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The Unruh Effect and Measurements in Relativistic Quantum Field Theory

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"Forty-two," said Deep Thought, with infinite majesty and calm.

Abstract

Lately intense discussion regarding measurements in relativistic quantum mechanics has been developed. We present the Fewster-Verch measuring scheme that allows to address problems of locality and causality; an explicit example is given and is used to model an Unruh-deWitt type of detector. The results are compared with the traditional approach of the UdW detector, which is also given in the work, and found to be the same under certain considerations. For the sake of completeness and to gain a deeper understanding of the phenomena, a standard as well as an algebraic derivation of the Unruh effect is given. The topics treated in this thesis may attract people from different fields such as QMT, QFT and AQFT. We hope the reader find this work as an useful introduction for those topics and foremost, an interesting review of the Unruh effect and the Fewster and Verch measurement scheme in relativistic quantum mechanics.

Resumen

Últimamente se ha desarrollado una intensa discusión sobre las mediciones en la mecánica cuántica relativista. Presentamos el esquema de medición de Fewster-Verch que permite abordar los problemas de localidad y causalidad; se da un ejemplo explícito y se utiliza para modelar un detector de tipo Unruh-deWitt. Los resultados se comparan con el enfoque tradicional del detector UdW, que también se da en el trabajo, y se encuentran los mismos resultados bajo ciertas consideraciones. En aras de la exhaustividad y para profundizar en la comprensión de los fenómenos, se ofrece una derivación estándar y algebraica del efecto Unruh. Los temas tratados en esta tesis pueden atraer a personas de diferentes campos como QMT, QFT y AQFT. Esperamos que el lector encuentre este trabajo como una introducción útil para esos temas y, sobre todo, una interesante revisión del efecto Unruh y del esquema de medición de Fewster y Verch en la mecánica cuántica relativista.

Contents

Introduction

At the beginning of the XX century David Hilbert, Paul Dirac, John von Neumann, and Hermann Weyl among many others, worked on the mathematical foundations of quantum mechanics. The probabilistic nature of Quantum Mechanics (QM) made measurement theory a deeper theory because observables cannot be ascribed definite values. Instead, the pairing between states and observables gives rise to probability distributions which are subject to correlations that cannot be explained by classical probability theory. One of the problems of measurement theory was the proposal of a measurement scheme consistent with a physical experiment and, at the same time, mathematically rigorous to work with it.

In 1932 von Neumann, in his work "Mathematische Begründung der Quantenmechanik" [1] proposed one of the first mathematical schemes for measurements where a system and a probe are coupled quantum systems giving an "experimental setup" giving birth to the so-called Quantum Measure Theory (QMT). QMT is a very robust[1] theory with many applications on non-relativistic quantum systems [2, 3]. Nevertheless, measurement schemes on relativistic systems have not been a great source of attention. The need for the development of a relativistic theory of quantum measurements cannot be overemphasized, as important Quantum Field Theory (QFT) phenomena such as the Unruh effect [4] or the Hawking effect [5] are still primary sources of debate.

One may think that QMT naturally extends to quantum field theories, but it turns out that assumptions of the locality of fields must be addressed. For instance, if not carefully worked, consecutive experiments may lead to a super-luminal transport of information, as in a measurement protocol proposed many years ago by Sorkin [6]. Accordingly, the purpose of this thesis is to introduce and analyze a measuring scheme proposed by Fewster and Verch in [7] that takes into account the principles of relativity, locality and causality within the algebraic formalism of relativistic quantum field theory. We then model a Unruh-deWitt detector within this framework as an example of indirect measures and compare those results with the traditional analysis of the Unruh-deWitt detector which, as emphasized above, is still a source of debate. We remark that much of the chapter 1 and Chapter 3 is focused in concepts related to the Unruh effect manly because of two reasons, the first one is to develop enough understanding of the effect that later will allow us to translate the concepts into the algebraic language and, the second is due to the beauty of the Unruh effect by its

[1] Even though the "measurement problem" in QM is far from being solved.

own.

Therefore, the reader with experience of the Unruh effect and the Unruh-deWitt detector may skip Chapter 1 as it is devoted to derive the Unruh effect and the Unruh-deWitt detector outcome which, in turn, allows us to gain grasp on the Unruh effect that helps us to interpret the results when the algebraic approach is introduced. In chapter 2, Sorkin's protocol is introduced, which suggest that quantum measurement theory is inherently incompatible with relativistic causality. The natural framework of Algebraic Quantum Field Theory (AQFT), also called local quantum physics, allows us to address problems of locality and causality; hence, in chapter 3, some aspects of the AQFT formalism are introduced and developed in order to address the measurement problem. Finally, In chapter 4 we introduce the Christopher J. Fewster and Reiner Verch (FV) measuring scheme [7] and show its advantages as it solves the Sorkin's protocol and can be used to model a Unruh-deWitt type of detector.

1 The Unruh Effect

The Unruh effect (Fulling–Davies–Unruh effect) is perhaps one of the most controversial results that Quantum Field Theory predicts [4, 8–11]. The Unruh effect states that an observer in accelerated motion at constant acceleration a will perceive a thermal bath at temperature $a/2\pi$ i.e., a source of particles. Despite of the fact that the Unruh effect was first described by Fulling in 1973, Paul Davies in 1975 and W. G. Unruh in 1976 for Minkowksi spacetime, many results have been derived in the same spirit. For example, the analog of the Unruh effect in Schwarzschild spacetime was first derived by Hartle and Hawking 1976 [5, 10, 12] in which vacua states (invariant under time translation in the Kruskal extension of Schwarzschild spacetime with mass M) are thermal states of temperature $1/8\pi M$. This result, in turn, motivates the close relation between the Unruh effect and the Gibbon-Hawking effect (1977), in which a temperature can be associated to each solution of the Einstein field equations that contains a causal horizon[2]. Many other results related to the Unruh effect can be found in [8] and are discussed in great detail.

Although experimental realizations of the Unruh effect have been proposed [13], the theoretical prediction of the Unruh effect is not yet experimentally confirmed. Several concepts of detectors have been designed; for instance the Unruh-DeWitt detector is the most relevant. Nevertheless, the interpretation of the effect that *different observers "measure" different things* could be a source of misconceptions[3]. Our main focus will not be to present how different results can be obtained [4]; instead, we formalize the study of how a "measurement" must be performed according to Fewster and Verch [7]. In order to understand the effect, we start with a standard derivation, following [8, 10, 12].

1.1 Quantum Field Theory on Curved Space Time (QFT on CST)

Definition 1.1.1. *A Lorentzian manifold M is a pseudo-Riemannian manifold equipped with a smooth Lorentzian metric g of signature $(+,-,\dots,-)$ and a fixed time orientation.*

Definition 1.1.2. *A space-time is a smooth, Lorentzian, Hausdorff, paracompact manifold of dimension n with at most finitely many connected components.*

In order to construct the Penrose diagram of a given space-time M we introduce the conformal transformation.

[2]These results where first observed in 1977 in which Gibbons and Hawking showed that acceptable deSitter–invariant vacuum state of the free scalar field in deSitter spacetime with Hubble constant H is a thermal state of temperature $H/2\pi$ of the theory inside the cosmological horizon with the de Sitter boost generator fixing the horizon as the Hamiltonian.

[3]Crispino et al. [8] argues that in fact, different results are different measures of some detector. Thus, inconsistencies arise from the fact that there is no definition of what measurements in QFT should be.

[4]For a further discussion look [8] section III.A.5.

Definition 1.1.3. *Given a metric*

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad dx^\mu dx^\nu := \frac{1}{2}(dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu),$$

a conformal transformation (of the metric) is the metric $\bar{g}$ described as

$$\bar{g}_{\mu\nu}(p) = \Omega^2(x)g_{\mu\nu}(x).$$

We can choose Ω such that all the distances on M have an upper bound and therefore, a diagram of the space-time can be drawn inside a compact region (say, on a A4 size sheet of paper).

Definition 1.1.4. *Let M be a space-time with line element*

$$ds^2 = N(x)^2dt^2 - G_{ab}(x)dx^a dx^b.$$

We say that the function N is the lapse function whereas G the metric on the space-like hyper-surface with constant t. The space-time is static if the lapse function and the metric G are independent of t.

Example 1.1.1. *Consider the $1+1$-dimensional Minkowski space-time with metric given by the line element*

$$ds^2 = dt^2 - dz^2 = dudv, \tag{1.1}$$

where u, v are the null coordinates given by $u = t - x$, $v = t + x$. Clearly the spacetime is static, furthermore, we can bound the metric

$$g_{\mu\nu} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}(du \otimes dv + dv \otimes du).$$

Meaning that we can rescale infinities "∞" into a finite number while preserving the causal structure. In this case, with the help of the transformations $u' = 2\arctan u$, $v' = 2\arctan v$. Thus, the line element in the new variables becomes

$$ds^2 = \frac{1}{4}\sec^2(u'/2)\sec^2(v'/2)du'dv'.$$

Hence, if we choose $\Omega^2(x) = (\sec^2(u'/2)\sec^2(v'/2)/4)^{-1}$ then, the conformal transform give us a Minkowski-like metric

$$\bar{g}_{\mu\nu} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}(du' \otimes dv' + dv' \otimes du'),$$

however, in this case over a compact region. The resulting figure is shown in figure 1 and, the diagram is known as the Penrose diagram of Minkowski spacetime.

The (real) scalar free field φ is the field that satisfies the Klein-Gordon (K-G) equation. On curved space times, the K-G equation can be extended by noticing that the d'Alambertian is the Laplace-Beltrami operator on M,

$$\Box\varphi = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = g^{ij}\left(\frac{\partial^2\varphi}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial\varphi}{\partial x^k}\right) = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^j}\left(\sqrt{-g}g^{ij}\frac{\partial\varphi}{\partial x^i}\right),$$

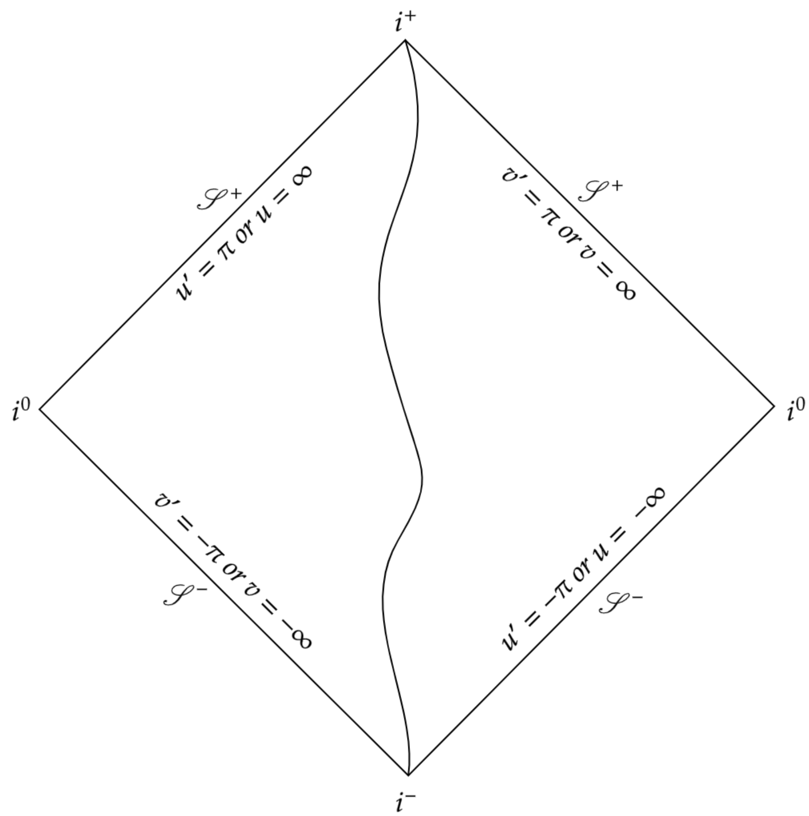

Figure 1: Penrose Diagram: Minkowski space-time: $\mathscr{J}^{\pm}$ are the event horizons

hence the D'Alambertian is well defined. It follows, by analogy if the Klein-Gordon equation in Minkowski space time, that the K-G in curved space times is:

$$\left[\Box + m^2 + \xi R(x)\right]\varphi(x) = 0, \quad \Box\varphi = g^{\mu\nu}\nabla_\mu\nabla_v\varphi, \tag{1.2}$$

where R is the Ricci curvature. The Lagrangian (Lagrangian density) is:

$$\mathscr{L}(x) = \frac{1}{2}[-\mathrm{g}(x)]^{1/2}\left\{g^{\mu\nu}(x)\varphi(x)_\mu\varphi(x)_\nu - \left[m^2 + \xi R(x)\right]\varphi^2(x)\right\}. \tag{1.3}$$

The case $\xi = 0$ or $\xi(n) = 1/4[(n-2)/(n-1)]$ are the minimally and conformally coupled case respectively. From now on, we work with the minimally coupled case,

$$\left[\Box + m^2\right]\varphi(x) = 0, \quad \Box\varphi = g^{\mu\nu}\nabla_\mu\nabla_v\varphi. \tag{1.4}$$

In order to exemplify the quantization, we introduce the notation

$$f\overset{\leftrightarrow}{\partial}_\mu g := f\left(\frac{\partial g}{\partial x^\mu}\right) - \left(\frac{\partial f}{\partial x^\mu}\right)g.$$

that allows us to define an "inner product" [5]. Indeed, let $\Sigma \subset M$ be a space-like hypersurface and suppose Σ is also a Cauchy surface and let $d\Sigma^\mu = n^\mu \, d\Sigma$ where n^μ is a future-directed orthonormal vector. We define a linear form by

$$(\varphi_1, \varphi_2) = -\mathrm{i} \int_\Sigma \varphi_1(x) \overleftrightarrow{\partial_\mu} \varphi_2^*(x) \left[-g_\Sigma(x)\right]^{\frac{1}{2}} \mathrm{d}\Sigma^\mu \tag{1.5}$$

which is often called the inner product between fields φ_1 and φ_2 when restricted to a "time slice". With these assumptions one can show (via the Gauss divergence theorem) that the scalar product is independent of Σ in the globally hyperbolic space-time M and thus, well-defined. A set of fundamental solutions $\{u_\omega\}_{\omega \in J}$ of (1.4) that satisfies

$$(u_\omega, u_{\omega'}) = \delta_{\omega,\omega'}, \quad (u_\omega^*, u_{\omega'}^*) = -\delta_{\omega,\omega'}, \quad (u_\omega, u_{\omega'}^*) = 0$$

and $\{u_\omega\}_{\omega \in J}$ constitutes a complete set is a orthonormal set of solutions with respect to the inner product. Thus, given a complete orthonormal set of solutions, every solution of (1.4) is given by:

$$\varphi(x) = \sum\!\!\!\!\!\!\int \left[a_\omega u_\omega(x) + a_\omega^\dagger u_\omega^*(x)\right] d\omega.$$

for some coefficents $\{a_i\}_{i \in J}$. Canonical quantization is carried by adopting a_i as operators that satisfy

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \quad \left[a_i, a_j\right] = 0, \quad \left[a_i^\dagger, a_j^\dagger\right] = 0, \tag{1.6}$$

and act in the "traditional" Fock space (the one generated by a one particle hilbert space $\mathcal{H}$) where the vacuum state is defined as the nonzero state that satisfies

$$a_i \left|0\right\rangle = 0$$

for all $i \in \omega$. For a detailed description, we refer to [14].

Example 1.1.2. *Let M be a flat space-time, i.e. Minkowski space-time we consider the real scalar field. In this case, we can choose a Cauchy surface as a slide of M with $t = const.$ i.e. $\eta^\mu = (1,0,0,0)$. Hence, the inner product takes the usual form*

$$(\varphi_1, \varphi_2) = \mathrm{i} \int d^3x \, \varphi_1(x) \overleftrightarrow{\partial_0} \varphi_2^*(x),$$

where $\varphi^* = \varphi$. We have that fields $\varphi(x)$ are described by

$$\varphi(x) = \int \frac{d^3p}{2E_p}\left[u_p(x)a(p) + u_p^*(x)a^\dagger(p)\right], \quad E_p = \sqrt{m^2 + p^2}.$$

Notice that we can find a and $a^\dagger$ as

$$a(p) = (u_p^*, \varphi), \quad a^\dagger(p) = (u_p^*, \varphi)^*. \tag{1.7}$$

[5] when restricted to an appropiate subspace

One can also show that $a(p)$ *is time-independent and the commutation relations*

$$[a(p), a^\dagger(q)] = i \int d^3x \, u_p^*(x) \overleftrightarrow{\partial_0} \, u_q(x) = 2E_p \delta(p-q),$$

$$[a(p), a(q)] = 0 = [a^\dagger(p), a^\dagger(q)].$$

holds. The vacuum state $|0\rangle$ *is realized as the state* $|0\rangle \neq 0$ *such that*

$$a_i |0\rangle = 0 \quad \textit{for all } i.$$

Consequently, the annihilation and creation operators a *and* $a^\dagger$ *define the usual Fock space and thus, the number operator* i *is* $N_i = a_i^\dagger a_i$ *with the total number operator given by*

$$N = \sum\!\!\!\!\!\!\int N_i.$$

The Hamiltonian is finally

$$H = \frac{1}{2} \int \frac{d^3p}{2E_p} E_p \big[a^\dagger(p)a(p) + a(p)a^\dagger(p)\big].$$

Remark 1.1.1. *In this example we started with a dynamics given by the K-G equation, define the observables (the fields) and, a posteriori, we defined the appropiate Hilbert space in which* a *and* $a^\dagger$ *acts. Then, and only then, we defined the Fock space. This way of construction will be the main focus of the theory (Algebraic Quantum Field Theory) presented in section 3.*

Remark 1.1.2. *In Minkowski space-time there is no ambiguity of how states* a *and* $a^\dagger$ *are chosen because* $u_p(x)$ *are uniquely determined by*

$$u_p(x) = \frac{1}{(2\pi)^{3/2}} e^{-ip\cdot x}.$$

Indeed, a vacuum state is chosen so that admits invariance under the action of the Poncairé group. However, no analogous can be described to curved space-times.

The Stone-von Neumann theorem [6] *implies that there is only one representation (up to unitary equivalent) of the CCR algebra e.g. the Schrödinger representation. In infinite dimensions inequivalent representations do exist. This fact is closely related to the non-uniqueness of a vacuum state for a free quantum field in a curved spacetime background.*

1.1.1 Vaccum states

Let $\varphi(x)$ be a field satisfying the K-G equation on a curved space-time (possibly flat). Consider two different orthonormal complete set of modes $\{\overline{u}_j(x)\}_{j\in I}$ and $\{u_j(x)\}_{j\in I}$. Then, a field $\varphi(x)$ can be written as

$$\varphi(x) = \sum\!\!\!\!\!\!\int a_i u_i(x) + a_i^\dagger u_i^*(x) = \sum\!\!\!\!\!\!\int \overline{a}_i \overline{u}_i(x) + \overline{a}_i^\dagger \overline{u}_i^*(x).$$

[6]If (V, σ) are finite dimensional, all irreducible representations of the Weyl algebra are unitarily equivalent

On the other hand, the vacuum states $|0\rangle$ and $|\bar{0}\rangle$ are defined as the states $|\bar{0}\rangle \neq 0 \neq |0\rangle$ such that for all annihilator operators $\bar{a}_j$ and a_j satisfies,

$$\bar{a}_j |\bar{0}\rangle = 0, \quad a_j |0\rangle = 0.$$

Naturally, we look how $|0\rangle$ and $|\bar{0}\rangle$ are related. Given two orthonormal basis $\{u_j\}_{j\in I}$ and $\{\bar{u}_j\}_{j\in I}$, we can write

$$\bar{u}_j = \sum_{i\in I} \left(\alpha_{ji} u_i + \beta_{ji} u_i^*\right), \quad \bar{u}_j^* = \sum_{i\in I} \left(\beta_{ji}^* u_i + \alpha_{ji}^* u_i^*\right), \tag{1.8}$$

for some α_{ij}, $\beta_{kl} \in \mathbb{C}$ naturally, we want α and β such that the annihilation and creation operators, associated to the two modes, follow the same CCRs or equivalently, that the transformations are linear. If $I = 1$, the transformation T from the modes u_i, u_i^* to the $\bar{u}_j$ is

$$T = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha \end{pmatrix} \implies T^{-1} = \frac{1}{|\alpha|^2 - |\beta|^2} \begin{pmatrix} \alpha^* & -\beta \\ -\beta^* & \alpha \end{pmatrix}. \tag{1.9}$$

By analogy, for general I we can write u_i as

$$u_i = \sum_{j\in I} \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*\right), \quad u_i^* = \sum_{j\in I} \left(-\beta_{ji}^* \bar{u}_j + \alpha_{ji} \bar{u}_j^*\right).$$

Hence, using the former equation and (1.8) we have the relations

$$\bar{u}_j = \sum_{i\in I} \left(\alpha_{ji} u_i + \beta_{ji} u_i^*\right), \quad u_i = \sum_{j\in I} \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*\right). \tag{1.10}$$

Using (1.10) in itself,

$$\begin{aligned} \bar{u}_j &= \sum_{i\in I} \left(\alpha_{ji} \left[\sum_{k\in I} \left(\alpha_{ki}^* \bar{u}_k - \beta_{ki} \bar{u}_k^*\right)\right] + \beta_{ji} \left[\sum_{k\in I} \left(-\beta_{ki}^* \bar{u}_k + \alpha_{ki} \bar{u}_k^*\right)\right]\right) \\ &= \sum_{i,k\in I} \left(\alpha_{ji} \alpha_{ki}^* \bar{u}_k - \alpha_{ji} \beta_{ki} \bar{u}_k^* - \beta_{ki}^* \bar{u}_k \beta_{ji} + \beta_{ji} \alpha_{ki} \bar{u}_k^*\right). \end{aligned}$$

Comparing the coefficient's of $\bar{u}_l$, the conditions of α and β are read to be

$$\sum_k \left(\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*\right) = \delta_{ij}, \quad \sum_k \left(\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}\right) = 0. \tag{1.11}$$

Definition 1.1.5. *The transformations* (1.10) *from the* $\overline{u}_j(x)$ *to the* $u_j(x)$ *modes with* α_{ij} *and* β_{ij} *satisfying condition* (1.11) *are called the Bogolyubov transformations whereas the coefficients* α_{ij} *and* β_{ij}, $i, j \in I$ *are called the Bogolyubov coefficients.*

Using equation (1.10) we can write φ as

$$\begin{aligned}
\varphi(x) &= \sum_{j\in I}\left[a_j u_j + a_j^\dagger u_j^*\right] = \sum_{j\in I}\left[\bar{a}_j\bar{u}_j + \bar{a}_j^\dagger\bar{u}_j^*\right] \\
&= \sum_{j\in I}\left[a_j\sum_{k\in I}\left(\alpha_{kj}^*\bar{u}_k - \beta_{kj}\bar{u}_k^*\right) + a_j^\dagger\sum_{k\in I}\left(-\beta_{kj}\bar{u}_k + \alpha_{kj}\bar{u}_k^*\right)\right] \\
&= \sum_{j\in I}\left[\underbrace{\sum_{k\in I}(a_j\alpha_{kj}^* - a_j^\dagger\beta_{kj})\,\bar{u}_k}_{\bar{a}_j} + \underbrace{\sum_{k\in I}(a_j^\dagger\alpha_{kj} - a_j\beta_{kj})\,\bar{u}_k^*}_{\bar{a}_j^\dagger}\right].
\end{aligned}$$

thus, we have

$$\bar{a}_k = \sum_{j\in I}\left(a_j\alpha_{kj}^* - a_j^\dagger\beta_{kj}\right) \quad \bar{a}_k^\dagger = \sum_{j\in I}\left(a_j^\dagger\alpha_{kj} - a_j\beta_{kj}\right). \tag{1.12}$$

We are now in position to see how the vacua $\left|\bar{0}\right\rangle$ and $|0\rangle$ are related. Notice that

$$\bar{a}_j\,|0\rangle = \sum_{k\in I}(a_j\alpha_{kj}^* - a_j^\dagger\beta_{kj})\,|0\rangle = -\sum_{j\in I}\beta_{kj}a_j^\dagger\,|0\rangle = -\sum_{j\in I}\beta_{kj}\,|1_j\rangle \neq 0,$$

likewise for $\left|\bar{0}\right\rangle$ we have

$$a_i\left|\bar{0}\right\rangle = \sum_{j\in I}\beta_{ji}^*\left|\bar{1}_j\right\rangle, \quad \bar{a}_j\,|0\rangle = -\sum_{j\in I}\beta_{kj}\,|1_j\rangle\,. \tag{1.13}$$

Hence, the expected value of the particle number operator are

$$\left\langle\bar{0}\right|N_i\left|\bar{0}\right\rangle = \sum_{j\in I}|\beta_{ij}|^2, \quad \langle 0|\,\overline{N}_i\,|0\rangle = \sum_{j\in I}|\beta_{ij}|^2.$$

Therefore, if $\beta_{kj} \neq 0$ for all $k, j \in I$ the modes $u_j(x)$ define a different vacuum state from the one defined by the modes $\bar{u}_j(x)$.

Remark 1.1.3. *If there exist $i, j \in I$ such that $\beta_{ij} \neq 0$ then, the vacuum $\left|\bar{0}\right\rangle$ contains particles in the u_j mode and vice-versa. Furthermore, given a set of modes $\{u_j\}_{j\in I}$ and if the modes $\{\bar{u}\}_{j\in I}$ are defined in the positive frequencies, that is*

$$\bar{u}_j = \sum_{i\in I}\alpha_{ji}u_i \tag{1.14}$$

then, the vacuum $|0\rangle$ and $\left|\bar{0}\right\rangle$ coincide. Notice that equation (1.13) tell us that $\beta_{kj} = 0$ for all $k, j \in I$ is a sufficient condition for both vacuums $\left|\bar{0}\right\rangle$ and $|0\rangle$ to be the same. A necessary condition for the vacua to be different is, as we said earlier, the existence of inequivalent representations of the CCR algebra.

1.1.2 The Rindler wedges

In order to discuss the Unruh effect, we pay special attention to accelerated observers. The equivalence principle of general relativity allows us to take a uniformly accelerating observer as an inertial observer under a gravitational field. Thus, allowing the study of an accelerated observer into a suitably curved space-time. It turns out that a uniformly accelerated observer on the Minkowski space-time can be seen as an observer in the so-called Rindler space-time; in this section, we introduce the concepts needed. Consider the Minkowski space-time ($ds^2 = dt^2 - dx^2 - dy^2 - dz^2$). The

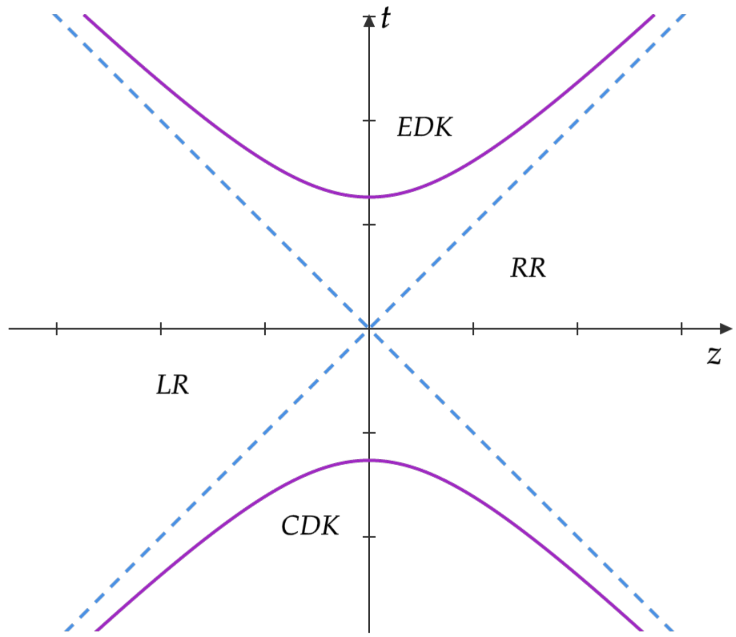

Figure 2: Rindler wedges right and left (RR and LL respectively) and the Kasner degenerate universe (EDK and CDK).

Killing vector $t\partial_z + z\partial_t$ defines the integral curves,

$$\begin{cases} t = \dfrac{\partial z(\beta)}{\partial \beta} \\ z = \dfrac{\partial t(\beta)}{\partial \beta} \end{cases}$$

with

$$z = \rho \cosh \beta, \quad t = \rho \sinh \beta.$$

Therefore, the metric is invariant under transformations of the form

$$\begin{cases} t \to t\cosh\beta + z\sinh\beta \\ z \to t\sinh\beta + z\cosh\beta \end{cases}.$$

We can take coordinates along the integral curves, i.e, $t = \rho\sinh\eta, z = \rho\cosh\eta$ then, the Killing vector in the new coordinates (η, ρ, x, y) will be

$$\begin{aligned} z\partial_t + t\partial_z = \rho\cosh\eta\partial_t + \rho\sinh\eta\partial_z &= \rho\cosh\eta((\sinh\eta)^{-1}\partial_\rho + (\rho\cosh\eta)^{-1}\partial_\eta) + \\ &\rho\sinh\eta((\cosh\eta)^{-1}\partial_\rho + (\rho\sinh\eta)^{-1}\partial_\eta) = \rho(\coth\eta + \tanh\eta)(\partial_\rho) + \partial_\eta \\ &= \partial_\eta. \end{aligned}$$

Consequently, world lines with fixed values ρ, x, y are trajectories (integral curves) of the boost transformations.

Remark 1.1.4. *Notice that the lines $t+z=0$ and $t-z=0$ naturally "separate" the space-time in four regions as shown in figure 2. Observe that $z^2 - t^2 = \rho^2\cosh^2\beta - \rho^2\sinh^2\beta = \rho^2 > 0$; hence, coordinates (η, ρ, x, y) cover the region $z^2 > t^2$. This region is known as the Rindler wedges, composed of the Right Rindler wedge (RR) for $z > |t|$ and the Left Rindler wedge (LR) for $-z > |t|$. In other words, the coordinates (η, ρ, x, y) cover the right Rindler and left Rindler wedges. The region $z^2 < t^2$ is known as the Kasner degenerate space-time (contracting and expanding). The Killing vector ∂_η is time-like on RR and the LR wedges, space-like in the Kasner degenerate space-time, and null on $t = \pm z$. Hence, the lines $t+z=0$ and $t-z=0$ indeed separate the space-time into four causally disjoint regions.*

The line element (of the RR and LR wedges) is then

$$\begin{aligned} ds^2 &= dt^2 - dx^2 - dy^2 - dz^2 \\ &= \rho(\cosh^2\beta - \sinh^2\eta)d\eta^2 \\ &\qquad + (\sinh^2\beta - \cosh^2\eta)d\rho^2 - dx^2 - dy^2 \\ &= \rho^2 d\eta^2 - d\rho^2 - dx^2 - dy^2. \end{aligned}$$

If we consider only the right (or left) Rindler wedge, we can make a further coordinate transformation, $\eta = a\tau$, $a > 0$ and $\rho = a^{-1}e^{a\xi}$. The overall transformation is

$$t = \rho\sinh\eta = \underbrace{a^{-1}e^{a\xi}}_{\rho}\sinh\underbrace{a\tau}_{\eta}, \quad z = \underbrace{a^{-1}e^{a\xi}}_{\rho}\cosh\underbrace{a\tau}_{\eta}, \quad a > 0.$$

Finally, since $d\eta^2 = a^2 d\tau^2$, $\quad d\rho^2 = e^{2a\xi}d\xi^2$ we have the line element for the right Rindler wedge

$$\begin{aligned} ds^2 = \rho^2 d\eta^2 - d\rho^2 - dx^2 - dy^2 &= a^{-2}e^{2a\xi}(a^2 d\tau^2) - a^{-1}e^{2a\xi}d\xi^2 - dx^2 - dy^2 && (1.15) \\ &= e^{2a\xi}(d\tau^2 - d\xi^2) - dx^2 - dy^2. && (1.16) \end{aligned}$$

Definition 1.1.6. *Let M be the Minkowski manifold $(\mathbb{R}^{1,3})$.*

The Right Rindler wedge (RR) is the manifold M restricted to $z > |t|$ together with the line element

$$ds^2 = e^{2a\xi}(d\tau^2 - d\xi^2) - dx^2 - dy^2, \tag{1.17}$$

where

$$t = a^{-1}e^{a\bar{\xi}}\sinh a\tau, \quad z = a^{-1}e^{a\xi}\cosh a\tau.$$

The Left Rindler wedge (LR) is the manifold M restricted to $-z > |t|$ together with the line element

$$ds^2 = e^{2a\bar{\xi}}(d\bar{\tau}^2 - d\bar{\xi}^2) - dx^2 - dy^2, \tag{1.18}$$

where

$$t = a^{-1}e^{a\bar{\xi}}\sinh a\bar{\tau}, \quad z = -a^{-1}e^{a\bar{\xi}}\cosh a\bar{\tau}.$$

We say that a Rindler observer is one whose wordline is parametrized with the Rindler coordinates (η, ρ, x, y).

Example 1.1.3. *Consider a trajectory of a Rindler observer on the right Rindler wedge,*

$$x^\mu = (a^{-1}e^{-a\xi}\sinh a\tau, a^{-1}e^{-a\xi}\cosh a\tau, 0, 0), \quad \xi, x, y, a \text{ const.}$$

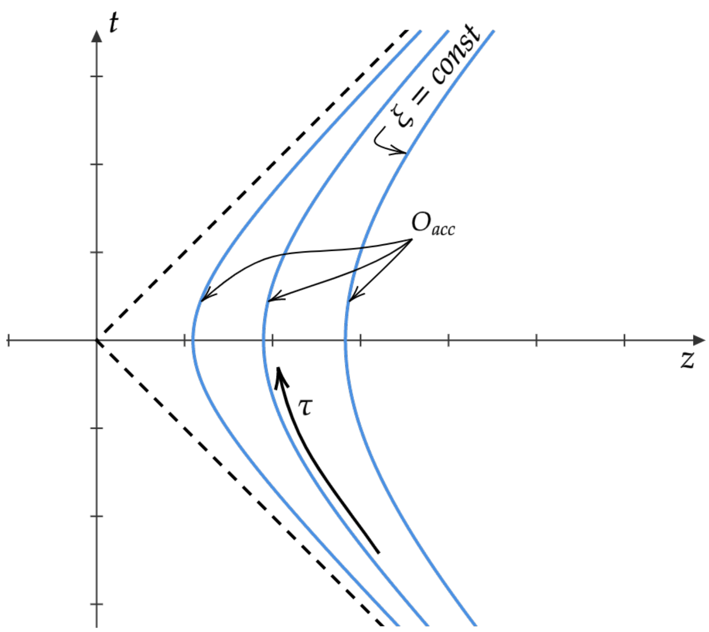

Figure 3: Right Rindler wedge with three different Rindler observers (O_{acc}).

