Notes on Fields

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I. GENERAL DISCRETE THEORIES AND THEIR GAUIGING

In order to talk about extended symmetries and their operators we begin by their introduction in general QFTs.[**?**
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A. Symmetries

Continuos symmetries on a field theory can be defined in terms of unitary operators $U(t)$ acting on operators local operators $\mathcal{O}$ by $U(t)\mathcal{O}(\vec{x},t)U^{-1}(t) = \mathcal{O}'(\vec{x},t)$. The symmetry, being generated by a charge $Q(t) = \int_{x_0=t} d^{d-1}xj^0(x)$, allows to write U as

$$U_\alpha(\Sigma_{d-1}) = \exp\left(i\alpha\int_{\Sigma_{d-1}} j^{d-1}\right) .$$

The fact the charge is conserved means the operator U is topological along deformations of a Cauchy slice Σ_t, $U(\Sigma_{d-1}) = U(\Sigma'_{d-1})$ where

$$\Sigma'_{d-1} - \Sigma_{d-1} = \partial\Sigma_d .$$

We call a codimension-1 operator U which is topological and invertible a **0-form symmetry**. The fact that U is invertible means that there exists another topological codimension-1 operator U' such that we have

$$U(\Sigma_{d-1})U'(\Sigma_{d-1}) = 1 .$$

$0-$form invertible symmetries may combine to form a $G^{(0)}$ group in which case we simply label the operators as $U_g \in G^{(0)}$ and the group multiplication describes the composition of symmetries:

$$U_g(\Sigma_{d-1})U_{g'}(\Sigma_{d-1}) = U_{gg'}(\Sigma_{d-1}) ,$$

which can be seen as a fusion rule $U_g \otimes U_{g'} = U_{gg'}$.

In general invertibility is non trivial, here it follows from unitarity, but a topological operator thus may be defined acting on local operators by linking with them,

$$U(S^{d-1})\mathcal{O}(x) = \mathcal{O}'(x) .$$

The picture is that for moving the topological symmetry past the local operator one must pay the price of converting it into the local operator $\mathcal{O}'(x)$. Once the operator has no topological restrinction then it can shrink to a point giving the above equality. Notice that the vector space of local operators at x forms a representation of $G^{(0)}$,

$$U_g\left(S^{d-1}\right)\mathcal{O}(x) = g\cdot\mathcal{O}(x) .$$

Example I.1. *For a $U(1)$ symmetry, we can use the Noether current j to find the operator $U(\alpha)$. In the presence of a local operator $\mathcal{O}(x)$ of charge $q \in \mathbb{Z}$ under $U(1)$, the continuity equation is modifed to*

$$\mathcal{O}(x)\partial_\mu j^\mu(x') = q\delta(x-x')\mathcal{O}(x) ,$$

which leads to

$$\begin{aligned} U_g\left(S^{d-1}\right)\mathcal{O}(x) &= \exp\left(i\alpha\int_{S^{d-1}} j^{d-1}\right)\mathcal{O}(x) ,\\ &= \exp\left(i\alpha\int_{D^d} dj^{d-1}\right)\mathcal{O}(x) ,\\ &= \exp\left(i\alpha\int_{D^d} q\delta^d(x)\right)\mathcal{O}(x) ,\\ &= \exp\left(iq\alpha\right)\mathcal{O}(x) . \end{aligned}$$

where D^d is the d-dimensional disk containing the point x whose boundary is S^{d-1}.

Similarly, a **p-form symmetry** is a codimension-$(p+1)$ operator $U(\Sigma_{d-p-1})s$ which is topological and invertible,

$$U(\Sigma_{d-p-1}) = U(\Sigma'_{d-p-1}), \quad U(\Sigma_{d-p-1})U^{-1}(\Sigma_{d-p-1}) = 1 \, .$$

for $\Sigma'_{d-p-1} - \Sigma_{d-p-1} = \partial\Sigma_{d-p}$. p-form ssymmetries may form a $G^{(p)}$ group. An important consequence of the linking is that higher-form symmetry groups are abelian as we can use topological deformations to change the ordering of two topological operators of codimension greater than one.

As we discussed above, a 0-form symmetry act on operators that naturally "capture". p-form symmetries similarly act on objects that cannot be trivially unlinked. For instance, a point can only be linked by a $d-1$ dimensional object as otherwise there is always a "space" to unlikthem by a topological move. Instead, it is not hard to see that a p-form symmetry can act only nontrivially to extended operators that are defined on a $q \geq 1$-dimensional submanifold M_q of spacetime for

$$q \geq p.$$

The simplest case to study is $q = p$. p-dimensional operators transform in representations of the p-form symmetry group $G^(p)$.

Consider a $p \geq 1$-dimensional extended operator $\mathcal{O}(M_p)$ placed along a p-dimensional submanifold M_p of spacetime. We can assume that $\mathcal{O}(M_p)$ is an irreducible p-dimensional operator i. e. there are no topological local operators that can be inserted at a point $x \in M_p$ except multiples of the identity local operator. Now, note that deforming $U_g(\Sigma_{d-p-1})$ across $\mathcal{O}(M_p)$ leaves behind a topological local operator $\mathcal{O}(x)$ at the intersection point x of M_p and Σ_{d-p};

$$U_g(\Sigma_{d-p-1})\mathcal{O}(M_p) = \mathcal{O}(x)\mathcal{O}(M_p)U_g(\Sigma'_{d-p-1}) \, .$$

Since the only possible topological local operators along M_p are multiples of identity, we can replace $\mathcal{O}(x)$ by a non-zero number in the above equation

$$U_g(\Sigma_{d-p-1})\mathcal{O}(M_p) = \phi(g) \times \mathcal{O}(M_p)U_g(\Sigma'_{d-p-1}), \quad \phi(g) \in \mathbb{C}^\times = \mathbb{C} - \{0\}.$$

Because of the fusion rule, the numbers $\phi(g)$ have to satisfy

$$\phi(g)\phi(g') = \phi(gg') \, ,$$

i.e. the numbers $\phi(g)$ furnish a one-dimensional representation of the p-form symmetry group $G^{(p)}$, and in particular the numbers $\phi(g)$ must be phase factors

$$\phi(g) \in U(1) \subset \mathbb{C}^\times.$$

Example I.2. *Consider a $U(1)$ gauge theory. The field strength $F = dA$ satisfies $dF = 0$ (integrating is topological) and thus it can be used as a current generate a $(d-3)-$form symmetry,*

$$U_g^{(m)}(\Sigma_2) = \exp\left(i\alpha \int_{\Sigma_2} F\right), \quad g = e^{i\alpha} \in U(1) \, .$$

Moreover, since $d * F = 0$, we can write the $1-$from symmetry

$$U_g^{(e)}(\Sigma_{d-2}) = \exp\left(i\alpha \int_{\Sigma_{d-2}} \star F\right), \quad g = e^{i\alpha} \in U(1) \, .$$

The former combines into the magnetic $G^{(d-3)}$ symmetry while the latter to the electric $G^{(1)} = U(1)$ group symmetries.

B. Background fields

Recall a background field for a continous 0-form symmetry group $G^{(0)}$ is a connection locally described by a 1-from A which may couple to the theory via a $\int A \wedge j^{d-1}$ term, where $j^{(d-1)}$ is the Noether current for the 0-form symmetry $G^{(0)}$. A background field B_{p+1} for a continous p-form symmetry can be analoougsly defined as a $(p+1)-$form B_{p+1}, whose field strength is $F_{d+2} = dB_{p+1}$, transforming as

$$B_{p+1} \to B_{p+1} + d\Lambda_p \, ,$$

for $p-$forms Λ on spacetime. The presence of the background modifies the partition function to be $Z[B_{p+1}]$.

More generally, one can define topological discrete p-Form symmetries described as a $(p+1)-$cochain acting on a BF-like theory

$$2\pi \int a_p \cup_\eta \delta b_{d-p-1},$$

where a_p is a $G^{(p)}$-valued p-cochain and b_{d-p-1} is a $\widehat{G}^{(p)}$-valued $(d-p-1)$-cochain and δ is a differential that maps q-cochains to $(q+1)$-cochains valued in the same abelian group and naturally defines the electric $G^{(p)}$ p-form symmetry and a magnetic $\widehat{G}^s s(p)$ $(d-p-1)$-form symmetry groups, both of which are associated to discrete versions of Noether currents.

The discrete gauge fields a_p and b_{d-p-1} are gauge Fields with gauge transformations acting as

$$a_p \to a_p + \delta a_{p-1}\ , \quad b_{d-p-1} \to b_{d-p-1} + \delta b_{d-p-2}\ ,$$

where a_{p-1} is an arbitrary $G^{(p)}$-valued $(p-1)$-cochain and b_{d-p-2} is an arbitrary $\widehat{G}^{(p)}$ valued $(d-p-2)$-cochain.

Analogously, we couple background fields B^e_{p+1} and B^m_{d-p} to the discrete theory by adding the term

$$2\pi \int \left(B^e_{p+1} \cup_\eta b_{d-p-1} + a_p \cup_\eta B^m_{d-p}\right)\ ,$$

where recall that a_p and $b_d - p - 1$ are dynamical gauge fields.

As discussed above, the cochain B_{p+1} is defined in terms of a triangulation $\mathcal{T}$ of spacetime. The topological defects live on a dual triangulation $\widehat{\mathcal{T}}$, see Figure 1. Now, the network of topological defects is constructed as follows. For a

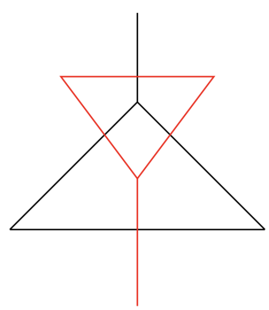


FIG. 1: A $d=2$ triangulation $\mathcal{T}$ in black and it's dual $\widehat{\mathcal{T}}$ in red.

$(p+1)$-simplex s_{p+1} of $\mathcal{T}$ we have

$$B_{p+1}(s_{p+1}) = g \in G^{(p)}.$$

Given this, insert the codimension-$(p+1)$ topological defect U_g along the dual $(d-p-1)$- simplex $\widehat{s}_{d-p-1}$ of $\widehat{\mathcal{T}}$. The partition function $Z[B_{p+1}]$ in the presence of a background feld is then the same as the correlation function of the corresponding network of topological defects. This is easy to demonstrate in a situation when we have a discrete Noether current j_{d-p-1} (which may not always be the case), as in such a situation the topological codimension-(p+1) operators constituting the $G^{(p)}$ p-form symmetry can be expressed as

$$U_g = \int g(j_{d-p-1}), \quad g \in G^{(p)},$$

where j_{d-p-1} which is a dynamical $\widehat{G}^{(p)}$-valued $(d-p-1)$-cochain. As discussed above, in such a case, turning on a background field $B_p + 1$ corresponds to adding to the action a coupling

$$2\pi \int B_{p+1} \cup_\eta j_{d-p-1}.$$

By Poincaré duality, computing partition function in the presence of this term is precisely equivalent to computing the correlation function of a network of U_g defects placed on the Poincaré dual codimension-$(p+1)$ chain to the cochain B_{p+1}.

