Lecture 6.1: Last time: more about spinors, with/without real forms
- related table of Clifford modules to Euclidean/Lorentzian spin groups
- Constructing the Dirac spinor over $\mathbb{C}$ using a polarization ($L \subseteq V_{\mathbb{C}}$ max$^{\mathbb{C}}$ isotropic)
"aka canonical quantization"
- Gamma matrices from Clifford multiplication
- built a pairing on the Dirac spinor $S = \Lambda^{\cdot}(L^{\vee})$ = functions on ΠL
$(s,t) \to (s \wedge t)_{top}$
(fermionic integration is strange: $\begin{cases} \int d\theta\, 1 = 0 \\ \int d\theta\, \theta = 1 \end{cases}$)

- $g(\Gamma(\overset{S}{s},\overset{S}{t}), \overset{V}{v}) = (s, \underset{\text{Clifford action}}{vt})$ spin pairing

This time: Finally write down all super~~symmetry~~ translations algebras! $N = \oplus S, \oplus S'$
V acts on itself by translations $\Rightarrow$ abelian Lie algebra
V is a rep. of spin(V) $\Rightarrow$ "supercharges" should be as well
By spin and statistics, supercharges sit in (some # of copies of) the spin rep.
(space of supercharges $= \Pi N$). The bracket $\Pi N \times \Pi N \to V$ should be symmetric, spin(V)-equivariant, built from Γ.

Dim(1): Spin$(V) = \mathbb{Z}_2$, acts by parity.
Simplest example: $V = \mathbb{R}$, $N = \mathbb{R}$, bracket = tensor product ($Q^2 = P$)
Extend this $\rightsquigarrow$ $N = \mathbb{R}^{n} = $ "S" $\otimes U$, $V = \mathbb{R}$, bracket = pairing ~~tensor product~~
U has a symmetric pairing
$(\{Q_i, Q_j\} = P\delta_{ij})$
$\Rightarrow$ New automorphisms: $SO(U)$

Dim 2: Lorentzian signature
$Cl(1,1) \cong M_2(Cl(0,0)) \cong M_2(\mathbb{R})$
$Cl(1,1)^{+} \cong Cl(1,0) = \mathbb{R} \oplus \mathbb{R}$ $\Rightarrow$ two real spin reps
To construct them $e_{\pm} = t \pm x \in V$. $\Lambda^0(L^{\vee}), \Lambda^1(L^{\vee})$ are our two spin reps
Clifford multiplication $\rightsquigarrow$ $\underset{s_+}{\Lambda^0(L^{\vee})} = S_+ \underset{e_-}{\overset{e_+}{\rightleftarrows}} \underset{s_-}{\Lambda^1(L^{\vee})} = S_-$

$\langle \Gamma(s_{\pm}, s_{\pm}), e_{+,-} \rangle = \begin{cases} (s_+, e_+ \cdot s_+) = (s_+, s_-) = 1 \Rightarrow \Gamma(s_+, s_+) = e_- \\ (s_+, e_- s_+) = 0 \\ (s_+, e_+ s_-) = (s_+, s_+) = 0 \end{cases}$

$(s_-, e_- s_-) = (s_-, s_+) = 1 \Rightarrow \Gamma(s_-, s_-) = e_+$
We can add just S_+ to the algebra!
$N = S_+^{\oplus d}$, $V = V_+ \oplus V_-$ $\boxed{!}$ (wave eq in 1+1 $\rightsquigarrow$ $-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} = 0$
$\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) = 0$)
$S_+^{\otimes 2} = V_-$, in coordinates: Q_-, $\overset{\Pi V_+}{P_+}$ $\overset{\Pi V_-}{P_-}$
$Q_-^2 = P_-$
The $d = 1,1$ looks like 2 indep. copies of SQM $P^2 = P_+ P_-$
$N = (N_+, N_-)$ "amount of extended susy"

→ Unique 2-dim real spin rep. $\quad Cl(1,2)^+ \cong Cl(1,1) \simeq M_2(\mathbb{R})$

$t, x_1, x_2 \rightsquigarrow e_\pm = t \pm x_1,\ x_2 \rightsquigarrow$ L (isotropic coordinates) generated by e_+, as before

$S = \Lambda^\bullet(L^\vee)$

$$S_- \in \Lambda^0(L^\vee) \underset{e_-}{\overset{e_+}{\rightleftarrows}} \Lambda^1(L^\vee) \ni S_+$$

$x_2 = +1$ on $\Lambda^0(L^\vee)$, $x_2 = -1$ on $\Lambda^1(L^\vee)$

- $g(\Gamma(s_-, s_-), e_+) = (s_-, e_+ \cdot s_-) = (s_-, s_+) = 1 \Rightarrow \Gamma(s_-, s_-) = e_-$
- $g(\Gamma(s_-, s_+), x_2) = (s_-, x_2 \cdot s_+) = -(s_-, s_+)$
- $g(\Gamma(s_+, s_-), x_2) = (s_+, x_2 \cdot s_-) = (s_+, s_-) = (s_-, s_+)$

$$\Gamma(x_2) = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix},\quad \Gamma(e_+) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\quad \Gamma(e_-) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

<u>Recap:</u> Constructing supersymmetric algebras in various dimensions

dim (Lorentzian)	minimal # of supercharges	chiral?	R-symmetry
0 (HW)	1	X	$GL(N)$
1 (easy one)	1	X	$SO(N)$
2	1 (Majorana Weyl)	✓	$SO(N_L) \times SO(N_R)$
3	2	X	$SO(N)$
4	4 (Majorana)	X	$U(N)$
6	8 Majorana-Weyl	✓	$Sp(N_L) \times Sp(N_R)$

10	16	✓	$SO(N_L) \times SO(N_R)$
11	32	X	$SO(N)$

← 8-periodic: $11 - 8 = 3$

<u>dim 3:</u> $Cl(1,2) \simeq Cl(1,1) \simeq M_2(\mathbb{R})$

$$\cup$$

$$Spin(1,2)_0 \simeq SL(2, \mathbb{R})$$

Unique spinor: $S = \mathbb{R}^2$. $\quad S \otimes S = \underset{\underset{\mathbb{R}^3}{\cong}}{Sym^2(S)} \oplus \underset{\underset{\mathbb{R} = \text{trivial}}{\cong}}{\Lambda^2(S)}$ (vectors are 2 matrices)

$\Gamma_S \simeq S^\vee$

$S \otimes S \simeq S \otimes S^\vee \simeq M_2(\mathbb{R})$

$(\operatorname{Hom}(S,S))$

$S \simeq S^\vee$

$\Leftrightarrow \mathbb{R} \simeq S \otimes S$

$S^\vee \simeq S$

$\Leftrightarrow \mathbb{R} \to S \otimes S$

$\det \begin{bmatrix} a & b+c \\ b-c & -a \end{bmatrix} = c^2 - b^2 - a^2$

$Sym^2(S) \simeq V \simeq \mathbb{R}^{1,2}$

To extend $S \otimes U$, $U = \mathbb{R}^N \rightsquigarrow Sym^2(S \otimes U) \to V$

Let's look $Sym^2(S \otimes U) = Sym^2(S) \otimes Sym^2(U) \oplus \Lambda^2(S) \otimes \Lambda^2(U)$

$$\underset{V}{\cong} \qquad \qquad \underset{\mathbb{R}}{\downarrow} \leftarrow \text{inner product}$$

<u>dim 4:</u> $Cl(1,3)^+ \simeq Cl(1,2) \simeq M_2(Cl(0,1)) \longrightarrow$ R-symmetry is $SO(N)$

$$= M_2(\mathbb{C})\,!$$

$$\cup$$

$$Spin(1,3)_0 \simeq SL(2, \mathbb{C})$$

$\det 1 \rightsquigarrow \Lambda^2 S \simeq \mathbb{C}$ (trivial rep.)

(HW)

$S = \mathbb{C}^2$, $\bar{S} = \bar{\mathbb{C}}^2 = S^\vee$ $\rightsquigarrow$ $S \otimes \bar{S} = \mathrm{Hom}(S,S) = M_2(\mathbb{C})$

$(S \otimes \bar{S})_0$ $\hat{=}$ traceless 2x2 complex matrices $\quad \begin{bmatrix}1\\0\end{bmatrix}[0\ -1]$

$\begin{bmatrix}a\\b\end{bmatrix} \longleftrightarrow [\,\cancel{\bar{a}}, \bar{b}\,] \xrightarrow{\sim} \begin{bmatrix}-\bar{b}\\ \bar{a}\end{bmatrix}$

$\begin{bmatrix}a\\b\end{bmatrix} \in S$, $[\,\bar{a}, \bar{b}\,] \in \bar{S}$, $\begin{bmatrix}-\bar{b}\\ \bar{a}\end{bmatrix} \in S$

$\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \longrightarrow \begin{bmatrix}0&0\\-1&0\end{bmatrix}$ | $SL(2,\mathbb{C})$ G hermitian 2x2 mtcs $\overset{\det}{\cong} \mathbb{R}^{1,3}$

$i\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}0&0\\i&0\end{bmatrix}$

$S \oplus \bar{S}$ = 4 complex rep of $Spin(1,3)$. $\rightsquigarrow \Lambda^2(S \oplus \bar{S}) \cong \Lambda^2(S) \oplus \Lambda^2(\bar{S}) + S \otimes \bar{S}$ (Dirac spinor) vector rep.

$(S \oplus \bar{S})_{\mathbb{R}}$ is a four real dimension (majorana)

Normally $\rightsquigarrow$ $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = P_{\alpha\dot{\beta}} = \gamma^\mu{}_{\alpha\beta} P_\mu$

Extended susy in 4d:

$S \otimes_{\mathbb{C}} U$, $U = \mathbb{C}^N$ $\quad (S \otimes_{\mathbb{C}} U \oplus \bar{S} \otimes_{\mathbb{C}} \bar{U})_{\mathbb{R}}$ $\quad U \otimes_{\mathbb{R}} \bar{U} \longrightarrow \mathbb{C}$ Hermitian inner product

<u>Dim 6</u> $\quad Cl(1,5)^+ \cong Cl(1,4) = M_2(Cl(0,3)) \cong M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$

$(Spin(1,5))_0 \cong SL(2,\mathbb{H}))$

$Cl(1,5)^+ \rightsquigarrow$ 2 chiral spinors $\quad S^+ \cong \mathbb{H}^2 \cong \mathbb{C}^4$

$S^- \cong \mathbb{H}^2 \cong \mathbb{C}^4$

Dirac construction: $\quad L \subseteq \mathbb{C}^6$, $L \cong \mathbb{C}^3$ $\quad S_+ = \Lambda^{even}(L^\vee) \cong \Lambda^0 \oplus \Lambda^2 \cong \mathbb{C} \oplus \Lambda^2 L^\vee$

$S_- = \Lambda^{odd}(L^\vee) \cong \Lambda^1 \oplus \Lambda^3 \cong L^\vee \oplus \Lambda^3 L^\vee$

$\Lambda^0 L^\vee \longleftarrow \Lambda^1 L^\vee$

$\Lambda^1 L^\vee$

$\Lambda^2 L^\vee \longleftarrow \Lambda^3 L^\vee$

$\cancel{\Lambda^3 L^\vee}$

S_+, S_-

$v \in L$: $t \in S_-$, $\cancel{s} t \in S_+ \Rightarrow s \in S_-$ | $(s, v \cdot t) \hat{=} q(\rho(v), v)$

$\Gamma: S_\pm \otimes S_\pm \longrightarrow V$

antisymmetrically!

$\dim(\Lambda^2 S_+) = 6 = \dim(V)$

$\mathbb{C}$ upper diagonal 4x4 matrix

in 6d $\quad Sym^2(S \otimes U) = Sym^2(S) \otimes Sym^2(U) \oplus \Lambda^2(S) \otimes \Lambda^2(U)$

$(S_+ \otimes_{\mathbb{C}} U)_{\mathbb{R}} \cong \mathbb{R}^8$ "symplectic-majorana-weyl" $\quad \Lambda^2 \downarrow V$ $\quad \downarrow$ symplectic form. $\mathbb{R}$

$\Lambda^2 S_+ \cong V$, $\Lambda^2 U \xrightarrow{\omega} \mathbb{R}$.

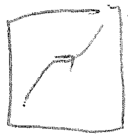
<u>Brief recap</u>: Supersymmetry $\leftarrow$ Bosons / Fermions : exchange statistics

Motivation: The Single-Particle Hilbert Space is a super vector space

- consequences of susy 1) BPS bound (energy is positive)

boson-fermion pairs degeneracy

They are exactly solvible.

- Quantization of fermionic degrees of freedom
(Fermionic phase spaces...are inner product spaces)
- Clifford algebra (canonical anticommutation relations)
  - classified all of them over $\mathbb{R}, \mathbb{C}$.
  - studied their rep. in 1 or 2, depending...
  - spinors, map $(\mathbb{R}^{0|1} \to \mathbb{R}^{1,d-1})$ $\rightsquigarrow$  spacetime

  "odd degrees of freedom arise as quanta of spinor fields".

