Notes on Fields

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I. STRING THEORY

Notes on the course of String theory.

Particle on the worldline $S = -m\int_0^T d\tau\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ difficult to quantize. Instead, introduce an einbein $e = e_\tau(\tau)$ "co-vector"

$$S[e,x] = \frac{1}{2}\int e^{-1}\dot{x}^2 - m^2 e$$

we can quantize in usual way. This is a $1+0-$dimensional QFT i.e. QM. Amplitudes: meaningful operators are integrated along the worldline "Vertex operators": $\mathcal{A} = \langle k_4|\, T[\int d\tau d\tau' e^{ik_3X(\tau')}e^{ik_2X(\tau)}]\,|k_1\rangle$ has a gauge symmetry of reparametrizations so we can gauge fix $\tau' = 0$,

$$\mathcal{A} = \langle k_4|\, T[e^{ik_3X(0)}\int d\tau e^{ik_2X(\tau)}]\,|k_1\rangle\,.$$

Strings are more rich:

A. Strings and Actions

Nambu-Goto describes action of string $S = \int dA = \int\sqrt{-h}$, $(\tau,\sigma) = \sigma^a$ coordinates and $h_{ab} = \partial_a X^\mu\partial_b X^\nu\eta_{\mu\nu}$. EOM:

$$0 = \frac{\delta S}{\delta X^\mu} = T\int d\sigma^2\partial_b(\sqrt{-h}h^{ab}\partial_a X^\mu)\delta X_\mu - T\int d\tau[\sqrt{-h}h^{ab}\partial_a X^\mu\delta X_\mu]_{\sigma=0}^{\sigma=\ell}$$

Second term is baundary term. First is just lapalacian in curved space time. Possible solutions of boundary terms:

- Closed string.
- Dirichlet: $\delta X^\mu = 0$
- Neumann $\pi^\mu_\sigma = 0$, π^μ_a is the conjugate momentum of X^μ.

Symmetries: Poincare & diffeo. Dificult to quantize.

We introduce γ instead and consider

$$S_p[X,\gamma] = -\frac{T}{2}\int d\tau d\sigma\sqrt{-\gamma}\left[\gamma^{ab}\partial_a X^\mu\partial_b X_\mu\right] \tag{1}$$

recovers N-G action integrating out γ but it also has **Weyl invariance** so $T = 0$ off shell and $T_{ab} = 0$ on-shell.

Gauge fixing: $\gamma_{ab} = e^{2\phi(\sigma)}\eta_{ab}$ is the **conformal gauge**. This allow us to simplify the action to

$$S_P = \frac{T}{2}\int d^2\sigma[\dot{X}^2 - X'^2]$$

which has usual $K-G$equation as EOM (On the worldshet! NOT ON SPACE TIME) and $T_{ab} = 0$ constraints for γ. Constraint's translate to

We can go to light-cone coordinates: $\sigma^\pm = \tau\pm\sigma$. EOM: $\partial_+\partial_- X = 0$ Constraints: $(\partial_\pm X^\mu)^2 = 0$. Conservation $\partial_\mp T_{\pm\pm} = 0$. We have then solutions of left and right movers:

$$X(\sigma^+,\sigma^-) = X_L(\sigma^+) + X_R(\sigma^-) \tag{2}$$

- Closed string: Periodicity of $\sigma\sim\sigma 2\pi$ requires two independent oscillator modes:

$$\boxed{X^\mu_L(\sigma^+) = \frac{x^\mu}{2} + \frac{\alpha'}{2}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}e^{-in\sigma^+}}$$

$$\boxed{X^\mu_L(\sigma^+) = \frac{x^\mu}{2} + \frac{\alpha'}{2}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-in\sigma^-}}$$

where x is the average coordinate of X and p is the average momentum given by the average of the charge $P^\mu \sim T\partial_\tau X^\mu$. Constraints are

$$0 = (\partial_+ X^\mu)^2 \sim \sum_q L_q e^{-iq\sigma^-}, \quad L_p = \frac{1}{2}\sum_m \alpha_{p-m}\cdot\alpha_m$$

and similarly for $-$, $\tilde{L}_q = 0$.

Notice L_p are fourier modes of T_{--}. In particular, for both L and $\tilde{L}$ they satisfy:

$$\tilde{L_0} = L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0}\alpha_{-n}\cdot\alpha_n = 0 \implies \boxed{m^2 = \frac{1}{4\alpha'}\sum_{n>0}\tilde{\alpha}_{-n}\tilde{\alpha}_n = \frac{1}{4\alpha'}N = \frac{1}{4\alpha'}\tilde{N}} \tag{3}$$

this is the level matching condition

- Open string:
  - Neumann (of length π): Only one set of oscilators. Similar formula. X goes as cos.
  - Dirichlet: Similar. Goes as sin.

1. *Quantization*

Impose CCR on X and $\pi = \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}$. Hilbert space is constructed from vacuum $|0\rangle$ and α_{-n} creations insertions. Not all states are physical, some are tachyonic, etc. Physical should obey the constraints in the weak sense (on correlations). This is equivalent to demanding:

$$\begin{cases} L_n\,|\text{Phys.}\rangle = 0 & n > 0 \\ (L_0 - a)\,|\text{Phys.}\rangle = 0 \end{cases} \tag{4}$$

as there is a normal ordering ambiguity for L_0 when we put the CCR. States are thus in two labels $|0; k^\mu\rangle$ labelling the vacuum and the oscilators in top of it k. This satisfy $L_n\,|0,k\rangle = 0$ for $n > 0$ and

$$L_0\,|0,k\rangle = \left(M^2 - \frac{1}{\alpha'}(-a+N)\right)|0,k\rangle = 0$$

so for $a > 0$ and $N = 0$, the **lowest state has a tachyon** (as $a > 0$).

Level 1: $\xi\alpha_{-1}^\mu\,|0,k\rangle = 0$ all $\alpha_{n\geq 2}$ annihilate so we check only L_1 and L_0. L_0 gives mass

$$M^2 = \frac{1-a}{\alpha'}$$

and L_1 $\xi\cdot k = 0$. Normalization requires $\xi^2 \geq 0$ we identify a massless transverse d.o.f. so we expect $a = 1$ in which case we have null states so we have to mod them out. At level 2 more symmetries are expected if $a = 1$ and $D = 26$.

2. *Light cone quantization*

Using light cone coordinates we can parametrize $X^\pm = X^0 \pm X^{D-1}$ to which themselves allow us to parametrize the boundary conditions.

Virasoro constraints then give $\partial_+ X^+ \partial_+ X^- + \sum_{i=1}^{D-2}(\partial_+ X^i)^2 = 0$ and $\partial_\pm X^\pm = \alpha' p^{+\mu}/2$ so

$$\alpha_n^- = \sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\sum_{-\infty}^{\infty}\sum_{i=1}^{D-2}\alpha_{n-m}^i\alpha_{m,i}, \implies M^2 = \frac{2}{\alpha'}\sum_{-\infty}^{\infty}\sum_{i=1}^{D-2}\tilde{\alpha}_{-m}^i\tilde{\alpha}_{m,i}$$

quantization now is carried out in the $D-2$ coordinates as we have effectively reduced. Only physical state condition surviving are zero modes

$$M^2 = \frac{4}{\alpha'}(N_{\text{transverse}-a})$$

Let us see the ambiguity a,

$$\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha_n^i\alpha_{-n}=\sum_{i=1}^{D-2}\sum_{n>0}...+\underbrace{\sum_{i=1}^{D-2}\sum_{n=1}^{\infty}n}_{-2a}$$

so using the zeta function analytic extension, $a=\frac{D-2}{24}$ which gives $a=1$ at $D=26$ as expected something happens.

B. CFTs

Virasoro algebra. For Virasoro primaries we have $z\to z'$

$$\mathcal{O}(z,\bar{z})\to\left(\frac{\partial z}{\partial z'}\right)^h\left(\frac{\partial\bar{z}}{\partial\bar{z'}}\right)^{\bar{h}}\mathcal{O}(z,\bar{z})$$

where $h=\frac{\Delta_0+\ell_0}{2}$ and $\bar{h}$ are the charges of $Vir_c\times\bar{Vir}_{\bar{c}}$. To generate actions we can use the conformal global Ward identities:

$$\varepsilon\delta_{L_n}\mathcal{O}(z,\bar{z})=-\varepsilon[L_n,\mathcal{O}(z,\bar{z})]=-\varepsilon(z^{1+n}\partial\mathcal{O}(z,\bar{z})+h(n+1)z^n\mathcal{O}(z,\bar{z}))$$

which can be easly derived from the OPE of a primary $\mathcal{O}(z,\bar{z})$ with T

$$T(w)\mathcal{O}(z,\bar{z})=\frac{h\mathcal{O}(z,\bar{z})}{(w-z)^2}+\frac{\partial\mathcal{O}(z,\bar{z})}{w-z}+\underbrace{\sum_{n\geq 3}\frac{L_{-n}^{(z)}}{(w-z)^n}\mathcal{O}(z,\bar{z})}_{\text{regular}}$$

and solving the residues i.e.

$$[L_n,\mathcal{O}(z,\bar{z})]=\frac{1}{2\pi i}\int_w dzw^{n+1}T(w)\mathcal{O}(z,\bar{z})$$

$$[L_n,L_m]=(n-m)L_{n+m}+A(n)\delta_{n+m,0} \tag{5}$$

Quasi-primaries are the ones the satisfy these relations for global primaries and not the rest. Conformal symmetry fixes $n-pt$ functions a lot.

- One point function begin translation invariance are constants, begin dilation invariant they must be 0.
- Two point functions depend on $f(|z_1-z_1|^2)$, correct scaling restricts to polynomials and special conformal transformations restrict dimensions.

$$\langle\mathcal{O}_\infty(z_1,\bar{z}_1)\mathcal{O}_\in(z_2,\bar{z}_2)\rangle=\frac{\delta_{h,\bar{h}}}{(z_1-z_2)^h(\bar{z}_1-\bar{z}_2)^{\bar{h}}}$$

- Three point functions are upto structure constants::

$$\langle\mathcal{O}_\infty(z_1,\bar{z}_1)\mathcal{O}_\in(z_2,\bar{z}_2)\mathcal{O}_\ni(z_3,\bar{z}_3)\rangle=\frac{\lambda_{123}}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_1+h_3-h_2}\times c.c.}$$

Currents:

$$\delta S=\int d\sigma^2 f(\omega)T_{\omega\omega}\bar{\partial}\bar{\varepsilon}=0$$

then current conservation $\bar{\partial}f(\omega)T_{\omega\omega}=0$ gives $T_{\omega\omega}=T_{\omega\omega}(\omega)$ and we have infinite conserved currents $j_f(\omega)=f(\omega)T_{\omega\omega}(z)$.

Example I.1. *From the cylinder ($w = \sigma^1 + i\sigma^2$) to the plane $z = e^{-iw}$, $L_n^{Cyl.} = \frac{1}{2\pi}\int_{Imw=cst} dwe^{-inw}T_{ww}$ and since $T(w) \to z^2T(z) + reg.$*

$$L_n^{Pln.} = \frac{1}{2\pi i}\int \frac{dz}{z} z^{n+2}T(z) \tag{6}$$

so

$$T(z) = \sum_{\mathbb{Z}} \frac{L_n}{z^{2+n}}$$

State-Op. correspondence, radial quantization, central charge, etc. T is not a primary, it transforms with Schwartzian derivative.

If there are no Weyl anomaly,

$$\langle \mathcal{O}_\infty \mathcal{O}_\infty ... \mathcal{O}_\backslash \rangle_{\Omega(x)^2\eta_{ab}} = \prod_{i=1}^{n} \Omega(x)^{-\Delta_i}(x_i)\langle \mathcal{O}_\infty \ldots \mathcal{O}_\backslash \rangle_\eta \tag{7}$$

Weyl anomaly is measured by $\langle T \rangle \sim cR$ so for plane cylinder this is fine but in string theory we require $c = 0$ to preserve the symmetry.

Example I.2. *Free boson CFT. Useful relations:*

$$\partial \frac{1}{\bar{z}} = 2\pi\delta^2(z)$$

$S = \frac{1}{2\pi\alpha'}\int d^z \partial X \bar{\partial} X$ *propagator satisfies*

$$\partial\bar{\partial}\langle X(w,\bar{w})X(z,\bar{z})\rangle = -\pi\alpha'\delta^2(z-w)$$

Hence,

$$XX \sim -\frac{\alpha'}{2}\log|z-w|^2$$

this are not CFT power laws so instead, look at ∂X which indeed behave as $h = 1$, $\bar{h} = 0$ and $\bar{\partial}X$ $h = 0$, $\bar{h} = 1$. We can find the stress tensor $T(z) = -\frac{1}{\alpha'} : \partial X \partial X(z)$ and compute OPEs with ∂X and wick contractions. In particular we can find the central charge by OPE of T with itself, $c = 1$.

State operator map. From the string oscilation modes we can find $\alpha_n \sim \int z^n \partial X$ this gives the relation

$$\alpha_{-n}|0\rangle \sim \partial^n X(0)|0\rangle, \quad n > 0$$

describing the state at level n and the operator $\partial^n X$. Here we didn't consider α_0 or k. To do so, consider the vertex operator

$$V_k(z,\bar{z}) =: e^{ikX(z,\bar{z})} : .$$

In this case, we can do the path integral for $X = x_0 + \underbrace{\tilde{X}}_{osc}$. The first integral gives a delta on the momentum (α_0 modes) and the second the result above:

$$\langle V_{k_1}(z,\bar{z})V_{k_2}(w,\bar{w})\rangle = \int \mathcal{D}x_0\mathcal{D}Xe^{-S[X]} \underbrace{e^{ik_1X(z,\bar{z})}e^{ik_2X(w,\bar{w})}}_{e^{i(k_1+k_2)x_0}e^{ik_1\tilde{X}(z,\bar{z})}e^{ik_2\tilde{X}(w,\bar{w})}} = \exp\left(-\frac{\alpha'}{4}\log|z-w|^2\right)\delta(k_1+k_2) = \frac{\delta(k_1+k_2)}{|z-w|^{\alpha'\frac{k_1+k_2}{2}}}$$

where the δ comes from the x_0 integral and the other term is straightforward to compute with the two point function of X. We conclude V_k has dimension $h = \bar{h} = \alpha'\frac{k_1+k_2}{4}$.

This allow us to conclude that $\alpha_{-n}|0\rangle \sim: \partial^n XV_k(0) : |0\rangle$ so the full state-operator map is:

$$\boxed{\alpha_{-n_1}^{m_1}...\alpha_{-n_N}^{m_N}|k\rangle \leftrightarrow: \partial^{n_1}X \ldots \partial^{n_N}XV_k(0)} \tag{8}$$

C. BRST

Recall

$$Z = \int \frac{\mathcal{D}\gamma \mathcal{D}X}{\text{Vol(Diff.} \times \text{Weyl)}} e^{-S_p[X,\gamma]}$$

to gauge fix we use the Fadeev-popov procedure where physical states now are elements of the cohomology of the BRST symmetry operator. Note that in addition of the gauge symmetries, we also need to fix transformations that leave the path integral invariant, for instance, conformal killing vectors and, for n point functions we have full translation symmetry of the 3 points (recall we can always put one at ∞, other at 0 and the last at 1). This gives:

$$\Delta_{FP}(\gamma,\sigma) \int \mathcal{D}\hat{\gamma} \int \mathcal{D}g \delta(\gamma - \hat{\gamma}^g) \prod_{i=1}^{N} \delta(\sigma_i - \tilde{\sigma}_1) = 1$$

Doing the full procedure in conformal gauge, we get the gauge fixed action

$$Z_{\hat{\gamma}} = \int \mathcal{D}x \mathcal{D}c \mathcal{D}b e^{-S_p[X,\hat{\gamma}] - \frac{1}{2\pi}\int d^2 z b \nabla_{\bar{z}} c + \bar{b} \nabla_z \bar{c}}$$

where b has $(h,\bar{h}) = (2,0)$ and c has $(1,0)$, the nabla reduces to the usual derivative for our gauge. The ghosts generate another CFT, the bc CFT,

$$b(z)c(w) \sim \frac{1}{z-w}$$

this is derived from the Ward identity of the symmetry $0 = \delta\langle b(z)\dots\rangle = \delta \int Db b(z)\dots e^{-S_{bc}}$. Stress tensor can be found to be $T = 2 : \partial c b(z) : + : c \partial b(z) :$ which gives $c = -26$.

Canceling the total central charge (bosonic + ghosts) gives an anomaly free theory. In particular it rquires $d = 26$.

We can compute $b(z) = \sum \frac{b_n}{z^{2+n}}$ and $c(z) = \sum \frac{c_n}{z^{-1+n}}$ modes (as $h = 2$ for b and $h = -1$ for c) with the usual residue integrals and, this gives:

$$\{b_n, c_m\} = \int_0 \frac{dw}{2\pi i} w^{-2+m} \int_w \frac{dz}{2\pi i} z^{1+n} b(z)c(w) = \delta_{n+m,0}$$

Inserting the b_n and c_n modes in the stress tensor gives the virasoro modes L_n where we need to introduce a osscilator normal ordering (as b and c are anticommuting and we impose b on the right always):

$$L_m = \sum_{n=-\infty}^{\infty} (2m-n) \not{:} b_n c_{m-n} \not{:} + \delta_{m,0} a^g.$$

Being anticommuting we can define the two states:

$$b_0|\downarrow\rangle = 0, \quad b_0|\uparrow\rangle = |\downarrow\rangle, \quad c_0|\downarrow\rangle = |\uparrow\rangle, \quad c_0|\uparrow\rangle = 0$$

and b_n, c_n for $n > 0$ kills both states. Note that since $L_0 \sim b_{-1}c_1c_0b_0 + ...$ kills all states,

$$-2\,|\downarrow\rangle = 2b_0c_1(-1)(b_{-1}c_0)\,|\downarrow\rangle = [L_1, L_{-1}]\,|\downarrow\rangle = 2L_0\,|\downarrow\rangle = 2a^g\,|\downarrow\rangle$$

so $a^g = -1$ and the energy of $|\downarrow\rangle$ (and $|\uparrow\rangle$) is -1. Since vacuum is invariant, we define $|0\rangle = b_{-1}\,|\downarrow\rangle$.

Ghost number current: $j^g = -\not{:} bc(z) \not{:}$ but this is not a primary as it gives a term on the T OPE of $-3/z^3$. This is related to

$$\nabla_a j^a = -\frac{3}{8} R^2$$

which integrated, for the sphere gives $\sum q = \frac{3}{2}\chi = 3$ so sum of charges is 3.

We need a theory such that the symmetry is not broken so we need correlation functions with $\sum q = 3$.

Example I.3.

$$\langle 0|\, c(z_1)c(z_2)c(z_3)\,|0\rangle = \sum_{p,q,r} \langle\downarrow|\, b_1 c_p \frac{1}{z_1^p} c_q \frac{1}{z_2^q} c_r \frac{1}{z_3^r} b_{-1}\,|\downarrow\rangle\, z_1 z_2 z_3 = z_{12} z_{23} z_{31}$$

where we used $\langle\downarrow|\uparrow\rangle = 1$

1. BRST symmetry

From the bosonic string, before we fix the gauge we have:

$$Z = \int \mathcal{D}x\mathcal{D}\gamma\mathcal{D}B\mathcal{D}c\mathcal{D}b e^{-S_p - S_{bc} - S_B}, \quad S_B = i\int d^2\sigma\sqrt{\gamma}B^{ab}(\gamma - \hat{\gamma})$$

This action present BRST symmetry

$$\delta_B \phi^i = -i\epsilon c^\alpha \delta_\alpha \phi^i$$

$$\delta_B B_A = 0$$

$$\delta_B b_A = \epsilon B_A$$

$$\delta_B c^\alpha = i\epsilon \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma$$

which can be generated from the current

$$j_B =: cT^X : + \frac{1}{2} : cT^g : + \frac{3}{2}\partial^2 c$$

whose charge is

$$\begin{aligned} Q_B &= \oint \frac{\mathrm{d}z}{2\pi i} j_B(z) \\ &= \sum_{n=-\infty}^{\infty} c_n L^X_{-n} + \sum_{m,n=-\infty}^{\infty} \left(\frac{m-n}{2}\right) \not{:} c_m c_n b_{-m-n} \not{:} + a^B c_0 \end{aligned}$$

$$\begin{aligned} \{Q_B, b_m\} &= \oint_{|z|<\epsilon} \frac{\mathrm{d}z}{2\pi i} \oint_{|w-z|<\epsilon} \frac{\mathrm{d}w}{2\pi i} z^{1-m} j_B(w) b(z) \\ &= \oint_{|z|<0} \frac{\mathrm{d}z}{2\pi i} z^{1-m} \left(T^X(z) + T^g(z)\right) = L^X_m + L^g_m \end{aligned}$$

Using the expression for Q_B in terms of oscillators we have

$$\{Q_B, b_0\}|0\rangle \otimes |\downarrow\rangle = a_B|0\rangle \otimes |\downarrow\rangle = (L^X_0 + L^g_0)|0\rangle \otimes ||\downarrow\rangle = (-1)|0\rangle \otimes |\downarrow\rangle$$

giving us $a^B = -1$. This also matches with an explicit computation of the anticommutator using the expresion of L and b_n c_n we already know. The BRST symmetry is unbroken iff $c = 0$ to which it allow us to define the cohomology and therefore the physical states,

$$Q_{BRST}\,|\text{Phys.}\rangle = 0, \quad b_0\,|\text{Phys.}\rangle = 0$$

where the second condition is the Siegel gauge which means physical states are built on top of the ghost vacuum state $|\downarrow\rangle$(as opposed to $|\uparrow\rangle$). It turns out this is the only physically sensible choice for our later interest in the computation of scattering amplitudes. We will not really explain why this is the case here, but as an aside note this apparent symmetry breaking isn't really one since b and c were never on the same footing to begin with, since they transform differently under conformal transformations.