II. GAUGING

Just like background felds describe networks of topological defects on spacetime, gauge transformations of background felds describe deformations of the topological defects including local rearrangements of the network of topological defects. The case for discrete symmetries is of important atention. Above we discussed that gauge transformations of background felds are equivalent to deformations of topological defects generating these symmetries. Therefore, gauging the symmetry is the procedure of promoting the background field B_{p+1} for a p-form symmetry to a dynamical $(p+1)$-form (or $(p+1)$-cochain) gauge field b_{p+1}

$$B_{p+1} \to b_{p+1}.$$

Concretely this means that we sum over all possible gauge-inequivalent backgrounds B_{p+1}. That is, we choose a representative B_{p+1} inside each equivalence class of background fields up to background gauge transformations, and sum over all such representatives. If the original theory is $\mathfrak{T}$ and the p-form symmetry group being gauged is $G^{(p)}$, the theory obtained after gauging is referred to as the gauged theory and denoted as

$$\mathfrak{T}/G^{(p)}$$

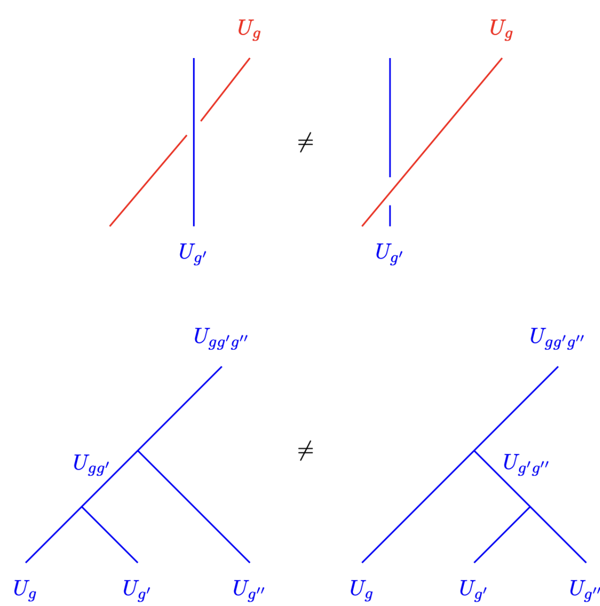
The partition function of the gauged theory is

$$\begin{aligned} Z_{\mathfrak{T}/G^{(p)}} &\propto \textstyle\sum_{[B_{p+1}]} Z_{\mathfrak{T}}\left[B_{p+1}\right], \qquad G^{(p)} \text{ discrete} \\ Z_{\mathfrak{T}/G^{(p)}} &\propto \textstyle\int \mathcal{D}\left[B_{p+1}\right] Z_{\mathfrak{T}}\left[B_{p+1}\right], \;\; G^{(p)} \text{ continuous}\,. \end{aligned}$$

Consistency of the topologicall symmetry requires

$$Z_{\mathfrak{T}}\left[B^g_{p+1}\right] = Z_{\mathfrak{T}}\left[B_{p+1}\right]$$

where B^g_{p+1} is related to B_{p+1} by a gauge transformation. Anomalies, however, prevent symmetries to be gauged, partition functions which are gauge dependent are called 't Hooft anomalous but it can be that anomalies arise as we turn more bacground operators. These are the mixed t' Hooft anomalies. A 't Hooft anomaly arises when such deformations are not topological and change the correlation functions. This is only possible when the topological defects or their junctions cross each other.



Indeed, a 't Hooft anomaly for a set of discrete symmetries arises when topological defects for these symmetries, or junctions between such topological defects, carry non-trivial charges under the symmetry. Addition of counterterms corresponds to redefining the topological operators and their junctions.

Example II.1. *Consider d-dimensional pure $U(1)$ gauge theory. In the presence of non-trivial backgrounds for both electric and magnetic higher-form symmetries, the action gets modified to*

$$\Delta S = 2\pi \int B_2^e \wedge \star F + 2\pi \int B_{d-2}^m \wedge F.$$

Performing a gauge transformation for the electric background $B_2^e \to B_2^e + d\Lambda_1^e$, modifies the action to

$$S \to S + 2\pi \int \Lambda_1^e \wedge d \star F \,.$$

The equation of motion for $\star F$ is thus modified, in the presence of magnetic background, to be

$$d \star F = -dB_{d-2}^m$$

which in turn leads to the gauge transformation of the partition function:

$$Z\left[B_2^e + d\Lambda_1^e, B_{d-2}^m\right] = \exp\left(-2\pi i \int \Lambda_1^e \wedge dB_{d-2}^m\right) \times Z\left[B_2^e, B_{d-2}^m\right].$$

Thus, we see that there is a mixed 't Hooft anomaly between the electric and magnetic higher-form symmetries of the Maxwell theory. As the electric symmetry is non-anomalous only if there are no external magnetic fields. We can also see the anomaly if we interchange the role of electric and magnetic pieces in the above computation. Performing a magnetic gauge transformation

$$B_{d-2}^m \to B_{d-2}^m + d\Lambda_{d-3}^m,$$

in the presence of an electric background leads to the following anomalous variation of the partition function

$$Z\left[B_2^e, B_{d-2}^m + d\Lambda_{d-3}^m\right] = \exp\left(2\pi i \int dB_2^e \wedge \Lambda_{d-3}^m\right) \times Z\left[B_2^e, B_{d-2}^m\right].$$

Analogously one can describe a discrete (higher-form) gauge theory mixed anomaly between both electric and magnetic symmetries. Performing a gauge transformation

$$B_{p+1}^e \to B_{p+1}^e + \delta\Lambda_p^e,$$

in the presence of B_{d-p}^m modifes the partition function as

$$Z\left[B_{p+1}^e + \delta\Lambda_p^e, B_{d-p}^m\right] = \exp\left(2\pi i\, s(p) \int \Lambda_p^e \cup_\eta B_{d-p}^m\right) \times Z\left[B_{p+1}^e, B_{d-p}^m\right],$$

where we have defined

$$s(n) := (-1)^n, \quad n \in \mathbb{Z}.$$

Similarly, performing a gauge transformation

$$B_{d-p}^m \to B_{d-p}^m + \delta\Lambda_{d-p-1}^m,$$

in the presence of B_p^e modifies the partition function as

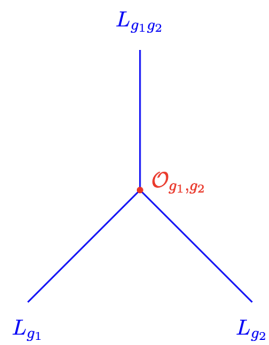
$$Z[B_{p+1}^e, B_{d-p}^m + \delta\Lambda_{d-p-1}^m] = \exp\left(-2\pi i \int B_{p+1}^e \cup_\eta \Lambda_{d-p-1}^m\right) \times Z\left[B_{p+1}^e, B_{d-p}^m\right].$$

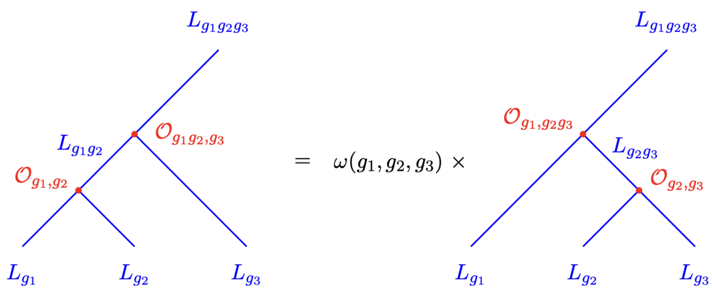
This 't Hooft anomaly has to do with the fact that the topological operators constituting the $G^{(p)}$ electric p-form symmetry are charged under the $\widehat{G}^{(p)}$ magnetic $(d-p-1)$-form symmetry, and vice versa. The gauge transformation corresponds to moving the topological operators for the magnetic symmetry such that they sweep out a $(p+1)$- chain Poincaré dual to the cochain Λ_{d-p-1}^m. Due to the presence of background $B_p + 1$, we have a network of topological operators for the electric symmetry inserted inside spacetime. Thus, under the above deformation, the magnetic operators cross the electric ones generating a phase described by intersections between the chain occupied by the electric operators with the chain along which the magnetic operators are deformed. By Poincaré duality, the product of these phases is computed as

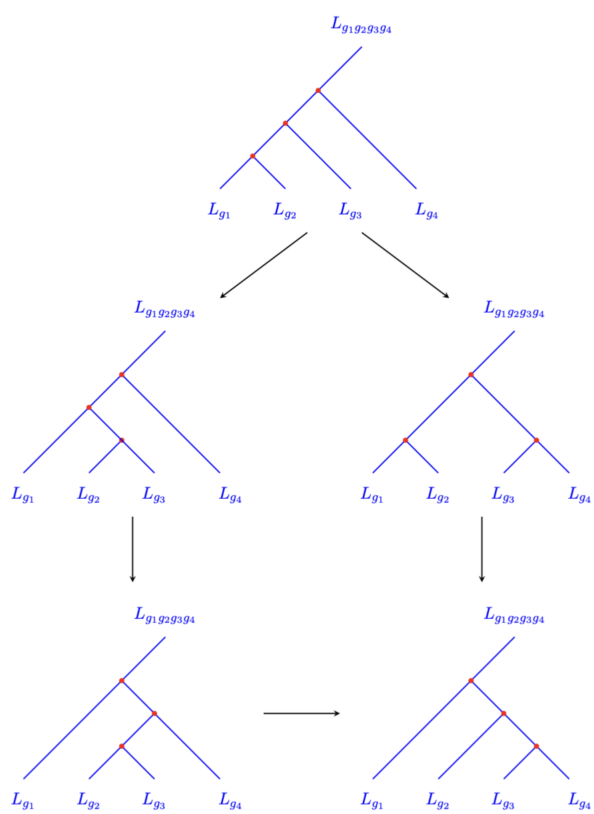
$$\exp\left(-2\pi i \int B_{p+1}^e \cup_\eta \Lambda_{d-p-1}^m\right)$$

which is the anomalous variation of the partition function discussed above.

’t Hooft Anomalies of Discrete 0-Form Symmetries in 2d are special as using the above description of the ’t Hooft anomaly in terms of movements of topological defects, we can completely characterize possible ’t Hooft anomalies of 0-form symmetries.







Consider a 0-form symmetry described by a discrete group $G^{(0)}$. That is, we have topological line operators $L_g, \quad g \in G^{(0)}$. Additionally choose topological local operators $\mathcal{O}_{g_1,g_2}$, at a junction where line operators L_{g_1} and L_{g_2} come together and transform into the line operator $L_{g_1g_2}$. See figure $\boxed{14}$. Consider a movement of topological lines shown in fgure $\boxed{15}$, which is sometimes referred to as an associator or as an F-move. Such a move might modify the correlation function by a complex factor

$$\omega(g_1, g_2, g_3) \in \mathbb{C}^\times ,$$

which characterizes the 't Hooft anomaly of the $G^{(0)}$ 0-form symmetry once counterterms have been taken into account. These numbers can be collected into a function

$$\omega : (G^{(0)})^3 \to \mathbb{C}^\times ,$$

Such a function is known as a $\mathbb{C}^\times$-valued 3-group-cochain on $G^(0)$. **Definition 4. 3: Group- Cochains** Let M be a group. An M-valued p-group-cochain α_p on $G^(0)$ is a function

$$\alpha : (G^{(0)})^p \to M \ .$$

There is a consistency condition, known as the pentagon identity, on ω arising by equating the two sets of moves shown in figure

$$\delta\omega(g_1, g_2, g_3, g_4) = \frac{\omega(g_2, g_3, g_4)\omega(g_1, g_2 g_3, g_4)\omega(g_1, g_2, g_3)}{\omega(g_1 g_2, g_3, g_4)\omega(g_1, g_2, g_3 g_4)} = 1 \ ,$$

which means that ω is a closed 3-group-cochain on $G^(0)$.

