In order to simplify the computation of $1/2 \sum_{spins} |M|^2$ funda, we need to work out the following invariants:

$$2q\cdot p = 2q\cdot k = m_t^2 - m_W^2 \qquad ; \qquad 2k\cdot p = m_t^2 + m_W^2$$

$\hookrightarrow (q-p)^2 = k^2$

$-2q\cdot p + m_t^2 = m_W^2$

where we made use of the approximation $m_W, m_t \gg m_b \simeq 0$. Thus, all in all, we get

$$\frac{1}{2}\sum_{spins} |M|^2 = \frac{g^2}{4}\frac{m_t^4}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)\left(1+2\frac{m_W^2}{m_t^2}\right)$$

Next, to compute the decay rate, we use the formula $\Gamma = \frac{1}{2m_t}\int d\Pi_f\, |M(p\to f)|^2$, where $d\Pi_f = \prod_i \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i}\delta^{(4)}(p-q-k)$ is the Lorentz invariant phase space measure.

$$\Gamma = \frac{g^2}{8}\frac{m_t^3}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)\left(1+\frac{2m_W^2}{m_t^2}\right)\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_W^2}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{\vec{q}^2+\cancelto{0}{m_b^2}}}\times(2\pi)^4\delta^{(4)}(p-q-k)$$

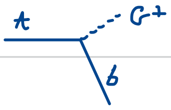
$$= \frac{g^2}{64\pi}\frac{m_t^3}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)^2\left(1+\frac{2m_W^2}{m_t^2}\right) \simeq \frac{g^2}{64\pi}\frac{m_t^3}{m_W^2}$$

in the limit $m_t \gg m_W$.

Therefore, in the limit $m_t \gg m_W$ we do find an m_t^2/m_W^2 enhancement w.r.t. the very naive estimate. This makes sense, after all we expect $\Gamma\downarrow$ if $m_W\uparrow$.

Let's next use the Goldstone equivalence theorem:

$$\Delta\mathcal{L} = -y_t\,\epsilon^{ab}(\overline{Q_L})_a (H^\dagger)_b\, t_R + h.c. \quad ; \quad H = \begin{pmatrix} G^+ \\ \frac{h+iG^0}{\sqrt{2}}\end{pmatrix}.$$

$\hookrightarrow \Delta\mathcal{L} = -y_t\,\bar{b}_L t_R G^+ \implies iM =$  $= -y_t\,\bar{u}(q)\left(\frac{1+\gamma_5}{2}\right)u(p)$;

Therefore $\frac{1}{2}\sum_{spins}|M|^2 = y_t^2\, q\cdot p.$

Next we perform the integral over phase space to get

$$\Gamma = \frac{y_t}{32\pi}m_t = \frac{g^2}{64\pi}\frac{m_t^3}{m_W^2}.$$

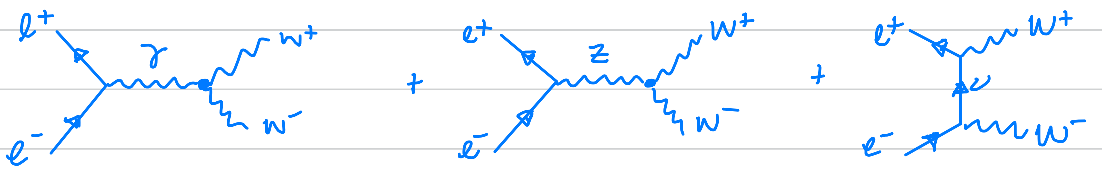
Nicely the result we get using the "Goldstone equivalence theorem" agrees with the exact tree-level computation we did of $t \to W^+ b$ in the limit $m_t \gg m_W$. This makes sense! In this limit it is favorable to produce energetic W's through the decay, and in particular it is more favorable to produce the longitudinal polarized ones because their polarization grows with the energy.

By the way this decay mode of the top is very important because $Br(t \to W^+ b) \simeq 1$, and therefore the top quark does not hadronize! Namely it decays through EW interactions before it has the time to reach longer distances where the QCD flux tubes acquire large energy and become energetically favorable to pair-produce on-shell quarks and gluons.

END OF L8

* Another instance of the Goldstone equivalence theorem:

Next we want to compute the following process



in the SM and using the equivalence theorem. First of all we note that naïvely these diagrams grow with the energy:

$$\frac{d\sigma}{d\cos\theta} \sim \frac{\pi\alpha^2}{4s} \left| \epsilon(k_+)\cdot\epsilon(k_-) \right|^2 \sim \frac{\pi\alpha^2}{4s}\left(\frac{s}{4m_W}\right)^2$$

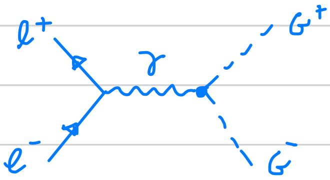
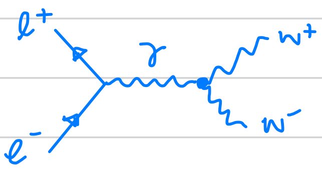
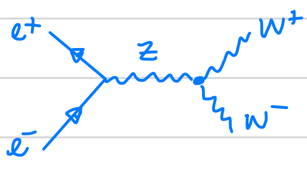
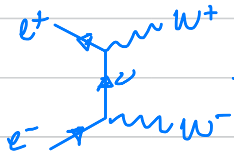
For longitudinals $\epsilon(k_+)\cdot\epsilon(k_-) = \frac{k_+\cdot k_-}{m_W^2}$ for $s = (k_+ + k_-)^2 \gg m_W^2$.

On the other hand,

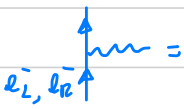
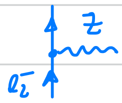
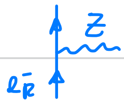
$$\frac{d\sigma}{d\cos\theta}(e^+e^- \to G^+G^-) = \frac{\pi\alpha^2}{4s}\sin\theta$$

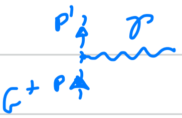
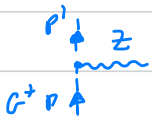
decays with $1/s$. How are these two claims compatible? We will see that there is a nice cancelation between the diagrams above and it turns out that $\frac{d\sigma}{d\cos\theta}(e^+e^- \to W^+W^-) \propto 1/s$!

We want to show

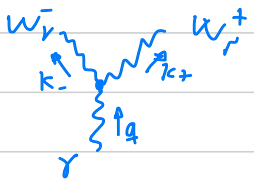
 $=$  $+$  $+$  $+\, \mathcal{O}\left(\frac{m_W^2}{E^2}\right)$

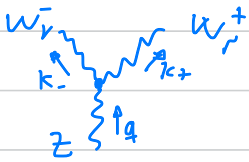
We will need the following Feynman rules:

 $= -ie\gamma^\mu$,  $= i\frac{e}{s_w c_w}\gamma_\mu\left(-\frac{1}{2}+s_W^2\right)$,  $= i\frac{e}{s_w c_w}\gamma_\mu\left(s_W^2\right)$,

 $= ie(p+p')^\mu$  $= i\frac{e}{s_w c_w}\gamma_\mu\left(\frac{1}{2}-s_W^2\right)$

as well as the usual triple gauge vertices

 $= ie\left[g^{\mu\nu}(k_- - k_+)^\lambda + g^{\mu\lambda}(-q-k_-)^\mu + g^{\lambda\mu}(q+k_+)^\nu\right]$

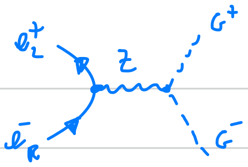
 $= igc_w\left[g^{\mu\nu}(k_- - k_+)^\lambda + g^{\mu\lambda}(-q-k_-)^\mu + g^{\lambda\mu}(q+k_+)^\nu\right]$

All right, we start by computing the left hand side of the equation above. The photon exchange contribution is given by

$$i\mathcal{M}(e^+e^-\to\gamma^*\to G^+G^-) = (-ie)\,\bar v\gamma_\mu u\,\frac{-ig^{\mu\nu}}{q^2+i\epsilon}\,(k_+-k_-)^\nu(-ie)$$

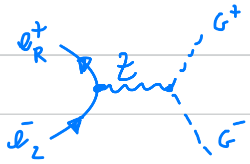
$$= ie^2\,\bar v\gamma_\mu u\,\frac{1}{q^2+i\epsilon}\,(k_+-k_-)^\mu .$$

Next we compute the Z-exchange contribution. We will distinguish between right and left handed electrons:



$$= \frac{ie}{c_W \cancel{s_W}} \bar{v}_R \gamma_\mu u_R \cancel{s_W^2} \frac{-ig^{\mu\nu}}{q^2 - m_Z^2 + i\epsilon} (k_+ - k_-)^\nu \frac{-ie}{c_W \cancel{s_W}} \left(\frac{1}{2} - s_W^2\right)$$

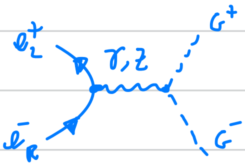
$$= -ie^2 \bar{v}_R \gamma_\mu u_R \frac{1}{q^2 - m_Z^2} (k_+ - k_-)^\mu \frac{1}{c_W^2} \left(\frac{1}{2} - s_W^2\right)$$



$$= \frac{-ie}{c_W s_W} \bar{v}_L \gamma_\mu u_L \left(\frac{1}{2} - s_W^2\right) \frac{-ig^{\mu\nu}}{q^2 - m_Z^2 + i\epsilon} (k_+ - k_-)^\nu \frac{-ie}{c_W s_W} \left(\frac{1}{2} - s_W^2\right)$$

$$= -ie^2 \bar{v}_L \gamma_\mu u_L \frac{1}{q^2 - m_Z^2} (k_+ - k_-)^\mu \frac{1}{c_W^2 s_W^2} \left(\frac{1}{2} - s_W^2\right)^2$$

Next we combine the photon and Z exchange amplitudes. For the right handed electrons we get



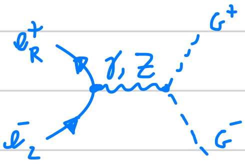
$$\simeq \frac{i e^2}{2 c_W^2} \bar{v}_R \gamma_\mu u_R \frac{1}{q^2} (k_+ - k_-)^\mu$$ which is just the amplitude

@ high energy

for e_R^- with $Y = -1$, to couple to a G^+ with $Y = 1/2$, through the $U(1)$ gauge boson B_μ with coupling constant $g' = e/c_W$.

This expression reflects the fact that e_R has no direct coupling to $SU(2)_L$.

Instead the high energy limit for the left handed amplitude:



$$\simeq ie^2 \left(\frac{1}{4 c_W^2} + \frac{1}{4 s_W^2}\right) \cdot \bar{v}_L \gamma^\mu u_L \frac{1}{q^2} (k_+ - k_-)^\mu$$ which is the sum of amplitudes with B_μ and A_3 exchange.

@ high energy

Next, let's compute the right hand side, i.e. the diagrams in the GWS theory.

For right handed electrons we only need to compute the γ and Z exchange diagrams:

$$iM(e_R^- e_L^+ \to W^+ W^-) = \bar{v}_R \gamma_\nu u_R \left[(-ie) \frac{-i}{q^2 + i\epsilon} (ie) + \frac{ie \cancel{s_W}}{\cancel{c_W}} \frac{-i}{q^2 - m_Z^2} \frac{ie \cancel{c_W}}{\cancel{s_W}}\right] \times$$

$$\times \left(g^{\mu\nu}(k_- - k_+)^\lambda + g^{\lambda\nu}(-q - k_-)^\mu + g^{\lambda\mu}(k_+ + q)^\nu\right) \epsilon^*_\mu(k_+) \epsilon^*_\nu(k_-)$$

Note that the last equation is valid in any gauge because $q^\lambda \bar{v}_R \gamma_\lambda u_R = 0$. Note also that the second line contains and enhancement at high energies for the longitudinal gauge bosons. Indeed, using $\epsilon_L^\mu = \frac{k^\mu}{m} + O(m/E_k)$, we have

$$\left(g^{\mu\nu}(k_- - k_+)^\lambda + g^{\lambda\nu}(-q-k_-)^\mu + g^{\lambda\nu}(k_+ + q)^\nu\right) \epsilon_\mu^*(k_+)\, \epsilon_\nu^*(k_-) = \frac{s}{2m_W^2}(k_+ - k_-)^\lambda + O(1)\cdot(k_+ - k_-)^\lambda$$

Using $\frac{1}{q^2} - \frac{1}{q^2 - m_Z^2} = -\frac{m_Z^2}{q^2(q^2 - m_Z^2)} \simeq -\frac{m_Z^2}{q^4}$, we get

$$i\mathcal{M}(e_R^- e_R^+ \to W^+ W^-) \simeq \bar{v}_R \gamma_\mu u_R \frac{i e^2 m_Z^2}{s} \frac{1}{2 m_W^2} (k_+ - k_-)^\lambda$$

which agrees with the high energy limit of $i\mathcal{M}(e_R^- e_L^+ \to G^+ G^-)$ that we computed above (use $m_Z^2/m_W^2 = 1/c_W^2$). Equivalence theorem at work!

