Pre-quantization (Report/still in development): Principal circle bundle over a fixed manifold M is a fibre bundle Q, $\pi: Q \to M$ equipped with a structure of group $S^1 = \{e^{i\varphi} \mid \varphi \in \mathbb{R}\}$.

Each fibre $\pi^{-1}(m)$ is a circle and there is a circle action $S^1 \times Q \to Q$ which is just multiplication on each fibre $M \cong Q/_{S^1}$.

Def: Let (P, ω) be a symplectic manifold. The symplectic manifold is quantizable iff there is a principal circle bundle $\pi: Q \to P$ and a one-form α on Q such that the pullback $\pi^* \omega = d\alpha$

Remark: Q is called the quantization manifold, α the connection, ω the curvature 2-form and on fibres $\alpha|_{\pi^{-1}(m)} = ds$

Ex: Let P be a momentum space, θ such that $\omega = -d\theta$ the canonical form then (P, ω) is quantizable since $Q = P \times S^1$ equipped with $\alpha = \theta \oplus ds$

$$\alpha\left(X, -i\frac{\partial}{\partial s}\right) = -i\,\theta(X)$$

Line bundle L, each fibre replaced by $\mathbb{C}$. The one-form α on Q gives a connection ∇ on L. admissible section φ of L: $\varphi: P \to L$ and $V \in TQ \rightsquigarrow V|_{TP} =: X$. Then $(\nabla_V \varphi)^a = X(\varphi^a) + \alpha^a_{bc} V^b \varphi^c$.

Theorem: (P, ω) be a symplectic manifold then there is a quantization map $f \to \hat{f}$ where $\hat{f}$ is an op. on the space of sections of L satisfying properties (1)-(4) (all with out Schrödinger EP) of the full quantization procedure. Explicitly,

$$\hat{f}: s \longmapsto -i \nabla_{X_f} s + fs.$$

Remark: 1) $L^2(P, \omega)$ 2) on a symplectic chart, $\theta = P_a dq^a$, $Q = P \times S^1$

$$X_f = \left(\frac{\partial f}{\partial p_a}, -\frac{\partial f}{\partial q^a}\right) \rightsquigarrow \theta(X_f) = p_a \frac{\partial f}{\partial p_a}. \text{ therefore } \cancel{\hat{f}s = -i X_f(s)}$$

$$\hat{f}s = -i X_f(s) + \left(f - p_a \frac{\partial f}{\partial p_a}\right) s$$

Proof: a) HW: $\hat{f}$ is symmetric op on $L^2(P, \omega^n)$

prequantization elements requirements:

1) $\widehat{f+g} = \hat{f} + \hat{g}$ ✓ 2) $\widehat{\lambda f} = \lambda \hat{f}$ ✓ 4) $\hat{1} = id_{L^2}$ ✓

3)
$$[\hat{f}, \hat{g}]s = \hat{f}(\hat{g}(s)) - \hat{g}(\hat{f}(s)) = -i\nabla_{X_f}(\hat{g}s) + f(\hat{g}s) - (f \leftrightarrow g)$$
$$= -\nabla_{X_f}\nabla_{X_g} s - i\nabla_{X_f}(gs) + f(-i\nabla_{X_g} s) + fgs - (f \leftrightarrow g)$$
$$= -(\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f})s - i\nabla_{X_f}(gs) + ig\nabla_{X_g} s$$
$$= -[\nabla_{X_f}, \nabla_{X_g}]s - i\underbrace{(X_f(g) - X_g(f))}_{X_{[f,g]}}s + i\nabla_{X_g}(fs) - if\nabla_{X_g} s$$
$$= -\nabla_{[X_f, X_g]} s - i\omega(X_f, X_g)s - i(\{g,f\} - \{f,g\})s$$
$$= \nabla_{X_{\{f,g\}}} s - i\{f,g\}s \Rightarrow -i[\hat{f}, \hat{g}]s = -i\nabla_{\ldots} s + \{f,g\}s$$

Use:
- $\nabla_{X_f}(gs) = X_f(g)s + g\nabla_{X_f} s$

Reminder:
- $i\omega(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$
- $[X_f, X_g] = -X_{\{f,g\}}$

$\theta(X_q)=0 \rightsquigarrow \boxed{\hat{q} = i\frac{\partial}{\partial p} + q\, id}$.

Consider $f =$ momentum, $\rightsquigarrow X_p = (1,0) = \frac{\partial}{\partial q} \rightsquigarrow \theta(X_p) = 1\cdot p$

$\Rightarrow \boxed{\hat{p} = -i\frac{\partial}{\partial q}} \Rightarrow [\hat q, \hat p] = -i\frac{i\partial^2}{\partial p \partial q} - i\, q\, id \frac{\partial}{\partial q} + i^2\frac{\partial^2}{\partial p\partial q} + i\, id = i\, id = \boxed{[\hat q, \hat p] = i}$

[Remark $f: P \to \mathbb{C}$. A submanifold $L \subset P$ is Lagrangian (sub)manifold if L is isotropic (ω vanishes on $T_\ell L \times T_\ell L$) and L is maximal ($\dim L = \frac{1}{2}\dim$)

Def: A real polarization of (P, ω) is a foliation F of P by Lagrangian submanifolds (as leaves).

Def: Let (P,ω) be a quantizable symplectic manifold and let F be a polarization. Let L be the line bundle obtained from the quantization manifold. Then the quantization Hilbert space is the space of L^2 sections of L that are constant on the leaves of F

Ex: $P = T^*C$, $L = P \times \mathbb{C}$ sections of $L \triangleq$ complex valued functions on P. The leaves of F are the lines space $T^*_c C$. This is a polarization and the intrinsic Hilbert space $\mathcal{H} = L^2(C, \mu)$]

Def: Let E and M be manifolds with $\dim m+n, m$ resp. $\pi: E \to M$ surjective smooth map. Let the fiber $\pi^{-1}(p) =: E_p$ carries the structure of a $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{C}$) V_p for each $p \in M$.

The quadruple $(E, \pi, M, \{V_p\}_{p\in M})$ is called a $\mathbb{K}$-vector bundle if for every $p \in M$ there exists an open neighborhood $\mathcal{U}$ of p in M and a diffeomorphism $\varphi: \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{K}^n$ s.t. the following diagram comm.

$$\begin{tikzcd} \pi^{-1}(\mathcal{U}) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & \# & \mathcal{U}\times\mathbb{K}^n \arrow[dl, "pr_1"] \\ & \mathcal{U} & \end{tikzcd}$$

and for every $q \in \mathcal{U}$ the map $pr_2 \circ \varphi|_{E_p} : E_p \to \mathbb{K}$ is a vector space isomorphism.

φ is called a local trivialization of E (the vector bundle)

$n \in \mathbb{N}$ is the rank of the vector bundle.

Ex/ Consider a) M is a smooth manifold of dim k. TM, $\pi: TM \to M$, $P_q \to q$

HW Construct a local trivialization of TM.

b) Most operations (E^*, $E \oplus F$, ...) can be carried out fiberwise.

c) T^*M d) Let $n = \dim M$, $k \in \{0, 1, \dots, n\}$. The kth exterior power of T^*M is the bundle of k-linear skew symmetric forms on TM $\Lambda^k T^*M$. This is a real vector bundle of rank $\binom{n}{k}$.

Def: A section in a vector bundle $(E, \pi, M, \{V_p\}_{p\in M})$ is a map $s: M \to E$ s.t. $\pi \circ s = id_M$. Ex/ • $E = M \times \mathbb{K}^n$ (global trivialization) $\rightsquigarrow \sigma \in \mathbb{K}^n$ valued functions on M.

• Sections of TM are called vectorfields on M

Let (U, φ) chart on M. For each $j \in I(n)$ the curves
$\gamma : I \subset \mathbb{R} \longrightarrow U \subset M$ represent a tangent vector $\frac{\partial}{\partial x^j}\Big|_p$
$t \longrightarrow \gamma(t) := \varphi^{-1}(\varphi(p) + t e_j)$

c) 1-forms d) k-form are sections of $\Lambda^k T^* M$

e) Let $|\Lambda M|_p$ be the set of all functions $\nu : \Lambda^n T^*_p M \to \mathbb{R}$ with
$\nu(\lambda \Omega) = |\lambda|_{\mathbb{R}} \, \nu(\Omega)$. Apparently $|\Lambda M|_p$ is one dim vector space and yields a vector bundle of rank one over M.

<u>Def</u> Sections over $|\Lambda M|$ are called <u>densities</u>. Given a local chart (U, φ) a smooth density $|dx|$, $|dx|(dx^1 \wedge \dots \wedge dx^n) = 1$ on U.
There is a unique linear map $\int_M : \mathcal{D}(M, |\Lambda M|) \longrightarrow \mathbb{R}$
called the integral for any $f \in \mathcal{D}(U, |\Lambda M|)$. $\int_M f\,|dx| := \int_{\varphi(U)} dx^1 \wedge \dots \wedge dx^n \, f \circ \varphi^{-1} (x^1, \dots, x^n)$

<u>Remark</u>. Consider sections of $E^* \otimes E^*$ relative to some coordinate syst.
Let $g = g_{ab}\, dx^a \otimes dx^b$ pseudo Riemannian metric. Then the induced ~~density~~ Riemannian density integral is
$$d\,\mathrm{vol}_g = \sqrt{|\det g|}\,|dx|.$$ For any $f \in \mathcal{D}(M, \mathbb{R})$ there is a canonical way to integrate it $\leadsto \int_M f \, d\,\mathrm{vol}_g$.

- Rule of differentiation sections in bundles is called a connection.

<u>Def</u>: Let $(E, \pi, M, \{V_p\}_{p \in M})$ be a $\mathbb{K}$-vector bundle.
A connection on E is a $\mathbb{R}$-bilinear map $\nabla : C^\infty(M, TM) \times C^\infty(M, E) \longrightarrow C^\infty(M, E)$
section that gives me the direction / section of the bundle to be differentiated
with the following properties: a) ∇ is $C^\infty(M, \mathbb{R})$ linear in the first argument.
b) ∇ is a derivation in it's 2^{nd} argument $(\nabla_{fX} s = f \nabla_X s)$
(It is $\mathbb{R}$-bilinear and $\nabla_X (fs) = X(f) s + f \nabla_X s$)

The value of $\nabla_X s$ at a point $p \in M$ depends only on $X(p)$ and the values of s on a curve representing $X(p)$.

<u>Remark</u> Let ∇ be a connection on a vector bundle E over M. Let $\gamma : [a,b] \to M$ smooth curve. Given $s_0 \in E_{\gamma(a)}$ there is a unique smooth solution $s : [a,b] \to E$, $t \to s(t) \in E_{\gamma(t)}$
satisfying $s_0 = s(a)$, of $\nabla_{\dot\gamma} s = 0_E$.
The map $P_a^b(\gamma) : E_{\gamma(a)} \to E_{\gamma(b)}$, $s_0 \to s(b)$ is called <u>parallel transport along γ</u> from $\gamma(a)$ to $\gamma(b)$.

<u>Remark</u>: Any connection ∇ on E induces a connection ∇ on E^*.
$$(\nabla_X \theta)(s) = X(\theta(s)) - \theta(\nabla_X s) \quad \text{for all } X \in C^\infty(M, TM),\ \theta \in C^\infty(M, E^*),\ s \in C^\infty(M, E)$$
$\leadsto$ define for usual multilinear generalization

<u>Remark</u>: If a vector bundle E carries a semi-Riemannian metric g then a connection on E is called metric compatible if the following rule holds.
$$\partial_X g(s, s') = g(\nabla_X s, s') + g(s, \nabla_X s')$$
directional derivative $\partial_X g(s,s') = X(g(s, s'))$ $\quad \forall X \in C^\infty(M, TM)$, $s, s' \in C^\infty(M, E)$

$\in C^\infty(M, TM)$ characterized by

$$\partial_{[X,Y]} f = \partial_X \partial_Y f - \partial_Y \partial_X f \qquad \forall f \in C^\infty(M, \mathbb{R})$$

The map $[\cdot,\cdot] : C^\infty(M, TM)^{\times 2} \to C^\infty(M, TM)$ is also a Lie bracket, it is skew $\mathbb{R}$-bilinear and satisfies Jacobi.

Remark: Let M be a n-dim manifold with a semi-Riemannian metric on TM. It can be shown that there exists a unique (metric compatible) connection ∇ on TM satisfying

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

called the Levi-Civita connection of (M, g)

Def: The Riemannian Curvature tensor of the Levi-Civita connection is the map $R: C^\infty(M,TM)^{\times 2} \times C^\infty(M,TM) \to C^\infty(M,TM)$

s.t. $(X, Y, V) \xrightarrow{R} R(X,Y)V := \nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X,Y]} V$

i.e. $R \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{Hom}_{\mathbb{R}}(TM, TM))$

Remark: The Ricci Curvature $\mathrm{ricc} \in C^\infty(M, T^*M \otimes T^*M)$ given by

$$\mathrm{ricc}(X, Y) = \sum_{a \in I(n)} \varepsilon_a g(R(X, e_a) e_a, Y)$$

where e_i are orthonormal wrt g and $\varepsilon_a = g(e_a, e_a) = \pm 1$

* Scalar Curvature: $\mathrm{scal} \in C^\infty(M, \mathbb{R})$, defined by $\mathrm{scal} = \sum_{a \in I(n)} \varepsilon_a \mathrm{ricc}(e_a, e_a)$ ∎

Remark: Let E and E' be two vector bundles with ∇, ∇' resp. They induce a connection D on $E \otimes E'$ by

$$D_X(s \otimes s') = \nabla_X s \otimes s' + s \otimes \nabla'_X s' \qquad \text{for all } X \in C^\infty(M, TM)$$

$s' \in C^\infty(M, E')$

Let $E \to M$ $\mathbb{K}$-vector bundle. Equip E and T^*M with connections ∇. (both denoted by ∇)

They induce connections on the tensor bundle $T^*M^{\otimes k} \otimes E$ and denote this as ∇ again

For $\varphi \in C^1(M, E)$, $\nabla \varphi \in C^0(M, T^*M \otimes E)$ i.e. $\nabla \varphi = \omega_\varphi^c \otimes e_c$

$$\nabla_X \varphi = \omega_\varphi^c(X) e_c$$

More generally: $\nabla^k \varphi \in C^0(M, T^*M^{\otimes k} \otimes E)$

we choose metrics on T^*M and E. This induces metrics on all tensor bundles $T^*M^{\otimes}$

Hence a (semi) norm of $\nabla^k \varphi$ can be defined on all points in M

For a subset $A \subset M$ and $\varphi \in C^k(M, E)$ we define the C^k-(semi)norm by

$$\|\varphi\|_{C^k(A)} := \max_{j \in I(k)} \sup_{p \in A} \|\nabla^j \varphi(p)\|_{T_p^*M^{\otimes j} \otimes E}$$

Ex/ If A is compact, the different choice of metrics and norms induce the same $C^k(A)$ norm.

