2d CFTs

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CONTENTS

A. CFTs in general dimension

I. 2D CFTS

Virasoro algebra.

We can define 2d CFTs on any Riemann surface. Due to Moore and Seiberg it's enough to demand consistency on sphere (genus 0) and torus (genus 1).

A. Weyl anomaly

II. BOOTSTRAP

- State-operator correspondance $\leftrightarrow$ radial quantization
- Primaries:

$$\mathcal{O} \to \Omega(x)^{-\Delta}\mathcal{O}(x')$$

it follows

$$\langle \mathcal{O}_1\mathcal{O}_2\dots\mathcal{O}_n\rangle_g = \prod_i^n \Omega(x)^{-\Delta_i}\langle \mathcal{O}'_1\mathcal{O}'_2\dots\mathcal{O}'_n\rangle_{g'=\Omega(x)^2 g}$$

- 1, 2 and 3 point functions fixed by conf. Symmetry (up to constant) and one point vanishing.

III. REPRESENTATIONS

Irreducible representations dont have null vectors. Construct them by quotient.

Example III.1. *Start with some dimension*

IV. STRING THEORY (GAIOTTO)

Point particle amplitudes:

$$A_\Gamma = \underbrace{\int d\ell}_{\text{Moduli}} \int_{X:\Gamma\to\mathbb{R}} \mathcal{D}X e^{-S[X,\ell]}$$

- Bad for regularization (singularities in vertex)

Schematically:

$$A_\Gamma = \int \prod_{(a,b)=\text{internal lines}} dp_{ab} \prod_{a=\text{internal vertices}} \prod_{(a,b)=\text{internal lines}} \frac{1}{p_{ab}^2+m^2}$$

$$\frac{1}{p_{ab}^2+m^2} = \int_0^\infty d\ell_{ab} e^{-\ell_{ab}p_{ab}^2-\ell_{ab}m^2}$$

$$\delta(\sum_b p_{ab}) = \int dx_a^d e^{ix_a \sum p_{ab}}$$

$$A_\Gamma = \prod_{a=\text{internal vertices}} \int dX_a^D e^{iX_a} \sum_{b=\text{external}} p_{ab} \prod_{(a,b)} G(x_a,x_b)$$

$$G(x_a,x_b) = \int_0^\infty d\ell_{ab} e^{\frac{(x_a-x_b)^2}{4\ell_{ab}}-\ell_{ab}m^2}$$

We will see how the delta conservation arrises from the non oscilatorry part of the path integral, the vertex operators insertion and the Strings:

$$A_\Gamma = \underbrace{\int d\ell}_{\text{Moduli}} \int_{X:\Gamma\to\mathbb{R}} \mathcal{D}X e^{-S[X,\ell]}$$

Example IV.1 (Klein-Gordon propagator from the worldline (path integral))**.** *Consider the action for a relativistic particle propagating between two points*

$$S = \frac{1}{2}\int_0^1 d\tau \left[e^{-1}\dot{X}^\mu\dot{X}_\mu + m^2 e\right] \quad (\dot{X} \equiv \partial_\tau X\,, \quad \mu = 1,\ldots,D)\,,$$

with Dirichlet boundary conditions $X(0) = x_0$ and $X(1) = x_1$. Here we have Wick rotated to a Euclidean worldline and a Euclidean D-dimensional target space. We proceed to show how the path integral formalism allow us to compute scattering amplitudes in the point particle and string cases.

Our starting point is the following formal expression for the path integral:

$$Z(X_0,X_1) = \int_{X(0)=X_0} \frac{[dXde]}{V_{\text{diff}}} \exp\left(-S_{\text{m}}[X,e]\right),$$

where the action for the"matter" fields X^μ is

$$S_m[X,e] = \frac{1}{2}\int_0^1 d\tau\, e\left(e^{-1}\partial X^\mu e^{-1}\partial X_\mu + m^2\right)$$

(where $\partial \equiv d/d\tau$). We have fixed the coordinate range for τ to be [0,1].

This action has invariance under diffeomorphisms $\zeta : [0,1] \to [0,1]$, under which the X^μ are scalars,

$$X^{\mu\zeta}(\tau^\zeta) = X^\mu(\tau),$$

and the einbein e is a"co-vector,"

$$e^{\zeta}(\tau^{\zeta}) = e(\tau)\frac{d\tau}{d\tau^{\zeta}}.$$

Think of this setup as 1d geometry with 1d metric $h_{\tau\tau} = m^2 e^2$. So the question is: how much can you fix using co-ordinate transformations in 1d geometry? Answer: There is no local geometry in 1d-all metrics are locally related by co-ordinate transformations. The only geometry can come from global features, and in 1d the only global feature is periodicity. The space can be either a circle or a line. If it is a circle, then the circumference $2\pi R = T = \int d\tau e(\tau)$ is the only diffeomorphism-invariant quantity.

Therefore, as usual, $V_{\rm diff}$ is the volume of this group of diffeomorphisms to which we should gauge fix and the e integral in the partition function runs over positive functions on $[0, 1]$, and the integral

$$l \equiv \int_0^1 d\tau\, e$$

is diffeomorphism invariant and therefore a modulus; the moduli space is $(0, \infty)$.

In order to make sense of the functional integrals of X we will need to define an inner product on the space of functions on $[0, 1]$, which will induce measures on the relevant function spaces. This inner product will depend on the einbein e in a way that is uniquely determined by the following two constraints:

1. *the inner product must be diffeomorphism invariant;*
2. *it must depend on $e(\tau)$ only locally, in other words, it must be of the form*

$$(f, g)_e = \int_0^1 d\tau\, h(e(\tau))f(\tau)g(\tau),$$

for some function h.

As we will see, these conditions will be necessary to allow us to regularize the infnite products that will arise in carrying out the functional integrals in (1), and then to renormalize them by introducing a counter-term action, in a way that respects the symmetries of the action (2). For f and g scalars, the inner product satisfying these two conditions is

$$(f, g)_e \equiv \int_0^1 d\tau\, efg.$$

We can express the matter action using this inner product:

$$S_{\rm m}[X, e] = \frac{1}{2}(e^{-1}\partial X^{\mu}, e^{-1}\partial X_{\mu})_e + \frac{lm^2}{2}.$$

We now wish to express the path integral of the partition function in a slightly less formal way by choosing a fiducial einbein e_l for each point l in the moduli space, and replacing the integral over einbeins by an integral over the moduli space times a Faddeev-Popov determinant $\Delta_{\rm FP}[e_l]$.