Now, consider the class of states described as oscillators acting on $|0\rangle \otimes |\downarrow\rangle$. BRST invariance is then

$$Q_B|\text{phys}\rangle = Q_B|\psi\rangle \otimes |\downarrow\rangle = \sum_{n=0}^{\infty} c_{-n}\left(L^X_n - \delta_{n,0}\right)|\psi\rangle \otimes |\downarrow\rangle$$

Hence to every physical state in Old Covariant Quantization there corresponds a BRST invariant state. We will not do it here, but it is possible to show the converse statement is also true, i.e. that every BRST cohomology class contains a representative of the form

D. Scattering Amplitudes

We compute scattering amplitudes of gauge invariant objects. These include topology and marked points

$$A^{(0)}(q_i) = \sum_{\text{topologies}} g_s^{-2\chi} \int \frac{[DX][D\gamma]}{\text{Vol diff} \times \text{Weyl}} e^{-S_P[\gamma,X]} \prod_{i=1}^{n} \int \mathrm{d}^2\sigma_i \sqrt{\gamma} \mathcal{V}_{q_i}(\sigma_i)$$

for the closed string amplitude on the sphere we have a residual Symmetries $SL(2,\mathbb{C})$. Fixing markings gives the Fadee-Popov

$$\Delta_{FP}(\gamma,\sigma_i) \int D\hat{\gamma} \int Dg\delta(\gamma - \hat{\gamma}^g) \prod_{i=1}^{3} \delta^{(2)}(\sigma_i - \hat{\sigma}_i^g) = 1$$

Computing the FP determinat and notting that $\delta(\delta\sigma_i) \to c(\hat{\sigma}_i)\bar{c}(\hat{\sigma}_i)$ which gives the full amplitude:

$$\begin{aligned} A^{(0)}(q_i) &= e^{-2\Phi_0} \int [DX][Db][Dc] e^{-S_P[X]-S_g[b,c]} \prod_{i=1}^{3} c(\hat{\sigma}_i)\bar{c}(\hat{\sigma}_i)\mathcal{V}_{q_i}(\hat{\sigma}_i) \prod_{j=4}^{n} \int \mathrm{d}^2\sigma_j \mathcal{V}_{q_j}(\sigma_j) \\ &= g_s^{-2} \left\langle \prod_{i=1}^{3} c(\hat{\sigma}_i)\bar{c}(\hat{\sigma}_i)\mathcal{V}_{q_i}(\hat{\sigma}_i) \prod_{j=4}^{n} \int \mathrm{d}^2\sigma_j \mathcal{V}_{q_j}(\sigma_j) \right\rangle \end{aligned}$$

Notice it has exactly 3 ghosts required for anomaly cancelation.

Example I.4. *Tachyon 4−pt function. We consider* $V_k(z,\bar{z}) = g_s e^{ik\cdot X(z,\bar{z})}$. *All the sectors decouple so we already computed the c ghost sector* $3-pt$ *function. We consider*

$$g_s^4 \langle : e^{ik_1 X(z_1,\bar{z_1})} : e^{ik_2 X(z_2,\bar{z_2})} : e^{ik_3 X(z_3,\bar{z_3})} : e^{ik_4 X(z_4,\bar{z_4})} : \rangle$$

to compute this we have computed the OPEs and resumed. Instead we consider the path integral as insertions of $\prod_1^4 V_k$ *are current insertions* $J = \sum_1^4 \delta(z - z_i)k_i$. *Using*

$$\begin{aligned} &\int [DX] e^{-S_P[X] + i\int \mathrm{d}^2 z J_\mu(z,\bar{z}) X^\mu(z,\bar{z})} \\ &= \delta^{(D)}\left(\int \mathrm{d}^2 z J^\mu(z,\bar{z})\right) \exp\left\{-\frac{1}{2}\int \mathrm{d}^2 z \int \mathrm{d}^2 z' J_\mu(z,\bar{z}) \langle \hat{X}^\mu(z,\bar{z}) \hat{X}^\nu(z',\bar{z}') \rangle J_\nu(z',\bar{z}')\right\} \end{aligned}$$

we get

$$g_s^4 \langle : e^{ik_1 X(z_1,\bar{z_1})} : e^{ik_2 X(z_2,\bar{z_2})} : e^{ik_3 X(z_3,\bar{z_3})} : e^{ik_4 X(z_4,\bar{z_4})} : \rangle = \delta^D(\sum k_i^\mu) e^{\frac{\alpha'}{2}\sum_{i<j} k_i k_j \log|z_{ij}|^2}$$

so all together:

$$A^{(0)}(k_i) = g_s^2 \delta^{(D)}(\sum_i k_i^\mu)|z_{12}|^2|z_{23}|^2|z_{31}|^2 \int \mathrm{d}^2 z_4 \prod_{i<j}^{4} |z_{ij}|^{\alpha' k_i\cdot k_j}$$

and after integration, we get the Virasoro-Shapiro amplitude:

$$A^{(0)}(k_i) \propto g_s^2 \delta^{(D)}(\sum_i^4 k_i^\mu) \frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)} \tag{9}$$

Analysis: In QFT, a pole in amplitude corresponds to a physical state e.g. for a ϕ^3 *theory,*

$$A(s,t,u) = -g^2\left(\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right)$$

indicating a state of mass m.

In our amplirude, we have

$$A^{(0)}(k_i) \sim \sum_{n=0}^{\infty} -\frac{R_{2n}(t)}{s - m_m^2}, \quad m_n^2 = \frac{4}{\alpha'}(n-1)$$

getting a pole for each level.

$R_{2n}(t) = \sum_{k=0}^n a_k^n c_{2k}^{\frac{D-3}{2}}(x)|_{s=m_n^2}$. *Unitarity requires* $a_k^n \geq$ *which requires* $D = 26$ *and* $m_0^2 = -\frac{4}{\alpha'}$

E. Background fields

Whenever we have a theory we should put all the allowed interactions. The string can be introduced self interactions, $\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)$ which allows to put in the Polyakov action

$$S_p = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{ab}\partial_a X^\mu\partial_b X^\nu G_{\mu\nu}(X)$$

for $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}$, $h_{\mu\nu} << 1$ then expanding allow us to see that coupling with $G_{\mu\nu}$

$$e^{-S_P} \to e^{-S_P}\left(1 - \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{\gamma}\gamma^{ab}\partial_a X^\mu\partial_b X^\nu h_{\mu\nu}(X) + \dots\right)$$

So to linear order the effect of a metric is consistent with making an insertion of a particular operator into the path integral. In particular, if we choose a plane wave perturbation,

$$h_{\mu\nu}(X) \propto s_{\mu\nu}(k)e^{ik\cdot X}$$

with $s_{\mu\nu}$ is a symmetric traceless tensor, we recognize the appearance of an insertion of the vertex operator for a graviton state in the path integral:

$$V_k = g_c\int \mathrm{d}^2\sigma\mathcal{V}_k(\sigma) = g_c\int \mathrm{d}^2\sigma\sqrt{\gamma}\gamma^{ab}s_{\mu\nu}\partial_a X^\mu\partial_b X^\nu$$
$$\Rightarrow_{\text{conformal gauge}}\int \mathrm{d}^2 z s_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu e^{ik\cdot X}$$

This is just the graviton by the state operator correspondance $\partial X^\mu\bar{\partial}X^\nu e^{ikX} \leftrightarrow \alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|k\rangle$ and the infinite coupling is precisely the self interacting part that generates the non-linear behavour of graviton (Deser computation of a massless spin 2 particle interacting with itself by it's stress-tensor and again and again.... $\implies$ GR). Infact, using Riemann Normal corrdinates we can write $G_{\mu\nu}(X) \sim \eta_{\mu\nu} - \frac{1}{3}Y^\rho Y^\tau\alpha' R_{\mu\rho\nu\tau}(X_0) + \dots$.

What about other couplings? At level 2 ($\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0\rangle$) we have had identified the graviton $G_{\mu\nu}$ the symmetric rep, the Kalb-Ramond field $B_{\mu\nu}$ the anti-simmetric (2-from) and the dilaton Φ as the trace. The Kalb-Ramond couples through the density $\varepsilon^{ab}\partial_a X^{[\mu}\partial_b X^{\nu]}B_{\mu\nu}$ and the dilaton through the only scalar $\alpha'\Phi(X)R^{(2)}$ which means Φ is subleading and integrating the zero mode of it gives a topological coupling $g_s = e^{-\phi\chi}$ depending on the Euler characteristic.

Demanding conformal invariance in the interacting theory requires $T = \sum\beta_i = 0$ so we are looking to a theory whose states are annihilated by T i.e. conformal invariant. Similarly to the graviton expansion, we can find the perturbation expansion which allow us to compute diagrams and their counter terms, we have β^G, β^B, β^Φ

$$\begin{aligned}
\beta^G_{\mu\nu}/\alpha' &= R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\omega}H^{\lambda\omega}_\nu + O(\alpha') \\
\beta^B_{\mu\nu}/\alpha' &= -\frac{1}{2}\nabla^\omega H_{\omega\mu\nu} + \nabla^\omega\Phi H_{\omega\mu\nu} + O(\alpha') \\
\beta^\Phi/\alpha' &= \frac{D-26}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \nabla_\omega\Phi\nabla^\omega\Phi - \frac{1}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + O(\alpha')
\end{aligned}$$

In order for $\beta = 0$ we define an action for the fields whose EOM gives $\beta = 0$,

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^D X\sqrt{-G}e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{2}H^2 + \frac{D-26}{3\alpha'}\right]$$

F. Superstrings and SCFT

Two types, target space and worldsheet. The former is a intrinsic SUSY formalism for "super fields", see lectures on SUSY. For the later, consider $d = 2$ fermions ψ_α which is a spinor of $2d$ Lorentz. Let ρ satisfiying the clifford algebra and the minimal representation is Weyl-Majorana (Chirality and reality resp.) so they have 1 degree of freedom. we use both chirality for parity,

$$S = \frac{1}{8\pi}\int d^2\sigma i\bar{\Psi}\rho^a\partial_a\Psi, \Psi = \begin{pmatrix}\psi_- \\ \psi_+\end{pmatrix}$$

after wick rotation,

$$S = \frac{1}{8\pi}\int \mathrm{d}^2w \left[\psi\bar{\partial}_{\bar{w}}\psi + \bar{\psi}\partial_w\bar{\psi}\right]$$

which gives $(h,\bar{h}) = (1/2, 0)$ for ψ and, since parity interchange them $\bar{\psi}$ has the opposite. We have interesting solutions for the EOM even for closed strings. Boundary gives $\psi(z) = \psi(e^{2\pi i}z)$ (Naveu-Schwartz) and $\psi(z) = -\psi(e^{2\pi i}z)$ (Ramond). For each left and right chiral fermions we could have either. Gives 4 possibilities.

Analysis: $S = \int d^2z\psi\partial\psi + h.c.$ gives OPE $\psi\psi \sim \frac{1}{z-w}$, $T = \frac{-1}{2} : \psi\partial\psi$ and computing central charge $c = \frac{1}{2}$. On the plane:

$$\psi(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{\psi_r}{z^{\frac{1}{2}+r}}, \quad \text{(N-S)}$$

$$\psi(z) = \sum_{r\in\mathbb{Z}} \frac{\psi_r}{z^{\frac{1}{2}+r}}, \quad \text{(Ramond)}$$

as ramond sector they need to pick up a phase. This gives $\{\psi_r, \psi_{-s}\} = \delta_{r,s}$. On the Ramond sector $\psi_0^2 = 1$ which gives ground state double degenerate.

The superstring is then the bosonic string with the fermionic string:

$$S = \frac{1}{4\pi}\int \mathrm{d}^2z \left[\frac{2}{\alpha'}\partial X^\mu\bar{\partial}X_\mu + \psi^\mu\bar{\partial}\psi_\mu + \bar{\psi}^\mu\partial\bar{\psi}_\mu\right]$$

SUSY is encoded in the super current

$$G(z) := i\sqrt{\frac{2}{\alpha'}}\psi_\mu\partial X^\mu$$

which is a primary of $h = 3/2$. Using the OPE's of the bosonic and fermionic string, we find the transformations of ψ and X under this symmetry.

$$\delta_\epsilon X^\mu(z,\bar{z}) = -\epsilon(z)\sqrt{\frac{\alpha'}{2}}\psi^\mu(z)\delta_\epsilon\psi^\mu(z) = \epsilon(z)\sqrt{\frac{2}{\alpha'}}\partial X^\mu(z)$$

In total we have $c = D + \frac{D}{2}$ from the bosonic and fermionic parts and

$$\begin{aligned} G(z)G(w) &= -\frac{2}{\alpha'} : \psi_\mu\partial X^\mu(z) :: \psi_\nu\partial X^\nu(z) : \\ &\sim -\frac{2}{\alpha'}\left[\frac{: \partial X^\mu(z)\partial X_\mu(w) :}{z-w} - \frac{\alpha' : \psi^\mu(z)\psi_\mu(w) :}{(z-w)^2} - \frac{\alpha'}{2}\frac{D}{(z-w)^3}\right] \\ &\sim \frac{D}{(z-w)^3} - \frac{2}{\alpha'}\frac{: \partial X^\mu\partial X_\mu(w) :}{z-w} - \frac{: \psi^\mu\partial\psi_\mu(w) :}{z-w} \\ &= \frac{D}{(z-w)^3} + 2\frac{T(w)}{z-w} \end{aligned}$$

And in general,

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

$$T(z)G(w) \sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}$$

$$G(z)G(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w}$$

Supper current mode decompostions:

$$G(z) = \sum_{r\in\mathbb{Z}} \frac{G_r}{z^{\frac{3}{2}+r}}, \quad \text{(N-S)}$$

$$G(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{G_r}{z^{\frac{3}{2}+r}}\,, \quad \text{(Ramond)}$$

which plugging in the oscilators of X and ψ gives the G_r oscilattors together with L_n. Note the definition of L has the $a^{NS/R}$ ambiguity which can be computed from redundancy of normal term and regularizing by using the relations: (OPEs implies them)

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n} \\
[L_m, G_r] &= \left(\frac{1}{2}m - n\right) G_{m+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r,-s}
\end{aligned}$$

For the NS, $a^N S\,|0\rangle = L_0\,|0\rangle \sim \{G_{1/2}, G_{-1/2}\}\,|0\rangle = 0$ as both annihilate vacuum. Hence, $a^{NS} = 0$. For the Ramond, since ψ_0 themselves fournish a representation of the clifford alegbra, then the vacuum is a $D-$dimensional spinor.

$$\psi_0^\mu|\alpha\rangle = \sum_\beta \frac{1}{\sqrt{2}}\Gamma^\mu_{\beta\alpha}|\beta\rangle$$

where Γ are the ordinary Γ matrices so

$$a^R|\alpha\rangle = L_0|\alpha\rangle = \frac{1}{2}\left[\{G_0, G_0\} + \frac{D}{8}\right]|\alpha\rangle = \frac{D}{16}|\alpha\rangle$$

where we plugged $c = 3D/2$.

Here constraint $G^{a\alpha} = 0$ automatically imply $T = 0$. We have $\gamma_{\mu\nu}$ coupling with the stress tensor and $\psi_{a\alpha}$ coupling with the supercurrent $G_{a\alpha}$.

1. BRST quantization

We have to gauge fix for the ψ action, this introduce superghosts.

$$S_{gf} = \frac{1}{2\pi}\int d^2z b\bar{\partial}c + \beta\bar{\partial}\gamma + c.c.$$

with $(h_b, h_c, h_\beta, h_\gamma) = (2, -1, 3/2, -1/2)$. It is easy to check that $c_{\beta\gamma} = 11$ so the critical dimension of the superstring is $c_T = D + D/2 - 26 + 11 = 0$ which gives $D = 10$.

For the spectrum we require

$$\begin{aligned}
[L_m - \tilde{a}^{\mathrm{R,NS}}\delta_{m,0}]|\mathrm{phys}\rangle &= 0 \quad m \geq 0 \\
G_r|\mathrm{phys}\rangle &= 0 \quad r \geq 0
\end{aligned}$$

$$\tilde{a}^{\mathrm{NS}} = \frac{1}{2}\,, \qquad \tilde{a}^{\mathrm{R}} = \frac{5}{8}\,.$$

and define the vacuum states $|k\rangle \otimes |0\rangle_{NS}$ and $|k\rangle \otimes |\alpha\rangle_R$ with

$$\begin{aligned}
\alpha_n^\mu|k\rangle &= \sqrt{2\alpha'}k^\mu\delta_{n,0}|k\rangle\,, \quad n \geq 0 \\
\psi^\mu_{\frac{1}{2}+n}|0\rangle_{\mathrm{NS}} &= 0\,, \quad n \geq 0 \\
\psi_n^\mu|\alpha\rangle_{\mathrm{R}} &= \delta_{n,0}\frac{1}{\sqrt{2}}\sum_\beta \Gamma^\mu_{\beta,\alpha}|\beta\rangle_{\mathrm{R}} \quad , \quad n \geq 0
\end{aligned}$$

How to determine the constants $\tilde{a}^{NR,R}$? Well, a trick for determining $\tilde{a}^{R}$ is as follows. The supercurrent constraints contain no normal-ordering constant (this follows from their expression in terms of modes). From the super Virasoro algebra we know that

$$G_0^2 = \left(L_0 - \frac{5}{8}\right)$$

where we plugged in $D = 10$. Consistency then implies that

$$\tilde{a}^{\mathrm{R}} = \frac{5}{8}.$$

Unfortunately in the NS sector no such trick is available. For the bosonic string the constraint was $(L_0 - 1)|\mathrm{phys}\rangle = 0$, where the -1 came from the contribution of the ghost sector. For the superstring, accounting also for the superghosts, this comes out to be -1/2. Alternatively, we could also have checked that for this particular value there are extra massless and null states in the spectrum. The L_0 constraint above can be stated as:

$$\alpha' m^2 = N - \frac{1}{2} \quad , \quad N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r \in \mathbb{N} - \frac{1}{2}} r \psi_{-r} \psi_r \; , \quad \text{NS sector}$$

$$\alpha' m^2 = N \quad , \quad N = \sum_{n=0}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=1}^{\infty} r \, \psi_{-r} \cdot \psi_r \; , \quad \text{Rsector}$$

Note that the N operator contains both bosonic as well as fermionic modes but still plays the same role, essentially computing the total scaling dimension.

2. Spectrum

For the NS sector for the first state ($|k\rangle \otimes |0\rangle_N$) the only non trivial physical condtion is L_0 which applying

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \not{\alpha}_{m-n} \cdot \alpha_n \not{} + \frac{1}{4} \sum_{r \in \mathbb{Z} + \frac{1}{2}} (2r - m) \not{\psi}_{m-r} \cdot \psi_r \not{} + \frac{1}{2} \delta_{m,0}$$

which indeed gives the above level-matching constraint. That is, the first state is still a tachyon.

For the first state $|\Psi\rangle = \xi_\mu \psi^\mu_{-1/2} |k\rangle \otimes |0\rangle_{NS}$, the physical conditions are

$$0 = \alpha' \left(N - \frac{1}{2}\right) = m^2$$
$$0 = G_{\frac{1}{2}} |\Psi\rangle = \alpha_0 \cdot \psi_{\frac{1}{2}} \left(\xi_\mu \psi^\mu_{-\frac{1}{2}} |k\rangle\right) = \xi \cdot \alpha_0 |k\rangle \sim \xi \cdot k$$

which gives a $m^2 = 0$ particle and transverse to k.

For the Ramond sector consider a linear combination of vacuua: $|\Omega\rangle_R = \sum_\alpha U^\alpha \, |k\rangle \otimes |\alpha\rangle_R$. applying G_0, $0 \sim \alpha_{\alpha,\beta k_\mu \Gamma^\mu_{\beta\alpha} U^\alpha |k,\beta\rangle} = \not{k} \cdot U \, |...\rangle$ so U is a massless spinor (satisfies Dirac equation).

Massless spinor in $D = 10$. We can have Majora-Wey and they have $2^{10/2-1-1} = 8$ components each. One dirac has $16 = 8_+ \oplus 8_-$ which each massless chiral spinor has $SO(8)$ little group. In even dimensions we define $\Gamma^{D+1} = \Gamma^0 ... \Gamma^{D-1}$ to define the chirality projections.

Fermionic parity:

$$(-1)^{\mathcal{F}} = \begin{cases} -e^{i\pi F}, & NS \\ \Gamma^{11} e^{i\pi F}, & R \end{cases} \tag{10}$$

where $F = \sum_{r>0} \psi_{-r} \psi_r$ counts fermionic insertions. We can use this projection to separate theory in two consistent ($[L_0, (-1)^{\mathcal{F}}]$) sectors the $+$ and $-$ eigenvalues this allow us to get rid of tachyons.

G. Type IIA and IIB closed superstrings

Spectrum is the tensor product of what we constructed on both $Vir \times \bar{Vir}$ with level matching $\frac{\alpha'}{4}m^2 = N - \frac{1}{2}$ on the NS sector and $\frac{\alpha'}{4}m^2 = N$ on Ramond sector.

- $NS \times NS$: We have a massless vector 8 d.o.f. (first bosonic excited state non-tachyonic) which gives

$$8 \otimes 8 = \underbrace{1}_{\text{dilaton}} + \underbrace{28}_{\text{KR}} + \underbrace{35}_{\text{graviton}}$$

  which is precisely what we had for the open string at level 2.

- $R \times NS$: By convention left ground state always + chiral. Then $8_+ \otimes 8 = 8_- \oplus 56_+$ which are the dilatino and gravitino respectively.

- $R \times R$: Two types:

  1. **Type IIA** $8_+ \otimes 8_- = 8 + 56$ vector and 3-form respectively.
  2. **Type IIB**: $8_+ \otimes 8_+ = 1 + 28 + 35_+$ that are scalar, $2-$form and a self dual $4-$form.

We restrict to those states satisfying:

$$\begin{aligned} &\text{Type IIA:} \qquad (-1)^{F_L} = 1 \quad , \quad (-1)^{F_R} = \begin{cases} -1 & \text{Ramond sector} \\ +1 & \text{NS sector} \end{cases} \\ &\text{Type IIB:} \qquad (-1)^{F_L} = 1 \quad , \quad (-1)^{F_R} = 1 \end{aligned}$$

The effect of this projection is as follows. In both cases, note that on the leftmoving and rightmoving sides we restrict to NS states with positive parity. Since the vacuum of the NS sector, the tachyon, had negative parity, it is therefore removed. So is every state in that sector which has an even number of fermionic modes acting on it. In the Ramond sector our choices are such that we keep the ground states (which were massless) in both sectors. We then mod out every state in the spectrum that has even numbers of fermion modes acting on them.

• It is a consistent truncation of the full theory. This means that there is no danger of starting with states in the truncated spectrum, evolving in time, and finding states we dropped out. This is simply because the Hamiltonian on the world sheet- namely $L_0 + \bar{L}_0 -$commutes with the projection, as it has an even number of fermionic modes.

• It removes the tachyon (finally!).

