Resolución de un sistema de ecuaciones no lineles mediante elementos finitos

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Considere el siguiente problema

$$\begin{cases} -\left[\left(2t^2+2\right)y'\right]'+ty=t & \text{in} \quad (0,1) \\ y(0)=0=y(1) \end{cases}$$

La forma bilineal asociada al problema es

$$f(u,v)=\int_0^1[(2t^2+2)u(t)'v(t)'+tu(t)v(t)]dt$$

Veamos que es definida positiva:
Como $t\in(0,1)$

$$f(u,u)=\int_0^1[(2t^2+2)(u(t)')^2+tu(t)^2]dt\geq\int_0^1 2(u(t)')^2dt\geq 0$$

Definamos una partición del intervalo $[0,1]$ como $h:=\frac{1}{N}$, $t_0=0$, $t_{j+1}:=t_j+h$, $t_N=1$
Consideremos las funciones (para $j\in\{1,...,N-1\}$)

$$e_j(t)=\begin{cases} N\left(t-t_{j-1}\right) & \text{si } t_{j-1}\leq t_j \\ N\left(t_{j+1}-t\right) & \text{si } t_j\leq t\leq t_{j+1} \\ 0 & \text{de lo contrario} \end{cases}$$

Queremos aproximar la solución y mediante estas funciones.

$$y_N=\sum_{j=1}^{N-1}K_je_j(t)$$

Para hallar estos coeficientes, veamos las restricciones dadas por la ecuación diferencial.

$$\begin{aligned}
-\int_0^1&\left[\left(2t^2+2\right)y_N'\right]'e_sdt+\int_0^1 ty_Ne_sdt=\int_0^1 te_sdt \\
&=\int_0^1\left(2t^2+2\right)y_N'e_s'dt+\int_0^1 ty_Ne_sdt \\
&=\int_0^1 te_sdt+\int_0^1 ty_Ne_sdt=\int_0^1 te_sdt=\int_{t_{s-1}}^{t_s}tN(t-t_{s-1})dt+\int_{t_s}^{t_{s+1}}tN(t_{s+1}-t)dt \\
&=\int_0^1\left(2t^2+2\right)\sum_{j=1}^{N-1}K_je_j(t)'e_s'dt+\int_{t_{s-1}}^{t_{s+1}}t(K_{s-1}e_{s-1}+K_se_s+K_{s+1}e_{s+1})e_sdt \\
&=N\left(\frac{t^3}{3}-t_{s-1}\frac{t^2}{2}\right)\Big|_{t_{s-1}}^{t_s}+N\left(t_{s+1}\frac{t^2}{2}-\frac{t^3}{3}\right)\Big|_{t_s}^{t_{s+1}} \\
&=N^2(\frac{8}{3}-\frac{2}{3}((t_s^3-t_{s-1}^3)k_{s-1}+(t_{s+1}^3-t_s^3)k_{s+1}))+\int_{t_{s-1}}^{t_{s+1}}t(K_{s-1}e_{s-1}+K_se_s+K_{s+1}e_{s+1})e_sdt \\
&=N\left(\frac{2t_s^3}{3}-\frac{t_s^2}{2}\left(t_{s-1}+t_{s+1}\right)+\frac{1}{6}\left(t_{s-1}^3+t_{s+1}^3\right)\right) \\
&=N^2(\frac{8}{3}-\frac{2}{3}((t_s^3-t_{s-1}^3)k_{s-1}+(t_{s+1}^3-t_s^3)k_{s+1}))+\int_{t_{s-1}}^{t_{s+1}}t(K_{s-1}N(t_s-t)+K_se_s+K_{s+1}N(t-t_s))e_sdt \\
&=N\left(\frac{2t_s^3}{3}-\frac{t_s^2}{2}\left(t_{s-1}+t_{s+1}\right)+\frac{1}{6}\left(t_{s-1}^3+t_{s+1}^3\right)\right)
\end{aligned}$$

