Dirac Structures and Classical Mechanics

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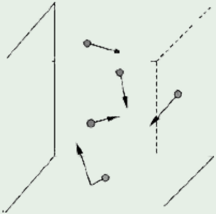
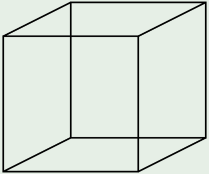


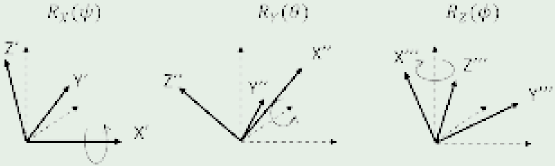
$$M = \left(\mathbb{R}^3\right)^N$$

$$M' = M \backslash \{\langle v_{1,1}, ..., v_{N,3}\rangle : \langle v_{i,1}, v_{i,2}, v_{i,3}\rangle = \langle v_{j,1}, v_{j,2}, v_{j,3}\rangle \; i \neq j\}$$

Classical Mechanics II

Example (Rigid body rotations)





$M = O^+(3) =$ Orthogonal linear transf. preserving orientation.

Classical Mechanics III

- Configuration $\to \varphi(t) \in M$
- State of the system $= \underbrace{M}_{Position} + \underbrace{\textbf{things}}_{Momentum} = S$

  Configurations $\pi : S \to M$ s.t.

$$= \pi\langle q^1, ..., q^{3N}, p_1, ..., p_{3N}\rangle = \varphi(t).$$

Remark

$q \in M \implies \dot{q} \in TM, \quad p \in ?$

State at a time t:

- $\varphi_{t,t_0} : S \to S, \quad \varphi_{t,t_0}(s) \leftarrow$ state at a time t
- $\varphi_{t,t_1} \circ \varphi_{t_1,t_0} = \varphi_{t,t_0} \implies \varphi_{s_2} \circ \varphi_{s_1} = \varphi_{s_1+s_2}$ φ describes a flow on $S = T * M$.

$$\boxed{\begin{array}{c}\textbf{Classical Mechanics Ingredients:}\\ \\ M,\ T^*M,\ \varphi,\ \pi\end{array}}$$

Equations of motion: Let M_1, M_2 and $\varphi : M_1 \to M_2$ a map,

- $T(\varphi \circ \psi) = T(\varphi) \circ T(\psi)$
- $\pi \circ T(\varphi) = \varphi \circ \pi$

$$\pi \circ T(\varphi_t)\xi = \varphi_t(\pi(\xi)).$$

The flow φ is generated by an infinitesimal generator X $(\dot{\varphi} = X(\varphi(t)))$.

Example

$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

$$\dot{\varphi}^1 = \varphi^2$$
$$\dot{\varphi}^2 = -\varphi^1$$

$\implies (\varphi^1)^2 + (\varphi^2)^2 = C^2.$

$\pi_{*\xi}(T(X)_\xi) = X_\xi$ hence

$$T(X)_{T(\alpha)}\langle v, w\rangle = \langle X_\alpha, dX_{\alpha(v)}(w)\rangle$$

i.e., with $\langle q_\alpha, \dot{q}_\alpha \rangle = \langle v, w \rangle$

$$\begin{cases} \dfrac{dq^i}{dt} = X^i \\ \dfrac{d\dot{q}^i}{dt} = \dfrac{\partial X^i}{\partial x^1}\dot{q}^1 + \ldots + \dfrac{\partial X^i}{\partial x^n}\dot{q}^n \end{cases} \tag{1}$$

and similarly for T^*M

The fundamental linear form. Given a $z \in T^*M$, let $\xi \in T_z T^*M$, the fundamental linear form θ is defined (in coordinates) as

$$\langle \xi, \theta_z \rangle = \langle \pi_* \xi, z \rangle$$

i.e.

$$\theta = \sum p_i dq^i$$

Let $\varphi : M_1 \to M_2$ diff. and θ_1, θ_2 the fundamental linear forms.

