Notes on Fields

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CONTENTS

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I. GENERAL DISCRETE THEORIES AND THEIR GAUIGING

nygaugingofdiscretesymmetriescanbeungauged,simplybyasubsequent gauging of the dual symmetry. This means that discrete gaugings do not change the full information of observables in a theory. However, as we have seen, discrete gaugings do mix different types of observables into each other, e.g. the exchange of twisted and charged sectors discussed in section 4.3.4. As a consequence, a physical observable may appear in different contexts in different gauge frames associated to a discrete symmetry. Symmetry topological field theories (TFTs), abbreviated SymTFTs or Symmetry TFTs, are powerful tools that detangle the physical observables from their behavior in different gauge frames. They also encode the 't Hooft anomalies of the symmetry and possible higher-group structure

II. BPZ EQUATIONS FROM FUSION RULES FOR ISING/MINIMAL MODELS

III. HW

$$\begin{aligned} Z_{\text{Ising}} &= Z_{0,\frac{1}{2}} + Z_{\frac{1}{2},\frac{1}{2}} + Z_{\frac{1}{2},0} \\ &= \frac{1}{2}\left[\left|\frac{\theta_2(0|\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_3(0|\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_4(0|\tau)}{\eta(\tau)}\right|\right] \\ &= |\chi_{1,1}(\tau)|^2 + |\chi_{2,1}(\tau)|^2 + |\chi_{1,2}(\tau)|^2 \end{aligned}$$

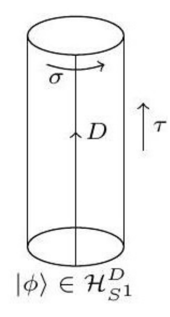
Conformal characters of identity, energy and spin operators are:

$$\begin{aligned} \chi_{1,1}(\tau) &= \frac{1}{2\sqrt{\eta(\tau)}}[\sqrt{\theta_3(0|\tau)} + \sqrt{\theta_4(0|\tau)}] \\ \chi_{2,1}(\tau) &= \frac{1}{2\sqrt{\eta(\tau)}}[\sqrt{\theta_3(0|\tau)} - \sqrt{\theta_4(0|\tau)}] \\ \chi_{1,2}(\tau) &= \frac{1}{\sqrt{2\eta(\tau)}}\sqrt{\theta_2(0|\tau)} \end{aligned}$$

indeed expandinf the characters in powers of q we have:

$$\begin{aligned} q^{\frac{1}{48}}\chi_1(\tau) &= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 5q^8 + 5q^9 + 7q^{10} + O\left(q^{481/48}\right) \\ q^{-\frac{1}{24}+\frac{1}{48}}\chi_\epsilon(\tau) &= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 4q^7 + 5q^8 + 6q^9 + 8q^{10} + O\left(q^{481/48}\right) \\ q^{-\frac{1}{24}}\chi_\sigma(\tau) &= 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 8q^9 + 10q^{10} + O\left(q^{481/48}\right) \end{aligned}$$

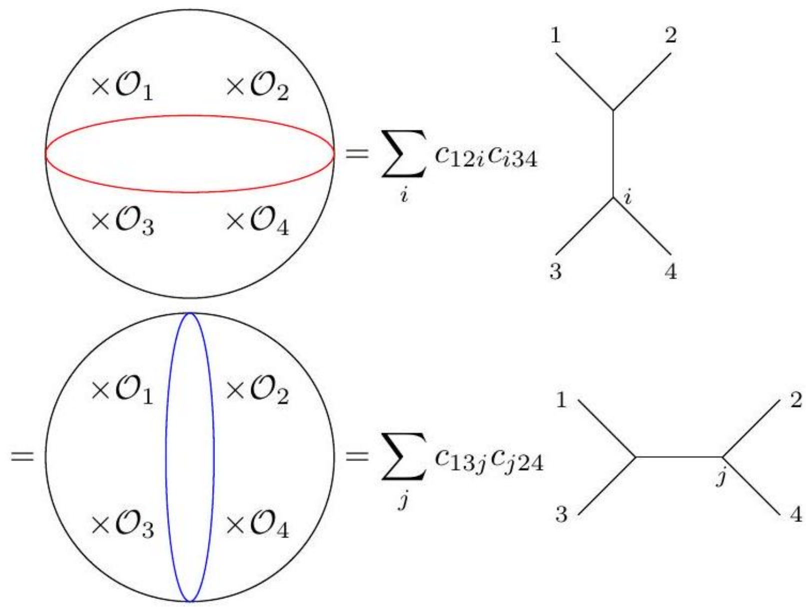
We consider $Z^{Ising}[g_1, g_2](\tau, \bar{\tau}) = \operatorname{tr}^{g_1}_{S^1}\, g_2 q^{L_0 - \frac{1}{48}} \bar{q}^{\bar{L}_0 - \frac{1}{48}}$



