Lectures SUSY (Solutions)

Poincaré algebra $\quad \eta = \mathrm{diag}(1,-1,-1,\dots)$

Lorentz transf $\rightsquigarrow$ $O(1,3)$ - group

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

(Metric invariant $\quad \Lambda^T \eta \Lambda = \eta$)

or, in coordinates $\quad (\Lambda^T)_\mu{}^\rho \eta_{\rho\sigma} \Lambda^\sigma{}_\nu = \eta_{\mu\nu}$

The Lorentz group has 6 generators

($4\cdot 4$ matrix $\rightarrow$ $4\cdot 4 - 1 = 15$ $\rightarrow$ $\frac{4\cdot 3}{2} = 6$
$\uparrow$ $GL(\mathbb{R},4)$ $\quad SL(\mathbb{R},4)$ $\quad O(4)$)

which are $\begin{cases} 3 \text{ boosts} & \leftarrow K_i = L_{0i} \\ 3 \text{ rotations} & \leftarrow R_i = \varepsilon_{ijk} L^{jk} \end{cases}$

$$\Lambda = \exp(-\omega_{\sigma\rho} L^{\sigma\rho}) = \mathbb{1} - \frac{1}{2}\omega_{[\mu\nu]} L^{[\mu\nu]}$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = -i (L_{\mu\rho}\eta_{\nu\sigma} + L_{\nu\sigma}\eta_{\mu\rho} - L_{\nu\rho}\eta_{\mu\sigma} - L_{\mu\sigma}\eta_{\nu\rho})$$

Poincaré: Add translations $P_\mu = -i\partial_\mu$

$$\boxed{[P_\mu, P_\nu], \quad [L_{\mu\nu}, P_\rho] = i(P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho})}$$

Reps: The Poincaré algebra has 2 Casimirs (commute with all others)

$$[C_i, P_\mu] = 0 = [C_i, L_{\mu\nu}]$$

$$\{C_1 = P^\mu P_\mu, \quad C_2 = W^\mu W_\mu$$

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} P_\nu L_{\rho\sigma}$$

$$[W^\mu, P^\nu] = 0 \qquad [W^\mu, W^\rho] = i\varepsilon^{\mu\nu\rho\sigma} W_{\rho}...$$

The little group:

massive

$$P_\mu P^\mu = m^2$$

rest frame: $P_\mu = (m, \vec{0})$

little group is SO(3)

massless

$$P^\mu P_\mu = 0$$

$$P_\mu = (E, 0, 0, E)$$

little group is SO(2)

$W^2 = -m^2 S(S+1)$, where L_2 eigenvalue S

$S(S-1)(\vec{L})^2$

$W^\mu = L_{12} P^\mu$

$\lambda = \hat{p} \cdot \vec{L}$

Definition of helicity.

Different fields in different Lorentz reps under Lorentz transf

<u>Scalar:</u> $\varphi(x) \xrightarrow{L} \varphi(x')$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2 \varphi^2}{2} \qquad [\varphi] = \frac{d-2}{2}$$

$$\left[\left(\frac{\varphi}{L}\right)^2\right] = [m^2 \varphi^2] \qquad \left[\frac{\varphi^2}{L^2}\right] =$$

$$\left[\frac{1}{L^2}\right] = [m] \sim [m^2 \varphi^2] = d = \left[\left(\frac{\varphi}{L}\right)^2\right]$$

$$[\varphi] = \frac{d-2}{2}$$

1 real dof.

<u>Dirac spinor:</u> $\Psi \to \Lambda_R \Psi$ $\qquad \Lambda = \exp\left(-\frac{1}{2} \omega^{\mu\nu} \Sigma_{\mu\nu}\right)$

$$\mathcal{L} = -i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m \bar{\Psi} \Psi + \dots$$

4 complex d.o.f.

$\leadsto \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \leftarrow$ 4 complex off shell $\xrightarrow{\text{Eqtn of motion}}$ 2 complex on shell

$(\not{\partial}\Psi = 0)$

$$[\Psi]_{\mathcal{F}} = [\Psi] = \frac{d-1}{2}$$

$$[\underbrace{\bar{\Psi}\Psi m}_{d-1+1}] = d \checkmark$$

$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \leftarrow$ Clifford algebra

$\Sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$(\gamma^5)^2 = \mathbb{1}$

$[\gamma_5, \Sigma_{\mu\nu}] = 0$, $\gamma_5^\dagger = \gamma_5$, $\{\gamma_\mu, \gamma_5\} = 0$

<u>Projectors:</u> $P_\pm = \frac{1}{2}(1 \pm \gamma_5) \leadsto$ irreps. $\begin{cases} P_+\Psi \leftarrow \text{left handed spinor} \\ P_-\Psi \leftarrow \text{right handed spinor} \end{cases}$

$$\gamma^\mu = \left(\begin{array}{c|c} 0 & \sigma^\mu \\ \hline \bar{\sigma}^\mu & 0 \end{array}\right), \quad \sigma^\mu = (\underset{\text{do}}{\mathbb{1}}, \sigma^i) \qquad \gamma_5 = \left(\begin{array}{c|c} \mathbb{1} & \\ \hline & -\mathbb{1} \end{array}\right)$$

$$\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$$

$$\Sigma^{\mu\nu} = \left(\begin{array}{c|c} \sigma^{\mu\nu} & \\ \hline & \bar{\sigma}^{\mu\nu} \end{array}\right) \qquad \sigma^i = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu}\right)$$
$$\bar{\sigma}^{\mu\nu} = \frac{i}{4}\left(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu}\right)$$

all $\Sigma^{\mu\nu}, \sigma^{\mu\nu}, \bar{\sigma}^{\mu\nu}$ satisfy Lorentz algebra!

<u>Weyl</u>: 2 complex off shell $\begin{pmatrix} \circ \\ \circ \end{pmatrix}$

2 real on shell

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad \begin{matrix} \alpha = 1,2 \\ \dot{\alpha} = 1,2 \end{matrix}$$

How to multiply $\gamma^\mu \Psi \rightsquigarrow$

$$\begin{pmatrix} 0 & \sigma^\mu \\ \hline \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

$$\Rightarrow \sigma^\mu{}_{\alpha\dot{\alpha}} = (\sigma^0, \sigma^i)$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_\alpha \Rightarrow (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (\sigma^0, -\sigma^i)$$

<u>Schematically</u>:

collection of indices

$$\begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix} \qquad \begin{pmatrix} & \dot{\alpha} \\ \dot{\alpha} & \end{pmatrix}$$

Ex.
$$\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^{\nu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\right)$$

$$\bar{\sigma}^{\mu\nu} = \frac{i}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\beta}} - \bar{\sigma}^{\nu}\sigma^{\mu}\right)$$

$\hookrightarrow$ $(\sigma^{\mu\nu})_{\alpha}{}^{\beta}$

$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}$

$$\Rightarrow \boxed{\begin{aligned} \psi_{\alpha} &\to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_{\alpha}{}^{\beta}\psi_{\beta} \\ \bar{\chi}^{\dot{\alpha}} &\to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}} \end{aligned}}$$

weyl transformation concrete.

note that

$$\begin{cases} \sigma^{\mu\nu} = \frac{1}{2i}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} & \text{(self dual)} \\ \bar{\sigma}^{\mu\nu} = -\frac{1}{2i}\varepsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} & \text{(antiself dual)} \end{cases}$$

we want to decompose an antisym tensor d... Self adjoint and ~~adjo~~ anti self adjoint part.

Vector field

$A_\mu \to A_\mu + \partial_\mu \lambda$
gauge

A_μ d.o.f

off shell $\quad 4 - 1 = 3$ ~~vector~~ real

on shell $\quad 3 - 1 = 2$ real

e.o.m

$$\frac{1}{g^2_{YM}} \int dx \, (\partial_\mu A)^2 \qquad [F_{\mu\nu}] = 2$$

$$[D_\mu, D_\nu] \quad \Rightarrow [g_{YM}^{-2}] = 0 \; 4d$$

Majorana rep

Reality condition on the Dirac

$$\Psi^c = \Psi \qquad (\text{or } \Psi^* = B\Psi)$$

charge conjugation

$$\Psi^c = C\bar{\Psi}^T = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix} \sim \boxed{\Psi = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix}}$$

Majorana

$C = i\gamma^0\gamma^2$

$\bar{\Psi} = \Psi^\dagger \gamma^0$

Algebras

$$so(1,3) \simeq su(2)_L \times su(2)_R$$

recall $\quad R_i = \varepsilon_{ijk} L^{jk}, \quad K_i = L_{0i}$

define $\quad J_i^\pm = \frac{1}{2}(R_i \pm i K_i) \sim (J_i^\pm)^\dagger = J_i^\mp$

$$[J_i^\pm, J_j^\pm] = i\varepsilon_{ijk} J_k^\pm \quad \leftarrow \text{two } su(2) \text{ algebras}$$

$[J_i^\pm, J_j^\mp] = 0$ $\leftarrow$ two $SU(2)$ ~~but~~ related

$\boxed{\text{by } (J_i^\pm)^\dagger = J_i^\mp}$

<u>Groups</u>

$SL(2,\mathbb{C}) \simeq SO(1,3)$

homo

$\forall M_i \in SL(2,\mathbb{C}) \rightsquigarrow \Lambda(M_1 M_2) = \Lambda(M_1)\Lambda(M_2) \in SO(1,3)$

<u>Desc</u>: $X_{\alpha\dot\beta} = x_\mu \sigma^\mu_{\alpha\dot\beta} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$

This object transforms as (in $SL(2,\mathbb{C})$)

$$X'_{\alpha\dot\beta} = M_\alpha{}^b X_{b\dot\alpha} \bar{M}^{\dot\alpha}{}_{\dot\beta}$$

$\bar{M} = M^\dagger \in SL($

$\rightsquigarrow \det(X'_{\alpha\dot\beta}) = \det(X_{b\dot\alpha})$

is invariant under $SL(2,\mathbb{C})$

$\det(X_{\alpha\dot\beta}) = x_0^2 - x_1^2 - x_2^2 - x_3^2 = x^\mu x_\mu$

Invariant of Lorentz group.

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}}$$

$$\mathrm{tr}(\sigma^{\mu}\bar{\sigma}^{\nu}) = 2\eta^{\mu\nu} \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} \rightarrow \left[\sigma^{\mu}\bar{\sigma}^{\nu} + \sigma^{\nu}\bar{\sigma}^{\mu}\right]_{\alpha}{}^{\beta} = 2\eta^{\mu\nu}\delta_{\alpha}{}^{\beta}$$

$$\gamma^{\mu}\gamma_{\mu} = 4\mathbb{1}_{4\times4} \rightarrow \sigma^{\mu}_{\alpha\dot{\beta}}\bar{\sigma}_{\mu}{}^{\dot{\gamma}\delta} = 2\delta_{\alpha}{}^{\delta}\delta^{\dot{\gamma}}_{\dot{\beta}}$$

<u>now, we can derive:</u>

$$\frac{1}{2}\mathrm{tr}(X\bar{\sigma}^{\mu}) = \frac{1}{2}x_{\nu}\underbrace{\mathrm{tr}(\sigma^{\nu}\bar{\sigma}^{\mu})}_{2\eta^{\nu\mu}} = x^{\mu}$$

Hence,

$$x^{\mu} \rightarrow \Lambda^{\mu}{}_{\nu}x^{\nu} \qquad X \rightarrow M X \bar{M}$$

$$\rightsquigarrow \frac{1}{2}\mathrm{tr}(M \underset{x_{\mu}\sigma^{\mu}}{X} \bar{M}\,\bar{\sigma}^{\mu}) = \underbrace{\frac{1}{2}\mathrm{tr}(M\sigma_{\nu}\bar{M}\bar{\sigma}^{\mu})}_{\Lambda(M)}x^{\nu}$$

$$\hookrightarrow \boxed{\Lambda^{\mu}{}_{\nu}(M) = \frac{1}{2}\mathrm{tr}(\bar{\sigma}^{\mu}M\sigma_{\nu}M^{\dagger})}$$

M and −M sent to the same Λ.

