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Universidad de Los Andes

The Harmonic Map Flow and The Eells-Sampson Theorem

Author:
Rafael F. Córdoba López

Supervisor:
Ph.D Jean Carlos Cortissoz Iriarte

Mathematics department

Faculty of science

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"Forty-two," said Deep Thought, with infinite majesty and calm.

Abstract

Contents

Introduction

Eells and Sampson introduced deformations of maps using the heat equation in their celebrated paper "Harmonic Mappings of Riemannian Manifolds" [1]. The work of Eells and Sampson has been very influential in geometric analysis as they not only introduce the concept of harmonic map, but also bring new tools to study classical problems ranging from the study of minimal surfaces to the Ricci flow. Indeed, Eells and Sampson's ideas were the starting point for Richard Hamilton to introduce the Ricci Flow [2]. For instance, if one computes in Riemannian normal coordinates around a point, the Ricci tensor takes the form $-\frac{1}{2}\Delta g^{ij} + o(3)$, thus, in some way, the Ricci flow is a nonlinear heat equation in the space of Riemannian metrics. On the other hand, we pay particular attention to the problem of finding 'extremal' representations of certain homotopy classes. Among them is the classical problem of finding a closed geodesic in every free homotopy class (in a Riemannian manifold).

Let M and M' be two Riemannian manifolds with metrics g and, g' respectively. Given a smooth map $f : M \to M'$, we can define an energy functional $E(f)$ which a measure of the mean square of the infinitesimal dispersion of the image points in M produced by f. The Euler-Lagrange equations of the map $E(f)$ define a vector field $\tau(f)$ called the tension field. We say that f is harmonic if $\tau(f)$ vanishes. Thus, the extremals of the energy functionals are those with

$$\tau(f) = 0. \tag{0.1}$$

For example, if we take M to be the circle S^1 then, $\tau(f) = 0$ is precisely the geodesic equation on M' and thus, if $F : S^1 \to M$ is harmonic then F is a closed geodesic. The deformation method of Eells and Sampson is the following system of nonlinear evolution equations:

$$\frac{\partial f_t}{\partial t} = \tau(f_t) \quad f_0 = f. \tag{0.2}$$

Since $\tau(f)$ is the Laplace-Beltrami operator applied on f then, (0.2) is a nonlinear heat equation. Consequently, we call the solutions of (0.2) a deformation of f.

In coordinates, equation (0.2) yields the system

$$\begin{cases} \dfrac{\partial u^\alpha}{\partial t} = g^{ij}\left[\dfrac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\dfrac{\partial u^\alpha}{\partial x^k} + \Gamma'^{\alpha}_{\beta\gamma}\dfrac{\partial u^\beta}{\partial x^i}\dfrac{\partial u^\gamma}{\partial x^j}\right] & u : M \times \mathbb{R}^+ \to M' \subset \mathbb{R}^m \\ u(p,0) = f(p) & p \in M,\ u = (u^1, ..., u^m) \end{cases} \tag{0.3}$$

where $\Gamma^k_{ij} = \Gamma^k_{ij}(p)$ and $\Gamma'^k_{ij} = \Gamma'^k_{ij}(f(p))$ are the Christoffel symbols of M and M' respectively. We will assume, to facilitate the exposition, that M' is embedded into $\mathbb{R}^n$ by the Nash embedding theorem. Consequently, the deformations of $f : M \to M'$ are, in fact, the solutions of a system of non-linear elliptic partial differential equations.

This thesis aims to study how $f : M \to M'$ can be deformed according to the non-linear heat equation, and how this deformation can lead to geometrical applications, in particular, to show the existence of a closed geodesic in every free homotopy class. In order to do so, we have several problems that we have to solve. First, we introduce the heat equation on a Riemannian manifold. Second, we prove the existence of solutions to the heat equation. Finally, we introduce a functional map E analogous to the energy of a physical system and use the deformations of a map f as a tool to see how E behaves. The deformations of f allow us to deduce geometrical properties on the target manifold. In particular, as we said above, we prove that every closed curve $f \in C^{\infty}(S^1, M)$ is homotopically equivalent to a geodesic under certain restrictions. That is, we prove the following theorem.

Theorem (Cartan-Hadamard)**.** *Let M be a compact Riemannian manifold with non-positive sectional curvature. Then, in every homotopy free class of closed curves there exists a closed geodesic.*

1 Riemannian Manifolds

To make this work more readable, we first introduce the basic definitions of Riemannian manifolds, differential operators, and some other definitions, theorems, and notation that allows us to have the necessary tools to study and prove the main theorem. Although we follow several books, many results can be found in a standard introduction to Riemannian geometry. In particular, we follow the books of Jöst and do Carmo [3, 4]. For a detailed discussion, we suggest consulting [3, 5, 6, 4].

1.1 Riemannian manifolds

Riemannian manifolds can be thought of as a generalization of the Euclidean spaces $\mathbb{R}^n$. Indeed, many results on Euclidean spaces can be generalized to Riemannian spaces. In order to study PDE's in Riemannian manifolds, one may formulate them in an invariant form because manifolds, in general, do not have a canonical coordinate chart (as $\mathbb{R}^n$ has) but rather, a fixed differential structure. Therefore, an invariant formulation makes the study of classical PDE's such as the heat equation, wave equation, Laplace equation, etc., on a Riemannian or pseudo-Riemannian manifold (of which Lorentzian manifolds are a particular case) possible. In this chapter, we introduce the Riemannian manifolds.

Throughout this work, a manifold is a smooth topological manifold, Hausdorff, second countable, locally Euclidean space with a maximal atlas [7].

Definition 1.1.1. *Let M be a n-dimensional manifold, $(\psi : M \to \mathbb{R}^n, U)$ a chart of M and $p \in U$. The (local) coordinates $(x^1(p), ..., x^n(p))$ of $p \in M$ with respect to the chart ψ are the coordinates of $\psi(p) \in \mathbb{R}^n$, i.e. $x^i(p) = \pi_i(\psi(p))$, where $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., n$ are the i-th projection functions.*

Remark 1.1.1. *Although the coordinates of p with respect to ψ are chart-dependent we may refer to them as coordinates around p and denote them simply as $\{x^i\}_{i=1}^n$.*

Definition 1.1.2 (Tangent and cotangent spaces)**.** *Let M be a n-dimensional manifold, $p \in M$ and $\{x^i\}_{i=1}^n$ the (local) coordinates of p. Denote the tangent space as T_pM and define a basis of T_pM by*

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, ..., \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

such that $\frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial}{\partial x}\bigg|_{\varphi p} f \circ \varphi^{-1} \in \mathbb{R}$ for every $f \in C^\infty(\mathbb{R}^n)$ Thus, we can write $v_p \in T_pM$ in coordinates as

$$v_p = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x^i}.$$

When there is no risk of confusion, we shall sometimes drop the index $p \in M$ and write the following *equivalent notation:*

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial}{\partial x^i} = \partial_i.$$

Likewise, we can define the dual of T_pM *denoted by* T_p^*M *called the cotangent space with vectors* $\omega_p \in T_p^*M$ *given by the coordinate's basis*

$$\omega_p = \sum_{i=1}^{n} \alpha_i dx^i,$$

with dx^i *defined so that*

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{ij}, \quad i, j \in \{1, ..., n\}$$

where δ_{ij} *is the Kronecker delta.*

Definition 1.1.3 (covariant and contravariant vectors)**.** *Let* V *be a vector space of dimension* n*, let* V^* *its dual. Consider two different basis* $B_1 = \{v_1, ..., v_n\}$ *and* $B_2 = \{u_1, ..., u_n\}$ *of* V*, and let* $\Lambda = (\Lambda_{ij})_{i,j}^n$ *the change of basis matrix from* B_1 *to* B_2*. Let* $\alpha \in V$ *with coordinates* α^i *on* B_1 *and* α'^i *on* B_2*.*

Then, there are two possibilities of change of variables,

$$\alpha'^j = \sum_{i=1}^{n} \left[\Lambda^{-1}\right]_i^j \alpha^i \Big\} \textit{Contravariant relation} \tag{1.1}$$

$$\alpha'^j = \sum_{i=1}^{n} \Lambda_i^j \alpha^i \Big\} \textit{Covariant relation} \tag{1.2}$$

We say that α *is contravariant if* α *transforms as the former case and covariant as the latter.*

Example 1.1.1. *Vectors in the tangent space are contravariant and vectors in the cotangent space are covariant. Indeed, let* M *be a* n *dimensional manifold with* $\{x^1, x^2, \ldots, x^n\}$ *and* $\{y^1, y^2, \ldots, y^n\}$ *two different local coordinates of* M*. The matrix of change of coordinates from* x^μ *to the* y^μ *of* T_pM *is the Jacobian* $J = \frac{\partial(y^1,y^2,\ldots,y^n)}{\partial(x^1,x^2,\ldots,x^n)}$ *thus,*

$$\frac{\partial}{\partial y^\mu} = \sum_{\nu=1}^{n} \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu} \Big\} \textit{Contravariant relation}$$

$$dy^\mu = \sum_{\nu=1}^{n} \frac{\partial y^\mu}{\partial x^\nu} dx^\nu \Big\} \textit{Covariant relation.}$$

Definition 1.1.4. *We will adopt the Einstein summation convention, that is, when a subindex and a superindex appear repeated in an expression the sum is implicit.*

Example 1.1.2 (Einstein summation)**.** *Let* $\omega = (w^1, ...w^n) \in \mathbb{R}^n$ *and let* $B = \{e_1, ..., e_n\}$ *the canonical basis of* $\mathbb{R}^n$ *then*

$$\omega = \sum_{i=1}^{n} w^i e_i = w^i e_i.$$

Definition 1.1.5 (Tangent bundle)**.** *The tangent bundle of* M *is the union of all its tangent spaces*

T_pM, $p \in M$. The tangent bundle is denoted as $TM := \bigcup_{p\in M} T_pM$.

Example 1.1.3. *The tangent bundle is a manifold. Indeed, we can extend a chart ϕ of the nth dimensional manifold M to the chart $\bar{\phi}$ on TM by $\bar{\phi} : TM \mapsto \mathbb{R}^{2n}$ with*

$$\bar{\phi}(p, v_p) = (\phi(p), \alpha^1, \alpha^2, \ldots, \alpha^n), \quad p \in M \quad v_p \in T_pM$$

where α^i are the coefficients of v_p, i.e. $v_p = \alpha^i \partial_i$.

Example 1.1.4 (Tangent bundle of the circle S^1)**.** *Let $M = S^1$. The dimension of S^1 is one and the dimension of TS^1 is two. Write $p \in S^1$ in coordinates $\{x, y\}$. Let us first find a coordinate basis on T_pS^1. By the dimensionality of T_pM, $v_p \in T_pM$ is $v_p = \alpha \frac{\partial}{\partial r} = \alpha\partial_r$, where $\frac{\partial}{\partial r}$ corresponds to the chosen coordinate vector field.*

On S^n the stereographic projection Π defined by $\Pi(x^0, ..., x^n) = (X^1, ..., X^n)$ with

$$X_i = \frac{x_i}{1 - x_0}$$

and the inverse of Π is $\Pi^{-1}(X^1, ..., X^n) = (x^0, x^1, ..., x^n)$ with

$$x_0 = \frac{s^2 - 1}{s^2 + 1}, \quad x_i = \frac{2X_i}{s^2 + 1}, \quad i = 1, ..., n, \quad s^2 = \sum_{i=1}^{n} X_i^2.$$

In particular, for S^1, $\Pi_N(x, y) = \frac{x}{1-y}$ with inverse $\Pi_N^{-1}(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right)$.

Given a test function $f \in C^\infty(\mathbb{R}^2)$ we find that:

$$\begin{aligned}
\frac{\partial}{\partial r} f &= \frac{d}{dt}(f \circ \Pi_N^{-1}(t)) \\
&= \frac{d}{dt}\left(f\left(\frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2}\right)\right) \\
&= \underbrace{\left(\frac{2-2t^2}{(1+t^2)^2}\frac{\partial}{\partial x} + \frac{4t}{(1+t^2)^2}\frac{\partial}{\partial y}\right)}_{\partial_r} f.
\end{aligned}$$

$$\partial_r = \left(\frac{2-2t^2}{(1+t^2)^2}\frac{\partial}{\partial x} + \frac{4t}{(1+t^2)^2}\frac{\partial}{\partial y}\right). \tag{1.3}$$

On the other hand, elementary differential equations tell us that the vector field

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

has circles as integral curves where $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is the canonical coordinate basis in $\mathbb{R}^2$. Thus, we can write any tangent vector to S^1 at (x, y) as $\beta(-y\partial_x + x\partial_y)$, $\beta \in \mathbb{R}$. On the other hand, the chart give us *the parametrization* $x = \frac{2t}{t^2+1}, \quad y = \frac{t^2-1}{t^2+1}$ *and therefore, we must have*

$$\beta(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) = \underbrace{\beta\left(-\frac{t^2-1}{t^2+1}\frac{\partial}{\partial x} + \frac{2t}{t^2+1}\frac{\partial}{\partial y}\right)}_{\partial_r}.$$

$$\partial_r = \beta\left(-\frac{t^2-1}{t^2+1}\frac{\partial}{\partial x} + \frac{2t}{t^2+1}\frac{\partial}{\partial y}\right). \tag{1.4}$$

Using (1.3) *and* (1.4) *we find that*

$$\begin{aligned}\left(\frac{2-2t^2}{(1+t^2)^2}\frac{\partial}{\partial x} + \frac{4t}{(1+t^2)^2}\frac{\partial}{\partial y}\right) &= \beta\left(-\frac{t^2-1}{t^2+1}\frac{\partial}{\partial x} + \frac{2t}{t^2+1}\frac{\partial}{\partial y}\right)\\ &= \beta\frac{t^2+1}{2}\underbrace{\left(\frac{2-2t^2}{(t^2+1)^2}\frac{\partial}{\partial x} + \frac{4t}{(t^2+1)^2}\frac{\partial}{\partial y}\right)}_{\frac{\partial}{\partial r}}\\ &= \alpha\frac{\partial}{\partial r} \implies \alpha = \beta\frac{t^2+1}{2}.\end{aligned}$$

Therefore, we have

$$\bar{\phi}\big(x, y, v_{(x,y)}\big) = \left(\frac{x}{1-y}, \frac{\beta}{2}\left(1 + \frac{x^2}{(1-y)^2}\right)\right),$$

where we use the fact that $\Pi_N(x,y) = \frac{x}{1-y} = t$. *The tangent bundle of* S^1 *is trivial since*

$$\psi : S^1 \times \mathbb{R} \mapsto TS^1,$$

with $\psi(\theta, \alpha) = (\theta, \alpha X(\theta))$ *it's a diffeomorphism. In other words, as* S^1 *admits a non-vanishing global vector field its tangent bundle is trivial.*

Definition 1.1.6. *A fiber bundle is a structure* $(E,\ B,\ \pi,\ F)$, *where* E, B, *and* F *are topological spaces and* $\pi : E \longrightarrow B$ *is a continuous surjection such that the following diagram commutes.*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "proj_1"] \\ U & \end{tikzcd}$$

where $proj_1 : U \times F \longrightarrow U$ *is the natural projection and* $\varphi : \pi^{-1}(U) \longrightarrow U \times F$ *is a homeomorphism.*

For any $p \in B$, *the preimage* $\pi^{-1}(\{p\})$ *is called the fiber over* p, *the space* B *is called the base space of the bundle,* E *the total space, and* F *the fiber. The map* π *is called the bundle projection.*

Definition 1.1.7. *Let* E *be a fiber bundle over a base space* B, $\pi : E \longrightarrow B$. *A section over* E *is a continuous map,* $\sigma : B \longrightarrow E$ *such that* $\pi(\sigma(x)) = x$ *for all* $x \in B$.

Definition 1.1.8. *Let* V *ve a vector space and* V^* *its dual. A* $(k,l)-$*tensor is a multi-linear function*

$T_l^k : V^k \times (V^*)^l \to \mathbb{R}$ where $V^k \times (V^*)^l = \underbrace{V \times \ldots \times V}_{k-times} \times \underbrace{V^* \times \ldots \times V^*}_{l-times}$. We say that T_l^k is k times covariant and l times contravariant.

Given a basis $\{e_i\}_{i=1}^n$ of V and $\{e^i\}_{i=1}^n$ its dual basis, we write a tensor T as:

$$T_l^k = T_{abc\ldots}^{\alpha\beta\gamma\ldots} e^a \otimes e^b \otimes e^c \otimes \cdots \otimes e_\alpha \otimes e_\beta \otimes e_\gamma \otimes \ldots$$

For instance, a $(3,2)-$tensor can be written as:

$$T_2^3 = T_{abc}^{\alpha\beta} e^a \otimes e^b \otimes e^c \otimes e_\alpha \otimes e_\beta.$$

Example 1.1.5. *Let M be a manifold, $p \in M$. Given a basis $\{\partial_i\}_{i=1}^n$ of T_pM and $\{dx^i\}_{i=1}^n$ its dual, a (k,l)-tensor in local coordinates takes the form:*

$$T = T_{abc\ldots}^{\alpha\beta\gamma\ldots} dx^a \otimes dx^b \otimes dx^c \otimes \cdots \otimes \partial_\alpha \otimes \partial_\beta \otimes \partial_\gamma \otimes \ldots$$

For instance, a $(3,2)-$tensor can be written as:

$$T_2^3 = T_{abc}^{\alpha\beta} dx^a \otimes dx^b \otimes dx^c \otimes \partial_\alpha \otimes \partial_\beta.$$

Definition 1.1.9. *Let M be a manifold, TM its tangent bundle and T^*M its dual, a $(k,l)-$tensor bundle over M,*

$$T_l^k(M) \simeq TM^{\otimes k} \otimes T^*M^{\otimes k} \left(= \underbrace{TM \otimes \ldots \otimes TM}_{k-times} \otimes \underbrace{T^*M \otimes \ldots \otimes T^*M}_{l-times} \right)$$

is a bundle whose fibers are $T_k^l : (T_pM)^k \times (T_p^*M)^l \to \mathbb{R}$ tensors.

Definition 1.1.10. *A $(k,l)-$tensor field $\mathcal{T}_l^k(M)$ is a section of a $(k,l)-$tensor bundle, that is,*

$$\mathcal{T}\Big|_{\underbrace{T_pM \times \ldots \times T_pM}_{k-times} \times \underbrace{T_p^*M \times \ldots \times T_p^*M}_{l-times}}$$

is a $T_l^k : (T_pM)^k \times (T_p^*M)^l \to \mathbb{R}$ tensor.

Definition 1.1.11. *Let M be a manifold. A Riemannian metric*

$$g : C^\infty(TM) \otimes C^\infty(TM) \to C^\infty(M)$$

is a smooth section of $T_0^2(M)$ such that its fibers

$$g_p = g\big|_{T_pM \otimes T_pM} : T_pM \otimes T_pM \to \mathbb{R}$$

define a scalar product. A Riemannian manifold is a manifold M equipped with a Riemannian metric

g.

Let $p \in M$, in local coordinates $\{x^i\}_{i=1}^n$ the scalar product of $v, w \in T_pM$ is defined as:

$$\begin{aligned} g_p(v,w) &= g_p(v^i\partial_i, w^i\partial_i) \\ &= g_{ij}(p)v^iw^j \quad \left(= \sum_{i,j=1}^n g_{ij}(p)v^iw^j\right). \end{aligned}$$

where $g_{ij}(p) := g_p(\partial_i, \partial_j)$.

We introduce some concepts of geometry and differential calculus on Riemannian manifolds. Let $f : M \to \mathbb{R}$ be a measurable function[1] and (U, ψ) be a chart of M ($\psi(p) = (x^1(p), ..., x^n(p))$), define the integral of f on U as

$$\int_U f := \int_U f(x)\sqrt{g(x)}\, dx^1 \dots dx^n$$

where $\sqrt{g(x)} := \sqrt{\det(g_{ij}(x^1, \dots, x^n))}$.

