) := \left\langle \Omega \middle| P^{\overline{\pi_\omega(A)}}(E)\Omega \right\rangle$$

and satisfies the moment problem:

$$\int_{\mathbb{R}} \lambda^n d\mu = \left\langle \Omega \middle| \overline{\pi_\omega(A)}^n \Omega \right\rangle = \langle \Omega | \pi_\omega(A^n)\Omega \rangle$$

because $\Omega \in D(\pi_\omega) \subset D(\pi_\omega(A^n)) \subset D(\pi_\omega(A)^n) \subset D(\overline{\pi_\omega(A)}^n)$.

- Unique if $\mathscr{A}$ is a C^*-algebra.

Definition

If ω is a non-normalized state, we will denote ω_B called the B-deformation of ω

$$\omega_B(A) := \omega(B^* AB)$$

for $B \in \mathscr{A}$. If $\omega_B(1) = \omega(B^* B) = 0$ we call ω_B a singular deformation.

$$(\mathcal{H}_{\omega,B}, D_{\omega,B}, \pi_{\omega,B}, \Omega_{\omega,B}) = (\overline{\pi_\omega(\mathscr{A})\Omega_{\omega,B}}, \pi_\omega(\mathscr{A})\Omega_{\omega,B}, \pi_\omega|_{D_{\omega,B}}, \pi_\omega(B)\Omega_\omega)$$

$$\text{Deformations} \quad \Longleftrightarrow \quad \text{states}$$

Theorem

Let $\mathscr{A}$ be a unital $*-$algebra, $A = A^* \in \mathscr{A}$ and ω a non-normalized state. Assume $\mu_{\omega,B}$ solves the moment problem for the deformed state ω_B uniquely. Then $\pi_\omega(A)$ is essentialy self-adjoint and $\pi_{\omega,B}(A)$ are also self-adjoint. Moreover, $\mu_{\omega,b}$ are induced by the PVM $P^{\overline{\pi_\omega(A)}}$ i.e.

$$\mu_{\omega,B}(E) = \left\langle \Omega_{\omega,b} \middle| P^{\overline{\pi_{\omega,B}(A)}}(E)\Omega_{\omega,B} \right\rangle$$

where $P^{\overline{\pi_{\omega,B}(A)}}(E) = P^{\overline{\pi_\omega(A)}}(E)|_{\mathcal{H}_{\omega,\mathcal{B}}}$

Example

Let $\mathscr{A}_{CCR,1}$ be the $*-$algebra generated by $I, Q, P : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$:

$$Q\psi(x) = x\psi(x), \quad P\psi(x) = -i\frac{d}{dx}\psi(x), \quad I\psi(x) = \psi(x).$$

And $A^* := A^\dagger|_{\mathscr{S}(\mathbb{R})}$ where $\dagger$ is the adjoint in $L^2(\mathbb{R}, dx)$ (e.g. $P = P^*$, $Q = Q^*$). Consider the state ω

$$\omega(A) = \int_{\mathbb{R}} \overline{\psi_0(x)}(A\psi_0)(x)dx, \qquad \underbrace{\psi_0(x)}_{GroundHO} = \pi^{-1/4}e^{-\frac{x^2}{2}}.$$

GNS $\implies \mathcal{H}_\omega = L^2(\mathbb{R}, dx)$, $\pi_\omega(A) = A|_{D_\omega}$, $\Omega = \psi_0$.

- $D_\omega \subset \mathscr{S}(\mathbb{R})$, the domain is dense in L^2 are **all finite linear combinations of Hermite functons** $\{\psi_n\}_n$ (eigenstates of the HO Hamiltonian).

- Using $I, A := \frac{1}{\sqrt{2}}(Q + iP)$, $A^* := \frac{1}{\sqrt{2}}(Q - iP)$ insted of I, Q, P, the elements correspond to annihilation and creation operators and

$$D_\omega = D_{\omega,B},\, \mathcal{H} = \mathcal{H}_{\omega,B}, \quad \pi_{\omega,B} = \pi_\omega$$

  The first identity holds because $\psi_\omega \in D_{\omega,B}$.