The four-velocity vector is

$$\begin{aligned} u^\mu &= \frac{dx^\mu}{d\tau} = (e^{-a\xi}\cosh a\tau, e^{-a\xi}\sinh a\tau, 0, 0) \\ \sqrt{u^\mu u_\mu} &= (g_{00}e^{-2a\xi}\cosh^2 a\tau + g_{11}e^{-2a\xi}\sinh^2 a\tau)^{1/2} \\ &= \sqrt{\cosh^2 a\tau - \sinh^2 a\tau} = 1 \end{aligned}$$

and the four-acceleration $a^\mu = \nabla_\tau u^\mu = \frac{\partial u^\mu}{\partial \tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta$,

$$\begin{aligned} a^0 &= e^{-a\xi} a \sinh a\tau + 0 \\ a^1 &= e^{-a\xi} a \cosh a\tau + 0 \\ a^i &= 0. \quad i = 1, 2. \end{aligned}$$

$$\sqrt{-a^\mu a_\mu} = \sqrt{e^{-2a\xi}e^{2a\xi}a^2\cosh^2 a\tau - e^{-2a\xi}e^{2a\xi}a^2\sinh^2 a\tau} = \sqrt{a^2} = a = const$$

Thus, the proper acceleration of Rindler observers is $a = const$ *as shown in the figure 3. Therefore, a uniformly accelerated detector with proper acceleration* a *will lie on a trajectory with* $\xi = 0$ *and* x, y *constant, making the Rindler spacetime the framework of the Unruh effect.*

1.2 The Unruh effect (Part I: A (1+1)D - model)

Having in mind that different vacua may exist, we will make an explicit computation of this phenomena in Rindler space-time which is essentially the Unruh effect. We present a standard derivation with some similarities to the one derived by W. G. Unruh (1976) [15].

Since Minkowski and Rindler spacetimes are static, let us first see what condition a mode satisfying the K-G equation must follow. Given a line element in a static space-time

$$ds^2 = N(x)^2 dt^2 - G_{ab}(x)dx^a dx^b,$$

notice that $\sqrt{-g} = \sqrt{-N(x)\prod_i G_{ii}} = \sqrt{N}\sqrt{-G}$ and $\Box\varphi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi)$ thus,

$$\begin{aligned} \Box\varphi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) = &\frac{1}{\sqrt{N}\sqrt{G}}\partial_t(\sqrt{N}\sqrt{G}N(x)\partial_t\varphi) \\ &+ \frac{1}{\sqrt{N}\sqrt{G}}\partial_a(\sqrt{N}\sqrt{G}(-G^{ab})g^{ab}\partial_b\varphi). \end{aligned}$$

Hence, the K-G equation (1.4) gives

$$\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) + m^2\varphi \\
&= \frac{1}{\sqrt{-g}}\partial_t(\sqrt{N}\sqrt{G}N(x)\partial_t\varphi) \\
&\qquad + \frac{1}{\sqrt{-g}}\partial_a(\sqrt{N}\sqrt{G}(-G^{ab})g^{ab}\partial_b\varphi) + m^2\varphi \\
&= \sqrt{N}\partial_t(\sqrt{N}\sqrt{G}N(x)\partial_t\varphi) \\
&\qquad + \sqrt{N}\partial_a(\sqrt{N}\sqrt{G}(-G^{ab})g^{ab}\partial_b\varphi) + N\sqrt{G}m^2\varphi) \\
&= \partial_t(N\sqrt{G}\partial_t\varphi) - \partial_a(N\sqrt{G}G^{ab}g^{ab}\partial_b\varphi) + N\sqrt{G}m^2\varphi.
\end{aligned}$$

Therefore, a mode satisfying the K-G equation in a static space-time must follow

$$\partial_t(N\sqrt{G}\partial_t\varphi) - \partial_a(N\sqrt{G}G^{ab}g^{ab}\partial_b\varphi) + N\sqrt{G}m^2\varphi = 0. \tag{1.19}$$

In order to address the Unruh effect, we consider an inertial observer in Minkowski space-time and an observer in accelerated motion. That is, we consider a field φ spanned by the annihilation and creation operators in Minkowski space-time and one spanned by the annihilation and creation operators in Rindler space-time.

In Minkowski spacetime, the K-G equation $(\partial_t^2 - \partial_a\partial^a)\hat{\varphi} = 0$ give us solutions of the form

$$\hat{\varphi}(x) = \int \frac{dk}{\sqrt{2\pi k}}(\hat{a}_k e^{-ikz} + \hat{a}^\dagger_{-k}e^{ikz}).$$

If we work only on the Rindler wedges, that is, $z > |t|$ for the RR and $-z > |t|$ on the LL then, writing $\hat{a}_k = (\hat{b}_k + \hat{b}_k^\dagger)/\sqrt{2}$ where $\hat{b}_k$ and $\hat{b}_k^\dagger$ are the creation and annihilator operators restricted to the region, we have:

$$\hat{\varphi}(x) = \int \frac{dk}{\sqrt{4\pi k}}\Big(\hat{b}_{-k}e^{-ik(t-z)} + \hat{b}_{+k}e^{ik(t+z)} + \hat{b}^\dagger_{-k}e^{-ik(t-z)} + \hat{b}^\dagger_{+k}e^{ik(t+z)}\Big)$$

with the usual commutator relations

$$[\hat{b}_{\pm k}, \hat{b}^\dagger_{\pm k'}] = \delta(k-k'), \quad [\hat{b}_{\pm k}, \hat{b}_{\pm k'}] = 0, \quad [\hat{b}^\dagger_{\pm k}, \hat{b}^\dagger_{\pm k'}] = 0, \tag{1.20}$$

with the other commutator relations vanishing because the LR and RR are causally disjoint. Let

$U = t - z$, $V = t + z$ and denote the fields on the RR and the LL by φ_+ or φ_- respectively then,

$$\begin{aligned}\hat{\varphi}(x) &= \int d^3k[\hat{b}_{+k}f_k(V) + \hat{b}^\dagger_k f^*_k(V)] + \int d^3k[\hat{b}_{-k}f_k(U) + \hat{b}^\dagger_{-k}f^*_k(U)] \\ &= \underbrace{\int d^3k \frac{1}{(4\pi k)^{3/2}}[\hat{b}_{+k}e^{-ikV} + \hat{b}^\dagger_k e^{ikV}]}_{\hat{\varphi}_+(V)} \\ &\qquad + \underbrace{\int d^3k \frac{1}{(4\pi k)^{3/2}}[\hat{b}_{-k}e^{-ikU} + \hat{b}^\dagger_{-k}e^{ikU}]}_{\hat{\varphi}_-(U)} \\ &= \hat{\varphi}_+(V) + \hat{\varphi}_-(U).\end{aligned}$$

where

$$\hat{\varphi}_+(V) = \int dk\left[\hat{b}_{+k}f_k(V) + \hat{b}^\dagger_{+k}f^*_k(V)\right], \quad \hat{\varphi}_-(U) = \int dk\left[\hat{b}_{-k}f_k(U) + \hat{b}^\dagger_{-k}f^*_k(U)\right] \tag{1.21}$$

with

$$f_k(V) = \frac{1}{\sqrt{4\pi k}}e^{-ikV}. \tag{1.22}$$

Since the RR and LR are causally disjoint, the fields φ_+ and φ_- are independent with each other, allowing us to work one at a time. Let us work with the left moving $\varphi_+(V)$, the other one is completely analog. As usual, the $\hat{b}_{+k}$ defines the Minkowski vacuum i.e. $|0_M\rangle \neq 0$ by $\hat{b}_{+k}|0_M\rangle = 0$ for all $k > 0$.

On the other hand, in the LR wedge with coordinates (τ, ξ, x, y), equation (1.19) give us

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \xi^2}\right)\hat{\varphi} = 0$$

by Lorentz covariance. Let $v = \tau + \xi$, $u = \tau - \xi$, $\overline{v} = \overline{\tau} - \overline{\xi}$ and $\overline{u} = \overline{\tau} + \overline{\xi}$, where (τ, ξ) and $(\overline{\tau}, \overline{\xi})$ are the RR and LR wedge coordinates defined earlier. Notice that the variables u, v and $\overline{v}$ and U, V are related by

$$U = t - z = -a^{-1}e^{-au}, \quad V = t + z = a^{-1}e^{av}, \quad V = -a^{-1}e^{-a\overline{v}}.$$

Again, we can separate the field $\varphi(x)$ into the left and right moving fields, in this case, $\hat{\varphi}_+(V)$ can be expressed in the LR wedge with the condition $V < 0 < U$ and in the RR wedge with $U < 0 < V$.

Hence, we have

$$\begin{aligned}\hat{\varphi}(x) &= \underbrace{\int d\omega \frac{1}{(4\pi\omega)^{1/2}}[\hat{a}^R_{+\omega}e^{-i\omega v} + \hat{a}^{R\dagger}_{+\omega}e^{i\omega v}]}_{\hat{\varphi}_+(V)} \\ &\qquad + \underbrace{\int d\omega \frac{1}{(4\pi\omega)^{1/2}}[\hat{a}^L_{-\omega}e^{-i\omega\overline{v}} + \hat{a}^{L\dagger}_{-\omega}e^{i\omega\overline{v}}]}_{\hat{\varphi}_-(U)} \\ &= \hat{\varphi}_+(V) + \hat{\varphi}_-(U),\end{aligned}$$

where

$$\hat{\varphi}_+(V) = \int d\omega \left[\hat{a}^R_{+\omega} g_\omega(v) + \hat{a}^{R\dagger}_{+\omega} g^*_\omega(v)\right], \quad \hat{\varphi}_-(U) = \int d\omega \left[\hat{a}^L_{-\omega} g_\omega(\overline{v}) + \hat{a}^{L\dagger}_{-\omega} g^*_\omega(\overline{v})\right] \tag{1.23}$$

with

$$g_\omega(v) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v}. \tag{1.24}$$

together with the usual commutator relations

$$[\hat{a}_{\pm\omega}, \hat{a}^\dagger_{\pm\omega'}] = \delta(\omega - \omega'), \quad [\hat{a}_{\pm\omega}, \hat{a}_{\pm\omega'}] = 0, \quad [\hat{a}^\dagger_{\pm\omega}, \hat{a}^\dagger_{\pm\omega'}] = 0. \tag{1.25}$$

We define the Rindler Vacuum $|0_R\rangle$ by $\hat{a}^R_{+\omega}|0_R\rangle = \hat{a}^L_{+\omega}|0_R\rangle = 0$ for all ω.

In summary, we have two different sets of creation operators $\{\hat{a}_{\pm\omega}\}_{\omega>0}$ and $\{\hat{b}_{\pm k}\}_{k>0}$. Let us look at the Bogolyubov coefficients; let $\alpha^R_{\omega k}, \beta^R_{\omega k}, \alpha^L_{\omega k}, \beta^L_{\omega k} \in \mathbb{C}$ be the Bogolyubov coefficients of the transformation from e^{-ikV} to $e^{-i\omega v}$, we have the relations

$$\theta(V) g_\omega(v) = \int \frac{d^3k}{(4\pi k)^3} \left(\alpha^R_{\omega k} e^{-ikV} + \beta^R_{\omega k} e^{ikV}\right), \tag{1.26}$$

$$\theta(-V) g_\omega(\overline{v}) = \int \frac{d^3k}{(4\pi k)^3} \left(\alpha^L_{\omega k} e^{-ikV} + \beta^L_{\omega k} e^{ikV}\right) \tag{1.27}$$

where we use the fact that $V > 0 > U$ lies on the RR wedge and $V < 0 < U$ in the left and θ is the Heaviside step function. Using the orthogonality of the eigenfunctions $e^{ikV}/2\pi$,

$$\begin{aligned}
\int \frac{dV}{2\pi} \theta(V) g_\omega(v) e^{ikV} &= \int \frac{dV}{2\pi} e^{ikV} \int \frac{d^3k}{(4\pi k)^3} \left(\alpha^R_{\omega k} e^{-ikV} + \beta^R_{\omega k} e^{ikV}\right) \\
&= \int\int \frac{d^3k}{(4\pi k)^3} \frac{dV}{2\pi} e^{ikV} \left(\alpha^R_{\omega k} e^{-ikV} + \beta^R_{\omega k} e^{ikV}\right) \\
&= \alpha^R_{\omega k} \frac{1}{(4\pi k)^{3/2}}.
\end{aligned}$$

Hence, using the relation between v and V, $g_\omega(v) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v} = \frac{1}{\sqrt{4\pi\omega}} (aV)^{-i\omega/a}$ then,

$$\alpha^R_{\omega k} = \sqrt{\frac{k}{\omega}} \int_0^\infty \frac{dV}{2\pi} (aV)^{-i\omega/a} e^{ikV}. \tag{1.28}$$

Notice equation (1.28) has divergence while integrating on k. For this reason, we introduce the cut-off

$$\delta(x) = \int \frac{dx}{2\pi} e^{ikx - \varepsilon|x|} = \frac{1}{2\pi i}\left[\frac{1}{k - i\varepsilon} + \frac{1}{k + i\varepsilon}\right].$$

Then, letting $V \to V + i\varepsilon$ and performing a Wick rotation, $V = ix/k$ we have

$$\begin{aligned}
\alpha^R_{\omega k} &= \frac{ie^{\pi\omega/2a}}{\sqrt{\omega k}} \left(\frac{a}{k}\right)^{-i\omega/a} \int_0^\infty \frac{dx}{2\pi} x^{-i\omega/a} e^{-x}\, dx \\
&= \frac{ie^{\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{-i\omega/a} \Gamma(1 - i\omega/a),
\end{aligned}$$

which gives the first Bogolyubov coefficient. By replacing e^{ikV} for e^{-ikV} the coefficients $\beta^R_{\omega k}$ are also given by

$$\begin{aligned}\beta^R_{\omega k} &= -\frac{ie^{-\pi\omega/2a}}{\sqrt{\omega k}}\left(\frac{a}{k}\right)^{-i\omega/a}\int_0^\infty \frac{dx}{2\pi}x^{-i\omega/a}e^{-x}\,dx \\ &= -\frac{ie^{-\pi\omega/2a}}{2\pi\sqrt{\omega k}}\left(\frac{a}{k}\right)^{-i\omega/a}\Gamma(1-i\omega/a).\end{aligned}$$

Similarly, one can find the coefficients $\alpha^L_{\omega k}$ and $\beta^L_{\omega k}$. Therefore, one can show that the Bogolyubov coefficients are given by

$$\alpha^R_{\omega k} = \frac{ie^{\pi\omega/2a}}{2\pi\sqrt{\omega k}}\left(\frac{a}{k}\right)^{-i\omega/a}\Gamma(1-i\omega/a), \tag{1.29}$$

$$\beta^R_{\omega k} = -\frac{ie^{-\pi\omega/2a}}{2\pi\sqrt{\omega k}}\left(\frac{a}{k}\right)^{-i\omega/a}\Gamma(1-i\omega/a), \tag{1.30}$$

$$\alpha^L_{\omega k} = -\frac{ie^{\pi\omega/2a}}{2\pi\sqrt{\omega k}}\left(\frac{a}{k}\right)^{i\omega/a}\Gamma(1+i\omega/a), \tag{1.31}$$

$$\beta^L_{\omega k} = \frac{ie^{-\pi\omega/2a}}{2\pi\sqrt{\omega k}}\left(\frac{a}{k}\right)^{i\omega/a}\Gamma(1+i\omega/a). \tag{1.32}$$

Furthermore, we immediately identify the relation

$$\beta^L_{\omega k} = -e^{-\pi\omega/a}\alpha^{R*}_{\omega k}, \quad \beta^R_{\omega k} = -e^{-\pi\omega/a}\alpha^{L*}_{\omega k}. \tag{1.33}$$

Now, define a new set of modes os the K-G equation, G_ω and $\overline{G}_\omega$ described by

$$G_\omega(V) := \theta(V)g_\omega(v) + \theta(-V)e^{-\pi\omega a}g^*_\omega(\bar{v}) \tag{1.34}$$
$$\overline{G}_\omega(V) := \theta(V)e^{-\pi\omega a}g^*_\omega(v) + \theta(-V)g_\omega(\bar{v}). \tag{1.35}$$

As we pointed out earlier (cf. Remark 1.1.3), a positive frequency solution gives rise to the same vacuum as the one defined by g_ω thus, we need only to find that G_ω is indeed a positive frequency solution of the K-G equation. A positive frequency solution of the K-G equation is analytic in the lower half plane on the complex V plane. Thus, solution $g_\omega(v) = (4\pi\omega)^{-1/2}(V)^{-i\omega/a}, V > 0$ should be continued to the negative real line avoiding the singularity $V = 0$ around a small circle in the lower half plane, leading to $(4\pi\omega)^{-1/2}e^{-\pi\omega/a}(-V)^{-i\omega/a}, V < 0$. Equation (1.34) is exactly this analytic continuation of $g_\omega(V)$ and therefore, G_ω must be solutions of positive frequency [7] and therefore, if the field is written with this basis, the operators associated with these frequencies define the same vacuum.

[7] This argument was the original proposed by Unruh [15]

Let us see the associated operators. Using G_ω and $\overline{G}_\omega$, equation (1.27) can be inverted as

$$\begin{aligned} \theta(V) g_\omega(v) &\propto G_\omega(V) - e^{-\pi\omega/a}\bar{G}^*_\omega(V), \\ \theta(-V) g_\omega(\bar{v}) &\propto \bar{G}_\omega(V) - e^{-\pi\omega/a} G^*_\omega(V). \end{aligned}$$

Next, if we replace back on φ,

$$\begin{aligned} \hat{\varphi}_+(V) &= \int_0^\infty d\omega\ \theta(V)[\hat{a}^R_{+\omega} g_\omega(v) + \hat{a}^{R\dagger}_\omega g^*_\omega(v)] + \theta(-V)[\hat{a}^L_{+\omega} g_\omega(\bar{v}) + \hat{a}^{L\dagger}_\omega g^*_\omega(\bar{v})] \\ &= \int_0^\infty d\omega\ \hat{a}^R_{+\omega}(G_\omega(V) - e^{-\pi\omega/a}\bar{G}^*_\omega(V)) + \hat{a}^{R\dagger}_\omega(G^*_\omega(V) - e^{\pi\omega/a}\bar{G}_\omega(V)) + \\ &\qquad \hat{a}^L_{+\omega}(\bar{G}_\omega(V) - e^{-\pi\omega/a} G^*_\omega(V)) + \hat{a}^{L\dagger}_\omega(\bar{G}^*_\omega(V) - e^{\pi\omega/a} G_\omega(V)) \\ &= \int_0^\infty d\omega\ G_\omega(V)(\hat{a}^R_{+\omega} - e^{\pi\omega/a}\hat{a}^{L\dagger}_\omega) + \bar{G}^*_\omega(V)(-e^{-\pi\omega/a}\hat{a}^R_{+\omega} + \hat{a}^{L\dagger}_\omega) \\ &\qquad + \bar{G}_\omega(V)(-e^{\pi\omega/a}\hat{a}^{R\dagger}_\omega + \hat{a}^L_{+\omega}) + G^*_\omega(V)(-e^{-\pi\omega/a}\hat{a}^L_{+\omega} + \hat{a}^{R\dagger}_\omega). \end{aligned}$$

Since $G_\omega(V)$ and $\bar{G}_\omega(V)$ are positive frequencies, Minkowski vacuum is defined as the unique state $|0_M\rangle$ such that for all ω,

$$(\hat{a}^R_{+\omega} - e^{\pi\omega/a}\hat{a}^{L\dagger}_\omega)\,|0_M\rangle = 0, \tag{1.36}$$

$$(\hat{a}^L_{+\omega} - e^{\pi\omega/a}\hat{a}^{R\dagger}_\omega)\,|0_M\rangle = 0. \tag{1.37}$$

Recall that

$$[\hat{a}^{R/L}_{+\omega}, \hat{a}^{R/L\dagger}_{+\omega'}] = \delta(\omega - \omega'), \quad [\hat{a}^{R/L}_{+\omega}, \hat{a}^{R/L}_{+\omega'}] = 0, \quad [\hat{a}^{R/L\dagger}_{+\omega}, \hat{a}^{R/L\dagger}_{+\omega'}] = 0.$$

Finally, we apply $\hat{a}^{R\dagger}_{+\omega}$ and $\hat{a}^{L\dagger}_{+\omega}$ into (1.36) and (1.37) respectively. Thus, the number of particles with frequency ω on the RR and LR are

$$\langle 0_M|\,\hat{a}^{R\dagger}_{+\omega}\hat{a}^R_{+\omega}\,|0_M\rangle = e^{-2\pi\omega/a}\,\langle 0_M|\,\hat{a}^{L\dagger}_{+\omega}\hat{a}^L_{+\omega}\,|0_M\rangle + e^{-2\pi\omega/a}. \tag{1.38}$$

The symmetry of the R and L operators imply that the number of particles on the right and left wedges must be equal and therefore, using (1.38) the expected value is

$$\langle 0_M|\,\hat{a}^{R\dagger}_{+\omega}\hat{a}^R_{+\omega}\,|0_M\rangle = \langle 0_M|\,\hat{a}^{L\dagger}_{+\omega}\hat{a}^L_{+\omega}\,|0_M\rangle = \frac{1}{e^{2\pi\omega/a} - 1}. \tag{1.39}$$

Hence, the expectation value of the Rindler-particle number is that of a Bose-Einstein particle in a thermal bath of temperature $T = a/2\pi$. It can also be expressed without discretization by

$$\hat{a}^{R/L}_{+f} = \int_0^\infty d\omega\, f(\omega)\hat{a}^{R/L}_{+\omega}$$

with $f(\omega)$ being the density of states ($\int_0^\infty |f(\omega)|^2 = 1$). Then, the expected value of the number of particles at a frequency f is

$$\langle 0_M | \hat{a}^{R/L\dagger}_{+f} \hat{a}^{R/L}_{+f} | 0_M \rangle = \int_0^\infty d\omega\, f(\omega) f^*(\omega) \langle 0_M | \hat{a}^{R/L\dagger}_{+\omega} \hat{a}^{R/L}_{+\omega} | 0_M \rangle$$
$$= \int_0^\infty d\omega \frac{|f(\omega)|^2}{e^{2\pi\omega/a} - 1}.$$

However, showing that the expectation values are those in a thermal state is not enough to conclude the nature of the state. A sufficient and necessary condition is that the probability of each, right and left Rindler energy eigenstate, corresponds to a canonical ensemble if the other Rindler wedge is disregarded. Equations (1.36) and (1.37) imply that $(\hat{a}^{R\dagger}_\omega \hat{a}^R_\omega - \hat{a}^{L\dagger}_\omega \hat{a}^L_\omega) |0_M\rangle = 0$ which confirms that number of particles in the right and left sides of Rindles wedges are the same. Allowing us to write

$$|0_M\rangle = N \prod_i \sum_{n_i=0}^\infty \frac{K_{n_i}}{n_i!} (\hat{a}^{R\dagger}_{+\omega_i} \hat{a}^{L\dagger}_{+\omega_i})^{n_i} |0_R\rangle, \quad K_{n_i+1} - e^{-\pi\omega_i/a} K_{n_i} = 0,$$

where N is the normalization constant. The recursive condition K_n give us $K_{n_i} = e^{-\pi n_i \omega_i/a} K_0$ and therefore, the Minkowski vacuum can be written as

$$|0_M\rangle = \prod_i C_i \sum_{n_i=0}^\infty e^{-\pi n_i \omega_i/a} |n_i, R\rangle \otimes |n_i, L\rangle, \quad C_i = \sqrt{1 - \exp(-2\pi\omega_i/a)}. \tag{1.40}$$

Then, the Minkowski vacuum has a density matrix (tracing out the left Rindler states)

$$\hat{\rho}_R = \prod_i \left(C_i^2 \sum_{n_i=0}^\infty e^{-2\pi n_i \omega_i/a} |n_i, R\rangle \langle n_i, R| \right) \tag{1.41}$$

which is exactly the density matrix of the system of free bosons with temperature $T = a/2\pi$. Similarly, the LR wedge give the same result. Thus, we can state the Unruh effect as follows:

The Unruh effect

The Unruh effect is the effect in which the Minkowski vacuum restricted to the left(right) Rindler wedge is a thermal state with temperature $T = a/2\pi$ with the boost generator normalized on $z^2 - t^2 = 1/a^2$ as the Hamiltonian.

1.3 The Unruh-DeWitt detector (UDW)

"Measure what can be measured, and make measurable what cannot be measured."

Galileo Galilei

The Unruh-DeWitt (UDW) detector, introduced by DeWitt in 1979 is a detector that tells if the system has detected a particle (in a thermal state) or not. That is, a two-level point monopole[8]. The

[8]A monopole (in analogy of electrodynamics) describes the system as coarser as it could be (and finer when

Unruh thermal bath is as isotropic as a thermal bath in equilibrium on a general static space-time, which means that Killing observers will see no net energy flux in any spatial direction. However, it is essential to clarify that predictions of the thermal state do not present ambiguity on the measurement but rather the interpretations of the measuring is what historically has been the source of discussion [8]. Audretsch and Müller [16] in 1994 argue that different results are, in fact, different measurements. Let us introduce the UDW detector formally.

We consider a two-level Unruh-DeWitt detector in a Minkowski space-time represented by the Hermitian operator μ_0 acting on a two-dimensional Hilbert space $\mathcal{H}_D$ whose basis elements are denoted as $|0\rangle$ and $|1\rangle$ and satisfies $\mu_0 |0\rangle = |1\rangle$, $\mu_0 |1\rangle = |0\rangle$.

Consider the Hamiltonian of the detector as $H_D |1\rangle = \frac{E}{2}, H_D |0\rangle = -\frac{E}{2} |0\rangle$ thus, the energy gap between the two states is E. The Hilbert space of the whole system (the system and the probe) is $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_D$ where $\mathcal{H}_S$ is the Hilbert space of the field. Associated with the Hilbert space $\mathcal{H}$ is the Hamiltonian of the system given by

$$H = \underbrace{H_S \otimes 1 + 1 \otimes H_D}_{H_0} + H_{\text{int}}, \quad H_{\text{int}} = \lambda\rho(\tau)\varphi(x(\tau)) \otimes \mu(\tau) \tag{1.42}$$

with $\mu(\tau) = e^{iH_D\tau}\mu(0)e^{-iH_D\tau}$, where τ is the proper time of the detector, $x(\tau)$ its world-line, λ the coupling constant and ρ a switch function compactly supported that controls the coupling region. Assuming the interaction picture then, the evolution operator U_{int} with the Dyson series is

$$\begin{aligned} U_{\text{int}} &= \mathcal{T} e^{-i\int d\tau H_{int}(\tau)} \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d\tau_1 \dots d\tau_n \mathcal{T}(H_{int}(\tau_1)\dots H_{int}(\tau_n)). \end{aligned}$$

In order to find the evolution operator, let us first take a look at the time-ordered "n-point function" $\mathcal{T}\langle 1|\mu(\tau_1)...\mu(\tau_n)|0\rangle$ where $\mu(\tau)$ is the evolution of the monopole given by the Heisenberg picture:

$$\mu(\tau) = e^{iH_D\tau}\mu(0)e^{-iH_D\tau}, \quad \mu(0) := \mu_0.$$

In this way, the time-order n-point function is:

$$\begin{aligned} \mathcal{T}\langle 1|\mu(\tau_1)...\mu(\tau_n)|0\rangle &= \mathcal{T}\langle 1| e^{iH_D\tau_1}\mu(0)e^{-iH_D\tau_1}...e^{iH_D\tau_n}\mu(0)e^{-iH_D\tau_n}|0\rangle \\ &= \mathcal{T}\langle 1| e^{iH_D\tau_1}\mu(0)\left(\prod_{i=1}^{n-1} e^{-iH_D(\tau_i-\tau_{i+1})}\mu(0)\right)e^{-iH_D\tau_n}|0\rangle \\ &= \mathcal{T}\langle 1| e^{i\frac{E}{2}\tau_1}\mu(0)\left(\prod_{i=1}^{n-1} e^{-iH_D(\tau_i-\tau_{i+1})}\mu(0)\right)e^{i\frac{E}{2}\tau_n}|0\rangle \\ &= \mathcal{T} e^{i\frac{E}{2}(\tau_1+\tau_n)}\langle 1|\mu(0)\left(\prod_{i=1}^{n-1} e^{-iH_D(\tau_i-\tau_{i+1})}\mu(0)\right)|0\rangle, \end{aligned}$$

where $\prod_{i=1}^{n} A_i = A_1A_2...A_n$. Notice that for each $\mu(0)$ there is a permutation interchanging the vectors

increasing the degree of the pole). For example, $\mu = \sigma^- + \sigma^+$ where $\sigma^\pm$ are ladder operators determines only two possible outcomes.

$|0\rangle$ with $|1\rangle$. Therefore, for n even, the contribution vanishes (as it give a $\langle 0|1\rangle$ value) and for n odd we have

$$
\begin{aligned}
&= \mathcal{T} e^{i\frac{E}{2}(\tau_1+\tau_n)} \langle 1|\, \mu(0) \left(\prod_{i=1}^{n-1} e^{-iH_D(\tau_i-\tau_{i+1})} \mu(0) \right) |0\rangle \\
&= \mathcal{T} e^{i\frac{E}{2}(\tau_1+\tau_n)} \langle 0| \left(\prod_{i=1}^{n-2} e^{-iH_D(\tau_i-\tau_{i+1})} \mu(0) \right) e^{-iH_D(\tau_{n-1}-\tau_n)} \mu(0)\, |0\rangle \\
&= \mathcal{T} e^{i\frac{E}{2}(\tau_1+\tau_n)} \langle 0| \left(\prod_{i=1}^{n-2} e^{-iH_D(\tau_i-\tau_{i+1})} \mu(0) \right) e^{-i\frac{E}{2}(\tau_{n-1}-\tau_n)} \,|1\rangle \\
&= \mathcal{T} e^{i\frac{E}{2}(\tau_1-\tau_{n-1}+2\tau_n)} \langle 0| \left(\prod_{i=1}^{n-2} e^{-iH_D(\tau_i-\tau_{i+1})} \mu(0) \right) |1\rangle \,.
\end{aligned}
$$

Inductively, we find

$$
\mathcal{T} \langle 1|\, \mu(\tau_1) ... \mu(\tau_n)\, |0\rangle = \mathcal{T} e^{i\frac{E}{2}(2\tau_1-2\tau_2+...-2\tau_{n-1}+2\tau_n)} \langle 1|\, \mu_0\, |0\rangle \tag{1.43}
$$

$$
= e^{iE(\tau_1-\tau_2+...-\tau_{n-1}+\tau_n)} \langle 1|\, \mu_0\, |0\rangle \,. \tag{1.44}
$$

Now, we are in position to compute the expected value of detecting a particle given an arbitrary state φ i.e., we write $|0,0_M\rangle =: |0\rangle \otimes |0_M\rangle$ for the initial state and $|1,\varphi\rangle$, where $|\varphi\rangle$ denotes an arbitrary state of the field, for the final state. Then, we have

$$
\begin{aligned}
\mathcal{A}_{0\to 1} &= \langle 1, \varphi|\, U_{int}\, |0, 0_M\rangle \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d\tau_1 \dots d\tau_n \Big[\lambda^n \rho(\tau_1) \dots \rho(\tau_n) \langle \varphi|\, \mathcal{T}(\varphi(x(\tau_1)) \dots \\
&\qquad \times \varphi(x(\tau_n)))\, |0_M\rangle \times \mathcal{T} \langle 1|\, e^{-iH_D(\tau_1-\tau_2)} \mu \dots e^{-iH_D(\tau_{n-1}-\tau_n)}\, |0\rangle \Big] \\
&= \sum_{n \text{ odd}}^{\infty} \lambda^n \frac{(-i)^n}{n!} \int d\tau_1 \dots d\tau_n \rho(\tau_1) \dots \rho(\tau_n) \\
&\qquad \times \langle \varphi|\, \mathcal{T}(\varphi(x(\tau_1)) \dots \varphi(x(\tau_n)))\, |0_M\rangle\, e^{iE(\tau_1-\tau_2+...-\tau_{n-1}+\tau_n)} \langle 1|\, \mu_0\, |0\rangle \,.
\end{aligned}
$$

Thus, the transition probability is given by tracing over the field final states $|\varphi\rangle$

$$
\begin{aligned}
P_{0\to 1} &= \int D\varphi |\, \langle 1, \varphi|\, U_{\text{int}}\, |0, 0_M\rangle\, |^2 \\
&= |\, \langle 1|\, \mu(0)\, |0\rangle\, |^2 \sum_{n,m\, odd}^{\infty} \lambda^{n+m} \frac{(-i)^{n-m}}{n!m!} \int d\sigma_1 \dots d\sigma_m d\tau_1 \dots d\tau_n \\
&\quad \times \Big[\rho(\sigma_1) \dots \rho(\sigma_m) \rho(\tau_1) \dots \rho(\tau_n) \times \langle 0_M|\, \mathcal{T}(\varphi(x(\sigma_1)) \dots \varphi(x(\sigma_n)))^\dagger \\
&\qquad \times \underbrace{\int d\varphi\, |\varphi\rangle \langle \varphi|}_{1}\, \mathcal{T}(\varphi(x(\tau_1)) \dots \varphi(x(\tau_n)))\, |0_M\rangle \\
&\qquad\qquad \times e^{Ei(\tau_1-\tau_2 \cdots +\tau_n)} e^{-Ei(\sigma_1-\sigma_2 \cdots +\sigma_m)} \Big]
\end{aligned}
$$

$$\begin{aligned}P_{0\to1} = |\langle 1|\,\mu(0)\,|0\rangle|^2 \sum_{n,m\,odd}^{\infty} \lambda^{n+m}\frac{(-i)^{n-m}}{n!m!}\int d\sigma_1\dots d\sigma_m d\tau_1\dots d\tau_n \\ \times\Big[\rho(\sigma_1)\dots\rho(\sigma_m)\rho(\tau_1)\dots\rho(\tau_n)\,\langle 0_M|\,\mathcal{T}(\varphi(x(\sigma_1))\dots\varphi(x(\sigma_n)))^\dagger \\ \times\mathcal{T}(\varphi(x(\tau_1))\dots\varphi(x(\tau_n)))\,|0_M\rangle\, e^{Ei(\tau_1-\tau_2\cdots+\tau_n)}e^{-Ei(\sigma_1-\sigma_2\cdots+\sigma_m)}\Big].\end{aligned}$$

That is,

$$P_{0\to1} = \lambda^2|\langle 1|\,\mu(0)\,|0\rangle|^2 \int d\sigma d\tau\rho(\tau)\rho(\sigma)\underbrace{\langle 0_M|\,\varphi(x(\sigma))\varphi(x(\tau))\,|0_M\rangle}_{\mathcal{W}(x(\sigma),x(\tau))}e^{Ei(\tau-\sigma)}+O(\lambda^4). \tag{1.45}$$

Definition 1.3.1. *Let $f: M\to\mathbb{C}$ be a smooth function supported on $\mathcal{O}\subset M$. $\mathcal{O}$ is a bounded open region in the space-time and $\varphi(x)$ a field. f is said to be a test function and the smeared field associated is*

$$\varphi(f) := \int dx\,\varphi(x)f(x).$$

Definition 1.3.2. *Notice that the correlation function of* $\mathbf{n}$ *smeared fields is a multilinear functional [17] $(f_1,...,f_n)\to\langle 0_M|\,\varphi(f_1)...\varphi(f_n)\,|0_M\rangle$. Using the Schwarz kernel theorem, the multilinear functional has a kernel W, such that*

$$\langle 0_M|\,\varphi(f_1)...\varphi(f_n)\,|0_M\rangle = \int f(x_1)...f(x_n)W(x_1,...,x_n)dx_1...dx_n.$$

W is called the n-point Wightman function of φ. Observe that the Wightman function is

$$W = \langle 0_M|\phi(x_1)...\phi(x_n)|0_M\rangle.$$

Remark 1.3.1. *$\mathcal{W}$ in equation* (1.45) *is the pull-back of the Wightman function of φ which is exactly the final result on [4],*

$$P_{0\to1} = \lambda^2\int d\tau d\tau' e^{-iE(\tau-\tau')}\rho(\tau)\rho(\tau')\,\mathcal{W}(\gamma(\tau),\gamma(\tau'))+O(\lambda^4) \tag{1.46}$$

for some smooth function $\tilde{\rho}$ compactly supported within a compact region $K\subset M$. This result will connect the measurement developed in section 4 and the Unruh detector because in section 4, we work with smeared fields $\varphi(f)$ rather than traditional fields $\varphi(x)$.

