Feynman rules for fermions

Recall $T(\psi(x)\bar\psi(y)) = \begin{cases} \psi(x)\bar\psi(y) & x^0>y^0 \\ -\bar\psi(y)\psi(x) & x^0<y^0 \end{cases}$

then $S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i(\not p+m)}{p^2-m^2+i\epsilon}\, e^{-ip\cdot(x-y)}$
$= \langle 0|T\psi(x)\bar\psi(y)|0\rangle$

$T(\psi\ldots) = :(\psi\ldots + \text{all possible contractions}):$

$\overline{\psi(x)\bar\psi(y)} = S_F(x-y)$

$\overline{\psi(x)\psi(y)} = \overline{\bar\psi(x)\bar\psi(y)} = 0$

$\overline{a(x)\,b(y)} = \begin{cases} [a^+(x), b^-(y)] & x^0>y^0 \\ [b^+(y), a^-(x)] & y^0>x^0 \end{cases}$

$\overline{\psi(x)\bar\psi(y)} = \begin{cases} \{\psi^+(x), \bar\psi^-(y)\} & x^0>y^0 \\ -\{\bar\psi^+(y), \psi^-(x)\} & x^0<y^0 \end{cases}$

what happens if I introduce a Yukawa field? Consider fermion + fermion $\to$ fermion + fermion

$\Delta H = \int d^3x\, g\bar\psi\psi\phi$.

$\hookrightarrow \langle p',k'|T\exp(-i\int d^4x\, g\bar\psi\psi\phi)|p,k\rangle$

$= 0 + 0 + \langle p',k'|\frac{1}{2}(-ig)\int d^4x\, \bar\psi_I\psi_I\phi_I\,(-ig)\int d^4y\, \bar\psi_I\psi_I\phi_I|p,k\rangle$

$\psi_I(x)|p,s\rangle = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum_{s'} a_{p'}^{s'} u^{s'}(p')\, e^{-ip'\cdot x}\left(\sqrt{2E_p}\, a_p^{s\dagger}|0\rangle\right)$

$= e^{-ip\cdot x} u^s(p)|0\rangle \quad \bar u(p') e^{ip'x} e^{-ipx} u(p)\; e^{ik'y}\bar u(k') e^{-iky} u^s(k)$

$= \frac{(-ig)^2}{2}\int\frac{d^4q}{(2\pi)^4}\int d^4x\, d^4y\, \frac{i}{q^2-m_\phi^2}\, e^{-iq(x-y)}$

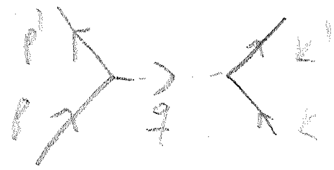
$= (-ig)^2\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m_\phi^2}(2\pi)^4\delta^4(p'-p+q)(2\pi)^4\delta^4(k'-k-q)\,\times$

$\bar u(p')u(p)\,\bar u(k')u(k)$

respect the order

$\langle p', k'|(-ig)\int d^4x\, \bar\psi\psi\phi\, (-ig)\int d^4y\, \bar\psi\psi\phi\, |p, k\rangle$

22

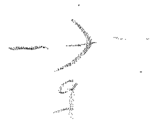


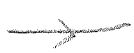
This gives as before $(-ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2-m_\phi^2} (2\pi)^4 \delta^4(p'-p+q) \times$ $(2\pi)^4 \delta^4(k'-k-q)\, \bar u(p') u(p)\, \bar u(k') u(k)$

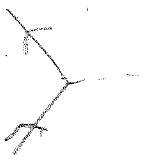
$\Rightarrow i\mathcal{M} = \frac{-ig^2}{q^2-m_\phi^2} \bar u(p') u(p) \bar u(k') u(k)$

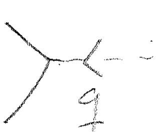
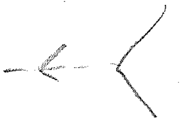
+ condition $p-p'=q=k'-k$

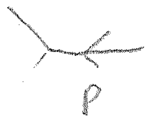
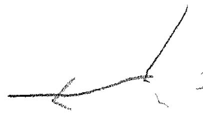
Using Feynman rules

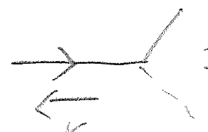
$\phi(x)\phi(y) =$  $= \frac{i}{q^2-m_\phi^2+i\epsilon}$

$\psi(x)\bar\psi(y) =$  $= \frac{i(\not p+m)}{p^2-m^2+i\epsilon}$

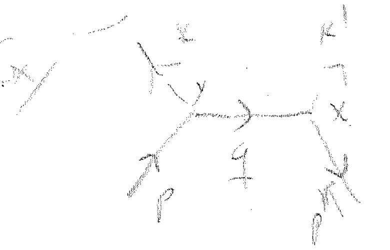
 $= -ig$

$\phi|q\rangle =$  $= 1$ $\qquad \langle q|\phi =$  $= 1$

$\psi|p, s\rangle =$  $= u^s(p)$ $\qquad \underbrace{\langle p, s|}_{\text{fermion}}\bar\psi =$  $= \bar u^s(p)$

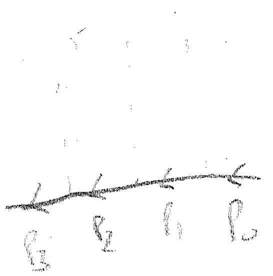
$\underbrace{\bar\psi|k, s\rangle}_{\text{anti fermion}} =$  $= \bar v^s(k)$ $\qquad \underbrace{\langle k, s|}_{\text{antifermion}}\psi =$  $= v^s(k)$

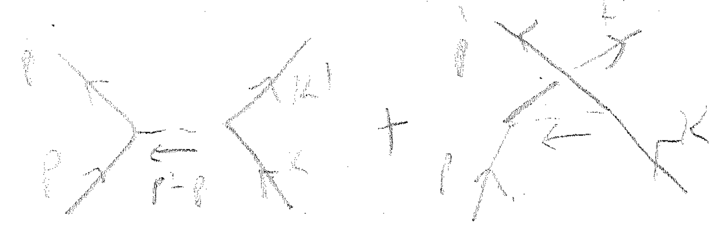
Yukawa theory never has symmetry factors the $\frac{1}{n!}$ of exponential cancels with the $n!$ exchange of vertices $\int d^4x_1 \dots \int dx_n$ and non of $\phi\bar\psi\psi$ are interchangeable so no symmetry factors.

 $= \langle k, k' | \int d^4x \, \phi \bar\psi\psi \int d^4y \, \phi \bar\psi\psi \, | p, p' \rangle$

$$\sim \int d^4x \int d^4y \, e^{ik'\cdot x} \bar{v}(p') e^{-ip'x} \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q}+m)}{q^2-m^2} e^{-iq(x-y)} u(p) e^{-ip\cdot y} e^{ik\cdot y}$$

Dirac indices contract together along fermion lines.

 $\sim \bar{u}(p_3) \frac{i(\not{p}_2+m)}{p_2^2-m^2} \frac{i(\not{p}_1+m)}{p_1^2-m^2} u(p_0)$

So $i\mathcal{M} =$ 

$$= (-ig)^2 \left(\bar{u}(p')u(p) \frac{i}{(p'-p)^2 - m_\phi^2} \bar{u}(k')u(k) \right.$$

$$\left. - \bar{u}(p')u(k') \frac{i}{(k-p')^2 - m_\phi^2} \bar{u}(k')u(p) \right)$$

In a nonrelativistic result

$u^s(p) = \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \to \bar{u}^{s'}(p')u^s(p) = 2m\delta^{ss'}$

distinguishable ptcs.