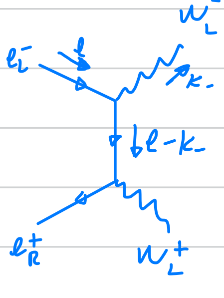
Next we compute the amplitude for the left handed electrons. We have the γ, Z and ν-exchange diagrams. The first two are given by

$$i\mathcal{M}(e_R^- e_L^+ \to W^+ W^-) = \bar{v}_L \gamma_\mu u_L \left[(-ie)\frac{-i}{q^2 + i\epsilon}(ie) + \frac{ie\left(-\frac{1}{2} + s_W^2\right)}{s_W \cancel{c_W}} \frac{-i}{q^2 - m_Z^2} \frac{ie\, \cancel{c_W}}{s_W}\right] \times$$
$$\times \left(g^{\mu\nu}(k_- - k_+)^\lambda + g^{\lambda\nu}(-q-k_-)^\mu + g^{\lambda\nu}(k_+ + q)^\nu\right) \epsilon_\mu^*(k_+)\, \epsilon_\nu^*(k_-)$$

Next we perform similar simplifications to what we did for the high energy limit of the e_R amplitude, and we are lead to

$$\simeq \bar{v}_L \gamma_\mu u_L\, i e^2 \left(\frac{m_Z^2}{s(s - m_Z^2)} - \underbrace{\frac{1}{2 s_W^2} \frac{1}{s - m_Z^2}}\right) \frac{s}{2 m_W^2} (k_+ - k_-)^\mu$$

Potentially dangerous term. It grows with the energy, thus it would lead to a loss of perturbativity. It better be cancelled by the neutrino exchange diagram:



$$= \left(\frac{ig}{\sqrt{2}}\right)^2 \bar{v}_L \gamma^\mu \frac{i(\cancel{l} - \cancel{k}_-)}{(l - k_-)^2} \gamma^\nu u_L(l) \underbrace{\epsilon_\mu^*(k_+)\, \epsilon_\nu^*(k_-)}_{\simeq \frac{k_+^\mu k_-^\nu}{m_W^2} + \dots}$$

$$\simeq -\frac{ig^2}{2} \bar{v}_L \frac{\cancel{k}_+}{m_W} \frac{\cancel{l} - \cancel{k}_-}{(l - k_-)^2} \frac{\cancel{k}_-}{m_W} u_L.$$

In order to simplify further the last expression we note that $u_L(\ell)$ satisfies the Dirac equation in momentum space: $(\not\ell - \not k_-)\not k_- u_L(\ell) = -(\not\ell - \not k_-)^2 u_L(\ell) = -(\ell - k_-)^2 u_L$

$\not\ell\, u_L(\ell) = 0 + \mathcal{O}(m_e)$

Therefore:

$$= \frac{i g^2}{2} \bar{v}_L \frac{\not k_+}{m_W^2} u_L(\ell) = \frac{i e^2}{2 s_W^2} \frac{1}{2 m_W^2} \bar{v}_L \gamma_\mu u(\ell) (k_+ - k_-)^\mu$$

$(k_+ + k_-)^\mu \bar{v}_L \gamma^\mu u(\ell) = q^\mu \underbrace{\bar{v}_L \gamma^\mu u_L(\ell)}_{J^\mu} = 0$

Thus, the neutrino exchange diagram does cancel the leading high energy term of the γ, Z-exchange diagrams!

In order to work out the ν-exchange diagram to the same order as the γ, Z-exchange, we need to keep more powers in the high energy expansion of ϵ_L^μ. This has the net effect of multiplying the γ, Z-exchange diagrams by $\left(1 + \frac{2 m_W^2}{s}\right)$ (after removing the leading high energy growing piece that we cancelled):

$$i\mathcal{M}(e_L^- e_R^+ \to W_L^+ W_L^-) = i e^2 \bar{v}_L \gamma^\mu u_L (k_+ - k_-)^\mu \frac{1}{s} \underbrace{\left(\frac{1}{2 c_W^2} - \frac{1}{4 c_W^2 s_W^2} + \frac{1}{2 s_W^2} \right)}_{\frac{2 s_W + 2 c_W - 1}{4 c_W^2 s_W^2} = \frac{1}{4 s_W^2} + \frac{1}{4 c_W^2} = \frac{1}{4 s_W^2 c_W^2}}$$

which is equal to $\mathcal{M}(e_L^- e_R \to G^+ G^-)$ that we computed!

The cancellation of the $e^+e^- \to W^+W^-$ naive high energy growth is a crucial feature of a theory of Spontaneous Symmetry Breaking.

At the beginning of these lectures we saw "SSB of gauge invariance" as a mechanism that can generate masses for gauge bosons. Now we have argued the opposite: that only the theories with SSB have amplitudes with a nice high energy behaviour, i.e. without strong coupling. We'll now see this feature in more generality, and expose the high energy behaviour at the level of the Lagrangian. The high energy behaviour is more obscure when working with the ϵ_L^μ's.

END OF L9

◉ THE HIGGS MECHANISM II

A dialogue with questions (Q) and answers (A).

Q1: The Higgs particle "h" is a scalar, but it is not <u>any</u> scalar... right? What makes the scalar "h" a Higgs boson? For instance, imagine adding an extra singlet scalar to the SM: $\mathcal{L} = \mathcal{L}_{SM} + \frac{1}{2}\partial_\mu s(x)\partial^\mu s(x) - V(s)$. The field "s" is a scalar but not a Higgs particle.

A1: Well, "h" is inside the Higgs doublet $H = (G^+, (h + iG_0)/\sqrt{2})$, which takes a non-zero VEV, that's why "h" is a Higgs boson!

Q2: Well... this does not sound very invariant... is there a more physical way in which we can describe the role of "h" by referring to observables and not to "mechanisms of Lagrangians written with dummy integration variables"?

A2: Yes! this is what today's lecture is about.

Next, for pedagogical purposes, to make the argument simpler I will set

* $g' = 0$, * $y_f = 0$, so that for the time being we can ignore fermions.
* Let's also imagine that we do not know about the Higgs boson because the energy that we have explored is $E_{cm} < m_h$.

We do know however of the existence of massive gauge bosons. Therefore the Lagrangian of this hypothetical universe is:

$$\mathcal{L}_{SU(2)} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} + m_W^2 \, \mathrm{Tr}[W_\mu W^\mu] \qquad \left(\text{where } W_\mu \equiv \frac{\sigma^a}{2} W^a_\mu\right)$$

$$\sim \frac{1}{2}(\partial W)^2 - \frac{1}{2} m_W^2 W^2 - g\, W^2 \partial W + g^2 W^4$$

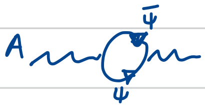
This theory makes perfect sense as an Effective Field Theory, i.e. observables computed with this Lagrangian are perfectly consistent as long as the energies that are probed are $E_{cm} < \Lambda$, where Λ is the physical cutoff.

By physical cutoff we mean the value of the energy at which new phenomena beyond the weakly coupled Lagrangian emerges.

<u>What is the value of Λ in this EFT?</u>

Recall that $\epsilon_L^\mu(P) = P^\mu/m_W + O(m_W/E)$, therefore perturbation theory Feynman diagrams in this theory can be trusted as long as $E \lesssim m_W/g \sim \Lambda$. This can be seen order by order computing Feynman diagrams, as we will show next.

(This is very different from e.g. QED! The E.M. coupling runs with the energy



, but only logarithmically $\alpha(\mu_f) = \dfrac{\alpha(\mu_i)}{1+\beta\frac{\alpha(\mu_i)}{4\pi}\log(\frac{\mu_f}{\mu_i})}$ @ 1-loop.)

Let's compute an elastic $2\to 2$ process involving the W_L^μ's:

$$\mathcal{M}(W_L + W_L \longrightarrow W_L + W_L) =$$

Individually each of these diagrams grows like E^4. The sum however grows like E^2.

$$\mathcal{M}(W_L + W_L \longrightarrow W_L + W_L) \simeq \frac{g^2}{4m_W^2}(s+t).$$

This is a milder growth, but still poses a problem of perturbativity at large $E \gg m_W$ energies. One manifestation of the loss of perturbativity (i.e. of our ability to compute observables by evaluating the first few Feynman diagrams) can be found in the naive violation of the unitary equation.

A quick recapitulation of unitary and the partial waves unitary bound:

Consider an elastic scattering in the C.o.M. frame

$$A(p_1) + A(p_2) \longrightarrow A(p_3) + A(p_4)$$

then,

$$\sigma_{tot}(AB \to AB) = \frac{1}{32\pi E_{cm}^2} \int d\cos\theta \, |M(\theta)|^2 .$$

To derive a useful bound, it is convenient to decompose the amplitude in partial waves. We can allways perform the decomposition

$$\mathcal{M}(\theta) = 16\pi \sum_{j=0}^{\infty}{}' a_j (2j+1) P_j(\cos\theta)$$

where $P_j(\cos\theta)$ are the Legendre Polynomials. They satisfy:

$$P_j(1) = 1 \qquad \& \qquad \int_{-1}^{1} P_j(x) P_i(x)\, dx = \frac{2}{2j+1} \delta_{ji} .$$

Therefore we can integrate over "$d\cos\theta$" and write

$$\sigma_{tot} = \frac{16\pi}{E_{cm}^2} \sum_{j=0}^{\infty}{}' (2j+1) |a_j|^2$$

p.w. partial waves.

Now, the optical theorem says

$$\mathrm{Im}\, M(AB \to AB \text{ at } \theta = 0) = 2 E_{cm} |\vec{p}_i| \sum_{X}{}' \sigma_{tot}(AB \to X) \geq 2 E_{cm} |\vec{p}_i| \, \sigma_{tot}(AB \to AB) .$$

Therefore

$$\sum_{j=0}^{\infty}{}' (2j+1)\, \mathrm{Im}(a_j) \geq \frac{2|\vec{p}_i|}{E_{cm}} \sum_{j=0}^{\infty}{}' (2j+1) |a_j|^2$$

It turns out that the sum can be dropped by considering angular momentum eigenstates (see e.g. Izykson & ZUBER) and we are lead to

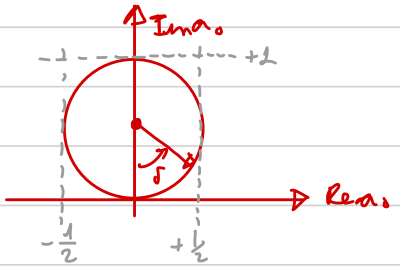
$$\mathrm{Im}\, a_j \geq |a_j| \quad , \text{ for } E_{cm} \gg m_A, m_B \, ; \ |\vec{p}_i| = \tfrac{1}{2} E_{cm}$$

(A nice way to write the p.w. unitary is using $S_\ell \equiv 1 + i\sqrt{\frac{s-4m_W^2}{s}}\, a_\ell$, then $|S_\ell| \leq 1$ for $s \geq 4m_W^2$.)