$SL(2,\mathbb{C})$ spinors

$$X_{\alpha\dot\beta} = \psi_\alpha \bar\chi_{\dot\beta}$$

$$\sigma^{\mu\nu\dagger} = \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)^\dagger = -\frac{i}{4}(\bar\sigma^\nu\sigma^\mu - \bar\sigma^\mu\sigma^\nu)$$

$$\boxed{(\sigma^{\mu\nu})^\dagger = \bar\sigma^{\mu\nu}}$$

The <u>fund rep</u> of $SL(2,\mathbb{C})$ $\quad SU(2)_L \times SU(2)_R$

$$\psi_\alpha \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^b \psi_\alpha \quad \left(\frac{1}{2},0\right)$$

left hand spinor

$$\psi_\alpha \to M_\alpha{}^\beta \psi_\alpha, \quad M \in SL(2,\mathbb{C})$$

<u>The conjugate rep</u>

$$\bar\chi^{\dot\alpha} \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar\sigma^{\mu\nu}\right)^{\dot\alpha}{}_{\dot\beta}\bar\chi^{\dot\beta} \quad \left(0,\frac{1}{2}\right)$$

right handed

$$\bar\chi^{\dot\alpha} \to \bar M^{\dot\alpha}{}_{\dot\beta}\bar\chi^{\dot\beta}, \quad \bar M = M^* \text{ of } SL(2,\mathbb{C})$$

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma \quad \sim$$

$$M_\alpha{}^\beta M_\gamma{}^\delta \varepsilon_{\beta\delta} = \det(M)\varepsilon_{\alpha\gamma}$$
$$= \varepsilon_{\alpha\gamma}$$

$\Rightarrow$ $\varepsilon_{\alpha\beta}$ is an invariant tensor of $SL(2,\mathbb{C})$.

$\Rightarrow$ use it to raise and lower indices.

$X^\mu X_\mu$ = invariant under $SO(1,3)$. Similarly: $\psi^\alpha \psi_\alpha$ = invariant under $SL(2,\mathbb{C})$

$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta$, $\psi^\alpha \to \psi^\beta (M^{-1})_\beta{}^\alpha$

<u>Similarly:</u> $\bar{\chi}_{\dot\alpha}\bar{\chi}^{\dot\alpha}$ = invariant.

$\bar{\chi}_{\dot\alpha} \to \bar{\chi}_{\dot\beta}(\bar{M}^{-1})^{\dot\beta}{}_{\dot\alpha}$

$(\psi_\alpha)^* = \bar{\psi}^{\dot\alpha}$ and $\bar{\psi}_{\dot\alpha} = (\psi_\alpha)^*$

<u>Recall:</u> $\psi_2\psi_1 = -\psi_1\psi_2$.

$\cancel{\psi\chi = \chi}$ $\psi\chi = \psi^\alpha\chi_\alpha = \varepsilon^{\alpha\beta}\psi_\beta\chi_\alpha$
$$= -\varepsilon^{\beta\alpha}\psi_\beta\chi_\alpha = +\varepsilon^{\beta\alpha}\chi_\alpha\psi_\beta$$

$$= \chi^\beta \psi_\beta = \chi\psi \Rightarrow \underline{\chi\psi = \psi\chi}$$

$$\bar{\chi}\bar{\psi} = \bar{\chi}_{\dot\alpha}\bar{\psi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar{\chi}_{\dot\alpha}\bar{\psi}_{\dot\beta}$$

$$= -\epsilon^{\dot\beta\dot\alpha}\bar{\chi}_{\dot\alpha}\bar{\psi}_{\dot\beta} = \epsilon^{\dot\beta\dot\alpha}\bar{\psi}_{\dot\beta}\bar{\chi}_{\dot\alpha}$$

$$\boxed{\bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi}}$$

$$(\psi\cdot\chi)^\dagger = (\psi^\alpha\chi_\alpha)^\dagger = \chi_\alpha^\dagger(\psi^\alpha)^\dagger = \bar{\chi}_{\dot\alpha}\bar{\psi}^{\dot\alpha}$$

$$= \bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi}$$

- $(\chi\sigma^\mu\bar{\psi})^\dagger = \psi\sigma^\mu\bar{\chi}$

- $\cancel{\bar{\chi}\sigma^\mu}$ $(\chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar{\psi}^{\dot\alpha})^\dagger = \psi^\alpha(\cancel{\sigma}^\mu_{\alpha\dot\alpha}$
$= \psi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar{\chi}$

- $\chi\sigma^\mu\bar{\psi} = \chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar{\psi}^{\dot\alpha} = -\bar{\psi}_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha}$
$= -\bar{\psi}^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\chi^\alpha$

- $\chi\sigma^\mu\bar{\psi} = \chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar{\psi}^{\dot\alpha}$, $\bar{\chi}\bar{\sigma}^\mu\psi = \bar{\chi}_{\dot\alpha}\bar{\sigma}$

we can generate all higher reps of

$SL(2,\mathbb{C})$ / Lorentz as products of $(\frac{1}{2},0)$ and

$(0,\frac{1}{2})$.

- $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$ = vector of Lorentz

$$\psi_\alpha \quad \bar{\chi}_{\dot\alpha} = \frac{1}{2}(\psi\sigma_\mu\bar{\chi})\sigma^\mu{}_{\alpha\dot\alpha}$$

- $(\frac{1}{2},0)\otimes(\frac{1}{2},0) = \underset{\text{scalar}}{(0,0)} \oplus \underset{\text{antisymmetric self dual tensor.}}{(1,0)}$

$$\psi_\alpha \chi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}(\psi\chi) + \frac{1}{2}(\psi\sigma_{\mu\nu}\chi)\sigma^{\mu\nu}{}_\alpha{}^\delta\epsilon_{\delta\beta}$$

Can we combine $SU(N_F)$, $SO(1,3)$ in a bigger group? No! (Coleman and Mandula)

Poincaré $\times$ (internal / Flavour)

$$[P_\mu, F] = 0 = [L_{\mu\nu}, F]$$

This using Lie algebra with

$$[T_a, T_b] = i f^c_{ab} T_c$$

← usual commutator

but what about

$$[T_a, T_b] = T_a T_b - (-1)^{F_a F_b} T_b T_a = i f^c_{ab} T_c$$

F is the fermion # $\begin{cases} F=0 & \text{Bosons} \\ F=1 & \text{Fermions.} \end{cases}$

Haag: allow $Q_\alpha, \bar{Q}_\alpha$ and nothing else

Minimal ($N=1$) s-sy algebra

$[L_{\mu\nu}, Q_\alpha] = ?$ (1)

$[Q_\alpha, P_\mu] = ?$ (2)

$\{Q_\alpha, Q_\beta\} = ?$ (3)

$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = ?$ (4)

$[Q_\alpha, F] = ?$ (5)

① $Q_\alpha \in SL(2,\mathbb{C})$ spinor $(\frac{1}{2},0)$

transforms as

$$Q_\alpha' = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta$$

Q_α is an operator that acts $Q_\alpha|\psi\rangle$

$U|\psi\rangle$, $\langle\psi|Q_\alpha|\psi\rangle \to \langle\psi|U^\dagger Q_\alpha U|\psi\rangle$

$$\Rightarrow \quad Q_\alpha' = U^\dagger Q_\alpha U, \quad U = e^{-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}}$$

$$Q_\alpha' = \left(1 - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta$$

$$= \left(1 + \frac{i}{2}\omega_{\mu\nu}\cancel{L^{\mu\nu}}\right) Q_\alpha \left(1 - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\right)$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta = [L^{\mu\nu}, Q_\alpha]$$

$$[Q_\alpha, L^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$$

$$[\bar{Q}^{\dot\alpha}, L^{\mu\nu}] = (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta}$$

(c) $[Q_\alpha, P^\mu] = c(\sigma^\mu)_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha}$

$$\left(\tfrac{1}{2},0\right)\otimes\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \cancel{\left(1,\tfrac{1}{2}\right)}\oplus\left(0,\tfrac{1}{2}\right)$$

Hang NO

use Jacobi

$$0 = [P^\mu,[P^\nu,Q_\alpha]] + [P^\nu,[Q_\alpha,P^\mu]] + [Q_\alpha,\underbrace{[P^\mu,P^\nu]}_{0}]$$

$$= -c(\sigma^\mu)_{\alpha\dot\alpha}[P^\nu,\bar{Q}^{\dot\alpha}] + c\,\sigma^\nu_{\alpha\dot\alpha}[P^\mu,\bar{Q}^{\dot\alpha}]$$

$$= c\,\sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\sigma}Q_\sigma - c\,\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\sigma}Q_\sigma$$

$$= c\underbrace{[\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu]_\alpha{}^\sigma}_{\neq 0}\underbrace{Q_\sigma}_{\neq 0}$$

$$\Rightarrow \boxed{c=0}$$

③ $\{Q_\alpha, Q_\beta\} = c' L_{\mu\nu} \sigma^{\mu\nu}{}_\alpha{}^{\beta'} \varepsilon_{\beta'\beta}$

$(\frac{1}{2},0) \otimes (\frac{1}{2},0) = (1,0) \oplus (0,0)$

Jacoby for $QQP \rightsquigarrow [Q,P]=0$

$[P,\{Q,Q\}]=0$

$\Rightarrow c'=0.$

<u>Minimal susy algebra</u>

$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu$ ④

$(\frac{1}{2},0), (0,\frac{1}{2}) : (\frac{1}{2},\frac{1}{2})$

⑤ $[F,Q]=0$ $\exists$ outer automorphism of the susy algebra

$Q_\alpha \to e^{+i\theta} Q_\alpha$

$\bar{Q}_{\dot\alpha} \to e^{-i\theta} \bar{Q}_{\dot\alpha}$

$\exists$ a $U(1)_R$ symmetry

$U_R^\dagger Q_\alpha U_R = e^{i\theta} Q_\alpha, \quad U_R = e^{-\theta R}$

$$\Rightarrow (1 + i\theta R)\, Q_\alpha\, (1 - i\theta R) = (1 - iq\theta)\, Q_\alpha$$

$$\Rightarrow \quad [R, Q_\alpha] = q\, Q_\alpha$$

$$[R, \bar{Q}_{\dot\alpha}] = -q\, \bar{Q}_{\dot\alpha}$$

Maximal, what happens in extended?

$(N > 1)$

$$Q_\alpha^I \quad \& \quad I = 1, \dots, N \qquad \bar{Q}_{\dot\alpha I}$$

$$[L^{\mu\nu}, Q^I_\alpha] = -\sigma^{\mu\nu}{}_\alpha{}^\beta\, Q^I_\beta$$

$$[L^{\mu\nu}, \bar{Q}^{\dot\alpha}{}_I] = -\bar{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}\, \bar{Q}^{\dot\beta}{}_I$$

$$[Q^I, P_\mu] = 0 = [\bar{Q}, P_\mu]$$

$$\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\} = 2\,\sigma^\mu_{\alpha\dot\beta} P_\mu\, \delta^I_J$$

$$\{Q^I_\alpha, Q^J_\beta\} = \underset{\text{anti}}{\epsilon_{\alpha\beta}}\, \underset{\text{antisymmetric}}{Z^{IJ}} + \cancel{(\sigma^{\mu\nu})_\alpha{}^\gamma \epsilon_{\gamma\beta}} K^{\ldots}$$

symmetric

use Jacobi you find that $[Z, J] = 0$

for all generators.