• The vertex operators corresponding to states in different GSO sectors are not mutually local. Essentially this means that it doesn't make sense to consider a theory where they simultaneously appear. By the state operator correspondence this maps to a `truncation in the states.`

• After the GSO projection the theory has spacetime supersymmetry. A basic check of this is that the number of bosonic and fermionic states in spacetime are the same. We can see this to be true in the massless sector:

$$\begin{aligned} &\qquad\quad \text{Bosons} \qquad\qquad\qquad\qquad \text{Fermions} \\ &\text{Type IIA:} (1+28+35) + (8+56) + 2\times(8+56) \\ &\text{Type IIB:} \underbrace{(1+28+35)}_{\text{NS}\times\text{NS}} + \underbrace{(1+28+35)}_{\text{R}\times\text{R}} + \underbrace{2\times(8+56)}_{\text{NS}\times\text{R}+\text{R}\times\text{NS}} \end{aligned}$$

The number of states indeed match. It can be checked that this is true at all levels This is not proof that the spacetime theory has supersymmetry, but it is a necessary condition. They are described at low energies by IIA/B SUGRA. String field theory gives a UV completion of the theory. In terms of fields we have:

$$\begin{aligned} IIA :& C_\mu, C_{\mu\nu\rho} \to F^{(2)}, F^{(4)} \\ IIB :& C^{(0)}, C^{(2)}_{\mu\nu}, C_{\mu\nu\rho\sigma} \to F^{(1)}_\mu, F^{(3)}_{\mu\nu\rho}, F^{(5)}_{\mu\nu\rho\sigma\delta} \end{aligned} \tag{11}$$

for $F^{(5)}$ self dual.

H. Compactifications and dualities

1. Kaluza-Klein reduction

Introduce a compactified coordinate y. This modifies the equation of motion to have discrete momenta in that direction: $\phi = e^{ip\cdot x + i\frac{n}{R}y}$ so $p^2 \sim m + (n/R)^2$ gives a mass spectrum as seen from a $D-2$ perspective. For $R >> 0$ you will need big energy to probe the theory while

$$\Phi(x,y) = \sum_{n\in\mathbb{Z}} \phi_n(x) e^{i\frac{ny}{R}}$$

action becomes

$$\int d^{D-1}x \int_0^{2\pi R} \mathrm{d}y \left[\partial_M \Phi(x,y)\partial^M \Phi(x,y) + m^2\Phi^2(x,y)\right] = 2\pi R \int d^{D-1}x \sum_{n\in\mathbb{Z}} \left[\partial_\mu\phi(x)\partial^\mu\phi(x) + \left(m^2 + \frac{n^2}{R^2}\right)\phi^2(x)\right]$$

so we descrived the local D dimensional theory in terms of a non-local $D-1$ theory. We can do the same for higher reps. Vectors beceome transversal and 2-forms allow us to write

$$\mathrm{d}s^2 = G_{MN}\mathrm{d}x^M\mathrm{d}x^N = G_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + G_{yy}(\mathrm{d}y + A_\mu\mathrm{d}x^\mu)^2$$

where interestingly the EH action in the coordinates expanded gives gravity, electromagentism and massless scalar all coupled together.

$$R^D \to R^{D-1} - 2e^{-\sigma}\nabla^2 e^\sigma - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}, \quad G_{yy} \equiv e^{2\sigma}$$

The symmetry under diffeomorphisms of the y coordinate has now mutated into the gauge symmetry of A_μ. Indeed, the transformation

$$\begin{aligned} y &\to y' = y + \lambda(x^\mu) \\ \mathrm{d}y &\to \mathrm{d}y' = \mathrm{d}y + \partial_\mu\lambda\mathrm{d}x^\mu \end{aligned}$$

can be absorbed by a change in σ and a shift

$$A_\mu \to A_\mu - \partial_\mu\lambda$$

Thus Kaluza-Klein reductions are telling us that seemingly distinct physical phenomena in one spacetime dimension may be simply consequences of not seeing the true structure of spacetime which may contain very small dimensions.

2. T duality for Closed Strings

We consdier a $X^{25} \sim X^{25} \pm 2\pi R$ so field is periodic. This means that X aquires a winding number w at the "end" $\sigma \to 2\pi + \sigma$ with total difference: $X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi\omega R$.

The consequence is that the boundary conditions are modified. First, as we saw earlier, momentum is quantized on n/R. Second is that the level oscilators, which before had $\tilde{\alpha} = \alpha$ is modified to $\tilde{\alpha} - \alpha \sim \omega R$. This modifies the level matching condition to:

$$m^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2) + \frac{n^2}{R^2} + \omega^2\frac{R^2}{\alpha'}. \tag{12}$$

This suggest there is a symmetry of the mass spectrum by taking $R \to \alpha'/R$ and $n \to \omega$. This is the "mapping big circles to small circles" which can be achived by $\alpha_0 \to -\alpha_0$. This suggest a full map

$$\begin{aligned} X_L^{25} &\to X_L^{25} \\ X_R^{25} &\to -X_R^{25} \\ R &\to \tilde{R} = \frac{\alpha'}{R}. \end{aligned} \tag{13}$$

This is the **T-duality** different sectors of the theory are related.

3. For Open strings

For open strings there is no winding for Neumann but there is for dirichlet boundary conditions. This give some equivalent (in addition to the momentum quantization). More precisely, for Neumann, there is only quantization of momentum. While dirichlet strings must end on some "D-brane" extened object which allws them to have a winding
Under T-duality we then map: Neumann $\leftrightarrow$ Dirichlet: $x_1 = \tilde{x}^{25}$ and $x_1^{25} - x_2^{25} = 2\pi\alpha' P^{25} = 2\pi\alpha' \frac{n}{R} = 2\pi n\tilde{R}$

4. D-branes

The dynamics of D-branes are described by the open strings themselves. For instance, you may recall that the open string had a massless vector excitation at the first level $\xi_\mu \alpha_{-1}^\mu |k\rangle$ for $\xi \cdot k = 0$ or $\xi_\mu \psi^\mu_{-\frac{1}{2}} |0\rangle_{NS}$ for superstrings.

We can parametrize the D_p-branes by the open strings

$$V_\mu \to \begin{cases} A_{\hat{\mu}} \equiv V_{\hat{\mu}}, & \hat{\mu} = 0, \dots p \\ \Phi_i \equiv V_i, & \hat{i} = p+1, \dots D \end{cases}$$

for Φ_i parametrizing the shape of the $D-$brane and A_μ might be interpreted as the gauge fields mediating endpoint interactions. Indeed,

$$S = S_p \pm \frac{i}{2\pi\alpha'} \int_{\partial M} d\tau \partial_\tau X^{\hat{\mu}} A_{\hat{\mu}}$$

This is the same as the coupling of a charged particle to a gauge field. Thus the tips of the open string act as "quarks" living in the D-brane worldvolume and sourcing a gauge field. The reason for why the tips should have opposite charges is that recall that the coupling to gauge field arises to ensure gauge invariance under gauge transformation of the Kalb-Ramond field, and in particular requires that the boundaries contribute with opposite charges

Multiple coincident branse can be added, say N. All states aquire extra quantum numbers $|k, \mathrm{Osc.}, i, j\rangle$ for $i, j = 1, ..., N$ for strings ending on i and starting on j. **Chan-paton** factors. Hence, now every state is N^2-fold degenerate. $|\lambda^a\rangle = \sum_{ij} \lambda^a |ij\rangle$ such that they are normalized to 1. This condition is just the $\mathrm{tr}\{\lambda^a \lambda^b\} = \delta^{ab}$ so λ^a are generators of $U(N)$ symmetry. Then A_μ^{ij} become non abelian.

Remark I.5. *An open string in adjoint cannot fuse to closed as vector does not have a singlet. 2 open strings can.*

Consider gauge field Background on compactified X^{25} direction (only space filling D-branes i.e. D_{24}). Let $A_{25} = \frac{1}{2\pi R} \mathrm{Diag}\{\theta_1, ..., \theta_N\}$ is a pure gauge but is not trivial as the couping with the string is non-vanishing.

What is the effect of this? Let us discuss the zero mode of the string, $x(\tau) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma X^{25}$, and consider modes with Chan-Paton factors ij going in opposite directions. Then

$$\frac{i}{2\pi\alpha'} \int \mathrm{d}\tau \partial_\tau X^{25}(\tau, \sigma = 0) \langle i | A_{25}(\tau, \sigma = 0) | i \rangle \to \frac{1}{2\pi\alpha'} \int \mathrm{d}\tau \; \dot{x} \frac{\theta_i}{2\pi R}$$

The action for this mode for such a string is

$$S_0 = \frac{1}{2\pi\alpha'} \int \mathrm{d}\tau \left[\frac{1}{2}\dot{x}^2 + i\frac{\theta_i - \theta_j}{2\pi R}\dot{x} \right]$$

In particular the canonical momentum is now

$$\pi := \frac{\delta S_0}{\delta \dot{x}} = \frac{1}{2\pi\alpha'} \left[\dot{x} + i\frac{\theta_i - \theta_j}{2\pi R} \right] = -i\left(p^{25} - \frac{\theta_i - \theta_j}{2\pi R} \right)$$

It is $i\pi$ that generates translations of the wavefunction in the x coordinate and thus in an eigenstate of momentum we have

$$\pi = -i\frac{n}{R} \Rightarrow p^{25} = \frac{n}{R} + \frac{\theta_i - \theta_j}{2\pi R}$$

Thus, switching on a background gauge field component on the circle modifies the momentum. When we go to the T-dual picture that component of the gauge field becomes the scalar matrix, $A_{25} = \Phi_{25}$ and the momentum becomes the winding which is now fractional, i.e. after moving from $\sigma = 0$ to $\sigma = \pi$ the coordinate has now changed by

$$\tilde{X}^{25}(\tau, \pi) - \tilde{X}^{25}(\tau, 0) = 2\pi n\tilde{R} + (\theta_i - \theta_j)\tilde{R}$$

Thus it must be the case that the D-brane i and the D-brane j are separated along X^{25} by an amount proportional to the diagonal elements of the Φ matrix.

Let us consider the low energy theory on a stack of D-branes. The massless modes of the open string are made up by the gauge field on the worldvolume of the D-branes and the scalar fields describing their transverse coordinates. These fields are described by a $U(N)$ gauge theory under which the scalars transform in the adjoint representation (i.e. they are matrices). As we switch on vev for the scalar fields the D-branes separate and the strings which stretch between them pick up a mass. Such strings have $i = j$, and thus the off-diagonal components of the gauge field pick up a mass. Thus the U(N) symmetry gets broken to $U(1)^N$. This is nothing but the Higgs phenomenon in the gauge theory.

Upshot:

T-duality maps big circles to small circles for the closed strings. This means a very big $R \to \infty$ compactified dimension is mapped to a one less dimension. $D_{D-1} \to D_{D-2} \to ... \to D_{D-p}$ until a point where the Everytime we do this we localize further dimensions so that the D-brane reduces in dimensionality. But then this suggests that before we started dualizing at all there was already a D-brane present, a spacefilling D25 brane. It is because of the presence of this brane that the purely Neumann open strings were allowed in the first place.

In the open string perspective, a space filling D_{25} brane has only gauge fields A after T duality it has $A^{\hat{\mu}}, \Phi$ Φ scalar. The existence of a vev on the scalar Φ is equivalent to say that there has been a higgsing as massless modes $A^{\hat{\mu}}$ aquire a $p = \frac{n}{R} + \Delta\theta/R$ mass term.

5. *Open string scattering*

As a check that indeed the A^μ modes constitute a gauge field, let us consider a scattering amplitude between gauge field and two tachyons whose vertex operators are:

$$\lambda^{a_1}\xi_\mu \partial_y X^\mu e^{ik_1\cdot X}, \quad \lambda^{a_{2,3}} e^{ik_{2,3}\cdot X}$$

with $k_1^2 = 0$ and $k_2^2 = k_2^2 = \frac{1}{\alpha'}$. As for the open string, the world sheet is conformal to a disk now (before the cylinder was conformal to the sphere which gave a g_s^{-2} and here the disk will give g_s^{-1} as $\xi = 1$), or equivalently the upper half-plane where the real line is where the gauge fields leave and interact with the tachyons. Conformal transformations allow us to put the three points along the real line (not six as we are on the disk and before was on the sphere) in any where but they cannot modify the ordering of the points on the real axis. Thus we must sum over these. There are two inequivalent orderings.

For the Chan-Paton factors, the boundary of the worldsheet couples to the gauge fields:

$$\langle \left(e^{\int_{y_n}^{y_1} A} \right)_{i_1 i_n} V^{a_n}_{i_n j_n}(y_n) \dots V^{a_2}_{i_2 j_2}(y_2) \left(e^{\int_{y_1}^{y_2} A} \right)_{j_2 i_1} V^{a_1}_{i_1 j_1}(y_1) \rangle$$

but we have turned off the fields so we get only $\mathrm{tr}\{\lambda^a \lambda^b \lambda^c\}$ or $b \leftrightarrow c$. for the two orientations. The computation of the vertex operators can be done with the OPE of ∂X with e^{ikX} and the resulting $\langle e^{ik_1 X} e^{ik_2 X} e^{ik_3 X} \rangle$ can be done with the current trick (we need the correlator of X in the half plane which can be done with the image method). Ghosts part is only $\langle c(z_1)c(z_2)c(z_3) \rangle$ as computed before. The amplitude using kinematics gives:

$$A(\xi_1, k_1, k_2, k_3) \propto \underbrace{\mathrm{Tr}[\lambda^{a_1}[\lambda^{a_2}, \lambda^{a_3}]]}_{f^{a_1 a_2 a_3}} \delta(\sum k_i) \xi \cdot (k_2 - k_3)$$

This is indeed the correct three point coupling between a gauge boson and two scalars charged in the adjoint of $SU(N)$

I. Effective action of $D-$branes

Using the T duality we can find the effective action of $D-$branes. The brane couples to the graviton G and Kalb-Ramond B fields and A (as a wilson line). The action has a gauge symmetry if both B and A transform to cancel them so the strength tensor of this symmetry must be present in the effective action. One can turn a non trivial gauge field that $A_2 = X^1 F_{12}$. Under T duality, since $\tilde{X}^2 \sim A \sim X^1 F$ then $G \sim F$ so they must be in the action. This gives:

$$S = -T_p \int \mathrm{d}^{p+1}\xi e^{-\Phi} \sqrt{-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}$$

Here $X^\mu(\xi)$ describe the embedding of the D-brane in spacetime, as a function of the $p+1$ worldvolume coordinate ξ. Accordingly the felds appearing above are

$$G_{ab}(\xi) = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}(X(\xi)) \,, \quad B_{ab}(\xi) = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}(X(\xi)) \,,$$

One D-brane expansion around it's effective field gives a scalar with a free abelian gauge boson.

$$S_{\text{eff}} = 4\pi^2(\alpha')^2 T_p \int \mathrm{d}^p x \left(\frac{1}{2}\partial\Phi^i \cdot \partial\Phi_i + \frac{1}{4}F_{ab}F^{ab}\right)$$

For N coincident branse there is no action but we can guess the effective action

$$S_{\text{eff}} = 4\pi^2(\alpha')^2 T_p \int \mathrm{d}^p x \, \mathrm{Tr}\left(D_a\Phi^i D^a\Phi_i + \frac{1}{4}F_{ab}F^{ab} + \frac{1}{4}[\Phi^i,\Phi^j][\Phi_i,\Phi_j]\right)$$

as some of the transverse components become scalars and the others are non-abelian. The ϕ^4 interaction comes from $[A, A]$ of the gauge field which should map to $[\phi, \phi]$ under T duality.

1. *Types of D-branes in supestrings*

Recall the EM duality. $A \to F \to * \to \tilde{A}$ for $d \to * \to d^{-1}$. An $A^{(p)}$ object couples to p objects and $*A$ to $D-p-4$. For types IIA we have :

- $C^{(1)} \leftrightarrow D_0$ brane (arrow means couples to),
- $C^{(3)} \leftrightarrow D_2$brane
- $*C^{(1)} = \tilde{C}^{(7)} \leftrightarrow D_6$brane
- $*C^{(3)} = \tilde{C}^{(5)} \leftrightarrow D_4$ brane

For types IIB we have :

- $C^{(0)} \leftrightarrow D_{-1}$ brane (arrow means couples to)
- $C^{(2)} \leftrightarrow D_1$brane
- $C^{(4)} \leftrightarrow D_3$brane
- $*C^{(0)} \leftrightarrow D_7$brane
- $*C^{(2)} \leftrightarrow D_5$brane note $C^{(4)}$ is the self dual. $T_d \sim \frac{1}{g_s}$

J. One loop amplitudes

We need to define CFTs on the torus. Note that for the point particle we have a similar story: On the torus $X(\tau) \sim X(\tau+1)$ and, since $e' d\tau' = e d\tau$, fixing gauge $e = \ell$ then gives

$$\tau' = c + \ell^{-1}\int_0^\tau d\tilde{\tau} e$$

. There is still some c residual gauge freedom undetermined so gauge is not completeley fixed (so we need to fix it in FP procedure) and, if we fix $\tau \in [0,1]$ then ℓ is not free, is given by the above equation. Equivalentely, if we fix $e = 1$ then period is fix to be $\ell = \int_0^\tau d\tilde{\tau} e$. No choice of both.

In the torous this is the idea of the "Teichumuller" parameter. We can fix periodicity but not the metric to be of $d\bar{w}dw$ form at the same time. Instead, fixing periodicity gives

$$ds^2 = |d\sigma^1 + \tau d\sigma^2|^2$$

or fixing metric gives a periodicity $(\sigma, \tau) \sim (\sigma, \tau) + 2\pi[m(1,0) + n(\tau_1, \tau_2)]$.

There are two CKV since I can shift origin and, I have modular invariance. This gives $PSL(2,\mathbb{Z})$ and τ goes only on the fundamental domain F_0 as they are the inequivalent configurations (due to $SL(2,\mathbb{Z})$ invariance).

The partition function with the usual game of gluing gives (and allowing possible gluing with twists $\langle\psi_2| Re^{-HT} |\psi_1\rangle$) for $H = L_0 + \tilde{L}_0 - \frac{c+\bar{c}}{24}$ (dilation operator is the generation of evolution on the plane and in the cylinder there is a nontrivial schwarzian), $R = e^{2\pi i\tau_1(L_0-\tilde{L}_0)}$ (Rotation of cilinder state of $2\pi\tau_1$ angle) and $T = 2\pi\tau_2$ (time in the cilinder from ψ_1 to ψ_2 state),

$$Z(\tau) := (q\bar{q})^{-\frac{c}{24}} \operatorname{Tr}\left[q^{L_0}\bar{q}^{\tilde{L}_0}\right] \qquad q := \exp(2\pi i\tau)$$

Example I.6. *We compute Z for the free boson. We devide the cmoputation in the modes α_n as they factorize and the anti-holo is exactly the same and $L_0 - \bar{L_0} = 0$ so the R is abscent. We have:*

$$\operatorname{tr}\{q^{L_0}\}_0 = iV_F \int \frac{dk^d}{(2\pi)^d} \langle k| e^{-2\pi\tau_2 L_0} |k\rangle$$

Using the state-operator correspndance we know $|k\rangle \leftrightarrow e^{ikX}$ so $h = \frac{1}{2}\alpha' k^2$ (remember we computed the OPE $T(z)e^{ikX} \sim he^{ikX}/z^2$) integrtating gives $iV_d(4\pi^2\alpha'\tau_2)^{-d/2}$.

Now for the α_n oscillators:

$$\operatorname{tr}\{q^{L_0}\}_n = \sum \langle 0| \alpha_n^m q^{L_0} \alpha_{-n}^m |0\rangle = \sum_{m=0}^{\infty} q^{mn} = \frac{1}{1-q^n}$$

Thus,

$$Z(\tau) = iV_D[(4\pi^2\alpha'\tau_2)^{-\frac{1}{2}}|\eta(\tau)|^{-2}]^D, \quad \eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n).$$

Example I.7. *simiarly, for the bc CFT,*

$$(q\bar{q})^{-\frac{c}{24}}\, Tr[q^{L_0}\bar{q}^{\bar{L}_0}] = 4(q\bar{q})^{\frac{1}{12}} \prod_{n=1}^{\infty} |1+q^n|^4$$

where the 4 comes from 4 ground states, the $\frac{1}{12}$ from $c = 26$ and q^{L_0} has -1 on ground state, and m series of oscilators stop at level q^2 as they are fermionic and 4 expnent as there are $b, c, \bar{b}, \bar{c}$.

This corresponds to anti periodic boundary conditions for the fermions. this is because when we take a fermion around a circle we pick up a minus sign. To see this, think about two fermions, holding one fixed and taking the other one for a loop, we pick up a sign since going around the loop must be equivalent to anticommuting the fermions. In the trace we are merely propagating the fermion with the Hamiltonian so we pick up this sign. But the b,cghosts arise from the Fadeev-Popov determinant and should have the periodicity of the coordinate transformations, which were periodic. To fix this we insert into the trace an operator $(-1)^F$ which by definition anticommutes with all ghost fields. In this case $Z_{bc} = 0$ to get a non-vanishing result we must insert ghosts violatin ghost number as we saw before the theory must have to be a full symmetry.

This is:

$$\langle c(w_1)b(w_2)\bar{c}(\bar{w}_3)\bar{b}(\bar{w}_4)\rangle_{\text{torus}} = (q\bar{q})^{\frac{13}{12}} \operatorname{Tr}\left[(-1)^F c_0 b_0 \bar{c}_0 \bar{b}_0 q^{L_0}\bar{q}^{\bar{L}_0}\right]$$

The zero mode insertions kill every ground state except $|\uparrow\uparrow\rangle$, so now

We get

$$Z(\tau) = (q\bar{q})^{\frac{1}{12}} \prod_{n=1}^{\infty} |1-q^n|^4 = |\eta(\tau)|^4$$

(10.31)

where the sign of the q^n has flipped because of the $(-1)^F$.