Let's then evaluate the partial wave unitary equation for our $W_L + W_L \to W_L + W_L$ amplitude. We have to compute $a_\ell = \frac{1}{32\pi}\int_{-1}^{+1} d\cos\theta \, \mathcal{M}(s,\theta)\, P_\ell(\cos\theta)$, in particular

$$a_0(W_L^+ W_L^- \to W_L^+ W_L^-) \simeq \frac{g^2}{32\pi\, 4m_W^2}\int_{-1}^{+1} d\cos\theta \left(s - \frac{s-4m^2}{2}(1-\cos\theta)\right) =$$

$$= \frac{s}{32\pi}\frac{g^2}{4m_W^2} \qquad \left[\text{recall } t = -\frac{(s-4m^2)}{2}(1-\cos\theta)\right]$$



$(\mathrm{Re}\, a_\ell)^2 + (\mathrm{Im}\, a_\ell - \frac{1}{2})^2 = \frac{1}{4}$

The loss of perturbativity can be estimated to be at $\pi \simeq \delta \simeq 2\,\mathrm{Re}(a_0)$, i.e. for

when real & Im are of the same order.

$\delta = \frac{1}{2}\left|\log\left(\frac{i+a_\ell}{-i+a_\ell^*}\right)\right| \simeq$
$\simeq 2\,\mathrm{Re}\, a_\ell$

$$\sqrt{s} \simeq \Lambda = 4\pi v \simeq 3\ \mathrm{TeV}.$$

where we introduced $v \equiv m_W^2\, 4/g^2$. Sometimes the estimate used is $\mathrm{Re}(a_0) \leq 1/2$, this is also a valid estimate, reflecting the theoretical uncertainty of the estimate.

We have found Λ by inspecting a particular process. Next we would like to expose this "EFT-like" behaviour in the Lagrangian description.
In order to do so we perform the following field redefinition in the Lagrangian above:

$$W_\mu \longrightarrow \frac{i}{g} U (D_\mu U)^\dagger = U W_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger$$

where $U(x) = e^{ig\,\tau^a \pi^a/m_W}$, $T^a \in SU(2)$. Then, the Lagrangian becomes

$$\mathcal{L}_{SU(2)} = -\frac{1}{4} W_{\mu\nu} W^{\mu\nu} + \frac{m_W^2}{g} \mathrm{Tr}\left[(D_\mu U)^\dagger (D^\mu U)\right]$$

$$= -\frac{1}{4} W_{\mu\nu} W^{\mu\nu} + \left[D_\mu U \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}\right]^\dagger \left[D_\mu U \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}\right]$$

with $D_\mu = \partial_\mu - i\,\vec{\tau}\cdot\vec{W}$ and $v \equiv 2m_W/g$. Several comments:

→ Note that we have introduced a new field $\pi^a(x)$! But we also declare a new gauge redundancy: $U \sim L(x)\, U\, L^\dagger(x)$
$W_\mu \sim L(x)\, D_\mu\, L^\dagger(x)$; with $L(x) \in SU(2)$.

→ "emergent" gauge invariance in this description. By taking the so called unitarity gauge we can set $\pi^a(x) = 0$ everywhere and then we are back to the original Lagrangian.

→ the second line make it clear that $SU(2)$ is broken: $T^a \langle U(0, \frac{v}{\sqrt{2}})^T \rangle \neq 0$.

The new Lagrangian is equivalent, but has a great advantage: it makes the high energy limit of the theory manifest! At high energies $W_L^\pm$ is described by the π^a fields.

This can be seen by taking the following limit:

$$m_W \to 0 \quad \& \quad g \to 0 \qquad \text{with} \qquad \frac{m_W}{g} \ \text{fixed}.$$

which is called the "decoupling limit". Then, the Lagrangian becomes:

$$\mathcal{L}_{SU(2)} = \frac{m_W^2}{g} \operatorname{Tr} \partial_\mu U^\dagger \partial^\mu U \quad \underbrace{- \frac{1}{4} W_{\mu\nu}^2}_{\text{decoupled.}}$$

The most important lesson is that this Lagrangian is not perturbative at energies $E \gtrsim m_W/g$. More accurately at $E = 4\pi m_W/g$ the loop expansion breaks down. Why so?

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$$\mathcal{L}_{SU(2)} = \frac{v^4}{4} \operatorname{Tr}\left[\partial_\mu \left(-i\frac{\pi \cdot \sigma}{v} - \frac{(\pi\cdot\sigma)^2}{2v^2} + i\frac{(\pi\sigma)^3}{3!v^3} \right) \partial^\mu \left(\frac{i\pi\cdot\sigma}{v} - \frac{(\pi\cdot\sigma)^2}{2v^2} - \frac{i(\pi\sigma)^3}{3!v^3} \right) \right] + O(v^{-4})$$

$$= \frac{1}{4} \partial_\mu \pi^a \partial^\mu \pi^b \underbrace{\operatorname{Tr}[\sigma^a\sigma^b]}_{2\delta^{ab}} + \xrightarrow{0}\cancel{\underbrace{\operatorname{Tr}[\sigma^a\sigma^b c^c]}_{\sim\epsilon^{abc}}} \partial_\mu(\pi^a\pi^b)\partial^\mu\pi^c \, O(1)/v^3$$

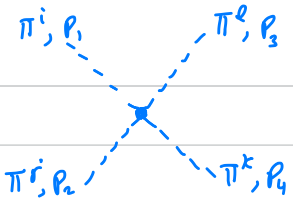
$$+ \frac{1}{v^2} \underbrace{\operatorname{Tr}[\sigma^a\sigma^b\sigma^c\sigma^d]}_{= 2\{\delta_{ab}\delta_{cd} \underbrace{-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}}_{\text{antisymmetric in } c \leftrightarrow d.}\}} \left\{ \frac{1}{4\times4} \partial_\mu(\pi^a\pi^b)\partial^\mu(\pi^c\pi^d) - \cancel{\frac{1}{3!4}\left(\partial_\mu\pi^a \partial_\mu(\pi^b\pi^c\pi^d) + \partial_\mu(\pi^a\pi^b\pi^c)\partial_\mu\pi^d\right)} \right\}$$

0, by Eqs. of motion

Thus, all in all:

$$\mathcal{L}_{SU(2)} = \frac{1}{2}\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \frac{1}{8v^2} \partial_\mu(\vec{\pi}\cdot\vec{\pi})\partial^\mu(\vec{\pi}\cdot\vec{\pi}) + O(\pi^6)$$

OK, let's now do some simple calculations with this Lagrangian.



$$= \frac{i}{8v^2} \langle \pi^i \pi^j | \int d^4x \; \partial_\mu (\pi^a \pi^a) \partial^\mu (\pi^b \pi^b) | \pi^k \pi^l \rangle$$

or

$$= \frac{1}{8v^2} \Big\{ \delta_{ij} (p_1+p_2)_\mu \, \delta_{kl} (p_3+p_4)^\mu \times 8$$

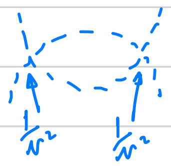
$$+ \delta_{kj}\delta_{il} (p_1+p_3)\cdot(p_2+p_4) \times 8$$

$$+ \delta_{ki}\delta_{jl} (p_1+p_4)\cdot(p_2+p_3) \times 8 \Big\} \times (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4)$$

Note $3 \times 8 = 4!$, as it should.

$$= \frac{1}{v^2} \Big\{ \delta_{ij}\delta_{kl}\, s + \delta_{il}\delta_{kj}\, t + \delta_{ik}\delta_{jl}\, u \Big\} \times \text{"4-momentum conservation"}$$

This computation is valid as long as $s/v^2 \ll 1$. Indeed:



$$\sim \frac{1}{16\pi^2} \frac{s^2}{v^4} \ll \frac{s}{v^2} \implies \frac{s}{(4\pi)^2 v^2} \ll 1 \quad \text{i.e. } E \ll 4\pi v \equiv \Lambda$$

tree level

We can project the computation into charged states:

$$P^{+-}_{ij}\, \pi^i \pi^j = \sum_{i,j} \sigma^+_i \sigma^-_j \, \pi^i \pi^j$$

with $\sigma^+ = (1, i)/\sqrt{2}$; $\sigma^- = (1, -i)/\sqrt{2}$.

$$\mathcal{M}(\pi^+\pi^- \to \pi^+\pi^-) = \sum_{i,j,k,l} P^{+-}_{ij} P^{+-}_{kl} \mathcal{M}(\pi_i \pi_j \to \pi_k \pi_l) = \frac{1}{v^2}(s+t)$$

$$= \frac{g^2}{4 m_W^2} (s+t)$$

like the $W_L^+ W_L^- \to W_L^+ W_L^-$ amplitude!

Now the "Higgs mechanism" comes in as a particularly simple way to UV complete this Lagrangian. By promoting

$$U \longrightarrow H \equiv U \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}$$

the Lagrangian becomes

$$\mathcal{L}_{SU(2)} \longrightarrow -\frac{1}{4} W_{\mu\nu} W^{\mu\nu} + (D_\mu H)^\dagger (D^\mu H)$$

which is UV complete because it does not have an offending growth at high energies. Indeed use the gauge equivalence to work with the variables $H = (G^+, \frac{h + iG_0}{\sqrt{2}})^T$ then the decoupling limit of $(D_\mu H)^\dagger D^\mu H$ is the free theory

$= 0$

[Notice that for the symmetry pattern $U(1) \otimes U(1) \longrightarrow U(1)_{QED}$ there is no loss of perturbativity at large energies. A theory with only photons and Z's would not necessarily need a Higgs to remain weakly coupled]

⊛ Bottom up SM Higgs model:

We write down the Lagrangian with all the d.o.f. with masses below the higgs boson mass

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{mass} \quad \text{with} \quad \mathcal{L}_0 = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \sum_{i=1}^{3} \bar{\Psi}_i \not{D} \Psi^{(i)}$$

and $$\mathcal{L}_{mass} = M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2} Z_\mu Z^\mu - \sum_{i,j} \left(\bar{u}_L^i M^u_{ij} u_R^j + \bar{d}_L^i M^d_{ij} d_R^j + \bar{e}_L^i M^e_{ij} e_R^j \right) + h.c.$$

where as usual $\Psi = (Q, u, d, L, e)$, and "i", "j" are generation indices.
$\mathcal{L}_0$ is invariant under $SU(2)_L \times U(1)_Y$ but $\mathcal{L}_{mass}$ is not. We expose the high energy behaviour of $\mathcal{L}_{mass}$ as follows:

$$\mathcal{L}_{mass} = \frac{v^2}{4} \mathrm{Tr}\left[(D_\mu \Sigma)^\dagger D_\mu \Sigma \right] - \frac{v}{\sqrt{2}} \sum_{i,j} (\bar{u}_L^i, \bar{d}_L^i)\, \Sigma \begin{pmatrix} \lambda^u_{ij}\, u_R^j \\ \lambda^d_{ij}\, d_R^j \end{pmatrix} + h.c.$$

where $D_\mu \Sigma = \partial_\mu \Sigma - i g \frac{\vec{\sigma}}{2} \cdot \vec{W} \Sigma + i g' \Sigma \frac{\sigma^3}{2}$, and we have introduced a gauge redundancy:

$$\Sigma' \sim U_L(x)\cdot\Sigma\cdot U_Y(x) \qquad \text{w/ } U_L\in SU(2)_L,\ U_Y\in U(1)_Y.$$

this transformation is non-linearly realized on the π fields ($\Sigma = \exp(i\,\sigma^a \pi^a/v)$):

$$\pi^a(x) \longrightarrow \pi^a(x) + \frac{v}{2}\alpha_L^a(x) - \frac{v}{2}\delta^{a3}\alpha_Y(x).$$

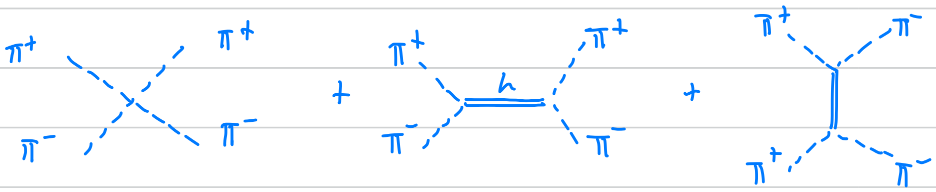
We saw that the Lagrangian above is an EFT with a physical cutoff $\Lambda \simeq 4\pi v$. Next we will try to enlarge the validity of this theory to energies $s > \Lambda$ by adding a new d.o.f. at a scale $m_h < \Lambda$.

$$\mathcal{L}_H = \frac{1}{2}(\partial_\mu h)^2 - V(h) + \frac{v^2}{4}\mathrm{Tr}\left[(D_\mu\Sigma)^\dagger(D^\mu\Sigma)\right]\left(1 + 2a\frac{h}{v} + b\frac{h^2}{v^2} + \dots\right)$$

$$- \frac{v}{\sqrt{2}}\sum_{i,j}\left(\bar{u}_L^i, \bar{d}_L^i\right)\Sigma\left(1 + c\frac{h}{v} + \dots\right)\begin{pmatrix}\lambda^u_{ij}\, u_R^j \\ \lambda^d_{ij}\, d_R^j\end{pmatrix} + h.c.$$

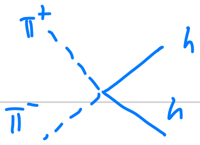
where $V(h) = m_h^2 h^2/2 + \dots$. The couplings a, b, c, are a priori arbitrary, but we can fix them by demanding perturbativity of $\pi + \pi$ scattering at high energy. For instance

$$\mathcal{M}(\pi^+\pi^- \to \pi^+\pi^-) =$$



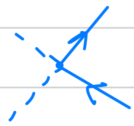
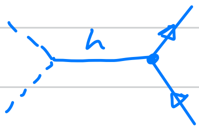
$$= \frac{1}{v^2}\left(s - a^2\frac{s^2}{s - m_h^2} + (s \leftrightarrow t)\right) = \frac{s+t}{v^2}(1 - a^2) + O\!\left(\frac{m_h^2}{E}\right)$$

Therefore $a = 1$ cancels the high energy growth! Thus the scalar "h" re-store unitarity, i.e. unitarizes, $\pi + \pi$ scattering at high energy.
Since we have introduced a new particle we need to check also inelastic channels involving the particle "h".