So $i\mathcal{M} = \frac{ig^2}{|\vec{p}'-\vec{p}|^2 + m_\phi^2} 2m\delta^{ss'} 2m\delta^{rr'} + O$

Compare with Born approx.

$$\langle p'| iT |p\rangle = -i\,\tilde{V}(\underline{q})\,(2\pi)\,\delta(E_{p'}-E_p)$$

so $$\tilde{V}(q) = \frac{-g^2}{|\underline{q}|^2+m_\phi^2} \rightsquigarrow V(x) = \int \frac{d^3q}{(2\pi)^3}\,\frac{-g^2}{|\underline{q}|^2+m_\phi^2}\,e^{iq\cdot x}$$

$$= \frac{-g^2}{4\pi^2 ir}\int_{-\infty}^{\infty} dq\,\frac{q\,e^{iqr}}{q^2+m_\phi^2}$$

$$\rightsquigarrow V(r) = -\frac{g^2}{4\pi}\,\frac{1}{r}\,e^{-m_\phi r}$$

(a Yukawa potential)

$$S=\int d^4x \underbrace{\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)}_{\mathcal{L}} \quad , \quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

a)

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right)=\partial_\mu\left(-\frac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)(\partial^\mu A^\nu-\partial^\nu A^\mu)\right)$$

$$=\partial_\mu\left(-\frac{2}{4}\left((\partial^\mu A^\nu-\partial^\nu A^\mu)+(\partial^\mu A^\nu-\partial^\nu A^\mu)\right)\right)$$

$$=-\partial_\mu F^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial A_\nu}=0 \rightsquigarrow \boxed{\partial_\mu F^{\mu\nu}=0}$$

And

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \rightsquigarrow \partial_\lambda F_{\mu\nu}=\partial_\lambda\partial_\mu A_\nu-\partial_\lambda\partial_\nu A_\mu$$

$$\rightsquigarrow \sum \varepsilon_{\mu\lambda\rho\sigma}\underbrace{\partial_\lambda F_{\rho\sigma}}_{\underbrace{\partial_\lambda\partial_\rho A_\sigma-\partial_\lambda\partial_\sigma A_\rho}_{\text{symmetric}}}=\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}=0$$

antisymmetric

$$-\partial_0 F^{0\nu}-\partial_i F^{i\nu}=\partial_\mu F^{\mu\nu} \quad \begin{cases}\nabla\cdot E=0 & , \nu=0\\ -\frac{\partial}{\partial t}E^i+\partial_j\varepsilon^{ijk}B_k=0 & , \nu=i\end{cases}$$

$$\vec\nabla\times\vec B-\frac{\partial\vec E}{\partial t}=0$$

$$\varepsilon_{0ijk}\partial_i F_{jk}=-\varepsilon_{0ijk}\partial_i\varepsilon_{jkl}B^l$$

$$=+2\delta_{il}\partial_i B^l \Rightarrow \boxed{\nabla\cdot B=0}$$

$$\varepsilon_{ijk0}\partial_j \overset{E_k}{F_{k0}}=\varepsilon_{ijk0}\partial_j E_k=0$$

$$\Rightarrow \boxed{\nabla\times E=0}$$

b) $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, $A_\mu(x) \to \Lambda_\mu^{\ \nu} A_\nu(\Lambda^{-1} x - a)$

$\Lambda^\nu_{\ \mu} = \left(e^{i\alpha^{\mu\nu} M_{\mu\nu}}\right)_\mu^{\ \nu} \rightsquigarrow \Lambda^{-1} x = \left(e^{-i\alpha^{\mu\nu}\omega_{\mu\nu}}\right)_\alpha^{\ \beta} x_\beta$

$\Lambda_\mu^{\ \nu} A_\nu(\Lambda^{-1} x - a) = A_\nu(\Lambda^{-1}x - a) + i\alpha^{\alpha\beta}(M_{\alpha\beta})_\mu^{\ \nu} A_\nu(x - a - i\alpha^{\mu\nu}\omega_{\mu\nu} \ldots$

$= A_\mu(x) + i\alpha^{\alpha\beta}(M_{\alpha\beta})_\mu^{\ \nu} A_\nu(x) + a^\alpha \partial_\alpha A_\mu(x)$

$+ i\alpha^{\alpha\beta}(M_{\alpha\beta})_\mu^{\ \nu} a^\sigma \partial_\sigma A_\nu(x) + \ldots$

$\hookrightarrow \delta_\alpha A_\nu = i\alpha^{\alpha\beta}(M_{\alpha\beta})_\mu^{\ \nu}\left(A_\nu(x) + a^\sigma \partial_\sigma A_\nu(x)\right)$

$+ a^\alpha \partial_\alpha A_\mu(x).$

Space-time translations $\rightsquigarrow \delta_\alpha A_\nu = \partial_\alpha A_\nu(x)$

$\Rightarrow J^\mu_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \delta_\alpha A_\nu = -F^{\mu\nu}\partial_\alpha A_\nu - \delta^\mu_{\ \alpha}\mathcal{L}$

$= -F^{\mu\nu}\partial_\alpha A_\nu + \frac{1}{4}\delta^\mu_{\ \alpha} F_{\lambda\kappa}F^{\lambda\kappa}$

and $\mathcal{L} \to \mathcal{L} + a^\alpha \partial_\alpha \mathcal{L} \to \delta_\alpha \mathcal{L} = \partial_\alpha \mathcal{L}$

$= \partial_\mu(\delta^\mu_\alpha \mathcal{L})$

we can add $\partial_\lambda K^{\lambda\mu\nu}$, K antisymmetric in λ,μ.

so $T^{\mu\alpha} = -F^{\mu\nu}\partial^\alpha A_\nu + \frac{1}{4}\eta^{\mu\alpha} F_{\lambda\kappa}F^{\lambda\kappa} + \partial_\lambda K^{\lambda\mu\alpha}$

$T^{\alpha\mu} = T^{\mu\alpha} = -F^{\alpha\nu}\partial^\mu A_\nu + \frac{1}{4}\eta^{\mu\alpha} F_{\lambda\kappa}F^{\lambda\kappa} + \partial_\lambda K^{\lambda\alpha\mu}$

so $F^{\alpha\nu}\partial^\mu A_\nu - F^{\mu\nu}\partial^\alpha A_\nu = \partial_\lambda(K^{\lambda\alpha\mu} - K^{\lambda\mu\alpha})$

$T^{\mu\nu} = F^{\mu\lambda}F^{\nu}_{\ \lambda} + \frac{1}{4}\eta^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa}$

Consider the complex field

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 |\phi|^2$$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \dot{\phi}^* , \quad (\pi)^* = \dot{\phi}$$

$$[\phi(x), \pi(y)] = i\delta^3(x-y) = [\phi^*(x), \pi^*(y)]$$

$$H = \int d^3x \; \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}$$

$$= \int d^3x \; |\pi|^2 + |\pi|^2 - |\pi|^2 + |\nabla\phi|^2 + m^2|\phi|^2$$

$$= \int d^3x \; |\pi|^2 + |\nabla\phi|^2 + m^2 \phi^2$$

$\nabla\phi \nabla\phi^*$

Eq. of motion

$$\dot{\pi}^*_{(y)} = \frac{\partial^2 \phi_{(y)}}{\partial t^2} = i[H, \pi_{(y)}] = i\int d^3x \; [|\nabla\phi|^2, \pi] + m^2 [|\phi|^2, \pi]$$

$$= i\int d^3x \; -(\partial_i \partial^i \phi)(+i\delta(x-y)) + m^2\phi(+i\delta(x-y))$$

$$= +\partial_i\partial^i\phi(y) - m^2\phi(y)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial t} - \frac{\partial^2}{\partial x^i \partial x_i} + m^2\right)\phi = 0$$

$$(\Box - m^2)\phi = 0$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + b_p^\dagger e^{ip\cdot x}\right)$$

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x}\right)$$

$[\phi(x), \pi(y)] = i\delta(x-y)$ then set $\left(\phi(x) = \int \frac{d^3 p}{(2\pi)^3}\, \Phi(p)\, e^{ip\cdot x}\right.$

$\left(\pi(x) = \int \frac{d^3 p}{(2\pi)^3}\, \Pi(-p)\, e^{+ip\cdot x}\right.$

$\hookrightarrow$ plugging in;

$$[\Phi(p), \Pi(q)] = (2\pi)^3 \delta^3(p-q)$$

and

$$a_p = \Pi(-p) - i\omega(p)\, \Phi(p)$$ or something like that

$$a_p^\dagger = \Pi^*(-p) + i\omega_p\, \Phi^*(p)$$ (same for a_p, b_p) $\phi \sim (a_p + b_p^\dagger)$

$\hookrightarrow$ $H = \int d^3x\; \pi^2 + (\nabla\phi)^2 + \mu^2\phi^2$

$$= \int d^3p\; \omega_p\, (a_p^\dagger a_p + b_p^\dagger b_p + \ldots)$$

Now $\phi \to e^{i\alpha}\phi$ is a symmetry. $\delta\mathcal{L} = 0$.

and $\delta\phi = i\phi \rightarrow \delta\phi^* = -i\phi^*$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi = \partial^\mu\phi^*(i\phi) - i\partial^\mu\phi\,\phi^*$$
$$= i(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi)$$

$$Q = \int d^3x\; i(\phi\dot\phi^* - \phi^*\dot\phi)$$
$$= \int d^3x\; \frac{i}{2}(\phi\pi - \phi^*\pi^*)$$

Now Consider $\mathcal{L} = \sum_{i=1}^{2} \partial_\mu \phi_i \partial^\mu \phi_i^* - m^2 |\phi_i|^2$

The symmetry arise $\phi_i \to e^{i\alpha} \phi_i$

But we Can write $\mathcal{L}$

$$\mathcal{L} = \partial_\mu (\phi_1^* \phi_2^*) \partial^\mu \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} - m^2 (\phi_1^* \phi_2^*) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Let $U \in M$ then Consider

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (\phi_1^* \phi_2^*) \to (\phi_1^* \phi_2^*) U^\dagger$$

$\hookrightarrow$ this is a symmetry iff $U^\dagger U = \mathrm{Id}$ i.e.