A. Dual Symmetries Obtained After Gauging

Consider the gauging of a discrete symmetry, which provides a $G^{(p)}$-valued gauge field b_{p+1} in the gauged theory $\mathfrak{T}/G^(p)$. Just as with any gauge feld we can consider Wilson operators $\mathcal{W}_R$ for $b_p + 1$ which are parametrized by irreducible representations R of $G^(p)$. These operators are $(p+1)$- dimensional operators of $\mathfrak{T}/G^{(p)}$. If $G^(p)$ is continuous, then $\mathcal{W}_R$ are non-topological operators. However, if $G^(p)$ is discrete, then $\mathcal{W}_R$ are topological operators since $b_p + 1$ is necessarily flat in this case. Given that the $\mathcal{W}_R$ are topological, they must generate a new symmetry in the gauged theory that is, the gauged theory $\mathfrak{I}/G^(p)$ obtained after gauging a discrete abelian $G^(p)$ p-form symmetry group admits a $(d-p-2)$-form symmetry given by the Pontryagin dual group

$$G^{(d-p-2)} = \widehat{G}^{(p)} \ .$$

Such symmetries are known as dual symmetries or quantum symmetries arising from the gauging of $G^(p)$. For $p > 0, G^(p)$ must be abelian, but for $p = 0, G^(0)$ can be non-abelian. Infact, they may comprise a non-invertible symmetry.

Indeed, note that there always exists an irreducible representation R of dimension bigger than one for a discrete non-abelian group $G^{(0)}$. As such, by dimensionality, it is impossible to find another representation R^* of $G^{(0)}$ such that

$$R \otimes R^* = \mathrm{id},$$

where on the RHS we have the trivial representation. Consequently, in a gauged theory $\mathfrak{T}/G^(0)$ we have topological Wilson line operators that are non-invertible. The symmetries generated by non-invertible topological operators are referred to as non-invertible symmetries. In the above situation, one says that $\mathfrak{Z}/G^(0)$ has a

$$\mathrm{Rep}(G^{(0)}) \ ,$$

non-invertible $(d-2)$-form symmetry.

If we gauge the resulting theory, it turns out that the resulting gauged theory is the original theory $\mathfrak{T}$, i.e. we have

$$\left(\mathfrak{T}/G^{(p)}\right)/G^{(d-p-2)} = \mathfrak{T}$$

and the dual of the dual symmetry $G^{(d-p-2)}$ is the original symmetry $G^{(p)}$. It can be quickly seen by noting that turning on a background B_{d-p-1} of the dual symmetry adds a coupling

$$2\pi \int B_{d-p-1} \cup_\eta b_{p+1}$$

After gauging B_{d-p-1}, and adding coupling to the new dual p-form symmetry background B_{p+1}^{new} , we have

$$2\pi \int b_{d-p-1} \cup_\eta b_{p+1} + B_{p+1}^{\text{new}} \cup_\eta b_{d-p-1}$$

Integrating out b_{d-p-1} sets

$$b_{p+1} = (-1)^p B_{p+1}^{\text{new}}$$

Thus we indeed return, up to a sign, to the original theory $\mathfrak{T}$, and find that the new dual symmetry is the original $G^{(p)}$ symmetry.

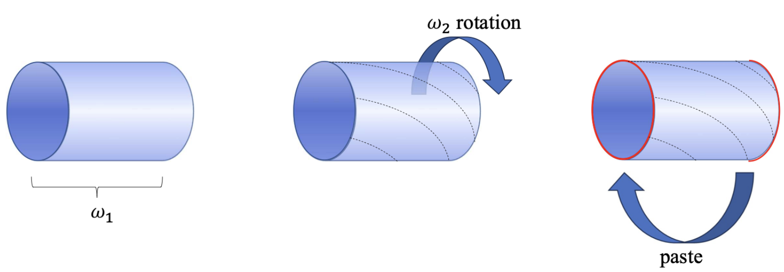
We conclude this section by remarking that discrete gaugings do not change the full information of observables in a theory. However, as we have seen, discrete gaugings do mix different types of observables into each other, e.g. the exchange of twisted and charged sectors discussed in section 4.3.4. As a consequence, a physical observable may appear in different contexts in different gauge frames associated to a discrete symmetry.

III. CFTS ON THE TORUS

[**?**] So far, we have looked at the Virasoro block expansion of the four-point function on the Riemann sphere. In fact, the Virasoro block expansion can be performed for N-point functions on a Riemann surface of arbitrary genus. Recalling that the conformal bootstrap is a relation between different ways of expanding a correlation function in the Virasoro block basis, we notice that the conformal bootstrap equations can be obtained from various correlation functions other than the four-point function. $\boxed{33}$ Here, as an example of the Virasoro block expansion other than the four-point function, we consider the Virasoro block expansion of the torus zero-point function, i.e. the partition function. The partition function can be constructed from a cylinder evolved in imaginary time by ω_1 with $H = 2\pi\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$(see the left of Figure $|1\dot{0}\rangle$. Since we are now considering time evolution on a cylinder with period 1, we add the Casimir energy $-2\pi\frac{c}{12}$ to the Hamiltonian as seen in (2.124). To create a torus, we need to join both ends of the cylinder, and in doing so, we can twist the cylinder (see the middle of Figure —10). This twist is realized by rotating by ω_2 with $P = 2\pi i(L_0 - \bar{L}_0)$. Finally, the torus is completed by joining both ends of the cylinder (i.e. taking the trace) (see the right of Figure $\boxed{1}$0). Setting $(\tau, \bar{\tau}) \equiv (-\omega_2 + i\omega_1, -\omega_2 - i\omega_1)$, the partition function can be written as follows, (3.39)

$$Z(\tau) \equiv \operatorname{tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}, \quad q \equiv \mathrm{e}^{2\pi i\tau}, \bar{q} \equiv \mathrm{e}^{-2\pi i\bar{\tau}}.$$

Here, $\mathcal{H}$ is the Hilbert space of the CFT $\sqrt{2.46}$). The parameter τ of the Riemann surface is called the moduli.



There are countless ways to take the time axis on the torus, and they should all be equivalent. The mapping that shifts to a different choice of the time axis is called the modular transforma- *tion*. The modular transformation is generated by the following two transformations.

T **Transformation**

$$T : \tau \to \tau + 1.$$

(3.40)

Let us see how the partition function changes under this transformation,

$$Z(\tau + 1) = \operatorname{tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \mathrm{e}^{2\pi i(L_0 - \bar{L}_0)}.$$

(3.41)

Since the eigenvalues of $L_0 - \bar{L}_0$ correspond to the spin, the invariance under the modular T transformation implies that the spin must be an integer. $\boxed{3}$4 S Transformation

$$S : \tau \to -\frac{1}{\tau}.$$

(3.42)

Roughly speaking, this corresponds to the exchange of two periods of the torus, that is, the angular cycle and the thermal cycle (see Figure —11). This exchange does not change the result of the path integral on the torus. When expressing the partition function in the operator formalism, we insert the identity operator, and by swapping the time axis and the space axis, it changes which cycle of the torus the identity operator is inserted along (see

Figure $\boxed{1}$2). However, the result of the calculation does not change depending on how the identity operator is inserted, so these are equivalent,

$$Z(\tau) = Z\left(-\frac{1}{\tau}\right).$$

(3.43)

This is exactly the torus partition function version of the conformal bootstrap equation in the same spirit as Figure 9.

By combining the S and T transformations, we can derive various conformal bootstrap equations from the partition function. These are specifically called modular bootstrap equations. **While the modular** T invariance merely implies the integrality of the spin, the modular S invariance non-trivially constrains the spectrum. Therefore, when referring to the modular bootstrap equation, it generally means (3.43).

The partition function can be expressed using Virasoro characters as follows,

$$Z(\tau) = \sum_p \chi_p(\tau)\overline{\chi_p(\tau)}.$$

(3.44)

Here, the Virasoro character is a special function defined by

(3.45)

$$\chi_i(\tau) \equiv \mathrm{tr}_{\mathcal{V}_i} q^{L_0 - \frac{c}{24}} = \sum_n d(n) q^{h_i + n - \frac{c}{24}},$$

where $d(n)$ is the number of descendant states at level n in the (irreducible) Verma module. If the Verma module does not contain null states, $d(n) = p(n)$. Note that this function is a special function determined solely by the Virasoro algebra, independent of the details of the theory. While (3.17) refers to the Virasoro block for the four-point function on the Riemann sphere, the Virasoro character is the Virasoro block for the zero-point function on the torus. Here are some comments on the modular bootstrap equation:

• The modular bootstrap equation constrains the spectrum, but it does not uniquely deter- mine the theory. In fact, there are examples of different CFTs having the same partition function [17]. To identify the theory (especially, the OPE coefficients of the theory), we need to solve the bootstrap equation for the four-point function.

• The modular bootstrap is not a subset of the conformal bootstrap for four-point functions but an independent condition. It is known that solving the conformal bootstrap for the four-point function and the one-point function on the torus is sufficient to completely determine the theory[18]. The reason for this is simply that from the fusion transformation of the four-point Virasoro block and the modular transformation of the torus onepoint block, one can construct any fusion transformation (transformation between bases of functions for N-point functions on any Riemann surface). This will be explained in **Section** $\boxed{7.5}$. • Unlike the conformal bootstrap equations for four-point functions or one-point functions on the torus, the modular bootstrap equation does not involve OPE coefficients. In this sense, the modular bootstrap equation is much easier to handle. Therefore, when investigating the possibility of the existence or universal properties of a given theory, one often considers the modular bootstrap first. In fact, many notable constraints have been **obtained just by considering the modular bootstrap**.

IV. BEHAVIOUR FOR LARGE CHANNEL DIMENSIONS AND PILLOW GEOMETRY

V. CFT PARTITION FUNCTION

Due to the state operator correspondance, the interpretation of correlation functions on the cylinder is straight forward. Starting from a state $|\psi_1\rangle$ at τ_1 we want to compute the amplitude of $|\psi_2\rangle$ at τ_2. This is

$$\langle\psi_2|\psi_1\rangle = \langle\psi_2|\, e^{-HT}\, |\psi_1\rangle$$

where H is the generator of time evolution i.e. dilations which in the cylinder is given by

$$H = L_0 - \frac{c}{24} + \bar{L}_0 - \frac{c}{24}$$

For the partition function thus we have

$$Z = \mathrm{tr}_{\mathcal{H}}\, e^{-HT}.$$

The torus is described by a parameter τ which belongs to the fundamental domain F_0. Doing the path integral we have been computing is presicely computing the partition function $Z = \mathrm{tr}_{\mathcal{H}}(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}})$. Let us define the character of irreducible representations $\mathcal{V}_\Delta$ by

$$\chi_\Delta(q) = \mathrm{tr}_{\mathcal{V}_{h,c}}(q^{L_0-\frac{c}{24}}) = q^{h-24}\sum_{n=0}^{\infty} d_n^{(h)} q^n$$

We conclude that the partition function on a torus can be decomposed into a bilinear form of characters,

$$Z = \sum_{(h,\bar{h})} n_{h\bar{h}}\chi_h(q)\chi_{\bar{h}}(\bar{q}) \ ,$$

for n_{ij} the multiplicities of ech $(h,\bar{h})$ representation. In general $N_{00} = 1$ as identity must be present and non degenerate.