Regarding to the essentially selfadjointness,

1. $\pi_{\omega,B}(Q^k)$ (and $\pi_{\omega,B}(P^k)$) are essentially selfadjoint for $k = 1, 2$ because of **Nelson's theorem** [1] and $\pi_\omega(Q^4) = \pi_{\omega,B}(Q^4)$ is essentialy selfadjoint.

In general, if $[a, a^+] = I$ with the other vanishing and $a\psi_0 = 0$. Then, $\pi_\omega(Q) = \pi_\omega(\frac{1}{\sqrt{2}}(A + A^*)) = \frac{1}{\sqrt{2}}(\pi_\omega(A) + \pi_\omega(A^*)) = \sqrt{2}^{-1}(a + a^+)$. Therefore,

$s_n^{(k)} := \omega(Q^{kn}) = \frac{1}{2^{kn/2}} \left\langle \psi_0 \middle| (a + a^+)^{kn} \psi_0 \right\rangle = \pi^{-1/2} \int_{\mathbb{R}} x^{kn} \boxed{e^{-x^2} dx}$.
Admits a solution, is unique? Using the Carleman's condition[2]

- $k = 1$: $s_{2n+1}^{(}1) = 0$ and $s_{2n}^{(}1) = 2^{-2n}(2n-1)!!$ satisfes the condition.
- $k = 2$: is determined by the Cramer's conditon
- $k = 3, 4$: is not determined by the krein condition. $k = 4$ is essentially selfadjoint but has many measures associated.

One can change the problem for measures defined on $[0, \infty)$ rather than $\mathbb{R}$ however, the <u>Stieltjes moment problem</u> is still undetermined.

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[1] a symmetric operator is essentially selfadjoint if it contains a dense subset of analytic vectors

[2] The moment problem is determined if $\sum_n m_{2n}^{-1/2n} = \infty$

Definition

An operator-valued map $Q : \Sigma \to \mathfrak{B}(\mathcal{H})$ is called positive-operator valued measure (POVM) if it satisfies the following two conditions:

- $\forall E \in \Sigma$, $Q(E) \geq 0$
- $\forall \psi, \varphi \in \mathcal{H}$ the map $Q_{\psi,\varphi} : E \in \Sigma \to \langle \psi | Q(E) \varphi \rangle \in \mathbb{C}$ defines a complex $\sigma-$additive measure.

A POVM Q is said to be normalized if $Q(\Omega) = I$. A normalized POVM is a standard PVM if and only if $Q(E)Q(F) = Q(E \cap F)$ for $E, F \in \Sigma$, so that $Q(E) \in \mathcal{L}(\mathcal{H})$.
Normalized POVMs are physically intepreted and called generalized observables, in the Hilbert-space formulation of quantum theory

The physical interpretation of $\langle \psi | Q(E) \psi \rangle$ is the probability that, measuring the generalized observable associated to the normalized POVM when the state is represented by the normalized vector ψ, the outcome belongs to the Borel set $E \subset \mathbb{R}$. Disadvantages:

- The logical interpretation of $Q(E)$ as an elementary YES-NO observable is lost
- The possibility to describe the post-measurement state with the standard Luders-von Neumann reduction postulate exploiting only the POVM (more information must be supplied),
- The fact that observables $Q(E)$ and $Q(F)$ are necessarily compatible

Theorem (Naimark's dilation theorem)

Let $Q : \Sigma \to \mathfrak{B}(\mathcal{H})$ be a normalized POVM. Then there exist a Hiilbert space $\mathcal{K}$ which includes $\mathcal{H}$ as a closed subspace, i.e. $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, and a PVM $P : \Sigma \to \mathcal{L}(\mathcal{H})$ s.t.

$$Q(E) = P_{\mathcal{H}} P(E)|_{\mathcal{H}}, \quad \forall E \in \Sigma$$

where $P_{\mathcal{H}} \in \mathcal{L}(\mathcal{K})$ is the orthogonal projection onto $\mathcal{H}$. The triple $(\mathcal{K}, P_{\mathcal{H}}, P)$ is called Naimark's dilation triple.

POVMs arise naturally when dealing with generalized extensions of symmetric operators.

Definition

A symmetric operator A on a Hilbert space is said to be maximally symmetric if there is no symmetric operator B on $\mathcal{H}$ s.t. $A \subset B$.

