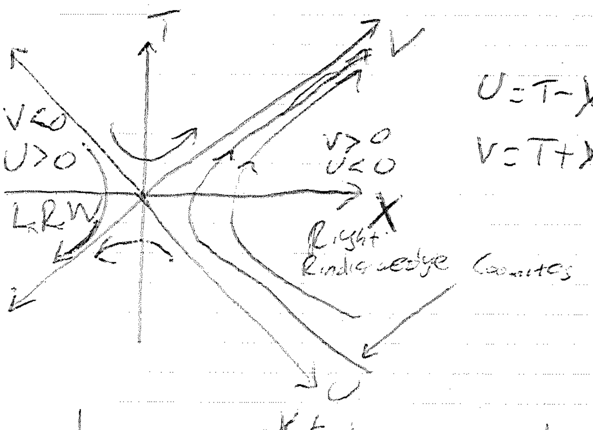
1-particle density matrix $\rho = S_1^* S_1 \overset{\text{today}}{\sim} e^{-\beta H t}$ $i\frac{\partial}{\partial t}$ → Time translation

$\tilde{a}^*(\xi) = a^*(S_1 f) + a(S_1 f) \rightsquigarrow$ inertial observer time Killing vector $\frac{\partial}{\partial T}$
↳ T

$\frac{\partial}{\partial T}\phi = i\omega\phi = i|\omega| J\phi$

positive frequency → a^*
negative " → a

$\rightsquigarrow$ accelerated observer Killing vector $\frac{\partial}{\partial t}$ ← boost generator
↳ t



$$\left.\begin{aligned} U &= T - X \\ V &= T + X \end{aligned}\right\} \quad \left.\begin{aligned} x &= \sqrt{-UV} = \sqrt{X^2 - T^2} \\ t &= \frac{1}{2\kappa}\ln\left(-\frac{V}{U}\right) \end{aligned}\right\} \rightsquigarrow \begin{aligned} U &= -x e^{-\kappa t} \\ V &= x e^{\kappa t} \end{aligned}$$

RRW } is globally hyperbolic (one can determine everything by evolution of a Cauchy surface which exists and is at $T=0$)
LRW }

$$dU = -e^{-\kappa t}dx + \kappa U dt = -\sqrt{\frac{-U}{V}}\,dx + (\kappa U dt)(1)$$

$$dV = e^{\kappa t}dx + \kappa V dt = \sqrt{\frac{V}{-U}}\,dx + \kappa V dt$$

$$\Rightarrow ds^2 = -dU dV = dx^2 + \kappa^2 \underbrace{UV}_{-x^2} dt^2 \cancel{-\kappa\sqrt{-UV}\,dx\,dt} \cancel{+\kappa\sqrt{-UV}\,dx\,dt}$$

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2$$ ← no explicit t → $\frac{\partial}{\partial t}$ is Killing (generator of translations on t)

$$\frac{\partial}{\partial t} = \frac{\partial U}{\partial t}\frac{\partial}{\partial U} + \frac{\partial V}{\partial t}\frac{\partial}{\partial V} = -\kappa U\frac{\partial}{\partial U} + \kappa V\frac{\partial}{\partial V} = \kappa\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right)$$

$$= \kappa\left(T\frac{\partial}{\partial X} + X\frac{\partial}{\partial T}\right) \leftarrow \text{boost} \qquad \left\|\frac{\partial}{\partial t}\right\|^2 =$$

wave
K-G eq. $\quad 0 = \left(-\frac{\partial^2}{\partial T^2} + \frac{\partial^2}{\partial X^2}\right)\phi = -\partial_U\partial_V\phi$, $\quad \phi = f(U) + g(V)$

positive freq solutions $\quad u_p(T,X) = c_p e^{-i(\omega T - pX)}$, $|p| = \omega$

For $p>0$ $\quad u_p = c_p e^{-ipU} = c_p e^{-i\omega U}$ ← right movers
For $p<0$ $\quad u_p = c_p e^{ipV} = c_p e^{-i\omega V}$ ← left movers

In Rindler coordinates assume t dependence as $e^{-i\sigma t}$

$$0 = \Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right) = \frac{1}{\kappa x}\left(\partial_x(\kappa x\,\partial_x\phi) + \partial_t\left(\frac{\kappa x}{(-\kappa^2 x^2)}\partial_t\phi\right)\right)$$

$$= \frac{1}{x}\left(\partial_x\phi + x\,\partial_x^2\phi\right) - \frac{1}{\kappa^2 x^2}\partial_t^2\phi$$

$$= \frac{1}{x^2}\left(x\partial_x(x\partial_x\phi) + \frac{\sigma^2}{\kappa^2}\phi\right) \rightsquigarrow \phi_\sigma = x^{\pm i\frac{\sigma}{\kappa}}$$

$|p|\kappa = \sigma$ ← positive frequency

↓
$$\phi = c_p e^{-i\sigma t} x^{ip} = c_p e^{-i(\sigma t - p\ln x)}$$

The two notions of positive frequency (Rindler and inertial) do not agree

Phase: $\sigma t - p\ln x = \frac{\sigma}{2\kappa}\ln\left(-\frac{V}{U}\right) - p\ln(x)$

for $p<0$ $\sigma = -\kappa p \sim \sigma t - p \ln x = -p \ln V \sim \phi = e^{-\frac{i\sigma}{\kappa}\ln(V)}$

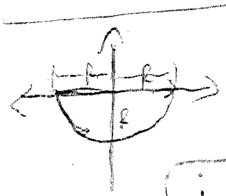
For modes on the left use symmetry $(U,V) \to (-U,-V)$ and take care of boost going to the past by complex conjugation

$$\sigma_L = \begin{cases} c_p\, e^{\frac{i\sigma}{\kappa}\ln U} & U>0 \\ 0 & U<0 \\ 0 & V>0 \\ c_p\, e^{-\frac{i\sigma}{\kappa}\ln(-V)} & V<0 \end{cases}$$

Fourier transforms: (cool on frequency modes for holomorphic functions)

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} du\, e^{i\omega u} f(u).$$

If $f(u)$ is holomorphic in $\operatorname{Im} u<0$ and bounded we can close the contour for $\omega<0$ and get 0.



$\Rightarrow \tilde{f}$ has only positive frequency modes.

