Lecture notes on
Statistical Field Theory

November 12, 2024

Abstract

Lectures on renormalization group flows.

Contents

1 Introduction

Recall the RG for Ising with long range interactions. Photo...

- Say that $S(x)$ is defined for all $x \in \mathbb{R}$ but $S(x) = 1$ for $x < 0$ and $x > L$. kinks live in $[0, L]$, even # of kink N.

Example 1.1. *Let us write* $J(r) = -\frac{d^2}{dr^2}u(r)$ *where* $u(r) \sim J \log r$ *as* $r \to \infty$ *we need to modify* $J(r)$ *at short distances so that a kink has finite energy.*

Hence, we can integrate by parts, $J(x - x') = \partial\partial' u(x - x')$*, to write* $\mathcal{H}$ *in terms of*

$$\partial S(x) = 2 \sum_{m=1}^{N} \sigma_m \delta(x - x_m) \tag{1}$$

where $\sigma_m = \pm 1$ *and they are not d.o.f.*

Integrating by parts,

$$\mathcal{H} = -\int \int_{x' > x} dx dx' u(x - x')(\partial S)(x) \partial S(x') = \int dx \int_x^\infty dx' \partial' \partial u(x - x')(SS - 1)$$

boundary terms vanish in our boundary conditions.

Using equation (1). Two cases $n > m$, $m = n$.

$$\mathcal{H} = -4 \sum_{n > m} u(x_n - x_m) \sigma_m \sigma_n - 2 \sum_m u(0).$$

- $m = n$ **term**

$$-4 \int \int_{x' > x} dx dx' u(x - x') \delta(x - x_m) \delta(x' - x_m) = -4u(0) \underbrace{\int \int_{x' > x} dx dx' \delta(x - x_m) \delta(x' - x_m)}_{A}$$

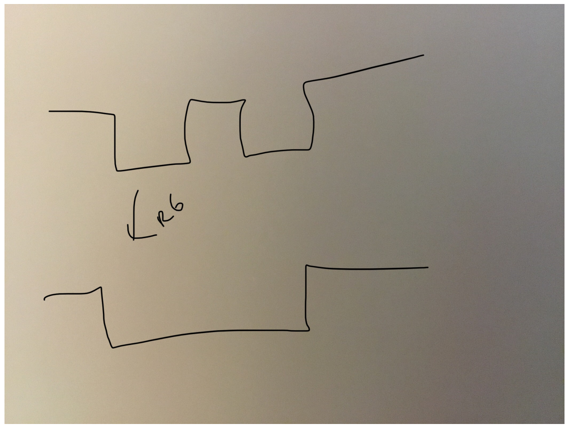
as A *is the same under* $x' \to x$ *rename we get* $2A = \int \int_{-\infty}^{\infty} dx dx' \delta(x - x_m) \delta(x' - x_m) = \left(\int_{-\infty}^{\infty} dx dx' \delta(x' - x_m) \right)^2 = 1$ *so that*

$$-4 \int \int_{x' > x} dx dx' u(x - x') \delta(x - x_m) \delta(x' - x_m) = -2u(0)$$

Energy of kink gas $u(0) = J \ln c$ *if* $c << 1$ $u(0)$ *is large and negative so second term is large energy for kink.*

Kink partition function

$$Z = \sum_N \int \int_{x_1 > 0,\, x_{m+1} - x_m > 1\,, x_N < L} \prod_m dx_m y^N \prod_{n > m} e^{4u(x_n - x_m)\sigma_m \sigma_n}$$



where $y = e^{2u(0)}$. When y is small, kink are far apart typically. So let's use long distance form $u(0) = J \log r$,

$$Z = \sum_N \int \int \prod_m dx_m y^N \prod_{n>m} (x_n - x_m)^{4J\sigma_n\sigma_m}$$

let us write $\alpha = 4J$. Kink gas partition for two couplings:

$$J(\alpha) \to \text{long distance interactions}$$

$$y = \text{kink "fugacity"}$$

both couplings will flow under RG.

1.1 5.4 RG for kink gas in 1D

RG procedure is a prototype of Kosterlitz-Thouless RG in $2D$ XY model.

- Here: pointlike kinks with log interactions.
- $2D$ XY: pointlike vortices with long interactions.

