Lecture notes on
Lie Groups, Lie Algebras and representations

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Abstract

Lie Groups, $SU(3)$, enveloping algebras and so on...

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2 Structure of semisimple Lie Algebras ($k = \mathbb{C}$)

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First idea: pick a semisimple element and decompose sl_n accordingly. Better ideas is to pick a maximal subspace in sl_n of commuting, semisimple element. One example is

$$h = \{ \begin{pmatrix} * & & .. \\ & * & \\ ... & & * \end{pmatrix} \}$$

We can decompose sl_n into simultaneous $\mathrm{ad}(h)$ eigenspaces for all $h \in h$.

$$g = \mathrm{sl}_n = \oplus_{\lambda \in h^*} g_\lambda$$

where $g_\lambda = \{x \in g | \forall h \in h,\, [h, x] = \lambda(h)x\}$

$$h = \{diag(\varepsilon_1, ..., \varepsilon_n) | \sum \varepsilon_i = 0\}$$

$$h^* = \oplus \mathbb{C}\epsilon_i / (\sum \epsilon_i)$$

where $\epsilon_i(\varepsilon_j) = \delta_{ij}$. Set of $\lambda \in h^*$ s.t. $g_\lambda \neq \{0\}$ and g_λ.

$$[\sum \varepsilon_i E_i i, \sum a_{ijE_{ij}}] = \sum_{ij} (\varepsilon_i - \varepsilon_j) a_{ij} E_{ij} =_? \lambda(\varepsilon_i, \varepsilon_j) \sum_{ij} a_{ij} E_{ij}$$

i.e. $\varepsilon_i - \varepsilon_j = \lambda(\varepsilon_i, \ldots, \varepsilon_n)$ s.t. $a_{ij} \neq 0$.

hence, either $a_{ij} = 0$ for all but one pair (i, j) with $i \neq j$ or $a_{ij} = 0$ for all $i \neq j$. This implies

$$g_0 = h$$

and

$$g_{\epsilon_i - \epsilon_j} = \mathbb{C}E_{ij}, \quad i \neq j$$

We have a nice piramid "Skeleton of sl_n"

$$
\begin{array}{r}
\epsilon_1 - \epsilon_n \\
\vdots \\
\epsilon_1 - \epsilon_3 \, \epsilon_2 - \epsilon_4 \, \ldots \epsilon_{n-2} - \epsilon_n \\
\epsilon_1 - \epsilon_2 \, \epsilon_2 - \epsilon_3 \, \ldots\ldots \epsilon_{n-1} - \epsilon_n \\
\hline \\
\epsilon_2 - \epsilon_1 \, \epsilon_3 - \epsilon_2 \, \ldots\ldots \epsilon_n - \epsilon_{n-1} \\
\vdots \\
\vdots \\
\epsilon_n - \epsilon_1
\end{array}
$$

What happens when we consider $[g_\lambda, g_\mu]$? Let $x_\lambda \in g_\lambda$, $x_\mu \in g_\mu$, $h \in h$.

$$[h, [x_\lambda, x_\mu]] = [[h, x_\lambda], x_\mu] + [x_\lambda, [h, x_\mu]] = [\lambda(h)x_\lambda, x_\mu] + [x_\lambda, \mu(h)x_\mu] = (\lambda + \mu)(h)[x_\lambda, x_\mu]$$

i.e. $[g_\lambda, g_\mu] \subset g_{\lambda+\mu}$. Since $\dim g_\sigma = 1$ if $\sigma \neq 0$ then if $\lambda, \mu, \lambda + \mu \neq 0$ this describes completely $[g_\lambda, g_\mu]$ up to scalar.