Now there is a bigger possibility of automorphisms.

$$R^I_{\ J}\, Q^J = Q'^I$$

$$\bar{Q}'_I = R^{*\ J}_{I}\, \bar{Q}_J$$

$RR^\dagger = \mathbb{1} \sim U(N)$ R-symmetry

Susy reps

1) $W^2 = W^\mu W_\mu$ is <u>not</u> a Casimir anymore

$[Q, W^2] \neq 0.$ $\quad W^2 = -m^2 s(s+1)$

$\quad W^\mu = L_2 p^\nu$

The spin will change within a multiple of $\frac{1}{2}$

Every susy state has positive energy.

$$\langle\psi|\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\}|\psi\rangle = 2\sigma^\mu_{\alpha\dot\beta}\,\delta^I_J\,\langle\psi|P_\mu|\psi\rangle$$

using $\bar{Q}_{\dot{I}} = (Q_I)^\dagger$

$$\langle\psi|QQ^\dagger + Q^\dagger Q|\psi\rangle = \underbrace{\|Q^\dagger|\psi\rangle\|^2 + \|Q|\psi\rangle\|^2}_{\geq 0}$$

$P_\mu = (E, \vec{p})$, $E_\psi \geq 0$ $\forall$ state ψ.

3) # of bosons and fermions

$n_B = n_F$.

(unless $E=0$) in a multiplet

$(-1)^F |B\rangle = |B\rangle$

$(-1)^F |F\rangle = -|F\rangle$

$tr((-1)^F) = \sum_{\text{all states}} \langle\psi|(-1)^F|\psi\rangle$

so result states

$tr((-1)^F) = 0$.

$(-1)^F Q|F\rangle = (-1)^F |B\rangle = |B\rangle = Q|F\rangle$

$= -Q(-1)^F|F\rangle$

$\Rightarrow \{Q, (-1)^F\} = 0$.

$tr(\underbrace{\{Q_\alpha, (-1)^F\}}_{0}\bar{Q}_{\dot\beta}) = 0$

$tr(Q(-1)^F\bar{Q} + (-1)^F Q\bar{Q})$

$= tr((-1)^F \{Q^I_\alpha, \bar{Q}_{\dot\beta J}\})$

$= \underbrace{2P_{\alpha\dot\beta}\delta^I_J}_{\neq 0} \underbrace{tr((-1)^F)}_{0} = 0$

Massless susy reps

$$p_{\alpha\dot\alpha} = \begin{pmatrix} p_0+p_3 & p_1-ip_2 \\ p_1-ip_2 & p_0-p_3 \end{pmatrix}$$

use the rest frame: $p_\mu = (E,\underbrace{0,0}_{SO(2)},E)$

$$\{Q_\alpha^I, \bar{Q}_{\dot\alpha J}\} = 2\sigma^\mu_{\alpha\dot\alpha} p_\mu \delta^I_J$$
$$= 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\alpha} \delta^I_J$$

$Q_2, \bar{Q}_{\dot 2}$ do not play any role for massless.

$Q_2|\psi\rangle = 0$, $\bar{Q}_{\dot 2}|\psi\rangle = 0$ $\langle\psi|\{Q_2, \bar{Q}_{\dot 2}\}|\psi\rangle$

Define $a_I = \frac{1}{\sqrt{4E}} Q^I_1$, $a_I^\dagger = \frac{1}{\sqrt{4E}} \bar{Q}_{\dot 1 I}$

Then $\{a_I, a_J^\dagger\} = \delta_{IJ}$ and
$\{a_I, a_J\} = \{a_I^\dagger, a_J^\dagger\} = 0$

Algebra of fermionic ~~of~~ harmonic oscillators!

this

$Z=0$ for massless multiplets!

$$[L_{12}, Q_1^I] = -\frac{1}{2} Q_1^I$$ ← a_I lowers helicity

$$[L_{12}, \bar{Q}_{1I}] = +\frac{1}{2} \bar{Q}_{1I}$$ ← $a_I^\dagger$ raises helicity.

Define a Clifford vacuum: $a_I|\Omega\rangle = 0 \ \forall I$

Built the multiplet by acting with $a_I^\dagger$

$|\lambda\rangle$

$a_I^\dagger|\lambda\rangle = |\lambda+\frac{1}{2}\rangle_I$ ← fermionic

$a_I^\dagger a_J^\dagger|\lambda\rangle = |\lambda+1\rangle_{[I,J]}$ ← all different (fermionic)

$a_1 a_2 = -a_2 a_1$

$a_1^\dagger \cdots a_N^\dagger|\lambda\rangle = |\lambda+\frac{N}{2}\rangle_{\text{singlet}}$

→ # of states with helicity?

$\lambda + \frac{K}{2} \leftarrow \binom{N}{K} \Longrightarrow$ Total number of states in this multi $\sum_{K=0}^{N}\binom{N}{K} = 2^N$

$\Rightarrow$ 2^{N-1} bosons and 2^{N-1} fermions.

In ~~high~~ physics all theories must obey CPT but CPT flips helicity! If I don't have λ and $-\lambda$ I add the CPT conjugate.

<u>$N=1$</u> ~~multiplets~~ <u>multiplets</u> | CPT conj.

$$\lambda = 0 \quad |0\rangle, \ |\tfrac{1}{2}\rangle \ \Big| \ |-\tfrac{1}{2}\rangle, \ |0\rangle$$

Complex scalar

and 2 degrees

weyl spinor <u>on shell</u>

(mass-zero multiplet)

"$N=1$ scalar multiplet"

shows the hier.

$$\lambda = \tfrac{1}{2} \quad |\tfrac{1}{2}\rangle, \ |1\rangle \ \Big\| \ |-1\rangle, \ |-\tfrac{1}{2}\rangle$$

two states with $\pm$!

weyl on shell

Physics gauge boson

$N=$ vector multiplet.

$\lambda = 1$ $\quad |1\rangle, \; |\tfrac{3}{2}\rangle \;\Big|\; |-\tfrac{3}{2}\rangle, \; |-1\rangle$

$N=1$ "gravitino"

$\lambda = \tfrac{3}{2}$ $\quad |\tfrac{3}{2}\rangle, \; |2\rangle \;\Big|\; |-2\rangle, \; |-\tfrac{3}{2}\rangle$

$N=1$ graviton.

<u>$N>1$</u> massless multiplet.

$\underline{N=2}$	multiplet	CPT conjugate
$\lambda = -\tfrac{1}{2}$	$\|-\tfrac{1}{2}\rangle, \|0\rangle_{IJ}, \|\tfrac{1}{2}\rangle$	$\|-\tfrac{1}{2}\rangle, \|0\rangle_{IJ}, \|\tfrac{1}{2}\rangle$
$\lambda = 0$	$\|0\rangle, \|\tfrac{1}{2}\rangle_I, \|1\rangle$	$\|-1\rangle, \|-\tfrac{1}{2}\rangle_I, \|0\rangle$
$\lambda = 1$	$\|1\rangle, \|\tfrac{3}{2}\rangle_I, \|2\rangle$	$\|-2\rangle, \|-\tfrac{3}{2}\rangle_I, \|-1\rangle$

$\lambda = 2$ $\quad |-\tfrac{1}{2}\rangle \qquad |\tfrac{1}{2}\rangle \leftarrow$ Weyl

$|+\tfrac{1}{2}\rangle \qquad |-\tfrac{1}{2}\rangle \leftarrow$ Weyl

2 doublets of complex scalars

$\lambda = 0$ $|0\rangle$ $|\tfrac{1}{2}\rangle_I$, $|1\rangle$ | $|-1\rangle$, $|-\tfrac{1}{2}\rangle_I$ $|0\rangle$

$(|1\rangle, |-1\rangle)$ A_μ gauge field on shell massless

$(|\tfrac{1}{2}\rangle_I, |-\tfrac{1}{2}\rangle_I)$ doublet of weyl spinor

$(|0\rangle, |0\rangle)$ complex scalar.

$\lambda = 1$ $|1\rangle$ $|\tfrac{3}{2}\rangle_I$, $|2\rangle$ | $|-2\rangle$, $|-\tfrac{3}{2}\rangle_I$ $|-1\rangle$

graviton $N=2$ sugra

gravitino

doublet

gauge boson

(hyper = $N=2$)

<u>$N=2$ massless as $N=1$</u>

(Hyper = $N=2$) = 2 $N=1$ wess-zumino multiplet

(degrees of freedom)

$N=2$ vector = 1($N=1$ vector) + 1($N=1$ WZ)

$N=2$ Sugra = 1($N=1$ sugra) + 1($N=1$ gravitino)

$I = 1,2,3,4$

<u>$\mathcal{N}=4$</u> <u>$\lambda=-1$</u>: $|-1\rangle, |-\frac{1}{2}\rangle_I, |0\rangle_{[I,J]}, |\frac{1}{2}\rangle_I$

$, |1\rangle$ $\cancel{\mathcal{S}}$

It is CPT self conjugate!

If you try <u>$\lambda = +1$</u> you get spin > 1

So no gauge theory any more.

If <u>$\mathcal{N}>4$</u> you will creek also.

<u>Then</u> $\mathcal{N}=4$ is <u>maximally supersymmetric</u>

vector multiplet in 4d.

<u>Review</u>:

$\mathcal{N}=1$ $\lambda=0$ $2_B + 2_F$: $\mathcal{N}=1$ Chiral matter, w2 fact

$2_B + 2_F$ $\mathcal{N}=2$ vector

$\mathcal{N}=2$ $4B + 4F$: $\mathcal{N}=2$ hyper (matter) after CPT

$4_B + 4_F$ $\mathcal{N}=2$ vector

$\mathcal{N}=4$ $\quad 8_F + 8_B$ $\quad$ $\mathcal{N}=4$ vector multiplet

(cpt self conjugate)

this is maximally susy vector in 4D.

Max susy quantity in 4D $\quad \mathcal{N}=8$

$|-2\rangle$, $|-\frac{3}{2}\rangle_I$, $|-1\rangle_{[I,J]}$, $|-\frac{1}{2}\rangle_{[I,J,K]}$ $\binom{8}{3}=56$

$\boxed{1}$ $\quad \binom{8}{1}=\boxed{8}$ $\quad \binom{8}{2}=28$ $\quad$ antisymmetrise

, $|0\rangle_{[I,J,K,L]}$, $|+\frac{1}{2}\rangle_{[IJK]}$

$\binom{8}{4}=70$

, $|+1\rangle_{[I,J]}$, $|\frac{3}{2}\rangle_I$, $|+2\rangle$.

$I = 1, \dots, 8$

we have

so $\quad |-\frac{3}{2}\rangle_I \quad |-1\rangle_{[IJ]} \quad |-\frac{1}{2}\rangle_{[IJK]} \quad |0\rangle$

$$1 + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \cdots$$

graviton $\quad$ gravitino $\quad$ gauge fields $\quad$ spinors $\quad$ scalars

Massive multiplets ($\mathcal{N}=1$ to start susy)

Go to rest frame $\quad P_\mu = (m, \underbrace{0,0,0}_{SO(3)})$

Use spin of $SO(3)$ to classify. Little group.

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot\beta}$$

$$\leadsto \{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$$

Define a vacuum $|\Omega\rangle = |m; s, s_3\rangle$

$$[Q_\alpha, R_{ij}] = (\sigma_{ij})_\alpha{}^\beta Q_\beta$$

$$[Q_\alpha, R_{\mu\nu}] = \frac{i}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta$$

$$[\bar{Q}^{\dot\alpha}, R_{\mu\nu}] = -\frac{i}{2}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \bar{Q}^{\dot\beta}$$

$\epsilon_{ijk} L_{ij} = R_k$

$\leadsto \quad Q_1 \quad\quad Q_2$

$\bar{Q}_{\dot 2} \quad\quad \bar{Q}_{\dot 1}$

~~raise spin~~ lower spin $\quad\quad$ raise spin. $\quad \leadsto$

$$a_\alpha = \frac{1}{\sqrt{2m}} Q_\alpha$$

$$a^\dagger_\alpha = \frac{1}{\sqrt{2m}} \bar{Q}^{\dot\alpha}$$

$$\{ a_\alpha, (a_\beta)^\dagger \} = \delta_{\alpha\beta}$$

$$\{ a_\alpha, a_\beta \} = 0 = \{ a_\alpha^\dagger, a_\beta^\dagger \}$$

<u>two</u> fermionic oscillators.