IV. GENERAL AXIOMATIC CONSTRAINTS

Let us study constraints coming from locality. No matter how we build our Riemann surface out of Hilbert spaces on circles, we will always get the same answer for CFT observables. It all boils down to consistency of cutting and gluing CFT observables on Riemann surfaces. Moore-Seiberg gave necessary and sufficient conditions for that:

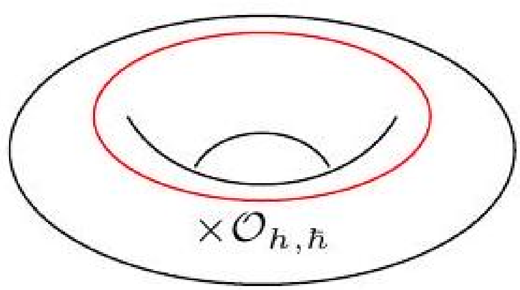
1. Quantization of spin.
2. Sphere 4 -point crossing



3. Modular covariance of the torus one-point function [3] $\langle \mathcal{O}_{h,\bar{h}} \rangle_\tau$, namely

$$\langle \mathcal{O} \rangle_{-1/\tau} = \tau^h \bar{\tau}^{\bar{h}} \langle \mathcal{O} \rangle_\tau .$$

where the factor is a Weyl factor coming from the $w \to w'/\tau$ coordinate change.



Question: if you write the modular invariance on arbitrary Riemann surfaces, is that enough to recover the sphere four-point function condition? Answer: that's a very good question; in some sense the genus 2 Riemann surface can be cut open into a four-point function.

Question: how trivial is the Moore-Seiberg result; is it deep? Answer: it depends on your particular taste. It is just about cutting and gluing.

Question: are there cases where the sphere condition is satisfied but not torus modular invariance. Answer: have to think.

1.7 Generalized global symmetries in $d = 2$ CFT

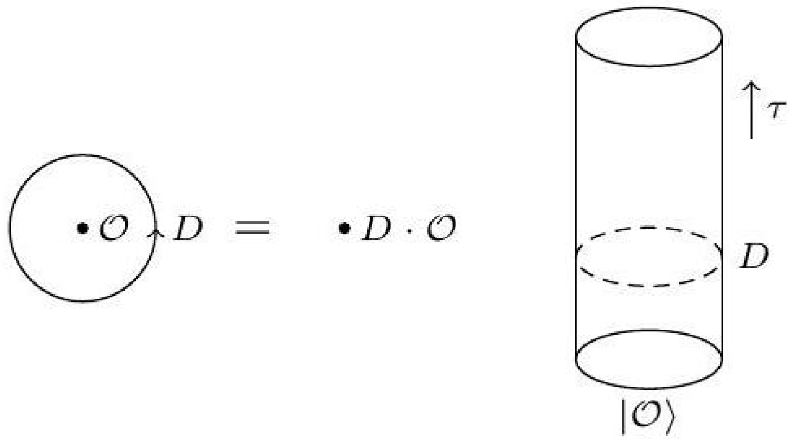
From our point of view, symmetries are the same as topological defect lines (TDL). Because we work in Euclidean signature with unitary Lorentz-invariant theories (actually CFT), so lines can be oriented in any direction.

By definition, topological defects obey

- Topological invariance
- Fusion with integer coefficients (see later for a reason based on locality)

$$D_i \uparrow D_j = \sum_k \uparrow D_k$$

The action on charged operators preserves $h, \bar{h}$ since the operator commutes with the stress-tensor. It can be depicted in radial quantization or on a cylinder,



Quantum dimension of a defect. Because the operator does not change the dimensions, and we have assumed we have a unique vacuum, the defect must simply rescale the vacuum by some number $\langle D\rangle = \langle 0|D|0\rangle$ called the quantum dimension,

$$D|0\rangle = \langle D\rangle|0\rangle.$$

In a unitary theory, $\langle D\rangle \geq 1$. If $\langle D\rangle = 1$ then D is invertible. If $\langle D\rangle > 1$ then D is not invertible.

1.7.1 Example: Ising CFT

What are the symmetries? In fact, symmetries are equivalent to Ward identities, so whenever you find Ward identities you should find the corresponding symmetry.

There is the obvious η symmetry, acting as $1 \to 1, \sigma \to -\sigma, \epsilon \to \epsilon$. Shows that $\langle\sigma\sigma\sigma\rangle = \langle\sigma\epsilon\epsilon\rangle = 0$ etc.

