Topics on 2d CFTs: The Boundary and the Defects

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These notes explore the interplay between two-dimensional conformal field theories (2D CFTs) and generalized symmetries. The motivation of this interplay is shown from the perspective of boundary and interface CFTs, and later on, we present an axiomatic approach to topological defect lines in CFTs. The Ising model is constructed from the standard 2D minimal model, while the defect lines it can accommodate are bootstrapped from the modular (defect) bootstrap. The discussion encompasses gauging discrete/non-invertible symmetries and their implications.

CONTENTS

In order to talk about extended symmetries and their applications to CFTs, let us first give a general discussion about extended symmetries and defects. For this purpose, we follow [4].

I. GENERALIZED SYMMETRIES

Continuous symmetries on a field theory can be defined in terms of unitary operators $U(t)$ acting on local operators $\mathcal{O}$ by $U(t)\mathcal{O}(\vec{x},t)U^{-1}(t) = \mathcal{O}'(\vec{x},t)$. The symmetry, being generated by a charge $Q(t) = \int_{x_0=t} d^{d-1}xj^0(x)$, allows us to write U as

$$U_\alpha(\Sigma_{d-1}) = \exp\left(i\alpha\int_{\Sigma_{d-1}} j^{d-1}\right) .$$

The fact that the current is conserved means the operator U is topological along deformations of a Cauchy slice Σ_t, $U(\Sigma_{d-1}) = U(\Sigma'_{d-1})$ where

$$\Sigma'_{d-1} - \Sigma_{d-1} = \partial\Sigma_d .$$

We call a codimension-1 operator U which is topological and invertible a **0-form symmetry**. The fact that U is invertible means that there exists another topological codimension-1 operator U' such that we have

$$U(\Sigma_{d-1})U'(\Sigma_{d-1}) = 1 .$$

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0−form invertible symmetries may combine to form a $G^{(0)}$ group in which case we simply label the operators as $U_g \in G^{(0)}$ and the group multiplication describes the composition of symmetries:

$$U_g(\Sigma_{d-1})U_{g'}(\Sigma_{d-1}) = U_{gg'}(\Sigma_{d-1}) \,,$$

which can be seen as a fusion rule $U_g \otimes U_{g'} = U_{gg'}$.

In general, invertibility is non-trivial; here, it follows from unitarity, but a topological operator thus may be defined acting on local operators by linking with them,

$$U(S^{d-1})\mathcal{O}(x) = \mathcal{O}'(x) \,.$$

The picture is that for moving the topological symmetry past the local operator, one must pay the price of converting it into the local operator $\mathcal{O}'(x)$. Once the operator has no topological restriction, then it can shrink to a point, giving the above equality. Notice that the vector space of local operators at x forms a representation of $G^{(0)}$,

$$U_g\left(S^{d-1}\right)\mathcal{O}(x) = g \cdot \mathcal{O}(x) \,,$$

if the transformation is non-trivial, then we say the operators are charged under the symmetry.

Example I.1. *For a $U(1)$ symmetry, we can use the Noether current j to find the operator $U(\alpha)$. In the presence of a local operator $\mathcal{O}(x)$ of charge $q \in \mathbb{Z}$ under $U(1)$, the continuity equation is modified to*

$$\mathcal{O}(x)\partial_\mu j^\mu(x') = q\delta(x-x')\mathcal{O}(x) \,,$$

which leads to

$$\begin{aligned} U_g\left(S^{d-1}\right)\mathcal{O}(x) &= \exp\left(i\alpha\int_{S^{d-1}} j^{d-1}\right)\mathcal{O}(x) \ = \exp\left(i\alpha\int_{D^d} dj^{d-1}\right)\mathcal{O}(x) \\ &= \exp\left(i\alpha\int_{D^d} q\delta^d(x)\right)\mathcal{O}(x) \,, \\ &= \exp\left(iq\alpha\right)\mathcal{O}(x) \,. \end{aligned}$$

where D^d is the d-dimensional disk containing the point x whose boundary is S^{d-1}.

Similarly, a **p-form symmetry** is a codimension-$(p+1)$ operator $U(\Sigma_{d-p-1})$ which is topological and invertible,

$$U(\Sigma_{d-p-1}) = U(\Sigma'_{d-p-1}), \quad U(\Sigma_{d-p-1})U^{-1}(\Sigma_{d-p-1}) = 1 \,.$$

for $\Sigma'_{d-p-1} - \Sigma_{d-p-1} = \partial\Sigma_{d-p}$. p-form symmetries may form a $G^{(p)}$ group. An important consequence of the linking is that higher-form symmetry groups are abelian as we can use topological deformations to change the ordering of two topological operators of codimension greater than one.

As we discussed above, a 0-form symmetry acts on operators that naturally "capture". p-form symmetries similarly act on objects that cannot be trivially unlinked. For instance, a point can only be linked by a $d-1$ dimensional object as otherwise there is always a "space" to unlink them by a topological move. Instead, it is not hard to see that a p-form symmetry can act only nontrivially to extended operators that are defined on a $q \geq 1$-dimensional submanifold M_q of spacetime for $q \geq p$.

The simplest case to study is $q = p$. p-dimensional operators transform in representations of the p-form symmetry group $G^{(p)}$. Consider a $p \geq 1$-dimensional extended operator $\mathcal{O}(M_p)$ placed along a p-dimensional submanifold M_p of spacetime. We can assume that $\mathcal{O}(M_p)$ is an irreducible p-dimensional operator i.e. there are no topological local operators that can be inserted at a point $x \in M_p$ except multiples of the identity local operator. Now, note that deforming $U_g(\Sigma_{d-p-1})$ across $\mathcal{O}(M_p)$ leaves behind a topological local operator $\mathcal{O}(x)$ at the intersection point x of M_p and Σ_{d-p};

$$U_g(\Sigma_{d-p-1})\mathcal{O}(M_p) = \mathcal{O}(x)\mathcal{O}(M_p)U_g(\Sigma'_{d-p-1}) \,.$$

Since the only possible topological local operators along M_p are multiples of identity, we can replace $\mathcal{O}(x)$ by a non-zero number in the above equation

$$U_g(\Sigma_{d-p-1})\mathcal{O}(M_p) = \phi(g) \times \mathcal{O}(M_p)U_g(\Sigma'_{d-p-1}), \quad \phi(g) \in \mathbb{C}^\times = \mathbb{C} - \{0\}.$$

Because of the fusion rule, the numbers $\phi(g)$ have to satisfy

$$\phi(g)\phi(g') = \phi(gg') ,$$

i.e. the numbers $\phi(g)$ furnish a one-dimensional representation of the p-form symmetry group $G^{(p)}$, and in particular, the numbers $\phi(g)$ must be phase factors

$$\phi(g) \in U(1) \subset \mathbb{C}^{\times} .$$

Example I.2. *Consider a $U(1)$ gauge theory. The field strength $F = dA$ satisfies $dF = 0$ and thus it can be viewed as a conserved current to generate a $(d-3)$-form symmetry,*

$$U_g^{(m)}(\Sigma_2) = \exp\left(i\alpha \int_{\Sigma_2} F\right), \quad g = e^{i\alpha} \in U(1) .$$

Moreover, since $d * F = 0$, we can also write a $1-$form symmetry

$$U_g^{(e)}(\Sigma_{d-2}) = \exp\left(i\alpha \int_{\Sigma_{d-2}} \star F\right), \quad g = e^{i\alpha} \in U(1) .$$

The former combines into a group forming the magnetic $G^{(d-3)}$ symmetry while the latter to the electric $G^{(1)} = U(1)$ group symmetries.

II. CONFORMAL FIELD THEORY IN 2D

A quantum field theory in two dimensions is special as it would admit 0- and 1-form symmetries which are now lines. We therefore turn our attention to 2d CFTs to elaborate on these concepts a bit further. The study of extended operators has been a long-standing area of research within CFTs. Pioneering work in boundary CFTs by Cardy and rational CFTs by Moore and Seiberg laid the foundation for the explicit construction of extended operators on what today are known as generalized symmetries, as they, in some sense, enrich the structure of CFTs. In this section, we will explore various applications and establish connections between these concepts. Recall that one can define a 2D CFT on a Riemann surface, up to a universal conformal anomaly. We start by considering the genus $g = 1$ case.