Definition 1.1.12. *A partition of unity $\{\rho_i\}_{i\in I}$ subordinated to the atlas $\{(\psi_i, U_i)\}_{i\in I}$ is a family of continuous functions $\rho_j : U_j \to [0,1]$ such that for all j there is an i, $\operatorname{supp}(\rho_j) \subset U_i$ and*

$$\sum_j \rho_j(x) = 1.$$

Then, the integral of a measurable function $f : M \to \mathbb{R}$ defined as

$$\int_M f = \sum_i \int_{U_i} \rho_i f \tag{1.5}$$

is well-defined, and it coincides with the definition of the integral when $M = \mathbb{R}^n$.

Likewise, the length of a smooth curve can be defined. Given $\gamma : [a,b] \to M$ a smooth curve, the length of γ is given by

$$L(\gamma) := \int_a^b \left\| \frac{d\gamma(t)}{dt} \right\| dt = \int_a^b \sqrt{g(\gamma_t, \gamma_t)}\, dt.$$

In local coordinates, if γ is wholly contained in a chart $\gamma(t) \to x(\gamma(t)) = (x^1(t), \dots, x^n(t))$ then, $g(\gamma_t, \gamma_t) = g_{ij}(x(\gamma(t)))\frac{dx^i(\gamma(t))}{dt}\frac{dx^j(\gamma(t))}{dt} = g_{ij}\dot{x}^i\dot{x}^j$ and therefore,

$$L(\gamma) = \int_a^b \sqrt{g(\gamma_t, \gamma_t)}\, dt = \int_a^b \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, dt.$$

Definition 1.1.13. *Let (M, g) and (N, h) be two Riemannian manifold. An isometric embedding is a smooth embedding $f : M \to N$ such that $g = f^*h$. In other words, the metric of M and the sub manifold $M \subset N$ coincide.*

[1] with respect to the Borel σ−algebra of M

Example 1.1.6. *Let $f : M \to N$ be an isometric embedding and $p \in M$, by definition,*

$$g_p(v, w) = h_{f(p)}(df(v), df(w))$$

holds. On the other hand, for all $v = v^i \frac{\partial}{\partial x^j}$ and $w = w^i \frac{\partial}{\partial x^i} \in T_pM$, then

$$g_p(v, w) = \sum_{i,j=1}^{n} v^i w^j g_{ij}(p) = \sum_{i,j,k,l=1}^{n} \frac{\partial f^k}{\partial y^j}\bigg|_p \frac{\partial f^l}{\partial y^i}\bigg|_p v^i w^j h_{kl}(f(p)),$$

where f^i are the i-th coordinate of f. Then, the condition of f being an isometric embedding is

$$g_{ij}(p) = f_i^k f_j^l h_{kl}(f(p)),$$

where $\frac{\partial f}{\partial x^j} := f_j$.

Theorem 1.1.2 (Nash embedding theorem)**.** *Any compact Riemannian manifold (M, g) without boundary can be isometrically embedded into R^n for n big enough.*

Proof. Look at [8]. ■

A notion of distance between two points on M can be defined as follows: Given $p, q \in M$, we let

$$d(p,q) = \operatorname*{Inf}_{\gamma}\{L(\gamma) : \gamma \text{ is a piece-wise smooth function, } \gamma(a) = p, \gamma(b) = q\}. \tag{1.6}$$

Proposition 1.1.1. *Let $p, q \in M$ and $C \subseteq M$ the set of points $p, q \in M$ such that there exist a piece-wise smooth function connecting them. Then, $C = M$.*

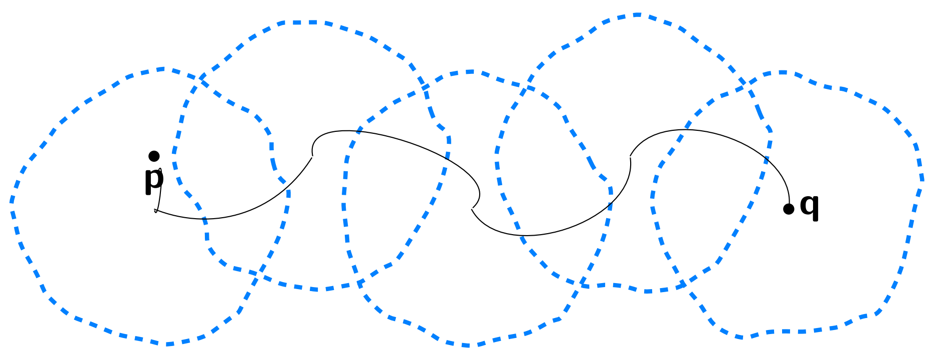

Figure 1: Open sets

Proof. By definition a manifold M is locally path connected, since M is connected, then it is path connected.

■

Proposition 1.1.2. *(M, d) defines a metric space.*

Proof. Proposition 1.1.1 tell us that d is defined on all M.

- $d(p, q) > 0$ for $p \neq q$ and $d(p, p) = 0$:

  We only have to find a lower bound on the length of any picewise smooth curve joining p and q in M. Let $p, q \in M$, $p \neq q$ and let U be an open set of M with $p, q \in U$ in which $x : U \longrightarrow \mathbb{R}^n$ is a chart. Let
  $$\overline{D}_\varepsilon(p) = \{y \in \mathbb{R}^n \ : \ ||x(p) - y|| \leq \varepsilon\}$$
  where $|| \cdot ||$ is the Euclidean norm. Choose ε such that q does not lie on $\overline{D}_\varepsilon(p)$. Since $\overline{D}_\varepsilon$ is compact and g_{ij} depends smoothly on the point p, then g_{ij} has its extremal points over $\overline{D}_\varepsilon(p)$. Therefore, there exist a $\lambda > 0$ such that for every $\eta \in \overline{D}_\varepsilon(p)$ then $g_{ij}\eta^i\eta^j \geq \lambda^2\delta_{ij}\eta^i\eta^j$. Consequently, if $\gamma([a', b']) \subseteq \overline{D}_\varepsilon(p)$,
  $$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{g(\gamma(t), \gamma(t))}\, dt \geq \int_{a'}^{b'} \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, dt \\ &\geq \lambda \underbrace{\int_{a'}^{b'} |\dot{x}|\, dt}_{\text{lenght of vector } x(\gamma(t))} \\ &\geq \lambda\varepsilon \geq 0. \end{aligned}$$

  If $p = q$ define $\gamma : [0, 1] \longrightarrow M$ as $\gamma(t) = p$, then $L(\gamma) = 0$.

- $d(p, q) = d(q, p)$:

  Let $\gamma : [a, b] \longrightarrow M$ with $\gamma(a) = p, \gamma(b) = q$ re-parametrize the curve by $\gamma(b - (t - a))$ then it's straightforward to see that the infimum of all such curves must be the same.

- $d(p, q) \leq d(p, r) + d(r, q) \quad p, q, r \in M$:

  Let γ, β be curves connecting p, r and r, q respectively. By joining them, we obtain a curve η that connects p, q and is continuous by the pasting lemma. Recall that, $\inf(f) + \inf(g) \leqslant \inf(f + g)$, therefore, taking infimum over the lengths of all γ, η gives the desired result.

∎

Definition 1.1.14. *We denote the (open) balls in M with respect to the metric d as*
$$B_\sigma(x) = \{z \in M : \ d(z, x) < \sigma\}, \quad \forall x \in M,$$
and the (open) balls in $\mathbb{R}^n$ with respect to the Euclidean norm $|| \cdot ||$ as
$$D_\sigma(x) = \{z \in \mathbb{R}^n : \ ||z - x|| < \sigma\}, \quad \forall x \in \mathbb{R}^n.$$

Proposition 1.1.3. *The topology induced by d on M coincides with the manifold topology.*

Proof. See [4]. ∎

1.2 Affine connections

Definition 1.2.1. *An affine connection is a bilinear $\nabla : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$, where $C^\infty(TM)$ are the smooth sections of the tangent bundle TM, such that for every $f, g \in C^\infty(M)$ and $X, Y, Z \in C^\infty(TM)$ the following conditions holds:*

1. $$\nabla_{fX+Y}Z = f\nabla_X Z + \nabla_Y Z, \quad f \in C^\infty(M),$$
2. $$\nabla_X(\alpha Y + Z) = \alpha \nabla_X Y + \nabla_X Z, \quad \alpha \in \mathbb{R},$$
3. $$\nabla_X(fZ) = X(f)Z + f\nabla_X Z.$$

In coordinates, let $X = \alpha^i \partial_i$ and $Y = \beta^j \partial_j$ then:

$$\nabla_X Y = \nabla_{\alpha^i \partial_i}(\beta^j \partial_j) = \alpha^i \nabla_{\partial_i}(\beta^j \partial_j) = \alpha^i \left[\partial_i(\beta^j)\partial_j + \beta^j(\nabla_{\partial_i}\partial_j)\right].$$

Definition 1.2.2. *Given local coordinates, the Christoffel symbols Γ^k_{ij} of the connection ∇ are defined as $\Gamma^k_{ij} e_k := \nabla_{e_i} e_j$.*

Therefore, the covariant derivative of Y along X in local coordinates is given by

$$\nabla_X Y = \alpha^i \partial_i(\beta^j)\partial_j + \alpha^i \beta^j \Gamma^k_{ij} \partial_k = \left[\alpha^i \partial_i(\beta^j) + \alpha^i \beta^k \Gamma^j_{ik}\right]\partial_j. \tag{1.7}$$

Example 1.2.1. *In $\mathbb{R}^3$ we can define the covariant derivative acting on a vector field ($Y = \alpha^i \partial_i$) as $\nabla_X Y := X(\alpha^i)\partial_i$.*

Let us verify the above conditions. Let $Z = \gamma^i \partial_i$,

1. $\nabla_{fX+Y}Z = [(fX+Y)\gamma^i]\partial_i = f\nabla_X Z + \nabla_Y Z,$
2. $\nabla_X(fZ) = X(f\gamma^i)\partial_i = X(f)Z + f\nabla_X Z,$
3. $\nabla_X(\alpha Y + Z) = \alpha\nabla_X Y + \nabla_X Z, \quad \alpha \in \mathbb{R}.$

Therefore, comparing with (1.7) we have that the Christoffel symbols are all zero.

Observe that an affine connection has two parts,

$$\nabla_X Y = \underbrace{\alpha^i \partial_i(\beta^j)\partial_j}_{\text{A}} + \underbrace{\alpha^i \beta^k \Gamma^j_{ik} \partial_j}_{\text{B}}.$$

The A factor is the usual derivative while the second is "how coordinates change along the vector field" due to the curvature. Indeed, we just saw that if $M = \mathbb{R}^n$ then there is no coordinate change and thus, $\Gamma_{ij}^k = 0$.

Example 1.2.2. *The Christoffel Symbols are not a* $(1,2)-$*tensor. Indeed, let* $\frac{\partial y^\mu}{\partial x^\nu}$ *the matrix change of coordinates from* y *to* x*. Then the Christoffel symbols transform as*

$$\bar{\Gamma}_{kl}^i = \frac{\partial y^i}{\partial x^m}\frac{\partial x^n}{\partial y^k}\frac{\partial x^p}{\partial y^l}\Gamma_{np}^m + \boxed{\frac{\partial^2 x^m}{\partial y^k \partial y^l}\frac{\partial y^i}{\partial x^m}},$$

consequently, the Christoffel symbols are not a tensor because of the extra term inside the rectangle.

Given a linear operator L, we can define the derivatives of linear operators by

$$\begin{aligned}\nabla_Z H :& \mathfrak{X}(M) \to \mathfrak{X}(M)\\ & X \mapsto \nabla_Z(H(X)) - H(\nabla_Z X)\end{aligned}$$

Proposition 1.2.1. *The derivative* $\nabla_Z H : \mathfrak{X}(M) \to \mathfrak{X}(M)$ *is linear.*

Proof. The only non-trivial part is to show that

$$(\nabla_Z H)fX = f(\nabla_Z H)(X), \quad f \in C^\infty(M).$$

Let $f \in C^\infty(M)$ then, we have

$$\begin{aligned}(\nabla_Z H)fX &= \nabla_Z(H(fX)) - H(\nabla_Z(fX))\\ & \nabla_Z(fH(X)) - H(f\nabla_Z X + Z(f)X)\\ &= f\nabla_Z(H(X)) + Z(f)H(X) - fH(\nabla_Z X) - Z(f)H(X)\\ &= f(\nabla_Z H)(X).\end{aligned}$$

■

One can define higher derivatives in the usual way by iterating derivatives, for example,

Example 1.2.3.

$$\begin{aligned}\nabla_Z^2 H(X) &= \nabla_Z(\nabla_Z(H(X)) - H(\nabla_Z X))\\ &= \nabla_Z(\nabla_Z(H(X))) - \nabla_Z(H(\nabla_Z X)) + H(\nabla_Z(\nabla_Z X)).\end{aligned}$$

Definition 1.2.3. *Let* (M, g) *be a Riemannian manifold. An affine connection is compatible with the metric if*

$$\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{1.8}$$

Example 1.2.4. *The covariant derivative defined in $\mathbb{R}^3$ is compatible with the Euclidean inner product.*

Definition 1.2.4. *Let*

$$[X,Y]_\nabla := \nabla_X Y - \nabla_Y X,$$

Notice that

$$\begin{aligned}[X,Y]_\nabla = \nabla_X Y - \nabla_Y X &= (\alpha^i \partial_i(\beta^j) + \alpha^i \beta^k \Gamma^j_{ik})\partial_j - (\beta^i \partial_i(\alpha^j) + \beta^i \alpha^k \Gamma^j_{ik})\partial_j \\ &= (\alpha^i \partial_i(\beta^j)\partial_j - \beta^i \partial_i(\alpha^j)\partial_j) + \alpha^i \beta^k (\Gamma^j_{ik} - \Gamma^j_{ki})\partial_j \\ &= [X,Y] + T(X,Y).\end{aligned}$$

Then, the torsion tensor $T : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is defined as $T(X,Y) = [X,Y]_\nabla - [X,Y]$. In coordinates define $\gamma^k_{ij} e_k = [e_i, e_j]$. The torsion is thus,

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} - \gamma^k_{ij}.$$

Example 1.2.5. *In $\mathbb{R}^3$ we have,*

$$\begin{aligned}[X,Y]_\nabla = \nabla_X Y - \nabla_Y X &= \alpha^i(\partial_i(\beta^j)\partial_j + \beta_j \underbrace{\nabla_{\partial_i}\partial_j}_{0 \text{ } In \text{ } \mathbb{R}^3}) - \beta^j(\partial_j(\alpha^i)\partial_i + \alpha^i \underbrace{\nabla_{\partial_j}\partial_i}_{0 \text{ } In \text{ } \mathbb{R}^3}) \\ &= \alpha^i \partial_i(\beta^j)\partial_j - \beta^i \partial_i(\alpha^j)\partial_j = XY - YX \\ &= [X,Y],\end{aligned}$$

which means that there is no torsion. This result is expected because we have:

$$T^k_{ij} = 0 = \Gamma^k_{ij} - \Gamma^k_{ji} - \gamma^k_{ij} + \underbrace{\gamma^k_{ij}}_{In \text{ } \mathbb{R}^3, \, \partial_i \text{ } conmute} \iff \Gamma_{ij} = \Gamma_{ji}.$$

Then, the torsion vanishes if and only if the Christoffel symbols are symmetric in its lower two indices.

Definition 1.2.5. *We say that a connection is torsion free if*

$$[X,Y]_\nabla = [X,Y].$$

Example 1.2.6. *If the connection is torsion free then, the Lie derivative coincides with the covariant derivative.*

$$\nabla_X Y = [X,Y] = \mathcal{L}_X Y.$$

Theorem 1.2.1 (Fundamental theorem of Riemannian geometry)**.** *Let (M,g) be a Riemannian manifold, there exist a unique affine connection which is both, compatible with the metric and torsion free. We call this connection the Levi-Civitá connection.*

Proof. Let ∇ an affine connection and $X, Y, Z \in \Gamma(TM)$, then we have

$$X\big(g(Y,Z)\big) = (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Hence we can write as

$$X\big(g(Y,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{1.9}$$

By the symmetry of g we then find:

$$\begin{aligned} X\big(g(Y,Z)\big) + Y\big(g(Z,X)\big) - Z\big(g(Y,X)\big) &= g(\nabla_X Y + \nabla_Y X, Z) \\ &\quad + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X). \end{aligned}$$

Using equation (1.9), the right-hand side is therefore equal to

$$2g(\nabla_X Y, Z) - g([X,Y],Z) + g([X,Z],Y) + g([Y,Z],X).$$

Putting all together,

$$g(Z, \nabla_Y X) \;=\; \frac{1}{2}[Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)]. \tag{1.10}$$

One can show in coordinates that equation (1.10) define the Christoffel symbols by

$$\Gamma^l_{jk} = \frac{1}{2} g^{lr} \left(\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{jk}\right), \tag{1.11}$$

and thus the affine connection is completely determined by means of equation (1.7). ■

Proposition 1.2.2. *Given the Levi-Cività connection, we can write the Christoffel symbols as*

$$\frac{1}{2} g^{mk}(2g_{ik,j} - g_{ij,k}) = \Gamma^m_{ij}. \tag{1.12}$$

Proof. By the symmetry in the m, k components, we have:

$$2g^{mk} g_{ik,l} = g^{mk} g_{ik,l} + g^{mk} g_{ik,l} = g^{mk} g_{ik,l} + g^{mk} g_{lk,i}.$$

Hence,

$$\begin{aligned} \frac{1}{2} g^{mk}(2g_{ik,j} - g_{ij,k}) &= \frac{1}{2} g^{mk}(g_{ik,j} + g_{jk,i} - g_{ij,k}) \\ &= \frac{1}{2} g^{mk}(g_{ik,j} + g_{jk,i} - g_{ij,k}) = \Gamma^m_{ij}. \end{aligned}$$

■

Definition 1.2.6. *Let $\gamma : [0,1] \to M$. The curve γ is a geodesic if*

$$\nabla_{\dot\gamma} \dot\gamma = 0,$$

where $\dot{\gamma}$ is the vector field of tangent vectors of γ. In coordinates,

$$0 = \nabla_{\dot{\gamma}}\dot{\gamma} = \dot{\gamma}^j \partial_j(\dot{\gamma}^k)\partial_k + \dot{\gamma}^j\dot{\gamma}^k\Gamma^l_{jk}\partial_l.$$

Observe that $\ddot{\gamma}^k = \frac{d\dot{\gamma}^k}{dt} = \partial_l\dot{\gamma}^k \underbrace{\frac{dx^l}{dt}}_{\dot{\gamma}^l} = \dot{\gamma}^l\partial_l\dot{\gamma}^k$ *and thus, a curve γ is geodesic if*

$$\ddot{\gamma}^k + \Gamma^k_{il}\dot{\gamma}^i\dot{\gamma}^l = 0. \tag{1.13}$$

Definition 1.2.7. *Let $\{dx^i\}_{i=1}^n$ be a basis of T_p^*M, define*

$$dx^\mu dx^\nu := \frac{1}{2}(dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu).$$

The inverse of $g_{\mu\nu}$ is defined as $(g^{-1})^{\mu\nu} = g^{\mu\nu}$ satisfying

$$g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho.$$

Let (M, g) be a Riemannian manifold. The computation of the Christoffel symbols of the Levi-Cività connection can be tedious. Fortunately for us, variational methods can make the work easier.

