Numerical Methods on Brachistochrone problem

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1 Introduction

The Brachistochrone problem, first proposed by Johann Bernoulli, attempts to find a curve, for which two given points, the time of descent of a mass from one point to the other is minimal. The aim of the following pages will be to give a simple overview of this problem with some examples with both a theoretical and numerical approach.

2 Theoretical Analysis

We will study this problem where the two given points are $(0,0)$ and $(1,1)$ which can be modified to any other two points by some dilatation and translation of the former ones. The value of g will be taken as $9.81m/s^2$

2.1 Time of descent

Given a smooth (monotone [1]) curve y that intercepts the points $(0,0)$ and $(1,1)$ and placing a particle of mass m at the point $(1,1)$, the time it takes to reach $(0,0)$ can be found as follows:

From the definition of velocity one obtains

$$||\vec{v}|| = \left|\left|\frac{d\vec{x}}{dt}\right|\right| \implies t(x) = \int_{(1,1)}^{(x,y(x))} \frac{||d\vec{x}||}{||\vec{v}||}$$

with $x \in [0,1]$, and

$$d\vec{x} = d(x,y) = \left(1, \frac{dy}{dx}\right)dx$$

The speed can be obtained from the energy conservation of the given system as

$$\frac{1}{2}m||\vec{v}||^2 + mgy = mg(1)$$
$$||\vec{v}|| = \sqrt{2g(1-y)}$$

then, it follows

$$t = \int_{(1,1)}^{(x,y(x))} \frac{||d\vec{x}||}{||\vec{v}||} = \int_1^x \frac{||(1, \frac{dy}{dx})dx||}{\sqrt{2g(1-y)}} = \int_x^1 \sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}}||-dx|| = \int_x^1 \sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}}dx$$

Thus, the total time is given by

$$t_{Total} = \int_0^1 \sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}}dx \tag{1}$$

2.2 Examples

The first curve that comes to mind when facing this problem is a straight line.
$y = x$ using (1), we have:

$$t_{Total} = \int_0^1 \sqrt{\frac{1}{g(1-x)}}dx = \frac{2}{\sqrt{g}} = 0.6385s$$

[1] We never use this, but we want that the particle arrives at (0,0)

Newton thought [2] that the answer to this problem was the curve describing a section of a circle
$y = -\sqrt{1-x^2}+1$

$$t_{Total} = \int_0^1 \sqrt{\frac{1+2x^2}{2g(1-x^2)(\sqrt{1-x^2})}}dx$$

Another guess one would have is a positive power of x.
$y = x^n$

$$t_{Total} = \int_0^1 \sqrt{\frac{1+n^2x^{2(n-1)}}{2g(1-x^n)}}dx$$

(we will consider the cases $n = 2$ and $n = 3$)
[Non of these curves are the real answer]

2.3 Numerical methods and the bound of its errors

To estimate the integral given in (1), we could use various numerical methods such as the midpoint method, the Simpson's rule, or Gaussian quadrature.

- **Midpoint method**

  Let $h = 1/n$, and $0 = x_0 < x_1 < ... < x_{n-1} < x_n = 1$ where $x_k = kh$ a partition of the interval $[0,1]$. Define $u_i = (x_{i+1} - x_i)/2$ as the midpoint between two consecutive elements of the partition. Then an approximation of the total time would be given by

$$t_{Total} = \int_0^1 \sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}}dx \sim \sum_{i=0}^{n-1} h\sqrt{\frac{1+(\frac{dy}{dx}(u_i))^2}{2g(1-y(u_i))}}$$

  We know that the error on this estimate is [3]

$$\left| \int_0^1 \sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}}dx - \sum_{i=0}^{n-1} h\sqrt{\frac{1+(\frac{dy}{dx}(u_i))^2}{2g(1-y(u_i))}} \right| \leq \frac{Mh^2}{24}$$

  where

$$M = \max_{x\in[0,1]} \left| \frac{d^2}{dx^2}\sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}} \right|$$

  Clearly, we face a problem because this value is unbounded at $x = 1$, thus we have to restrict our domain to $x \in [0, 1-\epsilon]$ where epsilon is a small positive value. Our partition would be of the interval $[0, 1-\epsilon]$ and its approximation will be given by

$$t_{Total} = \int_0^1 \sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}}dx \sim \sum_{i=0}^{n-1} h\sqrt{\frac{1+(\frac{dy}{dx}(u_i))^2}{2g(1-y(u_i))}} + E(y;\epsilon)$$

  Where the latter term is the estimate

$$E(y;\epsilon) \sim \int_{1-\epsilon}^1 \sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}}dx$$