2 Quantum Measurement Theory: From von Neumann to Sorkin's protocol

Many measurement schemes are based on the spirit of the von Neumann scheme [18]. The measurement presented in chapter 4 that allows measurements in QFT is no exception. In order to familiarize ourselves with measurement theory, we give a brief introduction. For this purpose, we follow the classical book of Busch [2, 19] and the concise exposition of Montes [18].

2.1 The von Neumann measurment process

Let $\mathcal{H}_S, \mathcal{H}_D$ be two Hilbert spaces. We want to measure an observable that lies in $S = (\mathcal{H}_S, \mathfrak{B}(\mathcal{H}_S))$, where $\mathcal{H}_S$ is the Hilbert space of the theory and $\mathfrak{B}(\mathcal{H})$ are the bounded operators acting on the elements of $\mathcal{H}_S$. The measure is then realized with the help of a external system denoted by $D = (\mathcal{H}_D, \mathfrak{B}(\mathcal{H}_D))$ which is said to be the probe. We want to measure an observable $A_s \in \mathfrak{B}(\mathcal{H})$ with the help of a probe that belongs to system D. Following [18], the measurement process of an observable $A_s \in \mathfrak{B}(\mathcal{H})$ can be summarized by the following rules:

1. The interaction takes a finite interaction time T.
2. The measurement is a quantum process, driven by an interaction Hamiltonian H_{int} and the system evolves in time with the Schrodinger equation.
3. The measurement updates the state of the probe system.
4. If the system S is in an eigenstate of the observable A_s and the interaction time is small enough, we can assume that the measurement interaction does not affect the value of the observable, i.e.
$$\frac{dA_s}{dt} = \frac{i}{\hbar}[H_{int}, A_s] = 0$$
5. T can be as brief as wanted.

Rules 1-5 are known as the von Neumann measurement scheme. Let us see how the measurement works. Prepare P in the state $\hat{\rho}_D$. The system S is in a state ρ_i, possibly unknown. The complete system is described in terms of the Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_D$. Thus, before the measurement, the state is $\hat{\rho} = \hat{\rho}_i \otimes \hat{\rho}_D$.

We want to infer A_s from the probe D and therefore, we want a correlation between A_S and some property B_D of the system D. The initial state $\hat{\rho}$ is evolved into the final one as usual in the interaction picture,
$$\hat{\rho}' = \hat{U}_{int}\hat{\rho}_i \otimes \hat{\rho}_D \hat{U}_{int}^{\dagger}, \quad \hat{U}_{int} = e^{-\frac{i}{\hbar}\int_0^T \hat{H}_{int}(t)\,dt}.$$

The final state of the probe is obtained by tracing out the system S, i.e., the marginal density

$$\hat{\rho}'_D = \mathrm{Tr}_S(\hat{\rho})'.$$

Finally, the measurement of the probe B_D allows us to identify the result (if possible), call it b_a of the probe with the state a of the system S and therefore, a measure of A_S with result a has been made. This, of course, is only possible if the interaction leads to a appropriate correlations between system and probe and the probe is suitable prepared.

Example 2.1.1. *The probe D is described by a quantum system of one degree of freedom with a canonical pair of observables (q_D, p_D), that can be thought of as position and momentum observables. The probe observable is chosen to be $B_D = p_D$ and the Hamiltonian of interaction is*

$$H_{int}(t) = \rho(t)\hat{A}_s \otimes \hat{p}_D, \quad \rho = \lambda \in \mathbb{C}$$

with ρ constant with support $\operatorname{supp}\rho = [0, T]$. Write $A_S = \sum_i a_i |a_i\rangle\langle a_i|$ then, the evolution $\hat{U}$ is given by

$$\hat{U}_{int} = \exp\left\{-\frac{i}{\hbar}\int_0^T \lambda \sum_i a_i |a_i\rangle\langle a_i| \otimes \hat{p}_M \, dt\right\} = \sum_i |a_i\rangle\langle a_i| \otimes e^{-\frac{i}{\hbar}\lambda T a_i \hat{p}_D}.$$

If we prepare the probe with a wave equation $\psi_0(q)$ and let $\rho_i = |\phi\rangle\langle\phi|$, $|\phi\rangle = \sum_i \alpha_i |a_i\rangle$ denote the initial state (possibly unknown) of the system S then, we have

$$\begin{aligned}\hat{U}_{int}(|\phi\rangle \otimes |\psi(q)\rangle) &= \sum_i \alpha_i |a_i\rangle e^{-\frac{i}{\hbar}\lambda T a_i \hat{p}_D} |\psi(q)\rangle = \sum_i \alpha_i |a_i\rangle \otimes |\psi(q - \lambda T a_i)\rangle \\ &= \sum_i \alpha_i |a_i\rangle \otimes |\psi_i(q)\rangle, \quad \psi_i(q) := \psi_0(q - \lambda T a_i),\end{aligned}$$

where we use the fact that $e^{-ix_0\hat{P}}$ is a translation operator, i.e. $e^{-ix_0\hat{P}}|\psi_0(q)\rangle = |\psi_0(q - x_0)\rangle$. Consequently, the state after the interaction will be $\hat{\rho}' = \sum_{i,j} \alpha_i\overline{\alpha_j} |a_i\rangle\langle a_j| \otimes |\psi_i(q)\rangle\langle\psi_j(q)|$ and hence, the state of the probe is updated to be

$$\hat{\rho}'_D = \mathrm{Tr}_S(\hat{\rho}') = \sum_i |\alpha_i|^2 |\psi_i(q)\rangle\langle\psi_i(q)|. \tag{2.1}$$

Remark 2.1.1. *A (projective) measurement of q_D will allow us to infer the value of A_D, provided the uncertainty Δq is small enough ($\Delta q << |\lambda T(a_i - a_j)|$), as shown in the figure 4.*

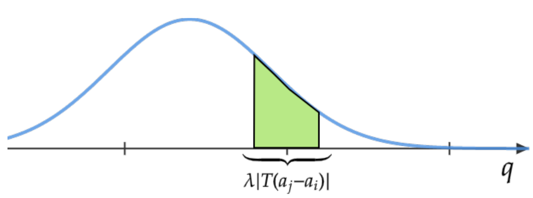

Figure 4: Probability distribution in function of the different values of q. If the uncertainty $|\lambda T(a_i - a_j)|$ around q_i is small enough so that doesn't overlap $q_{i\pm1}$ then, one can determine the value of q.

2.2 Measurements in Quantum Field Theory?

What hapens if we apply the measurment described above in a quantum field theory? Let us see by means of an example,

Example 2.2.1 (Sorkin's protocol)**.** *Sorkin's protocol is a measurement example on a quantum field that illustrates the problem discussed above. We will follow Sorkin's formulation in his article "Impossible measurements on quantum fields" [6]. We use standard terminology for geometric concepts such as causal sets, future and past directing curves etc. This terminology is briefly described (if needed) however, see Appendix E for details.*

Consider the following setup, let $\mathcal{O}_i$, $i = 1,2,3$ be three open regions in a space-time M. Denote J_i the casual set (past and future causal sets) of the region $\mathcal{O}_i$, for example, region J_1 is shown in the figure 5. And, suppose $\mathcal{O}_1$ be future-causally disjoint of $\mathcal{O}_3$ whereas $\mathcal{O}_2$ intersect some causal

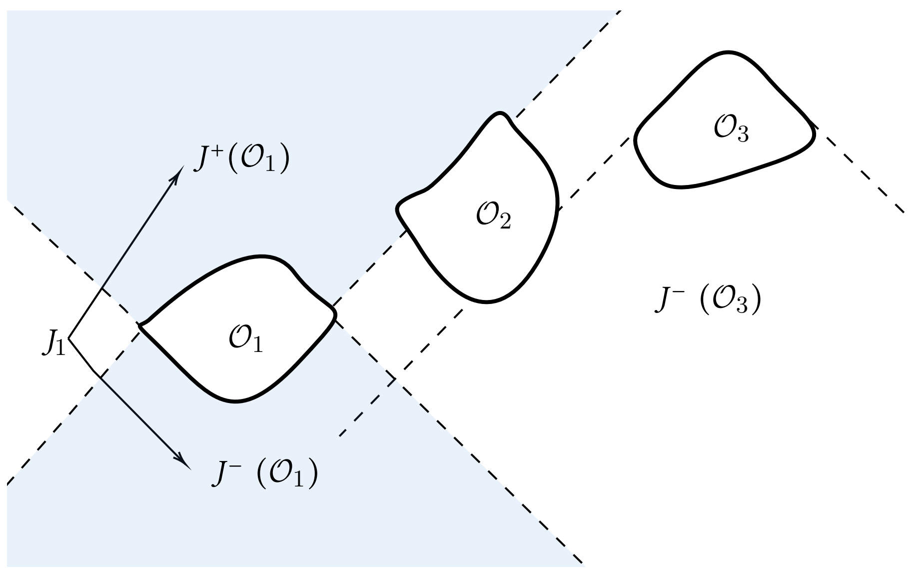

Figure 5: Sorkin's protocol: Regions $\mathcal{O}_1$ and $\mathcal{O}_2$ are future causal connected while regions $\mathcal{O}_1$ and $\mathcal{O}_3$ are not. Time is flowing upwards.

future/past of both $\mathcal{O}_1$ and $\mathcal{O}_3$, that is,

$$J_1^+ \cap \mathcal{O}_3 = \varnothing, \quad J_1^+ \cap \mathcal{O}_2 \neq \varnothing \neq \mathcal{O}_2 \cap J_2^-$$

where $J_i^{\pm}$ denotes the causal future (+) or causal past (-) of the region $\mathcal{O}_i$.

Let $\phi(x)$ be a free field. In the region $\mathcal{O}_1$ Alice performing an operation with the unitary $U = e^{i\lambda\phi(y)}$ and, in the region $\mathcal{O}_2$ Bob can perform a a projection $B = |b\rangle\langle b|, \quad |b\rangle = \alpha|0\rangle + \beta|1\rangle$. Start with the density state $\rho_0 = |0\rangle\langle 0|$. First, in the region $\mathcal{O}_1$ perform the evolution U, which gives the state ρ,

$$\rho_0 \longrightarrow \rho = U\rho_0 U^*. \tag{2.2}$$

Secondly, in the region $\mathcal{O}_2$ perform a quantum operation with the "projectos" Λ_i, $i = 1, 2$ defined as $\Lambda_1 = B, \quad \Lambda_2 = 1 - B,$

$$\rho \longrightarrow \rho' = \Lambda_1 \rho \Lambda_1^* + \Lambda_2 \rho \Lambda_2^*. \tag{2.3}$$

Finally, in the region $\mathcal{O}_3$, we measure $\phi(x)$,

$$\rho' \longrightarrow \omega_\rho(\phi(x)) := \mathrm{Tr}(\rho' \phi(x)). \tag{2.4}$$

Since $\mathcal{O}_1$ and $\mathcal{O}_3$ are causally disjoint, given an evolution U on region $\mathcal{O}_2$, the measurement of equation (2.4) *must be independent of any action of U and therefore, in this case, of λ. Equation* (2.5) *implies that the derivative $\frac{d}{dt}\Big|_{\lambda=0} \mathrm{Tr}(\rho'\phi(x)) =: \mathcal{G}$ should vanish because if otherwise, the region $\mathcal{O}_3$ would receive information of $\mathcal{O}_2$ and thus leading to super-luminal information transmission however, proposition 2.2.1 below shows that $\mathcal{G}$ does not vanishes.*

Remark 2.2.1. *Proposition 2.2.1 establishes an apparent paradox which has been the subject of intense discussion recently [7, 20–22]. It is also one of the main motivations of Fewster and Verch [7] to propose a measuring scheme in relativistic quantum field theory that is consistent with the principles of causality.*

It is know clear that QFT and measurements don't get along. QFT does not give an explicit information of localization properties and therefore, we could put assumptions that are too restrictive or even impossible. Particularly, the Sorkin's protocol assumed the observables to be localized in a finite regions but, it turns out that projectors, as used in region $\mathcal{O}_2$, could not be localized in the setup as showed in the figure 5, see [21] for details. Therefore, QFT struggles to distinguish fields that should be localized in a region of a space time rather than the whole space [23]. One way to handle this situation that solves the locality problems is the algebraic quantum field theory (AQFT) approach because it handles observables locally and has the benefits of not having a representation fixed as we pointed out in Remark 3.1.4. Therefore, AQFT will be introduced in section 3, as a theory that will allow us to define measurements on quantum fields in section 4 and manages thermal states (as in the Unruh effect) mathematically rigorous. Let us finish this section by probing the apparent paradox in the standard language of QFT.

Proposition 2.2.1. *The measurement of the updated initial state in the Sorkin's protocol,*

$$\mathcal{G} := \frac{d}{dt}\bigg|_{\lambda=0} \mathrm{Tr}(\rho'\phi(x))$$

does not vanish and therefore, Charlie receives some information of a casually disconnected observer (Alice).

Proof.

$$
\begin{aligned}
\mathcal{G} &= \frac{d}{dt}\bigg|_{\lambda=0} \langle 0| \underbrace{U^*}_{e^{-i\lambda\phi(y)}} B\phi(x)BU|0\rangle \\
&\quad + \frac{d}{dt}\bigg|_{\lambda=0} \langle 0|U^*(1-B)\phi(x)(1-B)U|0\rangle \\
&= \frac{d}{dt}\bigg|_{\lambda=0} \langle 0|\big[B\phi(x)B - i\lambda[\phi(y),B\phi(x)B] + o(\lambda^2)\big]|0\rangle + \\
&\quad \frac{d}{dt}\bigg|_{\lambda=0} \langle 0|\big[(1-B)\phi(x)(1-B) - i\lambda[\phi(y),(1-B)\phi(x)(1-B)] + o(\lambda^2)\big]|0\rangle \\
&= -i\langle 0|[\phi(y),B\phi(x)B]|0\rangle - i\langle 0|[\phi(y),(1-B)\phi(x)(1-B)]|0\rangle \\
&= -i\langle 0|[\phi(y),2B\phi(x)B - \phi(x)B - B\phi(x) + \phi(x)]|0\rangle \\
&= -i\langle 0|[\phi(y),2B\phi(x)B - \phi(x)B - B\phi(x)]|0\rangle
\end{aligned}
$$

where we use the Baker-Campbell-Hausdorff (BCH) formula,

$$
e^{-i\lambda\phi(y)}Ae^{i\lambda\phi(y)} = A - i\lambda[\phi(y),A] + o(\lambda^2)
$$

on the second line and that $[\phi(y),\phi(x)] = 0$ on the fourth line. Notice that $\overline{\langle 0|AB|0\rangle} = \langle 0|BA|0\rangle$ thus, $\langle 0|[A,B]|0\rangle = \langle 0|AB|0\rangle - \overline{\langle 0|AB|0\rangle} = 2i\,\mathrm{im}\,\langle 0|AB|0\rangle$ consequently,

$$
\begin{aligned}
\frac{d}{dt}\bigg|_{\lambda=0} \mathrm{Tr}(\rho'\phi(x)) &= 2\,\mathrm{im}\,\langle 0|\phi(y)(2B\phi(x)B - \phi(x)B - B\phi(x))|0\rangle \\
&= 2\,\mathrm{im}[\langle 0|\phi(y)2B\phi(x)B|0\rangle - \langle 0|\phi(y)\phi(x)B|0\rangle \\
&\quad - \langle 0|\phi(y)B\phi(x)|0\rangle].
\end{aligned}
$$

Denote $\psi(x) := \langle 0|\phi(x)|1\rangle$, since $B = |b\rangle\langle b|$, $|b\rangle = \alpha|0\rangle + \beta|1\rangle$, the first term gives us

$$
\begin{aligned}
2\,\mathrm{im}\,\langle 0|\phi(y)2B\phi(x)B|0\rangle &= 2\,\mathrm{im}\,\langle 0|\phi(y)2|b\rangle\langle b|\phi(x)|b\rangle\langle b|0\rangle \\
&= 4\,\mathrm{im}[\alpha^*(\langle 0|\phi(y)|1\rangle\beta + \langle 0|\phi(y)|0\rangle\alpha) \\
&\qquad \times (\langle 0|\phi(x)|0\rangle|\alpha|^2 + \langle 0|\phi(x)|1\rangle\beta\alpha^* \\
&\qquad\quad + \langle 1|\phi(x)|0\rangle\alpha\beta^* + \langle 1|\phi(x)|1\rangle|\beta|^2)] \\
&= 4\,\mathrm{im}[\alpha^*(\psi(y)\beta + 0)(0 + \psi(x)\beta\alpha^* + \overline{\psi}(x)\alpha\beta^*)] \\
&= 4\,\mathrm{im}[(\alpha^*\beta)^2\psi(y)\psi(x) + |\alpha\beta|^2\psi(y)\overline{\psi}(x)].
\end{aligned}
$$

Likewise, the third term is

$$
\begin{aligned}
2\,\mathrm{im}\,\langle 0|\phi(y)B\phi(x)|0\rangle &= 2\,\mathrm{im}\,\langle 0|\phi(y)|b\rangle\langle b|\phi(x)|0\rangle \\
&= 2\,\mathrm{im}\,|0\rangle\phi(y)\beta|1\rangle\alpha^*\langle 1|\phi(x)|0\rangle = \beta\alpha^*\psi(y)\overline{\psi}(x).
\end{aligned}
$$

And the second term is

$$\begin{aligned} 2\,\mathrm{im}\,\langle 0|\,\phi(y)\phi(x)B\,|0\rangle &= 2\,\mathrm{im}\,\langle 0|\,\phi(y)\phi(x)(\alpha\,|0\rangle + \beta\,|1\rangle)\alpha^* \\ &= 2\,\mathrm{im}\,|\alpha|^2\,\langle 0|\,\phi(y)\phi(x)\,|0\rangle + \mathrm{im}\,\alpha^*\beta\,\underbrace{\langle 0|\,\phi(y)\phi(x)\,|1\rangle}_{0} \\ &= -i|\alpha|^2\,\langle 0|\underbrace{[\phi(y),\phi(x)]}_{0}|0\rangle = 0. \end{aligned}$$

Thus, back in $\left.\frac{d}{dt}\right|_{\lambda=0} \mathrm{Tr}(\rho'\phi(x))$ we have

$$\left.\frac{d}{dt}\right|_{\lambda=0} \mathrm{Tr}(\rho'\phi(x)) = 2\,\mathrm{im}[2(\alpha^*\beta)^2\psi(y)\psi(x) + (2|\alpha\beta|^2 - \beta\alpha^*)\psi(y)\overline{\psi}(x)].$$

If we choose $\alpha = \beta = 1/\sqrt{2}$ then,

$$\left.\frac{d}{dt}\right|_{\lambda=0} \mathrm{Tr}(\rho'\phi(x)) = \mathrm{im}[\psi(y)\psi(x)] \neq 0. \tag{2.5}$$

■

Remark 2.2.2. *There are, however, measurements that present apparent causality violations. For instance consider two electrons, the measurement of a electron with spin up conditioned that we have measured a spin down in the other is completely determined (in the opposite direction) and therefore, even causally disconnected regions of space time can transmit some information between them. The Sorkin's problem is not of this type because no information is pre-conditioned, the state of the system is only updated (with out post-selection) by the "Kraus operators"* Λ_i.

3 Algebraic Quantum Field Theory

As we briefly discussed in Chapter 1, Algebraic Quantum Field Theory (AQFT) is a satisfactory theory to define thermal states and is also a natural way of dealing with locality and causality, making AQFT a natural starting point for our foundational quest of a QFT measurement scheme and its relation with the Unruh effect. In this chapter we introduce the basic tools and ideas of AQFT as well as its relation with the Unruh effect.

Locality is the principle of field theories and, on the other hand correlations, for example entanglement in quantum states, often relies on non-locality principles. Thus, we may want to separate the construction of states from the construction of the observables in order to make a richer theory. Indeed, Algebraic Quantum Field Theory seeks to separate those constructions and, for instance, allows us to have more states than those in the traditional approach (density matrices). Historically, the concepts of algebraic quantum field theory were formally introduced by Wightmann (1950s) with the axiomatic setting of fields supported in local regions and later generalized by Haag [24] in 1957 with the so-called Haag-Kastler assumptions in which any idea of fields is forgotten and rather, observables became objects in an abstract algebra. It was until the formulation of the Haag-Ruelle scattering theory that put the eye on these algebraic constructions and later on, it was then made mathematically precise by using the theory of operator algebras, mainly by Araki [25]. Also, AQFT enriched quantum statistical mechanics through Tomita-Takesaki modular theory [26, 27].

In this chapter, we give the basic tools as well as the main theorems in order to understand the formalism of AQFT, KMS states and their relation to the Unruh effect. For this purpose we follow several references, [24–30].

3.1 Algebraic formalism in a nutshell

We assume the reader is familiar with general mathematical definitions of functional spaces, algebras and representations. If not familiar consult Appendix A and D for further definitions and examples.

Definition 3.1.1 ($*$ and C^* alegbras)**.** *A $*-$algebra is a complex algebra equipped with an involution $*$. A C^*-algebra is a Banach $*-$algebra such that*

$$||A^*A|| = ||A||^2, \quad A \in \mathscr{A}.$$

The 'C' stands for 'closed', first defined by Segal in 1947 as a "uniformly closed, self-adjoint algebra of bounded operators on a Hilbert space" [31] which of course relates with the above definition by means of representation theory.

Definition 3.1.2. *Let $\mathscr{A}$ be a $*-$subalgebra of the bounded linear operators on $\mathcal{H}$ denoted by $\mathfrak{B}(\mathcal{H})$. We say that $\mathscr{A}$ is a von Neumann algebra if $\mathscr{A}$ is a unital algebra and*

$$(\mathscr{A}')' = \mathscr{A}, \quad \mathscr{A}' = \{B \in \mathfrak{B}(\mathcal{H}) : [B, A] = 0, \forall A \in \mathscr{A}\},$$

where $\mathscr{A}'$ is called the commutant of $\mathscr{A}$. In other words, a von Neumann algebra is a $*-$subalgebra of $\mathfrak{B}(\mathcal{H})$ whose double commutant is itself. In general, a von Neumann algebra need not be unital but will ease computations.

Definition 3.1.3. *A state on a (unital) $*-$algebra $\mathscr{A}$ is a linear functional $\omega : \mathscr{A} \to \mathbb{R}$ such that*

$$\omega(AA^*) \geq 0, \quad \forall A \in \mathscr{A} \quad \text{and } \omega(1) = 1.$$

ω is said to be mixed if it is a convex combination of at least two states $\omega_1 \neq \omega_2$. If ω is not mixed, then ω is said to be pure.

States and observables are independent constructions. The Gelfand-Naimark-Segal construction allows us to link these concepts in a suitable Hilbert space.

Theorem 3.1.1 (Gelfand-Naimark-Segal construction)**.** *Given a $*$-representation of a C^*-algebra on a Hilbert space $\mathcal{H}$, $\pi : \mathscr{A} \to \mathfrak{B}(\mathcal{H})$ and letting $v \in \mathcal{H}$ be a normalized cyclic vector for $\pi(\mathscr{A})$ (*span$\{\pi(\mathscr{A})v\}$ *is dense in the uniform topology) then, clearly*

$$\begin{aligned} \omega :& \mathscr{A} \to \mathbb{R} \\ & A \mapsto \langle \pi(A)v, v\rangle \end{aligned}$$

is a state of $\mathscr{A}$. Conversely, if ω is a state on a C^*-algebra $\mathscr{A}$, there exists a unique (up to unitary equivalent representations) representation $\pi_\omega : \mathscr{A} \to \mathfrak{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$ and a unit GNS vector $\Omega_\omega \in \mathcal{H}$ such that

$$\omega(A) = \langle \Omega_\omega, \pi(A)\Omega_\omega \rangle, \quad \forall A \in \mathscr{A}$$

$$\text{and} \quad \pi(\mathscr{A})\Omega_\omega \quad \text{is dense in } \mathcal{H}.$$

Proof. Define $\langle A, B\rangle := \omega\left(A^*B\right)$ and $I_\omega := \{A \in \mathscr{A} : \omega\left(A^*A\right) = 0\}$. Let $\mathcal{H} = (\overline{\mathscr{A}/I_\omega}, \langle \cdot, \cdot \rangle)$ be the completition of the quotient $\mathscr{A}/I_\omega$. Clearly $\mathcal{H}$ is a complex Hilbert space by the properties of ω, the representation π_ω is such that[9] $\pi_\omega(A)[B] = [AB]$ on $\mathscr{A}/I_\omega$ and, the unit vector is $\Omega_\omega := [1]$. Clearly, if $A \in \mathcal{A}$ we have

$$\langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle = \langle \Omega_\omega, [A]\rangle = \omega(A)$$

and $\pi_\omega(\mathscr{A})\Omega_\omega = \mathscr{A}/I_\omega$ is dense in $\mathcal{H}$. Now, suppose $(\mathcal{H}', \pi')$ is another $*-$representation of $\mathscr{A}$ and Ω' is their corresponding cyclic vector in this representation. In this case $U : \mathcal{H} \to \mathcal{H}'$ such that $U\pi_\omega(A)\Omega_\omega = \pi'(A)\Omega'$ is unitary since

$$\begin{aligned} \langle U\pi_\omega(A)\Omega_\omega, U\pi_\omega(B)\Omega_\omega \rangle &= \left\langle \pi'(A)\Omega', \pi'(B)\Omega' \right\rangle \\ &= \left\langle \Omega', \pi'\left(A^*\right)\pi'(B)\Omega' \right\rangle = \omega\left(A^*B\right) \\ &= \langle \Omega_\omega, \pi_\omega\left(A^*\right)\pi_\omega(B)\Omega_\omega \rangle \\ &= \langle \pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega \rangle . \end{aligned}$$

[9]Given $f : X \to Y$ and $G : X \to Y$ two continuos functions and suppose $f\mid_D = g\mid_D$ where D is a dense set of X. Then, is a well known fact that $f = g$. Therefore, we sometimes give definitions acting in a dense set rather than the domain.

It follows that $U^{-1}\pi'(A)U = \pi_\omega(A)$ and $U\Omega_\omega = \Omega'$.

Remark: I_ω is a (left)ideal. Let $A \in I_\omega$ and $B \in \mathscr{A}$; using the Cauchy-Schwartz inequality

$$\begin{aligned}|\omega\left((BA)^*BA\right)|^2 = |\omega\left(A^*B^*BA\right)|^2 = \left|\omega\left((B^*BA)^*A\right)\right|^2 \\ \leq \omega\left((B^*BA)^*\left(B^*BA\right)\right)\omega\left(A^*A\right) = 0,\end{aligned}$$

therefore, $BA \in I_\omega$. ■

The GNS theorem entails that there exists an equivalence between elements and states in the algebra with operators and states on a Hilbert space respectively.

Definition 3.1.4. *If an automorphism α is unitarily implemented*

$$\pi(\alpha(\rho)A) = U(\rho)\pi(A)U(\rho)^{-1}.$$

on the representation Hilbert space, we say that the representation is Poincaré covariant.

Theorem 3.1.2. *Let α be an automorphism of a unital $*-$algebra $\mathscr{A}$. If a state ω on $\mathscr{A}$ is invariant under α then, α is unitarily implemented in the GNS representation π_ω of ω that leaves the GNS vector Ω_ω invariant. Any group of automorphisms leaving ω invariant is unitarily represented in $\mathcal{H}_\omega$. The GNS representation of a Poincaré invariant state ω is always Poincaré covariant i.e.*

$$\omega(\alpha(\rho)A) = \omega(A), \quad \textit{for all } A \in \mathscr{A}(M),\ \rho \in \mathcal{P}$$

where $\mathcal{P}$ is the Poincaré group.

Proof. Since α is an automorphism, $\alpha I_\omega = I_\omega$ where I_ω is the GNS (left)ideal defined earlier. Therefore, U such that in the dense set $\mathscr{A}/I_\omega$ we have $U[A] = [\alpha(A)]$ is well-defined. The GNS vector $\Omega_\omega = [\mathbb{1}]$ is invariant and, U is unitary since

$$\begin{aligned}\langle U[A], U[B]\rangle &= \omega\left(\alpha(A)^*\alpha(B)\right) = \omega(\alpha(A^*B)) \\ &= \omega\left(A^*B\right) \\ &= \langle [A],[B]\rangle.\end{aligned}$$

Using the representation π_ω give us

$$\begin{aligned}\pi_\omega(\alpha(A))[B] &= [\alpha(A)B] = \left[\alpha\left(A\alpha^{-1}(B)\right)\right] \\ &= U\left[A\alpha^{-1}(B)\right] = U\pi_\omega(A)\left[\alpha^{-1}B\right] \\ &= U\pi_\omega(A)U^{-1}[B]\end{aligned}$$

that is, α is implemented unitarly. Finally, if β is another automorphism leaving ω invariant, V be its unitary implementation. Then

$$UV[A] = [\alpha(\beta(A))] = [(\alpha \circ \beta)(A)]$$

shows that UV implements $\alpha \circ \beta$ therefore, any group of automorphisms leaving ω invariant is unitarily represented. ∎

An important family of automorphisms are one-parameter group automorphisms because one can implement time evolution operations in the following way.

Definition 3.1.5. *Let $\mathscr{A}$ be a C^*−algebra. If τ_t is a one-parameter group of automorphisms such that $\tau(A) : \mathbb{R} \to \mathscr{A}, \quad \tau_t(A) \in \mathscr{A}$ is continuous for all $A \in \mathscr{A}$ then, the C^*−algebra $\mathscr{A}$ together with the one parameter group τ is said to be a C^*−dynamical system. In this case we write*

$$\tau_t(\omega)(A) := \omega(\tau_t(A))$$

and we say that a state ω on $\mathscr{A}$ is invariant under τ if

$$\omega(\tau_t(A)) = \omega(A), \quad \forall t \in \mathbb{R}, \forall A \in \mathscr{A}.$$

Definition 3.1.6. *Let $\mathscr{B}$ be a von Neumann algebra on a Hilbert space H. If τ_t is a one-parameter group of automorphisms such that $\tau(B)x : \mathbb{R} \to \mathcal{H}$ is continuous for all $B \in \mathscr{B}$ and $x \in \mathcal{H}$ then, the algebra $\mathscr{B}$ together with the one parameter group τ is said to be a W^*−dynamical system.*

Observe that a C^*−dynamical system is stronger than the W^*−dynamical system in the sense that the underlying von Neumann algebra endowed with the C^*− dynamics inherits the W^*−dynamics because the topology of the automorphisms on the von Neumann algebra is the weak topology whereas the topology on the C^* algebra is the strong topology.

Example 3.1.1. *Given a Hamiltonian H acting on a Hilbert space $\mathcal{H}$, the map*

$$\begin{aligned} \tau(A) :&\mathbb{R} \to \mathfrak{B}(\mathcal{H}) \\ &A \mapsto e^{iHt} A e^{-iHt} \end{aligned}$$

is a one parameter automorphism (inner) automorphisms in the algebra of bounded operators $\mathfrak{B}(\mathcal{H})$.

Theorem 3.1.3. *Let $\mathscr{A}$ be a C^*−algebra, $\tau : \mathscr{A} \to \mathrm{Aut}(\mathscr{A})$ a one parameter automorphism, ω an invariant state under τ and $\pi_\omega : \mathscr{A} \to \mathfrak{B}(\mathcal{H})$ the induced GNS representation. Then, there exists a unique unitary one-parameter group $U : \mathbb{R} \to \{A \in \mathfrak{B}(\mathcal{H}) : A^\dagger A = I = AA^\dagger\}$ such that $U_t \Omega_\omega = \Omega_\omega$ and $\pi_\omega(\tau_t(A)) = U_t \pi_\omega(A) U_{-t}$, $\forall t \in \mathbb{R}, A \in \mathscr{A}$.*

Proof. Look in [27]. ∎

Definition 3.1.7. *Let ρ be a density operator on a Hilbert space $\mathcal{H}$. Notice that $\omega_\rho : \mathfrak{B}(\mathcal{H}) \to \mathbb{C}$,*

$$\begin{aligned} \omega_\rho(A) &= \mathrm{Tr}(A\rho) \\ \omega_\rho(AA^*) &= \mathrm{Tr}(A\rho A^*) \geq 0, \quad \omega_\rho(1) = \mathrm{Tr}(\rho) = 1 \end{aligned}$$

is a state in the sense of definition 3.1.3. If $\mathscr{A} \subset \mathfrak{B}(\mathcal{H})$ is a C^*−algebra, the state $\omega_\rho : \mathscr{A} \to \mathbb{C}$ is said to be a normal state.

Remark 3.1.4. *The algebraic approach takes into account many subtleties that arise in QFT that are hidden in Quantum Mechanics (QM). For instance, in QM the algebra of canonical commutation relations is unique by the Stone-von Neumann theorem and the only states are characterized by density operators, i.e. normal states. However, in quantum field theories there are inequivalent irreducible representations of the canonical commutation relations, allowing the theory to have far more states than QM.*

Example 3.1.2 (Scalar field with external force)**.** *We follow [24] to depict that different (inequivalent) representations of a quantum field theory may exist. Consider a theory with a Lagrangian*

$$\mathcal{L} = \frac{1}{2}\left(\nabla_\mu \varphi\right)\nabla^\mu \varphi - \frac{1}{2}m^2\varphi^2 - \rho\varphi \tag{3.1}$$

where ρ is a time-independent real valued function. The field equation of this theory is

$$\left(\Box + m^2\right)\varphi = -\rho(x). \tag{3.2}$$

Using quantum mechanics, one can either use the Schrödinger picture or the interaction picture in order to find the dynamics of the system.

