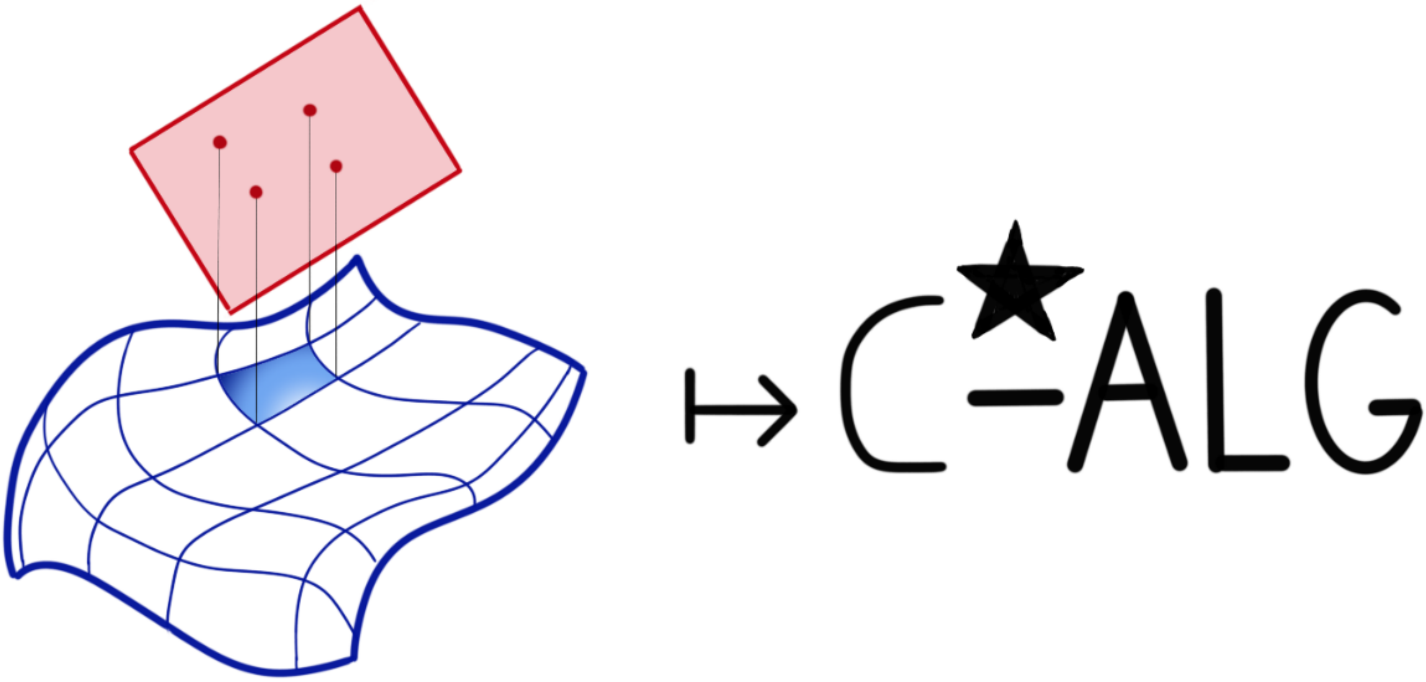

Quantum Field Theory in Curved Spacetimes

Lecture Notes:

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Date: May 12, 2022

Version: 1.01

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Preface

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1.1 Kinematical Aspects

The kinematical possible positions for a finite number of point particles in Galilean spacetime are represented by a smooth Riemannian manifold, the **configuration space** of the system. Defining Galilean spacetime is problematic as a relativistic theory, as there is no notion of spacetime distance, only distance on hypersurfaces of equal time.

Definition 1.1

*The **Configuration Space** $\mathcal{C}$ is a smooth Riemannian manifold.*

It is the space of all kinematically possible positions for a system of a finite number of point particle

♣

Points in configuration space are events in Galilean spacetime.

Definition 1.2

***Momenta** are represented by the cotangent space: at every position $q \in \mathcal{C}$ momenta are 1-forms in $T^\star_Q\mathcal{C}$.*

♣

Remark

Technical remark: For each position in $\mathcal{C}$ we replace the 0 co-vector $0^\star \in T^\star_q\mathcal{C}$ by $0^\star_q$.

What are states in this formulation? The answer is phase space:

Definition 1.3

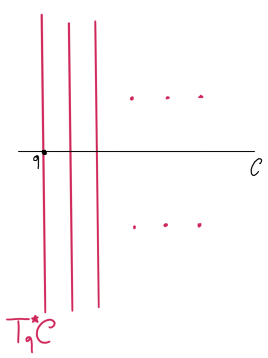
***Phase space** is the cotangent bundle of configuration space*

$$\mathbb{P} := T^\star\mathcal{C} := \bigcup_{q\in\mathcal{C}} T_q\mathcal{C} \tag{1.1}$$

It's the space of kinematically possible positions and momenta.

♣

The cotangent bundle is a "memory device" that associates to each point its cotangent space



with projection $\pi : T^\star\mathcal{C} \to \mathcal{C}$, $\pi^{-1}(q) = T^\star_q\mathcal{C}$.

$T^\star\mathcal{C}$ itself is a smooth manifold (so we can do analysis on it).

Coordinate neighborhoods: $n := \dim\mathcal{C}$, $\mathcal{U} \subset \mathcal{C}$, $\zeta = (u^1, \dots, u^n)$ coordinate chart on $\mathcal{U}$, then the corresponding coordinate vectors are denoted by

Physical statements mustn't depend on coordinate choice

$$\partial_1, \dots, \partial_n \tag{1.2}$$

and the dual coordinate 1-forms

$$\mathrm{d}u^1, \dots, \mathrm{d}u^n \tag{1.3}$$

The real valued functions on $\pi^{-1}(\mathcal{U}) \subset T^\star\mathcal{C}$ p_a ($a \in I(n)$), given by

$I(n)$ index set $1, \dots, n$

$$p_a(p) := p(\partial_a) \tag{1.4}$$

with p a one-form, gives the components of the momentum vector.

Introduce

$$q^a := u^a \circ \pi \ , \ \bar{\zeta} : \pi^{-1}(\mathcal{U}) \to \mathbb{R}^{2n} \tag{1.5}$$

(u^a coordinate function) defined by

$$\bar{\zeta} := (q^1, \dots, q^n, p_1, \dots, p_n) \tag{1.6}$$

this is a (smooth) coordinate representation of the cotangent bundle.

Definition 1.4

*Given a phase space $\mathbb{P}$, a **motion** γ of our system is a curve in $\mathbb{P}$, i.e.*

$$\gamma : I \subset \mathbb{R} \to \mathbb{P} \tag{1.7}$$

♣

Using the differential map:

t is really on I, but we're using thw whole real line.

$$\mathrm{d}\gamma_t : \underbrace{T_t}_{\text{at time } t} \mathbb{R} \to T_{\gamma(t)}\mathbb{P} \quad , \quad t \in \mathbb{R} \tag{1.8}$$

Characterized by, given $f : \mathbb{P} \to \mathbb{R}$:

$$\mathrm{d}\gamma_t\left(\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_t\right)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_t (f \circ \gamma) \tag{1.9}$$

"time derivative".

The corresponding velocity vector at time t is

$$\dot{\gamma}(t) = \mathrm{d}\gamma_t\left(\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_t\right) \in T_{\gamma(t)}\mathbb{P} \tag{1.10}$$

a motion γ is called an **integral curve** of a vector field V (smooth vector field on $\mathbb{P}$) provided:

$$\dot{\gamma}(t) = V_{\gamma(t)} \ , \ \ \forall t \in I \tag{1.11}$$

This was our kinematical setup: for dynamical features we need a **symplectic structure**.

1.2 Dynamical Aspects

We need an additional structure, which we will introduce first in a more abstract fashion.

In some sense Ω takes the responsibility of a metric.

Definition 1.5

*A **symplectic structure** (or symplectic form) on $\mathbb{P}$ is a non degenerate closed 2-form Ω on $\mathbb{P}$.*

*The pair $(\mathbb{P}, \Omega)$ is a **symplectic manifold**.* ♣

If $\mathcal{C}$ is a smooth manifold, $\mathbb{P}$ has a standard simple form:

Theorem 1.1

Let $\mathbb{P} = T^\star\mathcal{C}$ be the momentum phase space of a mechanical system with configuration space $\mathcal{C}$ as a base manifold.

Consider the mappings

$$\tau^\star_{\mathcal{C}} : \mathbb{P} \to \mathcal{C}\ , \quad T_{\tau^\star_{\mathcal{C}}} : T\mathbb{P} \to T\mathcal{C} \tag{1.12}$$

$\tau^\star_{\mathcal{C}}$ the projection.

Let $\alpha_q \in \mathbb{P}$ ($q \in \mathcal{C}$) and w_{α_q} a point in $T\mathbb{P}$ in the fiber over α_q (state of the system at position q).

Introduce

$$\theta_{\alpha_q} : T_{\alpha_q}\mathbb{P} \to \mathbb{R} \tag{1.13}$$

$$w_{\alpha_q} \mapsto \theta_{\alpha_q}(w_{\alpha_q}) := \alpha_q\left(T\tau^\star_{\mathcal{C}}(w_{\alpha_q})\right) \tag{1.14}$$

and

$$\theta_0 : \alpha_q \mapsto \theta_{\alpha_q} \tag{1.15}$$

then $\theta_0 \in \mathfrak{X}^\star(\mathbb{P})$ (1-form on $\mathbb{P}$) and $\Omega_0 := -\mathrm{d}\theta_0$ is a symplectic form on $\mathbb{P}$.

Proof We need to prove that θ_0 is a smooth 1-form on $\mathbb{P}$ and Ω_0 is symplectic (then θ_0 and Ω_0 are called **canonical forms** on $\mathbb{P}$).

1. Introduce 3 useful charts:

   Let $(\mathcal{U}, \varphi)$ be achart on $\mathcal{C}$ with $\varphi(\mathcal{U}) = \varphi' \subset \mathbb{R}^n$ ($n = \dim\mathcal{C}$), and let $(T^\star\mathcal{U}, T^\star\varphi)$, $T^\star\varphi : T^\star\mathcal{U} \to \mathcal{U}'\,(\mathbb{R}^n)^\star$ be the corresponding chart on $\mathcal{P}$.

   K Let $(TT^\star\mathcal{U}, TT^\star\varphi)$, $TT^\star\varphi : TT^\star\mathcal{U} \to \mathcal{U}' \times (\mathbb{R}^n)^\star \times \mathbb{R}^n \times (\mathbb{R}^n)^\star$ be the corresponding chart on T^P.

   Let $(T^\star T^\star\mathcal{U}, T^\star T^\star\varphi)$, $T^\star T^\star\varphi : T^\star T^\star\mathcal{U} \to \mathcal{U}' \times (\mathbb{R}^n)^\star \times (\mathbb{R}^n)^\star \times (\mathbb{R}^n)^\star$ be the corresponding chart on $T^\star\mathcal{P}$.

   Denote $\varphi(q) = x$, $T^\star_q\varphi(\alpha_q) = \alpha$, $T_{\alpha_q}T^\star\varphi(w_{\alpha_q}) = (e, \beta)$ ($w \in T_{\alpha_q}\mathbb{P}$).
2. Represent $\theta_{\alpha_q}(w_{\alpha_q})$ in these charts:

$$\alpha_q = \left(T^\star_q\varphi\right)^{-1}(\alpha) \tag{1.16}$$

coordinate representation of w_{α_q} $T_{\alpha_q}T^\star\varphi(w_{\alpha_q}) = (e, \beta)$ (e vector β one-form):

$$\theta_{\alpha_q}(w_{\alpha_q}) = \underbrace{T^\star_{\alpha_q}T^\star\varphi\circ\underbrace{\theta_0\circ\underbrace{\left(T^\star_q\varphi\right)^{-1}(\alpha)}_{\alpha_q}}_{\theta_{\alpha_q}}}_{\text{chart rep of } \theta_{\alpha_q}}\circ\underbrace{T_{\alpha_q}T^\star\varphi(w_{\alpha_q})}_{(e,\beta)} \tag{1.17}$$

with θ_0 the corresponding covector field in phase space $\alpha_q \mapsto \theta_{\alpha_q}$.

By definition

$$\begin{aligned}
\theta_{\alpha_q}(w_{\alpha_q}) &= \alpha_q\left(T_{\alpha_q}\tau^\star_{\mathcal{C}}(w_{\alpha_q})\right) && (1.18)\\
&= \alpha_q\left(T_{\alpha_q}\left(\tau^\star_{\mathcal{C}}\circ T^\star\varphi^{-1}\right)(e,\beta)\right) && (1.19)\\
&= \alpha_q\left(T_q\left(\varphi^{-1}\circ Pr_1(e,\beta)\right)\right) && (1.20)\\
&= \alpha_q\left(T_q\varphi^{-1}\left(TPr_1(e,\beta)\right)\right) && (1.21)\\
&= \alpha_q\left(T_q\varphi^{-1}(e)\right) = T^\star_q\varphi(\alpha_q)\cdot e && (1.22)\\
&= \alpha(e) && (1.23)
\end{aligned}$$

with Pr_1 the projection on the first factor.

Therefore θ_0 is given locally by

$$\left(T^\star T^\star\varphi\circ\theta_0\circ T^\star\varphi^{-1}\right)(x,\alpha)(x,\alpha,e,\beta) = \alpha(e) \tag{1.24}$$

3. (Symplectic structure) The local representation of θ_0 shows that θ_0 is smooth and hence $\theta_0 \in \mathfrak{X}^\star(\mathbb{P})$.

   Since $\Omega_0 = -\mathrm{d}\theta_0$, Ω_0 is clearly closed, non degenerate on $\mathbb{P}$, namely:

$$\Omega_0(x,\alpha)\left((x,\alpha,e_1,\beta_1),(x,\alpha,e_2,\beta_2)\right) = \beta_2(e_1) - \beta_1(e_2) \tag{1.25}$$

   hence $\mathbb{P}$ carries a symplectic structure. □

♡

Remark

The above local expression of Ω_0 is explicitly independent of (x,α), reflecting the fact that the natural charts of momentum phases space are so-called symplectic charts (clearer later).

How do these forms look in a particular coordinate system? **Canonical Coordiantes**:

Theorem 1.2

Let $\mathbb{P} = T^\star\mathcal{C}$, be the $2n$ dimensional momentum phase space of a mechanical system with configuration space $\mathcal{C}$, and Ω be a symplectic structure on $\mathbb{P}$.

Then $\mathrm{d}\Omega = 0$ if and only if there is a chart $(\mathcal{U},\varphi)$ at each $z \in \mathbb{P}$ such that $\varphi(z) = 0$, and with

$$\varphi(u) = \left(q^1(u),\ldots,q^n(u),p_1(u),\ldots p_n(u)\right) \quad , \quad u \in \mathcal{U} \tag{1.26}$$

we have that

$$\Omega|_{\mathcal{U}} = \sum_{a \in I(n)} \mathrm{d}q^a \wedge \mathrm{d}p_a \tag{1.27}$$

(constant)

Proof It's obvious that Ω is closed, so one direction is trivial. In the other direction:

1. Assum $z = 0$, let $\Omega_1(x) = \Omega(0)\ \forall x \in \mathbb{P}$ (constant form), set

$$\widetilde{\Omega} := \Omega_1 - \Omega \tag{1.28}$$

   and the convex linear combination

$$\Omega_t := \Omega + t\widetilde{\Omega} = t\Omega_1 + (1-t)\Omega\,, \quad t \in [0,1] \tag{1.29}$$

   So $\Omega_0 = \Omega$ and $\Omega_1 = \Omega(0)$.
   For each t, $\Omega_t(0) = \Omega(0)$ which is non degenerate by assumption.
   hence by oppenness of the set of linear transformations of $\mathbb{P}$ to $\mathbb{P}^\star$, there is a neighborhood of 0 on which Ω_t is non degenerate $\forall t \in [0,1]$, which we can assume to be a ball. Thus, by Poincaré lemma, $\widetilde{\Omega} = \mathrm{d}\omega$, for a one-form ω in this ball. Assume that $\omega(0) = 0$.
2. Define a smooth vector field X_t by

$$\iota_{X_t}\Omega_t = -\omega \tag{1.30}$$

   (interior product). This is possible because Ω_t is non degenerate.
   hence $\omega(0) = 0$ implies that $X_t(0) = 0$, then the local existence theorem implies that there is a ball around 0 on which the flow of X_t is defined for some time.
   Call this flow F_t (with initial condition $F_0 = \mathrm{id}$). (next: change in time of the flow is 0 implies Ω_t =const).
3.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(F_t^\star\Omega_t\right) = F_t^\star\left(\mathcal{L}_{X_t}\Omega_t\right) + F_t^\star\frac{\mathrm{d}}{\mathrm{d}t}\Omega_t \tag{1.31}$$

$$= F_t^\star\left(\mathrm{d}\iota_{X_t}\Omega_t\right) + F_t^\star\widetilde{\Omega} \tag{1.32}$$

$$= F_t^\star\left(-\mathrm{d}\omega + \widetilde{\Omega}\right) = 0 \tag{1.33}$$

   Therefore

$$F_1^\star\Omega_1 = F_0^\star\Omega_0 = \Omega \quad \square \tag{1.34}$$

♡

Remark

1. We introduce symplectic charts (canonical coordinates).

2. We have the local formulas in these coordinates

$$\theta_0 = \sum_{a \in I(n)} p_a \mathrm{d}q^a \tag{1.35}$$

$$\Omega_0 = \sum_{a \in I(n)} \mathrm{d}q^a \wedge \mathrm{d}p_a \tag{1.36}$$

Definition 1.6

Let $(\mathbb{P}, \Omega)$ be the momentum phase space equipped with a symplectic structure Ω of a mechanical system and

$$H : \mathbb{P} \to \mathbb{R} \tag{1.37}$$

*a given C^r function (r open for now), called the **Hamiltonian** of energy function.*

The vector field X_H determined by

$$\iota_{X_H}\Omega = \mathrm{d}H \tag{1.38}$$

*is called the **Hamiltonian vector field** with energy function H.*

*The triple $(\mathbb{P}, \Omega, H)$ (or sometimes $(\mathbb{P}, \Omega, X_H)$) is called a **Hamiltonian System**.* ♣

Remark

- The existence of X_H is guaranteed by the non-degeneracy of Ω.
- Say I have 2 Hamiltonians for the same Hamiltonian vector field, then they have the same differential, thus they differ by a constant (irrelevant).

Definition 1.7

Let $(\mathbb{P}, \Omega, H)$ be a Hamiltonian system.

*A **motion** of this system is an integral curve*

$$\gamma : I \subset \mathbb{R} \to \mathbb{P} \tag{1.39}$$

of the corresponding Hamiltonian vector field. ♣

Remark

A motion γ of $(\mathbb{P}, \Omega, H)$ satisfies $\dot{\gamma}(t) = X_H(\gamma(t))$ (velocity at point $\gamma(t)$).

Proposition 1.1

Let $(\mathbb{P}, \Omega, H)$ be a Hamiltonian system and let ($n = \dim \mathcal{C}$, $\mathcal{C}$ configuration space):

$$\left(q^1, \ldots, q^n, p_1, \ldots, p_n\right) \tag{1.40}$$

be canonical coordinates for Ω (constant on these in a specified neighborhood).

Then in these coordinates

$$X_H = \left(\frac{\partial H}{\partial p_a}, -\frac{\partial H}{\partial q^a}\right) \quad , \quad a \in I(n) \tag{1.41}$$

(dropping the point dependence)

*Thus $(q(\gamma(t)), p(\gamma(t)))$ is an integral curve of X_H if and only if **Hamilton's equations** hold:*

$$\begin{cases} \dot{q}\left(\gamma(t)\right) = \dfrac{\partial H}{\partial p_a}\left(\gamma(t)\right) \\ \dot{p}_a\left(\gamma(t)\right) = -\dfrac{\partial H}{\partial q^a}\left(\gamma(t)\right) \end{cases} \quad , \quad a \in I(n) \tag{1.42}$$

Proof

1. In canonical coordinates:
$$\mathrm{d}H = \sum_{a \in I(n)} \left(\frac{\partial H}{\partial q^a} \mathrm{d}q^a + \frac{\partial H}{\partial p_a} \mathrm{d}p_a \right) \tag{1.43}$$
furthermore,
$$\iota_{X_H} \Omega = \sum_{a \in I(n)} \iota_{X_H} \left(\mathrm{d}q^a \wedge \mathrm{d}p_a \right) \tag{1.44}$$
$$= \sum_{a \in I(n)} \Bigg[\underset{\uparrow}{\overset{\text{function in phase space}}{}} \iota_{X_H} \left(\mathrm{d}q^a \right) \quad \mathrm{d}p_a - \overset{\text{wedge}}{\uparrow} \quad \iota_{X_H} \left(\mathrm{d}p_a \right) \mathrm{d}q^a \Bigg] \tag{1.45}$$
equating this to $\mathrm{d}H$ gives
$$\iota_{X_H} \left(\mathrm{d}q^a \right) = \frac{\partial H}{\partial p_a} \tag{1.46}$$
$$\iota_{X_H} \left(\mathrm{d}p_a \right) = -\frac{\partial H}{\partial q^a} \quad , \quad a \in I(n) \tag{1.47}$$
2. Now
$$\iota_{X_H} \left(\mathrm{d}q^a \right) = X_H^a \tag{1.48}$$
$$\iota_{X_H} \left(\mathrm{d}p_a \right) = \left(X_H \right)_{n+a} \quad , \quad a \in I(n) \tag{1.49}$$
Thus
$$X_H = \left(\frac{\partial H}{\partial p_a}, -\frac{\partial H}{\partial q^a} \right) \tag{1.50}$$
Second part: by definition Hamilton's equations are dynamical equations for integral curves of Hamiltonian vector fields (canonical coordinates for Ω). $\square$

Proposition 1.2

Let $(\mathbb{P}, \Omega, H)$ be a Hamiltonian system with Energy function $H : \mathbb{P} \to \mathbb{R}$ (C^r), and let $\gamma : I \subset \mathbb{R} \to \mathbb{P}$ be an integral curve for the Hamiltonian vector field X_H.

then H is constant along γ (constant energy along a motion).

Proof

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\gamma(t)) = \mathrm{d}H_{\gamma(t)} \left(\dot{\gamma}(t) \right) \overset{\text{H eq.}}{=} \mathrm{d}H_{\gamma(t)} \left(X_H(\gamma(t)) \right) \tag{1.51}$$
$$\overset{def}{=} \left(\iota_{X_H} \Omega \right) \left(X_H \right)_{\gamma(t)} \tag{1.52}$$
$$= \Omega \left(X_H(\gamma(t)), X_H(\gamma(t)) \right) \overset{skew}{=} 0 \quad \square \tag{1.53}$$

Now we want to introduce an algebrai structure, important for a quantisation prescription:

Definition 1.8. Musical Maps

Let $(\mathbb{P}, \Omega, H)$ be a Hamiltonian system.

Introduce: the Ω-flat isomorphism

$$\Omega^\flat : T\mathbb{P} \to T^\star\mathbb{P} \tag{1.54}$$

$$\Omega^\flat(X) \cdot Y = \Omega(X, Y)\ \forall X, Y \in T\mathbb{P} \tag{1.55}$$

Since $\Omega^\flat$ is a vector bundle isomorphism we also define the Ω-sharp isomorphism

$$\Omega^\sharp := \left(\Omega^\flat\right)^{-1} \tag{1.56}$$

♣

Remark In this notation $X_H = \Omega^\sharp\,(\mathrm{d}H)$

Definition 1.9

Let $(\mathbb{P}, \Omega, H)$ be a Hamiltonian system, and let $f, g \in Fun(\mathbb{P})$ (smooth functions on $\mathbb{P}$) with $X_f := \Omega^\sharp(\mathrm{d}f) \in \mathfrak{X}(\mathbb{P})$ (smooth vector field).

The ***Poisson Bracket*** *of f and g is the function*

$$\{f, g\} := \Omega(X_f, X_g) \tag{1.57}$$

♣

This is a coordinate independent geometrical definition, later we will convince ourselves that it's the familiar notion.

There's a useful connection between Poisson brackets and Lie derivatives:

Proposition 1.3

Let $(\mathbb{P}, \Omega, H)$ be a Hamiltonian system $f, g \in Fun(\mathbb{P})$.

Then

$$\{f, g\} = -\mathcal{L}_{X_f} g = \mathcal{L}_{X_g} f \tag{1.58}$$

Proof $\mathcal{L}_{X_f} g = \iota_{X_f}(\mathrm{d}g)$. By definition $X_g = \Omega^\sharp(\mathrm{d}g)$, hence

$$\mathrm{d}g = \iota_{X_g}\Omega \tag{1.59}$$

therefore

$$\mathcal{L}_{X_f} g = \iota_{X_f}\iota_{X_g}\Omega = \Omega\left(X_g, X_f\right) \tag{1.60}$$

And by skew-symmetry

$$\iota_{X_f}\iota_{X_g}\Omega = -\iota_{X_g}\iota_{X_f}\Omega = -\iota_{X_g}\mathrm{d}f = -\mathcal{L}_{X_g} f \quad \square \tag{1.61}$$

♠

As there is also a connection between the flow and the Lie derivatives we also expect one between the Poisson brackets with the flow, which hints the following corollary:

Corollary 1.1

1. *For $f_0 \in Fun(\mathbb{P})$, $g \mapsto \{f_0, g\}$ is a derivation.*
2. *f is constant on the orbit of X_g if and only if $\{f, g\} = 0$ if and only if g is constant on*

the orbits of f.

Proof

1. Clear, because the Lie derivative is a derivation.
2. If F_t is the flow of X_f then

$$\frac{\mathrm{d}}{\mathrm{d}t}(g \circ F_t) = F_t^\star \mathcal{L}_{X_f} g = -F_t^\star \{f, g\} \tag{1.62}$$

which vanishes if and only if $\{f, g\} = 0$ ($F_t^\star$ is an isomorphism), and by skew-symmetry we get the other thing. □

♡

In particular,

$$\{H, H\} = 0 \tag{1.63}$$

trivially, which is an algebraic reformulation of the conservation of energy.

Corollary 1.2

In a symplectic chart

$$\{f, g\} = \sum_{a \in I(n)} \left(\frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a} \right) \tag{1.64}$$

Proof

$$\{f, g\} = \mathcal{L}_{X_g} f = \mathrm{d}f \cdot X_g \tag{1.65}$$

In canonical coordinates:

$$\mathrm{d}f = \sum_{a \in I(n)} \left(\frac{\partial f}{\partial q^a} \mathrm{d}q^a + \frac{\partial f}{\partial p_a} \mathrm{d}p_a \right) \tag{1.66}$$

And:

$$X_g = \Omega^\sharp(\mathrm{d}g) = \sum_{a \in I(n)} \left(\frac{\partial g}{\partial p_a} \frac{\partial}{\partial q^a} - \frac{\partial g}{\partial q^a} \frac{\partial}{\partial p_a} \right) \tag{1.67}$$

which then implies the statement. □

♡

Remark

Important Poisson Brackets are

$$\left\{q^a, q^b\right\} = 0\,, \quad \{p_a, p_b\} = 0\,, \quad \{q^a, p_b\} = \delta^a{}_b\,, \quad (a, b \in I(n)) \tag{1.68}$$

Sp expanding functions of q, p we can use this as a basis of the algebra of observables of the theory.

Remark

One can easily prove that $f \in Fun(\mathbb{P})$, F_t flow of a Hamiltonian vector field X_H

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ F_t) = \{f \circ F_t, H\} \tag{1.69}$$

i.e. H, the energy function, generates change in flow of a function in the state F_t (related to time translation symmetry via Noether's theorem).

The algebraic structure of observables is a **Lie Algebra**.

Chapter 2 Quantum Mechanics

We will now start the exploration of structures in quantum mechanics. We want to generalize $L^2(\mathbb{R}^3)$. What should be the state space?
Consider an arbitrary finite-dimensional (smooth) manifold $\mathcal{C}$ (configuration space of a classical system).
Our challenge is: it doesn't carry a canonical measure (on Riemannian manifolds there is always a **natural** measure).

Definition 2.1

Let $\mathcal{C}$ be a finite dimensional manifold.
Consider the set of all pairs (f, μ) where μ is a natural (absolutely continuous) measure (i.e. μ is equivalent to a Lebesgue measure in every chart of $\mathcal{C}$), and f is a complex measurable function such that

$$\int_{\mathcal{C}} \mathrm{d}\mu \, |f|^2_{\mathbb{C}} < \infty \tag{2.1}$$

Two pairs (f, μ), (g, ν) will be called equivalent provided that

$$f \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\nu}} = g \tag{2.2}$$

We denote the equivalence class of (f, μ) by the half density $f\sqrt{\mathrm{d}\mu}$.
The set of all such equivalence classes is denoted by $\mathcal{H}(\mathcal{C})$ ♣

The Hilbert space structure of $\mathcal{H}(\mathcal{C})$ can be defined as follows: choose a natural measure μ. Then the map

$$\mathcal{U}_\mu : f \mapsto f\sqrt{\mathrm{d}\mu} \tag{2.3}$$

is a bijection from $L^2\mathcal{C}, \mu$ onto $\mathcal{H}(\mathcal{C})$.
Question: Is the resulting structure independent of the choice of the natural measure?
Answer: Assume $(f, \mu) \sim (g, \nu)$, i.e. $\zeta := f\sqrt{\mathrm{d}\mu} = g\sqrt{\mathrm{d}\nu}$ in $\mathcal{H}(\mathcal{C})$.
Then

$$\|\mathcal{U}_\mu^{-1}\zeta\|^2_{L^2} = \int_{\mathcal{C}} \mathrm{d}\mu \, |f|^2_{\mathbb{C}} \overset{\text{abs. cont.}}{=} \int_{\mathcal{C}} \mathrm{d}\nu \frac{\mathrm{d}\mu}{\mathrm{d}\nu} |f|^2_{\mathbb{C}} \tag{2.4}$$

$$= \int_{\mathcal{C}} \mathrm{d}\nu \, |g|^2_{\mathbb{C}} \tag{2.5}$$

i.e. it's a coordinate independent construction although the natural measures L^2 have "local information".

