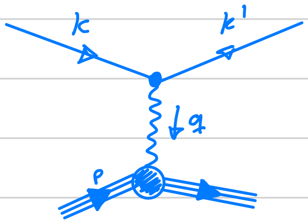
• ELEMENTS OF QCD AT COLLIDERS

We would like to describe how one can experimentally search for the Higgs boson and how it was actually discovered. Before that we need to understand how to describe collisions involving protons. Let's start reviewing one of the simplest experiments involving protons:

* Elastic e^-p^+ scattering:

Suppose the proton was elementary like the muon. Then we would expect e^-p scattering to look like $e^-\mu^+$ scattering. In fact, it does, at low energy. The leading Feynman diagram is given by the t-channel photon exchange diagram:



This is often called Coulomb scattering.

At low energy the proton looks elementary, and is indistinguishable from an elementary particle. It's structure is only revealed if probed with a highly energetic photon in the Coulomb scattering.

The x-section of two spin 1/2 particles is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha_e^2}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m_p^2}\sin^2\frac{\theta}{2}\right)$$

where E and E' are the electron's initial and final energies, $q^\mu = k^\mu - k'^\mu$ is the momentum transfer, θ is the angle between the outgoing and incoming electrons ($\theta = 0$ is forward). These quantities are related by

$$q^2 = -2k\cdot k' = -\left(4E'E\sin^2\frac{\theta}{2}\right)_{lab} \quad ,$$

where the formula is valid in the "lab" frame, where the proton is at rest.
and we have approximated $m_e = 0$.

That the proton is not elementary can be revealed by the following calculation and experiment. Instead of the QED vertex $ie\bar{u}\gamma^\mu u$, for the proton we shall use a form-factor

$$\Gamma^\mu(q) = F_1(q)\,\gamma^\mu + i\,\frac{\sigma^{\mu\nu}}{2m_p}\,q_\nu\,F_2(q^2)$$

and the vertex $ie\,\bar{u}\,\Gamma^\mu u$.

Then, repeating the tree-level coulomb scattering with this vertex, one finds:

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$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha_e^2}{4E^2\sin^4\frac{\theta}{2}}\,\frac{E'}{E}\left\{\left(F_1^2 - \frac{q^2}{4m_p^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m_p^2}\left(F_1+F_2\right)^2\sin^2\frac{\theta}{2}\right\}$$

which is known as the Rosenbluth formula.

If the proton only interacted through QED like $e^- Z^+$ scattering we could compute F_1 and F_2. For instance, up to one-loop level, one finds $F_2 \to 0$ for $|q| \gg m_Z$ and $F_1(q_1^2) - F_1(q_2^2) \approx -\frac{\alpha}{4\pi}\log\left(\frac{q_1^2}{q_2^2}\right)$, for $|q_1|, |q_2| \gg m_Z$. (and we may use on-shell renormalization $F_1(0) = 1$, i.e. $Q_Z = +1$).

The proton instead behaves very differently. It was found that a good fit is provided by

$$F_1(q^2) \sim \left(1 - \frac{q^2}{\mu}\right)^{-2},$$

where a new scale has emerged, $\mu \sim 0.7\,\text{GeV}$.

The form-factors is useful because they are the Fourier transform of scattering potentials in the Born approximation

$$F_1(q^2) = \int d^3x\; e^{i\vec{q}\cdot\vec{x}}\,V(x)$$

$$\left(\left(\frac{d\sigma}{d\Omega}\right)_{Born} = \frac{m_e^2}{4\pi^2}\left|\tilde{V}(\vec{k})\right|^2\right)$$

F.T. of $V(\vec{x})$

which implies

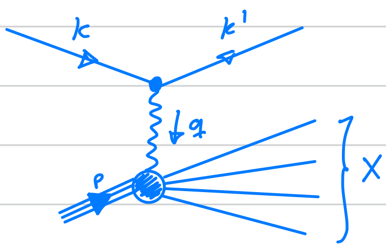
$$V(r) \sim \mu^3\,e^{-\mu r}.$$

Therefore the proton is characterised by an exponential shape of characteristic size $r_0 \sim \mu^{-1} \sim 1\,\text{fm}$, the size of the proton.

To learn more about the proton we need to go to much higher energies. One may expect to find a much more complicated fitting function. Instead what is find is an elementary scattering through the partons! That is, very high energy $e^- p^+$ scattering reveals point like constituents within the proton. We will explain next how this can be found out.

* Inelastic $e^- p^+$ scattering

So far we discussed elastic $e^- p^+$ scattering. When the center of mass (C.M.) energy is above the proton mass m_p, the proton can break apart.



Deep inelastic scattering (DIS)

Remarkably the physics simplifies in the deeply inelastic regime, and we will be able to make predictions.

In deriving the Rosenbluth formula for elastic scattering we reduced the photon-proton vertices to terms of the form $\bar{u}\gamma^\mu u$ and $\bar{u}\sigma^{\mu\nu}q_\nu u$ multiplied by F_1 and F_2 respectively. When the proton breaks apart, as in DIS, this parametrization is not valid. Instead, we need to parametrize photon-proton-X interactions, where "X" is anything the proton can break up into.

Therefore, it makes sense to parametrise the X-section (instead of the vertices) in terms of the momentum transfer q^μ and the proton momentum P.

In the lab frame, we define E and E' as the energies of the incoming and outgoing electron. We also define θ as the angle between $\vec{k}$ and $\vec{k}'$, so $\theta = 0$ is forward electron. The cross section can be written as

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha_e^2}{4\pi m_p q^4} \frac{E'}{E} L^{\mu\nu} W_{\mu\nu} ,$$

where $L_{\mu\nu}$ is the "leptonic tensor". For unpolarized scattering it is given by

$$L_{\mu\nu} = \frac{1}{2} Tr\left[\not{k}' \gamma^\mu \not{k} \gamma^\nu \right] = 2\left(k'^\mu k^\nu + k'^\nu k^\mu - k' \cdot k\, g^{\mu\nu} \right)$$

note that $L_{\mu\nu} = L_{\nu\mu}$. It comes from $L^{\mu\nu} = \frac{1}{2} \sum'_{spins} \bar{v}(k')\gamma^\mu u(k) \bar{u}(k) \gamma^\nu v(k')$, and the factor of "1/2" arises from taking the average over the initial electron's spin.

The "hadronic tensor" $W^{\mu\nu}$ includes an integral over phase space for all final state particles. It gives the rate $\gamma^* p^+ \to$ anything:

$$e^2 \epsilon_\mu \epsilon_\nu^* W^{\mu\nu} \equiv \frac{1}{2} \sum'_{X, spins} d\Pi_X (2\pi)^4 \delta^{(4)}(q + P - P_X) \left| \mathcal{M}(\gamma^* p^+ \to X) \right|^2$$

where ϵ_μ is the polarization of the off-shell photon. Since we are integrating over the final states, $W^{\mu\nu}$ can only depend on P^μ and q^μ. In unpolarized scattering, it is symmetric $W^{\mu\nu} = W^{\nu\mu}$. It should also satisfy the Ward identity $q_\mu W^{\mu\nu} = 0$, since the interaction is through a photon. Therefore, the most general tensor is:

$$W^{\mu\nu} = W_1 \cdot \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + W_2 \cdot \left(P^\mu - \frac{P \cdot q}{q^2} q^\mu \right)\left(P^\nu - \frac{P \cdot q}{q^2} q^\nu \right)$$

The Lorentz scalars W_1 and W_2 can depend on $P^2 = m_p^2$, q^2 and $P \cdot q$. The natural variables to use are

$$Q \equiv \sqrt{-q^2} > 0 \quad \text{and} \quad \nu \equiv \frac{P \cdot q}{m_p} = (E - E')_{lab}$$

Energy scale of the collision.

Energy lost by electron in the proton rest frame (lab)

An alternative to ν, is to use the dimensionless variable

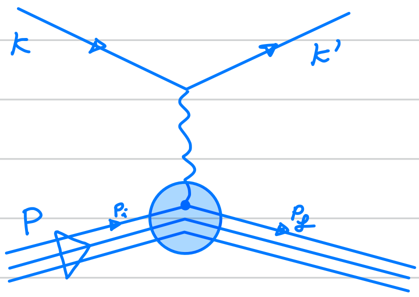
$$x \equiv \frac{Q^2}{2\,P\cdot q}$$

which is known as Bjorken x variable.
Using these definitions, we have

$$\left(\frac{d\sigma}{d\Omega\, dE'}\right)_{lab} = \frac{\alpha_e^2}{8\pi E^2 \sin^4\frac{\theta}{2}}\left(\frac{m_p}{2} W_2(x,Q)\cos^2\frac{\theta}{2} + \frac{1}{m_p} W_1(x,Q)\sin^2\frac{\theta}{2}\right)$$

As in the elastic case, everything is set such that we only need to know about the incoming and outgoing electron's momenta, and nothing about X. That is W_1 and W_2 are determined by measuring the energy and angular dependence of the outgoing electron.

<u>Parton model</u>: the defining assumption is that particles inside the proton, called "partons", are essentially free. Model originally by Feynman. Now we know that the partons are the valence quarks (uud), but also the gluons, and in principle any other SM particle.
To test the parton model we need to determine W_1 and W_2 when the electron scatters elastically off partons of mass m_q inside the proton. Parton scattering:



The circle represents the proton, and the lines inside the "partons" inside the proton.