- *In the Schrödinger picture one starts with the standard Fock space of the free scalar field with position and momentum satisfing the canonical commutation relations*

  $$[\varphi(x), \pi(y)] = i\delta(x-y)1,$$

  the interaction model is thus described by a Hamiltonian H_ρ in terms of the canonical variables and the evolution of the states are described by $e^{-iH_\rho t}$. Therefore, one can define

  $$\varphi_\rho(t,x) = e^{iH_\rho t}\varphi(x)e^{-iH_\rho t}$$

  as a state that follows the dynamics (3.2) *of the system.*

- *On the other hand, in the interaction picture the free field $\varphi_0(t,x)$ acts on the usual Fock space and the evolution is described by the evolution operator U_{int} with the Dyson series,*

  $$\begin{aligned}\hat{U}_{\text{int}} &= \mathcal{T}e^{-i\int d\tau \hat{H}_{int}(\tau)} \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d\tau_1 \dots d\tau_n \mathcal{T}(\hat{H}_{int}(\tau_1)\dots\hat{H}_{int}(\tau_n)).\end{aligned}$$

In either case, ultraviolet or infrared-like divergences may appear and thus, a need of re-normalization theory. One can also describe the dynamics in its algebraic nature. We start with equation (3.2) *and proceed with the canonical quantization from scratch. In this way, we do not use the Hilbert space of the free scalar field but rather, the one specific to the model. In this model, one can show that the* *canonical variables* φ_ρ *and* π_ρ *are given by*

$$\varphi_\rho(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}(a(k)+a(-k)^*)e^{ik\cdot x}$$
$$\pi_\rho(x) = -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega}{2}}(a(k)-a(-k)^*)e^{ik\cdot x}$$

with the free mode frecuency $\omega := \omega(k) = \sqrt{\|k\|^2+m^2}$ *and impose the Canonical Commutation Relations (CCRs),*

$$[a(k), a(k')^*] = (2\pi)^3\delta(k-k')1, \tag{3.3}$$

$$[a(k), a(k')] = 0 = [a(k)^*, a(k')^*]. \tag{3.4}$$

The Hamiltonian is then deduced from the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\nabla_\mu\varphi)\nabla^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \rho\varphi$$

as

$$H_\rho = \int \frac{d^3k}{(2\pi)^3}\left(\omega(k)a(k)^*a(k) + \frac{1}{\sqrt{2\omega(k)}}\left(\overline{\hat\rho(k)}a(k)+\hat\rho(k)a(k)^*\right)\right),$$

where $\hat\rho(k) = \int d^3x\rho(x)e^{-ik\cdot x}$ *and equivalently, if one redefines the annihilator and creation operators,*

$$H_\rho = \int \frac{d^3k}{(2\pi)^3}\omega(k)\tilde a(k)^*\tilde a(k) + E_\rho 1,$$

where $\tilde a$ *and the constant* E_ρ *are defined by*

$$\tilde a(k) = a(k) + \frac{\hat\rho(k)}{\sqrt{2\omega(k)^3}}1, \quad E_\rho = -\int \frac{d^3k}{(2\pi)^3}\frac{|\hat\rho(k)|^2}{2\omega(k)^2}.$$

The reader may notice that a Hilbert space is not yet specified. However, we want a representation in which H_ρ *is self-adjoint, the* $\tilde a(k)$ *satisfy the CCRs and there is a preferred vacuum state. For this purpose, notice that* $\tilde a(k)$ *satisfy the same CCRs as* $a(k)$ *and thus, we can write a vacuum vector* $\tilde\Omega$ *annihilated by all* $\tilde a(k)$*. This vacuum is the ground state of the Hamiltonian* H_ρ *and we may take this energy to be the zero energy state. The Fock space is then realized as usual,*

$$\mathcal{H} = \bigoplus_{n=0}^{\infty}\mathcal{H}^{\otimes n}$$

where $\tilde\Omega$ *is the vacuum vector of* $\mathcal{H}$ *and* H_ρ *is just the standard Hamiltonian of the free field on this space*

$$\tilde H_\rho = \int \frac{d^3k}{(2\pi)^3}\omega\tilde a(k)^*\tilde a(k).$$

The redefinition of annihilation and creation operators provide the crucial differentiation between the previous methods (Schr̈odinger and interaction picture). A calculation shows that the time inde-

pendent Heisenberg picture field is

$$\tilde{\varphi}_\rho(t,x) = e^{i\tilde{H}_0 t}\varphi_\rho(x)e^{-i\tilde{H}_0 t} = \tilde{\varphi}_0(t,x) + \varphi_\rho(x)1, \tag{3.5}$$

where

$$\tilde{\varphi}_\rho(x) = -\int \frac{d^3k}{(2\pi)^3}\frac{\hat{\rho}(k)}{\omega^2}e^{ik\cdot x} = -(V_Y \star \rho)(x)$$

is a time independet solution of the field equation and

$$\tilde{\varphi}_0(t,x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\left(\tilde{a}(k)e^{-i\omega t+ik\cdot x} + \tilde{a}(k)^* e^{i\omega t-ik\cdot x}\right)$$

is the free scalar field on the Fock space.

Remark 3.1.5.

- H_ρ *in the new variables is of the same form as the Hamiltonian in the free field defined with the new annihilator/creation operators.*
- *Given a field equation we found the generators.*
- *The Fock space was built a posteriori with the help of the vacuum vector once the operators* $\tilde{a}$ *and* $\tilde{a}^\dagger$ *are defined.*
- *The collection of fields has both, vector space and ring structure, that is, they form an algebra.*

Theorem 3.1.6 (Haag)**.** *If two representations are unitarily equivalent representations of scalar fields, and both representations contain a unique vacuum state, the two vacuum states are themselves related by the unitary equivalence. Hence neither field Hamiltonian can polarize the other field's vacuum.*

Definition 3.1.8. *States that are normal in one GNS representation and not in the other are said to be disjoint states.*

The example 3.1.2 illustrates that the vacuum state lies inside a renormalized Hilbert space $\mathcal{H}_{\text{renorm}}$ that differs from the Hilbert space $\mathcal{H}_{\text{free}}$ of the free field. Although an isomorphism is found between $\mathcal{H}_{\text{renorm}}$ and $\mathcal{H}_{\text{free}}$, mainly by the map $a \mapsto \tilde{a}$, Haag's theorem implies that no such mapping could deliver unitarily equivalent representations of the corresponding canonical commutation relations. In addition, Haag's theorem explains that disjoint states are essential in QFT. In fact, given a group G acting on the C^*−algebra of observables, and two G-invariant states ω_1 and ω_2 then either $\omega_1 = \omega_2$, or ω_1 and ω_2 are disjoint. Since vacuum states must be invariant with respect to the Poincaré group, it follows that vacuum states of both free and an interacting theory must be different. Therefore, by Haag's theorem, they cannot both be represented as density matrices in a single representation. This motivates why the algebraic description of quantum theories is so often used for relativistic quantum mechanics.

3.1.1 Thermal states, KMS states and the KMS condition

In the algebraic formalism, a thermal state can be seen as a specific class of states on a C^*−algebra characterized by Kubo (1957), Martin and Schwinger (1959) called the KMS-states. Let $\beta \in \mathbb{R}$, define $\mathfrak{D}_\beta$ as

$$\mathfrak{D}_\beta = \begin{cases} \{z \in \mathbb{C} \,:\, 0 < \operatorname{Im} z < \beta\}, & \beta \geq 0 \\ \{z \in \mathbb{C} \,:\, \beta < \operatorname{Im} z < 0\}, & \beta < 0 \end{cases},$$

if $\beta = 0$ then $\overline{\mathfrak{D}_\beta} = \mathbb{R}$ otherwise $\overline{\mathfrak{D}_\beta}$ is the closure of $\mathfrak{D}_\beta$.

Definition 3.1.9. *Let $\mathscr{A}$ together with τ a C^* or W^* dynamical system, ω a state and $\beta \in \mathbb{R}$. Suppose there exists a function $F_{A,B} : \overline{\mathfrak{D}_\beta} \to \mathbb{C}$, $\forall A, B \in \mathscr{A}$ analytic in $\mathfrak{D}_\beta$ such that*

$$F_{A,B}(t) = \omega(A\tau_t(B)), \tag{3.6}$$

$$F_{A,B}(t + i\beta) = \omega(\tau_t(B)A). \tag{3.7}$$

In that case, ω is said to be a (τ, β)−KMS state or equivalently, a KMS state with respect to the automorphism τ at a temperature β. Equations (3.6) and (3.7) are the KMS conditions and F is called the witness of ω being a (τ, β)−KMS state. A $(\tau, -1)$−KMS state is called a τ−KMS state. The importance of τ−KMS states is given in the following theorem:

Remark 3.1.7. *The motivation of the KMS condition is some times motivated by the condition*

$$\omega_\rho(AB) = \omega_\rho(BA) \tag{3.8}$$

that is already satisfied by normal states and the properties of trace. However, in general, the operator

$$A_z = e^{iHz} A e^{-iHz}, \quad z = t + i\gamma$$

will not be a bounded operator but the operator

$$A_z e^{-i\beta H} = e^{iHt} e^{-H\gamma} A e^{-iHt} e^{-H(\beta-\gamma)}$$

is a bounded operator of trace class if $0 \leq \gamma \leq \beta$ because all the operators above are bounded. Similarly, $e^{-i\beta H} A_z$ is of trace class if $-\beta \leq \gamma \leq 0$. and therefore, the KMS condition is a generalization of 3.8 in the space where it makes sense.

Theorem 3.1.8. *Let $\mathscr{A}$ together with τ a C^* or W^* dynamical system, ω a state and $\beta \in \mathbb{R}$. Then, τ induce a dynamical system on $\mathscr{A}$ by $\alpha : \mathbb{R} \to \operatorname{Aut}(\mathscr{A})$ where α is defined so that*

$$\begin{aligned} \alpha_t :& \mathscr{A} \to \mathscr{A} \\ & A \mapsto \alpha_t(A) := \tau_{-\beta t}(A). \end{aligned}$$

Furthermore, if ω is a (τ, β)−KMS state then ω is a α−KMS state. The converse is also true if $\beta \neq 0$.

Proof. Clearly $\mathscr{A}$ with α is a C^* or W^*-dynamical system. If ω is a (τ, β)-KMS state, let $F_{A,B}$ be the witness then, $G_{A,B}(z) := F_{A,B}(-\beta z)$ is a witness of ω being an α–KMS state. Conversely, if $\beta \neq 0$ and $G_{A,B}$ is a witness of ω being an α-KMS state $F_{A,B}(z) = G_{A,B}(-z/\beta)$ is a witness of ω being a (τ, β)-KMS state. ■

Theorem 3.1.9. *Let $\mathscr{A}, \tau$ be a C^* or W^*-dynamical system, $|\beta| > 0$ and, ω a (τ, β)–KMS state. Then, for all $A \in \mathscr{A}$ and $t \in \mathbb{R}$ we have $\omega(\tau_t(A)) = \omega(A)$.*

Proof. Let α be defined as above, $A \in \mathscr{A}$ and, $F_{1,A}$ be a witness to ω being a α-KMS state. Notice that,

$$\omega\left(\tau_t(A)\right) = \omega\left(\tau_t\left(\frac{A+A^*}{2} + i\frac{A-A^*}{2i}\right)\right) = \omega\left(\tau_t\left(\frac{A+A^*}{2}\right)\right) + i\omega\left(\tau_t\left(\frac{A-A^*}{2i}\right)\right).$$

That is, the condition $\omega(\tau_t(A)) = \omega(A)$ holds if and only if it holds for all self adjoint elements $A \in \mathscr{A}$. Therefore, we might assume that A is self-adjoint. In this case we have

$$\begin{aligned}
\overline{F_{1,A}(t)} = \overline{\omega\left(\alpha_t(A)\right)} &= \overline{F_{1,A}(t-i)} \\
&= \omega\left(\alpha_t(A)^*\right) \\
&= \omega\left(\alpha_t\left(A^*\right)\right) \\
&= \omega\left(\alpha_t(A)\right) = F_{1,A}(t-i)
\end{aligned}$$

It follows that $F_{1,A}\left(\overline{\mathfrak{D}_{-1}} \backslash \mathfrak{D}_{-1}\right) \subseteq \mathbb{R}$. By Liouville's theorem [10] $F_{1,A}$ is constant (which means α is also constant) and, since τ is a reparametrization of α we have

$$\omega\left(\tau_t(A)\right) = \omega(\tau_0(A)) = \omega(A).$$

■

An equivalent (and perhaps more transparent) KMS condition can be given,

Proposition 3.1.1. *Let $\mathscr{A}, \tau$ be a C^* or W^*-dynamical system, a state ω is a (τ, β)–KMS state if and only if the condition*

$$\omega(A\tau_{i\beta}(B)) = \omega(BA) \tag{3.9}$$

holds for all $A, B \in \mathscr{A}$.

Proof. Assume ω is a KMS state with respect to τ at a temperature β and let $F_{A,B}$ the witness of this. Evaluating (3.16) and (3.17) in $i\beta$ and 0 respectively gives us

$$\omega(A\tau_{i\beta}(B)) = \omega(BA).$$

[10] Every holomorphic function f for which there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$ is constant.

Conversely, if the condition (3.9) holds then, define $F_{A,B}(t) = \omega(A\tau_t(B))$. It follows that

$$\begin{aligned} F_{A,B}(t+i\beta) &= \omega(A\tau_{t+i\beta}(B)) = \omega(A\tau_{i\beta}(\tau_t(B))) \\ &= \omega(\tau_t(B)A). \end{aligned}$$

■

Definition 3.1.10. *Let $\mathscr{A}$ be a von Neumann algebra and ω a faithful and normal state on $\mathscr{A}$. Let π_ω be the GNS representation of $\mathscr{A}$ associated to ω and Δ the modular operator (see appendix D.2) associated to $(\pi_\omega(\mathscr{A}), \Omega_\omega)$. Then, Tomita's theorem define a one-parameter automorphism*[11] *group $\tau_t^\omega \in \operatorname{Aut}\mathscr{A}$ as*

$$\tau_t^\omega(A) = \pi_\omega^{-1}(\Delta^{it}\pi_\omega(A)\Delta^{-it}).$$

The 1−parameter group is called the modular automorphism group associated with the pair $(\mathscr{A}, \varphi)$.

Proposition 3.1.2. *Let $\mathscr{A}$ be a von Neumann algebra, ω a faithful and normal state. Then, ω is a KMS state at "temperature"* [12] *$\beta = -1$ with respect to the modular automorphism group associated with $(\mathscr{A}, \omega)$.*

Proof. Consider $(\mathcal{H}, \pi, \Omega)$ the GNS representation of $\mathscr{A}$ associated to ω.

$$\begin{aligned} \omega\left(A\tau_{-i}^\omega(B)\right) &= \langle \Omega|\pi(A)\pi\left(\pi^{-1}\left(\Delta^{1/2}\Delta^{1/2}\pi(B)\Delta^{-1/2}\Delta^{1/2}\right)\right)\Omega\rangle \\ &= \left\langle \Delta^{1/2}\pi\left(A^*\right)\Omega \mid \Delta^{1/2}\pi(B)\underbrace{\Delta^{-1/2}}_{J\Delta^{1/2}J}\Delta^{-1/2}\Omega\right\rangle \\ &= \left\langle \Delta^{1/2}\pi\left(A^*\right)\Omega \mid \Delta^{1/2}\pi(B)S(\Delta^{1/2}J\Omega)\right\rangle \\ &= \left\langle \Delta^{1/2}\pi\left(A^*\right)\Omega \mid \Delta^{1/2}\pi(B)\underbrace{J\Delta^{1/2}}_{S}\Omega\right\rangle \\ &= \left\langle \Delta^{1/2}\pi\left(A^*\right)\Omega \mid \Delta^{1/2}\pi(B)\Omega\right\rangle \\ &= \left\langle J\Delta^{1/2}\pi(B)\Omega \mid J\Delta^{1/2}\pi\left(A^*\right)\Omega\right\rangle \\ &= \langle \pi\left(B^*\right)\Omega \mid \pi(A)\Omega\rangle = \langle \Omega \mid \pi(BA)\Omega\rangle = \omega(BA). \end{aligned} \tag{3.10}$$

where we used $J\Delta^{1/2}J = \Delta^{-1/2} = SJ$ as a result of proposition D.2, J is the unique anti-linear transformation of $\mathcal{H}$ such that $J^2 = 1$ and $\langle Jf, Jg\rangle = \langle g, f\rangle$ and that $S(A\Omega) = A^*\Omega$. Hence, (3.9) holds. ■

Remark 3.1.10. *In the former proof, the condition*

$$\left\langle J\Delta^{1/2}\pi(B)\Omega \mid J\Delta^{1/2}\pi\left(A^*\right)\Omega\right\rangle = \langle \pi\left(B^*\right)\Omega \mid \pi(A)\Omega\rangle$$

is an equivalent KMS condition. Indeed, the condition uniquely determines the operator S acting by $S(\pi(B)\Omega) = \pi(B^*)\Omega$ which, in turn, establishes all the equalities in (3.10) *by proposition D.2.*

[11]Notice this is the converse of theorem 3.1.3.

[12]Some times we refer to the temperature at the value of $\beta = 1/kT$.

Proposition 3.1.3. *Let $\mathscr{A}$ be a von Neumann algebra, ω a state and let $(\mathcal{H},\pi,\Omega)$ be the GNS representation of $\mathscr{A}$ of ω. Then, the KMS condition can be formulated in the following equivalent ways:*

$$1.\ \left\langle \Delta^{1/2}A\Omega \middle| \Delta^{1/2}B\Omega \right\rangle = \langle B^*\Omega | A^*\Omega \rangle, \quad \forall A,B \in \mathscr{A}, \tag{3.11}$$

$$2.\ J\Delta^{1/2}A\Omega = A^*\Omega, \quad \forall A \in \mathscr{A}. \tag{3.12}$$

Proof. The first one follows from remark 3.1.10 and the later describes the operator S. ∎

KMS states are mathematically satisfactory states that characterize thermal states in quantum field theory. In order to link the abstract idea with physical applications, let us remember how thermal states are described. Let $\mathcal{H}$ be a Hilbert space and consider a system with Hamiltonian H in equilibrium with a heat bath without exchange of particles at inverse temperature β.

Definition 3.1.11. *The the density ρ_β defined by*

$$\rho_\beta = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$$

is said to be a $\beta-$Gibbs state and, has the property of being invariant under evolution of the system, that is, the state is in (thermal) equilibrium.

Remark 3.1.11.

- *The map $A \mapsto \mathrm{Tr}(\rho_\beta \mathrm{A})$ defines a state (ω_{ρ_β} as defined earlier) in the sense of 3.1.3.*
- *Assume that $\{|n\rangle : n \in \mathbb{N}\}$ are all the eigenvectors, associated with the eigenvalues E_n, of H. Then, the thermal vector is realized as a state*

$$|\beta\rangle = \frac{\sum_n e^{-\beta E_n/2} |n\rangle \otimes |n\rangle}{\sqrt{\sum_m e^{-\beta E_m}}} \tag{3.13}$$

in the Hilbert space $\mathcal{H}\otimes\mathcal{H}$. In this case,

$$|\beta\rangle\langle\beta| = \frac{\sum_{n,m} e^{-\beta(E_n+E_m)/2} |n\rangle\langle m| \otimes |n\rangle\langle m|}{\sum_m e^{-\beta E_m}} = \rho_\beta.$$

Theorem 3.1.12. *Let $(\mathfrak{B}(\mathcal{H}),\tau)$ be the W^*-dynamical system with Schrödinger evolution,*

$$\begin{aligned} \tau :& \mathbb{C} \to \mathrm{Aut}(\mathfrak{A}(\mathcal{H})) \\ & z \longrightarrow \tau_z : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}) \\ & \qquad\qquad\quad A \mapsto e^{iHz}Ae^{-iHz} \end{aligned}$$

then, a state $\omega \in \mathfrak{B}(\mathcal{H})$ is a $\beta-$Gibbs state if and only if it is a $(\tau,\beta)-$KMS state.

Proof. Look in [27]. ∎

Corollary 3.1.12.1. *A $\beta-$Gibbs state is invariant under evolution of the system.*

Proof. This is the result of the theorem 3.1.9 . ■

3.1.2 The KMS condition on the Unruh effect

In section 1, we saw that a Rindler observer identifies the Minkowski vacuum (1.40) as a thermal state and, by means of theorem 3.1.12, the state ρ in (1.41) must be a KMS state in the operator algebra of $\mathcal{H}$. Let us observe the explicit form of this result. We consider the discrete level system in section 1.2 equation (1.38) with a Hamiltonian H and denote the eigenvectors $|n, R\rangle$ by $|n\rangle$ with associated eigenvalues E_n. The expectation value of the operator A in a thermal state with temperature $T = \beta^{-1}$ is

$$\begin{aligned}
\langle A\rangle_\beta = \omega_\rho(A) = \mathrm{Tr}(\rho_\beta A) = \langle\beta| A^{(e)} |\beta\rangle \\
= \frac{\sum_n e^{-\beta E_n} \langle n| A |n\rangle}{\sum_m e^{-\beta E_m}} \\
= \frac{\mathrm{Tr}(e^{-\beta\hat{H}} A)}{\mathrm{Tr}(e^{-\beta\hat{H}})}
\end{aligned}$$

where $A^{(e)} = I \otimes A$. The time evolution operator is taken to be $\exp(-iH^{(e)}\tau)$.

Proposition 3.1.4. *The time evolution operator is given by*

$$\exp(-iH^{(e)}\tau) = I \otimes \exp(-iH\tau). \tag{3.14}$$

Proof. Let A, B be two operators and denote $e^A := \exp(A)$. Then,

$$e^A \otimes e^B = \sum_n \frac{A^n}{n!} \otimes \sum_m \frac{B^m}{m!} = \sum_{n=0}^{\infty} \underbrace{\sum_{k=0}^{n} \binom{n}{k} A^n \otimes B^k}_{L_n}.$$

If A and B commutes, the binomial theorem holds in the following form,

$$(A \otimes I + I \otimes B)^n = \sum_{k=0}^{n} \binom{n}{k} A^n \otimes B^k = L_n.$$

Thus,

$$e^A \otimes e^B = e^{A \otimes I + I \otimes B}.$$

Using the Baker-Campbell-Hasudorff formula, and since A and B commute,

$$e^A \otimes e^B = e^{A \otimes I + I \otimes B} = e^{A \otimes I} e^{I \otimes B}.$$

Then,

$$e^{-iH^{(e)}\tau} = e^{-i\tau(I \otimes H + I \otimes 0)} = e^{-i\tau 0} \otimes e^{-i\tau H} = I \otimes \exp\left(-i\hat{H}\tau\right).$$

■

Now, define an antiunitary involution $J^{(e)}$ acting as

$$J^{(e)}\alpha\,|n\rangle\otimes|m\rangle=\alpha^*\,|m\rangle\otimes|n\rangle\,,\quad \alpha\in\mathbb{C}. \tag{3.15}$$

In particular, it is not hard to see that equation (1.40) on section 1.2 implies that the two-dimensional model with only the left movers (or only the right movers) has an involution $J^{(e)}$ defined by requiring

$$\begin{aligned}J^{(e)}\,|0_M\rangle&=|0_M\rangle\,,\\ J^{(e)}\hat{a}^R_{+\omega}J^{(e)}&=\hat{a}^L_{+\omega},\\ J^{(e)}\hat{a}^{R\dagger}_{+\omega}J^{(e)}&=\hat{a}^L_{+\omega}.\end{aligned}$$

Observe that the operator $J^{(e)}$ commutes with the time evolution operator as

$$\begin{aligned}J^{(e)}\exp(-iH^{(e)}\tau)\alpha\,|n\rangle\otimes|m\rangle&=J^{(e)}\alpha\,|n\rangle\otimes\exp(-iH\tau)\,|m\rangle\\ &=\alpha^*\exp(-iH\tau)\,|m\rangle\otimes|n\rangle\\ &=\exp(-iH\tau)\alpha^*\,|m\rangle\otimes|n\rangle\\ &=\exp(-iH^{(e)}\tau)J^{(e)}\alpha\,|n\rangle\otimes|m\rangle\,,\end{aligned}$$

thus,

$$J^{(e)}\exp(-iH^{(e)}\tau)=\exp(-iH^{(e)}\tau)J^{(e)}. \tag{3.16}$$

Therefore, $J^{(e)}$ satisfies one (and therefore all) conditions of proposition D.2 with respect to the time evolution operator. Furthermore, uniqueness of $J^{(e)}$ follows from Tomita's theory and the polar decomposition (cf. Appendix D). Once the involution is found, let us take a look at the KMS-condition defined on equation (3.12). Let $A\in\mathfrak{B}(\mathcal{H})$ be given by a matrix A_{mn} as $A\,|n\rangle=\sum_m A_{mn}\,|m\rangle$ then,

$$\begin{aligned}\exp(-H^{(e)}\beta/2)A^{(e)}\,|\beta\rangle&=(1\otimes\exp(-H\beta/2))1\otimes A\,|\beta\rangle\\ &=(1\otimes\exp(-H\beta/2))1\otimes Ak\sum_n e^{-\beta E_n/2}\,|n\rangle\otimes|n\rangle\\ &=k\sum_{n,m}e^{-\beta E_n/2-\beta E_m/2}A_{mn}\,|n\rangle\otimes|m\rangle\\ &=k\sum_{n,m}e^{-\beta E_n/2-\beta E_m/2}J^{(e)}A^*_{mn}\,|m\rangle\otimes|n\rangle\\ &=k\sum_m e^{-\beta E_m/2}J^{(e)}A^{(e)\dagger}\,|n\rangle\otimes|n\rangle\\ &=J^{(e)}A^{(e)\dagger}\,|\beta\rangle\,,\end{aligned}$$

where k is the normalization constant of $|\beta\rangle$. If we apply J on both sides, it follows that

$$J^{(e)}\exp(-\hat{H}^{(e)}\beta/2)\hat{A}^{(e)}\,|\beta\rangle=\hat{A}^{(e)\dagger}\,|\beta\rangle\,, \tag{3.17}$$

establishing the KMS-condition (3.12) with respect to the time evolution, making the state ρ a

$2\pi/a$–KMS state.

Remark 3.1.13. *The former example allows us to define the Unruh effect in the algebraic theory,*

> *The Unruh effect*
>
> *The Unruh effect is the effect in which the Minkowski vacuum restricted to the right/left Rindler wedge is a $2\pi/a$–KMS state.*

Remark 3.1.14. *In the two-dimensional model of the Unruh effect the involution $J^{(e)}$ is in fact the PCT transformation,*

$$\varphi(t,z) \to \varphi(-t,-z).$$

This is no coincidence and we shall address this question in section 3.3.

3.2 Algebraic Quantum Field Theory

We motivated the algebraic tools in order to work in quantum field theory. Now, we present the so-called Algebraic Quantum Field Theory. For this purpose, we define a list of assumptions that a quantum field theory must have. Even though all of them are motivated by physical/structural ideas, we keep the study as broad as possible and thus, an algebraic theory need not fulfill all the assumptions. However, the ones marked with a (†) will not be essential in order to define the Fewester-Verch (FV) QMT scheme in section 4. Notations (and some examples) regarding causal/convex geometry in curved spacetimes can be found in the Appendix E if not defined in the main body of this document.

3.2.1 Assumptions of AQFT in CST (Haag-Kastler) [29, 33]

Definition 3.2.1. *Let M be a space-time. For each open region $\mathcal{O} \subseteq M$, an associated unital C^*−algebra of observables, which can be measured within the spacetime region $\mathcal{O}$ is denoted by*

$$\mathscr{A}(\mathcal{O}) = \{A \in \mathscr{A} : A \text{ is supported in } \mathcal{O}\}.$$

Definition 3.2.2. *We say that*

$$\mathscr{A} = \{\mathscr{A}(\mathcal{O}) : \mathcal{O} \subseteq M \text{ is causally open}\}$$

is the net of observable algebras.

Definition 3.2.3. *We call A a quasi-local algebra if it contains observables that can be uniformly approximated by local observables.*

Let $\mathscr{A}$ be a **non-trivial** net of observables, we require the following assumptions to hold.

A1 Isotony

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two open regions in the space-time M such that $\mathcal{O}_1 \subset \mathcal{O}_2 \subset M$ then,

$$\mathscr{A}(\mathcal{O}_1) \subset \mathscr{A}(\mathcal{O}_2).$$

A2 Einstein causality (Microcausality)

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two causally disjoint regions, that is, $\mathcal{O}_1 \subseteq \mathcal{O}_2^{\perp}$ then,

$$[\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2)] = 0.$$

A3 Compatibility

Let $\mathcal{O}$ be a open and causally convex region of space time i.e. $\mathcal{O}$ is equal to its causal hull. Then, there exists a morphism $\alpha_{M;\mathcal{O}}$ that is an embedding (unital injective $*$−homomorphism) from the local algebra $\mathscr{A}(\mathcal{O})$ onto the local algebra $\mathscr{A}(M)$ such that

$$\alpha_{M;\mathcal{O}}(\mathscr{A}(M\big|_{\mathcal{O}})) = \mathscr{A}(\mathcal{O})$$

where $\mathscr{A}(M\big|_{\mathcal{O}})$ is the subalgebra of observables localized in $\mathcal{O}$ of $\mathscr{A}(M)$ and such that if $\mathcal{O}_3 \subset \mathcal{O}_2 \subset \mathcal{O}_1$ then,

$$\alpha_{\mathcal{O}_1;\mathcal{O}_2} \circ \alpha_{\mathcal{O}_2;\mathcal{O}_3} = \alpha_{\mathcal{O}_1;\mathcal{O}_3}.$$

A4 Time-slice property

If M is globally hyperbolic and $N \subseteq M$ contains a Cauchy surface of M then, $\mathscr{A}(N) = \mathscr{A}(M)$.

A5 Poincaré covariance

To every transformation ρ in the identity connected component $\mathcal{P}_0$ of the Poincaré group, there is an automorphism $\alpha(\rho)$ of $\mathscr{A}(M)$ such that

$$\alpha(\rho) : \mathscr{A}(\mathcal{O}) \to \mathscr{A}(\rho\mathcal{O})$$

such that $\alpha(id) = id_{\mathscr{A}(M)}$ and $\alpha(\sigma) \circ \alpha(\rho) = \alpha(\sigma \circ \rho)$ for any $\sigma, \rho \in \mathcal{P}_0$.

Remark 3.2.1.

- *The isotony assumption conveys the idea that any observable measurable in a region $\mathcal{O}_1$ must be measurable in any region $\mathcal{O}_2$ containing $\mathcal{O}_1$. Also, isotony allow us to relax the construction of the net of observables algebras. Instead of defining a local algebra for each open set, it suffices* *to associate algebras to diamonds (e.g. in Minkowski space-time double cones) in the space-time and consider the sub algebras within them, a further explanation of these constructions is found in the examples.*

- *The main relativistic assumption of AQFT is contained in the Einstein Causality.*
- *One can restrict ourselves to an open subset of the space-time. In this case, the compatibility requirement states that the algebra can be seen on its own and, as the subalgebra of the whole space i.e.* $\mathscr{A}(M)$*. In both cases, they must coincide.*
- *The existence of a dynamical law is not yet implemented with requirements A1-A4 however, in order to determine the dynamics, it is enough to see the dynamics for a foliation of the space-time, that is, the time-slice assumption/property.*

3.2.2 Construction of an algebra from generators and relations

The task of specifying local algebras for all open regions by hand is, in general, very restrictive. We may equivalently present a construction of algebras in terms of generators and relations. Start by defining a unital $*-$algebra $\mathscr{U}$ spanned by elements $\varphi(f)$ and $\varphi(f)^*$ labeled by a smooth function $f \in C_0^\infty(L)$, $L \subseteq M$. The relations can be implemented by taking the quotient $\mathscr{U}/I$, where I is the $*-$ideal (an ideal invariant under $*-$operation) generated by the relations. The resulting algebra $\mathscr{A}(M) = \mathscr{U}/I$ is defined so that

$$[A][B] = [AB], \quad [A]^* = [A^*], \quad 1_{\mathscr{A}(M)} = [1_{\mathscr{U}}]$$

holds. Being I a $*-$ideal guarantees that these operations are well-defined.

Example 3.2.1 (The free scalar field, we follow [33])**.** *Consider the Lagrangian,*

$$\mathcal{L} = \frac{1}{2}(\nabla_\mu\varphi)\nabla^\mu\varphi - \frac{1}{2}m^2\varphi^2,$$

which has the field equation

$$P\varphi = 0, \quad P = \Box + m^2.$$

We want to describe the algebra $\mathscr{A}(M)$ *of this quantum field theory. Let* G_A *and* G_R *be the Green functions of the operator* P *and let* E_P^+ *and* E_P^- *be their corresponding retarded and advanced Green operators. To ensure convergence of these operators, we restrict the domain to* $C_0^\infty(M)$ *i.e. smooth* *functions of compact support. In this case, the operators* $E_P^{\pm}$ *and* $E_P := E_P^- - E_P^+$ *satisfy*

$$\mathrm{supp}(E_P^{\pm} f) \subseteq J^{\pm}(\mathrm{supp}\, f), \tag{3.18}$$

$$PE_P^{\pm} f = f \;\; \forall f \in C_0^{\infty}(\mathbb{R}) \tag{3.19}$$

$$E_P^{\pm} P f = f \;\; \forall f \in C_0^{\infty}(\mathbb{R}) \tag{3.20}$$

$$\ker E_P = PC_0^{\infty}(\mathbb{R}). \tag{3.21}$$

Therefore, $\varphi := E_P f := (E_P^- - E_P^+) f$ *solves the field equation and, in fact, every solution is of this form (see [34] for details). Define the operator* $E_P : C_0^{\infty}(M) \times C_0^{\infty}(M) \to \mathbb{C}$ *by*

$$E_P(f,g) = \int_M f(x) E_P g(x) dv = \int f(x) g(x) E_P(x,y) \tag{3.22}$$

where $E_P(x,y)$ *is the integral kernel of this operator. Notice that the operator* E_P *satisfies:*

$$E_P(x,y)|_{y^0=x^0} = 0 \quad \partial_{y^0} E_P(x,y)|_{y^0=x^0} = \delta(x-y).$$

The $*-$algebra construction. *Let* $N \subseteq M$*, the net of* $*-$*algebras* $\mathscr{A}(N)$ *are generated by a unit element together with the generators* $\varphi(f) = E_P f$, $f \in C_0^{\infty}(N)$ *such that:*

1. $f \to \varphi(f)$
2. $\varphi(\bar{f}) = \varphi(f)^*, \quad \forall f \in C_0^{\infty}(N)$
3. $\varphi(Pf) = 0, \quad \forall f \in C_0^{\infty}(N)$ *where* P *is the dynamics,* $P = \Box + m^2$.
4. $[\varphi(f), \varphi(h)] = iE_P(f,h)\mathbb{1}, \quad \forall f \in C_0^{\infty}(N)$,

$$E_P(f,h) := \int_M d\,\mathrm{vol}\, f E_P h.$$

Clearly, linear complex combinations of the unit lie in $\mathbb{C}$ *and, finite linear complex combinations of* $\varphi(f)$ *lie in* $S(Q^{\otimes n})$ *for some* n*. Thus, the underlying vector space of* $\mathscr{A}(M)$ *is isomorphic to the symmetric tensor vector space*

$$\mathbb{C} \oplus \bigoplus_{n=1}^{\infty} S(Q^{\otimes n}), \quad Q = C_0^{\infty}(M)/PC_0^{\infty}(M), \tag{3.23}$$

where $S(Q^{\otimes n})$ *is the symmetrized tensor product. In order to see that* $\mathscr{A}(M)$ *is non-trivial, we shall look whether the quotient space* Q *is non-trivial in the sense of vector space. In fact, any solution of the field equation can be written as the image under* E *of some element of* $C_0^{\infty}(\mathbb{R})$*, it follows from* *the isomorphism theorems (as vector spaces),*

$$\begin{aligned} Q &:= C_0^\infty(M)/PC_0^\infty(M) = C_0^\infty(M)/\ker E \cong \operatorname{im} E = \ker P \\ &\cong \{\varphi \in C_{sc}^\infty(M) : P\varphi = 0\} =: \mathrm{Sol}(M) \end{aligned} \tag{3.24}$$

where we used equation (3.18) *together with the definition of 'spatially compact' support i.e., functions that vanish in the causal complement of the set. Thus, Q is isomorphic to the space of smooth Klein-Gordon solutions with spatially compact support (*$\mathrm{Sol}(M)$*) and therefore, Q is nontrivial. We are in position to verify the assumptions of AQFT.*

A1 *If $\mathcal{O}_1 \subset \mathcal{O}_2$ then, clearly $f \in C_0^\infty(\mathcal{O}_1) \subset C_0^\infty(\mathcal{O}_1)$. It follows that $\mathscr{A}(\mathcal{O}_1) \subseteq \mathscr{A}(\mathcal{O}_2)$ and thus, isotony is satisfied.*

A2 *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are causally disjoint regions, let $f \in C_0^\infty(\mathcal{O}_1)$ and $g \in C_0^\infty(\mathcal{O}_2)$ then, $E_P g$ is supported in the union of the causal future and past of* $\operatorname{supp} g$ *and thus, the operator defined in* (3.22) *give $E_P(f,g) = 0$. It follows that all generators of $\mathscr{A}(\mathcal{O}_1)$ commute with all generators of $\mathscr{A}(\mathcal{O}_2)$ and thus, Einstein causality is satisfied.*

A3 *Let $N \subseteq M$, notice that the algebra defines a trivial morphism map $\alpha_{M;N} : \mathscr{A}(N) \to \mathscr{A}(M)$*

$$\alpha_{M;N}\varphi(f) = \varphi(f), \quad f \in C_0^\infty(N) \subseteq C_0^\infty(M)$$

in which every $\varphi(f)$ can be seen as an element of $\mathscr{A}(M)$ with generator f supported on $N \subseteq M$ satisfying the compatibility requirement.