$U \in SU(2)$ so we have $SU(2) \times U(1)$ symmetry.

$SU(2)$ is generated by Pauli matrices $e^{i\alpha^i \sigma_i}$ then

Intop of the $U(1)$ symmetries we have.

$$\delta_{\sigma_i} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = i\sigma_i \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \Rightarrow \delta_{\sigma_i} \phi_a = i\sigma_{iab} \phi_b$$

$$\hookrightarrow J_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta_{\sigma_i} \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a^*)} \delta_{\sigma_i} \phi_a^*$$

$$= i \partial^\mu \phi_a^* \sigma_{iab} \phi_b + (-i) \partial^\mu \phi_a (\sigma_{iab})^* \phi_b^*$$

$$\hookrightarrow Q = i \int d^3x \; \phi_b \sigma_{iab} \Pi_a - \phi_b^* (\sigma_i^*)_{ab} \Pi_a^*$$

Lorentz group ... by its Lie algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} + g^{\mu\sigma}M^{\nu\rho}\right)$$

Let $L^i = \frac{1}{2}\varepsilon^{ijk}M^{jk}$, $K^i = M^{0i}$

then $R(\Lambda)\phi = (-\theta\cdot L - \beta\cdot K)\phi$

infinitesimally.

$$[L^i, L^j] = \frac{1}{2}\varepsilon^{ilm}[M^{lm}, M^{pq}]\varepsilon^{jpq}$$

$$= \frac{i}{2}\varepsilon^{ilm}\varepsilon^{jpq}\left(-\delta^{mp}M^{lq} - \delta^{lp}M^{mq} - \delta^{mq}M^{lp} + \delta^{lq}M^{mp}\right)$$

$$= \frac{i}{2}\left(+\varepsilon^{ipl}\cancel{\varepsilon^{jpq}}M^{lq} - \varepsilon^{ipm}\cancel{\varepsilon^{jpq}}M^{mq} - \varepsilon^{ilq}\varepsilon^{jpq}M^{lp} - \varepsilon^{inq}\varepsilon^{jpq}M^{np}\right)$$

$$= -i\,\varepsilon^{inq}\varepsilon^{jpq}M^{np} = \frac{i\varepsilon^{ijk}}{2}\varepsilon^{klm}M^{lm}$$

so

$$[J^i_\pm, J^j_\pm] = \varepsilon^{ijk}J^k_\pm \qquad [J_+, J_-] = 0$$

where $J^i_\pm = \frac{1}{2}(L^i \pm iK^i)$ form SU(2) algebras.

The reps are

$(\frac{1}{2}, 0)$: $J_- = 0$, $J^i_+ = \frac{\sigma^i}{2}$

$L^i = iK^i$ ~ $\boxed{\frac{\sigma^i}{2} = L^i}$ and

$\hookrightarrow \boxed{K^i = -\frac{i\sigma^i}{2}}$ so.

A lorentz transf is

$$\psi_L \to e^{-i\theta^i \frac{\sigma^i}{2} - i(\beta^i \frac{-i}{2}\sigma^i)} \psi_L$$

$$= e^{-i\theta^i \frac{\sigma_i}{2} - \beta^i \frac{\sigma_i}{2}} \psi_L$$

$$\psi_R \to e^{-i\theta^i \frac{\sigma_i}{2} + \beta^i \frac{\sigma_i}{2}} \psi_R$$

$$\psi \to \underbrace{e^{-i\frac{\theta\cdot\sigma}{2}\otimes \mathbb{I}_2 - i\frac{\beta\cdot\sigma}{2}\otimes\sigma_3}}_{R(\Lambda)} \psi$$

more over, we $\sigma_i^T = -\sigma^2 \sigma_i \sigma^2$

$$\psi = \psi_L^T \sigma^2 = -\sigma^2 \psi_L \sigma^2$$

$$\psi_L^T \sigma_2 \to \left(\left(1 - \frac{i\theta\cdot\sigma}{2} + \frac{\beta\cdot\sigma}{2}\right)\psi_L\right)^T \sigma_2$$

$$= \psi_L^T \left(1 - i\frac{\theta\cdot\sigma^T}{2} + \frac{\beta\cdot\sigma^T}{2}\right)\sigma_2$$

$$\psi_L^T \sigma_2 \to \psi_L^T \sigma^2 \left(1 + \frac{i}{2}\theta\cdot\sigma + \frac{\beta\cdot\sigma}{2}\right) \leftarrow \text{rrow}$$

then we can put the $(\frac{1}{2}, \frac{1}{2})$ rep in matrix form

$$\psi = \psi_R \psi_L^T \sigma^2 \equiv V^\mu \bar{\sigma}_\mu = \begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} = V^\mu \bar{\sigma}_\mu$$

transfs wrt left and right the same.

spinor idx

$$\psi_{\alpha\dot{\alpha}}$$

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left(\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'_\nu - p_\nu)}{2m}\right)u(p)$$

$$= \frac{\bar{u}(p')}{2m}\left(\underbrace{g^{\mu\nu}}_{\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}}(p'_\nu + p_\nu) + \underbrace{i\sigma^{\mu\nu}}_{\frac{i}{2}[\gamma^\mu,\gamma^\nu]}(p'_\nu - p_\nu)\right)u(p)$$

$$= \frac{\bar{u}(p')}{2m}\left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}(p'_\nu + p_\nu) - \frac{1}{2}[\gamma^\mu,\gamma^\nu](p'_\nu - p_\nu)\right)u(p)$$

$$= \frac{\bar{u}(p')}{2\cdot 2m}\left(\gamma^\mu(\not{p}' + \not{p}) + (\not{p}' + \not{p})\gamma^\mu - (\gamma^\mu(\not{p}' - \not{p}) - (\not{p}' - \not{p})\gamma^\mu)\right)u(p)$$

$$= \frac{\bar{u}(p')}{2\cdot 2m}\left(2\gamma^\mu\not{p} + 2\not{p}'\gamma^\mu\right)u(p) = \frac{\bar{u}(p')}{2m}\gamma^\mu u(p)m + \frac{\bar{u}(p')}{2m}\gamma^\mu u(p)m$$

[Since $(\not{p} - m)u(p) = 0$
$\bar{u}(p')(\not{p}' - m) = 0$]

$$= \bar{u}(p')\gamma^\mu u(p)$$

Spinor Products

Let k_0^μ, k_1^μ satisfying $k_0^2 = 0$, $k_1^2 = -1$, $k_0 \cdot k_1 = 0$.

Let $u_{L0} = u_L(k_0)$, $u_{R0} = \not{k}_1 u_{L0}$

and let for any $p^2 = 0$, (any massless fermion)

$$\begin{cases} u_L(p) = \dfrac{1}{\sqrt{2p\cdot k_0}} \not{p}\, u_{R0} \\ u_R(p) = \dfrac{1}{\sqrt{2p\cdot k_0}} \not{p}\, u_{L0} \end{cases}$$

a) $\not{k}_0 u_{R0} = \not{k}_0 \not{k}_1 u_L(k_0) = \gamma^\mu \gamma^\nu k_{\mu 0} k_{\nu 1} u_L(k_0)$

$= (2g^{\mu\nu} - \gamma^\nu \gamma^\mu k_{\mu 0} k_{\nu 1}) u_L(k_0) = 2 \cancel{k_0 \cdot k_1} - \not{k}_1 \underbrace{\not{k}_0 u_L(k_0)}_{\text{Dirac eq.}}$

$= 0.$

For p similarly, $\not{p}\, u_L(p) = \dfrac{\not{p}^2}{\sqrt{2p\cdot k_0}} u_{R0}$

$= \dfrac{\gamma^\mu \gamma^\nu p_\mu p_\nu}{\sqrt{2p\cdot k_0}} u_{R0} = \dfrac{(2p^2 - \not{p}^2)}{\sqrt{2p\cdot k_0}} u_{R0} = \dfrac{-\not{p}\, u_{R0}}{\sqrt{2p\cdot k_0}}$

$$\Rightarrow \begin{cases} \not{p}\, u_L(p) = 0 \\ \not{p}\, u_R(p) = 0 \end{cases}$$

b) If $k_0 = (E, 0, 0, -E)$, $k_1 = (0, 1, 0, 0)$

$k_0^2 = E^2 - E^2 = 0$, $k_1^2 = -1$, and $k_0 \cdot k_1 = 0$.

u_{L0} is solution to $\not{k}_0 u_{L0} = (E\gamma_0 + E\gamma_3) u_{L0}$

In the Weyl rep.