Defining $\Delta_{\rm FP}$ by

$$1 = \Delta_{\rm FP}[e] \int_0^{\infty} dl \int [d\zeta]\, \delta[e - e_l^{\zeta}],$$

we indeed have, by the usual sequence of formal manipulations

$$Z(X_0, X_1) = \int_0^{\infty} dl \int_{X(0)=X_0} [dX]\Delta_{\rm FP}[e_l] \exp\left(-S_{\rm m}[X, e_l]\right).$$

To calculate the Faddeev-Popov determinant at the point $e = e_l$, we expand e about e_l for small diffeomorphisms ζ and small changes in the modulus:

$$e_l - e^{\zeta}_{l+\delta l} = \partial\gamma - \frac{de_l}{dl}\delta l,$$

where γ is a scalar function parametrizing small diffeomorphisms: $\tau^{\zeta} = \tau + e^{-1}\gamma$; to respect the fixed coordinate range, γ must vanish at 0 and 1. Since the variation of e is, like e, a co-vector, we will *for simplicity multiply it by e_l^{-1} in order to have a scalar, and then bring into play our innen product (7) in order to express the delta functional in (9) as an integral over scalar functions β :*

$$\Delta_{\mathrm{FP}}^{-1}[e_l] = \int d\delta l [d\gamma d\beta] \ \exp\left(2\pi i(\beta, e_l^{-1}\partial\gamma - e_l^{-1}\frac{de_l}{dl}\delta l)_{e_l}\right)$$

The integral is inverted by replacing the bosonic variables $\delta l, \gamma$, and β by Grassman variables ξ, c, and b:

$$\begin{aligned}\Delta_{\mathrm{FP}}[e_l] &= \int d\xi [dcdb] \ \exp\left(\frac{1}{4\pi}(b, e_l^{-1}\partial c - e_l^{-1}\frac{de_l}{dl}\xi)_{e_l}\right) \\ &= \int [dcdb] \ \frac{1}{4\pi}(b, e_l^{-1}\frac{de_l}{dl})_{e_l} \exp\left(\frac{1}{4\pi}(b, e_l^{-1}\partial c)_{e_l}\right).\end{aligned}$$

We can now write the path integral in a more explicit form:

$$\begin{aligned}Z(X_0, X_1) = \int_0^\infty dl \int_{X(0)=X_0} [dX] \int_{c(0)=c(1)=0} [dcdb] \ \frac{1}{4\pi}(b, e_l^{-1}\frac{de_l}{dl})_{e_l} \\ \times \exp\left(-S_{\mathrm{g}}[b, c, e_l] - S_{\mathrm{m}}[X, e_l]\right),\end{aligned}$$

where

$$S_g[b, c, e_l] = -\frac{1}{4\pi}(b, e_l^{-1}\partial c)_{e_l}.$$

At this point it becomes convenient to work in a specific gauge, the simplest being

$$e_l(\tau) = l.$$

Then the inner product becomes simply

$$(f, g)_l = l \int_0^1 d\tau \ fg.$$

In order to evaluate the Faddeev-Popov determinant, let us decompose b and c into nor- malized eigenfunctions of the operator

$$\Delta = -(e_l^{-1}\partial)^2 = -l^{-2}\partial^2 :$$

$$\begin{aligned}b(\tau) &= \frac{b_0}{\sqrt{l}} + \sqrt{\frac{2}{l}}\sum_{j=1}^\infty b_j \cos(\pi j\tau), \\ c(\tau) &= \sqrt{\frac{2}{l}}\sum_{j=1}^\infty c_j \sin(\pi j\tau),\end{aligned}$$

with eigenvalues

$$\nu_j = \frac{\pi^2 j^2}{l^2}.$$

The ghost action becomes

$$S_{\mathrm{g}}(b_j, c_j, l) = -\frac{1}{4l}\sum_{j=1}^\infty j b_j c_j.$$

The zero mode b_0 does not enter into the action, but it is singled out by the insertion appearing in front of the exponential in the Fadeev-Popov determinant:

$$\frac{1}{4\pi}(b, e_l^{-1}\frac{de_l}{dl})_{e_l} = \frac{b_0}{4\pi\sqrt{l}}.$$

The Faddeev-Popov determinant is, finally,

$$\begin{aligned}\Delta_{\mathrm{FP}}(l) &= \int \prod_{j=0}^{\infty} db_j \prod_{j=1}^{\infty} dc_j \frac{b_0}{4\pi\sqrt{l}} \exp\left(\frac{1}{4l}\sum_{j=1}^{\infty} j b_j c_j\right) \\ &= \frac{1}{4\pi\sqrt{l}} \prod_{j=1}^{\infty} \frac{j}{4l} \\ &= \frac{1}{4\pi\sqrt{l}} \det{}' \left(\frac{\Delta}{16\pi^2}\right)^{1/2},\end{aligned}$$

the prime on the determinant denoting omission of the zero eigenvalue.

Let us decompose $X^\mu(\tau)$ into a part which obeys the classical equations of motion,

$$X^\mu_{\mathrm{cl}}(\tau) = X_0 + (X_1 - X_0)\tau,$$

plus quantum fuctuations; the fluctuations vanish at 0 and 1, and can therefore be decomposed into the same normalized eigenfunctions of Δ as c was:

$$X^\mu(\tau) = X^\mu_{cl}(\tau) + \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} x^\mu_j \sin(\pi j \tau).$$

The matter action becomes

$$S_m(X_0, X_1, x_j) = \frac{(X_1 - X_0)^2}{2l} + \frac{\pi^2}{l^2} \sum_{j=1}^{\infty} j^2 x_j^2 + \frac{lm^2}{2},$$

and the matter part of the path integral

$$\begin{aligned}\int_{X(0)=X_0} [dX] \, \exp\left(-S_{\mathrm{m}}[X, e_l]\right) &= \exp\left(-\frac{(X_1 - X_0)^2}{2l} - \frac{lm^2}{2}\right) \int \prod_{\mu=1}^{D} \prod_{j=1}^{\infty} dx^\mu_j \exp\left(-\frac{\pi^2}{l^2} \sum_{j=1}^{\infty} j^2 x_j^2\right) \\ &= \exp\left(-\frac{(X_1 - X_0)^2}{2l} - \frac{lm^2}{2}\right) \overset{\prime}{\det}\left(\frac{\Delta}{\pi}\right)^{-D/2},\end{aligned}$$

where we have conveniently chosen to work in a Euclidean spacetime in order to make all of the Gaussian integrals convergent.

Putting together the results, and dropping the irrelevant constant factors multiplying the operator Δ in the infnite-dimensional determinants, we have:

$$Z(X_0, X_1) = \int_0^\infty dl \, \frac{1}{4\pi\sqrt{l}} \exp\left(-\frac{(X_1 - X_0)^2}{2l} - \frac{lm^2}{2}\right) \left(\overset{\prime}{\det}\Delta\right)^{(1-D)/2}.$$

We will regularize the determinant of Δ in the same way as it is done in Appendix A.1 of Polchinski, by dividing by the determinant of the operator $\Delta + \Omega^2$:

$$\begin{aligned}\frac{\det'\Delta}{\det'(\Delta + \Omega^2)} &= \prod_{j=1}^{\infty} \frac{\pi^2 j^2}{\pi^2 j^2 + \Omega^2 l^2} \\ &= \frac{\Omega l}{\sinh \Omega l} \\ &\sim 2\Omega l \exp\left(-\Omega l\right),\end{aligned}$$

where the last line is the asymptotic expansion for large Ω. The path integral becomes

$$\begin{aligned}&Z(X_0, X_1) \\ &= \frac{1}{4\pi (2\Omega)^{(D-1)/2}} \int_0^\infty dl \, l^{-D/2} \exp\left(-\frac{(X_1 - X_0)^2}{2l} - \frac{l(m^2 - (D-1)\Omega)}{2}\right).\end{aligned}$$

The inverse divergence due to the factor of $\Omega^{(1-D)/2}$ in front of the integral can be dealt with by a field renormalization, but since we will not concern ourselves with the overall normalization of the path *integral we will simply drop all of the factors that appear in front. The divergence coming from the Ω term in the exponent can be cancelled by a (diffeomorphism invariant) counterterm in the action,*

$$S_{\rm ct} = \int_0^1 d\tau\, eA = lA$$

The mass m is renormalized by what is left over after the cancellation of infinities,

$$m^2_{\rm phys} = m^2 - (D-1)\Omega - 2A,$$

but for simplicity we will assume that a renormalization condition has been chosen that sets $m_{\rm phys} = m$.

