Every particle H... ... ... super vector space

Physics	Math
States →	$\boxed{\text{Super}}$ Hilbert spaces
Observables →	$\boxed{\text{Super}}$ Algebras
Symmetries →	$\boxed{\text{Super}}$ Lie Algebras

Def: A <u>super vector space</u> V equipped with a splitting $V = V_+ \oplus V_-$

This is the same as ... of $\mathbb{Z}/2\mathbb{Z}$ on V (i.e. ... $M \in GL(V)$, $M^2 = 1$)

$Hom(V, W) = V^* \otimes W$ has a $\mathbb{Z}/2\mathbb{Z}$ ...

$= Hom(V,W)_+ \oplus Hom(V,W)_-$

..., <u>super linear maps</u> are super maps ... grading).

Remark: $Hom(V,V)$ is an algebra and in particular a super ...

$$\begin{array}{c|c} & v_+ \quad v_- \\ v_+ & \\ v_- & \end{array}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right) = \left(\begin{array}{c|c} AA' + BC' & AB' + BD' \\ \hline CA' + DC' & CB' + DD' \end{array}\right)$$

odd

even

<u>Def</u>: A <u>super algebra</u> is a super vector space $A = A_+ \oplus A_-$ with multiplication $A \otimes A \to A$, that preserves the grading i.e

$$A_{\pm} \times A_{\pm} \to A_+$$

$$A_{\mp} \times A_{-/+} \to A_-$$

Ex/ 1) Matrices, 2) Creation ops.

$$\mathrm{Sym}^*(V) = S^* V_+ \otimes \Lambda^* V_-$$

graded symmetric algebra ... symmetric algebra

2.1) $V = \mathbb{C}^2 = \underset{V_+}{\mathbb{C}} \oplus \underset{V_-}{\mathbb{C}} = \mathbb{C}^{1|1}$

$$\mathrm{Sym}^*(V) = \mathbb{C}[x, \phi] \rightsquigarrow \{\phi, \phi\} = 0$$
$$\Rightarrow \phi^2 = 0$$

$$\Rightarrow \mathrm{Sym}^*(V) = \text{poly on } x, \phi \,/\, \phi^2 = 0$$

Susy:

Every $\mathbb{1}$ particle Hilbert space should be a "super" Hilbert space.

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	Groups

Def: A super vector space is a vector space V equipped with a splitting $V = V_+ \oplus V_-$

Obs This is the same as giving a rep of $\mathbb{Z}/2\mathbb{Z}$ on V (i.e. $M \in GL(V)$, $M^2 = 1$)

"$\mathbb{Z}_2$-grading" "$\mathbb{Z}$"

Def $\mathrm{Hom}(V,W) = V^* \otimes W$ has a $\mathbb{Z}/2\mathbb{Z}$ action

$$= \mathrm{Hom}(V,W)_+ \oplus \mathrm{Hom}(V,W)_-$$

Hence, super linear maps are super maps equipped with a splitting.

Remark: $\mathrm{Hom}(V,V)$ is an algebra and in particular a super algebra [illegible]

$$
\begin{array}{c|c}
 & \begin{array}{cc} V_+ & V_- \end{array} \\
\begin{array}{c} V_+ \\ V_- \end{array} & \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)
\end{array}
\left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right) = \left(\begin{array}{c|c} AA' + BC' & AB' + BD' \\ \hline CA' + DC' & CB' + DD' \end{array}\right)
$$

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Def: A <u>super algebra</u> is a super vector space $A = A_+ \oplus A_-$ with multiplication $A \otimes A \to A$. that preserves the grading i.e

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Ex/

1) Matrices, 2) Creation ops.

$$Sym^*(V) = S^*V_+ \otimes \Lambda^* V_-$$

graded symmetric algebra

symmetric algebra

2) $V = \mathbb{C}^2 = \underset{V_+}{\mathbb{C}} \oplus \underset{V_-}{\mathbb{C}} = \mathbb{C}^{1|1}$

$$Sym^*(V) = \mathbb{C}[x, \phi] \rightsquigarrow \{\phi, \phi\} = 0$$

$$\Rightarrow \phi^2 = 0$$

$$\Rightarrow Sym^*(V) = \text{poly on } x, \phi \big/ \phi^2 = 0$$

3) Differential forms $\Omega^\bullet(\mathbb{R}^n)$ is an algebra

$\Omega^k(\mathbb{R}^n) \cong \mathbb{R}[x_+, dx_-] = \mathrm{Sym}^k(\mathbb{R}^{n|n})$

Ex 1) Does not commute or supercommute

4) $\Lambda^k \mathbb{C}^2 = \mathrm{Sym}^k(\mathbb{C}^{0|2}) \rightsquigarrow \mathbb{C}[\phi, \psi] = \mathrm{span}(1, \phi, \psi, \phi\psi)$

$\{\phi, \psi\} = 0$

2, 3) are super commutative

1) is not (in general)

Super commutator $[a, b] = ab - (-1)^{|a||b|} ba$

• $\mathbb{C}[\phi]$, $\phi^2 = 0$ $\rightsquigarrow \phi$ is odd

Given a sup matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

$\mathrm{Str}\, M = \mathrm{Str}\, A - \mathrm{Str}\, D$

why? • $\mathrm{tr}(AB) - \mathrm{tr}(BA) = \mathrm{tr}([A, B]) = 0$

Def: A super Lie algebra is an algebra with super commutator relations i.e.

• $[a, b] = (-1)^{ab} (-1) [b, a]$

• Jacobi:

$(-1)^{ca} [a, [b, c]] + (-1)^{ab} [b, [c, a]] + (-1)^{bc} [c, [a, b]] = 0$

Examples. If A is an Algebra we can define the bracket to be the commutator.

$$[A,B] \equiv AB - BA$$

Ex $su(2) \rightsquigarrow (1+\varepsilon A)v,\ \mathbb{C}\ni v,\quad u^\dagger = u^{-1}$

$$\rightsquigarrow (1+\varepsilon A)^{-1} = (1-\varepsilon A) \rightsquigarrow A^\dagger = -A\ -$$

Hence $su(2) = 2\times2$ - antihermitian, traceless matrices.

Note that the commutator of 2 hermitian is not hermitian

$$(AB-BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = BA - AB$$
$$= -(AB-BA)$$

Ex! Translations are generated by $\frac{\partial}{\partial x}$

$$f(x+\varepsilon) = f(x) + \varepsilon\frac{\partial}{\partial x} f(x) + \dots = \left(1+\varepsilon\frac{\partial}{\partial x}\right) f(x)$$

Corresponding (hermitian) observables is

$$P = -i\frac{\partial}{\partial x}$$

Traceless Antihermitian 2×2 matrices:

$$i\sigma_z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\quad i\sigma_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad i\sigma_x = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$\Rightarrow su(2)$ is a real Lie algebra.
even though we represent it using complex numbers

$$\begin{bmatrix} \cdot & \cdot \\ 0 & -i \end{bmatrix}\begin{bmatrix} \cdot & 1 \\ \cdot & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$\Rightarrow \quad [i\sigma_a, i\sigma_b] = -2\,\varepsilon_{abc}(i\sigma_c)$$

<u>Note that</u> $(i\sigma_2)^2 = -\mathbb{I} \rightarrow$ not traceless, not anti hermitian. NOT an element of $su(2)$.

<u>Recall</u>: the super commutator

$$[A,B] = AB - (-1)^{|A||B|}BA$$

lets see the Jacobi identity:

$$[a,[b,c]] = [a, bc - (-1)^{|b||c|}cb]$$

$$= abc - (-1)^{|b||c|}acb - (-1)^{|a||bc|}\left(bca - (-1)^{|b||c|}cba\right)$$

$$= abc - (-1)^{|b||c|}acb - (-1)^{|a||b||c|}bca + (-1)^{|b||c|(|a|+1)}cba$$

$$\Rightarrow J(a,b,c) = 0 \checkmark$$

$V = V_+ \oplus V_-$

$[V_+, V_+] \subseteq V_+$ (1)

$[V_+, V_-] \subseteq V_-$ (2)

$[V_-, V_-] \subseteq V_+$ (3)

- $a, b \in V_+$, $\psi \in V_-$

$$\overset{+}{[a}, \overset{+}{[b}, \overset{-}{\psi}]] + [b, [\psi, a]] + [\psi, [a, b]] = 0$$

$$\Rightarrow \quad [[a,b], \psi] = [a, [b, \psi]] - [b, [a, \psi]]$$

$$\Rightarrow \quad [[a,b], \bullet] = [a, [b, \bullet]] - [b, [a, \bullet]]$$

$\leftarrow$ V_- is a rep. of the Lie algebra of V_+!
(commutator acts as commutator rep)

In sum

$+ + + \Rightarrow V_+$ is a Lie algebra.

$+ + - \Rightarrow V_-$ is a rep. of V_+

$+ - - \Rightarrow [V_-, V_-] \to V_+$ is compatible w/ r

$- - - \Rightarrow [x, [x, x]] = 0 \quad \forall x \in V_-$.

Let's see + —. $a \in V_+$, $\psi, \phi \in V_-$ (4)

using jacobi:

$$[a, [\psi, \phi]] = [[a, \psi], \phi] + [\psi, [a, \phi]]$$

a acting on the commutator is the same as

the commutator acting on a.

$$\begin{tikzcd} V_- \otimes V_- \arrow[r, "{[a,-]\otimes 1 + 1 \otimes [a,-]}"] \arrow[d, "{[\cdot,\cdot]}"] \arrow[dr, phantom, "\#"] & V_- \otimes V_- \arrow[d, "{[\cdot,\cdot]}"] \\ V_+ \arrow[r, "{[a,-]}"] & V_+ \end{tikzcd}$$

i.e. $[\cdot,\cdot] : V_- \otimes V_- \to V_+$ is a map of V_+-rep.

Let's see — — :

$$[\psi, [\phi, \chi]] + [\phi, [\chi, \psi]] + [\chi, [\psi, \phi]] = 0$$

is the same as $x = \psi + \phi + \chi$

$\Rightarrow$ $\boxed{[x, [x, x]] = 0}$

Recall that linear trasf of a sup vector spce $\mathbb{R}^{1|1} = \left[\frac{v}{\psi}\right]$ $\Rightarrow$ $1|1 \times 1|1$ sup. matrces $\mathfrak{g}$

$$\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right], a,b,c,d \in \mathbb{R}.$$

Basis: $\alpha = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right]$, $\delta = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right]$ $\Big\}\ g_+$

$\beta = \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right]$, $\gamma = \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right]$ $\Big\}\ g_-$

Commutator relations:

$$\underbrace{\begin{aligned} [\alpha,\alpha] &= 0 \\ [\alpha,\delta] &= 0 \\ [\delta,\delta] &= 0 \end{aligned}}$$

g_+ is commutative.