Modular invariance on the torus

Now it is important to note that the shape of the torus does not uniquely determine τ, namely we can perform arbitrary 'modular' transformations,

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}\ ,\ ad - bc = 1\ ,$$

(127)

where a, b, c, d are integers, and τ' describes the same torus. Modular transformations form a group. Any modular transformation can be obtained as a composition of two basic transformations,

(128)

$$T : \tau \to \tau + 1 \quad , \quad S : \tau \to -\frac{1}{\tau} \quad .$$

The crucial point is that we want the partition function Z to be intrinsically attached to the torus, i.e. to be invariant under modular transformations. This modular invariance of the partition function, together with the form (126), turns out to yield very strong constraints on the operaton content.

T- transformation

We have here $q \to e^2 i\pi q$, and consequently

$$\chi_h(q) \to e^{2i\pi\left(h-\frac{c}{24}\right)}\chi_h(q)\ ,\ \chi_{\bar{h}}(q) \to e^{-2i\pi\left(\bar{h}-\frac{c}{24}\right)}\chi_{\bar{h}}(q)\ ,$$

$$\chi_h(q)\chi_{\bar{h}}(q) \to e^{2i\pi(h-\bar{h})}\chi_h(q)\chi_{\bar{h}}(q)\ .$$

(130)
Thus, the invariance under the T-transformation requires spins $h-\bar{h}$ of field
operators to be integers.
S- transformation
It is more diffcult to fnd the S-transformation, because the characters transform among themselves under it. Let us concentrate on the family of representations of the Virasaro algebra (116). Denote for short $\chi_r s = \chi_{h_{rs}}$. It can be shown that
(131)

$$\chi_{rs}\left(-\frac{1}{\tau}\right) = \sum_{r',s'} S_{rs,r's'}\chi_{r's'}(\tau) \ ,$$

with r', s' running over the same range as in (118). The matrix S is symmetric
and unitary

$$S_{rs,r's'} = (-1)^{rs'+r's+1}\sqrt{\frac{2}{pp'}}\sin\frac{\pi rr'p}{p'}\sin\frac{\pi ss'p'}{p} \ .$$

(132)
Likewise for the representations of the $\widehat{SU}(2)$ current algebra of level k, labeled by a spin j, with $0 \leq 2j \leq k$, formula (97) gives representations of the Virasoro algebra and the corresponding characters transform under the S transformation according to the unitary matrix

$$S_{jj'} = \sqrt{\frac{2}{k+2}}\sin\frac{\pi(2j+1)(2j'+1)}{k+2} \ .$$

(133)
Now we have all the ingredients to discuss the following problem: fnd all modular invariant partition functions Z of the form (126) with N's non negative integers, $N_{00} = 1$. Solving this problem for a given class of representations amounts to classifying conformal field theories of that class. I won't dwell on that any longer. Suffice it to say that this programme has been carried out for the 'minimal' representations of Virasoro and for the (re-) lated) cft's with a $SU(2)$ current algebra. The solution exhibits a beautiful structure that had not been anticipated: we refer the reader to the literature [Cappelli et al.]. This classification programme has been pursued lately for theories with a higher rank current algebra ($SU(3)$ in particular: see the recent work of T. Gannon).

VI. VERLINDE FORMULA

VII. MOORE-SEIBERG CONSTRUCTION

Since every two-dimensional Riemannian manifold is conformally flat, a 2D CFT is canonically defined on any Riemann surface, up to a universal conformal anomaly which introduces a dependence on the metric within a conformal class. Furthermore, the correlation function of any set of local operators inserted on a general Riemann surface in a given CFT is determined via the plumbing construction by the structure constants C_{ijk}. The consistency of the CFT on the Riemann surface, which may be constructed by different plumbings, leads to an important set of consistent conditions known as modular invariance. [4] A genus g surface as a smooth manifold can be constructed by gluing pair-of-pants, or threeholed spheres. A general genus $g > 1$ Riemann surface can be constructed by plumbing together $2g-2$ three-holed spheres. The genus one case can be constructed by plumbing the inner and outer boundaries of an annulus. The plumbing construction glues together a pair of circular boundaries of three-holed spheres (or two-holed discs) by an PSL$(2,\mathbb{C})$ map. A typical plumbing map takes the form

$$z' = q/z, \quad q \in \mathbb{C}^\times$$

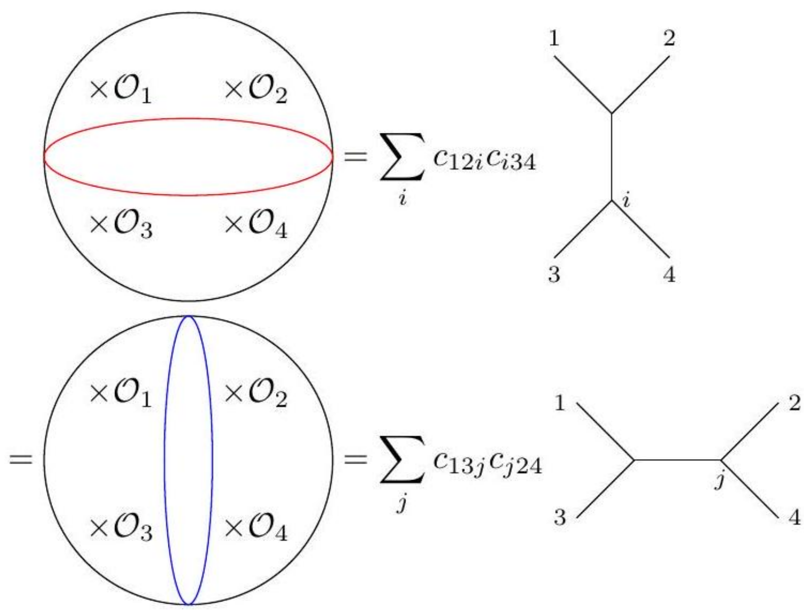
which identifies the boundary $|z| = r_1$ on one of the two-holed discs to the boundary $|z'| = r_2$ of the other two-holed disc, with $r_1 r_2 = |q|$. There is one complex modulus q associated with each of the $3g-3$ plumbing maps, giving rise to $3g-3$ complex structure moduli of the Riemann surface. The plumbing construction gives a precise way to construct the partition function, and more generally, correlation functions, on an arbitrary genus g Riemann surface, based on the 3-point function of arbitrarily Virasoro descendants on the Riemann sphere, which are determined by conformal Ward identities in terms of the structure constants C_{ijk}. Modular invariance amounts to the statement that all possible plumbing constructions of the same (punctured) Riemann surface lead to the same answer for the partition function (correlation function).

The equivalence of different plumbing constructions of the same Riemann surface would be guaranteed provided that all sphere 4-point functions can be equivalently decomposed in two different channels, and that all torus 1-point functions constructed by plumbing together the inner and outer boundaries of a 1-punctured annulus are independent of choice of how the torus is cut open into an annulus. The former is equivalent to the associativity of OPE as already mentioned. The latter amounts to the modular covariance of the torus 1-point function of all primaries. Namely, for every primary $\phi_k(z,\bar{z})$, the torus 1-point function $\langle\phi_k(z,\bar{z})\rangle_{T^2(\tau)} = f_k(\tau,\bar{\tau})$ is a modular form of weight $\left(h_k,\tilde{h}_k\right)$. That is,

$$f_k(\tau+1,\bar{\tau}+1) = f_k(\tau,\bar{\tau}), \quad f_k(-1/\tau,-1/\bar{\tau}) = (-i\tau)^{h_k}(i\bar{\tau})^{\tilde{h}_k} f_k(\tau,\bar{\tau})$$

...

The fact that one can obtain any closed orientable Riemann surfaces from different surgery procedures imposes consistency conditions on the fusion. The Moore-Seiberg constructions asserts that 4-point crossing on the sphere

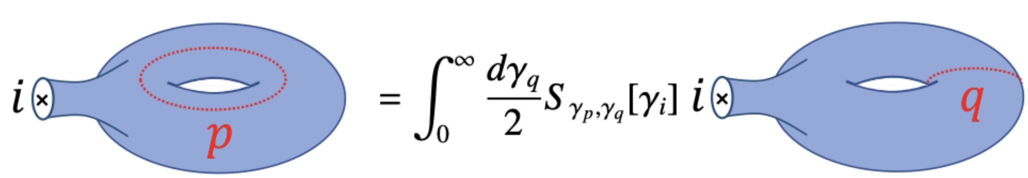


and 1-point modular covariance on the torus,

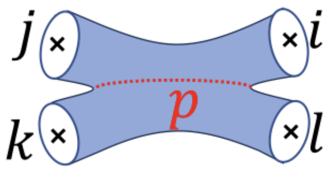
$$\langle\mathcal{O}\rangle_{-1/\tau} = \tau^{h}\bar{\tau}^{\hbar}\langle\mathcal{O}\rangle_{\tau},$$

where the factor is a Weyl factor coming from the $w \to w'/\tau$ coordinate change, are sufficient to ensure these "sewing constraints" of the theory on arbitrary orientable Riemannian surfaces.

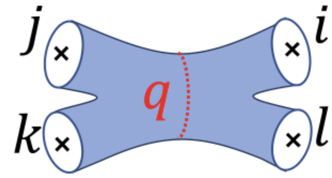
Indeed, from the fusion decompositions on the torous:



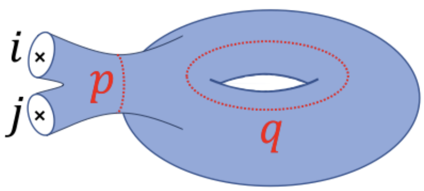
and from the sphere fusion:



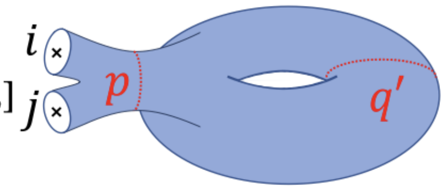
$$= \int_0^\infty \frac{d\gamma_q}{2} \mathbf{F}_{\gamma_p,\gamma_q} \left[\begin{array}{cc} \gamma_j & \gamma_i \\ \gamma_k & \gamma_l \end{array} \right]$$



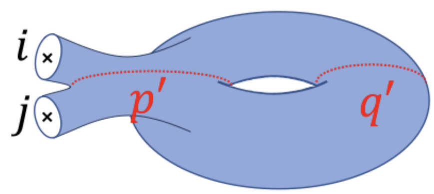
one can write the desired channel decomposition for any $n-$pt function. For instance, consider the 2-pt function on the torous. Using the fusion transformatoins, we can write



$$= \int_0^\infty \frac{d\gamma_{q'}}{2} S_{\gamma_q,\gamma_{q'}}[\gamma_p]$$



$$= \int_0^\infty \frac{d\gamma_{q'}}{2} \int_0^\infty \frac{d\gamma_{p'}}{2} S_{\gamma_q,\gamma_{q'}}[\gamma_p] \mathbf{F}_{\gamma_p,\gamma_{p'}} \left[\begin{array}{cc} \gamma_j & \gamma_i \\ \gamma_{q'} & \gamma_{q'} \end{array} \right]$$



.