Notation: $\mathcal{D}(M, E)$ denote space of compactly supported sections in E. (also called the test sections)

Def: Let $\varphi, \{\varphi_n\}_{n} \in \mathcal{D}(M, E)$. The sequence $(\varphi_n)_{n \in \mathbb{N}}$ converge to φ in $\mathcal{D}(M, E)$ if:

1) $\exists K \subset M$ compact s.t. $\mathrm{supp}\, \varphi_n \subset K$ for all but finite.
2) $(\varphi_n)_n$ converges to φ wrt to all C^k-norms i.e.

$$\|\varphi - \varphi_n\|_{C^k(K)} \xrightarrow{n \to \infty} 0 \quad \text{for each } n \in \mathbb{N} \text{ but finite.}$$

Def: Let W be a finite-dim $\mathbb{K}$-vector space. A $\mathbb{K}$-bilinear map $F: \mathcal{D}(M, E) \to W$

is called a distribution in E with values in W. if it is continuous, i.e. if $\varphi_n \to \varphi$ in $\mathcal{D}(M, E^*)$ then $F(\varphi_n) \to F(\varphi)$ in W.
The space of W-valued distributions in E is denoted by $\mathcal{D}'(M, E, W)$ ⌋

<u>Lemma:</u> Let F be a W-valued distribution in E, let $K \subset M$ be compact. There is a $k \in \mathbb{N}$ and constant $C > 0$ s.t. $\forall \varphi \in \mathcal{D}(M, E^*)$ with $\operatorname{supp}\varphi \subset K$
$\|F(\varphi)\| \leq C \|\varphi\|_{C^k(K)}$, such k is called the order of F over K.

<u>Proof:</u> 1) Assume doesn't hold, i.e. for any k we can find a nontrivial section $\varphi_k \in \mathcal{D}(M, E^*)$ with $\operatorname{supp}(\varphi_k) \subset K$ and s.t. $\|F(\varphi_k)\|_W \geq k \|\varphi_k\|_{C^k(K)}$

2) $\psi_k := \|F(\varphi_k)\|_W^{-1} \varphi_k \in \mathcal{D}(M, E^*)$ which has supp. $\subset K$. and
$\|\psi_k\| \leq k^{-1}$. Hence $j \leq k \leadsto \|\psi_k\|_{C^j(K)} \leq \|\psi_k\|_{C^k(K)} \leq \frac{1}{k} \Rightarrow \psi_k \to 0$ in $\mathcal{D}(M, E^*)$

3) Since F is a distribution $\Rightarrow F(\psi_n) \to F(0) = 0$ (linearity)
$\Rightarrow$ ~~$\|F(\varphi_n)\| \geq k \|\varphi_n\|_{C^k(K)}$~~
$\|F(\psi_n)\|_W = \frac{1}{\|F(\varphi_n)\|_W} \|F(\varphi_k)\|_W = 1$ a contradiction.

<u>Examples:</u> 1) Let $E \overset{\pi}{\to} M$, $p \in M$. The delta-distribution δ_p is an E_p^*-valued distr. in E. For $\varphi \in \mathcal{D}(M, E^*)$ it is given by $\delta_p(\varphi) = \varphi(p) \in E_p^*$

2) Every locally integrable section $f \in L^1_{loc}(M, E)$ can be interpreted as a $\mathbb{K}$-valued distribution in E by setting for every $\varphi \in \mathcal{D}'(M, E^*)$
$f(\varphi) := \int_M \varphi(f|_K)\, d\mathrm{vol}$ where $f|_K$ denotes the restriction of f to any $K \subset M$ compact.

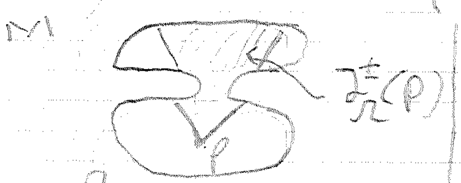
HW: what is the degree of δ and f?

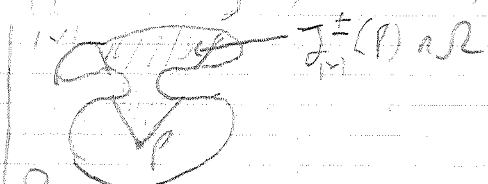
<u>Review</u> Causal relations (M, g), $q, p \in M$.

- <u>$p \ll q$</u> $\hat{=}$ if there exist a curve $\alpha : [t_p, t_q] \to M$ s.t. $\begin{cases} \alpha(t_p) = p \\ \alpha(t_q) = q \end{cases}$ and α is <u>timelike</u> i.e. $\|\dot{\alpha}(t)\|^2 < 0$ and $\dot{\alpha}(t)$ is future pointing (for all t).
- <u>$p \leq q$</u> $\hat{=}$ if we relax $<$ → $\leq$ in $p \ll q$.
- <u>Chronological future</u> of $p \in M$: $I_M^+(p) := \{q \in M \mid p \ll q\}$
  " of $A \subseteq M$: $I_M^+(A) := \bigcup_{p \in A} I^+(p)$
- <u>Causal future</u> of $p \in M$: $J_M^+(p) := \{q \in M \mid p \leq q\}$
- <u>Duality</u>: corresponding past relations by time reversing time-orientation

<u>Def:</u> A subset $\Omega \subset M$ in a time oriented Lorentzian mfd. is called <u>Causally compatible</u> if $\forall p \in \Omega$ $J_\Omega^\pm(p) = J_M^\pm(p) \cap \Omega$

Clearly $J_\Omega^\pm(p) \subset \{J_M^\pm(p) \cap \Omega\}$, however, the inverse is <u>not</u> trivial.





<u>Remark:</u> Suppose $\Omega \subset M$ is a causally compatible domain in M. For each $A \subseteq \Omega$ we have

Proof: • $\mathcal{I}^{\pm}_{\Omega}(p) \subset \mathcal{I}^{\pm}_{\Omega'}(p) \cap \Omega \subset \mathcal{I}^{\pm}_{\Omega''}(p) \cap \underbrace{\Omega' \cap \Omega}_{\Omega}$ Ω' compatible in $\Omega'' \Rightarrow \Omega$ compatible in Ω''.

$\exp_p(tv) = \gamma_v(t)$

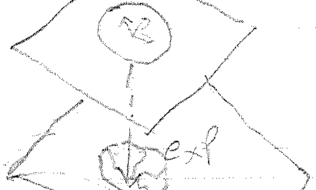
Let M be a semi-Riemannian mfd, assumed to be complete.
For all $p \in M$, the exp map $\exp_p : T_pM \to M$ constitute a single mapping $\exp : TM \to M$. Define $E : TM \to M \times M$

$v_p \to E(v_p) := (\pi(v_p), \exp_p(v))$

* When M is not complete exp and E have the same largest domain: the set $\mathcal{D}$ of all vectors in TM s.t. the geodesic γ_v is defined at least on the interval $[0,1]$.

Thus the largest domain of $\exp_p$ is $\mathcal{D}_p = \mathcal{D} \cap T_pM$

FW: ~~Exercise~~: The domain of exp is open in TM.
The domain $\mathcal{D}_p$ of $\exp_p$ is an open set of T_pM star shaped around the zero vector.



Def An open set $\mathcal{C}$ in a semi-Riemannian mfd. is called convex provided $\mathcal{C}$ is a normal nbhd to each of its points.

Remark: For any two points $p, q \in \mathcal{C}$ there is a unique geodesic (segment) $\sigma_{p,q} : [0,1] \to M$ from p to q that lies in $\mathcal{C}$.

Prop Each point o of M has a convex nbhd.

proof • Let $\xi = (x^1, \dots, x^n)$, $n = \dim(M)$ be a normal coordinate region on a nbhd V of $o \in M$. Let $N = \sum_{a \in I(n)} (x^a)^2$, for $\delta > 0$ sufficiently small, $V(\delta) = \{p \in V \mid N(p) < \delta\}$ is a nbhd of O diffeo under ξ to an open ball in $\mathbb{R}^n$.

• Now, lets see a candidate for a normal nbhd.
Consider the symmetric tensor B whose components relative to ξ are
$B_{ab} = \delta_{ab} - \sum \Gamma^c_{ab} x_c$. * B is positive definite at o.
since B is smooth, and by reducing δ further if necessary, B is positive definite on $V(\delta)$.
We assert that $\mathcal{U} = V(\delta)$ is a normal nbhd at each point $p \in \mathcal{U}$.

Addendum: E is a diffeo of a nbhd W of $0 \in T_oM$ in TM onto $V(\delta) \times V(\delta)$

• W is starshaped: Let $W_p = W \cap T_pM$, by construction, $E|_W$ is a diffeo onto $p \times \mathcal{U}$. We show that W_p is starshaped about $0 \in T_pM$. For $q \in \mathcal{U}$ and $q \neq p$, let $E^{-1}(p,q) = v$ and $\sigma = \gamma_v|_{[0,1]}$. provided σ lies in $\mathcal{U}$ it can be shown that

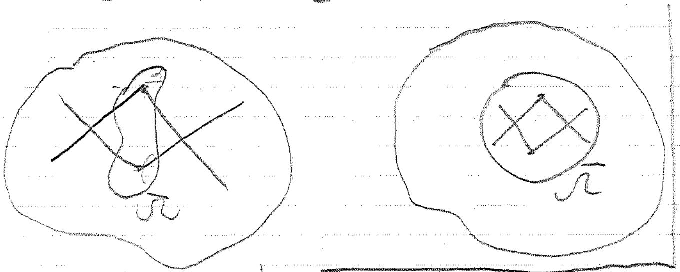
$tv \in W_p$ for $t \in [0,1]$. Hence W_p would qualified as starshaped.
Assume the opposite, i.e. σ leaves $\mathcal{U}$. Since, by assumption, $p, q \in \mathcal{U} = V(\delta) \Rightarrow N(p), N(q) < \delta$. Therefore there is a $t_* \in (0,1)$ s.t. $N \circ \sigma : \mathbb{R} \to \mathbb{R}$ has a maximum at t_* (why?).

Computation:
$$\frac{d}{dt} N(\sigma(t)) = 2 \sum_{a \in I} \dot{x}^a(\sigma(t))\, x^a(\sigma(t)),$$
$$\frac{d^2}{dt^2} N(\sigma(t)) = 2 \sum_{a \in I} \left[(\dot{x}^a)^2(\sigma(t)) + x^a(\sigma(t))\, \ddot{x}^a(\sigma(t)) \right]$$

* $\xi \circ \sigma$ is a co. rep of the geodesic σ
the geodesic equation gives:
$$\ddot{N}(\sigma(t)) = 2 \sum \left(\delta_{ab} - \Gamma^c_{ab}\, x_c(\sigma(t)) \right) \dot{x}^a(\sigma(t))\, \dot{x}^b(\sigma(t))$$
$$= 2 \sum B_{ab}(\sigma(t))\, \dot{x}^a \dot{x}^b \big|_{\sigma(t)}$$
$$= 2 B(\dot\sigma(t), \dot\sigma(t)) > 0$$

A contradiction. ~ at t_* supposed to be a maximum.

Definition: A domain Ω is called causal if $\bar{\Omega}$ is contained in a convex domain $\mathcal{C}$ and if for any $p, q \in \bar{\Omega}$ the intersection $J^+_{\mathcal{C}}(p) \cap J^-_{\mathcal{C}}(q)$ is compact and contained in $\bar{\Omega}$.



Convex but not causal.

Reminder: If M contains no closed time-like curves, we say the chronology condition holds on M.

Lemma: If M is compact, it contains a closed timelike curve.

Proof. By compactness $\{I^+(p), p \in M\}$ has finite subcover $\{I^+(p_1), \dots, I^+(p_n)\}$. We can assume it minimal. Now if p_1 would be in any $I^+(p_a)$, then $I^+(p_1) \subset I^+(p_a)$ contrary to minimality. Hence, we must have $p_1 \in I^+(p_1)$.

Definition: The strong causality condition holds at $p \in M$ provided that given any nbhd $\mathcal{U}$ of p there exists a nbhd $V \subset \mathcal{U}$ of p s.t. every causal curve (segment) with endpoints in V lies entirely in $\mathcal{U}$.

Remark: $\alpha : [a,b] \to M$ be a piecewise smooth segment in M.
The arc length of α is $L(\alpha) = \int_a^b dt\, \|\dot\alpha(t)\|$, where $\|\dot\alpha(t)\| = |\langle \dot\alpha(t), \dot\alpha(t) \rangle|^{1/2}$.
Let $\mathcal{U}$ be a normal nbhd of $o \in M$. The function $r : \mathcal{U} \to \mathbb{R}$, $p \mapsto \|\exp_o^{-1}(p)\|$ is called the radius function of M at o.
In normal coordinates $r = \left| -(x^0)^2 + \sum_a (x^a)^2 \right|^{1/2}$

Lemma: Let r be the radius function on a normal nbhd $\mathcal{U}$ of o. If σ is the radial geodesic from o ($\sigma(0)$) to p ($\sigma(1)$) $\in \mathcal{U}$ then $L(\sigma) = r(p)$.

$\Rightarrow L(\gamma) = \|\exp_o^{-1}(p)\| = \tau(p)$

Def. Let $p, q \in M$. The time separation from p to q, $\tau(p,q)$ is $\sup\{L(\alpha) : \alpha$ is future pointing causal curve segment from p to $q\}$.