A. The torus partition function

Due to the state operator correspondence, the interpretation of correlation functions on the cylinder is straightforward. Starting from a state $|\psi_1\rangle$ at τ_1, we want to compute the amplitude of $|\psi_2\rangle$ at τ_2. This is

$$\langle\psi_2|\psi_1\rangle = \langle\psi_2| e^{-HT} |\psi_1\rangle$$

where H is the generator of time evolution, i.e. dilations, which in the cylinder is given by

$$H = L_0 - \frac{c}{24} + \bar{L}_0 - \frac{c}{24}$$

and the c factors are interpreted as the Casimir energy. The partition function can thus be constructed from a cylinder evolved in imaginary time by ω_1 with H. To create a torus, we join both ends of the cylinder, allowing for a possible twist, which is realized by inserting a rotation $R = \exp\{2\pi i(L_0 - \bar{L}_0)\}$ of angle ω_2. This twist comes from the fact that on the torus, one can either fix the periodicity of the torus or the differential structure but not both at the same time; the parameter that measures different torii is called the moduli. Finally, the torus is made from gluing both ends of the cylinder as shown in Figure 1. This is nothing but taking the trace in the above picture for which, setting $(\tau, \bar{\tau}) \equiv (-\omega_2 + i\omega_1, -\omega_2 - i\omega_1)$, the partition function thus can be written as

$$Z(\tau) \equiv \mathrm{tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}, \quad q \equiv \mathrm{e}^{2\pi i\tau}, \bar{q} \equiv \mathrm{e}^{-2\pi i\bar{\tau}}.$$

Here, the parameter τ of the Riemann surface takes the role of the moduli.

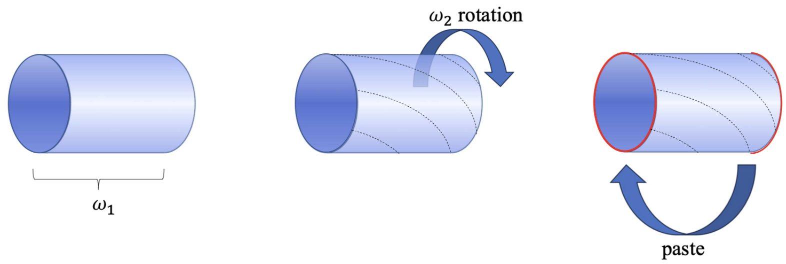


FIG. 1: Torus construction from gluing a cylinder. Image retrived from [2].

The fundamental domain of the group $PSL(2,\mathbb{Z})$ describes uniquely the orbits of equivalent torii and, in particular, this means that the partition function is invariant under these transformations. We consider two particular transformations, the translations T and the modular transformation S, given by:

$$T : \tau \to \tau + 1, \quad S : \tau \to -\frac{1}{\tau}.$$

where, for example, invariance under T requires

$$Z(\tau) = Z(\tau+1) = \mathrm{tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \mathrm{e}^{2\pi i (L_0 - \bar{L}_0)}.$$

which means that the spin must be an integer. The modular transformation imposes a rather stronger constraint,

$$Z(\tau) = Z\left(-\frac{1}{\tau}\right) \tag{1}$$

which in general puts constraints on the spectrum of the CFT –although it does not determine the underlying CFT as there are examples of CFTs with the same partition function–

By combining the S and T transformations, we can derive conformal bootstrap equations called modular bootstrap equations. Recall that on the sphere one can deduce consistency conditions from the 4-pt function by expanding in the different OPE channels leading to the conformal bootstrap. In the torus, consistency requires relations in the 0-pt function, i.e. the partition function.

Now, because the holomorphic and anti-holomorphic Virasoro representations decouple one can write the partition function using the Virasoro characters

$$Z = \sum_{(h,\bar{h})} n_{h\bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q}),$$

where

$$\chi_i(\tau) \equiv \mathrm{tr}_{\mathcal{V}_i} q^{L_0 - \frac{c}{24}} = \sum_n d(n) q^{h_i + n - \frac{c}{24}},$$

$d(n)$ counts the number of descendant states at level n in the irreducible Verma module and, n_{ij} are the multiplicities of each $(h, \bar{h})$ representation. In general $n_{00} = 1$, as the identity must be present and non-degenerate. In particular, in terms of the Virasoro characters, we define the $\mathcal{S}$ **modular matrix** to be defined by the S modular transformation on the Virasoro characters,

$$S : \chi_h(\tau) \mapsto \sum_j \mathcal{S}_{hj} \chi_j(\tau).$$

A fascinating result of the modular S matrix is that the modular $\mathcal{S}$ matrix diagonalizes the fusion rules. A result known as the Verlinde formula,

$$\mathrm{N}_{ij}^{k} = \sum_i \frac{\mathcal{S}_{il} \mathcal{S}_{jl} \mathcal{S}^*_{kl}}{\mathcal{S}_{l0}}. \tag{2}$$

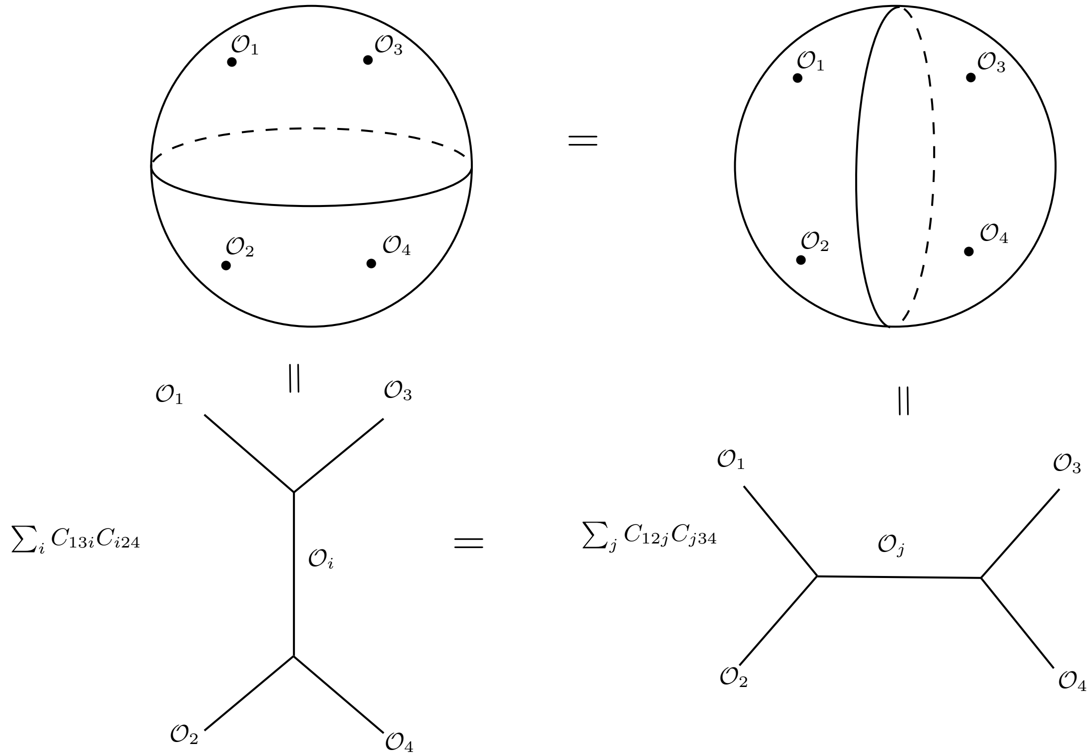


FIG. 2: Conformal bootstrap for the four-point function on the sphere

Indeed, if Equation (2) is satisfied, considering a vector v_a whose entries are given by $(v_a)_k = S_{ak}$, then

$$(\mathrm{N}_i v_a)_j = \sum_k \mathrm{N}_{ij}^k \mathcal{S}_{ka} = \frac{\mathcal{S}_{ia}}{\mathcal{S}_{0a}} \mathcal{S}_{ja} = \frac{\mathcal{S}_{ia}}{\mathcal{S}_{0a}} (v_a)_j \,.$$

We conclude v_a is an eigenvector with eigenvalue $n_{ia} = \frac{\mathcal{S}_{ia}}{\mathcal{S}_{0a}}$ and we define the **quantum dimension** of the TDL to be $d_a = n_{a0}$. In particular, note that

$$\chi_i(q) = \sum_j \left(\mathcal{S}^{-1}\right)_{ij} \chi_j(\tilde{q}) \xrightarrow{\tau \to i0^+} \left(\mathcal{S}^{-1}\right)_{i0} \chi_0(\tilde{q}) = d_i \chi_0(\tilde{q})$$

as the character is dominated by the vacuum.