A. Ising BPZ equations

$$\left(L_{-2} - \frac{4}{3} L_{-1}\right) \langle \sigma(\infty)\sigma(1)\sigma(z,\bar{z})\sigma(0)\rangle = 0 \tag{1}$$

$$\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\mathbb{I}|z) = \frac{1}{\sqrt{2}} \frac{\sqrt{1+\sqrt{1-z}}}{(z(1-z))^{\frac{1}{8}}}, \mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\epsilon|z) = \frac{2}{\sqrt{2}} \frac{\sqrt{1-\sqrt{1-z}}}{(z(1-z))^{\frac{1}{8}}}.$$

$$\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(0|z) = \frac{1}{\sqrt{2}} \frac{\sqrt{1+\sqrt{1-z}}}{(z(1-z))^{\frac{1}{8}}}, \quad \mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\epsilon|z) = \frac{2}{\sqrt{2}} \frac{\sqrt{1-\sqrt{1-z}}}{(z(1-z))^{\frac{1}{8}}}, \quad \mathcal{F}^{\epsilon\epsilon}_{\epsilon\epsilon}(0|z) = \frac{1-z+z^2}{z(1-z)}$$

$$\langle \sigma(\infty)\sigma(1)\sigma(z,\bar{z})\sigma(0)\rangle = |\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\mathbb{I}|z)|^2 + C^2_{\sigma\sigma\epsilon} |\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\epsilon|z)|^2$$

$$|\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\mathbb{I}|z)|^2 + C^2_{\sigma\sigma\epsilon} |\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\epsilon|z)|^2 = |\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\mathbb{I}|1-z)|^2 + C^2_{\sigma\sigma\epsilon} |\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\epsilon|1-z)|^2$$

$$\mathcal{F}^{21}_{34}(h_p|z) = \sum_q \mathbf{F}_{h_p,h_q} \left[\begin{array}{cc} h_2 & h_1 \\ h_3 & h_4 \end{array} \right] \mathcal{F}^{23}_{14}(h_q|1-z)$$

Using

$$\sqrt{1+\sqrt{1-z}} = \frac{1}{\sqrt{2}} \left(\sqrt{1+\sqrt{z}} + \sqrt{1-\sqrt{z}} \right), \sqrt{1-\sqrt{1-z}} = \frac{1}{\sqrt{2}} \left(\sqrt{1+\sqrt{z}} - \sqrt{1-\sqrt{z}} \right).$$

we find

$$\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\mathbb{I}|z) = \frac{1}{\sqrt{2}}\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\mathbb{I}|1-z) + \frac{1}{2\sqrt{2}}\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\epsilon|1-z), \mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\epsilon|z) = \frac{2}{\sqrt{2}}\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\mathbb{I}|1-z) - \frac{1}{\sqrt{2}}\mathcal{F}^{\sigma\sigma}_{\sigma\sigma}(\epsilon|1-z).$$

which allow us to deduce

$$C_{\sigma\sigma\varepsilon} = \frac{1}{2}$$

which completes the CFT data (with the normalization 1 of 2 pt functions).

The fusion matrix are

$$\mathbf{F}_{0,0}\begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} = 2\mathbf{F}_{0,\epsilon}\begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} = \frac{1}{2}\mathbf{F}_{\epsilon,0}\begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} = -\mathbf{F}_{\epsilon,\epsilon}\begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} = \frac{1}{\sqrt{2}}, \mathbf{F}_{0,0}\begin{bmatrix} \epsilon & \epsilon \\ \epsilon & \epsilon \end{bmatrix} = 2\mathbf{F}_{0,\sigma}\begin{bmatrix} \sigma & \sigma \\ \epsilon & \epsilon \end{bmatrix} = \frac{1}{2}\mathbf{F}_{\sigma,0}\begin{bmatrix} \epsilon & \sigma \\ \epsilon & \sigma \end{bmatrix} = -\mathbf{F}_{\sigma}$$

VIII. HW

For the Ising model we have $c = 1/2$, $1, \varepsilon, \sigma$ with $0, 1/2, 1/16$ h resp. This gives:

$$\chi_1 = q^{-\frac{1/2}{24}} \sum_{q^n}$$

$$\chi_{1/2} = q^{1/2-\frac{1/2}{24}} \sum_{q^n}$$

$$\chi_{1/16} = q^{1/16-\frac{1/2}{24}} \sum_{q^n}$$

$$\begin{aligned} Z_{\text{Ising}} &= Z_{0,\frac{1}{2}} + Z_{\frac{1}{2},\frac{1}{2}} + Z_{\frac{1}{2},0} \\ &= \frac{1}{2}\left[\left|\frac{\theta_2(0|\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_3(0|\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_4(0|\tau)}{\eta(\tau)}\right|\right] \\ &= |\chi_{1,1}(\tau)|^2 + |\chi_{2,1}(\tau)|^2 + |\chi_{1,2}(\tau)|^2 \end{aligned}$$

Conformal characters of identity, energy and spin operators are:

$$\begin{aligned} \chi_{1,1}(\tau) &= \frac{1}{2\sqrt{\eta(\tau)}}[\sqrt{\theta_3(0|\tau)} + \sqrt{\theta_4(0|\tau)}] \\ \chi_{2,1}(\tau) &= \frac{1}{2\sqrt{\eta(\tau)}}[\sqrt{\theta_3(0|\tau)} - \sqrt{\theta_4(0|\tau)}] \\ \chi_{1,2}(\tau) &= \frac{1}{\sqrt{2\eta(\tau)}}\sqrt{\theta_2(0|\tau)} \end{aligned}$$

indeed expandinf the characters in powers of q we have:

$$\begin{aligned} q^{\frac{1}{48}}\chi_1(\tau) &= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 5q^8 + 5q^9 + 7q^{10} + O\left(q^{481/48}\right) \\ q^{-\frac{23}{48}}\chi_\epsilon(\tau) &= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 4q^7 + 5q^8 + 6q^9 + 8q^{10} + O\left(q^{481/48}\right) \\ q^{-\frac{1}{24}}\chi_\sigma(\tau) &= 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 8q^9 + 10q^{10} + O\left(q^{481/48}\right) \end{aligned}$$

For defect insertions we consider We consider $Z^{Ising}[g_1, g_2](\tau, \bar\tau) = \mathrm{tr}_{\mathcal{H}^{g_1}_{S^1}}\left(g_2 q^{L_0-\frac{1}{48}}\bar q^{\bar L_0-\frac{1}{48}}\right)$ for g_1 a line defect twisting the Hilbert space and g_2 for the line defect in the transverse direction as shown in Figure 2.

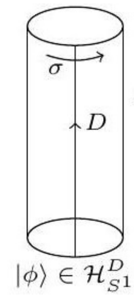


FIG. 2: Here we consider $D = g_1$.

For instance, the usual $\mathbb{Z}_2$ symmetry $\sigma \to -\sigma$, represented by η is seen by inserting only the line defect along the time direction ($g_1 = 0, g_2 = 1$). The fields 1 and ε are invariant (uncharged) so this leads to

$$Z^{\text{Ising}}[0,1](\tau,\bar{\tau}) = |\chi_0(\tau)|^2 + |\chi_{\frac{1}{2}}(\tau)|^2 - |\chi_{\frac{1}{16}}(\tau)|^2$$

Similarly, with the help of the modularity

$$Z^{\text{Ising}}_{T^2}[D,0](\tau,\bar{\tau}) = Z^{\text{Ising}}_{T^2}[0,D](-1/\tau,-1/\bar{\tau})\,. \tag{2}$$

and, since the characters satisfy

$$\begin{aligned}
\chi_{1,1}\left(-\frac{1}{\tau}\right) &= \frac{1}{2\sqrt{\eta(\tau)}}[\sqrt{\theta_3(0|\tau)} + \sqrt{\theta_2(0|\tau)}] \\
\chi_{2,1}\left(-\frac{1}{\tau}\right) &= \frac{1}{2\sqrt{\eta(\tau)}}[\sqrt{\theta_3(0|\tau)} - \sqrt{\theta_2(0|\tau)}] \\
\chi_{1,2}\left(-\frac{1}{\tau}\right) &= \frac{1}{\sqrt{2\eta(\tau)}}\sqrt{\theta_4(0|\tau)}
\end{aligned}$$

we find

$$Z^{\text{Ising}}[1,0](\tau,\bar{\tau}) = |\chi_\sigma(\tau)|^2 + \chi_1(\tau)\chi_\varepsilon(\bar{\tau}) + \chi_1(\bar{\tau})\chi_\varepsilon(\tau).$$

For the remaining partition function we can find it from the modular constraint

$$Z^{\text{Ising}}[1,1](\tau,\bar{\tau}) = \sum_{i,j\in\{0,\frac{1}{2},\frac{1}{16}\}} n_{ij}\chi_i(\tau)\chi_j(\bar{\tau}) = \sum_{i,j\in\{0,\frac{1}{2},\frac{1}{16}\}} n_{ij}\chi_i\left(-\frac{1}{\tau}\right)\chi_j\left(-\frac{1}{\bar{\tau}}\right)$$

for $n \in M_{3,3}(\mathbb{Z}_{\geq 0})$ which leads to the matrix

$$\begin{pmatrix} t+s & t & 0 \\ t & t+s & 0 \\ 0 & 0 & s \end{pmatrix} \implies n \in \text{Span}_{\mathbb{Z}_{\geq 0}}\left\{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

where we recognize the second solution as $Z^{\text{Ising}}_{T^2}[0,0]$ and the first gives

$$Z^{\text{Ising}}_{T^2}[1,1] = |\chi_1(\tau)|^2 + |\chi_\epsilon(\tau)|^2 + \chi_1(\tau)\chi_\varepsilon(\bar{\tau}) + \chi_1(\bar{\tau})\chi_\varepsilon(\tau).$$

All together, performing the gauging gives

$$Z^{\text{Ising}/\mathbb{Z}_2}(\tau,\bar{\tau}) = \frac{1}{2}\sum_{g_1,g_2\in\mathbb{Z}_2} Z^{\text{Ising}}[g_1,g_2](\tau,\bar{\tau}) = |\chi_0(\tau)|^2 + |\chi_{\frac{1}{2}}(\tau)|^2 + |\chi_{\frac{1}{16}}(\tau)|^2 = Z^{\text{Ising}}(\tau,\bar{\tau}).$$

We expected the resulting theory to have a residual dual symmetry however, rather surprisingly (or not as the $\mathbb{Z}_2$ symmetry is self dual...) we find the self-duality

$$Z^{\text{Ising}/\mathbb{Z}_2}(\tau,\bar{\tau}) = Z^{\text{Ising}}(\tau,\bar{\tau})$$

which trivially shows the "un-gauge" a.k.a the doulbe gauge procedure.

We will see how any CFT with such generalized symmetry arises the same self-duality property. Let D a topological defect and let us define D by the action of the operators of the theory. For the Ising in the torous we have

$$Z^{\text{Ising}}_{T^2}[0,D](\tau,\bar{\tau}) \equiv \mathrm{tr}_{\mathcal{H}_{S^1}}(Dq^{L_0-\frac{1}{48}}\bar{q}^{\bar{L}_0-\frac{1}{48}}) = d_1|\chi_0(\tau)|^2 + d_\epsilon|\chi_{\frac{1}{2}}(\tau)|^2 + d_\sigma|\chi_{\frac{1}{16}}(\tau)|^2$$

for d the charges of the operators under the action of D thus characterizing completelty the line defect by $(d_1, d_\varepsilon, d_\sigma)$.