$[\alpha,\beta] = \beta$
$[\delta,\beta] = -\beta$
$[\alpha,\gamma] = -\gamma$
$[\delta,\gamma] = +\gamma$

$[\beta,\beta] = 2\beta^2 = 0$
$[\gamma,\gamma] = 2\gamma^2 = 0$

$[\beta,\gamma] = \alpha + \delta$

Fourth Jacobi identity:

$$[\beta,[\cancel{\delta,\gamma}]^{0}] + [\delta,[\gamma,\beta]] + [\gamma,[\beta,\delta]] = 0$$

generators of $su(2)$. ← important in physics

We can complexify it! $su(2) \otimes_{\mathbb{R}} \mathbb{C} \cong sl($

$=$ all traceless cpx. matr

- Raising and lowering op $\in su(2) \otimes_{\mathbb{R}} \mathbb{C}$!

$$L_\pm = L_x \pm i L_y$$

- General sup. Lie algebras:

$$g = g_+ \oplus g_-, [\cdot, \cdot]$$

1. $[\mathfrak{g}_+, \mathfrak{g}_+] \to \mathfrak{g}_+$

 ($\mathfrak{g}_+$ is a (normal) bosonic Lie alg.

2. $[\mathfrak{g}_+, \mathfrak{g}_-] \to \mathfrak{g}_- \rightsquigarrow \mathfrak{g}_+ \to \mathrm{End}(\mathfrak{g}_-)$

 $\mathfrak{g}_-$ is a rep of $\mathfrak{g}_+$.

3. $[\mathfrak{g}_-, \mathfrak{g}_-] \to \mathfrak{g}_+$

 - must be compatible w/ $\mathfrak{g}_+$ symmetry
 - $[x, [x, x]] = 0 \quad \forall x \in \mathfrak{g}_-$

Reminder A rep of a Lie algebra $\mathfrak{g}_+$ on a vector space V is a map

$$\mathfrak{g}_+ \xrightarrow{\rho} \mathrm{End}(V) \quad \text{s.t.}$$

$$\rho([a,b]) = [\rho(a), \rho(b)] = \rho(a)\rho(b) - \rho(b)\rho(a)$$

- Ex. of super Lie algebras
- One ex. $\mathfrak{gl}(1|1)$

(super Lie algebra of $1|1 \times 1|1$ complex matrices w/ supercommutators)

$$\underbrace{\alpha = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right], \ \delta = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right]}_{\text{even}} \quad \underbrace{\beta = \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right], \ \varepsilon = \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right]}_{\text{odd}}$$

Commutation relations $\mathfrak{gl}(1|1,\mathbb{C}) \curvearrowright \mathbb{C}^{1|1}$

$+\ +$	$+\ -$	$-\ -$
$[\alpha,\delta]=0$	$[\alpha,\beta]=\beta$	$[\beta,\beta]=0$
$[\alpha,\alpha]=[\delta,\delta]=0$	$[\alpha,\gamma]=-\gamma$	$[\gamma,\gamma]=0$
	$[\delta,\beta]=-\beta$	$[\beta,\gamma]=\alpha+\delta$
	$[\delta,\gamma]=+\gamma$	

$\uparrow$ zero supertrace

Note: $\mathfrak{gl}(1|1)$ everything looks the same as $\mathfrak{gl}(2)$ except

$$[\beta,\gamma]=\alpha-\delta \quad \text{(in } \mathfrak{gl}(2)\text{)}$$

There's always a sub (super) Lie algebra consisting of elements of (super) trace 0.

$$\mathfrak{sl}(1|1) \subseteq \mathfrak{gl}(1|1)$$

$$\langle \beta, \gamma, \alpha+\delta \rangle$$

Commutation relations:

$$[\alpha+\delta,\beta]=[\alpha,\beta]+[\delta,\beta]=\beta-\beta=0$$

$$[\alpha+\delta,\gamma]=0$$

Recall $\mathfrak{sl}(2,\mathbb{C})$. $\mathrm{tr}(\ )=0$, $\beta,\gamma,\alpha-\delta$.

$\alpha-\delta=\begin{bmatrix}1 & \\ & -1\end{bmatrix}=h$ $\quad\rightarrow\quad$ $[\beta,\gamma]=\alpha-\delta$

$\beta=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=e$ $\quad$ $[\alpha-\delta,\beta]=2\beta$

$\gamma=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}=f$ $\quad$ $[\alpha-\delta,\gamma]=-2\gamma$

<u>Lesson</u>: $sl(1|1)$ rep can be <u>like</u> $sl(2)$ (6)

rep. except β, γ are like ladder ops.

they can be <u>a Serre element</u> because ladder ops.

are fermionic.

<u>Idea</u>. Consider the element we hope query

$$(\alpha - \delta \in sl(1|1))$$
$$\in sl(1|1)$$

$$V = \mathbb{C}^{1|1}$$

$$\operatorname{str}(\alpha - \delta) = 2$$

$$\Rightarrow \alpha - \delta = \left[\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right] : \begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} u \\ -v \end{bmatrix}$$

parity op. on $\mathbb{C}^{1|1}$.

Define $\alpha - \delta = T$. $\alpha + \delta = P$.

$$[\alpha, \beta] = \beta \qquad [T, \beta] = 2\beta$$
$$[\alpha, \gamma] = -\gamma \qquad [T, \gamma] = -2\gamma$$
$$[\delta, \beta] = -\beta \qquad [T, P] = 0$$
$$[\delta, \gamma] = \gamma \qquad [P, \beta] = [P, \gamma] = 0$$

$$[\beta, \gamma] = P, \quad P = [\,I\,] \in sl(1|1)$$

What about compact $su(1|1) \subseteq sl(1|1, \mathbb{C})$?

Note - Two notions of orthogonality.
+ Forget parity and ask for Hermitian inner product on V.

Pick a and b super structure on $\mathbb{C}^{1|1}$

$$\begin{bmatrix} u \\ \hline v \end{bmatrix} \bullet_1 \begin{bmatrix} u' \\ \hline v' \end{bmatrix} = \bar{u}u' + \bar{v}v'.$$

$$\begin{bmatrix} u \\ \hline v \end{bmatrix} \bullet_2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \pm \begin{bmatrix} c & b \\ c & a \end{bmatrix} \begin{bmatrix} u \\ \hline v \end{bmatrix} \bullet_2 \begin{bmatrix} u' \\ \hline v' \end{bmatrix}$$

Hermitian $\Rightarrow a, d \in \mathbb{R}, b = +i\bar{c}$

Anti-Hermitian $\Rightarrow a, d \in i\mathbb{R}, b = -\bar{c}$.

$$\mathfrak{su}(1|1) = \langle -i \begin{bmatrix} 1 & 0 \\ \hline 0 & 1 \end{bmatrix}, \overset{Q_1}{\begin{bmatrix} 0 & 1 \\ \hline i & 0 \end{bmatrix}}, \overset{Q_2}{\begin{bmatrix} 0 & 1 \\ \hline -i & 0 \end{bmatrix}} \rangle$$

$$[Q_1, Q_2] = 0, \quad Q_i^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

- or in more generality add "super" abstractly.

- Hermitian inner product?

$$g : V \otimes_{\mathbb{R}} V \to \mathbb{C}$$

- $\mathbb{C}$-linear in 2nd entry
- $\mathbb{C}$-conjugate linear in 1st.
- $g(v, w) = \overline{g(w, v)}$

s. pos. → replace with $(-1)^{|v||w|}$, $\overline{g(w,v)}$

Next [illegible] $\mathcal{Q}_2$ doesn't [illegible] this way. (a)

But e.g. $\left[\binom{u}{v}, \binom{u'}{v'}\right] \to \bar{u}u' + i\bar{v}v'$ does.

Absymmetry $su(1|1) = \langle P, Q_1, Q_2 \rangle$

$[P, Q_1] = [P, Q_2] = 0$

$[Q_1, Q_1] = [Q_2, Q_2] = 2P$

$[Q_1, Q_2] = 0$

$Q_1 + iQ_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}$

$= \begin{cases} 2, 0 & + \\ 2p & - \end{cases}$

Let's [illegible] of unitary rep.

I claim there are three.

(Each [illegible] distinguished by the value of P)

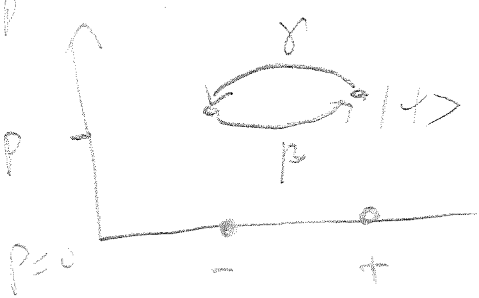
- Start with $|+\rangle$ annihilated by P.

$\gamma|-\rangle$ is zero or nonzero and has opp. parity.

$\gamma^2|+\rangle = 0$ automatically.

[illegible] $\{\gamma, \bar{\gamma}\}|+\rangle = 2p|+\rangle$

Two types of reps



nonzero momentum, one boson, one fermion

zero momentum, either bosonic or fermionic.

All reps (irreducible) are $\mathbb{C}^{1|1}, \mathbb{C}^{1|0}, \mathbb{C}^{0|1}$.

Various names:

- $su(1|1)$ - $N=2$ super QM, $Q_+^2 = Q_-^2 = P$

- $C\ell(2)/\sim$ - algebra of exterior derivatives on a Riemannian manifold.

Real form of a graded Lie algebra and their reps always require some care.

ex $\mathfrak{g} = su(2) = \langle L_x, L_y, L_z \rangle$

Suppose I want a rep. $\mathfrak{g}$ on a complex vector space $V = \mathbb{C}^n$.

Spin-j rep: $V = \mathbb{C}^{2j+1}$

Diagonalize L_z acting on V

<u>Spectrum</u>

$$-j, -j+1, \ldots, j-1, j$$

want to organize L_z action on $\mathfrak{g}$

$$[L_z, \cdot] : \mathfrak{g} \to \mathfrak{g}$$

$$[L_z, L_z] = 0, \quad L_\pm = L_x \pm i L_y$$

$$[L_z, L_\pm] = \pm L_\pm$$

<u>Problem</u> $L_\pm$ don't exist in $\mathfrak{g}$!

$\in \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C} = sl(2,\mathbb{C})$

fine if V's cpx, seems bad if not real!

<u>Ex</u> 3dim (adjoint) rep of $\mathfrak{g}$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ true rep on $\mathbb{R}^3$, or on $\mathbb{C}^3$ compatible w/ cpx conjugation

<u>eg</u> 2dim (spin $\frac{1}{2}$) rep of $\mathfrak{g}$ is the Pauli matrices. But this is <u>not</u> a real rep of $\mathfrak{g}$.

be ... due to time ... entirety.

L_x, L_y, L_z should be self adjoint.

$$(L_+)^\dagger = L_x^\dagger - i L_y^\dagger = L_-$$

su(...) ... $\underline{\simeq}$ ... the su(2), up to

some key offsets:

- ... physics conventions:

$$su(\ ) = \langle H, Q_+, Q_- \rangle$$

$$\{Q_+, Q_-\} = \{Q_-, Q_+\} = H$$

$$\{Q_+, Q_+\} = 0$$

$$[H, Q_+] = [H, Q_-] = 0$$

$[H, Q]$ ..., the ... are

the $\underline{\text{only}}$ ... symmetries.