Veamos el valor de

$$\int_{t_{s-1}}^{t_s}[t(t_s-t)(t-t_{s-1})]dt = \frac{1}{12}(t_s-t_{s-1})^3(t_{s-1}+t_s)$$

$$\int_{t_{s-1}}^{t_s}[t(t-t_{s-1})^2]dt = \frac{1}{12}(t_s-t_{s-1})^3(t_{s-1}+3t_s)$$

$$\int_{t_s}^{t_{s+1}}[t(t-t_s)(t_{s+1}-t)]dt = \frac{1}{12}(t_{s+1}-t_s)^3(t_{s+1}+t_s)$$

$$\int_{t_s}^{t_{s+1}}[t(t_{s+1}-t)^2]dt = \frac{1}{12}(t_{s+1}-t_s)^3(t_{s+1}+3t_s)$$

Así que

$$N(\frac{8}{3}-\frac{2}{3}((t_s^3-t_{s-1}^3)k_{s-1}+(t_{s+1}^3-t_s^3)k_{s+1}))+K_{s-1}N\frac{1}{12}(t_s-t_{s-1})^3(t_{s-1}+t_s)+K_sN\left[\frac{1}{12}(t_s-t_{s-1})^3(t_{s-1}+3t_s)\right.$$
$$\left.+\frac{1}{12}(t_{s+1}-t_s)^3(t_{s+1}+3t_s)\right.+K_{s+1}N\frac{1}{12}(t_{s+1}-t_s)^3(t_{s+1}+t_s)=\left(\frac{2t_s^3}{3}-\frac{t_s^2}{2}\left(t_{s-1}+t_{s+1}\right)+\frac{1}{6}\left(t_{s-1}^3+t_{s+1}^3\right)\right) \qquad progr$$

$$(\frac{8}{3}-\frac{2}{3}((t_s^3-t_{s-1}^3)k_{s-1}+(t_{s+1}^3-t_s^3)k_{s+1}))+K_{s-1}\frac{1}{12}h^2(t_{s-1}+t_s)+K_s\left[\frac{1}{12}h^2(t_{s-1}+3t_s)+\frac{1}{12}h^2(t_{s+1}+3t_s)\right]$$
$$+K_{s+1}\frac{1}{12}h^2(t_{s+1}+t_s)=\frac{2t_s^3}{3}-\frac{t_s^2}{2}\left(t_{s-1}+t_{s+1}\right)+\frac{1}{6}\left(t_{s-1}^3+t_{s+1}^3\right)$$

$$(\frac{8}{3}-\frac{2h^3}{3}((s^3-(s-1)^3)k_{s-1}+h^3((s+1)^3-s^3)k_{s+1}))+K_{s-1}\qquad\frac{1}{12}h^3(2s-1)+K_s\left[\frac{2s}{3}h^3\right]$$
$$+K_{s+1}\frac{1}{12}h^3(2s+1)=\frac{2h^3s^3}{3}-h^3s^3+\frac{h^3}{6}\left((s-1)^3+(s+1)^3\right)=h^3s$$

Tenemos finalmente la siguiente relación de coeficientes:

$$K_{s-1}\frac{1}{12}h^3(2s-1-8(s^3-(s-1)^3))+K_s\left[h^3(\frac{2s}{3}+(s+1)^3-s^3))\right]+K_{s+1}\frac{1}{12}h^3(2s+1)=h^3s-(\frac{8}{3})$$

Para calcular el error

$$\|y-I_Ny\|_a^2=\int_0^1[(2t^2+2)(y'-I_Ny)^2+t(y-I_Ny)^2]dt\le\int_0^1[4(y'-I_Ny)^2+(y-I_Ny)^2]dt$$
$$\le\left(\frac{h}{\pi}\right)^2\left(4+\left(\frac{h}{\pi}\right)^2\right)\int_0^1\left(y''(\tau)\right)^2d\tau$$