$\boxed{\text{i.e. Holomorphicity in } \operatorname{Im} u<0 \Rightarrow \text{pos freq}}$

Recall $\phi = e^{\frac{i\sigma}{\kappa}\ln(-U)}$, $p>0$, complex log is thus pos. freq. if we choose the branch cut in the upper half plane.

$U>0,\ p>0$ $\quad \ln U = \ln(-U) - i\pi$

$$\Rightarrow \sigma_L = c_p\, e^{\frac{i\sigma}{\kappa}\ln U} = c_p\, e^{\frac{\pi\sigma}{\kappa}} e^{\frac{i\sigma}{\kappa}\ln(-U)}$$

the "global" pos. freq. is $V_p = U_R + e^{-\frac{\pi\sigma}{\kappa}} U_L$

both on left and right

fraction of Rindler neg freq. in Minkowski pos. freq.

$$\frac{e^{-\frac{\pi\sigma}{\kappa}}}{1-e^{-\frac{\pi\sigma}{\kappa}}} = \frac{1}{e^{\frac{\pi\sigma}{\kappa}}-1}$$

Claim: "This is a Boltzmann factor"

$\sim e^{-\beta H} = e^{-\frac{\pi\sigma}{\kappa}}$, $\quad \sigma$ = Rindler freq. = Rindler energy = Rindler Hamiltonian

$\beta = \frac{1}{kT} = \frac{\pi}{\kappa} \sim T = \frac{\kappa}{k\pi}$ ← acceleration

$\boxed{\text{Mon Aug 1 Finux exam}}$

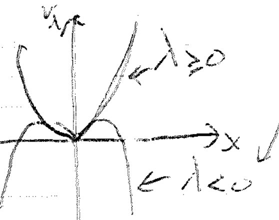
- particle in a box
- Harmonic oscillator
- H-atom

Instantons: Is perturbation theory good enough?

Look at convergence of $Z(\lambda) = \sum \frac{\lambda^k}{k!} Z^{(k)}(0)$ ← $Z^{(k)}(0)$ we know

E.g. $H = p^2 + V_\lambda(x) = p^2 + x^2 + \lambda x^4$

$x^2+\lambda x^4$



Instable $\sim$ Radius of convergence $r_c = 0$

Let's look a toy model 0-dim φ^4.

$$Z(\lambda) = \int_{-\infty}^{\infty} dx\, e^{-x^2-\lambda x^4}, \quad Z(0) = \sqrt{\pi}, \text{ converges iff } \lambda \geq 0$$

$$= \frac{e^{\frac{1}{8\lambda}} K_{1/4}\left(\frac{1}{8\lambda}\right)}{2\sqrt{\lambda}} \quad \text{← non perturbative}$$

$$= \int_{-\infty}^{\infty} dx\, e^{-x^2}\left(\sum \frac{(-\lambda)^k}{k!} x^{4k}\right) \simeq \sum \frac{(-\lambda)^k}{k!}\int dx\, x^{4k} e^{\dots}$$

↑ perturbative

$$\simeq \sum_{k}^{\infty} \frac{(-\lambda)^k}{k!} \frac{\partial^{2k}}{\partial\alpha^{2k}} I(\alpha)\Big|_{\alpha=1} (-1)^k, \quad I(\alpha) = \sqrt{\frac{\pi}{\alpha}}$$

$\sim$ $\frac{(2(k+1))!}{2^{2k}} = \dots \frac{4k!}{2^{4k}(2k)!} = \sqrt{\pi}\sum^{\infty} \frac{(-1)^k (4k)!}{k!(2k)!\,2^{4k}} \lambda^k$

use Stirling's formula $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \rightsquigarrow \frac{4k!}{k!(2k)!} \simeq \frac{\sqrt{2\pi}\sqrt{4k}\cdot 1}{\sqrt{2\pi}\sqrt{2\pi}\sqrt{k}\sqrt{2k}} \cdot \left(\frac{4k}{e}\right)^{4k} \cdot \left(\frac{e}{2k}\right)^{2k}\left(\frac{e}{k}\right)^k$

$\simeq \frac{1}{\sqrt{\pi k}} \frac{e^{-k}\,2^{8k}}{2^{2k}\,k^{-k}} = \frac{1}{\sqrt{\pi k}} \frac{e^{-k}\,2^{6k}}{1} k^k \simeq 4^{4k}\,k!$

$\Rightarrow$ have 0 convergence radius. bears $Z(\lambda) \sim \sum (-1)^k k!\,\lambda^k$

$Z(\lambda) = \int_{-\infty}^{\infty} dx\, e^{-x^2-\lambda x^4} \overset{u^2=\lambda x^2}{=} \frac{1}{\sqrt{\lambda}}\int_{-\infty}^{\infty} dx\, e^{-\frac{u^2+u^4}{\lambda}}$ ← This integral can be approx using steepest decent.

[steepest decent or in imaginary "stationary phase"

$\int dx\, A(x)\, e^{i\phi(x)\Lambda} \xrightarrow{\text{approx for } \Lambda \gg 1}$, ($A, \phi$ real, ~~continuous~~ smooth), A compact support

Let the critical points of ϕ be x_i, ie $\phi'(x_i) = 0$.

↳ $\int dx\, (A(x_i) + A'(x_i)(x-x_i) + \dots)\, \exp(i\Lambda(\phi(x_i) + 0 + \frac{1}{2}\phi''(x_i)(x-x_i)^2$

$= e^{i\Lambda\phi(x_i)} \frac{1}{\sqrt{\Lambda}}\int_{-\infty}^{\infty} dx\sqrt{\Lambda}\,(A(x_i) + A'(x_i)\frac{\sqrt{\Lambda}(x-x_i)}{\sqrt{\Lambda}} + \dots)(1 + \frac{i\Lambda}{6}\phi'''(x_i)(x-x_i)^3 + \dots)$
$\times (e^{\frac{i\Lambda}{2}\phi''(x_i)(x-x_i)^2})$

$= \frac{e^{i\Lambda\phi(x_i)}}{\sqrt{\Lambda}} A(x_i)\sqrt{\frac{2\pi}{i\phi''(x_i)}}\left(1 + O\left(\frac{1}{\Lambda}\right)\right)$ ← expanded around 1 x_i but each one contributes.