- $\tau(p,q) = \infty$ if the set of lengths is unbounded and $\tau(p,q) = 0$ if its empty i.e. $q \notin J_M^+(p)$. "q is the chronological future of p" "q precedes p"

Lemma 1) $\tau(p,q) > 0$ iff $p \ll q$.

2) If $p \leq q \leq r$, then $\tau(p,q) + \tau(q,r) \leq \tau(p,r)$

Proposition For $p<q$, if the set $J_M(p,q) := J_M^+(p) \cap J_M^-(q)$, is compact and the strong causality condition holds, then there is a causal geodesic from p to q of length $\tau(p,q)$

Def A connected time oriented Lorentzian mfd M is called globally hyperbolic provided the strong causality condition holds on it and for each $p<q$ in M $J_M(p,q)$ "the causal diamond of p, q" is compact

Remark Any pair of points that can be joined by a causal curve can be joined by a causal geodesic.

Def: The subset $\Omega \subset M$ is globally hyperbolic provided the strong causality condition holds on Ω and $p<q$ on Ω then $J_\Omega(p,q)$ is compact and contained in Ω.

Def. A subset S of M is called achronal (acausal) provided the causal relation $p \ll q$ ($p<q$) never holds for $p, q \in S$.

Def A Cauchy hypersurface in M is a subset S that is met exactly once by every inextendible timelike curve in M,

Def If A is an achronal subset of M, the future Cauchy development of A is the set $D^+(A) \subset M$ s.t. every past inextendible causal curve through $p \in M$ meets A

Remark $D(A) = D^+(A) \cup D^-(A)$

Theorem If A is an achronal set, the int $(D(A))$ is globally hyperbolic.

HW

Theorem Let M be a connected time-oriented Lorentzian manifold, then the following are equivalent:

a) M is globally hyperbolic.

b) There exists a Cauchy hypersurface in M.

c) M is isometric to a $\mathbb{R} \times S$ with metric

$-\beta\, dt \otimes dt + g_t$ where β smooth, positive definite function, g_t is a Riemannian metric on S depending smoothly on t, and each $\Sigma_t = \{t\} \times S$ is a smooth Cauchy hypersurface on M

Def: A smooth vector bundle is a quadruple (E, π, M, V) of rk $\dim V$, where

1) M is a smooth manifold called the base
2) V is a vector space called the typical fibre
3) E is a $(\dim M + \dim V)$-dimensional smooth manifold called the total space
4) $\pi: E \to M$ is a smooth surjective map called the projection

s.t.

a) Each fibre $E_p = \pi^{-1}(p)$ is a vector space isomorphic to the typical fibre.

b) For each base point $p \in M$ there is a nbhd U of p and a diffeo $\Psi: \pi^{-1}(U) \to U \times V$ s.t. $pr_1 \circ \Psi = \pi : \pi^{-1}(U) \to U$ $\quad E^{\prime\prime}$

c) Ψ acts as a vector space isomorphism on each fibre i.e. $E_p \to \{p\} \times V$ is an isomorphism between vector spaces.

Remark:

1) Any pair (U, Ψ) satisfying b) and c) is called a local trivialization of the vector bundle.
Any ~~local~~ collection of local trivializations covering M is called a vector bundle atlas.
2) Here M is a spacetime (always)

Ex: a) M is Minkowski, V vectorspace $(E = M \times V, \pi = pr_1, M, V)$
local trivialization: $(M, id_{\underset{E}{M \times V}})$ ← global trivialization

Remark $f: M \to V$, Let $M \times V$ a trivial vector bundle
define $\hat{f}: M \to M \times V$ • $f \to \hat{f}$ a bijection and $pr_1 \circ \hat{f} = id_M$
$p \to (p, f(p))$

Def: Let (E, π, M, V) be a vector bundle. A section of the vector bundle is a smooth function $\sigma: M \to E$ s.t. $\pi \circ \sigma = id_M$.
We denote the vectorspace of sections of the vector bundle by $\Gamma_0(M, E)$.

Def: Let (E, π, M, V) be a real vector bundle. A bosonic (fermionic) vector non degenerate bilinear form is a bilinear form nondegenerate spacetime M with a non-vanishing volume form vol_M.

Let $\langle , \rangle$ be a bosonic (fermionic) nondegenerate bilinear form on E s.t. each fibre is endowed with a non degenerate inner product. A non degenerate pairing between smooth sections and compactly smooth sections of E is a map from

$$(\cdot,\cdot) : \Gamma_0(M,E) \otimes \Gamma(M,E) \longrightarrow \mathbb{R}$$

$$\sigma \otimes \tau \longrightarrow \int vol_M \langle \sigma, \tau \rangle.$$

Def: Let (E,π,M,V), (F,π',M,W) two vector bundles over the same base M. A linear ~~p~~ partial differential operator of order at most k is a linear operator $L : \Gamma(M,E) \to \Gamma(M,F)$ satisfying: For each $p \in M$, U_p nbhd s.t. (U,Φ) trivializes E, (U,φ) ~~is a~~ trivializes F, (U,ϕ) is a local chart of M and there is collections:

$$\{ a_{j}, a_{j_1}, \dots, a_{j_1 \dots j_n} : j_a \in I(\dim M), a \in I(k) \}$$

of smooth $Hom(V,W)$-valued maps on $\phi(U)$ which allows to express L locally as follows

$$\Psi \circ (L\sigma) \circ \phi^{-1} = \sum_{a \in I(k)} \sum_{j_1 \dots j_a \in I(\dim M)} a_{j_1 \dots j_a} \partial_{j_1} \cdots \partial_{j_a} (\Phi \circ \sigma \circ \phi^{-1})$$

where ∂_b ($b \in I(\dim M)$) is the standard partial derivative acting on a vector ~~bundle~~ valued functions defined on the same open subset of $\mathbb{R}^{\dim M}$.

Def: Let (E,π,M,V) be a real bundle over a spacetime (M,g). A linear partial differential operator of exactly second order is called normally hyperbolic if, in a local trivialization, there is a collection $\{a, a_a : a \in I(\dim M)\}$ of smooth $Hom(V,V)$-valued maps on $\phi(U)$ such that P reads as follows:

For each section σ of E,

$$\Phi \circ (P\sigma) \circ \phi^{-1} = \Big\{ -\sum_{a,b \in I(\dim M)} g^{ab} \, id_V \, \partial_a \partial_b + \sum_{a \in I(\dim M)} a^a \partial_a + a \Big\} (\Phi \circ \sigma \circ \phi^{-1})$$

given a section J of E, called source, $P\sigma = J$ is called a wave equation.

Ex: (L^{+}, g), a trivial line bundle $M \times \mathbb{R}$. Sections = real valued funcs on M.

$$P : C^{\infty}(M) \to \overset{\text{sections}}{C^{\infty}(M)}$$

$$P := -\sum_{a,b \in I(4)} g^{ab}\, \partial_a \partial_b + \sum_{a \in I(4)} \left(\sum_{c,d \in I(4)} g^{cd}\, \Gamma^{a}_{cd} \right) \partial_a + m^2$$

Note: If you are in Minkowski and work in natural coordinates $\Gamma^{\mu}_{\nu\lambda} = 0$ locally $\rightsquigarrow$ usual wave eq.

Def: Let $L : \Gamma(M, E) \to \Gamma(M, F)$ be a linear diff op. Its formal adjoint $L^{*} : \Gamma(M, F) \to \Gamma(M, E)$ is a linear diff op. satisfying

$$(L^{*} u, v)_E = (u, L v)_F \quad \text{For all } u \in \Gamma(M, F),\ v \in \Gamma(M, E)$$

A linear diff op. $L : \Gamma(M, E) \to \Gamma(M, E)$ is formally self-adjoint if $L = L^{*}$.

Remark: E vector bundle, ∇ connection on E. This together with the Levi-Civita connection on $T^{*}M$ induces a connection on $T^{*}M \otimes E$, again denoted by ∇.

A connection d'Alembert operator $\Box^{\nabla}$ defined to be the composition of the maps

$$C^{\infty}(M, E) \xrightarrow{\nabla} C^{\infty}(M, T^{*}M \otimes E) \xrightarrow{\nabla} C^{\infty}(M, T^{*}M \otimes T^{*}M \otimes E) \xrightarrow{\mathrm{tr}_{g^{-1}} \otimes \mathrm{id}_E} C^{\infty}(M, E)$$

$$\Box^{\nabla} := -(\mathrm{tr}_{g^{-1}} \otimes \mathrm{id}_E) \circ \nabla \circ \nabla .$$

Lemma: Let $P : C^{\infty}(M, E) \to C^{\infty}(M, E)$ be a normally hyperbolic op. on a spacetime manifold M.
Then there exists a unique connection ∇ on E and a unique endomorphism field $B \in C^{\infty}(M, \mathrm{Hom}(E, E))$ s.t.

$$P = \Box^{\nabla} + B .$$

Proof: i) uniqueness: Let ∇' be an arbitrary connection on E.
For any section $s \in C^{\infty}(M, E) = \Gamma(M, E)$ and any function $f \in C^{\infty}(M)$ we have

$$\Box^{\nabla'}(f \cdot s) = (\Box^{\nabla'} f) \cdot s + f (\Box^{\nabla'} s) - 2 \nabla'_{\mathrm{grad} f}\, s$$

Suppose ∇ is s.t. $P = \Box^{\nabla} + B \implies B = P - \Box^{\nabla}$ is an endomorphism field

$$+ 2\nabla_{\operatorname{grad} f} s$$

At any point $x \in M$, any tangent vector $X \in T_x M$ $X = \operatorname{grad} f$ for a suitable f. Since $(*)$ holds for any connection on E, let's say $\tilde{\nabla}$ (i.e. satisfying $P = \square^{\tilde{\nabla}} + B$)

$$\Rightarrow \quad \nabla_X s = \tilde{\nabla}_X s \quad \text{for all smooth sections of } E \Rightarrow \nabla = \tilde{\nabla}.$$

- <u>Existence:</u> Let ∇' a connection on E. $P - \square^{\nabla'} = A' \cdot \nabla' + B'$ is a diff op of first order for some $A' \in C^\infty(M, \operatorname{Hom}(T^*M \otimes E, E))$ and $B' \in C^\infty(M, \operatorname{Hom}(E, E))$.

Introduce
$$\underbrace{\nabla_X}_{\text{new connection}} s := \nabla'_X s - \frac{1}{2} A'(\underbrace{X^\flat}_{\text{musical isomorphism}} \otimes s).$$
Choose ONB of TM : $\{e_a\}_{a \in I(m)}$ with $\varepsilon_a := g(e_a, e_a) = \pm 1$. $\nabla_{e_a} e_a = 0$

$$P(s) - B'(s) = \square^{\nabla'} s + A' \cdot \nabla' s = \sum_{a \in I(m)} \varepsilon_a \left\{ -\nabla'_{e_a} \nabla'_{e_a} s + A'(e_a^b \otimes \nabla'_{e_a} s) \right\}$$

$$\overset{\text{def}}{=} \square^{\nabla} s + \frac{1}{4} \underbrace{\sum_{a \in I(m)} \varepsilon_a \left\{ A'(e_a^b \otimes A'(e_a^b \otimes s)) - 2(\nabla_{e_a} A')(e_a^b \otimes s) \right\}}_{\square^{\nabla'} s + A' \cdot \nabla' s - \square^{\nabla} s =: Q(s)}$$

$$\sum \varepsilon_a \left\{ -\nabla'_{e_a}\left(\nabla_{e_a} s + \frac{1}{2} A'(e_a^b \otimes s) \right) + A'\left(e_a^b \otimes \left(\nabla_{e_a} s - \frac{1}{2} A'(e_a^b \otimes s)\right)\right) \right\}$$

$$= \sum \varepsilon_a \left\{ -\left(\nabla_{e_a}\left(\nabla_{e_a} s + \frac{1}{2} \cancel{A'(e_a^b \otimes s)}\right) \right) + \frac{1}{2} A'\left(e_a^b \otimes \left(\nabla_{e_a} s + \frac{1}{2} A'(e_a^b \otimes s)\right)\right) \right.$$
$$\left. + A'\left(e_a^b \otimes \left(\nabla_{e_a} s - \frac{1}{2} A'(e_a^b \otimes s)\right)\right) \right\}$$

$$= \square^{\nabla} s + \frac{1}{4} \sum \varepsilon_a \left(A'(e_a^b \otimes A'(e_a^b \otimes s)) - (\nabla_{e_a} A')(e_a^b \otimes s) 2 \right.$$
$$\left. \underbrace{+ \frac{3}{2} A'(e_a^b \otimes \nabla_{e_a} s) + \frac{}{3} A'(e_a^b \otimes A'(e_a^b \otimes s))}_{= 0} \right)$$

$Q(s)$ is of order 0! Here $P = \square^{\nabla'} + A' \cdot \nabla' + B' = \square^{\nabla} + \underbrace{Q + B'}_{B}$

<u>Theorem</u> Let (M, g) globally hyperbolic spacetime and Σ a spacelike smooth Cauchy surface of (M, g). Let u be the future directed timelike unit vector field on Σ.

Consider a vector bundle (E, Π, M, V) and a normally hyperbolic ~~express~~ operator $P = \Box^{\nabla} + B : \Gamma(M, E) \to \Gamma(M, E)$

For each initial data $u_0, u_1 \in \Gamma_0(\Sigma, E)$ and for each source $J \in \Gamma_0(M, E)$, the cauchy problem

$$P u = J \text{ on } M, \quad \underbrace{\nabla_n u = u_1}_{n \text{ normal to } \Sigma}, \quad u = u_0 \text{ on } \Sigma.$$

Admits a unique solution $u \in \Gamma(M, E)$.

the support of u, $\mathrm{supp}(u) \subseteq J_{M}(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(J))$

Moreover,
$$\Gamma_0(\Sigma, E) \times \Gamma_0(\Sigma, E) \times \Gamma_0(M, E) \to \Gamma(M, E)$$
$$(u_0, u_1, J) \mapsto u \text{ is linear and continuous.}$$

<u>Def:</u> Let E be a vector bundle over a globally hypobolic spacetime (M, g), and let $P : \Gamma(M, E) \to \Gamma(M, E)$ a linear diff. op. A linear map $G^{\pm} : \Gamma_0(M, E) \to \Gamma_0(M, E)$ is called an <u>advanced/retarded ~~op.~~ Green operator</u> for P if for each source $J \in \Gamma_0(M, E)$,

1) $P G^{\pm} f = f$;
2) $G^{\pm} P f = f$;
3) $\mathrm{supp}(G^{\pm} f) \subseteq J^{\pm}_{M}(\mathrm{supp}(f))$

<u>Def:</u> Let E be a vector bundle over (M, g), let $P : \Gamma(M, E) \to \Gamma(M, E)$ be a linear diff op. P is called <u>Green-hypobolic</u> if it admits advanced and retarded green operators.