$|\Omega\rangle = |m; s=0, s_3=0\rangle$ killed by $a_\alpha|\Omega\rangle = 0$

$a_\alpha^\dagger|\Omega\rangle = |m; s = \pm\frac{1}{2}, \mp\frac{1}{2}\rangle$

$a_1^\dagger a_2^\dagger |\Omega\rangle = |m, s=0, s_3=0\rangle$

one raise
and one lower

Start with $|\Omega\rangle = |m, j=s, \underbrace{j_3 = -s, \ldots, s}_{2s+1 \text{ states}}\rangle$

$a_\alpha^\dagger|\Omega\rangle = |m; j = s\pm\frac{1}{2}, \quad\rangle$ either $\begin{cases} 2(s+\frac{1}{2})+1 \text{ states} \\ \text{or} \\ 2(s-\frac{1}{2})+1 \text{ states} \end{cases}$

$a_1^\dagger a_2^\dagger|\Omega\rangle = |m; j=s, j_3 = -s \ldots s\rangle \Rightarrow 2s+1$ states.

<u>Comment #1</u>

For $s=0$ multiplet we have

$|\Omega\rangle = |s=0\rangle \quad a_1^\dagger a_2^\dagger|\Omega\rangle = |s=0\rangle$

$a_I^\dagger|\Omega\rangle = |s = \pm\frac{1}{2}, \pm\frac{1}{2}\rangle$

the ($\mathcal{N}=1$ massive matter multiplet)

$2_B + 2_F$ is the same as $\mathcal{N}=1$ massless multiplet

"means that I can add a mass term without adding extra degrees of freedom" (for free)

massive vector multiplet

Clifford vacuum from which I started

spin	$\|0\rangle$	$\|\tfrac{1}{2}\rangle$	$\|1\rangle$	$\|\tfrac{3}{2}\rangle$
0	2 bosons	1		
$\tfrac{1}{2}$	1 ← 1 weyl fermion	2 ← 2 weyl spinors	1	
1		1	2	1
$\tfrac{3}{2}$			1	2
2				1

↑ spins a multiplet can have

Comment The massive vector mult. has

$1_B + 4_F + 3_B$ dof

$=$ 1 massless vector + 1 massless chiral

spin 0 spin($\tfrac{1}{2}$) spin 1

[this is the $\mathcal{N}=1$ higgs mechanism]

Massive $\mathcal{N}>1$ multiplets

$$\{Q_\alpha^I, \bar{Q}_{\dot\beta J}\} = 2m \begin{pmatrix}1&0\\0&1\end{pmatrix}_{\alpha\dot\beta}\delta^I_J$$

$$\{Q_\alpha^I, Q_\beta^J\} = \varepsilon_{\alpha\beta} Z^{IJ} \quad , I,J = 1,\dots,\mathcal{N}.$$

$$\{\bar{Q}_I^{\dot\alpha}, \bar{Q}_J^{\dot\beta}\} = \varepsilon^{\dot\alpha\dot\beta}\bar{Z}_{IJ}$$

use the $U(\mathcal{N})$ to bring $Z^{IJ} = \begin{pmatrix}0&Z_1&&\\-Z_1&0&&\\&&0&Z_2\\&&-Z_2&0\\&&&&\ddots\end{pmatrix}$

$I \to i$

define
$$a^1_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha + \varepsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right)$$

$$b^1_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha - \varepsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right)$$

$$a^2_\alpha = \frac{1}{\sqrt{2}}\left(Q^3_\alpha + \varepsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right)$$

$$b^2_\alpha = \frac{1}{\sqrt{2}}\left(Q^3_\alpha - \varepsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right)$$

$a^r_\alpha, b^r_\alpha \quad , \quad r = 1,\dots,\frac{\mathcal{N}}{2}$

$$\{a^r_\alpha, (a^s_\beta)^\dagger\} = (2m - Z_r)\,\delta_{rs}\delta_{\alpha\beta}$$

$$\{b^r_\alpha, (b^s_\beta)^\dagger\} = (2m + Z_r)\,\delta_{rs}\delta_{\alpha\beta}$$

$$\{a^r_\alpha, a^s_\beta\} = 0 = \{b^r_\alpha, b^s_\beta\}$$

$$\{a^\dagger, a^\dagger\} = \{b^\dagger, b^\dagger\}$$

2N harmonic oscillators!

Impies $2m \geq |Z_r|$ ← BPS bound

- If all Z_r: $2m > |Z_r|$

we have 2N harmonic oscillators.

⇒ total # of states ($|\Omega\rangle = |m, s\rangle$)

$$= (2s+1)\sum_{k=0}^{2N}\binom{2N}{k} = 2^{2N}(2s+1)$$

with $n_B = n_F = 2^{2N-1}(2s+1)$

lets do $N=2$

$S=0$:

states	multiplicity
$\lvert 0\rangle$	1 spin 0
$A\lvert 0\rangle$	$\binom{4}{1}=4$ spin $\frac{1}{2}$
$AA\lvert 0\rangle$	$\binom{4}{2}=6$ spin 1 or 0
$AAA\lvert 0\rangle$	$\binom{4}{3}=4$
$AAAA\lvert 0\rangle$	$\binom{4}{4}=1$

we discard the massive (non-BPS) vector multiplet:

$$1\cdot(S=1)+4\cdot\left(S=\tfrac{1}{2}\right)+5\cdot(S=0)$$

$$3_B \leftarrow \begin{Bmatrix}1\\0\\-1\end{Bmatrix} \qquad 4\times 2_F \leftarrow \begin{Bmatrix}-\frac{1}{2}\\+\frac{1}{2}\end{Bmatrix} \qquad 5\cdot 1_B$$

$$= 8_B + 8_F$$

$$\underbrace{|0,0\rangle}_{1},\ \underbrace{\left|\tfrac{1}{2},\pm\tfrac{1}{2},0,0\right\rangle,\ \left|0,0,\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle}_{4}$$

$$\left|1,\substack{\pm1\\0},0,0\right\rangle,\ \left|0,0,1,\substack{\pm1\\0}\right\rangle,\ \left|\tfrac{1}{2},\pm\tfrac{1}{2},\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle$$

$$\left|1,\substack{\pm1\\0},\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle,\ \left|\tfrac{1}{2},\pm\tfrac{1}{2},1,\substack{\pm1\\0}\right\rangle,\ \left|1,\substack{\pm1\\0},1,\substack{\pm1\\0}\right\rangle$$

$N=2$ Higgs mechanism for non-BPS vector

$\hookrightarrow$ $8_B + 8_F = 1 * \begin{pmatrix} N=2 \text{ massless} \\ \text{vector} \end{pmatrix}$

$+ 1 * \begin{pmatrix} N=2 \text{ massless} \\ \text{hyper} \end{pmatrix}$

For $N=2$ but $2m = Z$ (BPS multiplets)

so we have only 2 oscillators (a goes away)

we can count $2^N (2s+1)$ states

$N=2$ BPS vector multiplets

$|\frac{1}{2}\rangle, \quad b^\dagger_{1,2}|\frac{1}{2}\rangle, \quad b^\dagger_1 b^\dagger_2 |\frac{1}{2}\rangle$

$= 1*(s=1) + 2*(s=\frac{1}{2}) + 1*(s=0)$

$= 3_B + 2 * 2_F + 1_B = 4_B + 4_F$

Same degrees of freedom for $\boxed{N=2 \text{ vector mult massless}}$

2nd reason (BPS) of Higgs mechanism.

the massless vector gets the scalar of its own multiplet.

HW For $N=4$ vector multiplet: there is no possible mass term! (related to the fact that $N=4$ vector is self conjugate)

Super Space: Enlarge Minkowski space x_μ (bosonic) by adding grassmann-valued $\theta_\alpha\ \bar\theta_{\dot\alpha}$ (generated by $Q_\alpha\ \bar Q_{\dot\alpha}$)

$4_F + 4_B$ ($= N=1$ superspace)

usual

$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} \quad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

we want $\theta_\alpha, \bar\theta_{\dot\alpha}$

$$\theta\theta = \theta^\alpha\theta_\alpha = \varepsilon^{\alpha\beta}\theta_\beta\theta_\alpha = 2\theta_2\theta_1$$
$$= -2\theta_1\theta_2$$

$$\bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = 2\bar\theta_{\dot 1}\bar\theta_{\dot 2}$$

$$\theta^\alpha\theta^\beta = -\tfrac{1}{2}\varepsilon^{\alpha\beta}\theta\theta\ , \quad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = +\tfrac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta$$

$$\theta_\alpha \theta_\beta = +\frac{1}{2}\, \epsilon_{\alpha\beta}\, \theta\theta$$

$$\bar\theta_{\dot\alpha} \bar\theta_{\dot\beta} = -\frac{1}{2}\, \epsilon_{\dot\alpha\dot\beta}\, \bar\theta\bar\theta$$

$$(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}\, \theta\theta\, \bar\theta\bar\theta\, \eta^{\mu\nu}$$

$$(\theta\psi)(\theta\chi) = -\frac{1}{2}(\theta\theta)(\psi\chi)$$

Super Fields

$$\Phi(x,\theta,\bar\theta) = \phi_0(x) + \cancel{\theta^{\alpha} f_{\alpha}(x)} + \cancel{\bar\theta_{\dot\alpha}\, g^{\dot\alpha}(x)}$$

$$+ \cancel{\theta^\alpha \bar\theta_{\dot\alpha}}\, h_\alpha{}^{\dot\alpha}(x) + \theta^\alpha\, \cancel{\theta^\beta\, \theta_\alpha}\, K_{\alpha\beta}(x)$$

$$+ \theta^\alpha\, \cancel{\theta^\beta\, \bar\theta_{\dot\alpha}}\, l_{\alpha\beta}{}^{\dot\alpha}(x) + \theta^\alpha \theta^\beta \bar\theta_{\dot\alpha} \bar\theta_{\dot\beta}\, m_{\alpha\beta}{}^{\dot\alpha\dot\beta}$$

$$+ \cancel{\bar\theta_{\dot\alpha}\, \bar\theta_{\dot\beta}\, n^{\dot\alpha\dot\beta}(x)} + \theta^\alpha \bar\theta_{\dot\alpha} \bar\theta_{\dot\beta}\, o_\alpha{}^{\dot\alpha\dot\beta}(x)$$

$$= \phi(x) + \theta\psi(x) + \bar\theta\bar\chi(x)$$

$$+ \theta\theta\, m(x) + \bar\theta\bar\theta\, n(x) + \theta\sigma^\mu\bar\theta\, A_\mu$$

$$+ \theta\theta\, \bar\theta\, \bar\lambda(x) + \bar\theta\bar\theta\, \theta\, \lambda(x) + \theta\theta\bar\theta\bar\theta\, D$$

Terms up to 4 fermions

A Super Field is a collection (= a multiplet) of ordinary fields. Φ has too many d.o.f. to be an IRREP. of $\mathcal{N}=1$ algebra.

How to shorten it?