2.1 Preparation for Observables

We will start with an extended remark: suppose I have a half density (state in the Hilbert space), $\zeta \in \mathcal{H}(\mathcal{C})$ is said to be C^∞ (smooth) with compact support provided

$$\zeta = f\sqrt{\mathrm{d}\mu} \tag{2.6}$$

where f is C^∞ with compact support and μ is the measure associated with a smooth $\dim\mathcal{C}$-form ω_μ on $\mathcal{C}$. These form a dense subspace D_0^∞ of $\mathcal{H}(\mathcal{C})$.
Let X be a smooth vector field on $\mathcal{C}$ with local flow F_t (t "time variable").
Consider $f\sqrt{\mathrm{d}\mu} \in D_0^\infty$. For sufficiently small times

$$\mathcal{U}_t\left(f\sqrt{\mathrm{d}\mu}\right) := (f\circ F_t)\sqrt{\mathrm{d}\left(\mu\circ F_t\right)} \tag{2.7}$$

is a well defined element of $\mathcal{H}(\mathcal{C})$ ($\mathcal{U}_0 = \mathrm{id}_{\mathcal{H}}$).
Introduce

$$\widehat{X}\left(f\sqrt{\mathrm{d}\mu}\right) := \frac{1}{i}\lim_{t\to 0}\frac{1}{t}\left[\mathcal{U}_t - \mathrm{id}_{\mathcal{H}(\mathcal{C})}\right]\left(f\sqrt{\mathrm{d}\mu}\right) \tag{2.8}$$

the **infinitesimal generator** of the evolution group. Note that we also have the measure μ, so we expect an extra term with respect to the normal construction.
This limit exists

μ absolutely continuous.

$$\mathcal{U}_t\left(f\sqrt{\mathrm{d}\mu}\right) = (f\circ F_t)\sqrt{\mathrm{d}\mu}\sqrt{\rho_t} \tag{2.9}$$

where

$$\rho_t := \frac{\mathrm{d}\left(\mu\circ F_t\right)}{\mathrm{d}\mu} \tag{2.10}$$

then

$$F_t^\star(\omega_\mu) = \rho_t\omega_\mu \tag{2.11}$$

Moreover

$$\begin{aligned}\lim_{t\to 0}\frac{\rho_t - 1}{t}\omega_\mu &= \lim_{t\to 0}\frac{1}{t}\left(F_t^\star - \mathrm{id}\right)\omega_\mu = \mathcal{L}_X\omega_\mu &(2.12)\\ &= \mathrm{div}_\mu X\omega_\mu &(2.13)\end{aligned}$$

i.e. the divergence relative to the measure ω_μ.
Hence

$$\begin{aligned}i\widehat{X}(f\sqrt{\mathrm{d}\mu}) &= \lim_{t\to 0}\frac{1}{t}\left[(f\circ F_t)\sqrt{\rho_t} - f\right]\sqrt{\mathrm{d}\mu} &(2.14)\\ &= \lim_{t\to 0}\frac{1}{t}\left[\left(f + tX(f) + O(t^2)\right)\left(\overset{\overset{1}{\uparrow}}{\sqrt{\rho_0}} + \frac{1}{2}t\dot{\rho}_0\right) - f\right]\sqrt{\mathrm{d}\mu} &(2.15)\\ &= \lim_{t\to 0}\frac{1}{t}\left[tX(f) + \frac{1}{2}t\dot{\rho}_0 f + O(t^2)\right]\sqrt{\mathrm{d}\mu} &(2.16)\\ &= \left(X(f) + \frac{1}{2}\left(\mathrm{div}_\mu X\right)f\right)\sqrt{\mathrm{d}\mu} &(2.17)\end{aligned}$$

Thus

$$\widehat{X}(f\sqrt{\mathrm{d}\mu}) := -i\left(X(f) + \frac{1}{2}\left(\mathrm{div}_\mu X\right) f\right)\sqrt{\mathrm{d}\mu} \tag{2.18}$$

$$= -i\left(X + \frac{1}{2}\mathrm{div}_\mu X \mathrm{id}_{D_0^\infty}\right)\left(f\sqrt{\mathrm{d}\mu}\right) \tag{2.19}$$

depends only on the equivalence class of (f, μ), i.e. it's an intrinsic construction.

2.2 Observables

We now want to discuss observables on any smooth manifold:

Proposition 2.1

The operator

$$\widehat{X} : D_0^\infty \to \mathcal{H}(\mathcal{C}) \tag{2.20}$$

$$\widehat{X}\left(f\sqrt{\mathrm{d}\mu}\right) = g\sqrt{\mathrm{d}\mu} \tag{2.21}$$

with

$$g = -i\left(X(f) + \frac{1}{2}\left(\mathrm{div}_\mu X\right) f\right) \tag{2.22}$$

is symmetric on its domain.

Proof Let $\zeta := f\sqrt{\mathrm{d}\mu}, \eta := g\sqrt{\mathrm{d}\mu} \in D_0^\infty$ states. We have

$$\langle \mathcal{U}_t\zeta, \mathcal{U}_t\eta\rangle_{\mathcal{H}(\mathcal{C})} = \int_{\mathcal{C}} \mathrm{d}\left(\mu \circ F_t\right)\overline{\left(f \circ F_t\right)}\left(g \circ F_t\right) \tag{2.23}$$

$$= \int_{\mathcal{C}} \mathrm{d}\mu \bar{f} g = \langle \zeta, \eta\rangle_{\mathcal{H}(\mathcal{C})} \tag{2.24}$$

Then applying $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}$:

$$\left\langle \widehat{X}\zeta, \eta\right\rangle_{\mathcal{H}(\mathcal{C})} = \left\langle \zeta, \widehat{X}\eta\right\rangle_{\mathcal{H}(\mathcal{C})} \quad \square \tag{2.25}$$

♠

Remark

You can even show that $\widehat{X}$ is essentially self adjoint on D_0^∞ if and only if the flow F_t of X is complete except on a set of measure 0.

What is momentum? We must adapt the concept of "directional derivative" as momentum.

Definition 2.2

For a vector field X on $\mathcal{C}$ we define

$$P(X) : T^\star(\mathcal{C}) \to \mathbb{R} \tag{2.26}$$

$$\alpha_q \mapsto \alpha_q\left(X_q\right) \tag{2.27}$$

and call it the ***classical momentum*** *corresponding to X.*

♣

We interpret this as the momentum in direction of X.

Remark

In coordinates $q^1, \dots, q^{\dim \mathcal{C}}$ on $\mathcal{C}$

$$q^1, \dots, q^{\dim \mathcal{C}}, p_1, \dots, p_{\dim \mathcal{C}} \tag{2.28}$$

on $\mathbb{P} = T^\star \mathcal{C}$, we have

$$P(X)(q,p) = \sum_{a \in I(\dim \mathcal{C})} p_a X^a_q \tag{2.29}$$

momentum in direction of X.

Notation: for any (smooth) function $f : \mathcal{C} \to \mathbb{R}$, introduce the corresponding position function (on phase space) by

$$\tilde{f} := f \circ \tau^\star_{\mathcal{C}} : \mathbb{P} \to \mathbb{R} \tag{2.30}$$

$$\tilde{f}(q,p) = f(q) \tag{2.31}$$

where $\tau^\star_{\mathcal{C}}$ is the projection on the first argument.

Proposition 2.2

For any 2 vector fields on $\mathcal{C}$ X and Y and any functions $f, g : \mathcal{C} \to \mathbb{R}$ we have

1.

$$\{P(X), P(Y)\} = -P([X,Y]) \tag{2.32}$$

2.

$$\left\{\tilde{f}, \tilde{g}\right\} = 0 \tag{2.33}$$

3.

$$\left\{\tilde{f}, P(X)\right\} = \widetilde{X(f)} \tag{2.34}$$

Proof

1. In coordinates

$$\begin{aligned}
\{P(X), P(Y)\} &= \left\{P_a X^a, P_b Y^b\right\} && (2.35)\\
&= \frac{\partial}{\partial q^a}(P_a X^a)\frac{\partial}{\partial P_a}\left(P_b Y^b\right) - (X \leftrightarrow Y) && (2.36)\\
&= P_a \frac{\partial X^a}{\partial q^c} Y^c - X^a P_c \frac{\partial Y^c}{\partial q^a} && (2.37)\\
&= -P_c\left(X^a \frac{\partial Y^c}{\partial q^a} - \frac{\partial X^c}{\partial q^a} Y^a\right) = -P_c \underset{\text{vector bracket}}{\overset{\uparrow}{[X,Y]^c}} && (2.38)\\
&= -P([X,Y]) && (2.39)
\end{aligned}$$

2. Trivial, only configuration space functions.
3. In coordinates, since f is a function on $\mathcal{C}$:

$$\left\{\tilde{f}, P(X)\right\} = \frac{\partial f}{\partial q^a}\frac{\partial}{\partial P_a}\left(P_b X^b\right) = X(f) = \widetilde{X(f)} \quad \square \tag{2.40}$$

(gradient in direction of X).

Definition 2.3

*Define the **classical momentum observable** $P(X)$ on $T^\star\mathcal{C}$ associated to the vector field X on $\mathcal{C}$ by*

$$P(X)(P_q) := P_q(X_q) \;\; , \;\; \forall q \in \mathcal{C} \tag{2.41}$$

*we shall call $\mathcal{P}_X := \widehat{X}$ the corresponding **quantum momentum observable**.*
*If f is a C^∞ function on $\mathcal{C}$, that is, a classical configuration observable, we define the corresponding **quantum position observable** to be the operator Q_f on $\mathcal{H}(\mathcal{C})$ given by*

$$Q_f\left(g\sqrt{\mathrm{d}\mu}\right) = fg\sqrt{\mathrm{d}\mu} \tag{2.42}$$

i.e. a new vector on state space, i.e. Q_f is multiplication by f.

2.3 Algebraic Perspective on Observables

We will now show the relations showing the importance of the sequence of quantum observables Q and P.

Theorem 2.1

Let $\mathcal{C}$ be a finite dimensional manifold with intrinsic Hilbert space $\mathcal{H}(\mathcal{C})$. Let X and Y be smooth vector fields on $\mathcal{C}$, and let f, g be smooth real valued functions on $\mathcal{C}$.
Then the following commutation relations hold on the domain $D_0^\infty \subset \mathcal{H}(\mathcal{C})$.

1. $$-i[\mathcal{P}_X, \mathcal{P}_Y] = -\widehat{[X, Y]} = -\mathcal{P}_{[X,Y]} \tag{2.43}$$

2. $$-i[Q_f, Q_g] = 0 \tag{2.44}$$

3. $$-i[Q_f, \mathcal{P}_X] = Q_{X(f)} \tag{2.45}$$

Proof

1. Let $\zeta = \varphi\sqrt{\mathrm{d}\mu} \in D_0^\infty$. Then

$$\mathcal{P}_X\zeta = -i\left(X(\varphi) + \frac{1}{2}(\mathrm{div}_\mu X)\,\varphi\right)\sqrt{\mathrm{d}\mu} \tag{2.46}$$

From this we calculate

$$[\mathcal{P}_X, \mathcal{P}_Y]\zeta = -\left\{X(Y(\varphi)) - Y(X(\varphi)) + \frac{1}{2}\left(X\left(\mathrm{div}_\mu Y\right) - Y\left(\mathrm{div}_\mu X\right)\right)\varphi\right\}\sqrt{\mathrm{d}\mu} \tag{2.47}$$

In terms of the volume form ω_μ

$$(\mathrm{div}_\mu[X,Y])\,\omega_\mu = \mathcal{L}_{[X,Y]}\omega_\mu = \mathcal{L}_X\mathcal{L}_Y\omega_\mu - \mathcal{L}_Y\mathcal{L}_X\omega_\mu \tag{2.48}$$

$$= \mathcal{L}_X\left(\mathrm{div}_\mu Y\omega_\mu\right) - \mathcal{L}_Y\left(\mathrm{div}_\mu X\omega_\mu\right) \tag{2.49}$$

$$= \left\{X\left(\mathrm{div}_\mu Y\right) - Y\left(\mathrm{div}_\mu X\right)\right\}\omega_\mu \tag{2.50}$$

Thus we have

$$-i[\mathcal{P}_X,\mathcal{P}_Y]\zeta = i\left\{[X,Y](\varphi) + \frac{1}{2}\left(\mathrm{div}_\mu[X,Y]\right)\varphi\right\}\sqrt{\mathrm{d}\mu} \tag{2.51}$$

$$= -\mathcal{P}_{[X,Y]}\zeta \tag{2.52}$$

2. Evident (multiplication is commutative)
3.

$$[Q_f,\mathcal{P}_X]\zeta = -i\left\{f\left(X(\varphi) + \cancel{\frac{1}{2}(\mathrm{div}_\mu X)}\varphi\right) - X(f\varphi) - \cancel{\frac{1}{2}(\mathrm{div}_\mu X)}f\varphi\right\} \tag{2.53}$$

$$= i\varphi X(f)\sqrt{\mathrm{d}\mu} = iQ_{X(f)}\zeta \tag{2.54}$$

♡

2.4 Kinetic Energy

We will now have an extended remark on kinetic energy, to guide our intuition.
Classical kinetic energy comes from a Riemannian metric g on configuration space $\mathcal{C}$.
The kinetic energy function on $T^\star\mathcal{C}$ associated with a metric g on $\mathcal{C}$ (setting masses to 1 for simplicity in some units):

$$K(p_q) = \frac{1}{2}p_q\left(g_q^{-1}(p_q)\right) \tag{2.55}$$

where $p_q \in T_q^\star\mathcal{C}$ and $g_q : T_q\mathcal{C} \to T_q^\star\mathcal{C}$ is the isomorphism induced by the Riemannian innser product on $T_q\mathcal{C}$.
The metric induces a smooth measure ω on $\mathcal{C}$, given by the volume form

$$\omega_q\left(v_1,\ldots,v_{\mathrm{dim}\mathcal{C}}\right) = \sqrt{\det\left(g_q\left(v_a,v_b\right)\right)} \tag{2.56}$$

By definition

$$\mathcal{L}_X\omega = \left(\mathrm{div}_\omega X\right)\omega \tag{2.57}$$

Recall that if $\Phi \in C^\infty(\mathcal{C})$, $\mathrm{d}\Phi$ is a one-form, and:

$$g^{-1}(\mathrm{d}\Phi) = \mathrm{grad}_g\Phi \tag{2.58}$$

is a vector field, the gradient of Φ relative to the metric g.
The **Laplace-Beltrami** operator is

$$\boldsymbol{\Delta}_g\Phi = \mathrm{div}_g\left(\mathrm{grad}_g\Phi\right) \tag{2.59}$$

The operator $\mathbf{\Delta}_g$ is symmetric with respect to the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ in $L^2(\mathcal{C},\omega)$.
Let φ and ψ be smooth functions with compact support

$$\langle \mathbf{\Delta}_g \varphi, \psi\rangle = \int_{\mathcal{C}} \omega \mathrm{div}_g\left(\mathrm{grad}_g \varphi\right)\overline{\psi} = \int_{\mathcal{C}} \omega \mathcal{L}_{\mathrm{grad}_g \varphi} \omega \overline{\psi} \tag{2.60}$$

$$= \int_{\mathcal{C}} \mathcal{L}_{\mathrm{grad}_g \varphi}\left(\omega\overline{\psi}\right) - \int_{\mathcal{C}} \omega \mathcal{L}_{\mathrm{grad}_g \varphi}\overline{\psi} \tag{2.61}$$

$$\overset{\text{stokes}}{=} - \int_{\mathcal{C}} \omega \mathcal{L}_{\mathrm{grad}_g \varphi}\overline{\psi} \tag{2.62}$$

the integral is really over the intersection of the supports of the functions.

and here we used the compact support.
But:

$$\mathcal{L}_{\mathrm{grad}_g \varphi}\overline{\psi} = \left(\mathrm{grad}_g \varphi\right)\left(\mathrm{d}\overline{\psi}\right) = \left(g^{-1}\left(\mathrm{d}\varphi\right)\right)\left(\mathrm{d}\overline{\psi}\right) \tag{2.63}$$

$$\overset{g\ \text{sym}}{=} \left(g^{-1}\left(\mathrm{d}\overline{\psi}\right)\right)\left(\mathrm{d}\varphi\right) = \mathcal{L}_{\mathrm{grad}\overline{\psi}}\varphi \tag{2.64}$$

We interchanged the roles of the conjugated function compared to the standard convention, it doesn't matter.

thus

$$\langle \mathbf{\Delta}_g \varphi, \psi\rangle = - \int_{\mathcal{C}} \omega \mathcal{L}_{\mathrm{grad}_g \overline{\psi}}\varphi = \langle \varphi, \mathbf{\Delta}_g \psi\rangle \tag{2.65}$$

i.e. it's symmetric, with domain D_0^∞ using the canonical identification of $L^2(\mathcal{C},\omega)$ with $\mathcal{H}(\mathcal{C})$.
It can be shown that $\mathbf{\Delta}_g$ is essentially self-adjoint if $\mathcal{C}$ is complete relative to the metric g.
We exploit it very often, e.g. for Fourier transformations, with orthonormal sets of eigen-functions.

Definition 2.4

If the classical potential energy is given by the function V on $\mathcal{C}$, we define the Hamilton operator to be

$$\mathcal{E} := -\frac{1}{2}\mathbf{\Delta}_g + Q_V \tag{2.66}$$

on $\mathcal{H}(\mathcal{C})$

♣

2.5 Quantisation

A foundational question even in research is the quantisation prescription. It is demanding both from a technical point of view and physical point of view.

Definition 2.5

*A full **quantisation** of a Hamiltonian system $(\mathbb{P},\Omega,X_H)$ with configuration space $\mathcal{C} = \mathbb{R}^n$ (first the special case) and energy function $H : T^\star\mathbb{R}^n \to \mathbb{R}$ is a map taking classical observables $f, g : T^\star\mathbb{R}^n \to \mathbb{R}$ to self-adjoint operators $\widehat{f}$, $\widehat{g}$ on $\mathcal{H} = L^2(\mathbb{R}^n)$ (with canonical natural measure) such that*

1.

$$\widehat{f+g} = \widehat{f} + \widehat{g} \tag{2.67}$$

$$\widehat{\lambda f} = \lambda\widehat{f}\ ,\quad \lambda\in\mathbb{R} \tag{2.68}$$

2.

$$\widehat{\{f,g\}} = -i\left[\widehat{f},\widehat{g}\right] \quad , \quad (\hbar = 1) \tag{2.69}$$

3.

$$\widehat{\mathrm{id}_{\mathbb{P}}} = \mathrm{id}_{\mathcal{H}} \tag{2.70}$$

4. *The position and momentum operator act irreducible on $\mathcal{H}$.* ♣

Remark

With the last point we're not including spin. For spin we need:

$4'$. The position and momentum operators are represented by a direct sum of finitely many copies of the Schroedinger representation.

More precisely: $\mathcal{H}$ is realised as the space of L^2 functions from $\mathbb{R}^n$ to h, a Hilbert space with $\dim h < \infty$ such that

$$\widehat{q}_a\phi(q) = q_a\phi(q) \tag{2.71}$$

$$\widehat{p}_a\phi(q) = -i\left(\partial_a\phi\right)(q) \tag{2.72}$$

Now we are also allowing for spin with this relaxation.

We know some observables have no classical anaolog, with quantisation prescription we might miss a lot of observables: we at least want to translate the classical ones to the quantum ones.

Theorem 2.2

Let $\mathfrak{A}$ be the Lie algebra of real valued polynomials on $\mathbb{R}^{2n}$ ($\mathbb{P}$), where the bracket is the Poisson bracket.

Let $\mathfrak{H} = L^2(\mathbb{R}^n, h)$ (h finite dimensional Hilbert space).

Then there is no (quantisation) map $f \mapsto \hat{f}$ from $\mathfrak{A}$ to the self-adjoint operators on $\mathfrak{H}$ that enjoyes the following properties:

0. *For each finite subset $\mathfrak{M} \subset \mathfrak{A}$ there is a dense subspace $D_{\mathfrak{M}} \subset \mathfrak{H}$ such that $\forall f \in \mathfrak{M}$, with $D_{\mathfrak{M}} \subset D_{\hat{f}}$ (domain of the observable $\widehat{f}$) and $\widehat{f}D_{\mathfrak{M}} = D_{\mathfrak{M}}$ ($D_{\mathfrak{M}}$ domain for a finite subset of classical observables $\mathfrak{M}$).*
1. $\widehat{f+g} = \widehat{f} + \widehat{g}$.
2. $\widehat{\lambda f} = \lambda\widehat{f}$, $\forall\lambda \in \mathbb{R}$
3. $\widehat{\{f,g\}} = -i\left[\widehat{f},\widehat{g}\right]$ *on* $D_{\mathfrak{M}}$.
4. $\widehat{\mathrm{id}_{\mathbb{P}}} = \mathrm{id}_{\mathfrak{H}}$.
5. $\widehat{q}_a =$ *multiplication by* q_a *and* $\widehat{p}_a = -i\partial_a$, ($\hbar = 1$).

Proof The proof is technical and not very insightful, lacking physical understanding (although technically clear), so we will not prove this. □ ♡

It turns out that property 5. is a problem. You can make it work with 0.-4..

Remark

A quantisation satisfying 1.-4. is called a **pre-quantisation**:

$\mathfrak{H} = L^2(\mathbb{R}^n, h)$ (h finite dimensional Hilbert space, for spin), 2 classical observables f, g, a pre-quantisation $\widehat{\cdot}$ of classical observables Lie algebra to quantum observables satisfying

1. $\widehat{f+g} = \widehat{f} + \widehat{g}$.
2. $\widehat{\lambda f} = \lambda \widehat{f}$.
3. $\widehat{\{f,g\}} = -i\left[\widehat{f}, \widehat{g}\right]$.
4. $\widehat{\mathrm{id}_{\mathbb{P}}} = \mathrm{id}_{\mathfrak{H}}$

missing the multiplication operator for position and derivative for momentum.

Preparation:

Let $\mathbb{P}$ be the momentum phase space of a Hamiltonian system.

By a **principal circle bundle** on $\mathbb{P}$ we need a fiber bundle $\pi : \mathcal{K} \to \mathbb{P}$ with structural group $\mathbb{S}^1 = \left\{\mathrm{e}^{is},\ s \in \mathbb{R}\right\}$, i.e. each fiber is a circle.

Definition 2.6

Let $(\mathbb{P}, \Omega, X_H)$ be a Hamiltonian system with symplectic structure Ω.

*We say $(\mathbb{P}, \Omega, X_H)$ is **quantisable** if and only if there is a principal circle bundle $\pi : \mathcal{K} \to \mathbb{P}$ over the momentum phase space $\mathbb{P}$ and a one-form α on $\mathcal{K}$ such that*

1. *α is invariant under $\mathbb{S}^1$.*
2. *$\pi^\star\Omega = \mathrm{d}\alpha$ (pull-back of Ω, 2-form on $\mathcal{K}$).*

♣

How does quantisation prescription work?

Remark

α is a connection one-form on $\mathcal{K}$, Ω is its curvature, then $\mathcal{K}$ is called the quantisation manifold.

Now we get more concrete, writing in some (generic) coordinate neighborhood. We can use the principal circle bundle to construct a complex **line bundle** over $\mathbb{P}$, with curvature form (with connection α)

$$\Omega(X, Y) = \boldsymbol{\nabla}_X \boldsymbol{\nabla}_Y - \boldsymbol{\nabla}_Y \boldsymbol{\nabla}_X - \boldsymbol{\nabla}_{[X,Y]} \tag{2.73}$$

For any section s of the line bundle L we have:

$$(\boldsymbol{\nabla}_X s)^a = X^b \partial_b s + X^b \alpha^a{}_{bc} s^c \tag{2.74}$$

Theorem 2.3

Let $(\mathbb{P}, \Omega, X_H)$ be a quantisable Hamiltonian system.

Then there is a quantisation map $f \mapsto \widehat{f}$, where $\widehat{f}$ is an operator on the space $\mathcal{S}$ of sections of L (associated line bundle), Hilbert space of L^2 sections on this bundle, satisfying 1.-4..

Remark

$\widehat{f} : s \mapsto -i\boldsymbol{\nabla}_{X_f} s + fs$ (X_f canonical vector field associated to f, $\Omega^\sharp(\mathrm{d}f)$). If Ω is exact,

$\Omega = -\mathrm{d}\theta$, so locally $\theta = \sum p_a \mathrm{d}q^a$.

Then we have

$$\widehat{f}s : -iX_f(s) + \left(f - \sum p_a \frac{\partial f}{\partial p_a}\right) s \tag{2.75}$$

where $s : \mathbb{P} \to \mathbb{C}$.

Proof

1. $\widehat{f}$ is symmetric because $X_f := \Omega^\sharp(\mathrm{d}f)$ preserves phase space volume.
2. 1., 2., 4. are trivial.
3.

$$\left[\widehat{f}, \widehat{g}\right] s = \tag{2.76}$$

$$= -i\left(\nabla_{X_f}(\widehat{g}(s)) - \nabla_{X_g}\left(\widehat{f}(s)\right)\right) + f(\widehat{g}(s)) - g\left(\widehat{f}(s)\right) \tag{2.77}$$

$$= -i\left[\nabla_{X_f}\left(-i\nabla_{X_g}s + g \cdot s\right) - \nabla_{X_g}\left(-i\nabla_{X_g}s + f \cdot s\right)\right] + \tag{2.78}$$

$$+ f \cdot \left(-i\nabla_{X_g}s + g \cdot s\right) - g\left(-i\nabla_{X_f}s + f \cdot s\right) \tag{2.79}$$

$$= -\left(\nabla_{X_f} - \nabla_{X_g} - \nabla_{X_f}\nabla_{X_g}\right)s - i\left(f\nabla_{X_g}s - g\nabla_{X_f}s\right) \tag{2.80}$$

$$= -\nabla_{[X_f, X_g]}s - i\Omega\left(X_f, X_g\right)s + \tag{2.81}$$

$$- i\left(\nabla_{X_g}(f \cdot s) - X_g(f)s - \nabla_{X_f}(g \cdot s) - X_f(g)s\right) \tag{2.82}$$

$$= -\nabla_{[X_f, X_g]}s - i\underbrace{\Omega(X_f, X_g)}_{\{f,g\}}s - i\left(\{g, f\} - \{f, g\}\right)s \tag{2.83}$$

$$=.\nabla_{[X_f, X_g]}s + i\{f, g\}s \tag{2.84}$$

Since $[X_f, X_g] = -X_{\{f,g\}}$, thus

$$\left[\widehat{f}, \widehat{g}\right](s) = \nabla_{X_{\{f,g\}}}s + i\{f, g\}s \tag{2.85}$$

$$= i\widehat{\{f, g\}}(s) \tag{2.86}$$

by the definition of the quantisation map. □

♡

The Hilbert space is not yet a concrete space, complex valued functions of position and momentum: we need to be able to cut out position and momentum. We know that the usual representations involve either position or momentum, for which we need to introduce the concept of **polarisation**.

2.6 Isotropic and Lagrangian subspaces

Towards pre-quantisation, we succeeded but we still have L^2 sections of the complex line bundle (associated to the principal circle bundle) functions on momentum phase space: in the standard formulation of quantum mechanics, the information that we can store in a state must be quantum compatible: if observables commute I can put simultaneous information of measurement in a state description.

For this we need polarisation, and before that we introduce some ingredients: isotropic and Lagrangian subspaces:

Definition 2.7

*Let (E, Ω) be a symplectic **vector space** and $F \subset E$ is a subspace.*

*The Ω**-orthogonal complement** of F is the subspace defined by*

$$F^{\perp} := \left\{ e \in E : \ \Omega(e, e') = 0, \ \forall e' \in F \right\} \tag{2.87}$$

We say:

1. *F is **isotropic** if $F \subset F^{\perp}$, i.e. $\Omega(e, e') = 0 \ \forall e, e' \in F$.*
2. *F is **Lagrangian** if F is isotropic and has an isotropic complement, i.e. $E = F \oplus F'$, where F' is isotropic as well.*

♣

Proposition 2.3

Let (E, Ω) be a symplectic vector space and $F \subset E$ a subspace.

Then the following operators are equivalent:

1. *F is Lagrangian.*
2. *$F = F^{\perp}$.*
3. *F is isotropic and $\dim F = \frac{1}{2} \dim E$.*

Proof 1. $\Longrightarrow$ 2.:

Since F is Lagrangian, we have $F \subset F'$ by definition.

We have to show $F \supset F^{\perp}$.

Let $e \in F^{\perp}$ and write $e = e_0 + e_1$, $e_0 \in F$, $e_1 \in F'$, with $E = F \oplus F'$.