The electron-parton scattering looks like $e^-\mu^+$ scattering. To evaluate it, we call p_i / p_f the parton initial/final momentum, so that

$p_i^\mu + q^\mu = p_f^\mu$ by momentum conservation. Squaring this equation we have:

$$m_q^2 + 2p_i\cdot q + q^2 = m_q^2 \implies 1 = \frac{Q^2}{2p_i\cdot q}.$$

The momentum of the parton is not directly measurable. But let's assume it carries a fraction ξ of the proton's momenta P, $p_i^\mu = \xi P^\mu$. Then

$$x = \frac{\xi Q^2}{2p_i\cdot q} = \xi .$$

In particular, if the parton model is correct, by measuring "x" we are measuring the parton's momentum fraction.

Next we compute $e^- q \to e^- q$ in perturbation theory. In particular we expect the form factors F_1, F_2, to have only a weak, logarithmic dependence on Q^2, when the initial momentum is fixed (i.e. at fixed "x"). The x-section (approximate) independence of Q^2 at fixed "x" is known as Bjorken scaling.

Another important ingredient of the parton model is the classical probabilities

$$f_i(\xi)\, d\xi$$

of the photon scattering into the parton "i" with momentum fraction ξ. These $f_i(\xi)$ are known as Parton Distribution Functions (PDFs).

The physical justification for the PDFs:
The time scales for interactions among proton constituents $\sim \Lambda_{QCD}^{-1} \sim m_p^{-1}$ is much slower than the time scale that the photon probes $\sim Q^{-1} \ll m_p^{-1}$. This separation of scales allows us to treat the wavefunctions within the proton as being decoherent, and hence the probabilistic interpretation. A proof of this result is beyond the scope of this course, and is the topic of the "factorization theorems".

After having introduced the PDF's, we can be much more precise about the predictions of the weakly interacting partons.

$$\sigma(e^- P^+ \to e^- X) = \sum_i \int_0^1 d\xi \, f_i(\xi) \, \hat{\sigma}(e^- p_i \to e^- X)$$

where partonic quantities are indicated with a hat $\hat{\sigma}$.

Next assuming partons weakly interact with the photon, we get the Rosenbluth formula with $F_1 = 1$, $F_2 = 0$. Before integrating over E', we have:

$$\left(\frac{d\hat{\sigma}(e^- q \to e^- q)}{d\Omega \, dE'}\right)_{lab} = \frac{\alpha_e^2 Q_i^2}{4E^2 \sin^4\frac{\theta}{2}} \left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_q^2} \sin^2\frac{\theta}{2}\right) \delta\left(E - E' - \frac{Q^2}{2m_q}\right)$$

where Q_i is the charge of the parton (i.e. quark). Our previous formula with $F_1 = 1$, $F_2 = 0$ is reproduced after integrating over dE'!

If we were not assuming weakly interacting quarks we would get generic Form-Factors, violating Bjorken scaling.

In order to get the DIS x-section we need to integrate over the incoming quark momentum. Since $p_i^\mu = \xi P^\mu$ and in the lab frame the proton is at rest, we have $m_q = \xi m_p$. Using $E - E' = \nu = Q^2 (2 m_p x)^{-1}$ it follows

$$\delta\left(E - E' - \frac{Q^2}{2m_q}\right) = \delta\left(\frac{Q^2}{2m_p x} - \frac{Q^2}{2m_p \xi}\right) = \frac{2m_p x^2}{Q^2} \delta(\xi - x).$$

Therefore the total x-section is given by

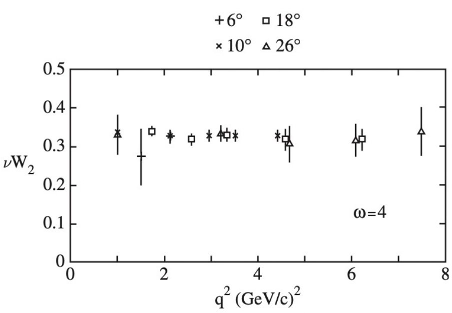
$$\left(\frac{d\sigma(e^- P \to e^- X)}{d\Omega \, dE'}\right)_{lab} = \sum_i f_i(x) \frac{\alpha_e^2 Q_i^2}{4E^2 \sin^4\frac{\theta}{2}} \left(\frac{2m_p}{Q^2} x^2 \cos^2\frac{\theta}{2} + \frac{1}{m_p} \sin^2\frac{\theta}{2}\right)$$

Comparing with our original parametrization in terms of the hadronic tensor, we have

$$W_1(x,Q) = 2\pi \sum_i' Q_i^2 f_i(x)$$

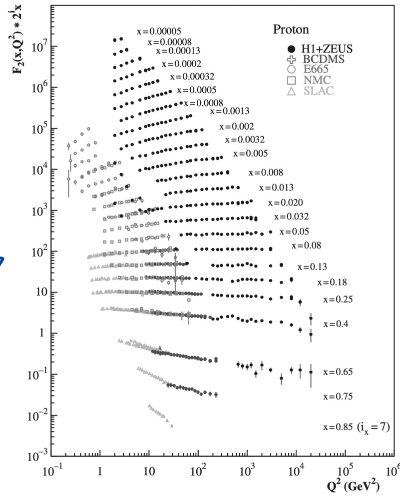
$$W_2(x,Q) = 8\pi \frac{x^2}{Q^2} \sum_i' Q_i^2 f_i(x).$$

This is a concrete prediction for Bjorken scaling! The quantities $W_1(x,Q)$ and $Q^2 W_2(x,Q)$ should be (approximately) independent of Q at fixed x. Recall, although quarks are not directly observable, the quantity $x = \frac{Q^2}{2m_p}(E-E')'$ can be measured. Below early and recent measurements of Bjorken scaling.



PDG, 2012

Friedman & Kendall (1972)



Another prediction of the parton model is

$$W_1(x,Q) = \frac{Q^2}{4x^2} W_2(x,Q) \quad \text{for } Q \gg m_p$$

which is known as the Callan-Gross relation. The proportionality factor can be traced back to the $Q^2/2m_q^2 = Q^2/2x^2m_p^2$ factor in the $e^-q \to e^-q$ parton amplitude, which is due to the quarks being free Dirac fermions. Thus the Callan-Gross relation tests that quarks have spin-$\frac{1}{2}$.

This relation is often presented in a different form. Using $y \equiv \frac{P\cdot q}{P\cdot k} = \frac{\nu}{E}$ (dimensionless), so that $dE'd\Omega = \frac{2m_p}{E'}\pi y \, dx\, dy$, we have

$$\frac{d\sigma(e^-P\to e^-X)}{dx\,dy} = \frac{2\pi\alpha^2}{Q^4} s \underbrace{\left(1+(1-y)^2\right)}_{\text{Callan-Gross relation.}} \sum_i' Q_i^2 \, x\, f_i(x)$$

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* PDFs Sum rules.

For the PDFs to admit a probability interpretation, they must satisfy various constraints. For example ~~if~~ the proton had exactly one down quark $\int_0^1 d\xi\, f_d(\xi) = 1$. In reality one can have virtual down-antidown quark pairs within the proton. However the down-quark number is conserved in QCD and QED. Therefore

$$\int_0^1 d\xi \left[f_d(\xi) - f_{\bar{d}}(\xi) \right] = 1$$

PDF for anti-down quark.

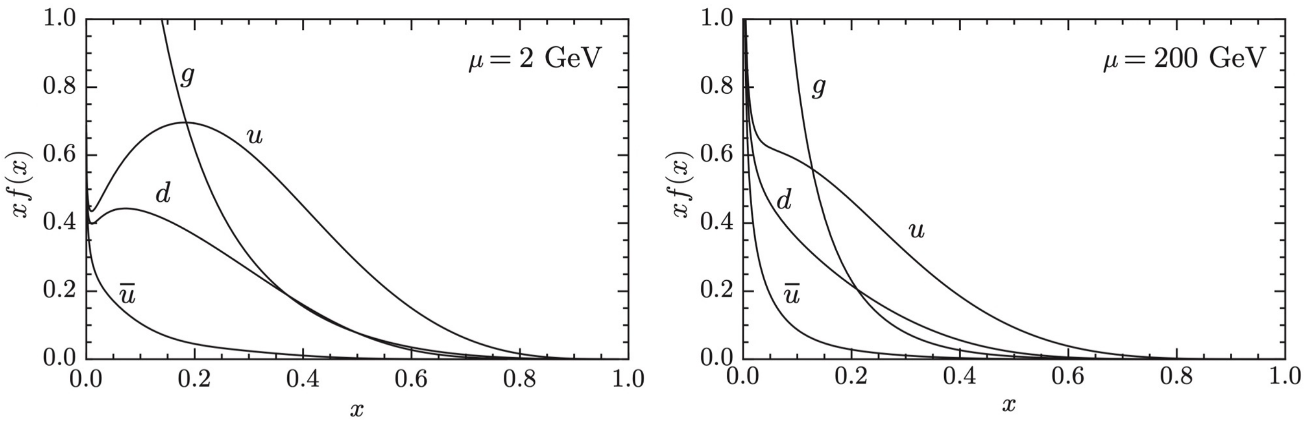
Similarly: $\int_0^1 d\xi \left[f_u - f_{\bar{u}}(\xi) \right] = 2$ because the total up-quark number is 2, and $\int_0^1 d\xi \left[f_s(\xi) - f_{\bar{s}}(\xi) \right] = 0$ because the strange-quark number is 0, and analogously for bottom and charm. There is no conserved quantum number for gluon so there is no sum rule for gluon. Finally, momentum conservation implies

$$\sum_j \int_0^1 d\xi\, \xi\, f_j(\xi) = 1.$$

Each of this sum rules correspond to a classically conserved current (up, down, strange number or momentum conservation).