Lets see how Q_α $\bar{Q}_{\dot\alpha}$ are in superspace

$P_\mu = -i\partial_\mu$ $\underset{\text{want}}{\rightsquigarrow}$ $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2 P_{\alpha\dot\beta}^{\mu}$

$$\rightsquigarrow \begin{cases} Q_\alpha = -i\partial_\alpha - ic(\sigma^\mu\bar\theta)_\alpha \partial_\mu & = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \\ \bar{Q}_{\dot\alpha} = -i\bar\partial_{\dot\alpha} - ic^*(\bar\theta\sigma^\mu)_\alpha \partial_\mu \end{cases}$$

real part of $c = 1 \Rightarrow \boxed{c=1}$

$$\left.\begin{aligned} Q_\alpha &= -i\partial_\alpha - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar{Q}_{\dot\alpha} &= -i\bar\partial_{\dot\alpha} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \end{aligned}\right\} \circledast$$

$\Rightarrow$ $\Phi(x+\delta x, \theta+\epsilon, \bar\theta+\bar\epsilon) = (1 + i\epsilon Q + i\bar\epsilon\bar{Q})\,\Phi(x,\theta,\bar\theta)$

$$\delta x^\mu = -i(\epsilon\sigma^\mu\bar\theta - \theta\sigma^\mu\bar\epsilon)$$

A superfield transforms under susy as

$$\delta_{susy} \Phi = i \left(\varepsilon Q + \bar{\varepsilon} \bar{Q} \right) \Phi$$

Note: A product of 2 superfields is superfield

$$\delta(\Phi \Psi) = (\delta \Phi) \Psi + \Phi (\delta \Psi)$$

$$= \left(i(\varepsilon Q + \bar{\varepsilon}\bar{Q}) \Phi \right) \Psi + \Phi\, i\left((\varepsilon Q + \bar{\varepsilon}\bar{Q}) \Psi \right)$$

$$= i(\varepsilon Q + \bar{\varepsilon}\bar{Q}) (\Phi \Psi) \checkmark$$

Note: $\partial_\alpha \Phi$ is not a good superfield.

$$\delta(\partial_\alpha \Phi) = i(\varepsilon Q + \bar{\varepsilon}\bar{Q}) \partial_\alpha \Phi$$

$$\neq i \partial_\alpha \left((\varepsilon Q + \bar{\varepsilon}\bar{Q}) \Phi \right) = \partial_\alpha (\delta \Phi)$$

Define a covariant derivative $[D, \delta] = 0$

$$D_\alpha = \partial_\alpha + i \sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i \theta^\beta \sigma^\mu_{\beta\dot{\alpha}} \partial_\mu$$

$\{D_\alpha, Q_\beta\} = 0 = \{D_\alpha, \bar{Q}_{\dot\beta}\}$

$\{\bar{D}_{\dot\alpha}, Q_\alpha\} = 0 = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$

$\{D_\alpha, \bar{D}_{\dot\alpha}\} = 2i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$

$\{D_\alpha, D_\beta\} = 0$ $\qquad$ Note $(D_\alpha)^\dagger = \bar{D}_{\dot\alpha}$

$\{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0$

<u>Shorten degrees of freedom</u>

$\bar{D}_{\dot\alpha}\Phi = 0$ $\qquad$ (also $D_\alpha\bar{\Phi} = 0$ antichiral)

Chiral superfields)

<u>observe!</u> $\cdot\ \bar{D}_{\dot\alpha}\theta_\alpha = 0$ and

$\cdot\ y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta, \quad \bar{D}_{\dot\alpha}y^\mu = 0$

(for the antichiral, $\begin{cases} D_\alpha\bar\theta_\beta = 0 \\ \bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar\theta \end{cases}$

$\Rightarrow D_{\alpha}\bar{y}^\mu = 0$)

$\Phi(y,\theta)$

is by def, a chiral superfield!

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\,\theta\psi(y) - \theta\theta F(y)$$

reverse the $y \to x$.

$$\Phi(y,\theta) = \Phi(x,\theta,\bar\theta) = \phi(x) + \sqrt{2}\,\theta\psi(x) - \theta\theta F(x)$$

$$+ i\,\theta\sigma^\mu\bar\theta\,\partial_\mu\phi(x) - \frac{1}{\sqrt{2}}\theta\theta\,\partial_\mu\psi(x)\sigma^\mu\bar\theta$$

$$- \frac{1}{4}\theta\theta\bar\theta\bar\theta\,\Box\phi(x)$$

<u>apply δ_{Susy} on components:</u>

$$\delta\phi = \sqrt{2}\,\varepsilon\psi$$

$$\delta\psi_\alpha = \sqrt{2}\,i(\sigma^\mu\bar\varepsilon)_\alpha\partial_\mu\phi - \sqrt{2}\,\varepsilon_\alpha F$$

$$\delta F = i\sqrt{2}\,\partial_\mu\psi\sigma^\mu\bar\varepsilon$$

$$\delta_{s\text{-}sy}\Phi = i(\varepsilon Q + \bar{\varepsilon}\bar{Q})\Phi$$

$$\delta\phi = i(\varepsilon Q + \bar{\varepsilon}\bar{Q})\Big(\phi(x) + \sqrt{2}\,\theta\psi(x) - \cancel{\sqrt{2}}\tfrac{\theta\theta}{\sqrt{2}} F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma\bar{\theta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x)\Big)$$

$$= \sqrt{2}\,\varepsilon\psi(x) \quad \checkmark$$

$$\delta\psi_\alpha = -\varepsilon_\alpha F\sqrt{2} + i\sqrt{2}\,\bar{\varepsilon}\sigma^\mu\partial_\mu\phi$$

$$\delta F = \sqrt{2}\,\bar{\varepsilon}\sigma^\mu\partial_\mu\psi$$

Rec:

$$[\phi] = 1 \Rightarrow [\Phi] = 1$$

$$[\psi] = \frac{3}{2} \Rightarrow [\theta] = -\frac{1}{2}$$

$$[F] = 2$$

this is an auxiliary field $[F\bar{F}] = 4$

cannot write proper kinetic term.

what if $F = 0$? $\leadsto$ $\delta F = 0 \Rightarrow \sigma^\mu\partial_\mu\psi = 0$

every fermion! $\boxed{\cancel{\partial}\psi = 0}$

~~$\phi \leftarrow$ HA~~

$\phi \leftarrow$ complex 2_B

$\psi \leftarrow$ complex 2_F

(on shell)

$\not{\partial}\psi = 0$

$= \boxed{\mathcal{N}=1 \text{ massless matter multiplet}}$

Integration

$$\int d\theta^1 \, d\theta^2 \, \theta^2 \theta^1 = 1$$

$$\int d^2\theta \, \theta\theta = 1 = \int d^2\bar{\theta} \, \bar{\theta}\bar{\theta}$$

$$\int d^2\theta = \frac{1}{4} \varepsilon^{\alpha\beta} \partial_\alpha \partial_\beta$$

$$\int d^2\bar{\theta} = -\frac{1}{4} \varepsilon^{\dot{\alpha}\dot{\beta}} \partial_{\dot{\alpha}} \partial_{\dot{\beta}}$$

$$\int d^2\theta \, d^2\bar{\theta} \, \theta\theta\bar{\theta}\bar{\theta} = 1 = \int d^4\theta \, \theta\theta\bar{\theta}\bar{\theta}$$

Susy Invariant actions

$$\delta \underbrace{\int d^4x \int d^2\theta\, d^2\bar\theta}_{\text{Full superspace}} F(x,\theta,\bar\theta)$$

$$= \int d^4x \int d^2\theta\, d^2\bar\theta \underbrace{\delta_{susy} F}_{i(\epsilon Q + \bar\epsilon \bar Q) F}$$

$$= \int d^4x\, d^4\theta\, i\Big(\epsilon^\alpha \partial_\alpha (\ ) + \bar\epsilon_{\dot\alpha} \bar\partial^{\dot\alpha} (\ ) - \partial_\mu (\ ) \Big) = 0$$

up to 3 θ's
so its 0 after integration.

Same

total derivative

So supersymmetry is guaranteed. How to construct actions? Take superfields and integrate them!

Wess-Zumino model

$\bar{D}_{\dot{\alpha}} \Phi = 0 \quad (D_\alpha \bar{\Phi} = 0)$

$[\Phi] = 1, \quad [\theta] = -\frac{1}{2}, \quad [d\theta] = \frac{1}{2}$

$[d^4\theta] = +2$

$$S = \int d^4x \underbrace{d^2\theta d^2\bar{\theta}}_{2} \underbrace{\bar{\Phi}\Phi}_{2}$$

- correct mass dim
- real
- supersymmetric.

$\int d^4x \, d^2\theta \, d^2\bar{\theta}$

$\left(\int d\theta_\alpha = \partial_\alpha \Rightarrow D_\alpha = \partial_\alpha + \partial_\mu \right.$

killed by the other integral

$\int d^4x \, d^2\bar{\theta} \, \bar{\Phi} D^2 \Phi$

we use $\left(D_\alpha \bar{\Phi} = 0 \right)$

$\bar{\Phi}$ is chiral.

$= \int d^4x \underbrace{d^2\bar{\theta}}_{\bar{D}^2} \bar{\Phi} \left[- F(x) \frac{i}{\sqrt{2}} \partial_\mu \psi(x) \sigma^\mu \bar{\theta} \right.$

$\left. - \frac{1}{4} \bar{\theta}\bar{\theta} \, \Box \phi(x) \right]$

$= \int d^4x \left[\bar{F} F + i (\partial_\mu \psi) \sigma^\mu \bar{\psi} - \bar{\phi} \Box \phi \right)$

$$= \int d^4x \left[\bar{F} F + i(\partial_\mu \psi)\sigma^\mu \bar{\psi} - \bar{\phi} \Box \phi \right]$$

Kinetic term auxiliary fields.

Kinetic term action weyl Fermion

Kinetic term of Scalar field

$$\rightsquigarrow \mathcal{L}_{Kinetic} = \int d^4x \, d^2\theta \, d^2\bar{\theta} \; \bar{\Phi} \Phi$$

$$= \int d^4x \left(\bar{\Phi} \Phi \right)$$

We need Interactions

$[d^4\theta] = 2 \qquad [\Phi] = 1 \quad , \quad [\Lambda] = 1$

cut off where it stop

$$\int d^4\theta \; K(\Phi, \bar{\Phi})$$

$$\hookrightarrow K(\Phi, \bar{\Phi}) - \sum_{n,m} c_{nm} \Phi^n \bar{\Phi}^m$$

$$[c_{n,m}] = \Lambda^{2-n-m}$$

Not Interaction of a renormalizable theory.

What about $\bar{\Phi} + \Phi$? real ✓ susy ✓ massive ✓

$$\hookrightarrow \int d^4x \, d^4\theta \, (\bar{\Phi} + \Phi) = \int d^4x \, d^2\bar{\theta} \left[-F - \frac{1}{\sqrt{2}} \partial_\mu \psi \sigma^\mu \bar{\theta}\right.$$

$$-\frac{1}{4}\bar\theta\bar\theta \,\Box\phi(x)\Big]$$

$$=\int d^4x \,\Box\phi \leftarrow \text{total derivatives}$$

$$=0.$$

$$K(\Phi,\bar\Phi)\to K(\Phi,\bar\Phi)+\Lambda(\Phi)+\bar\Lambda(\bar\Phi)$$

holomorphic.

is a symmetry of the theory.

"called Kähler transf"

$\hookrightarrow$ K is the Kähler potential.

<u>Note)</u> $\int d^4x\, d^2\theta\, d^2\bar\theta\; \Psi(x,\theta,\bar\theta)$

$$=\int d^4x\, d^2\theta\, \bar D^2\, \Psi(x,\theta,\bar\theta)$$

$$\bar D_{\dot\alpha}(\bar D^2\Psi)=0 \rightsquigarrow \bar D^2\Psi=\Phi \text{ is}$$

chiral! superfield

we can make susy actions by integrating chiral

$$\int d^4x\, d^2\theta\; \Phi(x,\theta,\bar\theta)$$

So in general Φ^n is chiral

$\hookrightarrow \int d^4x \, d^2\theta \, \Phi^n$ or in general

$$\int d^4x \, d^2\theta \, W(\Phi)$$

$$\hookrightarrow \bar{D}_{\dot\alpha} W(\Phi) = 0 = \frac{\partial W}{\partial \Phi} \overbrace{(\bar{D}_{\dot\alpha} \Phi)}^{0} + \underbrace{\boxed{\frac{\partial W}{\partial \bar\Phi}}}_{=0} \underbrace{(\bar{D}_{\dot\alpha} \bar\Phi)}_{\neq 0}$$

$\Rightarrow$ W holomorphic is the condition.

$$\left(\frac{\partial W}{\partial \bar\Phi} = 0 \right)$$

$W(\Phi)$ is called the super potential and it will have all the renormalizable interactions

In order to have a real $\mathcal{L}$.

$$\mathcal{L}_{int} = \int d^2\theta \, W(\Phi) + \int d^2\bar\theta \, \bar{W}(\bar\Phi)$$

<u>Note</u> $\left[\int d^2\theta\right] = 1 \qquad [W] = 3$

~~$[L]$~~ $\Rightarrow W(\Phi) = \ldots \Phi^3$ for renormalizable.