Notation:

$$\Delta := \{\lambda \in h^* | g_\lambda \neq \{0\}\} \backslash \{0\}$$

is called the root system of g with respect to h and an element in Δ is a root. In the piramid, $\Delta^+ = -\Delta^-$ being upper and lower triangles in the graph. Indeed,

Choosing

$$\Delta^+ := \{\epsilon_i - \epsilon_j | i < j\}$$

$$\Delta^- := \{\epsilon_i - \epsilon_j | i > j\}$$

we get $\Delta^+ = -\Delta^-$ and $(\Delta^\pm + \Delta^\pm) \cap \Delta \subset \Delta^\pm$ which implies

$$\oplus_{\lambda \in \Delta^+} g_\lambda$$

is a Lie subalgebra. but one would also get some splitting by choosing

$$\Delta^+ := \{\epsilon_i - \epsilon_j | \sigma(i) < \sigma(j)\}$$

, $\Delta^+ = -\Delta^-$ for any fixed $\sigma \in S_n$

Example 2.1. *$n = 3$ and $\sigma = (12)(3)$. So $\Delta^+ = \{\epsilon_2 - \epsilon_1, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3\}$*

$$\begin{pmatrix} 0 & 0 & * \\ * & 0 & * \\ & & 0 \end{pmatrix}$$

example: first row $*$ is $\epsilon_1 - \epsilon_3$.

2.3 Better picture of the root system

Recall that sl_n has a Killing form $\mathcal{K}_g$ which is invariant and nondegenerate.

Lemma 2.2. $\mathcal{K}_g(g_\lambda, g_\mu) = 0$ *if* $\lambda + \mu \neq 0$.

Proof. Pick $x_\lambda \in g_\lambda$, $x_\mu \in g_\mu$, $h \in h$.

$$\mathcal{K}_g([h, x_\lambda], x_\mu) + \mathcal{K}_g(x_\lambda, \underbrace{[h, x_\mu]}_{\mu(h)x_\mu}) = 0 = (\lambda + \mu)(h)\mathcal{K}_g(x_\lambda, x_\mu)$$

□

Corollary 2.3. *1.* $\mathcal{K}_g|_h$ *is nondegenerate.*

2. $g_\lambda^* = g_{-\lambda}$ *and in particular they are one dim.*

We can use the isomorphisme $h \sim h^*$ coming from 1. to transport $\mathcal{K}_g$ to nondegenerate form on h^*.

Digression 2.4. *V a vector space. $(,) : V \otimes V \to \mathbb{C}$ non degenerate. $\forall v \in V$, define $\lambda_v : u \mapsto (v, u)$ so the map $v \mapsto \lambda_v$*

$$V \to V^*$$

is an isomorphism.

This implies $g_\lambda \subset g \sim g^* - - - - >> g^*_{-\lambda}$

Lemma 2.5.

- *$span_{\mathbb{C}} = h^*$*
- *$span_{\mathbb{R}} = h^*_{\mathbb{R}}$ is a real form of h^* i.e. $h^*_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = h^*$*
- *$(h^*_{\mathbb{R}}, (,))$ is a euclidean vector space.*

Example 2.6. *Up to normalization*

$$\mathcal{K}_g(E_{ij}, E_{ji}) = 1$$

$$\mathcal{K}_g(E_{ii}, E_{jj}) = \delta_{ij}$$

all others are 0. $h^* = \oplus_{i=1}^{n-1} \mathbb{C}\alpha_i$ and $\alpha_i := \epsilon_i - \epsilon_{i+1}$, $h^*_{\mathbb{R}} = \oplus_{i=1}^{n-1} \mathbb{R}\alpha_i$ so

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\alpha_i, \alpha_j) = \begin{cases} 0 & |i - j| > 1 \\ -1 & |i - j| = 2 \\ 2 & i = j \end{cases}$$

Matrix of $(,)$ in basis $\{\alpha_i\}$ is

$$\begin{pmatrix} 2 & -1 & .. \\ -1 & ... & 0 \\ ... & & -1 \\ ... & -1 & 0 \\ & & 2 \end{pmatrix}$$

This is positive definite. Check.

Example 2.7. *For sl_3 $h^* = \mathbb{C}\alpha_1 \oplus \mathbb{C}\alpha_2$ so $(\alpha_i, \alpha_i) = 2$, $|\alpha_1| = \sqrt{2} = |\alpha_2|$, $(\alpha_1, \alpha_2) = -1$ so $\cos(angle(\alpha_1, \alpha_2)) = -1/2$ so $angle(\alpha_1, \alpha_2) = 2\pi/3$.*

$$\Delta = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$$

$$\begin{tikzcd}
 & \alpha_2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \arrow[r] \arrow[dl] & \alpha_2 + \alpha_1 \sim \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \arrow[dr] & \\
-\alpha_1 \sim \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & h & & \alpha_1 \sim \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 & -\alpha_1 - \alpha_2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \arrow[r] \arrow[ul] & -\alpha_2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix} \arrow[ur] &
\end{tikzcd}$$