The metric in coordinates takes the form of $g = g_{\mu\nu}dx^\mu \otimes dx^\nu$ and by symmetry, $dx^\mu dx^\nu = dx^\mu \otimes dx^\nu$ thus, $g = g_{\mu\nu}dx^\mu dx^\nu$. The energy functional of a curve $\gamma : [a, b] \to M$, $\mathcal{E}(\gamma)$ is defined as

$$\mathcal{E}(\gamma) = \frac{1}{2}\int_a^b \mathscr{E}(\gamma)\, dt = \frac{1}{2}\int_a^b ||\dot{\gamma}(\tau)||^2\, d\tau = \frac{1}{2}\int_a^b g_{ij}\dot{x}^i\dot{x}^j\, d\tau.$$

Proposition 1.2.3. *Let $\gamma : [a, b] \to M$ be a smooth curve, then, the following inequality holds.*

$$L(\gamma)^2 \leq 2(b-a)\mathcal{E}(\gamma), \tag{1.14}$$

with equality if and only if $||\dot{\gamma}||$ is constant. Therefore, if $\mathcal{E}$ is extremized then so is $L(\gamma)$.

Proof. By the Hölder inequality we have

$$\begin{aligned} L(\gamma) = \int_a^b 1 \cdot ||\dot{\gamma}||dt &\leq \left(\int_a^b 1^2 dt\right)^{1/2} \cdot \left(\int_a^b ||\dot{\gamma}||^2 dt\right)^{1/2} \\ &\leq (b-a)^{1/2}\left(\int_a^b ||\dot{\gamma}||^2 dt\right)^{1/2} \leq \sqrt{2}(b-a)^{1/2}\mathcal{E}(\gamma)^{1/2} \end{aligned}$$

with equality if and only if $c = ||\dot{\gamma}||$ for some $c \in \mathbb{R}$. ■

Proposition 1.2.4. *Let $L(t, x, \dot{x}) = \int \mathcal{L}\, d^n x$ be a functional, then the critical points of L satisfy the*

Euler-Lagrange (E-L) equations

$$\frac{\partial \mathcal{L}}{\partial x^i}(t,x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}^i}(t,x,\dot{x}) = 0 \quad i = 1,\dots,n. \tag{1.15}$$

We can use the E-L equations to find the critical points of $\mathcal{E}$.

Proposition 1.2.5. *Let $\gamma : [a,b] \to M$ be a smooth curve in a Riemannian manifold M and let $\{x^i\}_{i=1}^n$ the coordinates of γ. Then, the critical points of the energy functional $\mathcal{E}(\gamma)$ satisfy (in a chart)*

$$\ddot{x}^m + \frac{1}{2}g^{mk}(g_{ik,l} + g_{lk,i} - g_{il,k})\dot{x}^l\dot{x}^i = 0.$$

Since $\Gamma^m_{il} := \frac{1}{2}g^{mk}(g_ik,l + g_{lk,i} - g_{il,k})$ then, γ is a critical point of $\mathcal{E}$ if γ is a geodesic; that is if satisfies

$$\ddot{x}^m + \Gamma^m_{li}\dot{x}^l\dot{x}^i = 0.$$

Proof.

Observe that

$$\frac{d}{dt}\left(\frac{\partial \mathscr{E}}{\partial \dot{x}^k}\right) = \frac{d}{dt}\left(g_{ij}\delta^i_k\dot{x}^j + g_{ij}\dot{x}^i\delta^j_k\right) = \frac{d}{dt}\left(2g_{ij}\delta^j_k\dot{x}^i\right) = 2g_{ik}\ddot{x}^i + 2\frac{\partial g_{ik}}{\partial x^l}\dot{x}^l\dot{x}^i.$$

and thus,

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathscr{E}}{\partial \dot{x}^k}\right) = 2g_{ik}\ddot{x}^i + 2\frac{\partial g_{ik}}{\partial x^l}\dot{x}^l\dot{x}^i}, \quad \boxed{\frac{\partial \mathscr{E}}{\partial x^k} = \frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j}.$$

Therefore, the E-L equation are

$$2g_{ik}\ddot{x}^i + 2\frac{\partial g_{ik}}{\partial x^l}\dot{x}^l\dot{x}^i = g_{ij,k}\dot{x}^i\dot{x}^j.$$

Multiplying by g^{km} (we make use of $g_{ij}g^{jk} = \delta^k_i$) then,

$$\begin{aligned} g^{km}(2g_{ik}\ddot{x}^i + 2\frac{\partial g_{ik}}{\partial x^l}\dot{x}^l\dot{x}^i) &= g^{km}(g_{ij,k}\dot{x}^i\dot{x}^j), \\ 2\ddot{x}^m + 2g^{mk}g_{ik,j}\dot{x}^i\dot{x}^j &= g^{mk}g_{ij,k}\dot{x}^i\dot{x}^j, \\ \ddot{x}^m + \frac{1}{2}g^{mk}(2g_{ik,j}\dot{x}^i\dot{x}^j - g_{ij,k}\dot{x}^i\dot{x}^j) = 0, \end{aligned}$$

ans we arrive at

$$\ddot{x}^m + \Gamma^m_{ij}\dot{x}^i\dot{x}^j = 0.$$

■

Hence, to find the geodesics in a Riemannian manifold, instead of minimizing the functional L we can use the better behaved functional $\mathcal{E}$.

Example 1.2.7 (Geodesics on the sphere)**.** *Let $M = S^2$. We can embed M into $\mathbb{R}^3$. The metric in $\mathbb{R}^3$ is $g = dx^2 + dy^2 + dz^2$. In spherical coordinates M is parametrized as*

$$x = r\sin\theta\cos\phi, r \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

and hence,

$$\begin{aligned} dx &= \sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi \\ dy &= \sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi \\ dz &= \cos\theta dr - r\sin\theta d\theta. \end{aligned}$$

Since r is constant, $dr = 0$ and thus,

$$dx^2 + dy^2 + dz^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Therefore, the metric of S^2 as an embedding on $\mathbb{R}^3$ is

$$g = d\theta^2 + \sin^2\theta d\phi^2.$$

We apply the E-L equations to the energy defined as $\mathscr{E}(\gamma) = \dot\theta^2 + \sin^2\theta\dot\phi^2$ where $\gamma(t) = (\theta(t), \phi(t))$. Observe first that

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathscr{E}}{\partial\dot\theta}\right) &= \frac{d}{dt}(2\dot\theta) = 2\ddot\theta, \\ \frac{\partial\mathscr{E}}{\partial\theta} &= 2\sin\theta\cos\theta\dot\phi^2, \\ \frac{d}{dt}\left(\frac{\partial\mathscr{E}}{\partial\dot\phi}\right) &= \frac{d}{dt}(2\sin^2\theta\dot\phi), \\ \frac{\partial\mathscr{E}}{\partial\theta} &= 0. \end{aligned}$$

Therefore, the E-L equations are

$$\boxed{\ddot\theta = \sin\theta\cos\theta\dot\phi^2}, \quad \boxed{\frac{d}{dt}(2\sin^2\theta\dot\phi) = 0}.$$

Hence, there is a constant J such that $\sin^2\theta\dot\phi = J$. Assume $|J| \leq 1$, then J can be written as $J = \sin\theta_0$ for some $\theta_0 \in [0, 2\pi)$. On the other hand, the energy is extremized and thus, $\mathscr{E}$ is constant. Assume $\mathscr{E} = 1$ for simplicity, which gives us the condition

$$\dot\theta^2 = 1 - \sin^2\theta\dot\phi^2 = 1 - \sin^2\theta\frac{J^2}{\sin^4\theta} = 1 - \frac{\sin^2\theta_0}{\sin^2\theta}.$$

Let us solve the equation for θ. Since

$$\frac{d\theta}{dt} = \frac{\sqrt{\sin^2\theta - \sin^2\theta_0}}{\sin\theta}\frac{d\phi}{dt} = \frac{J}{\sin^2\theta},$$

using the chain rule in $\phi(\theta)$ we have

$$\begin{aligned}
\frac{d\phi}{d\theta} &= \frac{d\phi}{dt}\frac{dt}{d\theta} = \frac{J}{\sin^2\theta}\frac{\sin\theta}{\sqrt{\sin^2\theta - \sin^2\theta_0}} = \frac{J}{\sin\theta\sqrt{\sin^2\theta - \sin^2\theta_0}} \implies \\
\phi &= \int \frac{\sin\theta_0}{\sin\theta\sqrt{\sin^2\theta - \sin^2\theta_0}}\,d\theta = -\int \frac{du}{\sqrt{\frac{1}{\sin^2\theta} - (u^2+1)}} = \arccos\left(\cot\theta\cot\theta_0\right) + \phi_0 \\
&\implies \cos(\phi - \phi_0) = \cot\theta\cot\theta_0
\end{aligned}$$

Using the double angle formula,

$$\cos(\phi - \phi_0) = \cos\phi\cos\phi_0 + \sin\phi\sin\phi_0 = \cot\theta\cot\theta_0,$$

which implies that

$$x\cos\phi_0 + y\sin\phi_0 = z\cot\phi_0.$$

That is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \\ -\cot\phi_0 \end{pmatrix} = 0$$

i.e., the curve lies in the intersection of S^2 and a plane through the origin. We conclude that geodesics in S^2 are big circles.

On the other hand, one can also use the E-L equation in order to find the Christoffel symbols of the Levi-Cività connection of a metric.

Example 1.2.8. *The metric of a manifold M in spheric coordinates can be described as $g = dr^2 + f^2(r,\theta)d\theta^2$. The Lagrangian energy associated is $\mathcal{L} = \dot{r}^2 + f^2\dot{\theta}^2$. Therefore, with the E-L equations we find:*

$$\boxed{2\ddot{r} = ff_r 2\dot{\theta}^2}, \quad \boxed{(2ff_\theta\dot{\theta} + 2ff_r\dot{r})\dot{\theta} = ff_\theta\dot{\theta}^2 + f^2\ddot{\theta}}.$$

Hence, we have

$$\ddot{r} - ff_r\dot{\theta}^2 = 0 \tag{1.16}$$

$$\ddot{\theta} - \frac{f_\theta}{f}\dot{\theta}^2 - \frac{f_r}{f}\dot{r}\dot{\theta} - \frac{f_r}{f}\dot{\theta}\dot{r} = 0 \tag{1.17}$$

which give us the Christoffel symbols:

- $\Gamma^r_{rr} = 0$
- $\Gamma^r_{\theta r} = \Gamma^r_{r\theta} = 0$
- $\Gamma^r_{\theta\theta} = -ff_r$
- $\Gamma^\theta_{\theta\theta} = -\frac{f_\theta}{f}$
- $\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = -\frac{f_r}{f}$
- $\Gamma^\theta_{rr} = 0.$

Example 1.2.9 (The flat Torus T^2). *Let $I^2 = [0,1]^2 \subset \mathbb{R}^2$. We can define the torus T^2 as the quotient $T^2 = I^2/\sim$ with the equivalence relation $\sim$ defined as*

$$(x, y) \sim (x+1, y) \sim (x, y+1), \quad x, y \in I^2.$$

Clearly T^2 is flat and therefore, $\Gamma^{\eta}_{\mu\nu} = 0$.

1.3 Curvature: Riemann tensor and Ricci tensor.

Definition 1.3.1. *Let X, Y, Z be three vector fields, the Riemann curvature tensor is defined by the action:*

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z. \tag{1.18}$$

Example 1.3.1. *Given the basis $\{\partial_i\}_{i=1}^n$ and the Levi-Cività connection, the Riemann curvature tensor has*

$$R^a_{ijk}\partial_a := R(\partial_i, \partial_j)\partial_k.$$

Observe that

$$\nabla_i \nabla_j \partial_k = \nabla_i(\Gamma^l_{jk}\partial_l) = \Gamma^l_{jk,i}\partial_l + \Gamma^l_{jk}\Gamma^m_{il}\partial_m = (\Gamma^m_{jk,i} + \Gamma^l_{jk}\Gamma^m_{il})\partial_m.$$

Hence, the Riemann tensor takes the form

$$\begin{aligned} R(\partial_i, \partial_j)\partial_k &= (\Gamma^m_{jk,i} + \Gamma^l_{jk}\Gamma^m_{il})\partial_m - (\Gamma^m_{ik,j} + \Gamma^l_{ik}\Gamma^m_{jl})\partial_m = (\Gamma^m_{jk,i} + \Gamma^l_{jk}\Gamma^m_{il} - \Gamma^m_{ik,j} - \Gamma^l_{ik}\Gamma^m_{jl})\partial_m \\ &= (\Gamma^m_{jk,i} - \Gamma^m_{ik,j} + \Gamma^l_{jk}\Gamma^m_{il} - \Gamma^l_{ik}\Gamma^m_{jl})\partial_m. \end{aligned}$$

Therefore, the components of the Riemann tensor in the matrix form are

$$R^k_{\ell ij} = \Gamma^k_{j\ell,i} - \Gamma^k_{i\ell,j} + \Gamma^k_{im}\Gamma^m_{j\ell} - \Gamma^k_{jm}\Gamma^m_{i\ell}. \tag{1.19}$$

Definition 1.3.2. *The Ricci tensor is the trace of the Riemann tensor,*

$$\mathrm{Ric}(X,Y) = \mathrm{Tr}(Z \to R(Z,Y)X). \tag{1.20}$$

In coordinates, the Ricci tensor is defined as $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$.

Example 1.3.2 (The sphere S^3). *The sphere has an induced metric*

$$g_{ij} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{bmatrix},$$

the components of Riemann tensor are

- $R^1_{212} = \sin^2\theta$
- $R^1_{221} = -\sin^2\theta$
- $R^2_{121} = 1$
- $R^2_{112} = -1,$

with the other components vanishing. Finally, the Ricci tensor takes the form

$$R_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}.$$

Example 1.3.3 (The sphere S^2)**.** *The sphere has an induced metric*

$$g_{ij} = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & r^2\sin^2\theta\cos^2\phi \end{bmatrix},$$

the components of the Riemann tensor are

- $R^1_{212} = \sin^2\theta$
- $R^1_{313} = \sin^2\theta\sin^2\phi$
- $R^2_{323} = \sin^2\phi\sin^2\theta,$

with the other components vanishing. The Ricci tensor takes the form

$$R_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2\sin^2\theta & 0 \\ 0 & 0 & 2\sin^2\phi\sin^2\theta \end{bmatrix}.$$

Example 1.3.4 (The Torus T^2)**.** *Let $T^2 \subset \mathbb{R}^3$ given by the embedding*

$$\begin{aligned} x &= (R + r\cos\theta)\cos\varphi \\ y &= (R + r\cos\theta)\sin\varphi \\ z &= r\sin\theta \end{aligned}$$

with $\theta, \varphi = [0, 2\pi)$. The metric induced by the embedding is

$$g_{ij} = \begin{bmatrix} (R + rcos\phi)^2 & 0 \\ 0 & r^2 \end{bmatrix},$$

the components of the Riemann tensor are

- $R^1_{212} = \frac{r\cos\phi}{R+t\cos\phi}$
- $R^1_{221} = -\frac{r\cos\phi}{R+t\cos\phi}$
- $R^2_{121} = \frac{1}{r}\cos\phi(R + r\cos\phi)$
- $R^2_{112} = -\frac{1}{r}\cos\phi(R + r\cos\phi),$

with the other components vanishing. The Ricci tensor takes the form

$$R_{ij} = \begin{bmatrix} \frac{1}{r}\cos\phi(R + r\cos\phi) & 0 \\ 0 & \frac{r\cos\phi}{R+t\cos\phi} \end{bmatrix}.$$

1.4 Differential operators on a Riemannian manifold

Some known definitions from vector calculus can be extended to Riemannian manifolds. For instance, let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. the gradient of f is the vector field

$$\nabla f := \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}, \quad df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

Therefore, if $X = X^i \partial_i$ is a vector field on a Riemannian manifold M then,

$$df(X) = \langle \nabla f, X \rangle. \tag{1.21}$$

In Riemannian manifolds, we can define the gradient of $f : M \to \mathbb{R}$ such that equation (1.21) holds. That is, $\boldsymbol{\nabla} f$ and df are dual to each other with respect to the inner product of the Riemannian metric. In coordinates, we have

$$\nabla f := g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}, \quad ||\nabla f||^2 = g_{\alpha\beta} g^{i\alpha} \frac{\partial f}{\partial x^i} g^{j\beta} \frac{\partial f}{\partial x^j} = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}. \tag{1.22}$$

Remark 1.4.1. *We also use the symbol ∇ to denote the affine connection, so we can think of the gradient as an extension of the affine connection to functions. Hence, the use of the same symbol.*

The divergence of a vector field is defined in the same manner. Write the vector field X as $X = X^i \partial_i$. Then, the divergence in local coordinates is

$$\nabla \cdot X := \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g} X^j) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \left\langle X, \frac{\partial}{\partial x^i} \right\rangle \right). \tag{1.23}$$

Thus, the Laplacian in a Riemannian manifold (also called the Laplace-Beltrami operator) of $f : M \to \mathbb{R} \in C^2(M)$ is defined as

$$\Delta f := \nabla \cdot \nabla f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \left\langle \nabla f, \frac{\partial}{\partial x^i} \right\rangle \right) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^i} \right). \tag{1.24}$$

Proposition 1.4.1. *The Laplace-Beltrami operator coincides with,* $\mathrm{Tr}(f \to \nabla_\mu \nabla^\nu f)$.

Proof. Let $X = X^i \partial_i$ be a vector field on M. Given a Levi-Cività connection, the covariant derivative of X^μ is

$$\nabla_\mu X^\mu = \partial_\mu X^\mu + \Gamma^\mu_{\mu\nu} X^\nu, \quad \Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\lambda} g_{\mu\lambda,\nu}.$$

On the other hand, recall the identity $\frac{d}{dt}\det(A) = \det(A)\mathrm{Tr}\left(A^{-1}\frac{dA}{dt}\right)$ therefore,

$$\begin{aligned}\frac{\partial}{\partial x^\nu}\det(g) &= \det(g)\mathrm{Tr}(g^{-1}\frac{\partial}{\partial x^\nu}g)\\ &= \det(g)g^{\mu\lambda}\frac{\partial}{\partial x^\nu}g_{\mu\lambda}\\ &= 2\det(g)\Gamma^\mu_{\mu\nu}.\end{aligned}$$

Thus, we find that

$$\begin{aligned}\frac{1}{\sqrt{\det(g)}}\partial_\nu\sqrt{\det(g)} &= \partial_\nu\ln\left(\sqrt{\det(g)}\right)\\ &= \frac{1}{2}\partial_\nu\ln\det(g) = \frac{1}{2\det(g)}2\det(g)\Gamma^\mu_{\mu\nu}\\ &= \Gamma^\mu_{\mu\nu}.\end{aligned}$$

That is

$$\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{\det(g)}}\partial_\nu\sqrt{\det(g)}. \tag{1.25}$$

Back to the affine connection, using (1.25) we have

$$\begin{aligned}\nabla_\mu X^\mu &= \partial_\mu X^\mu + \frac{1}{\sqrt{\det(g)}}\partial_\mu\sqrt{\det(g)}X^\mu\\ &= \frac{1}{\sqrt{\det(g)}}\left(\sqrt{\det(g)}\partial_\mu X^\mu + \partial_\mu\sqrt{\det(g)}X^\mu\right)\\ &= \frac{1}{\sqrt{\det g}}\partial_\mu(\sqrt{\det g}X^\mu).\end{aligned}$$

Finally, let $f \in C^\infty(M)$. Choose the vector field $X^\mu = \nabla^\mu f$,

$$\nabla_\mu\nabla^\mu f = \frac{1}{\sqrt{\det g}}\partial_\mu\left(\nabla^\mu f\sqrt{\det(g)}\right)$$

is the Laplace-Beltrami operator. ■

Proposition 1.4.2. *Let $f \in C^\infty(M)$. The Laplace-Beltrami operator in local coordinates is*

$$\Delta f = g^{ij}\left(\frac{\partial^2 f}{\partial x^i\partial x^j} + \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right). \tag{1.26}$$

Proof. Using the affine connection,

$$\nabla_\mu\nabla^\nu f = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}f + \Gamma^k_{\mu\nu}\frac{\partial}{\partial x^k}f.$$

Taking trace,

$$\Delta f = g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} + \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\right).$$

Furthermore, if $f : M \to M' \subset \mathbb{R}^n$ one can show that

$$\Delta f = g^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f^\alpha}{\partial x^k} + \Gamma'^{\alpha}_{\beta\gamma} \frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\right), \tag{1.27}$$

where $\Gamma = \Gamma(p)$ and $\Gamma' = \Gamma'(f(p))$ are the Christoffel symbols of M and M' respectively. ■

1.4.1 Geodesics and the exponential map

Theorem 1.4.2. *Let $p \in M$, $v \in T_pM$ then, there exist $\varepsilon > 0$ and precisely one geodesic*

$$c : [0, \varepsilon] \to M$$

with $c(0) = p, \dot{c}(0) = v$, and which depends smoothly on p and v.