$\mathcal{M}(\pi^+\pi^- \to hh) =$ 

$+$  $= \frac{s}{v^2}(b - a^2) + O\left(\frac{m_h^2}{E^2}\right).$

which implies $b = a^2 = 1$! Finally, for $\pi\pi \to \psi\psi$ (which is equivalent to $W_L^+ W_L^- \to \psi\psi$ @ High energy) we have

$\mathcal{M}(\pi^+\pi^- \to \psi\bar{\psi}) =$  $+$ 

$= \frac{m_\psi \sqrt{s}}{v^2}(1 - ac) + O\left(\frac{m_h^2}{E^2}\right)$

which sets $ac = 1$.

The point $a = b = c = 1$ defines the Higgs model. In other words, π^a's toghether with "h" form a linear representation of $SU(2)_L \times SU(2)_R$. The restoration of unitarity at high energy, or restoration of the validity of perturbation theory, can thus be traced back to the renormalizability.

Note that $m_h \simeq 125\,\text{GeV} \ll 4\pi v \simeq 3\,\text{TeV}$. Thus the model is very weakly coupled.

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<u>A comment on custodial symmetry:</u>

Taking the unitary gauge $\langle \Sigma \rangle = \mathbb{1}$, the Lagrangian above reproduces:

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos\theta_W} = 1. \qquad (\text{Eq.}1)$$

This relation is consistent with experiment to a very good accuracy.

It follows from an accidental approximate symmetry of $\mathcal{L}_{mass}$:

$$\Sigma \to U_L \Sigma U_R^+,$$

which is spontaneously broken by $\langle \Sigma \rangle = \mathbb{1}$ to the diagonal subgroup

$$SU(2)_L \times SU(2)_L \to SU(2)_{L+R} \equiv SU(2)_C \leftarrow \text{"custodial"}$$

and explicitly broken by $g' \neq 0$ and $\lambda_{ij}^u \neq \lambda_{ij}^d$. In the limit $g' = \lambda^u - \lambda^d = 0$ the fields π^a's transform as a tripled under $SU(2)_C$ and $m_W = m_Z$. This equation is replaced by (Eq.1) for $g' \neq 0$ and loop corrections are small

and proportional to g' and $(\lambda_u - \lambda_d)$.
In fact the success of the $\rho = 1$ prediction @ tree-level justifies a posteriori the omission of the custodial breaking operator

$$v^2 \, \mathrm{Tr}\left[\Sigma^{\prime\dagger} D_\mu \Sigma' \sigma^3 \right]^2 ,$$

which is invariant under $SU(2)_L \times U(1)_Y$ but breaks the global $SU(2)_L \times SU(2)_R$.
Therefore ~~$SU(2)_c$~~ beyond the SM must be small.

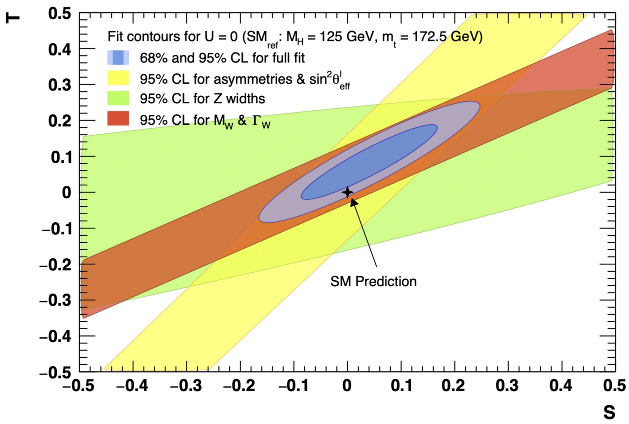
Few coments on EWPT: (oblique parameters only)

We saw that deviations from $\rho \simeq 1$ is a way to constrain new physics that violate $SU(2)_c$ (which is broken in the SM in a particular direction). It is useful to have additional ways to constrain and characterize new physics. To this end, Peskin and Takeuchi introduced the S, T and U parameters. These are useful to constraint BSM theories that couple "universally" (i.e. flavour diagonal) to the SM. Then all the new physics effects can be characterized by modifications of the W and Z propagators. It is useful to parametrize the leading deviations as Peskin-Takeuchi did:

$$\begin{cases} T \equiv \dfrac{1}{\alpha_e}\left(\dfrac{\Pi^{new}_{WW}(0)}{m_W^2} - \dfrac{\Pi^{new}_{ZZ}(0)}{m_Z^2} \right) = \dfrac{\rho - 1}{\alpha_e} , \\[2ex] S \equiv \dfrac{4 c_W^2 s_W^2}{\alpha_e}\left(\dfrac{\Pi^{new}_{ZZ}(m_Z^2) - \Pi^{new}_{ZZ}(0)}{m_Z^2} - \dfrac{c_W^2 - s_W^2}{c_W s_W} \dfrac{\Pi^{new}_{Z\gamma}(m_Z^2)}{m_Z^2} - \dfrac{\Pi^{new}_{\gamma\gamma}(m_Z^2)}{m_Z^2} \right) , \\[2ex] U \equiv \dfrac{4 s_W^2}{\alpha_e}\left(\dfrac{\Pi^{new}_{WW}(m_W^2) - \Pi^{new}_{WW}(0)}{m_W^2} - \dfrac{c_W}{s_W} \dfrac{\Pi^{new}_{Z\gamma}(m_Z^2)}{m_Z^2} - \dfrac{\Pi^{new}_{\gamma\gamma}(m_Z^2)}{m_Z^2} \right) - S . \end{cases}$$

with $\alpha_e = \alpha_e(m_Z)$. Here "new" means that we are subtracting the SM contribution to the various 2-point functions. Then $S = T = U = 0$ is the SM point, i.e. with no new physics. Current experiment gives

$$S = 0.04 \pm 0.11\,, \quad T = 0.09 \pm 0.14\,, \quad U = -0.02 \pm 0.11\,.$$



From ref. hep-ph/1803.01853. The plot shows the constraints on S & T parameters with $U = 0$. In practice S & T tend to give stronger constraints on BSM than U.

We can think of corrections to S and T away from the SM value as coming from higher dimensional operators. For instance the leading one arise at dimension-six level and are given by

$$\mathcal{O}_S = H^\dagger \sigma^i H W^i_{\mu\nu} B^{\mu\nu} \longrightarrow \Delta S = \frac{\alpha}{4 s_W^2} S$$

$$\mathcal{O}_T = \frac{1}{2} C_T^{ab}\, \mathrm{tr}\left[\sigma^a \Sigma^\dagger \overleftrightarrow{D}_\mu \Sigma\right] \mathrm{Tr}\left[\sigma^b \Sigma^\dagger \overleftrightarrow{D}_\mu \Sigma\right] = \frac{1}{2} \left(H^\dagger \overleftrightarrow{D}_\mu H\right)^2 \longrightarrow \Delta T = \alpha T$$

$3_R \times 3_L$ spurion.

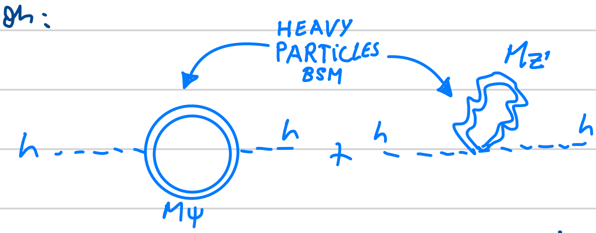
It is easy to construct BSM with $\Delta T = 0$ because it can be protected by with custodial symmetry, on the other hand we do not know of any symmetry to set $\Delta S = 0$.

(For instance, integrating out scalar triplets generates $\mathcal{O}_T$. How to generate "S" at at tree level? $S \sim C_W + C_B + C_{WB}$

$\mathcal{D}_\nu W^{\mu\nu}_a H^\dagger \overleftrightarrow{D}_\mu \sigma^a H$ $\qquad$ $\mathcal{D}_\nu B^{\mu\nu} H^\dagger \overleftrightarrow{D}_\mu H$)

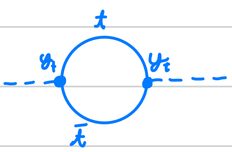
EW Hierarchy ~~problem~~ strategy :

In the bottom up construction of the SM we discover that the role of the Higgs is to UV complete $\mathcal{L}_0 + \mathcal{L}_{mass}$ at weak coupling. The higgs had to appear with a mass $m_h \ll 4\pi v$, and it turned out to be $m_h \simeq 125\,\text{GeV}$ thus $m_h/4\pi v \sim 0.04$, thus the $W_L^\pm$ amplitudes are very much perturbative in the SM. This is a success of theorists pure thought deductions!
Yet, we don't find it fully satisfactory because of the following observation:

$\delta m_h =$  $+ \dots = \alpha_{BSM} M_\psi + \alpha'_{BSM} M_{Z'}$

If the right hand side of this equation is not finely tune, then we would expect $m_h \sim M_\psi, M_{Z'}$.

There are various ways to characterize this problem in a more physical/invariant way using a condensed matter analogy (Wilson).
But here we take the particle physicist p.o.v.:

 $+ \dots = \frac{g_t^2}{16\pi^2} \Lambda^2$

physical cutoff where new physics comes in

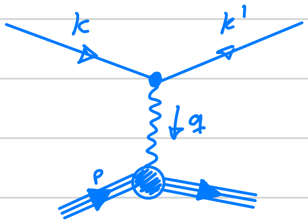
Every calculable model we know of $m_h \sim m_{NP}$ unless parameters in the UV theory are finely tuned.
More about it later on!

• ELEMENTS OF QCD AT COLLIDERS

We would like to describe how one can experimentally search for the Higgs boson and how it was actually discovered. Before that we need to understand how to describe collisions involving protons. Let's start reviewing one of the simplest experiments involving protons:

*** Elastic $e^- p^+$ scattering:**

Suppose the proton was elementary like the muon. Then we would expect $e^- p$ scattering to look like $e^- \mu^+$ scattering. In fact, it does, at low energy. The leading Feynman diagram is given by the t-channel photon exchange diagram:



This is often called Coulomb scattering.

At low energy the proton looks elementary, and is indistinguishable from an elementary particle. It's structure is only revealed if probed with a highly energetic photon in the Coulomb scattering.

The x-section of two spin 1/2 particles is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha_e^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m_p^2}\sin^2\frac{\theta}{2}\right)$$

where E and E' are the electron's initial and final energies, $q^\mu = k^\mu - k'^\mu$ is the momentum transfer, θ is the angle between the outgoing and incoming electrons ($\theta = 0$ is forward). These quantities are related by

$$q^2 = -2k\cdot k' = -\left(4E'E\sin^2\frac{\theta}{2}\right)_{lab} ,$$

where the formula is valid in the "lab" frame, where the proton is at rest. and we have approximated $m_e = 0$.