- $T^*(\varphi)^*\theta_2 = \theta_1 \quad (\pi_* T^*(\varphi)_* \xi = \varphi_* \pi_* \xi)$

- If $M_1 = M_2$, $(T^*(\varphi_t))^*\theta = \theta$

Using $\pi_*(T^*(X)_l = X_x)$,

$$D_{T^*(X)}\theta = 0$$

Note that $f_X(z) = \langle X_x, z\rangle$, $x = \pi(z)$ i.e. $f_X = \langle T^*(X), \theta\rangle$ since $\langle T^*(X)_z, \theta\rangle = \langle \pi_* T^*(X)_z, z\rangle = \langle X_z, z\rangle$.

Thus,

$$0 = D_{T^*(X)_w}\theta = d\langle \underbrace{T^*(X)}_{f_X}, \theta\rangle + \underbrace{\iota_{T^*(X)}d\theta}_{T^*(X)\lrcorner\theta}$$

$$\implies df_X = -\iota_{T^*(X)}d\theta$$

The fundamental exterior 2-form on T^*M

$$\Omega = d\theta$$

- $d\Omega = 0 = d^2\omega$
- Ω is non singular, i.e. if $\xi \in T_zT^*M$ s.t. $\iota_\xi\Omega = 0 \iff \xi = 0$
  Proof: $\theta = p_i dq^i \implies \Omega = dp_i \wedge dq^i$. $X = A^i\frac{\partial}{\partial q^i} + B^i\frac{\partial}{\partial p^i}$
  hence:

$$\iota_X\Omega = \sum B^i dq\hat{\imath} - A^i dp^i = 0 \iff A = B = 0.$$

Hence, there is a one-to-one correspondence $X \to \iota_X\Omega$ between diff. forms and vector fields,

$$\omega_X = \iota_X\Omega$$

$$\omega = \iota_{X_\omega}\Omega.$$

where $d\omega = 0 \iff D_{X_\omega}\Omega = d(\iota_{X_\omega}\Omega) = d\omega = 0$ This is a distinguished class of vector fields T^*M (corresponding to functions), vector fields on T^*M. These vector fields are called **Hamiltonian vector fields**.

- $aX_{dF} + bX_{dG} = X_{d(aF+bG)}$
- $[X_{dF}, X_{dG}]$ is Hamiltonian,

$$\begin{aligned} dD_X G = D_X(dG) = D_X(\iota_{X_{dG}}\Omega) = \iota_{D_X Y}\omega + \iota_Y D_X\Omega \\ = \iota_{D_X Y}\Omega = \iota_{[X,Y]}\Omega \\ \implies [X_{dF}, X_{dG}] = X_{d(X_{dF}G)} \end{aligned}$$

$[\cdot,\cdot]$ is a Lie Bracket on Hamiltonian vector fields, We can define a Poisson bracket $\{\cdot,\cdot\}$ by

$$\{F, G\} = X_{dF}G$$

hence,

$$[X_{dF}, X_{dG}] = X_{d\{F,G\}}$$

- $\{F, G\} = -\{G, F\}$,

$$\begin{aligned} X_{dF} G &= \langle X_{dF}, dG \rangle = \langle X_{dF}, \iota X_{dG} \Omega \rangle \\ &= \langle X_{dG} \wedge X_{dF}, \Omega \rangle \implies \{F, G\} = -\{G, F\} \end{aligned}$$

In coordinates $(\langle q_\alpha, p_\alpha \rangle)$

$$dF = \sum \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial p^i} dp^i \implies$$

$$X_{dF} = \sum \frac{\partial F}{\partial q^i} \frac{\partial}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial}{\partial q^i} \implies$$

$$\{F, G\} = \sum \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial G}{\partial q^i}$$

Proposition: If F, G are s.t. $X_{dF} G = 0$ then $X_{dG} F = 0$. Ïf G is constant along the solution curves of X_{dF} then F is constant along X_{dG}"

The moment function of Y satisfies:

$$-T^*(Y) = X_{df_Y}$$

using $df_X = -\iota_{T^*(X)} d\theta$ Hamiltonian mechanics:

- Evolution of the system is determined by a flow on T^*M
- The infinitesimal generator of the flow is a Hamiltonian vector field

  "There is a function H (energy) on T^*M s.t. X_{-dH} is the infinitesimal generator of the flow on T^*M"

In coordinates:

$$X_{-dH} = \sum \frac{\partial H}{\partial p^i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p^i},$$

the flow $\langle q^\alpha(t), p^\alpha(t)\rangle$ is an integral curve of the flow