Possible choices for Δ^+:

$$\begin{cases} \sigma = (1)(2)(3) & \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \\ \sigma = (13)(2) & -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2 \\ \sigma = (12)(3) & -\alpha_1, \alpha_1 + \alpha_2, \alpha_2 \\ \sigma = (23)(1) & -\alpha_1, , \alpha_1 + \alpha_2, \alpha_2 \\ \vdots \end{cases}$$

$\alpha_1 = \epsilon_1 - \epsilon_2 \in h^*$, $g_{\alpha_1} = E_{12}$,

$$[diag(\varepsilon_1, \varepsilon_2, \varepsilon_3), E_{12}] = (\varepsilon_1 - \varepsilon_2)E_{12}$$

Group of symmetries of the root system is D_3 which is isomorphic to S_3.

We may use the same picture to understand finite dimensional representations. Let V be a finite dimensional sl_n-representation. h acts by diagonalizable operators commuting between them. Therefore,

$$V = \oplus_{\lambda \in h^*} V_\lambda$$

where $V_\lambda = \{v \in V | \forall v \in h,\, h \cdot v = \lambda(h)v\}$ $wt(V) := \{\lambda \in h^* | V_\lambda \neq \{0\}\}$.

Lemma 2.8. $g_\lambda \cdot V_\mu \subset V_{\lambda+\mu}$

$g \otimes V \to V$ then $g_\lambda \otimes V_\mu \to V_{\lambda+\mu}$

Proof.

$$\begin{aligned} h \cdot (x_\lambda v_\mu) &= [h, x_\lambda] \cdot v_\mu + x_\lambda \cdot (h \cdot v_\mu) \\ &= \lambda(h) x_\lambda \cdot v_\mu + \mu(h) x_\lambda \cdot v_\mu \end{aligned}$$

□

Example 2.9. *sl_3. $V = \mathbb{C}$ with usual action. $wt(V) = \{\epsilon_1, \epsilon_2, -\epsilon_1 - \epsilon_2\}$ Indeed, $h = diag \epsilon_1, \epsilon_2, \epsilon_3$, $v^T = (v_1, v_2, v_3)$. We want to solve $h \cdot v = \lambda(h) v$*

$$\begin{cases} \lambda v_{\epsilon_1} = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \\ \lambda v_{\epsilon_2} = \begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix} \\ \lambda v_{-\epsilon_1 - \epsilon_2} = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix} \end{cases}$$

On the picture,
$\epsilon_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$, $\epsilon_2 = -\frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$ *and* $\epsilon_3 = -\frac{2}{3}\alpha_1 - \frac{1}{3}\alpha_2$.

Weights form a triangle pointing down inside the hexagon. This shows you the action of the rep.

Finally, the dual $wt(V^*) = -wt(V)$, the weights form a pointing up triangle.

2.4 General Case (statements and definitions)

Let g be a semisimple Lie Algebra over $\mathbb{C}$.

Definition 2.10. *A Cartan subalgebra is a Lie subalgebra $h \subset g$ s.t.*

1. *h is abelian*
2. *any $h \in h$ is semisimple*
3. *h is maximal with respect to 1. and 2.*

Theorem 2.11. *Chavalley Any two Cartan subalgebras are $Aut(g)-$conjugate.*

Definition 2.12. *$rank(g) = \dim h$ (for any Cartan subalgebra h)*

Example 2.13. *$rank(sl_n) = n - 1$*

How two construct Cartan subalgebras?

Definition 2.14. *An element $x \in g$ is regular if $g^x := \{y \in g | [x, y] = 0\}$ is of minimal dimension among all x.*

Fact: The function $x \mapsto \dim g^x$ is upper semicontinous. i.e. $\forall n$, $f^{-1}([0, n])$ is open. In particular, $g^{reg} = \{x | x \text{ is reg}\}$ is open in g.