B. Moore-Seiberg Construction

The modular bootstrap and the conformal bootstrap of four-point functions on the sphere are independent constraints on the spectrum of a CFT. The fact that one can obtain any closed orientable Riemann surfaces from different surgery procedures imposes consistency conditions on the fusion rules of $2d$ CFTs and, in fact, solving the conformal bootstrap for the four-point function on the sphere, as shown in Figure 2, and the one-point function on the torus,

$$\langle \mathcal{O} \rangle_{-1/\tau} = \tau^h \bar{\tau}^{\bar{h}} \langle \mathcal{O} \rangle_\tau,$$

where the factor is a Weyl factor coming from the $w \to w'/\tau$ coordinate change, are sufficient to completely determine the CFT on arbitrary orientable Riemannian surfaces. We follow closely [2, 5] on this construction. Graphs are also retrieved from [2].

The construction goes as follows: A general genus $g > 1$ orientable Riemannian surface can be constructed by plumbing together $2g - 2$ three-holed spheres. The plumbing construction glues together a pair of circular boundaries of three-holed spheres (or two-holed discs) by an $\mathrm{PSL}(2, \mathbb{C})$ map. A typical plumbing map takes the form

$$z' = q/z, \quad q \in \mathbb{C}^\times$$

which identifies the boundary $|z| = r_1$ on one of the two-holed discs to the boundary $|z'| = r_2$ of the other two-holed disc, with $r_1 r_2 = |q|$. There is one complex modulus q associated with each of the $3g-3$ plumbing maps, giving rise to $3g-3$ complex structure moduli of the Riemann surface.

The equivalence of different plumbing constructions of the same Riemann surface, and therefore of the consistency of the CFT on the surface, would be guaranteed provided that all sphere 4-point functions can be equivalently decomposed into two different channels, and that all torus 1-point functions constructed by plumbing together the inner and outer boundaries of a 1-punctured annulus are independent of the choice of how the torus is cut open into an annulus. The former is equivalent to the associativity of OPE, while the latter amounts to the modular covariance of the torus 1-point function of all primaries. Namely,

$$\langle \mathcal{O} \rangle_{-1/\tau} = \tau^{h} \bar{\tau}^{\bar{h}} \langle \mathcal{O} \rangle_{\tau},$$

for every primary $\mathcal{O}_{h,\bar{h}}$.

Let us see how one can construct such correlation functions. From the fusion transformation of the four-point Virasoro block and the modular transformation of the torus one-point block, one can construct any fusion transformation (transformation between bases of functions for N-point functions on any Riemann surface). Indeed, from the fusion decompositions on the torus, as shown in Figure 3, and from the sphere fusion on the sphere, as shown in Figure 4

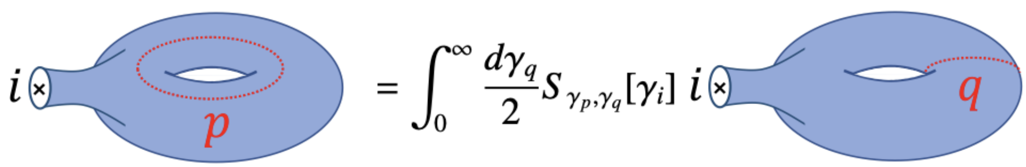


FIG. 3: Fusion of the torus.

one can write the desired channel decomposition for any $n-$pt function.

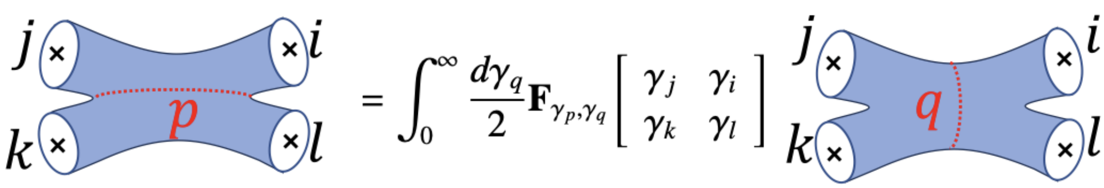


FIG. 4: Fusion on the sphere

For instance, consider the 2-pt function on the torus. Using the fusion transformations, one can write the channel decomposition, as shown in Figure 5. Similarly, the Virasoro block expansion can be performed for N-point functions on a Riemann surface of arbitrary genus using the plumbing construction.

III. DEFECT LINES IN $2d$ CFTS

Consider two, a priori different, CFTs, say CFT_1 and CFT_2. Now, one can locally describe the theory of CFT_1 while inserting a non-local operator that acts as the boundary of the CFT. In particular, one can consider the case where the two CFTs coexist, separated by a domain wall or boundary. For this theory to be consistent, we require the interface to fulfill certain relations. In particular, we require a non-local operator, called the interface or the domain wall, to glue the two CFTs according to

$$T_1(x) - \bar{T}_1(\bar{x}) = T_2(x) - \bar{T}_2(\bar{x}), \quad x \in \mathbb{R}.$$

The interface D is the operator $D : \mathcal{H}_1 \to \mathcal{H}_2$ mapping the Hilbert space of one theory to the other, while $\bar{D}$ is the reverse map. The existence of the gluing in particular may allow energy to flow through the defect depending on how

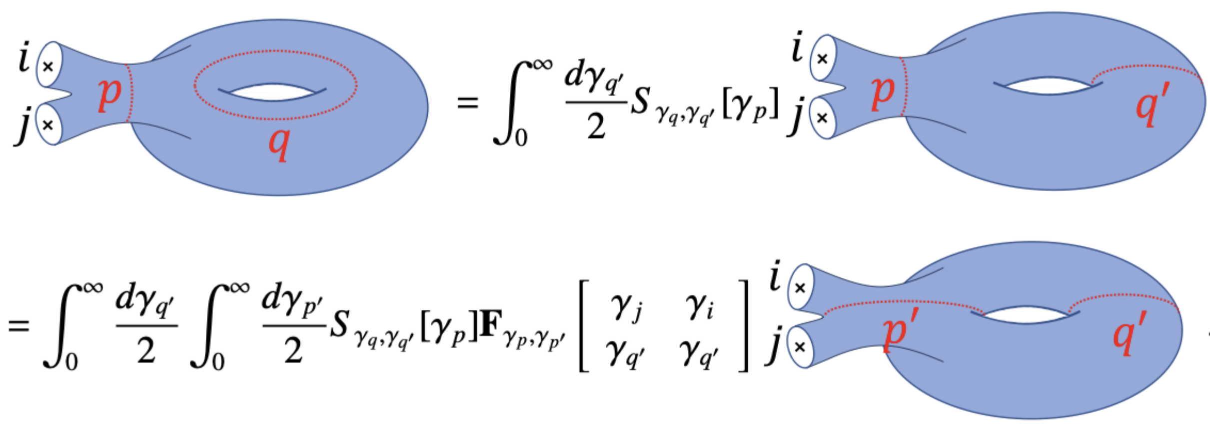


FIG. 5: 2-pt function channel transformation in the torus.

the gluing is done. For instance, if

$$T_1(x) - \bar{T}_1(\bar{x}) = 0$$

then no energy can be carried by the interface and thus energy cannot flow between the CFTs. Similarly, one can also require the defect D to commute with the Virasoro generators in which case the defect is free to move subject to topological deformations. In particular, this means that the gluing must satisfy

$$T_1(x) = T_2(x), \quad \bar{T}_1(\bar{x}) = \bar{T}_2(\bar{x}), \quad x \in \mathbb{R}. \tag{3}$$