$\rightsquigarrow \int dx\, A(x)\, e^{i\phi(x)\Lambda} = \sum_i \frac{e^{i\Lambda\phi(x_i)}}{\sqrt{\Lambda}} A(x_i)\sqrt{\frac{2\pi}{i\phi''(x_i)}}$]

$Z(\lambda) = \frac{1}{\sqrt{\lambda}}\int_{-\infty}^{\infty} dx\, e^{-\frac{u^2+u^4}{\lambda}}$, $\phi(u) = u^4 + u^2$ Classical eq of motion

$\phi'(u) = 0 = 4u^3 + 2u \rightsquigarrow u = 0, \pm\frac{i}{\sqrt{2}}$

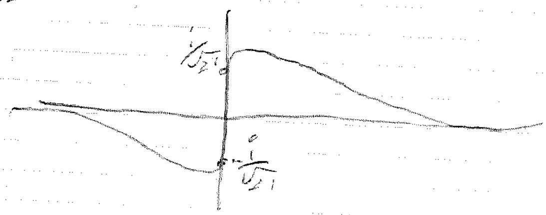
$\phi''(u) = 12u^2 + 2$ ↳ $\phi''(0) = 2$, $\phi''(\pm\frac{i}{\sqrt{2}}) = -4$

$\phi(0) = 0$
$\phi(\pm\frac{i}{\sqrt{2}}) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$\simeq \sum_{0,\pm\frac{i}{\sqrt{2}}} \frac{e^{i\Lambda\phi(x_i)}}{\sqrt{\Lambda}} A(x_i)\sqrt{\frac{2\pi}{i\phi''(x_i)}}$

$= \left(\frac{1}{\sqrt{\Lambda}}\sqrt{\frac{2\pi}{i2}} + O(\Lambda)\right) + \left(\frac{e^{-\frac{i}{4}\Lambda}}{\sqrt{\Lambda}}\sqrt{\frac{2\pi}{i\cdot(-4)}} + O(\Lambda)\right) + \left(\frac{e^{\frac{i}{4}\Lambda}}{\sqrt{\Lambda}}\sqrt{\frac{2\pi}{-i4}} + O(\Lambda)\right)$

usual perturbation theory.



essential singularity at $\lambda = 0$.
Taylor series is 0 (all its derivatives are 0)
Perturbative ~~solutions~~ expansions around imaginary solutions

" we expand around solutions of motion (S = action = $\phi \rightsquigarrow \phi' = 0$ is a extremal)

$\Rightarrow$ Imaginary solutions are important why?

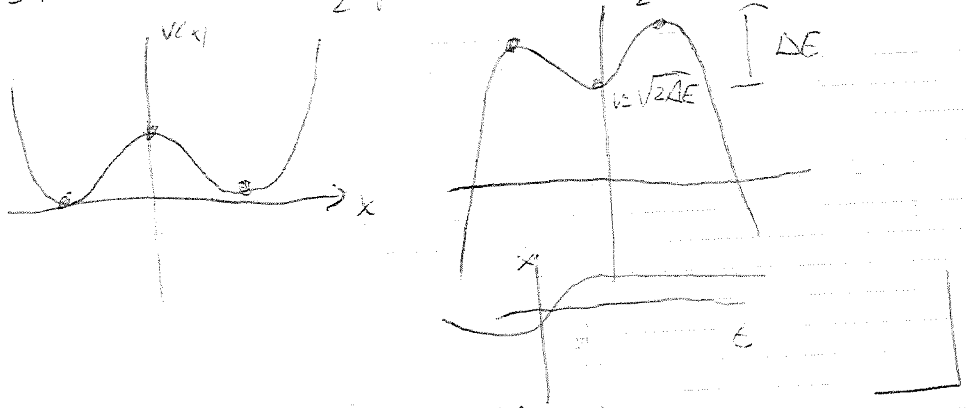
result.

E.g. QED: $\lambda = \alpha_s = \frac{1}{137}$

~~Instantons in Yang-Mills theory (4D)~~

We are looking for complex solutions to eq of motion
↑ Euclidean time (chose) $t = ix^4$

In QM $H = \frac{1}{2}p^2 + V(x) \rightsquigarrow \frac{1}{2}p^2 - V(x)$



$$S = -\frac{1}{2}\int d^4x \; tr\,(F^{\mu\nu}F_{\mu\nu})$$
$$\uparrow \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$A_\mu, F_{\mu\nu}$ Adjoint rep of some Lie algebra.

E.O.M (Remember all Euclidean)

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0.$$

We are interested in finite action solutions → Lagrangian has to go to 0 fast enough as $\|x\| \to \infty$
$\rightsquigarrow A_\mu$ has to be pure gauge asymptotically (it doesn't vanish in spatial...)

$$A_\mu(x) \longrightarrow i\,g(x)^{-1}\partial_\mu g(x) \quad \text{as } \|x\| \to \infty$$
$$g(x): \mathbb{R}^4 \to G$$

actually we only need $g(x)$ on the "sphere S^3 at infinity" parametrizing directions

Dual field strength $(*F)_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$

Nota bene N.B.: In euclidean 4-space $(**F)_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}(*F)_{\rho\sigma} = +F_{\mu\nu}$

"−" in Mink
"+" in Eucl

So we define the ± eigenspaces:

$$F^{sd} = \frac{1}{2}(F + *F) \rightsquigarrow *F^{sd} = F^{sd}$$
(↑ selfdual)
$$F^{asd} = \frac{1}{2}(F - *F) \rightsquigarrow *F^{asd} = -F^{asd}$$

$F = F^{asd} + F^{sd}$

$$0 \le (F \pm *F)^2 = F_{\mu\nu}F_{\mu\nu} \pm 2F_{\mu\nu}(*F)_{\mu\nu} + \underbrace{(*F_{\mu\nu})^2}_{F_{\mu\nu}F_{\mu\nu}} = 2(F_{\mu\nu}^2 \pm F_{\mu\nu}(*F)_{\mu\nu})$$

$$S = \int d^4x\,\mathcal{L} \ge \frac{1}{4}\int d^4x\,tr F_{\mu\nu}(*F)_{\mu\nu} \quad \text{with equality iff } F \pm *F = 0$$

If you keep $\int d^4x\; tr(F_{\mu\nu}(*F)_{\mu\nu})$ fixed, F^{sd} and F^{asd} are minimal action configurations.
(action = energy)

$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$, $\frac{1}{2} F_{\mu\nu} F_{\mu\nu}\, dvol = F \wedge * F$

$\frac{1}{2} F_{\mu\nu} (*F)_{\mu\nu}\, dvol = F \wedge F$; $dvol = *1$

$F\wedge A = A_\mu F_{\nu\rho}\, dx^\nu \wedge dx^\rho \wedge dx^\mu$

$A \wedge F = A_\mu F_{\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$

<u>Bianchi identity:</u> $DF = dF + \boxed{[A, F]} \leftarrow A\wedge F - F \wedge A$

$F = dA + A\wedge A$

The quantity to keep constant is $\int tr(F\wedge F)$, Claim $dK = tr(F \wedge F)$ where $K = tr(A \wedge dA + \frac{2}{3} A\wedge A \wedge A)$ "chern simons form"

$$\int_M tr(F\wedge F) = \int_M dK = \int_{\partial M} K \quad \leftarrow \text{depends only on asymptotic values of } A.$$

$\partial M = S^3$

$\int F\wedge F$ is constant under deformations of A inside compact sets.