Recall the partitoin function is

$$\int \frac{[\mathrm{d}\gamma]}{\text{vol diff} \times \text{Weyl}}[DX]\exp[-S_P[\gamma, X]]$$

Making sense of the division by the volume of the gauge group requires introducing the Fadeev-Popov determinant. In the end we are left with an integral over inequivalent metrics labeled by their moduli together with ghost insertions. cinsertions are related to conformal killing vectors: there were 6 for the sphere and here we have 2. A novelty in the torus case is that we can also end up with b insertions. Such insertions arise whenever the space of metrics is non-trivial, i.e. there are moduli. Here we have two moduli, τ and $\bar{\tau}$, so we expect two insertions. In general insertions take the form

$$B_m = \frac{1}{4\pi}\int \mathrm{d}^2 w\, b_{ww}(w)\partial_m \gamma_{\bar{w}\bar{w}}$$

for m a moduli (recall $|d\sigma^1 + \tau d\sigma|^2$). All together:

$$\begin{aligned} Z_{\text{torus}} &= \int_{F_0} \frac{\mathrm{d}\tau \mathrm{d}\bar{\tau}}{4\tau_2}\left\langle b(0)\bar{b}(0)c(0)\bar{c}(0)\right\rangle_{\text{torus}} \\ &= iV_D \int_{F_0} \frac{\mathrm{d}\tau\mathrm{d}\bar{\tau}}{4\tau_2^2}(4\pi^2\alpha'\tau_2)^{-\frac{D-2}{2}}|\eta(\tau)|^{-4\frac{D-2}{2}} \end{aligned}$$

Integrating over the fundamental domain eliminates the IR divergences (present in the particle as we probe arbitrary low energies).

2. General torous partition functions:

More generally the torus amplitude takes the following form. Assume there are D noncompact flat dimensions. The ghosts cancel the contributions from two of them and we are **left with**

$$Z_{\text{torus}} = iV_D \int_{F_0} \frac{\mathrm{d}\tau\mathrm{d}\bar{\tau}}{4\tau_2^2}\left(4\pi^2\alpha'\tau_2\right)^{-\frac{D-2}{2}} Z'_{\text{CFT}}(\tau)$$

where the prime on the CFT partition function means it includes only the non-zero modes of the transverse $D-2$ bosons, excludes ghosts, and has central charge 24,

$$Z'_{\text{CFT}}(\tau) = \sum_i q^{h_i - 1}\bar{q}^{\bar{h}_i - 1}$$

Now for a physical interpretation of the result. Consider the analogous vacuum one loop `amplitude for a point particle:`

$$\begin{aligned} Z &= \log[\det\left[\frac{1}{2}(P^2+m^2)\right]]^{-\frac{1}{2}}] = iV_D \int \frac{\mathrm{d}^D k}{(2\pi)^D}\int_0^\infty \frac{\mathrm{d}\ell}{2\ell}\exp\left[-(k^2+m^2)\ell/2\right] \\ &= iV_D \int_0^\infty \frac{\mathrm{d}\ell}{2\ell}(2\pi\ell)^{-D/2}\,\exp\left(-m^2\ell/2\right) \end{aligned}$$

this contain IR divergences as integral goes to arbirtrary low energies (ℓ is lenght of worldline)

Sum over particles with

$$m_i^2 = \frac{2}{\alpha'}(h_i + \bar{h}_i - 2), h_i = \bar{h}_i$$

and using $\delta_{h_i,\bar{h}_i}$ integral representation we get

$$\begin{aligned} Z &= iV_D \int_0^\infty \frac{\mathrm{d}\ell}{2\ell}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}(2\pi\ell)^{-D/2}\sum_i \exp\left[-\frac{\alpha'}{2}\ell(h_i+\bar{h}_i-2) + i\theta(h_i - \bar{h}_i)\right] \\ &= iV_D \int_R \frac{\mathrm{d}\tau\mathrm{d}\bar{\tau}}{4\tau_2}(4\pi^2\alpha'\tau_2)^{-D/2}\sum_i q^{h_i-1}\bar{q}^{\bar{h}_i-1} \end{aligned}$$

where we have set $q = \exp(2\pi i\tau)$ as before with

$$\tau := \theta + i\,\frac{\ell}{\alpha'}$$

We see that this is almost the same as the string theory result with a crucial difference: the `integration region is`

$$R: \quad \tau_2 > 0\,, \quad |\tau_1| < \frac{1}{2}$$

This includes the UV region $\tau_2 = \ell/\alpha' \to 0$ at the origin of the UV divergence. This should be contrasted with F_0 for string theory where $|\tau| \geq 1$.

For the bosonic string, there is in fact another source of divergence of the torus amplitude. Note that

$$\left|q^{h_i - 1}\bar{q}^{\bar{h}_i - 1}\right| = \exp\left(-2\pi\tau_2 m^2\right)$$

Since the integral over F_0 includes the region $\tau_2 \to \infty$, we have an exponential divergence if there are states with $m^2 < 0$, i.e. tachyons. This is the case for the bosonic string, but not for the superstring which has thus a perfectly finite one-loop amplitude. The torus amplitude illustrates a general loop result which is that when we approach any boundary of the moduli space of metrics the asymptotics are actually controlled by the lightest states, and thus long distance physics.

K. Other types of Superstrings and dualities between them

SUSY requires $d \leq 11$ as we need spin-2 at most. For a theory without gravity requires $d \leq 6$. This gives some SUGRA (low energy effective theories):

- In $10D$ SUGRA:
  1. Type IIA and $IIB \to \mathcal{N} = 2$.
  2. $\mathcal{N} = 1$: SUGRA type I (chiral) either: $SO(32)$ or $E_8 \times E_8$ coupled to $\mathcal{N} = 1$ SYM.
- In $D = 11$ SUGRA: There is a unique theory, this will be the low laying sector of M theory.

1. Type I superstrings

Unoriented strings, $\Omega : \sigma \to \ell - \sigma$ world sheet parity project the orient strings out.

- Closed string loses KR fields + massive states
- Open losses massels vector $\alpha_{-1}\,|k\rangle$

Chan paton then gives $\lambda^{a^T} = -\lambda^a$ so we get $SO(N)$ gauge symmetry (it is also possible $Sp(N)$). Tadpole cancelations on unoriented open strings with $SO(32)$.

2. Type I superstrings: Heterotic strings

SUSY on left bosonic on right. ...

3. M theory

M theory

L. AdS/CFT

Ads

II. EXACTLY SOLVABLE MODELS (2D CFTS)

Most of the notes here are from Sylvain Ribault course and lecture notes [?].

A. Preliminary definitions

The virasoro modes are the modes of the stress tensor in the cilinder:

$$L_n = \int \frac{dw}{2\pi} e^{-inw} T(w)$$

Going from the cilinder to the plane $z = e^{-iw}$ we get

$$L_n = \int \frac{dz}{2\pi i z} z^n z^2 T(z)$$

In particular, this means $T(z)V_\Delta(y) = \sum_n \frac{L_n}{(z-y)^{n+2}} V_\Delta(y)$. Note this chart covers $z = 0$ while taking $\tilde{z} = e^{inw}$ for w on the cylinder,

$$L_n = \int \frac{d\tilde{z}}{2\pi i \tilde{z}} \tilde{z}^{-n} \tilde{z}^2 T(\tilde{z})$$

covering ∞.

To write $\Delta_(r,s)$ for any $r, s \in \mathbb{N}^*$, we rewrite the conformal dimension in terms of the momentum P defined by

$$\boxed{c = 1 - 6\left(\beta - \beta^{-1}\right)^2}.$$

$$\boxed{\Delta = \frac{c-1}{24} + P^2} \quad \boxed{P_{(r,s)} = \frac{1}{2}\left(r\beta - s\beta^{-1}\right)}.$$

It follows that the null vector $L_{\langle r,s\rangle} V_{\Delta_{(r,s)}}$ is a primary state of dimension

$$\boxed{\Delta_{(r,s)} + rs = \Delta_{(r,-s)}}\ .$$

- For $\beta^2 \in \mathbb{C}\backslash\mathbb{Q}$, the Verma module $\mathcal{V}_\Delta$ has a null vector if and only if $\Delta = \Delta_{(r,s)}$ for some $r, s \in \mathbb{N}^*$, in which case it has only 1 null vector.
- For $\beta^2 \in \mathbb{Q}$ however, a Verma module can have several null vectors

If $\beta^2 = \frac{q}{p}$ with p, q coprime integers, we have the identity

$$\Delta_{(r+p,s+q)} = \Delta_{(r,s)}, \quad \Delta_{(r,s)} = \Delta_{(-r,-s)}.$$

Assuming the the null vector $L_\langle r_1, s_1\rangle V_{\Delta_{(r,s)}}$ generates a Verma module of the form $\mathcal{V}(r_2, s_2)$, then there exists another null vector $L_{\langle r_2,s_2\rangle} L_{\langle r_1,s_1\rangle} V_{\Delta_{(r,s)}}$ in the spectrum. All null vectors of Verma modules are of the type $L_\langle r_1, s_1\rangle V_{\Delta_{(r,s)}}$ or $L_{\langle r_2,s_2\rangle} L_{\langle r_1,s_1\rangle} V_{\Delta_{(r,s)}}$.

Remark II.1. *This happens if* $pq > 0$, or if $pq < 0$ *with* $\left|\frac{r_1}{|p|}\right| \neq \left|\frac{s_1}{|q|}\right|$ *and* $\left\lceil\frac{r_1}{|p|}\right\rceil \neq \left\lceil\frac{s_1}{|q|}\right\rceil$, *the dimension* $\Delta_{(r_1,-s_1)}$ *of this null vector is of the type* $\Delta_{(r_2,s_2)}$with $r_2, s_2 \in \mathbb{N}^*$, *leading to another null vector* $L_{\langle r_2,s_2\rangle} L_{\langle r_1,s_1\rangle} V_{\Delta_{(r,s)}}$.

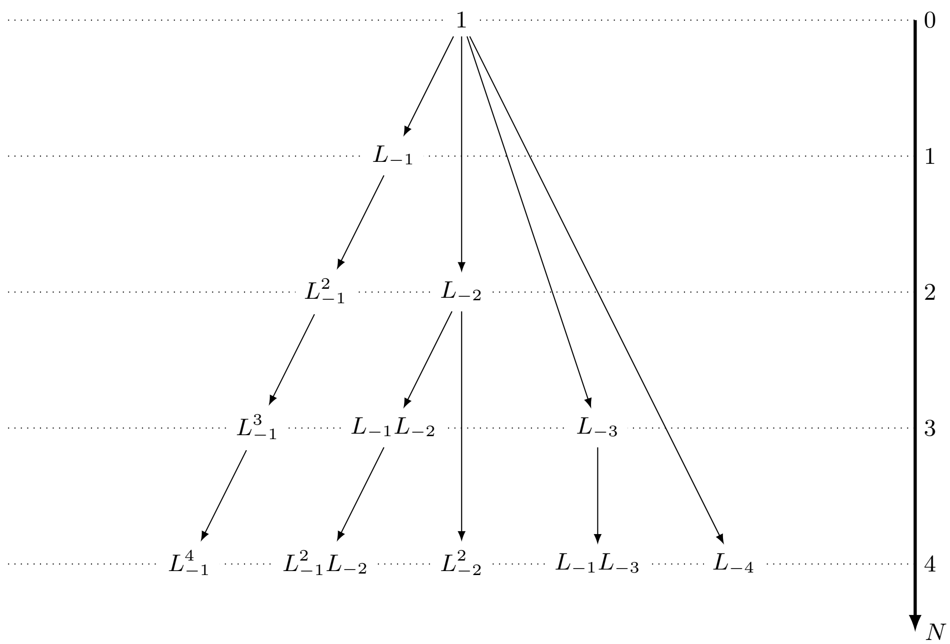
All of them are of this form: Indeed, given a null vector of the type $L_\langle r_3, s_3\rangle L_{\langle r_2,s_2\rangle} L_{\langle r_1,s_1\rangle} V_{\Delta_{(r_1,s_1)}}$,

$$\exists r_4, s_4 \in \mathbb{N}^*,\ L_{\langle r_3,s_3\rangle} L_{\langle r_2,s_2\rangle} L_{\langle r_1,s_1\rangle} V_{\Delta_{(r_1,s_1)}} = L_{\langle r_4,s_4\rangle} V_{\Delta_{(r_4,s_4)}}\ .$$

To see this, let $\epsilon, \eta \in \{+,-\}$ *be the signs such that* $P_{(-r_1,s_1)} = \epsilon P_{(r_2,s_2)}$ *and* $P_{(r_2,-s_2)} = \eta P_{(r_3,s_3)}$, *and let* $r_4 = |\epsilon r_1 - r_2 + \eta r_3|$ *and* $s_4 = |\epsilon s_1 + s_2 + \eta s_3|$, *then* $\sum_i = 1^3 r_i s_i = r_4 s_4$, *i.e. the 2 null vectors in Eq. have the same level.*

To summarize, numbers of null vectors depend a lot on β^2. If $\beta^2 < 0$, then $\{\Delta_{(r,s)}\}_{r,s\in\mathbb{N}^*}$ is bounded from above, and Eq. implies that a Verma module can only have finitely many null vectors. If $\beta^2 \in \mathbb{Q}_{>0}$ then by Eq.the existence of a null vector implies the existence of infinitely many others:

Value of β^2	$\mathbb{C}\backslash\mathbb{Q}$	$\mathbb{Q}_{>0}$	$\mathbb{Q}_{<0}$
Value of c	generic	$c \leq 1, c \in \mathbb{Q}$	$c \geq 25, c \in \mathbb{Q}$
#null vectors in $\mathcal{V}_{\Delta_{(r,s)}}$	1	∞	finite



(14)

FIG. 1: Sketch of Verma module by displaying all its basis states up to the level $N = 4$, with arrows representing the action of the Virasoro generators $L_{-1}, L_{-2}, L_{-3}, L_{-4}$: States such as $L_{-2}L_{-1}V = L_{-1}L_{-2}V - L_{-3}V$ are linear combination of elements of our basis, and are therefore not displayed.

1. Representations

Given a vector $V_{\Delta(r,s)}$ we generate a Verma module $\mathcal{V}_{\Delta(r,s)}$ by acting with creation modes (descendants) L. We know there will be a null vector at level $r \times s = N$, given by some combination of level N L operators, which we call $L_{(r,s)}V_{\Delta(r,s)}$. We therefore take the quotiend of the Verma module by its null vectors to take irreps. As for $\beta^2 \in \mathbb{Q}$ there will be at most (due to the commutation realations we can reduce two product in one) two kinds of null vectors, $L_{(r,s)}V_{\Delta(r,s)}$ and $L_{(r,s)}L_{(r',s')}V_{\Delta(r,s)}$.

Hence, we could write the degenerate quotient: Draw tree and relations of Delta(r,s) RC: Draw tree and relations of Delta(r,s)

$$R^d_{(r,s)} = \frac{\mathcal{V}_{\Delta(r,s)}}{V_{\Delta(r,s)+rs}}$$

and the irreducible, fully degenerate quotient (all null vectors vanish)

$$R^f_{(r,s)} = \frac{\mathcal{V}_{\Delta}}{V_{\Delta(r,s)+rs} + V_{\Delta(r_2,s_2)+r_2s_2}}$$

That is, in this irreducible represntation the Shapovalov form is non-degenerate.

2. OPE constraints

In 2d the conformal group gives a lot of information. Consider the OPE of 2 primary fields.

$$V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) = \sum_{L\in\mathcal{L}} C^{\Delta,L}_{\Delta_1,\Delta_2}(z_1,z_2)LV_\Delta(z_2)\,,$$

where $\mathcal{L}$ is the set of creation operators. Applying on both sides of $\frac{1}{2\pi i}\oint_{z_1,z_2} dy(y-z_2)^{n+1}T(y)$, where the integration contour encloses both z_1 and z_2 we can read some constraints the coefficients C. We find

$$\left(L_n^{(z_2)} + z_{12}^{n+1}\partial_{z_1} + (n+1)z_{12}^n\Delta_1\right)V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) = \sum_{L\in\mathcal{L}} C^{\Delta,L}_{\Delta_1,\Delta_2}(z_1,z_2)L_nLV_\Delta(z_2)\,.$$

For $n\geq 1$, we use $L_n^{(z_2)}V_{\Delta_2}(z_2)=0$, and perform the OPE again on the left-hand side. The action of the differential operator on the OPE coefficients is now known, and we find

$$\sum_{L\in\mathcal{L}} C^{\Delta,L}_{\Delta_1,\Delta_2}z_{12}^{|L|+n}(\Delta+|L|+n\Delta_1-\Delta_2)LV_\Delta = \sum_{L\in\mathcal{L}} C^{\Delta,L}_{\Delta_1,\Delta_2}z_{12}^{|L|}L_nLV_\Delta\,.$$

Extracting the coefficient of z_{12}^N for some $N\in\mathbb{N}$, we find

$$L_nW_N = \theta_{N,n}W_{N-n} \quad \text{with} \quad \begin{cases} W_N = \sum_{|L|=N} C^{\Delta,L}_{\Delta_1,\Delta_2}LV_\Delta \\ \theta_{N,n} = N-n+n\Delta_1-\Delta_2+\Delta \end{cases},$$

These linear equations for the OPE coefficients $C^{\Delta,L}_{\Delta_1,\Delta_2}$are called OPE Ward identities. Let us look for a solution $\left(f^{\Delta,L}_{\Delta_1,\Delta_2}\right)_{L\in\mathcal{L}} = \left(f^L\right)_{L\in\mathcal{L}}$.

Example II.2. *For $N=1=n$,*

$$W_1 = C^{\Delta,L_{-1}}_{\Delta_1,\Delta_2}L_{-1}V_\Delta(z_2)$$
$$W_0 = C^{\Delta,1}_{\Delta_1,\Delta_2}V_\Delta(z_2)$$

And therefore

$$L_1W_1 = \underbrace{2\Delta}_{S_{L_{-1},L_1}=2L_0} V_\Delta(z_2)C^{\Delta,L_{-1}}_{\Delta_1,\Delta_2} = (\Delta-\Delta_1-\Delta_2)C^{\Delta,1}_{\Delta_1,\Delta_2}V_\Delta(z_2)$$

Hence, we can then solve for $C^{\Delta,L_{-1}}$ as $C^{\Delta,1}$ can be reabsorbed in the overall constant of the OPE, provided the Shapovalov form at level 1 is non-degenerate i.e. it's non vanishing. Note that if $\Delta=0$ i.e. the identity appears in the OPE of $V_{\Delta_1}(z_1)V_{\Delta_2}(z_2)$ then $\Delta_1=\Delta_2$.

Similarly, for the second level, $N=2$, we require $n=1$ and $n=2$ as for this case we have a basis $\{L_{-1}^2, L_{-2}\}$ and hence two variables to find. The only extra W we need is

$$W_2 = C^{\Delta,L_{-1}^2}L_{-1}^2V_\Delta + C^{\Delta,L_{-2}}L_{-2}V_\Delta$$

which leads to the system

$$\begin{aligned} N=2=n, &\qquad L_1(C^{\Delta,L_{-1}^2}L_{-1}^2V_\Delta + C^{\Delta,L_{-2}}L_{-2}V_\Delta) = (\Delta+1+\Delta_1-\Delta_2)C^{\Delta,L_{-1}}_{\Delta_1,\Delta_2}L_{-1}V_\Delta(z_2) \\ N=2=n+1, &\quad L_2(C^{\Delta,L_{-1}^2}L_{-1}^2V_\Delta + C^{\Delta,L_{-2}}L_{-2}V_\Delta) = (\Delta+2\Delta_1-\Delta_2)C^{\Delta,1}V_\Delta(z_2) \end{aligned}$$

Acting by L_1 to get rid of the first equation RHS operator, we get the linear system:

$$\begin{pmatrix} S_{L_{-1}^2,L_1^2} & S_{L_{-2},L_1^2} \\ S_{L_{-1}^2,L_2} & S_{L_{-2},L_2} \end{pmatrix}\begin{pmatrix} C^{\Delta,L_{-1}^2} \\ C^{\Delta,L_{-2}} \end{pmatrix} = \begin{pmatrix} (\Delta+1+\Delta_1-\Delta_2)S_{L_1,L_1}C^{\Delta,L_{-1}} \\ (\Delta+2\Delta_1-\Delta_2)C^{\Delta,1} \end{pmatrix},$$

which has solutions if the second level Shapovalov form is invertible.

We conclude

$$V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) = \sum_{L\in\mathcal{L}} C^{\Delta}_{\Delta_1,\Delta_2}\left(V_\Delta + \frac{\Delta+\Delta_1-\Delta_2}{2\Delta}L_{-1} + C^{\Delta,L^2_{-1}}L^2_{-1} + C^{\Delta,L_{-2}}L_{-2} + O(|L|=3)\right)V_\Delta(z_2)\,,$$

Iterating this equation, we can compute L^*W_N for any creation operator $L\in\mathcal{L}_N$. On the other hand, $L^*W_N = \sum_{L'\in\mathcal{L}_N} S_{L,L'}f^{L'}V_\Delta$ where $S_{L,L'}$ is the Shapovalov form. This leads to the linear system

$$\forall L\in\mathcal{L}_N,\ \sum_{L'\in\mathcal{L}_N} S_{L,L'}f^{L'} = g^L \quad \text{with} \quad g^{L_{-n_1}L_{-n_2}\cdots L_{-n_k}}_{\Delta,\Delta_2,\Delta_1} = \prod_{i=1}^{k}\theta_{\sum_{j=i}^{k} n_j, n_i}\,.$$

Assuming the Shapovalov form is invertible at level N, the solution is

$$\boxed{f^{\Delta,L}_{\Delta_1,\Delta_2} = \sum_{|L'|=|L|} S^{-1}_{L,L'}(\Delta)g^{L'}_{\Delta,\Delta_2,\Delta_1}}\,.$$

For example, we have $g^{L_{-1}} = \theta_{1,1}$, $g^{L^2_{-1}} = \theta_{1,1}\theta_{2,1}$ and $g^{L_{-2}} = \theta_{2,2}$ with

$$\theta_{1,1} = \Delta+\Delta_1-\Delta_2 \quad , \quad \theta_{2,1} = \theta_{1,1}+1 \quad , \quad \theta_{2,2} = \Delta+2\Delta_2-\Delta_1\,.$$

3. Fusion

We want to encode the OPE structure in an algebraic (categoric operation) called fusion. We, therefore, define the fusion by the following rules:

Since at level 1 we have moded out everything but the identity and, the OPE of identity with V_Δ is proportional to V_Δ the fusion is defined by:

$$R^d_{(1,1)}\times\mathcal{V}_\Delta = \mathcal{V}_\Delta, \quad dimHom(\mathcal{V}_{\Delta_1}\times\mathcal{V}_{\Delta_2}, R^d_{(1,1)}) = \delta_{\Delta_1,\Delta_2}.$$

So $R^d_{(1,1)}$ acts as the identity and, $Hom(R,R')\equiv Hom(R\times R', R^d_{(1,1)})$. Moreover, we have $Hom(R\times R', R'')\equiv Hom(R\times R'', R')$ where here the dimension of this object is the **fusion multiplicity** of R'' in the fusion product $R\times R'$. To determine higher fusions we check them with a generator of rational representations.

$$R^d_{\langle r,\rangle s}\times\mathcal{V}_P = \sum_{i=-\frac{r-1}{2}}^{\frac{r-1}{2}}\ \sum_{j=-\frac{s-1}{2}}^{\frac{s-1}{2}} \mathcal{V}_{P+i\beta+j\beta^{-1}} \tag{15}$$

Let us define $c = 1-6(\beta-\beta^{-1})^2$ $\Delta = \frac{c-1}{24}+p^2$ and $p_{(r,s)} = \frac{1}{2}(\beta r-\beta^{-1}s)$. We define then the dimensions $\Delta_{(r,s)}$, $r,s\in\mathbb{N}^*$ For primaries of these type of dimension there are null vectors at $L_{(r,s)}\left|\Delta_{(r,s)}\right\rangle$ at level rs. To get irreps we mod out all the null vectors.