Surprisingly (at first) one has $\langle\epsilon\ldots\epsilon\rangle = 0$ whenever there is an odd number of ϵ, so there should be some symmetry guaranteeing that. But it cannot be just a $\mathbb{Z}_2$ charge of ϵ since $\langle\sigma\sigma\epsilon\rangle \neq 0$. It will be a non-invertible symmetry $\mathcal{N}$ sending $\epsilon \to -\epsilon\langle\mathcal{N}\rangle$ and $\sigma \to 0$ so as to allow $\langle\sigma\sigma\epsilon\rangle \neq 0$.

This leads us to go beyond groups and to discuss fusion categories.

2 TOPOLOGICAL DEFECTS AND FUSION CATEGORY

References for today:

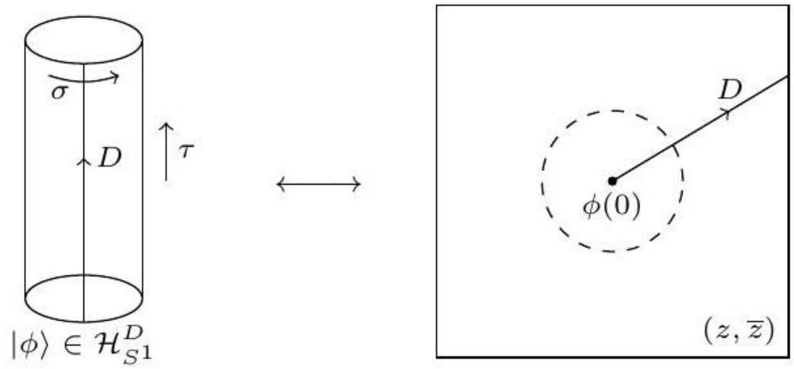
- https://arxiv.org/abs/1704.02330 Bhardwaj, Tachikawa,
- https://arxiv.org/abs/1802.04445 Chang, Lin, Shao, Wang, Yin.

The goal is to understand in what sense fusion categories are the natural object that comes up when studying generalized symmetries in 2d, just like groups arise when studying invertible symmetries.

2.1 Axiomatic approach to symmetries in 2d CFT

Here we focus on compact, unitary CFTs with a single ground state. Without symmetries the relevant axioms are Moore-Seiberg axioms (on fusion and braiding and torus S-move). We now want to refine these axioms by decorating them by topological defect lines.

We work in Euclidean signature. A given symmetry defect can be taken as wrapping the S^1 spatial direction, in which case it is simply a symmetry operator on the Hilbert space, or can be placed in the time direction at a point in space,

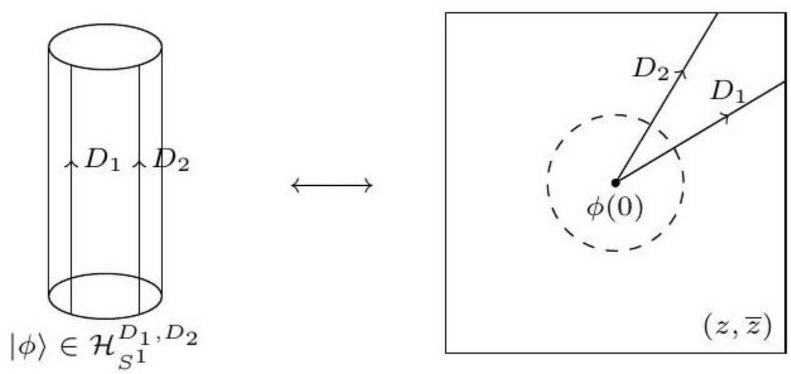


The Hilbert space in the presence of a defect D is denoted as $\mathcal{H}^D_{S^1}$ and called the D-twisted sector. Under the state-operator map it corresponds to an operator in the D-twisted sector.

Faithfulness condition. We assume the faithfulness condition which states that the only defect that acts trivially on all local operators is the identity defect. Equivalently, defects $D \neq 1$ cannot end topologically: otherwise you could cut it open and see that it acts trivially:

$$(\mathrm{O})\ D = \bigcirc D = \langle D \rangle \mathcal{O}$$

MULTI-DEFECT HILBERT SPACE.



The Virasoro action allows you to move the defects around. It introduces some factors so strictly speaking the Hilbert space $\mathcal{H}^{D_1,D_2,\cdots}_{S^1}$ depends on the separations and Hilbert spaces for different separations are easily isomorphic. The Hilbert space is also invariant (up to an important isomorphism) under cyclic permutations of the defect (to do things properly we need to include a marked point etc).