Donde la ultima desigualdad esta dada por análisis de Fourier.
Por otra parte, de la ecuacion diferencia tenemos que

$$|y''|=\left|\frac{ty-4ty'-t}{2t^2+2}\right|=\left|\frac{y-4y'-1}{2t+\frac{2}{t}}\right|\le\left|\frac{y-4y'-1}{2}\right|$$

dado que $t\in[0,1]$ Así tenemos:

$$|y-I_Ny|\le\|y-I_Ny\|_a^2\le\left(\frac{h}{\pi}\right)^2\left(4+\left(\frac{h}{\pi}\right)^2\right)\int_0^1\left|\frac{y-4y'-1}{2}\right|^2d\tau$$

Vamos a acotar la derivada, para ello veamos primero la ecuación diferencial problema y sea $u(t)=y(t)-1$ tenemos entonces:

$$\begin{cases}-\left[\left(2t^2+2\right)u'\right]'+tu=0 \quad \text{in} \quad (0,1)\\ u(0)=-1=u(1)\end{cases}$$

$$u'' + \frac{4t}{2t^2+2}u' - \frac{t}{2t^2+2}u = 0 \tag{1}$$

Usando la acotada de la derivada con $P = 2$, $Q = 1/2$, $M = Max[u]$, $b - a = 1 - 0$ entonces:

$$\max u' \leq e^{2(1-0)}|u(1) - u(0)|/|1-0| + \frac{M}{4}(e^2 - 1)$$

Por otra parte,

$$y(t) = \int_0^t u' d\tau \leq \frac{M}{4}\int_0^t e^2 - 1 d\tau \implies \max u + 1 \leq \frac{M(e^2-2)}{4}$$

Esto es:

$$M \leq \frac{4}{e^2 - 5}$$

Así tenemos,

$$|y - 4y' - 1|^2 = |u - 4u'| \leq |M - 4\frac{M(e^2-1)}{4}|^2 = M^2(e^2-2)^2 \leq (\frac{4}{e^2-5})^2(e^2-2)^2$$

luego tenemos:

$$|y - I_N y|^2 \leq \|y - I_N y\|_a^2 \leq \left(\frac{h}{\pi}\right)^2\left(4 + \left(\frac{h}{\pi}\right)^2\right)\int_0^1 \left|\frac{y - 4y' - 1}{2}\right|^2 d\tau \leq \left(\frac{h}{\pi}\right)^2\left(4 + \left(\frac{h}{\pi}\right)^2\right)\int_0^1 (\frac{2}{e^2-5})^2(e^2-2)^2 d\tau$$

$$\leq \left(\frac{h}{\pi}\right)^2\left(4 + \left(\frac{h}{\pi}\right)^2\right)(\frac{2}{e^2-5})^2(e^2-2)^2$$

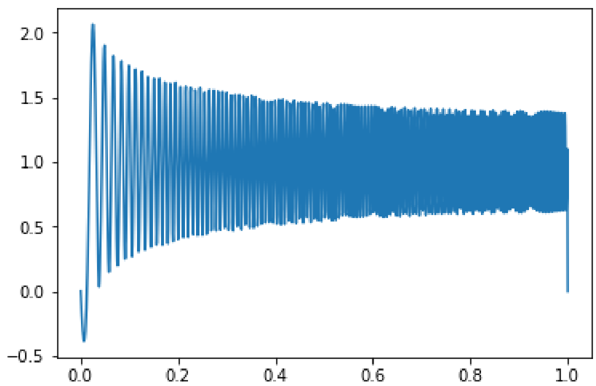
Si queremos un error de 10^{-4} se usa $h \leq 3.1142 \cdot 10^{-5}$, considernado $h = 1/N$, tenemos $N > 32111$ pasos.

c. Escriba un programa para resolver (1) numéricamente usando el método de elementos finitos con el N que encontró en el punto anterior. Grafique dicha solución aproximada