Lets compute dK, $dK = tr\left(dA\, dA + \frac{2}{3}\left(dA\, A A - A\, dA\, A + A A\, dA\right)\right)$

$\wedge$ implicit

$= tr\left((F - A^2)(F-A^2) + \frac{2}{3}\left((F-A^2) AA - A(F-A^2)A + A^2(F-A^2)\right)\right) = 0$

$= tr\left(F^2 - FA^2 - A^2 F + \frac{2}{3}(FA^2 - AFA + A^2 F)\right) = tr(F^2 - 2FA^2 + 2FA^2) = tr\, F^2$ $\square$

$tr(A^4) = 0$

$$\begin{aligned} tr(FA^2) &= \frac{1}{2} tr(F_{\mu\nu} A_\rho A_\sigma)\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= \frac{1}{2} tr(A_\sigma F_{\mu\nu} A_\rho)\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= -\frac{1}{2} tr(A_\sigma F_{\mu\nu} A_\rho)\, dx^\sigma \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho \\ &= -tr(AFA) \underset{\text{again}}{=} + tr(A^2 F) \end{aligned}$$

$$\int_M tr(F\wedge F) = \int_{S^3} \underbrace{tr(A\,dA + \tfrac{2}{3} A^3)}_{K} = \int_{S^3} tr\left(AF - A^3 + \tfrac{2}{3} A^3\right) \underset{F=0}{=} -\frac{1}{3}\int_{S^3} tr(A^3)$$

Keeping A fixed asymptotically, the action is minimized by $F = \pm *F$ (due to the Bogomolny inequality).

E.O.M. $\Rightarrow$ $D*F = 0 \overset{F=\pm *F}{\rightsquigarrow} DF = 0$ Bianchi $\Rightarrow$ EOM! "First order PDE for A" "second order PDE"

$\Rightarrow$ (A)SD eq = Vacuum!

$\int_{\mathbb{R}^4} tr(F\wedge F) = \int_{S^3} tr(g^{-1}dg\; g^{-1} dg\, g)$ is defined by a map $g: S^3 \to G$ (you can think of $g \in \pi_3(G)$)

Topologically $SU(2) = S^3$. $\rightsquigarrow \pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$ "winding number"

2:1 $\downarrow$

$SO(3) = \mathbb{RP}^3$

Dirac Matrices / Clifford algebras: Vector spaces equipped with a quadratic form Q

$$\gamma : V \to Cl(V) \qquad \gamma^\mu = \gamma(e_\mu)$$
$$\{\gamma(v), \gamma(v)\} = 2Q(v) \qquad \downarrow$$
$$\{\gamma(v), \gamma(w)\} = 2\langle v, w\rangle \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

For $Q(v) = v_1^2 + v_2^2 + v_3^2 + v_4^2$, $V = \mathbb{R}^4$ there's a rep of $Cl(\mathbb{R}^4)$ "weyl" rep.

$$\gamma^i = \begin{pmatrix} & \tau^i \\ \bar{\tau}^i & \end{pmatrix} = \begin{pmatrix} & i\sigma^i \\ -i\sigma^i & \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} & \tau^0 \\ \bar{\tau}^0 & \end{pmatrix} = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix},$$

all block-off diagonal

$$\omega = \gamma^1\gamma^2\gamma^3\gamma^4 \to \omega^2 = \gamma^1\gamma^2\gamma^3\gamma^4\gamma^1\gamma^2\gamma^3\gamma^4$$
$$= \gamma^2\gamma^3\gamma^4\gamma^2\gamma^3\gamma^4 (-1)^3$$
$$= \dots = (-1)^{3+2+1}\mathbb{1} = +\mathbb{1} \to \omega$$

- $\mathrm{Tr}\,\omega = \mathrm{tr}(\gamma^1\gamma^2\gamma^3\gamma^4) = \mathrm{tr}(\gamma^4\gamma^1\gamma^2\gamma^3) = (-)^3 \mathrm{tr}(\gamma^1\gamma^2\gamma^3\gamma^4)$ (cyclicity) $= -\mathrm{tr}\,\omega$

ω can only have ± 1 eigenvalues with same # of +'s and -'s i.e. $+1, +1, -1, -1$.

$\Rightarrow \mathrm{tr}\,\omega = 0 \to$

- $\omega\gamma^\mu = \gamma^1\gamma^2\gamma^3\gamma^4\gamma^\mu = -\gamma^\mu\omega$ } let v a ~~basis~~ eigenvector of ω with λ eigenvalue: $-\gamma^\mu\omega v = -\lambda\gamma^\mu v = \omega\gamma^\mu v$

$\Rightarrow \gamma^\mu v$ is eigenvector of ω with $-\lambda$. $\Rightarrow \gamma^\mu$ is block off diagonal in eigenbasis of ω. In fact, our basis above is an eigenbasis of ω

$$\omega = \begin{pmatrix} -\mathbb{1} & \\ & \mathbb{1} \end{pmatrix}$$

NB: Vectors multiplied by γ matrices are called spinors

split in two half dim $\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$ ← weyl spinors (flipped by γ + other things)

left, right handed.