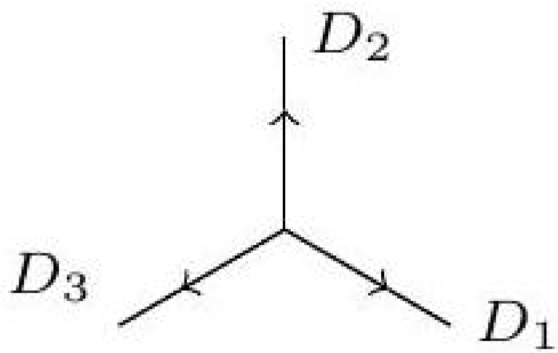
Besides the Hilbert space $\mathcal{H}^{D_1,D_2,\cdots}_{S^1}$ we also define

$$\mathcal{H}^{D_1+D_2}_{S^1} = \mathcal{H}^{D_1}_{S^1} \oplus \mathcal{H}^{D_2}_{S^1}$$

corresponding to the insertion of a direct sum of defects at the same place. This will be useful when discussing the fusion of defect.

Question: if you insert a non-topological defect do you break Virasoro? Answer: you break Vir $\times$ Vir to the diagonal subalgebra.

Topological junction. A topological junction is an operator of dimension $h = \bar{h} = 0$ inside $\mathcal{H}^{D_1\cdots D_n}_{S^1}$. The space of such junctions is denoted $V_{D_1\ldots D_n}$. An element $v \in V_{D_1\ldots D_n}$ is visualized on the plane as



In the invertible case defects are labeled by group elements g_i and $\dim V_{g_1 \ldots g_n}$ is 1 if $g_1 g_2 \ldots g_n = 1$ and is otherwise zero.

Dual defect. The dual defect $\bar{D}$ may have $D\bar{D}$ different from 1. The dual defect is simply defined as the orientation-reversed defect

$$\psi_{\bar{D}} = \psi_D$$

A defect is simple if $\dim V_{D\bar{D}} = 1$. As a consequence we can show that $D \neq D_1 + D_2$. Conversely when $\dim V_{D\bar{D}} \geq 2$ then we can always split D into pieces.

Fusion. When fusing defects we get a new defect, which can be decomposed into simple defects, so we can introduce notation in the case of simple defects:

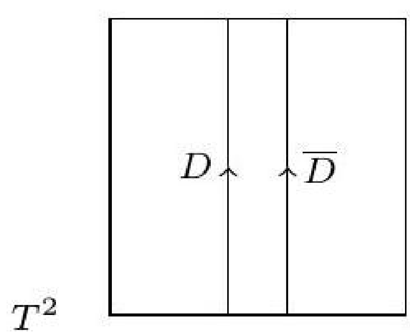
$$D_i D_j = \sum_k N_{ijk} D_k$$

One can check that $N_{ijk} = \dim V_{D_i D_j \bar{D}_k}$. This differs from the group-like multiplication law $D_g D_{g'} = D_{gg'}$ for invertible symmetries. Special case

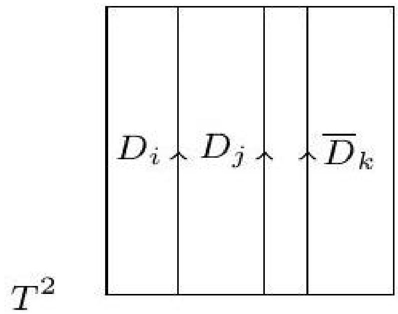
$$D_i \bar{D}_i = 1 + \ldots$$

Let us check that the leading term is 1 .

Using the thermal partition function. ..... missing discussion of how the thermal partition function allows one to show that the leading term in $D_i \bar{D}_i$ is $\dim V_{D\bar{D}} \cdots$.

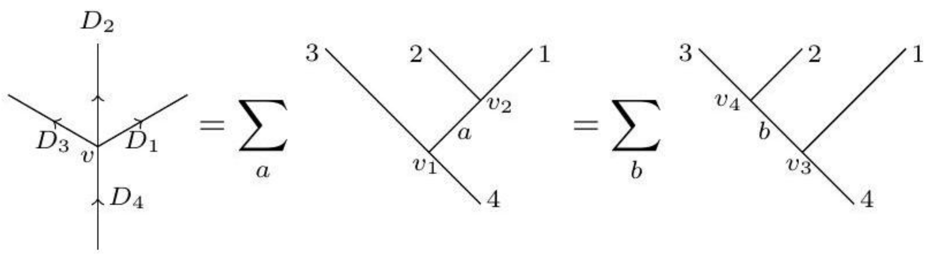


Likewise in an exercise we should prove $N_{ijk} = \dim V_{D_i D_j \bar{D}_k}$ using the lowtemperature limit of the torus partition function with three defect insertions