Se implemento un código en python de la siguiente manera:

```
import scipy as sp
import numpy as np
import matplotlib.pyplot as plt
from scipy import integrate

N=1000;
h=1./float(N)
t=np.linspace(0.,1.,N)

def gorrito(j):
    ceros=np.zeros(N)
    n=float(j)*h
    n\_atras=float(j-1)*h
    n\_adelante=float(j+1)*h
    if t[j-1]\<float(n) and t[j-1]\>=float(n\_atras) and j!=0:
        ceros[j-1]= float(N)*(t[j]-n\_atras)
    if t[j]\>=float(n) and t[j]\<float(n\_adelante)and j!=N:
         ceros[j]= float(N)*(n\_adelante-t[j])
    return ceros


def gorritoP(j):
    ceros=np.zeros(N)
    n=float(j)*h
    n\_atras=float(j-1)*h
    n\_adelante=float(j+1)*h
    if t[j-1]\<float(n) and t[j-1]\>=float(n\_atras)and j!=0:
```

```
            ceros[j-1]=1.0
        if t[j]\>=float(n) and t[j]\<float(n\_adelante) and j!=N:
            ceros[j]= -1.0
        return ceros


def soluciones(j):
    return np.trapz(gorrito(j)*t)


def coeficientes(i,j):
    fun=(-2*t*t-2)*gorritoP(i)*gorritoP(j)+t*gorrito(i)*gorrito(j)
    return np.trapz(fun)


A=np.zeros((N, N))
b=np.zeros(N)

for i in range(N):
    b[i]=soluciones(i)
    for j in range(N):
        A[i,j]=coeficientes(i,j)

A[0,0]=1.
A[N-1,N-1]=1.
c=np.linalg.solve(A,b)
sol=np.zeros(N)

for i in range(N):
    sol=c[i]*gorrito(i)+sol
```



Gráfica de la solución obtenida (N=1000) con el código presentado anteriormente

Ejemplo 2

Considere el sistema

$$\begin{cases} \Delta u = -2x(1-x) - 2y(1-y), & \text{en} \quad [0,1]^2 \\ u|_{\partial[0,1]^2} = 0 \end{cases}$$

La solución del problema es

$$u(x,y) = xy(1-x)(1-y)$$

Podemos considerar las carpitas

$$e_{i,j}(x,y) = e_i(x)e_j(y)$$

Queremos aproximar la solución como

$$u_{NM} = \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} K_{i,j} e_{i,j}$$

Para hallar estos coeficientes veamos las relaciones que deben satisfacer si consideramos la ecuación débil

$$\int_0^1 \int_0^1 \Delta u_{NM} e_{i,j} dxdy = -2 \int_0^1 \int_0^1 x(1-x) e_{i,j} dxdy - 2 \int_0^1 \int_0^1 y(1-y) e_{i,j} dxdy$$

Consideremos en partir simétricamente el área

$$\begin{aligned}
\int_0^1 \int_0^1 \sum_{l=1}^{N-1} \sum_{k=1}^{N-1} K_{l,k} e_{l,k} e_{i,j} dxdy &= -2 \int_0^1 e_j dy \int_0^1 x(1-x) e_i dx - 2 \int_0^1 e_i dx \int_0^1 y(1-y) e_j dy \\
&= -2h \left[\int_{t_{i-1}}^{t_i} t(1-t) N(t - t_{i-1}) dt + \int_{t_i}^{t_{i+1}} t(1-t) N(t_{i+1} - t) dt \right] \\
&- 2h \left[\int_{t_{j-1}}^{t_j} t(1-t) N(t - t_{j-1}) dt + \int_{t_j}^{t_{j+1}} t(1-t) N(t_{j+1} - t) dt \right] [] \\
&= -2 \left[-\frac{{t_{i-1}}^4}{12} + \frac{t_{i-1}^3}{6} + \frac{t_{i-1} t_i^3}{3} - \frac{t_{i-1} t_i^2}{2} - \frac{t_i^4}{2} + \frac{t_i^3 t_{i+1}}{3} + \frac{2t_i^3}{3} - \frac{t_i^2 t_{i+1}}{2} - \frac{t_{i+1}^4}{12} + \frac{t_{i+1}^3}{6} \right] \\
&- 2 \left[-\frac{{t_{j-1}}^4}{12} + \frac{t_{j-1}^3}{6} + \frac{t_{j-1} t_j^3}{3} - \frac{t_{j-1} t_j^2}{2} - \frac{t_j^4}{2} + \frac{t_j^3 t_{j+1}}{3} + \frac{2t_j^3}{3} - \frac{t_j^2 t_{j+1}}{2} - \frac{t_{j+1}^4}{12} + \frac{t_{j+1}^3}{6} \right] \\
&= \frac{5h^4}{3} + (i+j) h^3
\end{aligned}$$