Modding $L_{(r,s)}\left|\Delta_{(r,s)}\right\rangle$ gives a degenerate represnetation and modding the two $L_{(r_i,s_i)}$ gives the fully deegenerate $V^{d/f}$ respectively.

A good way to encode the inner prodduct of states is the **Shapalov form**.

$$\langle LV_\Delta|L'V_\delta\rangle = \begin{cases} s_{L,L'}(\Delta), & else \\ 0, & |L|\neq|L'| \end{cases}$$

Fields are classified as primaries and descendants. $L_{-1}=\partial$. Energy momentum is equivalent to the Virasoro algebra. For primaries,

$$T(y)V_\Delta = \frac{\Delta V_\Delta(z)}{(y-z)^2} + \frac{\partial V_\Delta(z)}{(y-z)} + \ldots$$

whre ... denote regular terms and,

$$T(y)T(z) = \frac{c/2}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{(y-z)} + ...$$

T is the descendant of the identity field: $L_{-2}I(z) = T(z) + ...$.

Ward identities. We can insert a virasoro mode L_n acting on y by:

$$\int_y \frac{dz}{2\pi i}(z-y)^{n+1}T(z) \iff T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n^{(z_2)}}{(y-z)^{n+2}}$$

Inserting on both $\int_{z_1,z_2}$ (to act on both) sides of the OPE $V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) = \sum_L C^{\Delta,L}_{\Delta_1,\Delta_2}(z_1,z_2)LV_\Delta(z")$ to find a conformal ward identity.

$$(L_n^{(z_2)} + (n+1)z_{12}^n\Delta_1 + z_{12}^{n+1}\partial_1)V_{\Delta_1}V_{\Delta_2} = \sum_L C^{\Delta,L}_{\Delta_1,\Delta_2}(z_1,z_2)L_n^{z_2}LV_\Delta \tag{16}$$

For global virasoro operators $n = 0, 1, -1$ we have the usual C scaling dependence, translation invariance and so on, only not determined is the coefficient.

$$V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) = \sum_L \underbrace{C^{\Delta,L}_{\Delta_1,\Delta_2}}_{Const.} z_{12}^{\Delta+|L|-(\Delta_1+\Delta_2)}LV_\Delta(z_2) \tag{17}$$

For $n > 0$ we can sistematically solve the equations by comparing coefficients of z. This allow us to find the constant f in the OPEs,

$$V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) = \sum_\Delta \underbrace{C^{\Delta}_{\Delta_1,\Delta_2}}_{Const.} z_{12}^{\Delta-(\Delta_1+\Delta_2)}\left(V_\Delta(z_2) + \sum_{L-\mathcal{L}} f^{\Delta,L}_{\Delta_1,\Delta_2} z_{12}^{|L|}LV_\Delta(z_2)\right) \tag{18}$$

This allow us to find the spectrum inside the OPE of two fields (up to the overal constant). For example, for the OPE of V_1 and V_2 to have $V_{<1,1>}$ then $V_1V_2 \sim C^\Delta_{\Delta_1\Delta_2}(2\Delta/(\Delta+\Delta_1-\Delta_2)V_\Delta + L_{-1}V_\Delta + ...)$ so, since $\Delta = 0$, we require $\Delta_1 = \Delta_2$ for this term to appear.

Fusion product is defined by the apperence or not of the OPE. For example, $V_{\Delta_{(2,1)}}$ is in the OPE of V_1V_2 iff $Res_{\Delta=\Delta_{(2,1)}} f^{\Delta,L^2_{-1}}_{\Delta_1,\Delta_2} = 0$ iff

$$(4\Delta_{(2,1)} + 2)(\Delta_{(2,1)} + 2\Delta_1 - \Delta_2) = 3(\Delta_{(2,1)} + \Delta_1 - \Delta_2)(\Delta_{(2,1)} + \Delta_1 - \Delta_2 + 1)$$

iff $p_2 = \pm p_1 \pm \beta/2$. This defines the fusion product $R^d_{\langle 2,1\rangle} \times \nu_p = \sum_\pm \nu_{p\pm\beta/2}$.

B. Spectrum and Models

We define different types of fields: A **Diagonal field** is a spinless field whose OPE with degenerate fields only producces spinless fields. A **Non-diagonal** field is either a nonzero spin field or is related to nonzero spins by taking OPEs with degenerate fields, we have

Name	Notation	Conditions	$(P,\bar{P})$	Representation
Degenerate	$V^d_{\langle r,s\rangle}$	$r,s\in\mathbb{N}^*$	$\left(P_{(r,s)},P_{(r,s)}\right)$	$\mathcal{R}^d_{\langle r,s\rangle}\otimes\bar{\mathcal{R}}^d_{\langle r,s\rangle}$
Fully degenerate	$V^f_{\langle r,s\rangle}$	$r,s\in\mathbb{N}^*$ $\beta^2\in\mathbb{Q}$	$\left(P_{(r,s)},P_{(r,s)}\right)$	$\mathcal{R}^f_{\langle r,s\rangle}\otimes\bar{\mathcal{R}}^f_{\langle r,s\rangle}$
Diagonal	V_P	$P\in\mathbb{C}$	(P,P)	$\mathcal{V}_P\otimes\bar{\mathcal{V}}_P$
Non-diagonal	$V_{(r,s)}$	$r,s\in\mathbb{C}$ $rs\in\frac{1}{2}\mathbb{Z}$	$\left(P_{(r,s)},P_{(r,-s)}\right)$	$\mathcal{V}_{P_{(r,s)}}\otimes\bar{\mathcal{V}}_{P_{(r,-s)}}$ (in general)

(19)

where (r, s) are the Kac indices. Existence of diagonal fields constraints the spectrum. Degenerate fields are special as they generate a tower of fields through their fusion. All togehter we have 7 possible sets of degenerate fields, which we characterize by whether or not they contain $V^d_{\langle 1,1\rangle}, V^d_{\langle 2,1\rangle}, V^d_{\langle 3,1\rangle}, V^d_{\langle 1,2\rangle}$ and $V^d_{\langle 1,3\rangle}$:

$$\emptyset \quad \left\{V^d_{\langle 1,1\rangle}\right\} \quad \left\{V^d_{\langle 1,3\rangle}\right\} \quad \left\{V^d_{\langle 1,2\rangle}\right\} \quad \left\{V^d_{\langle 3,1\rangle}, V^d_{\langle 1,3\rangle}\right\} \quad \left\{V^d_{\langle 3,1\rangle}, V^d_{\langle 1,2\rangle}\right\} \quad \left\{V^d_{\langle 2,1\rangle}, V^d_{\langle 1,2\rangle}\right\} \tag{20}$$

In particular, the set that is generated by $\{V^d_{\langle 2,1\rangle}, V^d_{\langle 1,2\rangle}\}$ is the full set of degenerate fields $\{V^d_{\langle r,s\rangle}\}_{r,s\in\mathbb{N}^*}$. In order to solve CFTs, the existence of degenerate fields matters much more than their presence in the spectrum. We will focus on 2 types of CFTs:

- CFTs with 2 degenerate fields, in the sense that both $V^d_{\langle 2,1\rangle}$ and $V^d_{\langle 1,2\rangle}$ exist.
- CFTs with 1 degenerate field $V^d_{\langle 1,2\rangle}$. Because the invariance under $\beta \to \beta^{-1}$ is broken by choosing $V^d_{\langle 1,2\rangle}$ rather than $V^d_{\langle 2,1\rangle}$, such CFTs depend on β^2 rather than on the central charge c.

For instance, existence of $V^d_{(2,1)}$ constraints $r \in \mathbb{Z}/2$ and $\sum r_i \in \mathbb{Z}$.

1. *Diagonal CFTs with 2 degenerate fields*

1. GMM: Spectrum is all diagonal degenerate fields.

$$\boxed{\mathcal{S}^{\text{GMM}} = \left\{V^d_{\langle r,s\rangle}\right\}_{r,s\in\mathbb{N}^*}} \ . \tag{21}$$

   for $\beta^2 \in \mathbb{C}^*\backslash\mathbb{Q}$. For $\beta^2 \in \mathbb{Q}$ there are relations among fields. This leads to finite spectrum models, the AMM. GMM only exist on the sphere as their tourous partition function are infinite, because $\Delta \to \infty$.

2. Liouville thoery: Let us consider the spectrum of a generalized minimal model with $\beta^2 \in \mathbb{R}_{>0}\backslash\mathbb{Q}$. The momentums $P_{(r,s)}$ are dense in the real line, any diagonal field with a real momentum can therefore be obtained as a limit of degenerate fields,

$$\forall P \in \mathbb{R}\ , \quad V_P = \lim_{\substack{r,s\to\infty \\ P_{(r,s)}\to P}} V^d_{\langle r,s\rangle}\ . \tag{22}$$

   The spectrum of primary fields is formally

$$\boxed{\mathcal{S}^{\text{Liouville}} = \frac{1}{2}\left\{V_P\right\}_{P\in\mathbb{R}}} \ . \tag{23}$$

   If we apply the limit to both fields in the OPE $V^d_{\langle r_1,s_1\rangle}V^d_{\langle r_2,s_2\rangle}$, the resulting OPE is formally given by the degenerate fusion rules with $r_1, s_1, r_2, s_2 \to \infty$, and we obtain

$$V_{P_1}V_{P_2} \sim \frac{1}{2}\int_{\mathbb{R}+i\epsilon} dP\ V_P\ . \tag{24}$$

   The regularized OPE (24) requires that we extend the spectrum to complex momentums. This will turn out to be possible, because correlation functions are analytic in P.

3. AMM: For $\beta^2 = q/p$ we get many relations among degenerate fields. If $q/p < 0$ however there are infinitely many fields as $V^f_{(p+1,1)}$ OPE with itself generates $V^f_{(kp+1,1)}$.

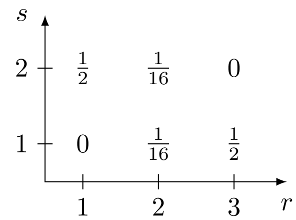
   We hope (and is the case) that for $q/p > 0$ then there are spectrum with finite number of fields. Using the OPE with $V^d_{(2,1)}$ and it's β^{-1} counter part we generate them all. These are defined in a **Kac Table**.

$$\boxed{\mathcal{S}^{\text{AMM}_{p,q}} = \frac{1}{2}\left\{V^f_{\langle r,s\rangle}\right\}_{(r,s)\in K_{p,q}} \quad \text{with} \quad 2 \le q < p \quad \text{and} \quad p, q \text{ coprime}} \ . \tag{25}$$

   The factor $\frac{1}{2}$ accounts for the $\mathbb{Z}_2$ symmetry (**??**). We assume $2 \le q$ for the Kac table to be non-empty, and $q < p$ because $\text{AMM}_{p,q} = \text{AMM}_{q,p}$.

For example, $\mathrm{AMM}_{4,3}$ has the central charge $c = \frac{1}{2}$ and describes some of the observables of the critical Ising model. Let us describe its 3 primary fields, and display their dimensions in the Kac table:

Field	$V^f_{\langle 1,1\rangle} = V^f_{\langle 3,2\rangle}$	$V^f_{\langle 1,2\rangle} = V^f_{\langle 3,1\rangle}$	$V^f_{\langle 2,1\rangle} = V^f_{\langle 2,2\rangle}$
Notation	I	ϵ	σ
Name	Identity	Energy	Spin
Dimension	0	$\frac{1}{2}$	$\frac{1}{16}$



(26)

Example II.3. *Indeed, from* $(p, q) = (4, 3)$ *we deduce* $c_{p,q} = 1 - 6(\beta - \beta^{-1})^2 = \frac{1}{2}$, $\beta^2 = q/p$. *Therefore, we have the relations (using* $p(r, s) = \frac{1}{2}(r\beta - s\beta^{-1})$*)*

$$\begin{aligned} V^f_{\Delta(1,1)} &= V^f_{\Delta(3,2)} = I\,(Identity) \to \Delta_{(r,s)} = \frac{c-1}{24} + p^2_{(r,s)} = 0 \\ V^f_{\Delta(1,2)} &= V^f_{\Delta(3,1)} = \varepsilon\,(Energy) \to \Delta = \frac{1}{2} \\ V^f_{\Delta(2,1)} &= V^f_{\Delta(2,2)} = \sigma\,(spin) \to \Delta = \frac{1}{16} \end{aligned}$$

Now, we compute the fusions:

$$I \times I = I, \quad I \times \varepsilon = \varepsilon, \quad I \times \sigma = \sigma$$

Since $V_{(2,1)} \times V_{(2,1)} = V_{(1,1)} + V_{(3,1)}$ *and also* $V_{(2,1)} \times V_{(2,1)} = V_{(2,2)} \times V_{(2,2)} = V_{(1,1)} + V_{(3,1)} + V_{(1,3)} + V_{(3,3)}$ *since the last two do not appear in both, we conclude:*

$$\begin{aligned} &\sigma \times \sigma = I + \varepsilon \\ &\varepsilon \times \sigma = \sigma \quad (V_{(1,2)} \times V_{(2,1)} = V_{(2,2)}) \\ &\varepsilon \times \varepsilon = I \quad (V_{(1,2)} \times V_{(3,1)} = V_{(3,2)}). \end{aligned}$$

In general, the one with the minimal number of fields is the one that appear.

2. Non-diagonal CFTs with 2 degenerate fields

1. DMM: We look to representations labeled by the Kac table ($\beta = q/p > 0$). Since they are non-diagonal, we require both left and right to conicide with dimensions from the Kac table. A simple ansatz is

$$(r, s) \in \left[\left(-\tfrac{p}{2}, \tfrac{p}{2}\right) \cap \left(\mathbb{Z} + \tfrac{p}{2}\right) \right] \times \left[\left(-\tfrac{q}{2}, \tfrac{q}{2}\right) \cap \left(\mathbb{Z} + \tfrac{q}{2}\right) \right]. \tag{27}$$

This amounts to taking advantage of the identity $\Delta_{(r,s)} = \Delta_{(r+\frac{p}{2}, s+\frac{q}{2})}$ for centering the Kac table at $(r, s) = (0, 0)$, which makes the table invariant under $(r, s) \to (r, -s)$.

- Spin condition requires (look at $V_{(r+p/2, s+q/2)}$) $p \in 2\mathbb{N}^*$ and $q \in 2\mathbb{N} + 1$ in which case $r \in 2\mathbb{Z}$.
- Since q is odd, all non-diagonal fields have $s \in \mathbb{Z} + \frac{1}{2}$ (as $s \in (-q/2, ...q/2)$), and the conservation $\sum s_i$ reduces to the conservation of diagonality:

$$D \times D = D \quad , \quad D \times N = N \quad , \quad N \times N = D. \tag{28}$$

Then, the diagonal sector is generated by taking OPEs of non-diagonal fields which gives $V_{(p/2, q/2)}$ OPE with itself gives a term $V_{(\underbrace{p-1}_{Odd}, \underbrace{q-1}_{Even})}$ so they generate all $V_{(2n+1, m)}$.

We therefore define a D-series minimal model $\mathrm{DMM}_{p,q}$ by the spectrum

$$\boxed{\mathcal{S}^{\mathrm{DMM}_{p,q}} = \frac{1}{2} \left\{ V^f_{\langle r,s\rangle} \right\}_{\substack{(r,s)\in K_{p,q} \\ r \equiv 1 \bmod 2}} \bigcup \frac{1}{2} \left\{ V_{(r,s)} \right\}_{\substack{(r,s)\in(-\frac{p}{2},\frac{p}{2})\times(-\frac{q}{2},\frac{q}{2}) \\ (r,s)\in 2\mathbb{Z}\times(\mathbb{Z}+\frac{1}{2})}}}\,, \tag{29}$$

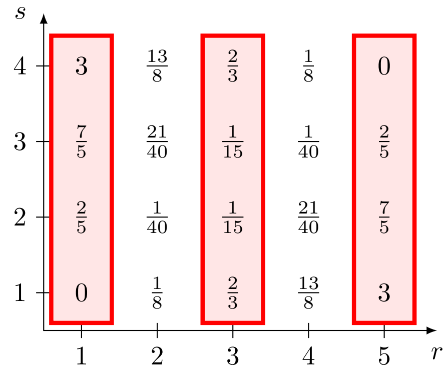
where the non-diagonal fields $V_{(r,s)}$ are fully degenerate. This is valid for

$$\boxed{p \in 2\mathbb{N}+6 \quad , \quad q \in 2\mathbb{N}+3 \quad , \quad p,q \text{ coprime}} \ , \tag{30}$$

where the lower bounds $p \geq 6$ and $q \geq 3$ ensure that the non-diagonal sector contains fields with nonzero spins.

Example II.4. *Consider $p = 6$, $q = 5$. The central charge is $c = 4/5$. The correspnding Kac table is:*

$$AMM_{6,5} \ \& \ DMM_{6,5} \ : \tag{31}$$



The 10 (independent) primary fields of $AMM_{6,5}$ are

$$V^f_{\langle 1,1\rangle}, V^f_{\langle 2,1\rangle}, V^f_{\langle 3,1\rangle}, V^f_{\langle 4,1\rangle}, V^f_{\langle 5,1\rangle}, V^f_{\langle 1,2\rangle}, V^f_{\langle 2,2\rangle}, V^f_{\langle 3,2\rangle}, V^f_{\langle 4,2\rangle}, V^f_{\langle 5,2\rangle} \ . \tag{32}$$

In $DMM_{6,5}$, only 6 of these fields are present (due to the $r = 2n+1$). For the non-diagonal we have $V_{(r,s)}$ where $(r,s) \in \{-2,0,2\} \times \{-3/2,-1/2,1/2,3/2\}$. For the zero spin ($r = 0$), denoting $(r,s) = \Delta_{(r,s)}$

$$\begin{aligned}
(0,-3/2) &\to (3,(-3+5)/2) = (3,1)\\
(0,-1/2) &\to (3,(-1+5)/2) \to (3,2)\\
(0,1/2) &\to (3,(1+5)/2) \to (3,2)\\
(0,3/2) &\to (3,(3+5)/2) \to (3,1)
\end{aligned}$$

hence the dimesions (on both left and right are:) $V_{(0,3/2)} \sim V^f_{(3,1)}$ and similarly for the others. Remember they live in the non-diagonal rep. As expected, the OPEs do not coincide, for example

$$V^f_{\langle 3,1\rangle} V^f_{\langle 3,1\rangle} \sim V^f_{\langle 1,1\rangle} + V^f_{\langle 3,1\rangle} + V^f_{\langle 5,1\rangle} \quad , \quad V^f_{\langle 3,1\rangle} V_{(0,\frac{3}{2})} \sim V_{(2,-\frac{3}{2})} + V_{(0,\frac{3}{2})} + V_{(2,\frac{3}{2})} \ . \tag{33}$$

Now, for the others non-diagonal (non-zero spin), we need to look at the conformal dimensions on both right and left sectors. For example, for $V_{(2,-3/2)}$ we have:

$$\begin{aligned}
L &: (2,-3/2) \to (5,1)\\
R &: (2,3/2) \to (5,4) \to (1,1)
\end{aligned}$$

The upshot is that the 6 diagonal fields ($r = 0$) are complemented by 6 non-diagonal fields. For each one of these 12 fields, let us indicate the left and right Kac indices of the corresponding degenerate representations:

Field	$V^f_{\langle 1,1\rangle}$	$V^f_{\langle 3,1\rangle}$	$V^f_{\langle 5,1\rangle}$	$V^f_{\langle 1,2\rangle}$	$V^f_{\langle 3,2\rangle}$	$V^f_{\langle 5,2\rangle}$
Indices	$(1,1)(1,1)$	$(3,1)(3,1)$	$(5,1)(5,1)$	$(1,2)(1,2)$	$(3,2)(3,2)$	$(5,2)(5,2)$

Field	$V_{(0,\frac{1}{2})}$	$V_{(0,\frac{3}{2})}$	$V_{(2,-\frac{3}{2})}$	$V_{(2,-\frac{1}{2})}$	$V_{(2,\frac{1}{2})}$	$V_{(2,\frac{3}{2})}$
Indices	$(3,2)(3,2)$	$(3,1)(3,1)$	$(5,1)(1,1)$	$(5,2)(1,2)$	$(1,2)(5,2)$	$(1,1)(5,1)$

(34)

$$V^f_{\langle r_1,s_1\rangle} V_{(r_2,s_2)} \sim \sum_{r \overset{2}{=} \max(r_2-r_1,r_1-r_2-p)+1}^{\min(r_1+r_2,p-r_1-r_2)-1} \ \sum_{s \overset{2}{=} \max(s_2-s_1,s_1-s_2-q)+1}^{\min(s_1+s_2,q-s_1-s_2)-1} V_{(r,s)} \ ,$$

$$V_{(r_1,s_1)} V_{(r_2,s_2)} \sim \sum_{r \overset{2}{=} |r_1-r_2|+1}^{p-|r_1+r_2|-1} \ \sum_{s \overset{2}{=} |s_1-s_2|+1}^{q-|s_1+s_2|-1} V^f_{(r,s)} \ .$$