It turns out that numerically $\int d\xi\, \xi \left(f_u(\xi) + f_d(\xi) \right) \simeq 0.38$. Therefore about 38% of the proton total momenta is in the valence quarks (u and d). The gluon takes 35% – 50% of the momentum, and the rest is on sea-quarks (i.e. s, c or b quarks and $\bar{d}, \bar{u}, \bar{c}, \bar{s}, \bar{b}$ anti-quarks).

PDF measurements from the MSTW group.



In practice PDF's are not only determined from DIS but also from many other high energy processes such as $p\bar{p}$ and pp collisions. A number of international research groups are dedicated to fitting PDFs.

* Factorization and the parton model from QCD.

For practical purposes the parton model is all what we need to perform perturbative QCD calculations at high energy colliders. The model build upon the concept of "factorization": PDFs are universal objects, and any scattering process involving protons can be computed with the same PDFs + an appropriate perturbative partonic crossection.

In the following lecture we would like to discuss the "proof" of factorization in the context of DIS using the Operator Product Expansion (OPE).

↳ this will lead to the identification of the moments of the PDFs with certain matrix elements of composite operators.

Intuitively by Factorization we mean $\sigma = \sigma_h \otimes \text{PDFs} + \mathcal{O}\left(\frac{\Lambda_{QCD}}{Q}\right)$,

hard scattering — convolution

where Q is the characteristic energy scale of the process. In the case of inclusive DIS, $Q = \sqrt{-(k-k')^2}$, the energy transfer, which we take large while keeping $x = Q^2/2p.q$ fixed.

↳ The strategy that we will follow will consist in relating the DIS crossection to a product of currents $J^\mu(x) J^\nu(y)$. We then rewrite this product in terms of local ops. $J^\mu(x) J^\nu(y) \sim \sum_n C_n(x-y)\, \mathcal{O}_n(x)$. In the DIS limit $Q^2 \to \infty$ at fixed Bjorken x will correspond to $x-y \to 0$ so that we we'll be able to keep the first terms of the OPE. It will turn out that $\text{PDFs} \sim \langle P | \mathcal{O} | P \rangle$!

* The operator product expansion (OPE):

The OPE is the position space version of the low momentum expansion that we do when deriving Effective Lagrangians.

For example, if we integrate the W boson at tree level

$$\mathcal{L}_W \sim g^2 \int d^4x\, d^4y\, \bar{\Psi}(x) \gamma^\mu \Psi(x)\, D^{\mu\nu}(x,y)\, \bar{\Psi}(y) \gamma^\nu \Psi(y)$$

where
$$D^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \frac{-g^{\mu\nu}}{p^2 - m_W^2} e^{ip(x-y)} = \frac{g^{\mu\nu}}{\Box_x + m_W^2} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}$$

is the W-boson propagator. For $\Box \sim p^2 << m_W^2$ we expand

$$\frac{g^2}{\Box + m_W^2} = G_F \left(1 - \frac{\Box}{m_W^2} + \left(\frac{\Box}{m_W^2}\right)^2 + \dots \right) \quad \text{with} \quad G_F \sim \frac{g^2}{m_W^2}$$

so that

$$\mathcal{L}_W \sim G_F \int d^4x \left[(\bar{\Psi}\gamma^\mu\Psi)_x (\bar{\Psi}\gamma^\mu\Psi)_x - (\bar{\Psi}\gamma^\mu\Psi)_x \frac{\Box}{m_W^2} (\bar{\Psi}\gamma^\mu\Psi)_x + \dots \right]$$

with all fields located at the same position. The effective Lagrangian it is therefore local.

The OPE states that as two operators get close, it's matrix element on any state can be reproduced by local operators:

$$\lim_{x \to y} O(x)\, O(y) = \sum_n C_n(x-y)\, O_n(x)$$

Wilson coefficients.

The equality holds at the level of operators and the Wilson coefficients are independent of the state. Thus one can determine the C_n's for a particular process and then use them at another process.

In the case of the 4-Fermi theory we can truncate the OPE because only a finite number of ops. are necessary at a given precision. In the case of DIS we will see that we need to keep an infinite power of ops. but the OPE will still be useful.

Often we will write the OPE as

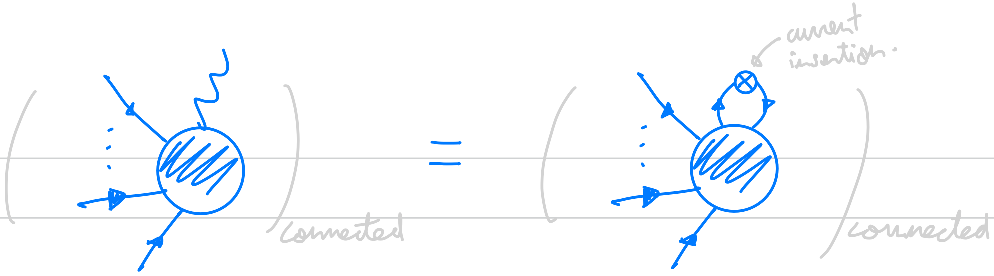
$$\int d^4x\, e^{iqx}\, O(x)\, O(0) = \sum_n C_n(q)\, O_n(0)$$

where the RHS operator is in position space but the Wilson coefficients are written in momentum space.

* DIS as a time ordered product of currents:

In order to apply the OPE to DIS we first need to express the hadronic tensor $W^{\mu\nu}$ in terms of matrix elements involving the electromagnetic current constructed out of quarks, i.e. $J^\mu(x) = \bar{\psi}(x)\gamma^\mu \psi(x)$, with ψ the Dirac field for quarks.

Recall that: S-matrix elements involving photons, in which polarizations of external photons are removed, are equal to time ordered products involving currents. In pictures:



In short, this equation follows because $\langle p'|J^\mu(x)|p\rangle\big|_{x=0} = \bar{u}_2(p')\gamma^\mu u_1(p)e^{i(p'-p)x}\big|_{x=0}$ $= \bar{u}(p')\gamma^\mu u(p)$.

Allright, for DIS we need

$$\mathcal{M}(\gamma^* p^+ \to X) = e\,\epsilon^\mu \langle X|J_\mu(0)|P\rangle$$

Comparing this equation with the definition of the hadronic tensor:

$$e^2 \epsilon_\mu \epsilon_\nu^* W^{\mu\nu} \equiv \frac{1}{2}\sum_{X,\,spins}' d\Pi_X (2\pi)^4 \delta^{(4)}(q+P-P_X)|\mathcal{M}(\gamma^* p^+ \to X)|^2$$

we deduce that

$$W_{\mu\nu}(\omega, Q) = \sum_X' d\Pi_X \langle P|J_\mu(0)|X\rangle\langle X|J_\nu(0)|P\rangle (2\pi)^4\delta^{(4)}(q+P-P_X)$$

$$= \sum_X' d\Pi_X \int d^4x\, e^{i(q+P-P_X)x}\langle P|J_\mu(0)|X\rangle\langle X|J_\nu(0)|P\rangle$$

where we defined $\omega \equiv 1/x = 2P\cdot q/Q^2$, the inverse of the Bjorken variable "x", in order to avoid confusion with spacetime variable "x^μ".

Next we use the translation generator to simplify:

$$\langle P|J_\mu(0)|X\rangle = \langle P|e^{-i\hat{P}\cdot x} J_\mu(x)\, e^{i\hat{P}\cdot x}|X\rangle = e^{-i(P-P_X)x}\langle P|J_\mu(x)|X\rangle$$

and arrive to:

$$W_{\mu\nu}(\omega, Q) = \sum_X' \int d\Pi_X \int d^4x\, e^{iq\cdot x}\langle P|J_\mu(x)|X\rangle\langle X|J_\nu(0)|P\rangle$$

$$= \int d^4x\, e^{iq\cdot x}\langle P|J_\mu(x)J_\nu(0)|P\rangle$$

where we used $\mathbb{1} = \sum_X' |X\rangle\langle X|$ to get an expression for the hadronic tensor that is independent of $|X\rangle$!

This expression is not yet useful because we'd like to apply OPE on time ordered products. To amend this issue we'll use optical theorem:

$$W_{\mu\nu} \sim |M(\gamma^* p \to X)|^2 \sim |\langle T\{J\}\rangle|^2$$

$$\sim \operatorname{Im} \mathcal{M}(\gamma^* p \to \gamma^* p) \sim \operatorname{Im}\langle T\{JJ\}\rangle$$

In more detail: $W_{\mu\nu} = 2\operatorname{Im} T^{\mu\nu}$ where $e^2 \epsilon_\mu \epsilon_\nu^* T^{\mu\nu}(\omega, Q) = \mathcal{M}(\gamma^* p \to \gamma^* p)$
Therefore

$$T_{\mu\nu}(\omega, Q) = i\int d^4x\, e^{iq\cdot x}\langle P|\, T\{J_\mu(x)\, J_\nu(0)\}\,|P\rangle$$

Thus instead of $W_{\mu\nu}$ we'll compute $2\operatorname{Im} T_{\mu\nu}$ through the OPE of $T\{J_\mu(x)\, J_\nu(0)\}$. It is customary to define

$$T_{\mu\nu}(\omega, Q) = T_1(\omega, Q)\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{T_2(\omega, Q)}{P\cdot q}\left(P^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right)$$

as we did before for the $W^{\mu\nu}$ tensor. Thus $W_1 = 2\operatorname{Im} T_1$ and $W_2 = 4\operatorname{Im} T_2/\omega Q^2$.