<u>Reminder:</u> A $\mathbb{K}$-linear map $F : \Gamma_0(M, E^*) \to W$ is called a <u>distribution in E with values on W</u> if F is continuous:

$$\text{if } \varphi_n \to \varphi \text{ in } \Gamma(M, E^*)$$
$$F[\varphi_n] \to F[\varphi].$$ (norm induced topology and all others are compatible)

$\Gamma'(M, E, W) =$ space of all W-valued distributions on E.

<u>Ex:</u> 1) $x \in M$. The <u>delta-distribution</u> δ_x is a E^*_x valued distribution in E. For $\varphi \in \Gamma(M, E^*)$,

$$\delta_x[\varphi] = \varphi(x)$$

$f \in \Gamma_{loc}(M, E)$

$$f[\varphi] := \int_M \mathrm{vol}_M \, \varphi(x, f(x))$$

$L : \Gamma(M, E) \to \Gamma(M, F)$ linear. There is a unique

$L^* : \Gamma(M, F^*) \to \Gamma(M, E^*)$ called the <u>formal adjoint</u> of L s.t. for any $\varphi \in \Gamma(M, E)$ and $\psi \in \Gamma(M, F^*)$

$$\int_M \mathrm{vol}_M \, \psi(L\varphi) = \int_M \mathrm{vol}_M (L^*\psi)(\varphi)$$

Any linear ~~map~~ diff op. $L : \Gamma(M, E) \to \Gamma(M, F)$ extends locally to a linear op.

W valued distributions on F

$$L : \Gamma'(M, E, W) \to \Gamma'(M, F, W)$$

$$(LT)[\varphi] := T[L^*\varphi] \quad \text{for any } \varphi \in \Gamma(M, F^*)$$

<u>Def:</u> Let E be a vector bundle over (M, g) and $P : \Gamma(M, E) \to \Gamma(M, E)$ be a normally hyperbolic diff op. A <u>fundamental solution</u> of P at $x \in M$ is a distribution $F \in \Gamma'(M, E, E_x^*)$ s.t. $PF = \delta_x$.

<u>Remarks</u> 1) $\varphi \in \Gamma_0(M, E^*)$

$$F[P^*\varphi] = \varphi(x)$$

2) If $\mathrm{supp}(F) \subset J_M^+(x)$ advanced fund solution

3) If $\mathrm{supp}(F) \subset J_M^-(x)$ retarded fund solution

<u>Proposition</u> Let E be a vector bundle over a globally hyperbolic st (M, g) and $P : \Gamma(M, E) \to \Gamma(M, E)$ a normally hyperbolic op. Let $F^{+/-}(x)$ be the advanced/retarded fundamental solutions for the adjoint op P^*. If $F^{+/-}(x)$ depends smoothly on $x \in M$ in the sense that $x \to F^{+/-}(x)[\varphi]$ is smooth for each test section φ and satisfies $P(F^{+/-}(\ )[\varphi]) = \varphi \Rightarrow$

$$(G^{+/-}\varphi)(x) := F^{-/+}(x)[\varphi] \quad \text{defines advanced/retarded Green}$$

op. for P.

Conversely, given $G^{+/-}$ for P the preceding equation defines fundamental solutions for P^* depending smoothly on $X \in M$

and satisfies $P(F^{+/-}(\cdot)[\varphi]) = \varphi(\cdot)$ for each test ~~func~~ section

Proof: 1) we show that $P(G^{\pm}\varphi) = \varphi$.

By def. $P(G^{+/-}\varphi) = P(F^{-/+}(\cdot)[\varphi]) = \varphi$

2) $G^{+/-}P\varphi = \varphi$ follows from the fact that $F^{+/-}(x)$ are fund. sol.

$$G^{+/-}(P\varphi)(x) = F^{-/+}(x)[P\varphi] = P^* F^{-/+}(x)[\varphi] = \varphi(x)$$

3) we show that $\mathrm{supp}(G^+\varphi) \subseteq J^+_M(\mathrm{supp}\,\varphi)$

Let $x \in M$ s.t. $(G^+\varphi)(x) \neq 0$.

Since $\mathrm{supp}(F^-(x)) \subset J^-_M(x)$, the support of φ must meet $J^-_M(x)$

Hence $x \in J^+_M(\mathrm{supp}(\varphi))$ and therefore $\mathrm{supp}(G^+\varphi) \subseteq J^+_M(\mathrm{supp}\,\varphi)$

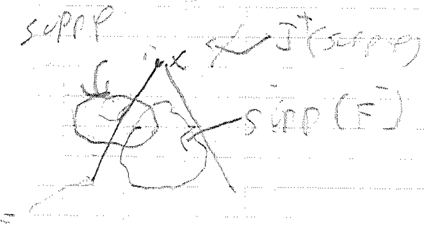
$(G^+\varphi)(x) = F^-(x)[\varphi] \neq 0$ (*) $P(G^+\varphi)(x) = \varphi(x) \neq 0$

1) In general, $\mathrm{supp}(F^-(x)) \subset J^-_M(x)$ ... $\Rightarrow$ 2)

2) $\mathrm{supp}\,\varphi \cap J^-_M(x) \neq \emptyset$

3) $x \in J^+(\mathrm{supp}\,\varphi) \Rightarrow \{\forall x \text{ s.t. } G^+\varphi \neq 0\} = \mathrm{supp}(G^+\varphi)$

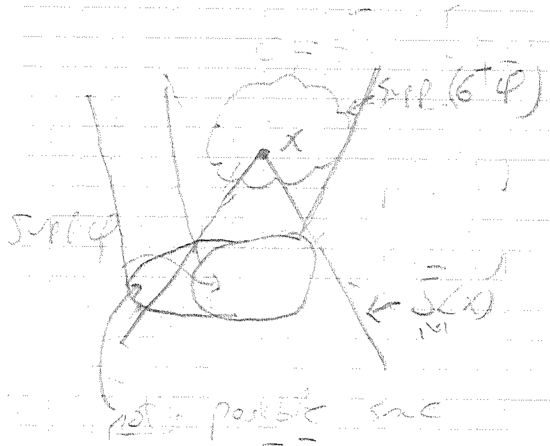
$\Rightarrow \mathrm{supp}(G^+\varphi) \subseteq J^+_M(\mathrm{supp}\,\varphi)$



Converse: Given G^+, x $F^-(x)[\varphi] = (G^+\varphi)(x)$

then $(G^+\varphi)(x) = F^-(x)[\varphi]$, we show $\mathrm{supp}(F^-(x)) \subseteq J^-_M(x)$

$\mathrm{supp}(G^+\varphi) \subseteq J^+_M(\mathrm{supp}\,\varphi)$



not possible since then $F^-(x)(\varphi) = 0$

$P(G^+\varphi)(x) = \varphi(x) = P(F^-(x)[\varphi]) \neq 0$

- $x \in \mathrm{supp}(F^-)$

$F^-(x)[\varphi] \neq 0$

$\|$

$G^+\varphi(x)$

- $G^+\varphi : M \to E$

$G^\pm : \Gamma \to \Gamma$

$F^\pm \in \Gamma'(M, E \otimes E_x^*)$

$\hookrightarrow F^\pm : \Gamma(M, E) \to E_x^*$

$F^\pm\varphi : M \to E_x^*$

Remarks: 1) ~~$\varphi \in \Gamma_0(M, E)$~~

~~$F[\varphi] = \varphi(x)$~~

2) ~~If $\mathrm{supp}(F) \cap J^{\pm}_M(\varphi)$ ... advanced / retarded ...~~

Corollary: Let E be a vector bundle over (M, g), let $P: \Gamma(M,E) \to \Gamma(M,E)$ be a normally hyperbolic op. Then there exist unique advanced/retarded green operator

$$G^{+/-}: \Gamma_0(M,E) \to \Gamma(M,E) \text{ for } P.$$

Lemma: Let E be a vector bundle over a globally hyperbolic space time (M,g) and $P: \Gamma(M,E) \to \Gamma(M,E)$ be a normally hyperbolic op. Let $G^{+/-}$ the green operators for P and $G^{*\,+/-}$ the green op. for the adjoint P^*.
Then

$$\int_M \mathrm{vol}\,(G^{*\,+/-}\varphi)(\psi) = \int_M \mathrm{vol}_M\, \varphi\,(G^{-/+}\psi)$$

for all $\psi, \varphi \in \Gamma_0(M, E^*)$

Remark: for all sections $u, v \in E$ with compact support, we have

$$(G^{*\,+/-}u, v) = (v, G^{-/+}v)$$

Proof: 1) $\mathrm{supp}(G^{+/-}\varphi) \cap \mathrm{supp}(G^{-/+}\psi) \subset J^{+/-}_M(\mathrm{supp}(\varphi)) \cap J^{-/+}_M(\mathrm{supp}\,\psi)$
is compact in a globally hyperbolic spacetime.

2) For the green op. we have

$$P G^{+/-} = \mathrm{id}_{\Gamma_0(M,E)}, \quad P^* G^{*\,+/-} = \mathrm{id}_{\Gamma_0(M,E^*)}$$

and therefore $(G^{*\,+/-}\varphi, \psi) = (G^{*\,+/-}\varphi, P G^{-/+}\psi) \overset{(1)}{=} (P^* G^{*\,+/-}\varphi, G^{-/+}\psi)$
$= (\varphi, G^{-/+}\psi)$

Remark: $\Gamma_{sc}(M,E)$ = set of all smooth sections of the vector bundle E for which there exists a compact $K \subset M$ s.t. $\mathrm{supp}(\varphi) \subset J_M(K)$.

vector subspace of $\Gamma(M,E)$

"sc" = spacelike compact.

Namely
If M is globally hyperbolic and $\varphi \in \Gamma_0 \Rightarrow$ for every Cauchy hypersurface the support of φ restricted to the hypersurface

is the intersection of the Cauchy hypersurface with the causal domain ($J^+ \cup J^-$ of K).

'sc' stands for space like compact.
If M is G.H and $\varphi \in \Gamma_{sc}(M,E)$ then for every C. Hyper $\Sigma \subset M$, the supp of $\varphi|_{\Sigma}$ is contained in $\Sigma \cap J_M(K)$.

<u>Def</u>: Let E be a vector bundle over a g.h.s.t. (M,g), let $P: \Gamma(M,E) \to \Gamma(M,E)$ be a green hyperbolic op.
Chose advanced/retarded green op $G^{\pm}$ for P.
Then, the linear map $G = G^+ - G^- : \Gamma_0(M,E) \to \Gamma_{sc}(M,E)$ is called the <u>causal propagator</u> for P defined by $G^{\pm}$.

<u>Theorem</u>: Let E be a v.b over a g.h.s.t. (M,g) and $P: \Gamma(M,E) \to \Gamma(M,E)$ be a ~~n.hyp~~ n.h. op. Let $G^{\pm}$ be adv/ret g.op. for P. Then the sequence of linear maps

$$0 \longrightarrow \Gamma_0(M,E) \xrightarrow{P} \Gamma_0(M,E) \xrightarrow{G} \Gamma_{sc}(M,E) \xrightarrow{P} \Gamma_{sc}(M,E)$$

is a chain complex which is exact every where.

<u>proof</u>:

1) Since $P G^{\pm} = \mathrm{id}_{\Gamma_0(M,E)}$ by def., it follows that $PG = 0$.
The second property of ret/adv green op., $G^{\pm} P = \mathrm{id}_{\Gamma_0(M,E)}$ gives $GP = 0$.
Moreover, $\mathrm{supp}(G^{\pm}\varphi) \subseteq J^{\pm}_M(\mathrm{supp}\,\varphi)$ for any $\varphi \in \Gamma_0$
hence, G maps $\Gamma_0(M,E) \longrightarrow \Gamma_{sc}(M,E)$
Hence, the sequence it's a chain complex.

2) The chain is exact: i.e. $\ker f = 0$.
1st exactness mean that $P: \Gamma_0(M,E) \to \Gamma_0(M,E)$ is injective
Let $\varphi \in \Gamma_0(M, \ker P)$ ~~...~~ $\Rightarrow \varphi = G^- P \varphi = 0$
$\Rightarrow \varphi = 0$.
2nd exactness: means that $\ker G = \mathrm{Image}\, P$
Let $\varphi \in \Gamma_0(M,E)$ with $G\varphi = 0$ i.e. $\varphi \in \ker G$.
i.e. $G^+\varphi = G^-\varphi$. Let $\psi = G^+\varphi \in \Gamma(M,E)$
$\mathrm{supp}(\psi) = \mathrm{supp}(G^+\varphi) \cap \mathrm{supp}(G^-\varphi) \subset J^+_M(\mathrm{supp}\,\varphi) \cap J^-_M(\mathrm{supp}\,\varphi)$
is compact $\Rightarrow \psi \in \Gamma_0(M,E)$

$$\Gamma_0 \xrightarrow{f} \Gamma_0 \xrightarrow{g} \Gamma_0$$

$\ker g = \mathrm{Img}\, f$

~~$\ker g$ ... $\Gamma_0 \xrightarrow{g} \Gamma_0$~~

Exercise: ... that $\Gamma_{sc}(M,E)$ means that G is surjective.

Let $\varphi \in \Gamma_{sc}(M,E)$ s.t. $P\varphi = 0$. W.l.o.g. assume that $\mathrm{supp}(\varphi) \subset I^+_M(K) \cup I^-_M(K)$ for a compact subset K of M.