$$\left(\begin{pmatrix} 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & E & 0 \\ 0 & 0 & 0 & -E \\ -E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \end{pmatrix}\right) u_{L0} = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E \\ 2E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} u_{L0} = 0 \qquad \begin{pmatrix} 0 \\ \alpha \\ x \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad u_{L0} = \begin{pmatrix} x \\ 0 \\ \vdots \\ \end{pmatrix} = \begin{pmatrix} 0 \\ x \\ \vdots \\ 0 \end{pmatrix}$$

Choose $\quad u_{L0} = \begin{pmatrix} 0 \\ \sqrt{2E} \\ 0 \\ 0 \end{pmatrix}$

$$v_{R0} = \not{K} u_{L0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{2E} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2E} \\ 0 \end{pmatrix}$$

$$u_L(p) = \frac{\not{p}}{\sqrt{2(p_0E + p_3E)}} \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2E} \\ 0 \end{pmatrix}$$

$$\not{p} = \begin{pmatrix} 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \\ p_0 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \vec{p}\cdot\vec{\sigma} \\ -\vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \begin{matrix} p_0+p_3 & p_1-ip_2 \\ p_1+ip_2 & p_0-p_3 \end{matrix} \\ \begin{matrix} p_0-p_3 & -p_1+ip_2 \\ -p_1-ip_2 & p_0+p_3 \end{matrix} & 0 \end{pmatrix}$$

$$= \frac{-\sqrt{2E}}{\sqrt{2E(p_0+p_3)}} \begin{pmatrix} p_0+p_3 \\ p_1+ip_2 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{p_0+p_3}} \begin{pmatrix} -(p_0+p_3) \\ -(p_1+ip_2) \\ 0 \\ 0 \end{pmatrix}$$

$$v_R(p) = \frac{1}{\sqrt{2E(p_0+p_3)}} \begin{pmatrix} 0 & \not{p} \\ \not{p} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{2E} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -p_1+ip_2 \\ p_0+p_3 \end{pmatrix} \frac{1}{\sqrt{p_0+p_3}}$$

Let $s(p,q) = \overline{u_R}(p)\, u_L(q)$

$t(p,q) = \overline{u_L}(p)\, u_R(q)$

$$s(p,q) = \begin{pmatrix} 0 & 0 & -p_1 - ip_2 & p_0 + p_3 \end{pmatrix} \frac{\gamma_0}{\sqrt{p_0+p_3}} \begin{pmatrix} -q_0 + q_3 \\ -(q_1 + iq_2) \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{q_0+q_3}}$$

$$= -\frac{(-p_1 - ip_2) \times (q_0 + q_3) + (p_0+p_3)(q_1 + iq_2)}{\sqrt{(p_0+p_3)(q_0+q_3)}} = -s(q,p)$$

$$t(p,q) = \begin{pmatrix} -p_0 - ip_3 & -(p_1 - ip_2) & 0 & 0 \end{pmatrix} \frac{\gamma_0}{\sqrt{p_0+p_3}} \begin{pmatrix} 0 \\ 0 \\ -q_1 + iq_2 \\ q_0 + q_3 \end{pmatrix} \frac{1}{\sqrt{q_0+q_3}}$$

$$= -\frac{-(p_0+p_3)(q_1 + q_2) + (p_1 - ip_2)(q_0+q_3)}{\sqrt{(p_0+p_3)(q_0+q_3)}}$$

$$= -\left(\frac{(p_0+p_3)(-q_1 - iq_2) + (p_1 + ip_2)(q_0+q_3)}{\sqrt{(p_0+p_3)(q_0+q_3)}}\right)^*$$

$$= s(q,p)^*$$

$$|s(p,q)|^2 = \frac{1}{(p_0+p_3)(q_0+q_3)} \times \left((p_0+p_3)(q_1 - iq_2) - (p_1 - ip_2)(q_0+q_3)\right) \times$$

$$\left((p_0+p_3)(q_1 - iq_2) - (p_1 - ip_2)(q_0+q_3)\right)$$

$$= \left(\frac{(q_1 + iq_2)}{q_0+q_3} - \frac{p_1 + ip_2}{p_0+p_3}\right)\left((p_0-p_3)(q_1 - iq_2) - (p_1 - ip_2)(q_0+q_3)\right)$$

$$= \frac{(q_1^2 + q_2^2)(p_0+p_3)}{q_0+q_3} - (q_1 - iq_2)(p_1 + ip_2) - (p_1 - ip_2)(q_1 - q_2) + (p_1^2 + p_2^2)\frac{(q_0+q_3)}{p_0+p_3}$$

$$= (q_1^2 + q_2^2)\frac{p_0+p_3}{q_0+q_3} + (p_1^2+p_2^2)\frac{q_0+q_3}{p_0+p_3} - 2(q_1 p_1 + q_2 p_2)$$

Since $p^2 = 0 = q^2$, $q_1^2 + q_2^2 = -q_3^2 + q_0^2$

$p_1^2 + p_2^2 = -p_3^2 + p_0^2$

$$= (q_0^2 - q_3^2)\frac{(p_0+p_3)}{(q_0+q_3)} + (p_0^2 - p_3^2)\frac{(q_0+q_3)}{p_0+p_3} - 2(q_1p_1 + q_2 p_2)$$

$$= (q_0 - q_3)(p_0+p_3) + (p_0-p_3)(q_0+q_3) - 2(q_1 p_1 + q_2 p_2)$$

$$= 2q_0 p_0 - 2q_3 p_3 - 2q_1 p_1 - 2q_2 p_2 = 2\, p \cdot q$$

So [illegible] the spin products are square root of [illegible]

The Dirac equation can be decomposed in two Left and right eq for spinors. In the weyl basis,

$$\gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \text{i.e.} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$(i\not{\partial} - m)\psi = 0$$

$$\| $$

$$(i\gamma^\mu \partial_\mu - m)\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \left(i\left(\begin{pmatrix} 0 & 0 & \partial_0 & 0 \\ 0 & 0 & 0 & \partial_0 \\ \partial_0 & 0 & 0 & 0 \\ 0 & \partial_0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & i\sigma\cdot\nabla \\ -\sigma\cdot\nabla & 0 \end{pmatrix} \right) - m \right) \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

which gives

$$i\partial_0 \psi_R - \sigma\cdot\nabla\psi_R - m\psi_L = 0$$

$$i\partial_0 \psi_L + \sigma\cdot\nabla\psi_L - m\psi_R = 0$$

what lagrangian (eq of motion generates Majorana Condition?

Majorana condition

$$C\psi^* = i\gamma_2 \psi^* \equiv \psi$$

$$\|$$

$$i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} \psi_L^* \\ \psi_R^* \end{pmatrix} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

$$\sigma_2 i\psi_R^* = \psi_L$$

$$(\sigma_2)^{-i}\psi_L^* = \psi_R$$

$$i\bar{\sigma}\cdot\partial\psi_R - mi\sigma_2\psi_R^* = 0$$

$$i\sigma\cdot\partial\psi_L + i\sigma_2 m\psi_L^* = 0$$

Is the eq Lorentz inv?

Recall $\gamma^\mu \to U(\Lambda)\gamma^\mu U(\Lambda^{-1}) = \Lambda^\mu{}_\nu \gamma^\nu$

$\Rightarrow$ in Weyl basis $\quad \gamma^\mu = \sigma_2 \otimes \sigma_1 = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$

$$U(\Lambda)\gamma^\mu U(\Lambda^{-1}) = \underbrace{\begin{pmatrix} u_1 & \\ & u_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma \\ \bar\sigma & 0 \end{pmatrix}\begin{pmatrix} u_1^{-1} & 0 \\ 0 & u_2^{-1} \end{pmatrix}} = \begin{pmatrix} 0 & \Lambda^\mu_\nu \sigma^\nu \\ \Lambda \bar\sigma^\nu & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma u_2^{-1} \\ \bar\sigma^\mu u_1^{-1} & 0 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & u_1 \sigma u_2^{-1} \\ u_2 \bar\sigma^\mu u_1^{-1} & 0 \end{pmatrix}$$

So $\quad u_2 \bar\sigma^\mu u_1^{-1} = \Lambda^\mu{}_\nu \bar\sigma^\nu$

i.e. $\quad \bar\sigma^\mu \partial_\mu \to \underbrace{\bar\sigma^\mu \Lambda^\nu{}_\mu}_{u_2(\Lambda)\bar\sigma^\nu u_1^{-1}} \partial_\nu$

and hence,

$$\bar\sigma\cdot\partial\chi = \bar\sigma^\mu\partial_\mu\chi \to u_2(\Lambda)\left[\bar\sigma^\nu \partial_\nu u_1^{-1} u_1 \chi\right]$$
$$u_2(\Lambda)\left(\bar\sigma\cdot\partial\chi\right)$$

To see how $\sigma_2 \psi^*$ transform,

recall $\psi \to \left(1 - \frac{i\theta\cdot\sigma}{2} - \frac{\beta\cdot\sigma}{2}\right)\psi$ infinitesimally.