* OPE for DIS:

We want to find the r.h.s. of

$$T\{J^\mu(x)\, J^\nu(0)\} = \sum_n{}' C_n(x)\, \mathcal{O}_n^{\mu\nu}(0)$$

and evaluate it on external proton states $|P\rangle$. It turns out that

$$i\int d^4x\, e^{iqx}\, T\{J^\mu(x)\, J^\nu(0)\} = \sum_q{}' Q_q^2\, t^{\mu_1\cdots\mu_n}(q)\, \mathcal{O}_q^{\mu_1\cdots\mu_n}(0) + O\left(\alpha_s, \frac{\Lambda_{QCD}}{Q}\right)$$

where the "Wilson coefficient" is

$$t^{\mu_1\cdots\mu_n}(q) = \sum_{n=2,4,\dots}^{\infty} \frac{(2q^{\mu_1})\cdots(2q^{\mu_n})}{Q^{2n}}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{2q^{\mu_3}\cdots 2q^{\mu_n}}{Q^{2n-2}} \times$$

$$\times\left(g^{\mu\mu_1} - \frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2} - \frac{q^\nu q^{\mu_2}}{q^2}\right).$$

and the local operators are given by

$$O_q^{\mu_1\cdots\mu_n} = \bar{\psi}_q(x)\,\gamma^{\mu_1}\, iD^{\mu_2}\cdots iD^{\mu_n}\,\psi_q(x) + \text{symmetrizations of } \mu_i - \text{traces.}$$

For instance:

$$O_q^{\mu_1\mu_2} = \bar{\psi}_q\left(i\gamma^{\mu}D^{\nu} + i\gamma^{\nu}D^{\mu} - \frac{i}{2}\, i g^{\mu\nu}\not{D}\right)\psi_q$$

We will not prove this OPE. Note that $O^{\mu_1\cdots\mu_s}$ has spin s and dimension $d = s+2$. What is common of the sum above among the operators is the "twist" $t = d - s = 2$. [You can read more about the derivation in e.g. Peskin & Schroeder]

END OF L14

* PDFs and Callan-Gross relation from DIS OPE:

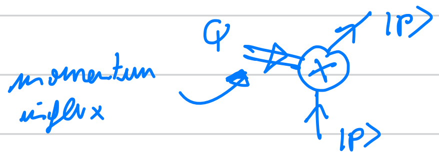
To use OPE in DIS, we need to evaluate the OPE in a proton state. By Lorentz invariance:

$$\sum_{\text{spins}} \langle p|\, O_q^{\mu_1\cdots\mu_n}\,|P\rangle = A_n \cdot 2P^{\mu_1}\cdots P^{\mu_n} - \text{traces.}$$

RENORM SCALE.

with A_n functions of μ. The traces give factors of $P^2 = m_p^2 \ll Q$ which are subleading w.r.t. contractions of P^{μ_i} with the Wilson coefficient, which gives factors of $q\cdot P = \frac{1}{2}\omega Q^2$. Therefore we drop the traces.

One can think of the last equation as form factor of the type



All in all, we get

$$T^{\mu\nu} = \sum_q{}' Q_q \left\{ \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{4}{Q^2\omega^2}\left(P^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right)\right\} \times \sum_{n=2,4,\dots}^{\infty}{}' \omega^n A_{q,n}$$

Therefore, comparing with

equation from above.

$$T_{\mu\nu}(\omega, Q) = T_1(\omega, Q)\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{T_2(\omega, Q)}{P\cdot q}\left(P^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right)$$

as we did before for the $W^{\mu\nu}$ tensor. Thus $W_1 = 2\,\mathrm{Im}\,T_1$ and $W_2 = 4\,\mathrm{Im}\, T_2/\omega Q^2$.

we obtain

$$T_1 = \frac{\omega}{2} T_2 = \sum_q{}' Q_q^2 \left(\sum_{n=2,4,\dots}{}' \omega^n A_{q,n}\right)$$

therefore, as promised we find the Callan-Gross relation $W_1 = \omega\, \mathrm{Im}\, T_2 = \frac{\omega^2 Q^2}{4} W_2$.

And because

$$W_1(x, Q) = 2\pi \sum_i{}' Q_i^2 f_i(x)$$

Eq. from above.

we find

$$f_q(x) = \frac{1}{\pi} \sum_{n=2,4,\dots}{}' x^{-n}\, \mathrm{Im}\, A_{q,n}$$

which gives a definition of the PDFs in QCD!

* Moments of the PDFs

A moment of order "n" of a distribution $f_i(x)$ is defined by:

$$C_{i,n} = \int_0^1 dx\, x^{n-1} f_i(x).$$

In terms of the PDF definition above, we have

$$C^{q}_{i,m} = \mathrm{Im}\, \frac{1}{\pi}\int_1^\infty dw \sum_n{}' w^{n-m-1} A_{q,n}.$$

2 Im = Disc.

$$= \sum_n{}' \frac{1}{2\pi i} \oint dw \; w^{n-m-1} A_{q,n} = \sum_n{}' \delta_{n,m} A_{q,n} = A_{q,m}$$

Therefore the A_n's are precisely the moments of the PDFs.

For example, for $m=2$

$$\mathcal{O}_q^{\mu_1\mu_2} = \bar{\psi}_q \left(i\gamma^\mu D^\nu + i\gamma^\nu D^\mu \right) \psi_q \; - \text{traces}.$$

Symmetrized energy momentum tensor.

The matrix element on a proton state gives

$$\sum_i{}' \langle P | \mathcal{O}_i^{\mu\nu} | P \rangle = \langle P | T^{\mu\nu}_{QCD} | P \rangle = 2 P^\mu P^\nu$$

here we sum over all partons, not only quarks.

Neglected traces and thus mp.

and therefore $\sum_i{}' A_{i,2} = 1.$

$$\sum_i{}' \int_0^1 dx \; x \; f_i(x) = 1.$$

$$\sum_i{}' C_{i,2} = \sum_i{}' \int dx \; x f_i(x)$$

Other sum rules can be derived evaluating other moments.

For $m=1$ we get

# of quarks q − # of antiquarks $\bar{q}$ in the proton.

$$\langle P, s | \mathcal{O}_q^\mu | P, s \rangle = \bar{u}_s(P) \gamma^\mu u_s(P) \cdot A_{q,1}$$

(Like the number operator).

summing over proton spin s, we get

$$\langle P | \mathcal{O}_q^\mu | P \rangle = 2 P^\mu A_{q,1}$$

$$= \begin{array}{ll} 2 & q = u \\ 1 & q = d. \end{array}$$

* Hard scattering processes in Hadron Colliders:

↳ High energy hadron scattering dominated by low momentum transfer cannot be treated using perturbative QCD.

↳ However, in some collisions two quarks or gluons exchange a large momentum transfer.

↳ Then, as in deep inelastic scattering, the parton scattering takes place rapidly, much faster than time scales associated to the wave functions of the partons inside the proton. Therefore low order QCD predictions are OK and a factorization formula (similar to DIS) applies.

↳ e.g. for $q + \bar{q}$ we'll get

any hadron state.

$$\sigma(p(P_1) + p(P_2) \mapsto X + Y) =$$

$$= \int_0^1 dx_1 \int_0^1 dx_2 \sum_i f_i(x_1) f_i(x_2) \cdot \sigma(q_i(x_1 P_1) + \bar{q}_i(x_2 P_2) \to Y)$$

↳ In the next section we will be interested in $\sigma(p + p \to h + X)$

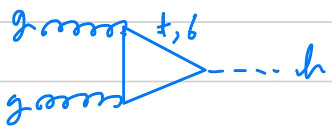
$\sqrt{s}$ (TeV)	Production cross section (in pb) for $m_H = 125$ GeV					
	ggF	VBF	WH	ZH	$t\bar{t}H$	total
7	$15.3^{+10\%}_{-10\%}$	$1.24^{+2\%}_{-2\%}$	$0.58^{+3\%}_{-3\%}$	$0.34^{+4\%}_{-4\%}$	$0.09^{+8\%}_{-14\%}$	17.5
8	$19.5^{+10\%}_{-11\%}$	$1.60^{+2\%}_{-2\%}$	$0.70^{+3\%}_{-3\%}$	$0.42^{+5\%}_{-5\%}$	$0.13^{+8\%}_{-13\%}$	22.3
13	$44.1^{+11\%}_{-11\%}$	$3.78^{+2\%}_{-2\%}$	$1.37^{+2\%}_{-2\%}$	$0.88^{+5\%}_{-5\%}$	$0.51^{+9\%}_{-13\%}$	50.6
14	$49.7^{+11\%}_{-11\%}$	$4.28^{+2\%}_{-2\%}$	$1.51^{+2\%}_{-2\%}$	$0.99^{+5\%}_{-5\%}$	$0.61^{+9\%}_{-13\%}$	57.1

The SM Higgs boson production x-section in pp collisions as a function of C.o.M. energy $\sqrt{s}$. See also the plot below.