That the proton is not elementary can be revealed by the following calculation and experiment. Instead of the QED vertex $ie\bar{u}\gamma^\mu u$, for the proton we shall use a form-factor

$$\Gamma^\mu(q) = F_1(q)\,\gamma^\mu + i\,\frac{\sigma^{\mu\nu}}{2m_p}\,q_\nu F_2(q^2)$$

and the vertex $ie\,\bar{u}\,\Gamma^\mu u$.

Then, repeating the tree-level coulomb scattering with this vertex, one finds:

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$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha_e^2}{4E^2\sin^4\frac{\theta}{2}}\,\frac{E'}{E}\left\{\left(F_1^2 - \frac{q^2}{4m_p^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m_p^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right\}$$

which is known as the Rosenbluth formula.

If the proton only interacted through QED like $e^- Z^+$ scattering we could compute F_1 and F_2. For instance, up to one-loop level, one finds $F_2 \to 0$ for $|q| \gg m_Z$ and $F_1(q_1^2) - F_1(q_2^2) \approx -\frac{\alpha}{4\pi}\log\left(\frac{q_1^2}{q_2^2}\right)$, for $|q_1|, |q_2| \gg m_Z$. (and we may use on-shell renormalization $F_1(0) = 1$, i.e. $Q_Z = +1$).

The proton instead behaves very differently. It was found that a good fit is provided by

$$F_1(q^2) \sim \left(1 - \frac{q^2}{\mu}\right)^{-2},$$

where a new scale has emerged, $\mu \sim 0.7$ GeV.

The form-factors is useful because they are the Fourier transform of scattering potentials in the Born approximation

$$F_1(q^2) = \int d^3x\; e^{i\vec{q}\cdot\vec{x}}\, V(x)$$

$$\left(\left(\frac{d\sigma}{d\Omega}\right)_{Born} = \frac{m_e^2}{4\pi^2}\left|\tilde{V}(\vec{k})\right|^2\right)$$

F.T. of $V(\vec{x})$

which implies

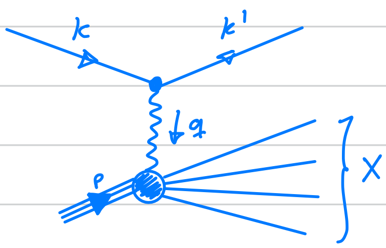
$$V(r) \sim \mu^3 e^{-\mu r}.$$

Therefore the proton is characterised by an exponential shape of characteristic size $r_0 \sim \mu^{-1} \sim 1\,\text{fm}$, the size of the proton.

To learn more about the proton we need to go to much higher energies. One may expect to find a much more complicated fitting function. Instead what is find is an elementary scattering through the partons! That is, very high energy $e^- p^+$ scattering reveals point like constituents within the proton. We will explain next how this can be found out.

* Inelastic $e^- p^+$ scattering

So far we discussed elastic $e^- p^+$ scattering. When the center of mass (C.M.) energy is above the proton mass m_p, the proton can break apart.



Deep inelastic scattering (DIS)

Remarkably the physics simplifies in this deeply inelastic regime, and we will be able to make predictions.

In deriving the Rosenbluth formula for elastic scattering we reduced the photon-proton vertices to terms of the form $\bar{u}\gamma^\mu u$ and $\bar{u}\sigma^{\mu\nu}q_\nu u$ multiplied by F_1 and F_2 respectively. When the proton breaks apart, as in DIS, this parametrization is not valid. Instead, we need to parametrize photon-proton-X interactions, where "X" is anything the proton can break up into. Therefore, it makes sense to parametrise the X-section (instead of the vertices) in terms of the momentum transfer q^μ and the proton momentum P.

In the lab frame, we define E and E' as the energies of the incoming and outgoing electron. We also define θ as the angle between $\vec{k}$ and $\vec{k}'$, so $\theta = 0$ is forward electron. The cross section can be written as

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha_e^2}{4\pi m_p q^4} \frac{E'}{E} L^{\mu\nu} W_{\mu\nu} ,$$

where $L_{\mu\nu}$ is the "leptonic tensor". For unpolarized scattering it is given by

$$L_{\mu\nu} = \frac{1}{2} \mathrm{Tr}\left[\not{k}' \gamma^\mu \not{k} \gamma^\nu \right] = 2\left(k'^\mu k^\nu + k'^\nu k^\mu - k' \cdot k \, g^{\mu\nu} \right)$$

note that $L_{\mu\nu} = L_{\nu\mu}$. It comes from $L^{\mu\nu} = \frac{1}{2} \sum_{spins}' \bar{v}(k')\gamma^\mu u(k) \bar{u}(k) \gamma^\nu v(k')$, and the factor of "1/2" arises from taking the average over the initial electron's spin.

The "hadronic tensor" $W^{\mu\nu}$ includes an integral over phase space for all final state particles. It gives the rate $\gamma^* p^+ \to$ anything:

$$e^2 \epsilon_\mu \epsilon_\nu^* W^{\mu\nu} \equiv \frac{1}{2} \sum_{X, spins}' d\Pi_X (2\pi)^4 \delta^{(4)}(q + P - p_X) \left| \mathcal{M}(\gamma^* p^+ \to X) \right|^2$$

where ϵ_μ is the polarization of the off-shell photon. Since we are integrating over the final states, $W^{\mu\nu}$ can only depend on P^μ and q^μ. In unpolarized scattering, it is symmetric $W^{\mu\nu} = W^{\nu\mu}$. It should also satisfy the Ward identity $q_\mu W^{\mu\nu} = 0$, since the interaction is through a photon. Therefore, the most general tensor is:

$$W^{\mu\nu} = W_1 \cdot \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + W_2 \cdot \left(P^\mu - \frac{P \cdot q}{q^2} q^\mu \right)\left(P^\nu - \frac{P \cdot q}{q^2} q^\nu \right)$$

The Lorentz scalars W_1 and W_2 can depend on $P^2 = m_p^2$, q^2 and $P \cdot q$. The natural variables to use are

$$Q \equiv \sqrt{-q^2} > 0 \qquad \text{and} \qquad \nu \equiv \frac{P \cdot q}{m_p} = (E - E')_{lab}$$

Energy scale of the collision.

Energy lost by electron in the proton rest frame (lab)

An alternative to ν, is to use the dimensionless variable

$$x \equiv \frac{Q^2}{2\,P\cdot q}$$

which is known as Bjorken x variable.

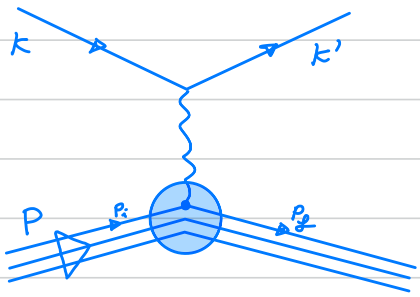
Using these definitions, we have

$$\left(\frac{d\sigma}{d\Omega\, dE'}\right)_{lab} = \frac{\alpha_e^2}{8\pi E^2 \sin^4\frac{\theta}{2}}\left(\frac{m_p}{2} W_2(x,Q)\cos^2\frac{\theta}{2} + \frac{1}{m_p} W_1(x,Q)\sin^2\frac{\theta}{2}\right)$$

As in the elastic case, everything is set such that we only need to know about the incoming and outgoing electron's momenta, and nothing about X. That is W_1 and W_2 are determined by measuring the energy and angular dependence of the outgoing electron.

<u>Parton model</u>: the defining assumption is that particles inside the proton, called "partons", are essentially free. Model originally by Feynman. Now we know that the partons are the valence quarks (uud), but also the gluons, and in principle any other SM particle.

To test the parton model we need to determine W_1 and W_2 when the electron scatters elastically off partons of mass m_q inside the proton. Parton scattering:



The circle represents the proton, and the lines inside the "partons" inside the proton.

The electron-parton scattering looks like $e^-\mu^+$ scattering. To evaluate it, we call p_i / p_f the parton initial/final momentum, so that

$p_i^\mu + q^\mu = p_f^\mu$ by momentum conservation. Squaring this equation we have:

$$m_q^2 + 2p_i\cdot q + q^2 = m_q^2 \implies 1 = \frac{Q^2}{2p_i\cdot q}.$$

The momentum of the parton is not directly measurable. But let's assume it carries a fraction ξ of the proton's momenta P, $p_i^\mu = \xi P^\mu$. Then

$$x = \frac{\xi Q^2}{2p_i\cdot q} = \xi.$$

In particular, if the parton model is correct, by measuring "x" we are measuring the parton's momentum fraction.

Next we compute $e^- q \to e^- q$ in perturbation theory. In particular we expect the form factors F_1, F_2, to have only a weak, logarithmic dependence on Q^2, when the initial momentum is fixed (i.e. at fixed "x"). The x-section (approximate) independence of Q^2 at fixed "x" is known as Bjorken scaling.

Another important ingredient of the parton model is the classical probabilities

$$f_i(\xi)\, d\xi$$

of the photon scattering into the parton "i" with momentum fraction ξ. These $f_i(\xi)$ are known as Parton Distribution Functions (PDFs).

The physical justification for the PDFs:
The time scales for interactions among proton constituents $\sim \Lambda_{QCD}^{-1} \sim m_p^{-1}$ is much slower than the time scale that the photon probes $\sim Q^{-1} \ll m_p^{-1}$. This separation of scales allows us to treat the wavefunctions within the proton as being decoherent, and hence the probabilistic interpretation. A proof of this result is beyond the scope of this course, and is the topic of the "factorization theorems".

After having introduced the PDF's, we can be much more precise about the predictions of the weakly interacting partons.

$$\sigma(e^- P^+ \to e^- X) = \sum_i \int_0^1 d\xi\, f_i(\xi)\, \hat{\sigma}(e^- p_i \to e^- X)$$

where partonic quantities are indicated with a hat $\hat{\sigma}$.

Next assuming partons weakly interact with the photon, we get the Rosenbluth formula with $F_1 = 1$, $F_2 = 0$. Before integrating over E', we have:

$$\left(\frac{d\hat{\sigma}(e^- q \to e^- q)}{d\Omega\, dE'}\right)_{lab} = \frac{\alpha_e^2 Q_i^2}{4E^2 \sin^4\frac{\theta}{2}} \left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_q^2}\sin^2\frac{\theta}{2}\right) \delta\left(E - E' - \frac{Q^2}{2m_q}\right)$$

where Q_i is the charge of the parton (i.e. quark). Our previous formula with $F_1 = 1$, $F_2 = 0$ is reproduced after integrating over dE'!
If we were not assuming weakly interacting quarks we would get generic Form-Factors, violating Bjorken scaling.

In order to get the DIS x-section we need to integrate over the incoming quark momentum. Since $p_i^\mu = \xi P^\mu$ and in the lab frame the proton is at rest, we have $m_q = \xi m_p$. Using $E - E' = \nu = Q^2 (2 m_p x)^{-1}$ it follows

$$\delta\left(E - E' - \frac{Q^2}{2m_q}\right) = \delta\left(\frac{Q^2}{2m_p x} - \frac{Q^2}{2m_p \xi}\right) = \frac{2 m_p x^2}{Q^2}\delta(\xi - x).$$

Therefore the total x-section is given by

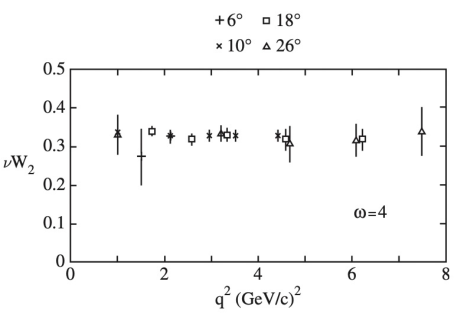
$$\left(\frac{d\sigma(e^- P \to e^- X)}{d\Omega\, dE'}\right)_{lab} = \sum_i f_i(x) \frac{\alpha_e^2 Q_i^2}{4E^2 \sin^4\frac{\theta}{2}} \left(\frac{2m_p}{Q^2} x^2 \cos^2\frac{\theta}{2} + \frac{1}{m_p}\sin^2\frac{\theta}{2}\right)$$

Comparing with our original parametrization in terms of the hadronic tensor, we have

$$W_1(x,Q) = 2\pi \sum_i Q_i^2 f_i(x)$$

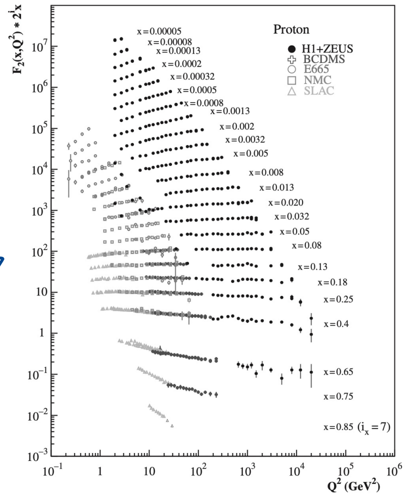
$$W_2(x,Q) = 8\pi \frac{x^2}{Q^2} \sum_i Q_i^2 f_i(x).$$

This is a concrete prediction for Bjorken scaling! The quantities $W_1(x,Q)$ and $Q^2 W_2(x,Q)$ should be (approximately) independent of Q at fixed x. Recall, although quarks are not directly observable, the quantity $x = \frac{Q^2}{2m_p}(E-E')^{-1}$ can be measured. Below early and recent measurements of Bjorken scaling.