We can now proceed to the integration over moduli space:

$$Z(X_0, X_1) = \int_0^\infty dl\, l^{-D/2} \exp\left(-\frac{(X_1 - X_0)^2}{2l} - \frac{lm^2}{2}\right).$$

The integral is most easily done after passing to momentum space:

$$\begin{aligned}
\tilde{Z}(k) &\equiv \int d^D X \ \exp\left(ik \cdot X\right) Z(0, X) \\
&= \int_0^\infty dl\, l^{-D/2} \exp\left(-\frac{lm^2}{2}\right) \int d^D X \ \exp\left(ik \cdot X - \frac{X^2}{2l}\right) \\
&= \left(\frac{\pi}{2}\right)^{D/2} \int_0^\infty dl \ \exp\left(-\frac{l(k^2 + m^2)}{2}\right) \\
&= \left(\frac{\pi}{2}\right)^{D/2} \frac{2}{k^2 + m^2};
\end{aligned}$$

neglecting the constant factors, this is precisely the momentum space scalar propagator.

Example IV.2. *Second way of deriving the propagator. Consider the action for a relativistic particle propagating between two points*

$$S = \frac{1}{2} \int_0^1 d\tau \left[e^{-1} \dot{X}^\mu \dot{X}_\mu + m^2 e\right] \quad (\dot{X} \equiv \partial_\tau X\,, \quad \mu = 1, \ldots, D)\,,$$

(1)

with Dirichlet boundary conditions $X(0) = x_0$ and $X(1) = x_1$. Here we have Wick rotated to a Euclidean worldline and a Euclidean D-dimensional target space.

(a) Gauge symmetry. Show that this theory is invariant under 1d diffeomorphisms $\tau \to \hat{\tau}(\tau)$. How does e transform?

(b) Gauge fixing. Explain why one can fix the gauge $e(\tau) = T$ where $T := \int d\tau e(\tau)$, and why there is no remaining gauge symmetry.

(c) Path integral. You may assume that the correct path integral measure is $\frac{\mathcal{D}e}{\rm gauge} = dT$ afte gauge fixing.

Expanding near the classical trajectory,

$$X = \bar{X} + \eta\,, \quad \bar{X} = x_0 + \tau(x_1 - x_0)\,,$$

(2)

and integrating over the fluctuation η, the propagator from X_1 to X_2 is given by the path integral

$$G(x_0, x_1) = \int_0^\infty dT \int \mathcal{D}\eta^\mu \exp\left(-\frac{1}{2}\int_0^1 d\tau \left[T^{-1} \dot{X}^\mu \dot{X}_\mu + m^2 T\right]\right).$$

. (3)

Do the Gaussian integral over η, with boundary conditions $\eta(0) = \eta(1) = 0$, to obtain

$$G(x_0,x_1)=\int_0^\infty dT\ \exp\left[-\frac{1}{2}T^{-1}(x_0-x_1)^2-\frac{1}{2}m^2T\right]\ \det{}'\left(-\frac{1}{T^2}\partial_\tau^2\right)^{-D/2}.$$

(4)
[You may assume that $\int\mathcal{D}\phi e^{-\frac{T}{2}\int_0^1 d\tau\phi\mathcal{O}\phi}=(\det{}'\mathcal{O})^{-1/2}$ *for a scalar feld* ϕ*.]*
(d) Functional determinant. By finding the eigenvalues of the operator $-\frac{1}{T^2}\partial_\tau^2$ *on the appropriate space of functions, evaluate this determinant to obtain*

$$G(x_0,x_1)\sim\int_0^\infty dT\ \exp\left\{\left[-\frac{1}{2}T^{-1}(x_0-x_1)^2-\frac{1}{2}m^2T\right]\right\}\prod_{n=1}^{\infty}\left(\frac{\pi^2n^2}{T^2}\right)^{-D/2},$$

where the overall numerical factor may be ignored. [Hint: the symbol det' means not to include the eigenvalue 0.]
(e) Zeta-function regularization. Using the fact that $\zeta(0)=-\frac{1}{2}$*, and the series expansion* $\zeta(s)=\sum_{n=1}^\infty n^{-s}$ *of the Riemann zeta function, evaluate this product to give*

$$G(x_0,x_1)\sim\int dT\exp\left[-\frac{1}{2}T^{-1}(x_0-x_1)^2-\frac{1}{2}m^2T\right]T^{-D/2}.$$

(5)
(f) Momentum space. Take the Fourier transform to obtain the momentum- space propagator that you already know for Klein-Gordon theory, $S=\int d^Dx\,(\partial_\mu\phi)^2$ *.*
Solution.
(a) Under a general diffeomorphism we have

$$\begin{aligned}&\tau\to\hat\tau(\tau)\,,\quad d\tau\to d\hat\tau=\hat\tau'(\tau)d\tau\,,\\&X(\tau)\to\hat X(\hat\tau)=X(\tau)\,,\quad\partial_\tau\to\partial_{\hat\tau}=\partial_\tau/\hat\tau'(\tau)\\&e(\tau)\to\hat e(\hat\tau)=e(\tau)/\hat\tau'(\tau)\,.\end{aligned}$$

(6)
Everything transforms covariantly and it is easy to check that each term in the action is invariant. (b) Think of this setup as 1d geometry with 1d metric $h_\tau\tau=m^2e^2$*. So the question is: how much can you fix using co-ordinate transformations in 1d geometry? Answer: There is no local geometry in 1d-all metrics are locally related by co-ordinate transformations. The only geometry can come from global features, and in 1d the only global feature is periodicity. The space can be either a circle or a line. If it is a circle, then the circumference* $2\pi R=T=\int d\tau e(\tau)$ *is the only diffeomorphism-invariant quantity. (c) • Why do we gauge fix? Because one should reduce to the physical degrees of freedom before quantizing. • Why is it a non-trivial fact that* $\frac{\mathcal{D}e}{\text{gauge}}=dT$*? Because, in general,*

non-trivial determinant $\mathcal{D}e=\mathcal{D}(\text{gauge})dT\ f(T)$, i.e. $\dfrac{\mathcal{D}e}{\text{gauge}}=dTf(T)$

one should expect a *non-trivial determinant* $\mathcal{D}e=\mathcal{D}(\text{gauge})dT\ f(T)$, i.e. $\dfrac{\mathcal{D}e}{\text{gauge}}=dTf(T)$ *the cor-*

corresponding to

rect gauge-invariant measure in the space of felds e. In this case $f(T)=1$*. Substituting the expansion (2) into the gauge-fixed action gives*

$$S=\frac{1}{2}\int_0^1 d\tau\big[T^{-1}(x_0-x_1)^2+m^2T+T^{-1}\dot\eta^2\big]\,,$$