Partition of unity subordinated to the open cover $\{I^+_M(K), I^-_M(K)\}$ of $\mathrm{supp}(\varphi)$, with $\varphi = \varphi_1 + \varphi_2$, $\mathrm{supp}(\varphi_1) \subset I^-_M(K) \subset J^-_M(K)$

$\mathrm{supp}(\varphi_2) \subset I^+_M(K) \subset J^+_M(K)$

For $\psi := -P\varphi_1 \overset{P\varphi=0}{=} P\varphi_2$ we see $\mathrm{supp}(\psi) \subset J^+_M(K) \cap J^-_M(K)$

Hence $\psi \in \Gamma_0(M,E)$. We show that $G^+\psi = \varphi_2$ for all $\theta \in \Gamma_0(\ldots)$ we have

$$(\theta, \underbrace{G^+P\varphi_2}_{G^+\psi}) = (G^{*-}\theta, P\varphi_2) = (P^* G^{*-}\theta, \varphi_2) = (\theta, \varphi_2) \quad \checkmark$$

Similarly: $G^-\psi = -\varphi_1$, so $G\psi = G^+\psi - G^-\psi = \varphi_1 + \varphi_2 = \varphi \in \mathrm{im}\, G$.

Proposition: Let E be a v.b. over a g.h.s.t. (M,g) and $P: \Gamma(M,E) \to \Gamma(M,E)$ be a n.h. op. Denote by $G^{+/-}$ the adv/retarded Green op. for P. Then $G^{\pm}: \Gamma_0(M,E) \to \Gamma_{sc}(M,E)$ and all maps of the preceding chain complex are sequentially continuous.

Remark: Let P be green-hyperbolic with P^* also g-h. Denote by $\mathcal{S}_{sc}(M)$ the space of solutions of $Pu = 0$ with spacelike compact support on M.

It can be shown that G induces an isomorphism of vector spaces from $\Gamma_0(M,E)/P(\Gamma_0(M,E))$ to $\mathcal{S}_{sc}(M)$

Proposition: Let E be a v.b. over (M,g) equipped with a bosonic nondeg. bilinear form.

Let $P: \Gamma(M,E) \to \Gamma(M,E)$ be a formally self adjoint green-hyperbolic op. Then, the causal propagator G for P fulfills:

$$(u, Gv) = -(Gu, v) \qquad u, v \in \Gamma_0(M,E) \text{ and}$$

the map $\quad \sigma: \mathcal{S}_{sc}(M) \otimes \mathcal{S}_{sc}(M) \to \mathbb{R}$

$$\sigma(u,v) := (f, Gh) \qquad \text{with } f, h \in \Gamma_0(M,E)$$

s.t. $u = Gf$, $v = Gh$, is a symplectic form.

Proof: 1) $P = P^*$ $\Rightarrow G^* = G$. Consider $(G^{\pm}u, v) = (u, G^{-/+}v)$ for $u, v \in \Gamma_0(M,E)$

Since $G = G^+ - G^-$ we conclude $(u, Gv) = -(Gu, v)$

2) Introduce a linear form τ on $\Gamma_0(M,E)$, defined by $\tau(u,v) := (u, G v)$

Consider $u \in \Gamma_0(M,E)$ s.t. $\tau(u,v) = 0 \quad \forall v \in \Gamma_0(M,E)$

Since $(\cdot,\cdot)$ is non degenerate $\Rightarrow$ $Gu = 0$ but then

$u \in P(\Gamma_0(M,E))$. $\Gamma_0/P(\Gamma_0) \cong \mathcal{S}_{sc}$ Hence, define

$$\sigma := \tau \circ (\mathfrak{J}^{-1} \otimes \mathfrak{J}^{-1}).$$

Ex: the real scalar field: G.h.s.t. (M,g) simple bundle.

$E = M \times \mathbb{R}$, $\Gamma(M,E) \simeq C^\infty(M)$ we recover all properties of scalar field.

Def: A <u>source-free real Klein-Gordon field</u> is a smooth section of $E = M \times \mathbb{R}$ s.t. the associated function $\Phi \in C^\infty(M)$ is a solution of the following Cauchy problem:

$$P\Phi = (\Box^\nabla + \xi R + m^2)\Phi = 0 \quad \text{on } M$$

$$\Phi = \Phi_0 \text{ on } \Sigma, \quad \nabla_n \Phi = \Phi_1 \quad \text{where } \Sigma \text{ smooth space like Cauchy surface with normal unit } n. \text{ future directed vector field}$$

$$\Phi_0, \Phi_1 \in C_0^\infty(M) \text{ are given initial data.}$$

Symplectic structure $\sigma : \mathcal{S}_{sc}(M) \otimes \mathcal{S}_{sc}(M) \to \mathbb{R}$

$$(\Phi_f, \Phi_h) \to (f, Gh)$$

$$\Phi_f = Gf, \quad \Phi_h = Gh.$$

<u>Quantization</u>

<u>Def:</u> Let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra and $\|\cdot\|$ be a norm on the $\mathbb{C}$-vector space $\mathcal{A}$.

Let $* : \mathcal{A} \to \mathcal{A}$, $a \to a^*$ be a $\mathbb{C}$-linear map. The triple $(\mathcal{A}, \|\cdot\|, *)$ is called a C^*-algebra, if the pair $(\mathcal{A}, \|\cdot\|)$ is complete and: $\forall a,b \in \mathcal{A}$

1) $(a^*)^* = a^{**} = a$ ($*$ is an involution),
2) $(ab)^* = b^* a^*$,
3) $\|ab\| \leq \|a\| \|b\|$ (sub multiplicativity)
4) $\|a^*\| = \|a\|$ ($*$ is an isometry)
5) $\|a^* a\| = \|a\|^2$ (C^*-property).

<u>Ex:</u> Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathcal{A} = \mathcal{L}(\mathcal{H})$ be the algebra of bounded op. on $\mathcal{H}$. Let

$$\langle ax, y\rangle = \langle x, a^* y\rangle \quad \forall x,y \in \mathcal{H}.$$

Proof (5): $\|a\|^2 = \sup_{\substack{x\in\mathcal{H} \\ \|x\|=1}} \|ax\|^2_{\mathcal{H}} = \sup_x \langle ax, ax\rangle = \sup_x \langle x, a^* a x\rangle$

$$\underset{\substack{\uparrow \\ \text{Cauchy-Schwarz ineq}}}{\leq} \sup_x \|x\|_H \|a^* a x\|_H = \|a^* a\| \overset{(3)}{\leq} \|a\|\|a^*\| \underset{(4)}{\leq} \|a\|^2$$

2) Let X be a locally compact Hausdorff space. Introduce $A = C_0(X) := \{ f: X \to \mathbb{C}$ continuous: $\forall \varepsilon > 0 \; \exists \underset{\text{compact}}{K_\varepsilon} \subset X$ s.t. $|f(x)|_{\mathbb{C}} < \varepsilon \;\; \forall x \in X \setminus K_\varepsilon \}$

$A =$ "algebra of continuous functions vanishing at infinity."

All $f \in C_0(X)$ are bounded, so we may define

$$\|f\| := \sup_{x\in X} |f(x)|_{\mathbb{C}}$$

Let $f^*(x) := \overline{f(x)}$ for any $x \in X$. The triple $(C_0(X), \|\cdot\|, *)$ is a commutative C^*-algebra.

3) Let M be a smooth manifold. Introduce $A = C_0^\infty(M) := C^\infty(M)$. $(C_0^\infty(M), \|\cdot\|)$ is not complete.

Remark: A C^*-algebra has atmost 1 unit. Assume $1, 1'$ are units. $1 = 1 \cdot 1' = 1'$.

For all $a \in A$ we have $1^* a = (1^* a)^{**} = (a^* 1^{**})^* = (a^* 1)^* = a^{**} = a$ and similarly for $a 1^* = a \Rightarrow 1^*$ is a unit $\Rightarrow 1 = 1^*$.

moreover, $\|1\| = \|1^* 1\| = \|1\|^2 \Rightarrow \|1\| = 0$ or $1 \rightsquigarrow \|\mathbb{1}\| = 1$ (otherwise trivial).

Let A be a C^*-algebra with unit $\mathbb{1}$. Write $A^\times$ be the set of invertible elements of A. If $a \in A^\times \rightsquigarrow a^* \in A^\times$.

$$a^* (a^{-1})^* = (a^{-1} a)^* = 1^* = 1$$

and similarly $(a^{-1})^* a^* = 1 \Rightarrow (a^*)^{-1} = (a^{-1})^*$.

Lemma: Let A be a C^*-algebra. Then the maps

1) $A \times A \to A$, $(a,b) \to a+b$
2) $\mathbb{C} \times A \to A$, $(\lambda, a) \to \lambda a$
3) $A \times A \to A$, $(a,b) \to a \cdot b$
4) $A^\times \to A^\times$, $a \to a^{-1}$
5) $A \to A$, $a \to a^*$

are continuous.

Proof: 4) $a_0 \in \mathcal{A}^\times$, for all $a \in \mathcal{A}^\times$ with $\|a - a_0\| < \varepsilon$ we have

$$\|a^{-1} - a_0^{-1}\| = \|a^{-1} a_0 a_0^{-1} - a^{-1} a\, a_0^{-1}\| = \|a^{-1}(a_0 - a) a_0^{-1}\|$$

$$\leq \|a^{-1}\| \, \|a_0 - a\| \, \|a_0^{-1}\| = \|a^{-1} - a_0^{-1} + a_0^{-1}\| \, \|a_0 - a\| \, \|a_0^{-1}\|$$

$$\leq (\|a^{-1} - a_0^{-1}\| + \|a_0^{-1}\|) \, \|a_0 - a\| \, \|a_0^{-1}\|$$

$$< (\|a^{-1} - a_0^{-1}\| + \|a_0^{-1}\|) \, \varepsilon \, \|a_0^{-1}\|$$

$$\|a^{-1} - a_0^{-1}\| < \frac{\varepsilon \|a_0^{-1}\|^2}{\underbrace{1 - \varepsilon \|a_0^{-1}\|}_{+}} \text{, choose } \varepsilon < \|a_0^{-1}\|^{-1}$$

$$< \varepsilon \|a_0^{-1}\|^2 \quad (a\text{-independent})$$

Def: Let $\mathcal{A}$ be a C^*-algebra with unit vector 1. For $a \in \mathcal{A}$ we call

$$\Gamma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : (\lambda 1 - a) \in \mathcal{A}^\times\}$$

the resolvent set of a and $\sigma_{\mathcal{A}}(a) := \mathbb{C} \setminus \Gamma_{\mathcal{A}}(a)$ the spectrum of a. For $\lambda \in \Gamma_{\mathcal{A}}(a)$, $(\lambda 1 - a)^{-1} \in \mathcal{A}$ is the resolvent of a at λ.

The number

$$\varrho_{\mathcal{A}}(a) := \sup\{|\lambda|_{\mathbb{C}} : \lambda \in \sigma_{\mathcal{A}}(a)\}$$

is called the spectral radius of a.

Ex: Let X be a compact Hausdorff space and $\mathcal{A} = C(X)$. Then $\mathcal{A}^\times = \{f \in C(X) \mid f(x) \neq 0 \text{ for all } x \in X\}$

$$\sigma_{C(X)}(f) = f(X) \subset \mathbb{C}, \qquad \Gamma_{C(X)}(f) = \mathbb{C} \setminus f(X)$$

$$\varrho_{C(X)}(f) = \max_{x \in X} |f(x)| .$$

Proposition: Let $\mathcal{A}$ be a C^* algebra with unit 1, let $a \in \mathcal{A}$. Then $\sigma_{\mathcal{A}}(a) \subset \mathbb{C}$ is a nonempty compact subset and the resolvent of a is continuous.

Further more, $\varrho_{\mathcal{A}}(a) = \lim_{n \to \infty} \|a^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|a^n\|^{1/n} \leq \|a\|$

Proof: 1) $\sigma_{\mathcal{A}}(a)$ is closed: Let $\lambda_0 \in \Gamma_{\mathcal{A}}(a)$. For each $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0|_{\mathbb{C}} < \|(\lambda_0 1 - a)^{-1}\|^{-1}$ the Neumann series

$\sum_{m=0} (\lambda_0 - \lambda)^m [(\lambda_0 1 - a)^{-1}]^{m+1}$ converges absolutely since

$$\|(\lambda_0-\lambda)^m[(\lambda_0 1-a)^{-1}]^{m+1}\| \leq |\lambda_0-\lambda|^m_{\mathbb{C}} \, \|[(\lambda_0 1-a)^{-1}]^{m+1}\|$$

$$\leq \|(\lambda_0 1-a)^{-1}\| \, \underbrace{\frac{\|(\lambda_0 1-a)^{-1}\|^m}{\|\lambda_0-\lambda\|^{-m}_{\mathbb{C}}}}_{<1 \text{ by the choise of } \lambda}$$

$\Rightarrow$ is a contraction. $\Rightarrow$ The Neumann series converges.

$$(\lambda 1-a)\sum_{m=0}^{\infty}(\lambda_0-\lambda)^m(\lambda_0 1-a)^{-m-1} = [(\lambda-\lambda_0)1-(\lambda_0 1-a)]\sum_{m=0}^{\infty}\dots$$

$$= -\sum_{m=0}^{\infty}(\lambda_0-\lambda)^{m+1}(\lambda_0 1-a)^{-m-1} + \sum_{m=0}^{\infty}(\lambda_0-\lambda)^m(\lambda_0 1-a)^{-m}$$

$= 1$ so $\lambda \in \Gamma_A(a)$ for $\lambda \in \mathbb{C}$ with $|\lambda-\lambda_0|_{\mathbb{C}} < \|(\lambda_0 1-a)^{-1}\|^{-1}$

Therefore $\Gamma_A(a)$ is open and $\sigma_A(a)$ is closed.