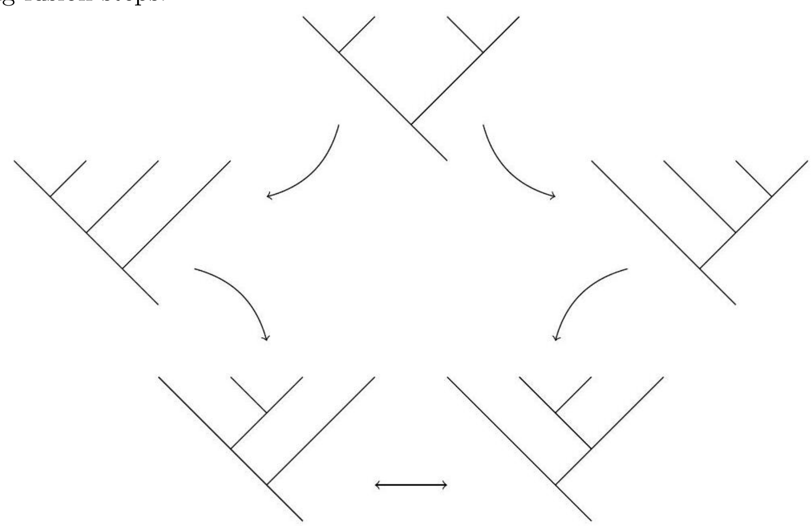
Topological junctions as morphisms. Topological junctions $v \in V_{D_1 D_2 D_3}$ are morphisms between $D_1 D_2$ and $\bar{D}_3$.

F-symbols (associators). In a general junction vector space there is no preferred basis. A several bases of $V_{D_1 D_2 D_3 \bar{D}_4}$ can be constructed from bases of three-fold junctions:



We have a unitary change of basis $\left(F_4^{321}\right)_{ab}$ mapping the basis of the form $v_1 \otimes v_2$ to that of the form $v_3 \otimes v_4$.

Pentagon identity. The fusion coefficients F have to obey an equation of the form $FF = \sum FFF$, obtained by performing the following fusion steps:

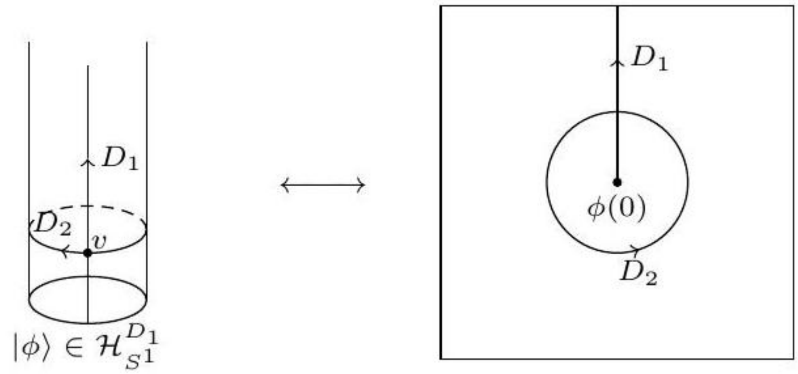


This is an analogue of the cocycle condition for $H^3(G, U(1))$ which classifies anomalies for the group G. In a sense, F-symbols capture the anomalies.

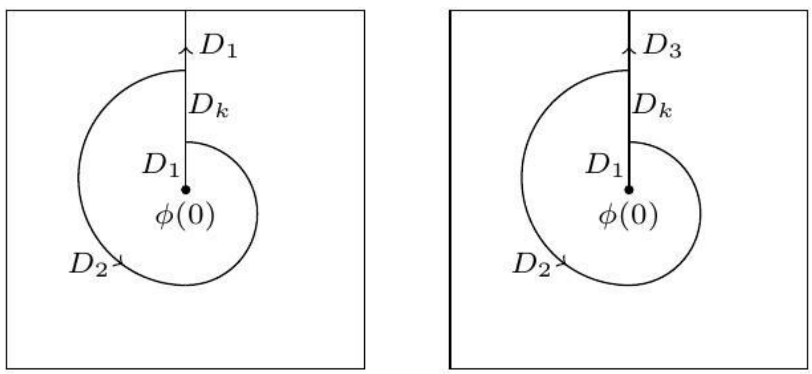
Should we continue with six-fold junctions etc? MacLane coherence theorem: the full set of consistency conditions is automatically satisfied once the pentagon identity is obeyed.

2.2 Symmetry action in defect Hilbert space

How can a symmetry labeled by a defect D_2 act on a twisted Hilbert space $\mathcal{H}_{S^1}^{D_1}$, namely on the D_1-twisted sector? We need a topological junction v between the operators.



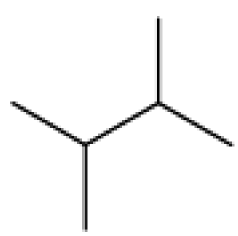
This can be resolved into an intermediate defect $D_k \in D_1 D_2$. There are normally multiple choices of this intermediate defect, hence $V_{D_1 D_2 \bar{D}_1 \bar{D}_2}$ is typically of dimension larger than 1 (in contrast to invertible symmetries). This means that there are multiple possible actions of D_2 on $\mathcal{H}_{S^1}^{D_1}$. This is called the Lasso action. A generalization is that the action can change $\mathcal{H}_{S^1}^{D_1}$ to $\mathcal{H}_{S^1}^{D_3}$ as in the second picture below. This generates the Tube algebra.