Permuting basis elements (parity transforms) of the original space is a right-hand $\Leftrightarrow$ left-hand if the permutation is odd $\Leftrightarrow$ flipping upper and lower half spinors

$\Leftrightarrow \epsilon^{\mu\nu\rho\sigma} \leftrightarrow -\epsilon^{\mu\nu\rho\sigma}$

$\Leftrightarrow$ self dual $\Leftrightarrow$ antiself dual

- $\gamma^\mu\gamma^\nu$ is block diagonal, $\gamma^\mu\gamma^\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) + \underbrace{\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)}_{\gamma^{\mu\nu} = \gamma^{[\mu}\gamma^{\nu]}}$

$$= g^{\mu\nu}\mathbb{1} + \tfrac{1}{2}\gamma^{\mu\nu}$$

$$\begin{pmatrix} & \tau^\mu \\ \bar{\tau}^\mu & \end{pmatrix}\begin{pmatrix} & \tau^\nu \\ \bar{\tau}^\nu & \end{pmatrix} = \begin{pmatrix} \tau^\mu\bar{\tau}^\nu & \\ & \bar{\tau}^\mu\tau^\nu \end{pmatrix} \qquad \begin{pmatrix} s.d.^* & \\ & asd^* \end{pmatrix}$$

upshot $\begin{cases} \tau^{[\mu}\bar{\tau}^{\nu]} \text{ is self dual} \\ \bar{\tau}^{[\mu}\tau^{\nu]} \text{ is antiselfdual} \end{cases}$ (convention) or vice versa.

$\Rightarrow F_{\mu\nu}$ of ADHM is antiself dual (other convention)

Recall $F = \pm * F$ implies e.o.m $D*F = 0$. Claim: First is the "square root" what if the YM theory we were discussing actually had fermions as well but these are all 0?

New action $S^{SYM} \int d^4x \left(tr(F^{\mu\nu}F_{\mu\nu}) + \bar{\Psi} \not{D} \Psi \right.$

$\left[\gamma^\mu(\partial_\mu + iA_\mu) \right.$ ← fundamental rep
I want this in the adjoint rep $\left.\right]$

Ψ in the adj rep of gauge group (ie algebra space)

$\gamma^\mu(\partial_\mu \delta^c_b + i A^a_\mu f^c_{ab})$.

Eq of motion ⤳ $\not{D}\Psi = 0$, $D*F = \bar{\Psi}\not{D}\Psi$.

Setting $\Psi = 0$ we retrieve $D*F=0$.

Turns out this SYM is supersymmetric for dim 3, 4, 6, 10.
(see Green Schw, witten Appendix 4A) Sug 1

$\delta_\varepsilon A_\mu = \frac{i}{2} \bar{\varepsilon} \gamma_\mu \Psi$ $\sim [\varepsilon Q, A]$

onshell ⤳ S^{SYM} action invariant under this transf.

fermionic parameter

$\delta_\varepsilon \Psi = -\frac{1}{4} F_{\mu\nu}\gamma^{\mu\nu}\varepsilon$ $\sim [\varepsilon Q, \Psi]$ ← superbracket

vectors, spinors

Can we have configurations of our original pure YM theory, that are susy?
$\Psi = 0$

$\delta_\varepsilon A \sim \Psi = 0$, what about $\delta(\Psi = 0)$, $\delta\Psi \sim F_{\mu\nu}\gamma^{\mu\nu}\varepsilon$ can be 0

if $F = \pm * F$ and ε is weyl spinor in the other half

i.e. F is asd ⤳ $\left(\begin{array}{c|c} \omega & 0 \\ \hline 0 & 0 \end{array}\right)$, $\varepsilon = \begin{pmatrix} 0 \\ \omega \end{pmatrix}$

$\Rightarrow \big(F = \pm * F \iff$ My configuration $A_\mu, \Psi = 0$ is supersymmetric

$\varepsilon Q \begin{pmatrix} A \\ \Psi \end{pmatrix} = 0 \Rightarrow \underset{=1}{\varepsilon^2} \underbrace{QQ}_{\text{Eq of motion}} \begin{pmatrix} A \\ \Psi \end{pmatrix} = 0$

Same explanation applies to another situation we encountered before extremal brane solutions (charged black hole) and equality $Q \geq |M|$

Saturated ⤳ "$M = Q$" + sum up   non-linear terms = 0   (1st order)
remaining linear equations ⤳ $\Delta H = 0$
harmonic

Bogomolny inequality

gravitino & photino variations.

Let's use ADHS to construct a 1-instanton config for SU(2)

$K = 1$

$B = (b_1, b_2)$, $C = (c_1, c_2)$ ← rotate basis of $\mathbb{H}^2$ so that $c = (0\ \ c)$

⤳ $M = B + Cx = (b_1, b_2 + cx)$ shift the origin of $\mathbb{R}^4$

add $-b_2 \frac{\bar{c}}{|c|^2} b_2 \rightsquigarrow (b_1, b_2 + c(-\frac{b_2}{c} + x)) = (b, cx) =: M$

Find $N = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ s.t. $M N = 0 = b n_1 + c x n_2$

$\Leftrightarrow \quad 0 = n_1 + \frac{c\bar{b}}{|b|^2} n_2 x \rightarrow n_1 = -\frac{c}{b} n_2 x$

$N = \begin{pmatrix} -\frac{c}{b} x \\ 1 \end{pmatrix} n \rightsquigarrow$ normalize $N \Rightarrow N^\dagger N = 1 = \bar{n}_2 (-\bar{x} \bar{c} \frac{\bar{b}}{|b|^2}, 1) \begin{pmatrix} -\frac{b}{|b|^2} c x \\ 1 \end{pmatrix}$

$= \bar{n}_2 \left(\frac{c^2}{b^2} x^2 + 1 \right) n_2$

$\rightsquigarrow n_2^2 = \frac{b^2}{c^2 x^2 + b^2} \rightsquigarrow n_2 = \overset{\text{one quaternion}}{u(x)} \sqrt{\frac{b^2}{c^2 x^2 + b^2}} \qquad = \left(\frac{c^2 x^2}{b^2} + 1 \right) n_2^2$

$A_\mu = N^\dagger \partial_\mu N$. notice that having $u(x) \leftarrow$ is related by gauge transf
so, up to gauge transf we can take $u(x) = 1$

$A_\mu(x) = N^\dagger \partial_\mu N = \sqrt{\frac{b^2}{c^2 x^2 + b^2}} \begin{pmatrix} -\frac{b}{b^2} c x \\ 1 \end{pmatrix}^\dagger \Bigg[\underbrace{\begin{pmatrix} -\frac{\bar{b}}{b^2} c \, \tau_\mu \\ 0 \end{pmatrix} n_2 + \begin{pmatrix} -\frac{b}{b^2} c x \\ 1 \end{pmatrix} \sqrt{\frac{b^2 c^2}{c^2 x^2 + b^2}}}_{\partial_\mu N}$