A4 *Let $\mathcal{O}_1 \subseteq \mathcal{O}_2$ such that $\mathcal{O}_1$ contains a Cauchy surface of $\mathcal{O}_2$. Choose $\varphi(f_2) = Ef_2 \in \mathcal{O}_2$ for some $f_2 \in C_0^\infty(\mathcal{O}_2)$. We want a $f_1 \in C_0^\infty(\mathcal{O}_1)$ such that $\varphi(f_2) = \varphi(f_1)$. Let $f_1 := P\chi\phi(f_2)$, where $\chi \in C^\infty(\mathcal{O}_2)$ and is defined so that χ vanishes on the future of one Cauchy surface of $\mathcal{O}_2$ contained in $\mathcal{O}_1$ and is 1 in the past of another (and thus, 1 in the Cauchy surface of $\mathcal{O}_1$). Then, f_1 is supported in $\mathcal{O}_1$ and $\varphi(f_2) = \varphi(f_1)$, which implies that $\mathscr{A}(\mathcal{O}_2) = \mathscr{A}(\mathcal{O}_1)$ establishing the time-slice property.*

A5 *Let $\rho \in \mathcal{P}_0$. Then the Poncairé covariance of $\Box + m^2$ and E can be used to show that the map of generators $\alpha(\rho)\varphi(f) = \varphi(\rho_* f)$, $(\rho_* f)(x) = f(\rho^{-1}(x))$, is compatible with the relations and extends to a well defined unit preserving $\star-$isomorphism*

$$\alpha(\rho) : \mathscr{A}(M) \to \mathscr{A}(M).$$

Clearly $\alpha(\rho)$ maps each $\mathscr{A}(\mathcal{O})$ to $\mathscr{A}(\rho\mathcal{O})$; as we also have $\alpha(\sigma) \circ \alpha(\rho) = \alpha(\sigma \circ \rho)$, condition 4 holds.

Therefore, the algebra $\mathscr{A}(M)$ generated by $\varphi(f)$ and its relations satisfies the AQFT assumptions.

Remark 3.2.2. *The real scalar field admits a global Z_2 gauge symmetry generated by $\varphi(f) \to -\varphi(f)$ and, if we consider a complex field, the algebra admits a further $U(1)$ gauge symmetry given on the* *generators by*

$$\eta_\alpha(\phi(f)) := e^{-i\alpha\varphi(f)}.$$

In either case, the "physically observable" algebra will be given by a further quotient consisting of all elements of $\mathscr{A}(\mathcal{O})$ that are invariant under $U(1)$ or $U(1) \rtimes Z_2$ gauge actions. Hence, the afore scalar field model a theory where fields $\varphi(x)$ and $-\varphi(x)$ can be physically distinguished for example, where opposite charge particles can be distinguished.

Example 3.2.2 (Real scalar field with external source, [33])**.** *Let $\rho \in D'(M)$ be a distribution such that $\overline{\rho(f)} = \rho(\overline{f})$.*

$$(\Box + m^2)\Phi_\rho = -\rho. \tag{3.25}$$

The $*-$algebra construction. *Let $N \subseteq M$, the net of $*-$algebras $\mathscr{A}(N)$ are generated by a unit element together with the generators*

$$\varphi_\rho(f) = \varphi(f) + \phi_\rho(f)1, \quad f \in C_0^\infty(N)$$

where $\phi_\rho \in D'(M)$ is any weak solution to $(\Box + m^2)\phi_\rho = -\rho$ and $\varphi(f)$ are the generators of the free field. Notice that the generators satisfy the relations

1. $f \to \phi_\rho(f)$ *is complex linear*
2. $\phi_\rho(f)^* = \phi_\rho(\overline{f})$
3. $\phi_\rho((\Box + m^2)f) + \rho(f)1 = 0$
4. $[\phi_\rho(f), \phi_\rho(g)] = iE(f,g)1$

Remark 3.2.3. *The algebras in the examples closely relate to each other, this is a general feature of AQFT which allow us to have a standard construction and, by hand, introduce further restrictions. However, the important construction is how the elements can be labeled by functions, in this case with the help of the Green operators.*

3.2.3 Quasifree states for the free scalar field

Given the relations and generators on an $*-$algebra $\mathscr{A}(M)$, an element $A \in \mathscr{A}(M)$ can be written as a polynomial

$$A = c_0 1 + \sum_{i_1} c_1^{i_i}\varphi(f_{i_1}^1) + \cdots + \sum_{i_1,\ldots,i_n} c_n^{i_1,\ldots,i_n}\varphi(f_{i_1}^n)\ldots\varphi(f_{i_n}^n) = c_i^{i_1,\ldots,i_k}\varphi(f_{i_1}^n\ldots f_{i_k}^n)$$

on the generators of $\mathscr{A}(M)$. Consequently, given a state $\omega : \mathscr{A}(M) \to \mathbb{C}$, its action on A is completely determined once the action of the $n-$point functions of ω,

$$\omega_n(f_1,\ldots,f_n) := \omega(\phi(f_1)\ldots\phi(f_n))$$

are known. Since $\mathscr{A}(M)$ is an $*-$algebra, we have not yet endowed $\mathscr{A}(M)$ with a topology and questions about continuity and convergence of the functionals $\omega_n(f_1,\ldots,f_n)$ may be discussed in the final topology of the ω_n's that is, where ω_n is continuous in the usual test function topology on $C_0^\infty(M)$. By the Schwartz kernel theorem [17], we can write the $n-$point function in terms of its distributional kernel:

$$\omega_n(f_1,\ldots,f_n) = \int_M \omega_n(x_1,\ldots,x_n) f_1(x_1)\ldots f_n(x_n) dv. \tag{3.26}$$

Hence, ω being normalized can be verified equivalent in the $n-$point functions by setting

$$1 = c_0 + \sum_{i_1} c_1^{i_i} \omega_1(f_{i_1}^1) + \cdots + \sum_{i_1,\ldots,i_n} c_n^{i_1,\ldots,i_n} \omega(f_{i_1}^n, \ldots f_{i_n}^n).$$

Despite this characterization, the positive requirement $\omega(A^*A) \geq 0$ is still unknown and needed to be verified for each observable A. Therefore, the construction of states and their corresponding GNS representations are nontrivial problems. For instance, given a "state" ω, the positive condition needs to verify $\omega(AA^*) \geq 0$ for all $A \in \mathscr{A}$ and, on the other hand, the GNS representation is needed to be specified by hand. Nevertheless, this task is greatly simplified if the state is one of the so-called quasifree states. Quasifree states (also called Gaussian states) became useful because they are completely determined by their two-point functions and their underlying Hilbert space is canonically built so that imitates the Fock space of the Minkowski vacuum (e.g. gives annihilator and creation operators). This constructions allows us to retrieve many results that are familiar from QFT (and in fact coincides with them).

Definition 3.2.4. *Let $\omega : \mathscr{A} \to \mathbb{C}$ be a state of the algebra $\mathscr{A}$. ω is said to be quasifree if its $n-$point functions satisfy*

$$\omega_n(f_1,\ldots,f_n) = \begin{cases} 0 & \text{for } n \text{ odd} \\ \sum_{\mathcal{G}_{2n}} \omega_2(x_{i_1}, x_{i_2}) \ldots \omega_2(x_{i_{n-1}}, x_{i_{i_n}}) & \text{for } n \text{ even} \end{cases}$$

where the sum is taken to all partitions of $\{1,\ldots,n\}$ into $n/2$ pairwise disjoint subsets of elements $\{i_1,i_2\},\{i_3,i_4\},\ldots,\{i_{n-1},i_n\}$ with $i_{2k-1} < i_{2k}$ for $k = 1,2,\ldots,n/2$.

Remark 3.2.4. *Since quasifree states are also characterized with the 2-point functions, we may write the two point function $\omega_2(f,g)$ for quasifree states as*

$$W(f,g) := \omega_2(f,g).$$

Therefore, a quasifree state can be written (that's why ω is also called Gaussian) as

$$\omega(e^{i\varphi(f)}) = e^{-W(f,f)/2}, \quad f \in C_0^\infty(M,R). \tag{3.27}$$

The importance of quasifree states relies on their representations. For this class of states the $*-$algebras and Weyl C^*-algebras give equivalent representations and therefore, as we saw earlier (or the appendix), many constructions are given explicitly.

Example 3.2.3. *The Minkowski vacuum state ω_0 is a quasifree state. Indeed, given a free field*

$\varphi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{\omega}}\left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right)$, the smeared field $\varphi(f)$ is given by

$$\varphi(f) = \int d^4x f(x)\varphi(x).$$

Therefore, the two-point function is given by

$$\begin{aligned}
W(f,g) &= \langle 0_M|\,\varphi(f)\varphi(g)\,|0_M\rangle \\
&= \langle 0_M|\int d^4y \frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega'}}f(y)\Big(a_{k'}e^{-ik'\cdot y} + a_k'^{\dagger}e^{ik'\cdot y}\Big) \\
&\qquad\qquad \times \int d^4x \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}f(x)\big(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\big)\,|0_M\rangle \\
&= \int d^4y \frac{d^3k'}{(2\pi)^3} d^4x \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}f(y)g(x)\,\langle 0_M|\,a_{k'}e^{-ik'\cdot y}a_k^\dagger e^{ik\cdot x}\,|0_M\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\overline{\hat{f}(k)}\hat{g}(k)
\end{aligned}$$

where $\omega = \sqrt{|k|^2 + m^2}$ and $\hat{g}(k) = \int d^4x e^{ik\cdot x}g(x)$.

One can define a one-particle Hilbert space in the following way[13]: Let $\mathcal{H}$ be the square integrable functions in the hyperboloid $H_m^+ = \{k\cdot k = m^2,\ k^0 > 0\}$ in $\mathbb{R}^4$ with respect to the measure

$$d\mu(S) = \frac{d^3k}{(2\pi)^3}\frac{\chi_S(k)}{2\omega}, \qquad \chi_S(k) = \begin{cases} 1 & k\in S \\ 0 & k\notin S \end{cases}.$$

Notice that the set of functions $\hat{f}$ supported on the hyperbolid is dense in $\mathcal{H}$; thus, we can define an inner product such that

$$\langle K\varphi(f), K\varphi(g)\rangle_{\mathcal{H}} = W(f,g)$$

where the map $K : \mathrm{Sol}_{\mathbb{R}}(M) \to \mathcal{H}$ is given by $KEg = \hat{g}|_{H_m^+}$, where $Eg := \varphi(g)$. In this case the Hilbert space is constituted as $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$. Then, a representation of $\varphi(g)$ is given by

$$\pi_0(\varphi(g)) = a(\hat{g}|_{H_m^+}) + a^*(\hat{g}|_{H_m^+}), \quad g\in C_0^\infty(M,\mathbb{R}),$$

where

$$a\left(\hat{g}|_{H_m^+}\right) = \underbrace{\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\overline{\hat{g}(k)}a(\boldsymbol{k})}_{\int d\mu\,\overline{\hat{g}(k)}a(\boldsymbol{k})}, \quad a^*\left(\hat{g}|_{H_m^+}\right) = \underbrace{\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\hat{g}(k)a^*(\boldsymbol{k})}_{\int d\mu\,\hat{g}(k)a^*(\boldsymbol{k})}. \tag{3.28}$$

If one chooses g such that $KEg = e^{ik\cdot r}$ we retrieve our usual free field

$$\pi_0(\varphi(g)) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\left(a(\boldsymbol{k})e^{-ik\cdot \mathrm{r}} + a^*(\boldsymbol{k})e^{ik\cdot\mathrm{r}}\right). \tag{3.29}$$

The above example gives an idea of how to build a (canonical) Hilbert space. In general, let W

[13] For further details such as existence and general features of quasifree states we refer to [35], particularly, Sec. 5.2.4

be the two-point function defined on $C_0^\infty(M)$. W obeys $W(f,g) - W(g,f) = iE(f,g)$ and therefore, let w be a skew-linear form on $\mathrm{Sol}(M)$ (defined in (3.24)) by the formula

$$w(Ef, Eg) = W(\overline{f}, g).$$

If the Cauchy-Scwartz inequality in w is saturated then, one can define a (complex) Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ where $\mathcal{H}$ is the completition of $\mathrm{Sol}_{\mathbb{R}}(M)$ in the norm $||\phi||_w := w(\phi,\phi)^{1/2}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is defined by

$$\begin{aligned}\langle\phi_1|\phi_2\rangle_{\mathcal{H}} &= \langle\phi_1|\phi_2\rangle_w + i\,\langle\phi_1|J\phi_2\rangle_w,\\ &= \mathrm{Re}\,w(\phi_1,\phi_2,) + i\,\mathrm{Im}\,w(\phi_1,\phi_2), \quad \phi_i \in \mathcal{H}\end{aligned}$$

where we used that multiplication by $-i$ is implemented by J because $w(\phi_1,\phi_2) = iw(\phi_1, J\phi_2)$. In this Hilbert space the inclusion of $\mathrm{Sol}_{\mathbb{R}}(M)$ into $\mathcal{H}$, $K : \mathrm{Sol}_{\mathbb{R}}(M) \to \mathcal{H}$, has dense image (being $\mathcal{H}$ the completition of $\mathrm{Sol}_{\mathbb{R}}(M)$) and therefore, the inner product is defined so that

$$\langle KEf|KEg\rangle_{\mathcal{H}} = W(f,g).$$

Accordingly, given a quasifree state ω, the representation $\pi_\omega : \mathscr{A}(M) \to \mathcal{H}$ is defined by the formula

$$\pi_\omega(\varphi(f)) = a(KEf) + a^*(KEf)$$

where $a(\psi)$ and $a^*(\psi)$ are the annihilation and creation operators on the Fock space which obey the CCRs

$$[a(\phi), a^*(\psi)] = \langle\phi|\psi\rangle_{\mathcal{H}}\,1$$

and, we also require that π_ω is C^*−linear,

$$\pi_\omega(\varphi(f)) := \pi_\omega(\varphi(\mathrm{Re}\,f)) + i\pi_\omega(\varphi(\mathrm{Im}\,f))a(KEf) + a^*(KEf)$$

for complex-valued $f \in C_0^\infty(M)$. Now, we check that the map π correctly represents the algebra.

- The representation obeys the field equation in the sense $\pi_\omega(\varphi(Pf)) = 0$
- The representation is hermitian by the quantization of the $*$−algebra $\mathscr{A}(M)$.
- For the CCRs, we compute

$$\begin{aligned}[\pi_\omega(\varphi(f)), \pi_\omega(\varphi(g))] &= [a(KEf), a^*(KEg)] + [a^*(KEf), a(KEg)]\\ &= \langle KEf|KEg\rangle_{\mathcal{H}}\,1 - \langle KEg|KEf\rangle_{\mathcal{H}}\,1\\ &= (W(f,g) - W(g,f))\mathbb{1}\\ &= iE(f,g)\mathbb{1}.\end{aligned}$$

- The Fock vacuum vector Ω_ω satisfies

$$\langle \Omega_\omega \mid \pi_\omega(\varphi(f_1)) \cdots \pi_\omega(\varphi(f_n)) \Omega_\omega \rangle = \omega(\varphi(f_1)\cdots\varphi(f_n)) . \tag{3.30}$$

  Indeed, using $\pi_0(\varphi(g)) = a(\hat{g}|_{H_m^+}) + a^*(\hat{g}|_{H_m^+})$, for an odd number n the right hand side of equation 3.30 vanishes as we require. If n is even, clearly for $n = 0, 2$ the equality holds and, for $n > 3$, using using the CCRs in the inductive step it is not hard to see that the equality holds.

Consequently, ω is seen to be a vector state on $\mathscr{A}(M)$ and $(\mathcal{F}(\mathcal{H}), \pi_\omega, \Omega_\omega)$ is its GNS representation whose dense domain $\mathscr{D}_\omega$ consisting of finite linear combinations of finite products of operators $a^*(KEf)$ acting on Ω_ω.

Remark 3.2.5. *Notice that one-particle states (elements of $\mathcal{H}$) can be identified with (complex linear combinations of) vectors generated from Ω_ω by a single application of the field as shown in the example. Due to remark 1.1.3, these vectors are identified as wavepackets of positive frequency modes relative to the choice of the quasifree state ω.*

3.2.4 Vacuum states and the detector problem

Let G be the symmetry group of M e.g., M is Minkowski and G the Poincaré group. A vacuum state in a algebra $\mathscr{A}$ must be at least translation invariant under the action of G but it may also coincide with the definition of being defined by the positive frequency modes that is, a vacuum state must have positive energy in every Lorentz frame. Neither of the above conditions are trivial nor satisfied with the above assumptions **A1-A5**; we first formalize the concept vacuum states and afterwards we introduce a further assumptions in order to define a vacuum in an AQFT. We follow [4, 29].

Let ω be a translationally invariant state. As we saw in the GNS construction, the translation group can be implemented unitarily by a strongly continuos function $U(x)$ which can also be written as $U(x) = e^{iP_\mu x^\mu}$ where P^μ are self-adjoint operators by Stone's theorem. In this case, the projection operator $E(\Delta)$ corresponding to the test of whether the result of 4−momentum measurement P^μ would be found to lie in a measurable set Δ. The assignment $\Delta \mapsto E(\Delta)$ is then a Projective Valued Measure (cf. Appendix B) and, one can write

$$U(x) = \int e^{ip_\mu x^\mu} dE(p) .$$

Definition 3.2.5. *The state ω is said to satisfy the spectrum condition if the support of E lies in the closed forward cone $\overline{V^+} = \left\{p \in \mathbb{R}^4 : p^\mu p_\mu \geq 0, p^0 \geq 0\right\}$*[14]*.*

Definition 3.2.6. *A vacuum state is a translationally invariant state obeying the spectrum condition, whose GNS vector is the unique translationally invariant vector (up to scalar multiples) in the GNS Hilbert space. The corresponding GNS representation is called the vacuum representation.*

A6† Weak additivity For any causally convex open region $\mathcal{O}$ the $*-$algebra is

[14]This implies that the spectrum of P^μ lies in $\overline{V^+}$.

generated by the algebras $\pi(\mathscr{A}(O+x))$ as x runs over $\mathbb{R}^4$.

Remark 3.2.6.

- *The algebra generated by translation is the algebra whose elements are of the form*

$$U(x_1)Q_1U^{-1}(x_1)\dots U(x_n)Q_nU^{-1}(x_n) = U(x_1)\,Q_1U(x_2-x_1)\,Q_2U(x_3-x_2)\cdots U(x_n-x_{n-1})\,Q_n$$

for $n \in \mathbb{N}$ and some $Q_1, ..., Q_n \in \pi(\mathscr{A}(\mathcal{O}+x))$, $x \in \mathbb{R}^4$. Weak additivity not only asserts that that this algebra is dense in $\mathscr{A}(M)$ but also that arbitrary observables can be built as limits of algebraic combinations of translates of observables in any given region $\mathcal{O}$ (as one would expect in a quantum field theory).

- *The translation group is Abelian and thus its mean μ is a translationally invariant, positive linear functional of bounded functions on the group G which is measurable with respect to the Haar measure. As usual, we can define an averaged state ρ which is of course invariant. Indeed, given a state ω of A define ρ as*

$$\rho(A) := \int \omega(\alpha_x A)\,d\mu(x).$$

Then, if α is an action of the translation group on $\mathscr{A}$, translation invariant states of $\mathscr{A}$ exist. In general, the group symmetry is not Abelian for example, the Lorentz group does not admit an invariant mean [29] and thus, the existence of invariant states is not guaranteed.

One immediate consequence (and perhaps the most important) of assumption **A6** is the Reeh-Schlider theorem,

Theorem 3.2.7 (Reeh-Schlieder)**.** *Let $\mathcal{O}$ be any causally convex bounded open region.*

(***a***) *The set $\{A\Omega : A \in \pi(\mathscr{A}(\mathcal{O}))\}$ is dense in $\mathcal{H}$ (the GNS vector is cyclic)*

(***b***) *If $A \in \pi(\mathscr{A}(\mathcal{O}))$ is such that $A\Omega = 0$ then, $A = 0$ (the GNS vector is separating).*

The existence of a cyclic and separating vector on a (von Neumann) algebra is the starting point of Tomita–Takesaki theory.

Proof.

(**a**) If Ω is not cyclic in $\pi(\mathscr{A}(\mathcal{O}))$ there exist a vector Ψ such that $\langle \Psi \mid A\Omega\rangle = 0$ for all $A \in \pi(\mathcal{A}(\mathcal{O}))$. Let $\mathcal{O}_1$ be a causally open region such that[15] $\overline{\mathcal{O}_1} \subset \mathcal{O}$ and, for any $Q_1, \dots, Q_n \in \pi(\mathcal{A}(\mathcal{O}_1))$ there exist an $\varepsilon > 0$ such that for all $\|x\| < \varepsilon$ we have $U(x)Q_iU(x)^{-1} \in \pi(\mathcal{A}(O))$ and

$$\langle \Psi \mid U(x_1)\,Q_1U(x_2-x_1)\,Q_2U(x_3-x_2)\cdots U(x_n-x_{n-1})\,Q_n\Omega\rangle = 0 \tag{3.31}$$

[15] This $\mathcal{O}_1$ always exist because the space is Hausdorff and regular.

for $\|x_1\| + \ldots + \|x_n\| < \varepsilon$. The spectrum condition implies that $U(x)$ can be extended to complex vectors. In this case, $U(x)$ is strongly continuous on $\mathbb{R}^4 + i\overline{V^+}$ and holomorphic on $\mathbb{R}^4 + i\,\mathrm{int}\,\overline{V^+}$. It follows that the function

$$F : (\zeta_1, \ldots, \zeta_n) \mapsto \langle \Psi \mid U(\zeta_1)\, Q_1 U(\zeta_2)\, Q_2 \cdots U(\zeta_n)\, Q_n \Omega \rangle$$

is continuous in $\left(\mathbb{R}^4 + i\,\mathrm{int}\,\overline{V^+}\right)^n \subset \left(\mathbb{C}^4\right)^n$ whose boundary value on $\left(\mathbb{R}^4\right)^n$ vanishes in $\left(\{\|x\| < \varepsilon/n : x \in \mathbb{R}^4\}\right)^n$. The 'edge of the wedge' theorem[16] [17] implies that F vanishes identically in $\left(\mathbb{R}^4 + i\,\mathrm{int}\,\overline{V^+}\right)^n$ i.e. condition (3.31) holds for all x_i, $i = 1, ..., n$. By weak additivity, the function $A \to \langle \psi | A\Omega \rangle$ vanishes in the dense set of translations and therefore vanishes in $\pi(\mathscr{A}(M))$ i.e. $\langle \Psi \mid \pi(\mathcal{A}(\mathbb{M}))\Omega \rangle = 0$ but Ω_ω is cyclic in $\mathcal{H}$ which means that $\Psi = 0$.

(b) Suppose that $A \in \pi(\mathcal{A}(\mathcal{O}_1))$ annihilates Ω and choose $\mathcal{O}_2 \subset \mathcal{O}_1^\perp$. For each $\Psi \in \mathcal{H}$ and all $B \in \pi(\mathcal{A}(\mathcal{O}_2))$ we have

$$\langle A^* \Psi \mid B\Omega \rangle = \langle \Psi \mid AB\Omega \rangle = \langle \Psi \mid BA\Omega \rangle = 0$$

using Einstein causality. The left most side implies that $A^*\Psi$ vanishes ($A^*\Psi$ is orthogonal to a dense) and, as $\Psi \in \mathcal{H}$ is arbitrary it follows $A = 0$.

■

Remark 3.2.8.

- *The Reeh-Schlieder (R-S) theorem tell us that any operation performed in $\mathcal{O}$ may be approximated as close as we want by realizing operations in a region elsewhere. Furthermore, the vacuum is separating for each local algebra which implies that every local event has a nonzero probability of occurring in the vacuum state, giving no possibility of existing a local number operators. This, in turn, tells us how deeply entangled states in relativistic QFT typically are, worse than non-locality in non-relativistic QM.*

- *The use of the spectrum condition may indicate that the R-S theorem is a purely relativistic result which many authors [4, 29] argue has no analog in non-relativistic QM or QFT. Indeed, we will see that the Unruh/Hawking effects are consequence of the Reeh-Schlieder theorem.*

- *As we pointed out in the first chapter, one may require a notion of a particle in which a detector registers an outcome of an experiment. The problem of defining a detector in this sense lie in the R-S theorem. A detector is a well-localized (in a bounded region K) observable C that registers 0 in the vacuum state and 1 in any other state. It follows, by the Reeh-Schlieder theorem, that C must be identically zero, therefore the detector cannot be defined as an observable! Nonetheless, the solution to this problem also lies in the R-S theorem, mainly we can approximate the localization of C with observables in $\mathscr{A}$ due to the cyclic property.*

[16] Holomorphic functions on two "wedges" with an "edge" in common are analytic continuations of each other provided they both give the same continuous function on the edge.

3.3 The Unruh effect (Part II: An algebraic approach)

Section 1 was devoted to derive the Unruh effect in a two-dimensional space. In this section, we derive a rigorous characterization of the Unruh effect using the full power of algebraic quantum field theory, which is essentially the result of the Bisognano-Wichmann theorem described below. Let $\mathscr{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, ω a faithful, normal state and let Ω the representation of the state ω. Recall that the state ω is a KMS state if Ω satisfy the KMS condition

$$J\exp\{(-\beta K/2)\}A\Omega = A^*\Omega, \quad \text{for all } A \in \mathscr{A}, \tag{3.32}$$

where J is the unique anti-linear transformation of $\mathcal{H}$ such that

$$J^2 = 1 \text{ and } (Jf, Jg) = (g, f).$$

Notice that once the KMS state is verified, equation (3.32) determines J and β uniquely therefore, we shall verify condition (3.32). Let M be Minkowski and denote by $M^{\pm}$ the right and left Rindler wedges. If $x = (x^0 = ct, ..., x^3) \in M$ then T and L defined by

$$\begin{aligned} T(t)x^\mu &= (x^0 + ct, x^1, x^2, x^3), \\ L(\tau)x^\mu &= (x^0 \cosh\tau + x^1 \sinh\tau, x^1 \cosh\tau + x^0 \sinh\tau, x^2, x^3), \end{aligned}$$

are the subgroups of the Poncairè group related to time-translation and Lorentz boosts respectively. Let us see how L acts on M^+. Given a point $x^\mu = (\xi, \tau', x^2, x^3) \in M^+$ then, we have

$$\begin{aligned} L(\tau')x^\mu &= (a^{-1}e^{a\xi}\sinh\tau\cosh\tau' + a^{-1}e^{a\xi}\cosh\tau\sinh\tau', a^{-1}e^{a\xi}\cosh\tau'\cosh\tau + a^{-1}e^{a\xi}\sinh\tau\sinh\tau', x^2, x^3) \\ &= (a^{-1}e^{a\xi}\sinh(\tau+\tau'), a^{-1}e^{a\xi}\cosh(\tau+\tau'), x^2, x^3) \end{aligned}$$

where we use the fact that $t = a^{-1}e^{a\xi}\sinh\tau'$ and $z = a^{-1}e^{a\xi}\cosh\tau'$. It follows that Lorentz boosts in the right Rindler wedge act by the equation

$$L(\tau)(\xi, \tau', x^2, x^3) = (\xi, \tau' + \tau, x^2, x^3) \in M^+. \tag{3.33}$$

Similarly, one can show the same for M^-. Thus, $M^{\pm}$ are stable under L and L is the time translation group of the Rindler wedges. In order to formulate the Unruh effect, let us state a few theorems related with the algebraic theory of the scalar field satisfying the assumptions A1-A7.

Theorem 3.3.1 (PCT theorem [17])**.** *There is a unique J_0 of $\mathcal{H}$ such that*

$$J_0\varphi(f_1)...\varphi(f_k)\Omega = \varphi(f_k^\dagger)...\varphi(f_1^\dagger)\Omega, \quad \forall f_1, ..., f_k \in \mathscr{S}(X)$$

where $f^\dagger(x) := f^*(-x) = \overline{f}(-x)$ and Ω is the vacuum state.

Theorem 3.3.2 (Bisognano-Wichmann [28])**.** *Let iK be the infinitesimal generator of $\hat{L} = U(L)$. Then,*

$$J_0\hat{\rho}e^{-\pi K}A\Omega = A^*\Omega, \quad \forall A \in \mathscr{A}(M^+),$$

where J_0 is the PCT conjugation, $\hat{\rho} = U(\rho)$ and ρ is the partial inversion $(x^2, x^3) \mapsto (-x^2, -x^3)$.

Proof. See in [28]; for a general result in curved space times look in [36]. ■

The Bisognano-Wichmann (BW) theorem tells us that the vacuum state Ω satisfies the KMS condition at a specified temperature with respect to the Lorentz boosts

$$\begin{cases} t \to t\cosh\beta + z\sinh\beta \\ z \to t\sinh\beta + z\cosh\beta \end{cases}$$

if Lorentz boost were the time translation group which inturn, generates the Hamiltonian. Therefore, if ψ is restricted to the right Rindler wedge then, by means of (3.33) Lorentz boosts correspond to time-translations $\tau \to \tau + \tau'$ for uniformly accelerated observers which generates the dynamics of dynamical system of the algebra $\mathscr{A}(M)$. It follows that the B-W theorem states the Unruh effect provided that the observer is restricted to the Rindler wedges.

Proposition 3.3.1. *The restriction of the state ψ to $\mathscr{A}(M^+)$ is a thermal one satisfying the following KMS condition with respect the time-translation group $\hat{L}^+ := U(L)$ for O_{acc}.*

$$Je^{-\pi K^+} A\psi = A^*\psi, \quad \forall A \in \mathscr{A}(M^+), \tag{3.34}$$

where iK^+ is the generator of the restriction of L in M^+, $J = J_0\hat{\rho}$, J_0 is the PCT operator and $\hat{\rho}$ the unitary representative of the partial inversion $(x^2, x^3) \mapsto (-x^2, -x^3)$.

Remark 3.3.3.

- *In this derivation it becomes clear that the conjugation operator J is the PCT operator J_0 corrected by the partial inversion $\hat{\rho}$. Thus, J is the PCT conjugation associated with the restricted inversion $(x^0, x^1) \mapsto (-x^0, -x^1)$. In the $1+1-$dimensional case, the partial inversion ρ is trivial and thus, J is the PCT operator as we saw in remark 3.1.14.*

- *Comparing equations* (3.34) *and* (3.32) *one finds that the temperature of the thermal state Ω satisfies*

$$\frac{K}{2kT} = \pi K.$$

This implies that the temperature of the thermal state is

$$T = \frac{1}{2\pi k}.$$

However, this temperature is based on the time τ instead of the proper time $\tau_\alpha = \xi\tau/c = c\tau/a$. Therefore, the temperature observed by O_{acc} will correspond to a re-scaling of the time τ to τ_α, i.e.,

$$T_\alpha = \frac{a}{2\pi kc} = \frac{a}{2\pi}$$

where we used natural units. This establishes a rigorous generalization of Unruh's result for fields in flat space-time.

- *The Unruh effect arises from the B-W theorems which, in turn, derives from the PCT and R-S theorems. Therefore, as long these theorems can be generalized to arbitrary quantum fields in spacetime then, the Unruh effect will still hold. This is, for example, the case of the Hawking effect and many other results.*

4 Quantum Measurements in Relativistic Quantum Field Theory

Having introduced the power of AQFT, we are now in position to discuss the Fewster and Verch measurement scheme. This scheme behaves in the spirit of the von Neumann measurement scheme, but with particular emphasis in taking account the locality of the theory. The measurement takes the following steps,

1. Prepare the system,
2. Make a measurement,
3. Measure the probe elsewhere.

In this chapter we introduce the relevant concepts in order to formalize the steps above and determine the correlation between the probe and the system. We strongly follow [7] for the general measurement scheme and, [21] for Sorkins protocol section. Notations (and some examples) regarding causal/convex geometry in curved spacetimes can be found in Appendix E if not defined in the main body.

4.1 A General Measurement Scheme on AQFT

4.1.1 The framework

Let M be a space-time. We take the probe and the system coupled in a compact region $K \subset M$. The probe and the system are equipped with an (algebraic) quantum field theory $\mathscr{B}$ and $\mathscr{A}$ respectively, e.g., both being free fields. Given the compact region K, define $M^{\pm} = M \backslash J^{\mp}(K)$.

We start with the uncoupled algebra of observables $\mathscr{U}(M) = \mathscr{A}(M) \otimes \mathscr{B}(M)$ and the coupled algebra $\mathscr{C}(M)$ equipped with the compatibility map (required in assumption **A3**) $\alpha_{M;N} \otimes \beta_{M;N}$ and $\gamma_{M;N}$ respectively. If there is no risk of confusion, we drop the sub-indices for ease of read.

The interaction takes place in region K and therefore, we shall assume that $\mathscr{C}$ reduces to $\mathscr{A} \otimes \mathscr{B}$ outside K. That is, for every $L \subset M \backslash K$ there exist an isomorphism $\chi_L : \mathscr{A}(L) \otimes \mathscr{B}(L) \to \mathscr{C}$ such that the following diagram commutes:

i.e. χ is compatible with the coupled and uncoupled structures.

Remark 4.1.1. *These regions are determined covariantly once the interaction region is specified and without reference to any observer's clock or time coordinate thus, making the following construction covariant.*

$$\begin{tikzcd}
\mathscr{A}(L') \otimes \mathscr{B}(L') \arrow[r, "\alpha_{L;L'} \otimes \beta_{L;L'}"] \arrow[d, "\chi_{L'}"'] & \mathscr{A}(L) \otimes \mathscr{B}(L) \arrow[d, "\chi_{L}"'] \\
\mathscr{C}(L') \arrow[r, "\gamma_{L;L'}"] & \mathscr{C}(L)
\end{tikzcd}$$

Figure 6: Compatibility of χ.

Assume the regions $M^{\pm}$ are big enough to have Cauchy surfaces then, by the time-slice property the morphisms α, β and γ are also isomorphism. We define

$$\kappa^{\pm} = \gamma^{\pm} \circ \chi^{\pm} : \mathscr{A}\left(M^{\pm}\right) \otimes \mathscr{B}\left(M^{\pm}\right) \to \mathscr{C}(M).$$

Definition 4.1.1. *Let $M, \mathscr{U}, \mathscr{C}, \alpha, \beta, \gamma$ and χ as above. Define $\tau^{\pm} : \mathscr{U}(M) \to \mathscr{C}(M)$ by*

$$\tau^{\pm} := \kappa^{\pm} \circ \left(\alpha^{\pm} \otimes \beta^{\pm}\right)^{-1} = \gamma^{\pm} \circ \chi^{\pm} \circ (\alpha^{\pm} \circ \beta^{\pm})^{-1}.$$

$\tau^{\pm}$ defines an isomorphism that identifies the uncoupled system with the coupled at early $(-)$ or late $(+)$ times; $\tau^{\pm}$ is said to be the retarded $(+)$ and advanced $(-)$ response maps.

Definition 4.1.2. *We define the scattering morphism $\Theta : \mathscr{U}(M) \to \mathscr{U}(M)$ as*

$$\Theta = (\tau^{-})^{-1} \circ \tau^{+}.$$

The scattering map determines the evolution after an interaction in the region K has been made.