Proof. The conditions are a *ODE* and by the Picard–Lindelöf [6] yields the local existence, uniqueness of solutions and smooth dependence of the solutions on the parameters p and v. ■

Theorem 1.4.3. *Let $p \in M$, the exponential map*

$$\begin{aligned} \exp_p : V_p &\to M, \qquad V_p := \{v \in T_pM \mid c_v \textit{defined on } [0,1]\} \\ v &\to c_v(1) \end{aligned}$$

maps a neighborhood of $0 \in T_pM$ onto a neighborhood of $p \in M$.

Proof. First notice that

$$d\exp_p(0) : T_pM \to T_0T_pM \sim T_pM.$$

a straightforward computation gives

$$\begin{aligned} d\exp_p(0)(v) &= \frac{d}{dt} c_{tv}(1)\bigg|_{t=0} = \frac{d}{dt} c_v(t)\bigg|_{t=0} = \dot{c}_v(0) \\ &= v \implies d\exp_p(0) = Id_{T_pM}. \end{aligned}$$

Hence, $d\exp(0)$ has maximal rank and by the inverse function theorem, there exist a neighborhood of $0 \in T_pM$ which is mapped diffeomorphically onto a neighborhood of $p \in M$. ■

Theorem 1.4.4. *The Riemannian normal coordinates centered at p are defined by the chart $\exp_p^{-1}$*

over U have

$$\begin{aligned} g_{ij}(0) &= \delta_{ij}, \\ g_{ij,k}(0) &= 0, \text{ and therefore, } \Gamma^i_{jk}(0) = 0 \quad \forall i,j,k \in \{1,\dots,n\}. \end{aligned}$$

Proof. Since T_pM is isomorphic to $\mathbb{R}^n$ then the Riemannian normal coordinates maps a orthonormal basis onto a Euclidean orthonormal basis. Now, let $v \in \mathbb{R}^n, t \in \mathbb{R}$ the line tv is mapped onto $c_{tv}(1) = c_v(t)$ for t small enough. Thus, the lines $x(t) = tv$ are geodesic and therefore,

$$\ddot{x}^\gamma + \Gamma^\gamma_{ij}\dot{x}^i\dot{x}^j = 0.$$

On the other hand, $\ddot{x}(t) = 0$ and consequently,

$$\Gamma^i_{jk}(tv)v^jv^k = 0, \quad i \in \{1,\dots,n\}.$$

Putting $t = 0$ and $v = \frac{1}{2}(e_l + e_m)$ gives, $\Gamma^i_{lm}(0) = 0$. By the symmetry of $g_{ij}(p)$ on 0 we have the result. ■

Definition 1.4.1. *The injectivity radius at a point x of a Riemannian manifold denoted by $\varepsilon(x) = i(x)$ given by*

$$i(x) = \sup_{r\in\mathbb{R}^+}\{r \mid \exp : B_r(x) \to M \textit{ is a diffeomorphism onto it's image}\}$$

is the largest radius for which the exponential map exp *is a diffeomorphism, where*

$$B_r(x) := \{v \in T_xM \mid g_p(v,0) < r\}.$$

The injectivity radius of a Riemannian manifold,

$$i(M) := \inf_{x\in M} i(x)$$

is the largest radius for which the exponential map $\exp : B_r(x) \to M$ *is a diffeomorphism for all* $x \in M$.

Example 1.4.1. *The injectivity radius of a sphere $\mathbb{S}^n$ is π, since* $\exp_p$ *maps the open ball of radius π in T_pM onto the complement of the antipodal point of p.*

Example 1.4.2. *The flat torus has injectivity radius of* $1/2$*, since the* $\exp_p$ *is injective in the interior of a square centered at* $0 \in T_pM$ *with side* 1*.*

Theorem 1.4.5. *Let M be a Riemannian manifold. The injectivity radius of M is positive. i.e.,* $i(M) > 0$.

Proof. See [4]. ■

Finally, we present a general version of the Arzelà–Ascoli theorem that will help us to prove the main theorem in chapter **??**.

Theorem 1.4.6 (Arzelà–Ascoli)**.** *Let M be a Riemannian manifold, suppose (M, d) is a compact metric space. Consider a sequence of continuous maps $\{g_n : M \to M'\}_{n\in\mathbb{N}}$ If the sequence is uniformly bounded and uniformly equicontinuous then, there exists a sub-sequence $\{g_{n_k} : M \to M'\}_{k\in\mathbb{N}}$ that converges uniformly.*

To prove this theorem we shall need a couple of lemmas,

Lemma 1.4.7. *Let M be a Riemannian manifold such that (M, d) is a compact metric space then, M is separable.*

Proof. Let $n \in \mathbb{N}$, the cover $\{B_{1/n}(p) : p \in M\}$ has a finite sub-cover $\{B_{1/n}(q) : q \in I_n \subset M\}$. Then, $A = \bigcup_{n=1}^{\infty} I_n$ is a countable set and for every $x \in M$ we have $d(x, a_n) < 1/n$ for some $a_n \in I_n \subset A$. Therefore, A is a dense countable subset of M. ∎

Lemma 1.4.8. *Embed M' into $\mathbb{R}^n$ for a suitable $n \in \mathbb{N}$. Let $\{g_n : M \to M'\}_{n\in\mathbb{N}}$ be a uniformly bounded sequence of functions that converges uniformly to $G : M \to \mathbb{R}^n$. Then, the image of G also lies within M'. That is, $G : M \to M' \subset \mathbb{R}^n$.*

Proof. Let $p \in M$, the sequence $\{g_n(p)\}_{n\in\mathbb{N}} \subset M'$ is uniformly bounded and therefore, there exist a ball $\overline{B_r}(p) \subseteq M'$ such that $\{g_n(p)\}_{n\in\mathbb{N}} \subseteq \overline{B_r}(p) \subseteq M' \subset \mathbb{R}^n$. Consequently, $G(p) \in \overline{\{g_n(p)\}_{n\in\mathbb{N}}} \subseteq \overline{B_r}(p) \subseteq M'$. ∎

Proof of theorem 1.4.6. By the Nash embedding theorem we can assume that $g_n : \mathbb{R}^n \to \mathbb{R}^m$. Because of lemma 1.4.8, if a subsequence converges uniformly then the limit function also lies within M'. Therefore, it is enough to prove that a uniformly convergent subsequence of $\{g_n\}_{n\in\mathbb{N}}$ exists.

Denote each coordinate of g as g^i, that is, $g_n(p) = (g_n^1(p), ..., g_n^m(p))$. Let $f_n := g_n^i$, the construction below tell us that each coordinate sequence $\{g_n^i\}_{n\in\mathbb{R}}$ has a uniform convergent subsequence.

Let $A = \{a_n\}_{n\in\mathbb{N}}$ be a countable dense set of M. Let $I_0 \subset \mathbb{N}$ infinite such that $\{f_b(a_0)\}_{b\in I_0}$ is a convergent subsequence and define $I_{i+1} \subset I_i$ infinite such that $\{f_b(a_{i+1})\}_{b\in I_{i+1}}$ is a convergent subsequence. The sets $I_i = 0, 1, ...$ exist because f_n is uniformly bounded and in this case, $\{f_n(a_i)\}_{n\in\mathbb{N}}$, $i \in \mathbb{N}$ is bounded, and it has a convergent subsequence. Let $I = \{n_k\}_{k\in\mathbb{N}}$ with n_k defined inductively as $n_0 \in I_0$ and $n_k \in I_k \setminus \{n_l : l < k\}$, $k > 1$. Notice that $I \setminus I_k = \{n_l : l < k\}$ is finite and thus, I coincides (up to finite elements) with a subsequence of I_k for all k. Then, the sequence $\{f_b(a_i)\}_{b\in I}$ converges for all $i \in \mathbb{N}$ and is Cauchy.

Let $\varepsilon > 0$, $x \in M$ and $a_i \in A$. Since f_n is equicontinuos there exist a $\delta > 0$ such that if $|x - a_i| < \delta$ then $|f_b(x) - f_b(a_i)| < \varepsilon/3$ for every $b \in I$. Since A is dense, there exist $a_i \in A$ such that $x \in B_\delta(a_i)$

and thus

$$\begin{aligned}|f_b(x) - f_d(x)| &\leq |f_b(x) - f_b(a_i)| + |f_b(a_i) - f_d(a_i)| + |f_d(a_i) - f_d(x)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + |f_b(a_i) - f_d(a_i)|.\end{aligned}$$

Since $\{f_b(a_i)\}_{b\in I}$ is Cauchy, choose $N > 0$ such that if $b, d > N$ then $|f_b(a_i)-f_d(a_i)| < \varepsilon/3$. Consequently, for every $x \in M$, there exists $N > 0$ such that if $b, d \in I$ and $b, d > N$ then

$$|f_b(x) - f_d(x)| \leq \varepsilon,$$

that is, $\{f_b(x)\}_{b\in I}$ is uniformly Cauchy and thus converges uniformly. This proves that every coordinate $\{g^i_n\}_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

Finally, given that every each coordinate sequence $\{g^i_n\}_{n\in\mathbb{R}}$ has a uniformly convergent subsequence, define $B_1 \subset \mathbb{N}$ be an infinite set such that $\{g^1_a\}_{a\in B_1}$ is uniformly convergent. Define inductively $B_{i+1} \subset B_i$, $i = 0, 1, ..., m-1$ infinite such that $\{g^{i+1}_a\}_{a\in B_{i+1}}$ is uniformly convergent. Since B_m is a subsequence of all B_i, $i = 1, ..., m-1$ then, all $\{g^i_a\}_{a\in B_m}$ are uniformly convergent and thus, so is $\{g_n\}_{n\in B_m}$. ■

2 The Heat Equation

Historically, the study of partial differential equations (PDE's) was developed along with the study of classical mechanics, in particular, the study of the heat equation, wave equation, Laplace equation, etc., enriched mathematics and physics.

The study of the heat equation was first introduced in the Euclidean space, in which it takes the form

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \tag{2.1}$$

On the other hand, in Riemannian manifolds, as a generalization of the Euclidean space, these PDEs can be naturally extended, and therefore, the study of the heat equation can be addressed in this context.

We must highlight that we want a coordinate-free definition of these equations, and therefore, our discussion of what the heat equation must be in terms of an invariant formulation. Indeed, we can see that equation (2.1) can be restated in terms of a differential operator D as

$$Du = 0, \quad D = \Delta - \frac{\partial}{\partial t}, \tag{2.2}$$

where $\Delta = \sum_i^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in the Euclidean space $\mathbb{R}^n$.Naturally, the differential operators on a Riemannian manifold lead us to a formulation in which Δ must be the Laplace-Beltrami operator. Hence, equation (2.2) is well-defined on a Riemannian (or Lorentzian) manifold. It is the purpose of this chapter to study some properties of the heat equation within the framework of a Riemannian manifold, assuming several theorems of the Euclidean case. We follow several references such as [**enc**, 3, 4, 1, 9]. Nevertheless, we refer the reader to [5, 10, 11, 12] for a detailed discussion of the Euclidean case.

2.1 The Heat equation on Manifolds

Consider a Riemannian manifold M of dimension n with Riemannian normal coordinates $\{x^i\}_{i=1}^n$ centered at $p \in M$. We saw in section 1.4 that the Laplace-Beltrami operator is

$$\Delta f(x) = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial f}{\partial x^i} \right) = g^{ij} \left(f_{ij} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right).$$

Therefore, the heat equation in local coordinates is

$$\left(\frac{\partial}{\partial t} - g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial}{\partial x^k}\right)\right) f = 0. \tag{2.3}$$

In particular, if we choose Riemannian normal coordinates centered at $p \in M$,

$$\Delta f(p) = \delta^{ij}\left(f_{ij}(p) - \Gamma^k_{ij}(p)\frac{\partial f}{\partial x^k}\bigg|_p\right) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}\bigg|_p.$$

Hence, equation (2.1) is recovered when Riemannian normal coordinates centered at p are introduced, although only at one point.

Notice also that if we consider the case of a Lorentzian manifold with signature $(1,3)$, and normal coordinates centered at p, the Laplace-Beltrami operator under coordinates $(x_0, x_1, ..., x_3)$ takes the form of

$$\Delta = \frac{\partial^2}{\partial x_0^2}\bigg|_p - \sum_{i=1}^n \frac{\partial^2}{(\partial x_i)^2}\bigg|_p,$$

which leads to the heat equation (only at p)

$$\left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x_0^2} - \sum_{i=1}^n \frac{\partial^2}{(\partial x_i)^2}\right) u(p,t) = 0. \tag{2.4}$$

A classical approach to the study of PDEs of this type is the study of the kernel or fundamental solution of the equation whose explicit form allows us to construct solutions and study their properties. In the case of the heat equation, one way of constructing this fundamental solution is via the parametrix method.

Eells and Sampson in [1] use the parametrix method to construct solutions for the heat equation type system. However, determining its properties requires computations that get pretty messy rather sooner than later (as we will see in the construction of the parametrix). Therefore, we depart from the approach of Eells and Sampson in the sense that we only use the parametrix method to prove existence. On the other hand, we introduce a practical idea to find a priori properties to prove continuity, differentiability, and smoothness; this method is the so-called Bernstein method.

2.2 An explicit solution of the Heat equation

The Duhamel principle tells us that a solution to the system

$$\begin{cases} \left(\Delta - \dfrac{\partial}{\partial t}\right) u(t,x) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(x,0) = \varphi(x) & \varphi : \mathbb{R}^n \to \mathbb{R} \end{cases} \tag{2.5}$$

is given by

$$u(t,x) = \int_{R^n} G(t, x-y)\varphi(y)dy, \quad G(t,x) = \frac{1}{(2\sqrt{\pi \cdot t})^n} \cdot e^{-\frac{|x|^2}{4t}}, \tag{2.6}$$

where G is the heat kernel. In the same fashion, we will construct a kernel G on the Riemannian case.

Notice that the heat kernel on equation (2.6) is defined globally. On the other hand, the kernel of equation (2.3) is chart dependent. To construct the heat kernel in a Riemannian manifold we will find a series of approximations. This procedure is known as the parametrix method for the heat equation.

Definition 2.2.1 (Parametrix of the Heat equation)**.** *A parametrix of the heat operator is a function* $H(t,x,y) \in C^\infty(\mathbb{R}^+ \times M \times M)$ *which satisfies the following conditions*

$$(\partial_t - \Delta)H \in C^0(\mathbb{R}^+ \cup \{0\} \times M \times M) \tag{2.7}$$

$$\lim_{t' \to 0} \int_M H(t', x, y) f(y)\, dy = f(x), \quad f : M \to \mathbb{R}. \tag{2.8}$$

2.2.1 The heat kernel

In order to construct the heat kernel we follow [9].

Let M be a Riemannian manifold, $x \in M$. The inyectivity radius $i(x)$ is a smooth function and therefore, if y is within a neighborhood V_x of x with $\exp_x(B_i(0)) = V_x$ we denote by $r(x,y) := d(x,y)$ the distance between x and y. Therefore, define U_i as

$$U_i := \{(x,y) \subset M \times M : y \in V_x, r(x,y) < i(x)\},$$

then, the map

$$G(t,x,y) \equiv (4\pi t)^{-\frac{n}{2}} e^{-\frac{r^2(x,y)}{4t}} \quad t \in \mathbb{R}^+, (x,y) \in U_i$$

is smooth $(G \in C^\infty(\mathbb{R}^+ \times U_{i(x)}))$.

Lemma 2.2.1. *Let* $f, g \in C^\infty(M)$ *Then*

$$\Delta(fg) = (\Delta f)g + 2\langle df, dg\rangle + f\Delta g.$$

Proof.

$$\begin{aligned}\Delta(fg) &= g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial}{\partial x^k}\right) fg \\ &= g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} g + \frac{\partial^2 g}{\partial x^i \partial x^j} f + 2\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k} g - \Gamma^k_{ij}\frac{\partial g}{\partial x^k} f\right) \\ &= (\Delta f)g + f(\Delta g) + 2\underbrace{g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}}_{\langle df, dg\rangle}.\end{aligned}$$

■

To define a sequence approximating the heat kernel, we will use a family of functions S_k on U_ε defined as

$$S_k(t,x,y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{r^2(x,y)}{4t}} \left(u_0(x,y) + \ldots + u_k(x,y)t^k\right), \quad k \in \mathbb{Z}^+,$$

where $u_j \in C^\infty(U_\varepsilon)$, $j = 0, 1, \ldots, k$ are smooth unknown functions. Introducing S_k on equation (2.3) we must have

$$\frac{\partial S_k}{\partial t} = G(t,x,y)\left[\left(-\frac{n}{2t} + \frac{r^2}{4t^2}\right)\left(u_0 + \ldots + t^k u_k\right) + \left(u_1 + 2u_2 t + \ldots + k u_k t^{k-1}\right)\right] \tag{2.9}$$

and, by lemma 2.2.1,

$$\Delta_y S = (\Delta G)\left(u_0 + \ldots + u_k t^k\right) - 2\left\langle dG, d\left(u_0 + \ldots + u_k t^k\right)\right\rangle + G\Delta\left(u_0 + \ldots + u_k t^k\right), \tag{2.10}$$

with

$$\begin{aligned}\left\langle dG, d\left(u_0 + \ldots + u_k t^k\right)\right\rangle &= \left\langle \frac{\partial G}{\partial r} dr + \frac{\partial G}{\partial \theta} d\theta, d\left(u_0 + \ldots + u_k t^k\right)\right\rangle \\ &= \left\langle \frac{\partial G}{\partial r} dr, \frac{\partial u_0}{\partial r} dr + \frac{\partial u_0}{\partial \theta} d\theta + \ldots + t^k \frac{\partial u_k}{\partial r} dr + t^k \frac{\partial u_k}{\partial \theta} d\theta\right\rangle \\ &= \left\langle \frac{\partial G}{\partial r} dr, \frac{\partial u_0}{\partial r} dr + \ldots + t^k \frac{\partial u_k}{\partial r} dr\right\rangle \\ &= \frac{\partial G}{\partial r}\left(\frac{\partial u_0}{\partial r} + \ldots + t^k \frac{\partial u_k}{\partial r}\right) \\ &= -\frac{r}{2t}\left(\frac{\partial u_0}{\partial r} + \ldots + t^k \frac{\partial u_k}{\partial r}\right) G.\end{aligned}$$

Theorem 2.2.2. *Let r be the radial coordinate of the exponential polar coordinates with center at p. Consider $T = \partial_r$, $D = \det\left(d\exp_p\right)$ and let Δ_S be the Laplacian over the sphere S of radius r_0. Then, if $q \in S$ and $f \in C^\infty(M)$, we have*

$$\Delta f(q) = -\nabla_T \nabla_T f(q) + \Delta_S f(q) - \left(\frac{n-1}{r} + D^{-1}\nabla_T D\right)\nabla_T f(q).$$

Proof. It is a routinary application of polar coordinates. See [9]. ■

Therefore, in normal coordinates ($\nabla_T = \partial_r$)

$$\Delta G = -\frac{\partial^2 G}{\partial r^2} - \left(\frac{n-1}{r} + D^{-1}\partial_r D\right)\frac{\partial G}{\partial r} = \left(\frac{n}{2t} - \frac{r^2}{4t^2}\right)G + \frac{r}{2t}\frac{D'}{D}G.$$

Since S_k must satisfy (2.9) and (2.10) we have

$$\left(\partial_t - \Delta_y\right) S_k = G(u_1 + \ldots + kt^{k-1}u_k + \frac{r}{2t}\frac{D'}{D}\left(u_0 + \ldots + t^k u_k\right) \tag{2.11}$$

$$+\frac{r}{t}\left(\frac{\partial u_0}{\partial r} + \ldots + t^k\frac{\partial u_k}{\partial r}\right) + \Delta_y u_0 + \ldots + t^k \Delta_y u_k\bigg). \tag{2.12}$$

Now, if we fix the following conditions for the u_i,

$$r\frac{\partial u_0}{\partial r} + \frac{r}{2}\frac{D'}{D}u_0 = 0 \implies u_0 = kD^{-\frac{1}{2}} \tag{2.13}$$

$$r\frac{\partial u_i}{\partial r} + \left(\frac{r}{2}\frac{D'}{D} + i\right)u_i + \Delta_y u_{i-1} = 0, \quad i = 1, \ldots, k, \tag{2.14}$$

then, equation (2.12) becomes

$$\left(\partial_t + \Delta\right) S_k = Gt^k \Delta_y u_k(x, y),$$

which is not exactly (2.3) but is "good enough" (content of theorem 2.2.4).