  For example, if $y = x^n$ this can be bounded as follows:

$$\begin{aligned} \int_{1-\epsilon}^1 \sqrt{\frac{1+(nx^{n-1})^2}{2g(1-x^n)}}dx &\leq \frac{1}{\sqrt{2g}} \int_{1-\epsilon}^1 \sqrt{\frac{1+n^2}{(1-(1-\epsilon)^n)}}dx \\ &\leq \sqrt{\frac{1+n^2}{2g(n\epsilon)}}(1-(1-\epsilon)) \\ &= \sqrt{\frac{\epsilon(1+n^2)}{2gn}} = E(x^n;\epsilon) \end{aligned}$$

[2] Cortissoz (2019) while giving a lecture.
[3] J. Cortissoz, Numerical Analysis Lecture Notes, Unpublished manuscript. (2019)

The success in a good numerical approximation by the discussed method is choosing a remarkably fine partition and a small enough epsilon such that the term $E(y;\epsilon)$ becomes more error-like than significant value-like on the total time of descent. In our case, letting epsilon become smaller implies that the bound on M will rise drastically as shown later. This means that the error made by the midpoint method will increase, but it can be countered by decreasing the value of h.

Given different epsilons, in this case 0.1 and 0.01, the values of the bound M change a lot.
For $y = x$ we have that

$$M = \max_{x\in[0,0.9]} \left| \frac{3}{4\sqrt{g(1-x)^5}} \right| \leq \frac{238}{\sqrt{g}} \leq 76 \qquad M = \max_{x\in[0,0.99]} \left| \frac{3}{4\sqrt{g(1-x)^5}} \right| \leq \frac{75000}{\sqrt{g}} \leq 23950$$

For $y = x^2$ we have that :

$$M = \max_{x\in[0,0.9]} \left| \frac{d^2}{dx^2}\sqrt{\frac{1+4x^2}{2g(1-x^2)}} \right| \leq \frac{271}{\sqrt{g}} \leq 87 \qquad M = \max_{x\in[0,0.99]} \left| \frac{d^2}{dx^2}\sqrt{\frac{1+4x^2}{2g(1-x^2)}} \right| \leq \frac{84005}{\sqrt{g}} \leq 26825$$

For $y = x^3$ we have that :

$$M = \max_{x\in[0,0.9]} \left| \frac{d^2}{dx^2}\sqrt{\frac{1+9x^4}{2g(1-x^3)}} \right| \leq \frac{322}{\sqrt{g}} \leq 105 \qquad M = \max_{x\in[0,0.99]} \left| \frac{d^2}{dx^2}\sqrt{\frac{1+9x^4}{2g(1-x^3)}} \right| \leq \frac{97245}{\sqrt{g}} \leq 31050$$

However, this is not the cleanest way of bounding these values.[4]

- Simpson's rule

  Let $h = 1/n$, and $0 = x_0 < x_1 < ... < x_{n-1} < x_n = 1$ where $x_k = kh$ a partition of the interval $[0,1]$.Let $u = \sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}}dx$. Then an approximation of the total time according to Simpson's rule would be given by

$$t_{Total} = \int_0^1 \sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}}dx \sim \frac{h}{6}(u_0 + 4u_1 + 2u_2 + 4u_3 + ... + 4u_{2n-1} + u_{2n})$$

and the error for one interval would be:

$$\left| \int_{x_i}^{x_{i+2}} udx - \frac{h}{6}(u_0 + 4u_1 + u_2) \right| \leq \frac{M}{72\sqrt{3}}h^4$$

where

$$M = \max_{x\in[0,1]} \left| \frac{d^3}{dx^3}\sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}} \right|$$

As in the midpoint rule, we face a similar problem because this value is unbounded at $x = 1$, thus we have to restrict our domain to $x \in [0, 1-\epsilon]$ where epsilon is a small positive value. Thus, similarly as before:

$$t_{Total} = \int_0^1 \sqrt{\frac{1+(\frac{dy}{dx})^2}{2g(1-y)}}dx \sim \frac{h}{6}(u_0 + 4u_1 + 2u_2 + 4u_3 + ... + 4u_{2n-1} + u_{2n}) + E(y;\epsilon)$$

[4] A little caveat must be made, we found these bounds by the use of third party numerical programs, thus they come as well with some error which we didn't took into account.

With an error[5] of $\frac{M}{192\sqrt{3}}h^3 + E(y;\epsilon)$

For $y = x$ we have that

$$M = \max_{x\in[0,0.9]} \left| \frac{d^3}{dx^3}\sqrt{\frac{2}{2g(1-x)}} \right| \leq \frac{5929.27}{\sqrt{g}} \leq 1893.07$$

$$M = \max_{x\in[0,0.99]} \left| \frac{d^3}{dx^3}\sqrt{\frac{2}{2g(1-x)}} \right| \leq \frac{18749986.59}{\sqrt{g}} \leq 5986410$$