- A maximally symmetric operator is necessarily closed (closure of a symmetric operator is symmetric)
- The converse holds if its deficiency indices is 0.
- If a maximally symmetric operator A on $\mathcal{H}$ satisfies $CA \subset AC$ for a conjugation $C : \mathcal{H} \to \mathcal{H}$, then A is selfadjoint.

Definition

Let A be a symmetric operator. A generalized symmetric (resp. selfadjoint) extension of A is a symmetric (resp. selfadjoint) operator B on a Hilbert space $\mathcal{K}$ s.t

1. $\mathcal{K}$ contains $\mathcal{H}$ as a closed subspace.
2. $A \subset B \in \mathcal{K}$
3. Every closed subspace $\mathcal{K}_0 \subset \mathcal{K}$ such that $\{0\} \neq \mathcal{K}_0 \subset \mathcal{H}^\perp$ does not reduce B

- Every non-selfadjoint symmetric operator (possibly maximally symmetric) always admits generalized selfadjoint extensions.
- A selfadjoint operator does not admit proper generalized symmetric extensions.

Naimark extended part of the spectral theory (usually formulated in terms of PVMs to POVMS). However, unless the symmetric operator is maximally symmetric, the POVM which decomposes it is not unique.

Theorem

For a symmetric operator A in the Hilbert space $\mathcal{H}$ the following holds:

1. There exist a normalized POVM $Q^{(A)}$: Borel-$\mathbb{R} \to \mathfrak{B}(\mathcal{H})$ s.t.

$$\langle \psi | A\varphi \rangle = \int_{\mathbb{R}} \lambda dQ^{(A)}_{\psi,\varphi}(\lambda), \ ||A\varphi||^2 = \int_{\mathbb{R}} \lambda^2 dQ^{(A)}_{\varphi,\varphi}(\lambda), \ \forall \psi \in \mathcal{H}, \varphi \in D(A$$

2. Every normalized POVM $Q^{(A)}$: Borel-$\mathbb{R} \to \mathfrak{B}(\mathcal{H})$ satisfaying 1. is of the form

$$Q^{(A)}(E) := P_{\mathrm{H}} P(E) \lceil_{\mathrm{H}} \quad \forall E \in \text{Borel-}(\mathbb{R})$$

for some Naimark's dilatation triple $(\mathcal{K}, P_{\mathcal{H}}, P)$ of $Q^{(A)}$ arising from a generalized selfadjoint extension $B = \int_{\mathbb{R}} \lambda dP(\lambda)$ of A in $\mathcal{K}$,

$$A = B \Big\lceil_{D(A)} \quad \text{and} \quad D(A) \subset \left\{ \psi \in \mathrm{H} \mid \lambda \in L^2\left(\mathbb{R}, Q^{(A)}_{\psi,\psi}\right) \right\}$$

3. A normalized POVM $Q^{(A)}$ is a PVM if and only if the selfadjoint operator B constructed out of Naimark's dilation triple of $Q^{(A)}$ can be chosen as a standard selfadjoint extension of A.

4. If A is closed, a normalized POVM $Q^{(A)}$ satisfies

$$A = B\Big|_{D(B)\cap H} \quad \text{and} \quad D(A) = \left\{\psi \in \mathsf{H} \mid \lambda \in L^2\left(\mathbb{R}, Q^{(A)}_{\psi,\psi}\right)\right\}.$$

5. If A is closed, then A is maximally symmetric if and only if there is a unique normalized POVM $Q^{(A)}$. In this case, (37) is valid for all choices of $(\mathrm{K}, P_{\mathrm{H}}, P)$ generating $Q^{(A)}$ as in (b).

6. If A is selfadjoint, there is a unique normalized POVM $Q^{(A)}$ satisfying 1. , and it is a *PVM*. In this case $\mathrm{K} = \mathrm{H}$, $Q^{(A)} = P$, and $A = B$ for all choices of $(\mathrm{K}, P_{\mathrm{H}}, P)$ generating $Q^{(A)}$.

7. If A is a symmetric operator in $\mathcal{H}$:

   -A and $\bar{A}$ admits the same class of POVMs for A and $\bar{A}$ respectively.