The fact that T generates diffeomorphisms implies that D is thus topological and the two CFTs must coincide, up to an isomorphism of algebras. Consider now a map $D : \mathcal{H}_1 \to \mathcal{H}_2$, where

$$\mathcal{H}_1 = \bigoplus_{h,\bar{h}} N^{(1)}_{h,\bar{h}} \mathcal{H}_h \otimes \mathcal{H}_{\bar{h}} \quad \text{and } \mathcal{H}_2 = \bigoplus_{h,\bar{h}} N^{(2)}_{h,\bar{h}} \mathcal{H}_h \otimes \mathcal{H}_{\bar{h}}$$

are the Hilbert space of the CFT_1, having $N^{(1)}_{h,\bar{h}}$ copies of $\mathcal{V}_h \otimes \mathcal{V}_{\bar{h}}$ and similarly for the CFT_2. Since both representations are irreducible, Equation (3) means that D must send a Verma module of the CFT_1 to the same Verma module, up to different copies, of the CFT_2 (otherwise the representations would be reducible), and so D and $\bar{D}$ are the interwiners of the different copies of the Verma modules in the two CFTs. Therefore, the set of Topological Defect Lines (TDLs) is:

$$\left\{ D_A = \sum_{i,r} D_A^{(h,\bar{h};sr)} P^{rs}_{h,\bar{h}} | \, A \in \sum_{h,\bar{h}} N^{(1)}_{h,\bar{h}} N^{(2)}_{h,\bar{h}} \right\}, \tag{4}$$

where $P^{rs}_{h,\bar{h}} : (\mathcal{H}_h \times \mathcal{H}_{\bar{h}})^s \to (\mathcal{H}_h \times \mathcal{H}_{\bar{h}})^s$ are the interwiners of $\mathcal{V}_h \otimes \mathcal{V}_{\bar{h}}$ mapping r copies of the Hilbert space to the s copies of $\mathcal{H}_2$ satisfying

$$L^{(1)}_n P^{rs}_{h,\bar{h}} = P^{rs}_{h,\bar{h}} L^{(2)}_n, \quad \bar{L}^{(1)}_n P^{rs}_{h,\bar{h}} = P^{rs}_{h,\bar{h}} \bar{L}^{(2)}_n.$$

Notice that in particular, the identity is a TDL. Few remarks are inorder.

1. Composition of two TDLs A and B, which we shall from now on call fusion, defines a new TDL $D_{A^\dagger B} : \mathcal{H}_1 \to \mathcal{H}_1$.

$$D_{A^\dagger D} = D_A^\dagger D_B = \sum_{h,\bar{h},r,r',s} \left(D_A^{(h,\bar{h};sr)} \right)^* \left(D_B^{(h,\bar{h};sr')} \right) P^{rr'}_{h,\bar{h}}$$

as † reverses the order.

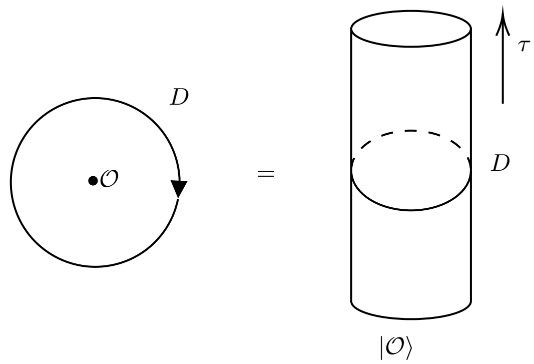


FIG. 6: Insertion of a space-like defect

2. Insertion of the defect on the partition function, as shown in Figure 6, gives

$$Z_A = \mathrm{tr}_{\mathcal{H}}\, D_A q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} = \sum_{h,\bar{h},r,r',s} D_A^{(h,\bar{h};sr)}\chi_h(q)\bar{\chi}_{\bar{h}}(\bar{q}). \tag{5}$$

   Thus, defining the action of the defect through the action in the characters of the Verma modules completely defines the action of the defect. Due to modular invariance, it also implies the existence of topological defects along the "time" direction, see Figure 7, where we recognize as a twisted Hilbert space in the presence of a defect D. This Hilbert space is denoted as $\mathcal{H}^D_{S^1}$ and is called the D-twisted sector.

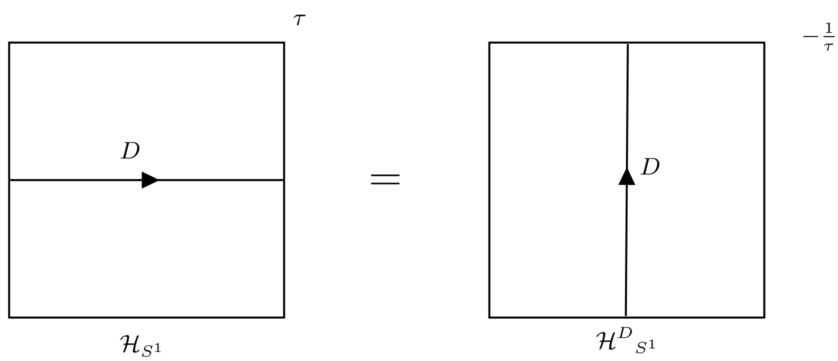


FIG. 7: Modularity of the defect insertion

3. In particular, insertion of these types of TDL amounts to

$$Z_D = \mathrm{tr}_{\mathcal{H}^D}\, q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} = \sum_{h,\bar{h}} n_D^{h,\bar{h}}\chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau})$$

   for some positive integers $n_D^{h\bar{h}}$ defining the twisiting

$$\mathcal{H}^D = \bigoplus_{h,\bar{h}} n_D^{h,\bar{h}}\mathcal{H}_h \otimes \mathcal{H}_{\bar{h}}.$$

We conclude by remarking that the introduction of TDL may generate local fields, corresponding to states in the twisted Hilbert space. These are called **defect fields** and are of four kinds, depending on how the defects can be attached:

1. **Junction fields**: Inserting at least three defects, junction fields map one defect into at least two others.

2. **Defect-changing fields**: Fields mapping a TDL A into another B.

3. **Twist fields**: They terminate a defect A, i.e. they are defect-changing fields from A into the identity TDL 1.
4. **Bulk fields**: Fields that are uncharged under the TDL, i.e. TDLs are invisible for the fields.

Example III.1. *Consider TDLs in a diagonal theory i.e.* $CFT_1 = CFT_2$ *and* $N_{h,\bar{h}} = \delta_{h,\bar{h}}$*. Since the interwinners are just projections on the Verma modules, we consider the type of TDL defined by*

$$D_a = \sum_h \frac{\mathcal{S}_{ah}}{\mathcal{S}_{0h}} P_h,$$

where $P_h = P^{11}_{h,\bar{h}}$ *in our previous notation, and* $\mathcal{S}$ *is the modular matrix. These types of TDLs are called Verlinde lines, and they are described by the underlying OPE fusion. Indeed, using* $\mathcal{S}$*, we compute:*

$$D_a D_b = \sum_{ij} \frac{\mathcal{S}_{ai}}{\mathcal{S}_{0i}} \frac{\mathcal{S}_{bj}}{\mathcal{S}_{0j}} \delta_{ij} P_j = \sum_{ijc} \frac{\mathcal{S}_{ai}\mathcal{S}_{bi}}{\mathcal{S}_{0i}} \frac{\mathcal{S}^*_{ic}\mathcal{S}_{cj}}{\mathcal{S}_{0j}} P_j \tag{6}$$

$$= \sum_{cj} \mathrm{N}^c_{ab} \frac{\mathcal{S}_{cj}}{\mathcal{S}_{0j}} P_j = \sum_c \mathrm{N}^c_{ab} D_c. \tag{7}$$

Notice that acting on primary states we have

$$D_a \left|\phi_i\right\rangle = D^i_a \left|\phi_i\right\rangle = \frac{\mathcal{S}_{ai}}{\mathcal{S}_{0i}} \left|\phi_i\right\rangle, \quad D_a|0\rangle = \frac{\mathcal{S}_{a0}}{\mathcal{S}_{00}} |0\rangle = \mathrm{d}_a|0\rangle,$$

where we used the quantum dimension of Section II A. We conclude that the expectation value of the TDL is precisely the **quantum dimension** *of the TDL,*

$$\langle D \rangle = d_D.$$

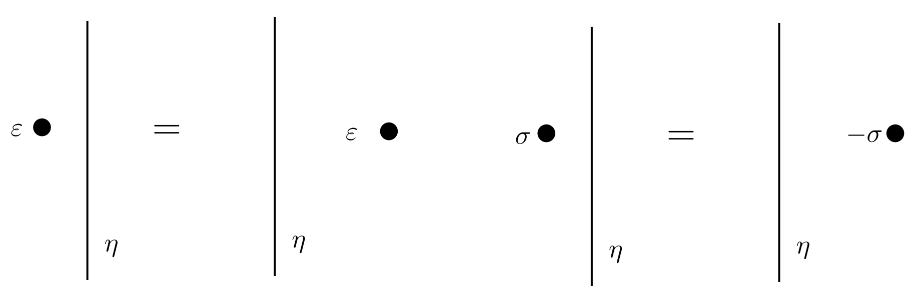
IV. GENERALIZED GLOBAL SYMMETRIES IN $d = 2$ CFT AND THEIR BOOTSTRAP

In $2d$ CFTs, since the action of TDL on charged operators preserves $(h, \bar{h})$ (since the operator commutes with the stress-tensor), from our point of view, we consider symmetries as topological defect lines (TDL) and every time we identify a symmetry we should ask what is the underlying TDL. Thus, we change the perspective to consider axiomatic CFTs in the presence of topological defect lines and ask the question: which kind of TDLs we can consistently accommodate? For this, we consider a set of axioms described by a so-called "symmetry enriched CFT" which will allow us to bootstrap different TDLs and their underlying symmetries.