Higgs discovery & it's current status

How do we search for the Higgs? What are the main decay of the Higgs?
Recall that $m_h < 2m_{W/Z} < 2m_t$. The Higgs main production modes from $\sigma(p+p \to h + X)$ are

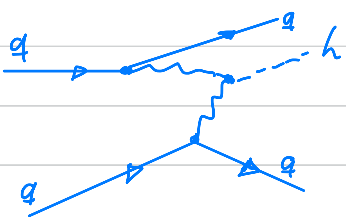
* Gluon fusion



$\sim 20\,pb = 20\cdot 10^{-12}\,b$ $(= 20\cdot 10^{-40}\,m^2)$ @ $\sqrt{s} \sim 8\,TeV$

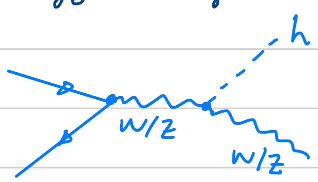
For comparison note that $\sigma(pp \to Z+X) \sim 40\cdot 10^{4}\,pb$
$\sigma(pp \to W+X) \sim 10^{5}\,pb$

* Vector Boson Fusion (VBF)



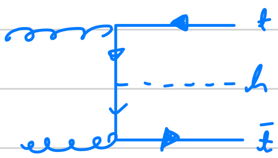
$\sim 2\,pb$ @ $\sqrt{s} \sim 8\,TeV$

* Higgsstrahlung or VH



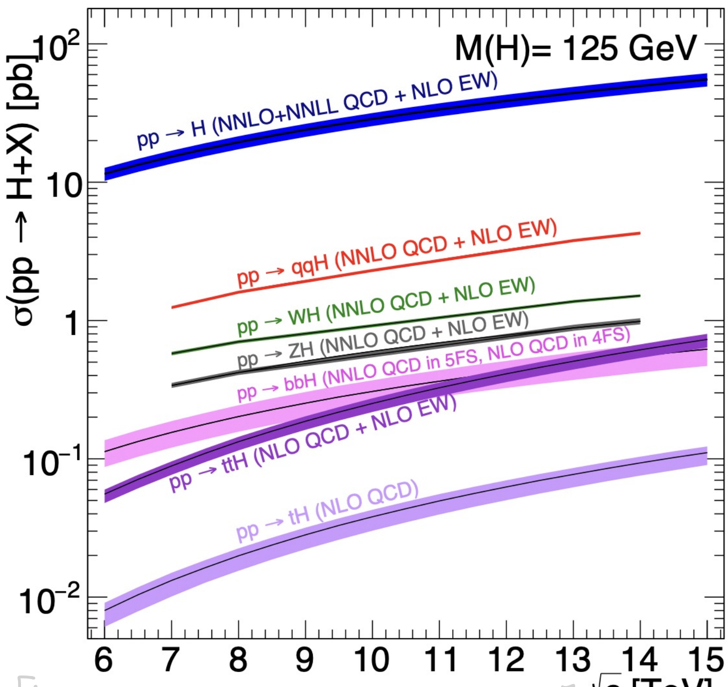
$\sim \sigma \to W/Z + H$
$\sim 0.8/0.3\,pb$ @ $\sqrt{s} \sim 8\,TeV$

* $t\bar{t}H$ channel



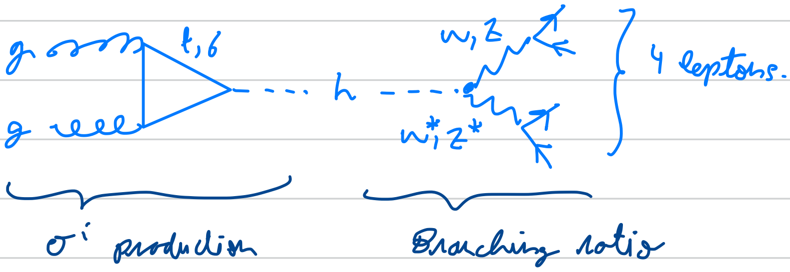
$\sim 0.05\,pb$ @ $\sqrt{s} \sim 8\,TeV$

SM production cross section as a function of the center of mass energy, $\sqrt{s}$, for proton + proton collisions.

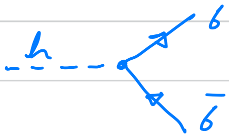
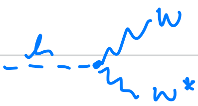
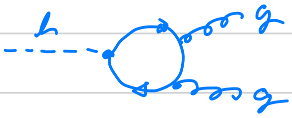
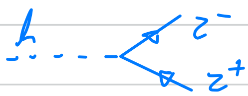
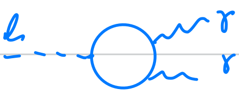


[LHC HIGGS XS WORKING GROUP 2016]

The Higgs boson has a very small width $\Gamma = 4\,MeV$ compared to it's mass. Therefore, we can factorise production and decay using the narrow width approximation. E.g.

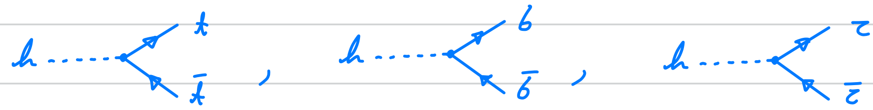


What are the Higgs main decay channels?

Decay channel	Diagram examples.	Branching ratio
$h \to b\bar{b}$		$\sim 57.5\%$
$h \to WW^*$		$\sim 21.5\%$
$h \to gg$		$\sim 8.5\%$
$h \to \tau^+\tau^-$		$\sim 6.3\%$
$h \to c\bar{c}$		$\sim 2.9\%$
$h \to ZZ$		$\sim 2.7\%$
$h \to \gamma\gamma$		$\sim 0.23\%$ ← discovery channel.
$h \to Z\gamma$		

END OF L15

All 3rd family higgs couplings have been measured.



Experimental results are often presented in terms of "signal strength":

observed

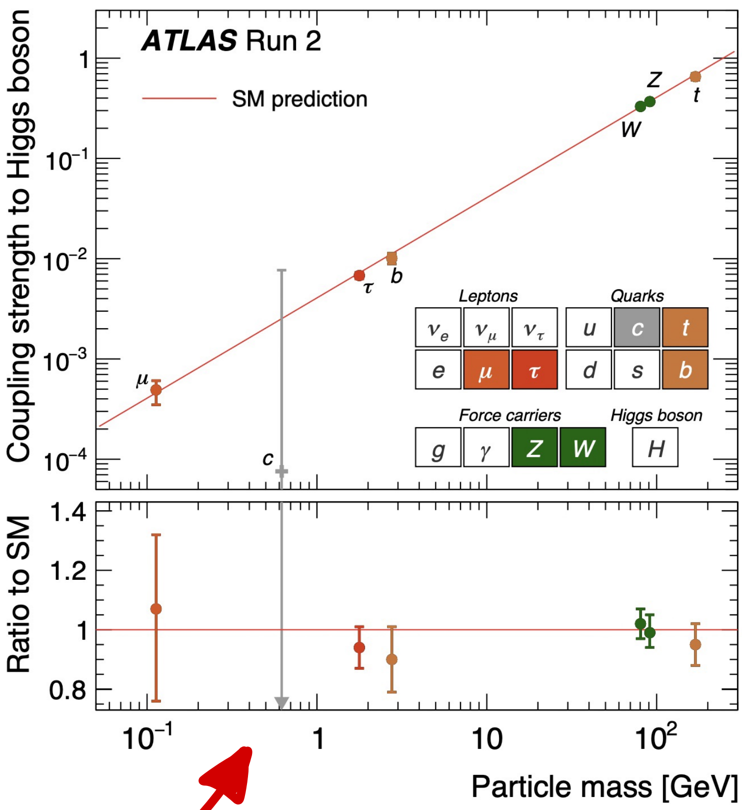
$$\mu_i^f = \frac{\sigma_i \cdot Br^f}{(\sigma_i \cdot Br^f)_{SM}} = \mu_i \times \mu_f \quad w/ \ \mu_i = \frac{\sigma_i}{\sigma_{SM}} \ \& \ \mu^f = \frac{Br^f}{(Br^f)_{SM}}$$

predicted

So far signal strengths of WW^*, ZZ^*, $\gamma\gamma$, $b\bar{b}$, $t\bar{t}$ and $\tau\bar{\tau}$ measured and agree with SM at $\lesssim 10\%$.

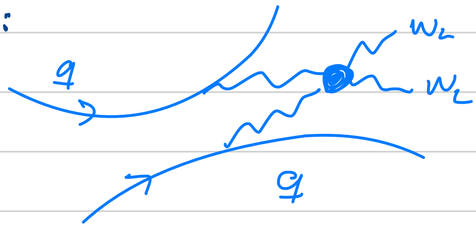
Couplings to u, d, s and e is not obvious we'll ever manage to measure in any foreseeable future. Couplings to $\mu\bar{\mu}$ will be will be conclusively stablished in the next 5-10 years at the LHC. The couplings $c\bar{c}$ is feasable and is the next important channel

The straight line shows the expected SM behaviour, where the interaction strength is proportional to the mass of the fermion, and to the squared mass for W & Z.



Improving precission of higgs couplings is an important experimental direction. Precission is not allways important: we do not care much of measuring much better the proton mass (why?). But measuring much better the Higgs

couplings is a direct test of the EWSB mechanism. This is the sector of the SM that is less poorly understood, experimentally.
Currently there are various important structural parts of the EWSB mechanism that have not been tested. For instance we do not know much about the Higgs potential beyond the quadratic piece responsible for the Higgs mass. Also the role of the higgs has not been tested. What is his role in life? Unitarize amplitudes. We'd like to test a process like:



Perhaps the LHC will manage to probe it in the next decade.

Neutrinos

It turns out that neutrinos are very light but not exactly massless. They do have a non-zero mass.
In the SM EFT we have a good explanation to why the neutrinos have a small non-vanishing mass. The leading operator of dimension $\Delta > 4$ is given by

$$\mathcal{L} = \frac{-y^{\nu}_{ij}}{\Lambda}\left(\bar{L}^i \tilde{H}\right)\left(\tilde{H} L^j\right)^{\dagger}$$

VIOLATES LEPTON #

dimension five operator

After EWSB we get a Majorana mass term for the neutrino

$$-\frac{y^{\nu}_{ij} v^2}{\Lambda}\left(\nu_L^i\right)^c \nu_L^j + h.c.$$

Majorana mass.

where $\nu_L^c = \nu_L^T \sigma_2$, the conjugate representation. We may expect $y \sim O(1)$ and $\Lambda >> v$, which would explain the smallness of the neutrino masses.