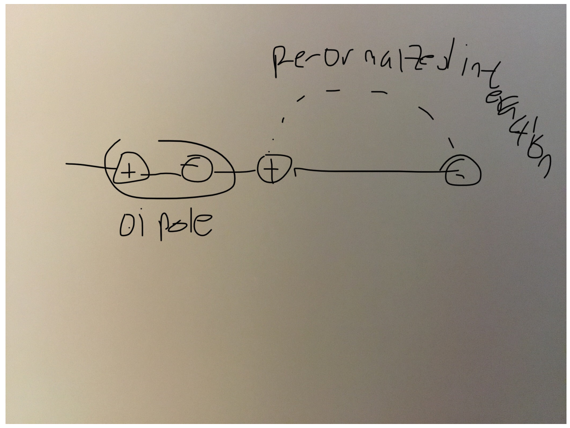
Steps:

1. Coarse graining: increase UV cutoff.

   Eliminate kink pairs with separation $\in [1, b]$. This renormalizes J screening effect.

2. As always, restore UV cutoff to 1 by simple rescaling $X = x/b$. Simplification: infinitesimal RG stop. $b = e^{d\tau} = 1 + d\tau$, when $d\tau << 1$, close together kinks pairs are very rare + we can consider effect of single dipole.

$$Z = \sum_N \int_{x_{m+1} > x_m + 1} dx_m W(\{x_m\}) = \sum_N \int_{x_{m+1} > x_m + 1} dx_m y^N \prod_{n>m} (x_n - x_m)^{\alpha\sigma_m\sigma_n}$$



Let's discuss the effect of step 2 first.
$\prod^N dx_i = \prod dx_i b^N$ and $\prod_{n>m}(x_n - x_m)^{\alpha\sigma_m\sigma_n} = \left(\prod_{n>m}(x_n - x_m)^{\alpha\sigma_m\sigma_n}\right) b^{\sum_{n>m}\alpha\sigma_m\sigma_n}$
Get factor

$$b^{N+\alpha\sum_{n>m}\sigma_m\sigma_n} = b^{N+\frac{\alpha}{2}\sum_{n\neq m}\sigma_m\sigma_n} = b^{N+\frac{\alpha}{2}} \sum_n \sigma_n \sum_m \sigma_m - \sum_m \sigma_m^2$$

and the first term vanishes by even number of kinks by boundary conditions. We conclude $b^{N(1-\frac{\alpha}{2})}$

So effect of step (2) is to renormalize y.

$$y_{new} = yb^{1-\frac{\alpha}{2}} = y + \underbrace{\left(1 - \frac{\alpha}{2}\right) y d\tau}_{dy}. \tag{2}$$

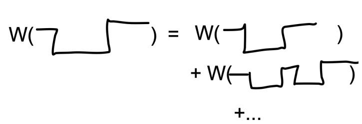
In fact, at lowest order in y, this is the only renormalization of y, nothing from step (1). We arrive to the equation:

$$\frac{dy}{d\tau} = \left(1 - \frac{\alpha}{2}\right) y + O(y^3) \tag{3}$$

Iterating RG corresponds to flow along τ i.e. increase τ. infinitesimal RG step $L \to L/e^{d\tau} = L/b$. Integrated flow $L \to L/e^{\tau}$ sends $y(0) \to y(\tau)$.

Step 1.

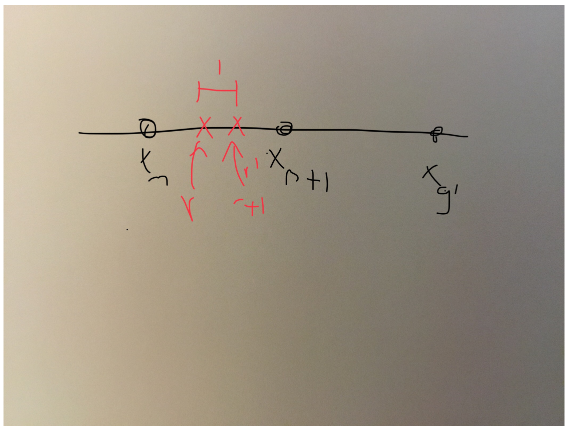
$$Z' = \sum_N \int_{\boxed{\Delta x > 1 + d\tau}} dx_m W'(\{x\})$$



to order $d\tau$, $W'(\{x\}) = W(\{x\}) + \Delta W(\{x\})$ where ΔW is the total weight of all configurations with one additinal close dipole.