Example IV.1. *For instance, consider the Ising model. The symmetry allows to write the Ward identities*

$$\langle \sigma\sigma\sigma \rangle = \langle \sigma\epsilon\epsilon \rangle = 0,$$

and more generally, any odd number of insertions of σ *vanish. The underlying reason is that we can define an associated TDL* η *acting on the fields by*



To define completely η *we require, in addition, the action on the identity. This reduces simply to* $1 \to \langle \eta \rangle 1$*, so in order to generate the* $\mathbb{Z}_2$ *symmetry,* η *must have quantum dimension* 1*. On the other hand, the Ward identity*

$$\langle \varepsilon \dots \varepsilon \rangle = 0$$

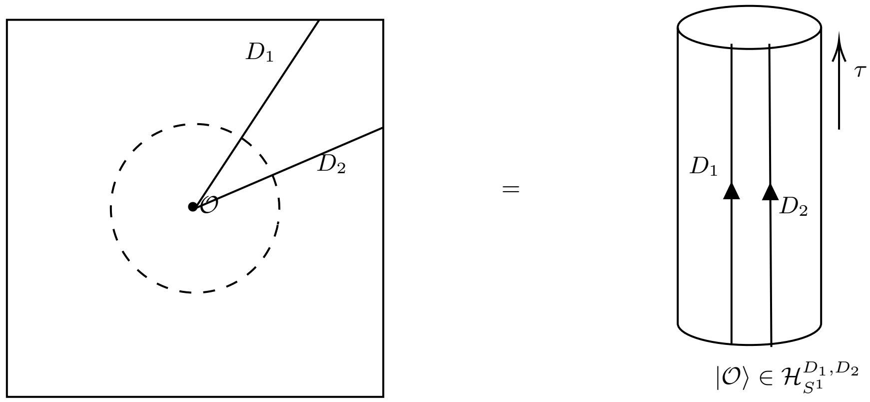


FIG. 8: Two defect insertions D_1 and D_2. After a topological move we see it is irrelevant the order of D_1 and D_2.

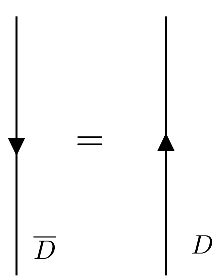
for odd insertions of ε, suggests there is an underlaying TDL guaranteeing that and, since $\langle\sigma\sigma\varepsilon\rangle \neq 0$, it cannot be a $\mathbb{Z}_2$ charge of ε. It turns out that this symmetry will be a non-invertible symmetry $\mathcal{N}$ acting by

$$\varepsilon \to -\varepsilon\langle\mathcal{N}\rangle, \quad \sigma \to 0$$

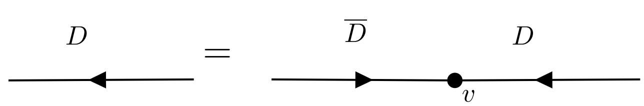
allowing $\langle\sigma\sigma\varepsilon\rangle \neq 0$.

Naturally, not every TDL (symmetry) is allowed. We are going to see how the space of allowed TDLs is greatly constrained by the (defect) modular bootstrap. Before we present the relevant axioms for CFTs interplaying with TDLs, let us introduce the relevant terminology and rules of the game.

First, for every TDL D, which we assume to act faithfully on all local operators (i.e. only defect that acts trivially is the identity defect), we define the dual defect $\bar{D}$ as the orientation-reversed defect.



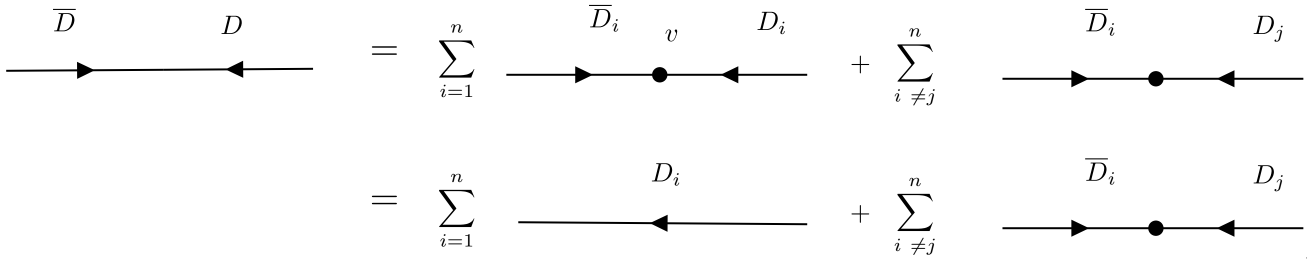
and we say a defect is **simple** if $\dim V_{D\bar{D}} = 1$, where $V_{D\bar{D}} \subset \mathcal{H}$ is the Hilbert space of junctions between D and $\bar{D}$. This means there is only one junction v, which is the identity field, as shown in the Figure:



We define the multi-defect Hilbert space $\mathcal{H}_{S^1}^{D_1,D_2,\ldots,D_j}$ as the Hilbert space describing the action of multi-defect insertions of simple topological defect lines, see Figure 8. For non-simple defect lines, we define the Hilbert space of the insertion as the direct sum of their underlying simple TDL Hilbert spaces. For instance, for $D = D_1 + D_2$ we define

$$\mathcal{H}_{S^1}^{D_1+D_2} := \mathcal{H}_{S^1}^{D_1} \oplus \mathcal{H}_{S^1}^{D_2}.$$

It follows that if D is not simple, then $D = \sum_{i=1}^{n} D_i$ for D_i simple lines, and we have:



This is translated in terms of junctions as $dim(D\bar{D}) \geq 1$ as the first term admits a unique $v = 1$. Similarly, we define the topological junctions $v \in V_{D_1 D_2 D_3}$ as morphisms between $D_1 D_2$ and $\bar{D}_3$ and, for higher junctions we write $V_{D_1 \ldots D_n} \subset \mathcal{H}$ as the space of n junctions. Finally, fusion of defects is defined as the fusion of simple defects. For the latter, we require the fusion rules:

$$D_i D_j = \sum_k N_{ij}^k D_k,$$

where N are the fusion coefficients and fusion satisfies commutativity, associativity, and bilinearity. Associativity is motivated by the fact that insertion of two TDLs, under topological moves, relates $D_1 D_2 = D_2 D_1$ as shown in Figure 8.

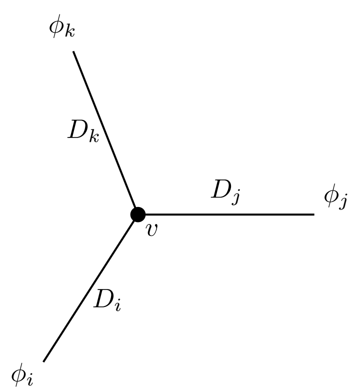
Finally, it is easy to see that $N_{ij}^k = \dim V_{D_i D_j \bar{D}_k}$ and, in particular, note that the symmetries associated to TDLs will not be invertible as fusion, in general, gives a non-simple TDL. If the symmetry is invertible, then one can label defects by group elements g_i whose junctions are $\dim V_{g_1 \ldots g_n} = 1$ if $g_1 g_2 \ldots g_n = 1$ and is otherwise zero.

Having unpacked the definitions of lines and junctions, we define a **symmetry enriched CFT** as a CFT enriched by a collection of topological defect lines $\{D_i\}$ such that:

1. Data: The data are those of a CFT, i.e. a spectrum $\mathcal{S}$ of local fields enriched with the structure of TDLs. This constitutes:

   (a) A collection of twisted Hilbert spaces $\mathcal{H}_{S^1}^{D_i \ldots D_j}$ describing the action of multi-defect insertions on the fields of the CFT.