By isotropy of F', i.e. $F' \subset F'^{\perp}$, $e_1 \in F^{1\perp}$, and similarly $e_1 = e - e_0 \in F^{\perp}$ because $e, e_0 \in F^{\perp}$.

Thus $e_1 \in F^{\perp} \cap F'^{\perp} = (F + F')^{\perp} = E^{\perp} = \{0\}$ (E is the parent vector space) because Ω is non-degenerate, hence $e_1 = 0_{F_{\perp}}$ and $F^{\perp} \subset F$.

2. $\Longrightarrow$ 3.:

Standard linear algebra arguments.

3. $\Longrightarrow$ 1.:

3. implies that $\dim F = \dim F^{\perp}$, since $F \subset F^{\perp}$ by isotropy then $F = F^{\perp}$.

We construct F' asfollows:

choose an arbitrary $v_1 \notin F$ and $F^{\perp} + V_1^{\perp} = (F \cap V_1)^{\perp} = E$.

Now pick $v_2 \in V_1^{\perp}$, $v_2 \notin F + V_1$. Let $V_2 := \text{span}(v_2)$.

Continue inductively until $F + V_k = E$.

By construction

$$F \cap V_k = \{0\}\,, \text{ so } E = F \oplus V_k \tag{2.88}$$

Also, by construction $V_2^{\perp} = (V_1 + \text{span}(v_2))^{\perp} = V_1^{\perp} \cap \text{span}(v_2)^{\perp} \supset \text{span}(v_1, v_2) = V_2$ (because $v_2 \in V_1^{\perp}$), so V_2 is isotropic.

Inductively V_k is isotropic as well, $F' = V_k$. □

♠

Remark

In literature this proposition is stated as Lagrangian subspaces are **maximal** isotropic subspaces (maximal dimension).

Example 2.1

1. Symplectic $2D$ vector spaces, any $1D$ subspace is isotropic hence Lagrangian ($1 = \frac{2}{2}$).
2. Let $E = \mathbb{R}^2 \times \mathbb{R}^2$ with $v = (v_1, v_2)$, and

$$\Omega\left((v_1, v_2), (w_1, w_2)\right) = \langle v_1, w_2 \rangle - \langle v_2, w_1 \rangle \tag{2.89}$$

   Where $\langle \cdot\, , \,\cdot \rangle$ the Eucledian innter product.

   A subspace spanned by linearly independent vectors v, w is Lagrangian if and only if

$$\langle v_1, w_2 \rangle = \langle v_2, w_1 \rangle \tag{2.90}$$

   For instance $\mathbb{R}^2 \times \{0\}$ and $\{0\} \times \mathbb{R}^2$ are Lagrangian subspaces as is (1,1,1,1),(0,1,0,1).
3. Let $E = V \oplus V^\star$ with

$$\Omega\left((v_1, \alpha_1), (v_2, \alpha_2)\right) = \alpha_2(v_1) - \alpha_1(v_2) \tag{2.91}$$

   Then $V \oplus \{0\}$ (0 form) and $\{0\} \oplus V^\star$ (0 vector) are Lagrangian (Ω_V vanishes on them) and the dimension is $\frac{1}{2}\dim E$.

We work with vector spaces only locally: at the end of the day we want to globalize this concept:

$$E = V \oplus V^\star \tag{2.92}$$

$V \oplus \{0\}$ and $\{0\} \oplus V^\star$ is a simple example but with Ω canonical defined in the example, it is a paradigmatic construction in the following sense:

Proposition 2.4

Let (E, Ω) be a symplectic vector space and $V \subset E$ a Lagrangian subspace.

Then there is a symplectic isomorphism

$$A : (E, \Omega) \to (V \oplus V^\star, \Omega_V) \tag{2.93}$$

where Ω_V is the symplectic structure discussed in the example, taking V to $V \oplus \{0\}$.

Proof

1. Let $E = V \oplus V'$, V' the complement: V' is isotropic and consider the map:

$$T : V' \to V^\star \tag{2.94}$$

$$T(e_1)e := \Omega(e, e_1) \ , \quad e_1 \in V', \ e \in V \tag{2.95}$$

   (which is the natural constructions with the structures we have).
2. T is an isomorphism:

$$T(e_1) = 0 \implies \Omega(e_1, e) = 0 \; \forall e \in V \tag{2.96}$$

V' isotropic then $\Omega(e_1, e) = 0 \; \forall e \in E$ implies (non-degeneracy) $e_1 = 0$.

Hence T is injective, $\dim V^\star = \dim V = \dim V'$, hence it's an isomorphism.

3. Symplectomorphism:

Let $A := \mathrm{id}_V \oplus T$, so we construct the pull-back

$$(A^\star \Omega_V)\left((e, e_1), (e', e_1')\right) = \tag{2.97}$$

$$= \Omega_V\left((e, Te_1), (e', Te_1')\right) \tag{2.98}$$

$$\overset{\text{def}}{=} \left(Te_1'\right)(e) - \left(Te_1\right)e' \tag{2.99}$$

$$\overset{\text{def}}{=} \Omega(e, e_1') - \Omega(e', e_1) \tag{2.100}$$

$$\underset{V, V' \text{ iso}}{\overset{\text{skew}}{=}} \Omega(e + e_1, e' + e_1') \tag{2.101}$$

Hence we have symplectic isomorphism. □

2.6.1 Globalization of Isotropy to symplectic manifolds

This is a preparation for the concept of polarisation: for the prequantisation framework to become more familiar, we need to cut down $\mathbb{P}$ for some sort of Schroedinger representation.

Tf is the differential of f.

Definition 2.8

Let $(\mathbb{P}, \Omega)$ be a symplectic manifold and $\iota : L \to \mathbb{P}$ an immersion. L is called an ***isotropic immersed submanifold*** *of $(\mathbb{P}, \Omega)$ if*

$$(T_e\iota)(T_e L) \subset T_{\iota(\ell)}\mathbb{P} \tag{2.102}$$

is an isotropic subspace $\forall \ell \in L$.

A submanifold $L \subset \mathbb{P}$ is called ***Lagrangian*** *if it is isotropic and there is an isotropic sub-bundle $E \subset T\mathbb{P}|_L$ such that*

$$T\mathbb{P}|_L = TL \oplus E \tag{2.103}$$

Now we make a few useful statements (without proof):

Proposition 2.5

Let $(\mathbb{P}, \Omega)$ be a symplectic manifold and $L \subset \mathbb{P}$ a submanifold, then L is Lagrangian if and only if L is isotropic and $\dim L = \frac{1}{2} \dim \mathbb{P}$.

Proposition 2.6

Let α be a 1-form on $\mathcal{C}$ (the configuration manifold) and $L \subset T^\star\mathcal{C}$ its graph.

Then L is a Lagrangian submanifold if and only if α is closed.

2.7 Polarisation

Polarisation allows us to cut down the state space introduced in the pre-quantisation framework into the "correct state-space".

Definition 2.9

*A real **polarisation** of the symplectic manifold $(\mathbb{P}, \Omega)$ is a foliation F of $\mathbb{P}$ by Lagrangian submanifolds (as leaves of the foliation).* ♣

Remark

The leaves L of a real polarisation are by definition isotropic, i.e. Ω vanishes on $T_pL \times T_pL$, $p \in L$, and are maximal, i.e. $\dim L = \frac{1}{2}\dim\mathbb{P}$.

Definition 2.10

Let $(\mathbb{P}, \Omega, X_H)$ be a quantisable Hamiltonian system and let F be a polarisation of the symplectic manifold $(\mathbb{P}, \Omega)$.
Let L be the line bundle obtained from the quantisation manifold.
*Then the **quantising Hilbert space** is the space of L^2 sections of L that are constant on the leaves of F.* ♣

Example 2.2 Let $\mathbb{P} := T^\star\mathcal{C}$, so $L = \mathbb{P} \times \mathbb{C}$.
Sections of L are just complex valued functions.
Let the leaves of F be the linear spaces $T^\star_q\mathcal{C}$.
This is a polarisation and

$$\mathcal{H} = L^2(\mathcal{C}) \tag{2.104}$$

So $\psi \in \mathcal{H}$ is just a configuration function.

2.8 Remark

We finish this chapter with an extended remark: we developed the structures relevant for this course, which is convenient for generalizing to infinite degrees of freedom.
Let $Q^1, \dots, Q^d, \mathcal{P}_1, \dots, \mathcal{P}_d$ ($d := \dim\mathcal{C}$) self-adjoint operators on $\mathcal{H}$ satisfying the Heisenberg commutation relations (technical problems arise from having unbounded operatos, which are relieved by considering 1-parameter groups).
Consider the 1-parameter (translation) group

unitary, we will see later

$$\mathcal{U}_a(t) := \exp\left(it\mathcal{P}_a\right) \tag{2.105}$$

$$\mathcal{V}^b(t) := \exp\left(itQ^b\right) \tag{2.106}$$

generated by $\mathcal{P}_a$ and Q^b ($a, b \in I(d)$).

Heisenberg commutation relations imply Weyl commutation relations

$$[\mathcal{U}_a(t), \mathcal{U}_b(t')] = \left[\mathcal{V}^b(t), \mathcal{V}^b(t')\right] = 0 \tag{2.107}$$

$$\mathcal{U}_a(t)\mathcal{V}^b(t') = \exp\left\{i\delta_a{}^b tt'\right\}\mathcal{V}^b(t')\mathcal{U}_a(t) \tag{2.108}$$

set $\mathbf{t} := (t_1, \dots, t_d) \in \mathbb{R}^d$ and write

$$\mathcal{U}(\mathbf{t}) = \mathcal{U}_1(t_1)\cdots\mathcal{U}_d(t_d) \tag{2.109}$$

and similarly for $\mathcal{V}$.

Then the Weyl relations become

$$\mathcal{U}(\mathbf{t}+\mathbf{t}') = \mathcal{U}(\mathbf{t})\mathcal{U}(\mathbf{t}') \tag{2.110}$$

and similar for V,

$$\mathcal{U}(\mathbf{t})\mathcal{V}(\mathbf{t}') = \exp\left\{-i\left\langle \mathbf{t}, \mathbf{t}'\right\rangle\right\}\mathcal{V}(\mathbf{t}')\mathcal{U}(\mathbf{t}) \tag{2.111}$$

Then

$$\mathbf{t} \mapsto \mathcal{U}(\mathbf{t}) \tag{2.112}$$

$$\mathbf{t} \mapsto \mathcal{V}(\mathbf{t}) \tag{2.113}$$

are representations of $\mathbb{R}^d$ on $\mathcal{H}$.

The Schroedinger representation is by definition the representation of $\mathbb{R}^d$ on $L^2\left(\mathbb{R}^d\right)$, i.e.

$$\left(\mathcal{U}(\mathbf{t})f\right)(\mathbf{q}) = f(\mathbf{q}-\mathbf{t}) \tag{2.114}$$

(translation)

$$\left(\mathcal{V}(\mathbf{t})f\right)(\mathbf{q}) = \exp\left\{i\left\langle \mathbf{t}, \mathbf{q}\right\rangle_{\mathbb{R}^d}\right\} f(\mathbf{q}) \tag{2.115}$$

The Schroedinger representation is in fact irreducible.

If you replace $L^2(\mathbb{R}^d)$ by $L^2\left(\mathbb{R}^d, h\right)$ (h complex Hilbert space to acoomodate for spin), we have a number of copies of the Schroedinger representation, with the number being the dimension of h.

Theorem 2.4. Stone-Von Neumann

Let $\mathcal{U}(\mathbf{t})$ and $V(\mathbf{t})$ be continuous unitary representations of $\mathbb{R}^d$ on $\mathcal{H}$ satisfying Weyl commutation relations.

Then there is a Hilbert space h and unitary map

$$T : \mathcal{H} \to L^2\left(\mathbb{R}^d, h\right) \tag{2.116}$$

that transforms $\mathcal{U}(\mathbf{t})$ and $\mathcal{V}(\mathbf{t})$ to Schroedinger representation.

This representation is irreducible if and only if h is one-dimensional. ♡

There are shortcomings of what we have done so far: namely, we have a theory in Galilean spacetime, no light-cone, causal relations, which we need for quantum fields in curved spacetimes. Causal relations don't need full dynamical spacetime: general concepts without having to solve general relativity, we need a more qualitative representation of general relativity rather than its kinematical aspects.

Chapter 3 Causality

3.1 Core Concepts

Motivation: we need to understand how to implement causal relations into a dynamical formulation of our field theory (and also at the kinematical level).
Everything will be qualitative, it's a new culture of thoughts in general relativity: statements do not require to solve the Einstein equations (Hawking, Penrose, ...).
At a foundational level, a **spacetime** is the collection of all (possible) events. Of course this set has some structure (otherwise it's useless).
We have to keep in mind that this theory breaks down eventually, that's why we look for a quantum theory of gravity: the theory of spacetime predicts its own breakdown (dynamically).

3.1.1 Spacetime

The mathematical model of **spacetime** is a pair $(\mathcal{M}, g)$, where $\mathcal{M}$ is a connected 4-dimensional (not important, but we stress it because we have no evidence of higher dimensions) Hausdorff C^∞ manifold. g is a Lorentz metric field on $\mathcal{M}$ (this is actually not enough, we need it also to be orientable, in particular time orientable).
Note that the Lorentz condition together with the Hausdorff condition imply paracompactness.

Remark

This doesn't fix the model.
Let $(\mathcal{M}, g)$, $(\mathcal{M}, g')$ be spacetimes, how can I discriminate?
They will be considered **equivalent** if they are **isometirc**, i.e. there is a diffeomorphism $\theta : \mathcal{M} \to \mathcal{M}'$ with

$$\theta_\star g = g' \tag{3.1}$$

(pull-back).
Consequently, a model of spacetime is an equivalence class of isometric pairs $(\mathcal{M}, g)$.

Definition 3.1

Let $(\mathcal{M}, g)$ be a spacetime and $p, q \in \mathcal{M}$ (events). Then

1. *$p \ll q$ means there is a future-pointing timelike curve in $\mathcal{M}$ from p to q.*
2. *$p < q$ means there is a future-pointing causal (includes timelike) curve in $\mathcal{M}$ from p to q.*
3. *$p \leq q$ means either $p < q$ or $p = q$.*

♣

$p \ll q$ implies $p < q$.

Definition 3.2

For a subset $\mathcal{A}$ of $\mathcal{M}$ spacetime, the subset

$$\mathcal{I}^+(\mathcal{A}) := \{q \in \mathcal{M} : \exists p \in \mathcal{A} \text{ with } p \ll q\} \tag{3.2}$$

*is called the **chronological future** of $\mathcal{A}$, and accordingly the relaxation:*

$$\mathcal{J}^+(\mathcal{A}) := \{q \in \mathcal{M} : \exists p \in \mathcal{A} \text{ with } p \leq q\} \tag{3.3}$$

*is called the **causal future** of $\mathcal{A}$.* ♣

Remark

1. $\mathcal{J}^+(\mathcal{A}) \supset \mathcal{A} \cup \mathcal{I}^+(\mathcal{A})$, and it also contains all the points connected by light-like curves.
2. (local version) For a single point $p \in \mathcal{M}$:
$$\mathcal{I}^+(p) = \{q \in \mathcal{M} : p \ll q\} \tag{3.4}$$
3. For a subset $\mathcal{A} \subset \mathcal{M}$
$$\mathcal{I}^+(\mathcal{A}) = \bigcup_{p \in \mathcal{A}} \mathcal{I}^+(p) \tag{3.5}$$
and similarly for $\mathcal{J}^+(\mathcal{A})$.

Correspondingly we have the past versions, for which we could just reverse the time orientation, but we make it explicit this time:

Definition 3.3

For a subset $\mathcal{A}$ of $\mathcal{M}$ spacetime, the subset

$$\mathcal{I}^-(\mathcal{A}) := \{q \in \mathcal{M} : \exists p \in \mathcal{A} \text{ with } q \ll p\} \tag{3.6}$$

*is called the **chronological past** of $\mathcal{A}$,*

$$\mathcal{J}^-(\mathcal{A}) := \{q \in \mathcal{M} : \exists p \in \mathcal{A} \text{ with } q \leq p\} \tag{3.7}$$

*is called the **causal past** of $\mathcal{A}$.* ♣

Example 3.1 Minkowski spacetime $\mathbb{R}^4_1 := \left(\mathbb{R}^4, \eta\right)$.

$$\mathcal{I}^+(p) = \left\{q \in \mathbb{R}^4_1 : \overrightarrow{pq} \text{ is timelike future-pointing}\right\} \tag{3.8}$$

$$\mathcal{J}^+(p) = \{p\} \cup \left\{q \in \mathbb{R}^4_1 : \overrightarrow{pq} \text{ is causal future-pointing}\right\} \tag{3.9}$$

(p and the future light-cone), with $\overrightarrow{pq}$ the vector from p to q.

Clearly $\mathcal{I}^+(p)$ is open, with closure

$$\overline{\mathcal{I}^+(p)} = \mathcal{J}^+(p) \tag{3.10}$$

Note that here we have a global concept of vectors.

Remark

1.

$$\mathcal{I}^+(\mathcal{A}) = \mathcal{I}^+(\mathcal{I}^+(\mathcal{A})) = \mathcal{I}^+(\mathcal{J}^+(\mathcal{A})) = \mathcal{J}^+(\mathcal{I}^+(\mathcal{I}^+(\mathcal{A}))) \subset \mathcal{J}^+(\mathcal{J}^+(\mathcal{A}))$$
$$= \mathcal{J}^+(\mathcal{A}) \quad (3.11)$$

2. Let $\mathcal{U} \subset \mathcal{M}$ open. Then $\mathcal{U}$ is a time oriented Lorentz manifold. Let $\mathcal{I}^+(\mathcal{A}, \mathcal{U})$ denote the chronological future in the manifold $\mathcal{U}$ of $\mathcal{A} \subset \mathcal{U}$. Then $\mathcal{I}^+(\mathcal{A}) \subset \mathcal{I}^+(\mathcal{A}) \cap \mathcal{U}$

Lemma 3.1

Let p, q be events in spacetime $\mathcal{M}$ with p in the chronological past of q.

Then there are neighborhoods $\mathcal{U}_p$ and $\mathcal{U}_q$ of p and q respectively, such that all events in $\mathcal{U}_q$ are in the chronological future of all events in $\mathcal{U}_p$.

Proof Let γ be a timelike curve from p to q.

Let $\mathcal{C}_q$ be convex neighborhood of q, and let $q^< \in \mathcal{C}^q$ on γ before q.

Dually, let $p^>$ be a point on γ between p and $q^<$ contained in a convex neighborhood $\mathcal{C}_p$ of p.

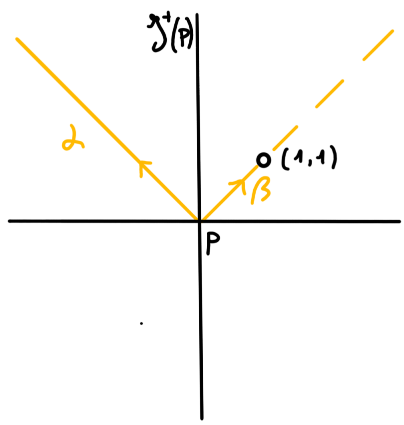
By causality (special relativity) in $T_{q^<}\mathcal{M}$, $\mathcal{I}^+(q^<, \mathcal{C}_q)$ is open in $\mathcal{C}$ (by definition of chronological future) and hence in $\mathcal{M}$. Similarly $\mathcal{I}^-(p^>, \mathcal{C}_p)$ is open in $\mathcal{M}$.

Hence $\mathcal{I}^\pm(q^</p^>, \mathcal{C}_{q/p})$ have the required properties of $\mathcal{U}_q$ and $\mathcal{U}_p$, respectively. □ ♡

Nice statement that links causality firmly to the topology of spacetime, the chronological future of any set in $\mathcal{M}$ is open.

Remark

For arbitrary $\mathcal{M}$ the sets $\mathcal{J}^+(p)$ need not be closed.
Consider $\mathbb{R}^2_1$ (2d Minkowski) with the point $(1,1)$ removed.
Consider p the origin:



There is no causal curve from p to points in the dashed line. The causal future is the chronological future and the curves α and β. The closure of the chronological future is no longer the causal future.

More precisely:

Corollary 3.1

If γ is a future pointing causal curve from a set $\mathcal{A}$ to a point $q \in (\mathcal{J}^+(\mathcal{A} \setminus \mathcal{I}^+(\mathcal{A})))$, then γ is a null geodesic that has no conjugate points before q and does not meet $\mathcal{I}^+(\mathcal{A})$.

Proof Won't prove here. ♡

Then

$$\mathcal{J}^+(\mathcal{A}) = \mathcal{A} \cup \mathcal{I}^+(\mathcal{A}) \cup \{\text{null geodesics from } \mathcal{A}\} \tag{3.12}$$

Lemma 3.2

For a subset $\mathcal{A}$ of $\mathcal{M}$ the following holds

1. $\operatorname{int}(\mathcal{J}^+(\mathcal{A})) = \mathcal{I}^+(\mathcal{A})$ *(interior).*
2. $\mathcal{J}^+(\mathcal{A}) \subseteq \overline{\mathcal{I}^+(\mathcal{A})}$*, with equality if and only if* $\mathcal{J}^+(\mathcal{A})$ *is a closed set.*

Proof See Hawking Ellis. ♡

3.2 Quasi-Limits

We now introduce a technicality used to learn deeper statements about causality relations.

Definition 3.4

Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be an infinite sequence of future pointing causal curves in a spacetime $\mathcal{M}$, and let $\mathcal{K}$ be a convex cover of $\mathcal{M}$.

A ***limit sequence*** *for $\{\gamma_n\}_{n\in\mathbb{N}}$* ***relative to*** *$\mathcal{K}$ is a finite or infinite sequence*

$$p = p_0 < p_1 < \cdots \tag{3.13}$$

of points in $\mathcal{M}$ such that

1. *For each p_a ($a \in \mathbb{N}$) there is a subsequence $\{\gamma_m\}$ and, for each m there are numbers (times)*

   $$s_{m_0} < s_{m_1} < \cdots < s_{m_a} \tag{3.14}$$

   such that

   (a).
   $$\lim_{m\to\infty} \gamma_m(s_{m_b}) = p_b \tag{3.15}$$
   for each $b \leq a$.

   (b). *For each $b < a$, the parts p_b, p_{b+1} and the segment $\gamma_m|_{s_{m_b}, s_{m_{b+1}}}$ are contained in a single set $\mathcal{C}_b \in \mathcal{K}$ for all $\mathcal{M}$ (will not prove that such things exist).*
2. *If $\{p_a\}$ is infinite, then it is non-convergent.*

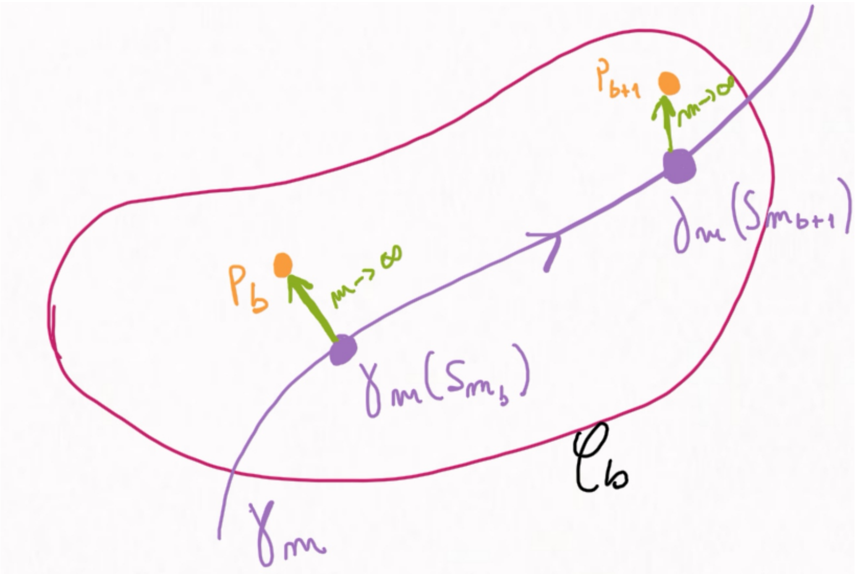
   If $\{p_a\}$ is finite, it has more than one point and no strictly longer sequence satisfies 1.. ♣

Remark

- 1.a. is a natural limit requirement.

- 1.b. accomplishes the reduction to individual convex sets, so global questions can in this way be reduced to local ones.
- 2. is a technicality, we will not bother for now.

With a picture in the convex set $\mathcal{C}_b$, we have:



as $m \to \infty$ $\gamma_m(s_{m_b}) \to p_b$ (and accordingly for $b+1$).

The idea is: we geodesically connect successive points, p_b in the causal past of p_{b+1} to obtain a *quasi-limit*.

Proposition 3.1

Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be an infinite sequence of future pointing causal curves such that $\{\gamma_n(0)\} \to p$ but $\{\gamma_n\} \not\to p$ (for any other parameter).

Then $\{\gamma_n\}$ has a limit sequence starting at p relative to any convex cover $\mathcal{K}$.

Proof Won't prove here.

Remark

If $\{p_a\}$ is a limit sequence for $\{\gamma_n\}$, let λ_a be the future pointing causal geodesic from p_a to p_{a+1} in a convex set $\mathcal{C}_a$ as required by 1.b..

Assembling these segments for all indices a gives a broken geodesic $\lambda = \sum_a \lambda_a$ called a **quasi-limit** of $\{\gamma_n\}_{n\in\mathbb{N}}$ **with vertices** p_a.

If $\{p_a\}$ is infinite, by 2. λ is future inextendible.

A quasi-limit of future inextendible curves is future inextendible of course.

Example 3.2 Consider $\mathbb{R}^2_1$, let γ_n be the timelike geodesic segment from $(0,0)$ to $\left(n+\frac{1}{n}, n\right)$ (with the origin as initial point).

In any limit sequence, vertices must lie on the null geodesic ray $\lambda(s) = (s,s)$, $s \geq 0$.

With initial conditions as specified, λ is the unique quasi-limit.

Consider now

$$\mathcal{M} := \mathbb{R}^2_1 \setminus (1,1) \tag{3.16}$$

The point $(1,1)$ is on the null geodesic ray as before.

There is a unique quasi-limit $\beta := \lambda|_{[0,1)}$: this is future inextendible. In this spacetime, we can't extend this ray further into the future, it is incomplete in $\mathcal{M}$.

We reach this point in a finite amount of time, we reach the *end of spacetime*.

3.3 Conditions

Definition 3.5

*If $\mathcal{M}$ contains no closed timelike curves, we say that the **chronological condition** holds on $\mathcal{M}$.* ♣

Remark

In relativistic physics, only spacetimes in which the chronlogical condition holds are considererd.

We can ask if a reasonable spacetime can be compact (topologically useful property)

Lemma 3.3

The chronological condition does not hold on compact $\mathcal{M}$.

Proof Consider the open covering $\{\mathcal{I}^+(p) : p \in \mathcal{M}\}$.

By compactness there is a finite subcover

$$\{\mathcal{I}^+(p_1), \dots, \mathcal{I}^+(p_k)\} \tag{3.17}$$

Assume $\mathcal{I}^+(p_1)$ is not contained in any later $\mathcal{I}^+(p_a)$, $a \in I(k)$ (if it was contained in some $\mathcal{I}^+(p_a)$ we could just discard it).

But then $p_1 \notin \mathcal{I}^+(p_a)$, otherwise $\mathcal{I}^+(p_1) \subset \mathcal{I}^+(p_a)$ for some a, thus $p_1 \in \mathcal{I}^+(p_1)$ (only a remaining), so we have a closed timelike curve through p. □ ♡

Spacetimes we are interested in cannot then be compact.

We lift the chronological condition to a causality condition:

Definition 3.6

*If $\mathcal{M}$ contains no closed causal curves, we say that the **causal condition** holds on $\mathcal{M}$.* ♣

Obviously the causality condition implies the chronological condition, but the inverse is false (find counterexamples as an exercise).

We can futher lift this to (local formulation)

Definition 3.7

*The **strong causality condition** holds at $p \in \mathcal{M}$ provided that given any neighborhood $\mathcal{U}$ of p, there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p such that every causal curve segment with endpoints in $\mathcal{V}$ lies entirely in $\mathcal{U}$.* ♣

Note *There are no causal curves starting arbitrarily close to p that are almost closed at p. Again, the strong causality condition implies the causality condition, but the converse is not true.*

Lemma 3.4

Suppose the strong causality condition holds on a compact subset $\mathcal{K}$ of $\mathcal{M}$.

If γ is a future inextendible causal curve that starts in $\mathcal{K}$ then γ eventually levaes $\mathcal{K}$ never to return.

Proof Assume the opposite.

Either γ remains in $\mathcal{K}$ or persistently return to $\mathcal{K}$.

Then there is a sequence $\{b\}$ in dom$\gamma := [0, B)$, $B \leq \infty$ (finite or not) such that $\{s_b\} \to B$ and $\{\gamma(s_b)\}$ is contained in $\mathcal{K}$.

Due to compactness of $\mathcal{K}$, there is a subsequence in $\{\gamma(s_b)\}$ that converges to a point $p \in \mathcal{K}$.

Since by assumption γ has no future endpoint (future inextendible) there must be another sequence $\{t_b\}$ converges to B such that $\{\gamma(t_b)\} \subset \mathcal{K}$ but does not converge to p.

By compactness, there is a subsequence in $\{\gamma(t_b)\}$ that converges to some other point than p in $\mathcal{K}$.