... / ... supersymmetry algebra

... "... super ... roots" of

... ...

This algebra is sometimes called "$N=2$ supersymmetric quantum mechanics" because H has two independent square roots (Q_1 and Q_2)

$N=1$ QM would be:

$\langle H, Q \rangle$, $\{Q, Q\} = H$

$[H, Q] = 0$.

$\mathbb{Z}_2$ super Lie algebra.

$\rightsquigarrow$ generalize for N arbitrary

$\langle H, Q_1, \ldots, Q_N \rangle$

$\{Q_i, Q_j\} = \delta_{ij} \cdot H$, $[H, Q_i] = 0$

In $d = 3+1$, there are four spacetime translations, hence $N=1$ has many supercharges (in fact, four of them)

These algebras always have invariant subalgebras, namely the ordinary translations $\langle H \rangle$

Representations of the superalgebra for a continuous parameter: the energy.

$$H|\psi\rangle = E|\psi\rangle$$

$$\psi(t) \sim e^{-iEt/\hbar}$$

So only some $SU(1,1)$ rep's will come in one parameter families

Let (W, g) be a real vector space with an inner product.

$$\mathfrak{g}_0 = \mathbb{R}$$

$$\mathfrak{g}_1 = W$$

$$[\cdot,\cdot] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 = W \otimes W \to \mathbb{R} = \mathfrak{g}_0$$

is the inner product g.

$[x,[x,x]] = 0 \quad \forall x \in \mathfrak{g}_1$ trivially

$$\mathcal{N} = \mathfrak{h}(W)$$

This superalgebra has nontrivial symmetries. Namely, all rotations of W preserving the inner product.

This is a general feature of SUSY algebras, often called R-symmetry!! here it's $SO(N)$.

For $N=2$ (Q_1, Q_2), R-symmetry is $SO(2)$ rotating Q_1 into Q_2.

$$\left.\begin{array}{l} \underset{+}{H},\ \underset{+}{R},\ \underline{Q_1},\ \underline{Q_2} \\ \{Q_i, Q_j\} = \delta_{ij} H \\ [H, R] = [H, Q_i] = 0 \end{array}\right\} g = (\ 1)$$

$$\left.\begin{array}{l} [R, Q_1] = i Q_2 \\ [R, Q_2] = -i Q_1 \end{array}\right\}$$

what if I add R-symmetry to the N-extended algebra?

$$g_+ = \mathbb{R}_H \oplus \underbrace{SO(N)}_{R\text{-Symmetry}} \quad \dim(g_+) = \frac{N(N-1)}{2} + 1$$

$$g_- = N \quad , \dim(g_-) = N$$

is also a super Lie algebra ($r(1|N)$)

Representations of $\mathfrak{su}(1|1)$.

$|\psi\rangle \in V \leftarrow$ complex super vector space

Ladder $Q_1 \pm i Q_2 = Q_\pm \in \mathfrak{sl}(1|1)$

$$\mathfrak{sl}(1|1) = \mathfrak{su}(1|1) \otimes \mathbb{C}$$

- $(Q_-)^\dagger = Q_+$
- $Q_+ |\psi\rangle = 0$

Can try to lower $|\psi\rangle$

$\Rightarrow$ $Q_- |\psi\rangle$ is either zero or nonzero.

$Q_-^2 |\psi\rangle$ is always zero!

$$Q_+ Q_- |\psi\rangle = (H - Q_- Q_+) |\psi\rangle$$

$$= H |\psi\rangle$$

$\leadsto$ $Q_- |\psi\rangle$ is non zero

① The rep is two-dim, spanned by

$|\psi\rangle, Q_- |\psi\rangle$.

$\nwarrow$ odd symmetry

$H |\psi\rangle = \lambda |\psi\rangle$, $\lambda > 0$. $\begin{cases} \text{one boson} \\ \text{one fermion} \end{cases}$

① The rep. is one dimensional (1)

$$Q_-|\psi\rangle = Q_+|\psi\rangle = H|\psi\rangle = 0$$

(trivial rep)

$|\psi\rangle$ could be bosonic <u>or</u> fermionic

$$\langle\psi|H|\psi\rangle = \langle\psi|\{Q_-, Q_+\}|\psi\rangle$$

$$= \langle\psi|Q_+Q_-|\psi\rangle + \langle\psi|Q_-Q_+|\psi\rangle$$

$$= \|Q_-|\psi\rangle\|^2 + \|Q_+|\psi\rangle\|^2$$

$$\geq 0, \quad = 0 \text{ iff } Q_-|\psi\rangle = Q_+|\psi\rangle = 0$$

<u>Can</u> $|\psi\rangle$ is homogeneous

$$[Q, F] = -Q$$

$$|\psi\rangle = |\psi_+\rangle + |\psi_-\rangle$$

$$\hookrightarrow F|\psi\rangle = |\psi_+\rangle - |\psi_-\rangle$$

$$\hookrightarrow Q|\psi_+\rangle = Q|\psi_-\rangle = 0$$

<u>homogeneous</u> ✓

Some notes on $N=2$ Supersymmetry

- The Hilbert space is a Fermi (mixing ... ?)

$L^2(\mathbb{R}) \ni \psi(x)$

$$H = -\frac{\partial^2}{\partial x^2} - p^2.$$

To allow for supersymmetry on the [illegible] space [illegible] about

$$H = L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1} \ni \begin{bmatrix} \phi(x) \\ \phi(x) \end{bmatrix}$$

$$H = -\frac{\partial^2}{\partial x^2} \otimes \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right]$$

Need to factor H into the anticommutator of two odd op.

On $\mathbb{C}^{1|1}$, we have

$$\beta = \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right], \quad \gamma = \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right]$$

$$\{\beta, \gamma\} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right]. \quad \gamma^+ = \beta.$$

For $\{A \otimes B, A' \otimes B'\} = (A \otimes B)(A' \otimes B') + (A' \otimes B')(A \otimes B)$

$$Q = \hat{p} \otimes \gamma \rightsquigarrow Q^+ = \hat{p} \otimes \beta$$

hence $$\{Q, Q^+\} = \frac{1}{2}\Big(\{\hat{p}, \hat{p}\} \otimes [\gamma, \beta] + [\hat{p}, \hat{p}] \otimes [\gamma, \beta]\Big)$$

$= \hat{p}^2 \otimes \{\gamma, p\} = \hat{p} \otimes \mathrm{id}$

$\hookrightarrow \quad Q = \left[\begin{array}{c|c} 0 & 0 \\ \hline -i\frac{\partial}{\partial x} & 0 \end{array}\right], \quad Q \begin{bmatrix} \varphi(x) \\ \hline 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \hline -i\frac{\partial \varphi}{\partial x} \end{bmatrix}$

$Q^{+} = \left[\begin{array}{c|c} 0 & -i\frac{\partial}{\partial x} \\ \hline 0 & 0 \end{array}\right]$

"A free ~~Pl~~ $N=2$ supersymmetry particle"

In fact, this works for a free particle in d dimensions.

Hilbert space?

$L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1} \ni \begin{bmatrix} \varphi(x) \\ \hline \phi(x) \end{bmatrix}$

realise as $\quad\varphi(x) + \phi(x)\,dx \in \Omega^{\bullet}(\mathbb{R})$

$\left.\begin{array}{l} \delta_Q(\phi(x)) = \dfrac{\partial \varphi}{\partial x}\,dx \\ \delta_Q \varphi = 0 \end{array}\right\}$ Q is the de Rham d operator on $\Omega^{\bullet}(\mathbb{R})$

What is Q^+? It's the adjoint to the de Rham differential with respect to the pairing

$$\langle \alpha, \beta \rangle = \int_0 \bar{\alpha} \wedge * \beta.$$

↳ This agrees with the def of the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$.

i.e. $Q^+ = d^+$, $Q = d$ w.r.t. inner product on de Rham forms.

where d is the exterior derivative.

$d = dx^i \frac{\partial}{\partial x^i}$ | $f(x)\,dx \to \frac{\partial f}{\partial x}\,dx$

$$Q^+ \sim f(x)\,dx \xrightarrow{Q^+} \frac{\partial g}{\partial x}.$$

Interestingly, the construction works for any smooth manifold M!

$$\mathcal{H} = \Omega^\bullet(M)_{L^2} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

$$\Omega^{\text{even}}(M) \oplus \Omega^{\text{odd}}(M)$$

$$\langle \alpha, \beta \rangle = \int_M \bar{\alpha} \wedge * \beta$$

↳ $H = \Delta$ ← Laplace opp. on diff forms.

Intuition: The Spectrum of the Hamiltonian
does not depend on the metric only, but
the BRST part

- Can I think more interactions w/o breaking supersymmetry?
- $(\mathcal{H}, H)$: some Quantum theory.
  Is it the quantization of a classical theory? Is there a Lagrangian/action principle?

On flat space, going up
$$L^2(\mathbb{R}^2) = L^2(\mathbb{R}_x) \otimes L^2(\mathbb{R}_y)$$
$$L^2(\mathbb{R}_x) \otimes \mathbb{C}[dx] \otimes L^2(\mathbb{R}_y) \otimes \mathbb{C}[dy]$$
$\Rightarrow$ Supersymmetric QM
$$L^2(\mathbb{R}^2) \otimes \mathbb{C}[dx, dy]$$
$$\mathbb{R}^{2|2}_{C}$$

[illegible] There's a particularly nice formulation of interactions [illegible] on:
$$\{Q, Q\} = \{Q^{\dagger}, Q^{\dagger}\} = 0 \quad \&$$
$$\{Q, Q^{\dagger}\} = 2H$$

I - s f... to take Q and replace it

by $e^{\lambda} d$, $\lambda \in \mathbb{C} \Rightarrow$

$$Q^{+} = Q^{+} e^{\lambda}$$ but it remains unchanged (up to scale)

What is the analog in functions on M?

$$Q \longrightarrow e^{-\lambda w} d e^{\lambda w}, \quad \lambda \in \mathbb{R}$$

$$Q^{+} \longrightarrow e^{\lambda w} d^{+} e^{-\lambda w}$$

$$Q^{2} = \underbrace{e^{-\lambda w} d^{2} e^{\lambda w}}_{} = 0$$

This ... a rep of $N=2$ algebra on $\Omega^{*}(M)$. But with a different Hamiltonian.