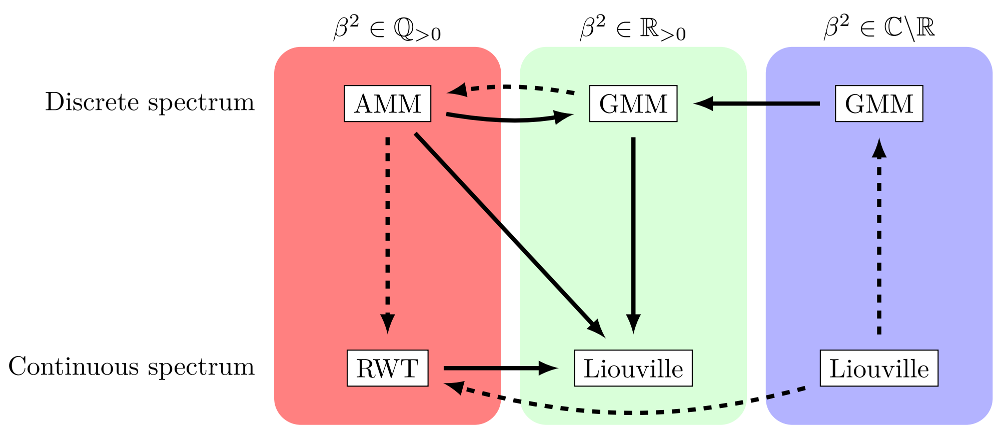
3. Summary of spectrums

For each CFT, we indicate the values of β^2, of the Kac indices of degenerate fields, of the indices of non-diagonal fields, and of the momentums of diagonal fields:

CFT	β^2	$V^{d/f}_{\langle r,s\rangle}$	$V_{(r,s)}$	V_P
GMM	$\mathbb{C}\backslash\mathbb{Q}$	$\mathbb{N}^*\times\mathbb{N}^*$	$-$	$-$
Liouville	$\mathbb{C}^*$	$-$	$-$	$\mathbb{R}$
$\widehat{\text{Liouville}}$	$\mathbb{C}^*$	$\mathbb{N}^*\times\mathbb{N}^*$	$-$	$\mathbb{C}$
AMM	$\frac{q}{p}>0$	$\mathbb{N}\times\mathbb{N}$ $(0,p)\times(0,q)$	$-$	$-$
RWT	$\frac{q}{p}>0$	$-$	$-$	$\mathbb{R}$
DMM	$\frac{q}{p}>0$ $p\in 2\mathbb{N}$	$(2\mathbb{N}+1)\times\mathbb{N}$ $(0,p)\times(0,q)$	$2\mathbb{Z}\times(\mathbb{Z}+\frac12)$ $(-\frac{p}{2},\frac{p}{2})\times(-\frac{q}{2},\frac{q}{2})$	$-$
GDMM	$\left\{\operatorname{Re}\beta^2>0\right\}$	$-$	$2\mathbb{Z}\times(\mathbb{Z}+\frac12)$	$\mathbb{R}$
$\widehat{\text{Loop}}$	$\left\{\operatorname{Re}\beta^2>0\right\}$	$\{1\}\times\mathbb{N}^*$	$\frac12\mathbb{N}^*\times\frac{1}{2r}\mathbb{Z}$	$\mathbb{C}$
$O(n)$	$\left\{\operatorname{Re}\beta^2>0\right\}$	$\{1\}\times(2\mathbb{N}+1)$	$\frac12\mathbb{N}^*\times\frac{1}{r}\mathbb{Z}$	$-$
$PSU(n)$	$\left\{\operatorname{Re}\beta^2>0\right\}$	$\{1\}\times\mathbb{N}^*$	$\mathbb{N}^*\times\frac{1}{r}\mathbb{Z}$	$-$
Potts	$\left\{\operatorname{Re}\beta^2>0\right\}$	$\{1\}\times\mathbb{N}^*$	$(\mathbb{N}+2)\times\frac{1}{r}\mathbb{Z}$	$\{P_{(0,s)}\}_{s\in\mathbb{N}+\frac12}$

(35)

Taking limits played an important role in deriving some of the CFTs. Let us summarize the limits that relate Liouville theory, (generalized) minimal models, and Runkel–Watts-type theories:



(36)

C. BPZ equations

We consider $4-pt$ function correlation functions $G(z)=\langle V_{(r,s)}(z)V_1(0)V_2(\infty)V_3(1)\rangle$ since $L_{(r,s)}V_{(r,s)}=0$ we then have a differential equation

$$\langle L_{(r,s)}V_{(r,s)}(z)V_1(0)V_2(\infty)V_3(1)\rangle=0$$

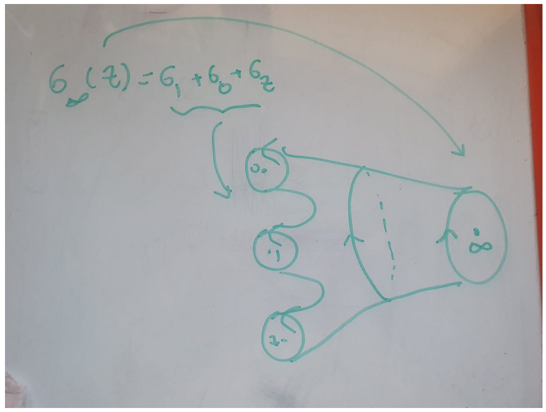
these are the BPZ equations. For $(2,1)$ $L_{(2,1)}\sim L_{-1}^2+\beta L_{-2}$ to determine $G(z)$ we thus need

$$\langle L_{-2}V_{(r,s)}(z)V_1(0)V_2(\infty)V_3(1)\rangle$$

this can be easley done by defining

$$G_{z_0}(z)=\int_{z_0}\frac{dz}{2\pi i}\frac{y(y-1)}{y-z}\langle T(y)V_{(2,1)}(z)V_1(0)V_2(\infty)V_3(1)\rangle$$

as then $G_z(z)=f(z,\Delta)G(z)+g(z)L_{-2}G(z)$ and



For computing G_∞ we compute

$$I_\infty(z) = \int_0 \frac{dw}{2\pi i w^3}\frac{(w-1)}{1-zw}\langle \underbrace{\underbrace{w^4 T(w)}_{T(z)}\, V_2(\infty)}_{w^4\left(\frac{\Delta_2 V_2(\infty)}{w^2}+\frac{\partial V_2(\infty)}{w}+\ldots\right)} \quad V_{(2,1)}(z)V_1(0)V_3(1)\rangle$$

$$= -\Delta_2 G(z)$$

where we computed I_∞ by a counterclockwise integration (as $1/z$ preserve orientation) the equality is satisfied for $G_\infty(z) = -I$ as reversing the orientations gives the picture equality.

Higher order $V_{(r,s)}$ insertion can be computed recursively by using $G_{z_0}(z)$ to find L_{-2} and $G^{(2)}_{z_0}(z) \sim \int dy \frac{y(y-1)(y-a)}{(y-z)^2}$ for finding L_{-3} in terms of L_{-1} and L_{-2}. a can be choosen so that L_{-2} do not appear. higher likewise.

Solutions to the second order BPZ are conformal blocks which by the fusion rules of $V_{(2,1)}$ we expected to be two. Each channel has a corresponding asymptotics of the conformal blocks $s \leftrightarrow z \to 0$ $t \leftrightarrow 1-z \to 0$ $u \leftrightarrow 1/z \to 0$ for which we can expand in whateber we want. If $f(z) = \varphi(z)^{-1}G(z)$, for some $\varphi = z^{\#}(1-z)^{\#}$, then plugging in f in the BPZ equation gives f to be a HyperGeometric function ${}_2F_1$.

The change of basis from one asymptotic to the other (i.e. from one channel to the other) are encoded in the F **degenerate fusing matrix** to which each conformal block are

$$\mathcal{F}^{(s)}_\varepsilon = (F\mathcal{F}^{(t)})_\varepsilon \tag{37}$$

as conformal blocks are labeled by $\varepsilon = \pm$ depending on the interchanging field.

D. Croosing symmetry

Consider the $4-pt$ function in the x channel: $\langle V_{(r,s)}(z)V_1(0)V_2(\infty)V_3(1)\rangle = \sum_\pm d^{(x)}_\pm |\mathcal{F}^{(x)}_\pm|$ Croosing symmetry then gives (compering coefficient of the basis by using the fusing matrix) the relation between $d^{(s)}(F\bar{F})_{\varepsilon\bar{\varepsilon}} = Id_{\varepsilon\bar{\varepsilon}}d^{(t)}$. When RHS is 0 gives a compatibility of d and using the expressin of the fusing matrix it gives all $d^{(s/t)}_\pm$ in terms of only one (for each channe) as the rations are known:

$$\frac{d^{(s)}_-}{d^{(s)}_+} = -\frac{F_{++}\bar{F}_{+-}}{F_{-+}\bar{F}_{--}} \quad , \quad \frac{d^{(t)}_+}{d^{(s)}_+} = \frac{F_{++}}{\bar{F}_{--}}\det \bar{F} \quad , \quad \frac{d^{(t)}_-}{d^{(s)}_+} = -\frac{\bar{F}_{+-}}{F_{-+}}\det F \, .$$

Having this relations we can study the LHS of the equation. Since $V_{(2,1)}V_1 \sim c_1^\pm V_1^\pm$ then $\frac{d^{(s)}_-}{d^{(s)}_+} = \frac{c_- C_{1-23}}{c_+ C_{1+23}}$ where the c_i terms depend only on the degenerate field. If we further consider $V_2 = V_{(2,1)}$ and $V_3 = V_1$ then we can explicitly determine c_i which plugging back gives a relation to

$$\boxed{\frac{C_{1-23}}{C_{1+23}} = (-)^{2s_2}\beta^{4\beta^2 r_1}\frac{\prod_{\pm,\pm}\Gamma\left(\frac{1}{2}-\beta\bar{P}_1 \pm \beta\bar{P}_2 \pm \beta\bar{P}_3\right)}{\prod_{\pm,\pm}\Gamma\left(\frac{1}{2}+\beta P_1 \pm \beta P_2 \pm \beta P_3\right)}},$$

known as the **shift equation**. We can let $\beta \to \beta^{-1}$ to get the associated shift equation for $V_{(1,2)}$.

Solutions are

$$C_{P_1,P_2,P_3} = \prod_{\pm,\pm,\pm} \Gamma_\beta^{-1}\left(\frac{\beta+\beta^{-1}}{2} \pm P_1 \pm P_2 \pm P_3\right)$$

and B can be found from $\langle VV\rangle = \langle IVV\rangle$ where I is identity field which has $\Delta = 0 \implies P = (\beta - \beta^{-1})/2$.

Note that shift equations determine solutions up to a factor $P_i \to P_i + \beta^{\pm}$. If β is rational then different periodic functions may exists but, notice that if β irrational then it must be a constant and since irrationals are dense then solutions is unique.

So this gives unique Liouville for $\beta^2 \in \mathbb{R}$. In minimal models the solution do not have to be defined in all p as in OPEs not all $p's$ exists. For instance $V_{(2,1)}^2 = V_{(1,1)} + V_{(3,1)}$ do not contain $V_{(2,1)}$. One can show this gives unique solution to AMM and GMM. This is for $c \in (-\infty, 1]$

Similar story for extending to $\beta \in i\mathbb{R}$ where a new function $\hat{\Gamma}_\beta$ can be defined. This allows a definition for Liouville theory.

III. QFT II

This is a review of some of the topics of QFT II.

A. BRST

Consider a guage field A

$$Z = \int \frac{\mathcal{D}A\mathcal{D}...}{volG} e^{-S[A,...]}$$

to gauge fix with gauge $f(A) = 0$, insert $1 = \Delta_{FP}(A)\int_G \mathcal{D}g\delta(f(A^g))$ then plugging in Z and performing a $A \to A^{-g}$ we get a gauge fixed path integral,

$$Z = \int \frac{\mathcal{D}A\mathcal{D}g\mathcal{D}...}{volG} \Delta_{FP}(A^{-g})\delta(f(A))e^{-S[A,...]} \tag{38}$$

To compute the F-P determinant note that inside Z we have

$$\Delta_{FP}^{-1}(A) = \int_G \mathcal{D}g\delta(f(A^g))\Big|_{f(A)=0} \implies \Delta_{FP}^{-1}(A) = \int_G \mathcal{D}\alpha\delta(\underbrace{\delta f}_{f(A)-f(A^g)})$$

Example III.1. *In Lorentz gauge $f(A) = \partial_\mu A^\mu$ then under a gauge $g = e^{i\alpha}$, since $A^g = A - iD_\mu\alpha$, we have*

$$\Delta_{FP}^{-1}(A) = \int_G \mathcal{D}\alpha\delta(i\partial_\mu D^\mu\alpha) \implies \Delta_{FP}(A) = \int \mathcal{D}c\mathcal{D}\bar{c}\,\exp\{i\bar{c}\partial_\mu D^\mu c\}$$

where we have used $\delta(x) = \int dpe^{ipx}$ and the anti-commuting path integral to invert the determinant.

1. R_ξ gauges

Notice that to evaluate Equation (38) requires to "put by hand" the gauge fixing $f(A) = 0$, how to compute path integrals with a restriction? No idea. We can, instead (and more generally), define a full action describing general gauges $f(A) = \partial_\mu A^\mu - \omega(x) = 0$ for an arbitrary ω. Since

$$e^{-\frac{i}{2\xi}\int(\partial_\mu A)^2} = \int \mathcal{D}fe^{-\frac{i}{2\xi}\int f^2}\delta(f - \partial_\mu A^\mu),$$

applying $\int \mathcal{D}\omega e^{-\frac{i}{2\xi}\int \omega^2}$ to both sides of Equation (38) and, since Z should be independent of the gauge we are choosing i.e. $Z[\omega] = Z$, we have

$$Z = \frac{1}{\mathcal{N}} \int \mathcal{D}\omega e^{-\frac{i}{2\xi}\int \omega^2} \int \frac{\mathcal{D}A\mathcal{D}g\mathcal{D}...}{volG} \Delta_{FP}(A^{-g})\delta(\partial_\mu A^\mu - \omega(x))e^{-S[A,...]} = \int \mathcal{D}\omega \int \frac{\mathcal{D}A\mathcal{D}g\mathcal{D}...}{volG} \Delta_{FP}(A^{-g})e^{-\frac{i}{2\xi}\int(\partial_\mu A)^2} e^{-S[A,...]},$$

for some normalization value $\mathcal{N}$. We conclude the gauge fixed integral has a total action $S_{tot} = S + S_{gf} + S_{gh}$. In the above example, we have

$$S_{tot} = S[A, \underbrace{\ldots}_{\text{phys.fields}}] + \underbrace{\int \bar{c}\partial_\mu D^\mu c}_{S_{\text{gh}}} - \underbrace{\frac{1}{2\xi}\int (\partial_\mu A)^2}_{S_{\text{gf}}}$$

2. BRST symmetry

Once the Fadeev-Popov procedure is caried out we have a lagrangian density

$$\mathcal{L}_{FP} = \mathcal{L} + \mathcal{L}_{Ghosts} + \mathcal{L}_{GaugeFixing}$$

Clearly fixing the gauge breaks gauge invariance on the total lagrangian however, a new symmetry has emerged. This goes by the name of BRST symmetry. Writing

$$\mathcal{L}_{g.f.} = e^{-\frac{i}{2\xi}\int f_\alpha^2(x)} = \int \mathcal{D}N e^{\frac{i\xi}{2}N\cdot N - if\cdot N}$$

where f_α is the gauge fixing term (e.g. for Lorentz gauge is $f_\alpha = \partial_\mu A^\mu_\alpha$) and I use the notation $f\cdot g = \int d^4x f(x)g(x)$. So that $\mathcal{L}_{tot} = \mathcal{L} + \mathcal{L}_{Ghosts} + \underbrace{\frac{\xi}{2}N^2(x) - f_\alpha N^\alpha}_{\mathcal{L}_{in}}$. This density is invariant under

$$\begin{aligned}
\delta_{\text{BRST}}A &= \theta Dc = \theta(dc - iA\wedge c) = \theta(dc - i[A,c]), \quad \theta \text{ is a grassman variable}\\
\delta_{\text{BRST}}\phi &= \theta i\pi(c)\phi\\
\delta_{\text{BRST}}\bar{c} &= \theta N\\
\delta_{\text{BRST}}c &= \theta\frac{i}{2}\{c,c\} = \theta ic\wedge c\\
\delta_{\text{BRST}}N &= 0
\end{aligned}$$

where $\pi(c)$ puts c in the appropiate rep. for ϕ and we define s such that $\delta_{\text{BRST}}\varphi = \theta s\varphi$ for every field φ. s is nilpotent and Leibniz (usual relations and from

$$\delta(\phi_1\phi_2) = \theta((s\phi_1)\phi_2 + (-1)^{gh(\phi_1)}\phi_1 s\phi_2)$$

then $s^2(\phi_1\phi_2) = 0$).

Since $\delta A = D^\mu\theta C$ and $\delta\phi = i\theta\pi(c)\phi$ BRST symmetry can be seen as a gauge transformation on θC parameter so $\mathcal{L}$ is invariant. Clearly gauge fixed action breaks gauge invariance but for the remaing terms (F-P and gauge fixing) we have

$$\mathcal{L}_{gh} + \frac{\xi}{2}N^2 - f_\alpha N = s\left(\frac{\xi}{2}\bar{c}N - f_\alpha\bar{c}\right)$$

so $\delta\mathcal{L} = \theta s\mathcal{L} = 0$by nilpotency of s. Hence the theory is BRST symmetric. We have an associated nilpoent charge Q.

3. Hilbert space

Nilpotent charge allow us to define a cohomological QFT. Types of states:

$$\begin{aligned}
&Q\,|\phi_1\rangle \neq 0\\
&Q\,|\phi_2\rangle = 0, \quad \& \quad \exists\,|\phi_1\rangle \text{ s.t. } |\phi_2\rangle = Q\,|\phi_1\rangle\\
&Q\,|\phi_3\rangle = 0, \quad \& \quad \nexists\,|\phi_1\rangle \text{ s.t. } |\phi_3\rangle = Q\,|\phi_1\rangle
\end{aligned}$$

I.e. ϕ_1 are one forms, ϕ_2 are exact and ϕ_3 are close but not exact i.e. they belong to the cohomology of s. Exact states $|\phi_2\rangle$ have null norm hence not physical and, physical states should be annihilated by Q (should not depend on choice of gauge).

Indeed, if vacuum is invariant under BRST,

$$0 = \langle [Q, T()\{\prod_i \phi_i\}]\rangle = \sum_k \langle T()\{\phi_1 ... \phi_{k-1} \prod_{i=1^{k-1}} [Q, \phi_k]_{-s_k} \phi_{k+1} ... \phi_n\}\rangle$$

then,

$$\langle T[Q, \phi_1]_{-s_1} \prod_{k\neq i} \phi_k\rangle = -\sum_{k\neq 1} \langle T()\{\phi_1 ... \phi_{k-1} \prod_{i=1^{k-1}} [Q, \phi_k]_{-s_k} \phi_{k+1} ... \phi_n\}\rangle$$

If ϕ_k, $k \neq 1$ are BRST close, RHS vanish and we have a Ward identity of one insertion of exact with all other closed vanish. We conclude Q should annihilate physical states and further they should be closed as

$$\langle \alpha|\beta\rangle = \langle \tilde{\alpha}|\beta\rangle$$

for $\alpha = \tilde{\alpha} + Q\phi_1$.

4. Unitarity

Unitary might be violated by unphysical states even if not present as physical external legs (due to the optical theorem). Check this do not happend. Consider

$$\langle \alpha| S^\dagger S |\beta\rangle = \sum_\phi \sum_{i=1}^{3} \langle \alpha| S^\dagger |\phi_i\rangle \langle \phi_i| S |\beta\rangle \underbrace{=}_{?} \langle \alpha| S^\dagger S |\beta\rangle$$

Since $[Q, S] = 0 = [Q, S^\dagger]$ then $\langle \alpha| S^\dagger Q = 0 QS |\beta\rangle$ then

$$\begin{aligned}
\langle \alpha| S^\dagger S |\beta\rangle &= \sum_\phi \sum_{i=1}^{3} \langle \alpha| S^\dagger |\phi_i\rangle \langle \phi_i| S |\beta\rangle \\
&= \sum_\phi \sum_{i=1}^{3} \langle \alpha| S^\dagger |\phi_i\rangle \langle \phi_i| S |\beta\rangle - \sum_\phi \sum_{i=1}^{3} \langle \alpha| QS^\dagger |\phi_i\rangle \langle \phi_i| SQ |\beta\rangle \\
&= \sum_\phi \sum_{i=1}^{3} \langle \alpha| S^\dagger (|\phi_i\rangle \langle \phi_i| - Q |\phi_i\rangle \langle \phi_i|) S |\beta\rangle
\end{aligned}$$

so we have an ambiguity for states ϕ_1 states (as Q kills all but phi_1 states) then only ϕ_2 and ϕ_3 must appear in the sum. Since $\langle \phi_2|\phi_3\rangle = 0$ then only states in the cohomology appear in the sum,

$$\langle \alpha| S^\dagger S |\beta\rangle = \sum_{\text{Phys.}} \langle \alpha| S^\dagger |\text{Phys.}\rangle \langle \text{Phys.}| S |\beta\rangle = \langle \alpha| S^\dagger S |\beta\rangle \tag{39}$$

B. Renormalization and β functions

Consider QED. We begin with a naive theory of A gauge fields and matter ψ. The theory has an interaction $-ig\bar{\psi}\not{A}\psi$which creates a flow as the operator is marginal in $d = 4$ (the mass dimension is $\frac{d-4}{2}$). The true theory thus is the renormalized theory which is defined by

$$\begin{aligned}
\sqrt{Z_2}\psi_R &= \psi \\
\sqrt{Z_3}\psi_R &= A \\
Z_1 g_R &= ig\bar{\psi}\not{A}\psi \\
m_R^2 &= m^2 + \delta m^2 \\
g_R Z_g &= g
\end{aligned}$$

The appropiate counter-terms are found by pluging $Z. = 1 + \delta.$ in the naive action and expanding. For instance, the fermion mass and wave renormalization are computed from the sunrise diagram which gives rise to a contribution $I \sim g_R^2 = g_R \mu^\epsilon$ where $4 - d = \varepsilon$ the counter term is the "propagator" counter term $\delta_2 p^2 - \delta m^2$ so from the p denendent part of the divergent diagram we find δ_2 to first order and δ from the other. This gives $\delta_2 \sim \frac{1}{\epsilon}$.