The Hausdorff property guarantees some neighborhood of p that contains no $\gamma(t_b)$.

Since $\{s_b\}$ and $\{t_b\}$ both converge to B, we can arrange for subseqences with $s_1 < t_1 < s_2 < t_2 < \cdots$.

Thus $\gamma|_{[s_k, s_{k+1}}$ are almost closed at p, which is a contradiction of our assumption. □ ♡

We get very insightful results without having to solve the dynamical equations.

3.4 Causal Geodesics

We will now introduce the main tools to construct causal geodesics from a point p to a point q

Lemma 3.5

Suppose the strong causality condition holds on a compact subset $\mathcal{K}$ of $\mathcal{M}$.

Let $\{\gamma_n\}$ be a (infinite) sequence of future pointing causal curve segments in $\mathcal{K}$ such that $\{\gamma_n(0)\} \to p$ (as $n \to \infty$) and $\{\gamma_n(1)\} \to q \neq p$.

Then there is a future pointing causal broken geodesic σ from p to q and a subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ such that

$$\lim_{m \to \infty} L(\gamma_m) \leq L(\sigma) \tag{3.18}$$

(L length)

Proof It can be shown that $\{\gamma_n\}$ has a limit sequence $\{p_a\}$ starting at p (we won't show this).

If $\{p_a\}$ is infinite, then the corresponding quasi-limit σ is a future inextendible causal curve starting at p.

By the preceeding lemma, σ must leave $\mathcal{K}$ and never return.

In particular, some vertex p_c is not in $\mathcal{K}$.

This implies that γ_n leave $\mathcal{K}$, which contradicts the assumption.

Thus the limit sequence must be finite.

The finite limit sequence starts at p and ends at

$$\lim_{m\to\infty}\gamma_m(1) = q \tag{3.19}$$

The quasi-limit with these vertices is a causal broken geodesic from $p \to q$ (this proves the existence).

By the 1.b. in the definition of the limit-sequence, consider a convex set $\mathcal{C}_a$.

The length of the a'th segment γ_m is at most the separation of its point in $\mathcal{C}_a$, i.e.

$$L\left(\gamma_m|_{[s_{m_a},s_{m_{a+1}}]}\right) \leq \left\|\overrightarrow{p_{m_a}p_{m_{a+1}}}\right\| \tag{3.20}$$

with $\overrightarrow{p_{m_a}p_{m_{a+1}}}$ the relative vector.

where $p_{m_a} := \gamma_m\left(s_{m_a}\right), \ldots$.

Therefore

$$L(\gamma_m) \leq \sum_{a\in I_b(k)} \left\|\overrightarrow{p_{m_a}p_{m_{a+1}}}\right\| := L_m \tag{3.21}$$

Since the norm of $\overrightarrow{p_{m_a}p_{m_{a+1}}}$ depends continuously on (p,q), the sequence $\{L_m\}$ converges to $\sum\left\|\overrightarrow{p_{m_a}p_{m_{a+1}}}\right\| = L(\sigma)$. $\square$

♡

3.5 Global Hyperbolicity

Definition 3.8

Let p, q be two points in $\mathcal{M}$.

*The **time separation** from p to q is*

$$\tau(p,q) := \sup\{L(\gamma):\ \gamma \text{ is a future pointing causal curve segment from } p \text{ to } q.\} \tag{3.22}$$

♣

$L(\gamma)$ length.

Remark

∞ as a value of time separation is allowed, and 0 is if q is not is in the causal past of p.

- If $\sup = \max$, τ is the "proper time of slowest trip in $\mathcal{M}$ from p to q".
- In Minkowski it's the Minkowski norm between p and q.

We will see a comparison between the time separation and Riemannian distance which is not so direct (the first is maximised, the second is minimised).

Because it involves time orientation, in general τ is not symmetric (only in trivial cases).

Lemma 3.6

1. *$\tau(p,q) > 0$ if and only if $p \ll q$.*
2. *$p \leq q \leq r$, then a reversed triangle inequality holds*

$$\tau(p,r) \geq \tau(p,q) + \tau(q,r) \tag{3.23}$$

Proof

1. If $\tau(p,q)$ is positive, then there is a future pointing causal curve γ from p to q with length > 0, thus it's a not null geodesic, so there is a fixed end point deformation of γ to a timelike curve.

   The inverse is clear.
2. Consider a future pointing causal curve segment α from p to q and β from q to r.

   $\forall$ *delta* > 0 we can choose α and β to have length within $\frac{\delta}{2}$ of the time separation of p and q, and q and r,

$$L(\alpha) = \tau(p,q) - \frac{\delta}{2} \tag{3.24}$$

$$L(\beta) = \tau(q,r) - \frac{\delta}{2} \tag{3.25}$$

   Hence

$$\tau(p,r) \geq L(\alpha+\beta) \geq L(\alpha) + L(\beta) = \tau(p,q) + \tau(q,r) - \delta \tag{3.26}$$

   thus the result.

   If there is no future pointing causal curve $p \to q$ or $q \to r$ the time separation is 0, $p \leq q \implies p = q$ so the results holds trivially. □

♡

Definition 3.9

Let $\mathcal{A}, \mathcal{B}$ be subsets of $\mathcal{M}$.

*The **time separation** of $\mathcal{A}$, $\mathcal{B}$ is*

$$\tau(\mathcal{A},\mathcal{B}) := \sup\{\tau(a,b) : \ a \in \mathcal{A}\,, \quad b \in \mathcal{B}\} \tag{3.27}$$

♣

Now we move towards the existence of the longest causal geodesic in Lorentzian spacetimes from p to q (probe of spacetime).

Definition 3.10

*The **causal diamond** of p and q is*

$$\mathcal{J}(q) := \mathcal{J}^+(p) \cap \mathcal{J}^-(q) \tag{3.28}$$

♣

Proposition 3.2

For $p < q$, if the set $\mathcal{J}(p,q)$ is compact and if the strong causality condition holds on $\mathcal{J}(p,q)$ then there is a causal geodesic from p to q of length $\tau(p,q)$.

Proof

Let $\{\gamma_n\}$ be a sequence of future pointing causal curves segments from p to q whose lengths converge to $\tau(p,q)$ (a priori could be infinite).

These are all in the **causal diamond** $\mathcal{J}(p,q)$, compact, so there is a future pointing causal broken geodesic σ connecting p and q with $L(\sigma) \geq \lim_{n\to\infty} L(\gamma_n)$ (proposition we proved).

But

$$L(\gamma_n) \xrightarrow{n\to\infty} \tau(p,q) \quad \Longrightarrow \quad \tau(p,q) \tag{3.29}$$

so $L(\sigma) = \tau(p,q)$ (it's a sup on curves, σ included).

If $L(\sigma) \geq \tau(p,q)$ we'd have a larger curve, hence we can also say that σ is unbroken. □ ♠

Definition 3.11

*$\mathcal{M}$ is **globally hyperbolic** provided:*

- *The strong causality condition holds.*
- *$\forall p < q$ the set $\mathcal{J}(p,q)$ is compact.* ♣

Then any pair of events that can be joined by a causal curve, can be joined by a causal geodesic (longest curve). This opens the way for a variety of geodesic constructions. We will always consider such spacetimes.

Definition 3.12

*A subset $\mathcal{S}$ of $\mathcal{M}$ is called **globally hyperbolic** provided*

1. *the strong causality condition holds on $\mathcal{S}$.*
2. *If $p, q \in \mathcal{S}$ with $p < q$, then $\mathcal{J}(p,q)$ is compact (which is trivial if $\mathcal{M}$ is globally hyperbolic) and contained in $\mathcal{S}$.* ♣

This is not an intrinsic property of $\mathcal{S}$, it relates a subset to the causality of $\mathcal{M}$.
By the last proposition there is a causal geodesic of $\mathcal{M}$ joining any $p < q$.

Lemma 3.7

If $\mathcal{U}$ is a globally hyperbolic open set in $\mathcal{M}$.

Then the causality relation $\leq$ of $\mathcal{M}$ is closed on $\mathcal{U}$, i.e. suppose I have a limit sequence $p_n \to p$, $q_n \to q$ with $p_n, q_n \in \mathcal{U}\ \forall n$, then each $p_n \leq q_n$ implies $p \leq q$.

Proof If $p_n = q_n$ for infinitely many n it's trivial.

If p_n in causal past of $q_n\ \forall n$, then $\gamma_n \in \{\gamma_n\}$ is a causal curve from $p \to q$ because $\mathcal{U}$ open iN $\mathcal{M}$, it contains points $p_<$ in the chronological past of p and $q_>$ in the chronological future of q.

We can of course suppose that the limit sequences are contained in open setes given by the chronological future of $p_<$ and chronological past of $q_>$.

Thus $\{\gamma_n\} \subset I(p_<, q_>)$ which is compact, then by the previous lemma we are done. $\square$ ♡

Suppose $\mathcal{M}$ itself is globally hyperbolic (always the case for us), $p, q \in \mathcal{M}$, $\mathcal{J}^+(p)$, $\mathcal{J}^-(q)$, $\mathcal{J}(p, q)$ are closed.

Definition 3.13

*A subset $\mathcal{A}$ is called **achronal** provided there is no timelike curve meeting $\mathcal{A}$ more than once. Alternatively: if $p \ll q$ never holds $\forall p, q \in \mathcal{A}$.* ♣

Example 3.3 Minkowski $4d$, any hyperplane of constant time is achronal.

Definition 3.14

*The **edge** of an achronal set $\mathcal{A}$ consists of all points $p \in \overline{\mathcal{A}}$ (closure with respect to $\mathcal{M}$) such that every neighborhood $\mathcal{U}$ of p contains a timelike curve from $\mathcal{I}^-(p, \mathcal{U})$, $\mathcal{I}^p(p, \mathcal{U})$, that does not meet $\mathcal{A}$.* ♣

The curve "goes around the edge, without penetrating $\mathcal{A}$".

Example 3.4 $\mathbb{R}^2_1$, $\mathcal{A} := \{(0, x) : 0 \leq x < 1\}$ is clearly achronal, and the 2 edge points are $(0, 0)$ and $(1, 1)$.

$\mathbb{R}^4_1$, any hyperplane of constant time is achronal, with empty edge.

Is having an empty edge bad? No, we will show that any edgeless achronal set is a hypersurface (not necessarily smooth), e.g. null-cone in Minkowski spacetime.

Definition 3.15

*A subspace $\mathcal{S}$ of a topological manifold $\mathcal{T}$ is a **topological hypersurface** provided that $\forall p \in \mathcal{S}$, $\exists \mathcal{U}$ neighborhood of p in $\mathcal{T}$ and a homeomorphism ϕ of $\mathcal{U}$ onto an open set in $\mathbb{R}^{\dim \mathcal{T}}$ such that*

$$\phi(\mathcal{U} \cap \mathcal{S}) = \phi(\mathcal{U}) \cap \Pi \tag{3.30}$$

where Π is a hyperplane in $\mathbb{R}^{\dim \mathcal{T}}$, $\dim \mathcal{S} = \dim \mathcal{T} - 1$. ♣

Example 3.5 $\Lambda^+(0) \subset \mathbb{R}^4_1$ (future null cone).

Homeomorphism:

$$(t, \mathbf{x}) \mapsto (t - |\mathbf{x}|, \mathbf{x}) \tag{3.31}$$

with $|\cdot|$ Eucledian norm, so we get the hyperplane $t = 0$.

Proposition 3.3

An achronal set $\mathcal{A}$ is a topological hypersurface if and only if $\mathcal{A}$ contains no edge points.

Proof Lengthy in one direction.

$\mathcal{A}$ is an achronal topological hypersurface implies no edge points is simple:

Let $p \in \mathcal{A}$, choose $\mathcal{U}$ coordinate neighborhood of p (connected).

$\mathcal{I}^-(p,\mathcal{U})$ and $\mathcal{I}^+(p,\mathcal{U})$ are open and connected, but disjoint and don't meet $\mathcal{A}$ (topological hypersurface).

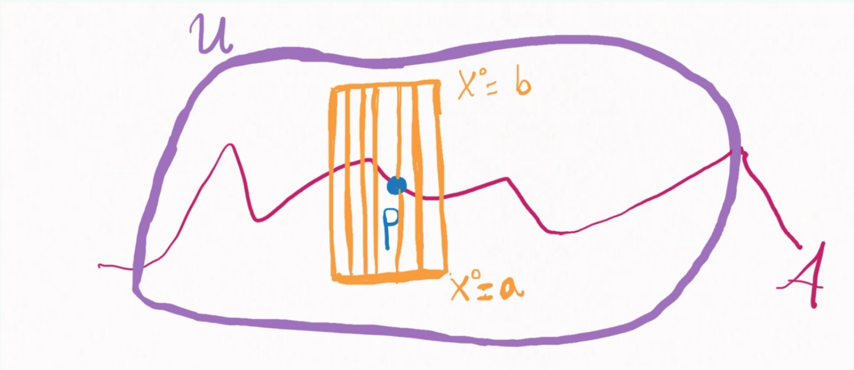
Any timelike curve through p meets both sets, contained in different components of $\mathcal{U} \setminus \mathcal{A}$ so p cannot be an element of the edge of $\mathcal{A}$.

<u>Other Direction</u>: the achronal set $\mathcal{A}$ supposedly contains no edge points, i.e. $\mathcal{A}$ and $edge(\mathcal{A})$ are disjoint.

Let $p \in \mathcal{A}$ and $\mathcal{U}$ neighborhood of p equipped with coordinate system ζ with $\frac{\partial}{\partial x^0}$ timelike future pointing.

We can find a smaller neighborhood $\mathcal{V}$ of p such that

1. $\zeta(\mathcal{V})$ has the form $(a-\delta, b+\delta) \times \mathcal{N} \subset \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^{\dim \mathcal{T}-1} = \mathbb{R}^n$
2. $x^0 = a$ "slice" of $\mathcal{V}$ is in $\mathcal{I}^-(p,\mathcal{U})$.
   $x^0 = b$ "slice" of $\mathcal{V}$ is in $\mathcal{I}^+(p,\mathcal{U})$.



For $\mathcal{U}$ sufficiently small, if $y \in \mathbb{R}^{n-1}$ the x^0-coordinate curve $s \mapsto \zeta^{-1}(s,y)$, $s \in [a,b]$ must meet $\mathcal{A}$ since by assumption p is not an edge point.

Achronality implies that the meeting point is unique: call its x^0-coordinate $h(y)$.

It sufficies to show that

$$h : \mathcal{N} \to (a,b) \tag{3.32}$$

is continuous, for then

$$\phi := \left(x^0 - h(x^1, \dots, x^{n-1}, x^1, \dots, x^{n-1}\right) \tag{3.33}$$

is a homeomorphism carrying $\mathcal{A} \cap \mathcal{V}$ to the slice $u^0 = 0$ of $\phi(\mathcal{V})$ in $\mathbb{R}^n$. Thus $\mathcal{A}$ is a topological hypersurface.

Let $\{y_n\}$, $y_n \to y$ in $\mathcal{N}$. Assume $h(y_n)$ doesn't converge to $h(y)$ (not continuous).

But the values of h are bounded, hence some subsequence $\{ym\}$ converges to some point $r \neq h(y)$ by assumption.

Consider $\zeta^{-1}(r,y) \in \mathcal{I}^-(q,\mathcal{V}) \cup \mathcal{I}^+(q,\mathcal{V})$, with $\mathcal{A} \ni q = \zeta^{-1}(h(y),y)$. The same is true for

$$\zeta^{-1}(h(y_n), y_n) \in \mathcal{A} \tag{3.34}$$

contrary to the achronality of $\mathcal{A}$, so $y_n \to y$, but $h(y_n) \not\to h(y)$ cannot hold, hence it's continuous. $\square$

Corollary 3.2

An achronal set $\mathcal{A}$ is a closed topological hypersurface if and only if edge$(\mathcal{A})$ is empty.

Proof $\mathcal{A}$ closed topological hypersurface. Then $\mathcal{A}$ and *edea*$(\mathcal{A})$ are disjoint by the previous proposition. But $edge\,(\mathcal{A}) \subset \overline{\mathcal{A}} = \mathcal{A}$ (closed), so $edge\,(\mathcal{A}) = \varnothing$.

On the other direction: suppose $edge\,(\mathcal{A})$ is empty, by the preceeding proposition $\mathcal{A}$ is a topological hypersruface.

Closed because of achronal identity $\overline{\mathcal{A}} \setminus \mathcal{A} \subset edge\,(\mathcal{A})$: proof of this:

$\overline{\mathcal{A}}$ is also achronal, $q \in \overline{\mathcal{A}} \setminus \mathcal{A}$. Null timelike curve through q can meet $\mathcal{A}$, so $q \in edge\,(\mathcal{A})$.

$\square$

Definition 3.16

*A subset F of $\mathcal{M}$ is called a **future set** provided*

$$\mathcal{I}^+(F) \subset F \tag{3.35}$$

Remark

F future set, its complement $\mathcal{M} \setminus F$ is a **past set**.

Corollary 3.3

The boundary of a future set is a closed achronal topological hypersurface.

Proof Let $p \in \partial F$. If $q \in \mathcal{I}^+(p)$, then $\mathcal{I}^-(q)$ is a neighborhood of p and hence contains a point of F.

Thus $q \in \mathcal{I}^+(F) \subset F$ and $\mathcal{I}^+(p) \subset F$.

Dually $\mathcal{I}^-(p) \subset \mathcal{M} \setminus F$.

It follows the $\mathcal{I}^+(\partial F) \cap \mathcal{I}^-(\partial F) = \varnothing$.

Hence ∂F is achronal.

Furthermore ∂F is closed, it has no edge points since $\mathcal{I}^+(p) \in int(F)$ and $\mathcal{I}^+(p) \in ext(F)$ (for $p \in \partial F$) thus the statement follows by the preceeding corollary. $\square$

Example 3.6 $\mathbb{R}^4_1$, null-cone anchored at p, boundary of $\mathcal{J}^+(p)$, is a closed topological hypersurface.

Definition 3.17

*$B \subset \mathcal{M}$ is **acausal** provided.*

$p \in \mathcal{J}^-(q)$ never holds for $p, q \in B$.

This is a stronger requirement than achronality, e.g. null geodesic in Minkowski is achronal, but not acausal.

A very important concept in classical field theory in spacetime (and quantum) is the dynamical problem as Cauchy initial value problem: need to clarify the notion of Cauchy hypersurface.

3.6 Initial Value Problems

Cauchy initial value problems are needed to set up the stage for **dynamical** problems, beyond topological.

Definition 3.18

A ***Cauchy Hypersurface*** *in $\mathcal{M}$ is a subset $\mathcal{S}$ that is met exactly once by every inextendible timelike curve in $\mathcal{M}$.* ♣

In particular, $\mathcal{S}$ is achronal.

Lemma 3.8

A Cauchy hypersurface is a closed, achronal, topological hypersurface and is met by every inextendible causal curve.

Proof Consider

1. $\mathcal{I}^-(\mathcal{S}), \mathcal{S}, \mathcal{I}^+(\mathcal{S})$. $\mathcal{S}$ is common boundary of $\mathcal{I}^-(\mathcal{S})$ and $\mathcal{I}^+(\mathcal{S})$.
   By the preceeding corollary, the boundary of a future set is a closed achronal topological hypersurface.
2. every inextendible causal curve γ.
   Assume γ does not meet $\mathcal{S}$.
   Without loss of generality, say $\gamma(0) \in \mathcal{I}^+(\mathcal{S})$ (initial condition).
   Then by assumption γ cannot be past extended to meet $\mathcal{S}$, i.e. γ is a past inextendible causal curve that does not meet $\mathcal{S}$. The domain is $[0, \infty)$. $\{\gamma(n)\}$ does not converge.
3. Related to $\mathcal{I}^+(\mathcal{S})$, $\ll$ of $\mathcal{I}^+(\mathcal{S})$.
   Let $p_0 \in \mathcal{I}^+(\gamma(0), \mathcal{I}^+(\mathcal{S}))$.
   Then $\gamma(0) \ll p_0$ and $\gamma(1) \ll p_0$.
   $\exists p_1$ such that $\gamma(1) \ll p_1 \ll p_0$.
   Get $\{p_n\}$ with $\gamma(n) \ll p_n \ll p_{n-1}, \forall n \geq 1$.
   Join each p_{n-1} to p_n by timelike segment.
   We have a past pointing timelike curve β in $\mathcal{I}^+(\mathcal{S})$, $\beta(0) = p$.
   $\{p_n\}$ does not converge (we discussed it in a previous section) thus β is inextendible.
   Any future pointing timelike curve, starting at the initial point of β, must remain in $\mathcal{I}^+(\mathcal{S})$: we have inextendible timelike curve that avoids $\mathcal{S}$.
   Assuming we have an inextendible causal curve avoiding $\mathcal{S}$ implies the existence of timelike curve that does not meet $\mathcal{S}$, and we have a contradiction. □

♡

Proposition 3.4

Let $\mathcal{S}$ be a Cauchy hypersurface in $\mathcal{M}$, and let X be a timelike vector field on $\mathcal{M}$.

If $p \in \mathcal{M}$, a maximal integral curve of X through p meets $\mathcal{S}$ at a unique point $\rho(p)$.

Then $\rho : \mathcal{M} \to \mathcal{S}$ is continuous map onto $\mathcal{S}$ pointwise fixed-

In particular, $\mathcal{S}$ is connected.

Proof Reminder: maximal integral curves of X are inextendible.

Let $\tilde{\psi} : \mathcal{D} \to \mathcal{M}$ be the flow of X, where $\mathcal{D}$ is an open set in $\mathcal{M} \times \mathbb{R}$.

$\mathcal{D}(\mathcal{S}) := (\mathcal{S} \times \mathbb{R}) \cap \mathcal{D}$ is a topological hypersurface in $\mathcal{D}$ (because $\mathcal{S}$ is in $\mathcal{M}$).

The restriction $\psi : \mathcal{D}(\mathcal{S}) \to \mathcal{M}$ is continuous, one-to-one (sine $\mathcal{S}$ is Cauchy).

$\mathcal{D}(\mathcal{S})$ and $\mathcal{M}$ are topological manifolds of the same dimension, so ψ is a homeomorphism.

The natural projection $\pi : \mathcal{S} \times \mathbb{R} \to \mathcal{S}$, open, continuous, onto.

But $\rho = \pi \circ \psi^{-1}$ since

$$\rho\left(\psi(p, t)\right) = \rho\left(\gamma_p(t)\right) = \gamma_p(0) = p\,, \quad \forall p \in \mathcal{S} \tag{3.36}$$

so ρ has the same properties as π.

Clearly $\rho|_{\mathcal{S}} = \mathrm{id}$. □

♠

Corollary 3.4

Any two Cauchy hypersurface in $\mathcal{M}$ are homeomorphic.

Proof Evident, exercise. □

♡

A central notion we will introduce now, it's also important for singularity theorems, which we will not go into, unfortunately, for which we need some dynamical input.

We have all the qualitative aspects though.

Definition 3.19

If $\mathcal{A}$ is an achronal subset of $\mathcal{M}$, the ***future Cauchy development*** *of $\mathcal{A}$ is the set $\mathcal{D}^+(\mathcal{A})$ of all points $p \in \mathcal{M}$, such that every past inextendible causal curve through p meets $\mathcal{A}$.*

♣

Remark

- $\mathcal{A}$ itself is in $\mathcal{D}^+(\mathcal{A})$.
- $\mathcal{D}^+(\mathcal{A})$ is that part of the causal future of $\mathcal{A}$ which is predictable from $\mathcal{A}$.
- Past Cauchy development $\mathcal{D}^-(\mathcal{A})$ defined dually.
- $\mathcal{D}^+\left(\mathcal{A}\right) \cup \mathcal{D}^-\left(\mathcal{A}\right)$ is called the Cauchy development of $\mathcal{A}$.

Example 3.7 $\mathcal{A}$ spacelike hypersurface $t = c$ in $\mathbb{R}^n_1$, $\mathcal{D}^+\left(\mathcal{A}\right)$ =causal future of $\mathcal{A}$ which consists of all points (coord syst) (t, x) such that $t \geq c$.

Dually for $\mathcal{D}^-\left(\mathcal{A}\right)$.

The Cauchy development of $\mathcal{A}$ is $\mathbb{R}^n_1$ itself.

Remark

An achronal set $\mathcal{A} \subset \mathcal{M}$ is a Cauchy hypersurface if and only if the Cauchy development of $\mathcal{A}$ is $\mathcal{M}$.

Can think of $\mathcal{D}(\mathcal{A})$ as the largest subset for which $\mathcal{A}$ plays the role of Cauchy hyperplane.

Note

- *$\mathcal{A}$ achronal, $\mathcal{D}^+(\mathcal{A}) \cap \mathcal{D}^-(\mathcal{A}) = \mathcal{A}$.*
- $\mathcal{D}^+(\mathcal{A}) \setminus \mathcal{A} = \mathcal{D}(\mathcal{A}) \cap \mathcal{J}^+(\mathcal{A})$

3.6.1 Cauchy Development

We now want to see the connection between the Cauchy development and global hyperbolicity.

Lemma 3.9

If $\mathcal{A}$ is achronal and $p \in int\mathcal{D}(\mathcal{A})$, then every inextendible causal curve through p meets both $\mathcal{I}^{\mp}(\mathcal{A})$.

Proof Exercise. □

♡

Theorem 3.1

If $\mathcal{A}$ is an achronal set, then $int(\mathcal{D})(\mathcal{A})$ is globally hyperbolic.

Proof We have to show

1. Strong causality condition holds on $\mathcal{D}(\mathcal{A})$.
2. If $p, q \in \mathcal{D}(\mathcal{A})$, with $p < q$ then
   (a). $\mathcal{J}(p, q)$ is compact.
   (b). $\mathcal{J}(p, q) \subset \mathcal{D}(\mathcal{A})$.

- Assume causality condition does not hold on $\mathcal{D}(\mathcal{A})$, then there is a causal curve γ anchored at a point $p \in \mathcal{D}(\mathcal{A})$. So transversing γ repeatedly, given an inextendible causal curve γ' which must then meet $\mathcal{A}$. But γ' meets $\mathcal{A}$. But γ' meets $\mathcal{A}$ repeatedly, which is a contradiction with achronality: there is no causal loop in $\mathcal{D}(\mathcal{A})$, so the causality condition must hold on $\mathcal{D}(\mathcal{A})$.
- Assume the strong causality condition does not holds on $\mathcal{D}(\mathcal{A})$.
  Then $\exists$ future pointing causal curve segments γ_n defined on $[0, 1]$ such that $\{\gamma_n(0)\}$ and $\{\gamma_n(1)\}$ both converge to p, but every γ_n leaves some fixed neighborhood of p.
  Thus $\{\gamma_n(0)\}$ and $\{\gamma_n(1)\}$ both converge to p, but every γ_n leaves some fixed neighborhood of p.
  Thus $\{\gamma_n\}$ has a future directed limit sequence $\{p_a\}$ starting at p.
  Suppose $\{p_a\}$ is finite, then it ends at $\lim \gamma_n(1) = p$. But then $p < p$, violating the causality condition.
  Thus $\{p_a\}$ must be infinite, and consequently the quasi-limit λ is future inextendible.
  λ enters $\mathcal{I}^+(\mathcal{A})$ hence remains there, so some vertex p_a is in $\mathcal{I}^+(\mathcal{A})$.
  Therefore there is a subsequence $\{\gamma_n\}$ and $s \in [0, 1]$ such that $\lim \gamma_n(s) = p_a$.
  We obtain (for a proposition we proved) that a past-oriented limit sequence $\{\bar{p}_a\}$ starting at p.
  If finite, it must end at p_a, thus $p_a \in \mathcal{J}^-(p)$, but $p_a \in \mathcal{J}^+(\mathcal{A})$ so we have a contra-

diction.

Thus it must be infinite, and we call the quasi-limit $\bar{\lambda}$.

In the previous section we showed that the quasi-limit meets $\mathcal{J}^-(\mathcal{A})$, thus restricting γ_m to $s \in [0,1]$ for some s this must meet $\mathcal{J}^-(\mathcal{A})$. Since γ_m is future pointng, and has a $\gamma_m(s)$ in $\mathcal{I}^+(\mathcal{A})$, we contradict achronality, so the strong causality condition must hold on $\mathcal{D}(\mathcal{A})$.

- If $p \leq q$ in *int* $(\mathcal{A})$, then $\mathcal{J}(p,q)$ is compact.

  If $p = q$, $\mathcal{J}(p,q) = \{p\}$, forbidden by the causality condition.

  Hence suppose $p < q$.

  Let $\{x_n\}$ be a sequence in $\mathcal{J}(p,q)$, must show that some subsequence converges in $\mathcal{J}(p,q)$.

  Let γ_n be a future pointing causal curve segment from p through x_n to q.

  Let $\mathcal{R}$ be a covering of $\mathcal{M}$ by convex open sets $\mathcal{C}$, such that $\overline{\mathcal{C}}$ is a compact set contained in a convex open set.

  For each p_a there is a subsequence $\{\gamma_m\}$ and for each m numbers,

$$s_{m_0} < s_{m_1} < \cdots < s_{m_a} \tag{3.37}$$

  such that

  - $\lim_{m\to\infty} \gamma_m(s_{m_b}) = p_b\ ,\quad \forall b \leq a.$
  - For each $b < a$ p_b, p_{b+1} and segments $\gamma_m|_{[s_{m_b}, s_{m_{b+1}}]}$ are contained in $\mathcal{C}_b$ of $\mathcal{R}$, $\forall m$.