Friedman & Kendall (1972)

PDG, 2012



Another prediction of the parton model is

$$W_1(x,Q) = \frac{Q^2}{4x^2} W_2(x,Q) \quad \text{for } Q \gg m_p$$

which is known as the Callan - Cross relation. The proportionality factor can be traced back to the $Q^2/2m_q^2 = Q^2/2x^2m_p^2$ factor in the $e^-q \to e^-q$ parton amplitude, which is due to the quarks being free Dirac fermions. Thus the Callan-Cross relation tests that quarks have spin-$\frac{1}{2}$.

This relation is often presented in a different form. Using $y \equiv \frac{P \cdot q}{P \cdot k} = \frac{\nu}{E}$ (dimensionless), so that $dE' d\Omega = \frac{2m_p}{E'} \pi y \, dx \, dy$, we have

$$\frac{d\sigma(e^-P \to e^-X)}{dx\,dy} = \frac{2\pi\alpha^2}{Q^4} s \underbrace{\left(1 + (1-y)^2\right)}_{\text{Callan - Cross relation.}} \sum_i Q_i^2 \, x \, f_i(x)$$

END OF L13

* PDFs Sum rules.

For the PDFs to admit a probability interpretation, they must satisfy various constraints. For example ~~if~~ the proton had exactly one down quark $\int_0^1 d\xi\, f_d(\xi) = 1$. In reality one can have virtual down-antidown quark pairs within the proton. However the down-quark number is conserved in QCD and QED. Therefore

$$\int_0^1 d\xi \left[f_d(\xi) - f_{\bar{d}}(\xi) \right] = 1$$

PDF for anti-down quark.

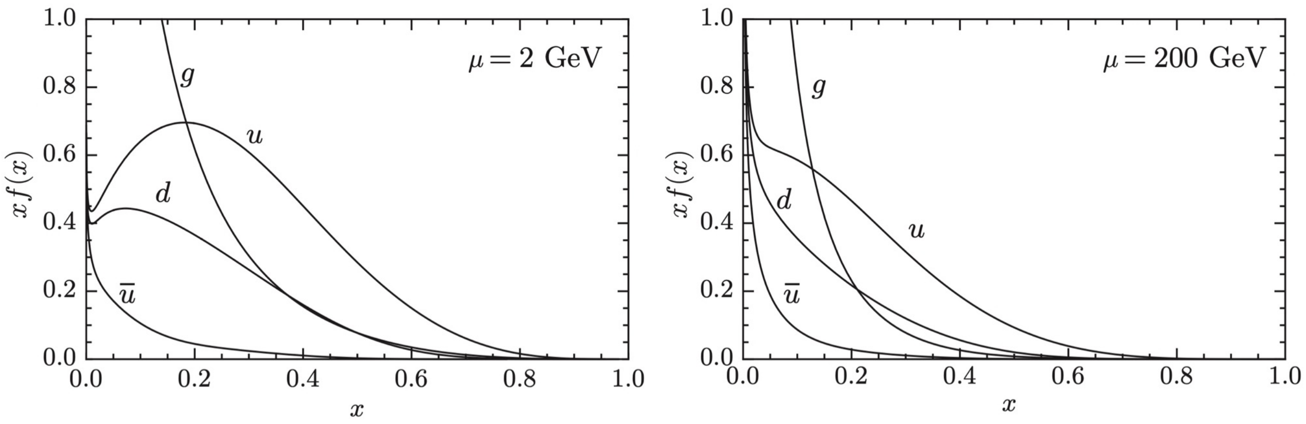
Similarly: $\int_0^1 d\xi \left[f_u - f_{\bar{u}}(\xi) \right] = 2$ because the total up-quark number is 2, and $\int_0^1 d\xi \left[f_s(\xi) - f_{\bar{s}}(\xi) \right] = 0$ because the strange-quark number is 0, and analogously for bottom and charm. There is no conserved quantum number for gluon so there is no sum rule for gluon. Finally, momentum conservation implies

$$\sum_i \int_0^1 d\xi\, \xi\, f_i(\xi) = 1.$$

Each of this sum rules correspond to a classically conserved current (up, down, strange number or momentum conservation).

It turns out that numerically $\int d\xi\, \xi \left(f_u(\xi) + f_d(\xi) \right) \simeq 0.38$. Therefore about 38% of the proton total momenta is in the valence quarks (u and d). The gluon takes 35% - 50% of the momentum, and the rest is on sea-quarks (i.e. s, c or b quarks and $\bar{d}, \bar{u}, \bar{c}, \bar{s}, \bar{b}$ anti-quarks).

PDF measurements from the MSTW group.



In practice PDF's are not only determined from DIS but also from many other high energy processes such as $p\bar{p}$ and pp collisions. A number of international research groups are dedicated to fitting PDFs.

* Factorization and the parton model from QCD.

For practical purposes the parton model is all what we need to perform perturbative QCD calculations at high energy colliders. The model build upon the concept of "factorization": PDFs are universal objects, and any scattering process involving protons can be computed with the same PDFs + an appropriate perturbative partonic crossection.

In the following lecture we would like to discuss the "proof" of factorization in the context of DIS using the Operator Product Expansion (OPE).

↳ this will lead to the identification of the moments of the PDFs with certain matrix elements of composite operators.

hard scattering — convolution

Intuitively by Factorization we mean $\sigma = \sigma_h \otimes \text{PDFs} + O\left(\frac{\Lambda_{QCD}}{Q}\right)$, where Q is the characteristic energy scale of the process. In the case of inclusive DIS, $Q = \sqrt{-(k-k')^2}$, the energy transfer, which we take large while keeping $x = Q^2/2p\cdot q$ fixed.

↳ The strategy that we will follow will consist in relating the DIS crossection to a product of currents $J^\mu(x) J^\nu(y)$. We then rewrite this product in terms of local ops. $J^\mu(x) J^\nu(y) \sim \sum_n C_n(x-y)\, \mathcal{O}_n(x)$. In the DIS limit $Q^2 \to \infty$ at fixed Bjorken x will correspond to $x-y \to 0$ so that we we'll be able to keep the first terms of the OPE. It will turn out that PDFs $\sim \langle P | \mathcal{O} | P \rangle$!

* The operator product expansion (OPE):

The OPE is the position space version of the low momentum expansion that we do when deriving Effective lagrangians.

For example, if we integrate the W boson at tree level

$$\mathcal{L}_W \sim g^2 \int d^4x\, d^4y\, \bar{\psi}(x) \gamma^\mu \psi(x)\, D^{\mu\nu}(x,y)\, \bar{\psi}(y) \gamma^\nu \psi(y)$$

where
$$D^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \frac{-g^{\mu\nu}}{p^2 - m_W^2} e^{ip(x-y)} = \frac{g^{\mu\nu}}{\Box_x + m_W^2} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}$$

is the W-boson propagator. For $\Box \sim p^2 << m_W^2$ we expand

$$\frac{g^2}{\Box + m_W^2} = G_F \left(1 - \frac{\Box}{m_W^2} + \left(\frac{\Box}{m_W^2}\right)^2 + \dots\right) \quad \text{with} \quad G_F \sim \frac{g^2}{m_W^2}$$

so that

$$\mathcal{L}_W \sim G_F \int d^4x \left[(\bar{\psi}\gamma^\mu\psi)_x (\bar{\psi}\gamma_\mu\psi)_x - (\bar{\psi}\gamma^\mu\psi)_x \frac{\Box}{m_W^2} (\bar{\psi}\gamma_\mu\psi)_x + \dots \right]$$

with all fields located at the same position. The effective lagrangian it is therefore local.

The OPE states that as two operators get close, it's matrix element on any state can be reproduced by local operators:

$$\lim_{x \to y} \mathcal{O}(x)\,\mathcal{O}(y) = \sum_n C_n(x-y)\,\mathcal{O}_n(x)$$

Wilson coefficients.

The equality holds at the level of operators and the Wilson coefficients are independent of the state. Thus one can determine the C_n's for a particular process and then use them at another process.

In the case of the 4-Fermi theory we can truncate the OPE because only a finite number of ops. are necessary at a given precision. In the case of DIS we will see that we need to keep an infinite power of ops. but the OPE will still be useful.

Often we will write the OPE as

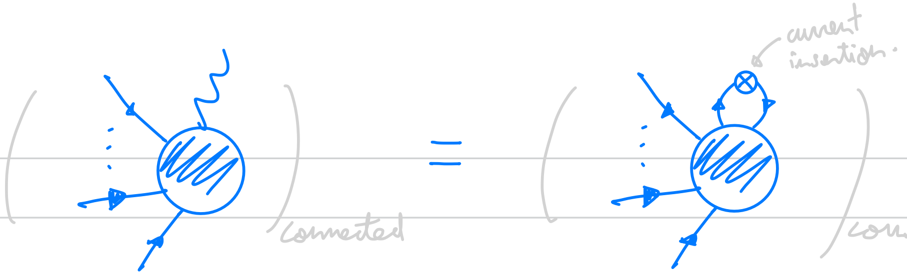
$$\int d^4x\, e^{iqx}\,\mathcal{O}(x)\,\mathcal{O}(0) = \sum_n C_n(q)\,\mathcal{O}_n(0)$$

where the RHS operator is in position space but the Wilson coefficients are written in momentum space.

* DIS as a time ordered product of currents:

In order to apply the OPE to DIS we first need to express the hadronic tensor $W^{\mu\nu}$ in terms of matrix elements involving the electromagnetic current constructed out of quarks, i.e. $J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$, with ψ the Dirac field for quarks.

Recall that: S-matrix elements involving photons, in which polarizations of external photons are removed, are equal to time ordered products involving currents. In pictures:



In short, this equation follows because $\langle p'| J^\mu(x) |p\rangle\big|_{x=0} = \bar{u}_2(p')\gamma^\mu u_1(p) e^{i(p'-p)x}\big|_{x=0}$ $= \bar{u}(p')\gamma^\mu u(p)$.

Allright, for DIS we need

$$\mathcal{M}(\gamma^* p^+ \longrightarrow X) = e\,\epsilon^\mu \langle X| J_\mu(0) |P\rangle$$

Comparing this equation with the definition of the hadronic tensor:

$$e^2 \epsilon_\mu \epsilon_\nu^* W^{\mu\nu} \equiv \frac{1}{2} \sum_{X,\,spins}\!\!\!' \; d\Pi_X (2\pi)^4 \delta^{(4)}(q+P-P_X) |\mathcal{M}(\gamma^* p^+ \to X)|^2$$

we deduce that

$$W_{\mu\nu}(\omega, Q) = \sum_X\!' \; d\Pi_X \langle P| J_\mu(0) |X\rangle\langle X| J_\nu(0) |P\rangle (2\pi)^4 \delta^{(4)}(q+P-P_X)$$

$$= \sum_X\!' \; d\Pi_X \int d^4x\, e^{i(q+P-P_X)x} \langle P| J_\mu(0) |X\rangle\langle X| J_\nu(0) |P\rangle$$

where we defined $\omega \equiv 1/x = 2P\cdot q/Q^2$, the inverse of the Bjorken variable "x", in order to avoid confusion with spacetime variable "x^μ".