In general there are not so many choices of d as they are constrained by modularity (c.f. Figure 3)

$$Z^{\text{Ising}}_{T^2}[D,0](\tau,\bar{\tau}) = Z^{\text{Ising}}_{T^2}[0,D](-1/\tau,-1/\bar{\tau})\,. \tag{3}$$

where the LHS is now a the insertion of the defect twisting the Hilbert space i.e. along the spatial direction. Thus $Z^{Ising}_{T^2}[0,D]$ counts operators in the twisted Hilberspace $\mathcal{H}^D_{S_1}$ where the insertion of the defect allow to mix inital to final states giving the decomposition:

$$Z^{\text{Ising}}_{T^2}[D,0](\tau,\bar{\tau}) \equiv \mathrm{tr}_{\mathcal{H}^D_{S^1}}(q^{L_0-\frac{1}{48}}\bar{q}^{\bar{L}_0-\frac{1}{48}}) = \sum_{i,j\in\{0,\frac{1}{2},\frac{1}{16}\}} n_{ij}\chi_i(\tau)\chi_j(\bar{\tau}).$$

Plugging in the modular Equation (3) yields the system of equations:

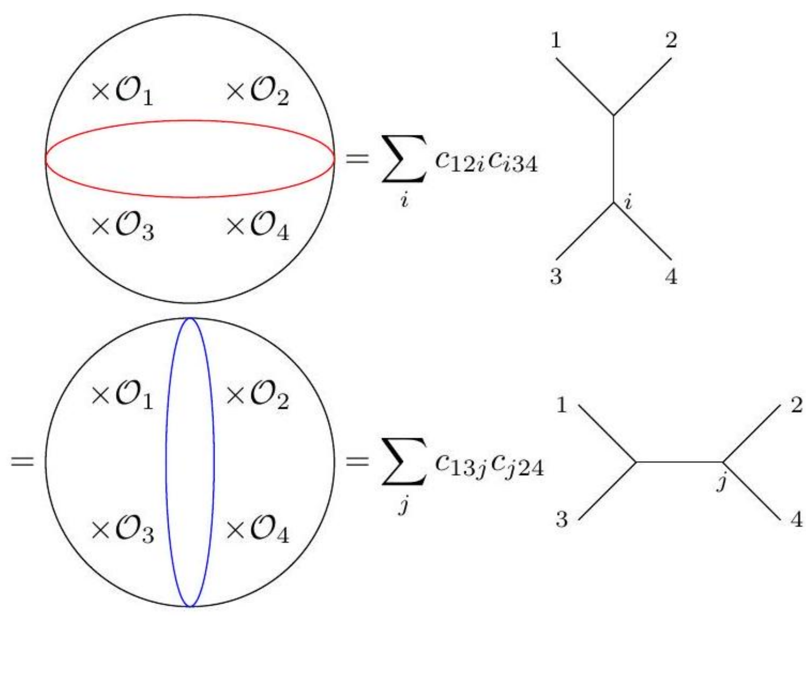
$$\begin{cases} d_\sigma = s \\ d_\varepsilon = t - \sqrt{2}u, \quad s,t,u\in\mathbb{Z} \\ d_1 = t + \sqrt{2}u \end{cases} \implies (d_1, d_\sigma, d_\varepsilon) \in \mathrm{Span}_{\mathbb{Z}}\{(1,1,1),(\sqrt{2},-\sqrt{2},0),(1,1,-1)\}. \tag{4}$$

Note that here we have written in the basis where we recognize the first solution element as the identity defect and the third as the $\sigma\to-\sigma$ η defect. The second element is the $\mathcal{N}$ non-invertible symmetry.

IX. GENERAL AXIOMATIC CONSTRAINTS

Let us study constraints coming from locality. No matter how we build our Riemann surface out of Hilbert spaces on circles, we will always get the same answer for CFT observables. It all boils down to consistency of cutting and gluing CFT observables on Riemann surfaces. Moore-Seiberg gave necessary and sufficient conditions for that:

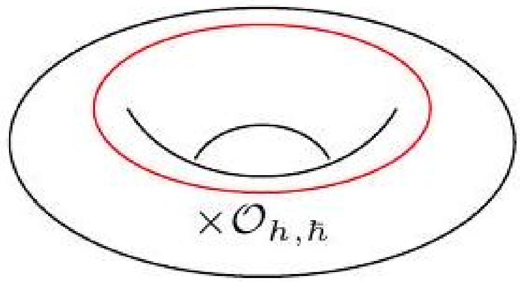
1. Quantization of spin.
2. Sphere 4 -point crossing



3. Modular covariance of the torus one-point function [3] $\langle \mathcal{O}_{h,\hbar} \rangle_\tau$, namely

$$\langle \mathcal{O} \rangle_{-1/\tau} = \tau^h \bar{\tau}^\hbar \langle \mathcal{O} \rangle_\tau .$$

where the factor is a Weyl factor coming from the $w \to w'/\tau$ coordinate change.



Question: if you write the modular invariance on arbitrary Riemann surfaces, is that enough to recover the sphere four-point function condition? Answer: that's a very good question; in some sense the genus 2 Riemann surface can be cut open into a four-point function.

Question: how trivial is the Moore-Seiberg result; is it deep? Answer: it depends on your particular taste. It is just about cutting and gluing.

Question: are there cases where the sphere condition is satisfied but not torus modular invariance. Answer: have to think.

1.7 Generalized global symmetries in $d = 2$ CFT

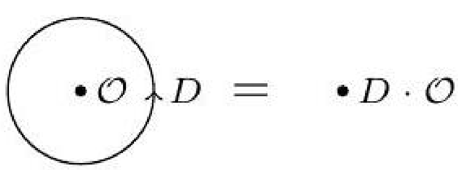
From our point of view, symmetries are the same as topological defect lines (TDL). Because we work in Euclidean signature with unitary Lorentz-invariant theories (actually CFT), so lines can be oriented in any direction.

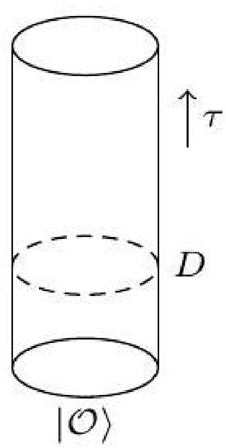
By definition, topological defects obey

- Topological invariance
- Fusion with integer coefficients (see later for a reason based on locality)

$$D_i \uparrow D_j = \sum_k \uparrow D_k$$

The action on charged operators preserves $h, \hbar$ since the operator commutes with the stress-tensor. It can be depicted in radial quantization or on a cylinder,





Quantum dimension of a defect. Because the operator does not change the dimensions, and we have assumed we have a unique vacuum, the defect must simply rescale the vacuum by some number $\langle D\rangle = \langle 0|D|0\rangle$ called the quantum dimension,

$$D|0\rangle = \langle D\rangle|0\rangle.$$

In a unitary theory, $\langle D\rangle \geq 1$. If $\langle D\rangle = 1$ then D is invertible. If $\langle D\rangle > 1$ then D is not invertible.

1.7.1 Example: Ising CFT

What are the symmetries? In fact, symmetries are equivalent to Ward identities, so whenever you find Ward identities you should find the corresponding symmetry.

There is the obvious η symmetry, acting as $1 \to 1, \sigma \to -\sigma, \epsilon \to \epsilon$. Shows that $\langle\sigma\sigma\sigma\rangle = \langle\sigma\epsilon\epsilon\rangle = 0$ etc.

Surprisingly (at first) one has $\langle\epsilon \dots \epsilon\rangle = 0$ whenever there is an odd number of ϵ, so there should be some symmetry guaranteeing that. But it cannot be just a $\mathbb{Z}_2$ charge of ϵ since $\langle\sigma\sigma\epsilon\rangle \neq 0$. It will be a non-invertible symmetry $\mathcal{N}$ sending $\epsilon \to -\epsilon\langle\mathcal{N}\rangle$ and $\sigma \to 0$ so as to allow $\langle\sigma\sigma\epsilon\rangle \neq 0$.

This leads us to go beyond groups and to discuss fusion categories.

2 TOPOLOGICAL DEFECTS AND FUSION CATEGORY

References for today:

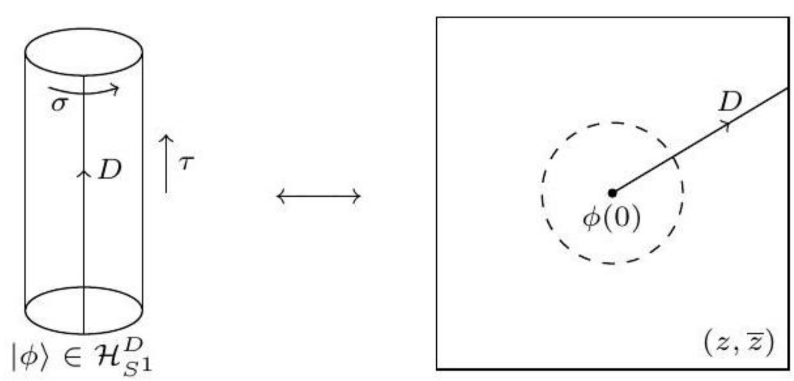
- https://arxiv.org/abs/1704.02330 Bhardwaj, Tachikawa,
- https://arxiv.org/abs/1802.04445 Chang, Lin, Shao, Wang, Yin.

The goal is to understand in what sense fusion categories are the natural object that comes up when studying generalized symmetries in 2d, just like groups arise when studying invertible symmetries.

2.1 Axiomatic approach to symmetries in 2d CFT

Here we focus on compact, unitary CFTs with a single ground state. Without symmetries the relevant axioms are Moore-Seiberg axioms (on fusion and braiding and torus S-move). We now want to refine these axioms by decorating them by topological defect lines.

We work in Euclidean signature. A given symmetry defect can be taken as wrapping the S^1 spatial direction, in which case it is simply a symmetry operator on the Hilbert space, or can be placed in the time direction at a point in space,

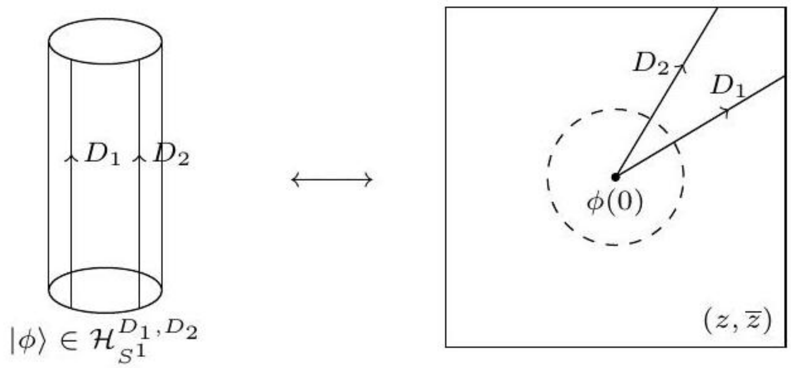


The Hilbert space in the presence of a defect D is denoted as $\mathcal{H}^D_{S^1}$ and called the D-twisted sector. Under the state-operator map it corresponds to an operator in the D-twisted sector.

Faithfulness condition. We assume the faithfulness condition which states that the only defect that acts trivially on all local operators is the identity defect. Equivalently, defects $D \neq 1$ cannot end topologically: otherwise you could cut it open and see that it acts trivially:

$$(\mathrm{O})\ D = \bigcirc D = \langle D\rangle \mathcal{O}$$

MULTI-DEFECT HILBERT SPACE.



The Virasoro action allows you to move the defects around. It introduces some factors so strictly speaking the Hilbert space $\mathcal{H}_{S^1}^{D_1,D_2,\dots}$ depends on the separations and Hilbert spaces for different separations are easily isomorphic. The Hilbert space is also invariant (up to an important isomorphism) under cyclic permutations of the defect (to do things properly we need to include a marked point etc).