For $y = x^2$ we have that

$$M = \max_{x\in[0,0.9]} \left| \frac{d^3}{dx^3}\sqrt{\frac{1+4x^2}{2g(1-x^2)}} \right| \leq \frac{6701.05}{\sqrt{g}} \leq 2139.4$$

$$M = \max_{x\in[0,0.99]} \left| \frac{d^3}{dx^3}\sqrt{\frac{1+4x^2}{2g(1-x^2)}} \right| \leq \frac{20986174.96}{\sqrt{g}} \leq 6700370$$

For $y = x^3$ we have that

$$M = \max_{x\in[0,0.9]} \left| \frac{d^2}{dx^2}\sqrt{\frac{1+9x^4}{2g(1-x^3)}} \right| \leq \frac{64232.41}{\sqrt{g}} \leq 20507.83$$

$$M = \max_{x\in[0,0.99]} \left| \frac{d^3}{dx^3}\sqrt{\frac{1+9x^4}{2g(1-x^3)}} \right| \leq \frac{24269265.062}{\sqrt{g}} \leq 7748580$$

3 Numerical Results

For some given examples in 2.2 and the given methods in 2.3 we found the value of the total time with its respective error. We will use the value of $g = 9.81m/s^2$.

3.1 Midpoint Method

If we just consider a mass falling from the point $(0.99, y(0.99))$, it can be easily distinguished (from the following data table) that the time it takes for the mass on the curve x^3 is less than x^2. The method was applied for the three curves shown on the left of the table with values of $h = 1/n$

$f\backslash n$	10	100	1000	10000	100000	1000000
x	0.50307895	0.57408623	0.5799137	0.58044266	0.58049497	0.5805002
x^2	0.45401009	0.52297636	0.52841578	0.52889916	0.52894684	0.5289516
x^3	0.43971175	0.51140163	0.51684801	0.51732217	0.51736882	0.51737348

To be sure about this claim, the values with their respective error should not overlap. Calculating the error of the midpoint method for the function x^2 we obtain $\Delta t = 1.095 \times 10^{-9}s$ and for the function x^3 is $\Delta t = 1,268 \times 10^{-9}s$. They are almost negligible for our purposes.

Now, the question is if this order holds for the whole interval $[0, 1]$, or whether the times from $[0.99, 1]$ taken into account flip the positions. Sadly, this cannot be concluded from just this scenario, because the values of $E(x^2; 0.01) = 0.0357$ and $E(x^3; 0.01) = 0.0412$ are sufficiently big to cause overlapping of the uncertainties of t_{Total}.

For the following functions, the values obtained by this method using n equal intervals that are a partition of $[0, 0.9999]$ are

$f\backslash n$	10	100	1000	10000	100000	1000000
x	0.51667964	0.61226813	0.62966831	0.63210433	0.63222158	0.63222793
x^2	0.47007693	0.56670489	0.5850998	0.58771653	0.58783692	0.58784295
x^3	0.45876434	0.56259547	0.58301958	0.58595815	0.58608883	0.58609495

Here we have $E(x, 0.0001) = 3.193 \times 10^{-3}$, $E(x^2, 0.0001) = 3.569 \times 10^{-3}$, $E(x^3, 0.0001) = 4,122 \times 10^{-3}$. And the values when $n = 10^6$ of the error caused by the midpoint method of x^2 and x^3 are $\Delta t = 1.09 \times 10^{-9}s$ and $\Delta t = 1.27 \times 10^{-9}s$ respectively.

[5] J. Cortissoz, Numerical Analysis Lecture Notes, Unpublished manuscript. (2019)

3.2 Simpson's Rule

The calculated values are obtained by using the Simpson's Rule for the total time on the interval $[0, 0.99]$. For this method, it can be easily distinguished again, that the time that it takes for the mas on the curve x^3 is less than x^2

$f\backslash n$	11	101	1001	10001	100001	1000001
x	0.55417709	0.57489812	0.57992088	0.58044273	0.58049497	0.58050019
x^2	0.51055433	0.52387719	0.52842374	0.52889924	0.52894684	0.52895160
x^3	0.50481022	0.51243917	0.51685718	0.51732226	0.51736882	0.51737348

In order to be sure of the previous claim, we must assure that the intervals in which the result of the method falls do not overlap. But if we choose n sufficiently as big as ~ 1000 the error is in the order of 10^{-5} therefore,this values can be negligible for our purposes.

Now, as in the midpoint method, we have to verify that our claim holds for the whole interval, but again, when the times $[0.99, 1]$ are taken into account, the uncertainties are sufficiently big to overlap.

3.3 Gaussian Quadrature

These are the values found using Gaussian quadrature (which was not discussed here, but still gives us some more confidence on the other methods)

$f\backslash n$	1	2	3	4	5
x	0.45152364	0.52702160	0.55900759	0.56948360	0.58797759
x^2	0.36866750	0.46828244	0.50552681	0.51759824	0.53825744
x^3	0.30168692	0.44925232	0.48985217	0.50562933	0.52864498