  There are Weyl, Majorana, Symplectic Majorana, Dirac, ... spinors
- Supersymmetry algebras for $d = 2,3,4,6$ various $\mathcal{N} \geq 1$
real forms ... (figured out how γ^μ matrices work out in general dim)

<u>Goal</u>: Understand susy field theories and interactions in higher spacetime dim

A particularly nice way "superspace"

~~History~~! There is a "No-Go" theorem by Coleman-~~Mandula~~, if a relativistic QFT has an Lie algebra of symmetries g that includes poincaré symmetries p (i.e. $P \subseteq g$) then ...(assumptions)... $g = P \times h$ and h acts internally.

<u>Doesn't</u> apply to super Lie algebras!

Fixed up by Haag-Łopuszański-Sohnius: only the super Lie algebras were [illegible] are allowed.

General structure of super-Poincaré algebras

- Choose the space of supercharges $W = \text{"}S \otimes U\text{"}$, $\{W, W\} \to T$ (Translations)

$$0 \to T \to \text{super } T \to \Pi W \to 0$$

$$P^\mu \quad M^\nu_\mu \quad \Pi W \oplus T \circlearrowright \mathrm{Spin}(1,d-1) \ (\text{locally})$$

$$\mathbb{R}$$

- poincaré is $T \rtimes \mathrm{spin}(1,d-1)$
- Super-poincaré is super $T \rtimes \mathrm{Spin}(1,d-1)$

$$[P^\mu, P^\nu] = 0$$
$$[M_\mu^\nu, M_\lambda^\rho] = \delta_\mu^\rho M_\lambda^\nu + \ldots$$
$$[P^\mu, M^\rho_\lambda] = \delta^\mu_\lambda P^\rho$$

Two ways of build supersymmetric field theories
- Components
- Superspace

If I think about Kaluza-Klein reduction. e.g. from 5d to 4d

Let's say I have a gauge field $A^\mu(x)$ in 5d. $\mathbb{R}^{1,3} \times S^1$ (y, θ)

$$A = A_\mu(x)\, dx^\mu = A_i(y,\theta)\, dy^i + A_4(y,\theta)\, d\theta$$

$$A_i \sim \sum A_i^{(n)}(y)\, e^{in\theta} \quad \text{circle small: very massive for } n \neq 0.$$

$$\xrightarrow{S^1\text{ small}} A_i^{(0)}(y)\, dy^i + A_4^{(0)}(y)\, d\theta$$

If I have an odd direction: $\mathbb{R}^{1|1}$ Scalar superfield: $f(y^i, \varepsilon)$

A super space is like a kind of brain but its exact in the sense that it doesn't have to be small to be unobservable $\longrightarrow f_0(y^i) + \varepsilon f_1(y^i)$

bosonic scalar, $\boxed{\varepsilon^2 = 0}$, fermionic scalar

Exponentiating supertranslations. Ex $d=1$, $N=2$.

Translations generated by P, symmetries Q_1, Q_2 with $\{Q_i, Q_j\} = 2\delta_{ij} P$ only non zero bracket. Consider a finite translation: $T_a = \exp(a \frac{\partial}{\partial x})$, $P = \frac{\partial}{\partial x}$ (use convention that P is anti hermitian) hence

$T_a: f(x) \to f(x+a)$

Could try (let's see Q_1 transitions in odd direction)

$\exp(\psi \frac{\partial}{\partial \varepsilon}) \longrightarrow f(\varepsilon) = f_0 + \varepsilon f_1$

$\parallel \qquad \varepsilon \to \varepsilon + \psi : (f_0 + \psi f_1) + \varepsilon \psi_1$

then $1 + \psi \frac{\partial}{\partial \varepsilon}$

BCH formula $\to e^A e^B = e^{A+B+\frac{1}{2}[A,B] + \ldots}$ for super translation algebras

$\Rightarrow e^{\psi Q_1} e^{\psi' Q_1} = e^{\psi Q_1 + \psi' Q_1 + \underline{\psi\psi'} P} \rightsquigarrow Q_1 = \frac{\partial}{\partial \varepsilon_1} + \varepsilon \frac{\partial}{\partial x}$

$Q_1 f = (\frac{\partial}{\partial \varepsilon} + \varepsilon \frac{\partial}{\partial x})(f_0 + \varepsilon f_1) = f_1 + \varepsilon f_0'$

Finite supertranslations: $\exp(\psi_1 Q_1) = 1 + \psi_1 Q_1 = 1 + \psi_1 \frac{\partial}{\partial \varepsilon_1} + \psi_1 \varepsilon_1 \frac{\partial}{\partial x}$

$f_0 + \varepsilon f_1 \longrightarrow f_0 + \varepsilon_1 f_1 + \psi_1 f_1 + \psi_1 \varepsilon f_0'$

i.e. $\qquad = f(x + \varepsilon_1 \psi_1, \varepsilon_1 + \psi_1)$

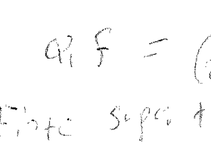
$f(x, \varepsilon_1) \xrightarrow{T_{\psi_1}} f(x + \psi_1 \varepsilon_1, \varepsilon_1 + \psi_1) \xrightarrow{T_{\psi_1'}} f(x + \psi_1 \varepsilon_1 + (\varepsilon_1 + \psi_1)\psi_1', \varepsilon_1 + \psi_1 + \psi_1')$

$= f(x + (\psi_1 + \psi_1')\varepsilon_1 + \underline{\psi_1 \psi_1'}, \varepsilon_1 + (\psi_1 + \psi_1'))$

Last time: Started discussing superfields (functions or fields on superspacetime)

Exam review: Wed 3.8, 14:00
Exam: Fri 5.8, 16:00
Einsicht: 9.8, 13:00

Spacetime



$\rightsquigarrow$ superspacetime is modeled on finite supertranslations,

BCH formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$ stops in the case of supertranslations

$(b-a)$ finite translations

Simple example of a superfield:
$N=2$ SQM $\rightsquigarrow$ 2 symmetries $\rightsquigarrow \{Q_i, Q_j\} = \delta_{ij} 2P$, $i,j=1,2$

Corresponding superspace: $T = \exp(t) \simeq \mathbb{R}^{1|2}$ $\quad \uparrow t = \mathbb{R}^{1|2}$

then T is equipped with one even and 2 odd coordinates.

A general function looks like $f(x, \theta_1, \theta_2) = \boxed{f_0(x) + f_1(x)\theta_1} + f_2(x)\theta_2 + f_{12}(x)\theta_1\theta_2$

2 bosonic fields, 2 fermionic fields

How do super symmetry transf act? $\exp(a \cdot P) \rightsquigarrow \exp(a \frac{\partial}{\partial x}): f(x, \theta_1, \theta_2) \to f(x+a, \theta_1, \theta_2)$

on odd: $\exp(\varepsilon_1 Q_1 + \varepsilon_2' Q_2) = \exp(\varepsilon_1 Q_1)\exp(\varepsilon_2' Q_2)\exp(-\varepsilon_1 \varepsilon_2' P)$

hence, $\exp(\varepsilon_1 Q_1) = \exp\left(\varepsilon_1\left(\frac{\partial}{\partial\theta_1} + \theta_1 \frac{\partial}{\partial x}\right)\right)$.

Check $\left\{\frac{\partial}{\partial\theta_i} + \theta_i\frac{\partial}{\partial x}, \frac{\partial}{\partial\theta_j} + \theta_j\frac{\partial}{\partial x}\right\} = 2\delta_{ij}\frac{\partial}{\partial x}$.

$$\Rightarrow Q_1 f = (\partial_{\theta_1} + \theta_1\partial_x)\left(f_0(x) + f_1(x)\theta_1 + f_2(x)\theta_2 + f_{12}\theta_1\theta_2\right)$$
$$= \left(f_1 + f_{12}\theta_2 + \frac{\partial f_0}{\partial x}\theta_1 + \frac{\partial f_2}{\partial x}\theta_1\theta_2\right)$$

$\left.\begin{array}{l}\delta f_0 = \varepsilon_1 f_1 \\ \delta f_1 = \varepsilon_1 \frac{\partial f_0}{\partial x}\end{array}\right\}$ familiar

$\left.\begin{array}{l}\delta f_2 = \varepsilon_1 f_{12} \\ \delta f_{12} = \varepsilon_1 \frac{\partial f_2}{\partial x}\end{array}\right\}$ strange

① what do action functionals look like?

② Integrate out extra boson f_{12} ("auxiliary field")

Note that f is a super field but $Q_1 f$ is not a superfield!

$Q_1(Q_1 f) = Pf = \frac{\partial f}{\partial x} \neq 0$!

The functions on a group G are a rep of G - by translating.
In fact, in two ways: left and right translations.

$h: f(g) \to f(gh^{-1})$ or $h(g): f(g) \to f(h^{-1}g)$

Right translations give other rep. of supersymmetry:

$$P \mapsto \underbrace{-\frac{\partial}{\partial x}}, \quad Q_i \mapsto \underbrace{-\frac{\partial}{\partial\theta_i} + \theta_i\frac{\partial}{\partial x}}_{\text{usually called } D_i \text{ or "supersymmetric cov derivatives"}}$$

$$\{D_i, Q_j\} = \left\{-\frac{\partial}{\partial\theta_i} + \theta_i\frac{\partial}{\partial x}, \frac{\partial}{\partial\theta_j} + \theta_j\frac{\partial}{\partial x}\right\} = -\delta_{ij}\frac{\partial}{\partial x} + \delta_{ij}\frac{\partial}{\partial x} = 0$$

- $\int_{\mathbb{R}^{1|0}} dx\, f(x) = \int_{\mathbb{R}^{1|0}} dx\, f(x+a)$ (translation invariance)

- $\int_{\mathbb{R}^{1|0}} dx\, f'(x) = 0$ if $f \xrightarrow[x\to\infty]{} 0$ (integration by parts)

- $\int_{\mathbb{R}^{2|0}} dx\,dy\, f(x,y) = \int_{\mathbb{R}^{1|0}} dx \left(\int_{\mathbb{R}^{1|0}} dy\, f(x,y)\right)$ (Fubini theorem)

Hence, it's enough to understand $\int d\theta\, f(\theta) = \int d\theta\,(f_0 + \theta f_1)$

$$\int d\theta\, f(\theta+\psi) = \int d\theta\,(\underbrace{f_0 + \psi f_1}_{f_0'} + \theta f_1)$$

Hence $\int d\theta\, f(\theta) \triangleq f_1$ otherwise it will not be trans inv.

↳ "Berezin integral" $\left(\int d\theta\,\theta = 1, \int d\theta\, 1 = 0\right)$

Change of coordinates • $\int dx\, f(x) = \int dy\, f(y) \frac{dx}{dy}$

• $\int d\theta\, f(\theta) = f_1$, $\psi := a\theta \rightsquigarrow f(\psi) = f_0 + a\theta f_1$, $\frac{\partial\psi}{\partial\theta} = a$

$\int d\psi\, f(\psi) = a\int d\theta\, f(\theta)$

$\int d\theta = \frac{\partial}{\partial\theta} \rightsquigarrow \int d\psi\, f(\theta) = \int d\psi \left(f_0 + \frac{f_1}{a}\psi \right) = \frac{f_1}{a} \Rightarrow$

$d\psi = \frac{d\theta}{a} = \left(\frac{\partial\psi}{\partial\theta}\right)^{-1} d\theta$ ← recall $\mathrm{sdet}\left(\frac{A}{\ |\ B}\right) = \frac{\det A}{\det B}$.

Try a kinetic potential for the super field f.

$$\int_{\mathbb{R}^{1|2}} dx\, d\theta_1 d\theta_2\, D_1 f\, D_2 f$$

$D_1 f = -f_1 - f_{12}\theta_2 + \frac{\partial f_0}{\partial x}\theta_1 + \frac{\partial f_2}{\partial x}\theta_1\theta_2$

$D_2 f = \cancel{f_2}$

$= -f_2 + f_{12}\theta_1 + \frac{\partial f_0}{\partial x}\theta_2 - \frac{\partial f_1}{\partial x}\theta_1\theta_2$

$$\Rightarrow \Big[\underbrace{f_1 \frac{\partial f_1}{\partial x} - f_2 \frac{\partial f_2}{\partial x}}_{\text{2-fermion kinetic terms}} + \underbrace{\left(\frac{\partial f_0}{\partial x}\right)^2}_{\text{" boson kinetic term}} - f_{12}^2 \Big]\theta_1\theta_2 + \cdots$$

↑ no derivatives ⤳ eq. of motion ⇒ $f_{12} = 0$

Last time: 2 possible approaches to susy field theories.

(1) "components" write down an ordinary theory ($S = \int_{\mathbb{R}^{d-1}} \mathcal{L}(\phi, \psi_A)$) ⤳ come up with some susy transformations

$(\delta\phi = \epsilon\psi,\ \delta\psi = \epsilon\partial\phi)$ — ϵ odd, ϕ boson, ψ fermion. Check by hand that $\delta S = 0$ i.e. there is susy.

e.g. 1 dim $\mathcal{N}=2$ susy (a long time ago)

(2) use super fields i.e. functions on the superspace and write $S = \int_{\mathbb{R}^{1,d-1|K}} dx\, d\theta\, \mathcal{L}(\Phi)$

K ← number of super charges depends on the dim of the spin rep.