Next, we find S_k. In order to determine $u_i, i = 1, ..., k$ we first solve

$$r\frac{\partial u_i}{\partial r} + \left(\frac{r}{2}\frac{D'}{D} + i\right)u_i = 0 \implies u_i = k(\theta, r)r^{-i}D^{-1/2}.$$

Hence, equation (2.14) under the u_i chosen as above is equivalent to

$$\frac{\partial k}{\partial r} = -D^{1/2}\left(\Delta u_{i-1}\right) r^{i-1}.$$

Therefore, integrating along a geodesic $\gamma(s)$ that connects x and y yields

$$u_i(x, y) = -r^{-i}(x, y)D^{-\frac{1}{2}}(y)\int_0^r D^{\frac{1}{2}}(\gamma(s))\Delta_y u_{i-1}(\gamma(s), y)s^{i-1}ds.$$

In this way, it is possible to define the $\{u_i \in C^\infty(U_\varepsilon)\}_{i=0}^k$ inductively with $u_0 = kD^{-\frac{1}{2}}$ and u_{i+1}, $i = 1, ..., k-1$ as above.

As is usual, to extend this family $\{u_i\}$ along with G to the whole manifold we define the bump function $\eta \in C^\infty(M \times M)$ as

$$\eta(x,y) = \begin{cases} 0 & (x,y) \in U_\varepsilon^c \\ 1 & (x,y) \in U_{\varepsilon/2} \\ \xi(x,y) & M \times M \backslash (U_\varepsilon^C \cup U_{\varepsilon/2}) \end{cases}.$$

where $\xi(x,y)$ connects smoothly these two regions. Therefore, it is possible to extend S_k onto $M \times M$ through $H_k = \eta S_k \in C^\infty(\mathbb{R}^+ \times M \times M)$.

Lemma 2.2.3. *$H_k(t,x,y) \in C^\infty(\mathbb{R}^+ \times M \times M), k > n/2$ is a parametrix of the equation* (2.3).

Proof. In order to see that condition (2.7) holds, we look at the regions $U_{\varepsilon/2}, U_\varepsilon \backslash U_{\varepsilon/2}$ and U_ε^c.

Clearly, on $\mathbb{R}^+ \times U_\varepsilon^c$ the function H_k vanishes and therefore, $(\partial_t - \Delta_y)H_k = 0$. In $\mathbb{R}^+ \times U_\varepsilon/2$ we have

$$(\partial_t - \Delta_y)H_k = (\partial_t - \Delta_y)S_k = \frac{1}{(4\pi t)^{n/2}} t^k e^{-\frac{x^2}{4t}} \Delta u_k \underbrace{\rightarrow}_{t \to 0} 0.$$

Finally on $\mathbb{R}^+ \times U_\varepsilon \backslash U_{\varepsilon/2}$

$$\begin{aligned}(\partial_t - \Delta_y)H_k &= \eta \partial_t S_k - \underbrace{\Delta_y(\eta S_k)}_{\text{Lemma}} = \eta(\partial_t - \Delta_y)S_k - 2\langle d\eta, dS_k\rangle + (\Delta_y \eta)S_k \\ &= \frac{1}{(4\pi t)^{n/2}} e^{-\frac{x^2}{4t}} \phi(t,x,y)\end{aligned}$$

where $\phi \in C^\infty(\mathbb{R}^+ \times M \times M)$ is $\xi \cdot G$.

Let us take a look at the second condition (2.8). Condition (2.8) means that

$$\lim_{t \to 0} \int_M \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \eta(x,y) \underbrace{(u_0 + \cdots + t^k u_k(x,y))}_{h(t,x,y)} f(y)\, dy = f(x).$$

Notice that

$$\begin{aligned} I(x) &= \lim_{t \to 0} \int_M \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \eta(x,y) u_i(x,y) f(y)\, dy \\ &= \lim_{t \to 0} \int_{B_{\varepsilon/2}(x)} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \eta(x,y) u_i(x,y) f(y)\, dy \\ &\quad + \lim_{t \to 0} \int_{B_{\varepsilon/2}(x)^c} \frac{1}{(4\pi t)^n/2} e^{-\frac{r^2}{4t}} \eta(x,y) u_i(x,y) f(y)\, dy, \end{aligned}$$

where the second integral tends to 0 as $t \to 0$. To understand the behavior of the first integral we perform a change of variables using the exponential map (as normal coordinates) and, therefore, it can be worked out as in the Euclidean case. That is,

$$\begin{aligned}
I(x) &= \lim_{t\to 0}\int_{B_{\varepsilon/2}(x)} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}}\eta(x,y)u_i(x,y)f(y)\,dy \\
&= \lim_{t\to 0}\int_{B_{\varepsilon/2}(0)\subset T_xM} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2(0,v)}{4t}} u_i(x, exp_x v)f(exp_x v)D(v)\,dv^1\ldots dv^n \\
&= \lim_{t\to 0}\int_{T_xM} \underbrace{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2(0,v)}{4t}}}_{\text{Heat kernel}} u_i(x, exp_x v)f(exp_x v)D(v)\,dv^1\ldots dv^n \\
&= \lim_{t\to 0}\int_{T_xM} G(t, x-y)u_i(x, exp_x v)f(exp_x v)D(v)\,d^n v,
\end{aligned}$$

where we extended u_i outside of $B_{\varepsilon/2}$ as $u_i = 0$ and G is the function defined on (2.6). Since $T_xM \simeq \mathbb{R}^n$, we use the usual properties of the heat kernel in $\mathbb{R}^n$ (see [11] for example), and therefore

$$\begin{aligned}
\lim_{t\to 0}\int_{T_xM} G(t, x-y)u_i(x, exp_x v)f(exp_x v)D(v)\,d^n v &= u_i(x \exp_x 0)f(\exp_x 0)D(0) \\
&= u_i(x,x)f(x)
\end{aligned}$$

uniformly, where we used the fact that $D(0) = 1$ and hence,

$$\lim_{t\to 0}\int_M \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}}\eta(x,y)u_0(x,y)f(y)\,dy = f(x),$$

$$\lim_{t\to 0}\int_M \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}}\eta(x,y)t^i u_i(x,y)f(y)\,dy = 0, \quad i > 0.$$

Finally, if we add everything up, in $h(t,x,y)$ we have

$$\begin{aligned}
\lim_{t\to 0}\int_M H_k(t,x,y)f(y)\,dy &= \lim_{t\to 0}\int_M \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \\
&\quad \times \eta(x,y)(u_0 + \cdots + t^k u_k(x,y))f(y)\,dy \\
&= f(x).
\end{aligned}$$

■

The existence of the heat kernel is the result of the following theorem. Let $K_k = (\partial_t - \Delta_y)H_k$. and let $Q_k = \sum_{\lambda=1}^{\infty}(-1)^{\lambda+1}K_k^{*\lambda}$ where $K_k^{*\lambda}$ is the convolution of K_k with itself λ-times.

Theorem 2.2.4. *For $k > 2 + \frac{n}{2}$ let $e(t,x,y) = H_k(t,x,y) - Q_k * H_k(t,x,y)$, i.e. ,*

$$e(t,x,y) = H_k(t,x,y) + \left[H_k * \sum_{\lambda=1}^{\infty}(-1)^{\lambda}K^{*\lambda}\right](t,x,y). \tag{2.15}$$

Then, $e(t,x,y) \in C^\infty(\mathbb{R}^+ \times M \times M)$ is independent of k and is the kernel of the heat equation.

Proof. See [9]. ■

Observe that the kernel is generated using the parametrix H_k and, therefore, expression (2.15) of the kernel can be approximated by a truncated series.

We have thus proven the existence of the heat kernel on a Riemannian manifold and therefore, we have the following result for the heat equation.

Corollary 2.2.4.1. *Let M be a Riemannian manifold and let $F : M \to \mathbb{R}$ be a smooth function. The initial value problem (IVP)*

$$\begin{cases} \Delta f(p,t) = \dfrac{\partial}{\partial t} f(p,t), \quad (p,t) \in M \times (0,T), \\ f(p,0) = F(p), \end{cases} \tag{2.16}$$

has a unique smooth solution $f : M \times (0,\infty) \to \mathbb{R}$.

Proof. It follows from the last result and the classical properties of kernels. ■

Corollary 2.2.4.2. *Let M be a Riemannian manifold and let $F : M \to \mathbb{R}^n$ be a smooth function. The initial value problem (IVP)*

$$\begin{cases} \Delta f^\alpha = \dfrac{\partial}{\partial t} f^\alpha, \quad f : M \times (0,T) \to \mathbb{R}^n,\ (p,t) \in M \times (0,T) \\ f(p,0) = F(p) \end{cases} \tag{2.17}$$

has a unique smooth solution $f : M \times (0,\infty) \to \mathbb{R}^n$.

Proof. The system is diagonal and each coordinate has solution by the above corollary. ■

2.2.2 The Bernstein method

Theorem 2.2.5. *Let M be a Riemannian manifold and consider $u : M \to \mathbb{R}$ a smooth function. The solution of the Initial Value Problem (IVP) of the heat equation associated with u,*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u \qquad \textit{in } M \times (0,T) \\ u(x,0) = u(x), \quad x \in M \end{cases} \tag{2.18}$$

have bounded derivatives on any compact subinterval $[\varepsilon, T-\delta]$ of $(0,T)$ with $\varepsilon, \delta > 0$, and the bounds only depend on u and on ε.

Proof. Let $G : M \to \mathbb{R}$ be the auxiliary function defined by

$$G := At\|\nabla u\|^2 + Bu^2, \quad \text{where } \|\nabla u\|^2 = g(\nabla u, \nabla u).$$

The function G allow us to establish a priori estimates of the maximum modulus of the derivatives of this solution of the required order (one and two).

Let us compute the Laplace operator one term at a time. For $||\nabla u||^2$ we have

$$\begin{aligned}\Delta||\nabla u||^2 &= -\nabla\cdot\nabla||\nabla u||^2 = -\nabla\cdot\nabla(g_{ij}(\nabla u)^i(\nabla u)^j)\\ &= -\nabla\cdot\nabla(g_{ij})(\nabla u)^i(\nabla u)^j - g_{ij}\nabla\cdot\nabla((\nabla u)^i(\nabla u)^j)\\ &= R(\nabla u,\nabla u) + 2\langle\nabla u,\nabla\Delta u\rangle + 2\langle\nabla^2 u,\nabla^2 u\rangle,\end{aligned}$$

where R is the Ricci tensor. For u^2 we have

$$\begin{aligned}\Delta u^2 &= -\nabla\cdot\nabla(u^2) = -\nabla\cdot(2u\nabla u) = -2\nabla u\nabla u - 2u\nabla\cdot\nabla u\\ &= -2||\nabla u||^2 + 2\langle u,\Delta u\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\Delta G &= At\Delta\langle\nabla u,\nabla u\rangle + B\Delta(u^2)\\ &= At\cdot R(\nabla u,\nabla u) + 2At\langle\nabla u,\nabla\Delta u\rangle + 2At\langle\nabla^2 u,\nabla^2 u\rangle - B2||\nabla u||^2 + 2B\langle u,\Delta u\rangle.\end{aligned}$$

Regarding the time derivative of G we compute

$$\begin{aligned}\frac{\partial G}{\partial t} &= A||\nabla u||^2 + At\frac{\partial}{\partial t}\langle\nabla u,\nabla u\rangle + 2Buu_t\\ &= A||\nabla u||^2 + At\left(\left\langle\nabla\underbrace{\frac{\partial u}{\partial t}}_{\Delta u},\nabla u\right\rangle + \left\langle\nabla u,\nabla\frac{\partial u}{\partial t}\right\rangle\right) + 2Buu_t\\ &= A||\nabla u||^2 + 2At\langle\nabla\Delta u,\nabla u\rangle + 2Buu_t\\ &= A||\nabla u||^2 + 2At\langle\nabla\Delta u,\nabla u\rangle + 2B\langle u,\Delta u\rangle.\end{aligned}$$

Therefore,

$$\frac{\partial G}{\partial t} = \Delta G - A\cdot t\cdot R(\nabla u,\nabla u) + (B2+A)||\nabla u||^2 - 2At\langle\nabla^2 u,\nabla^2 u\rangle.$$

Choose $A=-2B$ then, if M has negative curvature ($R<0$) we have

$$\frac{\partial G}{\partial t} \leq \Delta G \tag{2.19}$$

because the Hessian term $\langle\nabla^2 u,\nabla^2 u\rangle$ is non-negative.

Now, equation (2.19) tell us that G is a decreasing function in t which means that

$$G(p,t) = At||\nabla u||^2 + Bu^2 \leq C,$$

and since u^2 is always positive, $At||\nabla u||^2 \leq C'$, i.e.

$$||\nabla u||^2 \leq \frac{K}{t}, \tag{2.20}$$

on $(0, T)$. Hence, the first derivative is bounded. Finding the bounds on the second order derivatives is completely analogue using the auxiliary function $G : M \to \mathbb{R}$ defined by

$$G := At||\nabla^2 u||^2 + B||\nabla u||^2.$$

Notice that high order time derivatives can be bounded because a nth order derivative can be decomposed using the equivalence $\partial_t u = \Delta u$ into spatial derivatives which are all bounded by induction.

$$\frac{\partial^{2k}}{\partial x^{2k}} u = -\frac{\partial^{2(k-1)}}{\partial x^{n-2}\partial t} u = ... = (-1)^k \frac{\partial^k}{\partial t^k} u,$$
$$\frac{\partial^{2k+1}}{\partial x^{2k+1}} u = \frac{\partial}{\partial x}(-1)^k \frac{\partial^k}{\partial t^k} u = (-1)^k \frac{\partial^{k+1}}{\partial x \partial t^k} u.$$

■

Corollary 2.2.5.1 (Maximum principle)**.** *The following inequality holds,*

$$\sup_{x\in M,\, t\in[0,T]} ||u(x,t)|| \leq \sup_x ||u(x,0)||$$

and the system (2.18) *has unique solution.*

Proof. For the first part choose the auxiliary function $G(x) = \frac{1}{2}u^2$. The result is straight forward from equation (2.19).

For uniqueness assume that u, u' are two solutions of (2.18). Then $u - u'$ is a solution of

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v \\ v(x,0) = 0, \quad x \in M \end{cases} \tag{2.21}$$

thus,

$$\sup_{x\in M,\, t\in[0,T]} ||u'(x,t) - u(x,t)|| \leq \sup_x 0,$$

which means that $u' = u$. ■

Corollary 2.2.5.2. *Let M be a Riemannian manifold. Let $\{f^i : M \to \mathbb{R}\}_{i=1}^n$ be a set of smooth functions. The IVP*

$$\begin{cases} \dfrac{\partial u^i}{\partial t} = \Delta u^i & in\ M \times (0,\infty) \\ u^i(x,0) = u^i(x), & x \in M \end{cases} \tag{2.22}$$

has unique smooth solutions.

Proof. The diagonal system has solutions by corollary 2.2.4.2. Each u^i is unique by the above corollary and all u^i are smooth by the theorem 2.2.5. ∎

2.2.3 An a side: Lorentzian manifolds and the Heat equation

Just as the Riemmanian manifolds are a generalization of the Euclidean space $\mathbb{R}^n$ so are Lorentizian manifolds to the Minkowski space-time $\mathbb{R}^{1,3}$.

The metric on a Lorentzian manifold is given by the element of line $ds^2 = N(x)^2dt^2 - G_{ab}(x)dx^a dx^b$. We call N as the "lapse" function and G the metric on the spacelike hypersurface with constant t. The metric is static if the lapse function and the metric G are independent of t.

In physics, specially in a Minkowski spacetime ($g = (1, -1, \dots, -1)$, c=1) it is customary to work with the D'Alembertian $\square$ defined as

$$\square = g^{\mu\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu} = \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x^2} - \frac{\partial}{\partial y^2} - \frac{\partial}{\partial z^2}$$

Remark 2.2.6. *The d'Alembertian is exactly the Laplacian operator on a Minkowski space time and therefore the Lapalce-Beltrami operator allow us to write the d'Alambertian in curved spacetimes as*

$$\Delta = -\frac{1}{\sqrt{-g}}\partial_{x^i}\left(\sqrt{-g}g^{ij}\partial_{x^j}\right).$$

Hence, on a Lorentzian manifold, the Heat equation takes the form of

$$(\partial_s\varphi + \square\varphi) = 0, \tag{2.23}$$

where $\square$ is the L-B operator.

Remark 2.2.7. $\sqrt{-g} = \sqrt{N(x)\prod_i G_{ii}} = \sqrt{N}\sqrt{G}$ *and* $\square\varphi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi)$ *Hence, the Heat equation gives*

$$\left(\partial_s\varphi + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi)\right) = 0,$$

$$\partial_s\varphi = g^{ij}\left(\varphi_{ij} - \Gamma^k_{ij}\frac{\partial\varphi}{\partial x^k}\right).$$

In a static space time, we have

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) = \frac{1}{\sqrt{N}\sqrt{G}}\partial_t(\sqrt{N}\sqrt{G}N(x)\partial_t\varphi)$$
$$+ \frac{1}{\sqrt{N}\sqrt{G}}\partial_a(\sqrt{N}\sqrt{G}(-G^{ab})g^{ab}\partial_b\varphi),$$

thus,

$$\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) + m^2\varphi \\
&= \frac{1}{\sqrt{-g}}\partial_t(\sqrt{N}\sqrt{G}N(x)\partial_t\varphi) \\
&\qquad + \frac{1}{\sqrt{-g}}\partial_a(\sqrt{N}\sqrt{G}(-G^{ab})g^{ab}\partial_b\varphi) + m^2\varphi \\
&= \sqrt{N}\partial_t(\sqrt{N}\sqrt{G}N(x)\partial_t\varphi) \\
&\qquad + \sqrt{N}\partial_a(\sqrt{N}\sqrt{G}(-G^{ab})g^{ab}\partial_b\varphi) + N\sqrt{G}m^2\varphi) \\
&= \partial_t(N\sqrt{G}\partial_t\varphi) - \partial_a(N\sqrt{G}G^{ab}g^{ab}\partial_b\varphi) + N\sqrt{G}m^2\varphi
\end{aligned}$$

is the Heat equation in a static (curved) spacetime.