Proposition 4.1.1. *Let Θ be the scattering morphism of the interaction taking place in K; then, the following statements hold:*

(***a***) *If $K \subset \hat{K}$ with $\hat{K}$ compact then, the scattering morphism $\hat{\Theta}$ constructed in $\hat{K}$ is the same as Θ.*

(***b***) *If $L \subset K^{\perp}$ then Θ acts trivially on $\mathcal{U}(M;L)$*

(***c***) *If $L^{\pm} \subset M^{\pm}$ open casually convex and $L^{+} \subset D(L^{-})$. Then $\Theta(\mathcal{U}(M;L^{+})) \subset \mathcal{U}(M;L^{-})$*

Proof. All of them are a consequence of χ being a isomorphism and the definition of the Cauchy development.

(**a**) Notice that $\hat{M}^{\pm} = M \backslash J^{\mp}(\hat{K}) \subset M^{\pm}$. Thus, we can define $(\alpha^{\pm})^{-1} \circ \hat{\alpha}^{\pm} = \alpha_{M^{\pm}, \hat{M}^{\pm}}$ by the compatibility requirement and the time-slice property. Observe that

$$\kappa^{\pm} \circ \left(\alpha^{\pm} \otimes \beta^{\pm}\right)^{-1} \circ \left(\hat{\alpha}^{\pm} \otimes \hat{\beta}^{\pm}\right) = \gamma^{\pm} \circ \chi^{\pm} \circ \left(\alpha_{M^{\pm};\hat{M}^{\pm}} \otimes \beta_{M^{\pm};\hat{M}^{\pm}}\right) = \gamma^{\pm} \circ \gamma_{M^{\pm};\hat{M}^{\pm}} \circ \hat{\chi}^{\pm} = \hat{\gamma}^{\pm} \circ \hat{\chi}^{\pm}$$
$$= \hat{\kappa}^{\pm}$$

it follows that $\hat{\tau}^{\pm} = \tau^{\pm}$ and hence $\hat{\Theta} = \Theta$.

Suppose $L \subset M^{\pm}$ is an open causally convex set then,

$$\kappa^{\pm} \circ \left(\alpha_{M^{\pm};L} \otimes \beta_{M^{\pm};L}\right) = \gamma^{\pm} \circ \chi_{M^{\pm}} \circ \left(\alpha_{M^{\pm};L} \otimes \beta_{M^{\pm};L}\right) = \gamma^{\pm} \circ \gamma_{M^{\pm};L} \circ \chi_L = \gamma_{M;L} \circ \chi_L. \tag{4.1}$$

(b) Assume $L \subset M^{+} \cap M^{-}$, using (4.1) we have

$$\tau^{+} \circ (\alpha_{M;L} \otimes \beta_{M;L}) = \tau^{-} \circ (\alpha_{M;L} \otimes \beta_{M;L}),$$

and hence

$$\Theta \circ (\alpha_{M;L} \otimes \beta_{M;L}) = (\alpha_{M;L} \otimes \beta_{M;L}).$$

(c) We apply equation (4.1) to $L^{+} \subset M^{+}$ and $L^{-} \subset M^{-}$, giving

$$\tau^{+} \circ \left(\alpha_{M;L^{+}} \otimes \beta_{M;L^{+}}\right) = \gamma_{M;L^{+}} \circ \chi_{L^{+}}, \quad \tau^{-} \circ \left(\alpha_{M;L^{-}} \otimes \beta_{M;L^{-}}\right) = \gamma_{M;L^{-}} \circ \chi_{L^{-}}$$

the first of which asserts that $\tau^{+} \circ (\alpha_{M;L^{+}} \otimes \beta_{M;L^{+}})$ factors through $\gamma_{M;L^{+}}$, while the second implies that $(\tau^{-})^{-1} \circ \gamma_{M;L^{-}}$ factors through $\alpha_{M;L^{-}} \otimes \beta_{M;L^{-}}$. As $\gamma_{M;L^{+}}$ factors via $\gamma_{M;L^{-}}$ due to the assumption $L^{+} \subset D(L^{-})$ and the timeslice property, the two obsrvations combine to show that $\Theta \circ (\alpha_{M;L^{+}} \otimes \beta_{M;L^{+}})$ factors through $\alpha_{M;L^{-}} \otimes \beta_{M;L^{-}}$, as required.

■

Remark 4.1.2. *Part (a) shows that the scattering morphism is canonically associated with the theories $\mathscr{U}$ and $\mathscr{C}$ and the identifications between them, while parts (b) and (c) are locality properties. In particular, part (b) shows that, as one would expect, the coupling has no effect on observables localized in $L \subset K^{\perp}$.*

The first property captures the idea that the coupling has no effect in spacelike separated regions, whereas the second property indicates how Θ changes the localisation of observables.

4.1.2 The Measurement process

Given the dynamics of the system, we want to describe the measurements of the coupled $\mathscr{C}(M)$ in terms of the uncoupled observables. Suppose the system to be measured is in a state ω of $\mathscr{A}$ and prepare the probe in a state σ of $\mathscr{B}$. In terms of the response maps, we can identify this initial state as

$$\underset{\sim}{\omega}_{\sigma} = ((\tau^{-})^{-1})^{*}(\omega \otimes \sigma), \quad \underset{\sim}{\omega}_{\sigma}(X) = (\omega \otimes \sigma)((\tau^{-})^{-1}X)$$

which is uncorrelated at early times. Let $B \in \mathscr{B}(M)$ be an observable of the probe, then in the late times we can identify B with the observable

$$\tilde{B} = \tau^{+}(1 \otimes B) \in \mathscr{C}(M).$$

Therefore, the measurement in the probe after the interaction has been made (expectation value) is

$$\underset{\sim}{\omega}_\sigma(\tilde{B}) = (\omega \otimes \sigma)(\Theta(1 \otimes B)).$$

We want to identify observables in the system $\mathscr{A}$ as observables in $\mathscr{B}$ such that the measurement of the probe is interpreted as measurements on the system (provided that σ was prepared). That is, we want an identification $A \leftrightarrow B$ such that

$$\omega(A) = \underset{\sim}{\omega}_\sigma(\tilde{B}), \quad \text{for all states } \omega \text{ in } \mathscr{A}(M). \tag{4.2}$$

In order to determine A, consider the linear and continuous map $\eta_\sigma : \mathscr{A} \otimes \mathscr{B} \to \mathscr{A}$ such that

$$\begin{aligned} \eta_\sigma : \mathscr{A} \otimes \mathscr{B} &\to \mathscr{A} \\ A \otimes B \ &\mapsto \eta_\sigma(A \otimes B) = \sigma(B)A. \end{aligned}$$

Then, define A as

$$A = \varepsilon_\sigma(B) \coloneqq \eta_\sigma(\Theta(1 \otimes B)). \tag{4.3}$$

Taking expectation value of A,

$$\omega(A) = \omega(\eta_\sigma(\Theta(1 \otimes B))) = (\omega \otimes \sigma)(\Theta(1 \otimes B)) = \widetilde{\omega_\sigma}(\tilde{B}) \tag{4.4}$$

we see that A is what we have been looking for. Therefore, given an observable B, we can deduce the observable $A \coloneqq \varepsilon_\sigma(B)$ of the system.

Definition 4.1.3. *Let B and σ be an observable and a state on the probe, respectively. The observable $A \coloneqq \varepsilon_\sigma(B) \in \mathscr{A}(M)$ is said to be the induced system observable.*

Remark 4.1.3. *Let $C = D \otimes E \in \mathscr{A}(M) \otimes \mathscr{B}(M)$ and $A \in \mathscr{A}(M)$ then,*

$$\begin{aligned} A\eta_\sigma(C) &= A\sigma(E)D = \eta_\sigma(AD \otimes E) \\ &= \eta_\sigma((A \otimes 1)C), \\ \eta_\sigma(C)A &= \sigma(E)DA = \eta_\sigma(DA \otimes E) \\ &= \eta_\sigma(C(A \otimes 1)). \end{aligned}$$

The map η_σ is completely positive: its tensor products with any finite-dimensional matrix identity map preserve positivity

Proposition 4.1.2. *Prepare the system $\mathscr{B}$ in a state σ, and let $B \in \mathscr{B}$ and $A = \varepsilon_\sigma(B)$ the induced observable. If the states (of $\mathscr{A}$) separate*[17] *observables on $\mathscr{A}$, then A is the unique solution to (4.2). Furthermore, ε_σ is a completely positive linear map and the following statements holds,*

(*a*) $\varepsilon_\sigma(1) = 1$

(*b*) $\varepsilon_\sigma(B^*) = \varepsilon_\sigma(B)^*$ *and if B is self adjoint so is the induced observable.*

[17] ω is said to separate the observables $A, B \in \mathscr{A}(M)$ if $\omega(A) \neq \omega(B)$

(*c*) *$\varepsilon_\sigma(B^*B) - \varepsilon_\sigma(B)^*\varepsilon_\sigma(B)$ is a positive operator*

(*d*) *The measurement $\underset{\sim}{\omega}_\sigma(\widetilde{B})$ is less sharp*[18] *than the hypothetical measurement of the induced observable in the system state.*

(*e*) *For fixed B, the map $\sigma \mapsto \varepsilon_\sigma(B)$ is weak $*-$continuous .*

Proof. If $A, E \in \mathscr{A}(M)$ are two different solutions of the relation (4.2) then,

$$\omega(A) = \omega(E)$$

for all states ω in $\mathscr{A}(M)$, a contradiction. Therefore, the solution of (4.2) is unique.

(**a**) $\varepsilon_\sigma(1) = \eta_\sigma(\Theta(1 \otimes 1)) = \eta_\sigma(1 \otimes 1) = 1$.

(**b**) Note that $\eta_\sigma((A \otimes B)^*) = \eta_\sigma(A^* \otimes B^*) = \sigma(B^*)A^* = (\eta_\sigma(A \otimes B))^*$. It follows that $\varepsilon_\sigma(B^*) = \varepsilon_\sigma(B)^*$. Clearly self adjoint observables induce self adjoint observables.

(**c**) We need to show that η_σ is completely positive.

The positive operators of a $*-$ algebra are finite convex combinations of elements A^*A. Let $N > 0$ and consider an element $C \in M_N(\mathbb{C}) \otimes \mathscr{A}(M) \otimes \mathscr{B}(M)$,

$$C = \sum_r M_r \otimes A_r \otimes B_r.$$

We shall see that $X = (Id_N \otimes \eta_\sigma)(C^*C)$ is positive in $M_N \otimes \mathscr{A}(M)$.

$$\begin{aligned} X &= \sum_{r,s} (Id_N \otimes \eta_\sigma)(M_r^* M_s \otimes A_r^* A_s \otimes B_r^* B_s) \\ &= \sum_{r,s} \sigma(B_r^* B_s) M_r^* M_s \otimes A_r^* A_s. \end{aligned}$$

σ is a state and thus, $\sigma(B_r^* B_s)$ is a positive matrix which can be decomposed as

$$\sigma(B_r^* B_s) = \sum_i \overline{v_s^{(i)}} v_s^{(i)}$$

for mutually orthogonal vectors $v^{(i)}$. Then,

$$X = \sum_i \left(\sum_s v_s^{(i)} M_s \otimes A_s \right)^* \left(\sum_s v_s^{(i)} M_s \otimes A_s \right).$$

Therefore, η_σ is completely positive.

$C = \sum_r M_r \otimes B_r \in M_N(\mathbb{C}) \otimes \mathscr{B}(M)$,

$$(Id_N \otimes \varepsilon_\sigma)(C^*C) = (Id_N \otimes \eta_\sigma)(D^*D) \geq 0$$

where $D = \sum_r M_r \otimes \Theta(1 \otimes B_r)$.

[18] if the occurrence of the sharp measurement always implies the other's occurrence.

Finally, let $B \in \mathscr{B}(M)$ and define $C = \Theta(1 \otimes B) \in \mathscr{A}(M) \otimes \mathscr{B}(M)$, which can be decomposed as

$$C = \sum_r A_r \otimes B_r.$$

Then,

$$\varepsilon_\sigma(B^* B) - \varepsilon_\sigma(B)^* \varepsilon_\sigma(B) = \sum_{r,s} \Big(\sigma(B_r^* B_s) - \overline{\sigma(B_r)} \sigma(B_s) \Big) A_r^* A_s,$$

the factors in the parenthesis determine a positive matix due to the Cauchy-Schwarz inequality then the operator is positive by analogous argument to that used above.

(d) We shall see the variance,

$$\begin{aligned} \mathrm{Var}(\tilde{B}; \omega_\sigma) &= \vec{\omega}_\sigma(\tilde{B}^2) - \vec{\omega}_\sigma(\tilde{B})^2 = \omega(\varepsilon_\sigma(B^2)) - \omega(\varepsilon_\sigma(B))^2 \\ &\geq \omega(A^2) - \omega(A)^2 = \mathrm{Var}(A; \omega), \end{aligned}$$

where we used the fact that $\widetilde{B^2} = \widetilde{B}^2$. Thus, the actual measure is less sharp than the hypothetical measure.

(e) Since η_σ is $*-$continuous then so is ε_σ.

■

Remark 4.1.4. *The separability assumption may be relaxed by requiring only the induced observable to be separable. In this way, the condition of separability naturally arises, for example, in remark 2.1.1 when we require the measurement instrument to be able to distinguish small enough outcomes given a fixed q.*

The localization properties are important, if we measure an observable in the probe, the induced observable should be observable in the region of interaction, the following proposition probes this observation.

Proposition 4.1.3. *The induced observable need to be localized at least partially within the interaction region, that is, $\varepsilon_\sigma(A) \in \mathscr{A}(M; L), K \subset L$. Indeed, for each probe observable $B \in \mathscr{B}(M)$, the induced system observable $\varepsilon_\sigma(B)$ is localized in any connected open causally convex set containing K. Furthermore, if B is localized in $K^\perp$ then, the induced observable is trivial i.e., $\varepsilon_\sigma(B) = \sigma(B)1$.*

Proof. Let $L \subset K^\perp$, $B \in \mathscr{B}(M; L)$ a probe observable localized in L. Since $L \subset M^+ \cup M^-$ then Θ acts trivial and therefore, $\varepsilon_\sigma(B) = \eta_\sigma(\Theta(1 \otimes B)) = \sigma(B)1$. Thus, the induced observable is trivial. On the other hand, let $B \in \mathscr{B}(M), A \in \mathscr{A}(M; L)$. Notice that $\eta_\sigma(A \otimes 1) = 1$ then,

$$\begin{aligned} [\varepsilon_\sigma(B), A] &= [\eta_\sigma(\Theta(1 \otimes B)), A] \\ &= \eta_\sigma([\Theta(1 \otimes B), A \otimes 1]) \\ &= \eta_\sigma(\Theta[1 \otimes B, A \otimes 1]) = 0 \end{aligned}$$

as η_σ and Θ are linear and Θ acts trivially on $A \otimes 1$. Therefore, all induced observables must be localizable in any connected open causally convex set containing the coupling region K. ■

Having introduced the properties of the measurement scheme, let us see the interpretation. Let A and B be observables of the system and probe respectively, consider a joint measurement of A and B at late times. Analogously to QM, we write the probability of a joint effect being observed (at a prepared state σ) by

$$\mathrm{Prob}_\sigma(A \cap B, \omega) = (\omega \otimes \sigma)(\Theta(A \otimes B))$$

where we abuse the notation of $A \cap B$ to represent the probability when A and B are being observed. The conditional probability that A is observed given that B is observed is then

$$\mathrm{Prob}_\sigma(A|B, \omega) = \frac{\mathrm{Prob}_\sigma(A \cap B, \omega)}{\mathrm{Prob}_\sigma(B, \omega)} = \frac{\mathscr{I}_\sigma(B)(\omega)(A)}{\mathscr{I}_\sigma(B)(\omega)(1)}$$

where $\mathscr{I}_\sigma$ is defined for all $A \in \mathscr{A}(M)$ as

$$\mathscr{I}_\sigma(B)(\omega)(A) := (\omega \otimes \sigma)(\Theta(A \otimes B)) = \Theta^*(\omega \otimes \sigma)(A \otimes B).$$

In particular, using (4.4), the normalization factor is

$$\mathscr{I}_\sigma(B)(\omega)(1) = \omega(A).$$

Definition 4.1.4. *We call the map $\mathscr{I}_\sigma(B) : \mathscr{A}(M)^* \to \mathscr{A}(M)^*$ a pre-instrument*[19]*. If defined, the normalized post-selected state conditioned on B is given by*

$$\omega' = \frac{\mathscr{I}_\sigma(B)(\omega)}{\mathscr{I}_\sigma(B)(\omega)(1)}.$$

Remark 4.1.5. *Suppose a probe-effect B is tested when the system is in a state ω. The post-selected system state, conditioned on the effect observed, should correctly predict the probability of any system effect being observed given that B was observed. It is not clear whether the post-selected system state is indeed a state, the following proposition establishes the significance of calling ω' a state.*

Proposition 4.1.4. *The post-selected state ω' is a state in the system.*

Proof. Clearly ω is normalized and, since the effect B is positive, we can write $B = \sum_i C_i^* C_i$ for some finite set of elements $C_i \in \mathscr{B}(M)$. Then

$$\mathscr{I}_\sigma(B)(\omega)(A^*A) = \sum_i \Theta^*(\omega \otimes \sigma)((A \otimes C_i)^*(A \otimes C_i)) \geq 0,$$

it follows that $\omega(A^*A) \geq 0$ for all $A \in \mathscr{A}(M)$. ∎

Definition 4.1.5. *One can also realize a non-selective probe measurement where no filtering condition on the measurement outcome is imposed. Explicitly, the updated state resulting from the non-selective measurement is*

$$\omega'_{\mathrm{ns}}(A) = \mathscr{I}_\sigma(\mathbb{1})(\omega)(A) = (\Theta^*(\omega \otimes \sigma))(A \otimes \mathbb{1}) \tag{4.5}$$

[19] Given an EVM $E : \mathscr{X} \to \mathscr{A}(M)$, the composition of the pre-instrument with E is a instrument in the sense of [2] i.e., a measure $X \to \mathscr{I}_\sigma(E(X))$ on the σ-algebra of measurement outcomes.

In other words, ω'_{ns} is the partial trace of the state $\Theta^*(\omega \otimes \sigma)$ over the probe.

Remark 4.1.6. *The non-selective state depends only on the dynamics of the coupling, if $A \in A(M)$ is localised in the causal complement of the coupling region then $\omega_{ns}(A) = \omega(A)$ because Θ acts trivially. A non-selective measurement cannot influence the results of other experiments in causally disjoint regions. This is called parameter-independence in the context of Bell inequalities.*

Example 4.1.1. *Let A be a system observable localized in $K^\perp$ and B be a probe effect. Using that $\Theta(A \otimes 1) = A \otimes 1$ notice that*

$$A\varepsilon_\sigma(B) = A\eta_\sigma\Theta(1 \otimes B) = \eta_\sigma((A \otimes 1)\Theta(1 \otimes B)) = \eta_\sigma(\Theta(A \otimes B)).$$

The pre-instrument is then given by

$$\mathscr{I}_\sigma(B)(\omega)(A) = \omega(\eta_\sigma(\Theta(A \otimes B))) = \omega(A\varepsilon_\sigma(B)).$$

Therefore, the normalized post-selected state, conditioned on the effect being observed, for a localizable in $K^\perp$ is

$$\omega'(A) = \frac{\omega(A\varepsilon_\sigma(B))}{\omega(\varepsilon_\sigma(B))}.$$

It follows that $\omega'(A) = \omega(A)$ if and only if A is uncorrelated with $\varepsilon_\sigma(B)$ in ω.

Proposition 4.1.5. *If ω has a Reeh-Schlider property (e.g., the Minkowski vacuum state), then*

$$\omega'(A) = \omega(A) \iff \varepsilon_\sigma(B) = \omega(\varepsilon_\sigma(B))1$$

for observables localizable in $K^\perp$.

Proof.

$$\begin{aligned}\omega'\left(A_1^*A_2\right) - \omega\left(A_1^*A_2\right) &= \frac{\omega(A_1^*A_2\varepsilon_\sigma(B))}{\omega(\varepsilon_\sigma(B))} - \omega(A_1^*A_2) \\ &= \omega\left(\frac{A_1^*A_2\varepsilon_\sigma(B)}{\omega(\varepsilon_\sigma(B))} - A_1^*A_2\right) = \omega\left(A_1^*\left(\frac{\varepsilon_\sigma(B)}{\omega\left(\varepsilon_\sigma(B)\right)} - \mathbb{1}\right)A_2\right)\end{aligned}$$

Hence, if $\omega' = \omega$ the right hand side vanishes and, by the Reeh-Schlider property, it follows that $\varepsilon_\sigma(B) = \omega(\varepsilon_\sigma(B))1$ because A_1 and A_2 are arbitrary. Conversely, if $\varepsilon_\sigma(B) = \omega(\varepsilon_\sigma(B))1$ the left hand side vanishes i.e. $\omega' = \omega$. ∎

Remark 4.1.7. *Any nontrivial probe measurement necessarily alters the state in the causal complement of the coupling region in this highly correlated case. Fewster and Verch suggets that this property allows one to probe strong form of quantum entanglement between causally disjoint regions e.g. to detect EPR-like correlations.*

4.1.3 Successive measurements

Let $K_1, K_2 \subset M$ be two compact regions and consider a measurement scheme composed with a system and two probes whose algebras are $\mathscr{A}$, $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively. Denote the uncoupled system as $\mathscr{U}(M) := \mathscr{A} \otimes \mathscr{B}_1 \otimes \mathscr{B}_2$ and the coupling system as $\mathscr{C}(M)$, the scattering morphisms of each coupling $i = 1,2$ with the system are $\Theta_i : \mathscr{A}(M) \otimes \mathscr{B}_i(M) \to \mathscr{A}(M) \otimes \mathscr{B}_i(M)$. In this way, one can extend Θ_i into $\hat{\Theta}_i : \mathscr{U}(M) \to \mathscr{U}(M)$ by choosing $\hat{\Theta_1} := \Theta_1 \otimes_3 1_{\mathscr{B}_2}$ and $\hat{\Theta_2} := \Theta_2 \otimes_2 1_{\mathscr{B}_1}$, where $\otimes_i$ indicates the slot of where the second factor is inserted for example, $A \otimes B \otimes_2 C = A \otimes C \otimes B$.

In this scenario, the probe system consists of $\mathscr{B}_1 \otimes \mathscr{B}_2$ with coupling region $K_1 \cup K_2$. We assume further that the scattering morphism $\Theta : \mathscr{U} \to \mathscr{U}$ obeys the causal factorization,

$$\Theta = (\Theta_1 \otimes_3 1) \circ (\Theta_2 \otimes_2 1).$$

Theorem 4.1.8. *Consider two probes as described above with $K_2 \cap J^-(K_1) = \varnothing$. For all probe preparations σ_i of $\mathscr{B}_i(M)$ and all probe observables $B_i \in \mathscr{B}_i(M)$. If one can order K_1 and K_2 so that K_2 is later according to some observer then*

$$\mathscr{I}_{\sigma_2}(B_2) \circ \mathscr{I}_{\sigma_1}(B_1) = \mathscr{I}_{\sigma_1 \otimes \sigma_2}(B_1 \otimes B_2).$$

Furthermore, if there is no causal order between K_1 and K_2,

$$\mathscr{I}_{\sigma_2}(B_2) \circ \mathscr{I}_{\sigma_1}(B_1) = \mathscr{I}_{\sigma_1}(B_1) \circ \mathscr{I}_{\sigma_2}(B_2).$$

Remark 4.1.9. *This result implies that measurements act in a coherent fashion (according causality) allowing us to analyze experiments in their own causal structure; for example, Sorkin's protocol described in example 2.2.1.*

Proof. Let ω a state and A an observable of $\mathscr{A}$, we have

$$\begin{aligned}
\mathscr{I}_{\sigma_2}(B_2)\left(\mathscr{I}_{\sigma_1}(B_1)(\omega)\right)(A) &= \left(\mathscr{I}_{\sigma_1}(B_1)(\omega) \otimes \sigma_2\right)\left(\Theta_2(A \otimes B_2)\right) \\
&= \mathscr{I}_{\sigma_1}(B_1)(\omega)\left(\eta_{\sigma_2}\left(\Theta_2(A \otimes B_2)\right)\right) \\
&= \left(\Theta_1^*(\omega \otimes \sigma_1)\right)\left(\eta_{\sigma_2}\left(\Theta_2(A \otimes B_2)\right) \otimes B_1\right) \\
&= \left(\Theta_1^*(\omega \otimes \sigma_1) \otimes_2 \sigma_2\right)\left(\Theta_2(A \otimes B_2)\right) \otimes_3 B_1\right) \\
&= \left(\Theta_1^*(\omega \otimes \sigma_1) \otimes_3 \sigma_2\right)\left(\Theta_2(A \otimes B_2)\right) \otimes_2 B_1\right) \\
&= (\omega \otimes \sigma_1 \otimes \sigma_2)\left((\Theta_1 \otimes_3 1)\left(\Theta_2(A \otimes B_2) \otimes_2 B_1\right)\right) \\
&= (\omega \otimes \sigma_1 \otimes \sigma_2)\left(\left(\hat{\Theta}_1 \circ \hat{\Theta}_2\right)(A \otimes B_1 \otimes B_2)\right) \\
&= (\omega \otimes \sigma_1 \otimes \sigma_2)\left(\hat{\Theta}(A \otimes B_1 \otimes B_2)\right) \\
&= \mathscr{I}_{\sigma_1 \otimes \sigma_2}(B_1 \otimes B_2)(\omega)(A)
\end{aligned}$$

where we permuted the second and third tensor factor of both, elements and functionals, in line 5. ∎

Corollary 4.1.9.1. *Consider two probes as described above, $B_i \in \mathscr{B}_i(M)$ and probe preparation states $\sigma_i\,(i=1,2)$. Suppose B_1 has nonzero probability of being observed in system state ω and that B_2 has nonzero probability of being observed in system state ω_1, the post-selected system state conditioned on B_1 being observed in state ω. Then the post-selected state ω_{12} conditioned on B_2 being observed in state ω_1 coincides with the post-selected state ω_{12} conditioned on $B_1 \otimes B_2$ being observed in state ω.*

Proof. We shall compute

$$\omega_1' = \frac{\mathscr{I}_{\sigma_1}(B_1)(\omega)}{\mathscr{I}_{\sigma_1}(B_1)(\omega)(\mathbb{1})} \quad \text{and} \quad \omega_{12}'' = \frac{\mathscr{I}_{\sigma 2}(B_2)(\omega_1')}{\mathscr{I}_{\sigma_2}(B_2)(\omega_1')(\mathbb{1})}$$

conditioned on both effects being observed, post-selecting on the B_1 measurement at the intermediate step. Obviously, the normalization factors applied to ω_1 cancel in the formula for ω.

$$\omega_{12}'' = \frac{\mathscr{I}_{\sigma_2}(B_2)(\mathscr{I}_{\sigma_1}(B_1)(\omega))}{\mathscr{I}_{\sigma_2}(B_2)(\mathscr{I}_{\sigma_1}(B_1)(\omega))(\mathbb{1})} = \frac{\mathscr{I}_{\sigma_1\otimes\sigma_2}(B_1\otimes B_2)(\omega)}{\mathscr{I}_{\sigma_1\otimes\sigma_2}(B_1\otimes B_2)(\omega)(\mathbb{1})} = \omega_{12}'$$

and so we also have where the denominators are equal by setting $A = 1$. ∎

Remark 4.1.10. *If the K_i are causally disjoint then, the post-selection may be made in either order with the same result. Causal factorization formula must be verified for concrete models of system-probe interactions. Exactly as one would hope, we have shown that experiments conducted in causally disjoint regions [that is, for which the coupling regions are causally disjoint] may be conducted in ignorance of one another, or combined as a single overall experiment by coordinating their results.*

4.1.4 Example: Sorkin's protocol part III

Recall the arrangement of the Sorkin's protocol on example 2.2.1, Alice, Bob and Charlie each perform actions in the regions $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ that satisfy

1. $\mathcal{O}_2 \cap J^-(\mathcal{O}_1) = \varnothing$
2. $\mathcal{O}_3 \cap J^-(\mathcal{O}_2) = \varnothing$;
3. $\mathcal{O}_3$ is spacelike separated form $\mathcal{O}_1$;
4. $\mathcal{O}_3$ has compact closure $\overline{\mathcal{O}_3}$.

Suppose Alice chooses to make a non-selective measurement, Bob makes a non selective measurement, can Charly determine whether Alice performed the measurement?

In order to use the formalism described above set the system observables $\mathscr{A}$ to Charlie and the two probe theories for Alice and Bob. The coupling of the probes are localized in the regions $K_1 \subset \mathcal{O}_1$ and $K_2 \subset \mathcal{O}_2$ which defines the in-and-out regions $M_1^{\mp}, M_2^{\pm}$ respectively. Suppose also that ω is the prepared state in the system and $\sigma = \sigma_1 \otimes \sigma_2$ the joint probe state where σ_1 and σ_2 are the states of Alice and Bob and, let C be a system observable by Charlie.

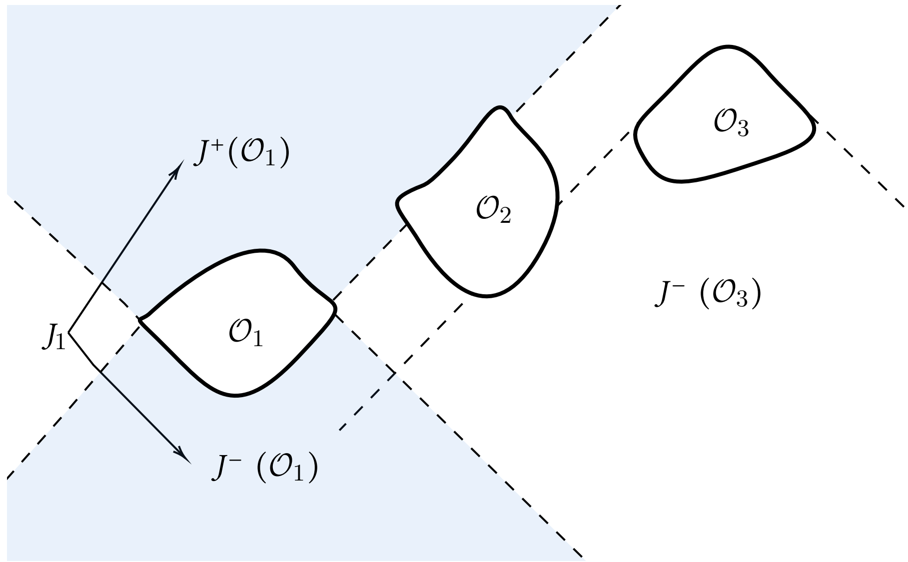

Figure 7: Sorkin's protocol: Alice is in region $\mathcal{O}_1$, Bob in region $\mathcal{O}_2$ and Charly on $\mathcal{O}_3$.

Once the measurement is made, the updated (non-selective) state will be

$$\omega'(C) = (\omega \otimes \sigma)(\Theta(C \otimes 1)).$$

In this case, if Alice and Bob each perform a measurement, the expectation value for Charlie's measurement is therefore given by[20]

$$\omega_{AB}(C) := (\omega \otimes \sigma_1 \otimes \sigma_2)\left(\left(\hat{\Theta}_1 \circ \hat{\Theta}_2\right)(C \otimes \mathbb{1} \otimes \mathbb{1})\right). \tag{4.6}$$

However, since Alice is causally disjoint of Charlie, the measurement should be as if Alice does not perform the experiment i.e.

$$\omega_B(C) = (\omega \otimes \sigma_2)(\Theta_2(C \otimes 1)).$$

The problem that arises in example 2.2.1 is that $\omega_{AB} \neq \omega_B$. The following theorem shows that the presented pathologies illustrated before in the traditional approach are non present within the algebraic formalism. In fact, it shows the equality $\omega_B(C) = \omega_{AB}(C)$ even if C is not an induced observable.

Theorem 4.1.11. *In the notation above, suppose the following assumptions hold:*

1. $K_2 \cap J^-(K_1) = \varnothing$;
2. $\mathcal{O}_3$ *is a region with compact closure;*
3. $\mathcal{O}_3 \cap J^-(K_2) = \varnothing$;
4. $\overline{\mathcal{O}_3}$ *is spacelike separated from* K_1.

Then, for all $C \in \mathscr{S}(O_3)$ *we have*

$$\left(\hat{\Theta}_1 \circ \hat{\Theta}_2\right)(C \otimes \mathbb{1} \otimes \mathbb{1}) = \hat{\Theta}_2(C \otimes \mathbb{1} \otimes \mathbb{1}). \tag{4.7}$$

[20] Asuming causal factorization that Alice realize the experiment after Bob.

Corollary 4.1.11.1. *Charlie's measurement is independent of Alice's experiment.*

Proof.

$$\begin{aligned}\omega_{AB}(C) &= (\omega \otimes \sigma_1 \otimes \sigma_2)\left(\left(\hat{\Theta}_1 \circ \hat{\Theta}_2\right)(C \otimes \mathbb{1} \otimes \mathbb{1})\right) \\ &= (\omega \otimes \sigma_2)\left(\Theta_2(C \otimes \mathbb{1})\right) \\ &= \omega_B(C).\end{aligned} \tag{4.8}$$

∎

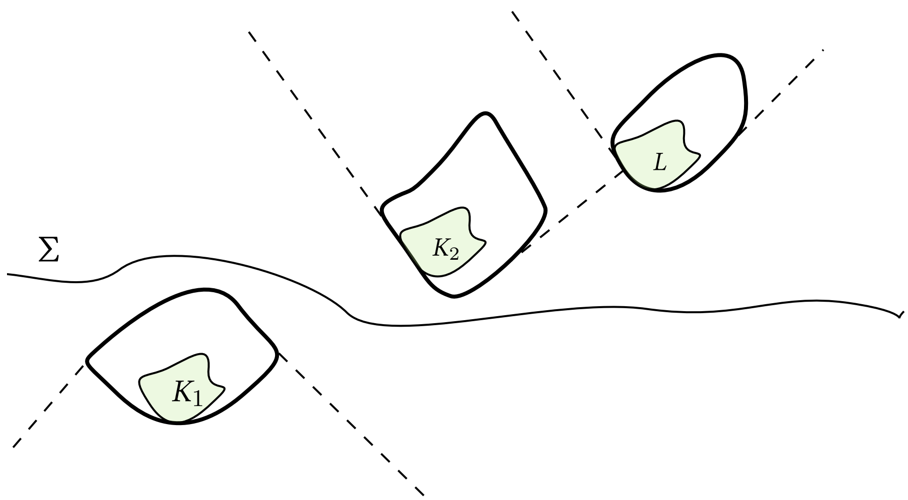

Figure 8: Diagram of the relative causal position of the compact sets K_1, K_2 and $L := \overline{\mathcal{O}_3}$ and the Cauchy surface Σ.