   -A admits a unique normalized POVM if and only $\bar{A}$ is maximally symmetric. In this case

$$D(\bar{A}) = \left\{\psi \in \mathrm{H} \mid \lambda \in L^2\left(\mathbb{R}, Q^{(A)}_{\psi,\psi}\right)\right\}$$

   -The unique normalized POVM as in (b) is a PVM if A is also essentially selfadjoint.

Definition

If A is a symmetric operator in the Hilbert space H, a normalized POVM $Q^{(A)}$ over the Borel σ-algebra over $\mathbb{R}$ which satisfies 1. of the Theorem is said to be associated to A or, equivalently, to decompose A.

Remark: A symmetric operator admits at least one normalized POVM which decomposes it, the converse is not true.

Every POVM over R can be weakly integrated determining a unique Hermitian operator over a natural domain. It is worth stressing that the result strictly depends on the choice of this domain and different alternatives are possible in principle.

Theorem

If $Q : \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathrm{H})$ is a normalized POVM in the Hilbert space H, define the subset $D\left(A^{(Q)}\right) \subset \mathrm{H}$,

$$D\left(A^{(Q)}\right) := \left\{\psi \in \mathrm{H} \mid \int_{\mathbb{R}} \lambda^2 dQ_{\psi,\psi}(\lambda) < +\infty\right\}.$$

The following facts are valid.

(a) $D\left(A^{(Q)}\right)$ is a subspace of H (which is not necessarily dense or non-trivial).

(b) There exists a unique operator $A^{(Q)} : D\left(A^{(Q)}\right) \to \mathrm{H}$ such that

$$\left\langle \varphi \mid A^{(Q)}\psi \right\rangle = \int_{\mathbb{R}} \lambda dQ_{\varphi,\psi}(\lambda), \quad \forall \varphi \in \mathrm{H}, \forall \psi \in D\left(A^{(Q)}\right)$$

(c) $A^{(Q)}$ is Hermitian, so that $A^{(Q)}$ is symmetric if and only if $D\left(A^{(Q)}\right)$ is dense.

(d) If $(\mathrm{K}, P_{\mathrm{H}}, P)$ is a Naimark's dilation triple of Q, then

$$A^{(Q)}\psi = P_{\mathrm{H}} \int_{\mathbb{R}} \lambda dP(\lambda)\psi, \quad \forall \psi \in D\left(A^{(Q)}\right)$$

(e) If there exists a Naimark's dilation triple $(\mathrm{K}, P_{\mathrm{H}}, P)$ of Q such that

$$\int_{\mathbb{R}} \lambda dP(\lambda) \left(D\left(A^{(Q)}\right)\right) \subset \mathrm{H}$$

then $A^{(Q)}$ is closed and

$$\left\|A^{(Q)}\psi\right\|^2 = \int_{\mathbb{R}} \lambda^2 dQ_{\psi,\psi}(\lambda), \quad \forall\psi \in D\left(A^{(Q)}\right)$$

Interpretation of $\pi_\omega(A)$ in terms of POVMs I

Let $\omega : \mathscr{A} \to \mathbb{C}$ be a non-normalized state on the unital $* -$ *algebra* and concider the symmetric operator $\pi_\omega(A)$, $A = A^*$ we interpret the operator as a generalized observable when it is not essentially selfadjoint

1. $\pi_\omega(a)$ and $\overline{\pi_\omega(a)}$ share the same class of associated normalized POVMs $Q^{(a,\omega)}$ so that
   they support the same physical information when interpreting them as generalized observables. More precisely, each of these POVMs endows those symmetric operators with the physical meaning of generalized observable in the Hilbert space H_ω. This is particularly relevant when $\pi_\omega(a)$ does not admit selfadjoint extensions;

Interpretation of $\pi_\omega(A)$ in terms of POVMs II

2. the above class of normalized POVMs however includes also all possible PVMs of all possible selfadjoint extensions of $\pi_\omega(a)$ provided they exist. Hence, the standard notion of quantum observable in Hilbert space is encompassed;
3. $Q^{(a,\omega)}$ is unique if and only if $\overline{\pi_\omega(a)}$ is maximally symmetric but not necessarily selfadjoint;
4. That unique POVM is a PVM if $\pi_\omega(a)$ is essentially selfadjoint.