(Each dipole gives a factor $d\tau$ from integrating over its separation)

$$\Delta W = \underbrace{\sum_m}_{m \text{ is which gap integrate in}} \underbrace{y^2}_{\text{fugacity dipole}} \underbrace{d\tau}_{\text{integral over separation}} W(\{x\}) \int_{x_m+1}^{x_{m+1}-2} dr \underbrace{I(r;\{x\})}_{\text{Interaction between dipole and oth}}$$



$$W' = W\left[1 + y^2 d\tau \sum_m \int drI\right] = W\exp\left\{y^2 d\tau \sum_m \int drI\right\}$$

Looking at I:

$$I = \prod_j \underbrace{|x_j - r|^{\alpha\sigma_j(-\sigma_m)}|x_j - r - 1|^{\alpha\sigma_j\sigma_m}}_{I_j}$$

consider $j > m$:

$$I_j = \left(\frac{x_j - r - 1}{x_j - r}\right)^{\alpha\sigma_j\sigma_m} \sim 1 - \frac{\alpha\sigma_j\sigma_m}{x_j - r}$$

$j \leq m$:

$$I_j \sim 1 + \frac{\alpha\sigma_j\sigma_m}{x_j - r}$$

so all together

$$I \sim 1 - \sum_{j>m} \frac{\alpha\sigma_j\sigma_m}{x_j - r} + \sum_{j\leq m} \frac{\alpha\sigma_j\sigma_m}{x_j - r}$$

expanded at 1/distance. Now we must integrate I over position of dipole.

$$W' = W\exp\left\{y^2 d\tau \sum_m \int drI\right\} = W\exp\left\{y^2 d\tau \sum_m \int dr1 - \sum ...\right\}$$

the first integral is putting the particle in gaps, $\int dr 1 = \sum_{\text{gaps}}[(\text{size of gap}) - 3]$ as dipole starts is grater than $x = 1$ and ends at least in $x = 2$. so $\int dr 1 = L^{-3N}$ contribution of F which leads to a renormalization of y at order y^3. On the other hand, contributions to renormalization in α comes from integrals of 1/dist. terms

$$-y^2 d\tau \int_{x_m+1}^{x_{m+1}} dr \frac{\alpha\sigma_j\sigma_m}{x_j - r} \sim -y^2 d\tau\alpha\sigma_j\sigma_m \log(x_j - x_m) - y^2 d\tau\alpha\sigma_j\sigma_{m+1}\log(x_j - x_{m+1})$$

where $\sigma_{m+1} = -\sigma_m$. This make sense as it decreases the value of the interaction (i.e. screening). To continue being more precise one should consider the interactions when the dipole is adjacent to the left, inside and outside the region and to the right inside and outside. A careful analysis gives they are indeed same. In the end,

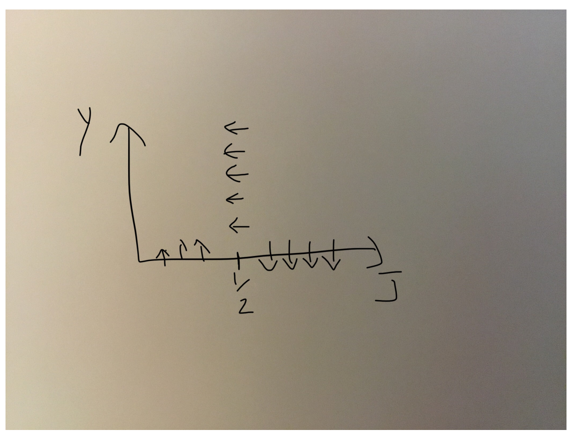
$$e^{\alpha\sigma_m\sigma_m \log(x_n - x_m)} e^{-4y^2 d\tau\alpha\sigma_m\sigma_n \log(x_n - x_m)}$$

which gives an effect on $\alpha : \alpha_{new} = \alpha - 4y^2 d\tau\alpha$,

$$\boxed{\frac{d\alpha}{d\tau} = -4y^2\alpha} \tag{4}$$

1.2 RG equations for 1D Ising with $1/r^2$ interactions

$$\boxed{\begin{aligned} \frac{dy}{d\tau} &= \left(1 - \frac{\alpha}{2}\right) y \\ \frac{d\alpha}{d\tau} &= -4\alpha y^2 \end{aligned}} \tag{5}$$



with $\alpha = 4J$ to lowest order in y.
somming to the $1/2 = J$ point, $J = \frac{1}{2} + x$, to quadratic order in x, y,

$$\frac{dy}{d\tau} = -2xy$$
$$\frac{dx}{d\tau} = -2y^2$$

$$y\frac{dy}{d\tau} = -2xy^2$$
$$x\frac{dx}{d\tau} = -2xy^2$$

which implies $\frac{d(x^2-y^2)}{d\tau} = 0$ so $x^2 - y^2$ =constant are RG flow lines.