Lemma 4.1.12 (Geometric lemma)**.** *Let K_1, K_2, L be compact subsets of M such that $K_2 \cap J^-(K_1) = \varnothing$ and $L \cap J^-(K_1) = \varnothing$. Then there exists a Cauchy surface Σ of M_1^+ such that*

$$\Sigma \subseteq M \backslash (J^-(K_1) \cup J^+(K_2) \cup J^+(L)). \tag{4.9}$$

Proof. Look in [21]. ∎

Taking K_1 and K_2 the coupling regions of the probe and $L = \overline{O_3}$, the former lemma gives

Lemma 4.1.13. *Let K_1, K_2 be compact subsets of M such that $K_2 \cap J^-(K_1) = \varnothing$. Then for every region O_3 with compact closure such that $O_3 \cap J^-(K_2) = \varnothing$ and $\overline{O_3} \subseteq K_1^{\perp}$ it holds that $O_3 \subseteq D(K_1^{\perp} \cap M_2^-)$.*

Proof. Using lemma 4.1.12 there exist a Cauchy surface for M_1^+ that satisfy the condition (4.9). Notice that Σ is disjoint from $J^-(K_1)$ and $J^-(\overline{O_3})$ is disjoint from $J^+(K_1)$ as $\overline{O_3} \subseteq K_1^{\perp}$ by assumption. In this case we have $T := J^-(\overline{O_3}) \cap \Sigma \subseteq K_1^{\perp} \cap M_2^-$.Now $O_3 \subseteq D(T) \subseteq D(K_1^{\perp} \cap M_2^-)$. ∎

Proof of Theorem 4.1.11. Using the lemma, $\mathcal{O}_3 \subseteq D(K_1^\perp \cap M_2^-)$. Since $C \otimes 1 \otimes 1$ must be localized in $\mathcal{O}_3$ then, $\hat{\Theta}_2(C \otimes 1 \otimes 1)$ must lie in $K_1^\perp \cap M_2^-$. On the other hand, since $K_1^\perp \cap M_2^- \subseteq K_1^\perp$ then, by proposition 4.1.1 $\hat{\Theta}_1$ acts trivially i.e.

$$\hat{\Theta}_1 \circ \hat{\Theta}_2(C \otimes 1 \otimes 1) = \hat{\Theta}_2(C \otimes 1 \otimes 1).$$

■

Remark 4.1.14. *The skeptical reader might noticed that no assumption on the algebra (or the dynamics) has been said. This is due to the spirit of the result i.e., a purely geometrical result derived from the principles of causality and locality of the theory. The only assumption of the dynamics used was causal factorization and, in practice, one need to check the factorization for each specific algebra. For the purpose of complentess, we verify this condition for a class of algebras, mainly those generated by Green-hyperbolic functions (thus, applying these results to the algebras illustrated earlier and the system introduced in the next section). In this case, one can show in Sorkin's protocol (example 2.2.1), that pathologies are still present when using a field $\varphi(x)$ instead of projector $|b\rangle\langle b|$; the above result (together with the following propositions) implies that the algebraic approach in this model fix those problems of Sorkin.*

Proposition 4.1.6. *Let P be a partial differential operator and suppose that Green hyperbolic operators Q_i $(i = 1, 2)$ agree with P outside compact sets K_i $(i = 1, 2)$ with $J^-(K_1) \cap J^+(K_2) = \varnothing$. Let $K = K_1 \cup K_2$ and suppose further that $Q = Q_1 + Q_2 - P$ is also Green-hyperbolic. Then the Green functions $E_i^\pm$ of Q_i and $E^\pm$ of Q are related by*

$$(1 - (Q - P)E^-)f = (1 - (Q_1 - P)E_1^-)(1 - (Q_2 - P)E_2^-)f, \quad f \in C_0^\infty(M^+)$$

$$(1 - (Q - P)E^+)f = (1 - (Q_2 - P)E_2^+)(1 - (Q_1 - P)E_1^+)f, \quad f \in C_0^\infty(M^-)$$

Proof. Notice that Q satisfies

$$Q = \begin{cases} Q_1, & \text{Outside of } K_2, \\ Q_2, & \text{Outside of } K_1, \\ P, & \text{Outside of } K. \end{cases}$$

If $f \in C_0^\infty(M^+)$, the equation $Q\varphi = f$ has unique solution on a future-compact support for φ. $\varphi = E^- f$ and thus, $Q_i\varphi = f - Q\varphi + Q_i\varphi$. Then,

$$\varphi = E^- f = E_i^-(Q_i\varphi) = E_i^- f - E_i^-(Q - Q_i)\varphi$$

and

$$(Q - Q_1)E^- f = (Q_2 - P)E^- f = (Q_2 - P)E_2^- f = (Q - Q_1)E_2^- f$$

It follows that

$$\begin{aligned} E^- f &= E_1^- f - E_1^-(Q - Q_1)E^- f = E_1^- f - E_1^-(Q_2 - P)E_2^- f \\ &= E_1^-(1 - (Q_2 - P)E_2^-)f. \end{aligned}$$

Then,

$$(Q_1 - P)E^- f = (Q_1 - P)E_1^- (1 - (Q_2 - P)E_2^-)f,$$

consequently,

$$\begin{aligned}(1 - (Q_1 - P)E_1^-)(1 - (Q_2 - P)E_2^-)f &= f - (Q_1 - P)E_1^-(1 - (Q_2 - P)E_2^-)f - (Q_2 - P)E_2^- f \\ &= f - (Q_1 - P)E^- f - (Q_2 - P)E^- f \\ &= (1 - (Q - P)E^-)f.\end{aligned}$$

The case $f \in C_0^\infty(M^-)$ is similar considering the advanced (+) Green operators and

$$(Q - Q_2)E^+ f = (Q_1 - P)E^+ f = (Q_1 - P)E_1^+ f = (Q - Q_2)E_1^+ f.$$

■

Corollary 4.1.14.1 (Causal factorization)**.** *The scattering morphism* Θ *obeys the causal factorization,*

$$\Theta = \Theta_1 \circ \Theta_2.$$

Proof. Suppose that there are two probe fields ψ_1 and ψ_2, coupled to φ (but not each other) with compactly supported functions ρ_i in regions K_i with $J^-(K_1) \cap J^+(K_2) = \varnothing$. Writing the corresponding free field equations as Q_1 and Q_2, the two probes together are described by $Q = Q_1 \oplus Q_2$, which is also Green-hyperbolic, while the coupling R is now $RF = \rho_1 F_1 + \rho_2 F_2$. The causal factorization formula gives

$$\hat{\Theta} = \hat{\Theta}_1 \circ \hat{\Theta}_2,$$

where $\hat{\Theta}_i$ is the scattering morphism for the dynamics given by $\hat{T}_i = P \oplus (Q_1 \oplus Q_2) + \tilde{R}_i$. Identifying the overall quantized theory with $\mathscr{A}(M) \otimes \mathscr{B}_1 \otimes \mathscr{B}_2$, we have

$$\hat{\Theta}_1 = \Theta_1 \otimes_3 1_{\mathscr{B}_2}, \quad \hat{\Theta}_2 = \Theta_2 \otimes_2 1_{\mathscr{B}_1}$$

where the Θ_i are the scattering morphisms for the quantized dynamics of $T_i = P \oplus Q_i + \tilde{R}_i$ relative to $P \oplus Q_i$. This establishes the causal factorization formula for system-probe models of this type. ■

4.2 Two Scalar Fields Coupled: An explicit example

The $*-$algebra quantization allows us to address the measurement process of an Unruh-DeWitt type detector. Indeed, by analyzing the system and probe modeled as two scalar fields we can address the detector's measurement (a free field and a two level system) in section 4.3. Consider two scalar fields coupled linearly with a Lagrangian of the form

$$\mathscr{L} = \underbrace{\frac{1}{2}\left[\Box\varphi - m_1\varphi^2 + \Box\psi - m_2\psi^2\right]}_{\mathscr{L}_0} \underbrace{-\rho\varphi\psi\cdot}_{\mathscr{L}_{\mathrm{int}}} \tag{4.10}$$

i.e. the fields ψ and φ satisfy the following equation,

$$\underbrace{\left[\Box + m_1^2\right]}_{P}\varphi + \underbrace{\left[\Box + m_2^2\right]}_{Q}\psi + \rho(\varphi + \psi) = 0, \tag{4.11}$$

where we define the operators P and Q as the Klein-Gordon operators and ρ is the scalar coupling between φ and ψ supported on a region K. Studies of the K-G equation ensures that that Green operators on a globally hyperbolic Lorentzian manifold behave as well as they are in Minkowski spacetime in the sense that there exist Green functions $E^{\pm}_{P/Q}$(retarded and advanced Green operators) related with the operators P and Q that are unique [37] and supported on the causal future or past of K (J^+ and J^- respectively)[21]. For the purpose of simplicity, combine both fields as a function $\Xi : M \to \mathbb{C}^2$ given by

$$\underbrace{\begin{pmatrix} P & R \\ R & Q \end{pmatrix}}_{T}\underbrace{\begin{pmatrix} \varphi \\ \psi \end{pmatrix}}_{\Xi} = 0, \quad T = (P \oplus Q) + \tilde{R},$$

where R is just multiplication to the left by ρ. In this way, one can show that the operator of the coupled (T) and, the uncoupled ($P \oplus Q$) system are also Green-hyperbolic operators [37]. Therefore, as we saw in the examples of the $\star-$algebra quantization, the algebras of the system $\mathscr{A}$, the probe $\mathscr{B}$, the uncoupled $\mathscr{U}$ and the coupled $\mathscr{C}$ theory are generated by elements $\varphi(f) \in \mathscr{A}$, $\psi(g) \in \mathscr{B}$, $\Xi_0(f \oplus h) \in \mathscr{U}$ and $\Xi(F) \in \mathscr{C}$ respectively, where $\Xi_0(f \oplus h) \in \mathscr{U}$ and $\Xi(F) \in \mathscr{C}$ are defined as

$$\Xi_0(f \oplus h) = \varphi(f) \otimes \mathbb{1}_{\mathscr{B}(M)} + \mathbb{1}_{\mathscr{A}(M)} \otimes \psi(h), \quad \Xi(F), F \in C_0^\infty(M; \mathbb{C}^2),$$

since $C_0^\infty(M;\mathbb{C}^2) \equiv C_0^\infty(M;\mathbb{C}) \oplus C_0^\infty(M;\mathbb{C})$. The quantization is carried with the relations,

The $\star-$algebra quantization. Let $N \subseteq M$, the net of $\star-$algebras $\mathscr{A}(N)$ are generated by a unit element together with the generators $\varphi(f)$, $f \in C_0^\infty(N)$ such that:

1. $f \to \varphi(f)$
2. $\varphi(\bar{f}) = \varphi(f)^*, \quad \forall f \in C_0^\infty(N)$
3. $\varphi(Pf) = 0$, $\forall f \in C_0^\infty(N)$ where P give the dynamics of the theory (e.g. P, Q, $P \oplus Q$ and T).
4. $[\varphi(f), \varphi(h)] = iE_P(f,h)\mathbb{1}, \quad \forall f \in C_0^\infty(N)$,

$$E_p(f,h) := \int_M d\,\mathrm{vol}\, f E_p h.$$

[21]This condition was essential to probe that the algebra of the scalar field in the examples is non-trivial

Notice that the compatibility diagram 6 commutes because both operators E_T and $E_{P\oplus Q}$ agree outside the support of ρ and therefore, similarly as the example 3.2.1, one can establish the existence of all algebras needed. In order to see the scattering morphism first note that Θ can be decomposed in the following diagram,

$$\Theta = (\tau^-)^{-1} \circ \tau^+,$$

$$\tau^+ : \mathscr{U}(M) \xrightarrow[\text{T-S}]{} \mathscr{U}(M^+) \xrightarrow[\gamma]{} \mathscr{C}(M^+) \xrightarrow[\text{T-S}]{} \mathscr{C}(M)$$

$$(\tau^-)^{-1} : \mathscr{C}(M) \xrightarrow[\text{T-S}]{} \mathscr{C}(M^-) \xrightarrow[\gamma]{} \mathscr{U}(M^+) \xrightarrow[\text{T-S}]{} \mathscr{U}(M)$$

where every map is an isomorphism either by the time-slice property (T-S) or the natural morphism $\gamma : \mathscr{A}(M) \to \mathscr{B}(m)$, $\gamma\varphi(f) := \psi(f)$. Hence, it is enough to see how Θ acts on a generator $\Xi(F)$ where F is compactly supported in M^+.

Proposition 4.2.1. *The scattering morphism $\Theta : \mathscr{U}(M) \to \mathscr{U}(M)$ acts as*

$$\Theta\,\Xi_0(F) = \Xi_0\left(F - (T - P \oplus Q)E_T F\right) = \Xi_0\left(F - \tilde{R}E_T^- F\right), \quad \tilde{R} = \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix} = T - P \oplus Q. \tag{4.12}$$

Proof. Choose f smooth supported on M^+. τ^+ acts by the chain of mappings,

$$\Xi_0(f) \to \Xi_0(f) \to \Xi(f) \to \Xi(f) \tag{4.13}$$

and, the map $(\tau^{-1})^{-1}$ acts as

$$\Xi(f) \to \Xi(g) \to \Xi_0(g) \to \Xi_0(g). \tag{4.14}$$

We want to find g supported in M^- such that $\Xi(f) = \Xi(g)$ because from (4.13) and (4.14), it follows that

$$\Theta\Xi_0(f) = \Xi_0(g).$$

Let us find g. For simplicity, we write $S := P \oplus Q$. We can write $E_T f$ as a sum of a part compactly supported on M^+ and another on M^-,

$$E_T f = \phi^+ + \phi^-, \quad \phi^\pm \in C^\infty_{sc,pc/fc}(M), \, \operatorname{supp}\phi^- \subset M^-.$$

Since f is compactly supported on M^+ then, $TE_T f = 0 = T\phi^+ + T\phi^-$ which implies

$$T\phi^- = -T\phi^+ \tag{4.15}$$

which is compactly supported because it has support on a spatially future and past compact. By the uniqueness of the Green functions, $E_T^- T\phi^- = \phi^-$ and, using the above equation,

$$E_T^+ T\phi^- = E_T^+(-T\phi^+) = -\phi^+.$$

It follows that

$$E_T f = \phi^- + \phi^+ = \phi^- - E_T^+ T\phi^- = \underbrace{(E_T^- - E_T^+)}_{E_T} T\phi^-,$$

which implies

$$\Xi(f) = \Xi(T\phi^-)$$

again, by uniqueness and the quantization of Ξ. In this way, we have

$$\Theta\Xi_0(f) = \Xi_0(T\phi^-). \tag{4.16}$$

Notice also that there are no further conditions on ϕ^- and therefore, we can choose ϕ^- such that $E_T^- f = \phi^- + \phi^0$ and ϕ^0 is a smooth compactly supported function and thus,

$$E_T f = \phi^- + \phi^0 - E_T^+ f.$$

Consequently,

$$\begin{aligned}
T\phi^- &= f - T\phi^0 \\
&= f - (T-S)\phi^0 - S\phi^0 = f - (T-S)\phi^0 - \underbrace{(T-S)\phi^-}_{0} - S\phi^0 \\
&= f - (T-S)(\phi^0 + \phi^-) - S\phi^0 \\
&= f - (T-S)(E_T^- f) - S\phi^0 - \underbrace{(T-S)}_{T=S}(E_T^+ f) \\
&= f - (T-S)(E_T^- - E_T^+) f - S\phi^0 = f - (T-S)E_T f - S\phi^0,
\end{aligned}$$

where the second line is because f is supported in M^+ and the third line because $T = S$ in the support of ϕ^-. Using (4.16), we finally have

$$\begin{aligned}
\Theta\,\Xi_0(f) &= \Xi_0(T\phi^-) = \Xi_0(f - (T-S)E_T f - S\phi^0) \\
&= \Xi_0(f - (T-S)E_T f) - \underbrace{\Xi_0(S\phi^0)}_{0} \\
&= \Xi_0(f - (T-S)E_T f) \\
&= \Xi_0(f - (T-S)E_T^- f).
\end{aligned}$$

■

Now, let $\psi(h) \in \mathscr{B}(M)$, $h \in C_0^\infty(M)$ be an observable on the probe. In order to find the induced observable, let us take a look at $\Theta(\mathbb{1} \otimes \psi(h))$ and, we shall assume h is supported on M^+. According to the morphism scattering equation (4.12),

$$\begin{aligned}\Theta\,\Xi_0(f\oplus h) &= \Xi_0\left(f\oplus h-\tilde{R}E_T^- f\oplus h\right) && (4.17)\\ &= \Xi_0(f\oplus h)-\Xi_0(\tilde{R}E_T^-(f\oplus h)) && (4.18)\\ &= \Phi\left(f^-\right)\otimes\mathbb{1}+\mathbb{1}\otimes\Psi\left(h^-\right) && (4.19)\end{aligned}$$

where f^- and h^- are defined as

$$\begin{pmatrix} f^- \\ h^- \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} - \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix} E_T^- \begin{pmatrix} f \\ h \end{pmatrix}. \tag{4.20}$$

In particular, if one choose $f = 0$,

$$\Theta\left(\mathbb{1}\otimes\psi(h)\right) = \Theta\Xi_0(0\oplus h) = \varphi\left(f^-\right)\otimes\mathbb{1}+\mathbb{1}\otimes\psi\left(h^-\right), \tag{4.21}$$

with $f^- = -RF_2$ and $h^- = h - RF_1$ where $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = E_T^-\begin{pmatrix} 0 \\ h \end{pmatrix}$. Thus, f^- is supported within $\operatorname{supp}\rho$ and h^- is compactly supported within $M^+\cup\operatorname{supp}\rho$. Finally, the induced observables are given by

$$\begin{aligned}\varepsilon_\sigma(\psi(h)) = \eta_\sigma(\Theta(\mathbb{1}\otimes\psi(h))) = \eta_\sigma(\varphi(f^-)\otimes\mathbb{1}+\mathbb{1}\otimes\psi(h^-)) &= \varphi(f^-)\sigma(\mathbb{1})+\sigma(\psi(h^-))\mathbb{1} && (4.22)\\ = \varphi(f^-)+\sigma(\psi(h^-))\mathbb{1}. &&& (4.23)\end{aligned}$$

In general, one can also find the variance and the moment-generating function. First note that

$$\begin{aligned}\varepsilon_\sigma(\psi(h)^n) &= \eta_\sigma((\varphi(f^-)\otimes\mathbb{1}+\mathbb{1}\otimes\psi(h^-))^n) = \eta_\sigma\left(\sum_{k=0}^n\binom{n}{k}\varphi(f^-)^{n-k}\otimes\psi(h^-)^k\right)\\ &= \sum_{k=0}^n\binom{n}{k}\varphi(f^-)^{n-k}\sigma(\psi(h^-)^k).\end{aligned}$$

Therefore, the induced observable of $e^{i\psi(h)}$ is

$$\begin{aligned}\varepsilon_\sigma(e^{i\psi(h)}) &= \varepsilon_\sigma\left(\sum_{n=0}^\infty\frac{i^n\psi(h)^n}{n!}\right) = \eta_\sigma\left(\sum_n^\infty\frac{i^n}{n!}\left(\varphi(f^-)\otimes\mathbb{1}+\mathbb{1}\otimes\psi(h^-)\right)^n\right)\\ &= \eta_\sigma\left(e^{i\varphi(f^-)}\otimes e^{i\psi(h^-)}\right) = \sigma\left(e^{i\psi(h^-)}\right)e^{i\varphi(f^-)}.\end{aligned}$$

This result allows us to take the variance,

$$\begin{aligned}\operatorname{Var}\left(\widetilde{\psi(h)};\omega_\sigma\right) =&\omega_\sigma\left(\widetilde{\psi(h)}^2\right)-\omega_\sigma(\widetilde{\psi(h)})^2\\ =&\omega\left(\varepsilon_\sigma\left(\psi(h)^2\right)\right)-\omega\left(\varepsilon_\sigma(\psi(h))\right)^2\\ =&\omega\left(\varphi\left(f^-\right)^2\right)+2\omega\left(\varphi\left(f^-\right)\right)\sigma\left(\psi\left(h^-\right)\right)+\sigma\left(\psi\left(h^-\right)^2\right)\\ &-\left(\omega\left(\varphi\left(f^-\right)\right)^2+2\omega\left(\varphi\left(f^-\right)\right)\sigma\left(\psi\left(h^-\right)\right)+\sigma\left(\psi\left(h^-\right)\right)^2\right)\\ =&\operatorname{Var}\left(\varphi\left(f^-\right);\omega\right)+\operatorname{Var}\left(\psi\left(h^-\right);\sigma\right)\end{aligned}$$

and, on the other hand, the moment-generating function is

$$\omega_\sigma\left(e^{i\widetilde{\psi(h)}}\right) = \omega_\sigma\left(\widetilde{e^{i\psi(h)}}\right) = \omega\left(\varepsilon_\sigma\left(e^{i\psi(h)}\right)\right) = \sigma\left(e^{i\psi\left(h^-\right)}\right)\omega\left(e^{i\varphi\left(f^-\right)}\right).$$

which determines all the n−point functions of the induced observable $\psi(h)$ as we remarked earlier.

4.2.1 Perturbative treatment of the detector response

Once our scheme is established, we shall make explicit computations with the aim to compare with the explicit computations of the UdW detector in 1.3; we might find f^- and h^- by using a perturvative analysis of (4.20). First, observe that equation $T\Xi = F$ can be rewritten to

$$(P \oplus Q)\Xi = F - \tilde{R}\Xi \tag{4.24}$$

and, one also has, solving for Ξ, $E_T^- F = \Xi$. Applying T_S^- to equation (4.24), $E_T^- F$ can be written as

$$\begin{aligned} E_T^- F &= E_S^- F - E_S^- \tilde{R} E_T^- F \\ &= E_S^- F - E_S^- \tilde{R} E_S^- F + E_S^- \tilde{R} E_S^- \tilde{R} E_T^- F, \end{aligned}$$

where we use the first line in itself in the second line. Then, replacing ρ by $\lambda\rho$ (R remains the same) we obtain a Born-like series for the operator $E_T^- F$,

$$E_T^- F = E_{P\oplus Q}^- F - \lambda E_{P\oplus Q}^- \tilde{R} E_{P\oplus Q}^- F + \lambda^2 E_{P\oplus Q}^- \tilde{R} E_{P\oplus Q}^- \tilde{R} E_{P\oplus Q}^- F + O\left(\lambda^3\right).$$

Therefore, using the Born series with $F = 0 \oplus h$, $h \in C_0^\infty(M)$ and using (4.20) then, f^- and h^- can be found to be

$$\begin{aligned} \begin{pmatrix} f^- \\ h^- \end{pmatrix} &= \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda \tilde{R} E_T^- \begin{pmatrix} 0 \\ h \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda \tilde{R} E_{P\oplus Q}^- \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda E_{P\oplus Q}^- \tilde{R} E_{P\oplus Q}^- \begin{pmatrix} 0 \\ h \end{pmatrix} \\ &\qquad + \lambda^2 E_{P\oplus Q}^- \tilde{R} E_{P\oplus Q}^- \tilde{R} E_{P\oplus Q}^- \begin{pmatrix} 0 \\ h \end{pmatrix} + O\left(\lambda^3\right) \\ &= \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda \begin{pmatrix} 0 & RE_Q^- \\ RE_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda \begin{pmatrix} 0 & E_Q^- \\ E_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 & RE_Q^- \\ RE_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} \\ &\qquad + \lambda^2 \begin{pmatrix} 0 & E_Q^- \\ E_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 & RE_Q^- \\ RE_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 & RE_Q^- \\ RE_P^- & 0 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} + O\left(\lambda^3\right) \\ &= \begin{pmatrix} -\lambda\rho E_Q^- h + O\left(\lambda^3\right) \\ h + \lambda^2 \rho E_P^- \rho E_Q^- h + O\left(\lambda^4\right) \end{pmatrix}. \end{aligned}$$

Assuming σ has vanishing one point function[22] and $h \in C_0^\infty(M^+)$, the induced observable of $\psi(h)^*\psi(h)$ is

$$\begin{aligned}\varepsilon_\sigma(\psi(h)^*\psi(h)) &= \eta_\sigma(\Theta(\mathbb{1}\otimes\psi(h)^*\psi(h)))\\ &= \varphi(f^-)^*\varphi(f^-) + \sigma(\psi(h^-)^*\psi(h^-))\mathbb{1}\end{aligned}$$

where f^- and h^- are given by the above equation. Define $h_1 = \rho E_Q^- h$, $h_2 = \rho E_P^- \rho E_Q^- h$ then, we have an explicit form of the induced observable,

$$\begin{aligned}\varepsilon_\sigma(\psi(h)^*\psi(h)) = \varphi(-\lambda h_1)^*\varphi(-\lambda h_1) + \sigma(\psi(h)^*\psi(h))\mathbb{1} + \sigma(\psi(h)^*\psi(\lambda^2 h_2))\mathbb{1}\\ + \sigma(\psi(\lambda^2 h_2)^*\psi(h))\mathbb{1} + \sigma(\psi(\lambda^2 h_2)^*\psi(\lambda h_2))\mathbb{1}.\end{aligned}$$

Taking expectation value, the measurement of the induced observable finally is

$$\begin{aligned}\omega(\varepsilon_\sigma(\psi(h)^*\psi(h))) &= \lambda^2\omega(\varphi(h_1)^*\varphi(h_1)) + \sigma(\psi(h)^*\psi(h))\\ &\quad + \lambda^2\sigma(\psi(h)^*\psi(h_2)) + \lambda^2\sigma(\psi(h_2)^*\psi(h)) + \lambda^4\sigma(\psi(h_2)^*\psi(h_2)) + O(\lambda^6)\\ &= \sigma(\psi(h)^*\psi(h)) + \lambda^2[\omega(\varphi(h_1)^*\varphi(h_1)) + 2\,\mathrm{Re}\,\sigma(\psi(h)^*\psi(h_2))] + O(\lambda^4)\\ &= S(\bar{h},h) + \lambda^2\left[W\left(\bar{h}_1,h_1\right) + 2\,\mathrm{Re}\,S\left(\bar{h},h_2\right)\right] + O\left(\lambda^4\right),\end{aligned}$$

where S and W are the so-called two-point functions of σ and ω respectively and h_1, h_2 are compactly supported within support ρ. Notice that in the absence of a coupling ($\rho = 0$), the expected value describes the spontaneous excitation of the probe and can be regarded as background noise.

4.3 The Unruh effect (Part III: The UdW detector in the FV scheme)

Having introduced the FV scheme as well as some elementary computations, we are in position to analyze the Unruh–DeWitt detector. As we derived earlier, the Unruh effect is the phenomenon in which a field undergoing accelerated object detects the vacuum state as a thermal state. The detector is then an object moving in accelerated motion (in Minkowski space-time) that measures if a (thermal) particle is detected or not. Within the FV scheme, let $\varphi(f)$ be a generic element of the algebra $\mathscr{A}(M)$ of the system, ω the vacuum state of $\mathscr{A}(M)$, γ the uniformly accelerated trajectory of the field φ and, $\psi(g)$ be a generic element of the algebra $\mathscr{B}(M)$ of observables in the probe.

The interaction between probe and system only takes place in the compact region K with a linear coupling ρ supported on K. We prepare the probe in a state σ and, recall that the measurement in the probe describes the measure in the induced observable $A := \varepsilon_\sigma(B)$ by the equation

$$\omega(A) = \underset{\sim}{\omega}(\tilde{B}).$$

[22]This assumption is justified that the state σ will be a quasifree state. Using the GNS representation $(\pi_\sigma, \Omega_\sigma)$, one get $\sigma(\varphi(h)) = \langle 0_M|\,\varphi(h)\,|0_M\rangle = 0$.

As we saw in section 3, we can assume that the GNS representation of the probe induced by σ is a Fock representation and, that h is chosen so that $\psi(h) = a(h)$ or at least $\psi(h)$ closely approximates (in the strong topology) to an annihilation operator in this representation. Accordingly, the operator $\psi(h)^*\psi(h) = N_f$ is the operator for the mode annihilated by $\psi(h)$ and, using our perturvative analysis,

$$\omega(\varepsilon_\sigma(N_f)) = S(\bar{h}, h) + \lambda^2\left[W\left(\bar{h}_1, h_1\right) + 2\,\mathrm{Re}\,S\left(\bar{h}, h_2\right)\right] + O\left(\lambda^4\right) \tag{4.25}$$

is the measure (in the probe) of the induced number of particles in the system, where $h_1 = \rho E_Q^- h$ and $h_2 = \rho E_P^- \rho E_Q^- h$.

We claim (as Fewster-Verch [7] did/suggest) that, under a judicious choice of h, this is in fact a (indirect) measurement of the Unruh-deWitt detector (carefully worked out in 1.3). Essentially, the correspondence of the measure $\underset{\sim}{\omega}(\tilde{B})$ with the induced observable implies that we are measuring the presence (or absence) of a particle at a state ω which is of course the quantity observed in the UdW detector. Moreover, since the two terms involving S can be regarded as background noise, the information of the system (at order 2) is stored in the two-point function $W(\overline{h}_1, h_1)$. In order to completely justify the claim i.e. that the measure (4.25) contains the information of the traditional analysis of the detector, we require a little bit more computational work. For this purpose it suffices to show in the $1+1$ dimensional case, however the computations in general cases are completely analogous.

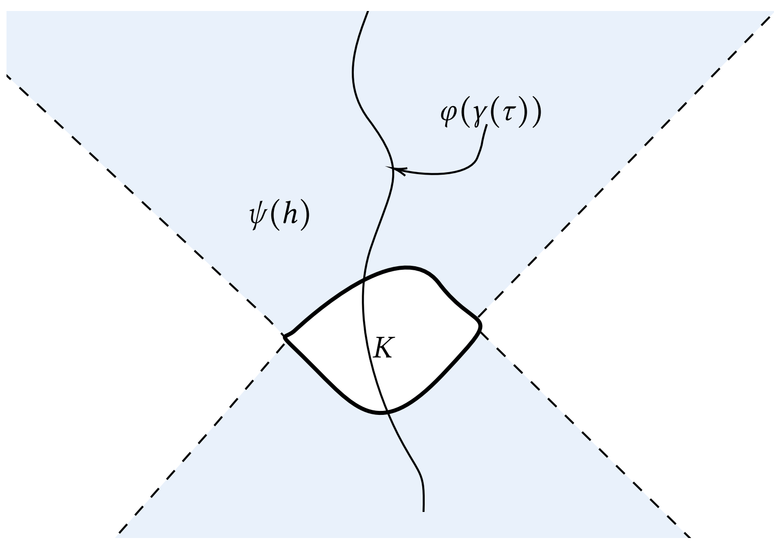

Figure 9: Schematic figure of the UdW detector: The field φ follows a time-like wordline $\gamma(\tau)$, the coupling takes place in the region K and the probe observables are described by $\psi(h)$.

Let $(\pi_\omega, \Omega_\omega)$ be the GNS representations of $\mathscr{A}$ with respect to the state ω. Notice that ω, since is a quasifree state (Minkowski vacuum), has the Minkowski vacuum $|0_M\rangle$ as the GNS vector. It follows that the two point function is given by the equation

$$\begin{aligned} W\left(\overline{h_1}, h_1\right) &= \omega(\varphi(\overline{h}_1)\varphi(h_1)) \\ &= \langle \Omega_\omega | \, \pi_\omega(\varphi(\overline{h}_1))\pi_\omega(\varphi(h_1)) \, | \Omega_\omega \rangle \end{aligned}$$

where, due to the example 3.2.3, the quasifree state ω allows the representation of the field $\varphi(h_1)$ to be given by

$$\pi_\omega(\varphi(h_1)) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\overline{\hat{h_1}(k)}a(\boldsymbol{k}) + \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\hat{h_1}(k)a^*(\boldsymbol{k}).$$

Observe that the model, described above by the equation (4.10), takes an interaction the form $\rho\varphi\psi$ for some ρ real valued. Nonetheless, the interaction of a Unruh-deWitt detector takes place along the uniformly accelerated trajectory γ given by $\rho(\tau)\varphi(\gamma(\tau))\psi(x))$, as shown on equation (1.42) and figure 9. In this sense, hereafter we take the UdW detector interaction $f \mapsto \rho(f)$ acting as a distribution. Accordingly, using the Schwartz kernel theorem, we can write $\rho(f)$ by the equation

$$\rho(f) = \int d\tau f(\gamma(\tau))\tilde{\rho(\tau)}, \tag{4.26}$$

for some compactly supported function $\tilde{\rho}$. Then, $\hat{h_1}$ (using equation (4.26)) is given by

$$\begin{aligned}\hat{h_1} &= \int d^4x h_1(x)e^{ik\cdot x} = \int d^4x\, d\tau\tilde{\rho}(\tau)\gamma^* E_Q^- h\, e^{ik\cdot x}\\ &= \int d\tau\, \tilde{\rho}(\tau)\gamma^*\Big(\int d^4x E_Q^- h e^{ik\cdot x}\Big),\end{aligned} \tag{4.27}$$

where γ^* is just the pullback (precomposition) with γ. In order to find an explicit form of h_1 let us take a look at $E_Q^- h$. Recall first that, in the $1+1$–d model, the Green retarded function is given by (see appendix F for details):

$$G_R(z,t) = i\Theta(t)\int \frac{dk}{2\pi}e^{ikz}\frac{1}{2E}\Big(e^{-iEt} - e^{iEt}\Big). \tag{4.28}$$

Therefore, taking h concentrated in a point of space[23] i.e. $h = \delta(x)$,

$$\begin{aligned}E_Q^- h &= \int dz'dt' G_R(z-z', t-t')h(z',t')\\ &= \int dz'dt' i\Theta(t-t')\int \frac{dk}{2\pi}e^{ik(z-z')}\frac{1}{2E}\Big(e^{-iE(t-t')} - e^{iE(t-t')}\Big)\delta(x')\\ &= i\Theta(t)\int \frac{dk}{2\pi}e^{ikz}\frac{1}{2E}\Big(e^{-iEt} - e^{iEt}\Big) = i\Theta(t)\delta(z)\frac{1}{2E}\Big(e^{-iEt} - e^{iEt}\Big).\end{aligned}$$

Consequently, the unkonwn factor on (4.27) is

$$\int d^2x\, E_Q^- h e^{ik\cdot x} = \frac{i}{2E}e^{-iEt}e^{ik\cdot x}$$

where we only use the negative sign in e^{iEt} to ensure convergence. It follows that $\hat{h_1}$, according to (4.27), is

$$\hat{h_1} = \frac{i}{2E}\int d\tau\tilde{\rho}(\tau)\gamma^*(e^{-iEt}e^{ik\cdot x}) = \frac{i}{2E}\int d\tau\tilde{\rho}(\tau)e^{-iE\tau}e^{ik\cdot\gamma(\tau)}.$$

[23] This assumption is justified by the fact that $\pi_\omega(\varphi(\delta(x))) = \varphi(x)$ as we require.

Returning to the representation $\pi_\omega(\varphi(h_1))$, we finally have

$$\begin{aligned}\pi_\omega(\varphi(h_1)) &= \int d\tau\tilde{\rho}(\tau)e^{-iE\tau}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\left(a(\boldsymbol{k})e^{-ik\cdot\gamma(\tau)} + a^*(\boldsymbol{k})e^{ik\cdot\gamma(\tau)}\right)\\ &= \frac{i}{2E}\int d\tau\tilde{\rho}(\tau)e^{-iE\tau}\varphi(\gamma(\tau)) \end{aligned} \tag{4.29}$$

and, putting all back together in the two point-function W,

$$\begin{aligned} W(\overline{h_1}, h_1) &= \langle 0_M|\,\overline{\pi_0(\varphi(h_1))}\pi_0(\varphi(h_1))\,|0_M\rangle\\ &= \langle 0_M|\int d\sigma d\tau \overline{\frac{i}{2E}\rho(\tau)e^{-iE\tau}\varphi(\gamma(\tau))}\frac{i}{2E}\rho(\sigma)e^{-iE\sigma}\varphi(\gamma(\sigma))\,|0_M\rangle\\ &= \frac{1}{4E^2}\int d\sigma d\tau\rho(\tau)\rho(\sigma)e^{-iE(\sigma-\tau)}\underbrace{\langle 0_M|\,\varphi(\gamma(\tau))\varphi(\gamma(\sigma))\,|0_M\rangle}_{W(\gamma(\sigma),\gamma(\tau))}\end{aligned}$$

where the function $\mathcal{W}(\gamma(\tau),\gamma(\tau'))$ is the same as described in (1.45). Consequently, if one disregard the background noise (the two functions involving S vanish) in the measure (4.25) we obtain

$$\omega\left(\varepsilon_\sigma\left(N_h\right)\right) = \lambda^2\int d\tau d\tau' e^{-iE\left(\tau-\tau'\right)}\rho(\tau)\rho\left(\tau'\right)W\left(\gamma(\tau),\gamma\left(\tau'\right)\right) + O\left(\lambda^4\right) \tag{4.30}$$

which is exactly the result on section 1.2. That is, the measure that the Unruh-DeWitt detector should give in the absence of any background noise.