Even if the symmetric operator $\pi_\omega(A)$ does not admit a selfadjoint extension, it can be considered a generalized observable with some precautions, since it admits decompositions in terms of POVMs which are generalized observables in their own right. However, in general, there are many POVMs associated with one given symmetric operator $\pi_\omega(A)$.

Expectation value interpretation of $\omega_B(A)$

Many measures solves the moment problem even if the operator is essentially selfadjoint. How do we reduce the number of this measures?

1. The measures μ and $\mu_{\omega,B}$ are not independent:
$$\int_{\mathbb{R}} \lambda^{2k+1} d\mu^{(a)}_{\omega}(\lambda) = \omega\left(a^{2k+1}\right) = \omega_{a^k}(a) = \int_{\mathbb{R}} \lambda d\mu^{(a)}_{\omega_{a^k}}(\lambda)$$
2. For every polynomial $p(A)$,
$$\omega_{b+c}(p(a)) + \omega_{b-c}(p(a)) = 2\left[\omega_b(p(a)) + \omega_c(p(a))\right]$$
$$\omega_{zb}(p(a)) = |z|^2 \omega_b(p(a))$$
(and the same with the measures)
3. Continuity:
$$\omega_{b+tc}(p(a)) \to \omega_b(p(a)) \quad t \to 0, \implies \int_{\mathbb{R}} p(\lambda) d\mu^{(a)}_{\omega_{b+tc}}(\lambda) \to \int_{\mathbb{R}} p(\lambda) d\mu^{(a)}_{\omega_b}(\lambda).$$

Restriction: Put f in place of p, where f are bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$ (e.g. characteristic functions).

This restriction is new: Polynomials are not necessarily dense in the relevant L^1 spaces, since the considered Borel measures have non-compact support in general.

Definition

The family of measures $\{\nu_\psi\}_{\psi \in D}$ is said to be consistent.

We now apply the summarized theory of POVMs to prove that the family of POVMs associated to $\pi_\omega(a)$ is one-to-one with the family of consistent classes of measures solving the moment problem for all ω_b. The proof consists of two steps. Here is the former.

If $Q^{(a,\omega)}$ is a POVM associated to $\pi_\omega(a)$ for $a^* = a \in \mathfrak{A}$ and for a non-normalized state $\omega : \mathfrak{A} \to \mathbb{C}$, let $\nu^{(a)}_{\omega_b}$ be the Borel measure defined by

$$\nu^{(a)}_{\omega_b}(E) := \left\langle \psi_{\omega_b} \mid Q^{(a,\omega)}(E)\psi_{\omega_b} \right\rangle \quad \text{if } E \in \mathscr{B}(\mathbb{R})$$

Theorem

*Consider the unital *-algebra $\mathfrak{A}$, a non-normalized state $\omega : \mathfrak{A} \to \mathbb{C}$, an element $a = a^+ \in \mathfrak{A}$ and the family of measures $\left\{\nu^{(a)}_{\omega_b}\right\}_{b\in\mathfrak{A}}$ defined above with respect to a normalized POVM $Q^{(a,\omega)}$ associated to $\pi_\omega(a)$. Then*

1. $\left\{\nu^{(a)}_{\omega_b}\right\}_{b\in\mathfrak{A}}$ *is a consistent family over $\mathcal{D}_\omega = \mathfrak{A}/G(\mathfrak{A},\omega)$.*
2. *Each $\nu^{(a)}_{\omega_b}$ is a solution of the moment problem (15) relative to (a, ω_b).*

Theorem

Consider the unital $*-$ algebra $\mathfrak{A}$, a non-normalized state $\omega : \mathfrak{A} \to \mathbb{C}, a = a^* \in \mathfrak{A}$, and a consistent class of measures $\left\{\mu^{(a)}_{\omega_b}\right\}_{b\in\mathfrak{A}}$ solutions of the moment problem relative to the pairs (a, ω_b) for $b \in \mathfrak{A}$. Then