Next we use the translation generator to simplify:

$$\langle P| J_\mu(0) |X\rangle = \langle P| e^{-i\hat{P}\cdot x} J_\mu(x) e^{i\hat{P}\cdot x} |X\rangle = e^{-i(P-P_X)x} \langle P| J_\mu(x) |X\rangle$$

and arrive to:

$$W_{\mu\nu}(\omega, Q) = \sum_X\!' \int d\Pi_X \int d^4x\, e^{iq\cdot x} \langle P| J_\mu(x) |X\rangle\langle X| J_\nu(0) |P\rangle$$

$$= \int d^4x\, e^{iq\cdot x} \langle P| J_\mu(x) J_\nu(0) |P\rangle$$

where we used $\mathbb{1} = \sum_X\!' |X\rangle\langle X|$ to get an expression for the hadronic tensor that is independent of $|X\rangle$!

This expression is not yet useful because we'd like to apply OPE on time ordered products. To amend this issue we'll use optical theorem:

$$W_{\mu\nu} \sim |M(\gamma^* p \to X)|^2 \sim |\langle T\{J\}\rangle|^2$$

$$\sim \text{Im}\, \mathcal{M}(\gamma^* p \to \gamma^* p) \sim \text{Im} \langle T\{JJ\}\rangle$$

In more detail: $W_{\mu\nu} = 2\,\text{Im}\, T^{\mu\nu}$ where $e^2 \epsilon_\mu \epsilon_\nu^* T^{\mu\nu}(\omega, Q) = \mathcal{M}(\gamma^* p \to \gamma^* p)$

Therefore

$$T_{\mu\nu}(\omega, Q) = i \int d^4x\, e^{iq\cdot x} \langle P| T\{J_\mu(x) J_\nu(0)\}|P\rangle$$

Thus instead of $W_{\mu\nu}$ we'll compute $2\,\text{Im}\, T_{\mu\nu}$ through the OPE of $T\{J_\mu(x) J_\nu(0)\}$. It is customary to define

$$T_{\mu\nu}(\omega, Q) = T_1(\omega, Q)\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{T_2(\omega, Q)}{P\cdot q}\left(p^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right)$$

as we did before for the $W^{\mu\nu}$ tensor. Thus $W_1 = 2\,\text{Im}\, T_1$ and $W_2 = 4\,\text{Im}\, T_2/\omega Q^2$.

* OPE for DIS:

We want to find the r.h.s. of

$$T\{J^\mu(x) J^\nu(0)\} = \sum_n{}' C_n(x)\, \mathcal{O}_n^{\mu\nu}(0)$$

and evaluate it on external proton states $|P\rangle$. It turns out that

$$i\int d^4x\, e^{iqx}\, T\{J^\mu(x) J^\nu(0)\} = \sum_q{}' Q_q^2\, t^{\mu_1\cdots\mu_n}(q)\, \mathcal{O}_q^{\mu_1\cdots\mu_n}(0) + O\left(\alpha_s, \frac{\Lambda_{QCD}}{Q}\right)$$

where the "Wilson coefficient" is

$$t^{\mu_1\cdots\mu_n}(q) = \sum_{n=2,4,\dots}^{\infty} \frac{(2q^{\mu_1})\cdots(2q^{\mu_n})}{Q^{2n}}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{2q^{\mu_3}\cdots 2q^{\mu_n}}{Q^{2n-2}} \times$$

$$\times \left(g^{\mu\mu_1} - \frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2} - \frac{q^\nu q^{\mu_2}}{q^2}\right).$$

and the local operators are given by

$$\mathcal{O}_q^{\mu_1\cdots\mu_n} = \bar{\psi}_q(x)\,\gamma^{\mu_1}\, iD^{\mu_2}\cdots iD^{\mu_n}\,\psi_q(x) + \text{symmetrizations of } \mu_i - \text{traces.}$$

For instance:

$$\mathcal{O}_q^{\mu_1\mu_2} = \bar{\psi}_q\left(i\gamma^{\mu}D^{\nu} + i\gamma^{\nu}D^{\mu} - \tfrac{1}{2}\, i g^{\mu\nu}\not{D}\right)\psi_q$$

We will not prove this OPE. Note that $\mathcal{O}^{\mu_1\cdots\mu_s}$ has spin s and dimension $d = s+2$. What is common of the sum above among the operators is the "twist" $t = d - s = 2$. [You can read more about the derivation in e.g. Peskin & Schroeder]

END OF L15

* PDFs and Callan-Gross relation from DIS OPE:

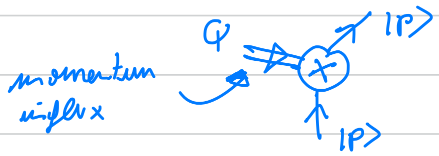
To use OPE in DIS, we need to evaluate the OPE in a proton state. By Lorentz invariance:

$$\sum_{\text{spins}} \langle p|\,\mathcal{O}_q^{\mu_1\cdots\mu_n}\,|p\rangle = A_n \cdot 2\, p^{\mu_1}\cdots p^{\mu_n} - \text{traces.}$$

RENORM SCALE.

with A_n functions of μ. The traces give factors of $p^2 = m_p^2 \ll Q$ which are subleading w.r.t. contractions of p^{μ_i} with the Wilson coefficient, which gives factors of $q\cdot P = \frac{1}{2}\omega Q^2$. Therefore we drop the traces.

One can think of the last equation as form factor of the type



All in all, we get

$$T^{\mu\nu} = \sum_q{}' Q_q \left\{ \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{4}{Q^2\omega^2}\left(p^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{P\cdot q}{q^2} q^\nu\right)\right\} \times \sum_{n=2,4,\ldots}^{\infty}{}' \omega^n A_{q,n}$$

equation from above.

Therefore, comparing with

$$T_{\mu\nu}(\omega, Q) = T_1(\omega, Q)\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{T_2(\omega, Q)}{P\cdot q}\left(p^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{P\cdot q}{q^2} q^\nu\right)$$

as we did before for the $W^{\mu\nu}$ tensor. Thus $W_1 = 2\,\mathrm{Im}\, T_1$ and $W_2 = 4\,\mathrm{Im}\, T_2/\omega Q^2$.

we obtain

$$T_1 = \frac{\omega}{2} T_2 = \sum_q{}' Q_q^2 \left(\sum_{n=2,4,\ldots}{}' \omega^n A_{q,n}\right)$$

therefore, as promised we find the Callan-Gross relation $W_1 = \omega\, \mathrm{Im}\, T_2 = \frac{\omega^2 Q^2}{4} W_2$.
And because

$$W_1(x, Q) = 2\pi \sum_i{}' Q_i^2 f_i(x)$$

Eq. from above.

we find

$$f_q(x) = \frac{1}{\pi} \sum_{n=2,4,\ldots}{}' x^{-n}\, \mathrm{Im}\, A_{q,n}$$

which gives a definition of the PDFs in QCD!

* Moments of the PDFs

A moment of order "n" of a distribution $f_i(x)$ is defined by:

$$C_{i,n} = \int_0^1 dx\, x^{n-1} f_i(x).$$

In terms of the PDF definition above, we have

$$C^q_{i,m} = \mathrm{Im}\,\frac{1}{\pi}\int_1^\infty dw \sum_n{}' w^{n-m-1} A_{q,n}.$$

2 Im = Disc.

$$= \sum_n{}' \frac{1}{2\pi i}\oint dw\; w^{n-m-1} A_{q,n} = \sum_n{}' \delta_{n,m} A_{q,n} = A_{q,m}$$

Therefore the A_n's are precisely the moments of the PDFs.

For example, for $m=2$

$$\mathcal{O}_q^{\mu_1\mu_2} = \bar\psi_q\left(i\gamma^\mu D^\nu + i\gamma^\nu D^\mu\right)\psi_q \;-\; \text{traces.}$$

Symmetrized energy momentum tensor.

The matrix element on a proton state gives

$$\sum_i{}' \langle P|\,\mathcal{O}_i^{\mu\nu}\,|P\rangle = \langle P|\,T^{\mu\nu}_{QCD}\,|P\rangle = 2P^\mu P^\nu$$

here we sum over all partons, not only quarks.

Neglected traces and thus mp.

and therefore $\sum_i{}' A_{i,2} = 1.$

$$\sum_i{}' \int_0^1 dx\; x\, f_i(x) = 1.$$

$$\sum_i{}' C_{i,2} = \sum_i{}' \int dx\; x f_i(x)$$

Other sum rules can be derived evaluating other moments.

For $m=1$ we get

# of quarks f − # of antiquarks f in the proton.

$$\langle P_s|\,\mathcal{O}_q^\mu\,|P_s\rangle = \bar u_s(P)\gamma^\mu u_s(P)\cdot A_{q,1}$$

(Like the number operator).

summing over proton spin s, we get

$$\langle P|\,\mathcal{O}_q^\mu\,|P\rangle = 2P^\mu A_{q,1}$$

$$= \begin{matrix} 2 & q=u \\ 1 & q=d. \end{matrix}$$

* Hard scattering processes in Hadron Colliders:

↳ High energy hadron scattering dominated by low momentum transfer cannot be treated using perturbative QCD.

↳ However, in some collisions two quarks or gluons exchange a large momentum transfer.

↳ Then, as in deep inelastic scattering, the parton scattering takes place rapidly, much faster than time scales associated to the wave functions of the partons inside the proton. Therefore low order QCD predictions are OK and a factorization formula (similar to DIS) applies.

↳ e.g. for $q + \bar{q}$ we'll get

any hadron state.

$$\sigma(p(P_1) + p(P_2) \mapsto X + Y) =$$

$$= \int_0^1 dx_1 \int_0^1 dx_2 \sum_i f_i(x_1) f_i(x_2) \cdot \sigma(q_i(x_1 P_1) + \bar{q}_i(x_2 P_2) \to Y)$$

↳ In the next section we will be interested in $\sigma(p + p \to h + X)$

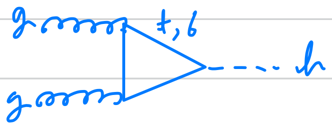
$\sqrt{s}$ (TeV)	Production cross section (in pb) for $m_H = 125\,\mathrm{GeV}$					
	ggF	VBF	WH	ZH	$t\bar{t}H$	total
7	$15.3^{+10\%}_{-10\%}$	$1.24^{+2\%}_{-2\%}$	$0.58^{+3\%}_{-3\%}$	$0.34^{+4\%}_{-4\%}$	$0.09^{+8\%}_{-14\%}$	17.5
8	$19.5^{+10\%}_{-11\%}$	$1.60^{+2\%}_{-2\%}$	$0.70^{+3\%}_{-3\%}$	$0.42^{+5\%}_{-5\%}$	$0.13^{+8\%}_{-13\%}$	22.3
13	$44.1^{+11\%}_{-11\%}$	$3.78^{+2\%}_{-2\%}$	$1.37^{+2\%}_{-2\%}$	$0.88^{+5\%}_{-5\%}$	$0.51^{+9\%}_{-13\%}$	50.6
14	$49.7^{+11\%}_{-11\%}$	$4.28^{+2\%}_{-2\%}$	$1.51^{+2\%}_{-2\%}$	$0.99^{+5\%}_{-5\%}$	$0.61^{+9\%}_{-13\%}$	57.1

The SM Higgs boson production x-section in pp collisions as a function of C.o.M. energy $\sqrt{s}$. See also the plot below.