$\Rightarrow$ Susy ... $M = \mathbb{R}$

$$Q_{\lambda}(f + g\,dx) = e^{-\lambda w} d\, f e^{\lambda w} + \cancel{g e^{\lambda w} dx}$$

$$= e^{-\lambda w}\left(\frac{\partial f}{\partial x} e^{\lambda w} + \lambda w' f e^{\lambda w} \right)$$

$$= dx\left(\frac{\partial f}{\partial x} + \lambda w' f \right)$$

$$\Rightarrow Q_{\lambda}(f + g\,dx) = df + \lambda\, dw\, f$$

$$\rightsquigarrow Q_\lambda = \begin{bmatrix} 0 & 0 \\ \frac{\partial}{\partial x} + \lambda\omega' & 0 \end{bmatrix} = \underbrace{\left(\frac{\partial}{\partial x} + \lambda\omega'\right)}_{P} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$Q_\lambda^\dagger = \begin{bmatrix} 0 & -\frac{\partial}{\partial x} + \lambda\omega' \\ 0 & 0 \end{bmatrix} = \left(-\frac{\partial}{\partial x} + \lambda\omega'\right) \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\langle \alpha, Q\beta \rangle = \langle Q^\dagger \alpha, \beta \rangle$

$\beta \in \Omega^{p-1}$ $\quad \Rightarrow Q^\dagger : \Omega^p \to \Omega^{p-1}$

$Q\beta \in \Omega^p$

$\Rightarrow \alpha \in \Omega^p$

$$H_\lambda = \frac{1}{2}\{Q_\lambda, Q_\lambda^\dagger\} = \left\{\frac{\partial}{\partial x} + \lambda\omega', -\frac{\partial}{\partial x} + \lambda\omega'\right\} \otimes \mathbb{1}$$

$$- \left[\frac{\partial}{\partial x} + \lambda\omega', -\frac{\partial}{\partial x} + \lambda\omega'\right] \otimes \sigma_z$$

commuts $\left[\frac{\partial}{\partial x}, x\right]$? , $\frac{\partial}{\partial x}(x\psi(x)) = \psi + x\psi'$

$\frac{\partial}{\partial x}$ $\quad x\psi' - x\psi' - \psi = \psi \Leftrightarrow \boxed{\left[\frac{\partial}{\partial x}, x\right] = 1}$

Since

$$H_\lambda = -\frac{\partial^2}{\partial x^2} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda^2 \omega'^2 \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda\omega'' \sigma_z$$

ω is called the "super potential"

($q_i = \theta_j$ & k Kähler potential)

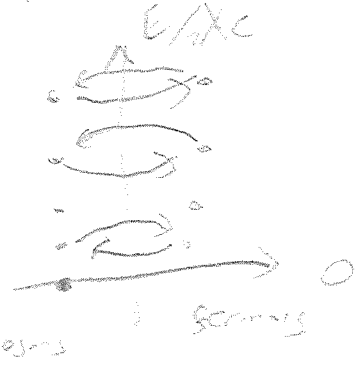
$$\begin{cases} U_{bos} = \lambda^2 \omega'^2 - \lambda \omega'' \\ U_{fer} = \lambda^2 \omega'^2 + \lambda \omega'' \end{cases}$$

How do we get a harmonic oscillator?

$$\omega = \pm c x^2/2 \Rightarrow \omega' = \pm c x, \quad \omega'' = \pm c$$

$$\begin{cases} U_{bos} = \lambda^2 c^2 x^2 - \lambda c \\ U_{fer} = \lambda^2 c^2 x^2 + \lambda c \end{cases}$$

Spectrum of SHO ($\hbar = 1$)



$$-\frac{\partial^2}{\partial x^2} + \lambda^2 c^2 x^2$$

is $(2n+1)\lambda c$

- I can always say for [illegible] them up $(+)/(-)$ ground state
- vacuum energy cancelled by [illegible] [illegible] [illegible]!

$$\begin{bmatrix} 0 & 0 \\ \partial - \lambda w' & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ f' - \lambda w' f \end{bmatrix}$$

$$Q^\dagger \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} -g' - \lambda w' g \\ 0 \end{bmatrix}$$

Bosonic: $f' = -\lambda w' f \Rightarrow f = e^{-\lambda w}$

$g' = \lambda w' g \Rightarrow g = e^{\lambda w}$

If we pick $w = -\frac{cx^2}{2} \Rightarrow f = e^{-\lambda c \frac{x^2}{2}}$

(Fermionic ground state would be

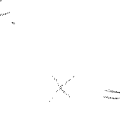
$g = e^{+\lambda c x^2/2}$, which is not in L^2)

Hence, if $w = -\frac{cx^2}{2}$, there is no fermionic ground state.

$w \to +\infty$ at large $x \Rightarrow$ ground state bosonic

$w \to -\infty$ at large $x \Rightarrow$ ground state fermionic

If M is a circle



$\Rightarrow$ w has a max and a min

1 local [illegible] minima $\Rightarrow$ at least (exactly) [illegible] vacuum.

$(Q^\dagger \alpha, \beta) = (\alpha, Q\beta)$

$= \int \bar{\alpha} \wedge \cancel{*} \beta = \pm \int \cancel{*} \bar{\alpha} \wedge d\beta$

$= \pm \int d * \alpha \wedge \beta = \pm \int \overset{Q^\dagger}{*d} * \bar{\alpha} \wedge \beta$

$= (Q^\dagger \alpha, \beta)$

But we can do the same inverse!

- Take any system whose ground state wavefunction is known.
- Shift H to place its energy at 0.

$\rightarrow -\frac{1}{\lambda} \log \psi(x) = W(x)$

the SQM whose bosonic part IS the system I started with.

Ex/ • SHO, $H = \frac{1}{2} p^2 + \frac{1}{2} x^2$, $E_n = (n + \frac{1}{2})$

$\rightsquigarrow \psi_0 = e^{-\frac{1}{2} x^2} \quad w(x) = -\frac{1}{\lambda} \log \psi_0 = \frac{1}{2} x^2$

$H_{bos} = \frac{1}{2} (w')^2 - \frac{1}{2} w'' = \underbrace{\frac{1}{2}(p^2 + x^2) + \frac{1}{2}}_{H} - \frac{1}{2}$

$H_{bos} = \frac{1}{2}(p^2 + x^2) - \frac{1}{2}$

$H_{ferm} = \frac{1}{2}(w)^2 - \frac{1}{2}w' = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \frac{1}{2}$

- Infinite square well

$\psi(x) = \begin{cases} \sin(\pi x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

$H_{bos} = \frac{p^2}{2} - \pi^2$, $\quad w = -\log \psi_0 = -\log \sin(\pi x)$

$w = -\pi \cot(\pi x)$, $\quad w' = \pi^2 \csc^2(\pi x)$

$\hookrightarrow$

$H_{bos} = \frac{1}{2}w^2 - \frac{1}{2}w' = \frac{1}{2}p^2 - \pi^2$

$H_{ferm} = \pi^2\left(\frac{1 + \cos^2 \pi x}{\sin^2 \pi x}\right) + \frac{1}{2}p^2$.

Canonical quantization:

- $S = \int dt\, L \rightarrow p_x = \frac{\partial L}{\partial \dot{x}} \rightarrow H = p_x \dot{x} - L$

- Poisson bracket on obs (functions on phase space)

  $\rightarrow$ simplectic form on TM

Turn a $\{\cdot,\cdot\}$ into a "quantum commutator" by

$\omega(X_\xi, -) = d\xi$ i.e.

$X_\xi = \omega^{b^{-1}} d\xi$

ex $\xi = p$ ← observable ⇝ $\omega(X_\xi, -) = dp$ (with $\omega = dp \wedge dx$)

$\Rightarrow X_\xi = \frac{\partial}{\partial x}$. $H = \frac{p^2}{2}$, $dH = p\,dp$.

$\rightsquigarrow \omega(X_H, -) = p\,dp \rightsquigarrow X_H = p\frac{\partial}{\partial x}$

Now, for fermions (free) $S = \int dt\, \theta\dot\theta$

$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = -\theta \rightsquigarrow \omega = dp \wedge d\theta = \underbrace{-d\theta \wedge d\theta}_{\text{non zero} \atop \text{cus symmetric}}$

• obs $\xi = \theta \rightsquigarrow$

$d\theta = \omega(X_\theta, -) \rightsquigarrow X_\theta = -\frac{\partial}{\partial \theta}$

Algebra of observables

$T(\Pi\mathbb{R}) \big/ (\{\theta, \theta\} = 1) = \mathbb{C}[\theta] \big/ \theta^2 = 1$

(and so for any fermionic phase space.

$T(\Pi V) \big/ (u \otimes v + v \otimes u - \omega(u, v))$

(for bosons $T(V) \big/ u \otimes v - v \otimes u = \omega(u, v)$)

However, in QM we have 1 rep, the Weyl algebra (17)

How the analogue to quantization for fermions?
($T\,R_\theta$ [illegible])

Remark: The algebra of (fermionic) obs. is called the Clifford algebra!!

$$Cl(V) = T(V) \big/ \langle u \otimes w + w \otimes u - 2g(u,w) \rangle$$

Given $V = \mathbb{R}^{p+q}$ w/ the inner product st it have signature (p,q) b/c not every fermionic degree of freedom is the same, even classically!

Ex/ $Cl(p,q) = Cl(V = \mathbb{R}^{p+q})$ $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{bmatrix}$ q

- $Cl(0,2) = \mathbb{R}(i,j) \big/ \left\langle \begin{matrix} i^2 = j^2 = -1 \\ ij = -ji \end{matrix} \right\rangle = \mathbb{H}$
- $Cl(0,0) = \mathbb{R}$
- $Cl(0,1) = T\mathbb{R} \big/ \langle e^2 = -1 \rangle = \mathbb{C}$

$Cl(1,0) = \mathbb{R}(e) \big/ \langle e^2 - 1 \rangle \cong$

$$\frac{1}{\sqrt{2}}(1\pm e)^2 = \underbrace{\frac{1}{\sqrt{2}}(1\pm e)}_{P_{+/-}} \rightsquigarrow P_{\pm} P_{\mp} = 0 \quad \checkmark$$

orthogonal projectors.

and, $v \in T(\mathbb{R}) / \langle e^2 = 1 \rangle \to v = a_+ P_+ + a_- P_-$

$w = b_+ P_+ + b_- P_-$

$$\Rightarrow \quad vw = a_+ b_+ P_+ + a_- b_- P_-$$

$$\Rightarrow \quad Cl(1,0) = \mathbb{R} \oplus \mathbb{R}$$

- $Cl(2,0) = \mathbb{R}\langle e, f \rangle \Big/ \begin{pmatrix} e^2 = 1 \\ f^2 = 1 \\ ef = -fe \end{pmatrix}$

<u>Generators:</u> $e^2 = 1$, $f^2 = 1$, $(ef)^2 = (-e)^2 (f^2)$

$= -1$

$$\Rightarrow \quad \underset{\mathbb{1}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \quad \underset{e}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}, \quad \underset{f}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \quad \underset{ef}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$

$$\Rightarrow \quad Cl(2,0) = M_2(\mathbb{R})$$

• $Cl(1,1) = \mathbb{R}\langle e,f\rangle \Big/ \left\langle \begin{matrix} e^2 = 1 = f^2 \\ ef = -fe \end{matrix} \right\rangle$ 18

$\rightsquigarrow (ef)^2 = 1 \quad \rightsquigarrow M_2(\mathbb{R})$

Classification of Clifford algebras

(A) • $Cl(k,0) \otimes Cl(0,2) = Cl(0,k+2)$

(B) • $Cl(0,k) \otimes Cl(2,0) = Cl(k+2,0)$

(C) • $\underbrace{Cl(p,q)}_{e_1 \dots e_k} \otimes \underbrace{Cl(1,1)}_{f_1, f_2} = \underbrace{Cl(p+1,q+1)}_{g_1 \dots g_k, g_{k+1}, g_{k+2}}$