$= \sqrt{\frac{b^2}{c^2 x^2 + b^2}} \left(\frac{c^2}{b^2} \bar{x} \tau_\mu \sqrt{\frac{b^2}{c^2 x^2 + b^2}} - \frac{c^2}{b^2} \frac{x^2 |b| c^2 x_\mu}{(c^2 x^2 + b^2)^{3/2}} - \frac{|b| c^2 x_\mu}{(c^2 x^2 + b^2)^{3/2}} \right) \qquad \frac{x(-1) \bar{x} \tau_\mu}{c^2 x^2 + b^2}$

$= \frac{c^2}{c^2 x^2 + b^2} \bar{x} \tau_\mu - \frac{(c^2 x^2 + b^2) b^2 c^2 x_\mu}{b^4 (c^2 x^2 + b^2)^2} = c^2 \frac{\bar{x} \tau_\mu - x_\mu}{c^2 x^2 + b^2}$

$= \frac{\bar{x} \tau_\mu - x_\mu}{x^2 + |b/c|^2}$

$\rightsquigarrow$ size $b/c \leftarrow$ centered at 0. we also fixed an SU(2) orientation by setting $u(x) = 1$ hence we have

$$\left. \begin{array}{l} 1 \leftarrow \text{size } (b/c) \\ \boxed{3} \leftarrow \text{fixed SU(2) orientation} \\ 4 \leftarrow \text{translations } x \to x + x_0 \end{array} \right\}$$

8 parameters or "moduli" Moduli space = solutions of $F = \pm * F$ modulo gauge equivalence

Moduli spaces can have singularities (here) if size $\to 0$ $A_\mu \sim \frac{1}{x}$

gauge is orientation $\Rightarrow$ 5 dim Moduli space.

For general K

Note In general $\dim \mathcal{M}_K = 8K - 3$ argument:

One can "sum" [necessarily] very far separated solutions since tails don't contribute N ...

Lets look at $\frac{1}{8\pi^2}\int F\wedge F = -\frac{1}{4\pi^2}\int_{\rho} A^3$

[At large x, $N \sim x$ has winding number 1 (x^K has winding number K)]

<u>Supersymmetic Quantum mechanics:</u>

You have an operator Q with $Q^2=0$ your Hamiltonian is $2H = \{Q,Q^\dagger\} = QQ^\dagger + Q^\dagger Q$

<u>Immediate consequences:</u> $\langle 2H\rangle = \langle\psi|2H|\psi\rangle = \langle\psi|QQ^\dagger|\psi\rangle + \langle\psi|Q^\dagger Q|\psi\rangle$

$= \|Q^\dagger|\psi\rangle\|^2 + \|Q|\psi\rangle\|^2 \geq 0.$

- H is positive operator (spectrum bounded from below)
  - saturated iff $Q^\dagger|\psi\rangle = Q|\psi\rangle = 0$ $(H|\psi\rangle = 0|\psi\rangle)$

At higher levels, let's assume $H\psi = E\psi$ with $E>0$

$[H,Q] = QQ^\dagger Q + \cancel{Q^\dagger QQ} - QQ^\dagger Q - \cancel{Q^\dagger QQ} = 0$

$\Rightarrow$ Q sends eigenspaces of H to themselves (and viceversa)

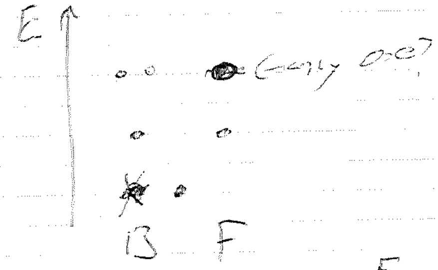
$Q: \mathcal{H}_E \to \mathcal{H}_E$, on $\mathcal{H}_E$, $\mathbb{1} = \frac{QQ^\dagger}{2E} + \frac{Q^\dagger Q}{2E} = \underbrace{P_B + P_F}_{\text{Projectors}}$

$\rightsquigarrow \mathcal{H}_E = \mathcal{H}_{E,B} \oplus \mathcal{H}_{E,F}$, $\mathcal{H}_{E,B} \xrightarrow{Q^\dagger} \mathcal{H}_{E,F}$, $\mathcal{H}_{E,F} \xrightarrow{Q} \mathcal{H}_{E,B}$

$\rightsquigarrow Q = \frac{QQ^\dagger Q}{2E}$

$P_B^2 = \frac{QQ^\dagger QQ^\dagger}{4E^2}$
$= \frac{Q(Q^\dagger Q + QQ^\dagger)Q^\dagger}{4E^2}$
$= \frac{Q\,2H\,Q^\dagger}{4E^2} = \frac{QQ^\dagger}{2E} = P_B$
and $P_B P_F = 0.$

upshot: ~~Boson~~ Besides the ground states all higher levels are split into isomorphic subspaces annihilated by Q and $Q^\dagger$



Distinguish the two halves: $(-1)^F = \int dE\, \frac{QQ^\dagger}{2E} - \frac{Q^\dagger Q}{2E}$

<u>Examples:</u> $Q = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ $\leftarrow$ square to 0.

$\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2 \rightsquigarrow Q = \begin{pmatrix} 0 & p+iw(x) \\ 0 & 0 \end{pmatrix}$, $Q^\dagger = \begin{pmatrix} 0 & 0 \\ p-iw & 0 \end{pmatrix}$

$2H = \begin{pmatrix} p^2+w^2-ipw+iwp & 0 \\ 0 & p^2+w^2+ipw-iwp \end{pmatrix} = \cancel{(p^2+w^2-i[p,w])\otimes\mathbb{1}}$

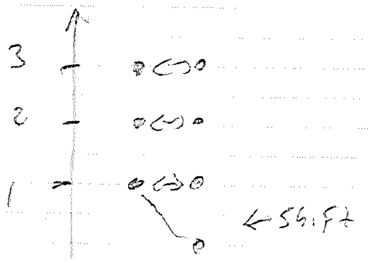
$= \begin{pmatrix} p^2+w^2+w' & 0 \\ 0 & p^2+w^2-w' \end{pmatrix} = (p^2+w^2)\otimes\mathbb{1} + w'\otimes\sigma_z$

- upper $V = w^2 + w'$
- lower $V = w^2 - w'$

Cases: $w=0$ ⇝ two copies of free particle

⇝ $H = \left(\frac{p^2}{2}+\frac{w^2}{2}\right)\otimes \mathbb{1} + \frac{w'}{2}\otimes\sigma_z$