– Susy transf. are automatic: $Q \sim \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$

If I use μ for spacetime and α for index on space W of susy's.

$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + \theta^\beta \gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu}$, $\{Q_\alpha, Q_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu$

A general element $Q \in W \rightsquigarrow Q = \epsilon^\alpha Q_\alpha$

$\{Q_\alpha, Q_\beta\} = \left\{\frac{\partial}{\partial\theta^\alpha} + \theta^\lambda \gamma^\mu_{\alpha\lambda}\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial\theta^\beta} + \theta^\delta\gamma^\nu_{\beta\delta}\frac{\partial}{\partial x^\nu}\right\} = \delta^\delta_\alpha \gamma^\nu_{\beta\delta}\frac{\partial}{\partial x^\nu} + \delta^\lambda_\beta \gamma^\mu_{\alpha\lambda}\frac{\partial}{\partial x^\mu}$

$= \gamma^\nu_{\beta\alpha}\frac{\partial}{\partial x^\nu} + \gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu} = 2\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu}$

There are also "covariant derivatives"

$D_\alpha = -\frac{\partial}{\partial\theta^\alpha} + \theta^\beta\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu} \rightsquigarrow \{Q_\alpha, D_\beta\} = 0$

$\{D_\alpha, D_\beta\} = 0$

Note that $Q\mathcal{L}(\Phi) \rightsquigarrow S = \int d\theta\, dx\, Q\mathcal{L}$

↑ has $\left(\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}\right)\cancel{\mathcal{L}}$

orthonormal basis $\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta} P$

$$\Phi = \Phi(x,\theta_1,\theta_2) = f_0(x) + f_1(x)\,\theta_1 + f_2(x)\,\theta_2 + f_{12}(x)\,\theta_1\theta_2$$

I need a term of the form $(f_0')^2\,\theta_1\theta_2$ ← kinetic energy, dug into S.

guess ~ $D_1\Phi\, D_2\Phi$.

$$(D_1\Phi) = \left(-\frac{\partial}{\partial\theta_1} + \theta_1\frac{\partial}{\partial x}\right)\Phi = \left(-f_1 - f_{12}\theta_2 + \theta_1\frac{\partial f_0}{\partial x} + \frac{\partial f_2}{\partial x}\theta_1\theta_2\right)$$

$$(D_2\Phi) = \left(-f_2 + f_{12}\theta_1 + \theta_2\frac{\partial f_0}{\partial x} - \frac{\partial f_1}{\partial x}\theta_1\theta_2\right)$$

$$\int d^2\theta\, D_1\Phi D_2\Phi = \underbrace{f_1\frac{\partial f_1}{\partial x} - f_2\frac{\partial f_2}{\partial x}}_{\text{Dirac language kinetic}} + \underbrace{\left(\frac{\partial f_0}{\partial x}\right)^2}_{\text{Boson}} - f_{12}^2$$

↳ EOM ~ $f_{12} = 0$

↳ $\mathcal{L} \sim f_{12}^2$ ~ $\frac{\partial \mathcal{L}}{\partial f_{12}} = 0 = f_{12}$

We constructed $N=2$ SQM using a "superpotential" $W(f_0)$

Mass term? Φ^2 is reasonable.

$$\int d^2\theta\,\Phi^2 = 2(f_1 f_2 + f_0 f_{12}) \sim \int d^2\theta\left((D_1\Phi)(D_2\Phi) + \frac{m}{2}\Phi^2\right)$$

$$= \left(\frac{\partial f_0}{\partial x}\right)^2 + f_1\frac{\partial f_1}{\partial x} - f_2\frac{\partial f_2}{\partial x} - f_{12}^2 + m f_0 f_{12} + m f_1 f_2$$

↳ EOM: $\frac{\partial \mathcal{L}}{\partial f_{12}} = 0 = -2f_{12} + m f_0$.

$$= \left(\frac{\partial f_0}{\partial x}\right)^2 + f_1\frac{\partial f_1}{\partial x} - f_2\frac{\partial f_2}{\partial x} + \frac{m^2}{4}f_0^2 + m f_1 f_2$$

Dirac mass (in a majorana basis)

More generally, could write $\int W(\Phi)$ for some W.

If $W(\Phi) = \Phi^\ell$ ~ $\int d^2\theta\, W(\Phi) = \ell f_0^{\ell-1} f_{12}\,\theta_1\theta_2 + \ell(\ell-1) f_0^{\ell-2} f_1 f_2\,\theta_1\theta_2$

$$S = \int_{\mathbb{R}^{1|2}} \left(D_1\Phi\, D_2\Phi + W(\Phi)\right) \sim \int d^2\theta\,(\text{kinetic}) - f_{12}^2 + \ell f_0^{\ell-1} f_{12} + \ell(\ell-1) f_0^{\ell-2} f_1 f_2$$

EOM for f_{12} ~ $2f_{12} = \ell f_0^{\ell-1} = W'(f_0)$

$$\sim (\text{kinetic}) + \frac{1}{4}\underbrace{(\ell f_0^{\ell-1})}_{W'(f_0)}{}^2 + \underbrace{\ell(\ell-1) f_0^{\ell-2}}_{W''(f_0)} f_1 f_2$$

How does one see the component susy transf?

$$Q_1 = \frac{\partial}{\partial\theta_1} + \theta_1\frac{\partial}{\partial x} \sim \begin{cases} Q_1\Phi = f_1 + f_{12}\theta_2 + \theta_1\frac{\partial f_0}{\partial x} + \frac{\partial f_2}{\partial x}\theta_1\theta_2 \\ Q_2\Phi = f_2 - f_{12}\theta_1 + \theta_2\frac{\partial f_0}{\partial x} - \frac{\partial f_1}{\partial x}\theta_1\theta_2 \end{cases}$$

⇒ $\delta_{Q_1} f_0 = f_1$ $\quad \delta_{Q_1} f_2 = f_{12}$

$\delta_{Q_1} f_1 = \frac{\partial f_0}{\partial x}$ $\quad \delta_{Q_1} f_{12} = \frac{\partial f_2}{\partial x}$

integrate out f_{12}

$$\boxed{\delta_{Q_1} f_0 = f_1 \quad \delta_{Q_1} f_2 = \frac{1}{2}W'(f_0) \quad \delta_{Q_1} f_1 = \frac{\partial f_0}{\partial x}}$$

In conclusion:

$$\delta_{Q_2} f_0 = f_2 \qquad \delta_{Q_1} f_0 = f_1$$

$$\delta_{Q_2} f_1 = -\frac{1}{2}\omega'(f_0) \qquad \delta_{Q_1} f_1 = \frac{\partial f_0}{\partial x}$$

$$\delta_{Q_2} f_2 = \frac{\partial f_0}{\partial x} \qquad \delta_{Q_1} f_2 = \frac{1}{2}\omega'(f_0)$$

Counting degrees of freedom

	how many fields (off shell)	how many degrees of freedom (on shell)
Scalar	1	1
gauge fields	d	$d-2$
Dirac fermion (massive)	$2^{\lfloor d/2 \rfloor}$	$2^{\lfloor d/2 \rfloor - 1}$
Auxiliary Scalar	1	0

1d $N=2$	on shell	off shell
scalar	1	1 $\leftarrow f_0$
fermi	1	2 $\leftarrow f_1, f_2$
aux	0	1 $\leftarrow f_{12}$

In 4-dim minimal ($N=1$) susy what might be expect? superspace $\mathbb{R}^{1,3|4}$
Minimal fermions have 4 real components (whether Majorana or Weyl)
$\Rightarrow$ On-shell 2-degrees of freedom

Chiral multiplet

	on shell	off shell
fermi	2	$4 = 2$ cpx
Scalars	2	$2 = 1$ cpx
Aux	0	$2 = 1$ cpx (complex)

	on shell	off shell
fermi	2	4
Vector	2	4

"vector multiplet"

Superfield
$\Phi(x^\mu, \theta^\alpha, \bar{\theta}^{\dot\alpha})$

Lorentz rep's.

Categories of Φ

$(\theta)^0$:	1	$+$ even
(θ)	4	$-$ odd
$(\theta)^2$	6	$+$
$(\theta)^3$	4	$-$
$(\theta)^4$	1	$+$

one scalar

one Dirac spinor

$\theta^\alpha \theta^\beta \qquad \theta^\alpha \bar\theta^{\dot\beta} \qquad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}$

1 scalar, 1 vector, 1 scalar

one Dirac spinor

$[S_+ \otimes S_- \cong V]$ "vectors as bispinors"

1 scalar

Real superfield
1 real scalar
1 Majorana
1 Real vector
1 complex (two real scalars)
1 Majorana
1 real scalar

To pick out subrepresentations of the free superfields need to impose constrains compatible w/ supersymmetry
Can act w/ D operator for example

Chiral superfield: $\bar{D}_{\dot\alpha}\Phi = 0$. If $D_\alpha \Phi = \bar{D}_{\dot\alpha}\Phi = 0 \Rightarrow \{D_1, \bar{D}_{\dot 2}\}\Phi = 0$

$$\left(-\frac{\partial}{\partial \bar\theta^{\dot\alpha}} + \theta^\beta \sigma^\nu_{\beta\dot\alpha}\frac{\partial}{\partial x^\nu}\right)\Phi = 0$$

If $D_\alpha\Phi = 0$, $\bar{D}_{\dot\alpha}\Phi = 0$

no interesting: $P\,\Phi = \frac{\partial \Phi}{\partial x_{12}}$

$\frac{\partial \Phi}{\partial x^\alpha} = 0 \sim$ no degrees of freedom

letting $y^\nu = x^\nu + \theta^\alpha \sigma^\nu_{\alpha\dot\beta}\bar\theta^{\dot\beta}$
$$\begin{cases} \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial \bar\theta^{\dot\alpha}} \\ D_\alpha = -\frac{\partial}{\partial \theta^\alpha} + 2\bar\theta\sigma\frac{\partial}{\partial y} \end{cases}$$

2 complex, 1 auxiliary

$\Rightarrow \Phi = \Phi(y^\nu, \theta^\alpha)$ In the literature, a single chiral multiplet

① $\boxed{\int_{\mathbb{R}^{1,3|4}} d^4x\, d^2\theta\, d^2\bar\theta \; K(\Phi, \bar\Phi) \quad \bar{K} = K}$

reality condition to be real!!

The condition $\bar{D}\Phi = 0$ gives other possibility:

② $\int d^4x\, d^2\theta \; W(\Phi) + h.c.$

$\uparrow$ holomorphic

"D-terms" "F-terms"

argument of susy

$\int d\theta\, Q(\sim)$

$\int d\theta \left(\frac{\partial}{\partial\theta} + \theta \frac{\partial}{\partial x}\right)$

$\uparrow$ kills as a θ $\uparrow$ total derivative.

What about $\int d^2\theta \underbrace{\left(\frac{\partial}{\partial\theta} + \bar\theta \frac{\partial}{\partial x}\right)}_{Q}(\sim)$

$\int d^2\theta \left(\frac{\partial}{\partial\bar\theta} + \theta \frac{\partial}{\partial x}\right)$

problem, but since we're integrating on Φ chiral elements

$\frac{\partial}{\partial\bar\theta} = \theta \frac{\partial}{\partial x}$ $\rightsquigarrow$ will be a total derivative ✓

Guess for kinetic term for chiral multiplets

$\int d^4x\, d^2\theta\, d^2\bar\theta\, (\bar\Phi \Phi)$

$\Phi = \phi(y) + \psi_\alpha(y)\theta^\alpha + F(y)\theta^2, \quad y = x + \theta\sigma\bar\theta \rightsquigarrow$ ~~$\Phi(x)$~~

$\rightsquigarrow \phi(y) = \phi(x) + \frac{\partial\phi}{\partial x^\mu}\theta\sigma^\mu\bar\theta + \frac{\partial^2\phi}{\partial x^\mu \partial x^\nu}(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)$

$= \nabla^2\phi\, \theta^2\bar\theta^2$ only one way to contract

$\rightsquigarrow \theta^\alpha \psi_\alpha(y) = \theta^\alpha \psi_\alpha(x) + \partial_\mu \psi_\alpha(x)\theta^\alpha\theta^\beta\sigma^\mu_{\beta\dot\beta}\bar\theta^{\dot\beta}$

$= \theta^2\partial_\mu\psi^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}$

$\rightsquigarrow F(y)\theta^2 = F(x)\theta^2$

Last time: Superfields, supermultiplets, and interactions in 4d $N=1$ susy. Super space: $\mathbb{R}^{1,3} \oplus \Pi S_{\mathbb{R}} = \mathbb{R}^{1,3|4}$

$\uparrow$ Majorana spinor

$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + \bar\theta^{\dot\beta}\gamma^\mu_{\alpha\dot\beta}\frac{\partial}{\partial x^\mu}$

$\left(D_\alpha = -\frac{\partial}{\partial\theta^\alpha} + \cdots\right)$

Two ways of getting irreducible supermultiplets

- Impose a chirality constraint $\Phi: \mathbb{R}^{1,3|4} \to \mathbb{C}$
  $\rightsquigarrow$ a matter type multiplet: $\bar{D}_{\dot\alpha}\Phi = 0$
  - 1 complex scalar
  - 1 Weyl fermion $\quad \Phi(y,\theta) = \phi(y) + \theta^\alpha\psi_\alpha(y) + F(y)\theta^2$
  - 1 complex auxiliary field
- Want to also get a multiplet w/ one vector, one Majorana fermion