2) Continuity of the resolvent:

$$\|(\lambda 1-a)^{-1} - (\lambda_0 1-a)^{-1}\| = \left\|\sum_{m=0}^{\infty}(\lambda_0-\lambda)^m(\lambda_0 1-a)^{-m-1} - (\lambda_0 1-a)^{-1}\right\|$$

$$\leq \sum_{m=1}^{\infty}|\lambda_0-\lambda|^m \, \|\lambda_0 1-a\|^{-m-1} \underset{\text{geometric series}}{=} \|(\lambda_0 1-a)^{-1}\| \frac{|\lambda_0-\lambda|_{\mathbb{C}}\|(\lambda_0 1-a)^{-1}\|}{1-|\lambda_0-\lambda|_{\mathbb{C}}\|(\lambda_0 1-a)^{-1}\|}$$

Since $\lambda_0 \in \Gamma_A(a)$, the above expression goes to zero for $\lambda \to \lambda_0$

3) $\rho_A(a) \leq \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$

Let $n \in \mathbb{N}$ be fixed s.t. $|\lambda|^n_{\mathbb{C}} > \|a^n\|_A$

Each $m \in \mathbb{N}_0$ can be written uniquely as $m = np+q$, $p, q \in \mathbb{N}_0$, $q \in [0, n-1]$

The series $\frac{1}{\lambda}\sum_{m=0}^{\infty}\left(\frac{a}{\lambda}\right)^m = \frac{1}{\lambda}\sum_{q=0}^{n-1}\left(\frac{a}{\lambda}\right)^q \sum_{p=0}^{\infty} \underbrace{\left(\frac{a^n}{\lambda^n}\right)}_{\|\cdot\|<1}{}^p$

converges absolutely and its limit is $(\lambda 1-a)^{-1}$:

$$(\lambda 1-a)\sum_{m=0}^{\infty}\lambda^{-m-1}a^m = \sum_{m=0}^{\infty}\lambda^{-m}a^m - \sum_{m=0}^{\infty}\lambda^{-m-1}a^{m+1} = 1$$

Hence, for $|\lambda|^n_{\mathbb{C}} > \|a^n\|$ the element $(\lambda 1-a)$ is invertible.

$\Rightarrow \lambda \in \Gamma_A(a)$. Therefore, $\rho_A(a) \leq \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$

The converse is also true. $\rho_A(a) \geq \limsup_{n\to\infty} \|a^n\|^{1/n} = \tilde{\rho}_A(a)$

Case 1: $\boxed{\tilde{\rho}(a)=0}$. If a were invertible, then

$$1 = \|1\| = \|a^{-n}a^n\| \leq \|a^n\| \, \|a^{-n}\| \rightsquigarrow 1 \leq \tilde{\rho}(a)\,\tilde{\rho}(a^{-1}) = 0$$

a contradiction $\Rightarrow a \notin A^\times$ and $0 \in \sigma_A(a) \neq \emptyset$.

$\rightsquigarrow \tilde{\rho}(a) = 0 \leq \rho_A(a)$

Case 2. $\tilde{\rho}(a) > 0$. Introduce $S := \{\lambda \mid |\lambda|_{\mathbb{C}} \geq \tilde{\rho}(a)\}$

We show $S \not\subset \Gamma_A(a)$ because ~~otherwise~~ if true then $\exists \lambda \in \sigma_A(a)$ s.t. $|\lambda|_{\mathbb{C}} \geq \tilde{\rho}(a)$ and thus

$$\rho_A(a) \geq |\lambda|_{\mathbb{C}} \geq \tilde{\rho}(a).$$

Assume ~~S~~ there is no λ and therefore $S \subset \Gamma_A(a)$

Let $\omega \in \mathbb{C}$ be an nth root of unity.

For $\lambda \in S$, $\lambda/\omega^k \in S$. Hence

$$\left(\frac{\lambda}{\omega^k}1 - a\right)^{-1} = \frac{\omega^k}{\lambda}\left(1 - \frac{\omega^k}{\lambda}a\right)^{-1} \text{ exists.}$$

$$R_n(a,\lambda) := \frac{1}{n}\sum_{k=1}^{n}\left(1 - \frac{\omega^k a}{\lambda}\right)^{-1}$$

We can show that $R_n(a,\lambda) = \left(1 - \frac{a^n}{\lambda^n}\right)^{-1} \; \forall \lambda \in S \subset \Gamma_A(a)$

$$\left\| \left(1 - \frac{a^n}{\tilde{\rho}_A(a)}\right)^{-1} - \left(1 - \frac{a^n}{\lambda^n}\right)^{-1} \right\| \leq |\tilde{\rho}(a) - \lambda|_{\mathbb{C}} \, \underbrace{\|a\| \sup_{z \in S} \|(z1-a)^{-1}\|^2}_{c}$$

For $|z|_{\mathbb{C}} \geq 2\|a\|$

$$\|(z1-a)^{-1}\| \leq \frac{1}{|z|}\sum_{n\in\mathbb{N}} \underbrace{\frac{\|a\|^n}{\|z\|^n}}_{\leq\left(\frac{1}{2}\right)^n} \leq \frac{2}{|z|} \leq \frac{1}{\|a\|}$$

Consider the annulus $\overline{B_{2\|a\|}(0)} - B_{\tilde{\rho}(a)}(0)$

$$\|R_n(a,\tilde{\rho}(a)) - R_n(a,\lambda)\| \leq c\,|\tilde{\rho}(a) - \lambda|$$

put $\lambda = \tilde{\rho}(a) + \frac{1}{k} \rightsquigarrow \|R_n(a,\tilde{\rho}(a)) - R_n(a, \tilde{\rho}(a)+\tfrac{1}{k})\| \leq \frac{c}{k} \xrightarrow{n\to\infty}$

$\lim_{n\to\infty} \|R_n(a,\tilde{\rho}(a)) - 1\| \quad$ rhs $\xrightarrow{n\to\infty} \frac{c}{k}$.

$n\to\infty$ $\quad\quad\quad\quad\quad\quad\quad\quad /k$

$\leadsto \limsup_{n\to\infty} \| R_n(a, \tilde\rho^{-1}(a)) - 1 \| = 0, \quad \forall k$

$\Rightarrow \dfrac{\|a^n\|}{\tilde\rho^n(a)} \xrightarrow{n\to\infty} 0$

On the other hand, $\|a^{n+1}\|^{\frac{1}{n+1}} \le \|a^n\|^{\frac{1}{n}} \Rightarrow$ sequence $\left(\|a^n\|^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ is monotonically non increasing.

$\leadsto \tilde\rho(a) = \limsup_{l\to\infty} \|a^l\|^{\frac{1}{l}} \le \|a^n\|^{\frac{1}{n}} \; \forall n\in\mathbb{N}$ thus

$1 \le \dfrac{\|a^n\|}{\tilde\rho^n(a)}$, a contradiction. $\leadsto s \notin \Gamma_A(a)$ $\boxtimes$

we end up with $\rho_A(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \le \|a\|$

Def: Let A be a C^*-algebra with unit. Then, $a\in A$ is called <u>normal</u> if $aa^* = a^*a$
an isometry if $a^*a = 1$
unitary if $a^*a = aa^* = 1$

Proposition: Let A be a C^*-algebra with unit, $a\in A$.
Then:
1) $\sigma_A(a^*) = \overline{\sigma_A(a)}$
2) If $a\in A^\times$, then $\sigma_A(a^{-1}) = \left(\sigma_A(a)\right)^{-1}$
3) If a is normal, then $\rho_A(a) = \|a\|$.
4) If a is an isometry, then $\rho_A(a) = 1$
5) If a is unitary, $\sigma_A(a) \subset S^1 \subset \mathbb{C}$
6) If a is self adjoint, $\begin{cases} \sigma_A(a) \subset [-\|a\|, \|a\|] \\ \sigma_A(a^2) \subset [0, \|a\|^2] \end{cases}$
7) If $P(z)$ is a poly with complex coefficients and $a\in A$ is arbitrary, then
$\sigma_A(p(a)) = p(\sigma_A(a)) := \{p(\lambda) : \lambda \in \sigma_A(a)\}$.

Proof: Reed & Simon

Corollary: Let $(\mathcal{A}, \|\cdot\|, *)$ be a C^*-algebra with unit. Then the norm is uniquely determined by $\mathcal{A}$ and $*$.

Proof: For $a \in \mathcal{A}$ the element a^*a is self adjoint. hence, by (3),

$\|a\|^2 = \|a^*a\| = \rho_{\mathcal{A}}(a^*a)$ which depends on $\mathcal{A}$ and $*$.

Def: Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. An algebra homomorphism $\pi: \mathcal{A} \to \mathcal{B}$ is called a *-morphism if $\forall a \in \mathcal{A}$ $\pi(a^*) = \pi(a)^*$

A map $\pi: \mathcal{A} \to \mathcal{A}$ is a *-automorphism if its invertible *-morphism

Corollary: $\mathcal{A}$ and $\mathcal{B}$ C^*-algebras with unit. Each unit-preserving *-morphism $\pi: \mathcal{A} \to \mathcal{B}$ satisfy:

$$\|\pi(a)\|_{\mathcal{B}} \leq \|a\|_{\mathcal{A}}$$ for all a, in particular, π is continuous.

Proof: For $a \in \mathcal{A}^\times$ we have $\begin{cases} \pi(a)\pi(a^{-1}) = \pi(aa^{-1}) = \pi(1_{\mathcal{A}}) = 1_{\mathcal{B}} \\ \pi(a^{-1})\pi(a) = 1_{\mathcal{B}} \end{cases}$

Hence, $\pi(a) \in \mathcal{B}^\times$ with $\pi(a)^{-1} = \pi(a^{-1})$.

If $\lambda \in \Gamma_{\mathcal{A}}(a)$, then $(\lambda 1_{\mathcal{B}} - \pi(a)) = \pi(\lambda 1_{\mathcal{A}} - a) \in \pi(\mathcal{A}^\times) \subset \mathcal{B}^\times$

i.e. $\lambda \in \Gamma_{\mathcal{B}}(\pi(a))$. Hence $\Gamma_{\mathcal{A}}(a) \subset \Gamma_{\mathcal{B}}(\pi(a))$ and $\sigma_{\mathcal{B}}(\pi(a)) \subset \sigma_{\mathcal{A}}(a)$

$\Rightarrow \rho_{\mathcal{B}}(\pi(a)) \leq \rho_{\mathcal{A}}(a)$

$$\|\pi(a)\|^2_{\mathcal{B}} = \|\pi(a)^*\pi(a)\|_{\mathcal{B}} = \rho_{\mathcal{B}}(\pi(a)^*\pi(a)) \overset{*\text{-automorphism}}{=} \rho_{\mathcal{B}}(\pi(a^*a))$$

$$\leq \rho_{\mathcal{A}}(a^*a) = \|a\|^2_{\mathcal{A}}$$

Corollary: Let $\mathcal{A}$ be a C^*-algebra with unit. Then each unit preserving *-automorphism $\pi: \mathcal{A} \to \mathcal{A}$ satisfies

$$\|\pi(a)\| = \|a\|$$

Def: A Weyl system of a symplectic vector space (V, Ω) consists of a C^* algebra $\mathcal{A}$ with unit and a map $W: V \to \mathcal{A}$ s.t. for all $\varphi, \psi \in V$ the following holds:

1) $W(0_V) = 1_{\mathcal{A}}$

2) $W(-\varphi) = W(\varphi)^*$,

3) $W(\varphi) \cdot W(\psi) = e^{-\frac{i}{2}\Omega(\varphi,\psi)} W(\varphi + \psi)$

(we didn't require W to be continuous.)

Example: Let (V, Ω) be an arbitrary symplectic vector space.

$H = \ell^2(V, \mathbb{C})$ endowed with the counting measure i.e.

functions vanishing everywhere except for countable many points and satisfying

$$\| f \|_{\ell^2} := \sum_{\varphi \in V} |f(\varphi)|_{\mathbb{C}} < \infty.$$

The Hermitian product on H is given by $(f, g)_{\ell^2} = \sum_{\varphi \in V} \overline{f(\varphi)} g(\varphi)$

Let $\mathcal{A} = \mathcal{L}(H)$ be the C^*-algebra of bounded ~~op.~~ linear op. on H. We define $W: V \to \mathcal{A}$ by

$$(W(\varphi) f)(\psi) := \exp\left(\frac{i}{2} \Omega(\varphi, \psi)\right\} f(\varphi + \psi)$$

(1) is obvious. (2) $(W(\varphi) f, g)_{\ell^2} = \sum_{\psi \in V} \overline{(W(\varphi) f)(\psi)} g(\psi)$

$$= \sum_{\psi \in V} \overline{\exp\left(\frac{i}{2} \Omega(\varphi, \psi)\right) f(\underbrace{\varphi + \psi}_{\chi})} g(\psi) = \sum_{\chi \in V} \overline{\exp\left\{\frac{i}{2} \Omega(\varphi, \chi - \varphi)\right\} f(\chi)} g(\chi - \varphi)$$

$$\overset{\text{skew}}{=} \sum_{\chi \in V} \overline{\exp\left[\frac{i}{2} \Omega(\varphi, \chi)\right]} \overline{f(\chi)} g(\chi - \varphi) = \sum_{\chi \in V} \overline{f(\chi)} \exp\left(\frac{i}{2} \Omega(-\varphi, \chi)\right) g(\chi - \varphi)$$

$$= (f, W(-\varphi) g)_{\ell^2} \Rightarrow W(\varphi)^* = W(-\varphi)$$

(3) $(W(\varphi)(W(\psi) f))(\chi) = \exp\left(\frac{i}{2} \Omega(\varphi, \chi)\right)(W(\psi) f)(\varphi + \chi)$

$$= \exp\left(\frac{i}{2} \Omega(\varphi, \chi)\right) \exp\left(\frac{i}{2} \Omega(\psi, \varphi + \chi)\right) f(\varphi + \chi + \psi)$$

$$= \exp\left(\frac{i}{2} \Omega(\psi, \varphi)\right) \exp\left(\frac{i}{2} \Omega(\varphi + \psi, \chi)\right) f(\varphi + \chi + \psi)$$

$$= \exp\left(-\frac{i}{2} \Omega(\varphi, \psi)\right) (W(\underline{\varphi + \psi}) f)(\chi) \checkmark$$

Let $CCR(V, \Omega)$ be the C^*-algebra of $\mathcal{L}(H)$ generated by the elements $W(\varphi)$, $\varphi \in V$.
Then $CCR(V, \Omega)$ together with the map W forms a Weyl system for (V, Ω)

Prop: Let $(\mathcal{A}, W)$ be a Weyl system of a symplectic vector space (V, Ω). Then,

1) $W(\varphi)$ is unitary for each $\varphi \in V$.
2) $\| W(\varphi) - W(\psi) \| = 2$ for all $\varphi, \psi \in V$, $\varphi \neq \psi$.
3) $\mathcal{A}$ is not separable unless $V = \{0_V\}$
4) The family $\{W(\varphi)\}_{\varphi \in V}$ is linearly indep.