$= \left(\frac{p^2}{2}+\frac{x^2}{2}\right)\otimes\mathbb{1} + \frac{x}{2}\otimes\sigma_z$ ← two harmonic oscillators but shifted in energy ($w=1$ frequency of oscillator)



ground state:

$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ $\quad Q\psi = \begin{pmatrix}0\\ (p+iw)\psi_2\end{pmatrix}=0$

⇝ $\psi_2' = w\psi_2$ ⇝ $\psi_2 = ce^{\int_0^x w(y)dy}$ $\quad Q^\dagger\psi = \begin{pmatrix}(p-iw)\psi_2\\ 0\end{pmatrix}=0$

$\psi_1 = ce^{-\int_0^x w(y)dy}$

we need $\psi_1,\psi_2\to 0$ as $x\to\pm\infty$ ⇒ at most one of this is square integrable

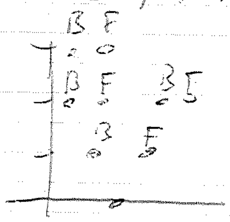
Since $\psi_2\to 0 \Rightarrow \int_0^x w(y)dy\to-\infty$ and therefore $\psi_1\to\infty$.

Only works for (poly) whose leading term is odd.

Recall Ground states $\underbrace{Q\psi=0=Q^\dagger\psi}_{\text{"first order eq"}} \Leftrightarrow \underbrace{H\psi=0}_{\text{"second order eq"}}$

Excited states split into bosonic and fermionic $\mathcal{H}_B\oplus\mathcal{H}_F$

$(-1)^F = \frac{QQ^\dagger Q}{2H} - \frac{Q^\dagger QQ^\dagger}{2H}$



Witten index, for any $\beta>0$

$I_W = \mathrm{tr}\left((-1)^F e^{-\beta H}\right)$ only ~~counts~~ ground states mod 2

to make it convergent — and is independent of β.

I_W continuous and $I_W\in\mathbb{Z}$ ⇒ I_W has to be constant upon continuous variations.

It has to be constant when you vary parameters ("coupling constants") in the Hamiltonian as long as Susy is preserved.

Hence calculate at "weak" coupling and change parameters to "strong" coupling

In particular if there is an odd number of $H\psi=0$ ground states at weak coupling, they will exist at any coupling.

In QFT, states with $Q\psi=0=Q^\dagger\psi$ are called "BPS-states" ("Bogomolny–Prasad–Sommerfield")

In many cases, this info about existence of BPS states is the only reliable info we have about the ...

EX: S-duality

Theory 1 $\equiv$ theory 2

with coupling g $\left(g=\frac{1}{e}\right)$ with coupling e

e.g. (Electromagnetism) $dF=0$ & $d{*}F=0$, take $F \leftrightarrow {*F}$

Trivially S-dual.

if $dF = j_m$ & $d{*}F = J_{e\,charge}$ $\leadsto$ $ge \in 2\pi\mathbb{Z}$ Dirac quantization

j_m: magnetic ~~charge~~ current (monopole)

Caveat: • $I_w \in \mathbb{Z}$ only works if $0 \notin$ cont spectrum of H.
• Make sure $H \to I_w$ is continuous in a suitable top.

$$Q\psi = 0 = Q^\dagger\psi \Longleftrightarrow H\psi = 0 \qquad H \geq 0$$

Einstein-maxwell $\leftrightarrow$ nonlinearities $=0$ $\leftrightarrow$ $M \geq Q$

$d{*}F=0$ $\leftrightarrow$ $F = {*F}$ $\leftrightarrow$ Bogomolny inequality

$$\int F\wedge F \geq \int (F \pm {*F})^2$$

Index theorem

$$\frac{1}{8\pi^2}\int tr\,(F\wedge F) = \text{index } \not{D}$$

Atiyah-Singer index theorem

$$\text{"="} \dim \ker \not{D} - \dim \ker \not{D}^\dagger$$

Riemann-Roch theorem } see class
Gauss-Bonnet th.

Fujikawa's proof via anomalies of chiral gauge theories.

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \chi \\ \xi^{\dagger} \end{pmatrix} \leadsto \mathcal{L} = i\chi^\dagger \sigma^\mu D_\mu \chi + i\xi^\dagger\bar\sigma^\mu D_\mu \xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

$\bar\psi\not{D}\psi$, $D_\mu = \partial_\mu + iA_\mu$

particle number is conserved $\leftarrow$ Noether charge of global symmetry

$$\left.\begin{aligned}\chi &\to e^{i\varphi}\chi \\ \xi &\to e^{-i\varphi}\xi\end{aligned}\right\} \leadsto \mathcal{L}' = \mathcal{L}. \checkmark$$

In the special case of massless particles (fermions) $(m=0)$

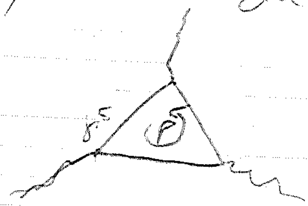
the mixed terms are not present and the symmetry is bigger as we can rotate χ and ξ independently

$$\left.\begin{aligned}\chi &\to e^{i\varphi_L}\chi \\ \xi &\to e^{+i\varphi_R}\xi\end{aligned}\right\} \leadsto \mathcal{L}' = \mathcal{L}, \quad \psi \to \psi' = e^{i(\varphi_L - \varphi_R)}\, e^{i(\varphi_L + \varphi_R)\gamma^5}\psi$$

electric charge, chiral charge

Chiral current $\rightsquigarrow$ $J_A = \bar\psi \gamma^\mu \gamma^5 \psi$ is also conserved for $m=0$

It turns out that this chiral symmetries does not survive quantization, it is anomalous



$\int d^4p \, \frac{\cdots}{p^3}$

linearly divergent; preserving $\partial_\mu J_A^\mu = 0$

Rather look at the path integral:

$$\underbrace{\int D\psi \, D\bar\psi}_{\text{what about the measure}} \; \underbrace{e^{i\int d^4x \, \mathcal{L}}}_{\text{inv. under the transf.}}$$

$\hookrightarrow$
$$\psi \to e^{iq\gamma^5}\psi = \psi' \qquad \bar\psi \to e^{-iq\gamma^5}\bar\psi = \bar\psi' \quad \rightsquigarrow \text{Jacob} = \frac{\delta\psi'(x)}{\delta\psi(y)} = e^{iq\gamma^5}\delta(x-y)$$

$$\rightsquigarrow \det(\text{Jacob})^2 = e^{2\,\mathrm{tr}\log \text{Jacob}} = e^{-2i\,\mathrm{trace}\int d^4x \; q\,\delta(x-x)\,\gamma^5)}$$

$$= \underbrace{e^{-2i\int d^4x \; q\,\mathrm{tr}(\delta(x-x)\,\gamma^5)}}$$

↑ functional trace ↑ spinor trace ↑ high.