    By the pigeon-hole principle, there is a $j < a$ such that for infinitely many m, x_m lies on the j-th segment:

$$x_m \in \gamma_m|_{[s_{m_b}, s_{m_{b+1}}]} \tag{3.38}$$

    By 1. the segments, hence x_m all lie on a single number of convex coverings of $\mathcal{M}$.

    By the properties of these convex sets, x_m converges to some x.

    x is somewhere between p and q, thus $x \in \mathcal{J}(p,q)$.

    We assumed that such finite sequence exists.

    Assume instead that every limit sequence relative to the convex cover $\mathcal{R}$ is infinite, with quasi-limit λ.

    λ future inextendible causal curve.

    COMPLETE

♡

Chapter 4 Classical Field Theory and Hyperbolic Operators

4.1 Natural Structure

Globally hyperbolic spacetimes ensure the existence both of families of hypersurfaces in which initial data can be stored and preferred evolution direction.

For physically reasonable systems, the structure of partial differential equations governing field dynamics has to be compatible with the causal structure of the underlying geometry.

Definition 4.1

*A **smooth vector bundle** is a quadruple $(E, \pi, \mathcal{M}, V)$ of rank $\dim V$, where*

1. *$\mathcal{M}$ is a smooth manifold called the **base manifold** (spacetime);*
2. *V is a vector space called the **typical fibre**;*
3. *E is a $(\dim \mathcal{M} + \dim V)$-dimensional smooth manifold called the **total space**;*
4. *$\pi : E \to \mathcal{M}$ is a smooth surjective map called the **projection** which satisfies the following conditions:*
   - *(a). Each fiber $E_p = \pi^{-1}(p)$, $p \in \mathcal{M}$, is a vector space isomorphic to V.*
   - *(b). For each base point $p \in \mathcal{M}$, there is a neighborhood $\mathcal{U}$ of p and a diffeomorphism*

$$\Psi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times V \tag{4.1}$$

such that

$$pr_1 \circ \Psi = \pi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \tag{4.2}$$

restricted to $\pi^{-1}(\mathcal{U})$

   - *(c). Ψ acts as a vector space isomorphism on each fiber $E_p \to \{p\} \times V$ is an isomorphism between vector spaces.*

♣

Remark

1. Any pair $(\mathcal{U}, \Psi)$, $\mathcal{U}$ neighborhood of a point $p \in \mathcal{M}$, satisfying (b) and (c) is called a **local trivialisation** of the vector bundle.

   Any collection of local trivialisation covering $\mathcal{M}$ is called **vector bundle atlas**.
2. Here $\mathcal{M}$ is always a spacetime (hence globally hyperbolic).

Example 4.1 $\mathcal{M}$ Minkowski spacetime, V vector space, $\mathcal{M} \times V$ as total space,

$$(\mathcal{M} \times V, pr_1, \mathcal{M}, V) \tag{4.3}$$

vector bundle. Local trivialisation $(\mathcal{M}, \mathrm{id}_{\mathcal{M}\times V})$ (global trivialisation, it's a globally trivial vector bundle).

Remark

Consider $f : \mathcal{M} \to V$ ($\mathcal{M}$ manifold, V vector space) is the typical way to introduce a field (with extra qualifications).
The trivial vector bundle $\mathcal{M} \times V$ (the quadruple is implicit) define a new function

$$\hat{f} : \mathcal{M} \to \mathcal{M} \times V \tag{4.4}$$

$$p \mapsto \hat{f}(p) := (p, f(p)) \tag{4.5}$$

bijection $f \to \hat{f}$ between vector valued function and vector bundle-valued function (a special class of vector bundle-valued functions):

$$pr_1 \circ \hat{f} = \mathrm{id}_{\mathcal{M}} \tag{4.6}$$

Definition 4.2

Let $(E, \pi, \mathcal{M}, V)$ be a vector bundle.
*A **section** of $(E, \pi, \mathcal{M}, V)$ is a smooth function $\sigma : \mathcal{M} \to E$ such that $\pi \circ \sigma = \mathrm{id}_{\mathcal{M}}$.*
We denote the vector space of sections of $(E, \pi, \mathcal{M}, V)$ by $\Gamma(\mathcal{M}, E)$. ♣

$\Gamma_0(\mathcal{M}, E)$ denotes the vector space of compactly supported sections.

4.2 Kinematical Structure and Dynamics

Definition 4.3

Let $(E, \pi, \mathcal{M}, V)$ be a real vector bundle.
*A **bosonic (fermionic) non-degenerate bilinear form** on E is a smooth real-valued map $\langle \cdot\,, \cdot \rangle$ on the vector bundle $E \otimes E$ such that for each $p \in \mathcal{M}$ the following properties hold:*

1. *On the fiber $E_p \otimes E_p$ over p, the map $\langle \cdot\,, \cdot \rangle$ is a symmetric (skew-symmetric) bilinear form-*
2. *If $v \in E_p$ is such that $\langle v, w \rangle = 0\ \forall w \in E_p$, then $v = 0_{E_p}$.* ♣

Remark

Smooth on $E \otimes E$, the bilinear form can be interpreted as a smooth section of $E^\star \otimes E^\star$ ($E^\star$ dual bundle).

Definition 4.4

Let E be a real vector bundle over a globally hyperbolic spacetime $\mathcal{M}$, with a non-vanishing volume form $vol_{\mathcal{M}}$.
Let $\langle \cdot\,, \cdot \rangle$ be a bosonic (fermionic) non-degenerate bilinear form on E such that each fiber is endowed with a non-degenerate inner product.
*A non-degenerate **pairing** between smooth sections and compactly supported sections of E*

is a map

$$(\cdot , \cdot) : \Gamma_0(\mathcal{M}, E) \otimes \Gamma(\mathcal{M}, E) \to \mathbb{R} \tag{4.7}$$

$$\sigma \otimes \tau \mapsto (\sigma, \tau) := \int_{\mathcal{M}} vol_{\mathcal{M}} \langle \sigma, \tau \rangle \tag{4.8}$$

♣

Compactness only needed in one factor.

Remark This can be extended to a subset of $\Gamma \otimes \Gamma \supset \Gamma_0 \otimes \Gamma_0$, such that $\mathrm{supp}(\sigma) \cap \mathrm{supp}(\tau)$ is compact.

We now have everything together to formulate the dynamical problem, for which we need linear partial differential operators.

Definition 4.5

Let $(E, \pi, \mathcal{M}, V)$ and $(F, \rho, \mathcal{M}, W)$ be two vector bundles over the same base manifold $\mathcal{M}$.
A ***linear partial differential operator of order at most*** *k is a linear operator $: \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, F)$ satisfying the following properties:*
For each $p \in \mathcal{M}$ there is a neighborhood $\mathcal{U} \subset \mathcal{M}$ of p such that

$$(\mathcal{U}, \Phi) \tag{4.9}$$

trivialises E, $(\mathcal{U}, \Psi)$ trivialises F, $(\mathcal{U}, \varphi)$ is a local chart of $\mathcal{M}$ and there is a collection:

$$\{\mathfrak{A}, \mathfrak{A}_{j_1}, \ldots, \mathfrak{A}_{j_1 \cdots j_k} : \ j_a \in I(\dim \mathcal{M}), \ \mathfrak{A} \in I(k)\} \tag{4.10}$$

of smooth $Hom(V, W)$-valued maps on $\varphi(\mathcal{U})$ which allows to express L locally as follows:

$$\Psi \circ (L\sigma) \circ \varphi^{-1} \tag{4.11}$$

$$= \sum_{a \in I(k)} \sum_{j_1 \cdots j_a \in I(\dim \mathcal{M})} \mathfrak{A}_{j_1 \cdots j_a} \partial_{j_1} \cdots \partial_{j_a} \left(\Phi \circ \sigma \circ \varphi^{-1} \right) \tag{4.12}$$

Where ∂_b ($b \in I(\dim \mathcal{M})$, is the standard partial derivative acting on vector valued functions defined on some open subset of $\mathbb{R}^{\dim \mathcal{M}}$

♣

Ugly definition, uses local representation.

Remark

We will say that L is of exactly order k if it's at most of order k but not at most of order $k-1$. We focus our attention on a special subclass of linear partial differential operators, allowing a generalization of the wave equation and with well-behaved initial date value problem (existence and uniquness of solutions, with initial data on Cauchy hypersurface).

We will use the preceeding definitions of $\mathcal{U}, \Phi, \varphi$:

Definition 4.6

Let $(E, \pi, \mathcal{M}, V)$ be a real-valued bundle over a spacetime $(\mathcal{M}, g)$.
A linear partial differential operator $\mathcal{P}$ of exactly second order is called ***normally hyperbolic*** *if, in a local trivialisation, there is a collection $\{\mathfrak{A}, \ \mathfrak{A}_a : \ a \in I(\dim \mathcal{M})\}$ of smooth $Hom(V, V)$-valued maps on $\varphi(\mathcal{U})$ such that $\mathcal{P}$ reads as follows:*

for each section σ of E:

$$\Phi \circ (\mathcal{P}\sigma) \circ \varphi^{-1} = \tag{4.13}$$

$$= \left\{ - \sum_{a,b \in I(\dim \mathcal{M})} g^{ab} \mathrm{id}_V \partial_a \partial_b + \sum_{a \in I(\dim \mathcal{M})} \mathfrak{A}_a \partial_a + a \right\} \left(\Phi \circ \sigma \circ \varphi^{-1} \right) \tag{4.14}$$

with g^{ab} the ab components of g inverse.

Given a section J of E, called ***source****,*

$$\mathcal{P}\sigma = J \tag{4.15}$$

is called a ***wave equation****.* ♣

Why does fundamental physics have this structure, of second order wave equation? This is a deep question, either too deep or we're too young to debate about it.

The idea of light-cones and causality is encoded in the kinetic operator. Thus the dispersion of information knows about causality, in fact it's implemented in the correct way.

Example 4.2 $(\mathcal{M}, g)$, trivial line bundle $\mathcal{M} \times \mathbb{R}$.

Sections $\simeq$ real-valued functions on $\mathcal{M}$,

$$\mathcal{P} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) \tag{4.16}$$

$$\mathcal{P} := - \sum_{a,b \in I(r)} g^{ab} \partial_a \partial_b + \sum_{a \in I(r)} \left(\sum_{c,d \in I(4)} g^{cd} \Gamma^a{}_{cd} \right) \partial_a + m^2 \tag{4.17}$$

where $\Gamma^a{}_{cd}$ is the Christoffel symbol.

Note that in inertial coordinates, the first term is the $\Box$ D'alembert operator and the second term vanishes.

Definition 4.7

Let $L : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, F)$ be a linear differential operator.

Its ***formal adjoint*** *$L^\star : \Gamma(\mathcal{M}, F) \to \Gamma(\mathcal{M}, E)$ is a linear differential operator satisfying:*

$$(L^\star u, v)_E = (u, Lv)_F \tag{4.18}$$

for each $u \in \Gamma(\mathcal{M}, F)$ and $v \in \Gamma(\mathcal{M}, E)$ with $\mathrm{supp}(u) \cap \mathrm{supp}(v)$ compact.

A linear differential operator $L : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E)$ is ***formally self-adjoint*** *if it coincides with its formal adjoint: $L = L^\star$.* ♣

Remark

E vector bundle, ∇ connection on E.

This together with the Levi-Civita connection on $T^\star\mathcal{M}$ induces a connection on $T^\star\mathcal{M} \otimes E$ (the $T^\star\mathcal{M}$ gives the direction of the derivative), again denoted by ∇.

The **connection-d'Alembert operator** $\Box^{\nabla}$ defined to be minus the composition of the following 3 maps:

$$C^{\infty}(\mathcal{M}, E) \xrightarrow{\nabla} C^{\infty}(\mathcal{M}, T^{\star}\mathcal{M} \otimes E) \xrightarrow{\nabla} C^{\infty}(\mathcal{M}, T^{\star}\mathcal{M} \otimes T^{\star}\mathcal{M} \otimes E) \tag{4.19}$$

$$\xrightarrow{\mathrm{tr}_{g^{-1}} \otimes \mathrm{id}_E} C^{\infty}(\mathcal{M}, E) \tag{4.20}$$

$$\Box^{\nabla} := -\left(\mathrm{tr}_{g^{-1}} \otimes \mathrm{id}_E\right) \circ \nabla \circ \nabla \tag{4.21}$$

The metric encodes the causal relations.

Lemma 4.1

Let $\mathcal{P} : C^{\infty}(\mathcal{M}, E) \to C^{\infty}(\mathcal{M}, E)$ be a normally hyperbolic operator on a spacetime manifold $\mathcal{M}$.

Then there is a unique connection ∇ on E and a unique endomorphism field $B \in C^{\infty}(\mathcal{M}, Hom(E, E))$ such that

$$\mathcal{P} = \Box^{\nabla} + B \tag{4.22}$$

(wave equation= prototype equation for normally hyperbolic operators)

Proof

1. Uniqueness:

   Let ∇' be an arbitrary connection on E.

   For any section $s \in C^{\infty}(\mathcal{M}, E) = \Gamma(\mathcal{M}, E)$ and any function $f \in C^{\infty}(\mathcal{M})$ we have

$$\Box^{\nabla'}(f \cdot s) = \left(\Box^{\nabla'} f\right) \cdot s + f \cdot \left(\Box^{\nabla'} s\right) - \overbrace{2\nabla'_{\mathrm{grad} f} s}^{\text{mixed terms}} \tag{4.23}$$

   Suppose ∇ is such that (existence) $\mathcal{P} = \Box^{\nabla} + B$.

   Then $B = \mathcal{P} - \Box^{\nabla}$ is an endomorphism field and

$$f \cdot \left(Ps - \Box^{\nabla} s\right) = \mathcal{P}(f \cdot s) - \Box^{\nabla}(f \cdot s) \tag{4.24}$$

$$= \mathcal{P}(f \cdot s) - f\left(\Box^{\nabla} s\right) + 2\nabla_{\mathrm{grad} f} s \tag{4.25}$$

   At any point $x \in \mathcal{M}$ every tangent vector $X \in T_x\mathcal{M}$:

$$X = \mathrm{grad} f \tag{4.26}$$

   for a suitable function f.

   For any other connection $\tilde{\nabla}$ on E, we find for $\tilde{\nabla}$ satisfying

$$\mathcal{P} = \Box^{\tilde{\nabla}} + B \tag{4.27}$$

$$\nabla_X s = \tilde{\nabla}_X s \tag{4.28}$$

   for all smooth sections, thus uniqueness.

2. Existence:

   ∇' connection on E.

Since $\mathcal{P}$ and $\Box^{\nabla'}$ are both normally hyperbolic operators,

$$\underbrace{\mathcal{P} - \Box^{\nabla'}}_{\text{first order diff. operator}} = A' \circ \nabla' + B' \tag{4.29}$$

for some $A' \in C^\infty(\mathcal{M}, Hom(T^\star\mathcal{M} \otimes E, E))$ and $B' \in C^\infty(\mathcal{M}, Hom(E, E))$.

Introduce, for any vector field X and section s

$$\nabla_X s := \nabla'_X s - \frac{1}{2} A' \left(X^\flat \otimes s\right) \tag{4.30}$$

new connection ∇ on E.

Given an orthonormal (Lorentz) basis of $T\mathcal{M}$ $\{e_a\}_{a \in I(n)}$ with

$$\varepsilon_a := g(e_a, e_a) = \pm 1 \tag{4.31}$$

and by the principle of equivalence

$$\nabla_{e_a} e_a = 0 \tag{4.32}$$

so

$$\mathcal{P}(s) - B'(s) = \Box^{\nabla'} s + A' \circ \nabla' s \tag{4.33}$$

$$= \sum_{a \in I(n)} \varepsilon_a \left\{ -\nabla'_{e_a} \nabla'_{e_a} s + A' \left(e_a^\flat \otimes \nabla'_{e_a} s \right) \right\} \tag{4.34}$$

$$\stackrel{def}{=} \Box^{\nabla} s + \underbrace{\frac{1}{4} \sum_{a \in I(n)} \varepsilon_a \left\{ A' \left(e_a^\flat \otimes A' \left(e_a^\flat \otimes s \right) \right) - 2 \left(\nabla_{e_a} A' \right) \left(e_a^\flat \otimes s \right) \right\}}_{Q(s) := \Box^{\nabla'} s + A' \circ \nabla' s - \Box^{\nabla} s} \tag{4.35}$$

$Q(s)$ is of order zero! Hence

$$\mathcal{P} = \Box^{\nabla'} + A' \circ \nabla' + B' = \Box^{\nabla} + \underbrace{Q + B'}_{B :=} \tag{4.36}$$

i.e. can always get rid of the first order term. $\Box$

♡

Theorem 4.1

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime and Σ a spacelike smooth Cauchy hypersurface of $(\mathcal{M}, g)$.

Let n be the future directed timelike unit normal vector-field on Σ.

Consider a vector bundle $(E, \pi, \mathcal{M}, V)$ and a normally hyperbolic operator

$$\mathcal{P} = \Box^{\nabla} + B : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E) \tag{4.37}$$

where ∇ and B are uniquely determined by the previous lemma.

For each initial data $u_0, u_1 \in \Gamma_0(\Sigma, E)$ and each source $J \in \Gamma_0(\mathcal{M}, E)$, the Cauchy problem

$$\mathcal{P} u = J \tag{4.38}$$

on $\mathcal{M}$, with

$$\nabla_n u = u_1\ , \quad u = u_0\ , \quad \text{on } \Sigma \tag{4.39}$$

admits a unique solution $u \in \Gamma(\mathcal{M}, E)$.
Furthermore $\mathrm{supp}(u)$ *satisfies*

$$\mathrm{supp}(u) \subseteq J^{+}_{\mathcal{M}}(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(J)) \tag{4.40}$$

Moreover the map

$$\Gamma_0(\Sigma, E) \times \Gamma_0(\Sigma, E) \times \Gamma_0(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E) \tag{4.41}$$

$$(u_0, u_1, J) \mapsto u \tag{4.42}$$

is linear and continuous. ♡

This is the core of the solution theory of the wave equation, we will not prove it, as the proof is long, and we can use Green operators to completely characterize the space of solutions in a constructive way.

4.3 Green Operators and Fundamental Solutions

Definition 4.8

Let E be a vector bundle over a globally hyperbolic spacetime $(\mathcal{M}, g)$ and let $\mathcal{P} : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E)$ be a linear differential operator.
A linear map

$$G^{\pm} : \Gamma_0(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E) \tag{4.43}$$

is called an ***advanced/retarded Green operator*** *for $\mathcal{P}$ if for each source $J \in J_0(\mathcal{M}, E)$:*

1. $\mathcal{P}G^{\pm}f = f$;
2. $G^{\pm}\mathcal{P}f = f$;
3. $\mathrm{supp}(G^{\pm}f) \subseteq \mathcal{J}^{\pm}_{\mathcal{M}}(\mathrm{supp}(f))$

♣

Definition 4.9

Let E be a vector bundle over $(\mathcal{M}, g)$ and let $\mathcal{P} : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E)$ be a linear differential operator.
$\mathcal{P}$ is called ***Green-hyperbolic*** *if it admits advanced and retarded Green operators.* ♣

We're interested in such $\mathcal{P}$ of course.

Reminder:
A $\mathbb{K}$-linear map $F : \Gamma_0(\mathcal{M}, E^{\star}) \to W$, W vector space, is called a **distribution in E with values in W** if it is continuous:

$$\varphi_n \to \varphi \ \text{ in } \Gamma(\mathcal{M}, E^{\star}) \tag{4.44}$$

$$F[\varphi_n] \to F[\varphi] \tag{4.45}$$

(W assumed to be finite dimensional, so all norms in W give the same topology in W)
Moreover, distributions in E act as test-sections in $E^{\star}$.

$\Gamma'(\mathcal{M}, E, W)$ space of W-valued distributions in E.

Example 4.3

1. $x \in \mathcal{M}$. The **delta-distribution** δ_x is an $E_x^\star$-valued distribution in E.
   For $\varphi \in \Gamma(\mathcal{M}, E^\star)$
$$\delta_x[\varphi] := \varphi(x) \in E^\star \tag{4.46}$$
2. Locally integrable $f \in L^1_{loc}(\mathcal{M}, E)$ interpreted as $\mathbb{K}$-valued distribution in E, for any $\varphi \in \Gamma_0(\mathcal{M}, E^\star)$
$$f[\varphi] := \int_{\mathcal{M}} vol_{\mathcal{M}} \varphi(x, f(x)) \tag{4.47}$$

$L : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, F)$ linear differential operator, there is a unique
$$L^\star : \Gamma(\mathcal{M}, F^\star) \to \Gamma(\mathcal{M}, E^\star) \tag{4.48}$$
called the **formal adjoint** of L such that for any $\varphi \in \Gamma(\mathcal{M}, E)$ and $\psi \in \Gamma(\mathcal{M}, F^\star)$:
$$\int_{\mathcal{M}} vol_{\mathcal{M}} \psi(L\varphi) = \int_{\mathcal{M}} vol_{\mathcal{M}} (L^\star\psi)\varphi \tag{4.49}$$
Any linear differential operator $L : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, F)$ extends canonically to a linear differerntial operator $L : \Gamma'(\mathcal{M}, E, W) \to \Gamma'(\mathcal{M}, F, W)$:
$$(LT)[\varphi] := T[L^\star\varphi] \tag{4.50}$$
$\forall T \in \Gamma'(\mathcal{M}, E, W), \forall \varphi \in \Gamma(\mathcal{M}, F^\star)$.

Definition 4.10

Let E be a vector bundle over a globally hyperbolic spacetime $(\mathcal{M}, g)$ and $\mathcal{P} : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E)$ be a normally hyperbolic differential operator.

*A **fundamental solution** of $\mathcal{P}$ at $x \in \mathcal{M}$ is a distribution $F \in \Gamma'(\mathcal{M}, E, E_x^\star)$ such that*
$$\mathcal{P}F = \delta_x \tag{4.51}$$
♣

Remark

1. $\varphi \in \Gamma_0(\mathcal{M}, E^\star)$,
$$F[\mathcal{P}^\star\varphi] = \varphi(x) \tag{4.52}$$
2. If $\mathrm{supp}(F) \subset \mathcal{J}^+_{\mathcal{M}}(x)$, F is called advanced fundamental solution (retarded accordingly).

Proposition 4.1

Let E be a vector bundle over a globally hyperbolic spacetime $(\mathcal{M}, g)$ and $\mathcal{P} : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E)$ be a normally hyperbolic differential operator.

Let $F^\pm(x)$ be a family of advanced/retarded fundamental solutions for the adjoint operator $\mathcal{P}^\star$.

If $F^\pm(x)$ depends smoothly on x in the sense that $x \mapsto F^\pm(x)[\varphi]$ is smooth for each test section φ, and satisfies the differential equation
$$\mathcal{P}\left(F^\pm(\,\cdot\,)[\varphi]\right) = \varphi(\,\cdot\,) \tag{4.53}$$

then

$$\left(G^{\pm}\varphi\right)(x) := F^{\mp}(x)[\varphi] \tag{4.54}$$

defines advanced/retarded Green operators for $\mathcal{P}$.

Conversely, given Green operators $G^{\pm}$ for $\mathcal{P}$, the preceeding equation defines fundamental solution for the adjoint operator $\mathcal{P}^{\star}$, depending smoothly on $x \in \mathcal{M}$ and satisfies

$$\mathcal{P}\left(F^{\pm}(\,\cdot\,)[\varphi]\right) = \varphi(\,\cdot\,) \tag{4.55}$$

for each test function φ.

Proof $(\implies)$

1. We show that $\mathcal{P}\left(G^{\pm}\varphi\right) = \varphi$:
   By definition
   $$\mathcal{P}\left(G^{\pm}\varphi\right) = \mathcal{P}\left(F^{\mp}(\,\cdot\,)[\varphi]\right) = \varphi \tag{4.56}$$
2. $G^{\pm}\mathcal{P}\varphi = \varphi$ follows from the fact that $F^{\pm}(x)$ are fundamental solutions:
   $$G^{\pm}\left(\mathcal{P}\varphi\right)(x) = F^{\mp}(x)[\mathcal{P}\varphi] = \mathcal{P}^{\star}F^{\mp}(x)[\varphi] = \varphi(x) \tag{4.57}$$
3. We show that $\mathrm{supp}(G^{+}\varphi) \subseteq \mathcal{J}^{+}_{\mathcal{M}}(\mathrm{supp}\varphi)$.
   Let $x \in \mathcal{M}$ such that $\left(G^{+}\varphi\right)(x) \neq 0$.
   Since $\mathrm{supp}F^{-}(x) \subset \mathcal{J}^{-}_{\mathcal{M}}(x)$ by definition, $\mathrm{supp}\varphi$ must meet $\mathcal{J}_{\mathcal{M}}(x)$.
   Hence $x \in \mathcal{J}^{+}_{\mathcal{M}}(\mathrm{supp}\varphi)$ and therefore
   $$\mathrm{supp}\left(G^{+}\varphi\right) \subseteq \mathcal{J}^{+}_{\mathcal{M}}\left(\mathrm{supp}\varphi\right) \tag{4.58}$$
   and similar for G^{-}.
   $(\impliedby)$ similar, exercise. □

♠

Corollary 4.1

Let E be a vector bundle over a globally hyperbolic spacetime $(\mathcal{M}, g)$ and let

$$\mathcal{P} : \Gamma\left(\mathcal{M}, E\right) \to \Gamma\left(\mathcal{M}, E\right) \tag{4.59}$$

be a normally hyperbolic operator.

Then there exists unique advanced/retarded Green operators

$$G^{\pm} : \Gamma_0\left(\mathcal{M}, E\right) \to \Gamma\left(\mathcal{M}, E\right) \tag{4.60}$$

for $\mathcal{P}$.

Proof We will not prove it here. □

♡

Lemma 4.2

Let E be a vector bundle over a globally hyperbolic spacetime $(\mathcal{M}, g)$ and let

$$\mathcal{P} : \Gamma\left(\mathcal{M}, E\right) \to \Gamma\left(\mathcal{M}, E\right) \tag{4.61}$$

be a normally hyperbolic operator.

Let $G^{\pm}$ be the associated Green operators for $\mathcal{P}$ and $G^{\star\pm}$ the Green operators for the adjoint operator $\mathcal{P}^{\star}$.
Then

$$\int_{\mathcal{M}} vol_{\mathcal{M}} \left(G^{\star\pm}\varphi\right) \cdot \psi = \int_{\mathcal{M}} vol_{\mathcal{M}} \varphi \cdot \left(G^{\mp}\psi\right) \tag{4.62}$$

(adjoint relates to flipping advanced and retarded operators), $\forall \varphi, \psi \in \Gamma_0(\mathcal{M}, E)$.

Remark *For all sections $u, v \in E$ with compact support, we have (pairing)*

$$\left(G^{\star\pm}u, v\right) = \left(u, G^{\mp}v\right) \tag{4.63}$$

Proof

1. $\text{supp}\left(G^{\pm}\varphi\right) \cap \text{supp}\left(G^{\mp\psi}\right) \subset \mathcal{J}^{\pm}_{\mathcal{M}}\left(\text{supp}(\varphi)\right) \mathcal{J}^{\pm}_{\mathcal{M}}\left(\text{supp}(\psi)\right)$ is compact in any globally hyperbolic spacetime.
2. For the Green operators we have

$$\mathcal{P}\mathcal{G}^{\pm} = \text{id}_{\Gamma_0(\mathcal{M},E)} \tag{4.64}$$

$$\mathcal{P}^{\star}\mathcal{G}^{\star\pm} = \text{id}_{\Gamma_0(\mathcal{M},E^{\star})} \tag{4.65}$$

and therefore

$$\left(G^{\star\pm}\varphi, \psi\right) = (G^{\star\pm}\varphi, \overbrace{\mathcal{P}G^{\mp}}^{\text{id}}\psi) = \left(\mathcal{P}^{\star}\mathcal{G}^{\star\pm}\varphi, G^{\mp}\psi\right) \tag{4.66}$$

$$= \left(\varphi, G^{\mp}\psi\right) \quad \square \tag{4.67}$$

♡

Remark

$\Gamma_{sc}\left(\mathcal{M}, E\right)$ is the set of all smooth sections φ of the vector bundle E for which there exists a compact subset $\mathcal{K} \subset \mathcal{M}$ such that $\text{supp}\varphi \subset \mathcal{J}_{\mathcal{M}}\left(\mathcal{K}\right)$, it's avector subspace of $\Gamma\left(\mathcal{M}, E\right)$ (sc stands for *spacelike compact*).

If $\mathcal{M}$ is globally hyperbolic and $\varphi \in \Gamma_{sc}\left(\mathcal{M}, E\right)$, then for every Cauchy hypersurface $\Sigma \subset \mathcal{M}$ $\text{supp}\varphi|_{\Sigma} \subset \Sigma \cap \mathcal{J}_{\mathcal{M}}\left(\mathcal{K}\right)$.

Definition 4.11

Let E be a vector bundle over a globally hyperbolic spacetime $\left(\mathcal{M}, g\right)$ and let

$$\mathcal{P} : \Gamma\left(\mathcal{M}, E\right) \to \Gamma\left(\mathcal{M}, E\right) \tag{4.68}$$

be a Green-hyperbolic operator.

Choose advanced/retarded Green operators $G^{\pm}$ for $\mathcal{P}$.