<u>Ex</u>: $W(\Phi) = \frac{1}{2} m \Phi^2 + \frac{1}{3!} g \Phi^3$

$$\underbrace{\int d^2\theta}_{D^2} W(\Phi) = \underbrace{-m \phi F}_{\substack{\text{act on the} \\ \text{same } \Phi}} + m\psi\psi + \frac{1}{3!} g \phi^2 F$$

$$+ g \phi \psi\psi$$

<u>In general</u>

$$\int d^2\theta \, W(\Phi) = -\left(\frac{\partial W}{\partial \phi}\right) F - \frac{1}{2} \frac{\partial^2 W}{\partial \phi \partial \phi} \psi\psi$$

$$\mathcal{L}_{total} = \mathcal{L}_{kinetic} + \mathcal{L}_{int}$$

$$= \ldots + F\bar{F} - \left(\frac{\partial W}{\partial \phi}\right) F - \frac{1}{2} \frac{\partial^2 W}{\partial \phi \partial \phi} \psi\psi$$

$$- \left(\frac{\partial \bar{W}}{\partial \bar{\phi}}\right) \bar{F} - \frac{1}{2} \frac{\partial^2 \bar{W}}{\partial \bar{\phi} \partial \bar{\phi}} \bar{\psi}\bar{\psi}$$

$$= \ldots + \left| F + \frac{\partial W}{\partial \phi} \right|^2 - \left| \frac{\partial W}{\partial \phi} \right|^2 + \text{Yukawa}.$$

E.o.m: (F)

$$\begin{cases} \bar F = -\dfrac{\partial W}{\partial \phi} \\[2ex] F = -\dfrac{\partial \bar W}{\partial \bar\phi} \end{cases}$$

$$\rightsquigarrow \mathcal{L} = (\partial\phi)^2 + \bar\psi\bar\sigma^\mu\partial_\mu\psi - V_{scalar} + \text{Yukawa}$$

$$V_{scalar} = F\bar F = \left|\frac{\partial W}{\partial\phi}\right|^2 \geq 0$$

- potential always positive or 0 in susy theories.

- note: mass term has to be the same.

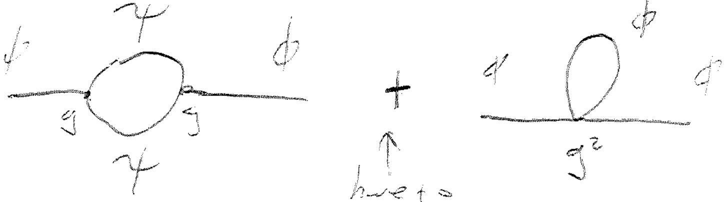
$$|m|^2\phi\bar\phi + m\psi\psi + \bar m\bar\psi\bar\psi$$

$$\cancel{\int d^2\theta\, W(\Phi) = -m\phi F + m\psi\psi + \frac{1}{2}\ldots}$$

$$= m\phi$$

- note: $\underbrace{g\,\psi\psi\,\phi}_{\text{Yukawa}} + \underbrace{g^2(\phi\bar\phi)^2}_{\text{potential}}$

the Yukawa and potential come with the same coupling constant!



$= 0\,\Lambda^2 + \#\log\Lambda$

fermion loop with a (-1)
cancels the ϕ loop

A comment on R-symmetry

$R[\theta]=1,\quad R[d\theta]=-1$

$R[\mathcal{L}]=0 \;\Rightarrow\; \boxed{R[W]=2.}$

$\int d^2\theta\, d^2\bar{\theta}\, K(\Phi,\bar{\Phi}) \qquad R[K]=0$

$\Rightarrow \boxed{c_{n,m}=\delta_{n,m}}$

has to be monomials (co

$$\mathcal{L}_{\text{matter}} = \int d^4\theta\, K(\Phi, \bar{\Phi}) + \int d^2\theta\, W(\Phi) + h.c$$

most general lagrangian!

allow Φ_i, with $i = 1, \ldots, n$

[Φ captures the $N=1$ multiplet]

with Φ non-linear sigma model.

$$V_{\text{scalar}}(\phi, \bar{\phi}_i) = \sum_{i=1}^{n} \left| \frac{\partial W}{\partial \phi_i} \right|^2 \geq 0$$

$$\mathcal{L}_{\text{kinetic}} = g_{i\bar{j}}(\phi_i, \bar{\phi}_{\bar{j}})\, \partial\phi^i \partial\bar{\phi}^{\bar{j}}$$

$$g_{i\bar{j}} = \frac{\partial^2 K(\phi, \bar{\phi})}{\partial\phi^i \partial\bar{\phi}^{\bar{j}}} \quad \boxed{\text{Kähler-metric}}$$

$$K \to K + \Lambda(\phi) + \bar{\Lambda}(\bar{\phi})$$

leave the metric invariant.

Real or vector superfield (for gauge interactions)

$$V = V^{\dagger} = \bar{V}$$

$$V(x,\theta,\bar\theta) = C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x)$$

$$+ \theta\sigma^\mu\bar\theta A_\mu(x) - \frac{1}{2}\theta\theta(M(x)+iN(x))$$

$$- \frac{1}{2}\bar\theta\bar\theta(M(x)-iN(x)) + i\theta\theta\bar\theta\left(\bar\lambda(x)+\frac{i}{2}\bar\sigma^\mu\partial_\mu\chi\right)$$

$$- i\bar\theta\bar\theta\theta\left(\lambda(x)+\frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right)$$

$$+ \frac{1}{2}\theta\theta\bar\theta\bar\theta\left(D(x)-\frac{1}{2}\Box C(x)\right)$$

this multiplet has $8_F + 8_B$ d.o.f.

we want to get $N=1$ vector multiplet which has $2_B + 2_F$.

$\rightarrow$ gauge fix : $4_B + 4_F$

$\rightarrow$ go on shell : $2_B + 2_F$

Note: $\Phi + \bar\Phi$ is also real.

$\rightsquigarrow V \rightarrow V + \cancel{\Phi+\bar\Phi}\ \Phi + \bar\Phi$ where $\bar D_{\dot\alpha}\Phi = 0$, $D_\alpha\bar\Phi = 0$

In components:

$$\Phi = \xi + \theta\psi_\xi + \theta\theta F_\xi$$

$$C \to C + 2\,\mathrm{Re}\,\xi$$

$$\chi \to \chi - i\sqrt{2}\,\psi_\xi$$

$$iM \to M = 2\,\mathrm{Im}(F_\xi)$$

$$N \to N + 2\,\mathrm{Re}[F_\xi]$$

$$D \to D$$

$$\lambda \to \lambda$$

gauge transf of a gauge field.

$$A_\mu \to \boxed{A_\mu - 2\partial_\mu(\mathrm{Im}\,\xi)}$$

So Take Φ so that

$$C \to 0$$
$$\chi \to 0$$
$$M \to 0$$
$$N \to 0$$

"gauge fix"

$$V_{WZ}(x,\theta,\bar\theta) = i\theta\theta\,\bar\theta\bar\lambda - i\bar\theta\bar\theta\,\theta\lambda + \theta\sigma^\mu\bar\theta A_\mu + \frac{1}{2}\theta\theta\bar\theta\bar\theta D$$

As promised it has $4_B + 4_F$ d.o.f.

Resumen

1) Rep of susy alg

$\mathcal{N}=1$ matter/chiral can get a mass term for free as they have same degrees of freedom.

vector eats ~~an~~ on $\mathcal{N}=1$ chiral and became massive.

$\mathcal{N}=2$ can only have mass if $\exists$ $Z\neq0$ s.t. BPS bound $Z=2m$.

Vector: $\mathcal{N}=2$ BPS massless eats 1 of its own real scalars an becomes massive

Non-BPS vector needs an extra hyper to become massive.

$\mathcal{N}=2$ no way to turn on a mass.

only way to add renormalizable recets

$$\mathcal{L}_{matter} = \int d^4\theta \, \underbrace{K(\bar{\Phi}, \Phi)}_{\text{only pick up kinetic term } \bar{\Phi}\Phi} + \int d^2\theta \, W(\Phi) + h.c.$$

$\bar{D}_{\dot{\alpha}} \Phi = 0$
chiral

- W has mass dim = 3
- R symmetry 2

with Φ^i $i = 1, \dots, n$

Gauge Interactions: $V = V^\dagger = \bar{V}$

$$V(x, \theta, \bar{\theta}) = C(x) + \theta \chi(x) + i\bar{\theta}\bar{\chi}(x)$$
$$+ \theta\sigma^\mu\bar{\theta} A_\mu + i\theta\theta(M(x) + iN(x))$$
$$- i\bar{\theta}\bar{\theta}(M(x) - iN(x)) + i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) + \tfrac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi\right)$$
$$- i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) + \tfrac{1}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta} D(x)$$

$$8_B + 8_F \xrightarrow[\text{SX}]{\text{gauge}} 4_F + 4_B \xrightarrow{\text{on shell}} 2_F + 2_B$$

$V \to V + \Xi + \bar{\Xi}$ with $\bar{D}_{\dot{\alpha}} \Xi = 0$

$D \to D$

$\Xi = \xi + \theta\psi_\xi + \theta^2 F_\xi$

$\lambda \to \lambda$

$A_\mu \to A_\mu - 2\partial_\mu(\operatorname{Im} \xi)$ $\Rightarrow$ W-Z gauge

$$\rightsquigarrow V_{WZ} = i\,\bar\theta\bar\theta\,\theta\lambda - i\,\theta\theta\,\bar\theta\bar\lambda + \theta\sigma^\mu\bar\theta A_\mu + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x)$$

$$V_{WZ}^2 = \frac{1}{2}\theta\theta\bar\theta\bar\theta A^\mu A_\mu$$

and $V_{WZ}^n = 0$, $n > 2$

$$\boxed{\text{!}\quad \delta_{susy} V_{WZ} \neq V_{WZ}}$$ ← supersymmetry breaks

<u>Abelian Field Strength.</u>

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A_\mu \to A_\mu + \partial_\mu \xi$

$F'_{\mu\nu} = F_{\mu\nu}$.

$$V \to V + \Xi + \bar\Xi$$

$$W_\alpha = -\frac{1}{4}\bar D\bar D D_\alpha V \quad , \quad \bar W_{\dot\alpha} = -\frac{1}{4} DD\bar D_{\dot\alpha} V$$

$$W_\alpha \to -\frac{1}{4}\bar D\bar D D_\alpha (V + \Xi + \bar\Xi) \qquad D_\alpha \bar\Xi = 0$$

$$= W_\alpha - \frac{1}{4}\bar D\bar D D_\alpha \Xi$$

$$= W_\alpha - \frac{1}{4}\bar D_{\dot\beta}\bar D^{\dot\beta} D_\alpha \Xi$$

$$= W_\alpha + \frac{1}{4}\bar D^{\dot\beta}\bar D_{\dot\beta} D_\alpha \Xi$$

$$= W_\alpha + \frac{1}{4}\bar{D}^{\dot\beta}\underbrace{\{\bar{D}_{\dot\beta}, D_\alpha\}}_{\sim\ \partial_{\dot\beta\alpha}}\Phi$$

$$= W_\alpha + \frac{1}{4}\bar{D}^{\dot\beta}\partial_{\dot\beta\alpha}\Phi$$

$$= W_\alpha + \partial_{\dot\beta\alpha}(\underbrace{\bar{D}^{\dot\beta}\Phi}_{=0})$$

$W'_\alpha = W_\alpha$ ✓ ← we can use for the superfield strength.