Remark 4.3.1.

- *This result allows one to model a UdW type of detector by means of "experimental system" (probe and system) which can be used to realize a physical experimental set up.*
- *The functions considered in h_1 turns out to not be given in the Hilbert space (it is not square integrable) as expected, because, by the virtue of the Reeh-Schlider theorem, one cannot model a detector as an observable in the probe algebra; one only find an approximation. Particularly, by approximating the Dirac delta function δ by square integrable functions.*
- *Notice that the perturvative approach in the traditional analysis is derived from the Dyson series while the perturvative series in this section is a derived by a analysis of the Green operator E_T. It may not seem not clear why those two approaches coincide; an idea of this fact is remarked in [38] where Ruepp remarks that the scattering map Θ acts by*

  $$\Theta(\mathbb{1}\otimes\varphi(h)) = \mathrm{S}^\dagger \star (\mathbb{1}\otimes\varphi(h)) \star \mathrm{S}$$

  where $\mathrm{S} = \mathcal{T}\exp\left(\frac{-i\lambda}{\hbar}\int_M \rho(x)\psi(x)\varphi(x)\mathrm{d}V_M(x)\right)$ *and the operators $\star$ is the star-product of a "deformation quantization", see [39] for details.*

Summary and Outlook

It is important to notice that the algebraic formalism is a powerful tool that has many triumphs, raging from defining thermal states (that are mathematically satisfactory) to deriving a consistent AQFT (with a few assumptions i.e, assumptions **A1-A5**). At the same time, the algebraic formalism allows us to have a deeper understanding of relativistic quantum mechanics while allowing us to be able to identify many structures that are hidden in QFT as we remarked in 3.1.4. This, in turn allows relativistic theories to be handled by their true nature, locality and causality[24] which is essential to formulate a measuring scheme in relativistic quantum field theory.

Regarding the measurement problem, it seems rather strange that (traditional) measurement schemes in QMT do not involve notions of causality. The FV scheme is a novel approach in the sense that takes into account causality and locality by modifing the von Neumann measuring scheme presented in Chapter 2 and taking locality explicitly. Thereby, proposing a natural (besides the algebraic formulations) scheme that is very much based in the spirit of QMT. The FV has many advantages,the explanation of the Sorkin protocol and its application to the UdW detector are some triumphs that are coined to the algebraic approach. However, the FV scheme has also disadvantages, the principal and an operational one, is that the question of finding a probe observable corresponding to a given system observable is not addressed as one only knows the converse (given a probe observable we identify a induced observable).

An important result, and perhaps the most important of this work, is the connection between the UdW type detector and the FV scheme realized in the last section. We remark that results, between the traditional analysis and the FV scheme, agree under suitable approximations. Therefore, the FV scheme and the UdW model will (possible) allow one to propose a physical experiment that measures the Unruh effect; which will be essential to support the source of discussion of the results of the Unruh effect [4, 8, 13, 40] discussed earlier on Chapter 1.

Having said that, we summarized that this work mainly focuses on three parts. The Unruh effect, the algebraic formalism and the FV measurement scheme. We addressed the main point in each chapter (with particular detail in the FV scheme) in order to understand the theory. However we remark that many important features that are not related too much with our approach, are not covered in this work. We list some of the aspects that the reader may want to further consult.

1. The standard derivation of the Unruh effect along with its UdW detector,
    (a) There are many types of Unruh-DeWitt detector that, as we said before, differ in their outcomes. Detectors in inertial, Rindler, static or circularly moving observers can be found in [8] for further reading.

    The discussion of each motion in the FV scheme, according to Fewster and Verch [7],

    "[...] should, in particular, shed light on the localization properties of induced observables of measurements conducted with Unruh–deWitt detectors in arbitrary motion, and lead

[24] That is why AQFT is sometimes called Local Quantum Physics

to a structural understanding of the relation of the coupling and the spacetime structure to the state of the detector (the probe) after measurement." Therefore, we regard the computations of this clame as as possible future work.

(b) Hawking effect and some generalizations of the Unruh effect can be found in [5, 8, 28, 36].

2. The algebraic formalism and the AQFT, We showed the general features of AQFT; however, physical theories require more specific tools. We list a few concepts that were not used in this work, nevertheless they are important.

    (a) Applications of KMS theory and Tomita-Takesaki [26, 27].

    (b) Scattering theory in AQFT [35] chapter 1.3

    (c) Superslection rules in AQFT [29, 35].

Appendices

A Definitions of functional spaces

Definition A.1. *A complete normed space is said to be a Banach space*

Definition A.2. *A map $h : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is said to be a skew-linear form if it is conjugate linear in the first argument and linear in the second; if h is positive definite and symmetric $(f, g) = (g, f)^*$ it is called a inner product*

Definition A.3. *If $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is complete it is called a Hilbert space*

Definition A.4. *A linear map A between normed spaces X and Y is called a operator with domain $D(A)$, $D(A)$ is usually required to be dense on X.*

Definition A.5. *The kernel and range of an operator A on a Hilbert space are defined by*

$$\mathrm{Ker}(A) = \{f \in D(A)\,|\,Af = 0\}$$
$$\mathrm{Ran}(A) = \{Af\,|\,f \in D(A)\} = AD(A)$$

Definition A.6. *A is Bounded if the norm of A is finite,*

$$||A|| = \operatorname*{Sup}_{||f||_X=1} ||Af||_Y < \infty$$

The set of bounded operators from X to Y id denoted by $\mathfrak{B}(X, Y)$.

Remark A.1.

- *A linear operator A is bounded if and only if it is continuous.*
- *Let $A \in \mathfrak{B}(X, Y)$ and let Y be a Banach space. If $D(A)$ is dense, there is a unique (continuous) extension of A to X which has the same norm. Indeed, a bounded operator maps Cauchy sequences to Cauchy sequences, thus $Af = \lim Af_n$ as $f_n \in D(A)$, $f \in X$. This definition is independent of the sequence.*

Definition A.7. *Let $\mathcal{H}$ be a separable Hilbert space and $\{e_j\}_{j\in J}$ an orthonormal basis of $\mathcal{H}$. A bounded linear operator $A \in \mathfrak{B}(\mathcal{H})$ is of trace class if*

$$\mathrm{Tr}(|A|) := \sum_{j\in J} \left\langle e_j \middle| (A^*A)^{1/2} e_j \right\rangle < \infty.$$

We denote the family of trace class operators as $\mathfrak{T}(\mathcal{H})$.

Definition A.8. *Let $\mathcal{H}$ a Hilbert space. An operator $A : D(A) \to \mathcal{H}$ is closed if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$ such that $x_n \xrightarrow{n\to\infty} x$ and $Ax_n \xrightarrow{n\to\infty} y$ then, $x \in D(A)$ and $Ax = y$.*

Definition A.9. *A operator $A : D(A) \to \mathcal{H}$ is closable if it has a closed extension $B : D(B) \to \mathcal{H}$ where $D(A) \subset D(B)$.*

B Definitions on QMT

Definition B.1. *Let Ω be a nonempty set and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A countably additive mapping $E : \mathcal{F} \to \mathfrak{B}(\mathcal{H})$ is called a POVM or semispectral measure if $0 \le E(\Delta) \le \mathbb{1}$ for all $\Delta \in \mathcal{F}$ (i.e. each $E(\Delta)$ is an effect) and $E(\Omega) = \mathbb{1}$.*

Definition B.2. *A POVM $E : \mathcal{F} \to \mathfrak{B}(\mathcal{H})$ is called a projection-valued measure (PVM) or spectral measure if in addition $E(\Delta)^2 = E(\Delta)$ for all $\Delta \in \mathcal{F}$ or (equivalently) $E(\Delta)E(\Delta') = 0$ whenever $\Delta \cap \Delta' = \varnothing$.*

Remark B.1. *A POVM is also referred to as an unsharp observable while a PVMs are sharp observables.*

Remark B.2. *The spectral theorem allow us to write every Hermitian operators on $\mathcal{H}$ as the integral of the identity with respect to the induced measure,*

$$\hat{A} = \int_{\mathbb{R}} x d\hat{\mu}_A.$$

Theorem B.3 (Naimark)**.** *Every POVM $\hat{M}$ defined on a Hilbert space $\mathcal{H}$ can be extended to a PVM in the sense that there exist a PVM $\hat{\mu}$ defined on $\mathcal{H}' \supset \mathcal{H}$ such that*

$$\hat{M}(E) = P_{\mathcal{H}} \hat{\mu}(E) \Big|_{\mathcal{H}}$$

where $P_{\mathcal{H}}$ is the projection into $\mathcal{H}$.

Definition B.3. *A quantum operator is a linear map $\Phi : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{G})$ between trace-class operators of a Hilbert space $\mathcal{H}$ into a trace-class operators of a Hilbert space $\mathcal{G}$ satisfying that $\mathrm{Tr}_{\mathcal{G}}(\Lambda(S)) \le 1$ and Λ is completely positive.*

Theorem B.4 (Kraus)**.** *Let $\mathcal{H}$ and $\mathcal{G}$ be two Hilbert spaces of dimension n and m respectively and Φ be a quantum operation between $\mathcal{H}$ and $\mathcal{G}$. Then, there are matrices $\{B_i\}_{i \in I}$ such that,*

$$\Phi(\rho) = \sum_{i \in I} B_i \rho B_i^* \tag{B.1}$$

Conversely, any map

$$\rho \mapsto \sum_{i \in J} B_i \rho B_i^*, \quad \sum_i B_i^* B_i = 1 \tag{B.2}$$

is a quantum operation.

Definition B.4. *The matrices $\{B_i\}_{j \in J}$ defined on equation* (B.2) *are called Kraus operators.*

Definition B.5. *Let $\mathscr{A}$ be a $*-$algebra. An element $A \in \mathscr{A}$ is an effect if A and $1 - A$ are both positive.*

Definition B.6. *Let (X, Σ) be a measureable space and $\mathscr{A}$ a $*-$algebra. Any map $E : \Sigma \to \mathscr{A}$ is called an Effect-Valued Measure (EVM) if E has the properties of a measure, that is,*

1. *$E(X)$ is positive for all $X \in \Sigma$.*

2. $E(\varnothing) = 0$, $E(X) = 1$

3. $\hat{\mu}$ *is additive.*

and takes its values in the effects of $\mathscr{A}$.

Then one interprets the elements of X *as potential outcomes of the measurement and for each* $X \in \Sigma$, $\omega(E(X))$ *is the probability that a value lying in* Σ *is measured in the state* ω.

B.1 Algebraic quantum mechanics

In Quantum Mechanics (QM), (pure) states are rays on a (separable) Hilbert space $\mathcal{H}$ and observables are self-adjoint operators in $\mathcal{H}$. Given a self-adjoint operator A acting on $\mathcal{H}$ and a normalized vector $\varphi \in \mathcal{H}$, the probabilistic nature of QM allows us to assign a probability measure μ such that the probability of finding the observable A on one of the values of Δ is

$$\mathrm{Prob}(A \in \Delta; \varphi) = \int_\Delta \mu(t) = \langle \varphi | P_A(\Delta) \varphi \rangle ,$$

where P_A is the spectral projection of the operator A associated with the Borel set $\Delta \subset \mathbb{R}$. Notice that the map $\Delta \to P_A(\Delta)$ determines a PVM.

C Canonical Quantization

The quantum system, associated to a dynamical system, is carried by adopting

1. There exists a Hilbert space $\mathcal{H}$ for a quantum system and the state of the system is required to be described by a vector (or state) $|\varphi\rangle$ and, two states $|\varphi\rangle$ and $c\,|\varphi\rangle$ describe the same state. The state can also be described as a ray representation of $\mathcal{H}$.

2. A physical quantity A in classical mechanics is replaced by a Hermitian operator $\hat{A}$ acting on $\mathcal{H}$. The operator $\hat{A}$ is often called an observable. The result obtained when A is measured is one of the eigenvalues of $\hat{A}$.

3. The Poisson bracket in classical mechanics is replaced by the **commutator**

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

multiplied by $-i/\hbar$. The unit in which $\hbar = 1$ will be employed here. The fundamental commutation relations are

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \quad [\hat{q}_i, \hat{p}_j] = i\delta_{ij}$$

Hamilton's equations became

$$\frac{d\hat{q}_i}{dt} = \frac{1}{i}[\hat{q}_i, \hat{H}] \quad \frac{d\hat{p}_i}{dt} = \frac{1}{i}[\hat{p}_i, \hat{H}]$$

When a classical quantity A is independent of t explicitly, A satisfies the same equation as Hamilton's equation. By analogy, for $\hat{A}$ which does not depend on t explicitly, one has Heisenberg's equation of motion:

$$\frac{d\hat{A}}{dt} = \frac{1}{i}[\hat{A}, \hat{H}]$$

4. Let $|\varphi\rangle \in \mathcal{H}$ be an arbitrary state. Suppose one prepares many systems, each of which is in this state. Then, observation of A in these systems at time t yields random results in general. Then the expectation value of the results is given by

$$\langle A\rangle_t = \frac{\langle\varphi|\hat{A}(t)|\varphi\rangle}{\langle\varphi|\varphi\rangle}$$

5. For any physical state $|\varphi\rangle \in \mathcal{H}$, there exists an operator for which $|\varphi\rangle$ is one of the eigenstates.

D The $*$, C^* and von Neumann algebras

We state some concepts and well known facts about mathematical algebras in operator theory. We follow [29, 30].

Definition D.1. *An associative algebra $\mathscr{A}$ is a set equipped with the maps addition $+ : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$, scalar multiplication $\lambda \in \mathbb{C}$ $\lambda : \mathscr{A} \to \mathscr{A}$ and multiplication $\cdot : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$*

$$+(A,B) := A + B, \quad \lambda(A) := \lambda A \quad \textit{and} \quad \cdot(A,B) := AB.$$

The algebra is both, a complex vector space and a ring, $V = (\mathscr{A}, +, \{\lambda\}_{\lambda\in\mathbb{C}})$ and $R = (\mathscr{A}, +, \cdot)$ respectively. An algebra is said to be unital if the ring structure has a unit. An involution on an algebra $\mathscr{A}$ is a map $* : \mathscr{A} \to \mathscr{A}$ $*(A) := A^*$ such that

$$(\lambda A + B)^* = \lambda^* A^* + B^*, \quad (AB)^* = B^* A^* \quad \text{and} \quad (A^*)^* = A,$$

where λ^* is the complex conjugate of λ. Also, we define a norm $||\cdot|| : \mathscr{A} \to \mathbb{R}^+$ in which the multiplication $\cdot$ is a continuous map. That is,

$$||AB|| \leq ||A||||B||.$$

If $(V, ||\cdot||)$ is a Banach vector space the algebra is said to be a Banach algebra.

Definition D.2 ($*$ and C^* algebras)**.** *An $*-$algebra is an algebra equipped with an involution $*$. An C^*-algebra is a Banach $*-$algebra such that*

$$||A^*A|| = ||A||^2, \quad A \in \mathscr{A}.$$

The 'C' stands for 'closed', first defined by Segal in 1947 as a "uniformly closed, self-adjoint algebra of bounded operators on a Hilbert space" [31] which of course relates with the above definition by *means of representation theory.*

Definition D.3. *Let $\mathscr{A}$ be a $*-$subalgebra of $\mathfrak{B}(\mathcal{H})$. We say that $\mathscr{A}$ is a von Neumann algebra if $\mathscr{A}$ is a unital algebra and*

$$(\mathscr{A}')' = \mathscr{A}, \quad \mathscr{A}' = \{B \in \mathfrak{B}(\mathcal{H}) : [B, A] = 0, \ \forall A \in \mathscr{A}\}.$$

$\mathscr{A}'$ is called the commutant of $\mathscr{A}$ and thus, a von Neumann algebra is a $*-$subalgebra of $\mathfrak{B}(\mathcal{H})$ whose double commutant is itself. In general, a von Neumann algebra need not be unital but will ease computations.

Observe that a von Neumann endows the topology on $\mathfrak{B}(\mathcal{H})$ and thus, we introduce some of them.

Definition D.4. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. The uniform topology on $\mathfrak{B}(\mathcal{H})$ is defined by the norm $||\cdot|| : \mathfrak{B}(\mathcal{H}) \to \mathbb{R}^+$,*

$$||A|| = \sup\{ \underbrace{||Av||}_{\langle Av, Av\rangle^{1/2}} : v \in \mathcal{H}, ||v|| \le 1\}, \quad A \in \mathfrak{B}(\mathcal{H}), \tag{D.1}$$

The sequence $\{A_i\}_{i\in\omega}$ converges (uniformly) to $A \in \mathfrak{B}(\mathcal{H})$ if and only if

$$\lim_{i\to\infty} ||A_i - A|| = 0.$$

Definition D.5. *The weak topology on $\mathfrak{B}(\mathcal{H})$ is defined by the family of seminorms $\{f_{u,v} : u, v \in \mathcal{H}\}$ where*

$$f_{u,v}(A) = \langle u, Av\rangle.$$

In other words, the weak topology is the initial topology of the family $\{f_{u,v} : u, v \in \mathcal{H}\}$.

The sequence $\{A_i\}_{i\in\omega}$ converges (weakly) to $A \in \mathfrak{B}(\mathcal{H})$ if and only if

$$\lim_{i\to\infty} f_{u,v}(A_i) = f_{u,v}(A), \quad \forall u, v \in \mathcal{H}.$$

Definition D.6. *The strong topology on $\mathfrak{B}(\mathcal{H})$ is defined by the family of norms $\{f_v : v \in \mathcal{H}\}$ with*

$$f_v(A) = ||Av||.$$

In other words, the strong topology is the initial topology with respect to the family $\{f_v : v \in \mathcal{H}\}$. The sequence $\{A_i\}_{i\in\omega}$ converges (strongly) to $A \in \mathfrak{B}(\mathcal{H})$ if and only if

$$\lim_{i\to\infty} f_u(A_i) = f_u(A), \quad \forall u \in \mathcal{H}.$$

Definition D.7. *The ultra weak topology on $\mathfrak{B}(\mathcal{H})$ is defined in terms of the family $\{f_v : v \in \mathcal{T}(\mathcal{H})\}$, where $\mathcal{T}(\mathcal{H})$ is the set of positive operators with trace 1 on $\mathcal{H}$ and*

$$f_v(A) = \mathrm{Tr}(vA).$$

The sequence $\{A_i\}_{i\in\omega}$ converges (ultra-weak) to $A \in \mathfrak{B}(\mathcal{H})$ if and only if

$$\lim_{i\to\infty} f_u(A_i) = f_u(A), \quad \forall u \in \mathcal{H}.$$

If $\mathcal{H}$ is finite dimensional the four topologies coincide. Indeed, it is enough to take basis elements. In general, the ultraweak topology is not comparable with the strong topology, the norm topology is the finest of them all and the weak topology is coarser than both, the ultraweak topology and the strong topology.

Theorem D.1 (von Neumann double commutant theorem)**.** *$\mathscr{A}$ is weakly closed if and only if the double commutant of $\mathscr{A}$ is $\mathscr{A}$.*

D.1 States

Definition D.8. *A state on a (unital) $*-$algebra $\mathscr{A}$ is a linear functional $\omega : \mathscr{A} \to \mathbb{R}$ such that*

$$\omega(AA^*) \geq 0, \quad \forall A \in \mathscr{A} \qquad \textit{and } \omega(1) = 1.$$

ω is said to be mixed if it's sum of at least two states $\omega_1 \neq \omega_2$. If ω is not mixed, ω is pure.

Definition D.9. *Let $\mathscr{A}$ be an C^*-algebra. A representation (on a Hilbert space $\mathscr{H}$) of $\mathscr{A}$ is $*-$homomorphism $\pi : \mathscr{A} \to \mathfrak{B}(H)$. The representation π is reducible if $\pi(\mathscr{A})(\mathcal{H})$ has non-trivial invariant subspaces and, π is faithful if π is an isomorphism.*

Definition D.10. *Let $\pi : \mathscr{A} \to \mathfrak{B}(\mathcal{H}_1), \phi : \mathscr{A} \to \mathfrak{B}(\mathcal{H}_2)$ be two representations of $\mathscr{A}$ on some Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$.*

1. *π and ϕ are equivalent if there is $U : \mathcal{H}_1 \to \mathcal{H}_2$ unitary such that $U\pi(A) = \phi(A)U, \forall A \in \mathscr{A}$. ϕ is a subrepresentation of π if there is an isometry $V : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\pi(A)V = V\phi(A), \forall A \in \mathscr{A}$.*
2. *π and ϕ are quasiequivalent if the von Neumann algebras $\pi(\mathscr{A})'', \phi(\mathscr{A})''$ are $*-$isomorphic.*
3. *π and ϕ are disjoint if they are not quasiequivalent.*

Definition D.11. *Let $\mathcal{H}$ be a Hilbert space, $S \subset \mathfrak{B}(\mathcal{H})$ and $G \subset \mathcal{H}$. Then G is said to be cyclic for S if $\operatorname{span}\{SG\}$ is dense with respect to the uniform topology. G is separating for S if $\forall A \in S, AG = 0$ implies $A = 0$.*

Definition D.12. *Two projections P_1 and P_2 in a von Neumann algebra $\mathscr{A}$ are said to be equivalent if there is a $V \in \mathscr{A}$ such that $V^*V = E, VV^* = F$, in that case we write $E \sim F$.*

Definition D.13. *Given two von Neumann algebras $\mathscr{A}_1$ and $\mathscr{A}_2$, the generated von Neumann algebra $\mathscr{A}$ is the algebra*

$$\mathscr{A} = \bigcap_{\mathscr{B}\in\mathfrak{B}} \mathscr{B}, \quad \mathfrak{B} = \{\mathscr{B} : \mathscr{B} \textit{ is a von Neumann algebra and } \mathscr{A}_1 \subseteq \mathscr{B}, \mathscr{A}_2 \subseteq \mathscr{B}\}$$

Definition D.14. *Let $\mathscr{A}$ a von Neumann algebra. In general the algebra is not closed under commutant, thus, the center of the algebra is the elements that commute with the algebra that lies on the algebra, that is, $Z(\mathscr{A}) = \mathscr{A} \cap \mathscr{A}'$. Clearly (in a unital von Neumann algebra) $\mathbb{C}I \subset \mathscr{A}$, thus, $Z(\mathscr{A}) \neq \varnothing$. If $Z(A) = \mathscr{C}I$ the algebra is called a factor, the algebra generated by $\mathscr{A}$ and its commutant is $\mathfrak{B}(\mathcal{H})$. A projection that lies on the center of the algebra is called a central projection.*

Definition D.15. *A faithful normalized trace on a von Neumann algebra $\mathscr{A}$ is a state ρ on $\mathscr{A}$ such that*

$$\rho(AB) = \rho(BA), \quad \forall A, B \in \mathscr{A}. \tag{D.2}$$

$$\rho(A^*A) = 0 \iff A = 0, \quad A \in \mathscr{A}. \tag{D.3}$$

D.2 Tomita-Takesaki theory [25, 26]

Theorem D.2. *Let φ be a state on a von Neumann algebra $\mathscr{A}$ acting on a Hilbert space $\mathcal{H}$. Then φ is normal if and only if there exist a density (positive trace-class) operator ρ, on $\mathcal{H}$ with $\operatorname{Tr}\rho = 1$, such that*

$$\varphi(A) = \operatorname{Tr}(\rho A) \quad \forall A \in \mathscr{A},$$

where Tr *is the trace operator.*

Definition D.16. *Let $\mathscr{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, $\Omega \in \mathcal{H}$ a separating and cyclic vector. The operators S_0, F_0 are defined such that*

$$S_0(m\Omega) = A^*\Omega, \quad F_0(m\Omega) = A'^*\Omega \quad \forall A \in \mathscr{A}, A' \in \mathscr{A}'.$$

F_0, S_0 are neither bounded nor defined on the whole Hilbert space. However, they are defined on a dense subset.

Proposition D.1. *The operators S_0 and F_0 are closable. Denote $S = \operatorname{Cl}(S_0)$ and $F = \operatorname{Cl}(F_0)$ their closed extensions.*

Theorem D.3 (Polar decomposition)**.**

Corollary D.3.1. *There exist a unique, positive, self adjoint operator Δ, and a unique anti-unitary operator J such that*

$$S = J\Delta^{1/2}.$$

Definition D.17. *We call Δ the modular operator associated with $\mathscr{A}, \Omega$ and J its correspondent modular conjugation.*

Proposition D.2. *The following relations hold,*

1. $\Delta = FS$
2. $F = J\Delta^{-1/2}$
3. $\Delta^{-1} = SF$
4. $J = J^*$
5. $\Delta^{-1/2} = J\Delta^{1/2}J$
6. $S = J\Delta^{1/2}$
7. $J^2 = 1.$

Theorem D.4 (Tomita-Takesaki theorem)**.** *Let $\mathscr{A}$ be a von Neumann algebra with cyclic and separating vector Ω, let Δ the modular operator and J the associated modular conjugation. Then,*

$$J\mathscr{A}J = \mathscr{A}'$$

holds. Moreover,

$$\Delta^{it}\mathscr{A}\Delta^{it} = \mathscr{A}.$$

Definition D.18. *A state ϕ on a von Neumann algebra $\mathscr{A}$ is faithful if $\phi(A) > 0$ for all non-zero $A \in \mathscr{A}$.*

Tomita-Takesaki theory has meaningful applications in both classical and quantum physics, for a detailed explanation we refer to [27].

Lemma D.5. *Let $\mathscr{A}$ be a von Neumann algebra, and φ a faithful, normal state on $\mathscr{A}$. Consider π_φ, the GNS representation of $\mathscr{A}$ associated to φ. Then, the cyclic vector Ω_φ is separating for $\pi_\varphi(\mathscr{A})$.*

Proof. Fix $\pi_\varphi(A), A \in \mathscr{A}$. Suppose $\pi_\varphi(A)\Omega_\varphi = 0$. Then,

$$\begin{aligned} 0 = \|\pi_\varphi(A)\Omega_\varphi\|^2 &= \left\langle \pi_\varphi(A)\Omega_\varphi \middle| \pi_\varphi(A)\Omega_\varphi \right\rangle \\ &= \left\langle \Omega_\varphi \middle| \pi_\varphi(A)^* \pi_\varphi(A)\Omega_\varphi \right\rangle \\ &= \left\langle \Omega_\varphi \middle| \pi_\varphi(A^*A)\Omega_\varphi \right\rangle \\ &= \varphi(A^*A). \end{aligned}$$

Since φ is faithful, $A^*A = 0$ and hence $A = 0$. ∎

E Convex geometry in CST

Given a fixed time orientation in the manifold (space-time), time-like or light-like vectors can be classified as future or past pointing. We define some concepts of causal convex geometry. We follow [7, 37]. Let M be space-time,

Definition E.1. *A subset $\Sigma \subset M$ is called a Cauchy hypersurface if every inextensible timelike curve in M meets Σ exactly once.*

Remark E.1. *Any Cauchy hypersurface of M is a submanifold of dimension* 1*.*

Definition E.2. *If M possesses a Cauchy hypersurface then M is called a globally hyperbolic space-time.*

Example E.1. *Minkowski, Friedmann, Schwazschild and de Sitter spacetimes are all global hyperbolic.*

Definition E.3. *Let $x \in M$, the causal future/past $J^\pm(x)$ of x is the set of all points that can be reached from x by smooth future/past directed causal curves (including x itself).*

Let $S \subset M$, we write $J^{+/-}(S) := \bigcup_{x \in S} J^{+/-}(x)$ and, $J(S) = J^+(S) \cup J^-(S)$. If S is closed, then so is $J^{\pm}(S)$ (connectedness)

Definition E.4. *Let $S \subset M$, S is called strictly past compact (pc) if S is closed and there is a compact $K \subset M$ such that $A \subset J^+(K)$.*

If A is a strictly past compact then, there exist K compact such that $J^+(A) \subset J^+(J^+(K)) = J^+(K)$. Thus, $J^+(A)$ is strictly past compact.

Definition E.5. *Let $A \subset M$, A is called a future compact (fc) if $A \cap J^+(x)$ is compact for all $x \in M$.*

Example E.2. *If Σ is a Cauchy hypersurface, then $J^-(\Sigma)$ is future compact.*

Definition E.6. *If $A \subset M$ is past compact and future compact then we call A temporally compact.*

Example E.3. *Let M be Minkowski space and let $C \subset M$ be an open cone with tip 0 containing the closed cone $J^-(0) \backslash \{0\}$. Then $A = M \backslash C$ is past compact but not strictly past compact. Indeed, for each $x \in M$, the set $J^-(x) \cap A = J^-(x) \backslash C$ is compact. But A is not strictly past compact because the intersection of A and space-like hyperplanes is not compact.*

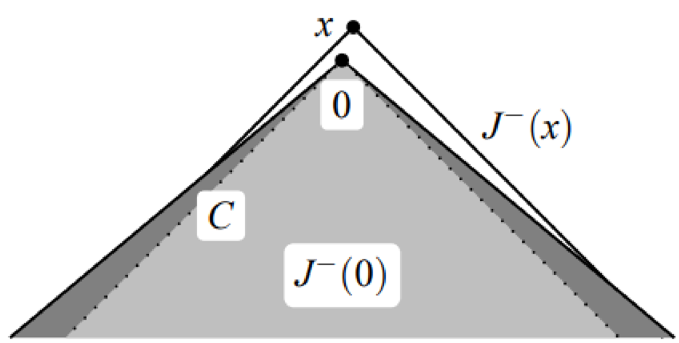

Figure 10: Cones $J^-(0)$ and the open cone C. Figure retrieved from [37].

Definition E.7. *The causal hull of S is the intersection $J^+(S) \cap J^-(S)$. Set of all points that lie on causal curves with both endpoints in S.*

$A \subset M$ is causally convex if it is equal to its causal hull. (contains every causal curve that begins and ends in it) The causal hull of S is the intersection of all causal convex sets containing S

Definition E.8. *The space-time is globally hyperbolic if and only if it is devoid of closed causal curves and the causally hull of ant compact is compact.*

Definition E.9. *A Cauchy surface is a set intersected exactly once by every in-extendable smooth time-like curve. Every Lorentizan space-time possessing a Cauchy surface is globally hyperbolic and every globally hyperbolic space-time may be foliated into Cauchy surfaces that are, additionally, smooth space-like hypersurfaces.*

Any open causally convex subset of a globally hyperbolic space-time is itself globally hyperbolic when equipped with the induced metric and time-orientation.

Definition E.10. *The causal complement of a set S is defined as $S^{\perp} = M \backslash J(S)$.*

Sets S and $T \subset S^{\perp}$ are causally disjoint if there is no causal curve joining S and T.

In a globally hyperbolic space-time, the causal future and past of an open set are open, while those of a compact set are closed. K compact, $K \perp$ open, $K \subseteq K^{\perp\perp}$ is closed possibly not compact and in general $K^{\perp\perp}$ is not the causal hull of K. (In Minkowski space-time, if K is a time-like curve segment they coincide but if K is a subset of a constant time hyper-surface they differ)

Let $S, T \subset \mathbf{M}$, if $J^{+}(S) \cup J^{-}(T) = \varnothing$ or $J^{+}(T) \cup J^{-}(S) = \varnothing$ then, there is a Cauchy surface of $\mathbf{M}$ lying to the future of T and the past of S thus, establishing a causal ordering in which S is later than T. Also, if S and T are causally disjoint it is possible to order S both later and earlier than T.

Definition E.11. *If $J^{+}(S) \cup J^{-}(T) = \varnothing$ or $J^{+}(T) \cup J^{-}(S) = \varnothing$ we say that S and T are causally orderable.*

Definition E.12. *The future/past Cauchy development $D^{+\backslash -}(S)$ is the set of points p so that every past/future-inextendable pice-wise smooth casual curve through p meets S*

Definition E.13. *The spaces of smooth functions with supports that are future, past, or spatially compact are designated by subscripts fc, pc, sc. For example, $C^{\infty}_{sc,fc}(M)$ consists of smooth functions on M whose supports are both spatially compact and future compact.*

F Green retarded function for the K-G equation in $1+1$–d

The Klein-Gordon equation in $1+1$-D is given by the equation:

$$(\partial_t^2 - \partial_z^2 + m^2)\phi = f(z,t). \tag{F.1}$$

The Green's function satisfy

$$(\partial_t^2 - \partial_z^2 + m^2)G(z,t) = \delta(z)\delta(t). \tag{F.2}$$

Proposition F.1. *The Green retarded function of the K-G equation in $1+1$-d is given by*

$$G(z,t) = i\Theta(t)\int \frac{dk}{2\pi} e^{ikz} \frac{1}{2\sqrt{k^2+m^2}}\left(e^{-i\sqrt{k^2+m^2}t} - e^{i\sqrt{k^2+m^2}t}\right) \tag{F.3}$$

Lemma F.1.

$$\lim_{\epsilon\to 0^+} F_\epsilon(t) := \lim_{\epsilon\to 0^+}\int \frac{d\omega}{2\pi} e^{-i\omega t}\frac{1}{\omega-\omega_0+i\epsilon} = -ie^{-i\omega_0 t}, \quad \omega_0 \in \mathbb{R} \tag{F.4}$$

Proof. Let $\omega = Re^{i\phi}$ with $\phi \in (\pi, 2\pi)$. Notice that $(t<0)$ $|e^{-i\omega t}| = |e^{-iRe^{i\phi}t}| = |e^{R\sin\phi t}| \overset{R\to\infty}{\to} 0$. Therefore the integral (F.4) coincides (in the limit $R\to\infty$) with the closed integral in the lower half circle (for $t<0$). Using the residue theorem,

$$F(t) = \oint \frac{d\omega}{2\pi} e^{-i\omega t}\frac{1}{\omega-\omega_0+i\epsilon} = -i\mathrm{Res}\left(\frac{e^{-i\omega t}}{\omega-\omega_0+i\epsilon}, \omega_0 - i\epsilon\right) = -ie^{-i\omega_0 t}e^{-\epsilon t} \tag{F.5}$$

where we put a minus as we integrate in anticlockwise orientation. Taking limits gives the desired result. ■

Proof of proposition H.1. In order to find G_R, the equation (F.2) in the Fourier space is given by

$$\hat{G}(k,\omega) = \frac{1}{(-\omega^2 + k^2 + m^2)} = -\frac{1}{2\sqrt{k^2+m^2}}\Big(\frac{1}{\omega - \sqrt{k^2+m^2}} - \frac{1}{\omega + \sqrt{k^2+m^2}}\Big). \tag{F.6}$$

which gives the function

$$G(z,t) = -\int \frac{dz}{2\pi} e^{ikz} \frac{1}{2\sqrt{k^2+m^2}} \int \frac{d\omega}{2\pi} e^{-i\omega t}\Big(\frac{1}{\omega - \sqrt{k^2+m^2}} - \frac{1}{\omega + \sqrt{k^2+m^2}}\Big). \tag{F.7}$$

The Green retarded function is given by shifting both poles into the lower half plane and, using the lemma,

$$G(z,t) = i\Theta(t) \int \frac{dk}{2\pi} e^{ikz} \frac{1}{2\sqrt{k^2+m^2}}\Big(e^{-i\sqrt{k^2+m^2}t} - e^{i\sqrt{k^2+m^2}t}\Big) \tag{F.8}$$

where $\Theta(t)$ is introduced due to the convergence of the integrals, for $t > 0$, in the lemma. ■

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