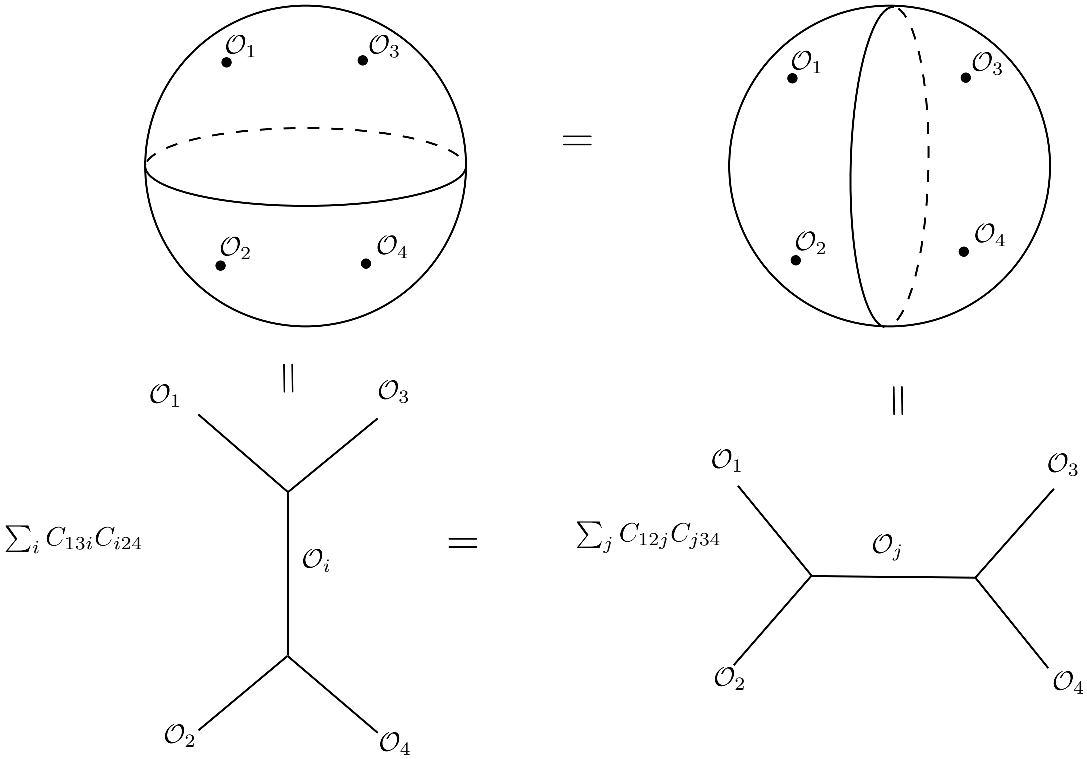
   (b) The collection of three-point functions of operators attached to defects,



   for v a topological junction i.e. an operator of dimension $h = \bar{h} = 0$ inside $\mathcal{H}_{S^1}^{D_1 \ldots D_n}$, $\{\phi_i\}$ local fields of the underlaying CFT and, $\{D_i\}$ TDLs.

2. Bootstrap conditions (locality): The axioms of locality and coherence are those of Moore and Seiberg compatible with the presence of TDLs,

   (a) Sphere four-point crossing in the presence of topological defects,



(b) Modular covariance of the torus one-point function with defects,

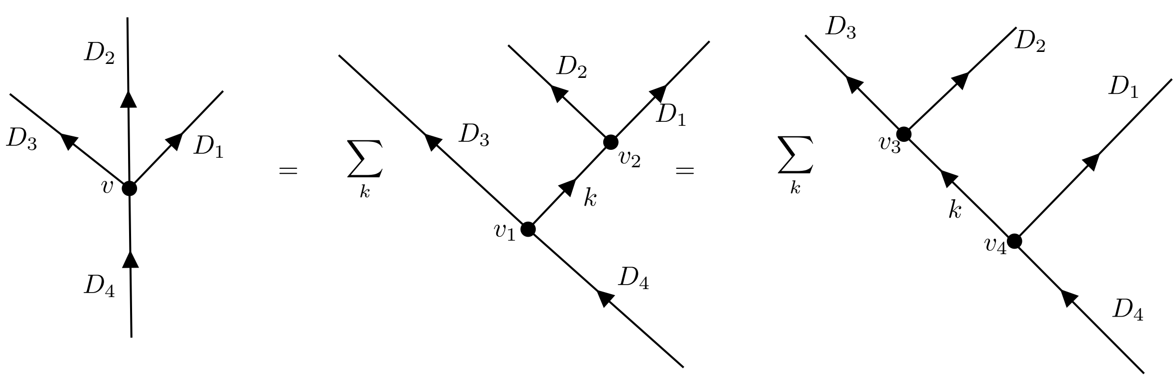
$$\langle \mathcal{O} \rangle_{\mathcal{H}^D,-1/\tau} = \tau^h \bar{\tau}^{\bar{h}} \langle \hat{D}\mathcal{O} \rangle_{\mathcal{H},\tau},$$

for every primary $\mathcal{O}_{h,\bar{h}}$ and D TDL.

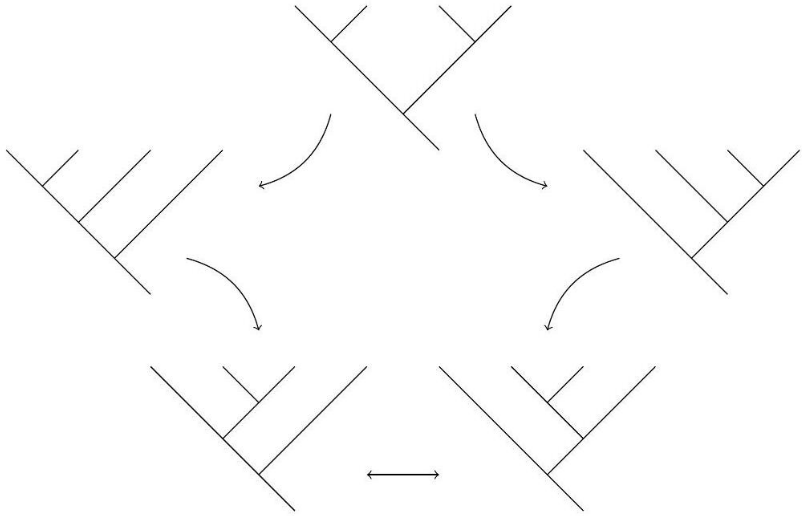
Few remarks are in order:

1. Note that the data of a symmetry enriched CFT includes the usual $\mathcal{H}_{S^1}$ and three point functions as we could set $D = 1$.

2. In a general junction Hilbert space there is no canonical basis. Due to the fact that many bases of $V_{D_1 D_2 D_3 \bar{D}_4}$ can be constructed from bases of three-fold junctions,

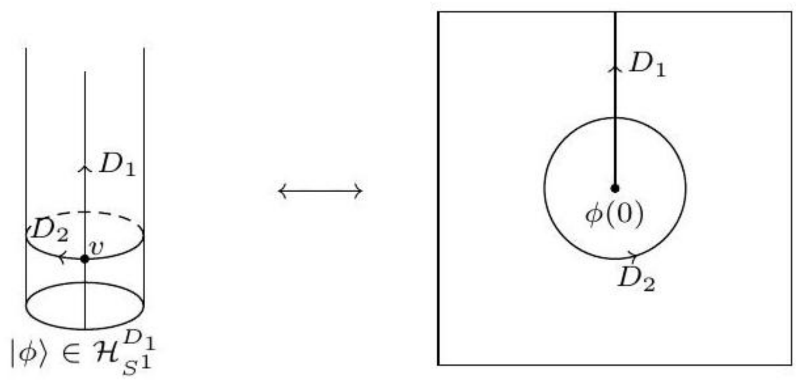


there exists a unitary change of basis $\left(F_4^{321}\right)_{ab}$ mapping the basis $v_1 \otimes v_2$ to $v_3 \otimes v_4$. It can be shown that the fusion coefficients F have to obey an equation of the form $FF = \sum FFF$, obtained by performing the following fusion steps:

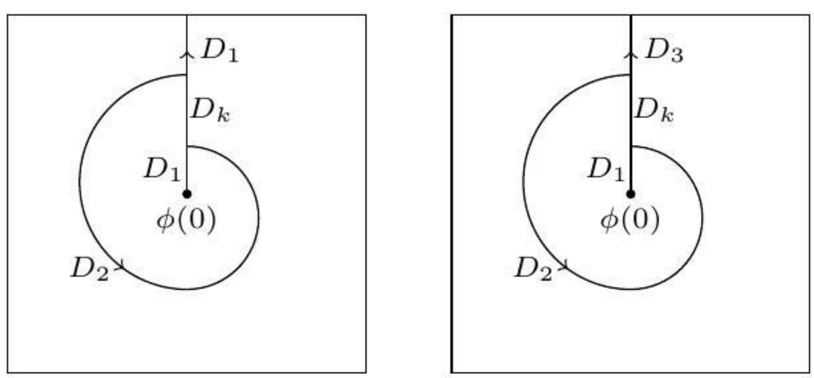


This is the "cocycle" condition which classifies anomalies from preventing of gauging the underlaying symmetry[4].