Inside the real superfield:

Power of θ	content
0	$\mathbb{R}$ scalar
1	$\mathbb{R}$ fermion (Majorana)
2	$\mathbb{R}$ vector fields A, one cpx scalar
3	Majorana fermion λ
4	$\mathbb{R}$ scalar D

"W-Z gauge"

Φ chiral $\rightsquigarrow i(\Phi - \bar\Phi)$ is not chiral, $\begin{cases} \mathrm{Im}\,\Phi \\ \Phi + \bar\Phi \\ \mathrm{Re}\,\Phi \end{cases}$

Expansion of the chiral superfield: Take taylor series of $\Phi(y) \to \Phi(x)$

$\theta \downarrow \searrow \bar\theta$

$$\phi(x)$$
$$\theta\psi_\alpha(x)$$
$$\theta^2 F(x) \qquad \theta^\alpha \bar\theta^{\dot\beta} \partial_{\alpha\dot\beta} \phi(x)$$
$$\theta^2 \bar\theta^{\dot\beta} \partial_{\alpha\dot\beta} \psi_\alpha(x)$$
$$\theta^2 \bar\theta^2 \partial^2 \phi(x)$$

and cmplx conjugates

$\phi(x) + \bar\phi(x) \in$ real

$$\left\langle \begin{matrix} & \theta\psi & & \bar\theta\bar\psi & \\ \theta^2 F & & \theta^\alpha\bar\theta^{\dot\beta}(\partial_{\alpha\dot\beta}(\phi - \bar\phi)) & & \bar\theta^2 F \\ \theta^2\bar\theta \not{\partial}\psi & & & & \theta^2\bar\theta\not{\partial}\bar\psi \\ & & \theta^2\bar\theta^2(\partial^2(\phi+\bar\phi)) & & \end{matrix} \right\rangle$$

wess-zumino gauge to transform this $\downarrow$ to this

then $V \sim V + (\Phi + \bar\Phi)$

real superfield $V: \mathbb{R}^{1,3|4} \to \mathbb{R}$.

Digression: in any gauge theory, there's a current $J = \frac{\delta \mathcal{L}}{\delta A}$ $\leftarrow$ gauge field

minimal coupling $\mathcal{L} \sim A_\mu J^\mu$, $\partial_\mu J^\mu = 0$

what's the super version? $\rightsquigarrow A \to V$ real superfields G w/ $D^2 G = \bar D^2 G = 0$ give "current type" multiplets

Vector multiplet	on shell	off shell
A_μ	2	4
λ	2	4
D	0	1
Inp	0	−1

!

The simplest interacting theory is the Wess-Zumino model: chiral superfields + superpotential interaction

D-term: $\int d^4x\, d^2\theta\, d^2\bar\theta\, K(\Phi, \bar\Phi)$, simplest $K = \bar\Phi\Phi$

Use $\rightsquigarrow \theta^2 \bar\theta^{\dot\beta} \partial_{\alpha\dot\beta}\psi_\alpha(x)\,(\bar\theta \bar\psi_\alpha(x))$ $\leftarrow$ like terms

$$\rightsquigarrow \int d^4x \left(\bar\phi \partial^2 \phi + \bar\psi \not{\partial} \psi + F\bar F \right)$$

Mass is not here $\uparrow$ not required by susy. To get mass, get F-term

F-term: $\int d^4x\, d^2\theta\, W(\Phi) + h.c.$, simplest $W = \Phi^2$

chiral / chiral need

look $\rightsquigarrow \int d^4x\, d\theta\, \psi^2 + F\phi + h.c.$

$\mathcal{L} \sim (F\bar F + F\phi + \bar F \bar\Phi)$

$\uparrow$ mass but F is auxiliary. EOM $\rightsquigarrow F = -\bar\Phi$

$\rightsquigarrow$ generate the mass term $\phi\bar\phi$ after integrating out F.

Using Super fields

Two types of superfields: - Chiral (cpx w/a constraint)
- Vector (real w/a gauge invariance)

Two types of interactions terms: - F-terms (integrate sth chiral over $\int d^2\theta + h.c.$

- D-terms (integrate $\int d^2\theta d^2\bar\theta$)

For a theory of only chiral superfields

$$\boxed{\mathcal{L} = K(\Phi^i, \bar\Phi^{\bar j})\Big|_D + W(\Phi^i)\Big|_F + \bar W(\bar\Phi^{\bar i})\Big|_{\bar F}.}$$

To get the standard kinetic term, take $K = \bar\Phi^{\bar i}\Phi^i$ = quadratic.

$\Phi^i : \mathbb{R}^{3|4} \to \mathbb{C}^n$

$K : \mathbb{C}^n \to \mathbb{R}$ Kähler
a real Fcn potential)

$W : \mathbb{C}^n \to \mathbb{C}$
a holomorphic function

Because of the Structure of the superfield expansion finitely many derivatives of K appear in the component field expansion.

Φ^i	$\bar\Phi^{\bar i}$
ϕ^i	$\bar\Phi^i$
$\theta^\alpha \psi^i_\alpha$	$\bar\theta^{\dot\alpha}\bar\Psi^i_{\dot\alpha}$
$\theta^2 F$ $\theta^\alpha\bar\theta^{\dot\beta}\partial_{\alpha\dot\beta}\phi^i$	$\theta^\alpha\bar\theta^{\dot\beta}\partial_{\alpha\dot\beta}\bar\Phi^i$ $\bar\theta^2\bar F^i$
$\theta^2\bar\theta^{\dot\beta}\partial_{\alpha\dot\beta}\psi^{i\alpha}$	$\bar\theta^2\theta^\alpha\partial_{\alpha\dot\beta}\bar\Psi^{i\dot\beta}$
$\theta^2\bar\theta^2\partial^2\phi^i$	$\theta^2\bar\theta^2\partial^2\bar\Phi^i$

$$\mathcal{L} = K(\bar\Phi^i, \Phi^i)\Big|_D = \frac{\partial^2 K}{\partial\Phi^i\partial\bar\Phi^{\bar j}}\Big(\partial\phi^i\partial\bar\phi^{\bar j} + \bar\psi^{\bar j}\not\partial\psi^i + \psi^i\not\partial\bar\psi^{\bar j} + \bar F^{\bar j}F\Big)$$

$$+ \frac{\partial^3 K}{\partial\Phi^i\partial\Phi^j\partial\bar\Phi^{\bar k}}\Big(\psi^i\psi^j\bar F^{\bar k} + \bar\psi^{\bar k}(\not\partial\phi^i)\psi^j\Big) + h.c.$$

$$+ \frac{\partial^4 K}{\partial\Phi^i\partial\Phi^j\partial\bar\Phi^{\bar k}\partial\bar\Phi^{\bar l}}\Big(\psi^i\psi^j\bar\psi^{\bar k}\bar\psi^{\bar l}\Big)$$

Claim: I can just remember $g_{i\bar j}(\Phi) = \dfrac{\partial^2 K}{\partial\Phi^i\partial\bar\Phi^{\bar j}}$ ← looks like metric tensor on the target space $\mathbb{C}^n$

Terms involving F:

- $g_{i\bar j}F^iF^{\bar j} + \dfrac{\partial g_{i\bar j}}{\partial\Phi^k}\psi^i\psi^k\bar F^{\bar j} + \dfrac{\partial g_{i\bar j}}{\partial\bar\Phi^{\bar k}}\bar\psi^{\bar j}\bar\psi^{\bar k}F^i$

EOM: $\frac{\partial \mathcal{L}}{\partial F^i} = g_{i\bar{j}} \bar{F}^{\bar{j}} + \frac{\partial g_{i\bar{k}}}{\partial \bar{\Phi}^{\bar{l}}} \bar{\psi}^{\bar{k}} \bar{\psi}^{\bar{l}} = 0 \Rightarrow \bar{F}^{\bar{j}} = -\underbrace{g^{i\bar{j}} \frac{\partial g_{i\bar{k}}}{\partial \bar{\Phi}^{\bar{l}}}}_{= -\Gamma^{\bar{j}}_{\bar{k}\bar{l}}} \bar{\psi}^{\bar{k}} \bar{\psi}^{\bar{l}}$

$$\frac{\partial^4 K}{\partial \phi^i \partial \phi^j \partial \bar{\Phi}^{\bar{k}} \partial \bar{\Phi}^{\bar{l}}} = \frac{\partial^2 g_{i\bar{k}}}{\partial \Phi^j \partial \bar{\Phi}^{\bar{l}}} *$$

$\uparrow$

$$\frac{\partial g_{i\bar{j}}}{\partial \Phi^k} = \frac{\partial^3 K}{\partial \phi^i \partial \phi^k \partial \bar{\Phi}^{\bar{j}}} = \frac{\partial g_{k\bar{j}}}{\partial \Phi^i} \quad \text{(Kähler identity)}$$

Also get terms of the form: $(\Gamma^{\bar{j}}_{\bar{k}\bar{l}} \bar{\psi}^{\bar{k}} \bar{\psi}^{\bar{l}})(\Gamma^i_{mn} \psi^m \psi^n) g_{i\bar{j}}$

Combine to produce 4-fermi interactions of the form

$R_{i\bar{j}k\bar{l}} \psi^i \psi^k \bar{\psi}^{\bar{j}} \bar{\psi}^{\bar{l}}$

~~Supersymmetry Breaking~~

$W = \phi^1((\phi^2)^2 - \phi^3)$

What about going to gauge fields?

$V: \mathbb{R}^{1,3|4} \longrightarrow \mathbb{R} \qquad V \sim V + (\Psi + \bar{\Psi}),\ \Psi$ chiral i.e. $\Psi: \mathbb{R}^{1,3|4} \to \mathbb{C}$, $\bar{D}_{\dot{\alpha}} \Psi = 0$

Anzats: $\Phi \to e^{i\Psi} \Phi$. Problem:

$K: \bar{\Phi}\Phi \to \bar{\Phi} e^{q\bar{\Psi}} e^{q\Psi} \Phi$

Solution:

$$\bar{\Phi} e^{-qV} \Phi = \bar{\Phi}\Phi - q \bar{\Phi} V \Phi + \frac{q^2}{2} \bar{\Phi} V^2 \Phi$$

In WZ gauge stops at V^2!

More on R-symmetry, holomorphy and non-renormalization theorems

↳ A symmetry that acts non-trivially on the supercharges (a tangent to the vector space)

- Doesn't commute with supersymmetry but is an automorphism of super translations.

(similar to Lorentz group but it doesn't act in the bosonic spacetime)

In the context of 4-d $N=1$ theories, R-symmetry is $U(1)$

$Q_\alpha,\ \bar{Q}_{\dot{\alpha}},\ P_{\alpha\dot{\alpha}} \qquad \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = P_{\alpha\dot{\alpha}}$

R-charge	R-charge	R-charge
$+1$	-1	0
$\downarrow$	$\downarrow$	
$e^{i\theta} Q_\alpha$	$e^{-i\theta} \bar{Q}_{\dot{\alpha}}$	

- R-symmetry may or not may be broken (without breaking susy) in any given theory.
- Determines R-charges of components but not homogeneously

Ex. Chiral superfield $\Phi(x, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x) + \dots$

R-char +1 . +1

$$\mathcal{L} = (\bar{\Phi}\Phi)\Big|_D + W(\Phi)\Big|_F + \overline{W(\Phi)}\Big|_{\bar{F}}$$

$\bar{\Phi}\Phi$: $-r$, r → 0

$W(\Phi)|_F = \int d^2\theta\, W(\Phi)$: $d^2\theta$ is $+2$ → must have -2

In a theory of 1 Chiral S.F. Field

$\Rightarrow W(\Phi)$ must be a monomial (IF R-symmetry is unbroken)

↳ ϕ^n in particular homogeneous

$nr = -2 \rightsquigarrow r = -\frac{2}{n}$ / Sum of other? impossible!