So $\psi^* \to \left(1 + \frac{i\theta\cdot\sigma^*}{2} + \frac{\beta\cdot\sigma^*}{2}\right)\psi^*$

and $\sigma_2\psi^* \to \underbrace{\left(1 - \frac{i\theta\cdot\sigma}{2} - \frac{\beta\cdot\sigma}{2}\right)}_{\text{generator of } \Lambda_L}\sigma_2\psi^*$

$\Rightarrow \sigma_2\psi^* \to \Lambda_L(\sigma_2\psi^*)$

and therefore the Majorana eq of motion is covariant. Indeed,

This indeed generates the KG eq. Indeed,

$i\bar{\sigma}\cdot\partial\psi - im\sigma^2\psi^* = 0$

$(-i\bar{\sigma}^*\cdot\partial\psi^* + im\sigma_2^*\psi = 0$

$m\sigma_2\psi^* = \bar{\sigma}\cdot\partial\psi \Rightarrow \psi^* = \frac{1}{m}\sigma_2\bar{\sigma}\cdot\partial\psi$

$\Rightarrow -(\bar{\sigma}^*\cdot\partial)\frac{1}{m}\sigma_2(\bar{\sigma}\cdot\partial)\psi + im\sigma_2^*\psi = 0$

$\Rightarrow -\sigma_2^*(\bar{\sigma}^*\cdot\partial)\sigma_2(\bar{\sigma}\cdot\partial)\psi + m^2\psi = 0$

$-\sigma_2$

$\sigma_2(\bar{\sigma}^{*\nu}\sigma_2\bar{\sigma}^{\mu})\partial_\mu\partial_\nu\psi + m^2\psi = 0$

In the [illegible] [illegible] the $\bar{\sigma}^{\lambda} = (-I, \sigma_1, \sigma_2, \sigma_3)$

and since $\{\sigma_i, \sigma_j\} = 2\delta_{ij} \rightsquigarrow$ the [illegible] [illegible] [illegible] the [illegible] and [illegible] and [illegible], is easy to see

$$\sigma_2 \bar{\sigma}^{*\mu} \sigma_2 \sigma^{\nu} = -\bar{\sigma}^{\mu} (\sigma_2^2) \sigma^{\nu} = -\bar{\sigma}^{\mu} \sigma^{\nu}$$

then $-\bar{\sigma}^{\mu} \sigma^{\nu} \partial_{\mu} \partial_{\nu} = \partial_t^2 - \sum \partial_i^2$.

and we recover $(\Box + m^2)\chi = 0$.

The [illegible] eq [illegible] [illegible] [illegible] [illegible] from the action

$$S = \int d^4x \left[\chi^{\dagger} \bar{\sigma} \cdot \partial \chi + \frac{im}{2} \left(\chi^{T} \sigma_2 \chi - \chi^{\dagger} \sigma_2 \chi^{*} \right) \right]$$

$$S^{*} = \int d^4x \left(-\chi^{\dagger} \bar{\sigma} \overleftarrow{\partial} \chi - \frac{im}{2} \left(\chi^{\dagger} \sigma_2^{*} \chi^{*} - \chi^{T} \sigma_2^{*} \chi \right) \right)$$

$$\chi^{\dagger} \bar{\sigma} \partial \chi \qquad + \frac{m}{2} \left(\chi^{T} \sigma_2 \chi - \chi^{\dagger} \sigma_2 \chi^{*} \right)$$

$$\frac{\delta S}{\delta \chi} = i \bar{\sigma} \cdot \partial \chi - \frac{im}{2} \sigma_2 \chi^{*} 2) \qquad -(\chi^{\dagger} \sigma_2 \chi^{*})$$

[illegible] we [illegible] [illegible] [illegible] the Dirac equation

$\psi = [illegible]$ with $\psi_1 = \chi_1$

$\psi_2 = \sigma^2 \chi_2^{*}$

what are the symmetries in Lagrangian?

$$\mathcal{L} = \bar{\psi}(i\not{\partial}-m)\psi$$

$$= \begin{pmatrix} \chi_1^\dagger & -\chi_2^T\sigma_2 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\gamma_0} \begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix} \begin{pmatrix} \chi_1 \\ i\sigma_2\chi_2^* \end{pmatrix}$$

$$= i\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + i\chi_2^T\bar{\sigma}^{\mu *}\partial_\mu\chi_2^* - im(\chi_2^T\sigma_2\chi_1 - \chi_1^\dagger\sigma_2\chi_2^*)$$

$$= i\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + i\chi_2^\dagger\bar{\sigma}^\mu\partial_\mu\chi_2 - im(\chi_2^T\sigma_2\chi_1 - \chi_1^\dagger\sigma_2\chi_2^*)$$

where $\alpha = i\chi_2^T\bar{\sigma}^{\mu *}\partial_\mu\chi_2^*$ is a number so

(for grassman numbers $(\alpha\beta)^* = \beta^*\alpha^* = -\alpha^*\beta^*$)

$$(i\chi_2^T\bar{\sigma}^{\mu *}\partial_\mu\chi_2^*)^* = -(-i\chi_2^\dagger\bar{\sigma}^\mu\partial_\mu\chi_2)$$

$$= i\chi_2^\dagger\bar{\sigma}^\mu\partial_\mu\chi_2$$

This lagrangian has symmetry $U(1)$,

$\psi \to e^{i\alpha}\psi$

$$\begin{pmatrix} \chi_1 \\ i\sigma_2\chi_2^* \end{pmatrix} \to e^{i\alpha}\begin{pmatrix} \chi_1 \\ i\sigma_2\chi_2^* \end{pmatrix} = \begin{pmatrix} e^{i\alpha}\chi_1 \\ i\sigma_2(e^{-i\alpha}\chi_2)^* \end{pmatrix}$$

So for $\chi_1 \to e^{i\alpha}\chi_1 \rightsquigarrow \delta\chi_1 = i\alpha\chi_1$

$\chi_2 \to e^{-i\alpha}\chi_2 \quad \delta\chi_2 = -i\alpha\chi_2$

$$J^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_1)}\delta\chi_1(\alpha) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_2)}\delta\chi_2 = -(i\chi_1^\dagger\bar{\sigma}^\mu(i\chi_1) + i\chi_2^\dagger\bar{\sigma}^\mu(-i\chi_2))$$

$$= -\chi_2^\dagger\bar{\sigma}^\mu\chi_2 - \chi_1^\dagger\bar{\sigma}^\mu\chi_1)$$

To quote the [illegible]

$$\{ \chi_a(x), \chi_b^\dagger(y) \} = \delta_{ab} \delta^2(x-y)$$

violates this in the Dirac lagrangian if we set $\chi_2 = \chi_1$

then

$$\mathcal{L} = 2i\, \chi^\dagger \bar\sigma^\mu \partial_\mu \chi + \frac{im}{2}\left(\chi^\dagger \sigma_2 \chi - \chi^\dagger \sigma_2 \chi^* \right)$$

i.e. $\psi_R = i\sigma_2 \psi_L^*$

$$\psi_L(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_r \left(u_L^r(p)\, c_r(p)\, e^{-ip\cdot x} + v_L^r(p)\, d_r^\dagger(p)\, e^{ipx} \right)$$

$$\psi_R(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_r \left(u_R^r(p)\, c_r(p)\, e^{-ipx} + v_R^r(p)\, d_r^\dagger(p)\, e^{ipx} \right)$$

$$u_R^r(p)\, c_r(p) = i\sigma_2\, v_L^{*r}\, d_r$$

$\Rightarrow c_r = d_r$

$$u^r = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^r \\ \sqrt{p\cdot\bar\sigma}\, \xi^r \end{pmatrix} = \begin{pmatrix} u_L^r \\ u_R^r \end{pmatrix}, \quad v_r = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^r \\ -\sqrt{p\cdot\bar\sigma}\, \xi^r \end{pmatrix}$$

$$\Rightarrow \sqrt{p\cdot\bar\sigma}\, \xi^r = i\sigma_2 v_L^{*r} = i\sigma_2 \sqrt{p\cdot\sigma}^*\, \xi^{r*} \rightarrow v_L^r = -i\sigma_2 \sqrt{p\cdot\sigma}\, \xi^{r*}$$

$$\chi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p\cdot\sigma}}{\sqrt{2E_p}} \sum_r \left(\xi^r a_r(p) e^{-ipx} + (-i\sigma_2)\, \xi^{*r} a_r^\dagger(p)\, e^{ipx} \right)$$