Moreover, the coupling g has a beta function we wish to compute. To do so we consider the relation

$$g = Z_g g_R \mu^{1varepsilon/2}$$

as introduced for computing the divergence in the sunrise Feynmann diagram. Since g is constant, deriving both sides of $g\mu^{-\varepsilon/2} = Z_g g_R$ with respect to μ and multipliying by μ gives:

$$\begin{aligned}
\mu \frac{d}{d\mu} g\mu^{-\varepsilon/2} &= \mu \frac{d}{d\mu}(Z_g g_R) \\
-\frac{\varepsilon}{2} g\mu^{-\varepsilon/2} &= \beta(g_R) Z_g + \frac{dZ_g}{dg_R}\beta(g_R) g_R.
\end{aligned}$$

Solving for β gives

$$\begin{aligned}
\beta &= -\frac{\varepsilon}{2}(1+\delta g) g (1 - \delta g - g\frac{d\delta g}{dg}) \\
&= -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} g\delta g + \frac{\varepsilon}{2} g\delta g + \frac{\varepsilon}{2} g(\delta g)^2 + \frac{\varepsilon}{2} g^2 \frac{d\delta g}{dg} + \frac{d\delta g}{dg} g^2 \delta g \frac{d\delta g}{dg} \\
&= \underbrace{-\frac{\varepsilon}{2} g}_{O(\varepsilon)} + \underbrace{\frac{\varepsilon}{2} g^2 \frac{d\delta g}{dg}}_{O(\varepsilon^0)} + \underbrace{\frac{\varepsilon}{2}((\delta g)^2 g + g^2 \delta g \frac{d\delta g}{dg})}_{O(\varepsilon^{-1})}
\end{aligned}$$

as $\delta g \sim \varepsilon^{-2}$. We need to find the $Z_g(g)$ dependence. To do so, notice $Z_g = \frac{Z_1}{Z_2 Z_3^{1/2}} = 1 + \delta_1 - \delta_2 - \frac{1}{2}\delta_3$ as $g\bar{\psi}\not{A}\psi = \underbrace{Z_g Z_2 Z_3^{1/2}}\bar{\psi}_R \not{A}_R \psi_R$ where Z_1 can be computed from the vertex renormalization of the "Cocktail" diagram. Computing the vertex renormalization, the wave function renormalization of bith ψ and A then gives

$$\beta(g) = -\frac{\varepsilon}{2} g + \frac{\varepsilon}{2} g^2 \frac{d\delta g}{dg} = -\frac{\varepsilon}{2} g - \frac{g^3}{16\pi^2}\left(\frac{11}{3} c_A - \frac{4}{3} n_f T_D\right) \tag{40}$$

where we omited the g_R subscript. Asymptotic freedom!

C. Goldstone theorem

1. Path integral derivation

Consider

$$Z[J] = e^{iW[J]} = \int \mathcal{D}\varphi e^{\frac{i}{\hbar}(S[\varphi] - J\cdot\varphi)}$$

where I use the notation $J \cdot \varphi = \int d^4x J(x)\varphi(x)$. Under a symmetry variation, we have:

$$e^{iW[J]} = \int \mathcal{D}\varphi e^{\frac{i}{\hbar}(S[\varphi] - J\cdot(\varphi + i\theta\cdot T\varphi))} = e^{iW[J]} + \int \mathcal{D}\varphi e^{\frac{i}{\hbar}(S[\varphi] - J\cdot\varphi)} iJ\cdot\theta\cdot T\varphi$$

where we expaned the second term in J. This leads to

$$0 = J \cdot T \underbrace{\phi(J, x)}_{\langle\varphi\rangle_J}$$

Now, notice $\frac{\delta W[J]}{\delta J(x)} = \phi(J,x)$ (compute it from Z) and, defining the Legendre transform of $W[J]$ (to define a quantum full action i.e. an action with all quantum interactions) $\Gamma(\phi) = W[J] - J\cdot\phi$,

$$\frac{\delta\Gamma}{\delta\phi(x)} = \left(\int dy \frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\phi(x)}\right) - \frac{\delta J}{\delta\phi}\cdot\phi - J(x) = -J(x)$$

where the first and second term vanish by the above relation of J.

Hence, our Ward identity gives:

$$0 = \frac{\delta\Gamma}{\delta\phi} \cdot T\phi$$

and varying with respect to ϕ,

$$0 = \frac{\delta^2\Gamma}{\delta\phi\delta\phi} \cdot T\phi + \underbrace{\frac{\delta\Gamma}{\delta\phi}}_{J} \cdot T\delta$$

evaluating at $J = 0$, if there is a non vanishing vev $\phi(0, x) = v$ (by Lorentz invariance it is constant),

$$0 = \Gamma_{,ab} T_{bc} v^c$$

where $\Gamma_{,ab} = \frac{\delta^2\Gamma}{\delta\phi^a\delta\phi^b}$. Finally, notice

$$\delta^a_b = \frac{\delta\phi^a}{\delta\phi^b} = \frac{\delta^2 W[J]}{\delta\phi^b\delta J^a} = \int dy \frac{\delta^2 W[J]}{\delta J^c(y)\delta J^a}\frac{\delta J^c(y)}{\delta\phi^b} = -\int dy \frac{\delta^2 W[J]}{\delta J^c(y)\delta J^a}\frac{\delta^2\Gamma}{\delta\phi^c(y)\delta\phi^b}$$

hence, in fourier space $i(\tilde{\Gamma}_2)^{-1}_{ab} = \tilde{W}_{,ab}$ and is easy to check $\tilde{W}_2$ is the propagator (in fourier space). We conclude, evaluating the previous equation at $p^2 = 0$

$$0 = M_{ab} T_{bc} v^c \tag{41}$$

where M is the mass matrix. Hence, diagonalizing, for each broken generator we must have a massless field.

2. *Spectral functions and Cluster decomposition*

We proceed to show similarly the Goldstone theorem using the spectral function of the one point function.

We consider the quantity $\langle\Omega|\,\delta\phi\,|\Omega\rangle$, let Q the charge of the associated symmetry,

$$\begin{aligned}
\langle\Omega|\,[Q, \phi_n(x)]\,|\Omega\rangle &= \int d^3y\,\langle\Omega|\,[J^0(y), \phi(x)]\,|\Omega\rangle = \int d^3y {-\partial_t}^y \int d\mu^2\rho(\mu^2)\Delta^+(x-y, \mu^2) + (\text{Other part of the commutator}) \\
&= \int d^3y \int d\mu^2 \frac{\delta(x-y)}{2}\rho(\mu^2) + (\text{Other part of the commutator}) \\
v = \langle\Omega|\,(T\cdot\phi)_n(x)\,|\Omega\rangle &= \int d\mu^2\rho(\mu^2)
\end{aligned}$$

as the other part of commutator gives the same. Since ρ is positive definite, we conclude that if there is a symmetry broken (LHS is non vanishing), then the spectrum must contain a massles particle, namely $\rho_n \sim \delta(\mu^2)v$.

Notice here we used several things.

- First, ∂^z_t. Indeed,

$$\partial^z_t\Delta^+(z,\mu) = \delta(z) = \partial^z_t \frac{i}{(2\pi)^3}\int d^4p\,\delta(\mu^2 - p^2)\theta(p^0)e^{ip\cdot z} = \frac{i}{(2\pi)^3}\int d^4p\,\underbrace{\delta(\mu^2 - p^2)\theta(p^0)}_{\delta(p_0 - E_p)/(2p_0)}(-ip_0)e^{ip\cdot z} = \frac{\delta^3(z)}{2}$$

- $\langle\Omega|\,J^\mu(y)\phi(x)\,|\Omega\rangle \sim \int d\mu^2\rho_n(\mu^2)\Delta^+(x-y, \mu^2)$

$$\begin{aligned}
\langle\Omega|\,J^\mu(y)\phi(x)\,|\Omega\rangle &= \sum_N \langle\Omega|\,J^\mu(0)\,|N\rangle\langle N|\,\phi(0)\,|\Omega\rangle = \frac{i}{(2\pi)^3}\int d^4p\,\rho^\mu_n(p^2)e^{ip\cdot(x-y)} \\
&= \frac{i}{(2\pi)^3}\int d^4p\,p^\mu\theta(p^0)\rho_n(p^2)e^{ip\cdot(x-y)} = \frac{i}{(2\pi)^3}\int d\mu^2\rho_n(p^2)\Delta^+(\mu^2, x.y)
\end{aligned}$$

- Δ^+ is nothing but a solution to the $K-G$equation. It is easy to see that current conservation on $\langle\Omega| [J^\mu(y), \phi(x)] |\Omega\rangle$ (as then $\partial_\mu \langle\Omega| [J^\mu(y), \phi(x)] |\Omega\rangle \sim \int d\mu^2 \mu^2 \rho_n(\mu^2)$) requires $\rho_n(\mu^2) = 0$ for $\mu^2 \neq 0$.

It remains only one ingridient to the picture. How to introduce a non vanishing vev?

Example III.2. *Consider a complex $\lambda\Phi^4$ theory,*

$$\partial\Phi^*\partial\Phi + m^2|\Phi|^2 - \frac{\lambda}{4}(\Phi^*\Phi)^2 \tag{42}$$

. $V = \frac{\lambda}{4}(\Phi^*\Phi)^2 - m^2|\Phi|^2$ *whose minimum is at* $|\Phi| = 0$ *or* $\sqrt{\frac{2m^2}{\lambda}} = \phi_0$, *this is the "Mexican hat" picture. Hence, we have a continous vacua*

$$\langle\Phi\rangle = e^{i\theta}\phi_0 = \phi_\theta$$

We introduce a field reparametrization

$$\Phi(x) = \left(\phi_0 + \frac{1}{\sqrt{2}\sigma(x)}\right)e^{i\left(\theta + \frac{\pi(x)}{f_\pi}\right)} \tag{43}$$

which makes evident the π fields will not aquire a mass as any interaction potential must have a derivative for the field π to appear. f_π is fixed by canonically normalize fields.

D. Standard Model

In the standard model, at high energies since the mass of Up and Down quarks are close, we have an approximate symmetry $SU(2)_L \times SU(2)_R$ mixing them. However, in nature we only see $SU(2)$ so we need to break it spontaneously (SSB). Quark condensates $\bar{u}u$ acquire can be given a vev independently which breaks both left and right symmetries. Instead, to preserve the diagonal $SU(2)$ we consider they aquire same vev. so

$$\langle\Omega| \bar{Q}Q |\Omega\rangle = 2v^3 \tag{44}$$

We build a model where this symmetry is present and spontaneously broken to the diagonal. Consider Σ transforming for two independent $u_{L/R}$ by $\Sigma \to u_L \Sigma u_R^\dagger$

$$\mathcal{L} = \mathrm{tr}\left\{\Sigma\Sigma^\dagger\right\}$$

has the full symmetry and is broken by a mass term.

$$\mathcal{L} = \mathrm{tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\} + m^2\,\mathrm{tr}\left\{\Sigma\Sigma^\dagger\right\} - \frac{\lambda}{4}\,\mathrm{tr}\left\{\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger\right\}$$

then $\Xi = \Sigma\Sigma^\dagger$ transforms as $\Xi \to u_L \Xi u_L^\dagger$ for $SU(2)_L \times SU(2)_R$ and it's invarinat under $e \times SU(2)_R$. The minimum of the potential gives is found at $\Xi = \frac{2m^2}{\lambda} id$ which is an element of the orbit $SU(2)_L$ of solutions.

Vacuum is thus broken to $U(2)_L \times U(2)_R \to U(2)_D$. To identify the Goldstone bosons, we consider polar decomp.

$$\Sigma(x) = \rho(x)U(x)$$

and shift $\rho = (v + p(x))/2$. We have: 4 real fields parametrizing U (being $SU(2)$ element) which are Goldstone bosons and $4+4$ generators broken to 4 so 4 remainig massive bosons. Parametrizing the $U(x)$ matrix with τ_a $U(2)$ basis we found the $\pi-$ons,

$$\mathcal{L}_{\chi p.t.} = \frac{f_\pi^2}{4}\,\mathrm{tr}\left\{\partial_\mu U\partial^\mu U^\dagger\right\} + L_1\,\mathrm{tr}\left\{(\partial_\mu U\partial^\mu U^\dagger)(\partial_\mu U\partial^\mu U^\dagger)\right\}..., \quad U = \exp\left\{2i\frac{\pi^a\tau_a}{f_\pi}\right\} \tag{45}$$

Higger terms, interactions with electromagnetism, etc. $\mathcal{L}_{\chi p.t.}$ is non renormalizable. as f_π^{-2} has dimensions of ℓ^2.

Spurionic method. To break the symmetrt we introduce $M = \mathrm{diag}\{m_u, m_d\}$

$$M \to u_R M u_L^\dagger, \Delta\mathcal{L}_{\chi p.t.} = k\,\mathrm{tr}\left\{MU + M^\dagger U^\dagger\right\} = 2k(m_u + m_d) + k\left(-\frac{m_u + m_d}{f_\pi^2}\pi_a\pi^a + ...\right)$$

estimates of k,

$$\langle\bar{u}u\rangle = \langle\bar{d}d\rangle = V^3 \implies \mathcal{L}_m \sim V^3(m_u + m_d) = 2k(m_u + m_d)$$

then $k = \frac{V^3}{2}$ so $m_\pi^2 = V^3(m_u + m_d)/f_\pi^2$.

1. General broken global symmetries

Consider $\mathcal{L}$ invariant under G and broken to $H \subset G$ (i.e. $h \cdot \langle\Omega|\,\Psi\,|\Omega\rangle\rangle = \psi_0$ for all $h \in H$) then from the defining condition of the broken generators, define $\tilde{\psi}$

$$\tilde{\psi}_n(x) t_a^{nm} \psi_{0n} = 0 \tag{46}$$

and then write, for some appropiate $\gamma(x) : M \to G$ and representation π,

$$\psi(x) = \pi(\gamma(x))\tilde{\psi}(x) \tag{47}$$

Proposition 1. *Such decomposition exists for any choice of ψ i.e. $\pi^{-1}(\gamma(x))\psi(x)$ satisfies* (46)

Proof. Consider $V_\psi(g) = (\psi(x), \pi(g)\psi_0)$ where we used an invariant product $(,)$ on rep. space. Since V_ψ is continous and real on G and bounded, it acquires it's maxima at, say, x. Let $\gamma(x) \in G$ be it's maximum and $\delta\pi(\gamma(x)) = \pi(gamma)\pi(i\varepsilon^a t_a)$. Then $0 = \delta_g V_\psi(g) = (\psi(x), \delta(\pi(\gamma))\psi_0) = i\varepsilon^a(\pi^{-1}(\gamma)\psi(x), \pi(t^a)\psi_0)$. which is true for all variatons, then:

$$0 = (\underbrace{\pi^{-1}(\gamma)\psi(x)}_{\tilde{\psi}(x)}, \pi(t^a)\psi_0)$$

defines $\tilde{\psi}$.

Next, we parametrize γ. As γ is not uniquely defined ($V_\psi(g) = V_\psi(gh)$, as $\psi(h)\psi_0 = \psi_0$) then we require a maremetrization of the cosets G/H. Make an adapted choice of generators g. Let $\{s_\gamma\}$ generators of $\mathfrak{h} \subset \mathfrak{g}$ and extend to a basis of $\mathfrak{g}$ (at least around $[H] \in G/H$). The resulting basis (without the H) gives a parametrization of Goldstone bosons □

Example III.3. $U(2) \times U(2)$ *revisited.*

$$\Sigma \to e^{i\tau^a}\Sigma e^{-i\tau^b} = \Sigma + i\tau^a\Sigma - \Sigma i\tau^b + ... \implies \delta\Sigma_{ij} = i(\tau^a_{ik}\delta_{lj} - \delta_{ik}\tau^b_{lj})\Sigma_{kl}$$

Recall $\langle\Sigma\rangle \sim id$ and, since the symmetry is spontanouslt broken ($V \neq 0$), then preserved generators satisfy

$$i(\tau^a_{ik}\delta_{lj} - \delta_{ik}\tau^b_{lj})\langle\Sigma\rangle = 0 \implies \tau^a_{ij} = \tau^b_{ij} \tag{48}$$

so a basis of broken generators is $\tau^a_{ik}\delta_{lj}$ such that $\tau^a_{ik}\delta_{lj}\langle\Sigma\rangle$ does not vanish identically. Hence, the symmetry transformation must be diagonal and so massive modes need to be orthogonal to these directions.

Remark III.4. *For real $\pi(it^a)$ and ψ we will drop π and more over, given a complex representation iT^a acting on χ complex field, we can always decompose into double dimension real (where unitary becomes orthogonnal) by*

$$it^a = \begin{pmatrix} ReiT^a & -ImiT^a \\ ImiT^a & ReiT^a \end{pmatrix} = \begin{pmatrix} -ImT^a & -ReT^a \\ ReT^a & -ImT^a \end{pmatrix}, \quad \psi(x) = \begin{pmatrix} Re\chi \\ Im\chi \end{pmatrix}.$$

Remark III.5. *We can express the euclidian innerproduct such that the real image on the cplx representation directly in terms of complex representation,*

$$(\tilde{\psi}, it^a\psi) = (Re\tilde{\chi}, Im\tilde{\chi})it^a \begin{pmatrix} Re\chi \\ Im\chi \end{pmatrix} = Re\tilde{\chi}ReiT^a\chi + Im\tilde{\chi}ImiT^a\chi = Re\tilde{\chi^\dagger}iT^a\chi = Im\tilde{\chi}^\dagger T^a\chi.$$

Hence, the variation of V_ψ gives

$$\begin{aligned} & 0 = Im\tilde{\Sigma}^*_{ij}\tau^a_{ik}\delta_{lj}\langle\Sigma\rangle_{kl} \\ \implies & 0 = Im\tilde{\Sigma}^*_{il}\tau^a_{il} = Imtr(\tilde{\Sigma}^\dagger\tau^a) \\ & = \frac{1}{2i}Tr\big(\tilde{\Sigma}^\dagger\tau^a - (\tau^a)^\dagger\tilde{\Sigma}\big) = \frac{1}{2i}Tr\big(\tau(\tilde{\Sigma}^\dagger - \tilde{\Sigma})\big). \end{aligned}$$

Hence, traces vanishes if and only if $\tilde{\Sigma}$ is hermitian. This is consistent with our result before (doing a polar decomposition on

$$\Sigma(x) = \frac{1}{2}\Big(vid + \rho(phi(x))e^{i\phi}e^{2i\frac{\pi^a(x)\tau_a}{f_\pi}}\Big)$$

and identifying the Goldstones as the unitary part and then defining π through parametrization). We thus map to Σ via

$$\tilde{\Sigma} \to \Sigma = \underbrace{e^{i\phi}e^{2i\frac{\pi^a(x)\tau_a}{f_\pi}}}_{\pi(\gamma(x))}\tilde{\Sigma}$$

Back to general formalism, as LL is invariant under $\psi \to \pi(g)\psi$, $g \in G$ (though ψ_0 is not), then $\mathcal{L}$ in terms if $\psi(x) = \pi(\gamma(x))\tilde{\psi}(x)$ will depend on $\gamma(x)$, and hence on the Goldstone bosons, only via derivatives.

2. Unitary gauge

Notice that the constraint

$$0 = \tilde{\phi}_n(x)t_a^{nm}v_m \tag{49}$$

is a gauge fixing condition and so we can eliminate all the Goldstone dependance. This is the unitary gauge. Indeed, since the dependence of $D_\mu\phi_n = \partial_\mu\phi_n - igA_\mu^a t_{nm}^a\phi_n(x)$ then, upon shifting $\phi = \tilde{\phi} + v$ where $\tilde{\phi}$ does not have a vev, then the unitary gauge (49) vanish some of the terms in the kinetic $D_\mu\phi D^\mu\phi$. Infact, at second order,

$$\mathcal{L}_{\phi,quad} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\underbrace{g^2t_{nm}^a t_{nl}^b v_m v_l}_{\mu_{ab}^2} A_{a\mu}A_b^\mu$$

where the mass matrix is symmetric, real and semidefinite positive so we conclude $c_a t^a v = 0 \iff c_a\mu_{ab}^2 c^b = 0t$ so gauge field components become massive iff associated generator is broken.

The resulting theory has massive propagators that have superficial degree of divergence non-renormalizable, propagator:

$$\pi = -(p^2 - \mu^2)^{-1}(g_{\mu\nu} - \mu^{-2}p_\mu p_\nu)$$

Use R_ξ gauges $f(A) = \partial_\mu A^\mu + i\xi g\tilde{\phi}t_a V$ where as we saw before the action term of this gauge fixing will give $\frac{1}{2\xi}(f(A))^2$ recovering the Unitarity gauge at $\xi \to \infty$. Working out the propagator it gives

$$\pi_{\mu\nu} = -\frac{1}{p^2 - \mu^2}\left(g_{\mu\nu} - (1-\xi)\frac{1}{p^2 - \mu^2\xi}p_\mu p_\nu\right)$$

which has fine behavour $1/p^2$ except at $\xi \to \infty$ but in that case the ghost decouple.

E. Weak interactions

We consider electro-weak symmetry breaking $SU(2) \times U(1)_Y \to U(1)_{EM}$ with $SU(2)$ and $U(1)$ gauge strength tensors W and B respectively. We SSB into electromagnetism by H **on the defining rep of** $SU(2)$ **and hypercharge** q_Y. Notice we can write

$$\delta H = \sum_1^2 i\varepsilon^a\tau_a H + i\varepsilon^+(\tau^3 + q_y Id)H + i\varepsilon^-(\tau^3 - q_Y id)H$$

for $\varepsilon^\pm = \frac{1}{2}(\varepsilon^3 \pm \varepsilon)$. **Choosing** $q_Y = \frac{1}{2}$ we can find $(\tau^3 - q_y Id) = diag\{1,0\}$ and $(\tau^3 - q_y Id) = diag\{0,1\}$. Taking $\langle H\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ breaks $\tau^1, \tau^2, (\tau^3 - id/2)$. Unbroken $(\tau^3 + id/2) = 0$ generates $U(1)_{EM}$.