$\phi^n + \phi^m$: $rn = -2$, $rm = -2$ $\Rightarrow \boxed{m = n}$

Using EoM of $F \rightsquigarrow F = \frac{\partial W}{\partial \phi}$: F is $r+2$, $\frac{\partial W}{\partial\phi}$ is $(n-1)r$ $\rightsquigarrow r = -\frac{2}{n}$

In a more general theory $W(\phi_i)$ is quasi-homogeneous:

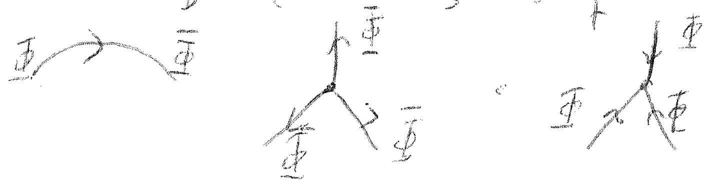
$W(e^{t r_i}\Phi_i) = e^{2t}\, W(\Phi_i)$ for some choice of charges r_i w/ unbroken R-symmetry

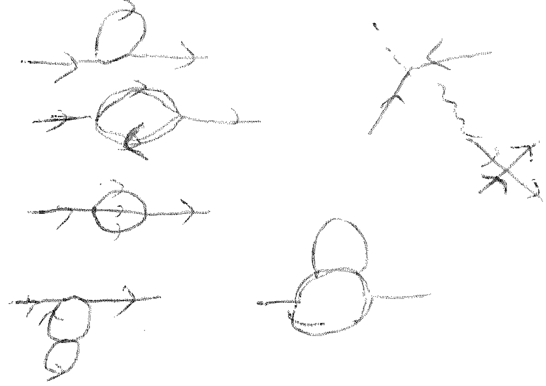
Recall: On the HW we consider $W = \frac{1}{2} m \Phi^2 + \frac{1}{3}\lambda\Phi^3$

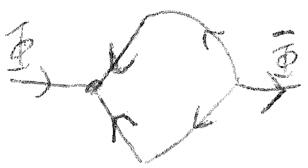
Not homogeneous $\Rightarrow$ no R-symmetry

Thus a nice way to organize these...

$$\mathcal{L} = \bar{\Phi}\Phi\Big|_D + \left(\frac{1}{2} m \Phi^2 + \frac{1}{3}\lambda\Phi^3\right)\Big|_F + h.c.$$







Wave function renormalization (Kähler potential)

The D-term does get renormalized!

$\bar{\Phi}\Phi \rightsquigarrow K\bar{\Phi}\Phi$, $K \sim \ln\Lambda$

$m_{phys} = \frac{m}{K}$, $\lambda_{phys} = \frac{\lambda}{K^{3/2}}$ ← all renormalizations are multiplicative

- Purely multiplicative.

$m_{phys} = Z_m m$, $\lambda_{phys} = Z_\lambda \lambda$ (Z_m, Z_λ not independent)

• Another way to see this uses R-symmetry

Imagine promoting coupling constants to chiral super fields

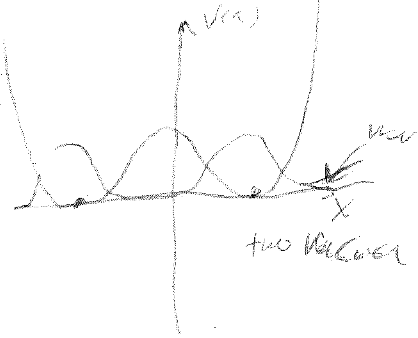
	R-charge	Flavour-charge
Φ	r	1
$\bar{\Phi}$	$-r$	-1
m	$-2-2r$	-2 ←

	R charge	flavour charge
m	$-2-2r$	-2
(W, Φ) [illegible]	-2	0
λ	$-2-3r$	-3
$\lambda \Phi / m$	0	0

$\rightarrow W_{eff} = \frac{1}{2} m \Phi^2 f\left(\frac{\lambda \Phi}{m}\right)$ ← holomorphic function

$= W!$

One major consequence of non renormalization for W is the existence of moduli spaces of vacua in supersymmetric theories.



Two fields x, y $\quad V(x,y) = x^2 y^2$ [illegible] "true" QM vacu

In S-Sy it doesn't happen: $W(\Phi_i) \rightsquigarrow V(\phi_i) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2$ ← $F_i \bar{F}_i$

Zero iff $\frac{\partial W}{\partial \phi_i} = 0 \Rightarrow$ i.e. it's at a critical point in fields space. $\rightsquigarrow$ {vacua} = Crit(W)

$W(\Phi_1, \Phi_2) = \frac{1}{2} m \Phi_2^2 + \frac{1}{2} \lambda \Phi_2 \Phi_1^2$

$2r_2 = -2 \Rightarrow \boxed{r_2 = -1}$

$2r_1 + r_2 = -2$

$2r_1 - 1 = -2 \rightarrow \boxed{r_1 = -\frac{1}{2}}$ ↳ good ✓

Crit? $\rightsquigarrow \frac{\partial W}{\partial \Phi_2} = m \Phi_2 + \lambda \Phi_1^2 = 0$

$\frac{\partial W}{\partial \Phi_1} = \frac{2}{2} \lambda \Phi_2 \Phi_1 = 0$

↓ only at $\Phi_1 = \Phi_2 = 0$

but also if $m \to 0$, $\begin{cases} \Phi_1 = 0 \\ \Phi_2 = \text{anything} \end{cases}$ ← continuously many vacua!

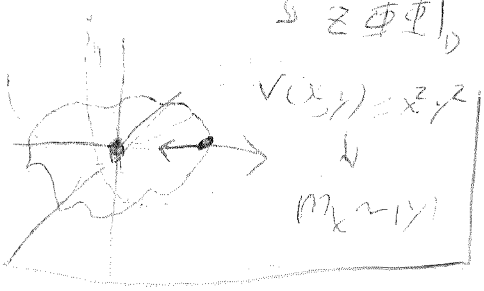
Lecture 11.2

Recall Z_m and Z are not holo.

↳ $\mathcal{L} \sim \bar{\Phi}\Phi \big|_D + W(\Phi)\big|_F \rightsquigarrow \Phi \to Z^{\frac{1}{2}} \Phi$

$\rightsquigarrow Z \bar{\Phi}\Phi \big|_D$ ↳ $Z^{\ldots}$ [illegible]

Foundations and gauge physics



Non perturbative corrections can (and do) occur

Ex: SQM ($\mathcal{N}=2$) w/ target space $\mathbb{R}$

$W = \phi^3 \rightsquigarrow V \sim \left| \frac{\partial W}{\partial \phi} \right|^2 \sim \phi^4$

↳ $\phi^4 - m^2 \phi^2 + c$

$= \left(\phi^2 - \frac{m^2}{2} \right)^2$

$\left(\frac{\partial W'}{\partial \phi} \right)^2$

w/ $\phi^3 - m^2 \phi$

← {vacua} $\stackrel{?}{=}$ Crit(W)

More on this later.

Action for super QED

Remember, V is a gauge field $\rightsquigarrow V$: $\Phi \to e^{\ldots} \Phi$ then $\bar{\Phi}\Phi\big|_D$ is not [illegible]

+ transforms

then our recipe guess $\bar{\Phi}\Phi|_D \to \bar{\Phi} e^{+\bar{\Lambda}} e^{-2V} e^{\Lambda} \Phi = \bar{\Phi} e^{-2(V+\bar{\Lambda}+\Lambda)} \Phi$ ✓

Remember

$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$, W is: - supergauge inv.

- chiral : $\bar{D}_{\dot\beta} W_\alpha = 0$

- fermionic $\underset{\text{boson}}{\bar{D}^2}\ \underset{\text{fermion}}{D_\alpha}\ \underset{\text{boson}}{V} \to W$ fermionic and $W_\alpha W^\alpha \neq 0$ (Lorentz, chiral scalar superfield)

want to write : $W^\alpha W_\alpha|_F + h.c.$

V in W-Z gauge in coords $(\underset{x_{\alpha\dot\alpha} - \theta_\alpha \bar\theta_{\dot\alpha}}{y_{\alpha\dot\alpha}}, \theta^\alpha, \bar\theta^{\dot\alpha})$

$\bar{D}_{\dot\alpha} = \frac{\partial}{\partial \bar\theta^{\dot\alpha}}$

$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - 2\bar\theta^{\dot\alpha} \underset{\partial_{\alpha\dot\alpha}}{\frac{\partial}{\partial y_{\alpha\dot\alpha}}}$

$V = \theta^\alpha \bar\theta^{\dot\alpha} A_{\alpha\dot\alpha}(y) + \theta^2 \bar\theta^{\dot\alpha} \bar\lambda_{\dot\alpha}(y) + \bar\theta^2 \theta^\alpha \lambda_\alpha(y) + \frac{1}{2}\theta^2\bar\theta^2 (D - \partial_{\alpha\dot\alpha} A^{\alpha\dot\alpha})$.

$\rightsquigarrow W_\alpha(y, \theta\bar\theta) = \lambda_\alpha(y) + \frac{1}{2}\theta_\alpha D + \theta^\beta \underline{\underline{F_{\alpha\beta}}} + \theta^2 \partial_{\alpha\dot\beta} \bar\lambda^{\dot\beta}$ 3 dim

$F_{\mu\nu} \leftrightarrow (F_{\alpha\beta}, F_{\dot\alpha\dot\beta})$ $\qquad \varepsilon^{\dot\alpha\dot\beta}(\partial_{\beta\dot\beta} A_{\alpha\dot\alpha} + \partial_{\alpha\dot\beta} A_{\beta\dot\alpha})$

Maxwell action $F \wedge \star F = F_{\mu\nu} F^{\mu\nu} \leftarrow$ QED action

$F\wedge F = \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \leftarrow$ "theta angle"

$\star : \Lambda^k V \to \Lambda^{d-k} V$, $\star^2 = (-1)^{k(d-k)}$ signature $\Big\}$ $\Lambda^2 V \rightsquigarrow \underset{\star = i}{\underset{\text{"}Sym^2(S_+)\text{"}}{(\Lambda^2 V)_+}} \oplus \underset{\star = -i}{\underset{\text{"}Sym^2(S_-)\text{"}}{(\Lambda^2 V)_-}}$

$V = S_+ \otimes S_-$, $\Lambda^2(S_+ \otimes S_-) = \underset{\emptyset}{\Lambda^2 S_+} \otimes Sym^2 S_- \oplus Sym^2 S_+ \otimes \Lambda^2 S_-$

$W^\alpha W_\alpha|_F = \lambda^\alpha \partial_{\alpha\dot\beta}\bar\lambda^{\dot\beta} + D^2 + (F_{\alpha\beta})^2$

$\hookrightarrow F\wedge\star F + i F\wedge F$

$F_{0i} = E_i$, $F_{jk} = \varepsilon_{ijk} B_i$

$\theta : F = \star F \Rightarrow$ self dual : $B = \pm i E$

F is self dual $\rightsquigarrow F\wedge F = \frac{1}{4}(F - i\star F)(F - i\star F) = \frac{1}{4}(F\wedge F - \underset{-F\wedge F}{\star F \wedge \star F} - 2i F\wedge\star F)$

$= \frac{1}{2}(F\wedge F - i F\wedge\star F)$

Then we have our ingredients to the SQCD!

$\mathcal{L} = (\tau W_\alpha W^\alpha)|_F + h.c. + (\bar\Phi_i e^{-2q_i V} \Phi_i)_D + W(\Phi_i)|_F + h.c. + \xi V|_D$

Maxwell term; gauge inv kinetic term; gauge inv. superpotential; gauge invariant; FI term (Fayet-Iliopoulos)

$\frac{\tau}{g_{YM}^2} F\wedge\star F + \theta F\wedge F + \lambda\partial\bar\lambda + D^2$

$\underline{\partial\bar\Phi\partial\Phi} + \underline{\bar\psi\partial\psi} + \underline{\bar F F} + A\phi\partial\bar\phi + A\partial\phi\bar\phi \cdot A\bar\psi\psi + \bar\lambda\bar\psi\phi + \bar\phi\lambda\psi$

$+ D\bar\phi\phi + \underline{A^2\bar\phi\phi}$ $\Big\}$ $\sim (D\phi)^2$ = kinetic energy term

$W \to \left|\frac{\partial W}{\partial\phi}\right|^2 + \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j + \ldots$

$V|_D$ $+ \xi D$

Terms involving D: $\frac{1}{2}D^2 + \sum D\, q_i \bar{\phi}_i \phi_i + \xi D =$

now $D = \left(\sum q_i |\phi_i|^2 + \xi\right)$

charges

EOM
$\frac{1}{2}D^2 + cD$
$\Rightarrow D = -c$
$\Rightarrow \frac{1}{2}D^2 + cD = -\frac{1}{2}c^2$

Lecture 8.1

Supersymmetric gauge theories:

- Last time ~> Abelian susy gauge theory

V: real superfield (vector multiplet) ~> A, $(\lambda, \bar{\lambda})$, D

$\{\bar{\Phi}_i; q_i\}$ charged chiral multiplets ~> φ_i, ψ_i, F_i

ψ_i chiral fermions

g: electromagnetic coupling constant

ξ: Fayet-Iliopoulos parameters

$W(\Phi_i)$ gauge inv. holomorphic superpotential

$$\mathcal{L} = \sum_i \underbrace{\bar{\Phi}_i e^{2q_i V} \Phi_i \Big|_D}_{\text{Scalar kinetic terms}} + \xi V\Big|_D + W^\alpha W_\alpha \Big|_F + \text{h.c.} + W(\Phi_i)\Big|_F + \text{h.c.}$$

"flavors"; Fayet-Iliopoulos terms; "Super maxwell" term; superpotential interactions

This is where the kinetic term for λ is "hidden" we could have guessed it since its superpartner A is there

Since we have chiral fermions coupled to gauge fields, we should also check that the theory is free of gauge anomalies

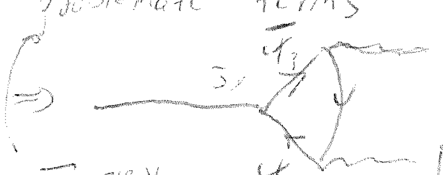
anomalys $\Rightarrow \partial_\mu J^\mu \neq 0$ in the QFT theory".

What sort of diagram would I imagine I have to compute?