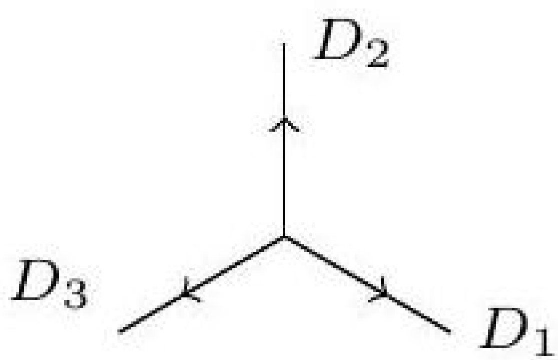
Besides the Hilbert space $\mathcal{H}_{S^1}^{D_1,D_2,\dots}$ we also define

$$\mathcal{H}_{S^1}^{D_1+D_2} = \mathcal{H}_{S^1}^{D_1} \oplus \mathcal{H}_{S^1}^{D_2}$$

corresponding to the insertion of a direct sum of defects at the same place. This will be useful when discussing the fusion of defect.

Question: if you insert a non-topological defect do you break Virasoro? Answer: you break Vir $\times$ Vir to the diagonal subalgebra.

Topological junction. A topological junction is an operator of dimension $h = \bar{h} = 0$ inside $\mathcal{H}_{S^1}^{D_1\dots D_n}$. The space of such junctions is denoted $V_{D_1\dots D_n}$. An element $v \in V_{D_1\dots D_n}$ is visualized on the plane as



In the invertible case defects are labeled by group elements g_i and $\dim V_{g_1\dots g_n}$ is 1 if $g_1 g_2 \dots g_n = 1$ and is otherwise zero.

Dual defect. The dual defect $\bar{D}$ may have $D\bar{D}$ different from 1. The dual defect is simply defined as the orientation-reversed defect

$$\psi_{\bar{D}} = \psi_D$$

A defect is simple if $\dim V_{D\bar{D}} = 1$. As a consequence we can show that $D \neq D_1 + D_2$. Conversely when $\dim V_{D\bar{D}} \geq 2$ then we can always split D into pieces.

Fusion. When fusing defects we get a new defect, which can be decomposed into simple defects, so we can introduce notation in the case of simple defects:

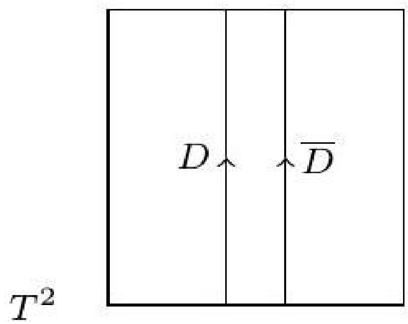
$$D_i D_j = \sum_k N_{ijk} D_k$$

One can check that $N_{ijk} = \dim V_{D_i D_j \bar{D}_k}$. This differs from the group-like multiplication law $D_g D_{g'} = D_{gg'}$ for invertible symmetries. Special case

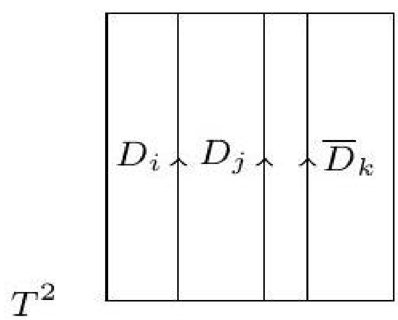
$$D_i \bar{D}_i = 1 + \dots$$

Let us check that the leading term is 1 .

Using the thermal partition function. ..... missing discussion of how the thermal partition function allows one to show that the leading term in $D_i\bar{D}_i$ is $\dim V_{D\bar{D}} \cdots$.

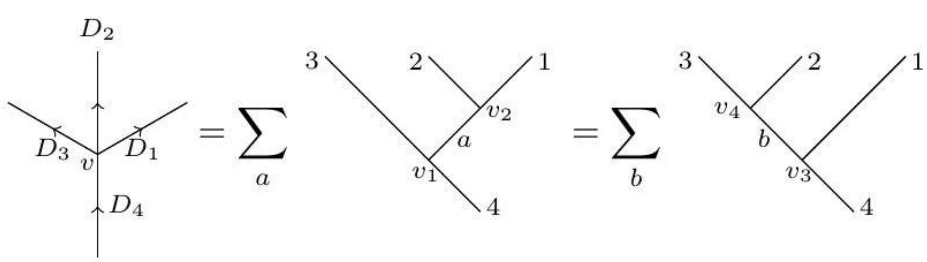


Likewise in an exercise we should prove $N_{ijk} = \dim V_{D_i D_j \bar{D}_k}$ using the lowtemperature limit of the torus partition function with three defect insertions



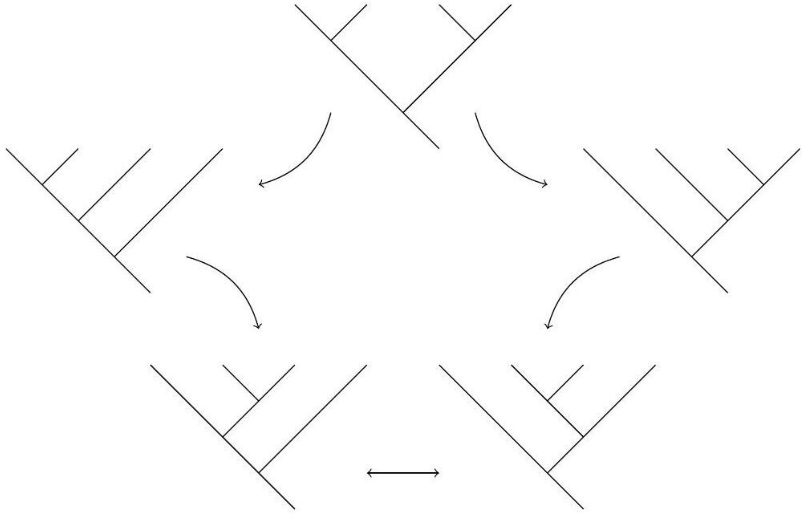
Topological junctions as morphisms. Topological junctions $v \in V_{D_1 D_2 D_3}$ are morphisms between $D_1 D_2$ and $\bar{D}_3$.

F-symbols (associators). In a general junction vector space there is no preferred basis. A several bases of $V_{D_1 D_2 D_3 \bar{D}_4}$ can be constructed from bases of three-fold junctions:



We have a unitary change of basis $\left(F_4^{321}\right)_{ab}$ mapping the basis of the form $v_1 \otimes v_2$ to that of the form $v_3 \otimes v_4$.

Pentagon identity. The fusion coefficients F have to obey an equation of the form $FF = \sum FFF$, obtained by performing the following fusion steps:

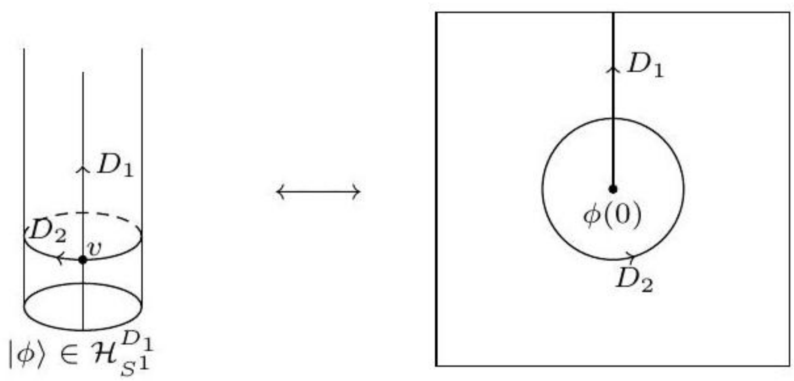


This is an analogue of the cocycle condition for $H^3(G, U(1))$ which classifies anomalies for the group G. In a sense, F-symbols capture the anomalies.

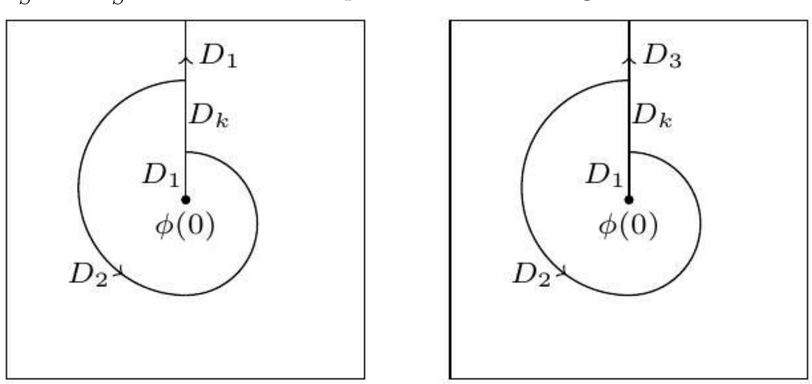
Should we continue with six-fold junctions etc? MacLane coherence theorem: the full set of consistency conditions is automatically satisfied once the pentagon identity is obeyed.

2.2 Symmetry action in defect Hilbert space

How can a symmetry labeled by a defect D_2 act on a twisted Hilbert space $\mathcal{H}_{S^1}^{D_1}$, namely on the D_1-twisted sector? We need a topological junction v between the operators.



This can be resolved into an intermediate defect $D_k \in D_1D_2$. There are normally multiple choices of this intermediate defect, hence $V_{D_1D_2\bar{D}_1\bar{D}_2}$ is typically of dimension larger than 1 (in contrast to invertible symmetries). This means that there are multiple possible actions of D_2 on $\mathcal{H}_{S^1}^{D_1}$. This is called the Lasso action. A generalization is that the action can change $\mathcal{H}_{S^1}^{D_1}$ to $\mathcal{H}_{S^1}^{D_3}$ as in the second picture below. This generates the Tube algebra.



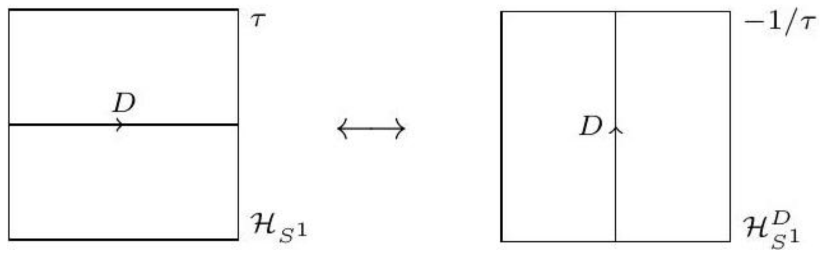
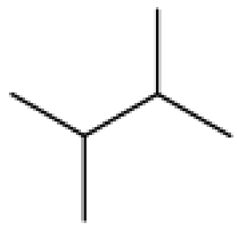


FIG. 3: Modularity of the defect insertion

2.3 Definition of symmetry-enriched CFT

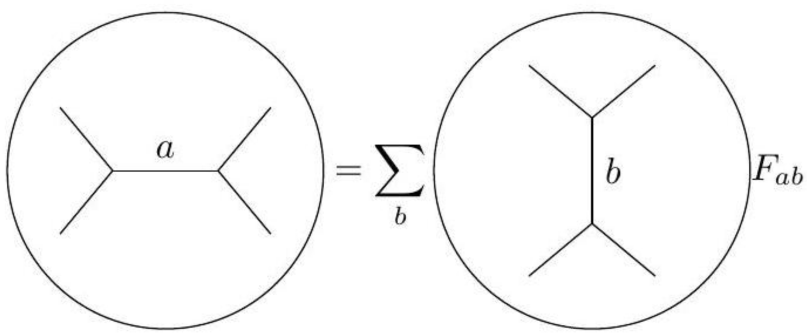
A CFT enriched by a collection of topological defect lines $\{D_i\}$ is given by

- Data: $\mathcal{H}_{S^1}^{D_i \ldots D_j}$ and three-point functions of operators attached to defects,



This includes the usual $\mathcal{H}_{S^1}$ and three point functions.

- Bootstrap conditions (locality). Sphere four-point crossing in the presence of topological defects,



Modular covariance of the torus one-point function with defects.