<u>Supersymmetry</u>

Let $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma}\cdot\partial \chi + F^* F$

2 scalar fields (complex) 1 weyl fermion ↑ auxiliary field

Consider $\phi \to \phi - i\epsilon^T \sigma^2 \chi \to \delta\phi = -i\epsilon^T \sigma^2 \chi$

$$\delta\chi = \epsilon F + \sigma\cdot\partial\phi\, \sigma^2 \epsilon^*$$

$$\delta F = -i\epsilon^\dagger \bar{\sigma}\cdot\partial\chi$$

then $\delta(\partial_\mu \phi^* \partial^\mu \phi) = i(\partial_\mu \chi^\dagger \sigma^2 \epsilon^*)\partial^\mu \phi + (\partial_\mu \phi^*)(-i\epsilon^T \sigma^2 \partial^\mu \chi)$

$$\delta(\chi^\dagger (\bar{\sigma}\cdot\partial)\chi) = (F^* \epsilon^\dagger + \epsilon^T \sigma^2 \phi^* \sigma\cdot\overleftarrow{\partial})\, i(\bar{\sigma}\cdot\partial)\chi$$
$$+ \chi^\dagger i\bar{\sigma}^\mu(\epsilon\partial_\mu F + \sigma^\nu \sigma^2 \epsilon^* \partial_\mu\partial_\nu \phi)$$
$$= i\underline{F^* \epsilon^\dagger (\bar{\sigma}\cdot\partial)\chi} + i\partial_\mu(\epsilon^T \sigma^2 \sigma^\nu \bar{\sigma}^\mu (\partial_\nu \phi^*)\chi)$$
$$- i\epsilon^T \sigma^2 \sigma^\nu \bar{\sigma}^\mu (\partial_\mu\partial_\nu \phi^*)\chi$$
$$+ i\chi^\dagger \bar{\sigma}^\mu \epsilon\partial_\mu F + i\chi^\dagger \bar{\sigma}^\mu \sigma^\nu \sigma^2 \epsilon^* \partial_\mu\partial_\nu \phi$$

$$\delta(F^* F) = i(\partial_\mu \chi^\dagger)\bar{\sigma}^\mu \epsilon F - \underline{iF^* \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi}$$

Hence, $\delta\mathcal{L} = i\partial_\mu(\chi^\dagger \sigma^2 \epsilon^* \partial^\mu \phi) - i\partial_\mu \chi^\dagger \cancel{\sigma^2 \epsilon^* \partial_\mu \partial^\mu \phi}$

$$+ \partial_\mu(-i\epsilon^T \sigma^2 \chi\, \partial^\mu \phi^*) + i\epsilon^T \sigma^2 \cancel{\chi\, \partial_\mu \partial^\mu \phi^*}$$
$$+ i\partial_\mu(\epsilon^T \sigma^2 \sigma^\nu \bar{\sigma}^\mu \partial_\nu \phi^*)\chi) - i\epsilon^T \sigma^2 \cancel{\sigma^\nu \bar{\sigma}^\mu (\partial_\mu\partial_\nu \phi^*)}\chi$$
$$+ \partial_\mu(i\chi^\dagger \bar{\sigma}^\mu \epsilon F) - i(\partial_\mu \chi^\dagger)\cancel{\bar{\sigma}^\mu} \epsilon F + i\chi^\dagger \bar{\sigma}^\mu \cancel{\sigma^\nu} \sigma^2 \epsilon^* \partial_\mu \partial_\nu \phi$$
$$+ i\partial_\mu \chi^\dagger \cancel{\bar{\sigma}^\mu} \epsilon F = i\partial_\mu F^\mu \quad ✓$$

if we add

$$\Delta\mathcal{L} = \left(m\phi F + \frac{1}{2} i m \chi^T \sigma^2 \chi\right) + c.c.$$

$$\delta(\Delta\mathcal{L}) = m\left(-i\epsilon^T\sigma^2\chi\right)F + m\phi\left(-i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi\right)$$

$$+\frac{1}{2} i m\left(\epsilon^T F + \epsilon^\dagger(\sigma^2)^T i\bar{\sigma}^{\mu T}\partial_\mu\phi\right)\sigma^2\chi$$

$$+\frac{1}{2} i m \chi^T\sigma^2\left(\epsilon F + i\bar{\sigma}\cdot\partial\phi\,\sigma^2\epsilon^*\right) + c.c.$$

$$= -\frac{1}{2} i m F\left(\epsilon^T\sigma^2\chi - \chi^T\sigma^2\epsilon\right) - i m\phi\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi$$

$$\doteq -im\,\partial_\mu\left(\phi\epsilon^\dagger\bar{\sigma}^\mu\chi\right) + c.c.$$

where $(\sigma^2)^T = -\sigma^2$, $\sigma^2(\sigma^\mu)^T\sigma^2 = \bar{\sigma}^\mu$

$$\epsilon^T\sigma^2\chi = \chi^T\sigma^2\epsilon$$

$$\epsilon^\dagger\bar{\sigma}^\mu\chi = -\chi^T(\bar{\sigma}^\mu)^T\epsilon^*$$

Now, solving the eqn of motion for F,

$$F^* + m\phi = 0 \Rightarrow$$

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \frac{1}{2}\left(im\chi^T\sigma^2\chi + c.c.\right)$$

scalar and spinor has same mass.

A general model with interactions is

$$\mathcal{L} = \partial_\mu\phi_i^*\partial^\mu\phi_i + \chi_i^\dagger i\bar{\sigma}^\mu\partial_\mu\chi_i + F_i^* F_i$$

$$+\left[F_i\frac{\partial W[\phi]}{\partial\phi_i} + \frac{i}{2}\frac{\partial^2 W(\phi)}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\chi_j + c.c.\right]$$

To see supersymmetry invariance → means to see

$$\delta\left(F_i \frac{\partial W}{\partial \phi_i} + \frac{1}{2} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \chi_j + c.c. \right]$$

$$= -i \epsilon^\dagger \bar\sigma^\mu (\partial_\mu \chi_i) \frac{\partial W}{\partial \phi_i} - F_i \underbrace{\frac{\partial^2 W}{\partial \phi_i \partial \phi_j} (-i \epsilon^T \sigma^2 \chi_j)}$$

$$+ \frac{1}{2} \frac{\partial^3 W}{\partial \phi_i \partial \phi_j \partial \phi_k} (-i \epsilon^T \sigma^2 \chi_k) \chi_i^T \sigma^2 \chi_j$$

$$+ \frac{1}{2} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \Big(\overline{\big((\epsilon^T F_i + \epsilon^\dagger (\sigma^\mu)^T (\sigma^\mu)^T \partial_\mu \phi_i \big) \sigma^2 \chi_j}$$

$$\underline{+ \chi_i^T \sigma^2 (\epsilon F_j + \sigma^\mu \partial_\mu \phi_j \sigma^2 \epsilon^*) \Big) } \Big] + c.c.$$

Totally symmetric × totally antisymmetric = 0.

sum to 0.

$$= -i \epsilon^\dagger \bar\sigma^\mu \partial_\mu \chi_i \frac{\partial W}{\partial \phi_i} + i \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \epsilon^\dagger (\sigma^2)^T (\sigma^\mu)^T (\partial_\mu \phi_i) \sigma^2 \chi_j$$

$$= -i \partial_\mu \left(\epsilon^\dagger \bar\sigma^\mu \chi_i \frac{\partial W}{\partial \phi_i} \right) + i \epsilon^\dagger \bar\sigma^\mu \chi_i \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \partial_\mu \phi_j - i \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \epsilon^\dagger \bar\sigma^\mu (\partial_\mu \phi_i) \chi_j$$

$$= -i \partial_\mu \left(\epsilon^\dagger \bar\sigma^\mu \chi_i \frac{\partial W}{\partial \phi_i} \right) \quad ✓$$

So this is a SUSY Lagrangian.

Ex/ $W(\phi) = \frac{g}{3} \phi^3$, $n = 1$

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar\sigma^\mu \partial_\mu \chi + F^* F + (g F \phi^2 + i g \phi \chi^T \sigma^2 \chi + c.c.)$$

$$\Rightarrow F + g \phi^{*2} = 0$$

$$\Rightarrow \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi - g^2 (\phi^* \phi)^2$$
$$+ ig\left(\phi \chi^T \sigma^2 \chi - \phi^* \chi^\dagger \sigma^2 \chi^* \right)$$

Lagrangian of a massless complex scalar and weyl spinor with ϕ^4 and yukawa interactions.