2.3 Definition of symmetry-enriched CFT

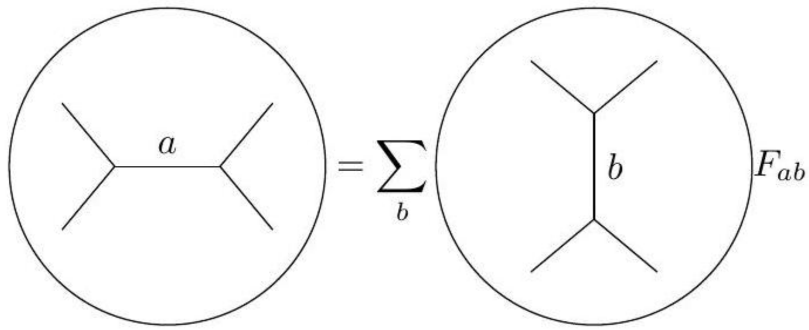
A CFT enriched by a collection of topological defect lines $\{D_i\}$ is given by

- Data: $\mathcal{H}_{S^1}^{D_i \dots D_j}$ and three-point functions of operators attached to defects,



This includes the usual $\mathcal{H}_{S^1}$ and three point functions.

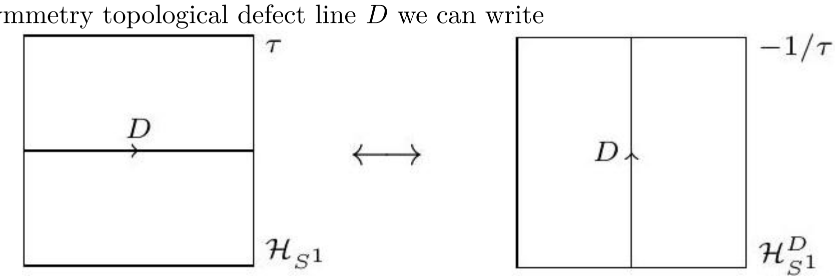
- Bootstrap conditions (locality). Sphere four-point crossing in the presence of topological defects,



Modular covariance of the torus one-point function with defects.

2.4 Modular invariance of the symmetry enriched CFT

Recall that for a symmetry topological defect line D we can write



This leads to the following relation

$$\mathrm{Tr}_{\mathcal{H}_{S^1}}\left(\widehat{D} q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right) = \mathrm{Tr}_{\mathcal{H}^D_{S^1}}\left(\widetilde{q}^{L_0-c/24}\bar{\widetilde{q}}^{\bar{L}_0-c/24}\right)$$

where $\widetilde{q} = e^{2\pi i(-1/\tau)}$.

Ising model. See homework. The Hilbert space on the LHS here is a direct sum of representations of Virasoro so

$$\begin{aligned} \mathrm{LHS} &= A_1 \left|\chi_0\right|^2 + A_{1/2}\left|\chi_{1/2}\right|^2 + A_{1/16}\left|\chi_{1/16}\right|^2 \\ \mathrm{RHS} &= \sum_{i,j} n_{ij}\chi_i(\widetilde{q})\chi_j(\bar{\widetilde{q}}), \end{aligned}$$

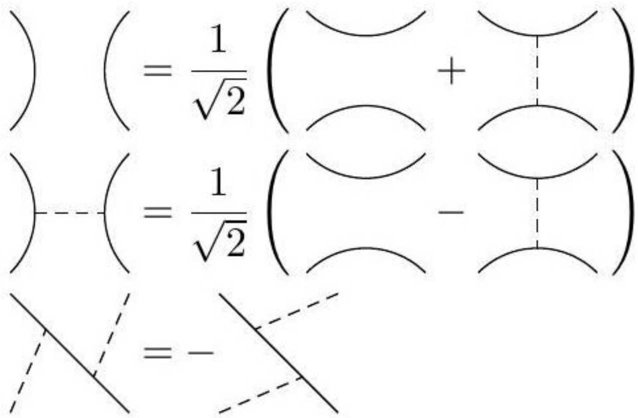
where the A_i are not yet quantized and the n_{ij} are non-negative integers. Using the modular transformations of Virasoro characters we find

- $(A_i) = (1,1,1)$ corresponds to the identity defect $D = 1$;
- $(A_i) = (1,1,-1)$ corresponds to the $D = \eta$ defect;
- $(A_i) = (\sqrt{2}, -\sqrt{2}, 0)$ corresponds to the $D = \mathcal{N}$ defect.