It could be the case that there exists a right handed sterile neutrino. namely imagine the existence of a lepton which is neutral under $SU(2)_L \times U(1)_Y$ and is a right handed Fermion. Then, we can write the following terms in the SM lepton sector:

$$\Delta\mathcal{L} = -y^{e}_{ij}\,\bar{L}^i H e_R^j - y^{\nu}_{ij}\,\bar{L}^i \tilde{H}\,\underbrace{\nu_R^j}_{\text{singlet a.k.a. sterile}} - i\,\underbrace{M_{ij}}_{\text{singlet}}\left(\nu_R^i\right)^c \nu_R^j + h.c.$$

After EWSB the Yukawa interactions provide a Dirac mass term for neutrinos, while the last term provides a Majorana type of mass.

The majorana type of masses violates lepton number. Thus we could forbid those masses imposing such symmetry (or B-L, ...).
With neutrinos we often go back and forth with Dirac v.s. Weyl

Recall that Dirac spinors are usually written in terms of two independent left- and right-handed Weyl spinors $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$.
But we can also construct Dirac out of a single Weyl.

$$\psi_L \equiv \begin{pmatrix} \nu_L \\ i\sigma_2 \nu_L^* \end{pmatrix} \quad ; \quad \psi_R = \begin{pmatrix} i\sigma_2 \nu_R^* \\ \nu_R \end{pmatrix}$$

transforms like right handed

transforms like left handed.

Then we can write "Dirac" and "Majorana" mass terms in a unified notation:

$$\mathcal{L}_{\nu, \text{mass}} = -m \bar{\psi}_L \psi_R - \frac{M}{2} \bar{\psi}_R \psi_R + h.c.$$

$$= -(\bar{\psi}_L, \bar{\psi}_R) \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

where for simplicity we suppressed the family index. The last equation follows from $\Delta\mathcal{L}$ (in previous page) after EWSB.

The eigenvalues of this mass matrix are $\sqrt{m^2 + M^2/4} \pm \frac{1}{2} M$. So that in the limit $M \gg m$:

$$m_{\text{light}} = \frac{m^2}{M^2} \quad ; \quad m_{\text{heavy}} \simeq M$$

The scale M may indeed be much larger than the EW scale, $v \ll M$. In the limit $v \gg M$, the last equation reduces to

$$-\frac{y^\nu_{ij} v^2}{\Lambda} (\nu_L^i)^c \nu_L^j + h.c. \quad + M \nu_R^c \nu_R + O\left(\frac{m^2}{M}\right) \bar{\psi}_L \psi_R + h.c.$$

small mixing

Heavy and sterile, so who cares?

Thus, at low energy is equivalent to the dimension five "Weinberg" operator.

Regardless of whether neutrinos are Majorana or Dirac, or whether the masses arise from dimension 3 $(\nu_R^c \nu_R)$, 4 $(\bar{L}\hat{H}\nu_R)$ or 5 $(\bar{L}\hat{H}\tilde{H}L)$, the only neutrinos that we can detect are left-handed. Because left-handed neutrinos only interact through the weak force, it is more natural to work in the "flavor basis" than in the "mass basis".

We denote by $\nu_{Le}, \nu_{L\mu}, \nu_{L\tau}$ the left handed electron, muon and tau neutrinos. In the flavor basis the W couplings are diagonal

$$\mathcal{L}_{\nu W} = -\frac{g}{\sqrt{2}}\left(\bar{e}_L W^\mu \gamma_\mu \nu_{Le} + \bar{\mu}_L W^\mu \gamma_\mu \nu_{L\mu} + \bar{\tau}_L W^\mu \gamma_\mu \nu_{L\tau} + h.c.\right).$$

while the mass matrices are not diagonal. The mass eigenstates (ν_1, ν_2, ν_3) are related to $(\nu_{Le}, \nu_{L\mu}, \nu_{L\tau})$ by a unitary transformation. Then

$$\mathcal{L}_{\nu W} = -\frac{g}{\sqrt{2}} U^{ij}\left(\bar{e}_{Li} W^\mu \gamma_\mu \nu_{Lj} + h.c.\right)$$

where $\nu_{Le} = U^{e1}\nu_{L1} + U^{e2}\nu_{L2} + U^{e3}\nu_{L3}$ and so on. The matrix U is called Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. It is the lepton analog of CKM matrix. It can be written in a verisimilar form:

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{23}c_{13} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{12}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix}$$

Here we do not assume $m_1 < m_2 < m_3$ ("normal" hierarchy of neutrino masses). Like in the CKM matrix, we have 3 mixing angles θ_{12}, θ_{13} and θ_{23} (which of course are different from the CKM mixing angles). The number of phase depends on the Dirac vs. Majorana nature of the neutrinos. If neutrinos are Dirac, there is a single phase (exact analog to CKM). If neutrinos have a Majorana component then there are two possible additional phases α_{12} and α_{31}.

END OF L16

The counting of physical parameters for Dirac is: $2\times(9\,\text{real}+9\,\text{im.})$ from the Yukawa matrices Y_e, Y_ν, we remove $3\times(3\,\text{real}+6\,\text{im})$ with the rotations, $(L, e_R, \nu_R) \longrightarrow (U_L L, V_R^e e_R, V_R^\nu \nu_R)$ where U_L, V_L^e and V_R^ν are unitary matrices, except for a phase corresponding to an unbroken generator $(L, e_R, \nu_R) \to e^{i\varphi}(L, e_R, \nu_R)$.

Counting for Majorana. In this case we have the operators $Y_e^{ij}\bar{L}_i H e_R^j$ and the Weinberg operator $C^{ij} L^i L^j HH$, where C_{ij} is a 3×3 symmetric complex matrix. Therefore we have $(9\,\text{real}+9\,\text{im})+(6\,\text{real}+6\,\text{im})$ parameters, minus $2\times(3\,\text{real}+6\,\text{im})$, from the rotations $(L, e_R) \to (U_L L, V_R^e e_R)$. Therefore $15-6=9$ real parameters for the 6 leptonic masses and the 3 mixing angles, and $15-12=3$ physical phases δ, α_{12}, α_{23}.

* Measurements of neutrino masses and mixing angles:

It is hard to measure neutrino masses and mixings. Neutrino masses were first observed by Sudbury Neutrino Observatory (SNO) in 2001, which measured that $\sim 35\%$ of neutrinos coming from the sun are electron neutrinos while $\sim 65\%$ are of ν_μ and ν_τ type. This solved the so called "solar neutrino problem": We expect a source of ν_e neutrinos from the sun (flavour basis), but we only observed a fraction of the expected number. Explanation: the flavour eigenstate ν_e oscillates as they propagate in through space, the propagators are diagonal only in the mass eigenstate basis.

There are other "types" of neutrinos with names coming from their origin. Apart from "solar neutrinos", we have observed "atmospheric neutrinos", which come from cosmic rays. Cosmic rays hit nuclei from the atmosphere and produce pions which mostly decays through $\pi^- \longrightarrow \mu^- \bar{\nu}_\mu \to (e^- \bar{\nu}_e \nu_\mu)\bar{\nu}_\mu$. Recall that this branching fraction is ~ 0.9999! Therefore one roughly expects a $2:1$ ratio of muon to electron neutrinos coming from

the atmosphere. Deviations from this ratio constrain the neutrino mixing angles and masses.
Neutrino oscillations are also measured using "reactor neutrinos" (which mostly produce $\bar{\nu}_e$) and "accelerator neutrinos" (typically producing ν_e)

It turns out that neutrino oscillations are only sensitive to difference in squares of neutrino masses. Solar neutrinos give

$$\Delta m_{21}^2 = m_2^2 - m_1^2 = (7.50 \pm 0.20) \cdot 10^{-5}\, eV^2$$

while atmospheric oscillations give

$$\Delta m_{32}^2 = |m_3^2 - m_2^2| = 0.0032 \pm 0.00012\, eV^2.$$

This differences are consistent with both:

$$m_1 < m_2 < m_3 \qquad \text{(normal hierarchy)}$$

or

$$m_3 < m_2 < m_1 \qquad \text{(inverted hierarchy)}.$$

	Normal Ordering	Inverted Ordering
$\sin^2\theta_{12}$	$0.310^{+0.013}_{-0.012}$	$0.310^{+0.013}_{-0.012}$
$\sin^2\theta_{23}$	$0.563^{+0.018}_{-0.024}$	$0.565^{+0.017}_{-0.022}$
$\sin^2\theta_{13}$	$0.02237^{+0.00066}_{-0.00065}$	$0.02259^{+0.00065}_{-0.00065}$
δ/π	$1.23^{+0.22}_{-0.16}$	$1.57^{+0.13}_{-0.14}$
$\Delta m^2_{21}/10^{-5}\text{eV}^2$	$7.39^{+0.21}_{-0.20}$	$7.39^{+0.21}_{-0.20}$
$\Delta m^2_{3\ell}/10^{-3}\text{eV}^2$	$2.528^{+0.029}_{-0.031}$	$-2.510^{+0.030}_{-0.031}$

Results from a global fit analysis '19.
Updated fits can be found in e.g. nu-fit.org.
Presently the data shows a preference for the normal ordering hierarchy and a non-zero value of δ.
We note that θ_{12} and θ_{23} are large, while θ_{13} is much smaller and by chance $\theta_{12} \simeq \theta_C$.