3. In terms of our axioms, an action of a defect D_2 acting on the twisted Hilbert space $\mathcal{H}_{S^1}^{D_1}$,



is described by some topological junction $v \in V_{D_1 D_2 \bar{D}_1 \bar{D}_2}$ which in turn can be described in terms of an intermediate insertion of a defect $D_k \in D_1 D_2$. This is called the **Lasso action** and is pictured as



.

There are $\dim V_{D_1 D_2 D_k}$ choices of v (or equivalently there are as many choices of D_k as resolutions of a junction with four vertex), hence $V_{D_1 D_2 \bar{D}_1 \bar{D}_2}$ is typically of dimension larger than 1 (in contrast to invertible symmetries). This means that there are multiple possible actions of D_2 on $\mathcal{H}_{S^1}^{D_1}$ as shown in the left and right pictures above.

A. Bootstrap and their gauging

Similarly to the CFT in the torus, consistency of modularity in the presence of a defect insertion leads to the modular bootstrap on symmetry-enriched CFTs.

$$\mathrm{Tr}_{\mathcal{H}_{S^1}}\left(\widehat{D}q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right)=\mathrm{Tr}_{\mathcal{H}^D_{S^1}}\left(\widetilde{q}^{L_0-c/24}\overline{\widetilde{q}}^{\bar{L}_0-c/24}\right), \tag{8}$$

where $\widetilde{q}=e^{2\pi i(-1/\tau)}$. Moreover, because the TDLs commute with the Virasoro generators, they do not change the dimensions of states. Since we assumed a CFT with a unique vacuum, the action of the defect must simply rescale the vacuum by some number $\langle D\rangle=\langle 0|D|0\rangle$ which we identify as the quantum dimension,

$$D|0\rangle=\langle D\rangle|0\rangle.$$

In particular, the quantum dimension is useful to determine the invertibility of the symmetry. If $\langle D\rangle=1$ then D is invertible (as in that case $N=\delta_{DD'}$ for some D') as we saw in the Ising model of Example IV.1 and, if $\langle D\rangle>1$ then D is not invertible. Let us concretely show how the defect modular bootstrap allows us to bootstrap TDLs in the familiar Ising model.

1. *Symmetry-enriched CFT: The Ising model revisited*

Consider the the Ising model i.e. a CFT with charge $c=1/2$ and spectrum $\{1,\varepsilon,\sigma\}$ of weights $h=0,1/2,1/16$ respectively. The Virasoro characters are computed to be

$$\chi_1(\tau)=\frac{1}{2\sqrt{\eta(\tau)}}[\sqrt{\theta_3(0|\tau)}+\sqrt{\theta_4(0|\tau)}],\quad \chi_\varepsilon(\tau)=\frac{1}{2\sqrt{\eta(\tau)}}[\sqrt{\theta_3(0|\tau)}-\sqrt{\theta_4(0|\tau)}],\quad \chi_\sigma(\tau)=\frac{1}{\sqrt{2\eta(\tau)}}\sqrt{\theta_2(0|\tau)}$$

for θ_n the modular theta functions and η the Dedekind eta function. These characters count the number of the descendants at each level. For instance, for the first 10 leveles, expanding the characters in powers of q they give

$$\begin{aligned}
q^{\frac{1}{48}}\chi_1(\tau)&=1+q^2+q^3+2q^4+2q^5+3q^6+3q^7+5q^8+5q^9+7q^{10}+O\left(q^{481/48}\right)\\
q^{-\frac{23}{48}}\chi_\epsilon(\tau)&=1+q+q^2+q^3+2q^4+2q^5+3q^6+4q^7+5q^8+6q^9+8q^{10}+O\left(q^{481/48}\right)\\
q^{-\frac{1}{24}}\chi_\sigma(\tau)&=1+q+q^2+2q^3+2q^4+3q^5+4q^6+5q^7+6q^8+8q^9+10q^{10}+O\left(q^{481/48}\right).
\end{aligned}$$

The partition function is thus given by

$$Z_{\text{Ising}}=|\chi_1(\tau)|^2+|\chi_\sigma(\tau)|^2+|\chi_\varepsilon(\tau)|^2.$$

For defect insertions we consider the action under TDL insertions,

$$Z^{Ising}[g_1,g_2](\tau,\bar{\tau})=\mathrm{tr}_{\mathcal{H}^{g_1}_{S^1}}\left(g_2q^{L_0-\frac{1}{48}}\bar{q}^{\bar{L}_0-\frac{1}{48}}\right)$$

for g_1 a TDL defect twisting the Hilbert space and g_2 for the TDL in the transverse direction. As we saw earlier, the action of a TDL is equivalent to the action on the characters. From now on, we consider the usual $\mathbb{Z}_2$ symmetry $\sigma\to-\sigma$ represented by the TLD η as seen by inserting the line along the time direction ($g_1=0,g_2=1$ where 1 represents an insertion and 0 otherwise). The fields 1 and ε are invariant (uncharged) so this leads to

$$Z^{\text{Ising}}[0,1](\tau,\bar{\tau})=|\chi_1(\tau)|^2+|\chi_\sigma(\tau)|^2-|\chi_\varepsilon(\tau)|^2.$$

Similarly, with the help of the modularity

$$Z^{\text{Ising}}_{T^2}[D,0](\tau,\bar{\tau})=Z^{\text{Ising}}_{T^2}[0,D](-1/\tau,-1/\bar{\tau})\,. \tag{9}$$

and, since the characters satisfy

$$\begin{aligned}
\chi_1\left(-\frac{1}{\tau}\right)&=\frac{1}{2\sqrt{\eta(\tau)}}[\sqrt{\theta_3(0|\tau)}+\sqrt{\theta_2(0|\tau)}]\\
\chi_\varepsilon\left(-\frac{1}{\tau}\right)&=\frac{1}{2\sqrt{\eta(\tau)}}[\sqrt{\theta_3(0|\tau)}-\sqrt{\theta_2(0|\tau)}]\\
\chi_\sigma\left(-\frac{1}{\tau}\right)&=\frac{1}{\sqrt{2\eta(\tau)}}\sqrt{\theta_4(0|\tau)}
\end{aligned}$$

we find

$$Z^{\text{Ising}}[1,0](\tau,\bar\tau) = |\chi_\sigma(\tau)|^2 + \chi_1(\tau)\chi_\varepsilon(\bar\tau) + \chi_1(\bar\tau)\chi_\varepsilon(\tau)$$

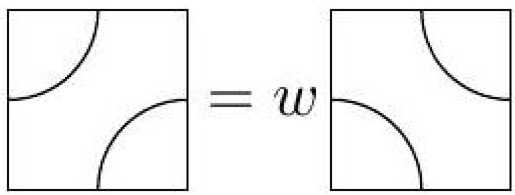
which gives the relation how the defect twisting acts on the characters. Moreover, the insertion of both types of defects can be found from the modular constraint

$$Z^{\text{Ising}}[1,1](\tau,\bar\tau) = \sum_{i,j\in\{0,\frac{1}{2},\frac{1}{16}\}} n_{ij}\chi_i(\tau)\chi_j(\bar\tau) = \sum_{i,j\in\{0,\frac{1}{2},\frac{1}{16}\}} n_{ij}\chi_i\left(-\frac{1}{\tau}\right)\chi_j\left(-\frac{1}{\bar\tau}\right)$$

for $n \in M_{3,3}(\mathbb{Z}_{\geq 0})$ which leads to the matrix

$$\begin{pmatrix} t+s & t & 0 \\ t & t+s & 0 \\ 0 & 0 & s \end{pmatrix} \implies n \in \text{Span}_{\mathbb{Z}_{\geq 0}}\left\{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}.$$

Both solutions determine action of lines in the twisted sector. In particular, since there are no topological junctions (no defect transforming junctions), there are two resolutions, simply related by a phase w,



.