Generally for the operator J_μ: $\mathcal{L} = A_\mu J^\mu + \ldots$, so looking at "problematic terms"

$J_\mu = \sum_i q_i \bar{\psi}_i \gamma_\mu \psi_i + \ldots$ (minimally coupled)

↳ we are interested in terms containing chiral ~~fields~~ fermions.



$\sum \bar{\Phi}_i e^{2q_i V} \Phi_i \Big|_D$

$q_i \bar{\psi}_i A \psi_i$

"triangle diagram"

$\langle \partial_\mu J^\mu \rangle = (\#'s)\, F \wedge F$
$\sim q_i^3\, \# F \wedge F$

3 vertices

$J = \frac{\delta \mathcal{L}}{\delta A}$ - 3-form

$\Rightarrow dJ = 4$-form

There is also a "gravitational anomaly" $\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} = T_{\mu\nu}$

↳ contributing to the non-conservation of $T_{\mu\nu}$ ($\partial_\mu T^{\mu\nu} \neq 0$)

The gravitational anomaly only vanishes iff $\sum q_i = 0$

An obvious nontrivial solution to both anomalies:

chiral fields in pairs (Φ_i^+, Φ_i^-) with charge $\pm q_i$ ~> the theory will be symmetric with tildeness

~> "super QED"

in the literature $\Phi, \tilde{\Phi}$

What about non-abelian gauge theories? Pick a gauge group

Φ_i {charge specified} $\otimes R_i$, R_i is a rep. of G.

$$\Phi_i \to e^{\chi} \Phi_i$$

$\uparrow R_i$ $\quad$ $G \circ R_i$

we want the covariant Kinetic terms to make sense.

Generally: $A_\mu \in \Omega^1 \otimes \mathfrak{g}$

$\lambda \in \Omega^0 \otimes \mathfrak{g}$ map$(\lambda) \in \Omega^0(\ldots)$

infinitesimally: $A \to A + d\lambda + [\lambda, A]$

finitely: $A \to u^{-1} A u + u^{-1} du$

abelian: $\quad 0 + e^{-\lambda} d\lambda e^{\lambda}$

$$\Phi_i e^{V} \Phi_i \to \bar{\Phi}_i \underbrace{e^{\bar{\chi}} e^{\tilde{V}} e^{\chi}}_{e^{V}} \Phi_i, \text{ i.e. } e^{V} \to e^{\tilde{V}} = e^{-\bar{\chi}} e^{V} e^{\chi}$$

omit 2_i because the charge is specified by the rep of Φ_i

to find the form of this we use BCH,

$$e^{\tilde{V}} = e^{V - \chi - \frac{1}{2}[V,\chi] - \bar{\chi} - \frac{1}{2}[\bar{\chi}, V] + \cdots}$$

[• Note: we have commutators in the group algebras, so this time commutators don't vanish, we just choose to work in linear order $O(\chi)$,

$$e^{\tilde{V}} = \underbrace{e^{V - (\chi + \bar{\chi}) - \frac{1}{2}[V,\chi] - \frac{1}{2}[\bar{\chi}, V] + \cdots}}_{\text{we can go to W-Z gauge, as before.}}$$]

• D transforms under the adjoint rep, so the question is how do I get a gauge inv. term from sth like that? (Just tracing stuff won't work)

Let's consider SQED with fields $\Phi_\pm$ of charge ± 1

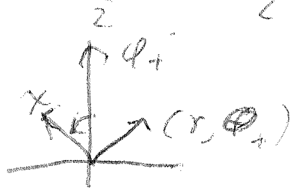
<u>D terms</u> $\quad \frac{1}{2} D^2 + JD + D \bar{\varphi}_+ \varphi_+ - D \bar{\varphi}_- \varphi_-$

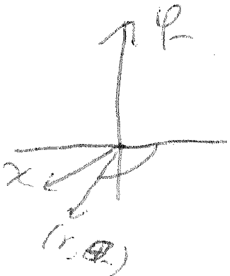
E.O.M. for D: $\quad D = -(J + |\varphi_+|^2 - |\varphi_-|^2)$

$$\leadsto V(\Phi) = \frac{1}{2} D^2 = \frac{1}{2}(|\varphi_+|^2 - |\varphi_-|^2 - J)^2$$

If $J = 0$ $\quad |\varphi_+| = |\varphi_-|$

$(\varphi_+, \varphi_-) \in \mathbb{C}^2$





To fix gauge I could impose:

$$(r, \theta_+, \theta_-) \sim (r, \theta_+ + \chi, \theta_- - \chi)$$

different charges

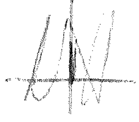
$$\parallel$$

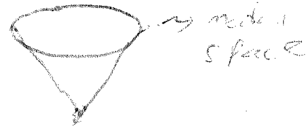
$$(r, \alpha + \chi, \alpha - \chi)$$

$\alpha = \dfrac{\theta_+ + \theta_-}{2}$ $\leadsto$ we can think of this as a gauge inv. parameter

$\in [0, \pi)$

$$\varphi_\pm = \varphi_\pm^{(0)} + J_\pm$$





Lecture 12: Moduli spaces of vacua (mostly in gauge theory)

Example: $\mathcal{N}=1$ SQED (abelian gauge theory) with one flavour ($\Phi_\pm$ chiral multiplet, $q_\pm = \pm 1$)

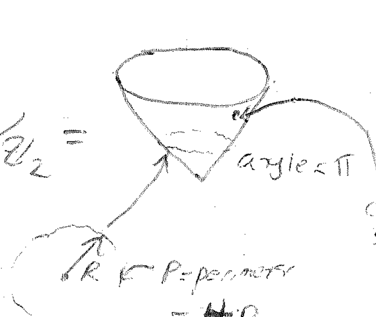
FI parameter $\quad \sum q_i = 0$, $\quad \sum q_i^3 = 0$

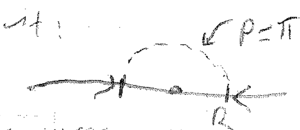
If $W=0 \Rightarrow$ potential for scalars comes from D-terms.
$$\mathcal{L} \sim \frac{1}{2}D^2 + D\bar{\Phi}_+\phi_+ - D\bar{\phi}_-\phi_- + \xi D \Rightarrow D = \bar{\phi}_-\phi_- - \bar{\phi}_+\phi_+ - \xi$$

Before taking quotient by gauge symmetries, (ϕ_+, ϕ_-)

$\mathbb{C}^2 \supset \{D=0\}$

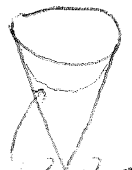
$$\mathcal{M} = \{D=0\}/U(1) = \mathbb{C}/\mathbb{Z}_2 =$$





one massless cpx scalar $\Downarrow$ one massless Weyl fermions

if $\xi \neq 0 \Rightarrow \phi_+ = \phi_- = 0$ is not a vacuum $\Rightarrow$ gauge symmetry is broken



$|\phi_+|^2 = |\phi_-|^2 + \xi$

Recall $\{D=0\} = \{(\phi_+, \phi_-) \mid |\phi_+|^2 = |\phi_-|^2\}$, parametrize $\phi_+ = re^{i\theta_+}$, $\phi_- = re^{i\theta_-}$

Photon mass
$$\bar{\Phi}_+ e^V \Phi_+ + \bar{\Phi}_- e^{-V}\Phi_- \supset (\bar{\phi}_+\phi_+ + \bar{\phi}_-\phi_-)A^\mu A_\mu$$
$$= 2|\phi_+|^2 + \xi \Rightarrow m_\gamma \sim \sqrt{|\phi|^2+\xi}$$

$$\frac{\theta_+ + \theta_-}{2}$$

$\Rightarrow$ The moduli space is always a (potentially singular) $\mathbb{C}$x manifold (parametrized by chiral superfields)

$\Rightarrow$ In fact it always has a Kähler metric! (D-terms of the massless scalars are the Kähler potential)

$$K = \bar{\Phi}_+\Phi_+ + \bar{\Phi}_-\Phi_- = 2|\phi_+|^2 \quad \text{(if D-term condition is satisfied)}$$
on the space $\{D=0\}$

Rewrite in terms of gauge inv. observables.
$$\phi_+\phi_- = M \Rightarrow M\bar{M} = \bar{\phi}_+\bar{\phi}_-\phi_+\phi_- = \left(\frac{K}{2}\right)^2 \Rightarrow K = 2\sqrt{\bar{M}M}$$
$$g_{M\bar{M}} = \frac{\partial^2 K}{\partial M \partial \bar{M}} = \frac{\partial}{\partial \bar{M}}\frac{\bar{M}}{\sqrt{\bar{M}M}} = -\frac{1}{2}\frac{\bar{M}M}{(\bar{M}M)^{3/2}} + \frac{1}{(\bar{M}M)^{1/2}} = \frac{1}{2}\frac{1}{(\bar{M}M)^{1/2}}$$

In the low energy limit "deep IR", only massless d.o.f. are relev...

- The statement about K ~~being~~ defining a singular metric when d.o.f become massless isn't just true for gauge theories
- Have to remember that K can get quantum corrections!

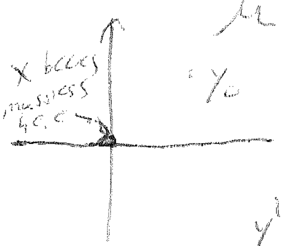
Consider a theory with 2 chiral superfields X, Y

$W = \frac{1}{2} X^2 Y$, $\mathcal{M} = \mathrm{crit}(W) = \{ \cancel{XY = \frac{1}{2} X^2 = 0} \}$

$\overset{\parallel}{\frac{\partial W}{\partial \phi_i} = 0} = \{ XY = \frac{1}{2} X^2 = 0 \} \Leftarrow X = 0$, $Y = \text{anything} = Y_0 + Y'$

Expand around $(0, Y_0) \Rightarrow W = \frac{1}{2} Y_0 X^2 + \frac{1}{2} Y' X^2$

mass term for X.



There are loop diagrams in this theory that correct the Kähler potential ...

$\sim \int d^4 p \, \frac{1}{p+m} \cdot \frac{1}{p+m} \sim \int p \, dp \sim \Lambda^2 - m^2$ diagram for g

For K: $\sim \int \frac{1}{p} dp \sim \log\left(\frac{\Lambda}{m}\right)$

Recall

$$\begin{pmatrix} & \phi & \\ \theta\psi & & \\ \theta^2 F & \theta\bar{\theta}\partial\phi & \\ \theta^2\bar{\theta}\partial\psi & & \\ & \theta^2\bar{\theta}^2 \Box\phi & \end{pmatrix}$$

metric for Y'

$g_{Y\bar{Y}} \, \partial Y \partial \bar{Y}$

$\bar{Y} \cdots\cdots \bar{Y}$

$Y \psi_X \psi_X + Y X F_X$

$\overset{\parallel}{\bar{X} + \bar{Y}\bar{X}}$

$Y \psi_X \psi_X \to Y X \bar{X} + Y X \bar{Y} \bar{X}$

$K \sim \left(1 + \# \log\left(\frac{\Lambda}{m_X}\right) + \cdots\right) \bar{Y} Y$

$= \left(1 + \frac{\#}{2} \log\left(\frac{\Lambda^2}{\cancel{m}|Y|^2}\right)\right) \bar{Y} Y$!

$g_{Y\bar{Y}} \sim \frac{\partial^2 K}{\partial Y \partial \bar{Y}} = 1 + \# \log\left(\frac{\Lambda}{|Y|}\right) - \frac{\bar{Y} Y}{\bar{Y} Y} \cdot \frac{\#}{2}$

$= \left(1 - \frac{\#}{2}\right) + \# \log\left(\frac{\Lambda}{|Y|}\right)$

Recitation 10

a) $\mathcal{M} = \{ v \in V : D(v) = 0 \}/U(1) = V/\mathbb{C}^\times$

$|\langle \phi_+ \rangle|^2 - |\langle \phi_- \rangle|^2 = r$ D-term $\Big|$ $\mathcal{M} = \{ v \in \mathbb{C}^2 \mid |v_1|^2 - |v_2|^2 = r \}/U(1)$

$\langle \phi_+ \rangle = v_1$, $\langle \phi_- \rangle = v_2$ $\pi[v] = \{ \mathbb{C}^\times v \mid v \in \mathbb{C}^2 \setminus \Delta_r \}$ $r \in \mathbb{R}$.

show that $\mathcal{M} \cong (\mathbb{C}^2 \setminus \Delta_r)/\mathbb{C}^\times$ $\boxed{GIT}$

- For $r > 0$ $\Delta_r \subset \mathbb{C}^2$ set of elements not containing solution to D-term eqn $\Delta_r = \{ (0, v_2) \in \mathbb{C}^2 \}$.