Parity $\quad P\, a_p^{s\dagger}|0\rangle = a_{-p}^{s\dagger}|0\rangle$

we want $\quad P\, a_p^{s\dagger}\, P = \eta_a\, a_{-p}^{s} \quad P\, b_p^s\, P = \eta_b\, b_{-p}^s$

$\eta_a^2 = 1$

$$P\psi(x)P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_r \left(U^r(p)\, a_r(-p)\,\eta_a\, e^{-ip\cdot x} + \eta_b^*\, b_s^\dagger(-p)\, V^s(p)\, e^{ip\cdot x} \right)$$

$e^{-i\omega t + i\underline{p}\cdot \underline{x}}$

In other words

$$P\psi(\underline{x})P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_r \left(U^r(p)\, a_r(-p)\,\eta_a + \eta_b^*\, b_s^\dagger(-p)\, V^s(-p) \right) e^{i\underline{p}\cdot\underline{x}}$$

Recall $\quad U(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\bar p\cdot\bar\sigma}\,\xi \\ \sqrt{\bar p\cdot\sigma}\,\xi \end{pmatrix} = \gamma^0 U(\bar p)$

$$V(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\bar p\cdot\bar\sigma}\,\xi \\ -\sqrt{\bar p\cdot\sigma}\,\xi \end{pmatrix} = -\gamma^0 V(\bar p)$$

So $\quad P\psi(\underline{x})P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_r \left(\gamma^0 U(\bar p)\, a_r(-p)\,\eta_a - \gamma^0\eta_b^*\, b_s^\dagger(p)\, V(\bar p) \right) e^{ip\cdot x}$

$$\overset{-\eta_b^* = \eta_a}{=} \gamma_0\eta_a \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_r \left(u(p)\, a_r + b_s^\dagger(-p)\, v(p) \right) e^{-ip\cdot x}$$

$$= \eta_a\gamma_0\, \psi(-\underline{x})$$

So P is represented. Indeed,

$$P\bar\psi P = P\psi^\dagger\gamma^0 P = P\psi^\dagger P\gamma^0 = (P\psi P)^\dagger\gamma^0$$

$$= (\eta_a\gamma^0\psi(-x))^\dagger\gamma^0 = \eta_a^*\,\bar\psi(-x)\,\gamma^0$$

and hence $\quad P\underbrace{\bar\psi\psi}_{\text{scalar}}P = |\eta_a|^2\, \bar\psi(-x)\,\gamma^0\gamma^0\,\psi(-x) = \bar\psi\psi(-\underline{x})$

as expected.

Time reversal $\psi(t,\underline{x}) \to \psi(-t,\underline{x})$

want to be a symmetry so $[H,T]=0$

$$\psi(t,\underline{x}) = e^{iHt}\psi(\underline{x})\, e^{-iHt}$$

$$\hookrightarrow \quad T\psi(x)T = e^{iHt}\, T\psi(x)T\, e^{-iHt}$$

$$\hookrightarrow \quad T\psi(x)T|0\rangle = e^{iHt}\, T\psi(x)T|0\rangle$$ ← negative frequency terms

$$\psi(-t,\underline{x})|0\rangle = \underbrace{e^{iHt}\psi(\underline{x})}_{\text{sum of positive frequency states}}|0\rangle$$

so T is not unitary it's antiunitary.

we take $T(\text{constants}) = (\text{constants})^* T$, $T=T^{\dagger}$.

so $T e^{iHt} = e^{-iHt} T$

set $\xi^s = (\xi(\uparrow), \xi(\downarrow))$, $s=1,2$ let

$-i\sigma^2\xi^{s*} = \xi^{-s} = (\xi(\downarrow), -\xi(\uparrow))$. recall $\sigma^i\sigma^2 = \sigma^2(-\sigma^{i*})$

If $(n_i\sigma^i)\xi = \xi$ then

$$\underline{n}\cdot\underline{\sigma}(-i\sigma^2\xi^*) = -i\sigma^2(-\underline{n}\cdot\underline{\sigma})^*\xi^* = i\sigma^2\xi^*$$
$$= -(-i\sigma^2\xi^*)$$

so flips 2 successive spins return to original.

$$u^{-s}(\tilde{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,(-i\sigma^2\xi^{s*})\\ \sqrt{p\cdot\bar{\sigma}}\,(-i\sigma^2\xi^{s*})\end{pmatrix} = \begin{pmatrix}-i\sigma^2\sqrt{p\cdot\sigma^*}\,\xi^{s*}\\ -i\sigma^2\sqrt{p\cdot\bar{\sigma}^*}\,\xi^{s*}\end{pmatrix}$$

$$= -i\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}(u^s(p))^* = -\gamma^1\gamma^3(u^s(p))^*$$

similarly $v^{s}(\tilde{p}) = -\gamma^1\gamma^3(v^s(p))^*$

so using

$$a_p^{-s} = (a_p^2, -a_p^1), \quad b_p^{-s} = (b_p^2, -b_p^1)$$

$$T a_p^s T = a_{-p}^{-s}, \quad T b_p^s T = b_{-p}^{-s}$$

plugging in in $\psi(\underline{x})$

$$T \psi(t, \underline{x}) T = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s T\left(a_p^s u^s(p) e^{-ip\cdot x} + b_p^{s\dagger} v^s(p) e^{ip\cdot x} \right) T$$

anti-unitary

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_{-p}^{-s} (u^s(p))^* e^{ip\cdot x} + b_{-p}^{-s\dagger} (v^s(p))^* e^{-ip\cdot x} \right)$$

$$= \gamma^1\gamma^3 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_{\bar{p}}^{-s} u^{-s}(\bar{p}) e^{i\bar{p}(t,-x)} + b_{\bar{p}}^{-s\dagger} v^{-s}(\bar{p}) e^{-i\bar{p}(t,-x)} \right)$$

$$= \gamma^1\gamma^3 \psi(-t, \underline{x})$$

$$T\bar{\psi}T = T\psi^\dagger(x) T (\gamma^0)^* = (T\psi(t,\underline{x})T)^\dagger (\gamma^0)^*$$

$$= \psi^\dagger(-t,\underline{x}) \gamma^0\gamma^0\gamma^3\gamma^1(\gamma^0)^* = \bar{\psi}(-t,\underline{x})(-\gamma^1\gamma^3)$$

$$\rightarrow T\bar{\psi}\psi T = T\bar{\psi}TT\psi T = \bar{\psi}(-t,\underline{x})(-\gamma^1\gamma^3)(\gamma^1\gamma^3)\psi(-t,\underline{x})$$

$$= \bar{\psi}\psi(-t,\underline{x}) \text{ as expected.}$$

Charge Conjugation

$$C a_p^s C = b_p^s$$

$$C b_p^s C = a_p^s.$$

Lets see V interms of U.

$$(V^s(p))^* = \begin{pmatrix} \sqrt{p\cdot\sigma}\,(-i\sigma^2\xi^*) \\ -\sqrt{p\cdot\bar\sigma}\,(-i\sigma^2\xi^*) \end{pmatrix}^* = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\bar\sigma^*}\,\xi^* \\ i\sigma^2\sqrt{p\cdot\sigma^*}\,\xi^* \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

Hence,
$$u^s(p) = -i\gamma^2 (V^s(p))^*$$
$$V^s(p) = -i\gamma^2 (U^s(p))^*.$$

Hence,

$$C\psi(x)C = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \Big(-i\gamma^2 b_p^s (V^s(p))^* e^{-ipx} - i\gamma^2 a_p^{s\dagger} (U^s(p))^* e^{ipx}\Big)$$

$$= -i\gamma^2\psi^*(x) = -i\gamma^2 (\psi^\dagger)^T = -i(\bar\psi\gamma^0\gamma^2)^T.$$

$$C\bar\psi(x)C = (C\psi^\dagger C\gamma^0 = (-i\gamma^2\psi)^T\gamma^0 = (-i\gamma^0\gamma^2\psi)^T$$

$$C\bar\psi\psi C = (-i\gamma^0\gamma^2\psi)^T(-i\bar\psi\gamma^0\gamma^2)^T$$

$$= -\gamma^0_{ab}\gamma^2_{bc}\psi_c\bar\psi_d\gamma^0_{de}\gamma^2_{ea}$$

$$= \bar\psi_d\gamma^0_{de}\gamma^2_{ea}\gamma^0_{ab}\gamma^2_{bc}\psi_c = -\bar\psi\gamma^2\gamma^0\gamma^0\gamma^2\psi$$

$$= \bar\psi\psi.$$

Scalar field with a Source

$$H = H_{free} + \int d^3x \left(-j(t,\underline{x})\,\phi(x)\right)$$

then $\phi(x) = \phi_0(x) + i\int d^4y\, D_R(x-y)\, j(y)$

$$= \phi_0(x) + i\int d^4y \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\,\theta(x^0-y^0)\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) j(y)$$

If the time is long enough such that $x \gg T$, when j stops (or is turned off) $\rightsquigarrow$ $\phi(x) = \phi_0(x) + i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(\tilde{J}(p)e^{-ipx} + \tilde{J}^*(p)e^{ipx}\right)$

So $\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(\left(a_p + \frac{i}{\sqrt{2E_p}}\tilde{J}(p)\right)e^{-ip\cdot x} + h.c\right)$

$$\rightsquigarrow H = \int \frac{d^3p}{(2\pi)^3} E_p \left(a_p^\dagger - \frac{i}{\sqrt{2E_p}}\tilde{J}^*(p)\right)\left(a_p + \frac{i}{\sqrt{2E_p}}\tilde{J}(p)\right)$$

$$\rightsquigarrow \langle 0|H|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}|\tilde{J}(p)|^2$$

If we interpret $\frac{|\tilde{J}(p)|^2}{2E_p}$ as the probability density of particles then

$$\int dN = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\tilde{J}(p)|^2$$

, only those modes of $\tilde{J}$ at resonance with $p^2 = m^2$ waves are effective to create particles.

why is this true?