Equations of motion:

$$\frac{\partial H}{\partial p^i} = \frac{dq^i(t)}{dt}$$
$$-\frac{\partial H}{\partial q^i} = \frac{dp^i(t)}{dt}$$

Trivial consequence of $\{F, G\} = -\{G, F\} \implies$

$$X_{-dH}H = 0 \leftarrow \text{ Conservation law}$$

H is a constant along trajectories of the system

Let X_{-dH}, F s.t.

$$X_{-dF}H = 0$$

then F is a constant on the trajectories of the flow. Prototype of momentum conservation.

The kinetic energy is a function on T^*M associated to the Riemannin metric $(\cdot, \cdot)$ hence, $K = \frac{1}{2}(\ell, \ell)$.

Example:

Particle of mass m, $p = m\dot{q}$

$$||(\dot{q}_x, \dot{q}_y, \dot{q}_z)||^2 = m\dot{q}_x^2 + m\dot{q}_y^2 + m\dot{q}_z^2$$

the map $T_x R^3 \to T_x^* R^3$ sends

$$\langle q_\alpha, \dot{q}_\alpha \rangle \to \langle q_\alpha, p_\alpha \rangle$$

$\implies K = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right)$ The function U is assumed to be

$U = \bar{U} \circ \pi$ where $\bar{U}$ is a function on M. The form $F = -d\bar{U}$ is called the force field with potential U. In coordinates,

$$\langle \xi, F \rangle = -\langle \xi, d\overline{U} \rangle$$

Using $H = U + K = \frac{1}{2m} \sum p_\alpha^2 + U$,

$$\frac{\partial H}{\partial p^i} = \frac{p^i}{m} \equiv \frac{dq^i}{dt}$$
$$-\frac{\partial H}{\partial q^i} = F \equiv \frac{dp^i}{dt} = \dot{p}.$$

Frame Title I

Definition

A 2-form $\omega \in \Omega^2(M)$ is called **symplectic** if it is nondegenerate, i.e.

$$\omega^{\#} : TM \to T^*M$$
$$X \to \iota_X \omega$$

is an isomorphism ($\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$, ω_{ij} invertible) and $d\omega = 0$. The pair (M, ω) is a symplectic 2-form, called a **symplectic manifold**.

Hamiltonian formalism:

For any function $f \in C^\infty(M)$, there is an associated **hamiltonian vector field** X_f uniquely defined by the condition

$$\iota_{X_f}\omega = df.$$

Frame Title II

In other words, $X_f = (\omega^{\#})^{-1}(df)$. There is an induced bilinear operator

$$\{\cdot,\cdot\} : C^{\infty}(M) \times C^{\infty}(M) \to C^{\infty}(M),$$

konwn as the **Poission bracket**, that measures the rate of change of a function g along the Hamiltonian flow of f,

$$\{f,g\} := \omega(X_g, X_f) = \mathcal{L}_{X_f} g$$

- $d\omega(X_f, X_g, X_h) = \{f, \{g,h\}\} + \{h, \{f,g\}\} + \{g, \{h,f\}\} = 0$
- Stisfies the Leibniz rule $\{f, gh\} = \{f,g\}h + \{f,h\}g$.

Frame Title III

From the Leibniz rule, the Possion bracket is defined by a bivector field $\pi \in \Gamma(\Lambda^2 TM)$, uniquely determined by

$$\pi(df, dg) = \{f, g\} = \omega(X_g, X_f)$$

and locally, $\pi = \frac{1}{2}\pi^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$.

The bivector field π defines a bundle map

$$\begin{aligned} \pi^{\#} : T^*M &\to TM \\ \alpha &\to \iota_\alpha \pi, \end{aligned}$$

such that $X_f = \pi^{\#}(df)$. Since $df = \omega^{\#}(X_f) = \omega^{\#}(\pi^{\#}(df))$, we see that ω and π are related by

$$\omega^{\#} = (\pi^{\#})^{-1}$$

Recall: A symplectic structure

1. non-degenerate $\underbrace{\textit{closed}\, 2 - \textit{form}}_{\textit{symplectic form}}$ or,
2. non-degenerate $\underbrace{\textit{Poission bivectorfield}}_{\textit{Poisson Structure}}$