↳ has to be regulated.

[Heat eq. $\partial_t T = -\Delta T$
$T(t=0,x) = \delta(x)$ $\rightsquigarrow$ $T(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$]

We need to regulate δ since its not a function. use $f: [0,\infty) \to [0,1]$

s.t. $f(0) = 1$, $\lim_{s\to\infty} f(s) = 0$
$\lim_{s\to\infty} s f'(s) = 0 = \lim_{s\to 0} s f'(s)$ } Any such f, e.g. $f = e^{-s}$

$$\delta^4(x-y) \rightsquigarrow f\left(-\frac{\slashed{D}^2}{M^2}\right) \leftarrow (\partial_\mu - iA_\mu)\gamma^\mu, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}$$

↖ some parameter   regulator goes away when $M \to \infty$

$$\delta^4(x-y) \overset{M\to\infty}{=} \int \frac{d^4k}{(2\pi)^4} \, f\left(-\frac{\slashed{D}^2}{M^2}\right) e^{ik\cdot(x-y)} = \int \frac{d^4k}{(2\pi)^4} \, e^{ikx} f\left(-\frac{(\slashed{D}+i\slashed{k})^2}{M^2}\right)$$

$$= \int \frac{d^4k}{(2\pi)^4} \, e^{ikx} f\left(-\frac{\slashed{D}^2 + 2ik\slashed{D} - k^2}{M^2}\right), \quad \text{recall } \slashed{D}^2 = \gamma^\mu\gamma^\nu D_\mu D_\nu$$

$$= \left(\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\}_+ + \gamma^{[\mu}\gamma^{\nu]}\right) D_\mu D_\nu$$

[$\slashed{D}^2 = D^2 \pm \frac{ig}{2}\gamma^{\mu\nu}F_{\mu\nu}$]

$k\slashed{D}$: $k^\mu D_\mu + \gamma^{[\mu}\gamma^{\nu]}[k_\mu, D_\nu]$ ← 0

$$= \left(\tfrac{2}{2} g_{\mu\nu} D^\mu D^\nu + \gamma^{[\mu}\gamma^{\nu]} \tfrac{1}{2}[D_\mu, D_\nu]\right)$$

$$= M^4 \int \frac{d^4k}{(2\pi)^4} \, e^{ikxM} f\left(\frac{k^2 + 2ik^\mu D_\mu/M - D^2/M^2 + \frac{1}{2}g\gamma^{\mu\nu}F_{\mu\nu}}{\cdots}\right)$$

Insert this into $tr(O(\ldots))$

Taylor expand f, since it looks divergent.

But there is also the trace over γ matrices.

Before, we computed $tr\, \gamma^5 = 0$. So the term $tr(O(M^4)\gamma^5)$
$= 0$.

Also $tr(\gamma^5 \gamma^\mu) = 0$ and in fact $tr(\underbrace{\gamma^5 \gamma \gamma \gamma \ldots}_{\text{odd \# of factors}}) = 0$.
is 0.

• $tr(\gamma^5 \gamma^{[\mu}\gamma^{\nu]}) = 0$.

$\gamma^1\gamma^2 \ldots \gamma^5 \gamma^\mu \gamma^\nu - \gamma^\mu \ldots \gamma^5 \gamma^\nu \gamma^\mu$

since $\ldots$

• $tr(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = tr(\gamma^5 \gamma^{\mu\nu\rho\sigma}) = \varepsilon^{\mu\nu\rho\sigma}$ ← only non zero
since need 4 γ matrices

Upshot: the only term surviving $tr(\gamma^5 \ldots)$ and $M \to \infty$ is

$\frac{\partial f}{\partial x} \sim \quad Tr(\delta(x,x)\gamma^5) = \frac{g^2}{8} \int \frac{d^4k}{(2\pi)^4} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} f''(k^2)$

Area 3sphere

$\left[= \int \frac{d^4k}{(2\pi)^4} f''(k^2) = \frac{A(3)}{(2\pi)^4} \int dk\, k^3 f''(k^2) = \frac{A(3)}{2(2\pi)^4} \int ds\, s f''(s) \right.$

$= -\frac{A(3)}{2(2\pi)^4} \int ds\, f'(s) = \frac{A(3)}{2(2\pi)^4} \Big]$

↑ no boundary terms

$= -\frac{g^2}{8} \frac{A(3)}{(2\pi)^4} \underbrace{\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}}_{F\wedge F}$

$tr(f(\frac{\cancel{D}}{M^2})\gamma^5)$ lets look at the op $\cancel{D}$ over compactified euclidean ~~of~~ space time. Mathematics tells us that it is elliptic
↑ (like laplace)

has a discrete spectrum.

Let's take eigen functions (spinors) $\cancel{D}\varphi_n = \lambda_n \varphi_n$.

• $\cancel{D}\gamma^5 \varphi_n = -\gamma^5 \cancel{D}\varphi_n = -\gamma^5 \lambda_n \varphi_n = -\lambda_n \gamma^5 \varphi_n \Rightarrow (\gamma^5\varphi_n, -\lambda_n)$

$tr(f(\frac{\cancel{D}^2}{M^2})\gamma^5) = \sum_n \langle \varphi_n | f(-\lambda_n^2/M^2) | \varphi_n \rangle$

$= \sum_n \langle \varphi_n | \varphi_n \rangle = n_+ - n_-$ ← # 0 modes of $\cancel{D}$ with + chirality ... − chirality

cancellation for $\lambda_n \neq 0$ as we have $|\varphi_n\rangle$ and $\gamma^5|\varphi_n\rangle$

$\Rightarrow \frac{1}{16\pi^2} \int F\wedge F = \dim(\ker(\sigma^\mu \partial_\mu)) - \dim \ker(\bar{\sigma}^\mu \partial_\mu) \in \mathbb{Z}$

↑ topological ↑ analytic ↖ 4 dim of Atiyah-Singer theorem