Proof Define

$$g_i \to e_i \otimes f_1 f_2 \qquad 1 \le i \le k$$
$$g_{k+1} \to 1 \otimes f_1$$
$$g_{k+2} \to 1 \otimes f_2$$

Hence

$$g_i^2 = e^2 \otimes (f_1 f_2)^2 = e_i^2 \otimes -f_1^2 f_2^2$$
$$= \begin{cases} -e_i^2 \otimes 1 & \text{(A)} \\ e_i^2 \otimes 1 & \text{(C)} \end{cases}$$

we have that?

k	$Cl(k,0)$	$Cl(0,k)$	$Cl(k,\mathbb{C})$
0	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{C}$	$\mathbb{C}\otimes\mathbb{C} = \mathbb{C}\oplus\mathbb{C}$
2	$M_2(\mathbb{R})$	$\mathbb{H}$	$M_2(\mathbb{C})$
3	$M_2(\mathbb{C})$	$\mathbb{R}\oplus\mathbb{R}\otimes\mathbb{H}$ $\mathbb{H}\oplus\mathbb{H}$	$M_2(\mathbb{C})\oplus M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$
5	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$	$M_4(\mathbb{C})$	
6	$M_4(\mathbb{H})$ $M_4(\mathbb{H})$	$M_8(\mathbb{H})$	
7	$M_8(\mathbb{C})$	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	
8			

8 periodic

2 periodic

a) $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} = \mathbb{C}\oplus\mathbb{C}$

b) $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H} = M_2(\mathbb{C})$

c) $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H} = M_4(\mathbb{R})$

$Cl(1,\mathbb{C}) = \mathbb{C}[e]/e^2-1$ → any rep $\mathbb{Z}$ → eigenvalues ± 1 (19)

$$e \longrightarrow \begin{bmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

⇒ some copies of [illegible] rep where $e \to +$ and some other number where $e \to -$

Projections $\frac{1}{2}(1 \pm e)$

$Cl(2,\mathbb{C}) = \mathbb{C}\langle e_1, e_2 \rangle \Big/ \left\langle \begin{matrix} e_1^2 = e_2^2 \\ \{e_1, e_2\} = 0 \end{matrix} \right\rangle$

try $x \mapsto x$, $p \mapsto \frac{\partial}{\partial x}$

Look [illegible] alg.

$(a e_1 + b e_2)^2 = a^2 + b^2 \to a = 1, b = \pm i$

Observe

$e_\pm = \frac{1}{2}(e_1 \pm i e_2) \to \{e_+, e_-\} = \mathbb{1}$

$e_\pm^2 = 0$ ⇝ Ideal: try to act on functions on $\mathbb{C}$ $\quad \mathbb{C}[\epsilon]/\langle \epsilon^2 \rangle$

⇒ $e_+ \to \epsilon$

$e_- \to \frac{\partial}{\partial \epsilon}$

In general modules for $Cl(n, \mathbb{C})$

$(T\mathbb{C}^n, g)$ pick a polarization, a subspace $L \subseteq \mathbb{C}^n$

- $g(l_1, l_2)$, $l_1, l_2 \in L$ vanishing.
- maximal ("complete")

$\mathcal{H} = \mathcal{O}(TL)$

- If n is even, $\mathbb{C}^n = L \oplus L^\vee$

↳ $e_\pm^{[illegible]} = e_{even} \pm e_{odd}$

If n is odd, one vector basis is left over, choose the parity $\pm$.

In this construction, we know how vectors act on the Clifford module: $V \hookrightarrow Cl(V)$

$\sum v^i e_i \mapsto \sum v^i e^i \rightarrow \sum v_i e_i + v^i \frac{\partial}{\partial e}$

How does this have to act with spinors? $V \times S \xrightarrow{F} S$

- The Spin group acts on every Clifford module!
- In fact, Spin $\hookrightarrow Cl(V)$

How to construct it?

① The Clifford algebra is a superalgebra.

- has a parity op. π

[$Cl(V) = \Sigma$ ... super ... the complex conjugation]

② $Cl(p, q+1)^+ \cong Cl(p,q)$

$e_i \rightarrow$ ...

③ ... modules

$1 \otimes \otimes V_2 \rightarrow V_2 \otimes V_1$

④ ... group of ... elements ... Cl

$Cl(V)^\times$

⑤ $Cl(V)^\times$ acts on $Cl(V)$ by the ...

$y: x \rightarrow \pi(y) x y^{-1}$

(19)

$Cl(1,\mathbb{C}) = \mathbb{C}[e]/e^2-1$ in any rep e has eigenvalues ± 1

$$e \longrightarrow \begin{bmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \Rightarrow$$ some subspace of irrep where $e \to +1$ and some other subspace where $e \to -1$

Projections $\frac{1}{2}(1 \pm e)$

$Cl(2,\mathbb{C}) = \mathbb{C}\langle e_1, e_2 \rangle \Big/ \left\langle \begin{matrix} e_1^2 = e_2^2 \\ e_1 e_2 + e_2 e_1 = 0 \end{matrix} \right\rangle$

try $x \mapsto \lambda$, $p \mapsto \frac{\partial}{\partial x}$

Look for ... alg.

$(a e_1 + b e_2)^2 = a^2 + b^2 \to a = 1,\ b = \pm i$

Define $e_\pm = \frac{1}{2}(e_1 \pm i e_2) \to \{e_+, e_-\} = \mathbb{I}$

$e_\pm^2 = 0 \rightsquigarrow$ Idea: try to ... on $\mathbb{C}[e_+]/\langle e_+^2 \rangle$

$\Rightarrow e_+ \mapsto e_+$

$e_- \mapsto \frac{\partial}{\partial e_+}$

In general ... $Cl(n,\mathbb{C})$

$(T\mathbb{C}^n, g)$ ... a ... polarization ... a Lagrangian $L \subset \mathbb{C}^n$

$- g(\ell_1, \ell_2)$, $\ell_1, \ell_2 \in L$ ...

- maximal ("compact")

$\mathcal{X} = \wedge(TL)$. If n is even, $\mathbb{C}^n = L \oplus L^\vee$

$\hookrightarrow e^{\dots}_{\pm} = e_{2i-1} \pm e_{2i-2}$

If n is odd, one vector basis is left over, choose the parity ...

In this construction, we know how vectors act on the Clifford module: $V \hookrightarrow Cl(V)$

$\sum c_i e_i \mapsto \sum c_i e_i \to \sum x_i \frac{\partial}{\partial e_i} + \ldots$

What does this have to do with spinors?

$V \times S \xrightarrow{f} S$

- The Spin group acts on every Clifford module!
- In fact, $Spin(V) \hookrightarrow Cl(V)$

How to construct it?

① The Clifford algebra is a superalgebra.

- it has a parity op. π.

[$Cl(V) = \ldots$]

② $Cl(p, q-1)^{+} \cong Cl(p, q)$

$e_i \to f_i \ldots \quad e_i^2 \to (f_i \ldots)^2 = f_i^2$

③ The ... module

$V_1 \otimes V_2 \to V_2 \otimes V_1$

④ Forms a group of ... $Cl(V)$

$Cl(V)^{\times}$

⑤ $Cl(V)^{\times}$ acts on $Cl(V)$ by the formula

$y : x \to \pi(y) x y^{-1}$

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⑥ We can consider the subgroup $\Gamma \subseteq Cl(V)^\times$ that preserves $V \subseteq Cl(V)$

It naturally comes with a map $\rho: \Gamma \to GL(V)$

⑦ There's a "norm map" $N: Cl(V) \to Cl(V)$

N maps Γ to $\mathbb{R}^\times$. $\quad \gamma \Rightarrow \ldots$

Proof $\quad \gamma \in \Gamma, \ x, x' \in V$

$\pi(\gamma)\, x\, \gamma^{-1} = x' = x'^t = (\gamma^{-1})^t \ldots (\pi(\gamma))^t$

$\Rightarrow \gamma^t \pi(\gamma)\, x = x\, \pi(\gamma)^t \gamma$

$\Rightarrow \pi(\gamma)^t \gamma = N(\gamma) \in \ker(\rho)$

Claim: $\ker(\rho)$ = nonzero constants.

Suppose $\gamma = a + b e_1 \ldots \in \ker(\rho) \to \pi(\gamma) x = x \gamma$

$\Rightarrow (a - b e_1 \ldots) x = x (a + b e_1 \ldots) \Rightarrow b = 0.$

⑧ $Spin(V) = \ker(N) \subseteq \Gamma$. ...

$\rho: Spin(V) \to SO(p)$

$Cl(2,0)^+ = \langle 1, e_1 e_2 \rangle \simeq Cl(0,1) \simeq \mathbb{C}$

$a + b e_1 e_2$.

$N(a + b e_1 e_2) = (a - b e_2 e_1)(a + b e_1 e_2)$

$= a^2 + b^2$

$\Gamma = \cos\theta + \sin\theta\, e_1 e_2$

$V \hookrightarrow Cl(0,2)$

$(\cos\theta - \sin\theta\, e_1 e_2)(v_1 e_1 + v_2 e_2)(\cos\theta + \sin\theta\, e_1 e_2)$

$= (v_1 \cos(2\theta) - v_2 \sin(2\theta))\, e_1 + \cdots$

Example: – construct the spin group

$Spin(V) \subseteq Cl(V)^+$

set of elements that preserve $V \subseteq Cl(V)$

under the "twisted adjoint rep."

and have unit norm

– the elements in $Spin(V)$ are those

that can be written as products of (even number of)

unit vectors in V.

In the adjoint rep, $y \in Cl(V)^\times : x \mapsto \alpha(y) x y^{-1}$

$y \in V$ a unit vector.

$\Rightarrow \quad y^2 = -1 \quad y^{-1} = -y$

$x \in V$ so $x = \alpha y + \beta z$, $z \perp y$.

$\Rightarrow \quad x \mapsto (-y)(\alpha y + \beta z)(-y) = -\alpha y + \beta z$

In the example $Spin(0,2) \subseteq Cl(0,2)^+ \cong Cl(0,1)$

$\cong \mathbb{C}$

Ex: how $\rho : Cl^\times(V,Q) \to GL(Cl(V,Q))$ acts?