2.4 Modular invariance of the symmetry enriched CFT

Recall that for a symmetry topological defect line D we can write
This leads to the following relation

$$\mathrm{Tr}_{\mathcal{H}_{S^1}}\left(\widehat{D} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}\right) = \mathrm{Tr}_{\mathcal{H}^D_{S^1}}\left(\widetilde{q}^{L_0 - c/24} \bar{\widetilde{q}}^{\bar{L}_0 - c/24}\right)$$

where $\widetilde{q} = e^{2\pi i(-1/\tau)}$.
Ising model. See homework. The Hilbert space on the LHS here is a direct sum of representations of Virasoro so

$$
\begin{aligned}
\text{LHS} &= A_1 \left|\chi_0\right|^2 + A_{1/2} \left|\chi_{1/2}\right|^2 + A_{1/16} \left|\chi_{1/16}\right|^2 \\
\text{RHS} &= \sum_{i,j} n_{ij} \chi_i(\widetilde{q}) \chi_j(\overline{\widetilde{q}}),
\end{aligned}
$$

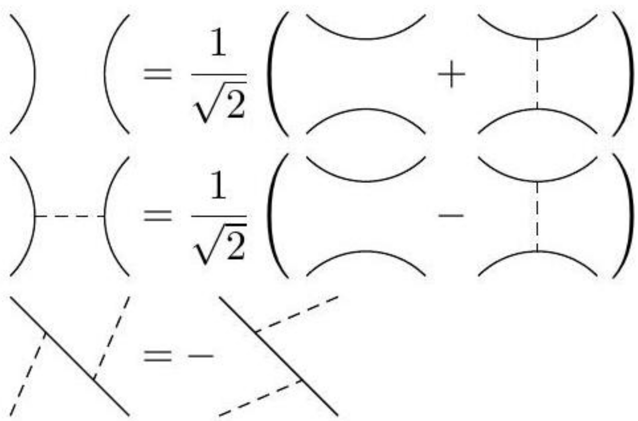
where the A_i are not yet quantized and the n_{ij} are non-negative integers. Using the modular transformations of Virasoro characters we find

- $(A_i) = (1, 1, 1)$ corresponds to the identity defect $D = 1$;
- $(A_i) = (1, 1, -1)$ corresponds to the $D = \eta$ defect;
- $(A_i) = (\sqrt{2}, -\sqrt{2}, 0)$ corresponds to the $D = \mathcal{N}$ defect.

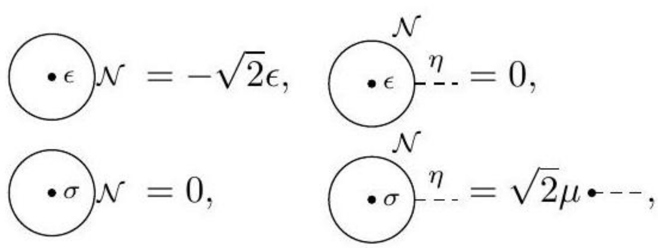
The fusion rule for $\mathcal{N}^2$ can be found by squaring these eigenvalues A_i and reexpressing them in the basis of other solutions (A_i). The same can be done for all fusion rules and we find

$$
\mathcal{N}^2 = 1 + \eta, \quad \eta^2 = 1, \quad \mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N}.
$$

Ising F-symbols.



Action on local operators. Then we will find



where $\sqrt{2}$ is the quantum dimension $\langle \mathcal{N} \rangle = \sqrt{2}$, and μ is a primary operator in the twisted sector:

$$
\mathcal{H}^{\eta}_{S^1} = \left\{ \psi_{1/2,0}, \widetilde{\psi}_{0,1/2}, \mu_{1/16,1/16} \right\}
$$

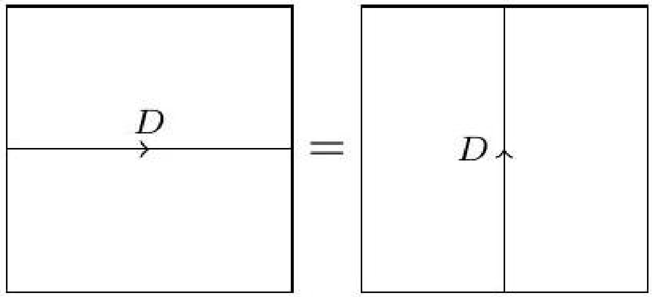
Passing TDL through local operators.

$$
\epsilon \cdot |_{\mathcal{N}} = \ |_{-\epsilon\bullet} \qquad \sigma \cdot |_{\mathcal{N}} = \ |_{\mathcal{N}} |_{\mathcal{N}}
$$

(Unrelated?) claim: the presence of $\mathcal{N}$ means that the CFT is self-dual under the $\mathbb{Z}_2$ orbifold.

2.5 Dynamic consequences of non-invertible symmetry

Assume that you have a UV theory T_{UV} with D symmetry and you deform it by a (marginally relevant or) relevant operator $\mathcal{O}_{h,\bar{h}}$ with $\bar{h} = h$ and $\Delta \leq 2$ then perform the RG flow to the IR theory T_{IR}. Assume also that the operator $\mathcal{O}$ commutes with the defect D (namely the deformation preserves the symmetry).
Claim 1 (Theorem). If $\langle D\rangle \notin \mathbb{Z}$ then T_{IR} cannot be trivially gapped: we either get a CFT or spontaneous symmetry breaking.
Proof. We have



If the IR is trivially gapped then there is a unique ground state so the left-hand side is $\langle 0|D|0\rangle = \langle D\rangle$. The right-hand side is a trace of 1 over the defect Hilbert space, which has to be an integer. Contradiction. If there were multiple ground states then D can act differently on different ground states and somehow this resolves the problem.

Example 1. Tricritical Ising model $c = 7/10$. The symmetry is Ising $\boxtimes$ Fib where the "Ising" symmetry is the usual $\{1, \eta, \mathcal{N}\}$ and the "Fib" symmetry is generated by W with $W^2 = 1 + W$, of quantum dimension $\langle W\rangle = (1+\sqrt{5})/2$.

Deforming this CFT by $\epsilon'_{3/5,3/5}$, which commutes with $\mathcal{N}$, gives an RG flow whose low-energy limit is either gapless (necessarily Ising by c monotonicity) or gapped with at least three vacua. Both cases arise depending on the sign of the deformation, as can be shown using integrability.

Question: what operator tracks the RG flow arriving into the Ising model? Answer: it is an irrelevant operator, which turns out to be the $T\bar{T}$ operator constructed from the stress-tensor, which thus automatically commutes with all of the symmetries, including $\mathcal{N}$.

Example 2. The $1+1$ dimensional $SU(N)$ massless adjoint QCD also has a huge amount of non-invertible symmetries. This leads to a gapped phase with $\sim 2^N$ vacuum degeneracies. Heuristic explanation: the adjoint fermions are described by the WZW model Spin $\left(N^2-1\right)_1$; gauging $SU(N)$ roughly amounts to taking a coset, which suggests the TQFT Spin $\left(N^2-1\right)_1/SU(N)_N$, which has a ton of topological defects.

3 TOPOLOGICAL INTERFACES AND GENERALIZED GAUGING

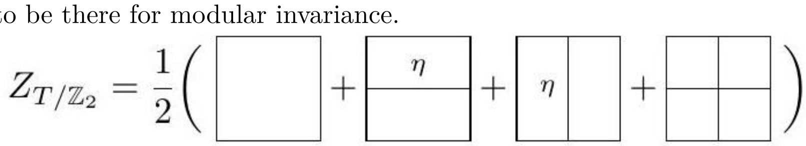
See Fuchs-Runkel-Schwaigert https://arxiv.org/abs/hep-th/0204148,
..., Diatlyk-Luo-Weller-Wang https://arxiv.org/abs/2311.17044

3.1 Gauging procedure

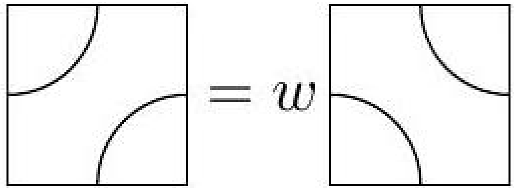
Gauging a usual abelian group symmetry. To gauge a $\mathbb{Z}_2$ symmetry of a theory T (which is like an orbifold in string theory), two steps:

- project to the $\mathbb{Z}_2$-even sector;
- include $\mathbb{Z}_2$-twisted sectors.

For instance, the torus partition function is a sum of four terms: the first two terms are a trace in the usual Hilbert space but with a projection $(1+\eta)/2$ onto the $\mathbb{Z}_2$-even sector; the second two are the projection but in the twisted sector. All terms have to be there for modular invariance.



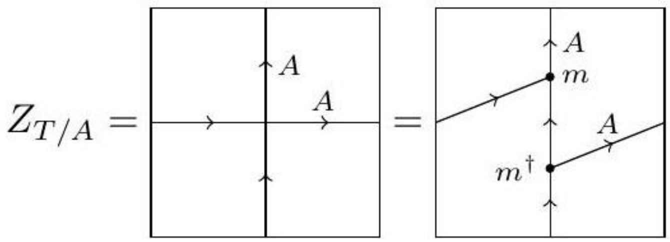
The last diagram is subtle and has to be resolved to be defined. We have no topological junction for a group symmetry so there are two resolutions simply related by a phase w (equal to ± 1 in the $\mathbb{Z}_2$ case)



If we pick one of these resolutions, then we have to worry about modular invariance of Z. If $w \neq 1$ we will find that the partition function is not S-invariant regardless of what we do. There is an anomaly preventing you from gauging.

Example: the Ising model has $w = +1$ while $SU(2)_1$ has $w = -1$ for the $\mathbb{Z}_2$ center symmetry.

Gauging topological defect lines. Pick a general topological defect line A. (For instance, to reproduce the previous gauging we would take $A = 1 + \eta$, or more generally for a group we would take the projector $A = \sum_{g \in G} g$.) Then the sum of partition functions we had before is reproduced by inserting a complete network of A defects,

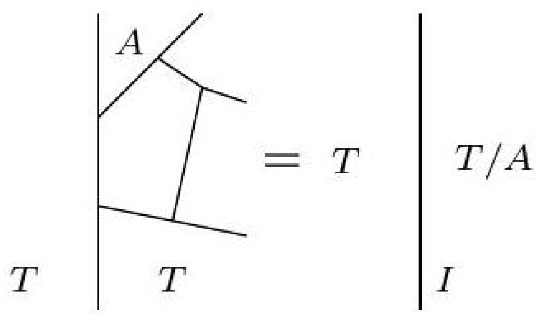


Data for generalized gauging: $\left(A, m, m^\dagger, u, u^\dagger\right)$ with $u, u^\dagger$ end-points of A and $m, m^\dagger$ three-fold junctions. To avoid gauge anomaly, data has to form a symmetric special Frobenius algebra object. These latter data $m, m^\dagger$ capture "discrete torsion", $1+1$ dimensional SPT phases. (We also assume A is self-dual but it is not clear how much work this assumption is making.)

Abstractly: gauging is decorating the observables in T with a network of $\left(A, m, m^\dagger\right)$ with a mesh that is fine enough.

3.2 Half-gauging and topological interfaces

Suppose you have a symmetry and a choice of $\left(A, m, m^\dagger\right)$. Then by gauging over a half-space you can make an interface between the theory T and T/A :



Gauging just in a small slab (such as a time interval) defines a topological

X. ADS/CFT

Modern Approach to 2D Conformal Field Theory [?]