Neutrinos can oscillate in vacuum. These oscillations can be seen to be a consequence of the non-zero value of their mass.

* Mass-induced flavour oscillations in vacuum

Neutrinos are produced in a definite flavour, but oscillate into different mass eigenstates through propagation. Say that through a charged weak current we produce a charged lepton l_α, and it's corresponding neutrino ν_α. Then this neutrino is a superposition

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} U_{\alpha i}^* |\nu_i\rangle$$

where ν_i are the mass eigenstates. After traveling a certain distance L (note that for relativistic neutrinos $L \simeq ct$), the state evolves into

$$|\nu_\alpha(t)\rangle = \sum_{i=1}^{3} U_{\alpha i}^* |\nu_i(t)\rangle$$

this neutrino can subsequently interact through a charged current, producing a charged lepton of type β, $\nu_\alpha(t) N' \longrightarrow l_\beta N$, with probability

$$P_{\alpha\beta} = |\langle \nu_\beta | \nu_\alpha(t)\rangle|^2 = \Big| \sum_{i,j=1}^{3} U_{\alpha i}^* U_{\beta j} \langle \nu_j | \nu_i(t)\rangle \Big|^2.$$

Next we assume that $|\nu\rangle$ is a plane wave, $|\nu_i(t)\rangle = e^{-iE_i t}|\nu_i(0)\rangle$, with $E_i = \sqrt{p_i^2 + m_i^2}$. In all practical cases $p_i \simeq p_j \equiv p \simeq E$, and we can write

$$E_i \simeq p + \frac{m_i^2}{E}$$

this would not be true if we try to derive an oscillation formula for charged leptons.

and use the orthogonality of the mass eigenstates $\langle \nu_i | \nu_j \rangle = \delta_{ij}$.
And we arrive to

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\sum_{i<j}^{3} \mathrm{Re}[U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}] \sin^2 X_{ij} + 2\sum_{i<j}^{3} \mathrm{Im}[U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}] \sin 2X_{ij}$$

where

$$X_{ij} = \frac{(m_i^2 - m_j^2)L}{4E} = 1.267 \frac{\Delta m_{ij}^2}{eV^2} \frac{L/E}{m/MeV}.$$

If we had done an analogous derivation for the antineutrino, we would

have ended with a similar expression but with $U \leftrightarrow U^*$ exchanged.
Therefore, the second term of the last equation is CP conserving while the last term is not. The last term depends on the "Dirac" phase δ.
Note however that the Majorana phases cancel out in the probability. This is expected because neutrino oscillations do not violate the total lepton number.

In practice the spectrum is not monochromatic, and we need to take into account that the detector has some definite efficiency, and cross-section for the process of neutrino detection. Averaging over these:

$$\langle P_{\alpha\beta}\rangle = \frac{\int dE \frac{d\Phi}{dE}\sigma(E)\epsilon(E) P_{\alpha\beta}(E)}{\int dE \frac{d\Phi}{dE}\sigma(E)\epsilon(E)} =$$

$$= \delta_{\alpha\beta} - 4\sum_{i<j}^{3} \mathrm{Re}[U_{\alpha i}U_{\beta i}^* U_{\alpha j}^* U_{\beta j}]\langle\sin^2 X_{ij}\rangle + 2\sum_{i<j}^{3} \mathrm{Im}[U_{\alpha i}U_{\beta i}^* U_{\alpha j}^* U_{\beta j}]\langle\sin 2X_{ij}\rangle$$

Clearly if $E/L \gg \Delta m_{ij}^2$ ($L \ll L_{0,ij}^{osc}$) then $\sin^2 X_{ij} \ll 1$ and there is no effect from the oscillation phase. Also note that if $L \gg L_{0ij}^{osc}$, many oscillations take place and $\langle \sin^2 X_{ij}\rangle \simeq 1/2$.

To interpret the full experimental data set, at present it is of course necessary to take into account that there are three neutrinos.
For many experiments, and due to historical reasons, it is useful to consider neutrino oscillations in the case of only two active neutrinos.
Then the neutrino oscillation formula reduces to

$$P_{\alpha\beta} = \delta_{\alpha\beta} - (2\delta_{\alpha\beta} - 1)\sin 2\theta \sin^2 X$$

In limit, there is no CP violation. Furthermore, the previous formula is invariant under $\Delta m^2 \to -\Delta m^2$, and $\theta \to \frac{\pi}{2} - \theta$. Therefore, from a measurement of e.g. $P_{e\mu}$, we cannot tell if ν_e resides in the heavy or light neutrino mass eigenstate. This symmetry or ambiguity is broken when considering mixing of three flavours and/or when the neutrino oscillation occurs when traversing regions of dense matter instead of vacuum.

* Neutrinoless Double-Beta Decay.

The most sensitive probe to whether neutrinos are Dirac or Majorana states is the neutrinoless double beta decay $(0\nu\beta\beta)$:

\# protons

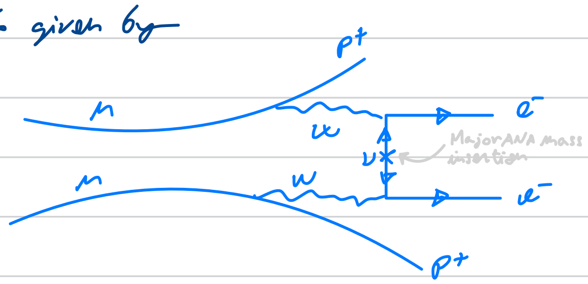
$$(A, Z) \longrightarrow (A, Z+2) + e^- + e^-$$

mass number

to be compared to usual beta decay $(A, Z) \longrightarrow (A, Z+1) + \bar\nu_e + e^-$. The corresponding amplitude is proportional to

$$M_{\alpha\beta} \propto (\bar e \gamma_\alpha (1-\gamma_5)\nu_e)(\bar e \gamma_\beta (1-\gamma_5)\nu_e) \propto \sum_i (U_{ei})^2 (\bar e \gamma_\alpha (1-\gamma_5)\nu_i)(\bar e \gamma_\beta (1-\gamma_5)\nu_i)$$

Diagramatically, the process is given by



and is non zero so long as $\langle \nu_i(x)\nu_i^T(y)\rangle \neq 0$, i.e. as the neutrino contains a majorana component, the self-two-point function is proportional to the Majorana mass. If this process is observed, neutrinos can not be Dirac states. The process above lets to a half-life

$$\left(T^{0\nu}_{1/2}\right)^{-1} = G \times |M|^2 \times \left(\frac{m_{ee}}{m_e}\right)^2 ,$$

where G is the phase space integral taking into account the final atomic state, M is the nuclear matrix element, and m_{ee} is the "effective Majorana mass", given by

$$m_{ee} = \left| \sum_i m_i U_{ei}^2 \right|$$

$$= \begin{cases} m_{light}\, c_{12}^2\, c_{13}^2 + \dots & \text{in NO} \\ m_{light}\, s_{13}^2 + \dots & \text{in IO} \end{cases}$$

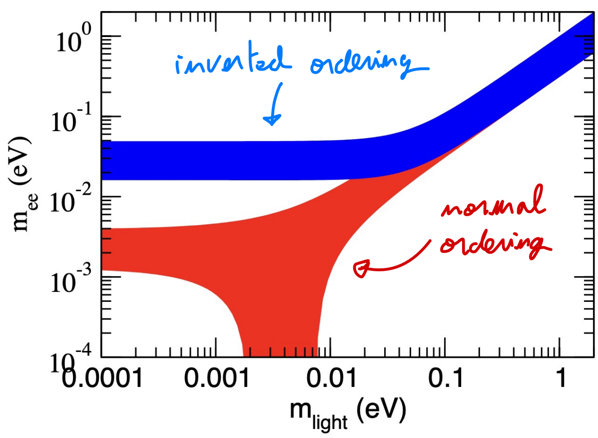
see e.g. PDG for the exact formula.

(using PDG definition of phases). The allowed range of this formula is plotted below.

Current best bounds on mee are given by
Current best bound comes from KamLAND-Zen experiment, which uses 13 Tons of Xe to search for $0\nu\beta\beta$ of ^{136}Xe. The non-observation sets a bound

$$m_{ee} < 61 - 165 \text{ meV}.$$

See also the table below:



Isotope	Lower Bound on $T^{0\nu}_{1/2}$ (yr)	Upper Bound on $\|m_{ee}\|$ (meV)	Collaboration
^{76}Ge	$8.0 \cdot 10^{25}$	$120 \div 260$	GERDA
^{130}Te	$1.5 \cdot 10^{25}$	$110 \div 520$	CUORE
^{136}Xe	$1.07 \cdot 10^{26}$	$61 \div 165$	KAMLAND Zen
^{136}Xe	$3.5 \cdot 10^{25}$	$93 \div 286$	EXO 200

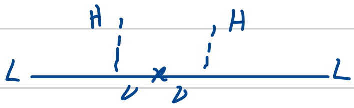
Lower and upper bounds on $T^{0\nu}_{1/2}$ and $|m_{ee}|$, respectively. The quoted ranges reflect the uncertainty in the nuclear matrix elements.

* Tree-level origins of the Weinberg operator

We have seen the "see-saw" mechanism in the context of the SM EFT: the lightness of the active neutrinos (i.e. ν_L's) is explained if we are happy accepting that Λ is large. This is realized through the Weinberg operator $LLHH/\Lambda$. We can ask our-selves what type of physics at the UV scale can give rise to the Weinberg operator. This question has a simple answer if the operator $LLHH$ is generated at tree-level.