Standard higgsing $D_\mu H = \partial_\mu H - igW^a\tau_a H - \frac{1}{2}ig'B_\mu H$,

$$\mathcal{L}_H = (D_\mu H)^\dagger(D_\mu H) + m^2 H^\dagger H - \lambda(H^\dagger H)^2$$

in the unitarity gauge (recall H being a complex defining rep of $SU(2)$ has 4 real d.o.f.)

$$H = g(x)\begin{pmatrix} 0 \\ \frac{\tilde{h}}{\sqrt{2}} \end{pmatrix}$$

with g parametrized by dimension of $SU(2) \oplus U(1)-$ dim $U(1) = 3$ parameters and $\tilde{h}$ being real scalar whose vev is v. Shifting $\tilde{h} = v - h$ and noting h is electrically uncharged (the coupling $(\tau^3 + q_y id)H$ do not contain h) then we can find the mass matrix for gauge fields W^1, W^2

- $W^{1,2} \to m_w^2 = \frac{v^2}{4}g^2$

For the other fields, recall the unbroken generator is $T = (tau^3 + id/3)$ and the remaing broken is $\bar{T} = (tau^3 - id/3)$, writing $D_\mu H \sim \frac{1}{2}\big(gW^3 + g'B\big)T + (g'B - gW^3)\bar{T}$, we define $Z = -gW^3 + g'B$ and $A = gW^3 + g'B$ for Z will acquire a mass and A will remain masless. Going from $W^3, B \to Z, A$ gauge fields, we define the Weinberg angle

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix}\begin{pmatrix} W^3 \\ B \end{pmatrix}$$

to which we see $\tan\theta_w = \frac{g'}{g}$ and, in the W^1, W^2, Z and A basis we see

$$m_Z = \frac{m_w}{\theta_w}.$$

To express W electromagnetic charge, we go in the we go to W^+, W^-, Z, A basis ($W = W^3\tau_3 + W^+\tau_+ W^-\tau_-$) with $\tau^\pm = \frac{1}{\sqrt{2}}(\tau^1 \pm i\tau^2)$. As a transformation of W gives:

$$W^+\tau_+ \to W^+\tau_+ + i\varepsilon\underbrace{[\tau_3, W^+\tau_+]}_{W^+\tau_+}$$

we conclude W has $+1$ charge and similarly, W^- has -1 charge.
Many interactions. Self interactions, cubic interactions, Z with $W^\pm$, quartic interactions,

- Gauge interactions: $W^+W^-(A/Z)$, W^+W^-xy where x, y any gauge field (interaciton that conserves charge)...
- Gauge-boson interaction: No HWZ interaction. $W^+W^-h, ZZh, W^+W^-hh, ZZhh$
- Higgs-Higgs interaction: Three vertex, for vertex

Fermions: L =Doublets, could be of any generations e.g.

$$L = \begin{pmatrix} \nu_{e_L} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu_L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau_L} \\ \tau_L \end{pmatrix}$$

for first generations. Quarks:Q same in up down doublets of same generation e.g.

$$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}$$

Electroweak neutral:

$$e^i = e_R, \mu_R, \tau_R, \quad \nu^i = \nu_{e_R}, \nu_{\mu_R}, \nu_{\tau_R}, \quad u^i = u_R, c_R, t_R, \quad d^i = d_R, s_R, b_R$$

$$\mathcal{L} = i\bar{L}_i(\partial\!\!\!/ - igW\!\!\!/^a\tau_a - ig'Y_L B\!\!\!/)L_i + i\bar{Q}_i(\partial\!\!\!/ - igW\!\!\!/^a\tau_a - ig'Y_Q B\!\!\!/)Q_i i\bar{e}_i(\partial\!\!\!/ - ig'Y_e B\!\!\!/)e_i \\ +(\nu \leftrightarrow e) + (u \leftrightarrow e) + (d \leftrightarrow e) \tag{50}$$

here we have implicit $L = \frac{1}{2}(1-\gamma^5)L$ and similarly for e_R and we have not included copling of quarks to $SU(3)$. Since

$$\begin{pmatrix} W^3 \\ B \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix}$$

we have $e = g\sin\theta_w = g'\cos\theta_w$, $\tau^3 = \frac{1}{2}diag\{1,-1\}$. Looking at the left handed fermion, the electron is charged -1 then $y_L = -\frac{1}{2}$ which gives 0 charge to neutrino, as expected. Quarks can be found simiarly.

1. *Fermion mass*

Introducing a mass term for the fermions is not possible as it would break the chirality. We use higgs

$$\mathcal{L}_{\text{Yukawa}} = -y_e\bar{H}He + h.c.$$

note that charges add up, $Y_h = 1/2$, $Y_L = -1/2$ ($\bar{L}$ has the opposite) and $Y_e = -1$. This term generates masses for e ($\frac{1}{\sqrt{2}}y_e v$) but not ν as $\langle H\rangle \sim (0,1)^T$ and analogously for down with

$$\mathcal{L} = y_d\bar{Q}Hd$$

To give mas for up and neutrinos we notice $\bar{2} = 2$ representation of $SU(2)$ so there exists A s.t. $A(i\tau^a)^*A^{-1} = i\tau^a$. Suing $\{\sigma^a,\sigma^b\} = 2\delta^{ab}$ and σ^2 is onlt imaginary valued pauli matrix

$$\sigma^2H^* \to \sigma^2(i\tau^aH)^* = i\tau^a\sigma^2H^*$$

so $\tilde{H} = i\sigma^2H^*$ has

$$\langle\tilde{H}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix}$$

which allow us to write

$$\mathcal{L} = -y_\nu\bar{L}\tilde{H}\nu_R - y_q\tilde{Q}\tilde{H}u$$

We conclude

$$\mathcal{L}^{\text{Quark}_{\text{Mass}+\bar{q}qh}} = -\frac{v}{\sqrt{2}}(\bar{d}^i_L Y^d_{ij} d^i_R + ...) \tag{51}$$

we dont want off diagonal terms. We invoke $Y = uMK^\dagger$ where u and k are unitary and M diagonal (use spectral theorem for H hermitian $Y = HV = UDU^\dagger V$). This allow us to go to the **mass basis** $u^\dagger d_L = d^m_L$ and $d^m_R = k^\dagger_d d_R$ and like wise for the others. This has mixed generations for terms that contain a τ^a generator as $K^\dagger_u\tau K_u \neq= id$ this happens only on left W^+, W^- as right are W neutral and W^3 is diagonal.

$$\mathcal{L} \sim i(\bar{\tilde{u}}_L\,\bar{\tilde{d}}_L)^i(\partial\!\!\!/ - igW\!\!\!/^3\tau_3 - ig'Y_Q B\!\!\!/)\begin{pmatrix} \tilde{u}_L \\ \tilde{d}_L \end{pmatrix} + i\bar{\tilde{u}}_R D\!\!\!\!/\,\tilde{u}_R + (u \leftrightarrow d) + i\bar{\tilde{u}}_L gW\!\!\!/^+ \underbrace{u^\dagger_u u_d}_{V_{CKM}} \tilde{d}_L + (d \leftrightarrow u)$$

V_{CKM} is unitary so it has $(2(n^2-n)/2 + n = n^2)$ real parameters. In particular, for 3 generations, we can parametrize by 3 angles ($O(3)$) and 6 phases. but mass terms (where we defined the u's and d's) has $U(1)^6$ symmetry,

$$(\tilde{d}^i_L, \tilde{d}^i_R) \to e^{i\alpha_i}(\tilde{d}^i_L, \tilde{d}^i_R) \\ (\tilde{u}^i_L, \tilde{u}^i_R) \to e^{i\beta_i}(\tilde{u}^i_L, \tilde{u}^i_R) \tag{52}$$

but diagonal $\alpha_i = \beta_j = v$ leaves V_{CKM} invariant so we can eliminate 5 phases. In total, 3 angles + 1 phase. One can show phase induce CP violation while for $SU(2)$ only two generations, all phases can be absorved: 1 angle + 3 phases and from $U(1)^{2\times 2}$ symmetry (up and down and two colours)4 phases -1 diagonal $= 3$ phases so we get only one angle.

F. Anomalies

Recall that conservation of currents at classical level translates to

$$\partial_\mu \langle\Omega| T\{J^\mu(x)\dots\} |\Omega\rangle = \text{Contact terms from } \partial \text{ acting on the time ordered } \Theta \tag{53}$$

An anomaly exists if there are no regularization procedure which respects the symmetry while preserving the symmetry. Dimensional regularization is polyvalent but it highly depends on d and thus γ^5.

1. *Chiral anomaly*

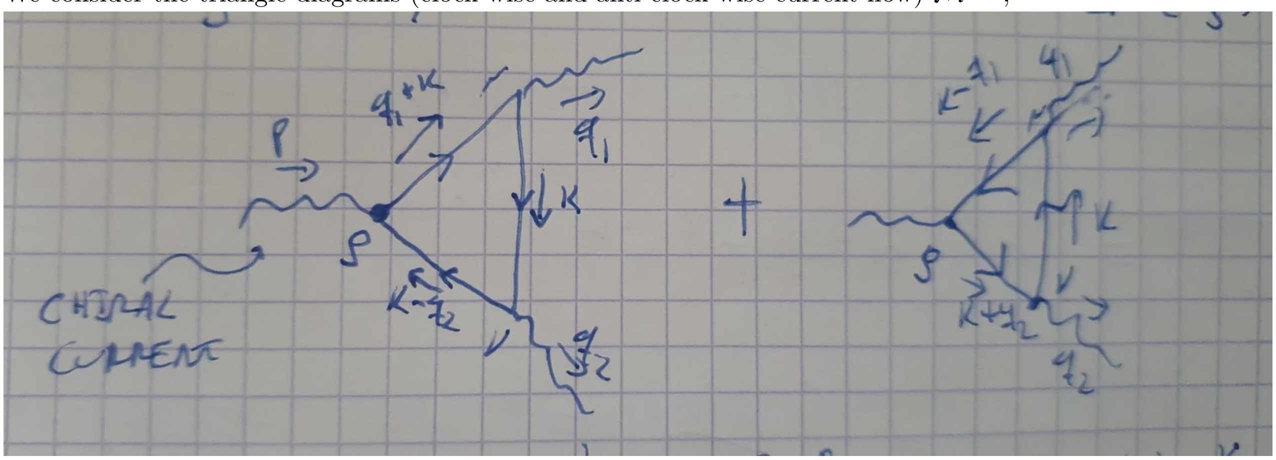
The non-conservation of axial current (at $m = 0$) must be proportinal to a parity violating pseudo-tensor. Only at our disposal is $\varepsilon_{\mu_1 \dots \mu_4}$. Has to be contracted with polarization tensors of gauge fields or momenta.

Anomalous diagram with smallest number of external legs will have only gauge fields then $\underbrace{n}_{\text{Ext.legs}} + \underbrace{(n-1)}_{Indep.momenta} \geq 4$ thus $n \geq 5$ so $n = 3$. We consider

$$\langle\Omega| T\{\underbrace{J_5^\rho(x)}_{\text{Chiralcurrent}} J^\mu(y) J^\nu(z)\} |\Omega\rangle$$

where recall $\partial_\rho J_5^\rho(x) = 2im\bar{\psi}\gamma^5\psi$.

We consider the triangle diagrams (clock wise and anti-clock wise current flow) $\mathcal{M}^{\rho\mu\nu}$,



Demanding current conservation on J currents we arrive to an expression that seems to vanish:

$$q_{1\mu}\mathcal{M}^{\rho\mu\nu} = -4i\varepsilon^{\mu\nu\sigma\rho} \int \frac{d^4k}{(2\pi)^4} \frac{(k-q_1)_\mu (k+q_2)_\sigma}{(k+q_2)^2(k-q_1)^2} - \frac{(k-q_2)_\mu(k+q_1)_\sigma}{(k+q_1)^2(k-q_2)^2} = \int \frac{d^4k}{(2\pi)^4} F^{\nu\rho}(k+q_2-q_1) - F^{\nu\rho}(k)$$

for a change of variable $k = \tilde{k} + q_1 - q_2$ on the first integral.

We should be careful because this diagram has a linear divergence. To do so, consider $\Delta(a) = \int_{\mathbb{R}} dx f(x+a) - f(x)$ for $f(x) \sim c_\pm$ as $x \to \pm\infty$. Taylor expanding, we find $\Delta(a) = a(f(\infty) - f(-\infty))$. For $4d$, and linear divergences $F^\mu(k_E) \sim Ak_E^\mu / k_E^4$, after wick rotating, taylor expandindg and applying stokes,

$$\begin{aligned}
\Delta^\mu(a) &= \int \frac{d^4k}{(2\pi)^4} F^\mu(k+a) - F^\mu(k) \\
&= i\int \frac{a \cdot S_\infty}{(2\pi)^4}(F^\mu(k_E) + \frac{1}{2}a^\rho\partial_\rho F^\mu(k_E) + \dots) \\
&= \lim_{|k_E|\to\infty} i\int (a\cdot\hat{k}_E)\frac{|k_E|^3}{(2\pi)^4}F^\mu(k_E)d\Omega_4 \\
&= \lim_{|k_E|\to\infty} ia_\rho \int k_E^2 k_E^\rho \frac{Ak_E^\mu}{(2\pi)^4}F^\mu(k_E)d\Omega_4 = \frac{i}{32\pi^2}Aa^\mu
\end{aligned}$$

where we used $k_E^\rho k_E^\mu = \frac{1}{4} g^{\mu\nu} k_E^2$.

Since $F^{\nu\rho} \to \underbrace{-4i\varepsilon^{\mu\nu\sigma\rho}(q_1+q_2)_\sigma}_{A} \frac{k_\mu}{(k^2)^2}$ and, $a = q_2 - q_1$, for our triangle diagrams we have

$$q_{1\mu}\mathcal{M}^{\rho\mu\nu} = \frac{i}{4\pi^2}\varepsilon^{\mu\nu\sigma\rho} q_{1\sigma} q_{2\rho} \neq 0 \tag{54}$$

telling that current is non-conserved. Let us see the axial current $p_\rho \mathcal{M}^{\rho\mu\nu}$. Similaryl for $F^q_{\sigma\rho}(k) = \frac{k_\sigma q_\rho}{k^2(k^2+q^2)^2} \sim \frac{k_\sigma q_\rho}{k^4}$ at $k >> 1$, we find:

$$\begin{aligned} p_\rho \mathcal{M}^{\rho\mu\nu} &= 4i\varepsilon^{\mu\nu\rho\sigma} \int \frac{d^4k}{(2\pi)^4} F^{q_2}_{\sigma\rho}(k-q_2) + F^{q_1}_{\sigma\rho}(k) - F^{q_1}_{\sigma\rho}(k-q_1) - F^{q_2}_{\sigma\rho}(k) \\ &= 4i\varepsilon^{\mu\nu\rho\sigma}\frac{i}{32\pi^2}[\underbrace{q_{2\rho}}_{A}\underbrace{(-q_2)_\sigma}_{a} + q_{1\rho} q_{1\sigma}] = 0 \end{aligned}$$

so axial current is conserved.This is bad for the gauge symmetry $U(1)V$ and good for $U(1)_A$. Can we have things other way around? To implement the "forbiden" shift that the current required to vanish we can just implement it directly from the Feynmann diagram writing $k \to k + b_1 q_1 + b_2 q_2$ we know $b_1 = 1$ $b_2 = -1$ already gives a conserved current. More generally, plugging in $q_1 \mathcal{M}^{\rho\mu\nu} = 0$ iff $b_1 - b_2 = 0$. And the axial current is conserved if $b1 = b2$. This conditions are incompatible. So no choice of b's allows for both to be conserved. We must choose $b_1 - b_2 = 2$ to preserve gauge symmetry and, then the chiral symmetry is anomalous:

$$p_\rho \mathcal{M}^{\rho\mu\nu} = \frac{1}{4\pi^2}\varepsilon^{\mu\nu\sigma\rho} q_{1\sigma} q_{2\rho} \tag{55}$$

We can express the anomaly as a vev of a theory with external current coupled to J^μ, $\Delta S = -eJ \cdot A$ which gives

$$\begin{aligned} \partial_\rho \langle J_5^\rho(x)\rangle_A &= \frac{e^2}{2}\int d^4y d^4z \langle \partial_\rho J_5^\rho(x) J^\mu(y) J^\nu(z)\rangle A_\mu(y) A_\nu(z) + O(e^4) \\ &= -\frac{e^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = -\frac{e^2}{32\pi^2} F \wedge F \end{aligned} \tag{56}$$

2. Anomalies in chiral gauge theories

We consider the 3-current pt-function $\langle J^\mu(x) J^\nu(y) J^\rho(z)\rangle$ in a theory with chiral (massless) fermions. The triangle diagram gives $\mathcal{M} \sim \mathrm{tr}\Big\{\gamma^\mu p_L \frac{\not k}{k^2}\gamma^\nu p_L \frac{\not p}{p^2}\gamma^\rho p_L \frac{\not q}{q^2}\Big\}$ which gives:

$$\mathcal{M}^{\rho\mu\nu} = \frac{1}{2}\mathcal{M}_V^{\rho\mu\nu} \pm \frac{1}{2}\mathcal{M}_A^{\rho\mu\nu}$$

for the vector an axial triangles and $\pm$ is either right or left fermions. In the case of non-abelian gauge symmetry (fermions couple with Yang-mills theory) then each orientation of the diagram gives:

- For the clock wise: $\mathrm{tr}\{T^a T^b T^c\} = \frac{1}{2} i f^{abd}\,\mathrm{tr}\{T^d T^c\} + \frac{1}{2}\mathrm{tr}\{\{T^a, T^b\} T^c\} = \frac{1}{2} i f^{abd} T(\pi) 2\frac{1}{2}\delta^{dc} + \frac{1}{4} d_\pi^{abc}$ where the first term gives rise to contact terms (totally anti-symmetric) while the second to the anomaly.
- For the opposite fermion current: $\mathrm{tr}\{T^a T^c T^b\}$ similarly.

For $SU(N)$ we can specify completely $d_T^{abc} = A(\pi) d_N^{abc}$ where N is in the defining representation and A is the **anomaly coeffcicent**.

All together,

$$\langle \partial_\mu J^{a,\mu}\rangle = \frac{1}{2}\left(\sum_{\pi\in\pi_L} A(\pi) - \sum_{\pi\in\pi_R} A(\pi)\right)\frac{g^2}{128\pi^2} d_N^{abc}\varepsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma} \tag{57}$$

so an anomaly is abscent if either there is an anomaly coefficient cancelation or the defining representation coefficient d_N is 0.

When d_N vanish? Consider π real i.e. $\exists U$ s.t. $\pi^*(\tau^a) = U\pi(\tau^a)U^{-1}$. Since $iT^a = \pi(i\tau^a)$ is real, $\pi(i\tau^a)^* = -i(T^a)^T$ as T is hermitian. Then

$$d_\pi^{abc} = 2\,\mathrm{tr}\{\{T^a, T^b\}T^c\} = 2\,\mathrm{tr}\{U\{T^a, T^b\}T^c U^{-1}\} = 2\,\mathrm{tr}\{\{-(T^a)^T, (-T^b)^T\}(-T^c)^T\} = -d_\pi^{abc}$$

so $d_\pi = 0$. Only complex representations can contribute to anomaly.

3. Gauge anomaly in SM

For the SM we have $G = SU(3) \times SU(2) \times U(1)_Y$. The following triangle diagrams exists (here $A \times B \times C$ means a trinagle with vertes of currents of symmetry A, B, C):

1. $U(1)^3$:

$$\langle \partial_\mu J_Y^\mu \rangle \sim (\sum_{\pi \in \pi_L} Y_\pi^3 - \sum_{\pi \in \pi_R} Y_\pi^3) \sim \sum_{\text{Gen.}} [(2Y_L^3 - Y_e^3 - Y_\nu^3) + \underbrace{3}_{\text{colors}}(2Y_Q^3 - Y_u^3 - Y_d^3)]$$

   pluging in the charges:

$$= \sum_{\text{Gen.}} \left[\underbrace{(2\left(-\frac{1}{2}\right)^3 - (-1)^3 - 0^3)}_{-\frac{5}{4}} + \underbrace{3\left(2\left(\frac{1}{6}\right)^3 - \frac{2}{3}^3 - \left(-\frac{1}{3}\right)^3\right)}_{\frac{5}{4}} \right] = 0$$

2. $SU(2)^3$: $2 = \bar{2}$ so $d = 0$.
3. $SU(3)^3$: $SU(3)$ is not chiral in SM.
   Mixed anomalies:
4. $SU(N) \times U(1)^2$: $d^{a11} = 2\,\mathrm{tr}\{T^a 1, 1\} = 0$
5. $SU(N) \times G$: $d^{a\beta\gamma} = 2\,\mathrm{tr}\big\{T^a\{T^\beta, T^\gamma\}\big\} = 2\,\mathrm{tr}\{T^a\}\,\mathrm{tr}\big\{\{T^\beta, T^\gamma\}\big\}$=0
6. $SU(3)^2 \times U(1)$: $\sim \sum_{\text{Gen.}} \delta^{ab}(2Y_Q - Y_u - Y_d)3 = 0$
7. $SU(2)^2 \times U(1)$: $\sum_{left} 2\,\mathrm{tr}\big\{Yt^a, t^b\big\} = 2\delta^{ab} \sum_{\text{Gen.}} (2\underbrace{Y_L}_{-\frac{1}{2}} + 3 \cdot 2\underbrace{Y_Q}_{\frac{1}{6}}) = 0$

Note that being anomaly free of $U(1)_Y^3$ and $SU(2)U(1)^2$ forces relations between leptons and quark hypercharges. Same number of generations in lepton and quarks sector.

IV. CLASSICAL AND QUANTUM CFTS

Notes of [1].

Chiral Algebra Definition: A closed subalgebra of the operator product algebra in a conformal field theory, formed by holomorphic fields (this due to operator product expansion is also holomorphic) containing at least the unit operator and the stress tensor.

examples. A is the enveloping algebra of Virasoro. A somewhat more involved example is given by Kac-Moody algebras. The super Virasoro algebra has a (super) Lie algebra structure but its enveloping algebra is not a chiral algebra by the above definition. We should project out of it all the half integral weight operators.

Perhaps the simplest example of a non- trivial algebra $\mathcal{A}$ is the "rational torus". This algebra is generated by ∂X and $e^{\pm}i\sqrt{2N}X$ where N is an integer. It can be understood by the procedure above as a sub-algebra of the enveloping algebra of various Kac-Moody algebras (e.g. of $SU(2N)$ level I). Its representations can easily be found. Since it includes $U(1)$ KM as a subalgebra, its representations are also representations of $U(1)$ KM. These are labeled by a real number $k-$the momentum. The presence of the operator $e^{\pm}i\sqrt{2N}X$ excludes most of the momenta. Only $k = \frac{m}{\sqrt{2N}}$ with m integer is local relative to $e^{\pm}i\sqrt{2N}X$. Furthermore, different values of m which correspond to different KM representations are combined into the same iτreducible representations of this chiral algebra. There are only $2N$ irreducible representations: $m = -N+1, -N+2, ... - 1, 0, 1, ...N$. The representationm $m = 0$ includes the identity operator and the whole chiral algebra. It is the basic representation.

Definition of chiral vertex operators

[1] G. W. Moore and N. Seiberg, Commun. Math. Phys. **123**, 177 (1989).