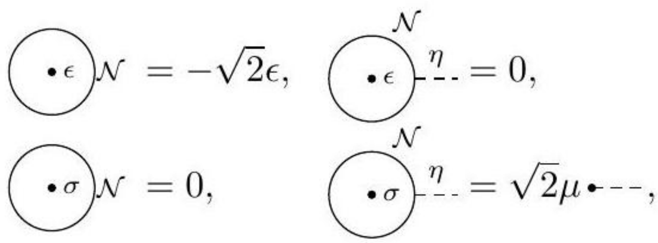
The fusion rule for $\mathcal{N}^2$ can be found by squaring these eigenvalues A_i and reexpressing them in the basis of other solutions (A_i). The same can be done for all fusion rules and we find

$$\mathcal{N}^2 = 1 + \eta, \quad \eta^2 = 1, \quad \mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N}.$$

Ising F-symbols.



Action on local operators. Then we will find



where $\sqrt{2}$ is the quantum dimension $\langle \mathcal{N} \rangle = \sqrt{2}$, and μ is a primary operator in the twisted sector:

$$\mathcal{H}^{\eta}_{S^1} = \left\{ \psi_{1/2,0}, \widetilde{\psi}_{0,1/2}, \mu_{1/16,1/16} \right\}$$

Passing TDL through local operators.

$$\epsilon \cdot |_{\mathcal{N}} = |_{-\epsilon \bullet} \quad \sigma \cdot |_{\mathcal{N}} = |_{\mathcal{N}} |_{\mathcal{N}}$$

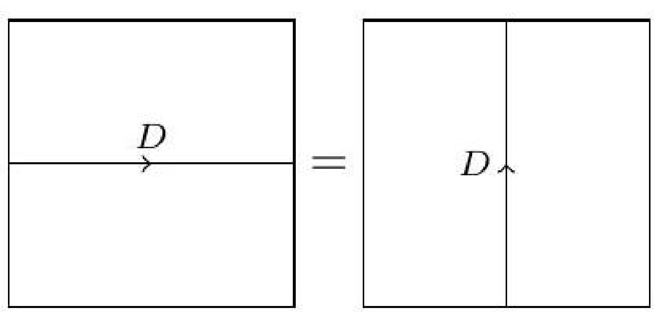
(Unrelated?) claim: the presence of $\mathcal{N}$ means that the CFT is self-dual under the $\mathbb{Z}_2$ orbifold.

2.5 Dynamic consequences of non-invertible symmetry

Assume that you have a UV theory $T_{\rm UV}$ with D symmetry and you deform it by a (marginally relevant or) relevant operator $\mathcal{O}_{h,\bar{h}}$ with $\bar{h} = h$ and $\Delta \leq 2$ then perform the RG flow to the IR theory $T_{\rm IR}$. Assume also that the operator $\mathcal{O}$ commutes with the defect D (namely the deformation preserves the symmetry).

Claim 1 (Theorem). If $\langle D \rangle \notin \mathbb{Z}$ then $T_{\rm IR}$ cannot be trivially gapped: we either get a CFT or spontaneous symmetry breaking.

Proof. We have



If the IR is trivially gapped then there is a unique ground state so the left-hand side is $\langle 0|D|0\rangle = \langle D \rangle$. The right-hand side is a trace of 1 over the defect Hilbert space, which has to be an integer. Contradiction. If there were multiple ground states then D can act differently on different ground states and somehow this resolves the problem.

Example 1. Tricritical Ising model $c = 7/10$. The symmetry is Ising $\boxtimes$ Fib where the "Ising" symmetry is the usual $\{1, \eta, \mathcal{N}\}$ and the "Fib" symmetry is generated by W with $W^2 = 1 + W$, of quantum dimension $\langle W \rangle = (1 + \sqrt{5})/2$.

Deforming this CFT by $\epsilon'_{3/5,3/5}$, which commutes with $\mathcal{N}$, gives an RG flow whose low-energy limit is either gapless (necessarily Ising by c monotonicity) or gapped with at least three vacua. Both cases arise depending on the sign of the deformation, as can be shown using integrability.

Question: what operator tracks the RG flow arriving into the Ising model? Answer: it is an irrelevant operator, which turns out to be the $T\bar{T}$ operator constructed from the stress-tensor, which thus automatically commutes with all of the symmetries, including $\mathcal{N}$.

Example 2. The $1 + 1$ dimensional $SU(N)$ massless adjoint QCD also has a huge amount of non-invertible symmetries. This leads to a gapped phase with $\sim 2^N$ vacuum degeneracies. Heuristic explanation: the adjoint fermions are described by the WZW model Spin $\left(N^2 - 1\right)_1$; gauging $SU(N)$ roughly amounts to taking a coset, which suggests the TQFT Spin $\left(N^2 - 1\right)_1 / SU(N)_N$, which has a ton of topological defects.