If $x, y \in V \Rightarrow \rho(x) y = \pi(x) y x^{-1}$

Since $x = e_i \Rightarrow \pi(x) = -x$

$\Rightarrow$

$\rho(x) y = -(xy) x^{-1} = -(-yx + 2\langle x,y\rangle) x^{-1}$

$= y - 2\langle x,y\rangle x^{-1}$

$x^2 = +\langle x,x\rangle 1 \Rightarrow \left(+\frac{x}{\langle x,x\rangle}\right)\cdot x = 1$

$\overset{!!}{=} x^{-1} \Rightarrow$

$$\boxed{\rho(x) y = y - \frac{2\langle x,y\rangle}{\langle x,x\rangle} x}$$

"Reflection along hyperplane orthogonal to x"

$\rho(y) y = y - \frac{2\langle y,y\rangle}{\langle y,y\rangle} y$

$= -y$

$\cos\theta + \sin\theta\, e_1 e_2$ $\qquad Cl(0,2)^+ \subseteq Cl(0,2)$

$(\alpha e_1 + \beta e_2)$. ~~is a unit vector~~ $\qquad Spin(0,2) \xrightarrow{2:1} SO(2)$

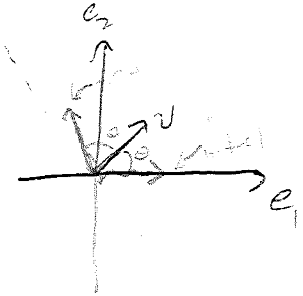
$(\cos\phi\, e_1 + \sin\phi\, e_2)$ $\qquad e^{i\theta} \to e^{i2\theta}$

Is a unit vector; Take the product of two

such $\rightsquigarrow (\cos\phi\, e_1 + \sin\phi\, e_2)(e_1 \cos\phi' + e_2 \sin\phi')$

$= \cos\phi\cos\phi'\, e_1^2 + \sin\phi\sin\phi'\, e_2^2 + (\sin\phi\cos\phi' - \cos\phi\sin\phi')\, e_1 e_2$

$= -\cos(\phi-\phi') - \sin(\phi-\phi')\, e_1 e_2$



Recall

k	$Cl(0,k)$	$Cl(k,\mathbb{C})$
1	$\mathbb{C}$	$\mathbb{C}\oplus\mathbb{C}$
2	$\mathbb{H}$	$M_2(\mathbb{C})$
3	$\mathbb{H}\oplus\mathbb{H}$	$M_2(\mathbb{C})\oplus M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$
5	$M_4(\mathbb{C})$	$M_4(\mathbb{C})\oplus M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$
7	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	$M_8(\mathbb{C})\oplus M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$

For spinors in Euclidean signature

$Spin(0,d) \subseteq Cl(0,d)^+ = Cl(0,d-1)$

$Pin(0,1) = \mathbb{Z}/4 \qquad Spin(V) \xrightarrow{2:1} SO(V)$

$Pin(1,0) = \mathbb{Z}/2 \times \mathbb{Z}/2 \qquad Pin(V) \xrightarrow{2:1} O(V)$

$Spin(0,1) = Spin(1,0) = \mathbb{Z}/2$

"Convention $-+++$ mostly plus"

but we that $N(v) = -g(v,v)$

Lorentzian Signature:

$\mathrm{spin}(1, d-1) \subseteq Cl(1, d-1)^+ \cong Cl(1, d-2)$

$\cong M_2(Cl(0, d-3))$

k	Euclidean d		Lorentzian	$\dim_{\mathbb{C}}$
1	2	4	4	2=1+1
2	3	4	5	2
3	4	8	6	4=2+2
4	5	8	7	4
5	6	16	8	8=4+4
6	7	16	9	8
7	8	32	10	16=8+8
8	9	32	11	16

Last time we constructed the Dirac spinors by choosing a polarization $L \subseteq V_{\mathbb{C}}$ ("complete set of commuting observables")

$$\dim L = \left\lfloor \frac{\dim V}{2} \right\rfloor$$

$$S = \Lambda^{\bullet}(L) \qquad \dim S = 2^{\left\lfloor \frac{\dim V}{2} \right\rfloor}$$

If $\dim V$ = even, splits as $\Lambda^{\mathrm{even}}(L) \oplus \Lambda^{\mathrm{odd}}(L)$
(as reps of

Working over the complex numbers, we need two ingredients to get to "gamma matrices"
- we have Clifford mult. $V \times S \to S \rightsquigarrow S^\vee \times S \to V' \cong V$

– want to arrive at a map $S \times S \to V$
to use to construct supersymmetry algebras

How is $S^\vee \cong S$? $S = \Lambda^\bullet(L) \Rightarrow S^\vee \cong \Lambda^\bullet(L^\vee)$

$= \mathbb{C}[\theta]$ $\qquad \mathbb{C}\left[\frac{\partial}{\partial\theta}\right]$

Need to act w/ lowering op. on $\Lambda^{top}(L)$!

$\Rightarrow$ A pairing on S. $(\sigma, \tau) \mapsto (\sigma \wedge \tau)_{top}$

e.g. $(\theta_1, \partial_{\theta_1}) \mapsto (\theta_1 \wedge \partial_{\theta_1})_{top} = \theta_1 \wedge \theta_2$ assume $\theta_1\theta_2$ case

Clifford algebras:

$$Cl(V) = T(\Pi V) \Big/ \left(u\otimes v + v\otimes u - 2g(u,v)\right) \qquad (V, g)$$

$Cl(p,q)$:

~~$Cl(1,1)$~~ $\begin{pmatrix} 1_{p} & \\ & -1_{q} \end{pmatrix} \to$

$Cl(0,2) \rightsquigarrow Cl(0,2) = Cl(\mathbb{R}^{0|2})$

$e_1, e_2 \rightsquigarrow \quad e_1^2 = -1$

$e_2^2 = -1$

$e_1 e_2 + e_2 e_1 = 0 \rightsquigarrow e_1 e_2 = -e_2 e_1$

$i = e_1$, $j = e_2$, $(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1^2 e_2^2 = -1$

$e_1 \not\perp e_2 \Rightarrow k = e_1 e_2, \; k^2 = -1$

$\Rightarrow$

$Cl(0,2) = \left\langle \alpha + \beta j + \gamma i + \delta k \;\middle|\; i^2 = j^2 = k^2 \right\rangle = \mathbb{H}$

$Cl(0,0) = \mathbb{R}$, $\quad Cl(0,1) = \mathbb{C}$, $e_1^2 = -1$ $\quad Cl(1,0)$: $e_1^2 = 1 \rightsquigarrow e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$Cl(1,0) = \mathbb{R}(e)/\langle e^2 = 1 \rangle = \langle \alpha \mathbb{1} + e \beta \rangle$$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightsquigarrow \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \langle \alpha\beta \quad \langle \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

$$\Rightarrow \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \rightsquigarrow \text{better basis} = \mathbb{R} \oplus \mathbb{R}$$

Or $e^2 = 1 \rightsquigarrow (1 \pm e)^2 = (1 \pm e)(1 \pm e)$

$$= 1 \pm e \pm e + e^2$$

$$= 2(1 \pm e)$$

$$\rightsquigarrow p = \frac{(1 \pm e)}{2} \qquad p^2 = \frac{1}{4} 2(1 \pm e) = \frac{1}{2}(1 \pm e) =$$

$$\frac{(1 \pm e)}{2} \frac{(1 \mp e)}{2} = \frac{(1 - e^2)}{4} = 0 \checkmark \text{ orthogonal proj.}$$

$$\rightsquigarrow \quad v \in \mathbb{R}(e) \rightsquigarrow v = p_+ v + p_- v$$

$$= v_+ e_+ + v_- e_-$$

$$Cl(2,0) = \langle \mathbb{R}(e_1, e_2) / e_1^2 = 1, \; e_2^2 = 1, \; e_1 e_2 = -e_2 e_1 \rangle$$

$$M_2(\mathbb{R}) \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_1 e_2 = e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e_1 e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$Cl(1,1) = \mathbb{R}(e,f)$$

$e_4^2 = 1$

$f^2 = -1$

$ef = -fe$

$(ef)^2 = ef\,ef = -e^2f^2 = +1 \to$

$$\mathbb{R}(e,f) = \langle \mathbb{1},\ \underset{e^2=1}{e},\ \underset{(ef)^2=1}{ef},\ \underset{f^2=-1}{f}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$Cl(1,1) \cong M_2(\mathbb{R})$$

$$Cl(p,q+2) = Cl(q,p) \otimes Cl(0,2)$$

$$Cl(p+2,q) = Cl(q,p) \otimes Cl(2,0)$$

$e_1 \ldots, e_q$ $\quad f_1 \ldots f_p$ $\quad e_{q+1}, e_{q+2}$

$(e_i)^2 = +1$ $\quad (f_i)^2 = -1$ $\quad (e_{q+1})^2 = 1$

$\to\ e'_i = f_i \otimes e_{q+1} e_{q+2}$

$\to (e'_i)^2 = f_i^2 \otimes (e_{q+1} e_{q+2})^2 = -1 \otimes -e_{q+1}^2 e_{q+2}^2$

$e'_{p+i} = 1 \otimes e_{q+i}$ ✓ $\quad = \mathbb{1}$

Table

k	$Cl(k,0)$	$Cl(0,k)$	$Cl(k,\mathbb{C})$
0	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{C}$	$\mathbb{C}\oplus\mathbb{C}$
2	$M_2(\mathbb{R})$	$\mathbb{H}$	
3	$M_2(\mathbb{C})$	$\mathbb{H}\oplus\mathbb{H}$	
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$	
5	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$	$M_4(\mathbb{C})$	
6	$M_4(\mathbb{H})$	$M_8(\mathbb{R})$	
7	$M_8(\mathbb{C})$	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$ ✓	
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$	
9	$M_{16}(\mathbb{R})+M_{16}(\mathbb{R})$	$M_{2}(\mathbb{C})$	

$$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H} = \mathbb{C}(\alpha+\beta\hat{i}+\gamma\hat{j}+\delta\hat{k})$$

$$= \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \sim M_2(\mathbb{C})$$

$$\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H} = \begin{pmatrix}\mathbb{H}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}(\ ) & (\ )\\ (\ ) & (\ )\end{pmatrix} \in M_4($$

General techniques: $Cl(n,\mathbb{C}) = \mathbb{C}^n$ $e_i^2 = 1$ $\{e_i, e_j\} = 0$

$$e_\pm^{(k)} = e_{2k+1} \pm i e_{2k+2}$$

projectors ~> gets us $\lfloor n/2 \rfloor$ elements.

e.g. e_1, e_2, e_3, e_4 ~> $\begin{cases} e_\pm^{(0)} = e_1 \pm i e_2 \\ e_\pm^{(1)} = e_3 \pm i e_4 \end{cases}$

"$e_\pm^{(k)}$" are projectors into the polarization

~> $L \subseteq \mathbb{C}^n$ where $L = \langle e_{+}^{(k)}, \ldots \rangle$

$L^\vee = \langle e_\pm^{(k)\vee}, \ldots \rangle$

$$\Downarrow$$

$$\frac{\partial}{\partial e_\pm^{(k)}}$$

~> $\mathbb{C}^n = L \oplus L^\vee$

If n is odd ~> $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$

~> $\mathbb{C}^n = L \oplus L^\vee$ + projector? ??

we have then $\forall v \in V$ ~>

$$v = \sum_{k=0}^{\lfloor n/2 \rfloor} v_k^+ e_+^{(k)} + v_k^- e_-^{(k)} \quad \leftarrow \text{fine} \checkmark$$

this exp then gives us

$$V \longrightarrow Cl(V)$$

$$v \longrightarrow \sum v_k^+ e_+^{(k)} + v_k^- \frac{\partial}{\partial e_+^k} \quad \leftarrow \text{this one you know}$$

$Cl(p,q) = Cl(p,q)_0 + Cl(p,q)_1$

$\uparrow$

$v = i\, e_1 e_2 \cdots e_i\, f_1 \cdots f_j$

$Cl(p,q) \ni \prod_i^{l} e_i \prod_j^{2m} f_j$

$Cl(p,q+1)^{+} \cong Cl(p,q)$

$\boxed{e_1 \cdots e_p}\, f_1, \cdots f_{q+1}$ $\qquad \boxed{e_1 \cdots e_p}\, f_1 \cdots f_q$

$\rightsquigarrow \quad \underset{\text{even}}{e_i} \to \underset{\text{even}}{f_i f_{p+q+1}} \quad \rightsquigarrow \quad e_i^2 = (f_i f_{p+q+1})(f_i f_{p \cdots}$

$= -f_i^2 f_{p+q+1}^2 = f_i^2$

$= -1$

$f_j = \underbrace{f_i f_j}_{\downarrow}\ \underbrace{f_l f_m} \cdots$ l copies at most

Clifford and Spinors?