Proof: 1) $\begin{cases} W(\varphi)^* W(\varphi) = W(-\varphi) W(\varphi) = \exp\left(-\frac{i}{2} \sigma(\varphi, \varphi)\right) W(0) \\ W(\varphi) W(\varphi^*) = 1_{\mathcal{A}} \qquad\qquad = W(0_V) = 1_{\mathcal{A}} \end{cases}$

Hence, $W(\varphi)$ is unitary for each φ.

2) Let $\varphi, \psi \in V$ $\varphi \neq \psi$. For an arbitrary $\chi \in V$ we consider

$$W(\chi) W(\varphi - \chi) W(\chi)^{-1} = W(\chi) W(\varphi - \psi) W(\chi)^* = \exp\left\{-\frac{i}{2} \sigma(\chi, \varphi - \psi)\right\} W(\chi + \varphi - \psi) W(-\chi)$$

$$= \exp\left(-\frac{i}{2} \sigma(\chi, \varphi - \psi)\right\} \exp\left\{-\frac{i}{2} \sigma(\chi + \varphi - \psi, -\cancel{\chi})\right\} W(\varphi - \psi)$$

$$= \exp\left(-\frac{i}{2} \sigma(\chi, \varphi - \psi)\right) W(\varphi - \psi)$$

Hence, the spectra satisfies:

$$\sigma_{\mathcal{A}}(W(\varphi - \psi)) = \sigma_{\mathcal{A}}(W(\chi) W(\varphi - \psi) W(\chi)^{-1})$$

$$= \exp\left(-\frac{i}{2} \sigma(\chi, \varphi - \psi)\right) \sigma_{\mathcal{A}}(W(\varphi - \psi))$$

since $\varphi - \psi \neq 0_V$ the real numbers $\sigma(\chi, \varphi - \psi)$ run through all of $\mathbb{R}$ as χ runs through V.

$\Rightarrow$ The spectrum of $W(\varphi - \psi)$ is $U(1)$-invariant.

$\Rightarrow$ $\sigma_{\mathcal{A}}(W(\varphi - \psi)) \subseteq S^1$ but in fact equal.

$\quad$ in general with $W(\varphi)$ unitary

$\Rightarrow$ $\sigma_{\mathcal{A}}\left(\exp\left\{-\frac{i}{2} \sigma(\varphi, \psi)\right\} W(\varphi - \psi)\right) = S^1$ $\Rightarrow$

$\sigma_{\mathcal{A}}\left(\exp\left(+\frac{i}{2} \sigma(\psi, \varphi)\right) W(\varphi - \psi) - 1_{\mathcal{A}}\right)$ is the unit circle centered at $1_{\mathcal{A}}$. Therefore, $\| \exp\left\{\frac{i}{2} \sigma(\psi, \varphi)\right\} W(\varphi - \psi) - 1_{\mathcal{A}} \|$

$$= \rho\left(\exp\left\{\frac{i}{2} \sigma(\psi, \varphi)\right\} W(\varphi - \psi) - 1_{\mathcal{A}}\right)$$

$$= 2.$$

Now, $W(\varphi) - W(\psi) = W(\psi)\left(W(\psi)^* W(\varphi) - 1_{\mathcal{A}}\right) = W(\psi)\left(\exp\left(\frac{i}{2} \sigma(\psi, \varphi)\right) W(\varphi - \psi) - 1_{\mathcal{A}}\right)$

$\Rightarrow \| W(\varphi) - W(\psi) \|^2 = \| \left(\exp\left(\frac{i}{2} \sigma(\psi, \varphi)\right) W(\varphi - \psi) - 1_{\mathcal{A}}\right)^* \underline{W(\psi)^*}$

$$= \| \exp\{ \tfrac{i}{2} \sigma(\psi, \varphi)\} W(\varphi - \psi) - 1_A \|$$

$$= \rho_A^2 \left(\exp\left(\tfrac{i}{2} \sigma(\psi, \varphi)\right) W(\varphi-\psi) - 1_A \right) = 4.$$

~~proof~~

(3) Since $\sigma_A(W(\varphi)) = S^1$ and $\| W(\varphi) - W(\psi) \| = 2$ for any $\varphi, \psi \in V$, $\varphi \neq \psi$ the balls of radius 1 centered at $W(\varphi)$ form an uncountable collection of mutually disjoint open sets.

(4) Let $\varphi_a \in V$, $a \in I(n) \subset \mathbb{N}$ be pairwise different. Let $\sum_{a \in I(n)} \alpha_a W(\varphi_a) = 0$.

We show that $\alpha_a = 0$ for all $a \in I(n)$ by induction on n.

$n=1$ trivial. W.l.o.g. assume $\alpha_n \neq 0$ hence

$$W(\varphi_n) = \sum_{a \in I(n-1)} -\frac{\alpha_a}{\alpha_n} W(\varphi_a). \quad \text{Therefore} \quad 1_A = W(\varphi_n)^* W(\varphi_n) = W(-\varphi_n) W(\varphi_n)$$

$$\rightsquigarrow 1_A = \sum_{a \in I(n-1)} -\frac{\alpha_a}{\alpha_n} W(-\varphi_n) W(\varphi_a) = \sum_{a \in I(n-1)} \underbrace{-\frac{\alpha_a}{\alpha_n} \exp\left(-\tfrac{i}{2} \sigma(-\varphi_n, \varphi_a)\right)}_{\beta_a} W(\varphi_a - \varphi_n)$$

$$= \sum_{a \in I(n-1)} \beta_a \varphi(\varphi_a - \varphi_n)$$

For an arbitrary $\psi \in V$ we obtain

$$1_A = W(\psi) 1_A W(-\psi) = \sum_{a \in I(n-1)} \beta_a W(\psi) W(\varphi_a - \varphi_n) W(-\psi)$$

$$= \sum_{a \in I(n-1)} \beta_a \exp\left(-i \sigma(\psi, \varphi_a - \varphi_n)\right) W(\varphi_a - \varphi_n)$$

$$\rightsquigarrow \beta_a \exp\left(-i\sigma(\psi, \varphi_a - \varphi_n)\right) = \beta_a \quad \forall a \in I(n-1)$$

Since $\beta_a \neq 0 \rightsquigarrow \sigma(\psi, \varphi_a - \varphi_n) = 0$ but σ is non degenerate.

A contradiction $\boxtimes$.

<u>Remark</u>: 1) Let (A, W) a Weyl system of a symplectic vector space (V, σ).

The linear span of $W(\varphi)$ $(\varphi \in V)$, denoted by $\mathrm{span}(W(V)) \subset A$ is closed under multiplication and under $*$.

Let $(\tilde{A}, \tilde{W})$ another Weyl system of (V, σ). There is a unique linear map

$$\pi : \mathrm{span}(W(V)) \longrightarrow \mathrm{span}(\tilde{W}(V))$$

$$\pi(W(\varphi)) = \tilde{W}(\varphi)$$

In fact, Π is a $*$-isomorphism.

$$\begin{tikzcd} & \mathrm{span}(\tilde{W}(V)) \subset \mathcal{A} \\ V \arrow[ur, "\tilde{W}"] \arrow[dr, "W"'] \arrow[r, phantom, "\#"] & \\ & \mathrm{span}(W(V)) \subset \mathcal{A}. \arrow[uu, "\Pi"'] \end{tikzcd}$$

2) On span $(W(V))$ we define the norm:

$$\left\| \sum_{\varphi \in V} c_\varphi W(\varphi) \right\|_1 := \sum_{\varphi \in V} |c_\varphi|_{\mathbb{C}} \quad \leftarrow \text{Not a } C^*\text{-norm.}$$

But for every C^*-norm $\|\cdot\|_0$ on span$(W(V))$:

$$\|\underset{\in \mathrm{span}(W(V))}{a}\|_0 = \left\| \sum_{\varphi \in V} c_\varphi W(\varphi) \right\|_0 \leq \sum_{\varphi \in V} |c_\varphi|_{\mathbb{C}} \|W(\varphi)\|_0 \overset{\text{unitarity}}{\leq} \|a\|_1$$

<u>Lemma</u>: Let $(\mathcal{A}, W)$ be a Weyl system of a symplectic vector space (V, Ω).

Then $\|a\|_{max} := \sup\{ \|a\|_0 : \|\cdot\|_0 \text{ is a } C^*\text{-norm on span}(W(V))\}$ defines a C^*-norm on span$(W(V))$.

<u>Proof</u>:

Triangle inequality:

$$\|a+b\|_{max} = \sup\{\|a+b\|_0 : \|\cdot\|_0 \text{ is a } C^*\text{-norm on span}(W(V))\}$$
$$\leq \sup\{\|a\|_0 + \|b\|_0 : \ldots\}$$
$$\leq \sup\{\|a\|_0 : \ldots\} + \sup\{\|b\|_0 : \ldots\} \leq \|a\|_{max} + \|b\|_{max}.$$

<u>Def</u>: A Weyl system $(\mathcal{A}, W)$ of a symplectic space (V, Ω) is called a <u>CCR-representation</u> of (V, Ω) if $\mathcal{A}$ is generated as a C^*-algebra by the elements $W(\varphi)$, $\varphi \in V$. In this we call $\mathcal{A}$ a <u>CCR-algebra</u> of (V, Ω).

<u>Theorem</u>: Let (V, Ω) be a symplectic vector space and $(\mathcal{A}_1, W_1)$, $(\mathcal{A}_2, W_2)$ two CCR-reps of (V, Ω).

There is a unique $*$-isomorphism $\Pi: \mathcal{A}_1 \to \mathcal{A}_2$ s.t. the diagram commutes

$$\begin{tikzcd} & \mathcal{A}_1 \arrow[dd, "\Pi"] \\ V \arrow[ur, "W_1"] \arrow[dr, "W_2"'] \arrow[r, phantom, "\#"] & \\ & \mathcal{A}_2 \end{tikzcd}$$

CCR(V, Ω)

Let $S : V_1 \to V_2$ a symplectic linear map, i.e.

$$\sigma_2(S\varphi, S\psi) = \sigma(\varphi, \psi) \quad \forall\, \varphi, \psi \in V,$$

then there exists a unique injective $*$-morphism

$$CCR(S) = CCR(V_1, \sigma_1) \to CCR(V_2, \sigma_2) \quad \text{s.t.}$$

$$\begin{tikzcd} V_1 \arrow[r, "S"] \arrow[d, "w_1"'] & V_2 \arrow[d, "w_2"] \\ CCR(V_1, \sigma_1) \arrow[r, "CCR(S)"'] & CCR(V_2, \sigma_2) \end{tikzcd}$$

Remark: Because of unicity of the map,
$$\begin{cases} CCR(id_V) = id_{CCR(V,\sigma)} \\ CCR(S_2 \circ S_1) = CCR(S_2) \circ CCR(S_1) \end{cases}$$

We constructed a functor $CCR : \underbrace{Symp\,Vec \to C^*\text{-}Alg}_{\text{categories}}$

Definition: The category $\underline{Glob\,Hyp}$ has objects triples (M, E, P) where M is a g.h.s.t., $E \to M$ a real vector bundle with nondegenerate inner product, and P is a formally self adjoint normally hyperbolic operator acting on sections in E.

Let (M_1, E_1, P_1) and (M_2, E_2, P_2) two objects in $Glob\,Hyp$.
A morphism $(M_1, E_1, P_1) \to (M_2, E_2, P_2)$ in this category is a pair (f, F) where $f : M_1 \to M_2$ is a time-orientation preserving isometric embedding so that $f(M_1) \subset M_2$ is a causally compatible open subset.
Furthermore, $F : E_1 \to E_2$ is a vector bundle homeomorphism over f which is fiberwise an isometry.
The diagram

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\#"] & E_2 \arrow[d] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$

Moreover, F has to preserve the normally hyperbolic op i.e.

$$\begin{tikzcd} \Gamma_0(M_1, E_1) \arrow[r, "P_1"] \arrow[d, "ext"'] \arrow[dr, phantom, "K"] & \Gamma_0(M_1, E_1) \arrow[d, "ext"] \\ \Gamma_0(M_2, E_2) \arrow[r, "P_2"] & \Gamma_0(M_2, E_2) \end{tikzcd}$$

where $ext(\varphi)$ denotes the extension of

$$F \circ \varphi \circ f^{-1} \text{ to all of } M_2 \text{ by } 0.$$

Def: Let Lor Fund denote the category whose objects are five triplets (M, E, P, G^+, G^-) where M is a time oriented lorentzian mfd, E a real vector bundle over M with non degenerate inner product, P is a formally self adjoint op. normally hyperbolic op acting on sections on E, $G^{+/-}$ its advanced/retarded green op for P.

Let $X_1 := (M_1, E_1, P_1, G_1^{+/-})$ and $X_2 := (M_2, E_2, P_2, G_2^{+/-})$ two objects of Lor Fund.
If M_1 is not hyperbolic (globally), then we let the set of morphisms from X_1 to X_2 be empty unless $X_2 = X_1$ in which case we put $Mor(X_1, X_2) := \{(id_{M_1}, id_{E_1})\}$.

Remark: If M_1 is globally hyperbolic, then $Mor(X_1, X_2)$ consists of all pairs (f, F) with the same properties as those of the morphism in $Glob\,Hyp$

$$\begin{array}{ccc} \Gamma_0(M_1, E_1) & \xrightarrow{ext} & \Gamma_0(M_2, E_2) \\ {\scriptstyle G_1^{+/-}} \big\downarrow & & \big\downarrow {\scriptstyle G_2^{+/-}} \\ C^\infty(M_1, E_1) & \longleftarrow & C^\infty(M_2, E_2) \end{array}$$

where $ext := F \circ \varphi \circ f^{-1} \in \Gamma_0(f(M_1), E_2)$

$res = F^{-1} \circ \varphi \circ f \in \Gamma_0(M_1, F^{-1}(E_2))$ ← restriction of E_1 to $F^{-1}(E_2)$.