Let $v \in \mathbb{C}^2 \setminus \Delta_r$ arbitrary. Claim: $\mathbb{C}^\times v$ contains exactly one solution to D-term eqn.

consider cv, $c \in \mathbb{C}^\times$, plug into D term eq.

$$|cv_1|^2 - |c^{-1} v_2|^2 = |c|^2 |v_1| - |c|^{-2} |v_2|^2 = r$$

$$\rightsquigarrow \quad |c|^4 |v_1| - r|c|^2 - |v_2|^2 = 0$$

$$|c|^2 = \frac{r}{|v_1|^2} \pm \sqrt{\frac{r^2}{4|v_1|} + \frac{|v_2|^2}{|v_1|^2}} \leftarrow \text{only take } + \text{ since } |c|^2 > 0$$

Always has a real solution for $v \in \mathbb{C}^2 \setminus \Delta_r$ which fixes $|c|^2$ $\Rightarrow$ c up to a phase $\Rightarrow \mathcal{M} \cong (\mathbb{C}^2 \setminus \Delta_r)/\mathbb{C}^\times$

Its analogous for $r < 0$ $|c|^2 = \frac{+r}{|v_1|^2} \pm \sqrt{\frac{r^2}{4|v_1|^2} + \frac{|v_2|^2}{|v_1|^2}}$

For $r = 0$ $\{ v \in \mathbb{C}^2 \mid |v_1|^2 - |v_2|^2 = 0 \}/U(1) = \mathbb{C}^2/\mathbb{C}^\times$

$\mathbb{C}^2/\mathbb{C}^\times \ni [v] = \{ cv = (cv_1, c^{-1} v_2) \mid c \in \mathbb{C}^\times \}$

$\downarrow$

$\mathbb{C}/\mathbb{Z}_2$ look for $(c, \tilde{v}) \in \mathbb{C}^\times \times \mathbb{C}$ s.t. $\left. \begin{array}{l} cv_1 = \tilde{v} \\ c^{-1} v_2 = \tilde{v} \end{array} \right\} \frac{v_2}{v_1} = c^2$

$\Rightarrow \tilde{v} = \pm \sqrt{\frac{v_2}{v_1}}\, v_1$ $\rightsquigarrow$ each $[v]$ contains elements $(\pm \tilde{v}, \pm \tilde{v})$

$\Rightarrow \mathbb{C}^2/\mathbb{C}^\times \xrightarrow{\sim} \mathbb{C}/\mathbb{Z}_2$,

$[v]_{\mathbb{C}^\times} \longrightarrow [\tilde{v}]_{\mathbb{Z}_2}$ (ignoring $(0,0) \in \mathbb{C}^2$, which is not well defined)

b) $D = |X_+|^2 - |X_-|^2 + 2|Y_+|^2 - 2|Y_-|^2 = 0$

look for $(c, \bar{\nu})$ do the sec. as in a)

c) Generators of invariant monomials:

$A_1 := X_+ X_-$, $A_2 = Y_+ Y_-$, $B_1 := Y_+ X_- X_-$, $B_2 = Y_- X_+ X_+$

not indep $(A_1^2 A_2 = B_1 B_2) \to \dim \mathcal{M} = \#\text{gen} - \#\text{constraints} = 3$

d) <u>Gauge linear sigma models:</u> $V = (\mathbb{R}^{1,1}, g)$ Minkowski, $V_{\mathbb{C}} = \mathbb{C}^2 \simeq L \oplus L^\vee \cong \mathbb{C} \oplus \mathbb{C}^\vee$ maximally isotropic

$S = \Lambda^\bullet \mathbb{C}^\vee$ Dirac spinors $\quad \Gamma : S \otimes S \to V$ bracket $\quad g(\Gamma(\phi, \psi), v) = (\phi, v \cdot \psi)$

$\phi, \psi \in S$, $v \in V_{\mathbb{C}}$

choose basis $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$ of $\mathbb{R}^{1,1}$ $\leadsto$ induces a basis of the complexified spacetime $V_{\mathbb{C}} = \mathbb{C} \oplus \mathbb{C}^\vee \in \{\tilde{e}_1, \tilde{e}_1^\vee\}$

Let $\psi, \phi \in \Lambda^\bullet \mathbb{C}^\vee$, $\left.\begin{array}{l}\psi = \psi_0 + \psi_1 \tilde{e}_1^\vee \\ \phi = \phi_0 + \phi_1 \tilde{e}_1^\vee\end{array}\right\}$ $g(\Gamma(\phi,\psi), e_1) = \Gamma(\phi,\psi)_1$

$= (\phi, \tilde{e}_1 \lrcorner \psi)$

$= (\phi, \psi_1)$

$= \phi_1 \psi_1$

$= (\Gamma(\psi, \phi))_1$

$g(\Gamma(\phi,\psi), e_2) = -(\Gamma(\phi,\psi))_2 = (\phi, \tilde{e}_1^\vee \wedge \psi)$

$\tilde{e}_1^\vee$ acts with wedges, $\tilde{e}_1$ acts with $\lrcorner$

$= \phi_0 \psi_0 = -\Gamma(\psi, \phi)_2$

$S = \Lambda^\bullet \mathbb{C}^\vee = \mathbb{C} \oplus \mathbb{C}^\vee$ $\leadsto$ algebraic (chiral) don't think

want $\mathcal{N} = (2,2)$: $\quad S = \mathbb{C} \otimes (\mathbb{R}^2_A, \delta) \oplus \mathbb{C}^\vee \otimes (\mathbb{R}^2_V, \delta)$ chiral

$\leadsto$ R-symmetry $SO(2)_A \times SO(2)_V \simeq U(1)_A \times U(1)_V$

Choose ONB $\{e_{A_1}, e_{A_2}\}$ and $\{e_{V_1}, e_{V_2}\}$ of $\mathbb{R}^2_A, \mathbb{R}^2_V$

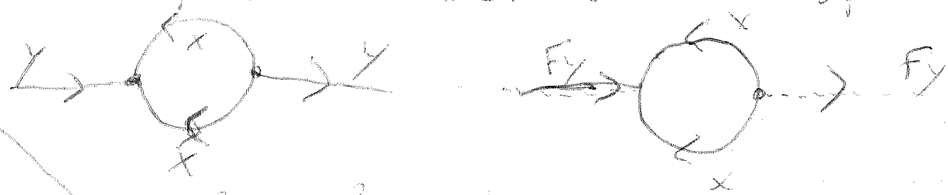
$\leadsto Q_1 = 1 \otimes e_{1A} \qquad Q_2 = 1 \otimes e_{2A}$

$Q_3 = \tilde{e}_1^\vee \otimes e_{1V}$, $Q_4 = \tilde{e}_1^\vee \otimes e_{2V}$

Lecture: One loop corrections to the Kähler potential

(D-term for chiral superfields = metric on moduli spaces)

$W = \frac{1}{2} X^2 Y$ (vev)

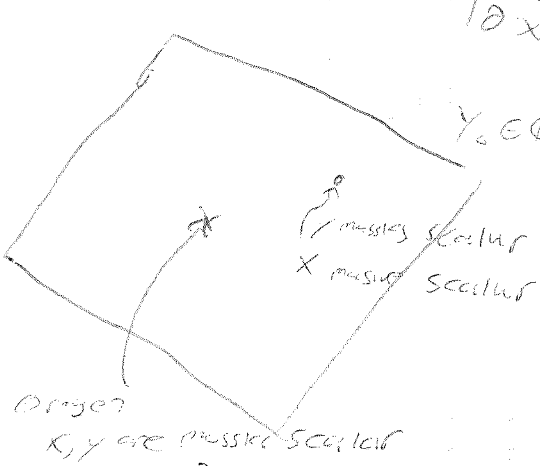


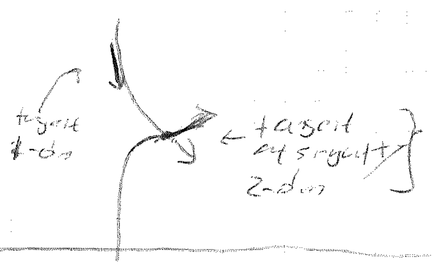
$\bar{Y} Y|_D = K(\bar{Y}, Y) \rightarrow \frac{1}{2} X^2 Y + \frac{1}{2} X^2 Y_0$ (Y_0 mass of X) $\sim \int \frac{p^3 dp}{(p^2+Y_0)^2} \sim \log\left(\frac{\Lambda}{Y_0}\right)$

$\ni \bar{F}_y F_y$

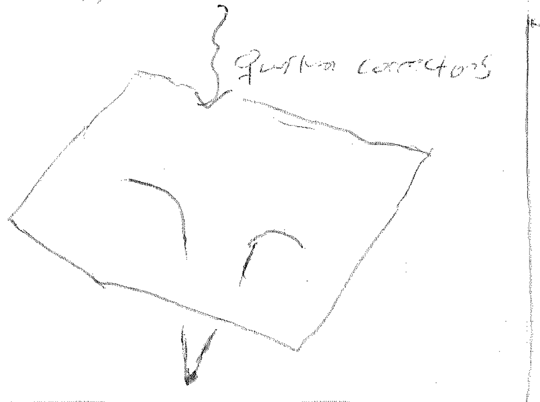
$(X=0, Y=Y_0)$

$\left|\frac{\partial W}{\partial X}\right|^2 \sim \frac{1}{2} Y_0^2 X^2$





singularities add a degree of freedom.



Moduli spaces in QCD (non-abelian gauge theory)

$$\frac{\{D=0\}}{G} = \frac{\text{no D-term}}{G_{\mathbb{C}}}$$

Ex: $G = U(1)$, $G_{\mathbb{C}} = \mathbb{C}^\times$

$$\mathbb{C}^2 / \mathbb{C}^\times = \frac{\{D=0\}}{U(1)}$$

- Pick a gauge group $G = SU(N_c)$, N_c = "number of colours"

$\mathfrak{g}$ = adj rep, $\dim(\mathfrak{g}) = N_c^2 - 1$ | $C = \mathbb{C}^{N_c} \sim G \curvearrowright C$

Vector multiplet valued in $\mathfrak{g}$: $(A_\mu^a, \lambda^a, \bar{\lambda}^a, D^a)$

- Matter: V = some other rep of G. Chiral multiplets valued in V.

(Ex: $V = C$ fundamental rep.)

- To guarantee anomaly cancellation, take reps + conjugate reps (e.g. $C \oplus \bar{C}$). sufficient but not necessary.

SQCD in the literature means "N_f fundamental flavours" (N_f number of flavours)

Flavor space $F = \mathbb{C}^{N_f}$

$\Rightarrow$ Matter fields are: $\left\{ \Phi_+ \in \overset{i,j}{F} \otimes \overset{m,n}{C} \right.$ ... $\bar{\Phi}_- \in \bar{F} \otimes \bar{C}$

V: $A_\mu^a \; \lambda^a \; \bar{\lambda}^a \; D^a$ ($\mathfrak{g}$-index)

$v \in \mathfrak{g}$ ... $\Phi^{im} \; \psi^{im} \; F^{im}$

$\bar{\Phi}_+ e^{-V} \Phi_+ + \dots$

$\bar{F} \otimes \bar{C} \quad G \quad F \otimes C$

$\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$

$\mathbb{R} \xrightarrow{\exp} \mathbb{R}^+$ (non compact)

$i\mathbb{R} \xrightarrow{\exp} U(1)$ (compact)

$\hat{\alpha} \longrightarrow \exp(\alpha) = e^\alpha$

$G = \mathbb{C}^\times$.

$\mathbb{C}^\times = GL(1,\mathbb{C}) \supseteq U(1)$
$\supseteq GL(1,\mathbb{R})_+$

$GL(n,\mathbb{C}) \supseteq U(n)$, $\supseteq GL(n,\mathbb{R})$

Re the VEVs; $\mathbb{C}^{2N_F N_C} \cong (F \otimes C) \oplus (\bar{F} \otimes \bar{C}) / SL(C)$

In general I break $SU(N_C)$ to $SU(N_C - N_F)$

A bit (more) on moduli spaces in "super-QCD"

V - $\mathfrak{g}$-gauge field

$G = SU(N_C)$, $C = \mathbb{C}^{N_C}$, N_F = "Flavours", $F = \mathbb{C}^{N_F}$ $\begin{cases} \Phi_+ \in C \\ \Phi_- \in \bar{C} \end{cases}$

$\rightsquigarrow \begin{cases} \Phi_+ \in C \otimes F \\ \Phi_- \in \bar{C} \otimes \bar{F} \end{cases}$

To compute the moduli space: space of possible VEVs $= \mathbb{C}^{2N_C N_F} = \underset{\Phi_+}{(C\otimes F)} \oplus \underset{\Phi_-}{(\bar{C}\otimes\bar{F})}$

- impose D-term conditions
- divide out the action of the gauge group $SU(N_C)$

... or, divide out by $G_\mathbb{C} = SL(N_C)$ (*)

The phase space of a (classical) dynamical system is a symplectic mfld

symmetries act by $G \to Symp((\mathbb{R}^{2n}, \omega))$

differential form $\rightsquigarrow \mathfrak{g} \to \underset{\text{symplectic vector fields}}{Vect^{sympl}_{\mathbb{R}}}(\mathbb{R}^{2n}, \omega)$ $\left(\begin{array}{l} X \in Vect(\mathbb{R}^{2n}) \\ \text{is symplectic when} \\ \mathcal{L}_X \omega = 0 \end{array} \right)$

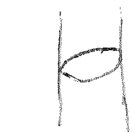
Pf $i_X\omega$ is closed $\Leftarrow$ $X \in vect^{sympl}$ $\quad d i_X \omega + i_X d\omega$ (exact, closed, ω is symplectic)