(7)
where we dropped a cross-term $(x_0-x_1)\dot\eta$ *since it is a total derivative The* η*-dependent part of the path-integral (3) is then*

$$\int\mathcal{D}\eta\exp\left(-\frac{T}{2}\int_0^1 d\tau\ \eta[-\frac{1}{T^2}\partial_\tau^2]\eta\right)=\det{}'\left(-\frac{1}{T^2}\partial_\tau^2\right)^{-D/2}.$$

(8)

- *Note the measure* $\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int_0^1 d\tau e\phi\mathcal{O}\phi\right) = \int \mathcal{D}\phi \exp\left(-\frac{T}{2}\int_0^1 d\tau\phi\mathcal{O}\phi\right)$, *withthe factor of* T, *is the natural diffeomorphism- invariant one.*

(d) det'$\mathcal{O}$ *is just the product of the non-zero eigenvalues of* $\mathcal{O}$ *in the appropriate space of functions, in this case those vanishing at 0 and 1. The eigenfunctions of* $-\frac{1}{T^2}\partial_\tau^2$ *in this space are the classical solution* $\bar{x}$ *and* $\sin(n\pi\tau)$ *with* $n \in \mathbb{Z}_>0$. *The eigenvalues are 0 and* $n^2\pi^2/T^2$ *We remove 0 and take the product of the others.*

(e) Evaluate

$$
\begin{aligned}
\prod_{n=1}^{\infty}\left(\frac{\pi^2 n^2}{T^2}\right)^{-D/2} &= \text{(const)}\prod_{n=1}^{\infty} T^D \\
&\sim \exp\log\prod T^D \\
&= \exp\Big((\log T^D)\sum 1\Big) \\
&= \exp\big(\zeta(0)\log T^D\big) \\
&= \exp\left(-\frac{1}{2}\log T^D\right) \\
&= T^{-D/2}\,.
\end{aligned}
$$

(f) We simply use

$$\int \mathrm{d}^D x\, e^{ip\cdot x} e^{-\frac{x^2}{2T}} = (2\pi T)^{\frac{D}{2}} e^{-\frac{T}{2}p^2}$$

(9)

To get

$$G(p) = \int \mathrm{d}^D x e^{ip\cdot x} G(x,0) \propto \int_0^{\infty} \mathrm{d}T\, e^{-\frac{T}{2}(p^2+m^2)} \propto \frac{1}{p^2+m^2}$$

(10)

Scattering amplitudes from the worldline (canonical quantization). Consider the action for a relativistic point particle on an open path in D-dimensional Minkowski space,

(11)

$$S[e,X] = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \left[e^{-1}\dot{X}^2 - m^2 e\right]\,.$$

(a) Constraint. Write down the equation of motion from varying e. Show that it is equivalent to the vanishing of the Hamiltonian,

(12)

$$\mathcal{H} = \frac{1}{2}[e^{-1}\dot{X}^2 + m^2 e] = 0\,.$$

Since e is an auxiliary field (it has no derivatives), this equation will be imposed as a constraint on physical states, $A|\psi\rangle = 0$. *(b) Vertex operators. Show that the'vertex operators',*

(13)

$$V(k) = \int d\tau\, e(\tau)\, \exp\left[ik\cdot X(\tau)\right]\,,$$

are diffeomorphism-invariant for any momentum vector k. (c) Quantization. Explain why we can gauge-fix $e(\tau) = 1/m$ *on the infinite line (with freedom to choose the coefficient* 1*). In this gauge the* *canonical momentum is $\mathcal{P} = m\dot{X}$. (i) Convince yourself that the Schrodinger equation gives the time evolution*

$$X(\tau) = x + \tau\,\frac{1}{m}\,p\,, \quad \mathcal{P}(\tau) = p\,.$$

(14)

We impose the canonical commutation relation $[x^\mu, p^\nu] = i\eta^{\mu\nu}$. (ii) Show, in this gauge, that $V(k)$ carries momentum k^μ,

$$[p^\mu, V(k)] = k^\mu V(k)\,.$$

(15)

Eigenstates $|k\rangle$ of p form a basis for the Hilbert space, with $p^\mu|k\rangle = k^\mu|k\rangle$. (iii) Show that $|k\rangle$ is physical ($\mathcal{H}|k\rangle = 0$) precisely on the mass-shell $k^2 + m^2 = 0$. We may think of $|k\rangle \sim V(k)|vac\rangle$, so the'physical' vertex operators $V(k)$ are those with $k^2 + m^2 = 0$. (d) Scattering amplitude. Let us consider the tree-level 4-point scattering amplitude $\mathcal{A}(k_1, k_2, k_3, k_4)$ in ϕ^3 theory. We regard this as the transition amplitude for a particle of momentum $k_1 \to -k_4$, while emitting particles of momentum k_2 and k_3,

$$\langle -k_4|\mathrm{T}\big\{V(k_3)V(k_2)\big\}|k_1\rangle = \int_{-\infty}^{\infty}\frac{d\tau'}{m}\int_{-\infty}^{\infty}\frac{d\tau}{m}\langle -k_4|\mathrm{T}\big\{\exp\left[ik_3\cdot X(\tau')\right]\exp\left[ik_2\cdot X(\tau)\right]\big\}|k_1\rangle$$

where T denotes time ordering. There is a residual gauge symmetry: constant time translations $\tau \to \tau + c$. We use it to fx the position of $\tau' = 0$,

$$\int_{-\infty}^{\infty}\frac{d\tau}{m}\langle -k_4|\mathrm{T}\big\{\exp\left[ik_3\cdot X(0)\right]\exp\left[ik_2\cdot X(\tau)\right]\big\}|k_1\rangle\,.$$

(17)

By splitting the integral into $\int_{-}\infty^0$ and $\int_0^\infty$, show that this amplitude is proportional to

$$\left(\frac{1}{(k_1+k_2)^2+m^2} + \frac{1}{(k_1+k_3)^2+m^2}\right)\delta(k_1+k_2+k_3+k_4)\,.$$

(18)

You may wish to apply the Baker-Campbell-Hausdorff formula to the operator $\exp\left[ik_2\cdot X(\tau)\right] = \exp\left[ik_2\cdot x + \frac{i\tau}{m}k_2\cdot p\right]$. *Discuss why this approach misses the contribution $\frac{1}{(k_1+k_4)^2+m^2}$ from one of the three channels.*

Solution.