Def: we define a functor

$$SOLVE : Glob\,Hyp \to Lor\,Fund,$$

$$SOLVE(M, E, P) := (M, E, P, G^{+/-}),$$

$SOLVE(f, F) := (f, F)$, for all triples (M, E, P) in $Glob\,Hyp$ and all morphisms $(M_1, E_1, P_1) \to (M_2, E_2, P_2)$ in $Glob\,Hyp$.

Let $(M, E, P, G^{\pm})$ ...

using $G = G^{+} - G^{-} : \Gamma_0(M,E) \to C^\infty(M,E)$ we define

$$\tilde{\Omega} : \Gamma_0(M,E) \times \Gamma_0(M,E) \to \mathbb{R}$$
$$(\varphi, \psi) \mapsto \tilde{\Omega}(\varphi,\psi) := \int_M \mathrm{vol}_M \langle G\varphi, \psi \rangle_E$$

<u>Skew symmetry</u>: $\int_M \mathrm{vol}_M \langle G^{+/-}\varphi, \psi \rangle = \int_M \mathrm{vol}_M \langle \varphi, G^{-/+}\psi \rangle$

However, $(\Gamma_0(M,E), \tilde{\Omega})$ is <u>not</u> a symplectic vector space!

Since $\langle G\varphi, \psi\rangle = 0 \ \forall \psi \in \Gamma(M,E) \Rightarrow \varphi \in \ker(G) \neq \{0\}$

In fact, $\ker G = P(\Gamma_0(M,E))$

$$0 \longrightarrow \Gamma_0(M,E) \xrightarrow{P} \boxed{\Gamma_0(M,E) \xrightarrow{G}} C^\infty_{sc}(M,E) \xrightarrow{P} C^\infty_{sc}(M,E)$$

is a complex exact.

On the quotient $\left(V(M,E,G) := \Gamma_0(M,E)/\ker(G), \ \Omega\right)$ induced by the ~~presymplectic~~ form $\tilde{\Omega}$.

<u>Remark</u>: Let $X_a := (M_a, E_a, P_a, G_a^{+/-})$, $a \in I(2)$ be objects in the category LorFund, and $(f, F) \in \mathrm{Mor}(X_1, X_2)$ be a morphism.

Then $\mathrm{ext}: \Gamma_0(M_1,E_1) \to \Gamma_0(M_2, E_2)$ maps $\ker(G_1)$ to $\ker(G_2)$ and induces a symplectic map $V(M_1, E_1, G_1) \xrightarrow{\text{symplectic}} V(M_2, E_2, G_2)$

<u>Remark</u> Functor $\mathrm{SYMPL} : \mathrm{LorFund} \to \mathrm{SymplVec}$,

$\mathrm{Sympl}(M, E, P, G^{+/-}) := V(M, E, G := G^{+} - G^{-})$

Morphism (f, F) is mapped to the symplectic map induced by ext_0

$$\begin{tikzcd}
\mathrm{GlobHyp} \arrow[d, "\mathrm{SOLVE}"'] & \mathrm{SymplVec} \arrow[d] \\
\mathrm{LorFund} \arrow[ur, "\mathrm{SYMPL}"] & C^*\text{-}\mathrm{Alg}
\end{tikzcd}$$

!

Def: A set $\mathcal{J}$ is called a <u>directed set with weak orthogonality relation</u> if it carries a partial order $\leq$ and a symmetric relation $\perp$ between its elements such that

1) for all $\alpha, \beta \in \mathcal{J}$ there exists a $\gamma \in \mathcal{J}$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$
2) for every $\alpha \in \mathcal{J}$ there is a $\beta \in \mathcal{J}$ with $\alpha \perp \beta$,
3) if $\alpha \leq \beta$ and $\beta \perp \gamma$ then $\alpha \perp \gamma$.

If in addition

(4) if $\alpha \perp \beta$, $\alpha \perp \gamma$ then there is $\delta \in \mathcal{J}$ s.t. $\beta \leq \delta$, $\gamma \leq \delta$ and $\alpha \perp \delta$, then $\mathcal{J}$ is called a <u>directed set with orthogonality relations</u>

Def: A <u>quasi-local C^*-algebra</u> is a pair $(\mathcal{A}, \{\mathcal{A}_\alpha\}_{\alpha \in \mathcal{J}})$ of a C^*-algebra and a family $\{\mathcal{A}_\alpha\}_{\alpha \in \mathcal{J}}$ of C^*-subalgebras, where $\mathcal{J}$ is a directed set with orthogonality relation s.t.

1) $\mathcal{A}_\alpha \subset \mathcal{A}_\beta$ whenever $\alpha \leq \beta$.
2) $\mathcal{A} = \overline{\bigcup_{\alpha \in \mathcal{J}} \mathcal{A}_\alpha}$
3) The algebras $\mathcal{A}_\alpha$ have a common unit.
4) If $\alpha \perp \beta$, the commutators of $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are trivial.

Def: A morphism between two quasi-local C^*-algebras $(\mathcal{A}, \{\mathcal{A}_\alpha\}_{\alpha \in \mathcal{J}})$ and $(\mathcal{M}, \{\mathcal{M}_\alpha\}_{\alpha \in \mathcal{L}})$ is a pair (φ, Φ) where $\Phi: \mathcal{A} \to \mathcal{M}$ is a unit preserving C^* morphism and $\varphi: \mathcal{J} \to \mathcal{L}$ is a map s.t.

1) φ is monotonic,
2) φ preserves $\perp$
3) $\Phi(\mathcal{A}_\alpha) \subset \mathcal{M}_{\varphi(\alpha)}$ for all $\alpha \in \mathcal{J}$

Remark: $\begin{cases} \text{Objects: quasi-local } C^*\text{-algebras} \\ \text{Morphisms: } (\varphi, \Phi) \end{cases}$

category: $\mathrm{QuasiLocAlg}_{(\text{weak})}$ of quasi-local C^*-alg.

Goal: Associate to any object $(M, E, P, \sigma^{\pm})$ in the category LocFund a weak quasi-local C^*-algebra.

$\mathcal{J} := \{ U \subset M : U \text{ is open, relatively compact, causally compatible, globally hyperbolic} \}$

$\cup \{\varphi, M\}$.

Partial ordering: inclusion $\subset$ and ...

orthogonality: $\mathcal{U} \perp \mathcal{U}' :\Leftrightarrow J_M(\overline{\mathcal{U}}) \cap \overline{\mathcal{U}'} = \phi$

causal past and future

Lemma: The set $\mathcal{J}$ defined above is a directed set with weak orthogonality relation.

Lemma: Let M be g.h.s.t. then $\mathcal{J}$ is a directed ~~set~~ with orthogonality relation $\perp$.

object: $(\mathcal{U}, E|_{\mathcal{U}}, P, G^{+/-}_{\mathcal{U}})$ for each $\mathcal{U} \in \mathcal{J}$, $\mathcal{U} \neq \phi$.

For $\mathcal{U}_1 \subset \mathcal{U}_2$ the inclusion induces a morphism $\iota_{\mathcal{U}_2, \mathcal{U}_1}$ in the category $\mathrm{Lor\,Fund}$, given by the embeddings

$$\mathcal{U}_1 \hookrightarrow \mathcal{U}_2 \quad \text{and} \quad E|_{\mathcal{U}_1} \hookrightarrow E|_{\mathcal{U}_2}$$

Let $\alpha_{\mathcal{U}_2, \mathcal{U}_1}$ denote the morphism

$$(\mathrm{CCR} \circ \mathrm{SYMPL})(\iota_{\mathcal{U}_2, \mathcal{U}_1}) \quad \text{in } C^*\text{-}\mathrm{Alg}$$

$$(V_{\mathcal{U}}, \Omega_{\mathcal{U}}) := \mathrm{SYMPL}(\mathcal{U}, E|_{\mathcal{U}}, P, G^{+/-}_{\mathcal{U}})$$

$$A_{\mathcal{U}} := \alpha_{M,\mathcal{U}}(\mathrm{CCR}(V_{\mathcal{U}}, \Omega_{\mathcal{U}})).$$

notice $A_{\mathcal{U}}$ is a C^*-subalgebra of $\mathrm{CCR}(V_{\mathcal{U}}, \Omega_{\mathcal{U}})$

$$A_M := \bigcup_{\mathcal{U} \in \mathcal{J}} A_{\mathcal{U}}$$

Lemma: Let $(M, E, P, G^{+/-})$ be an object in $\mathrm{Lor\,Fund}$.

Then $(A_M, \{A_{\mathcal{U}}\}_{\mathcal{U} \in \mathcal{J}})$ is a weak quasilocal C^*-algebra.

Proof: (2), (3) ✓

(1) by functorality we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{CCR}(V_{\mathcal{U}}, \Omega_{\mathcal{U}}) \arrow[r, "\alpha_{M,\mathcal{U}}"] \arrow[d, "\alpha_{\mathcal{U}',\mathcal{U}}"'] \arrow[dr, phantom, "\#"] & \mathrm{CCR}(V_M, \Omega_M) \\
\mathrm{CCR}(V_{\mathcal{U}'}, \Omega_{\mathcal{U}'}) \arrow[ur, "\alpha_{M,\mathcal{U}'}"'] &
\end{tikzcd}$$

Since $\alpha_{\mathcal{U}',\mathcal{U}}$ is injective, $A_{\mathcal{U}} \subset A_{\mathcal{U}'}$

(4) Let $\mathcal{U}, \mathcal{U}' \in \mathcal{J}$ be causally indep (i.e. $\mathcal{U} \perp \mathcal{U}'$)

let $\varphi \in \Gamma_0(\mathcal{U}, E)$ and $\psi \in \Gamma_0(\mathcal{U}', E)$

From $\mathrm{supp}(G\varphi) \subset J_{M_1}(\mathcal{U})$ it follows that $\mathrm{supp}(G\varphi) \cap \mathrm{supp}\,\psi = \phi$

Hence $(G\varphi, \psi)_{M_1} = 0$.

For the symplectic form Ω on $\Gamma_0(M, E)/_{\ker(G)}$ this implies $\Omega(\varphi, \psi) = 0$

This gives property (3) of a Weyl-system:

$$W(\varphi)\, W(\psi) = W(\varphi + \psi) = W(\psi)\, W(\varphi).$$

then $[A_{\mathcal{U}}, A_{\mathcal{U}'}] = 0$.

<u>Remark:</u> we associate a morphism in QuasiLoc A_{LG} weak) to any morphism on $\mathrm{Lor}\,\mathrm{Fund}$)

$$X_a := (M_a, E_a, P_a, G_a^{+/-}).$$

If M_a is g.h.s.t., then $\mathrm{Mor}(X_1, X_2)$ consists of all pairs (f, F) where $f: M_1 \to M_2$ is a time orientation preserving isometric embedding s.t. $f(M_1) \subset M_2$ is causally compatible open subset.

$F: E_1 \to E_2$ is a vector bundle homomorphism over f which is fiberwise an isometry. Let J_a and let $(A_{M_1}, \{A_{\mathcal{U}}\}_{\mathcal{U} \in J_1})$ and $(\mathfrak{m}_{M_1}, \{A_{\mathcal{U}}\}_{\mathcal{U} \in J_2})$ be the corresponding weak quasilocal C^*-algebra

Consider the morphism $\Phi = CCR \circ \mathrm{SymPL}(f, F): CCR(V_{M_1}, \Omega_{M_1}) \to CCR(V_{M_2}, \Omega_{M_2})$

$$\begin{tikzcd} \mathcal{U}_1 \arrow[r, hook] \arrow[d, "f|_{\mathcal{U}_1}"'] \arrow[dr, phantom, "\not\exists"] & \mathcal{U}_2 \arrow[d, "f|_{\mathcal{U}_2}"] \\ f(\mathcal{U}_1) \arrow[r, hook] & M_2 \end{tikzcd}$$

we find: $\Phi(A_{\mathcal{U}_1}) \doteq \mathfrak{m}_{f(\mathcal{U}_1)}$

This implies $\Phi(A_{M_1}) \subset \mathfrak{m}_{M_2}$. Therefore $(\varphi, \Phi|_{A_{M_1}})$ is a morphism in QuasLoc A_{LG}.

<u>Theorem</u> The assignment $(M, E, P, G^{+/-}) \to (A_M, \{A_{\mathcal{U}}\}_{\mathcal{U} \in J})$ and $(f, F) \to (\varphi, \Phi|_{A_M})$ yields a functor

$$\mathrm{Lor\,Fund} \to \mathrm{QuasiLoc}\,A_{LG}\ \mathrm{weak}$$

<u>Corollary:</u>

$$\begin{tikzcd} \mathrm{GlobHyp} \arrow[r, "\mathrm{SOLVE}"] \arrow[d] & \mathrm{Lor\,Fund} \arrow[d, dash] \\ \mathrm{QuasLoc}\,A_{LG} \arrow[r, "\mathrm{Inclusion}"] & \mathrm{QuasLoc}\,A_{LG}\ \mathrm{weak} \end{tikzcd}$$

denote by $(\mathcal{A}_M, \{\mathcal{A}_U\}_{U\in J})$ the corresponding quasi-local C^*-algebra.

Then $\mathcal{A}_M = CCR \circ SYMPL \circ SOLVE(M, E, P)$

Proof: Let $\bar{\mathcal{A}} := CCR \circ SYMPL \circ SOLVE(M, E, P)$.

By def. $\mathcal{A}_M \subset \bar{\mathcal{A}}$.

For the other, $(M, E, P, G^\pm) := SOLVE(M, E, P)$. Then

$SYMPL(M, E, P, G^\pm)$ is $V_M = \Gamma_0(M, E)/\ker(G)$

$\bar{\mathcal{A}}$ is generated by $\mathcal{E} = \{W([\varphi]) : \varphi \in \Gamma_0(M, E)\}$

and since $W([\varphi]) \in \mathcal{A}_U$ hence $\mathcal{E} \subset \bigcup_{U\in J} \mathcal{A}_U \subset \mathcal{A}_M$