Then the linear map

$$G := G^{+} - G^{-} : \Gamma_0\left(\mathcal{M}, E\right) \to \Gamma_{sc}\left(\mathcal{M}, E\right) \tag{4.69}$$

is called the ***causal propagator*** *for $\mathcal{P}$ defined by $G^{\pm}$.*

♣

Theorem 4.2

Let E be a vector bundle over a globally hyperbolic spacetime $(\mathcal{M}, g)$ and let

$$\mathcal{P} : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E) \tag{4.70}$$

be a normally hyperbolic operator. Let $G^{\pm}$ be advanced/retarded Green operators for $\mathcal{P}$. Then the sequence of linear maps:

$$0 \to \Gamma_0(\mathcal{M}, E) \xrightarrow{\mathcal{P}} \Gamma_0(\mathcal{M}, E) \xrightarrow{G} \Gamma_{sc}(\mathcal{M}, E) \xrightarrow{\mathcal{P}} \Gamma(\mathcal{M}, E) \tag{4.71}$$

is a chain complex which is exact everywhere.

Proof

1. Since $\mathcal{P}G^{\pm} = \mathrm{id}_{\Gamma_0(\mathcal{M},E)}$ by definition, it follows that

$$\mathcal{P}G = \mathcal{P}\left(G^{+} - G^{-}\right) = \mathcal{P}G^{+} - \mathcal{P}G^{-} = 0 \tag{4.72}$$

   The second property of $G^{\pm}$, $G^{\pm}\mathcal{P} = \mathrm{id}_{\Gamma_0(\mathcal{M},E)}$, gives $G\mathcal{P} = 0$.

   Moreover, by definition,

$$\mathrm{supp}\left(G^{\pm}\varphi\right) \subseteq \mathcal{J}^{\pm}_{\mathcal{M}}(\mathrm{supp}(\varphi)) \ , \quad \forall\varphi \in \Gamma_0(\mathcal{M}, E) \tag{4.73}$$

   Hence G maps $\Gamma_0(\mathcal{M}, E)$ to $\Gamma_{sc}(\mathcal{M}, E)$.

2. Exactness:

   exactness of the first Γ_0 means that

$$\mathcal{P} : \Gamma_0(\mathcal{M}, E) \to \Gamma_0(\mathcal{M}, E) \tag{4.74}$$

   is injective.

   Let $\varphi \in \Gamma_0(\mathcal{M}, E)$ with $\mathcal{P}\varphi = 0$. Then

$$\varphi = G^{+}\mathcal{P}\varphi = G^{+}0 = 0 \tag{4.75}$$

   which proves the assertion.

   Exactness at the second Γ_0 means

$$\ker G = \mathrm{Im}\,\mathcal{P} \tag{4.76}$$

   Let $\varphi \in \Gamma_0(\mathcal{M}, E)$ with $G\varphi = 0$, so $\varphi \in \ker G$.

   In other words $G^{+}\varphi = G^{-}\varphi$.

   Set $\psi := G^{+}\varphi \in \Gamma(\mathcal{M}, E)$ with

$$\mathrm{supp}(\psi) = \mathrm{supp}\left(G^{+}\varphi\right) \cap \mathrm{supp}\left(G^{-}\varphi\right) \subset \mathcal{J}^{+}_{\mathcal{M}}(\mathrm{supp}(\varphi)) \cap \mathcal{J}^{-}_{\mathcal{M}}(\mathrm{supp}(\varphi)) \tag{4.77}$$

   because $\varphi \in \ker G$, which is compact by global hyperbolicity, so $\psi \in \Gamma_0(\mathcal{M}, E)$.

   From $\mathcal{P}\psi = \mathcal{P}(G^{+}\varphi) = \varphi$, we have $\varphi \in \mathcal{P}(\Gamma_0(\mathcal{M}, E))$ hence exactness ($\ker G = \mathrm{Im}\,\mathcal{P}$).

3. Exactness at the first $\Gamma_{sc}(\mathcal{M}, E)$ means that G is surjective.

   Let $\varphi \in \Gamma_{sc}(\mathcal{M}, E)$ such that $\mathcal{P}\varphi = 0$.

   Without loss of generality, assume that $\mathrm{supp}(\varphi) \subset \mathcal{I}^{+}_{\mathcal{K}} \cup \mathcal{I}^{-}_{\mathcal{K}}$ for a compact subset $\mathcal{K}$ of $\mathcal{M}$.

Take a partition of unity subordinated to the open covering $\{\mathcal{I}_{\mathcal{K}}^{+}, \mathcal{I}_{\mathcal{K}}^{-}\}$ of supp (φ), write $\varphi = \varphi_1 + \varphi_2$, with

$$\text{supp}\,(\varphi_1) = \mathcal{I}_{\mathcal{K}}^{-} \subset \mathcal{J}_{\mathcal{K}}^{-} \tag{4.78}$$

$$\text{supp}\,(\varphi_2) = \mathcal{I}_{\mathcal{K}}^{+} \subset \mathcal{J}_{\mathcal{K}}^{+} \tag{4.79}$$

For $\psi := -\mathcal{P}\varphi_1 \overset{\mathcal{P}\varphi=0}{=} \mathcal{P}\varphi_2$ we see that

$$\text{supp}\,(\psi) \subset \mathcal{J}_{\mathcal{M}}^{-}(\mathcal{K}) \cap \mathcal{J}_{\mathcal{M}}^{+}(\mathcal{K}) \tag{4.80}$$

Hence $\psi \in \Gamma_0(\mathcal{M}, E)$ by compactness of $\mathcal{K}$.

We show that $G^{+}\psi = \varphi_2$: for all $\theta \in \Gamma_0(\mathcal{M}, E)$ we have

$$(\theta, \underbrace{G^{+}\mathcal{P}\varphi_2}_{G^{+}\psi}) = \left(G^{\star -}\theta, \mathcal{P}\varphi_2\right) = \left(\mathcal{P}G^{\star -}\theta, \varphi_2\right) \tag{4.81}$$

$$= (\theta, \varphi_2) \tag{4.82}$$

where we used the adjoint relations between G^{+} and G^{-} and the adjoint identity for $G^{\pm}$.

Similarly $G^{-}\psi = -\varphi_1$, so $G\psi = G^{+}\psi - G^{-}\psi = \varphi_1 + \varphi_2 = \varphi \in \text{Im}\, G$. □ ♡

Proposition 4.2

Let E be a vector bundle over a globally hyperbolic spacetime $(\mathcal{M}, g)$ and let

$$\mathcal{P} : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E) \tag{4.83}$$

be a normally hyperbolic operator. Denote by $G^{\pm}$ the advanced/retarded Green operators for $\mathcal{P}$.

Then $G^{\pm} : \Gamma_0(\mathcal{M}, E) \to \Gamma_{sc}(\mathcal{M}, E)$ and all maps of the preceeding chain complex are sequentially continuous.

Proof Will not prove here. ♠

Remark

Let $\mathcal{P}$ be Green-Hyperbolic with Green-hyperbolic formal adjoint. Denote by $\mathcal{S}_{sc}(\mathcal{M})$ the space of solutions of $\mathcal{P}u = 0$ with spacelike compact support on $\mathcal{M}$.

It can be shown that G induces an isomorphism $\mathcal{I}$ of vector spaces

$$\Gamma_0(\mathcal{M}, E) / \mathcal{P}\left(\Gamma_0(\mathcal{M}, E)\right) \xrightarrow{\sim} \mathcal{S}_{sc}(\mathcal{M}) \tag{4.84}$$

Proposition 4.3

Let E be a vector bundle over a globally hyperbolic spacetime $(\mathcal{M}, g)$ equipped with a bosonic non-degenerate bilinear form. Let let

$$\mathcal{P} : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E) \tag{4.85}$$

be a formally self-adjoint Green-hyperbolic operator.

Then the causal propagator G for $\mathcal{P}$ fulfills

$$(u, Gv) = -Gu, v \quad \forall u, v \in \Gamma_0(\mathcal{M}, E) \tag{4.86}$$

and the map

$$\sigma : \mathcal{S}_{sc}(\mathcal{M}) \otimes \mathcal{S}_{sc}(\mathcal{M}) \to \mathbb{R} \tag{4.87}$$

$$\sigma(u, v) := (f, Gh) \tag{4.88}$$

with $f, h \in \Gamma_0(\mathcal{M}, E)$ such that $u = Gf$, $v = Gh$, is a symplectic form.

Proof $\mathcal{P}$ Green-hyperbolic, then Green operators are unique and coincide with the Green operators of the formal adjoint.

1. $(G^{\pm}u, v) = (u, G^{\mp}v)$ for $u, v \in \Gamma_0(\mathcal{M}, E)$.
   $G = G^+ - G^- \implies (u, Gv) = -(Gu, v)\,\forall u, v \in \Gamma_0(\mathcal{M}, E)$.
2. Symplectic structure:
   introduce bilinear form τ on $\Gamma_0(\mathcal{M}, E)$, defined by

$$\tau(u, v) := (u, Gv) \quad \forall u, v \in \Gamma_0(\mathcal{M}, E) \tag{4.89}$$

Consider $u \in \Gamma_0(\mathcal{M}, E)$ such that $\tau(u, v) = 0\,\forall v \in \Gamma_0(\mathcal{M}, E)$.
Since $(\cdot, \cdot)$ is non-degenerate, $Gu = 0$, thus $u \in \mathcal{P}(\Gamma_0(\mathcal{M}, E))$, so τ induces non-degenrate skew-symmetric bilinear form on the quotient

$$\Gamma_0(\mathcal{M}, E) / \mathcal{P}(\Gamma_0(\mathcal{M}, E)) \overset{\mathcal{I}}{\simeq} \mathcal{S}_{sc}(\mathcal{M}) \tag{4.90}$$

so

$$\sigma := \tau \circ \left(\mathcal{I}^{-1} \otimes \mathcal{I}^{-1}\right) \tag{4.91}$$

does the work. □

Example 4.4 Real Scalar Field.

Globally hyperbolic spacetime $(\mathcal{M}, g)$.

Simple bundle: $E = \mathcal{M} \times \mathbb{R}$, bosonic bilinear form: fiberwise multiplication.

$$\Gamma(\mathcal{M}, E) \simeq C^\infty(\mathcal{M}) \tag{4.92}$$

Definition 4.12

*A **source-free real Klein-Gordon field** is a smooth section of $E = \mathcal{M} \times \mathbb{R}$ such that the associated smooth function $\Phi \in C^\infty(\mathcal{M})$ is a solution of the following Cauchy problem*

$$\mathcal{P}\Phi = \left(\Box^{\nabla} + \zeta R + m^2\right)\Phi \text{ on } \mathcal{M} \tag{4.93}$$

with $\zeta \in \mathbb{R}$, R ricci scalar associated to g.

R is present in interactions involving loops, as counterterm in renormalization.

With R be a constant or not, we have a "spacetime dependent mass" for Φ.

$$\Phi = \Phi_0 \text{ on } \Sigma, \quad \nabla_n\Phi = \Phi_1 \text{ on } \Sigma \tag{4.94}$$

with Σ smooth spacelike Cauchy hyperbolic of $\mathcal{M}$ with future directed unit normal vector field n, and $\Phi_0, \Phi_1 \in C_0^\infty(\mathcal{M})$ are the given initial data on Σ. ♣

Symplectic structure:

$$\sigma : \mathcal{S}_{sc}(\mathcal{M}) \otimes \mathcal{S}_{sc}(\mathcal{M}) \to \mathbb{R} \tag{4.95}$$

$$\sigma(\Phi_g, \Phi_h) = (f, Gh) \tag{4.96}$$

$f, h \in C_0^\infty(\mathcal{M})$ such that $\Phi_f = Gf$ $\Phi_h = Gh$.

This is it for treatment of classical field theory: there are many subtle aspects we didn't go through, for theory building and phenomenology.

Chapter 5 Quantisation

5.1 $C^\star$-Algebras

We initially extracted all the strucutres that are relevant for the quantisation of fields on curved spacetimes.

Now we will introduce the preliminary concepts and applications of $C^\star$-algebras.

This approach is known as local quantum physics, in the framework of algebraic quantum field theory in curved spacetimes.

The idea is to associate to each spacetime region (reasonable) an algebra of observables that can be measured in this region. The two main aspects are then: the observables form an algebra, and the observables can be associated to these regions.

A quantisation will then be a functor from the category of globally hyperbolic spacetimes, with formally self-adjoint normally hyperbolic operators, to the category of $C^\star$-algebras.

Normally self adjoint operators here are included as part of the geometrical description determining the equations of motion.

Usually fields are <u>not</u> observables, they play the role of *book keepers*.

Definition 5.1

Let $\mathfrak{A}$ be an associative $\mathbb{C}$-algebra and $\|\cdot\|$ be a norm on the $\mathbb{C}$-vector space $\mathfrak{A}$.

Let $\star : \mathfrak{A} \to \mathfrak{A}$, $a \mapsto a^\star$, be a $\mathbb{C}$-antilinear map.

*The triple $(\mathfrak{A}, \|\cdot\|, \star)$ is called a $C^\star$**-algebra**, if the pair $(\mathfrak{A}, \|\cdot\|)$ is complete and the following conditions hold $\forall a, b \in \mathfrak{A}$:*

1. $a^{\star\star} = (a^\star)^\star = a$ *(involution)*
2. $(ab)^\star = \star a^\star$
3. $\|ab\| \leq \|a\|\|b\|$ *(submultiplicativity)*
4. $\|a^\star\| = \|a\|$ *(isometry)*
5. $\|a^\star a\| = \|a\|^2$ *($C^\star$ property)*

♣

Example 5.1

1. Let $(H, \langle\cdot\,,\,\cdot\rangle)$ be a complex Hilbert space, $\mathfrak{A} := \mathcal{L}(H)$ be the algebra of bounded operators on H.

   Let $\|\cdot\|$ be the usual operator norm, i.e.

$$\|a\| := \sup_{x\in H,\ \|x\|_H=1} \|ax\|_H \tag{5.1}$$

   Let $a^\star$ be the operator adjoint to a, i.e.

$$\langle ax, y\rangle = \langle x, a^\star y\rangle \ \forall x, y \in H \tag{5.2}$$

   <u>Proof that it's a $C^\star$-algebra</u>

The only not obvious point is 5.:

$$\|a\|^2 \overset{def}{=} \sup_{x\in H,\ \|x\|_H=1} \|ax\|_H^2 \overset{def}{=} \sup_{x\in H,\ \|x\|_H=1} \langle ax, ax\rangle = \sup_{x\in H,\ \|x\|_H=1} \langle x, a^\star ax\rangle \tag{5.3}$$

$$\overset{Cauchy-Schwarz}{\leq} \sup_{x\in H,\ \|x\|_H=1} \|x\|_h \|a^\star ax\|_H = \|a^\star a\| \overset{3.}{\leq} \|a^\star\|\|a\| \overset{3.}{=} \|a\|^2 \tag{5.4}$$

thus $\|a^\star a\| = \|a\|^2$. □

2. Let X be a locally compact Hausdorff space.

   Introduce

$$\mathfrak{A} = C_0(X) := \Big\{ f : X \to \mathbb{C} \text{ continuous} :\ \forall \varepsilon > 0\ \exists \mathcal{K}_\varepsilon \subset X \text{ compact}$$
$$\text{such that } |f(x)|_{\mathbb{C}} < \varepsilon,\ \forall x \in X \setminus \mathcal{K}_\varepsilon \Big\} \tag{5.5}$$

   i.e. the **algebra of continuous functions vanishing at infinity**.

   All $f \in C_0(X)$ are bounded, so we define

$$\|f\| := \sup_{x\in X} |f(X)|_{\mathbb{C}} \tag{5.6}$$

   Let $f^\star(x) := \overline{f(x)}$ (complex conjugate), for any $x \in X$.

   Then the triple $(C_0(X), \|\cdot\|, \star)$ is a **commutative** $C^\star$-algebra.

3. Let $\mathcal{M}$ be a smooth manifold. Introduce

$$\mathfrak{A} = C_0^\infty(\mathcal{M}) := C^\infty(\mathcal{M}) \cap C_0(\mathcal{M}) \tag{5.7}$$

   i.e. the algebra of smooth functions vanishing at infinity.

   This satisfies everything except $(C_0^\infty(\mathcal{M}), \|\cdot\|)$ is not complete: completion techniques solve this.

Remark

A $C^\star$-algebra has at most one unit element 1.

Assume the opposite: let $1'$ be another unit. Then

$$1 = 1 \cdot 1' = 1' \tag{5.8}$$

$\forall a \in \mathfrak{A}$ we have

$$1^\star a \overset{1.}{=} (1^\star a)^{\star\star} \overset{2.}{=} (a^\star 1^{\star\star})^\star \overset{1.}{=} (a^\star 1)^\star = a^{\star\star} \overset{1.}{=} a \tag{5.9}$$

and similar for $a1^\star = a$.

Therefore $1^\star$ is also a unit, and by uniqueness $1 = 1^\star$, i.e. the unit is self-adjoint.

Moreover

$$\|1\| = \|1^\star 1\| = \|1\|^2 \implies \|1\| = \begin{cases} 1 \\ 0 \end{cases} \tag{5.10}$$

if it were 0, we would have $1 =$ the 0 element, so we have the trivial algebra, so

$$\|1\| = 1 \tag{5.11}$$

Let $\mathfrak{A}$ be a $C^\star$-algebra with unit 1.

Write $\mathfrak{A}^\times$ for the set of invertible elements in $\mathfrak{A}$.

If $a \in \mathfrak{A}^\times$, then also $a^\star \in \mathfrak{A}^\times$, since

$$a^\star \left(a^{-1}\right)^\star \overset{2.}{=} \left(a^{-1}a\right)^\star = 1^\star = 1 \tag{5.12}$$

(and similarly $\left(a^{-1}\right)^\star a^\star = 1$). So $a^\star$ is invertible with

$$\left(a^\star\right)^{-1} = \left(a^{-1}\right)^\star \tag{5.13}$$

Keep in mind the idea of a theory for observables

Lemma 5.1

Let $\mathfrak{A}$ be a $C^\star$-algebra. Then the maps

1. $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, $(a, b) \mapsto a + b$
2. $\mathbb{C} \times \mathfrak{A} \to \mathfrak{A}$, $(\alpha, a) \mapsto \alpha a$
3. $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, $(a, b) \mapsto a \cdot b$
4. $\mathfrak{A}^\times \to \mathfrak{A}^\times$, $a \mapsto a^{-1}$
5. $\mathfrak{A} \to \mathfrak{A}$, $a \mapsto a^\star$

are continuous.

Proof The maps associated to 1. and 2. are continuous for all normed vector spaces.

3. Let $a_0, b_0 \in \mathfrak{A}$, $\forall a, b \in \mathfrak{A}$ such that

$$\|a - a_0\| < \epsilon\,, \quad \|b - b_0\| < \epsilon \tag{5.14}$$

(take the same $\varepsilon > 0$, up to taking the inf), we have

$$\|a \cdot b - a_0 \cdot b_0\| = \|ab - a_0 b + a_0 b - a_0 b_0\| \tag{5.15}$$
$$\leq \|a - a_0\|\|b\| + \|a_0\|\|b - b_0\| \tag{5.16}$$
$$= \|a - a_0\|\|b - b_0 + b_0\| + \|a\|\|b - b_0\| \tag{5.17}$$
$$\leq \|b - b_0\| \left(\|a - a_0\| + \|a_0\|\right) + \|a - a_0\|\|b_0\| \tag{5.18}$$
$$< \varepsilon\left(\varepsilon + \|a_0\|\right) + \varepsilon\|b_0\| \tag{5.19}$$

4. Let $a_0 \in \mathfrak{A}^\times$. $\forall a \in \mathfrak{A}^\times$ with $\|a - a_0\| < \varepsilon$ we have

$$\left\|a^{-1} - a_0^{-1}\right\| = \left\|a^{-1}a_0 \cdot a_0^{-1} - a^{-1}aa_0^{-1}\right\| \tag{5.20}$$
$$= \left\|a^{-1}\left(a_0 - a\right)a_0^{-1}\right\| \leq \left\|a^{-1}\right\|\|a_0 - a\|\left\|a_0^{-1}\right\| \tag{5.21}$$
$$= \left\|a^{-1} - a_0^{-1} + a_0^{-1}\right\|\|a_0 - a\|\left\|a_0^{-1}\right\| \tag{5.22}$$
$$\leq \left(\left\|a^{-1} + a_0^{-1}\right\| + \left\|a_0^{-1}\right\|\right)\|a_0 - a\|\left\|a_0^{-1}\right\| \tag{5.23}$$
$$< \left(\left\|a^{-1} - a_0^{-1}\right\| + \left\|a_0^{-1}\right\|\right)\varepsilon\left\|a_0^{-1}\right\| \tag{5.24}$$

thus

$$\left\|a^{-1} - a_0^{-1}\right\|\left(1 - \varepsilon\left\|a_0^{-1}\right\|\right) < \varepsilon\left\|a_0^{-1}\right\|^2 \tag{5.25}$$

choosing $\varepsilon < \left\|a_0^{-1}\right\|^{-1}$ we have $1 - \varepsilon\left\|a_0^{-1}\right\| > 0$

$$\left\|a^{-1} - a_0^{-1}\right\| < \frac{\varepsilon}{1 - \varepsilon\left\|a_0^{-1}\right\|}\left\|a_0^{-1}\right\|^{-1} \tag{5.26}$$

with the right hand side being a independent, hence continuity.

5. Clear, isometry. □

♡

5.2 Spectral Properties

Definition 5.2

Let $\mathfrak{A}$ be a $C^\star$-algebra with unit element 1.

For $a \in \mathfrak{A}$ we call

$$r_{\mathfrak{A}}(a) := \left\{\lambda \in \mathbb{C} :\ (\lambda 1 - a) \in \mathfrak{A}^\times\right\} \tag{5.27}$$

*the **resolvent set of** a, and*

$$\sigma_{\mathfrak{A}}(a) := \mathbb{C} \setminus r_{\mathfrak{A}}(a) \tag{5.28}$$

*the **spectrum of** a.*

*For $\lambda \in r_{\mathfrak{A}}(a)$, $(\lambda 1 - a)^{-1} \in \mathfrak{A}$ is called the **resolvent of** a **at** λ.*

The number

$$\rho_{\mathfrak{A}}(a) := \sup\left\{|\lambda|_{\mathbb{C}}\ :\ \lambda \in \sigma_{\mathfrak{A}}(a)\right\} \tag{5.29}$$

*is called the **spectral radius of** a.*

♣

Example 5.2 Let X be a compact Hausdorff space and $\mathfrak{A} = C(X)$.

Then

$$\mathfrak{A}^X = \{f \in C(X) :\ f(x) \neq 0\,,\ \forall x \in X\} \tag{5.30}$$

$$\sigma_{C(X)}(f) = f(X) \subset \mathbb{C} \tag{5.31}$$

$$r_{C(X)}(f) = \mathbb{C} \setminus f(z) \tag{5.32}$$

$$\rho_{C(X)}(f) = \max_{x \in X} |f(x)|_{\mathbb{C}} \tag{5.33}$$

Proposition 5.1

Let $\mathfrak{A}$ be a $C^\star$-algebra with unit 1, and let $a \in \mathfrak{A}$.

Then $\sigma_{\mathfrak{A}} \subset \mathbb{C}$ is a nonempty compact subset and the resolvent of a is continuous.

Moreover

$$\rho_{\mathfrak{A}}(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \|a\| \tag{5.34}$$

where by a^n we denote the n-fold multiplication $a \cdot a \cdot \dots \cdot a$, n times.

Proof

1. $\sigma_{\mathfrak{A}}(a)$ is closed.

   let $\lambda_0 \in r_{\mathfrak{A}}(a)$, i.e. $(\lambda 1 - a) \in \mathfrak{A}^{\times}$.

   For $\lambda \in \mathbb{C}$ with

   $$|\lambda - \lambda_0|_{\mathbb{C}} < \left\|(\lambda_0 1 - a)^{-1}\right\|^{-1} \tag{5.35}$$

   the Neumann series

   $$\sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m \left[(\lambda_0 1 - a)^{-1}\right]^{m+1} \tag{5.36}$$

   converges absolutely, since

   $$\left\|(\lambda_0 - \lambda)^m \left[(\lambda_0 - 1 - a)^{-1}\right]^{m+1}\right\| \leq |\lambda_0 - \lambda|_{\mathbb{C}}^m \left\|\left[\lambda_0 1 - a)^{-1}\right]^{m+1}\right\| \tag{5.37}$$

   $$\leq \left\|(\lambda_0 1 - a)^{-1}\right\| \underbrace{\frac{\left\|(\lambda_0 1 - a)^{-1}\right\|^m}{|\lambda_0 - \lambda|^{-m}}_{\mathbb{C}}}_{<1} \tag{5.38}$$

   the Neumann series converges in $\mathfrak{A}$ by completeness

   $$(\lambda 1 - a) \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1} \tag{5.39}$$

   $$= [(\lambda - \lambda_0)1 + (\lambda_0 1 - a)] \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1} \tag{5.40}$$

   $$= -\sum_{m=0}^{\infty} (\lambda - \lambda_0)^{m+1} (\lambda_0 1 - a)^{-m-1} + \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m} \tag{5.41}$$

   $$= 1 \tag{5.42}$$

   (all the terms cancel but the first one of the second sum)

   Thus $\lambda \in r_{\mathfrak{A}}(a)$, for $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0|_{\mathbb{C}} < \left\|(\lambda_0 1 - a)^{-1}\right\|^{-1}$, so $r_{\mathfrak{A}}(a)$ is open, thus $\sigma_{\mathfrak{A}}(a)$ is closed.

2. Continuity of the resolvent.

Consider λ satisfying the above condition, (λ and λ_0 "close")

$$\left\| (\lambda 1 - a)^{-1} - \underbrace{(\lambda_0 1 - a)^{-1}}_{resolvent} \right\| \tag{5.43}$$

$$= \left\| \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1} - (\lambda_0 1 - a)^{-1} \right\| \tag{5.44}$$

$$\leq \sum_{m=1}^{\infty} |\lambda_0 - \lambda|_{\mathbb{C}}^m \left\| (\lambda_0 1 - a)^{-1} \right\|^{m+1} \tag{5.45}$$

$$\overset{geom.\ series}{=} \left\| (\lambda_0 1 - a)^{-1} \right\| \frac{|\lambda_0 - \lambda|_{\mathbb{C}} \left\| (\lambda_0 1 - a)^{-1} \right\|}{1 - |\lambda_0 - \lambda| \left\| (\lambda_0 1 - a) \right\|} \tag{5.46}$$

$$= |\lambda_0 - \lambda|_{\mathbb{C}} \frac{\left\| (\lambda_0 1 - a)^{-1} \right\|^2}{1 - |\lambda_0 - \lambda|_{\mathbb{C}} \left\| (\lambda_0 1 - a)^{-1} \right\|} \tag{5.47}$$

Since $\lambda_0 \in r_{\mathfrak{A}}(a)$, the above expansion goes to 0 for $\lambda \to \lambda_0$, thus continuity.

3. $\rho_{\mathfrak{A}}(a) \leq \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \liminf_{n \to \infty} \|a^n\|^{\frac{1}{n}}$.
   Let $n \in \mathbb{N}$ fixed, and let $|\lambda|_{\mathbb{C}}^n > \|a^n\|_{\mathfrak{A}}$.
   Each $m \in \mathbb{N}_0$ can be written uniquely in the form

$$m = pn + q\ , \quad p, q \in \mathbb{N}_0\ , \quad q \in [0, n-1] \tag{5.48}$$

   The series

$$\frac{1}{\lambda} \sum_{m=0}^{\infty} \left(\frac{a}{\lambda} \right)^m = \frac{1}{\lambda} \sum_{1=0}^{n-1} \left(\frac{a}{\lambda} \right)^q \sum_{p=0}^{\infty} \Big(\underbrace{\frac{a^n}{\lambda^n}}_{\|\cdot\|<1} \Big)^p \tag{5.49}$$

   converges absolutely and its limit is $(\lambda 1 - a)^{-1}$:

$$(\lambda 1 - a) \sum_{m=0}^{\infty} \lambda^{-m-1} a^m = \sum_{m=0}^{\infty} \lambda^{-m} a^m - \sum_{m=0}^{\infty} \lambda^{-m-1} a^{m+1} \tag{5.50}$$

$$= 1 \tag{5.51}$$

   and the reverse holds, for $|\lambda|_{\mathbb{C}}^n > \|a^n\|_{\mathfrak{A}}$, $(\lambda 1 - a)$ is invertible: $\lambda \in r_{\mathfrak{A}}(a)$.
   Therefore $\rho_{\mathfrak{A}}(a) \leq \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \liminf_{n \to \infty} \|a^n\|^{\frac{1}{n}}$.

4. We show that

$$\rho_{\mathfrak{A}}(a) \geq \limsup_{n \to \infty} \|a^n\|^{\frac{1}{n}} := \tilde{\rho}(a) \tag{5.52}$$

   - Case 1: $\tilde{\rho}(a) = 0$. If a is invertible, then

$$1 = \|1\| = \left\| a^{-n} a^n \right\| \leq \|a^n\| \left\| a^{-n} \right\| \tag{5.53}$$

     but then $1 \leq \tilde{\rho}(a)\tilde{\rho}(a^{-1}) = 0$, which is a contradiction: thus $a \in \mathfrak{A}^\times$ implies $0 \in \sigma_{\mathfrak{A}}(a)$ non-empty, $\tilde{\rho}(a) = 0 \leq \rho_{\mathfrak{A}}(a)$.
   - Case 2: $\tilde{\rho}(a) > 0$:
     Introduce $S := \{\lambda \in \mathbb{C} : \ |\lambda|_{\mathbb{C}} \geq \tilde{\rho}(a)\}$. We will show that $S \not\subset r_{\mathfrak{A}}(a)$ because if it is true then there is $\lambda \in \sigma_{\mathfrak{A}}(a)$ such that $|\lambda_{\mathbb{C}}| \geq \tilde{\rho}(a)$, then $\rho_{\mathfrak{A}}(a) \geq |\lambda|_{\mathbb{C}} \geq \tilde{\rho}(a)$.
     Assume the opposite: there is no such λ and therefore $S \subset r_{\mathfrak{A}}(a)$.