(a) Varying e gives $-e^-2\dot{X}^2 - m^2 = 0$. (b) One can think of $e = \sqrt{-\det h}$ where h is a 1d metric, so $V(k)$ is the integral of a scalan against the diffeomorphism-invariant measure $d\tau\sqrt{-\det h}$. (c) (i) In this gauge $\mathcal{H} = \frac{1}{2m}(\mathcal{P}^2 + m^2)$. The Schrodinger equation (equivalently the equation of motion from varying X) gives $\dot{P} = i[\mathcal{H}, P] = 0, \dot{X} = i[\mathcal{H}, X] = \frac{1}{m}\mathcal{P}$. (ii) Note that $[p, X(\tau)]$ is independent of time since $[p,p] = 0$. So $[p^\mu, V(k)] = m^{-1}\int d\tau [p^\mu, \exp\left[ik\cdot X(\tau)\right] = m^{-1}\int d\tau\; k^\mu \exp\left[ik\cdot X(\tau)\right] = k^\mu V(k)$ by stan- } dard arguments. (iii) $\mathcal{H}|k\rangle = \frac{1}{2m}(\mathcal{P}^2 + m^2)|k\rangle = \frac{1}{2m}(k^2+m^2)|k\rangle$ vanishes only when $k^2 + m^2 = 0$. (d) Splitting up the integral as indicated, the time ordering gives

$$m^{-1}\langle -k_4|\int_{-\infty}^{0}\exp\left[ik_3\cdot X(0)\right]\exp\left[ik_2\cdot X(\tau)\right] + \int_{0}^{\infty}\exp\left[ik_2\cdot X(\tau)\right]\exp\left[ik_3\cdot X(0)\right]|k_1\rangle$$

Apply the BCH formula to get exp $\left[ik_2\cdot X(\tau)\right] = \exp\left[ik_2\cdot x\right]\exp\left[\frac{i\tau}{m}k_2\cdot p\right]\exp\left[\frac{i\tau}{2m}k_2^2\right]$ in the

Apply the BCH formula to get exp $\left[ik_2\cdot X(\tau)\right] = \exp\left[ik_2\cdot x\right]\exp\left[\frac{i\tau}{m}k_2\cdot p\right]\exp\left[\frac{i\tau}{2m}k_2^2\right]$ in the *first*

Apply the BCH formula to get $\exp\left[ik_2\cdot X(\right]$

term and exp $\left[ik_2\cdot X(\tau)\right] = \exp\left[\frac{i\tau}{m}k_2\cdot p\right]\exp\left[ik_2\cdot x\right]\exp\left[-\frac{i\tau}{2m}k_2^2\right]$ *in the second term. Act with the operators containing p on $|k_1\rangle$ and $\langle -k_4|$ to get*

$$m^{-1}\Big(\int_{-\infty}^{0}d\tau e^{i\frac{\tau}{m}(k_2\cdot k_1+\frac{1}{2}k_2^2)} + \int_{-\infty}^{0}d\tau e^{i\frac{\tau}{m}(-k_2\cdot k_4-\frac{1}{2}k_2^2)}\Big)\langle -k_4|e^{i(k_2+k_3)\cdot x}|k_1\rangle\,.$$

Up to a possible normalization, $\langle -k_4|e^{i(k_2+k_3)\cdot x}|k_1\rangle = \delta(k_1+k_2+k_3+k_4)$*. The integrals may be evaluated with an* $i\varepsilon$ *prescription* $\tau \to \tau(1 \pm i\varepsilon)$ *to obtain the desired result. Only the 2 of 3 Feynman diagrams are included here (the ones with a certain topology) - in QFT one needs to count up different topologies by hand. NB the difference in string theory: only one topology at each loop order.*

Example IV.3. *Scattering amplitudes from the worldline (canonical quantization). Consider the action for a relativistic point particle on an open path in D-dimensional Minkowski space,*
(11)

$$S[e,X] = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \left[e^{-1}\dot{X}^2 - m^2 e\right] .$$

(a) Constraint. Write down the equation of motion from varying e. Show that it is equivalent to the vanishing of the Hamiltonian,
(12)

$$\mathcal{H} = \frac{1}{2}[e^{-1}\dot{X}^2 + m^2 e] = 0 .$$

Since e is an auxiliary field (it has no derivatives), this equation will be imposed as a constraint on physical states, $A|\psi\rangle = 0$*. (b) Vertex operators. Show that the'vertex operators',*
(13)

$$V(k) = \int d\tau\, e(\tau)\, \exp\left[ik \cdot X(\tau)\right] ,$$

are diffeomorphism-invariant for any momentum vector k. (c) Quantization. Explain why we can gauge-fix $e(\tau) = 1/m$ *on the infinite line (with freedom to choose the coefficient 1). In this gauge the canonical momentum is* $\mathcal{P} = m\dot{X}$*. (i) Convince yourself that the Schrodinger equation gives the time evolution*

$$X(\tau) = x + \tau\, \frac{1}{m}\, p\, , \quad \mathcal{P}(\tau) = p\, .$$

(14)
We impose the canonical commutation relation $[x^\mu, p^\nu] = i\eta^{\mu\nu}$*. (ii) Show, in this gauge, that* $V(k)$ *carries momentum* k^μ*,*

$$[p^\mu, V(k)] = k^\mu V(k) .$$

(15)
Eigenstates $|k\rangle$ *of p form a basis for the Hilbert space, with* $p^\mu|k\rangle = k^\mu|k\rangle$*. (iii) Show that* $|k\rangle$ *is physical* ($\mathcal{H}|k\rangle = 0$) *precisely on the mass-shell* $k^2 + m^2 = 0$*. We may think of* $|k\rangle \sim V(k)|vac\rangle$*, so the'physical' vertex operators* $V(k)$ *are those with* $k^2 + m^2 = 0$*. (d) Scattering amplitude. Let us consider the tree-level 4-point scattering amplitude* $\mathcal{A}(k_1, k_2, k_3, k_4)$ *in* ϕ^3 *theory. We regard this as the transition amplitude for a particle of momentum* $k_1 \to -k_4$*, while emitting particles of momentum* k_2 *and* k_3*,*

$$\langle -k_4|\mathrm{T}\big\{V(k_3)V(k_2)\big\}|k_1\rangle = \int_{-\infty}^{\infty}\frac{d\tau'}{m}\int_{-\infty}^{\infty}\frac{d\tau}{m}\langle -k_4|\mathrm{T}\big\{\exp\left[ik_3\cdot X(\tau')\right]\exp\left[ik_2\cdot X(\tau)\right]\big\}|k_1\rangle$$

where T denotes time ordering. There is a residual gauge symmetry: constant time translations $\tau \to \tau + c$*. We use it to fx the position of* $\tau' = 0$*,*

$$\int_{-\infty}^{\infty}\frac{d\tau}{m}\langle -k_4|\mathrm{T}\big\{\exp\left[ik_3\cdot X(0)\right]\exp\left[ik_2\cdot X(\tau)\right]\big\}|k_1\rangle .$$

(17)

By splitting the integral into $\int_-\infty^0$ and $\int_0^\infty$, show that this amplitude is proportional to

$$\left(\frac{1}{(k_1+k_2)^2+m^2}+\frac{1}{(k_1+k_3)^2+m^2}\right)\delta(k_1+k_2+k_3+k_4)\,.$$

(18)

You may wish to apply the Baker-Campbell-Hausdorff formula to the operator $\exp\left[ik_2\cdot X(\tau)\right]=\exp\left[ik_2\cdot x+\frac{i\tau}{m}k_2\cdot p\right]$. *Discuss why this approach misses the contribution* $\frac{1}{(k_1+k_4)^2+m^2}$ *from one of the three channels.*

Solution.