$\bar{D}_{\dot\alpha} W_\alpha = 0$ is chiral superfield (W_α has $\bar{D}\bar{D}DV$

↳ $y^\mu = x^\mu + \theta\sigma^\mu\bar\theta$ ⇝ $\begin{cases} \bar{D}_{\dot\alpha} y^\mu = 0 \\ \bar{D}_{\dot\alpha}\theta_\alpha = 0 \end{cases}$

$$W_\alpha(y,\theta) = i\lambda_\alpha(y) + \theta_\alpha D(y)$$

$$+ i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu\partial_\mu\bar\lambda)_{\alpha}(y)$$

$$D_\alpha = -\frac{\partial}{\partial\theta^\alpha} + \underbrace{\bar\theta^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}}$$

$$\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} D_\alpha V = \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}(\theta\sigma^\mu\bar\theta\bar\theta\partial_\mu A_\nu)$$

↑ gauge

$$= \bar{D}_{\dot\alpha}(\theta\theta\sigma^\mu\sigma^\nu\bar\theta\bar\theta\partial_\mu A_\nu)$$

$[\lambda] = \frac{3}{2}$

$\downarrow$

$[W_\alpha] = \frac{3}{2}$

Since W_α is chiral $\int d^2\theta\, W_\alpha$ is susy invariant

$\left[\int d^2\theta\right] = +1 \rightsquigarrow \left[\int d^2\theta\; W^\alpha W_\alpha\right] = 4$

and $\int d^2\theta\; W^\alpha W_\alpha$ + h.c. $\leftarrow$ real ✓

$$\int d^2\theta\; W^\alpha W_\alpha = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$- 2i\,\lambda\sigma^\mu \partial_\mu \bar{\lambda} + D^2$$

$$\mathrm{tr}(\sigma^{\mu\nu})(\sigma^{\rho\sigma}) = \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})$$

$$-\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}$$

<u>Non abelian fields strength</u>

Given group G, $r = \mathrm{rank}(G)$

$F_{\mu\nu} = F_{\mu\nu}^a T^a \quad a = 1, \ldots, r$

$F_{\mu\nu} \to U^{-1} F_{\mu\nu} U$ transf. covariantly!

$\mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) =$ invariant.

~~By~~ we did for the abelian shifting
$V \to V + \Phi + \bar{\Phi}$)

we will do

$$e^V \to e^{i\bar{\Lambda}} e^V e^{-i\Lambda}$$

$$\left(V \to V + \frac{i}{2}(\Lambda - \bar{\Lambda}) + \frac{i}{2}[V, \Lambda + \bar{\Lambda}] + \cdots\right.$$

$+$

$$W_\alpha = -\frac{1}{4} \bar{D}\bar{D}(e^{-V} D_\alpha e^V)$$

$$\bar{W}_{\dot{\alpha}} = -\frac{1}{4} DD(e^{+V} \bar{D}_{\dot{\alpha}} e^{-V})$$

$$W_\alpha \to -\frac{1}{4} \bar{D}\bar{D}\Big[\underbrace{(e^{i\Lambda} e^{-V} e^{-i\bar{\Lambda}})}_{e^{-V}} D_\alpha (e^{i\bar{\Lambda}} e^V e^{-i\Lambda})$$

$\Gamma D_\alpha \bar{\Lambda} = 0$
Commutes

$$= -\frac{1}{4} \bar{D}\bar{D}\Big[e^{i\Lambda} e^{-V} (D_\alpha \cancel{e^{-i\bar{\Lambda}} e^{i\bar{\Lambda}}} e^V e^{-i\Lambda})$$

$$= -\frac{1}{4} e^{i\Lambda} \bar{D}\bar{D}(e^{-V} D_\alpha e^V) e^{-i\Lambda} + D_\alpha e^{-i\Lambda})$$

$$= e^{i\Lambda} W_\alpha e^{-i\Lambda} + e^{i\Lambda} \underbrace{\bar{D}\bar{D} D_\alpha}_{\{\bar{D}_{\dot{\beta}}, D\} = \partial_{\beta\alpha} \sim \partial_{\beta\alpha} D} e^{-i\Lambda}$$

Hence $\quad W_\alpha \to \overset{U^{-1}}{e^{i\Lambda}}\, W_\alpha\, \overset{U}{e^{-i\Lambda}}$

$\quad \overline{W}_{\dot\alpha} \to e^{i\bar\Lambda}\, \overline{W}_{\dot\alpha}\, e^{-i\bar\Lambda}$

- transforms covariantly ✓

- W_α ← chiral $\overset{\text{gauge}}{\rightsquigarrow}$ chiral
  $\overline{W}_{\dot\alpha}$ ← antichiral $\overset{\text{gauge}}{\rightsquigarrow}$ antichiral

$\overline{W}_\alpha \sim \underset{\text{antichiral}}{e^{\bar\Lambda}}\ \underset{\text{antichiral}}{\overline{W}}$

$\Rightarrow$ antichiral

- $tr(W^\alpha W_\alpha)$ invariant!

$+\ tr(\overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha})$ ✓ invariant.

In WZ gauge:

$$e^{V_{WZ}} = 1 + V_{WZ} + \frac{1}{2} V_{WZ}^2 + \underline{\underline{0}}$$

Working in WZ gauge.

$$W_\alpha = -\frac{1}{4} \overline{D}\overline{D} \left[(1 - V + \frac{1}{2} V^2)\, D_\alpha (1 + V + \frac{1}{2} V^2) \right]$$

$$= -\frac{1}{4} \overline{D}\overline{D} D_\alpha V - \frac{1}{8} \overline{D}\overline{D} \underbrace{D_\alpha V^2}_{V D_\alpha V + (D_\alpha V) V} + \frac{1}{4} \overline{D}\overline{D} V \cdot D_\alpha V$$

$$= W_\alpha^{\text{abelian}} + \frac{1}{8} \overline{D}\overline{D} [V, D_\alpha V]$$

$$\frac{1}{8}\bar{D}\bar{D}[V, D_\alpha V] = \frac{}{2}(\sigma^{\mu\nu}\theta)_\alpha [A_\mu, A_\nu]$$

$$- \frac{i}{2}\theta\theta\sigma^\mu_{\alpha\dot\beta}[A_\mu, \bar\lambda^{\dot\beta}]$$

$$W_\alpha = -i\lambda(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y)$$

$$+ \theta\theta(\sigma^\mu D_\mu \bar\lambda(y))$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$-\frac{i}{2}[A_\mu, A_\nu]$$

$$D_\mu = \partial_\mu - \frac{i}{2}[A_\mu, \cdots]$$

<u>Am in all</u>

$$\boxed{\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{YM}}}$$ holomorphic coupling.

$$\mathcal{L}^{img}_{SYM} = \frac{1}{32\pi}\operatorname{Im}\left(\tau\int d^2\theta\, W^\alpha W_\alpha\right)$$

$$= \frac{1}{g^2_{YM}}\operatorname{tr}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar\lambda + \frac{1}{2}D^2\right]$$

$$+ \frac{\theta}{32\pi^2}\operatorname{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}), \quad \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

rescale $V \to 2g_{YM}V$, $A_\mu \to 2g_{YM}A_\mu$, $D \to 2g_{YM}D$

$\lambda \to 2g_{YM}\lambda$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\, g_{YM} [A_\mu, A_\nu]$$

$$D_\mu = \partial_\mu - i\, g_{YM} [A_\mu, \ ]$$

$N=1$ gauge-matter interactions

Φ^i $\quad i = 1, \dots, n \leftarrow n = \dim(R)$ $\quad R$, st R rep of $\underset{\text{com group}}{\underset{\uparrow}{G}}$

$T^a \longrightarrow (T^a_R)^i \leftarrow$ generator $\leftarrow \Lambda = \Lambda_a T^a_R$

$$\Phi \to e^{i\Lambda} \Phi \quad , \quad \bar{\Phi} \to \bar{\Phi} e^{-i\bar{\Lambda}} \Big)$$

$\bar{\Phi}$ transf with R^*.

$$\bar{\Phi} \Phi \to \bar{\Phi} \underbrace{e^{-i\bar{\Lambda}} e^{i\Lambda}}_{\neq 1} \Phi$$

$$\bar{\Phi} e^V \Phi \to \bar{\Phi} \cancel{e^{-i\bar{\Lambda}}} \left(\cancel{e^{i\bar{\Lambda}}} e^V \cancel{e^{-i\Lambda}}\right) \cancel{e^{i\Lambda}} \Phi$$

$$\overset{\shortparallel}{\bar{\Phi} \Phi} \quad \text{gauge invariant!}$$

$$\underline{\mathcal{L}_{matter}} = \int d^4\theta \, \underbrace{\bar{\Phi} e^V \Phi}_{K(\bar{\Phi}, e^V \Phi)} + \int d^2\theta \, \underbrace{W(\Phi)}_{\substack{W \text{ as long} \\ \text{as its gauge} \\ \text{invariant.}}} + h.c.$$

$W(\Phi)$ gauge invariant and renormalizable.

<u>Ex:</u>

$G = SU(N)$, $N > 3$, R = Fundamental

$\Rightarrow$ there's nothing you can write down that is invariant and renormalizable.

$N = 3$

$\exists$ 1 thing you can write.

$\cancel{}\ \epsilon_{ijk}\, \Phi^i \Phi^j \Phi^k$

nothing else!

$SU(2)$

$$\bar{\Phi} e^{V} \Phi \Big|_{\theta^2\bar\theta^2} = \underbrace{\bar{\Phi}\Phi}_{\text{above}} + \bar{\Phi} V \Phi + \frac{1}{2} \bar{\Phi} V^2 \Phi$$

$$\bar{\Phi} V \Phi \Big|_{\theta\theta\bar\theta\bar\theta} = \frac{i}{2} \bar{\phi} A_\mu \partial_\mu \phi - \frac{i}{2} (\partial^\mu \bar{\phi}) A_\mu \phi$$

$$- \frac{1}{2} \bar{\psi} \bar{\sigma}^\mu A_\mu \psi + \frac{i}{\sqrt{2}} \bar{\phi} \lambda \psi - \frac{i}{2} \bar{\psi} \bar{\lambda} \phi$$

$$+ \frac{1}{2} \bar{\phi} D \phi$$

$$\langle \bar\Phi V^2 \Phi \rangle \Big|_{\theta\theta\bar\theta\bar\theta} = \frac{1}{2} \bar{\phi} A^\mu A_\mu \phi$$

$$\int d^4\theta\, \bar{\Phi} e^V \Phi = (D_\alpha \phi)(D^\alpha \phi) - i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi$$

$$+ \bar{F} F + \frac{1}{\sqrt{2}} \bar{\Phi} \lambda \psi - \frac{1}{\sqrt{2}} \bar{\psi} \bar{\lambda} \phi + \frac{1}{2} \bar{\Phi} D \phi$$

$$D_\mu = \partial_\mu - \frac{i}{2} A_\mu^a T_R^a$$

$$\bar{\Phi} \lambda \psi = \bar{\phi}_i (T_R^a)^i{}_j \lambda_a \psi^j$$

$$V \to 2gV \qquad \int d^4\theta\, \bar{\Phi} e^{2gV} \Phi = (D\phi)(D\phi)$$

$$-\sqrt{2} g \bar{\Phi} \lambda \psi$$

$$+ g \bar{\Phi} D \phi$$

$$D_\mu = \partial_\mu - ig A_\mu^a (T_R^a)$$

[Fayet Iliopoulos term

we have almost all we need but the most general will be with $U(1)$

$$\cancel{\text{M}}\quad V \to V + i(\Lambda - \bar{\Lambda})$$

$$\xi_\alpha \int d^2\theta\, V^\alpha \to \text{gauge invariant.}$$
]

$$\mathcal{L}_{\text{matter}} = \mathcal{L}_{SYM} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{FI}$$

$$= \frac{1}{32\pi} \operatorname{Im}\left(\tau \int d^2\theta \operatorname{tr} W^\alpha W_\alpha\right) + \cancel{2g\,\xi_A} + \int d^4\theta\, \bar{\Phi} e^V \Phi$$

$$+ \int d^2\theta\, W(\Phi) + h.c.) + \sum_A 2g\,\xi^A \int d^4\theta\, V_A .$$

$$= \frac{1}{g_{YM}^2} \operatorname{tr}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \lambda \sigma^\mu D_\mu \bar{\lambda} + \frac{1}{2} D^2 \right]$$

$$+ \frac{\theta_{YM}}{32\pi^2} \operatorname{tr}(\tilde{F}_{\mu\nu} F^{\mu\nu})$$

$$+ \sum \xi^A D_A + \overline{(D_\mu \phi)} (D^\mu \phi)$$

$$+ i \psi \sigma^\mu D_\mu \bar{\psi} + \bar{F} F + i \sqrt{2}\, \bar{\phi} \lambda \psi$$

$$- i \sqrt{2}\, \bar{\psi} \bar{\lambda} \phi + \bar{\phi} D \phi + \frac{\partial W}{\partial \phi^i} F_i$$

$$+ \frac{\partial \bar{W}}{\partial \bar{\phi}_i} \bar{F}_i + \frac{1}{2} \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} \psi^i \psi^j + \frac{1}{2} \frac{\partial^2 \bar{W}}{\partial \bar{\phi}_i \partial \bar{\phi}_j} \bar{\psi}_i \bar{\psi}_j$$