2.3 The non-linear Heat equation

In this section we generalize the heat equation to general mappings $f : M \longrightarrow M'$ between manifolds and we show that they can be solved given a smooth initial data. In this procedure we use the Nash embedding theorem so that we can embed (isometrically) M' into $\mathbb{R}^n$ for a suitable $n \in \mathbb{N}$ and therefore, we assume $f : M \longrightarrow M' \subseteq \mathbb{R}^n$.

The Laplace operator for f is defined as

$$\begin{aligned}
\Delta f^\alpha = \mathrm{Tr}\,\nabla\nabla f &= \mathrm{Tr}\left[\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k} + \Gamma'^k_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\right] \\
&= g^{ij}\left[\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k} + \Gamma'^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\right],
\end{aligned}$$

where $\Gamma^k_{ij} = \Gamma^k_{ij}(p)$ and $\Gamma'^k_{ij} = \Gamma'^k_{ij}(f(p))$ are the Christoffel symbols of M and M' respectively. Therefore, the general heat equation takes the form

$$\begin{cases}
\dfrac{\partial u^\alpha}{\partial t} = g^{ij}\left[\dfrac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\dfrac{\partial u^\alpha}{\partial x^k} + \Gamma'^\alpha_{\beta\gamma}\dfrac{\partial u^\beta}{\partial x^i}\dfrac{\partial u^\gamma}{\partial x^j}\right] & u : M \times \mathbb{R}^+ \longrightarrow M' \subset \mathbb{R}^m \\
u(p,0) = f(p) & p \in M,\ u = (u^1, ..., u^m)
\end{cases} \tag{2.24}$$

which is of course a non-linear system of PDE's.

The problem of embedding the Manifold M' into a larger one, call it B, is that introduces an ambient space. Solutions of (0.3) in general, may not lie within the manifold M' but instead in the ambient space. We will see that if $M' \subset B \subset \mathbb{R}^m$, the Laplacian Δ_B on B coincides with the Laplacian of $\Delta_{M'} f$ on M' and therefore, the solutions will be constrained to live within the Manifold M' provided that the image of the initial condition f lies entirely on M'. We follow Hamilton's idea [12] for this purpose.

Theorem 2.3.1. *Let $f : M \to M' \subseteq B \subset \mathbb{R}^m$ be a map between compact Riemannian manifolds M and M'. Then,*

$$\Delta_B f = \Delta_{M'} f.$$

Lemma 2.3.2. *If M' is a submanifold of B and $f : M \to M \subseteq B$ then $\Delta_{M'} f = \pi_* \Delta_B f$ where $\pi_* : T_y B \to T_y M'$ is the orthogonal projection.*

Proof. Let $\dim M' = m$ and $\dim B = k$, $k > m$. Choose coordinates $\{z^1, ..., z^m, z^{m+1}, ..., z^k\}$ around $y \in M'$ so that locally M' has $\{z^{m+1} = z^k = 0\}$. Now,

$$\Delta_B f^\alpha = g^{ij} \left\{ \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f^\alpha}{\partial x^k} + \Gamma'^\alpha_{\beta\gamma} \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j} \right\}.$$

Since $f : M \to M'$ we have $f^i = 0, i = m+1, ..., k$. Therefore, $\Delta_B f^\alpha = \Delta_{M'} f^\alpha$, $\alpha = 1, ..., m$. Thus, $\Delta_{M'} f = \pi_* \Delta_B f$. ■

Proof of theorem 2.3.1. Since $f : M \to M'$ we have $\Delta_B f \in T_y M'$ so $\pi_* \Delta_B f = \Delta_B f$. Thus, the lemma gives the desired result, $\Delta_B f = \Delta_{M'} f$. ■

Theorem 2.3.3. *Let $f : M \times [0, T] \to B$. Assume*

1. *f is continuous and its first space derivatives $\frac{\partial f^\alpha}{\partial x^i}$ exist and are continuous on $M \times [0, T]$.*
2. *f is smooth in the interior of $M \times [0, T]$, and that there it satisfies the heat equation*

$$\frac{\partial f}{\partial t} = \Delta_B f.$$

If $f(M, 0) \subseteq M'$ then $f(M \times [0, T]) \subseteq M'$ and $f : M \times [0, T] \to M'$ satisfies the heat equation

$$\frac{\partial f}{\partial t} = \Delta_{M'} f.$$

Proof. If the image of f does not always remain in M', we can restrict ourselves to a smaller interval $M \times [0, t]$, $0 < t \leq T$ such that the image of f does not always remain in M. Assume $f' : M \times [0, t] \to B$ is another solution of the heat equation and has the same initial value as f, so by uniqueness $f' = f$. Thus, $\Delta_B f \in T_{p'} M'$ and using the previous theorem, we have $\Delta_B f = \Delta_{M'} f$. ■

Theorem 2.3.3 means that the image of u in (0.3) already lies in M' and thus, it is enough to look at solutions $u : M \to \mathbb{R}^m$ for which the image of the initial data lies in M'. In order to prove existence of this solutions we will take four steps,

1. Solve the non coupled system,

$$\begin{cases} \dfrac{\partial u^\alpha}{\partial t} = g^{ij}\left[\dfrac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \dfrac{\partial u^\alpha}{\partial x^k}\right] & u : M \times \mathbb{R}^+ \to M' \subset \mathbb{R}^m \\ u(p,0) = f(p) & p \in M,\ u = (u^1, ..., u^m) \end{cases} \tag{2.25}$$

2. Solve the weakly coupled linear system,

$$\begin{cases} \dfrac{\partial u^\alpha}{\partial t} = g^{ij}\left[\dfrac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \dfrac{\partial u^\alpha}{\partial x^k}\right] + v_\beta^{\alpha\gamma}\dfrac{\partial u^\beta}{\partial x^\gamma} & u : M \times \mathbb{R}^+ \to M' \subset \mathbb{R}^m \\ u(p,0) = f(p) & p \in M,\ u = (u^1, ..., u^m) \end{cases} \tag{2.26}$$

3. Solve the complete system (0.3) for small times.

4. Extend the solution for arbitrary time.

First step:

Proposition 2.3.1. *The system* (2.25) *has unique smooth solutions.*

Proof. The system (2.25) is exactly the system on equation (2.22),

$$\begin{cases} \dfrac{\partial u^\alpha}{\partial t} = \Delta_M u^\alpha & u^\alpha : M \times \mathbb{R}^+ \to \mathbb{R} \\ u(p,0) = f(p) & p \in M,\ u = (u^1, ..., u^m) \end{cases} \tag{2.27}$$

Therefore, the system (2.25) has a unique smooth solution because the diagonal system in corollary 2.2.5.2 has a unique smooth solution and, by theorem 2.3.3, $u : M \times \mathbb{R}^{\geq 0} \to M' \subseteq \mathbb{R}^m$. ■

Second step:

In order to prove existence of solutions to system (2.26) we use the continuity method showed in [10] together with the Schauder estimates illustrated on [14].

The invertibility of a bounded linear mapping may be deduced from the invertibility of a similar mapping through the following theorem, which is known as the method of continuity.

Theorem 2.3.4 (Method of continuity)**.** *Let $\mathcal{B}$ be a Banach space, $\mathcal{V}$ a normed linear space and let $L_0, L_1 \in \mathfrak{B}(\mathcal{B}, \mathcal{V})$ where $\mathfrak{B}(\mathcal{B}, \mathcal{V})$ are the bounded linear mappings from $\mathcal{B}$ into $\mathcal{V}$. For each $t \in [0,1]$, set*

$$L_t = (1-t)L_0 + L_1,$$

and suppose there exist a constant C such that

$$||x||_{\mathcal{B}} \leq C||L_t x||_{\mathcal{V}}, t \in [0,1]. \tag{2.28}$$

Then L_1 maps $\mathcal{B}$ onto $\mathcal{V}$ if and only if L_0 maps $\mathcal{B}$ onto $\mathcal{V}$.

Proof. Suppose that L_s is onto for some $s \in [0,1]$. Equation (2.28) implies that $L_t \in [0,1]$ is one-to-one and hence, the inverse mapping $L_s^{-1} : \mathcal{V} \to \mathcal{B}$ exists. For $t \in [0,1]$ let us see that $x \to L_t x$ is onto that is, given any $y \in \mathcal{V}$ the equation $L_t x = y$ has solution. This equation is equivalent to

$$\begin{aligned} L_s x &= y + (L_s - L_t)x \\ &= y + (t-s)L_0 x - (t-s)L_1 x, \end{aligned}$$

which in turn, is equivalent to the equation

$$x = L_s^{-1} y + (t-s)L_s^{-1}(L_0 - L_1)x.$$

Define the mapping $T : \mathcal{B} \to \mathcal{B}$ by $Tx = L_s^{-1} y + (t-s)L_s^{-1}(L_0 - L_1)x$. Clearly, T is a contraction if

$$|s-t| < \delta = C(||L_0|| + ||L_1||)^{-1}.$$

Notice that x is a fixed point of T and, being T a contraction in a banach space, x is unique and exist. Hence, the mapping L_t is onto for all t such that $|s-t| < \delta$. Furthermore, if we take a partition of the interval $[0,1]$ into subintervals of length less than δ, we can propagate the surjectivity of L_t and therefore, the mapping L_t is onto for all $t \in [0,1]$ provided L_t is onto for any fixed $t \in [0,1]$, in particular for $s = 0$ and $t = 1$. ■

where the norms are

$$|u|_{k,\alpha}^T := |u|_{k+\alpha}^T + \sum_{j=0}^{k} |u|_j^T,$$

where

$$\begin{aligned} |u|_j^T &:= \sum_{2r+s=j} \left|D_t^r D_x^s u\right|_{\Omega_T} \\ |u|_{k+\alpha}^T &:= |u|_{k+\alpha,x}^T + |u|_{\frac{k+\alpha}{2},t}^T \\ |u|_{k+\alpha,x}^T &:= \sum_{2r+s=k} \left|D_t^r D_x^s u\right|_{\alpha,x}^T \\ |u|_{\frac{k+\alpha}{2},t}^T &:= \sum_{0<k+\alpha-2r-s<2} \left|D_t^r D_x^s u\right|_{\frac{k+\alpha-2r-s}{2}}{}^T{}_{,t} \end{aligned}$$

and for $0 < \beta < 1$

$$|u|_{\beta,x}^{T} := \sup_{(x,t),(x',t)\in\Omega_T} \frac{|u(x,t) - u(x',t)|}{|x-x'|^{\beta}}$$
$$|u|_{\beta,t}^{T} := \sup_{(x,t),(x,t')\in\Omega_T} \frac{|u(x,t) - u(x,t')|}{|t-t'|^{\beta}}.$$

which all are taken in the appropriate Banach space defined below.

Theorem 2.3.5. *Let $f : M \to M'$ be a smooth function between Riemannian manifolds. The system*

$$\begin{cases} \dfrac{\partial u^{\alpha}}{\partial t} = g_{ij}\left[\dfrac{\partial^2 u^{\alpha}}{\partial x^i \partial x^j} - \Gamma_{ij}^{k}\dfrac{\partial u^{\alpha}}{\partial x^k}\right] + v_{\beta}^{\alpha\gamma}\dfrac{\partial u^{\beta}}{\partial x^{\gamma}} & u : M \times (0,T) \to M' \subset \mathbb{R}^m \\ u(p,0) = f(p) & p \in M,\ u = (u^1, ..., u^m) \end{cases} \tag{2.29}$$

has a solution for $u : M \times (0,T) \to M' \subset \mathbb{R}^m$.

Proof. We can "parametrize" the coupling with $\mu \in [0,1]$ as

$$\begin{cases} \mathcal{A}_{\mu}u \equiv -\dfrac{\partial u^{\alpha}}{\partial t} + g_{ij}\left[\dfrac{\partial^2 u^{\alpha}}{\partial x^i \partial x^j} - \Gamma_{ij}^{k}\dfrac{\partial u^{\alpha}}{\partial x^k}\right] + \mu v_{\beta}^{\alpha\gamma}\dfrac{\partial u^{\beta}}{\partial x^{\gamma}} = f \\ u|_{t=0} = u_0. \end{cases}$$

Let $\Omega_T = M \times [0,T)$ and suppose $u : \Omega \to M' \subset \mathbb{R}^m$ satisfies the system (2.25) then, the Schauder estimates of the weakly coupled heat equation are a well known bounds [14] of u given by,

Schauder estimates

$$|u|_{2,\alpha}^{T} \leq C\left(|f|_{\alpha}^{T} + |u_0|_{2,\alpha}\right)$$

In order to use the so-called continuity method, consider the $t-$Hölder (Banach) space $C_t^{k,\alpha}(\overline{\Omega}_T)$ which is the set of continuous functions in $\overline{\Omega}_T$ together with all the derivatives of the form $D_t^r D_x^s$ for $2r + s < l$ and, define the subspace of the $t-$Hölder space that satisfy the initial condition u_0,

$$\mathcal{E}_t^{k,\alpha}(\overline{\Omega}_T) = \{u :\ u \in C_t^{k,\alpha}(\overline{\Omega}_T), u|_{t=0} = u_0\}.$$

The family of operators $\{\psi_{\mu}\}_{\mu\in[0,1]}$,

$$\begin{aligned} \psi_{\mu} : \mathcal{E}_t{}^{k,\alpha}(\overline{\Omega}_T) &\to C_t^{0,\alpha}(\overline{\Omega}_T) \\ u &\mapsto \mathcal{A}_{\mu}u \end{aligned}$$

satisfy

$$\psi_t = (1-t)\psi_0 + \psi_1.$$

In order to use the continuity method, we need to see that at least one operator is invertible and, the inequality (2.28) holds. Notice that the map ψ_0 is the uncoupled system (2.29),

$$\begin{cases} -\dfrac{\partial u^\alpha}{\partial t} + g_{ij}\left[\dfrac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\dfrac{\partial u^\alpha}{\partial x^k}\right] = 0 \\ u|_{t=0} = 0 \end{cases}$$

hence, ψ_0 is invertible. On the other hand, by the maximum principle we can drop the second term in the Schauder estimates and thus,

$$|u|^T_{2,\alpha} \leq C|f|^T_{0,\alpha} = C|\psi_\mu u|^T_{0,\alpha}.$$

Since f and its derivatives up to order k are bounded on $\overline{\Omega_T}$ then, we can use the $t-$Hölder norms,

$$||u||_{2,\alpha} \leq C||\psi_\mu u||_{0,\alpha}.$$

Therefore, if we choose $\psi_\mu : \mathcal{E}^{2,\alpha}_t(\overline{\Omega}_T) \to C^{0,\alpha}_t(\overline{\Omega}_T)$ theorem 2.3.4 imply that the operator ψ_1 is onto and thus, invertible. Notice that u has continuous derivatives up to order 2. ■

Third step:

To prove existence of short time of the non linear system (0.3) we use inverse function theorem.

Theorem 2.3.6. *Let $f : M \to M'$ be a smooth function, There is an $\varepsilon > 0$ such that*

$$\begin{cases} \dfrac{\partial u^\alpha}{\partial t} = g_{ij}\left[\dfrac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\dfrac{\partial u^\alpha}{\partial x^k} + \Gamma'^\alpha_{\beta\gamma}\dfrac{\partial u^\beta}{\partial x^i}\dfrac{\partial u^\gamma}{\partial x^j}\right] & u : M \times [0,\varepsilon) \to M' \subset \mathbb{R}^m \\ u(p,0) = f(p) & p \in M,\ u = (u^1, ..., u^m) \end{cases} \tag{2.30}$$

has solution for $u : M \times (0,\varepsilon) \to M' \subset \mathbb{R}^m$.

Proof. Let $W = \{(F,f)\} \subseteq C(M \times [0,1]) \bigoplus C(M)$.

Define $\Omega(u) : C(M \times [0,1]) \to W$, as

$$\Omega(u) = \left(u^1_t - \Delta u^1 + K^{ij}_{\alpha\beta}\frac{\partial u^1}{\partial x^i}\frac{\partial u^1}{\partial x^i}, ..., u^m_t - \Delta u^m + K^{ij}_{\alpha\beta}\frac{\partial u^m}{\partial x^i}\frac{\partial u^m}{\partial x^i}, u\Big|_{t=0} - f\right), \quad K^{ij}_{\alpha\beta} = K^{ji}_{\alpha\beta}.$$

Notice that a solution of (2.30) has $\Omega(u) = (0, ..., 0, 0)$. The map $D_u\Omega : C(M \times [0,1]) \to T_{\Omega(u)}W$,

$$D_u\Omega(v) = \left(v^1_t - \Delta v^1 + 2K^{ij}_{\alpha\beta}\frac{\partial u^1}{\partial x^i}\frac{\partial v^1}{\partial x^i}, ..., v^m_t - \Delta v^m + 2K^{ij}_{\alpha\beta}\frac{\partial u^m}{\partial x^i}\frac{\partial v^m}{\partial x^i}, v\Big|_{t=0}\right)$$

is parabolic and by step two is invertible (the solution exist and has continuous derivatives up to order 2). Then, the Inverse Function Theorem implies the existence of a solution of the system (2.30) for a short time and, since u satisfy the heat equation

$$\frac{\partial u}{\partial t} = \Delta u,$$

the Bernstein method imply that u is smooth. ■

Fourth step:

Theorem 2.3.7. *Let $f : M \to M'$ be a smooth function, the system*

$$\begin{cases} \dfrac{\partial u^\alpha}{\partial t} = g^{ij}\left[\dfrac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \dfrac{\partial u^\alpha}{\partial x^k} + \Gamma'^{\alpha}_{\beta\gamma}\dfrac{\partial u^\beta}{\partial x^i}\dfrac{\partial u^\gamma}{\partial x^j}\right] & u : M \times \mathbb{R}^+ \to M' \subset \mathbb{R}^m, \\ u(p,0) = f(p) & p \in M,\ u = (u^1, ..., u^m), \end{cases} \tag{2.31}$$

has smooth solutions.

Proof. Take u solution for small times, the Bernstein method of section 2.2.2 uses invariant operators and thus, the conclusion also apply for $u : M \to M'$ making the solutions of the system smooth and unique provided that u is continuous.

On the other hand, consider v solution of the system:

$$\begin{cases} \dfrac{\partial v^\alpha}{\partial t} = g_{ij}\left[\dfrac{\partial^2 v^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \dfrac{\partial v^\alpha}{\partial x^k} + \Gamma'^{\alpha}_{\beta\gamma}\dfrac{\partial v^\beta}{\partial x^i}\dfrac{\partial v^\gamma}{\partial x^j}\right] & v : M \times [0,\varepsilon') \to M' \subset \mathbb{R}^m, \\ v(p,0) = u(p,\varepsilon) & p \in M,\ u = (u^1, ..., u^m), \end{cases} \tag{2.32}$$

and define $\overline{u} : M \times [0, \varepsilon + \varepsilon') \to M'$ by

$$\overline{u} = \begin{cases} u & t \in [0,\varepsilon) \\ v & t \in [\varepsilon, \varepsilon + \varepsilon'). \end{cases}$$

Notice that the bounded derivatives of u implies that the limit $\lim_{t\to\varepsilon} u$ exist. It is not difficult to check that, for any derivative $\partial_\alpha u$, $\lim_{t\to\varepsilon} \partial_\alpha u$ also exists. Hence, by the pasting lemma the function $\overline{u}$ is continuous together with all its derivatives hence, smooth. Notice that $\overline{u}$ satisfy the same IVP (2.30) thus $u = \overline{u}$. ■

The results on this chapter are summarized in the following theorem.