Spin(V) act on $Cl(V)$ as $Cl(V) \hookleftarrow Spin(V)$

Construction: Ingredients:

① Clifford algebra is a super algebra

$\rightsquigarrow$ $Cl(V) = Cl(V)^+ \oplus Cl^-(V)$

$\exists$ $\Pi \in$ parity. op.

e.g.

$Cl(0,\mathbb{C}) : \mathbb{C} \cong \langle 1, i \rangle$

$\Pi(1) = +1$ $\longrightarrow$ $1^2 = 1$

$\Pi(i) = -i$ $\qquad$ $i^2 = -1$

Π is complex conjugation.

② $Cl(V)^+ \cong Cl(p, q+1)^+ \simeq Cl(p,q)$

$(f_i f_{p+q+1} \longleftarrow e_i)$

③ There's a "transpose"

$v_1 \otimes v_2 \cdots \otimes v_r \xrightarrow{t} v_r \otimes \cdots \otimes v_1$

Let $Cl(V)^\times$ = invertible $Cl(V)^\times Cl(V)$

$Cl(V)^\times \curvearrowright Cl(V)$ by $y : x \longrightarrow \Pi(y)\, x\, y^{-1}$

~~1~~ twisted adjoint $^{\vee}$

(6) Consider $\Gamma \subseteq Cl(V)^{\times}$ s.t.

$y: x \to \pi(y)\, x\, y^{-1}$, $\forall y \in \Gamma$ preserves

$V \subseteq Cl(V)$ i.e. it provides a ~~map~~ rep.

$\rho: \Gamma \to GL(V)$ well defined.

(ρ = twisted adjoint action, $\rho_y(x) = \pi(y)\, x\, y^{-1} \in V$

(7)

there's a map $N: Cl(V) \to Cl(V)$

$$y \to \pi(y^t)\, y$$

that maps Γ to $\mathbb{R}^{\times}$

(8) Define $Spin(V) = Ker(N) \subseteq \Gamma$ "elements with norm 1"

i.e $\rho: Spin(V) \to SO(V)$

spin acts on V! (as rotations)

<u>EX:</u> $Cl(0,2)$. $\rightsquigarrow$ $Cl(0,2)^+ \cong Cl(0,1) = \mathbb{C}$

$ \; a+b \qquad\qquad a + b e_1 e_2$

$N(a+be_1e_2) = N(w) = \pi(a+be_1e_2)^t (a+be_1e_2)$

$= (a+be_2e_1)(a+be_1e_2) = (a^2 - b(e_1e_2)^2$

i.e.

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightsquigarrow \begin{pmatrix} a\cos(2\theta) - b\sin(2\theta) \\ a\cos(2\theta) + b\sin(2\theta) \end{pmatrix}$$

Spin acts as

$$\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a\cos 2\theta - b\sin 2\theta \\ \end{pmatrix}$$

$\boxed{\text{rotations twice as fast!}}$

i.e. ~~$\theta = \alpha$~~ ~~α ...~~ $\alpha \in [0, 2\pi)$

and $\alpha \pm \pi$ have the same ~~function~~

(double cover!)

Ex: $\mathrm{Spin}(0,d) \subseteq C\ell(0,d)^+ \simeq C\ell(0,d-1)$

$\mathrm{Spin}(V) \xrightarrow{2:1} SO(V)$

$\mathrm{Pin}(V) \xrightarrow{2:1} O(V)$

$\mathrm{Pin}(0,1) = \mathbb{Z}/4 \qquad \mathrm{Spin}(0,1) = \mathbb{Z}/2$

$\mathrm{Pin}(1,0) = \mathbb{Z}/2 \times \mathbb{Z}/2 \qquad \mathrm{Spin}(1,0) = \mathbb{Z}/2$

$$\ker(N) = \{\underbrace{a^2+b^2=1}_{a+be_1e_2}\} \subseteq Cl(V)$$

$$\mathrm{spin}(V) = \{\cos\theta + \sin\theta\, e_1e_2 \mid \theta \in [0, 2\pi)\} \subseteq \Gamma_?$$

$\rho : \mathrm{spin}(V) \to Cl(V)$ image in $SO(V)$

$$\rho_\theta(x) = \pi(y_\theta)\, x\, y_\theta^{-1} = (\cos\theta + \sin\theta\, e_1e_2)\, x\, (\cos\theta + \sin\theta\, e_1e_2)^{-1}$$

$$(\cos\theta - \sin\theta\, e_1e_2)(\cos\theta + \sin\theta\, e_1e_2)$$
$$= \cos^2\theta - \sin^2\theta\,(e_1e_2)^2 = 1$$

$$\rho_\theta(x) = (\cos\theta + \sin\theta\, e_1e_2)\, \underset{\uparrow}{x}\, (\cos\theta - \sin\theta\, e_1e_2)$$

$$(ae_1 + be_2)$$

$$= (\cos\theta + \sin\theta\, e_1e_2)(a\cos\theta\, e_1 + b\cos\theta\, e_2 + \sin\theta\, a\, e_2 - \sin\theta\, b\, e_1)$$

$$= (a\cos^2\theta\, e_1 + b\cos^2\theta\, e_2 + \sin\theta\cos\theta\, a\, e_2$$
$$- \sin\theta\cos\theta\, b e_1 + a\sin\theta\cos\theta\, e_2$$
$$- \sin\theta\cos\theta\, b e_1 - a\sin^2\theta\, e_1 + \sin^2\theta\, b\, e_?$$

$$= a\cos 2\theta\, e_1 - b\sin(2\theta)\, e_1 + a\sin(2\theta)\, e_2 + \cos 2\theta \ldots$$
$$= (a\cos 2\theta - b\sin 2\theta)\, e_1 + (a\sin 2\theta + b\cos 2\theta)\, e_2$$

$Spin(V) \cong \mathbb{R} \oplus \mathbb{R} \leftarrow$ two $\underline{\text{real}}$ spin reps.

$\cancel{e_\pm(t \pm x)}$ $\quad L = \langle e_+ \rangle$, $\mathbb{C} = L \oplus L^\vee$

$e_\pm = e_1 \pm e_2 = t \pm x \rightsquigarrow S = \Lambda^\bullet(L^\vee)$

$= \Lambda^0(L^\vee) \oplus \Lambda^1(L^\vee)$ ← two spin reps.

Let $s_+ \in \Lambda^0(L^\vee)$ (even), $s_- \in \Lambda^1(L^\vee)$ (odd)

$$(e_\pm)^2 = (t \pm x)^2 = (t \pm x)(t \pm x)$$
$$= t^2 \pm \cancel{tx} \pm \cancel{xt} + x^2$$
$$= -1 + 1 = 0$$

Elements in $\Lambda^0(L^\vee) \rightsquigarrow s_+ = f$

$\Lambda^1(L^\vee) \rightsquigarrow s_- = g \frac{\partial}{\partial e_+}$

$e_+ s_+ = (t+x) f$

$e_+ s_- = (t-x) g \frac{\partial}{\partial e_+}$

$\langle \Gamma(s_+, s_+), e_+ \rangle = (s_+, e_+ \cdot s_+)$

$= (s_+, s_-) =$

Def of Γ matrices!

pairing on S: $(\sigma,\tau) \to (\sigma\wedge\tau)_{top}$

e.g. $(e_1, \frac{\partial}{\partial e_1})$ → clifford action! $V \curvearrowright S$ (spinor)

$$g(\Gamma(s,t), v) = (s, v\cdot t)$$

$s, t \in S$, $v \in V$; (V,g); $\Gamma : S\times S \to V$

Start with $V \rightsquigarrow$ basis: $\{e_i\}_{i=1}^n$

"assume n is even"

$\rightsquigarrow$ Define $e_{\pm}^{(i)} = (e_{2i+1} \pm i e_{2i+2})/2, \quad i = 0, \dots, \frac{n-2}{2}$

there are $\left(\frac{n}{2}\right)$ $e_{\pm}^{(i)}$'s. ~~$2(i+1) = 2\left(\frac{n-2}{2}+1\right) = n$~~

~~$\rightsquigarrow \mathbb{C}^n \neq e^i$~~ Let $L = \langle e_+^i \rangle_{i=1}^n$

$\rightsquigarrow L^\vee = e_+^{\vee(i)} \rightsquigarrow \mathbb{C}^n = L \oplus L^\vee$

$$S = \Lambda^\bullet V) \leftarrow \dim S = \sum_{i=0}^{n/2} \binom{n/2}{i} = 2^{\lfloor n/2 \rfloor}$$

$S = \Lambda^{even}(V) + \Lambda^{odd}$ the map $S\times S \xrightarrow{\Gamma} V$

Ex Dim 2 Lorentz $\left(V, g = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}\right)$

$\rightsquigarrow$ basis $t, x, \quad t^2 = +1, \; x^2 = -1.$

$Cl(V) = Cl(1,1) = M_2(Cl(0,0)) = M_2\, C\mathbb{R}$

graded spin $\cong Cl \quad \mathbb{R}\oplus\mathbb{R} \Longrightarrow$

[illegible] space [illegible] space $\mathcal{H}$ [illegible] $\mathcal{H}$-space [illegible] (up to phase) [illegible] $\Rightarrow$ actual states $\mathcal{H}/\mathbb{C}^*$

Suppose the symmetry group G acts on $\mathcal{H}$

via $\rho : G \longrightarrow GL(\mathcal{H})$ [illegible] on $\mathcal{H}/\mathbb{C}^*$ $\hat{\rho} : G \to GL(\mathcal{H})/\mathbb{C}^*$

[illegible] Q: consider proj rep [illegible] groups!

[illegible]: a [illegible] extension of a group G by a [illegible] group N

$$1 \longrightarrow N \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

↑ sequence is exact

i.e. $\hat{G}/N \cong G$, $\hat{G}$ is a cover of G.