Example 2.15. $g = sl_n$

$$x = \begin{pmatrix} Id_n 1 a_1 & & .. \\ & Id_2 a_2 & \\ ... & & Id_n a_n \end{pmatrix}$$

$a_i \neq a_j$ for $i \neq j$. Then

$$g^x = \begin{pmatrix} \boxed{*} & & .. \\ & \boxed{*} & \\ ... & & \boxed{*} \end{pmatrix}$$

$\dim g^x$ *is minimal if $n_1 = ... = n_k = 1$. When it happens, it is equal to $n - 1$.*

Fact: For sl_n, $n - 1$ is the minimal dimension for g^x.

$$\{diag(a_1, ..., a_n) | a_i \neq a_j \text{ for } i \neq j\} \subset g^{reg}$$

Jordan block are in g^{reg}. Exercise: x regular $\iff$ $\exists$ at most one Jordan block for every eigenvalue λ.

Proposition 2.16. *$g^{reg} \cap g^{ss}$ is open and dense in g. Exercise: For sl_n, $g^{reg} \cap g^{ss}$ is the set of matrices with characteristic polynomial with simple roots.*

Theorem 2.17.

1. *$\forall x \in g^{reg} \cap g^{ss}$, g^x is cartan subalgebra*
2. *Any cartan subalgebra is of this form.*

Example 2.18. *sl_n, let $x = diag(a_1, ..., a_n)$ $a_i \neq a_j$. then*

$$g^x = \{ \begin{pmatrix} * & & .. \\ & * & \\ ... & & * \end{pmatrix} \}$$

2.4.1 Root Space Decomposition

Fix a Cartan subalgebra h. For $\alpha \in h^*$,

$$g_\alpha := \{x \in g | \forall h \in h, [h, x] = \alpha(h) x\}$$

$$g = \oplus_{\lambda \in h^*} g_\lambda$$

Definition 2.19.

$$\Delta := \{\alpha \in h^* \{0\} | g_\alpha \neq 0\}$$

is the root system of g with respect to h.

Theorem 2.20.

1. *$g_0 = h$ ($\Longrightarrow$ $\dim g_0 = rh(g)$)*
2. *$\forall \alpha \in \Delta, \dim g_\alpha = 1$*
3. *$[g_\lambda, g_\mu] \subset g_{\lambda+\mu}$ $\forall \lambda, \mu$.*

2.4.2 Metric structure on h^*

Exercise:$span_{\mathbb{C}} = h^*$ (hint: $Z(g) = \{0\}$).

Let $\mathcal{K}_g$ be the Killing form on g.

Lemma 2.21. *1. $\mathcal{K}_g(g_\lambda, g_\mu) = 0$ if $\lambda + \mu \neq 0$*

2. $\mathcal{K}_g|_h$ is non-degenerate.

Using $h \tilde{\to} h^*$ we can transport $\mathcal{K}_g$ to a non-degenerate form on h^* Notation: $t_\alpha \in h$ corresponds to $\alpha \in h^*$ $\mathcal{K}_g(t_\alpha, h) = \alpha(h)$ $\forall h \in h$.

Lemma 2.22.

$$[g_\alpha, g_{-\alpha}] = \mathbb{C}t_\alpha$$

and $\mathcal{K}_g(t_\alpha, t_\alpha) \neq 0$

Proposition 2.23. *$h^*_{\mathbb{R}} := span_{\mathbb{R}}$ is a Euclidean space and a real form of h^*.*

Example 2.24. *There exists a Lie Algebras called g_2 of rank 2 whose root system is:*

$$\begin{tikzcd}
 & & & 2\alpha_2 + 3\alpha_1 & & & \\
\alpha_2 \arrow[urrr] \arrow[rr] \arrow[ddd] & & \alpha_2 + \alpha_1 \arrow[rr] \arrow[dl] & & \alpha_2 + 2\alpha_1 \arrow[rr] \arrow[dr] & & \alpha_2 + 3\alpha_1 \arrow[ulll] \arrow[ddd] \\
 & -\alpha_1 & & 2 \arrow[ulll] \arrow[rr] & & \alpha_1 & \\
 & & & & & & \bullet \\
\bullet \arrow[drrr] & & \bullet \arrow[ul] \arrow[rr] & & \bullet \arrow[ur] & & \bullet \\
 & & & \bullet \arrow[urrr] & & &
\end{tikzcd}$$

$\dim g_2 = 14$