Let $w \in \mathbb{C}$ be an n-th root of unity.

For $\lambda \in S$, $\frac{\lambda}{\omega^k} \in S$. Hence

$$\left(\frac{\lambda}{\omega^k}1 - a\right)^{-1} = \frac{\omega^k}{\lambda}\left(1 - \frac{\omega^k a}{\lambda}\right)^{-1} \tag{5.54}$$

exists. Define

$$R_n(a,\lambda) := \frac{1}{n}\sum_{k=0}^{\infty}\left(1 - \frac{\omega^k a}{\lambda}\right)^{-1} \tag{5.55}$$

We can show that

$$R_n(a,\lambda) = \left(1 - \frac{a^n}{\lambda^n}\right)^{-1} \quad , \quad \forall \lambda \in S \overset{!}{\subset} r_{\mathfrak{A}}(a) \tag{5.56}$$

$$\begin{aligned}
&\left\|\left(1 - \frac{a^n}{\tilde{\rho}^n(a)}\right)^{-1} - \left(1 - \frac{a^n}{\lambda}\right)^{-1}\right\| \\
&\qquad \leq |\tilde{\rho}(a) - \lambda|_{\mathbb{C}} \, \overbrace{\underbrace{\|a\| \sup z \in S \left\|(z1 - a)^{-1}\right\|^2}_{\text{finite for } z \in r_{\mathfrak{A}}(a)}}^{c}
\end{aligned} \tag{5.57}$$

For $|z| \geq 2\|a\|$ we have

$$\left\|(z1 - a)^{-1}\right\| \leq \frac{1}{|z|}\sum_{n\in\mathbb{N}} \underbrace{\frac{\|a\|^n}{|z|^n}}_{\leq\left(\frac{1}{2}\right)^n} \leq \frac{2}{|z|} \leq \frac{1}{\|a\|} \tag{5.58}$$

Consider the annulus $\overline{B}_{2\|a\|}(0) \setminus B_{\tilde{\rho}(a)}(0)$.

Outside $\left\|(z1 - a)^{-1}\right\| \leq \frac{1}{2}$, inside bounded by continuity

$$\begin{aligned}
&\|R_n(a,\tilde{\rho}(a)) - R_n(a,\lambda)\| \leq c\,|\tilde{\rho}(a) - \lambda|_{\mathbb{C}} \ , \\
&\qquad\qquad \forall n \in \mathbb{N},\ \forall \lambda \in S \subset r_{\mathfrak{A}}(a)
\end{aligned} \tag{5.59}$$

set $\lambda = \tilde{\rho}(a) + \frac{1}{k}$: the left hand side then:

$$\left\|R_n(a,\tilde{\rho}(a)) - R_n(a,\tilde{\rho}(a) + k^{-1})\right\| \tag{5.60}$$

$$= \lim_{n\to\infty}\|R_n(a,\tilde{\rho}(a)) - 1\| \tag{5.61}$$

and the right hand side as $n \to \infty$ goes to $\frac{c}{k}$, thus

$$\limsup_{n\to\infty}\|R_n(a,\tilde{\rho}(a)) - 1\| \leq \frac{c}{k} \ , \ \forall k \in \mathbb{N} \tag{5.62}$$

i.e. the left hand side goes to 0, $R_n(a,\tilde{\rho}(a)) \xrightarrow{n\to\infty} 1$,

$$\frac{\|a^n\|}{\tilde{\rho}^n(a)} \xrightarrow{n\to\infty} 0 \tag{5.63}$$

On the other hand

$$\left\|a^{n+1}\right\|^{\frac{1}{n+1}} \leq \|a^n\|^{\frac{1}{n}} \tag{5.64}$$

i.e. the sequence $\left\{\|a^n\|^{\frac{1}{n}}\right\}_{n\in\mathbb{N}}$ is monotonically non-increasing:

$$\tilde{\rho}(a) = \limsup_{\ell\to\infty}\left\|a^{\ell}\right\|^{\frac{1}{\ell}} \leq \|a^n\|^{\frac{1}{n}} \ , \ \ \forall n \in \mathbb{N} \tag{5.65}$$

thus

$$1 \leq \frac{\|a^n\|}{\tilde{\rho}^n(a)}, \ \forall n \in \mathbb{N} \tag{5.66}$$

which contradicts equation 5.63: thus $S \not\subset r_{\mathfrak{A}}(a)$.

We end up with

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq \rho_{\mathfrak{A}}(a) \leq \liminf_{n\to\infty} \|a^n\|^{\frac{1}{n}} \tag{5.67}$$

which implies that

$$\rho_{\mathfrak{A}}(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \overset{submultiplicativity}{\leq} \|a\| \tag{5.68}$$

where the limit is well defined.

The spectrum is non-empty: if it were, $\rho_{\mathfrak{A}}(a) = -\infty$, but from the equation above it is ≥ 0. $\square$

Definition 5.3

Let $\mathfrak{A}$ be a $C^\star$-algebra with unit 1.

Then $a \in \mathfrak{A}$ is called:

- ***normal**, if $aa^\star = a^\star a$,*
- ***isometry**, if $a^\star a = 1$,*
- ***unitary**, if $a^\star a = aa^\star = 1$ (normal and isometry).*

Proposition 5.2

Let $\mathfrak{A}$ be a $C^\star$-algebra with unit 1, and let $a \in \mathfrak{A}$.

Then the following holds:

1. $\sigma_{\mathfrak{A}}(a^\star) = \overline{\sigma_{\mathfrak{A}}(a)}$
2. *If $a \in \mathfrak{A}^\times$, then $\sigma_{\mathfrak{A}}(a^{-1}) = (\sigma_{\mathfrak{A}}(a))^{-1}$*
3. *If a is normal, then $\rho_{\mathfrak{A}}(a) = \|a\|$*
4. *If a is an isometry, then $\rho_{\mathfrak{A}}(a) = 1$*
5. *If a is unitary, then $\sigma_{\mathfrak{A}}(a) \subset \mathbb{S}^1 \subset \mathbb{C}$*
6. *If a is self adjoint, $\sigma_{\mathfrak{A}}(a) \subset [-\|a\|, \|a\|]$ and $\sigma_{\mathfrak{A}}(a^2) \subset [0, \|a\|^2]$*
7. *If $P(z)$ is a polynomial with complex coefficients and $a \in \mathfrak{A}$ arbitrary, then*

$$\sigma_{\mathfrak{A}}(P(a)) = P(\sigma_{\mathfrak{A}}(a)) = \{P(\lambda) : \lambda \in \sigma_{\mathfrak{A}}(a)\} \tag{5.69}$$

Proof We will skip the proof.

Corollary 5.1

Let $(\mathfrak{A}, \|\cdot\|, \star)$ be a $C^\star$-algebra with unit 1.

Then the norm $\|\cdot\|$ is uniquely determined by $\mathfrak{A}$ and $\star$.

Proof

$\forall a \in \mathfrak{A}$, $a^\star a$ is self adjoint and by 3. of the preceeding proposition

$$\|a\|^2 = \|a^\star a\| = \rho_{\mathfrak{A}}\left(a^\star a\right) \tag{5.70}$$

which depends only on $\mathfrak{A}$ and $\star$.

♡

Note that having this algebraic formulation allows us to elevate from the specific representation, even though at the end of the day we need a representation for the calculations.

Definition 5.4

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $C^\star$-algebras.

*An algebra homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is called a $\star$-**morphism** if $\forall a \in \mathfrak{A}$ the following holds:*

$$\pi\left(a^\star\right) = \pi^\star(a) \tag{5.71}$$

*a map $\pi : \mathfrak{A} \to \mathfrak{A}$ is called a $\star$-**automorphism** if it as an invertible $\star$-morphism.*

♣

Corollary 5.2

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $C^\star$ algebras with units.

Each unit-preserving $\star$-morphism $\pi : \mathfrak{A} \to \mathfrak{B}$ satisfies

$$\|\pi(a)\|_{\mathfrak{B}} \leq \|a\|_{\mathfrak{A}} \tag{5.72}$$

$\forall a \in \mathfrak{A}$. In particular, π is continuous.

Proof

For $a \in \mathfrak{A}^\times$, we have $\pi(a)\pi(a^{-1}) = \pi(aa^{-1}) = \pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.
And similarly $\pi(a^{-1})\pi(a) = 1_{\mathfrak{B}}$, so that $\pi(a) \in \mathfrak{B}^\times$, with $\pi(a)^{-1} = \pi(a^{-1})$.
If $\lambda \in r_{\mathfrak{A}}(a)$, then

$$\lambda 1_{\mathfrak{B}} - \pi(a) = \pi\left(\lambda 1_{\mathfrak{A}} - a\right) \in \pi\left(\mathfrak{A}^\times\right) \subset \mathfrak{B}^\times \tag{5.73}$$

i.e. $\lambda \in r_{\mathfrak{B}}(\pi(a))$ thus $r_{\mathfrak{A}}(a) \subset r_{\mathfrak{B}}(\pi(a))$, and $\sigma_{\mathfrak{B}}(\pi(a)) \subset \sigma_{\mathfrak{A}}(a)$, then

$$\rho_{\mathfrak{B}}\left(\pi(a)\right) \leq \rho_{\mathfrak{A}}(a) \tag{5.74}$$

and

$$\|\pi(a)\|_{\mathfrak{B}}^2 = \|\pi(a)^\star \pi(a)\|_{\mathfrak{B}} = \rho_{\mathfrak{B}}\left(\pi(a)^\star \pi(a)\right) \tag{5.75}$$

$$= \rho_{\mathfrak{B}}(\pi(a^\star a)) \leq \rho_{\mathfrak{A}}(a^\star a) = \|a\|_{\mathfrak{A}}^2 \quad \square \tag{5.76}$$

♡

Corollary 5.3

Let $\mathfrak{A}$ be a $C^\star$-algebra with unit.

Then each unit preserving $\star$-automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ satisfies

$$\|\pi(a)\| = \|a\|\ , \quad \forall a \in \mathfrak{A} \tag{5.77}$$

Proof

$$\|\pi(a)\| \leq \|a\| \leq \left\|\pi^{-1}(\pi(a))\right\| \leq \|\pi(a)\| \quad \square \tag{5.78}$$

Remark

This can be extended to π injective but not onto; this extension is not a consequence of this corollary (not clear that the image of the $\star$-morphism is a $C^\star$-algebra).

5.3 Weyl Systems and Canonical Commutation Relations

Definition 5.5

A ***Weyl system*** *of a symplectic vector space* (V, Ω) *consists of a* $C^\star$*-algebra* $\mathfrak{A}$ *with unit and a map*

$$W : V \to \mathfrak{A} \tag{5.79}$$

such that $\forall \varphi, \psi \in V$ *the following holds:*

1. $W(0_V) = 1_{\mathfrak{A}}$
2. $W(-\varphi) = W(\varphi)^\star$
3. $$W(\varphi) \overset{\mathfrak{A}\text{ multiplication}}{\overset{\uparrow}{\cdot}} W(\psi) = \exp\left\{-\frac{i}{2}\Omega(\varphi, \psi)\right\} W(\varphi + \psi) \tag{5.80}$$

No requirement for continuity? We will see.

Example 5.3 It's hard to appreciate it now, but the following example is a natural realisation.
Let (V, Ω) be an arbitrary symplectic vector space and $H = L^2(V, \mathbb{C})$, i.e. elements of H have square integrable functions $f : V \to \mathbb{C}$, endowed with the counting measure, i.e. the functions vanish everywhere except for countably many points, and satisfy

$$\|f\|^2_{L^2} := \sum_{\varphi \in V} |f(\varphi)|^2_{\mathbb{C}} < \infty \tag{5.81}$$

The Hermitian product on H is given by

$$(f, g)_{L^2} := \sum_{\varphi \in V} \overline{f(\varphi)} g(\varphi) \tag{5.82}$$

Let $\mathfrak{A} = \mathcal{L}(H)$ be the $C^\star$-algebra of bounded linear operators on H.
We define $W : V \to \mathfrak{A}$ by

$$(W(\varphi) f)(\psi) := \exp\left\{\frac{i}{2}\Omega(\varphi, \psi)\right\} f(\varphi + \psi) \tag{5.83}$$

$\varphi, \psi \in V$, $f \in H$.

1. $W(0_V) = \mathrm{id}_H$ clearly.

2.

$$(W(\varphi)f,g)_{L^2} = \sum_{\psi\in V} \overline{(W(\varphi)f)(\psi)}g(\psi) \tag{5.84}$$
$$= \sum_{\psi\in V} \overline{\exp\left\{\frac{i}{2}\Omega(\varphi,\psi)\right\} f(\varphi+\psi)}g(\psi) \tag{5.85}$$
$$\overset{\chi:=\varphi+\psi}{=} \sum_{\chi\in V} \overline{\exp\left\{\frac{i}{2}\Omega(\varphi,\chi-\varphi)\right\} f(\chi)}g(\chi-\varphi) \tag{5.86}$$
$$\overset{skew}{=} \sum_{\chi\in V} \overline{\exp\left\{\frac{i}{2}\Omega(\varphi,\chi)\right\}}\,\overline{f(\chi)}g(\chi-\varphi) \tag{5.87}$$
$$= \sum_{\chi\in V} \overline{f(\chi)} \exp\left\{\frac{i}{2}\Omega(-\varphi,\chi)\right\} g(\chi-\varphi) \tag{5.88}$$
$$= (f, W(-\varphi)g)_{L^2} \tag{5.89}$$

$\forall f,g \in H$, thus $W(\varphi)^\star = W(-\varphi)$.

3. We compute

$$(W(\varphi)(W(\psi)f))(\chi) = \tag{5.90}$$
$$= \exp\left\{\frac{i}{2}\Omega(\varphi,\chi)\right\}(W(\psi)f)(\varphi+\chi) \tag{5.91}$$
$$= \exp\left\{\frac{i}{2}\Omega(\varphi,\chi)\right\}\exp\left\{\frac{i}{2}\Omega(\psi,\varphi+\chi)\right\} f(\varphi+\chi+\psi) \tag{5.92}$$
$$= \exp\left\{\frac{i}{2}\Omega(\psi,\varphi)\right\}\exp\left\{\frac{i}{2}\Omega(\varphi+\psi,\chi)\right\} f(\varphi+\chi+\psi) \tag{5.93}$$
$$= \exp\left\{-\frac{i}{2}\Omega(\varphi,\psi)\right\}(W(\varphi+\psi)f)(\chi) \tag{5.94}$$

as required.

Let $CCR(V,\Omega)$ be the $C^\star$-algebra of $\mathcal{L}(H)$ generated by the elements $W(\varphi)$, $\varphi\in V$. Then $CCR(V,\Omega)$ together with the map W forms a Weyl system for (V,Ω).

Proposition 5.3

Let $(\mathfrak{A},W)$ (the other information is implicit) be a Weyl system of a symplectic vector space (V,Ω).

Then

1. *$W(\varphi)$ is unitary for each $\varphi\in V$.*
2. *$\|W(\varphi)-W(\psi)\| = 2$, $\forall\varphi,\psi\in V$, $\varphi\neq\psi$.*
3. *$\mathfrak{A}$ is not separable unless $V=\{0_V\}$.*
4. *The family $\{W(\varphi)\}$ is linearly independent.*

Proof

1.

$$W(\varphi)^\star W(\varphi) \overset{2.}{=} W(-\varphi)W(\varphi) \overset{3.}{=} \underbrace{\exp\left\{-\frac{i}{2}\Omega(-\varphi,\varphi)\right\}}_{1} W(\underbrace{-\varphi+\varphi}_{0_V}) = 1_{\mathfrak{A}} \tag{5.95}$$

and similarly $W(\varphi)W(\varphi)^\star = !_{\mathfrak{A}}$ hence $W(\varphi)$ is unitary $\forall \varphi \in V$.

2. Let $\varphi, \psi \in V$, $\varphi \neq \psi$. For an arbitrary $\chi \in V$ we have

$$W(\chi)W(\varphi-\psi)W(\chi)^{-1} = W(\chi)W(\varphi-\psi)W(\chi)^\star \tag{5.96}$$
$$= \exp\left\{-\frac{i}{2}\Omega(\chi,\varphi-\psi)\right\} W(\chi+\varphi-\psi)W(-\chi) \tag{5.97}$$
$$= \exp\left\{-\frac{i}{2}\Omega(\chi,\varphi-\psi)\right\} \exp\left\{-\frac{i}{2}\Omega(\chi+\varphi-\psi,-\chi)\right\} W(\varphi-\psi) \tag{5.98}$$
$$= \exp\left\{-i\Omega(\chi,\varphi-\psi)\right\} W(\varphi-\psi) \tag{5.99}$$

Hence the spectrum satisfies (invariance under this similarity transformation)

$$\sigma_{\mathfrak{A}}(W(\varphi-\psi)) = \sigma_{\mathfrak{A}}(W(\chi)W(\varphi-\psi)W(\chi)^{-1}) \tag{5.100}$$
$$= \exp\left\{i\Omega(\chi,\varphi-\psi)\right\} \sigma_{\mathfrak{A}}(W(\varphi-\psi)) \tag{5.101}$$

Since $\varphi-\psi \neq 0$ by assumption, the real number $\Omega(\chi,\varphi-\psi)$ runs through all of $\mathbb{R}$ as χ runs through V, so the spectrum of $W(\varphi-\psi)$ is $U(1)$-invariant.

$W(\varphi-\psi)$ is unitary, so we know that it's spectrum is non empty and (by 1.)

$$\sigma_{\mathfrak{A}}(W(\varphi-\psi)) \subset \mathbb{S}^1 \tag{5.102}$$

and also $U(1)$ invariant, hence

$$\sigma_{\mathfrak{A}}(W(\varphi-\psi)) = \mathbb{S}^1 \tag{5.103}$$

Thus

$$\sigma_{\mathfrak{A}}\left(\exp\left\{-\frac{i}{2}\Omega(\varphi,\psi)\right\} W(\varphi-\psi)\right) = \mathbb{S}^1 \tag{5.104}$$

and

$$\sigma_{\mathfrak{A}}\left(\exp\left\{-\frac{i}{2}\Omega(\varphi,\psi)\right\} W(\varphi-\psi) - 1_{\mathfrak{A}}\right) \tag{5.105}$$

is the unit circle centered at $1_{\mathbb{C}}$.

By a previous proposition, a normal implies $\rho_{\mathfrak{A}}(a) = \|a\|_{\mathfrak{A}}$. Unitarity implies normality, thus

$$\left\|\exp\left\{\frac{i}{2}\Omega(\psi,\varphi)\right\} W(\varphi-\psi) - 1_{\mathfrak{A}}\right\| \tag{5.106}$$
$$= \rho_{\mathfrak{A}}\left(\exp\left\{\frac{i}{2}\Omega(\varphi,\psi)\right\} W(\varphi-\psi) - 1_{\mathfrak{A}}\right) \tag{5.107}$$
$$= 2 \tag{5.108}$$

by the definition of $\rho_{\mathfrak{A}}$.

Rewriting

$$W(\varphi) - W(\psi) = W(\psi)\left(W(\psi)^\star W(\varphi) - 1_{\mathfrak{A}}\right) \tag{5.109}$$
$$= W(\psi)\left(\exp\left\{\frac{i}{2}\Omega(\psi,\varphi)\right\} W(\varphi-\psi) - 1_{\mathfrak{A}}\right) \tag{5.110}$$

We have

$$
\begin{aligned}
&\|W(\varphi)-W(\psi)\|^2 && (5.111)\\
=&\|(W(\varphi)-W(\psi))^\star\,(W(\varphi)-W(\psi))\| && (5.112)\\
=&\left\|\left(\exp\left\{\frac{i}{2}\Omega(\psi,\varphi)\right\}W(\varphi-\psi)-1_{\mathfrak{A}}\right)^\star\underbrace{W(\psi)^\star W(\psi)}_{1_{\mathfrak{A}}}\right. && (5.113)\\
&\qquad\left.\cdot\left(\exp\left\{\frac{i}{2}\Omega(\psi,\varphi)\right\}W(\varphi-\psi)-1_{\mathfrak{A}}\right)\right\| && (5.114)\\
=&\left\|\exp\left\{\frac{i}{2}\Omega(\psi,\varphi)\right\}W(\varphi-\psi)-1_{\mathfrak{A}}\right\|^2 && (5.115)\\
=&\rho_{\mathfrak{A}}^2\left(\exp\left\{\frac{i}{2}\Omega(\psi,\varphi)\right\}W(\varphi-\psi)-1_{\mathfrak{A}}\right) && (5.116)\\
=&2^2=4 && (5.117)
\end{aligned}
$$

and by definite positivity of the norm, we have the assertion.

3. Since $\sigma_{\mathfrak{A}}(W(\varphi))=\mathbb{S}^1$ and $\|W(\varphi)-W(\psi)\|=2\ \forall\varphi,\psi\in V,\ \varphi\neq\psi$, the balls of radius 1 centered at $W(\varphi)$ form an uncountable collection of mutually disjoint open subsets, thus the $C^\star$-algebra of this Weyl system is not separable (unless $V=\{0_V\}$).
4. Let $\varphi_a\in V$, $a\in I(n)$ $(n\in\mathbb{N})$ be pairwise different and let

$$\sum_{a\in I(n)}\alpha_a W(\varphi_a)=0_{\mathfrak{A}} \tag{5.118}$$

for some $\alpha_a\in\mathbb{R}$. We show that $\alpha_a=0\ \forall a\in I(n)$ by induction on n. The $n=1$ case is trivial.

Assuming the inductive hypothesis on $n-1$, without loss of generality assume (by contraddiction) that one coefficient is non-zero, say $\alpha_n\neq 0$. Hence

$$W(\varphi_n)=\sum_{a\in I(n-1)}-\frac{\alpha_a}{\alpha_n}W(\varphi_a) \tag{5.119}$$

Therefore

$$
\begin{aligned}
1_{\mathfrak{A}}&=W(\varphi_n)^\star W(\varphi)=W(-\varphi_n)W(\varphi_n) && (5.120)\\
&=\sum_{a\in I(n-1)}-\frac{\alpha_a}{\alpha_n}W(-\varphi_n)W(\varphi_a) && (5.121)\\
&=\sum_{a\in I(n-1)}\underbrace{-\frac{\alpha_a}{\alpha_n}\exp\left\{-\frac{i}{2}\Omega(-\varphi_n,\varphi_a)\right\}}_{\beta_a}W(\varphi_a-\varphi_n) && (5.122)\\
&=\sum_{a\in I(n-1)}\beta_a W(\varphi_a-\varphi_n) && (5.123)
\end{aligned}
$$

For an arbitrary vector $\psi \in V$ we obtain

$$1_{\mathfrak{A}} = W(\psi) \cdot 1_{\mathfrak{A}} \cdot W(-\psi) \tag{5.124}$$

$$= \sum_{a \in I(n-1)} \beta_a W(\psi) W(\varphi_a - \varphi_n) W(-\psi) \tag{5.125}$$

$$\overset{proof of 2.}{=} \sum_{a \in I(n-1)} \beta_a \exp\left\{-i\Omega(\psi, \varphi_a - \varphi_n)\right\} W(\varphi_a - \varphi_n) \tag{5.126}$$

equate both representations of $1_{\mathfrak{A}}$, and with the inductive hypothesis on the first $n-1$ φ_a's,

$$\beta_a = \beta_a \exp\left\{-i\Omega(\psi, \varphi_a - \varphi_n)\right\} \ , \quad \forall a \in I(n-1) \tag{5.127}$$

suppose some $\beta_a \neq 0$, then

$$\Omega(\psi, \varphi_a - \varphi_n) = 0_{\mathbb{C}} \tag{5.128}$$

but $\varphi_a \neq \varphi_n$ and Ω is non degenerate, while ψ is arbitrary, so we have a contradiction: $\alpha_a = 0 \ \forall a \in I(n)$. □

♠

Remark

1. Let $(\mathfrak{A}, W)$ be a Weyl system of a symplectic vector space (V, Ω).
   The linear span of $W(\varphi)$ $(\varphi \in V)$, denoted by

$$\operatorname{span}(W(V)) \subset \mathfrak{A} \tag{5.129}$$

   is closed under multiplication and under $\star$.
   Let $\left(\widetilde{\mathfrak{A}}, \widetilde{W}\right)$ be another Weyl system of (V, Ω).
   There is a unique linear map

$$\pi : \operatorname{span}(W(V)) \to \operatorname{span}(\widetilde{W}(V)) \tag{5.130}$$

$$W(\varphi) \mapsto \pi(W(\varphi)) = \widetilde{W}(\varphi) \tag{5.131}$$

   (natural map to define with the structures defined above).
   Since π is given by a bijection on the bases of two spans, it's a linear isomorphism.
   In fact, π is a $\star$-isomorphism, i.e. there is a unique $\star$-isomorphism such that the following diagram commutes

$$\begin{tikzcd}
 & span(\widetilde{W}(V)) \\
V \arrow[ur, "\widetilde{W}"] \arrow[dr, "W"'] & \\
 & span(W(V)) \arrow[uu, "\pi"']
\end{tikzcd}$$

2. On span$(W(V))$ we define the norm

$$\left\| \sum_{\varphi\in V} c_\varphi W(\varphi) \right\|_1 := \sum_{\varphi\in V} |c_\varphi|_{\mathbb{C}} \tag{5.132}$$

This norm is <u>not</u> a $C^\star$-norm. But for every $C^\star$-norm $\|\cdot\|_0$ on span$(W(V))$:

$$\|a\|_0 = \left\| \sum_{\varphi\in V} c_\varphi W(\varphi) \right\|_0 \overset{triangle}{\leq} \sum_{\varphi\in V} |c_\varphi|_{\mathbb{C}} \, \|W(\varphi)\|_0 \overset{unitarity}{=} \sum_{\varphi\in V} |c_\varphi|_{\mathbb{C}} \tag{5.133}$$

$$= \|a\|_1 \tag{5.134}$$

5.4 Quantisation Functors

Lemma 5.2

Let $(\mathfrak{A}, W)$ be a Weyl system of a symplectic vector space (V, Ω).

Then

$$\|a\|_{max} := \sup\{\|a\|_0 \ : \ \|\cdot\|_0 \ C^\star\text{-norm on } span(W(V))\} \tag{5.135}$$

defines a $C^\star$-norm on $span(W(V))$ (still bounded from above by $\|a\|_1$)

Proof

- sup is finite (bounded by $\|\cdot\|_1$)
- We show only the "most difficult" part of showing that it's a norm: the triangle inequality.

  For $a, b \in span(W(V))$

$$\|a+b\|_{max} = \sup\{\|a+b\|_0 : \ \|\cdot\|_0 \ C^\star \text{ norm}\} \tag{5.136}$$

$$\leq \sup\{\|a\|_0 + \|b\|_0 : \ \|\cdot\|_0 \ C^\star \text{ norm}\} \tag{5.137}$$

$$\leq \sup\{\|a\|_0 : \ \|\cdot\|_0 \ C^\star \text{ norm}\} + \sup\{\|b\|_0 : \ \|\cdot\|_0 \ C^\star \text{ norm}\} \tag{5.138}$$

$$= \|a\|_{max} + \|b\|_{max} \tag{5.139}$$

  similar for the other properties. □

♡

Remark

Suppose you have such Weyl system, take the completion of $Span(W(V))$ with respect to $\|\cdot\|_{max}$: it turns out that this completion is simple, there are no non-trivial closed 2-sided $\star$-ideals.

Definition 5.6

A Weyl system $(\mathfrak{A}, W)$ of a symplectic vector space (V, Ω) is called a ***CCR-representation*** *of (V, Ω) if $\mathfrak{A}$ is generated as a $C^\star$-algebra by the elements $W(\varphi)$, $\varphi \in V$.*

In this case we call $\mathfrak{A}$ a ***CCR-algebra*** *of (V, Ω).*

♣

Remark

For any Weyl system we can replace $\mathfrak{A}$ by a $C^\star$-subalgebra generated by $W(\varphi)$, $\varphi \in V$ to obtain a CCR representation.

Theorem 5.1

Let (V, Ω) be a symplectic vector space and let $(\mathfrak{A}_1, W_1)$ and $(\mathfrak{A}_2, W_2)$ be two CCR-representations of (V, Ω).

Then there is a unique $\star$-isomorphism $\pi : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that the following diagram commutes

$$\begin{tikzcd} & \mathfrak{A}_1 \arrow[dd, "\pi"] \\ V \arrow[ur, "W_1"] \arrow[dr, "W_2"'] & \\ & \mathfrak{A}_2 \end{tikzcd}$$

Proof No proof.

Remark

There is a unique CCR-representation up to $\star$-isomorphism.

Notation: $CCR(V, \Omega)$ is the "paradigmatic example" we gave, which is then unique up to isomorphism.