(a) There is a unique normalized POVM $Q^{(a,\omega)} : \mathscr{B}(\mathbb{R}) \to \mathcal{B}\left(\mathrm{H}_\omega\right)$ such that, if $b \in \mathfrak{A}$,

$$\mu^{(a)}_{\omega_b}(E) = \left\langle \psi_{\omega_b} \mid Q^{(a,\omega)}(E)\psi_{\omega_b} \right\rangle \quad \forall E \in \mathscr{B}(\mathbb{R})$$

(b) $Q^{(a,\omega)}$ decomposes $\pi_\omega(a)$ so that, in particular,

$$\mathcal{D}_\omega = D\left(\pi_\omega(a)\right) \subset \left\{\psi \in \mathrm{H}_\omega \mid \int_{\mathbb{R}} \lambda^2 dQ^{(a,\omega)}_{\psi,\psi}(\lambda) < +\infty\right\}$$

(c) $Q^{(a,\omega)}$ decomposing $\pi_\omega(a)$ is unique if and only if $\overline{\pi_\omega(a)}$ is maximally symmetric. In this case

$$D\left(\overline{\pi_\omega(a)}\right) = \left\{\psi \in \mathrm{H}_\omega \mid \int_{\mathbb{R}} \lambda^2 dQ^{(a,\omega)}_{\psi,\psi}(\lambda) < +\infty\right\}$$

and that unique $Q^{(a,\omega)}$ is a PVM if and only if $\overline{\pi_\omega(a)}$ is selfadjoint. In that case $Q^{(a,\omega)}$ coincides with the *PVM* of $\pi_\omega(a)$. (d) Let us define

$$Q^{(a,\omega_b)}(E) := P_{\omega_b} Q^{(a,\omega)}(E) \upharpoonright_{\mathrm{H}_{\omega_b}},$$

where $P_{\omega_b} : \mathrm{H}_\omega \to \mathrm{H}_\omega$ is the orthogonal projector onto H_{ω_b}. It turns out that, for $b \in \mathfrak{A}$, (i) $Q^{(a,\omega_b)}$ is a normalized POVM in H_{ω_b}. (ii) It holds

$$\left\langle \psi_{\omega_b} \mid Q^{(a,\omega_b)}(E)\psi_{\omega_b}\right\rangle = \mu^{(a)}_{\omega_b}(E) \quad \forall E \in \mathscr{B}(\mathbb{R})$$

(iii) $Q^{(a,\omega_b)}$ decomposes $\pi_{\omega_b}(a)$

Conclusion I

- Dealing with $*$-algebras algebraic observable and quantum observable do not necessarily agree.
    1. Physical interpretation of those algebraic observables which do not produce quantum observables in some GNS representation
- In the C^*-algebraic setting, an algebraic observable always defines quantum observables in every GNS representation.

  The use of $*$-algebras that are not C^*-algebras is mandatory in some important cases as perturbative QFT.

Conclusion II

- POVM provides a notion of generalized observable that can be used in the interpretation of $\pi_\omega(A)$ when it is not essentially selfadjoint for an algebraic observable A.
- POVMs have a nice interplay with the moment problem of the pair (a, ω) and those of the deformations (a, ω_b) , $b \in \mathfrak{A}$.
    1. The moment problem should be tackled for accepting the popular intepretation of $\omega(a)$ as the expectation value of the algebraic observable a.
    2. The moment problem admits a unique (spectral) solution if $\mathfrak{A}$ is a C^*-algebra.
    3. For an algebraic observable a in a $*-$algebra $\mathscr{A}$, if the moment problems of the class of (a, ω_b) , $b \in \mathfrak{A}$, admit unique solutions, then $\pi_\omega(a)$ is essentially selfadjoint (quantum observable) and the POVM of $\pi_\omega(a)$ is a PVM.

Conclusion III

4. The class of POVMs decomposing a given GNS representation $\pi_\omega(a)$ of an algebraic observable a are one-to-one with a class of special, physically meaningful, families of solutions of the moment problems for the pairs (a, ω_b).

5. There is only one such family if and only if $\pi_\omega(a)$ admits a unique POVM, i.e., it is maximally symmetric. From this viewpoint, a maximally symmetric $\pi_\omega(a)$ seems to define a good generalization of a quantum observable.

Bibliografía I

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