3 TOPOLOGICAL INTERFACES AND GENERALIZED GAUGING

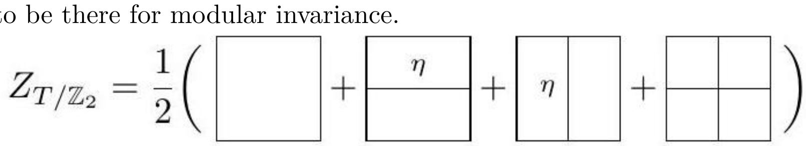
See Fuchs-Runkel-Schwaigert https://arxiv.org/abs/hep-th/0204148,
..., Diatlyk-Luo-Weller-Wang https://arxiv.org/abs/2311.17044

3.1 Gauging procedure

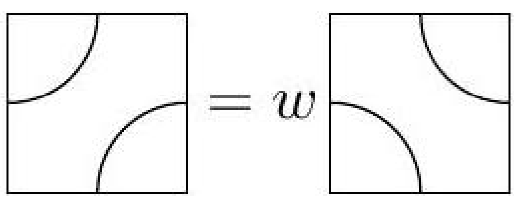
Gauging a usual abelian group symmetry. To gauge a $\mathbb{Z}_2$ symmetry of a theory T (which is like an orbifold in string theory), two steps:

- project to the $\mathbb{Z}_2$-even sector;
- include $\mathbb{Z}_2$-twisted sectors.

For instance, the torus partition function is a sum of four terms: the first two terms are a trace in the usual Hilbert space but with a projection $(1 + \eta)/2$ onto the $\mathbb{Z}_2$-even sector; the second two are the projection but in the twisted sector. All terms have to be there for modular invariance.



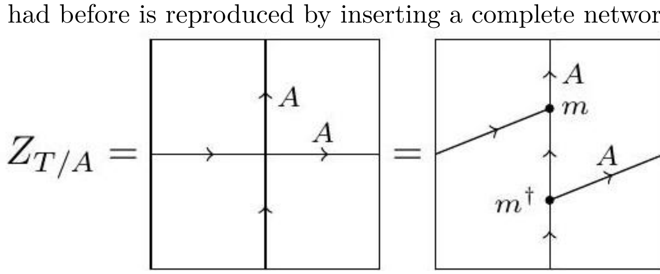
The last diagram is subtle and has to be resolved to be defined. We have no topological junction for a group symmetry so there are two resolutions simply related by a phase w (equal to ± 1 in the $\mathbb{Z}_2$ case)



If we pick one of these resolutions, then we have to worry about modular invariance of Z. If $w \neq 1$ we will find that the partition function is not S-invariant regardless of what we do. There is an anomaly preventing you from gauging.

Example: the Ising model has $w = +1$ while $SU(2)_1$ has $w = -1$ for the $\mathbb{Z}_2$ center symmetry.

Gauging topological defect lines. Pick a general topological defect line A. (For instance, to reproduce the previous gauging we would take $A = 1 + \eta$, or more generally for a group we would take the projector $A = \sum_{g \in G} g$.) Then the sum of partition functions we had before is reproduced by inserting a complete network of A defects,

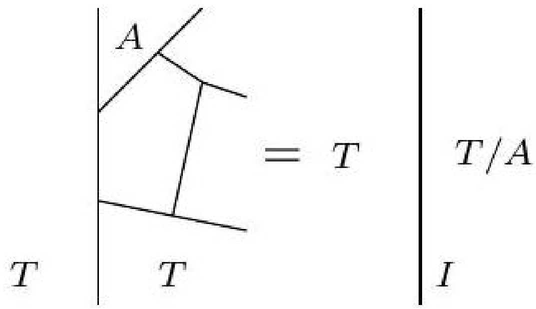


Data for generalized gauging: $\left(A, m, m^{\dagger}, u, u^{\dagger}\right)$ with $u, u^{\dagger}$ end-points of A and $m, m^{\dagger}$ three-fold junctions. To avoid gauge anomaly, data has to form a symmetric special Frobenius algebra object. These latter data $m, m^{\dagger}$ capture "discrete torsion", $1+1$ dimensional SPT phases. (We also assume A is self-dual but it is not clear how much work this assumption is making.)

Abstractly: gauging is decorating the observables in T with a network of $\left(A, m, m^{\dagger}\right)$ with a mesh that is fine enough.

3.2 Half-gauging and topological interfaces

Suppose you have a symmetry and a choice of $\left(A, m, m^{\dagger}\right)$. Then by gauging over a half-space you can make an interface between the theory T and T/A :



Gauging just in a small slab (such as a time interval) defines a topological

[] G. W. Moore and N. Seiberg, Commun. Math. Phys. **123**, 177 (1989).