Corollary 5.4

Let (V_1, Ω_1) and (V_1, Ω_2) be two symplectic vector spaces and let $S : V_1 \to V_2$ be a symplectic linear map, i.e.

$$\Omega_2(S\varphi, S\psi) = \Omega_1(\varphi, \psi) \ , \ \forall \varphi, \psi \in V_1 \tag{5.140}$$

Then there exists a unique $\star$-morphism

$$CCR(S) : CCR(V_1, \Omega_1) \to CCR(V_2, \Omega_2) \tag{5.141}$$

such that

$$\begin{tikzcd} V_1 \arrow[r, "S"] \arrow[d, "W_1"'] & V_2 \arrow[d, "W_2"] \\ CCR(V_1, \Omega_1) \arrow[r, "CCR(S)"'] & CCR(V_2, \Omega_2) \end{tikzcd}$$

commutes.

Remark

Uniqueness implies $CCR(\mathrm{id}_V) = \mathrm{id}_{CCR(V,\Omega)}$,

$$CCR(S_2 \circ S_1) = CCR(S_2) \circ CCR(S_1) \tag{5.142}$$

We have constructed a functor:

$$CCR : \textsc{SymplVec} \to C^\star - \textsc{Alg} \tag{5.143}$$

from the category of symplectic vector spaces, with symplectic vector spaces as objects and symplectic linear maps as morphisms, to the category with $C^\star$-algebra as objects and $\star$-morphisms as morphisms.
If $(V_1, \Omega_1) = (V_2, \Omega_2)$ the induced $\star$-automorphisms are called Bogoliubov transformations.
What is then the relation to spacetime?

Definition 5.7

*The category **GlobHyp** has objects triples $(\mathcal{M}, E, \mathcal{P})$, where $\mathcal{M}$ is a globally hyperbolic spacetime, $E \to M$ is a real vector bundle with non-degenerate inner product, and $\mathcal{P}$ is a formally self-adjoint normally hyperbolic operator acting on sections in E.*
Let $(\mathcal{M}_1, E_1, \mathcal{P}_1)$ and $(\mathcal{M}_2, E_2, \mathcal{P}_2)$ be two objects in GlobHyp. A morphism: $(\mathcal{M}_1, E_1, \mathcal{P}_1) \to (\mathcal{M}_2, E_2, \mathcal{P}_2)$ in this category is a pair (f, F) where $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a time-orientation preserving isometric embedding so that $f(\mathcal{M}_1) \subset \mathcal{M}_2$ is a casually compatible open subset in $\mathcal{M}_2$.
Furthermore, $F : E_1 \to E_2$ is a vector bundle homomorphism over f which is fiberwise an isometry.
The diagram

$$\begin{tikzcd}
E_1 \arrow[r, "F"] \arrow[d] & E_2 \arrow[d] \\
\mathcal{M}_1 \arrow[r, "f"'] & \mathcal{M}_2
\end{tikzcd}$$

commutes.
Moreover, F has to preserve the normally hyperbolic operator, i.e.

$$\begin{tikzcd}
\Gamma_0(\mathcal{M}_1, E_1) \arrow[r, "\mathcal{P}_1"] \arrow[d, "\mathrm{ext}"'] & \Gamma_0(\mathcal{M}_1, E_1) \arrow[d, "\mathrm{ext}"] \\
\Gamma_0(\mathcal{M}_2, E_2) \arrow[r, "\mathcal{P}_2"'] & \Gamma_0(\mathcal{M}_2, E_2)
\end{tikzcd}$$

commutes, where $\mathrm{ext}(\varphi)$ denotes the extension of

$$F \circ f^{-1} \tag{5.144}$$

to all of $\mathcal{M}_2$ by 0. ♣

It's implicit that $\dim \mathcal{M}_1 = \dim \mathcal{M}_2$ and $rankE_1 = rankE_2$.
In the following we will restrict to the category of Lorentzian manifolds, not necessarily globally hyperbolic, even though in our case we will mostly deal with the globally hyperbolic ones, with normally hyperbolic operators and globally fundamental solutions.
Note that with the absence of global hyperbolicity we cannot rely on the existence and uniqueness of Green operators, you have to supply it in the objects:

Definition 5.8

Let LorFund *denote the category where objects are five-tuples $(\mathcal{M}, E, \mathcal{P}, G^+, G^-)$ (or shortly $(\mathcal{M}, E, \mathcal{P}, G^\pm)$) where $\mathcal{M}$ is a time oriented Lorentzian manifold, E a real vector bundle over $\mathcal{M}$ with non degenerate inner product, $\mathcal{P}$ is a formally self adjoint normally hyperbolic operator acting on sections in E, and $G^\pm$ are advanced/retarded Green operators for $\mathcal{P}$, respectively.*
Let $X_1 := (\mathcal{M}_1, E_1, \mathcal{P}_1, G_1^\pm)$ and $X_2 := (\mathcal{M}_2, E_2, \mathcal{P}_2, G_2^\pm)$ be two objects in LorFund.
If $\mathcal{M}_1$ is not globally hyperbolic, then we let the set of morphisms from X_1 to X_2 be empty unless $X_1 = X_2$, in which case we put $Mor(X_1, X_2) = \{(\mathrm{id}_\mathcal{M}, \mathrm{id}_{E_1})\}$.
If $\mathcal{M}_1$ is globally hyperbolic, then we take the corresponding morphisms in the category GlobHyp.

♣

Remark

If $\mathcal{M}_1$ is globally hyperbolic then $Mor(X_1, X_2)$ consists of all pairs (f, F) with the same properties of the morphisms in GlobHyp:

$$\begin{tikzcd}
\Gamma_0(\mathcal{M}_1, E_1) \arrow[r, "ext"] \arrow[d, "G_1^\pm"'] & \Gamma_0(\mathcal{M}_2, E_2) \arrow[d, "G_2^\pm"] \\
C^\infty(\mathcal{M}_1, E_1) & C^\infty(\mathcal{M}_2, E_2) \arrow[l, "res"]
\end{tikzcd}$$

commutes, where the extension $ext := F \circ \varphi \circ f^{-1} \in \Gamma_0(f(\mathcal{M}_1), E_2)$ and the restriction $res := F^{-1} \circ \varphi \circ f \in \Gamma_0(\mathcal{M}_\infty, \mathcal{F}^{-\infty}(\mathcal{E}_\in)$ is the restriction of E_1 to $F^{-1}(E_2)$.

Is there a way to embed the first category into the second one? We need the following functor:

Definition 5.9

We define a functor

$$SOLVE : \text{GlobHyp} \to \text{LorFund} \tag{5.145}$$

$$(\mathcal{M}, E, \mathcal{P}) \mapsto (\mathcal{M}, E, \mathcal{P}, G^\pm) \tag{5.146}$$

$$(f, F) \mapsto (f, F) \tag{5.147}$$

$\forall(\mathcal{M}, E, \mathcal{P})$ object in GlobHyp *and for all morphisms $(\mathcal{M}_1, E_1, \mathcal{P}_1) \xrightarrow{(f,F)} (\mathcal{M}_2, E_2, \mathcal{P}_2)$ in* GlobHyp.

♣

$G^{\pm}$ are the (existent and) unique Green operators for $\mathcal{P}$, granted by global hyperbolicity.
This is a clear definition, nothing deep going on.
Let $(\mathcal{M}, E, \mathcal{P}, G^{\pm})$ be an object in the category LorFund.
Using $G := G^{+} - G^{-} : \Gamma_0(\mathcal{M}, E) \to C^{\infty}(\mathcal{M}, E)$ we define

$$\widetilde{\Omega} : \Gamma_0(\mathcal{M}, E) \times \Gamma_0(\mathcal{M}, E) \to \mathbb{R} \tag{5.148}$$

$$(\varphi, \psi) \mapsto \widetilde{\Omega}(\varphi, \psi) := \int_{\mathcal{M}} vol_{\mathcal{M}} \langle G\varphi, \psi \rangle_E \tag{5.149}$$

which is a bilinear and skew symmetric form.
Bilinearity is clear, skew-symmetry follows from

$$\int_{\mathcal{M}} vol_{\mathcal{M}} \langle G^{\pm}\varphi, \psi \rangle_E = \int_{\mathcal{M}} vol_{\mathcal{M}} \langle \varphi, G^{\mp}\psi \rangle \tag{5.150}$$

However, $(\Gamma_0(\mathcal{M}, E), \widetilde{\Omega})$ is <u>not</u> a symplectic vector space!
Suppose the pairing $(G\varphi, \psi) = 0 \; \forall \psi \in \Gamma(\mathcal{M}, E)$, does this imply $\varphi = 0$?
Not in general, $\varphi \in \ker(G)$ contains more than the zero section in general.
In fact $\ker(G) = \mathcal{P}(\Gamma_0(\mathcal{M}, E))$. We know this because

$C^{\infty}_{sc}(\mathcal{M}, E)$ space-like compact.

$$0 \to \Gamma_0(\mathcal{M}, E) \xrightarrow{\mathcal{P}} \Gamma_0(\mathcal{M}, E) \xrightarrow{G} C^{\infty}_{sc}(\mathcal{M}, E) \xrightarrow{\mathcal{P}} C^{\infty}_{sc}(\mathcal{M}, E) \tag{5.151}$$

is a complex, with $\ker G = \operatorname{Im} \mathcal{P}$.
Therefore on the quotient

$$V(\mathcal{M}, E, G) := \Gamma_0(\mathcal{M}, E) / \ker(G) \tag{5.152}$$

$\widetilde{\Omega}$ induces a symplectic form denoted by Ω.

Lemma 5.3

Let $X_a := (\mathcal{M}_a, E_a, \Gamma_a, G_a^{\pm})$, $a \in I(2)$ be objects in LorFund, and $(f, \psi) \in Mor(X_1, X_2)$ be a morphism.
Then $ext : \Gamma_0(\mathcal{M}_1, E_1) \to \Gamma_0(\mathcal{M}_2, E_2)$ maps $\ker(G_1)$ to $\ker(G_2)$ and induces a symplectic map

$$V(\mathcal{M}_1, E_1, G_1) \xrightarrow{sympl} V(\mathcal{M}_2, E_2, G_2) \tag{5.153}$$

♡

Remark

The construction is clear, we have constructed a functor:

$$SYMPL : \text{LorFund} \to \text{SymplVec} \tag{5.154}$$

$$(\mathcal{M}, E, \mathcal{P}, G^{\pm}) \mapsto SYMPL(\mathcal{M}, E, \mathcal{P}, G^{\pm}) = V(\mathcal{M}, E, G := G^{+} - G^{-}) \tag{5.155}$$

A morphism (f, F) is mapped to the symplectic map induced by the extension.

Summarizing: we have started with GlobHyp, so we brought spacetime back in the game then included LorFund which "might be useful", to which with the CCR functor we then associate an observable algebra through the symplectic vector space (with the natural symplectic form), i.e.

$$\begin{tikzcd}
\textsc{GlobHyp} \arrow[d, "SOLVE"'] & & \textsc{SymplVec} \arrow[d] \\
\textsc{LorFund} \arrow[urr, "SYMPL"] & & C^\star - \textsc{Alg}
\end{tikzcd}$$

Hipp Hipp Horray! We have obtained the construction of the quantisation functor: a functorial connection between globally hyperbolic spacetimes and $C^\star$-algebras.

5.4.1 Quasi-local $C^\star$-algebras

We will now focus on quasi-local $C^\star$-algebras, which is a natural concept, dedicated to the description of the action of measurements by taking certain spacetime properties of measurement devices.
In this introductory course, it's sufficient to assume to work with GlobHyp, with the existence and uniqueness of fundamental solutions.
With the quantisation functor we get elements of $C^\star$-algebras which are the observables related to the field whose wave equation is given by a normally hyperbolic operator.
Furthermore, "reasonable subsets" (we will qualify it) of spacetime are time oriented Lorentzian manifolds, equipped with the restriction of the normally hyperbolic operator: each of these subsets, $\mathcal{U}$, has its $C^\star$-algebra, thus observables that can be measured in the spacetime region $\mathcal{U}$.
We know about the causal structure on these reasonable open subsets, because any measurements will have compact support, which is important for causality itself.

Definition 5.10

A set $\mathcal{I}$ is called a ***directed set with weak orthogonality relation*** *if it carries a partial order $\leq$ and a symmetric relation $\perp$ between its elements such that the following 3 properties hold:*

1. $\forall \alpha, \beta \in \mathcal{I}, \exists \gamma \in \mathcal{I}$ *with* $\alpha \leq \gamma$ *and* $\beta \leq \gamma$.
2. $\forall \alpha \in \mathcal{I}, \exists \beta \in \mathcal{I}$ *with* $\alpha \perp \beta$.
3. *If* $\alpha \leq \beta$ *and* $\beta \perp \gamma$, *then* $\alpha \perp \gamma$.

   If in addition
4. *If* $\alpha \perp \beta$ *and* $\alpha \perp \gamma$, *then* $\exists \delta \in \mathcal{I}$ *such that* $\beta \leq \delta$, $\gamma \leq \delta$ *and* $\alpha \perp \delta$.

Then $\mathcal{I}$ is called a ***directed set with orthogoanlity relation****.* ♣

In the following we will use these as index sets for our quasi-local $C^\star$-algebras.

Definition 5.11

*A **quasi-local $C^\star$-algebra** is a pair $(\mathfrak{A}, \{\mathfrak{A}_\alpha\}_{\alpha\in\mathcal{I}})$ of a $C^\star$-algebra and a family $\{\mathfrak{A}_\alpha\}_{\alpha\in\mathcal{I}}$ where $\mathcal{I}$ is a directed set with orthogonality relation such that*

1. $\mathfrak{A}_\alpha \subset \mathfrak{A}_\beta$ *whenever* $\alpha \leq \beta$.
2. $\mathfrak{A} = \overline{\bigcup_{\alpha\in\mathcal{I}} \mathfrak{A}_\alpha}$ *(closure with respect to the norm of* $\mathfrak{A}$*).*
3. *The algebras of* $\mathfrak{A}_\alpha$ *have a common unit.*
4. *If* $\alpha \perp \beta$*, then the commutators of* $\mathfrak{A}_\alpha$ *and* $\mathfrak{A}_\beta$ *are trivial.* ♣

The last property reminds already of the often called microcausality relation of the commutation relations in quantum field theory over Minkowski spacetime: we just have to define orthogonality here.

Definition 5.12

A morphism between two quasi-local $C^\star$-algebras $(\mathfrak{A}, \{\mathfrak{A}_\alpha\}_{\alpha\in\mathcal{I}})$ and $(\mathfrak{M}, \{\mathfrak{M}_\beta\}_{\beta\in\mathcal{J}})$ is a pair (φ, Φ) where $\Phi : \mathfrak{A} \to \mathfrak{M}$ is a unit preserving $C^\star$-morphism and $\varphi : \mathcal{I} \to \mathcal{J}$ is a map such that:

1. φ *is monotonic, i.e. if* $\alpha_1 \leq \alpha_2$ *(in* $\mathcal{I}$ *then* $\varphi(\alpha_1) \leq \varphi(\alpha_2)$ *in* $\varphi(\mathcal{I}) \subset \mathcal{J}$.
2. φ *preserves* $\perp$*, i.e. if* $\alpha_1 \perp \alpha_2$ *in* $\mathcal{I}$*, then* $\varphi(\alpha_1) \perp \varphi(\alpha_2)$.
3. $\Phi(\mathfrak{A}_\alpha) \subset \mathfrak{M}_{\varphi(\alpha)}, \forall \alpha \in \mathcal{I}$. ♣

Remark

Objects: quasi-local $C^\star - algebras$, morphisms: see above, form a category QuasiLocAlg of quasi-local $C^\star$-algebras.
This is in fact the interesting category for observation.

The concept of **weak quasi-local $C^\star$-algebras** follows accordingly by letting $\perp$ denote a weak orthogonality relation. This induces the category of weak quasi-lcoal $C^\star$-algebras, with the same data but $\text{QuasiLocAlg}_{\text{weak}}$.

<u>Goal</u>: we want to associate to any object $(\mathcal{M}, E, \mathcal{P}, G^\pm)$ in the category LorFund a weak quasi-local $C^\star$-algebra.
Set:

$$\mathcal{I} := \left\{\mathcal{U} \subset \mathcal{M} : \mathcal{U} \text{ open, relatively compact, causally compatible, globally hyperbolic}\right\} \cup \{\varnothing, \mathcal{M}\} \tag{5.156}$$

Partial ordering on this set (of course) is the inclusion $\subset$, and orthogonality relation:

$$\mathcal{U} \perp \mathcal{U}' \iff \mathcal{J}_\mathcal{M}(\overline{\mathcal{U}}) \cap \overline{\mathcal{U}'} = \varnothing \tag{5.157}$$

i.e. elements of $\mathcal{I}$ are orthogonal if and only if they're causally independent subsets of $\mathcal{M}$!

Lemma 5.4

The set $\mathcal{I}$ defined above is a directed set with weak orthogonality relation.

Proof

1. $\mathcal{M} \in \mathcal{I} \implies \forall \alpha, \beta \in \mathcal{I}, \alpha \subset \mathcal{M}, \beta \subset \mathcal{M}$.

The rest is left as an exercise for the reader. ♡

Lemma 5.5

Let $\mathcal{M}$ be globally hyperbolic.

Then $\mathcal{I}$ is a directed set with orthogonality relation.

Proof

Exercise. ♡

We are now in position to associate a weak quasi-local $C^\star$-algebra to any object in the category LorFund, just consider $\mathcal{I}$ as above and for a non-empty open set $\mathcal{U} \in \mathcal{I}$ we consider the restriction of the vector bundle E to this (relatively compact) open subset in the base manifold of the vector bundle, with the corresponding restriction of sections.
The restriction of $G^\pm$ on $\mathcal{U}$, $G^\pm_\mathcal{U}$ for $\mathcal{P}$ on $\mathcal{U}$ gives an object $(\mathcal{U}, E|_{\mathcal{U},\mathcal{P},G^\pm_\mathcal{U}}$ for each $\mathcal{U} \in \mathcal{I} \setminus \varnothing$.
For $\mathcal{U}_1 \subset \mathcal{U}_2$ the inclusion induces a morphism $\iota_{\mathcal{U}_2,\mathcal{U}_1}$ in the category LorFund, given by the embedding

$$\mathcal{U}_1 \hookrightarrow \mathcal{U}_2\,, \quad E|_{\mathcal{U}_1} \hookrightarrow E|_{\mathcal{U}_2} \tag{5.158}$$

Let $\alpha_{\mathcal{U}_2,\mathcal{U}_1}$ denote the morphism

$$(CCR \circ SYMPL)(\iota_{\mathcal{U}_2,\mathcal{U}_1}) \tag{5.159}$$

in the category $C^\star - $ Alg (injective, unit preserving $\star$-morphism).
Let $\mathcal{U} \in \mathcal{I} \setminus \varnothing$,

$$(V_\mathcal{U}, \Omega_\mathcal{U}) := SYMPL(\mathcal{U}, E|_\mathcal{U}, \mathcal{P}, G^\pm_\mathcal{U}) \tag{5.160}$$

is the localized version of the symplectic vector space.
Finally,

$$\mathfrak{A}_\mathcal{U} := \alpha_{\mathcal{M},\mathcal{U}}(CCR(V_\mathcal{U}, \Omega_\mathcal{U})) \tag{5.161}$$

$\mathfrak{A}_\mathcal{U}$ is $C^\star$-subalgebra of $CCR(V_\mathcal{U}, \Omega_\mathcal{U})$,

$$\mathfrak{A}_\mathcal{M} = \bigcup_{\mathcal{U} \in \mathcal{I} \setminus \{\varnothing, \mathcal{M}\}} \mathfrak{A}_\mathcal{U} \tag{5.162}$$

Lemma 5.6

Let $(\mathcal{M}, E, \mathcal{P}, G^\pm)$ be an object in the category LorFund.

Then $(\mathfrak{A}_\mathcal{M}, \{\mathfrak{A}_\mathcal{U}\}_{\mathcal{U}\in\mathcal{I}})$ is a weak quasi-local $C^\star$-algebra.

Proof We have to show the 4 properties of the definition:

it is clear by construction that $\mathfrak{A}_\mathcal{M} = \overline{\bigcup_{\mathcal{U}\in\mathcal{I}} \mathfrak{A}_\mathcal{U}}$, since $\mathcal{M} \in \mathcal{I}$.

By construction it is also clear that all algebras $\mathfrak{A}_\mathcal{U}$ have a common unit $W(0_V)$, $0_V \in V$

zero-vector of the symplectic manifold associated to the Lorentzian manifold, thus we get 2. and 3..

1. By functoriality we have the following commutative diagram

$$\begin{tikzcd} CCR(V_{\mathcal{U}}, \Omega_{\mathcal{U}}) \arrow[r, "\alpha_{\mathcal{M},\mathcal{U}}"] \arrow[d, "\alpha_{\mathcal{U}',\mathcal{U}}"] & CCR(V_{\mathcal{M}}, \Omega_{\mathcal{M}}) \\ CCR(V_{\mathcal{U}'}, \Omega_{\mathcal{U}'}) \arrow[ur, "\alpha_{\mathcal{M},\mathcal{U}'}"'] & \end{tikzcd}$$

Since $\alpha_{\mathcal{U}',\mathcal{U}}$ is injectiv, $\mathfrak{A}_{\mathcal{U}} \subset \mathfrak{A}_{\mathcal{U}'}$.

4. Let $\mathcal{U}, \mathcal{U}' \in \mathcal{I}$ be causally independent ($\mathcal{U} \perp \mathcal{U}'$). We want to prove that their $C^{\star}$-algebras have trivial commutator.

Let $\varphi \in \Gamma_0(\mathcal{U}, E)$ and $\psi \in \Gamma_0(\mathcal{U}', E)$.

From $supp(G\varphi) \subset \mathcal{J}_{\mathcal{M}}(\mathcal{U})$ (which tells us that the Green operators "solve the wave equation", encoding the "generalized lightcone") it follows that $supp(G\varphi) \cap supp(\psi) = \varnothing$, hence the pairing $(G\varphi, \psi)_{\mathcal{M}} = 0$.

For the symplectic form Ω on $\Gamma_0(\mathcal{M}, E)/_{\ker(G)}$ this implies that $\Omega(\varphi, \psi) = 0$. This gives for property 3. of a Weyl system

$$W(\varphi) \cdot W(\psi) \overset{\Omega(\varphi,\psi)=0}{=} W(\varphi + \psi) = W(\psi) \cdot W(\varphi) \tag{5.163}$$

Therefore the algebra generators restricted to $\mathcal{U}$ commute with those of $\mathcal{U}'$,

$$[\mathfrak{A}_{\mathcal{U}}, \mathfrak{A}_{\mathcal{U}'}] = 0 \quad \square \tag{5.164}$$

♡

Remark

We associate a morphism in QuasiLocAlg$_{\text{weak}}$ to any morphism in LorFund.

Let $X_a := (\mathcal{M}_a, E_a, \mathcal{P}_a, G_a^{\pm})$, $a \in I(2)$.

If $\mathcal{M}_1$ is globally hyperbolic, then $Mor(X_1, X_2)$ consists of all pairs (f, F) where $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a time orientation preserving isometric embedding such that $f(\mathcal{M}_1) \subset \mathcal{M}_2$ is a causally compatible open subset.

$F : E_1 \to E_2$ is a vector bundle homomorphism over f which is fiberwise an isometry.

Let $\mathcal{I}_a$, $a \in I(2)$, index set for X_a and let $(\mathfrak{A}_{\mathcal{M}_1}, \{\mathfrak{A}_{\mathcal{U}}\}_{\mathcal{U}\in\mathcal{I}_1})$ and $(\mathfrak{M}_{\mathcal{M}_2}, \{\mathfrak{M}_{\mathcal{U}}\}_{\mathcal{U}\in\mathcal{I}_2})$ be the corresponding weak quasi-local $C^{\star}$-algebras.

$f(\mathcal{U}_1) \in \mathcal{I}_2$ $(\mathcal{U}_1\mathcal{I}_1)$ by definition of LorFund, define

$$\varphi : \mathcal{I}_1 \to \mathcal{I}_2 \tag{5.165}$$

$$\mathcal{M}_1 \mapsto \mathcal{M}_2 \tag{5.166}$$

$$\mathcal{U}_1 \mapsto f(\mathcal{U}_1) \subset \mathcal{M}_2 \tag{5.167}$$

φ is monotonic and preserves causal independence.

Consider the morphism:

$$\Phi = CCR \circ SYMPL(f, F) : CCR(V_{\mathcal{M}_1}, \Omega_{\mathcal{M}_1}) \to CCR(V_{\mathcal{M}_2}, \Omega_{\mathcal{M}_1}) \tag{5.168}$$

from the commutative diagram

$$\begin{tikzcd} \mathcal{U}_1 \arrow[r, "\iota"] \arrow[d, "f|_{\mathcal{U}_1}"'] & \mathcal{U}_2 \arrow[d, "f|_{\mathcal{U}_2}"] \\ f(\mathcal{U}_1) \arrow[r, "\iota"'] & \mathcal{M}_2 \end{tikzcd}$$

$\Phi(\mathfrak{A}_{\mathcal{U}_1}) \overset{!}{=} \mathfrak{M}_{f(\mathcal{U}_1)}$ (exercise).

This implies that $\Phi(\mathfrak{A}_{\mathcal{M}_1}) \subset \mathfrak{M}_{\mathcal{M}_2}$.

Therefore $(\varphi, \Phi|_{\mathfrak{A}_{\mathcal{M}_1}})$ is a morphism in $\textsc{QuasiLocAlg}_{\textsc{weak}}$.

Theorem 5.2

The assignment of $(\mathcal{M}, E, \mathcal{P}, G^{\pm}) \mapsto (\mathfrak{A}_{\mathcal{M}}, \{\mathfrak{A}_{\mathcal{U}}\}_{\mathcal{U} \in \mathcal{I}})$ and $(f, F) \mapsto (\varphi, \Phi|_{\mathfrak{A}_{\mathcal{M}}})$ yields a functor

$$\textsc{LorFund} \to \textsc{QuasiLocAlg}_{\textsc{weak}} \tag{5.169}$$

Proof See the above remark (and verify that $(\mathrm{id}_{\mathcal{M}}, \mathrm{id}_E) \mapsto (\mathrm{id}_{\mathcal{I}}, \mathrm{id}_{\mathfrak{A}_{\mathcal{M}}})$). ♡

Corollary 5.5

$$\begin{tikzcd} \textsc{GlobHyp} \arrow[r, "SOLVE"] \arrow[d] & \textsc{LorFund} \arrow[d] \\ \textsc{QuasiLocAlg} \arrow[r, "\text{inclusion}"'] & \textsc{QuasiLocAlg}_{\textsc{weak}} \end{tikzcd}$$

commutes.

Proof

Exercise. ♡

Lemma 5.7

Let $(\mathcal{M}, E, \mathcal{P})$ be an object in the category $\textsc{GlobHyp}$ and $(\mathfrak{A}_{\mathcal{M}}, \{\mathfrak{A}_{\mathcal{U}}\}_{\mathcal{U} \in \mathcal{I}})$ the corresponding quasi-local $C^{\star}$-algebra. Then

$$\mathfrak{A}_{\mathcal{M}} = CCR \circ SYMPL \circ SOLVE(\mathcal{M}, E, \mathcal{P}) \tag{5.170}$$

Proof

Let $\overline{\mathfrak{A}} := CCR \circ SYMPL \circ SOLVE(\mathcal{M}, E, \mathcal{P})$.

By definition $\mathfrak{A}_{\mathcal{M}} \subset \overline{\mathfrak{A}}$.

For the other inclusion: let

$$(\mathcal{M}, E, \mathcal{P}, G^{\pm}) := SOLVE(\mathcal{M}, E, \mathcal{P}) \tag{5.171}$$

Then $SYMPL(\mathcal{M}, E, \mathcal{P}, G^{\pm})$ is $V_{\mathcal{M}} = {}^{\Gamma_0(\mathcal{M}, E)}\!/\!_{\ker(G)}$ with the usual symplectic form induced by G.
$\overline{\mathfrak{A}}$ is generated by

$$\mathcal{E} = \{W([\varphi]) : \ \varphi \in \Gamma_0(\mathcal{M}, E)\} \tag{5.172}$$

with W the Weyl map for the "paradigmatic example" of Weyl system we gave. $[\varphi$ is the equivalence class of φ in ${}^{\Gamma_0(\mathcal{M}, E)}\!/\!_{\ker(G)}$.
It can be shown that $\forall\varphi$, $\exists$ relatively compact, globally hyperbolic, causally compatible $\mathcal{U}$ containing the compact set which describes the support properties of φ.
Thus $W([\varphi]) \in \mathfrak{A}_{\mathcal{U}}$.
Hence $\mathcal{E} \subset \bigcup\limits_{\mathcal{U}\in\mathcal{I}} \mathfrak{A}_{\mathcal{U}} \subset \mathfrak{A}_{\mathcal{M}}$, from which the assertion follows. □

♡

The quantisation program is now complete!
We put observables and algebraic structures imprinted on these at the core of a physically motivated quantization program.
We are not interested in the fields themselves, but the observables.
How does this compare to the usual construction of QFT (i.e. axiomatic approach, Haag-Kastler axioms)?
As you would expect, it turns out that you can prove as a theorem, that the weak quasi-local $C^{\star}$-algebras that we associated to any object in LorFund fulfils the Haag-Kastler aximos!
This modern take on the quantisation procedure uses the modern language of category theory to put observables in the core (Haag-Kastler axioms do that, without this modern, powerful language).

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