◉ HIGGS DISCOVERY & IT'S CURRENT STATUS

How do we search for the Higgs? What are the main decay of the Higgs?
Recall that $m_h < 2m_{W/Z} < 2m_t$. The Higgs main production modes from $\sigma(p+p \to h+X)$ are

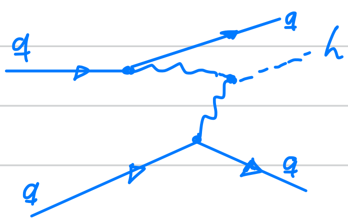
* Gluon fusion



$\sim 20\,\text{pb} = 20\cdot 10^{-12}\,\text{b}$ $\left(= 20\cdot 10^{-40}\,\text{m}^2\right)$ @ $\sqrt{s} \sim 8\,\text{TeV}$

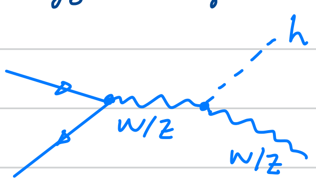
For comparison note that $\sigma(pp \to Z+X) \sim 40\cdot 10^{4}\,\text{pb}$
$\sigma(pp \to W+X) \sim 10^{5}\,\text{pb}$

* Vector Boson Fusion (VBF)



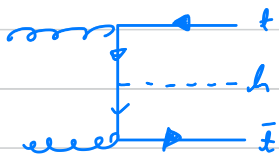
$\sim 2\,\text{pb}$ @ $\sqrt{s} \sim 8\,\text{TeV}$

* Higgsstrahlung or VH



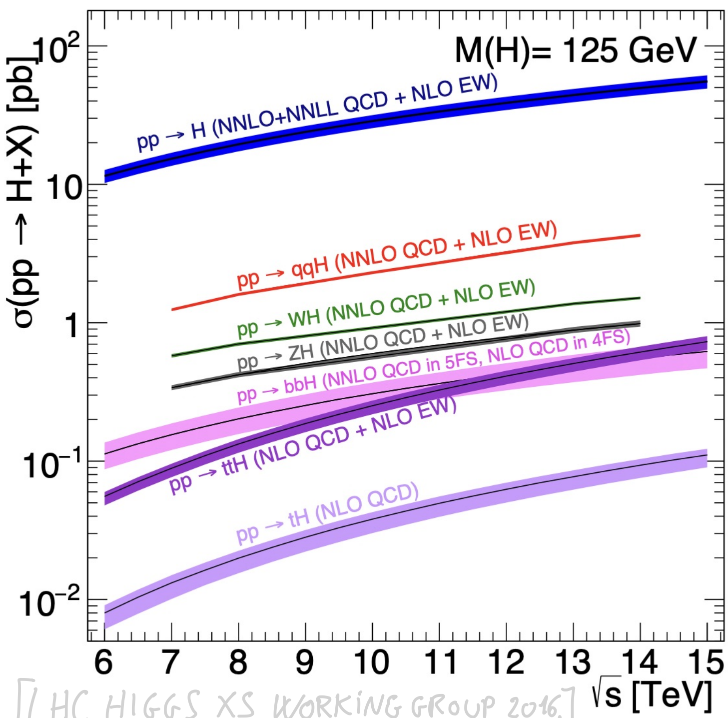
$\sim \sigma \to W/Z + H$
$\sim 0.8/0.3\,\text{pb}$ @ $\sqrt{s} \sim 8\,\text{TeV}$

* $t\bar{t}H$ channel



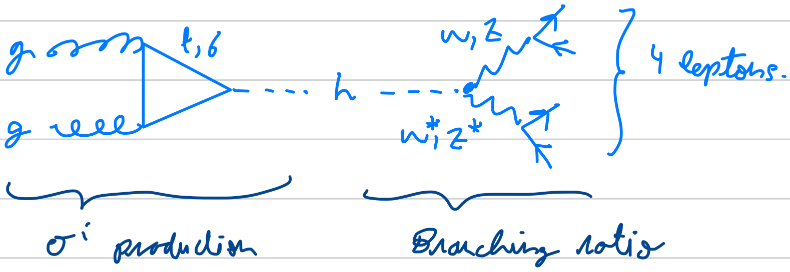
$\sim 0.05\,\text{pb}$ @ $\sqrt{s} \sim 8\,\text{TeV}$

SM production cross section as a function of the center of mass energy, $\sqrt{s}$, for proton + proton collisions.

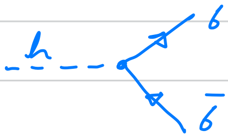
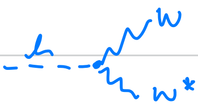
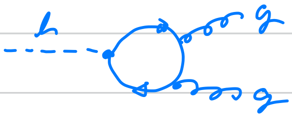
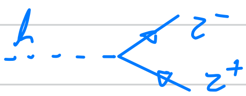
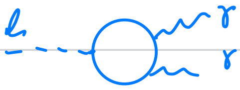


[LHC HIGGS XS WORKING GROUP 2016]

The Higgs boson has a very small width $\Gamma = 4\,MeV$ compared to it's mass. Therefore, we can factorise production and decay using the narrow width approximation. E.g.

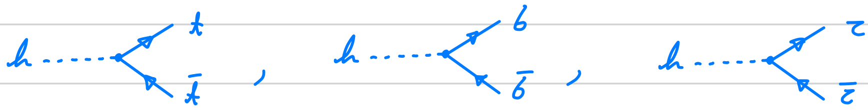


What are the Higgs main decay channels?

Decay channel	Diagram examples.	Branchin ratio
$h \to b\bar{b}$		$\sim 57.5\%$
$h \to WW^*$		$\sim 21.5\%$
$h \to gg$		$\sim 8.5\%$
$h \to \tau^+\tau^-$		$\sim 6.3\%$
$h \to c\bar{c}$		$\sim 2.9\%$
$h \to ZZ$		$\sim 2.7\%$
$h \to \gamma\gamma$		$\sim 0.23\%$ ← discovery channel.
$h \to Z\gamma$		

END OF L16

All 3rd family higgs couplings have been measured.

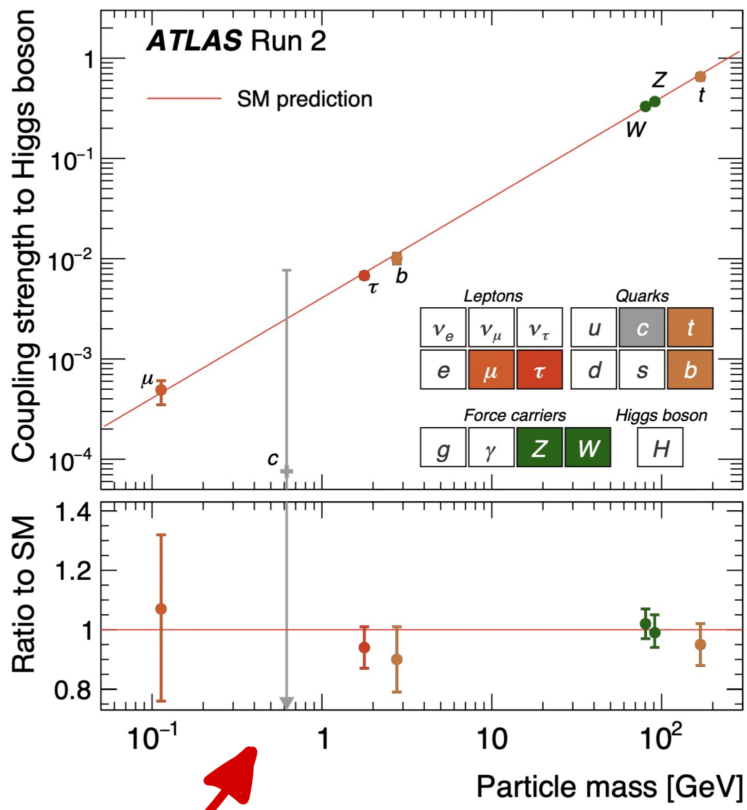


Experimental results are often presented in terms of "signal strength":

$$\mu_i^f = \frac{\overbrace{\sigma_i \cdot Br^f}^{\text{observed}}}{\underbrace{(\sigma_i \cdot Br^f)_{SM}}_{\text{predicted}}} = \mu_i \times \mu_f \quad w/\ \mu_i = \frac{\sigma_i}{\sigma_{SM}} \ \&\ \mu^f = \frac{Br^f}{(Br^f)_{SM}}$$

So far signal strengths of WW^*, ZZ^*, $\gamma\gamma$, $b\bar{b}$, $t\bar{t}$ and $\tau\bar{\tau}$ measured and agree with SM at $\lesssim 10\%$.

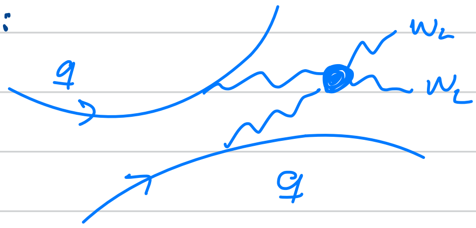
Couplings to u, d, s and e is not obvious we'll ever manage to measure in any foreseeable future. Couplings to $\mu\bar{\mu}$ will be will be conclusively stablished in the next 5-10 years at the LHC. The couplings $c\bar{c}$ is feasable and is the next important channel



The straight line shows the expected SM behaviour, where the interaction strength is proportional to the mass of the fermion, and to the squared mass for W & Z.

Improving precission of higgs couplings is an important experimental direction. Precission is not allways important: we do not care much of measuring much better the proton mass (why?). But measuring much better the Higgs

couplings is a direct test of the EWSB mechanism. This is the sector of the SM that is less poorly understood, experimentally.
Currently there are various important structural parts of the EWSB mechanism that have not been tested. For instance we do not know much about the Higgs potential beyond the quadratic piece responsible for the Higgs mass. Also the role of the higgs has not been tested. What is his role in life? Unitarize amplitudes. We'd like to test a process like:



Perhaps the LHC will manage to probe it in the next decade.

Neutrinos

It turns out that neutrinos are very light but not exactly massless. They do have a non-zero mass.
In the SM EFT we have a good explanation to why the neutrinos have a small non-vanishing mass. The leading operator of dimension $\Delta > 4$ is given by

$$\mathcal{L} = \frac{-y^{\nu}_{ij}}{\Lambda} (\bar{L}^i \tilde{H})(\tilde{H} L^j)^{\dagger}$$

dimension five operator

After EWSB we get a Majorana mass term for the neutrino

$$-\frac{y^{\nu}_{ij} v^2}{\Lambda} (\nu_L^i)^c \nu_L^j + h.c.$$

majorana mass.

where $\nu_L^c = \nu_L^T \sigma_2$, the conjugate representation. We may expect $y \sim O(1)$ and $\Lambda >> v$, which would explain the smallness of the neutrino masses.

It could be the case that there exists a right handed sterile neutrino. namely imagine the existence of a lepton which is neutral under $SU(2)_L \times U(1)_Y$ and is a right handed Fermion. Then, we can write the following terms in the SM lepton sector:

$$\Delta\mathcal{L} = - y^{e}_{ij} \bar{L}^i H e_R^j - y^{\nu}_{ij} \underbrace{\bar{L}^i \tilde{H}}_{\text{singlet}} \underbrace{\nu_R^j}_{\text{singlet a.k.a. sterile}} - i M_{ij} (\nu_R^i)^c \nu_R^j + h.c.$$

After EWSB the Yukawa interactions provide a Dirac mass term for neutrinos, while the last term provides a Majorana type of mass.

The majorana type of masses violates lepton number. Thus we could forbid those masses imposing such symmetry.
With neutrinos we often go back and forth with Dirac v.s. Weyl

Recall that Dirac spinors are usually written in terms of two independent left- and right-handed Weyl spinors $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$.
But we can also construct Dirac out of a single Weyl.

$$\Psi_L \equiv \begin{pmatrix} \nu_L \\ i\sigma_2 \nu_L^* \end{pmatrix} \quad ; \quad \Psi_R = \begin{pmatrix} i\sigma_2 \nu_R^* \\ \nu_R \end{pmatrix}$$

transforms like left handed

transforms like right handed

Then we can write "Dirac" and "Majorana" mass terms in a unified notation:

$$\mathcal{L}_{\nu, mass} = -m \bar{\Psi}_L \Psi_R - \frac{M}{2} \bar{\Psi}_R \Psi_R + h.c.$$

$$= -(\bar{\Psi}_L, \bar{\Psi}_R) \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$$

where for simplicity we suppressed the family index. The last equation follows from $\Delta\mathcal{L}$ (in previous page) after EWSB.

The eigenvalues of this mass matrix are $\sqrt{m^2 + M^2/4} \pm \frac{1}{2}M$. So that in the limit $M \gg m$:

$$m_{light} = \frac{m^2}{M^2} \quad ; \quad m_{heavy} \simeq M$$

The scale M may indeed be much larger than the EW scale, $v \ll M$. In the limit $M \gg m$, the last equation reduces to

small mixing

$$-\frac{y^{\nu}_{ij} v^2}{\Lambda} (\nu_L^i)^c \nu_L^j + h.c. + M \nu_R^c \nu_R + O\left(\frac{m^2}{M}\right) \bar{\Psi}_L \Psi_R + h.c.$$

Heavy and sterile, so who cares?

Thus, at low energy is equivalent to the dimension five "Weinberg" operator.