If $i_X \omega$ = exact $= d\lambda \rightsquigarrow X$ is "Hamiltonian"

here $\mathfrak{g} \to \underset{\text{observables (Poisson algebra)}}{C^\infty(\mathbb{R}^{2n}, \omega)}$ is the moment map assign the generator to the observable i.e.

generator of translations $\longrightarrow$ momentum $p \in C^\infty(\mathbb{R}^{2n})$

moment map allows one to kill things in both ways, symmetries and observables



T^*S^1 $\ni (p_\theta, \theta)$ periodic

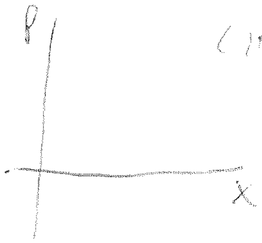
$\omega = d\theta \wedge dp_\theta$

$X = \frac{\partial}{\partial p_\theta} \rightsquigarrow i_X \omega = -d\theta$
$= d\lambda$
$\rightsquigarrow \lambda = -\theta \leftarrow$ not periodic

Example of symplectic but not hamiltonian

$=\}$ is truly a symplectic space

Hence μ-terms valued in $\mathfrak{g}^\vee$ $\Rightarrow$ def a moment map for action of gauge symmetry



$(\mathbb{R}^2, d\rho\wedge dx) \rightsquigarrow \mathbb{C}$

$\mathbb{R}: \frac{\partial}{\partial x} \Rightarrow$ moment map is ρ

$\mathbb{R}^2/\!/\mathbb{R} \simeq \mathbb{C}/\mathbb{C}$

It's nice to think of (ϕ_+, ϕ_-) as a pair of linear maps:

$\phi_+ \in C\otimes F \simeq \mathrm{Hom}(\bar F, C)$

$\phi_- \in \bar C\otimes \bar F \simeq \mathrm{Hom}(C, \bar F)$

$\bar F \underset{\phi_-}{\overset{\phi_+}{\rightleftarrows}} C \circlearrowright SL(C)$

$\bar F \xrightarrow{\phi_+} C \xrightarrow{\phi_-} \bar F$ is gauge invariant

indep of coimages of C.

$\underset{M}{\phi_+\phi_-} \in \bar F\otimes F$ is a gauge inv. observable "meson" quark and antiquark

There are N_f^2 maps $\bar F \to \bar F$ (if $N_c > N_f$)

If $N_c < N_f$ $\rightsquigarrow$ $N_f \times N_f$ matrices of rank $\le N_c$

On the symplectic side $\dim_{\mathbb{C}}(\mathcal{M}) = 2N_cN_f - \{\#$ broken gauge generators$\}$

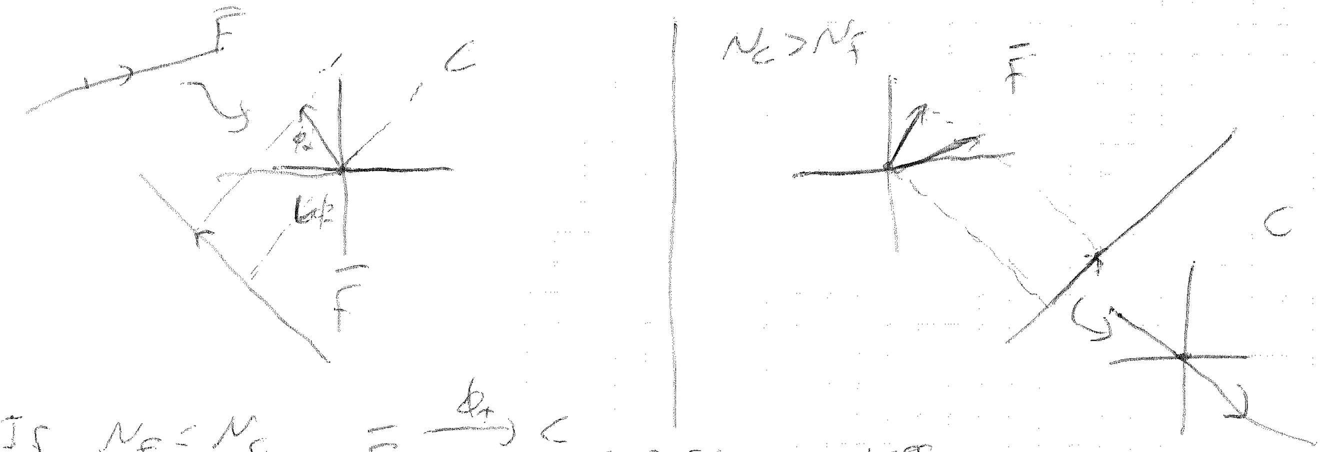
$N_f < N_c$: break $SU(N_c) \rightsquigarrow SU(N_c - N_f)$

$(N_c^2 - 1) - ((N_c - N_f)^2 - 1) = 2N_cN_f - N_f^2$

$\Rightarrow \dim_{\mathbb{C}}(\underset{\text{Conical space}}{\mathcal{M}}) = N_f^2$

If $N_f \ge N_c$: break $SU(N_c)$ completely (generically)

$\Rightarrow \dim_{\mathbb{C}}(\mathcal{M}) = 2N_cN_f - N_c^2 + 1$



If $N_f = N_c$ $\quad \bar F \underset{\phi_-}{\overset{\phi_+}{\rightleftarrows}} C$ are square matrices

$\Rightarrow \det(\phi_\pm) = B_\pm$ is a gauge inv op. (baryon like particles 3 quarks $\leftarrow$ 3 color)

N_c^2 $\quad$ $\left.\begin{array}{l}\phi_+\phi_- = M \\ B_\pm = \det(\phi_\pm)\end{array}\right\}$ rel: $\det M = B_+B_-$

2

$\to \mathbb{C} \to \Lambda^*(N_f) \longrightarrow \Lambda^*(\mathbb{C}) \to$ [illegible]

$$\binom{N_f}{N_c} = \frac{N_f!}{N_c!(N_f - N_c)!}$$

$N=2$ multiplets in 4d:

$N=2$	$N=1$ (guessing)
vector	vector + chiral
hypermultiplet	chiral + chiral

vector: $A, \binom{\lambda}{\bar\lambda}, D$

$$\delta_\varepsilon A_\nu = \varepsilon \gamma_\nu \lambda$$

$$\delta_\varepsilon \lambda = \varepsilon F_{\mu\nu}$$

(using eqt of motion? D)

supersymmetry

Kähler space: $g(u, Jv) = \omega(u, v)$ $\mathbb{C}$, $(z, \bar z) \to$ $\left.\begin{array}{l} dz = dx + i\,dp \\ d\bar z = dx - i\,dp \end{array}\right\}$ $dz \wedge d\bar z = 2i(dp \wedge dx)$

hyper Kähler space: $\mathbb{C}^2$ $(x_1\ x_2\ x_3\ x_4)$

~~$g(u, Jv) = \omega(u, v)$~~

$J_1 = g^{-1}\omega_1$

$$\omega_1 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$
$$\omega_2 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$$
$$\omega_3 = dx_1 \wedge dx_4 + dx_2 \wedge dx_3$$

"Dimensional reduction"

To dimensionally reduce, I just set some number of translations to zero.

$$t^{6d}_{N=(1,0)} = \underbrace{\mathbb{R}^{1,5}}_{+} \oplus \underbrace{(S_+ \otimes \mathbb{C}^2)_{\mathbb{R}}}_{-}$$

$\mathbb{C}^2$ ← symplectic

S_+: complex 4d chiral spin rep.

eight real supercharges

reduce to 4d | quotient map

$$t^{4d}_{N=2} = \mathbb{R}^{1,3}$$

$Spin(k) \times Spin(d-k) \leq Spin(d)$ Spinors of $Spin(d)$ decompose as tensor products of spinors of $Spin(k)$ and $Spin(d-k)$

How does chirality work? 6d Dirac spinor is

$$\Lambda^\bullet(L^\vee) \simeq \Lambda^\bullet(\mathbb{C}^3)^\vee, \quad S_+ = \Lambda^{even}, \quad S_- = \Lambda^{odd}$$

$\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$ ← 2d pol, 4d polarization

$\Rightarrow \Lambda^\bullet(\mathbb{C}^3)^\vee \cong \Lambda^\bullet(\mathbb{C}^2)^\vee \otimes \Lambda^\bullet(\mathbb{C})^\vee$

$\Lambda^{even}(\mathbb{C}^3)^\vee \cong \Lambda^{even}(\mathbb{C}^2)^\vee \otimes \Lambda^{even}(\mathbb{C})^\vee + \Lambda^{odd}(\mathbb{C}^2)^\vee \otimes \Lambda^{odd}(\mathbb{C})^\vee$

$\Rightarrow S_+^{6d} = S_+^{(4d)} \otimes S_+^{(2d)} \oplus S_-^{(4d)} \otimes S_-^{(2d)}$

~~odd~~ pairing $S_+^{(4d)} \otimes \mathbb{C}^2 \otimes S_+^{(2d)} \oplus \underbrace{S_-^{(4d)} \otimes \mathbb{C}^2 \otimes S_-^{(2d)}}_{\text{cpx conjugate}}$

R-Symmetry:

6d $\quad Sp(1) \cong SU(2)$

4d $\quad U(2)$

$\qquad \cong SU(2) \times U(1) / \mathbb{Z}_2$

The 4d R-symmetry consists of 6d R-symmetry together with transverse rotations.

Given this, I might expect 4d $N=2$ multiplets to arise by dimensionally reducing 6d $N=1$ multiplets.

In the example of the vector multiplet:

6d Gauge field		Chiral fermions		auxiliary field	
onshell	off shell	off	on	onshell	offshell
4	6	8	4	0	2+(1)

when I reduce, I have to decompose fields as reps of $Spin(1,3) \times Spin(2) \subseteq Spin(1,5)$. vector $\rightsquigarrow$ vector + 2 real scalars (w/ spin(2) charge)

After reduction, I effectively have
- one gauge field
- 2 physical scalars ($\mathbb{R}$)
- eight fermionic real d.o.f.

Counting works out too: 1 vector$^{(N=1)}$ + 1 $N=1$ chiral

$\mathcal{E}^{4d}_{N=1} \quad \mathcal{E}^{6d}_{N=(1,0)}$ How could I get more? SUSY?

$\downarrow \qquad\qquad \downarrow$ 4d $N=4$? (1 $N=1$ vector + 3 chirals)

$\mathcal{E}^{2d}_{N=2} \quad \mathcal{E}^{4d}_{N=2}$ (1 $N=2$ vector + 1 $N=2$ hyper)

real supercharges: 4 $\quad$ 8 $\quad$ 16

$\mathcal{E}^{4d}_{N=1} \rightsquigarrow \mathcal{E}_{N=2} \rightsquigarrow \mathcal{E}_{N=4}$

$$t^{4d}_{N=1} \hookrightarrow t^{4d}_{N=2} \hookrightarrow t^{4d}_{N=4}$$

$$t_{6d\, N=(1,0)} \qquad t_{6d\, N=(1,1)}$$

two options (other with $N=(2,0)$) but it doesn't lift higher

Count d.o.f.: vector multiplet:

fermions:		vector	
off	on	off	on
16	8	$\boxed{10}$	8

Reduce degrees of freedom: 10d vector → 4d vector + 6 scalars (real)

vector of Spin(6)

10 dim R-symmetry: SO(1) (trivial).

Reduction: Spin(6)

4d expect: U(4)?

Fermions transform as $S_+^{(10d)} = S_+^{4d} \otimes S_+^{6d} \oplus \left(\underline{S}^{4d} \otimes \underline{S}^{6d}\right)$

conjugate to

• $S_+^{(6d)} \simeq \mathbb{C}^4 \rightsquigarrow \underline{S}^{(6d)} \simeq \overline{\mathbb{C}^4}$

we are seeing the equivalence $Spin(6) \simeq SU(4)$

$$\frac{6 \cdot 5}{2} = 15 \qquad 4^2 - 1 = 15$$

Dynkin diagrams.

The R-symmetry or almost all of it arises by dimensional reduction.

Is the additional $U(1)$ important? Anyway it doesn't act on super Yang-Mills theory, which is $N=4$ theory w/o gravity.

⇒ For practical purposes we only discussed $SU(4)$-R-symmetry.

Could write down a (complicated) action for 4d $N=4$ by reducing!

10 d guess:

$$\mathcal{L} = \mathrm{Tr}(F \wedge \star F) + \mathrm{tr}(\lambda \cancel{D} \lambda)$$

$\partial + A$

reduce:

$A \to (A, \phi)$

$d A \wedge \star A + \star dA \wedge [A, A] + |[A,A]|^2$

$d\phi \wedge \star\phi + d\phi\, A \phi + A\phi A\phi$

combine into $D\phi \wedge \star D\phi$

$$\lambda \cancel{\partial} \lambda + \lambda A \lambda + \underbrace{\lambda \phi \lambda}_{\text{Yukawa couplings.}} + |[\phi,\phi]|^2 \leftarrow \text{scalar potential}$$

<u>Lesson</u>: 4d $N=4$ is <u>not</u> 4d $N=1$ gauge theory minimally coupled to 3 adjoint chirals.

need to add the superpotential $W =$