ω is non-degenerate if the map (bundle map)

$$\begin{aligned} \omega^{\#} : TM &\to T^*M \\ X &\to i_X\omega \end{aligned}$$

is an isomorpism (or in coordinates (ω_{ij}) is invertible). Hamilton formalism:

1. For any $f \in C^\infty(M)$ there is a X_f defined by

$$i_{X_f}\omega = df$$

   i.e.

$$X_f = (\omega^\#)^{-1}(df)$$

2. There is a op. $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ called the **Poission bracket** defined by

$$\{f, g\} := \omega(X_g, X_f) = \mathcal{L}_{X_f} f$$

Results:

1. $\{f, g\} = -\{g, f\}$
2. $Jac_{Poiss} = d\omega = 0$

The pair $(C^\infty(M), \{\cdot,\cdot\})$ is called a **Poisson algebra**. Poisson algebra= Lie algebra+$\{\cdot,\cdot\}$ is compatible with the associative, commutative product via Leibniz,

$$\{f, gh\} = \{f, g\}h + \{f, h\}g$$

(Immediatly verified by the Lie derivative).

One can equivalently define a bivectorfield $\pi \in \Gamma(\Lambda^2 TM)$ such that

$$\pi(df, dg) = \{f, g\} = \omega(X_g, x_f)$$

π defines a bundle map

$$\pi^\# : T^*M \to TM$$
$$\alpha \to i_\alpha \pi$$

in such a way that $X_f = \pi^\#(df)$.

Remark

Comparing $\omega^\#$ *and* $\pi^\#$*,*

$$\omega^\# = (\pi^\#)^{-1}.$$

Hence, one can either choose π *or* ω *to describe the system. Likewise,* π *is non degenerate if the bundle map* $\pi^\#$ *is an iso. (or* (π_{ij}) *is invertible).*

- *We say that* π *is* **Poission** *if* $\{f, g\} = \pi(df, dg)$ *satisfies the Jacobi identity:*

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = \{f, \{g, h\}\} + c.p = 0$$

Remarks: We asked for

1. *A lie Algebra* ✓
2. $\{\cdot, \cdot\}$ *compatible with the product via Leibniz*
3. *Skew symmetric* $\pi \in \Gamma(\Lambda TM)$

There is a correspondence of Bivectorfields $\leftrightarrow$ *non-degenerate 2-forms s.t. pi is Poission iff* ω *is closed.*

non-degenerate π		non-degenerate ω
$Jac_\pi = 0$	$\Leftrightarrow$	$d\omega = 0$

Poisson manifolds

Contrary at symplectic forms, if (M, π) is a Poisson manifold, any function $f \in C^\infty(M)$ defines a unique Hamiltonian

$$X_f = \pi^\#(df),$$

i.e. $(C^\infty(M), \{\cdot, \cdot\})$ is a Poisson algebra.

Dirac structures [Courant, Weinstein '88, Courant'90] I

"A way to treat both types of degenerate symplectic structures in a unified manner" The presymplectic and Poisson structures are subbundles of the **generalized tangent**

$$\mathbb{T}M = TM \bigoplus T^*M$$

defined by the graphs of $\pi^\#$, $\omega^\#$ and additional geometric structures.

- A bilinear form $\langle\cdot,\cdot\rangle$ non-degenerate, symmetric on $\mathbb{T}M$ defined by
$$\langle (X,\alpha),(Y,\beta)\rangle = \beta(X) + \alpha(Y)$$
- A Courant bracket $[\![\cdot,\cdot]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ defined by
$$[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \frac{1}{2}d(i_Y\alpha - i_X\beta) + H(X,Y,\cdot))$$

Dirac structures [Courant, Weinstein '88, Courant'90] II

A **Dirac structure** on M is a vector subbundle $L \subseteq \mathbb{T}M$ satisfying:

1. $L = L^{\perp}$, resp. to $\langle , \rangle$.
2. $[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L)$ i.e. L is involutive w.r.t the Courant bracket.