(a) Varying e gives $-e^{-}2\dot{X}^2-m^2=0$. (b) One can think of $e=\sqrt{-\det h}$ where h is a 1d metric, so $V(k)$ is the integral of a scalan against the diffeomorphism-invariant measure $d\tau\sqrt{-\det h}$. (c) (i) In this gauge $\mathcal{H}=\frac{1}{2m}(\mathcal{P}^2+m^2)$. The Schrodinger equation (equivalently the equation of motion from varying X) gives $\dot{P}=i[\mathcal{H},P]=0,\dot{X}=i[\mathcal{H},X]=\frac{1}{m}\mathcal{P}$. (ii) Note that $[p,X(\tau)]$ is independent of time since $[p,p]=0$. So $[p^\mu,V(k)]=m^{-1}\int d\tau[p^\mu,\exp\left[ik\cdot X(\tau)\right]=m^{-1}\int d\tau\, k^\mu\exp\left[ik\cdot X(\tau)\right]=k^\mu V(k)$ by stan- } dard arguments. (iii) $\mathcal{H}|k\rangle=\frac{1}{2m}(\mathcal{P}^2+m^2)|k\rangle=\frac{1}{2m}(k^2+m^2)|k\rangle$ vanishes only when $k^2+m^2=0$. (d) Splitting up the integral as indicated, the time ordering gives

$$m^{-1}\langle -k_4|\int_{-\infty}^0\exp\left[ik_3\cdot X(0)\right]\exp\left[ik_2\cdot X(\tau)\right]+\int_0^\infty\exp\left[ik_2\cdot X(\tau)\right]\exp\left[ik_3\cdot X(0)\right]|k_1\rangle$$

Apply the BCH formula to get exp $\left[ik_2\cdot X(\tau)\right]=\exp\left[ik_2\cdot x\right]\exp\left[\frac{i\tau}{m}k_2\cdot p\right]\exp\left[\frac{i\tau}{2m}k_2^2\right]$ in the

Apply the BCH formula to get exp $\left[ik_2\cdot X(\tau)\right]=\exp\left[ik_2\cdot x\right]\exp\left[\frac{i\tau}{m}k_2\cdot p\right]\exp\left[\frac{i\tau}{2m}k_2^2\right]$ in the *first*

Apply the BCH formula to get $\exp\left[ik_2\cdot X(\right]$

term and exp $\left[ik_2\cdot X(\tau)\right]=\exp\left[\frac{i\tau}{m}k_2\cdot p\right]\exp\left[ik_2\cdot x\right]\exp\left[-\frac{i\tau}{2m}k_2^2\right]$ *in the second term. Act with the operators containing p on $|k_1\rangle$ and $\langle -k_4|$ to get*

$$m^{-1}\Big(\int_{-\infty}^0 d\tau e^{i\frac{\tau}{m}(k_2\cdot k_1+\frac{1}{2}k_2^2)}+\int_{-\infty}^0 d\tau e^{i\frac{\tau}{m}(-k_2\cdot k_4-\frac{1}{2}k_2^2)}\Big)\langle -k_4|e^{i(k_2+k_3)\cdot x}|k_1\rangle\,.$$

Up to a possible normalization, $\langle -k_4|e^{i(k_2+k_3)\cdot x}|k_1\rangle=\delta(k_1+k_2+k_3+k_4)$. The integrals may be evaluated with an $i\varepsilon$ prescription $\tau\to\tau(1\pm i\varepsilon)$ to obtain the desired result. Only the 2 of 3 Feynman diagrams are included here (the ones with a certain topology) - in QFT one needs to count up different topologies by hand. NB the difference in string theory: only one topology at each loop order.

V. REMARKS ON GAUGE FIXING THE STRING ACTION

When you gauge fix you endup coverinf your surface with an atlas of complex dimension 1. The complex structure is a gauge invariant point. One can endow the surface in many ways (i.e. many ways to choose the complex structure) so

$$A_\Sigma=\int_{\mathcal{M}[\Sigma]=\text{Cplx. structures on}\Sigma}d\mu\int\mathcal{D}Xe^{-S}$$

i.e. the first integral is the "integral over moduli space"

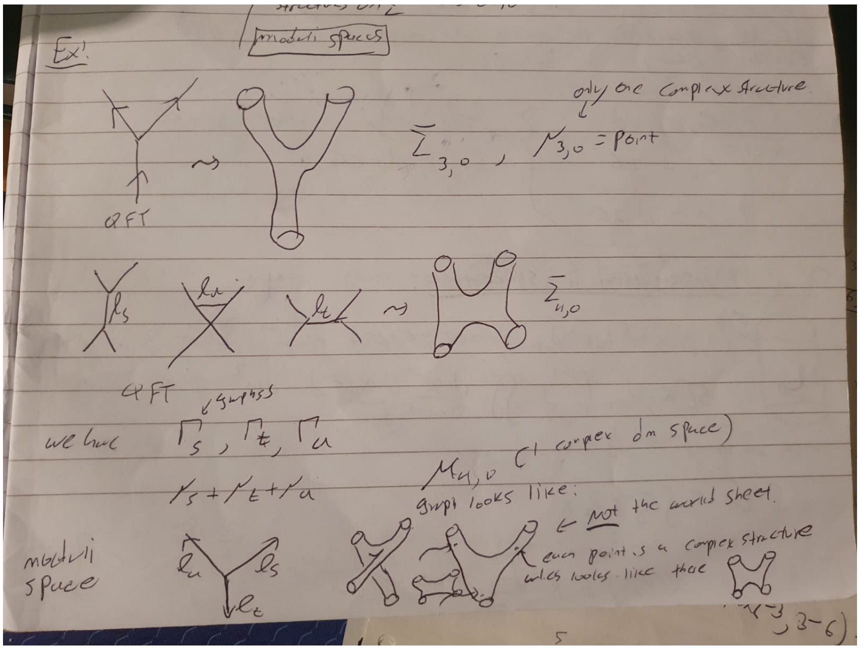
A. Cohmological QFT

Cohmological QFTs are theories with Grassman odd symmetry Q ($Q^2=0$) observables are those **closed** that are not exact.

$$\text{physical obs.}=\frac{Q-\text{closed}}{Q-\text{exact}}$$

as correlations are preserved iff Q acts on them. We can describe operators X^μ and $QX=C$ (secretly dx^μ) as Q is secretly d, with $QC=0$. So $f_{\mu_1\ldots\mu_n}(X)C^{\mu_1}..C^{\mu_n}$ can be integrated on

$$\int dx_1\ldots dx_ndC_1\ldots dC_nf_{\mu_1\ldots\mu_n}(X)C^{\mu_1}..C^{\mu_n}\equiv\langle f\rangle=\int f$$



so we intrinsically had to integrate over the C variables to find the f correlation. Studying $H(Q)$ is studying the topology of the manifold as we study correlation function.