Theorem 2.3.8. *Let $f : M \to M'$ be a smooth map between Riemannian manifolds M and M'. Then, the initial value problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u & u : M \times \mathbb{R}^+ \to M' \subset \mathbb{R}^m \\ u(p,0) = f(p) & p \in M \end{cases} \tag{2.33}$$

has unique smooth solutions.

2.3.1 Homotopy

Definition 2.3.1. *Let X and Y be two topological spaces. A family of maps $f_t : X \longrightarrow Y, t \in I$ is said to be a homotopy if the associated map $F : X \times I \longrightarrow Y$ given by $F(x,t) = f_t(x)$ is continuous. One says that $f, g : X \longrightarrow Y$ are homotopic if there exist a homotopy f_t, $t \in [0,1]$ such that $f_0 = f$ and $f_1 = g$.*

The existence of the solutions of the Heat equation motivates the following definition:

Definition 2.3.2. *Let $u : M \times \mathbb{R}^+ \longrightarrow M'$ be a solution of the system in theorem 2.3.8. Fix $t \in \mathbb{R}^+$ and define $u_t : M \longrightarrow M'$ by $u_t(p) = u(p,t)$, $p \in M$. The smooth function $u : \mathbb{R}^+ \times M \longrightarrow M'$ is called a deformation of f via the heat equation. Theorem 2.3.8 states that a deformation of $f \in C^\infty(M, M')$ always exists.*

Remark 2.3.9. *In these terms, a deformation $f_t : M \times [0,T) \longrightarrow M'$ is a homotopy.*

Definition 2.3.3. *A retraction of X into $A \subseteq X$ is a continuous map $r : X \longrightarrow X$ such that $r(X) = A$ and $r|_A = 1_X$ where 1_X is the identity on X. A deformation retraction is a homotopy from the identity map 1_X to a retraction of X onto $A \subseteq X$.*

Definition 2.3.4. *Let M be a submanifold of the Riemannian manifold (N, g) with $\iota : M \longrightarrow N$ the inclusion map so that $p \in M$. Then $i_*(T_pM) \subset T_pN$. We define $T_pM^\perp \subset T_pM$ as*

$$T_pM^\perp = \{v \in T_pN : g(v, i_*(w)) = 0 \ \forall w \in T_pM\}.$$

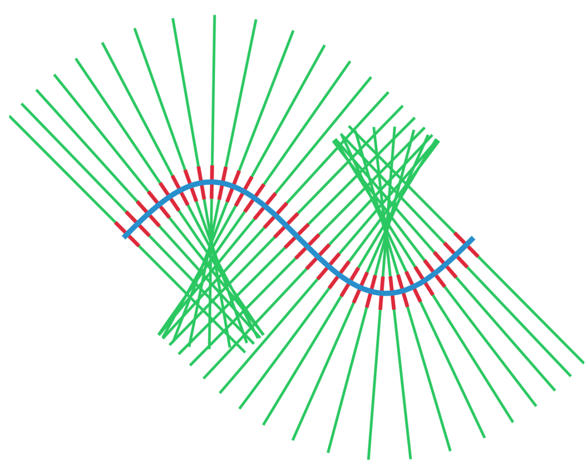

Figure 2: A tubular neighborhood, retrived from [15]: The submanifold M is in blue, U in red and the normal bundle is in green.

Definition 2.3.5. *Let*

$$E = \bigcup_{p \in M} T_pM^\perp,$$

and $\psi : E \to M$ *such that* $v \in T_pM^\perp \subset E$ *and* $\psi(v) = p$. *Then,* ψ *is the normal bundle of* M *in* N.

Example 2.3.1 (The normal bundle of S^{n-1})**.** *Let* $p \in S^{n-1}$, T_pS^{n-1} *is the tangent plane of the sphere at a point* p *and thus,* $(T_pS^{n-1})^\perp \simeq \mathbb{R}$ *are the normal vectors of* S^{n-1}.

Definition 2.3.6. *Let* M *be a submanifold of* B. *We say that* (f, ξ) *is a tubular neighborhood of* M *if* $\xi = (p, E, M)$ *is a vector bundle, where* p *is the projection of the bundle and* $f : E \to B$ *is an embedding such that*

- $f(E)$ *is open,*
- $f|_M = 1_M$, *where we identify* M *with the zero section.*

Since $f(E)$ *is open, we call say that* $U := f(E)$ *is "the" tubular neighborhood of* M, *see figure 2. Notice that there exists a retraction* $r : U \to M$ *defined by* $r = p \circ f^{-1}$.

.

3 Harmonic Mappings

In section 1, we defined the energy map $\mathcal{E}$ of a curve $\gamma : [0,1] \to M$. Furthermore, given a metric g, we can define an energy functional that allows us to compute the Christoffel symbols in a simplified way and thus, identify the geometry of M. In this section, we define an energy functional E of a map $f : M \to M'$ between Riemannian manifolds in the sense of Eells and Sampson. The energy functional E is a generalization of $\mathcal{E}$ and takes a similar role to the physical energy and may be applied to deduce further geometric information of M as Eells and Sampson suggest.

3.1 The energy of a map

Definition 3.1.1. *Let (M,g) and (M',g') be two Riemannian manifolds of dimension n and m respectively. Denote by $\{x^i\}$,$\{y^i\}$ a set of local coordinates of M and M' respectively and take $T = T_{ij}dx^i \otimes dx^j$ a 2-covariant tensor field on M. We can define a inner product of two 2-covariant tensor fields T and T' on M at p as*

$$\langle T, T' \rangle_p = T_{ij} T'_{pq} g^{ip} g^{jq}. \tag{3.1}$$

Proposition 3.1.1. *The inner product of two 2-covariant tensor fields is indeed a inner product.*

Proof. Linearity and symmetry are clear. Choose a Riemannian normal coordinates centered at p.

$$\langle T, T \rangle_p = T_{ij} T_{pq} \delta^{ip} \delta^{jq} = \sum_{i,j=1}^{n} T_{ij}^2 > 0.$$

■

Let $f : M \to M'$ be a smooth mapping between Riemannian manifolds M and M'.

Definition 3.1.2. *The energy density of f at $p \in M$ denoted by $e[f](p)$ is*

$$e[f](p) := \frac{1}{2} \langle g(p), (f^* g')(p) \rangle_p. \tag{3.2}$$

Let $\{x^\alpha\}$ and $\{y^\beta\}$ be local coordinates in M and M' respectively. The energy density takes the explicit form:

$$e[f](p) = \frac{1}{2}\langle g(p), (f^*g')(p)\rangle_p = \frac{1}{2}\langle g_{ij}, (f^*g')_{pq}\rangle_p \tag{3.3}$$

$$= \frac{1}{2}(g_{ij}\underbrace{(f^*g')_{pq}}_{g'_{\alpha\beta}f_p^\alpha f_q^\beta} g^{ip}g^{jq}) = \frac{1}{2}(g_{ij}g'_{\alpha\beta}f_p^\alpha f_q^\beta g^{ip}g^{jq}) \tag{3.4}$$

$$= \frac{1}{2}(g^{pq}g'_{\alpha\beta}f_p^\alpha f_q^\beta) \quad . \tag{3.5}$$

Remark 3.1.1. *The energy density of f is allways non negative. Choose normal coordinates centered at $p \in M$. Then,*

$$e(f) = \frac{1}{2}\sum_{i,\alpha}[f_i^\alpha(p)]^2 \geq 0.$$

Definition 3.1.3. *The energy of the map f is the integral over M of $e[f]$, i.e.,*

$$E(f) = \frac{1}{2}\int_M g^{ij}f_i^\alpha f_j^\beta g'_{\alpha\beta}\sqrt{g}\,dx^1 \wedge \cdots \wedge dx^n = \frac{1}{2}\int_M g^{ij}f_i^\alpha f_j^\beta g'_{\alpha\beta} * 1. \tag{3.6}$$

Proposition 3.1.2. *The energy functional E is a generalization of $\mathcal{E}$ defined on section 1. That is, given a $f : [0,1] \to M$ then, the energies $E(f)$ and $\mathcal{E}(f)$ coincide.*

Proof. If $M = [0,1]$, denote $\{t\}$ the coordinate of $[0,1]$ then, $g = dt^2$ and

$$E(f) = \frac{1}{2}\int_M g^{ij}f_i^\alpha f_j^\beta g'_{\alpha\beta} * 1 = \frac{1}{2}\int_0^1 \frac{df^\beta}{dt}\frac{df^\alpha}{dt}g'_{\alpha\beta}(f(t))dt = \mathcal{E}(f).$$

■

Remark 3.1.2. *The energy map $E(f)$ measures how much f scatter points of M onto M'.*

Indeed, let $r' : M' \times M' \to \mathbb{R}^{\geq 0}$ such that $r'(a,b)$ is the geodesic distance between $a, b \in M'$. Let $p, q \in M$ and $f(p) = p'$, $f(q) = q'$. Fix $p \in M$, consider the map $h : M \to \mathbb{R}$ such that $h(q) = r'(f(p), f(q))$.

If p' and q' are near enough then $h(q) = r(p', q') = \exp_{p'}(v)$ is the geodesic distance of p' and q' for some $v \in T_p \in M$. Theorem 1.4.2 implies that $h : M \to \mathbb{R}$ is smooth. In local coordinates $\{y^\alpha = f^\alpha(x)\}$ in M' and $\{x^\alpha\}$ in M we have

$$\Delta h(y(x)) = g^{ij}\left[\frac{\partial^2 h}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial h}{\partial x^k}\right].$$

Therefore,

$$\Delta h = g^{ij}\left[\frac{\partial}{\partial x^i}\left(\frac{\partial h}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i}\right) - \Gamma_{ij}^k\frac{\partial h}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^k}\right] = g^{ij}\left[\frac{\partial^2 h}{\partial y^\alpha \partial y^\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + \frac{\partial h}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k\frac{\partial h}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^k}\right]$$
$$= g^{ij}\frac{\partial^2 h}{\partial y^\alpha \partial y^\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + \frac{\partial h}{\partial y^\alpha}\Delta y^\alpha.$$

Now, let $\frac{\partial h}{\partial y^\alpha} = O(r')$, $\frac{\partial^2 h}{\partial y^\alpha \partial y^\beta} = 2g'_{\alpha\beta}(p') + O(r')$ *whence,*

$$\Delta h = 2g^{ij}g'_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} + O(r') + \frac{\partial h}{\partial y^\alpha}\Delta y^\alpha = 4e(f) + O(r').$$

$$\overline{u_\varepsilon}(p) = \int_{B_\varepsilon} \frac{u}{V(B_\varepsilon)} * 1, \quad B_\varepsilon = \{q \in M : f(p,q) \leq \varepsilon\}.$$

On the other hand, a Maxwell relation [1] gives us

$$\overline{u_\varepsilon}(p) = u(p) + \frac{\varepsilon^2}{2(n+2)}\Delta u(p) + O(\varepsilon^2).$$

Using the Maxwell relation with $u = h$, $h(p) = 0$ *then,*

$$\overline{h}(p) = \frac{2\varepsilon^2}{(n+2)}e(f)_p + O(\varepsilon^2).$$

Finally, integrating over M *we have*

$$\int_M \int_M \frac{h^2(p,q)}{V(B_\varepsilon)}\chi(p) * 1 = \frac{2\varepsilon^2}{n+2}E(f) + O(\varepsilon^2), \quad \chi(p) = \begin{cases} 1 & p \in B_\varepsilon \\ 0 & p \notin B_\varepsilon \end{cases} \tag{3.7}$$

then, the energy map $E(f)$ *measures how much* f *scatter points on* M *onto* M'.

Definition 3.1.4. *Let* M, M' *be two Riemannian manifolds. Denote by* $\mathscr{H}(M, M')$ *the set of smooth maps from* M *to* M'.

Definition 3.1.5. *Let* $f \in \mathscr{H}(M, M')$ *and* $\pi : TM' \longrightarrow M'$ *denote the tangent bundle. Then, the vector fields* u *along* f *are maps* $u : M \longrightarrow TM'$ *such that* $\pi \circ u = f$.

Proposition 3.1.3. *The set of vector fields forms a vector space* $\mathscr{H}(f)$ *with point-wise operations. Furthermore, the product on* $\mathscr{H}(f)$ *defined by*

$$\langle u, v\rangle_f = \int_M \langle u(p), v(p)\rangle_{f(p)} * 1 \quad , v, u \in \mathscr{H}(f) \tag{3.8}$$

is an inner product.

Proof. The vector fields along f inherits the vector space structure of $T_{p'}M'$ pointwise and thus, they define a vector space. On the other hand, in the inner product, linearity follows from the linearity of the integral and the inner product $\langle \cdot, \cdot \rangle_{f(p)}$. Symmetry follows also from the symmetry of the inner product. Finally,

$$\langle u, u\rangle_f = \int_M \underbrace{\langle u(p), u(p)\rangle_{f(p)}}_{\geq 0} *1 \geq 0$$

with equality if and only if $\langle u(p), u(p)\rangle_{f(p)} = 0$ that is, when $u = 0$. ∎

Definition 3.1.6. *Let v be a vector field along f. The directional derivative of $E(f)$ in the direction v is*

$$\nabla_v E(f) := \frac{d}{dt} E(\underbrace{exp_{f(p)}(tv(p))}_{f_t})\Big|_{t=0}.$$

Definition 3.1.7. *The Euler-Lagrange operator applied to a map f defines a vector field $\tau(f)$ along f which is the contravariant representative of the differential of E at f, i.e.*

$$\nabla_v E(f) = -\langle \tau(f), v\rangle_f, \quad \forall v \in \mathscr{H}(f).$$

Definition 3.1.8. *The tension field of f, $\tau(f)$ is defined such that the equality*

$$\frac{dE(f)}{dt} = -\int_M \left\langle \tau(f), \frac{df}{dt} \right\rangle * 1 \tag{3.9}$$

holds.

Definition 3.1.9. *We say that f is harmonic if its tension field $\tau(f)$ vanishes.*

3.2 Closed curves on M ($f : S^1 \to M$)

Consider $M = S^1$ and $f : S^1 \to M'$ a smooth curve on M'. Let $\{f_t\}_{t\in\mathbb{R}}$ be a deformation of f. Letting $x^1 = \theta$ and keeping in mind that $g = d\theta^2$ then, the energy density of f is

$$e[f] = \frac{1}{2} f_\theta^\alpha f_\theta^\beta g'_{\alpha\beta},$$

and the energy of the map f is:

$$E(f) = \frac{1}{2}\int_{S^1} f_\theta^\alpha f_\theta^\beta g'_{\alpha\beta} * 1 = \frac{1}{2}\int_{S^1} f_\theta^\alpha f_\theta^\beta g'_{\alpha\beta}\, d\theta = \frac{1}{2}\int_{S^1} \left\| \frac{df}{d\theta} \right\|^2 d\theta.$$

Let us take a look at the tension field of closed curves $f : S^1 \to M'$.

An application of the Stokes theorem tells us that if ξ is a tensor field in TM'. Then the integral of the divergence vanishes ($\int_{M'} \xi_j^j * 1 = 0$) in a manifold without boundary. Thus, if $\xi^j = g^{ij} \frac{\partial f_t^\alpha}{\partial x^i} \frac{\partial f_t^\beta}{\partial t} g'_{\alpha\beta} = \frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial f_t^\beta}{\partial t} g'_{\alpha\beta}$ we compute as follows:

$$\begin{aligned} \int \xi_j^j * 1 &= \int_{S^1} \frac{\partial}{\partial \theta}\left(\frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial f_t^\beta}{\partial t} g'_{\alpha\beta} \right) \underbrace{*1}_{d\theta} \\ &= \int_{S^1} \frac{\partial^2 f_t^\alpha}{\partial \theta^2} \frac{\partial f_t^\beta}{\partial t} g'_{\alpha\beta} + \frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial^2 f_t^\beta}{\partial t \partial \theta} g'_{\alpha\beta} + \frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial f_t^\beta}{\partial t} \underbrace{\frac{\partial g'_{\alpha\beta}}{\partial y^\gamma} \frac{\partial f_t^\gamma}{\partial \theta}}_{\frac{\partial g'_{\alpha\beta}}{\partial \theta}} \, d\theta = 0. \end{aligned}$$

Therefore,

$$\int_{S^1} \frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial^2 f_t^\beta}{\partial t \partial \theta} g'_{\alpha\beta}\, d\theta = -\int_{S^1} \frac{\partial^2 f_t^\alpha}{\partial \theta^2} \frac{\partial f_t^\beta}{\partial t} g'_{\alpha\beta} + \frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial f_t^\beta}{\partial t} \frac{\partial g'_{\alpha\beta}}{\partial y^\gamma} \frac{\partial f_t^\gamma}{\partial \theta}\, d\theta. \tag{3.10}$$

Next, we compute $\frac{dE(f_t)}{dt}$. Since we have a parameter t in f let us introduce the notation $\frac{\partial f_t^\alpha}{\partial \theta} = f_{t,\theta}^\alpha$.

$$\begin{aligned}
\frac{dE(f_t)}{dt} &= \frac{1}{2}\int_{S^1} \frac{d}{dt}(f_{t,\theta}^\alpha f_{t,\theta}^\beta g'_{\alpha\beta})\, d\theta = \frac{1}{2}\int_{S^1} 2\frac{\partial^2 f_t^\alpha}{\partial \theta \partial t} \frac{f_t^\beta}{\partial \theta} g'_{\alpha\beta} + \frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial f_t^\beta}{\partial \theta} \frac{\partial g'_{\alpha\beta}}{\partial y^\gamma} \frac{\partial f_t^\gamma}{\partial t}\, d\theta \\
&= \frac{1}{2}\int_{S^1} -2\left(\frac{\partial^2 f_t^\alpha}{\partial \theta^2} \frac{\partial f_t^\beta}{\partial t} g'_{\alpha\beta} + \frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial f_t^\beta}{\partial t} \frac{\partial g'_{\alpha\beta}}{\partial y^\gamma} \frac{\partial f_t^\gamma}{\partial \theta}\right) + \frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial f_t^\beta}{\partial \theta} \frac{\partial g'_{\alpha\beta}}{\partial y^\gamma} \frac{\partial f_t^\gamma}{\partial t}\, d\theta \\
&= \frac{1}{2}\int_{S^1} -2\frac{\partial^2 f_t^\gamma}{\partial \theta^2} \frac{\partial f_t^\nu}{\partial t} g'_{\gamma\nu} - 2\frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial f_t^\beta}{\partial \theta} \frac{\partial g'_{\alpha\nu}}{\partial y^\beta} \frac{\partial f_t^\nu}{\partial t} + \frac{\partial f_t^\alpha}{\partial \theta} \frac{\partial f_t^\beta}{\partial \theta} \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} \frac{\partial f_t^\nu}{\partial t}\, d\theta \\
&= \frac{1}{2}\int_{S^1} -2 f_{t,\theta\theta}^\gamma f_{t,t}^\nu g'_{\gamma\nu} - 2 f_{t,\theta}^\alpha f_{t,\theta}^\beta g'_{\alpha\nu,\beta} f_{t,t}^\nu + f_{t,\theta}^\alpha f_{t,\theta}^\beta g'_{\alpha\beta,\nu} f_{t,t}^\nu\, d\theta \\
&= \frac{1}{2}\int_{S^1} -2 f_{t,\theta\theta}^\gamma f_{t,t}^\nu g'_{\gamma\nu} - 2 f_{t,\theta}^\alpha f_{t,\theta}^\beta g'_{\alpha\nu,\beta} g'^{\nu} g'_{\gamma} f_{t,t}^\nu + f_{t,\theta}^\alpha f_{t,\theta}^\beta g'_{\alpha\beta,\nu} g'^{\nu} g'_{\gamma\nu} f_{t,t}^\nu\, d\theta \\
&= -\int_{S^1} \left(f_{t,\theta\theta}^\gamma + \frac{1}{2} g'^{\gamma\nu}(2g'_{\alpha\nu,\beta} - g'_{\alpha\beta,\nu}) f_{t,\theta}^\alpha f_{t,\theta}^\beta\right) f_{t,t}^\nu g'_{\gamma\nu}\, d\theta \\
&= -\int_{S^1} \left(f_{t,\theta\theta}^\gamma + \Gamma'^{\gamma}_{\alpha\beta} f_{t,\theta}^\alpha f_{t,\theta}^\beta\right) f_{t,t}^\nu g'_{\gamma\nu}\, d\theta,
\end{aligned}$$

where to go from the first line to the second one we use equation (3.10), and the last line we use the representation of the Christoffel symbols found in equation (1.12). Let us analyze our result. The Laplace operator is defined as :

$$\Delta f^\gamma := -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial f^\gamma}{\partial x^j}\right).$$