Remarks:

- 1. is equivalent to $\langle , \rangle|_L = 0$ and $rank(L) = dim(M)$
- The Courant bracket satisfies

$$\mathrm{Jac}_{[\![,]\!]} = [\![[\![a_1, a_2]\!], a_3]\!] + c.p. = \frac{1}{3} d \langle [\![a_1, a_2]\!], a_3 \rangle$$

  i.e. it is NOT a Lie bracket.

- A subbundle $L \subset \mathbb{T}M$ satisfiying 1. is called **Lagrangian subbundle** of $\mathbb{T}M$

Dirac structures [Courant, Weinstein '88, Courant'90] III

- 2. can be equivalently written by (using 1.)

$$\langle [\![a_1, a_2]\!], a_3 \rangle = 0.$$

- For any lagrangian subbundle L,

$$\Upsilon_L(a_1, a_2, a_3) := \langle [\![a_1, a_2]\!], a_3 \rangle$$

defines an elemeent $\Upsilon_L \in \Gamma(\Lambda^3 L^*)$ called the **Courant tensor** of L.

Dirac structures [Courant, Weinstein '88, Courant'90] IV

Example: Any bivector field π defines a lagrangian subbundle by

$$L_\pi = \{(\pi^\#(\alpha), \alpha) \,|\, \alpha T^* M\}$$

Hence, $(a_i = (\pi^\#(df_i), df_i))$

$$[\![a_1, a_2]\!] = [X_{f_1}, X_{f_2}] \bigoplus \mathcal{L}_{X_{f_1}} df_2 - \mathcal{L}_{X_{f_2}} df_1 + \frac{1}{2} d(X_{f_2}(df_1) - X_{f_1}(df_2))$$

Recall that

$$\mathcal{L}_{X_{f_1}} df_2 - \mathcal{L}_{X_{f_2}} df_1 = d(\mathcal{L}_{X_{f_1}} f_2 - \mathcal{L}_{X_{f_2}} f_1) = 2d\{f_1, f_2\}$$

,

$$\frac{1}{2} d(i_{X_{f_1}} - i_{X_{f_2}} df_1) = \frac{1}{2} d(\mathcal{L}_{X_{f_1}} f_2 - \mathcal{L}_{X_{f_2}} f_1) = d\{f_1, f_2\}$$

Dirac structures [Courant, Weinstein '88, Courant'90] V

Hence,

$$
\begin{aligned}
\Upsilon_{L_\pi}(a_1, a_2, a_3) &= \langle [\![a_1, a_2]\!], a_3 \rangle = df_3([X_{f_1}, X_{f_2}]) + d\{f_1, f_2\}(X_{f_3}) \\
&= X_{f_1}\{f_2, f_3\} - X_{f_2}\{f_1, f_2\} \\
&= Jac_{\{\cdot,\cdot\}}(f_1, f_2, f_3).
\end{aligned}
$$

So 2. is satisfied iff π is Poisson.i.e. L_π is a Dirac structure.
Similarly, the graph

$$L_\omega = \{(X, \omega^{\#}(X) \,|\, X \in TM\}$$

has

$$\Upsilon_{L_\omega}(a_1, a_2, a_3) = d\omega(X_1, X_2, X_3)$$

i.e. L_ω is a Dirac structure iff ω is presymplectic. [Casallas].

Hamiltonian vector fields: Let L be a Dirac structure on M. A function $f \in C^{\infty}(M)$ is called addmisible if there is a vector field X_f s.t.

$$(X, df) \in L.$$

In this case X is called the Hamiltonian relative to f.

- All the addmisible functions are always a Poisson algebra.

Morphisms: One can either identify the structures $\varphi : M_1 \to M_2$ either with the pullback or the pushforward

$$\varphi^* \omega_2 = \omega_1$$

$$\varphi_* \pi = \pi$$

however, the morphisms are not equivalent.

Example

Concider $\omega_{\mathbb{R}^2} = dq^1 \wedge dp_1$, $\pi_{\mathbb{R}^2} = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q^1}$ and $\omega_{\mathbb{R}^4} = dq^1 \wedge dp_1 + dq^2 \wedge dp_2$, $\pi_{\mathbb{R}^4} = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q^1} + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q^2}$ The projection $(q^1, p_1, q^2, p_2) \to (q^1, p_1)$ satisfies $\varphi_* \pi_1 = \pi_2$ but not $\varphi^* \omega_2 = \omega_1$ and the inclusion $(q^1, p_1) \to (q^1, p_1, q^2, p_2)$ satisfies $\varphi^* \omega_2 = \omega_1$ but not $\varphi_* \pi_1 = \pi_2$.