1. *Menaing of gohsts*

Instead of considering orbits as 1 point, we introduce ghosts along the orbits such that locally they cancel out the bosonic directions, i.e. they reproduce the true degrees of freedom.

Obsercables of the form $\mathcal{O}(X,C) = \mathcal{O}_{\mu_1\ldots\mu_n}C^{i_1}\ldots C^{i_n}$ wich can be seen as forms by $C \to dx$, $Q \to d$, then

$$\int_N \omega = \int_M \omega \wedge df_1\delta(f_1) \wedge \ldots \wedge df_n\delta(f_n)$$

where $N_{d-1} \subset M_d$ defined by $\{f_\alpha(x) = 0\}_{\alpha=1^n}$.

One representation is

$$df_n\delta(f_n) = \int dB^i e^{iB^i f_\alpha} \int db^\alpha e^{b^\alpha df_\alpha}$$

so

$$\int dxdCdBdbe^{iB^\alpha f_a + b^a \partial_i f_a^{c^i}} \mathcal{O}\mathcal{O}\mathcal{O} \tag{1}$$

still has Q symmetry

$$\begin{cases} Qb = -Bi \\ QB = 0 \end{cases}$$

Example V.1. *Consider*

$$\int_{\mathbb{R}} \frac{\partial f}{\partial x}\delta(f(x)) = \sum_{x_i,s.t.\ f(x_i)=0} sign(\frac{\partial f}{\partial x})$$

is a topological invariant. One if the graph goes from $-\infty$ *to* ∞ *and* 0 *if it dont.*

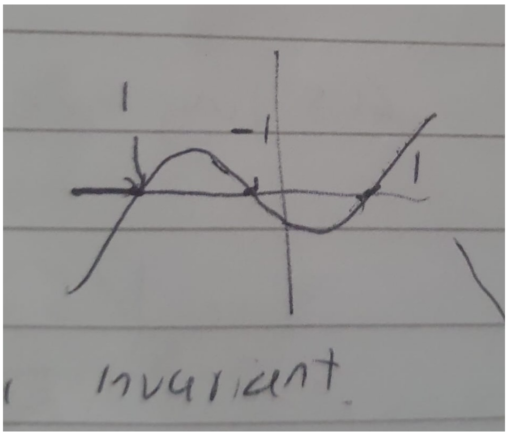


Figure 1. Image of manifold with a courve in the principla bundle. One curve intersecting the orbits perpendicular is fine. A curve with some topologicall twists is also fine due to the example above.

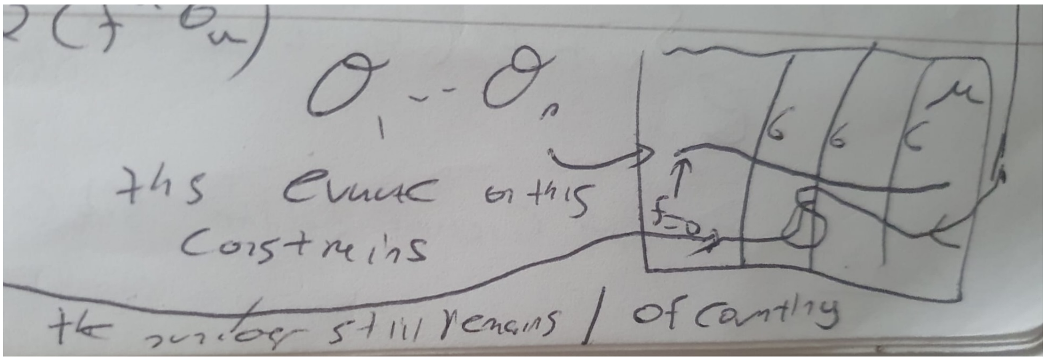


Figure 2. Image of manifold with a courve in the principla bundle. One curve intersecting the orbits perpendicular is fine. A curve with some topologicall twists is also fine due to the example above.

Let M be a manifold, G a Lie group acting on M, t^a generators of the lie algebra,

$$QX^i = C^a v^i_a(x) \implies Qf(X) = C^a v^i_a \frac{\partial f}{\partial x^i} \tag{2}$$

$$Q^2 X = 0 = QC^a v^i_a(x) = C^a C^b \partial_j v^j_{[a} v^i_{b]} = C^a C^b \partial_j f^c_{ab} v^i_c \tag{3}$$

$$QC^a = f^a_{bc} C^b C^c \tag{4}$$

$$Q^2 C^a = f^a_{[b\alpha} f^\alpha_{de]} C^b C^d C^e = 0 \text{due to jacobi} \implies \text{self consistent} \tag{5}$$

so we can talk about integrals in the space of orbits. If i want to gauge fix I put a bunch of constraints $\{f_\alpha(x) = 0\} \to B$ then a correlation function is

$$Z \to \int dX dc dB db e^{Q f^a b_a} \mathcal{O}_1 \dots \mathcal{O}_n$$

this evaluate on this constraints:

$$\int dx dc dB db e^{Q(f^a b_a)} \mathcal{O}_1 ... \mathcal{O}_n = \int_{\mathcal{M}/G} \implies Z = \int dx ... e^{Q(f_a b^a) + S[x]} \mathcal{O}_1 \dots \mathcal{O}_n = \int_{\mathcal{M}/G} e^{S[x]} \mathcal{O}_1 \dots \mathcal{O}_n \tag{6}$$

B. String theory?

If $X(t,x)$,

$$\int dxdcdc_t dtdBdbe^{iB^a f_a(x,t)+b^a\partial_i f_a(x,t)c^i+b^a\partial_t f_a(x,t)c^t}\mathcal{O}_1...\mathcal{O}_n \tag{7}$$

we always locally gauge fix to be flat the metric using Weyl symmetry but we cannot globally.

$$\int DX^\mu Dh^{ab}Dc^a Dc_{\text{Weyl}}Db^{ab}DB^{ab}\underbrace{d\mu}_{(1)}\underbrace{dC_\mu}_{(2)} \tag{8}$$

where (1) = possible states of complex structures (finite dimensional) and (2) partners. Note C^a are functions on the surface Σ to the tangent bundle.

$$QX^\mu = C^a\nabla_a X^\mu, \quad Qh_{ab} = \nabla_a C_b \nabla_b C_a + h_{ab}C^{Weyl}, \quad Qb^{ab} = B^{ab}$$

$$QC^{\text{Weyl}} = C^a\partial_a C^{Weyl}, \quad QC^a = C^b\nabla_b C^a$$

s o $Q(b^{ab}(h_{ab}-\hat{h}_{ab}(\mu))) = B^{ab}(h_{ab}-\hat{h}_{ab}(\mu)) - b^{ab}(\nabla_a c_b + \nabla_b c_a) - b^{ab}(h_{ab}C^{Weyl}) + b^{ab}\frac{\partial \hat{h}_{ab}}{\partial\mu}\hat{C}^\mu$

Now, integrate out $B, h, b^{ab}, h_{ab}, C^{Weyl}, \hat{C}^\mu$

$$\int \mathcal{D}XDb^{ab}DC^a e^{S[X,\hat{h}_{ab}(\mu)]+S_{gf}[\hat{h}_{ab}(\mu)]}\prod b^{ab}\frac{\partial\hat{h}_{ab}}{\partial\mu^i}\underbrace{d\mu^i}_{\text{Cplx. stru.}} \tag{9}$$

Where the X integral computes the correlation function of some field theory. **Caution!** When you integrate out fields you replace them with the EOM so for h we have a contribution from the action S_X and from $Q(b^{ab}(h_ab-\hat{h}_{ab}))$ i.e. the term $B^{ab}(h_{ab}-\hat{h}_{ab}(M))$. This sets

$$\frac{\delta S}{\delta h_{ab}} = B^{ab} = T^{ab}$$

Example V.2. *In the finite case,*

$$\int dxdcdBdbe^{\underbrace{iB^a f_a(x)+b^a\partial_i f_a c^i + B^2}_{-Q(b^a f_a + bB)}} = \int dxdcdbe^{-f^2+b^a\partial_i f_a c^i}$$

as B is exact. but now the transformation of b i.e. $Qb = f$ but $Q^2 b = \partial_i f c^i$ this only vanishes because equation of motion set $\partial_i f = 0$ but not identically.