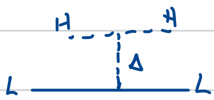
There are three possibilities. We denote the quantum numbers of the new heavy states by (r_3, r_2, r_1) under $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$.

Type I see-saw: $(1,1,0)$ heavy right handed ν



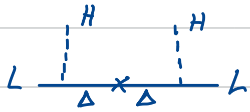
Type II see-saw: $(1,3,1)$: $\mathcal{L} = y L \sigma_a L \Delta^a + \mu H \sigma^a H \Delta^{*a} + M^2 \Delta^a \Delta^a$

Heavy complex scalar triplet.



Type III see-saw: $(1,3,0)$: $\mathcal{L} = \lambda T^a \ell \sigma_a H + M T^a T^a$

heavy triplet fermion



The Weinberg operator could also be generated radiatively.

END OF L17

N.B. Lecture 18 will be dedicated to the Gross-Neveu model. Our understanding of QCD is also based on lessons from "toy models". In this model we'll be able to calculate the expectation value of a composite operator, in perturbation theory of large N expansion. Lecture 19 will be entirely dedicated to answer questions from the whole course. Each student should bring at least one question.

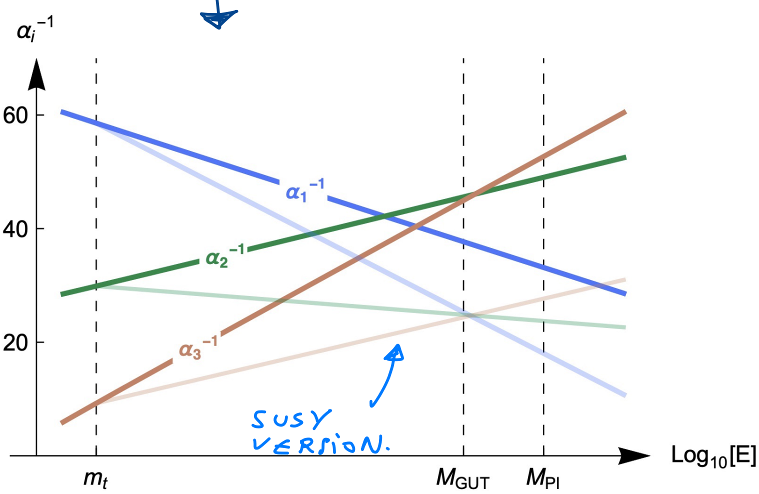
• SHADOWS OVER THE SM

We have insisted many times in this course on the point of view that the SM should be considered an EFT with a putative physical cutoff Λ. At the energy scale Λ new physics, possibly in the form of new BSM weakly interacting particles (but not necessarily), is expected.
The big question is what is the value of v/Λ??

There are various theoretical hints within the SM and GR that point towards the presence of new physics:

a) The three gauge couplings of the SM approximately coincide at an energy scale $\sim 10^{16}$ GeV. This feature of the couplings running might be signaling that, around that scale, the SM gauge groups are embedded into a bigger group with a single gauge coupling. These type of theories are called "Grand unified theories" GUTs. In the supersymmetric version of the SM couplings unify very precisely.

I encourage you to try to reproduce this plot. Here for the SM i have used $\alpha_i = \{\frac{5}{3}g'^2, g^2, g_s^2\}/(4\pi)$



Most popular GUTs are $SU(5)$ and $SO(10) \supset SU(5) \supset SU(3) \otimes SU(2) \otimes U(1)$.

In order to reproduce the plot above, take into account:

$$\{\beta_{g'^2}, \beta_{g^2}, \beta_{g_s^2}\} = \left\{\frac{41}{3} g'^4, -\frac{19}{3} g^4, -14 g_s^4\right\} \frac{1}{16\pi^2} + O(2\text{-loop})$$

$$\overset{\text{SUSY}}{\underset{\text{or}}{=}} \{22 g'^4, 2g^4, -6g_s^4\} \frac{1}{16\pi^2} + O(2\text{-loop})$$

and

$$\{g'_0, g_{2\,0}, g_{s\,0}\} \simeq \{0.12, 0.42, 1.35\} \quad @\ m_t$$

b) The massive nature of the neutrinos is easily explained in the SM EFT via the dimension-five higher dimensional operator $\sim y_\nu LLHH/\Lambda$. If "y_ν" is taken of $O(1)$, then in order to explain the neutrino mass scale $m_\nu \lesssim eV$, then the cutoff Λ of the Weinberg operator must be around the GUT scale $\Lambda \sim 10^{15}$ GeV.

c) Gravity is very weak at the electroweak scale because it couples to the SM degrees of freedom via the coupling $\sqrt{g_N}\, E = E/M_{Pl}$. However, gravitational interactions become non-perturbative at energies $E \simeq M_{Pl} \simeq 10^{18}$ GeV, the so called reduced Planck mass, which is not far from the GUT scale.

$$\mathcal{L}_{GR} = \frac{1}{2}(\partial\gamma)^2 + \sqrt{g_N}\,(\partial\gamma)^2\gamma + \sqrt{g_N}\, T\cdot\gamma + O(g_N)$$

where recall that $\quad S_{GR} = \frac{1}{16\pi^2 G_N}\int d^4x \sqrt{-g}\, R(g)$ w/

$g = \eta + \sqrt{g_N}\,\gamma$ and $g_N = 8\pi G_N = M_{Pl}^{-1} = 10^{-18}$ GeV.

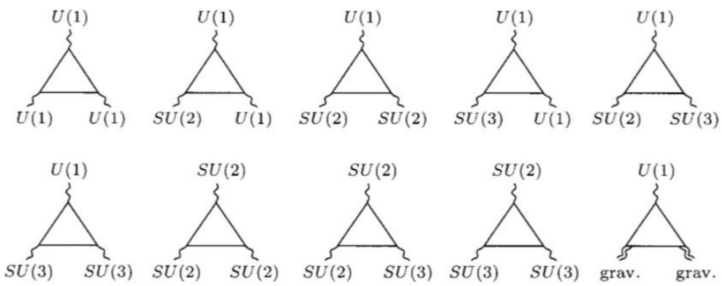
It is surprising that these three different hints within the SM + GR EFT point to similar high scale energies. These are energy scales much higher than the EW scale and it is inconceivable an experiment here on Earth that would directly probe such humongous scales.

In fact, there are various other pieces of "circumstantial evidence" that support the picture that there is no new physics directly coupled to the SM until vastly higher energy scale:

a) The accuracy of the SM predictions are an unprecedented success in science, and none of the predictions is contradicted by any experiment. Flavour and CP tests of the SM suggest that BSM flavour is either not generic or appears at much higher scales.

b) None of the SM couplings hits a Landau pole before the Planck scale. Furthermore, the Higgs quartic coupling is negative at high energies but small enough in absolute value to ensure the stability of the electroweak vacuum under quantum tunneling.

c) All possible gauge and gravitational anomalies cancel.



In other words, it is theoretically and experimentally consistent to set $\Lambda = M_{Pl}$! Closely related to the theoretical hints, there are theoretical or experimental puzzles (it is not a sharp dichotomy). These are unsettling features of the SM that we would like to explain and gain further experimental insight on them. For instance, we would like to have a deeper understanding of:

* hierarchy of fermion masses
* origin of the gauge groups
* and of the # of families
* the Higgs potential
* ...
* why electric charge appears to be quantized
* why $\theta_{QCD} << O(1)$
* Dark matter, particle nature?
* Inflation before big bang, related to part. phys?

However, note that models that address these conundrums exist with $\Lambda >> v$. Thus, explaining these open questions does not necessarily require new physics around the E.W. scale.
Luckily, this is not the whole story. There is a theoretical puzzle that seems to require new physics coupled to the SM with $O(1)$ coupling strength and close to the e.w. scale. This is called the EW hierarchy problem and is associated fine-tuning problem.

The EW hierarchy problem consist in giving a satisfactory answer to "why $v << M_{Pl}$"?

The meaning of "satisfactory" is of course a bit personal tipically what it is meant is that the proposed theory has couplings and ratios of energy scales of order $O(1)$, or dictated by symmetries, but otherwise generic.

Perhaps the best way to define the EW fine tuning problem is through the following observation: in any calculable UV completion of the SM conceived so far, the Higgs mass value turns out to be around the same energy scale of the UV completion scale

$$m_h^2 = g_{SM}^2 v^2 \pm g_{BSM}^2 \Lambda^2/(16\pi^2)$$

in order to accomodate $m_h \simeq v$, $\sum_i g_{BSM,i}^2 \sim O(16\pi^2 v^2/\Lambda^2)$. Therefore if $\Lambda \gg v$, then $\sum_i' g_{BSM_i}^2$ must be finely tuned.

The need for new physics not far from EW scale to avoid fine-tuning is not a theorem. However it is supported by the EFT analysis that, in the SM, the Higgs mass term is not forbidden by any symmetry.
The fine-tuning problem can be taken as a motivation for new physics at the EW scale. The argument says: if the full BSM theory is natural, then $\Lambda \sim 4\pi v$.

However, there are instances in physics that disfavor the naturalness "principle"/ "strategy" as a guide. For example, the cosmological constant problem seems to be totally at odds with the standard EFT analysis of naturalness $\langle T_{\mu\nu}\rangle \sim -(10^{-3}eV)^4 g_{\mu\nu} \ll -(M_{Pl})^4 g_{\mu\nu}$. (Indeed the prototypical part. phys. solution would be to have SUSY softly broken with $M_{SOFT} \lesssim 10^{-3}eV$).

All in all, the motivations / expectations / strategies / arguments for new physics are not sharp. This is OK! we should be open minded.

* Further techical coments on the EW hierarchy problem.