Recall: The evolution of the system is given by the Poisson bracket,

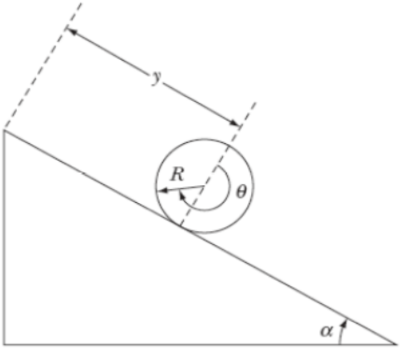
$$\frac{df}{dt} = \{f, H\}$$

The introduction of constrains will reduce the space of admissible functions.

Idea [Dirac'60s]: Describe the dynamics of the system (with constrains) in a bracket (Dirac bracket). $\Longrightarrow$ Dirac $H-$twisted structures.

- Constrains $\phi_r(q_i, p_i) = 0 \leftarrow$ first class constrain.
- Constrains $\phi_r(q_i, p_i) \geq 0$ second class constrain.

Example: The rolling (without slip) cylinder.



$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - mg(l - y)\sin\alpha,$$

$$\phi = y - R\theta = 0.$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y}, \quad \frac{\partial L}{\partial y} = mg\sin\alpha, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}mR^2\ddot{\theta}, \quad \frac{\partial L}{\partial \theta} = 0$$

E-L:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \sum \lambda^i \frac{\partial \phi^i}{\partial q^i}$$

Hence,

$$\ddot{y} = \frac{3g\sin\alpha}{2} \implies \lambda = -\frac{mg\sin\alpha}{2} = F$$

$$\ddot{\theta} = \frac{g\sin\alpha}{2R}.$$

Let L_H be a Dirac structure over the configuration space, a set of constrins ϕ^i are indep. if

$$\{f, \sum \phi^i\} = 0.$$

Example

As above, $\phi = y - R\theta \implies$

$$\{\phi, H\} = \frac{P_x}{m} - \frac{4P_\theta R}{mR^2} = 0\checkmark$$

In this context, Dirac introduced

$$\dot{q}_i = \sum \frac{\partial H}{\partial p_i} + \lambda^r \frac{\partial \phi_r}{\partial p_i}$$
$$\dot{p}_i = -\sum \frac{\partial H}{\partial q_i} - \lambda^r \frac{\partial \phi_r}{\partial q_i}, \quad \phi_r = 0.$$

One can then write

$$\dot{q}_i = \{p_i, H\} + \lambda^r \{q_i, \phi_r\}$$
$$\dot{p}_i = \{p_i, H\} + \lambda^r \{q_i, \phi_r\}$$

and therefore, observables:

$$\dot{f} = \sum \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = \{f, H\} + \lambda^r \{f, \phi_r\}$$

One then could define a Dirac bracket

$$\{\cdot,\cdot\}_D : C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$$

by

$$\{f, g\}_D = \{f, g\} - \{f, \xi_\mu\}\Delta^{\mu\nu}\{\xi_\nu, g\}$$

where ξ_μ are secondary constrains.

Example

$\phi = y - R\theta = 0$ then the symplectic form is

$$\omega_0 = dp_i \wedge dq^i = dp_y \wedge dy + dp_\theta \wedge d\theta.$$

To carry the constrains we "perturve" the form into

$$\omega = \phi\omega_0$$

which in turn introduces a Dirac $H-$twisted structure with $H = d\omega$. The algebra of admissible functions is then given by f satisfying

$$\begin{aligned} L_{X_f}\omega &= L_{X_f}\left((y - R\theta)\left(dp_y \wedge dy + dp_\theta \wedge d\theta\right)\right) = 0 \\ &= X_f(y - R\theta)\left(dp_y \wedge dy + dp_\theta \wedge d\theta\right) \\ &\qquad\qquad + (y - R\theta)L_{X_f}\left(dp_y \wedge dy + dp_\theta \wedge d\theta\right) = 0 \\ &= \{f, \phi\} = 0 \end{aligned}$$

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