In our case ($M = S^1$), we have, $\Delta f^\gamma = \frac{1}{1}\frac{\partial}{\partial \theta}\left(1 \cdot 1 \frac{\partial f^\gamma}{\partial \theta}\right) = \frac{\partial^2 f^\gamma}{\partial \theta^2}$ which is our first term. In fact, one can show that this is no coincidence. If we carry the same computations with M arbitrary, we have

Proposition 3.2.1. *Let $f_t : M \times [0,T] \to M'$ be a smooth deformation of f . Then,*

$$\frac{dE(f_t)}{dt} = -\int_M \underbrace{(\Delta f_t^\gamma + g^{\delta\mu}\Gamma'^{\gamma}_{\alpha\beta} f_{t,\delta}^\alpha f_{t,\mu}^\beta)}_{\tau(f)} f_{t,t}^\nu g'_{\gamma\nu} * 1 = -\int_M \left\langle \tau(f), f_{t,t}^\nu \right\rangle * 1, \tag{3.11}$$

where $\Gamma'^{\gamma}_{\alpha\beta} = \Gamma'^{\gamma}_{\alpha\beta}(f_t(\theta))$ are the Christoffel symbols on M'. Consequently, the tension field is well defined and takes the form

$$\tau(f)^\gamma(p) = \Delta f^\gamma(p) + g^{ij}(p)\Gamma'^{\gamma}_{\alpha\beta}(f(p)) f_i^\alpha(p) f_j^\beta(p). \tag{3.12}$$

Remark 3.2.1. *A deformation $f_t : M \times [0,T) \to M'$ of f with respect to the heat equation satisfies*

$$\frac{\partial f_t}{\partial t} = \tau(f_t), \quad f_0(p) = f(p).$$

Also, f is a extremal of the energy map if and only if f is harmonic.

Proposition 3.2.2. *The following maps are harmonic:*

1. *The constant map,*
2. *The identity map,*
3. *If M' is flat, f is harmonic if and only if each of its component functions are harmonic*

Proof. In coordinates, f is harmonic if and only if the following equation holds,

$$g^{ij}\left[\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k} + \Gamma'^{\alpha}_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\right] = 0. \tag{3.13}$$

1. Clearly a constant map satisfies (3.13).
2. If f is the identity $\Gamma = \Gamma'$ and the second derivatives vanishes. It follows that
$$g^{ij}\left[-\Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k} + \Gamma^{\alpha}_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\right] = g^{ij}\left[-\Gamma^\alpha_{ij} + \Gamma^\alpha_{ij}\right] = 0.$$
3. If M' is flat, the condition (3.13) is
$$g^{ij}\left[\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k}\right] = 0.$$
Which is a diagonal system of equations. Therefore, the equality holds if and only if each f^α are harmonic in the sense of maps $f^\alpha : M \to \mathbb{R}$.

■

Proposition 3.2.3. *Take $M = S^1$ and M' arbitrary then the following statements are equivalent.*

1. *f_t is harmonic,*
2. *f_t is a closed geodesic,*
3. *$dE(f_t)/dt = 0$.*

Proof.

3. $\implies$ 1. The E-L equations applied on dE/dt give $\tau(f_t) = 0$. Hence, f_t is harmonic.

1. $\implies$ 3. Put $\tau(f_t) = 0$ on equation (3.11).

1. $\iff$ 2. Notice that
$$\tau(f_t)^\gamma(\theta) = \frac{d^2 f_t^\gamma}{d\theta^2} + \Gamma'^{\gamma}_{\alpha\beta}\frac{df_t^\alpha}{d\theta}\frac{df_t^\beta}{d\theta} = 0$$

is exactly the geodesic equation (1.13). ■

Proposition 3.2.4. *Let $f_t : M \times [0,T] \longrightarrow M'$ be a smooth deformation of f . Then,*

$$\frac{d^2E(f_t)}{dt^2} = 2\int_M g^{ij}\left\langle \nabla_{\partial_i}\left(\frac{\partial f_t}{\partial t}\right), \nabla_{\partial_j}\left(\frac{\partial f_t}{\partial t}\right)\right\rangle * 1 - \int_M g^{ij} R'_{\alpha\mu\beta\upsilon} f^\alpha_{ti} f^\beta_{tj} \frac{\partial f^\mu_t}{\partial t}\frac{\partial f^\nu_t}{\partial t} * 1 \tag{3.14}$$

where $R'_{\alpha\mu\beta\upsilon}$ is the Riemann tensor on M'.

Proof. Observe that

$$\nabla_{\partial_i}\left(\frac{\partial f_t}{\partial t}\right)^\alpha = \frac{\partial^2 f^\alpha_t}{\partial x^i \partial t} + \Gamma'^\alpha_{\mu\nu}\frac{\partial f^\mu_t}{\partial x^i}\frac{\partial f^\nu_t}{\partial t}, \quad \tau(f_t)^\alpha = g'^{ij}\left(\frac{\partial^2 f^\alpha_t}{\partial x^i \partial x^j} + \Gamma'^\alpha_{\mu\nu}\frac{\partial f^\mu_t}{\partial x^i}\frac{\partial f^\nu}{\partial x^j}\right),$$

the rest is an application of the Stokes theorem analogue to the procedure used in Proposition 3.2.1 and the use of equation (1.19) for the Riemann tensor. ■

Remark 3.2.2. *If $R'_{\alpha\mu\beta\upsilon}$ is nonpositive then $E''(f_t) < 0$.*

Proposition 3.2.5. *Let $E(t)$ be the function $E : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined as $E(t) := E(f_t)$ where f_t is the deformation of $f \in C^\infty(M, M')$ and similarly for $E'(t) := E'(f_t)$. Then, $E(t)$ is uniformly continuous and $E'(t)$ is continuous provided that the sectional curvature of M' is nonpositive.*

Proof. By Proposition 2.2.5, there exists a $k_1, k_2 > 0$ such that $|\frac{\partial^2 f^\gamma_t}{\partial t^2}| < k_1$ and $|\frac{\partial f^\nu_t}{\partial t}| < k_2$. Let $k = k_1 k_2$ and M the volume of the manifold M. Given $\varepsilon > 0$, choose $\delta = \varepsilon/(Mk)$. If $l, m \in \mathbb{R}^+$ satisfy $|l - m| < \delta$ then, using that $\tau(f) = \Delta f = \frac{\partial f_t}{\partial t}$,

$$\frac{dE(f_t)}{dt} = -\int_M \frac{\partial f^\gamma_t}{\partial t}\frac{\partial f^{\nu\,\nu}_t}{\partial t} g'_{\gamma\nu} * 1$$

hence,

$$\begin{aligned}
|E'(m) - E'(l)| &= \frac{1}{2}\left|\int_M \left(\frac{\partial f^\gamma_t}{\partial t}\frac{\partial f^\nu_t}{\partial t}\bigg|_{t=m} - \frac{\partial f^\gamma_t}{\partial t}\frac{\partial f^\nu_t}{\partial t}\bigg|_{t=l}\right) g'_{\gamma\nu} * 1\right| \\
&= \frac{1}{2}\left|\int_M \int_{t_l}^{t_m} 2\frac{\partial^2 f^\gamma_t}{\partial t^2}\frac{\partial f^\nu_t}{\partial t}\, dt\, g'_{\gamma\nu} * 1\right| \\
&\le \frac{1}{2}\int_M \left|\int_{t_l}^{t_m} 2\frac{\partial^2 f^\gamma_t}{\partial t^2}\frac{\partial f^\nu_t}{\partial t}\, dt\right| |g'_{\gamma\nu}| * 1 \le \frac{1}{2}\int_M \left|\int_{t_l}^{t_m} 2k\, dt\right| |\det(g')| * 1 \\
&\le k|t_m - t_l| M < \varepsilon.
\end{aligned}$$

Therefore, $E'(t)$ is uniformly continuous. On the other hand, let $\varepsilon > 0$ and $W > 0$ such that $|E'(t)| < W$ for all t (which exist because $E(t) > 0$ and $E''(t) < 0$ whenever $R_{\alpha\beta\eta\delta} < 0$) then, choose $\delta = \varepsilon/W$. If $l, m \in \mathbb{R}^+$ satisfy $|l - m| < \delta$ hence,

$$|E(m) - E(l)| = \left|\int_{t_l}^{t_m} E'(t)dt\right| \le \int_{t_l}^{t_m} W dt \le |t_m - t_l| W < \varepsilon.$$

That is, $E(t)$ is uniformly continuous. ∎

Theorem 3.2.3. *Let M be a compact Riemannian manifold with metric g and non-positive Riemannian curvature. Let $f : S^1 \to M$ be a closed curve in M and $\{f_t \mid f_t : S^1 \to M'\}_{t \geq 0}$ be a deformation of f such that $f_0 = f$. Then, there is a increasing subsequence $\{t_k \in \mathbb{R}^+\}_{k \in \mathbb{Z}^+}$ such that f_{t_k} converges uniformly (along with its derivatives) to a smooth mapping $F : S^1 \to M$. Furthermore, f and F lies in the same free homotopy class.*

Lemma 3.2.4. *If $\{f_n : M \to M'\}_{k \in \mathbb{N}}$ converges uniformly to $F : M \to M'$ then, there exist a $K > 0$ such that for all $n > K$, f_{t_n} and F are homotopic.*

Proof. By the Nash embedding theorem, there exist an $n \in \mathbb{N}$ such that M' is embedded in $\mathbb{R}^n$. Let $m \in \mathbb{N}$ define $h_m : [0,1] \times M \to \mathbb{R}^n$ by

$$h_m(t,p) = tF(p) + (1-t)f_m(p).$$

Since f and F are continuous, then $h_m : [0,1] \times M \to \mathbb{R}^n$ is continuous and thus h_m is a homotopy in the ambient space $\mathbb{R}^n$. Let U be a tubular neighborhood of M' in $\mathbb{R}^n$, Proposition 2.3.6 implies that there exist a deformation retract $r : U \to M'$. Since $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly to F, choose $K > 0$ such that for all $m \geq K$ then $h_m([0,1] \times M) \subseteq U$. Finally, define $H : [0,1] \times M \to M'$ by $H = r \circ h_K$ which is continuous (h_K and r are continuous), $H(0,p) = r(f_K(p)) = f_K(p)$ and $H(1,p) = r(F(p)) = F(p)$. Therefore, H is the desired homotopy. ∎

Proof of the theorem 3.2.3. In order to find a convergent subsequence, it is enough to apply the Arzelà-Ascoli theorem 1.4.6. Let us see that f_t satisfy the hypotesis of the theorem. Clearly $\{f_t\}_{t \in \mathbb{R}^+}$ is uniformly equicontinuous because f_t is uniformly continuous in the variable t by proposition 2.2.3. Let us see that $\{f_t\}_{t \in \mathbb{R}^+}$ is uniformly bounded. Let $x, y \in M, z \in M'$ and $\gamma_t : [0,1] \to M'$ be a picewise smooth function such that $\gamma(0) = f_t(x), \gamma(1) = f_t(y)$. Then, using inequality (1.14) we have

$$d(f_t(x), z) \leq \int_0^1 \sqrt{g_{ij}\dot{\gamma_t}^i \dot{\gamma_t}^j}\, d\tau \leq 2E(\gamma_t) \leq 2\kappa,$$

with $\kappa = \sup_{t \in \mathbb{R}^+} E(\gamma_t) = \max_{t \in \mathbb{R}^+} E(\gamma_t) < \infty$ because M' is compact.

Therefore, the family $\{f_t \mid f_t : M \to M'\}_{t \geq 0}$ is uniformly equicontinuos and uniformly bounded. Theorem 1.4.6 assures that there exist a subsequence $\{f_{t_k} \mid f_{t_k} : M \to M'\}_{k \in A}$ that converges uniformly. However, the limit function may not be smooth. We proceed with a diagonal argument.

For each i define the sets A_i, $i \in \mathbb{N}$ inductively such that $A_0 := A$ with A as above and, given A_i define $A_{i+1} \subseteq A_i$ infinite such that $\{f_{t_k}^{(i+1)} \mid f_{t_k}^{(i+1)} : M \to M'\}_{t_k \in A_{i+1}}$ converges uniformly, which exist by the above argument. Define $B = \{a_i\}_{i \in \mathbb{N}}$ with $a_0 \in A_0$ and $a_{i+1} \in A_{i+1} \backslash \{a_k : k < i+1\}$. Then, B is a subsequence (up to finite elements) of each A_i, consequently,

$$\{f_{t_k}^{(i)} \mid f_{t_k}^{(i)} : M \to M'\}_{t_k \in B}$$

converges uniformly for each $i \in \mathbb{N}$ and therefore, the limit function, which we shall denote by F,

$$\lim_{t_k \to \infty, t_k \in B} f_{t_k} := F$$

is smooth. Lemma 3.2.4 implies that given $k \in B$ big enough f_{t_k} is homotopic to F and thus, f_t, f and F lie in the same free homotopy class. ■

It remains to prove that F is geodesic.

Theorem 3.2.5. *The limit function F of the deformation $\{f_{t_k} \,|\, f_t : S^1 \to M'\}_{t_k \in B}$ is a geodesic provided that the sectional curvature of M' is non positive.*

Proof. We want to see that $E'(F)$ vanishes which implies that F is geodesic by proposition 3.2.3. Recall that

$$E(f_t) > 0,\ t \in \mathbb{R}^+, \quad \frac{dE(f_t)}{dt} < 0, \quad \frac{d^2E(f_t)}{dt^2} > 0.$$

Let us see that

$$\lim_{k\to\infty} \frac{dE(f_{t_k})}{dt} = 0.$$

We proceed by contradiction. Let $\varepsilon > 0$ suppose that for all N there is an $n > N$ such that $|E'(f_{t_n})| > \varepsilon$. Since E and its derivative E' are uniformly continuous, choose δ such that

$$|t - t_n| < \delta \implies |E'(t) - E'(t_n)| \le \frac{\varepsilon}{2}.$$

Since $|E'(f_t)| > 0$ take $t \in [t_n, t_n + \delta]$,

$$\begin{aligned} |E'(t)| &= |E'(t_n) - (E'(t_n) - E'(t))| \\ &\ge |E'(t_n)| - |E'(t_n) - E'(t)| \ge \varepsilon - \frac{\varepsilon}{2} \ge \frac{\varepsilon}{2}, \end{aligned}$$

and therefore,

$$\begin{aligned} \left| \int_a^{t_n+\delta} E'(t)dt - \int_a^{t_n} E'(t)dt \right| &= \left| \int_{t_n}^{t_n+\delta} E'(t)dt \right| \\ &\ge \int_{t_n}^{t_n+\delta} |E'(t)|\, dt \ge \int_{t_n}^{t_n+\delta} \frac{\varepsilon}{2} dt = \frac{\varepsilon\delta}{2} > 0. \end{aligned}$$

However, since E is continuous,

$$\begin{aligned} \lim_{n\to\infty} \left| \int_a^{t_n+\delta} E'(t)dt - \int_a^{t_n} E'(t)dt \right| &= \lim_{n\to\infty} |E(t_n+\delta) - E(t_n)| \\ &= \left| \lim_{n\to\infty} E(t_n+\delta) \right| - \left| \lim_{n\to\infty} E(t_n) \right| \\ &= |\alpha| - |\alpha| = 0, \end{aligned}$$

where we can interchange limits with the absolute value because $E(t)$ is a decreasing function. We arrive to a contradiction and consequently,

$$\lim_{k\to\infty} E(f_{t_k}) = 0 \text{ implies that } \lim_{t\to\infty} \frac{dE(f_{t_k})}{dt} = 0.$$

Hence,

$$\lim_{k\to\infty} \frac{dE(f_{t_k})}{dt} = \frac{d}{dt}\left(E\left(\lim_{k\to\infty} f_{t_k}\right)\right) = \frac{dE(F)}{dt} = 0$$

where the first equality is because $E'(f_t)$ is continuous on t. That is, F is a closed geodesic and F is homotopic to f by Lemma 3.2.4. ■

The result of the section can be summarized in the following theorem:

Theorem 3.2.6 (Eells and Sampson)**.** *Let M and M' two Riemannian manifolds. Assume M is compact and M' has non-positive sectional curvature then, in every homotopy free class of closed curves, there exists a closed geodesic.*

Corollary 3.2.6.1.

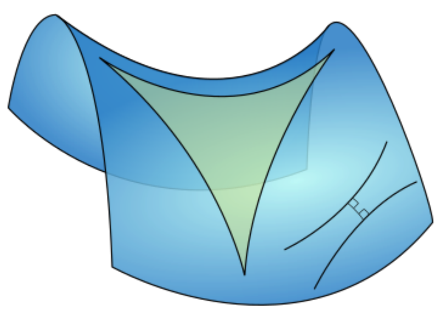

Figure 3: A compact (Triangle) surface immersed in a anti deSitter sapace. Retrived from [17]

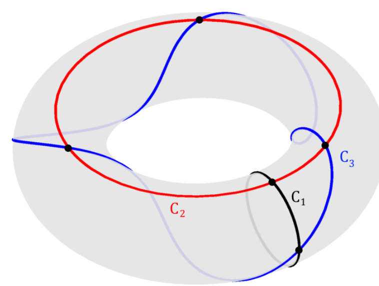

Figure 4: Three non-homotopic closed curves in the non-flat torus. Retrived from [18].

1. *Every closed curve in a flat torus is homotopic to a closed geodesic.*

2. *Every closed curve in a compact region of an anti deSitter space is homotopic to a closed geodesic.*

3. *Every closed curve in the cube $[0, 1]^n$, $n \in \mathbb{N}$ is homotopic to a constant curve.*

Final comments and outlook

In this work we prove the Cartan-Hadamard theorem which exemplify how the energy map, in the sense of Eells and Sampson, carries geometric information of the manifold. In the process we made use of several techniques of geometric analysis that will allow us to investigate further properties. In the main theorem we only use the fact that $f : S^1 \to M$ is a closed curve and that S^1 is compact. Further work may be applied on the study of minimal surfaces with $f : S^2 \to M$ and, in general $f : M \to M'$ provided M is compact. In particular, we may apply this ideas in the analysis of geodesics in Lorentzian manifolds as well as the study of the following topics

1. Study of closed timelike curves on a Lorentzian manifold,
2. Can a timelike curve be deformed into a spacelike curve?
    (a) If affirmative, what does this mean about causality?
    (b) If negative, what is the difference of a timelike curve and a spacelike curve (in terms of an energy map)?
3. How we relax the condition of theorem, of the sectional curvature being non positive?

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