CFT of the Ghost and so on… $J_{gh} = bc$

$$\{Q,b\} = T = T_X + T_{gh}$$

$$T_{gh} = c\partial b + 2\partial cb, \quad J_{BRST} = cT_x + bc\partial c \implies Q = \int J_{BRST} + \bar{J}_{BRST}$$

$$\{Q, b_n\} = L_n$$

$Q^2 \sim (d-26)$
Physical states:

$$L_n|n\rangle\otimes|0\rangle = \{Q,b_n\}|n\rangle\otimes|0\rangle = 0$$

so physical states are those in cohmology. It is sufficient as $|\tilde{m}\rangle = L_n|m\rangle$, $m<0$ reduces to

$$Q(b_n|\tilde{m}\rangle\otimes|0\rangle) = L_n|\tilde{m}\rangle\otimes|0\rangle + b_nQ|\tilde{m}\rangle\otimes|0\rangle = ... = L_n|\tilde{m}\rangle\otimes|0\rangle \tag{10}$$

So physical states are indeed in BRST cohmology.

VI. COMPLEX STRUCTURES

Stereographic projection of the plane to the sphere. Plane to cylinder gives$z = e^s$,

$$\frac{dzd\bar{z}}{|z|^2} = dsd\bar{s}$$

and sphere to cilinder

$$\frac{dsd\bar{s}}{(e^{(s+\bar{s})/2} + e^{-(s+\bar{s})/2})^2}.$$

What if we take plane with three marked points (punctured)

$$dzd\bar{z}(1 + \frac{1}{|z-z_1|^2} + \frac{1}{|z-z_2|^2} + \frac{1}{|z-z_3|^2})$$

and, if I remove one point the topology is that of a 3 punctured sphere. In general

$$dzd\bar{z}(\sum_{ij} \frac{1}{|z-z_i|^2|z-z_j|^2}) \tag{11}$$

In general, if you start from a n punctured sphere, diffeomorphism that map the metric to itself? rotations. But what about complex structures allowed? We already chose that by taking z coordinates so we can find $z = z(z')$ global a diffeo i.e. that maps the punctured sphere to a punctured sphere in other places. For this we set $z = \frac{az'+b}{cz'+d}$ this work fine in the sphere (only dosent work in infinity as it sends to 1 points) we can take $ad - bc = 1$ which then defines an $PSL(2, \mathbb{C})$ action.

Example VI.1. *$SL(2, \mathbb{C})$ $z \to \lambda z$ (rescaling) so these are rescaling, rotations (if λ is aphase), etc. These are the transf that lives the metric conformally flat. If we take a generic function $z = f(z')$ it dosent map to the sphere to sphere $1-1$, $Mobius$ transformation are special in this sense. Maps are really sensitive to holomorphy to not have poles and behave well.*

For example, suppose there is an invertible map from plane to sphere (which can't be among other reasons because S^2 is compact) this is equivalent to say to find a function on the sphere holomorphic and with no poles. Using Liouville's theorem, this function is bounded and holomorphic $\implies$ constant function $\implies$ non invertible.

Vector fields on the sphre that are globally well defined and holo are $\{\partial, z\,\partial, z^2\partial\}$ and their respective conjugates. This Mobius transform moves three points: $z' = \frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}$ this gives $z_2 \to 1$ $z_1 \to 0$ $z_3 \to \infty$. For 4-pt functions the best is $z_2 \to 1$ $z_1 \to 0$ $z_3 \to \infty$ and $z_4 \to z_4'$.

$$z' = \frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}$$

so for 3 points we can always map in the above way, no freedom. For 4 points we have 1 freedom.
Hence,

$$\mathcal{M}_{0,3} = \{\cdot\}$$

$$\mathcal{M}_{0,4} = S^2 - \{0, 1, \infty\}$$

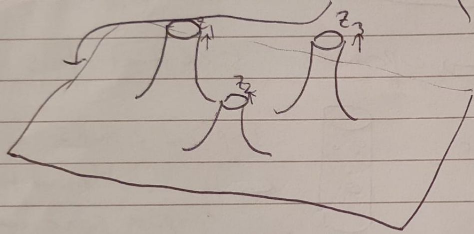
so $\mathcal{M}_{0,4}$ is a "pair of pants" $=$ sphere with 3 distinguished points.
For theories like QFT, the different s, t and u channels. So moduli is just a line.

VII. THE MOORE-SEIBERG CONSTRUCTION ON RCFTS

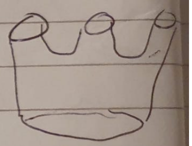
VIII. MODULAR BOOTSTRAP AND LINE DEFECTS

$$\frac{ds\,d\bar{s}}{\left(e^{\frac{s+\bar{s}}{2}}+e^{-\frac{(s+\bar{s})}{2}}\right)^2}$$

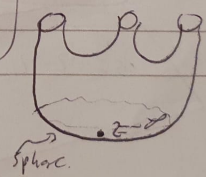
what if we take $dz\,d\bar{z}\left(1+\frac{1}{|z-z_1|^2}+\frac{1}{|z-z_2|^2}+\frac{1}{|z-z_3|^2}\right)$



If I remove 1 ⇝



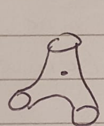
or $dz\,d\bar{z}\left(\sum \frac{c_{ij}}{|z-z_i|^2|z-z_j|^2}\right)$

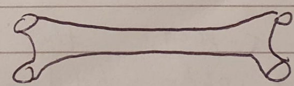
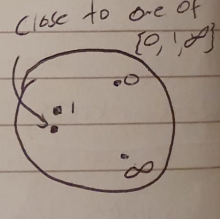


So for 3 points we can always map in this way so no freedom. In 4-points we have 1 freedom.

Hence $\mathcal{M}_{0,3}$ = point , $\mathcal{M}_{0,4} = S^2 - \{0,1,\infty\}$

↑ moduli space of sphere 3-punctures

So $\mathcal{M}_{0,4} \iff$  ← sphere without 3 points with the choice of choosing the forth.

For theories (like QFT)  ⇝ 

this differentiate S, T or U channel.