EOM:

$$F \qquad \bar{F}_i = -\frac{\partial W}{\partial \phi^i}$$

$$\bar{F} \qquad F^i = -\frac{\partial \bar{W}}{\partial \bar{\phi}_i}$$

$$D \qquad D^a = -\bar{\phi}_i T^a \phi^i - \xi^a$$

$$D = T^a D$$

Scalar Potential

$$V_{scalar}(\phi, \bar{\phi}) = F \bar{F} + \frac{1}{2} D^2 \geq 0$$

$$= \sum \left| \frac{\partial W}{\partial \phi^i} \right|^2 + \sum_a \left| \bar{\phi}_i (T^a)^i_j \phi^j + \xi^a \right|^2$$

Theories with $N=2$ susy

different ones

$\boxed{N=1}$ $\lambda=0$ $\leftarrow$ { WZ ~~multiplet~~ ~~matter~~
{ matter multiplet

$|0\rangle, |+\tfrac{1}{2}\rangle$ | CPT $|-\tfrac{1}{2}\rangle, |0\rangle$

$2_F + 2_B$ $\leftarrow$ 1 weyl massless on shell
1 complex scalar.

$\lambda=\tfrac{1}{2}$ $\leftarrow$ { Vector multiplet

CPT

$|\tfrac{1}{2}\rangle, |1\rangle,$ | $|-1\rangle, |-\tfrac{1}{2}\rangle$

A_μ $\leftarrow$ gauge boson on shell

D.O.F A_μ: Initially $A_\mu \leftarrow 4$ (one each μ)

$\rightsquigarrow$ $A_\mu \to A_\mu + \partial_\mu \chi$ $\leftarrow$ gauge

$\to$ D.O.F $4 - 1 = 3$

On shell

$3 - 1 = \boxed{2}$ $\rightsquigarrow$

So

1 gauge boson onshell

1 weyl fermion

$\lambda = 1$ $\leftarrow$ { $\mathcal{N}=1$ gravitino

$|1\rangle, |\tfrac{3}{2}\rangle$ | cpt $|-\tfrac{3}{2}\rangle, |-1\rangle$

gravitino

$\lambda = \frac{3}{2}$ $|\tfrac{3}{2}\rangle, |2\rangle$ | cpt $|-2\rangle, |-\tfrac{3}{2}\rangle$

"$\mathcal{N}=1$ sugra" $\mathcal{N}=1$ graviton

$\boxed{\mathcal{N}=2}$ $\lambda = -\frac{1}{2}$ $\leftarrow$ name. "hyper"

$|-\tfrac{1}{2}\rangle, |0\rangle_I, |\tfrac{1}{2}\rangle$ | cpt $|-\tfrac{1}{2}\rangle, |0\rangle_I, |\tfrac{1}{2}\rangle$

1 weyl

2 complex scalar. 1 weyl

$\rightsquigarrow$ $\mathcal{N}=2$ hyper $=$ ($\mathcal{N}=1$ WZ) + ($\mathcal{N}=1$ WZ)

$\lambda = 0$ "$N=2$ vector"

$$\underbrace{|0\rangle}_{}, \ \underbrace{|\tfrac{1}{2}\rangle_I}_{}, \ |1\rangle \ \Big| \ \overset{CPT}{|-1\rangle}, \ |-\tfrac{1}{2}\rangle_I, \ |0\rangle$$

A_μ gauge boson

2 weyl spinor

1 complex scalar

$\rightsquigarrow \boxed{N=2 \text{ vector}} = (N=1 \text{ WZ}) + \boxed{N=1 \text{ vector}}$

$\lambda = 1$ "sugra"

$$|\phi\rangle, \ |\tfrac{3}{2}\rangle_I, \ |2\rangle \ \Big| \ \overset{CPT}{|-1\rangle}, \ |-\tfrac{3}{2}\rangle_I, \ |-2\rangle$$

1 gauge boson

2 gravitino

2 graviton

$(N=2 \text{ sugra}) = (N=1 \text{ sugra}) + (N=1 \text{ gravitino})$

~~Massive~~

Massless

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu = 2E\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_{\alpha\dot\beta}$$

$\hookrightarrow \{Q_1, \bar{Q}_{\dot 1}\} = 2E \leftarrow$ others 0.

$\hookrightarrow \uparrow \boxed{1}$ algebra of fermionic things

$\hookrightarrow a = \frac{Q_1}{\sqrt{2E}},\ a^\dagger = \bar{Q}_{\dot 1}/\sqrt{2E}$ ✓

Massive

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu = 2M\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_{\alpha\dot\beta}$$

$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2M \leftarrow$

$\Uparrow$

$\boxed{2}$ algebra of fermionic op.

$$a_\alpha = \frac{1}{\sqrt{2M}} Q_\alpha, \quad a^\dagger_\alpha = \frac{\bar{Q}^{\dot\alpha}}{\sqrt{2M}},$$

where raise ~~things~~ spin $\begin{cases} Q_1 & Q_2 \\ \bar{Q}_{\dot 2} & \bar{Q}_{\dot 1} \end{cases}$ $\Big\}$ lower spin

where $|\Omega\rangle$ is defined $\boxed{a_\alpha|\Omega\rangle = 0}$

$\boxed{\text{Ex}}$ $|\Omega| m; S=0, S_3=0\rangle$ ← little group measure is $SO(3)$

$a_\alpha^+|\Omega\rangle = |m, S=\pm 1, S_3=\pm\frac{1}{2}\rangle$

(depending on the α.

$a_1^+ a_2^+ |\Omega\rangle = |m; 0, 0\rangle$

raise, lower

$\boxed{S=0}$ $|\Omega\rangle = |S=0\rangle$.

$|0\rangle$

$a_I^+|\Omega\rangle = |\frac{1}{2}, \pm\frac{1}{2}\rangle$ or $|0\rangle$.

$\cancel{a^+_{[I,J]}|\Omega\rangle}$

$a_1^+ a_2^+ |\Omega\rangle = |0\rangle$

$\hookrightarrow$ $|0\rangle, |\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle, |0\rangle$

same as ($N=1$ WZ) $\rightsquigarrow$

mass can be added for free ← without adding extra degrees of freedom

(massless case is simpler)

Spin	$\|0\rangle$	$\|\frac{1}{2}\rangle$	$\|1\rangle$	$\|\frac{3}{2}\rangle$
0	2	1		
$\frac{1}{2}$	1 weyl fermion	2 weyl	1	
~~1~~ 1		1	2	1
$\frac{3}{2}$			1	2
2				1

massive vector multiplet

$|\frac{1}{2}\rangle$, $a_2^\dagger|\frac{1}{2}\rangle = |0\rangle$, $a_1^\dagger|\frac{1}{2}\rangle = |1\rangle$

$a_1^\dagger a_2^\dagger|\frac{1}{2}\rangle = |\frac{1}{2}\rangle$ $= |1\rangle$

2 weyl

1 scalar

1 scalar

$s = \frac{1}{2} \leftarrow$ [N=2 massive vector multiplet]

$\hookrightarrow$ $|0\rangle$, $|\frac{1}{2}\rangle$, $|\frac{1}{2}\rangle$, $|1\rangle$ $\leftarrow$ ~~Higgs~~

1B, 4F, 3F $= 4_B + 4_F$

$\Rightarrow$

$\mathcal{N}=1$ massive vector $=$ $\mathcal{N}=1$ massless vector
$\quad\quad\quad\quad\quad\quad \parallel$

$(|0\rangle, |\tfrac{1}{2}\rangle) = \cancel{\mathcal{H}}$ $\quad (|\tfrac{1}{2}\rangle, |1\rangle)$

$(|\tfrac{1}{2}\rangle, |1\rangle)$ $\quad$ + $\mathcal{N}=1$ massless chiral

$(|0\rangle, |\tfrac{1}{2}\rangle)$

[This is the $\mathcal{N}=1$ Higgs mechanism]

what about $\mathcal{N}>1$ massive?

$$\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot\beta} \delta^I_J$$

$$\{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta} Z^{IJ}$$

$$\{\bar{Q}^{\dot\alpha}_I, \bar{Q}^{\dot\beta}_J\} = \varepsilon^{\dot\alpha\dot\beta} \bar{Z}_{IJ}$$

$$Z^{IJ} = \begin{pmatrix} 0 & z_1 & & & \\ -z_1 & 0 & & & \\ & & 0 & z_2 & \\ & & -z_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Define

$$\begin{cases} a^1_\alpha = \frac{1}{\sqrt{2}}\left(Q_\alpha + \varepsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) \\ b^1_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha - \varepsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) \end{cases}$$

jumps

$$a^2_\alpha = \frac{1}{\sqrt{2}}\left(Q^3_\alpha + \varepsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right)$$

$$b^2_\alpha = \frac{1}{\sqrt{2}}\left(Q^3_\alpha - \varepsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right)$$

So there are a^r_α, b^r_α, $r = 1, \dots, N/2$

$$\rightsquigarrow \{a^r_\alpha, (a^s_\beta)^\dagger\} = (2m - Z_r)\,\delta_{rs}\delta_{\alpha\beta}$$

$$\{b^r_\alpha, (b^s_\beta)^\dagger\} = (2m + Z_r)\,\delta_{rs}\delta_{\alpha\beta}$$

with the rest vanishing!

So we have $\underset{\substack{\uparrow \\ \text{one each} \\ a,\, b}}{2r} \cdot \underset{\substack{\uparrow \\ \alpha = 1,2}}{2} =$ $\boxed{2N \text{ harmonic oscillators}}$

we say $2m \geq |Z_r|$ is the BPS bound

- If $2m > |Z_r| \rightsquigarrow 2N$ harm oscillators
- if $2m = |Z_r| \rightsquigarrow N$ harm oscillators.

\# of states: $|\Omega\rangle = |\Omega\rangle = |m, s\rangle$

$$(2s+1)\sum_{n}^{2N}\binom{2N}{n} = 2^{2N}(2s+1)$$

\# of states

$$(2s+1)\sum_{n}^{N}\binom{N}{n} = 2^{N}(2s+1)$$

Called the "BPS multiplets"

<u>$N=2$</u> BPS multiplets:

$$|\tfrac{1}{2}\rangle, \quad b_1^{+}|\tfrac{1}{2}\rangle, \quad b_1^{+}b_2^{+}|\tfrac{1}{2}\rangle$$

$$|0\rangle, |1\rangle \qquad |\tfrac{1}{2}\rangle$$

$= 2\times 2F + B\times 3B + 1B = 4_B + 4_F$.

Same as $\boxed{N=2 \text{ vector multiplet massless}}$