Suppose $\hat{\rho} : \hat{G} \to GL(V)$ is [illegible] irrep, N acts via $\hat{\rho}$ [illegible] of the identity (Schur's lemma)

$\rho : G \to GL(V)/\mathbb{C}^*$

$$g \longmapsto (v \to [\hat{\rho}(g)](v))$$

[illegible] projective rep. [illegible] $\hat{g} \in gN$ and

$[\cdot] : GL(V) \to GL(V)/\mathbb{C}^*$

[illegible] rep of covering groups yield proj. rep [illegible] underlying group!!

a)

$\tilde{I}\tilde{J} = 0 \quad \tilde{J}_1 = \tilde{J}_2 = 0 \longrightarrow \tilde{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\tilde{I} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

pauli matrices

$u + v\tilde{J} \Rightarrow \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$

c) $Cl(n)^{\times} = \mathbb{H} \setminus \{0\}, \quad N: \Gamma \longrightarrow \mathbb{R}$

$\Gamma \cong \mathbb{H}$

$(a + bI + cJ + dIJ) \to (a - bI - cJ + dIJ)$

$\times (a + bI + cJ + dIJ)$

$= a^2 + b^2 + c^2 + d^2$

$Cl(0,2) \cong \mathbb{H} \cong \left\{ \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in M_2(\mathbb{C}) \right\}$

$Pin(0,2) = \{ v \in \Gamma \mid N(v) = 1 \}$

$= \{ v \in \mathbb{H} \mid \|v\| = 1 \} = \left\{ \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; |u|^2 + |v|^2 = 1 \right\}$

$= |u|^2 + |v|^2$

$\det M = 1 \quad = SU(2)$

$Cl(0,2) \xrightarrow[f]{\cong} Cl(0,3)^{+}$

$f(I) = e_1 e_3$

$f(J) = e_2 e_3$

$f(K) = e_1 e_3 e_2 e_3 = +e_1 e_2$

$, \; e_i \in \mathbb{R}^3$

$\rho : Cl(0,3)^{+} \times \mathbb{R}^3 \to \mathbb{R}^3$

$(a + b e_1 e_3 + c e_2 e_3 + d e_1 e_2) , \; x^i e_i)$

$\to \frac{1}{a^2 + b^2 + c^2 + d^2} (a + bI + cJ + dK)(x^i e_i)(a - Ib - cJ - dK)$

Ke

compare with $I \; I \; I$ to see if a cardan map ?

$I \quad J$

$\mathbb{C}^2 \cong \mathbb{R} \times \mathbb{C} \to \operatorname{Im} \mathbb{H} \to M_2(\mathbb{C})$

$(x, y, z) \to (xi, y + zi) \to xI + yJ + zIJ$

$\to \begin{pmatrix} xi & y + zi \\ -y + zi & -xi \end{pmatrix}$

$e_1 \to I$

$e_2 \to J$

$e_3 \to K$

$\tilde{\rho} : \underbrace{Cl(0,2)}_{\cong \mathbb{H}} \times \operatorname{Im} \mathbb{H} \to \operatorname{Im} \mathbb{H}$

$(q, \; xI + yJ + zK) \to \frac{1}{|q|} q(xI + yJ + zK)\bar{q}$

d) compose $\tilde{\rho}$ with f and g from (a) and use $\mathbb{R}^3 \cong \operatorname{Im} \mathbb{H}$

from c)

$\Rightarrow \tilde{\rho}' : SU(2) \times \mathbb{R}^3 \to \mathbb{R}^3$

$(A, x) \to A x A^{-1} = A x A^{+}$

e) Show $H \otimes_{\mathbb{R}} H \cong \mathrm{End}(\mathbb{R}^4)$

$$H \otimes_{\mathbb{R}} H \cong \left\{ \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in M_2(\mathbb{C}) \right\}^{\otimes_{\mathbb{R}} 2}$$

$$\cong (\mathbb{R}\, U(2))^{\otimes_{\mathbb{R}} 2}$$

basis $= \{ e_0 = 1, e_i = \sigma_i \mid i \in \{1,2,3\} \}$

$\Rightarrow$ basis $\tilde{B} := \{ e_i \otimes e_j \mid e_i, e_j \in B \}$

$\Rightarrow H \otimes_{\mathbb{R}} H$ has dim $|B| \times |B| = 16 = \# \tilde{B}$

as a real vector space.

f) $Cl(1,3)^{+} \cong Cl(\ ,2)$

generators e, f_1, f_2 of $Cl(\ ,2)$

$$(e f_1 f_2)^2 = e f_1 f_2 e f_1 f_2 = e^2 f_1 f_2 f_1 f_2 = -f_1 f_1 f_2 f_2 = -1$$

$e f_1 f_2 e = e^2 f_1 f_2$

$e f_1 f_2 f_1 = f_1 e f_1 f_2$ $\Rightarrow$ $\sigma: Cl(1,2) \to M_2(\mathbb{C})$

$e f_1 f_2 f_2 = f_2 e f_1 f_2$

$1 \longrightarrow 1$

$e \to \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$f_1 \to i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$f_2 \to i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$V: \Gamma \to \mathbb{R}$

$Cl(\ ,2) \cong Cl(1,3)^{+}$

$(a - be - cf_1 + df_2) \to (a - be - cf_1 - df_2)(a + be + cf_1 + df_2)$

$= a^2 - b^2 + c^2 + d^2$

h) $SL(2,\mathbb{C}) \times \mathbb{R}^{1,3} \to \mathbb{R}^{1,3}$

$$(A, x) \to A\varphi(x)A^{+} = AXA^{+} = X'$$

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix}$$

via $\varphi: \mathbb{R}^{1,3} \simeq \mathrm{Herm}(2,\mathbb{C})$

Show that $\psi: SL(2,\mathbb{C}) \to SO^{+}(1,3)$ is <u>surjective</u> with $\ker(\pm 1)$

$$\eta(x',x') = \det(X') = \det(AXA^{+}) = \det(X) = \eta(x,x)$$

$\Rightarrow \psi(SL(2,\mathbb{C})) \subset SO(1,3)$

now $SO(1,3)$ has two connected components

but $1 \in SO(1,3)^{+}$, $\dim(SL(2,\mathbb{C})) = \dim(SO^{+}(1,3))$

$\Rightarrow \psi$ is surjective

kernel $AXA^{+} \overset{!}{=} X \to \ker(\psi) = \{1, -1\}$

HW 8 SUSY:

- $\rho : Cl(V)^\times \times Cl(V) \longrightarrow Cl(V)$

  $(x, y) \longrightarrow \pi(y) x y^{-1}$

- $\Gamma \subseteq Cl(V)^\times$ fixing $V \subset Cl(V)$ i.e.

  $\Gamma = \{ y \in Cl(V) \mid \rho(y, x) = x \ \forall x \in V \}$

  $\Rightarrow \rho : \Gamma \longrightarrow GL(V)$ is a rep. of Γ

- The norm map $N : \Gamma \longrightarrow \mathbb{R}^\times \cong Z_{Cl(V)} \cdot \mathbb{R}^\times$

  $y \longrightarrow \pi(y) y$

- $Spin(V) = \{ x \in \Gamma \mid N(x) = 1 \}$

  $\hookrightarrow \rho : Spin(V) \longrightarrow \underbrace{GL(V)}_{SO(V)} \subset Cl(V)$

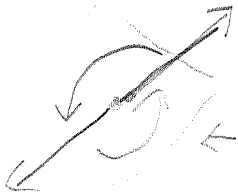
  2:1 map with kernel ± 1.

- $Spin(V)$ = elements of $Cl(V)$ such that

  $x \in Spin(V) \Rightarrow x = v_1 v_2 \cdots v_k$, $N(v_i) = \pm 1$, $v_i \in V$.

Hence, in this rep.

Each unit vector is sent to its reflection through the hyperplane orthogonal to it



c) ... $\mathrm{Spin}(3) \subset Cl(0,3)^{+} \cong Cl(0,2) = \mathbb{H}$

Identify $\mathbb{H}$ as $\mathbb{C}^2$

$\varphi(a+bI+cJ+dK) = (a+bi) + (c+di)J =: u + vJ$

$\varphi : \mathbb{H} \longrightarrow \mathbb{C}^2$

for I, $b=1$, $a=c=d=0$

$\hookrightarrow \varphi(I) = i + 0J = i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\varphi(J) = 0 + J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\varphi(K) = 0 + iJ \Rightarrow -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\varphi(J) = J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Leftarrow J^2 = -1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\to \varphi(a+bI-cJ+dK)$

$$= \begin{pmatrix} a+bi & ci-d \\ ci+d & a-bi \end{pmatrix}$$

$$\begin{pmatrix} a-bi & ci+d \\ ci-d & a+bi \end{pmatrix}$$

$$N(a+bI-cJ+dK) = \pi\begin{pmatrix} a+bi & ci-d \\ ci+d & a-bi \end{pmatrix}$$

$$= (a+bi)(a-bi) - (ci+d)(ci-d)$$

$(a-bi)(ci-d)+(ci+d)(a+bi)$

$$= (a^2+b^2-(-c^2-d^2)$$

$+(ci\,a - cb + da + db)$

$\varphi^{\pi}(a+bI+cJ-dK) = \varphi(a-bI-cJ-\ldots$

$\to a^2-b^2-c^2-d^2$

c) $\hat{i} \leftarrow I \to e_1 e_3$, $J \to e_2 e_3$

$K = IJ = e_1 e_3 e_2 e_3 \to e_1 e_2$

$\varphi(a - [illegible]) = a - b e_1 e_3 + c e_2 e_3 + d e_1 e_2$

$[illegible] = a - b e_1 e_3 - c e_2 e_3 - d e_1 e_2$

$\to [illegible])(\alpha + \beta e_1 e_3 + \gamma e_2 e_3 + \delta e_1 e_2)\,\varphi(u)$

$= (a - b e_1 e_3 - c e_2 e_3 - d e_1 e_2)(\alpha - [illegible])$

$= \pi\varphi(u)\big(\alpha a + \alpha b e_1 e_3 - \alpha c e_2 e_3 + \alpha d e_1 e_2$
$+ \beta a e_1 e_3 - \beta b - \beta c e_1 e_2 - d\beta e_2 e_3$
$+ \gamma a e_2 e_3 - \gamma b e_1 e_2 - \gamma c - \gamma c e_1 e_3$
$+ \delta a e_1 e_2 - \delta b e_2 e_3 - \delta c e_1 e_3 - \delta d\big)$

$= \pi\varphi(u)\big((\alpha a - \beta b - \gamma c - \delta d) + (\alpha b + \beta a - \gamma d - \delta c) e_1 e_3$
$+ (\alpha c - d\beta + \gamma a - \delta b) e_2 e_3$
$+ (\alpha d - \beta c - \gamma b + \delta a) e_1 e_2$

⋮

[illegible] y^{-1})

$$y^{-1} = \frac{(a - bi - cj - dk)}{a^2+b^2+c^2+d^2}$$

$$y\,y^{-1} = \frac{(a - bi - cj - dk)(a + bi + cj + dk)}{a^2+b^2+c^2+d^2}$$

$$= \frac{1}{a^2+b^2+c^2+d^2}\left(a^2+b^2+c^2+d^2 + abi - bai + \ldots\right) = 1$$

d)