Here, both solutions show $\omega = 1$ as the matrices are symmetric. We recognize the second solution as $Z^{\text{Ising}}_{T^2}[0,0]$ giving the first solution to be

$$Z^{\text{Ising}}_{T^2}[1,1] = |\chi_1(\tau)|^2 + |\chi_\epsilon(\tau)|^2 + \chi_1(\tau)\chi_\varepsilon(\bar\tau) + \chi_1(\bar\tau)\chi_\varepsilon(\tau).$$

Notice that we can also compute the matrix $\mathcal{S}$ from the Verlinde formula to be:

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

from which we have

$$\mathcal{N}_{ij}^k = \sum_{l=0}^{2} \frac{S_{jl}S_{il}(S^{-1})_{lk}}{S_{0l}}$$

giving

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_\varepsilon = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

which leads to the fusion rules

$$\varepsilon\times\sigma = \sigma, \quad \sigma\times\sigma = 1+\varepsilon, \quad \varepsilon\times\varepsilon = 1$$

as expected.

Now, let us consider a general TDL D and let us define D by the action of the operators of the theory. For the Ising on the torus, we have

$$Z_{T^2}^{\text{Ising}}[0,D](\tau,\bar\tau) \equiv \mathrm{tr}_{\mathcal{H}_{S^1}}(Dq^{L_0-\frac{1}{48}}\bar{q}^{\bar{L}_0-\frac{1}{48}}) = d_1|\chi_0(\tau)|^2 + d_\epsilon|\chi_{\frac{1}{2}}(\tau)|^2 + d_\sigma|\chi_{\frac{1}{16}}(\tau)|^2, \tag{10}$$

for d, the charges of the operators under the action of D. Thus, characterizing completely the line defect is done by specifying the coefficients $(d_1, d_\varepsilon, d_\sigma)$. In general, modularity (c.f. Figure 7) constrains greatly the values of d as for the insertion of the defect, we must have

$$Z_{T^2}^{\text{Ising}}[D,0](\tau,\bar\tau) = Z_{T^2}^{\text{Ising}}[0,D](-1/\tau,-1/\bar\tau)\,. \tag{11}$$

where the LHS is now a the insertion of the defect twisting the Hilbert space which can be written as

$$Z_{T^2}^{\text{Ising}}[D,0](\tau,\bar\tau) \equiv \mathrm{tr}_{\mathcal{H}_{S^1}^D}(q^{L_0-\frac{1}{48}}\bar{q}^{\bar{L}_0-\frac{1}{48}}) = \sum_{i,j\in\{0,\frac{1}{2},\frac{1}{16}\}} n_{ij}\chi_i(\tau)\chi_j(\bar\tau).$$

Plugging Equation (10) in the modular Equation (11) yields the system of equations:

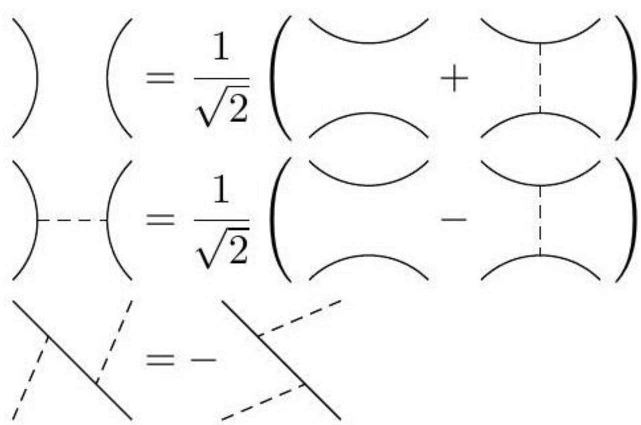
$$\begin{cases} d_\sigma &= s \\ d_\varepsilon &= t-\sqrt{2}u, \quad s,t,u\in\mathbb{Z} \\ d_1 &= t+\sqrt{2}u \end{cases} \implies (d_1,d_\sigma,d_\varepsilon)\in \mathrm{Span}_{\mathbb{Z}}\{(1,1,1),(\sqrt{2},-\sqrt{2},0),(1,1,-1)\}. \tag{12}$$

Note that here we have written in the basis where we recognize the first defect as the identity defect and the third as the η defect (whose quantum dimensions are 1). The remaining solution is the anticipated $\mathcal{N}$ non-invertible symmetries, whose quantum dimension is $\sqrt{2}$ as we expected it to be greater to 1.

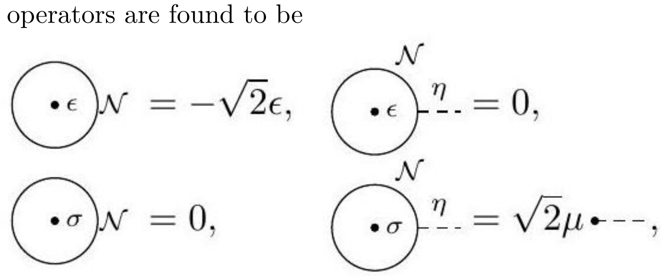
To see that this new bootstrapped defect $\mathcal{N}$ is a non-invertible symmetry we compute the fusion rule for $\mathcal{N}^2$. This can be found by squaring the eigenvalues d_i and reexpressing them in the basis of other solutions $\{d_i\}$, as shown by the Verlinde formula. The same can be done for all types of fusion and we find:

$$\mathcal{N}^2 = 1+\eta, \quad \eta^2 = 1, \quad \mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N}.$$

Notice, in particular, that these fusion composes a dimension 2 space, which means, as we saw earlier, they cannot be invertible. We can compute the Ising F-symbols by looking at the resolutions of 4-junctions with 3-junctions. Writing η as a dashed line and $\mathcal{N}$ as a solid line, we find



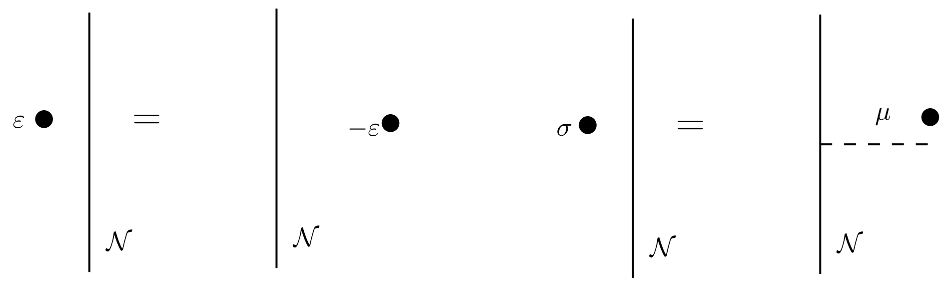
Finally, the action on local operators are found to be



where $\sqrt{2}$ is the quantum dimension $\langle\mathcal{N}\rangle = \sqrt{2}$, and μ is a primary operator in the twisted sector:

$$\mathcal{H}^{\eta}_{S^1} = \left\{\psi_{1/2,0}, \widetilde{\psi}_{0,1/2}, \mu_{1/16,1/16}\right\}.$$

Moreover, passing the non-invertible TDL through local operators gives



and $1 \to d_{\mathcal{N}} 1$, where $d_{\mathcal{N}}$ is the quantum dimension of the TDL $\mathcal{N}$.

V. FINAL REMARKS

We have embarked on a study of the extension of two-dimensional conformal field theories (2D CFTs) in the presence of extended operators known as topological defect lines. In this context, the bootstrap has proven to be an extremely important tool, enabling us to constrain the possible symmetries of our theory. The introduction of extended operators, as viewed from the boundary perspective, can be interpreted as long-range interactions where projections, commonly referred to as topological defect lines (TDLs), can become dynamical. When we dynamically introduce these operators, i.e., we simply sum over all such symmetry generators, we gauge the symmetry. This process leads to intriguing concepts of discrete gauging and anomalies, which undoubtedly play a significant role in this picture. One important feature that has caused great attention is the apparent violation of crossing symmetry in the presence of non-invertible symmetries [6] modifying the crossing symmetry up to an overall normalization. This can be resolved by the fact that crossing symmetry relates two theories differing by a space-time global transformation. As we saw earlier, extended symmetries must transform covariantly -being projections-, thus giving a modified crossing relation as recently suggested by Copetti and collaborators [6].

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