Notice that the prob of no creating particles is

$$|\langle 0|0\rangle_{out}|^2 = \lim_{t\to\infty} |\langle 0| e^{-iHt} e^{iH_0 t}|0\rangle|^2$$

$$= |\langle 0| T\exp\{-i\int d^4x\, H_{int}\}|0\rangle|^2$$

$$= |\langle 0| T\exp\{i\int d^4x\, j(x)\phi_I(x)\}|0\rangle|^2$$

$$\langle 0|T\exp\{i\int d^4x\, j(x)\phi_I(x)\}|0\rangle = 1 + \langle 0|\{\cancel{\phi_I}^{\,0}\}|0\rangle$$

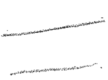
$$+ \frac{i^2}{2!}\int d^4x\, d^4y\; j(x)\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle j(y) + O(j^4)$$

$$= 1 - \frac{1}{2}\int d^4x\, d^4y\; j(x)\int\frac{d^3p}{(2\pi)^3}\frac{e^{-ip(x-y)}}{2E_p} j(y) + O(j^4)$$

$$= 1 - \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} J(-p)J(p) + O(j^4)$$

$$= 1 - \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|J(p)|^2 = 1 - \frac{1}{2}\lambda + O(j^4)$$

So $P(0) = |1 - \frac{1}{2}\lambda + O(j^4)|^2 = 1 - \lambda + O(j^4)$

$= \frac{1}{0!} + \frac{1}{1!2}$  $+ \frac{1}{2!2^2}$  $+$  $+ \dots$

Symmetry factors: each j is paired with an out.

So for $2n$ vertices (n propagators) there is $2^{2n/2} = 2^n$ ways to pair. Additionally the $\overline{\underline{x}}$ integrating of propagators gives a symmetry of $n!$ then

$$\langle 0|T|0\rangle = 1 - \frac{\lambda}{2} + \left(\frac{\lambda}{2}\right)^2\frac{1}{3!}\dots = e^{-\frac{\lambda}{2}}$$

So finally $P(0) = |e^{-\lambda/2}|^2 = e^{-\lambda}$

To compute $P(\text{one part with } \underline{k})$

we see $\langle k | T\{\exp(i\int d^4x\, j\, \phi_I(x))\} |0\rangle$

$= \langle k | 1 - i\int d^4x\, j(x)\, \phi_I(x) |0\rangle + O(j^2)$

$= i\int d^4x\, j(x) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \langle 0 | a_k (a_p e^{-ipx} + a_p^\dagger e^{ipx}) |0\rangle$

$= i\int d^4x\, j(x) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (2\pi)^3 \delta(k-p)\, e^{-ipx}$

$= i\int d^4x \frac{j(x)}{\sqrt{2E_k}} e^{-ikx} = \frac{i}{\sqrt{2E_k}} \tilde{J}(k)$

$\hookrightarrow P(\text{one part } \underline{k}) = \frac{|\tilde{J}(k)|^2}{2E_k} = \left|\frac{i}{\sqrt{2E_k}} \tilde{J}(k)\right|^2$

then the prob of $\underline{\text{one}}$ particle is

$P(\text{one}) = \int \frac{d^3k}{(2\pi)^3} \frac{|\tilde{J}_k|^2}{2E_k} = \lambda.$

write $\langle k | T\{\exp(i\int d^4x\, j\, \phi_I(x))\} |0\rangle$

$= \langle k|0\rangle + i\int d^4x\, j(x) \langle 0 | a_k\, \phi_I(x) |0\rangle$

$+ i^2 \int d^4x\, d^4y\, j(x) \langle 0 | \underbrace{T\{a_k \phi_I(x)\, \phi_I(y)\}}_{= a_k \phi_I \phi_I + \dots + a_k^{\dots}} |0\rangle\, j(y)$

Prob of 1 particle $= |\longrightarrow \times (1 - \longrightarrow + \dots - [illegible] + \dots)|^2$

$$= |i\sqrt{\lambda}\, e^{-\lambda/2}|^2 = \lambda e^{-\lambda}.$$

The prob to create n particles is first

$$\langle 0|\, a_k^n\, T\exp\left[i\int d^4x\, j(x)\, \phi_I(x)\right]|0\rangle$$

the first term is for contracted of order n this gives $(i\sqrt{\lambda})^n$

to sum

$$P(n) = \frac{1}{n!}\left|(\ldots)^n \times \left(1 - \ldots + \ldots - \ldots + \ldots\right)\right|^2$$

$$= \frac{1}{n!}\lambda^n e^{-\lambda}$$

← n distinct particles final state.

Decay of scalar:

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \Phi)\partial^\mu \Phi - \frac{1}{2}M^2 \Phi^2}_{\mathcal{L}_{free}} + \underbrace{\frac{1}{2}\partial\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2}_{\mathcal{L}_{free}} \underbrace{-\mu\Phi\phi\phi}_{\text{interact.}}$$

$$H_I = \int d^3x\, \mu\,\Phi\phi\phi$$

to decay we want

$$\langle p_1, p_2 | T | p\rangle = \lim_{T\to\infty}\left(\langle p_1, p_2 | T\exp\left[-i\int_{-T}^{T} dt\, H_I\right]|p\rangle\right)$$

connected amputated

$$= (2\pi)^4\,\delta^4(p-p_1-p_2)\, i\mathcal{M}(\Phi_p \to \phi_{p_1}\phi_{p_2})$$

$$\simeq -i\mu\int d^4x\, \langle p_1, p_2 | T\phi\phi\Phi|p\rangle$$

$$= -i\mu\int d^4x \left(\langle p_1, p_2|\phi\phi\Phi|p\rangle + \langle p_1, p_2|\phi\phi\Phi|p\rangle\right)$$

$$= -i\mu\int d^4x \left(2e^{ip_2\cdot x}e^{ip_1\cdot x}e^{-ip\cdot x}\right)$$

$$= -2i\mu\,\delta^4(p_1+p_2-p)(2\pi)^4$$

$$\boxed{\begin{aligned}\phi_I(x)|p\rangle &= e^{ip\cdot x}|0\rangle \\ \langle p|\phi_I(x) &= \langle 0|e^{ipx}\end{aligned}}$$

So $\mathcal{M} = -2\mu$

The decay is

$$d\Gamma = \frac{1}{2m_A}\left(\prod_f \frac{d^3 p_f}{(2\pi)^3\, 2E_f}\right) |\mathcal{M}(m_A \to \{p_f\})|^2 (2\pi)^4 \delta^4(p_A - \textstyle\sum p_f)$$

Since we have 2 identical bosons

$$\Gamma_\Phi = \frac{1}{2}\frac{1}{2m_\Phi}\int \frac{d^3 p_1}{(2\pi)^3}\frac{1}{2E_1}\frac{d^3 p_2}{(2\pi)^3}\frac{1}{2E_2}\, \mu^2 (2\pi)^4 \delta^4(p_1+p_2-P)$$

$$\delta^4(p_1+p_2-P) = \delta^3(\underline{p}_1+\underline{p}_2)\,\delta(M-E_{p_1}-E_{p_2})$$

but they are 2 mass-shell bosons.

$$m^2 + \underline{p}_1^2 = E_{p_1}^2.$$

$$\text{so}\quad \Gamma_\Phi = \frac{\mu^2}{m_\Phi}\int \frac{d^3 p_1}{(2\pi)^3}\frac{2\pi}{E_{p_1}^2}\,\delta(M - 2E_{p_1}) = \frac{\mu^2}{8\pi M}\left(1-\frac{4m^2}{M^2}\right)^{\frac{1}{2}}$$

$$\text{so}\quad \tau = \Gamma^{-1} = \frac{8\pi M}{\mu^2}\left(1-\frac{4m^2}{M^2}\right)^{-\frac{1}{2}}$$

to first order.