Si tomamos $K_{l,k} = K_l K_k$

$$\int_0^1 \int_0^1 \sum_{l=1}^{N-1} \sum_{k=1}^{N-1} K_{l,k} e_{l,k} e_{i,j} dxdy = \int_0^1 (K_{i-1} e_{i-1} + K_i e_i + K_{i+1} e_{i+1}) e_i dx \int_0^1 (K_{j-1} e_{j-1} + K_j e_j + K_{j+1} e_{j+1}) e_j dx$$

Calculando

$$\int_{t_{s-1}}^{t_s} [(t_s - t)(t - t_{s-1})] dt = \frac{1}{6} h^3$$

$$\int_{t_{s-1}}^{t_s} [(t - t_{s-1})^2] dt = \frac{1}{3} h^3$$

$$\int_{t_s}^{t_{s+1}} [(t - t_s)(t_{s+1} - t)] dt = \frac{1}{6} h^3$$

$$\int_{t_s}^{t_{s+1}} [(t_{s+1} - t)^2] dt = \frac{1}{3} h^3$$

tenemos que

$$\begin{aligned}
\int_0^1 \int_0^1 \sum_{l=1}^{N-1} \sum_{k=1}^{N-1} K_{l,k} e_{l,k} e_{i,j} dxdy &= \left[K_{i-1} \frac{h}{6} + K_i \frac{2h}{3} + K_{i+1} \frac{h}{6} \right] \left[K_{j-1} \frac{h}{6} + K_j \frac{2h}{3} + K_{j+1} \frac{h}{6} \right] \\
&= \frac{5h^4}{3} + (i+j) h^3
\end{aligned}$$

El error va a aumentar al doble de lo que aumentaba si solo consideramos una solucion

$$\begin{aligned}
8 \Bigg(\bigg(& 1024\sqrt{3} z^5 - 1\,76 \cdot 3^{\frac{3}{2}} z^3 - 3^{\frac{7}{2}} z \ln \left(\left| 16 \cdot 3^{\frac{3}{2}} z^2 + 4 \cdot 3^{\frac{3}{2}} l^2 - 2 \cdot 3^{\frac{3}{2}} l + 3^{\frac{3}{2}} \right| \right) + \sqrt{192 z^2 + 9} \left(352 z^3 - 6z \right) \arctan \left(\frac{4\sqrt{3} l + \sqrt{3}}{\sqrt{192 z^2 + 9}} \right) \quad [\\
& + \sqrt{192 z^2 + 9} \left(352 z^3 - 6z \right) \arctan \left(\frac{4\sqrt{3} l - \sqrt{3}}{\sqrt{192 z^2 + 9}} \right) + \Big(-1024\sqrt{3} z^5 + 176 \cdot 3^{\frac{3}{2}} z^3 \\
& + 3^{\frac{7}{2}} z \ln \left(16 \cdot 3^{\frac{3}{2}} z^2 + 4 \cdot 3^{\frac{3}{2}} l^2 + 2 \cdot 3^{\frac{3}{2}} l + 3^{\frac{3}{2}} \right) \quad [\\
& - 9216 \arctan \left(\frac{\sqrt{3} l}{4z} \right) z^4 + 144 \arctan \left(\frac{\sqrt{3} l}{4z} \right) z^2 + 27 \arctan \left(\frac{\sqrt{3} l}{4z} \right))) \quad [\\
& / z \left(16384 z^6 + 6912 z^4 + 864 z^2 + 27 \right)
\end{aligned}$$