Black holes

~~Chandrasekhar limit~~ — Static & Stationary.

- Chandrasekhar limit

- Killing vectors

- Tetrads ⇝ spherical symm metric

$(ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2)$

? (ADM) mass

— orbits of test particles (schwarzschild)

↳ null coordinates

↳ no causal curve leaves r < 2m

BH!

Critical coordinates

↳ Horizons

↳ white hole?

—

Study graph page (12b).

↓ Penrose conformal

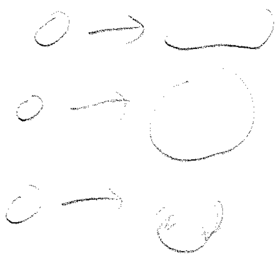
Study graph page (3)

— Causal future/past $J^{\pm}(\mathcal{U})$ — event horizon

Chronology future/past

- Killing horizon
  - ↳ surface gravity
    - ↳ Schwarzschild.
- Reissner-Nordström sol (charged)
- Kerr-Newman sol (Rotating)
  - ↳ Naked singularity
  - ↳ Ergo sphere
- no hair theorem
- Horizon area's
- Penrose process.
- Numerical relativity. { Post newtonian / Quasi-normal modes
- Global techniques.
- Penrose theorem (2nd law)
  - Proof:
-
  - ↳ Horizon naked singularities?
- Geometric optics = "Null energy condition"

- Focusing eq. | meaning of



— Unruh effect

↳ algebraic things

↳ Bogoliubov transf.

— observers.

— positive frequency solutions.

Instantons — perturbation theory motivation QFT

— Instantons in Yang–Mills theory

Chandrasekhar limit

$$E_{tot} =$$

Grav ~ $\frac{M}{r}$

$$E_{Grav} = (\pm) \int^{?} V \, d^3 r = \int$$

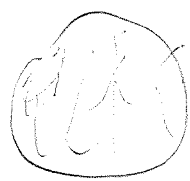
$$E = -\int F \cdot dr = -\int \frac{M}{r^2} m \, r^2 dr$$

$$E = -\frac{M}{R} = +\frac{M}{r} \Big|_R^{\infty}$$

$$E = \sqrt{c^2 p^2 + m^2 c^4}$$

$$= \int \frac{M}{r^2} dr = \int F \cdot dr$$

$$E_{tot} = -\frac{M^2}{R} + K_{part}$$



$=$ # atoms x Energy per atom

$= \frac{M}{m_p}$

Relativistic $E = mc^2$

Relativistic $E = p^2$

$$E_{tot} = \int F_{grav} \cdot dr + \frac{1}{2} m v^2$$

$$= -\frac{M^2}{R} + \frac{\langle p \rangle^2}{2M}$$

the gravit
the R
the

~~$\langle p \rangle = \hbar n$~~

~~$\langle p \rangle$~~

$\langle p \rangle^3 \langle x \rangle^3 \geq \hbar^3$

$\Rightarrow \frac{\langle p \rangle^3}{n} \geq \hbar^3$

$\Rightarrow n = \frac{\langle p \rangle^3}{\hbar^3}$

n = # particles density

Chandrasekhar limit

How much ~~can~~ mass can a star (white dwarfs? (approx. sol. mass)) have ~~to~~ so that it doesn't collapse? (composed by electrons)

take $E = E_{Grav} + E_{kinetic}$

Tolman Oppenheimer-Volkoff limit

Analogous but for "neutron stars".

Killing Fields — Generators of Isometrics

$\hookrightarrow$ $g_{\alpha\beta}(x) = g_{\alpha\beta}(\phi_t(x))$ (1)

where ϕ_t is the flow generated by $K \leftarrow$ Killing vector field $\left(\cdot \frac{d\phi_t^\mu(x)}{dt} = K^\mu \quad \cdot \phi_0^\mu(x) = x^\mu \right)$

$\downarrow$

Condition (1) translates to

$$\lim_{t\to 0} \frac{g_{\alpha\beta}(x) - g_{\alpha\beta}(\phi_t(x))}{t} = \boxed{\mathcal{L}_K g = 0}$$

or, in coordinates $\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0$

$$\nabla_\alpha K_\beta + \nabla_\beta K_\alpha \overset{||}{=} 2\nabla_{(\alpha} K_{\beta)} = 0$$

$g_{\alpha\beta}(x) = g_{\alpha\beta}(\phi_t(x))$

$= \tilde{g}_{\alpha\beta}(\tilde{x}) = \frac{\partial \tilde{x}}{\partial x}\frac{\partial \tilde{x}}{\partial x} \tilde{g}_{\alpha\beta}(x)$ ($\tilde{g}_{\alpha\beta}(x) \overset{\shortparallel}{=} g_{\alpha\beta}(x)$)

$\tilde{x} = \phi_t(x) = x^\mu + K^\mu t + O(t^2)$

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choose local coordinates: $K^\mu e_\mu = K^\varphi \partial_\varphi$

$\Rightarrow$ ∂_φ is a Killing vector $\Leftrightarrow$ provided $\partial_\varphi g_{\mu\nu}$.

- Commutator of two KVF's is Killing

$$\mathcal{L}_{[K_1, K_2]} g = [\mathcal{L}_{K_1} \overset{0}{\overset{\shortparallel}{g}}, \mathcal{L}_{K_2} \overset{0}{\overset{\shortparallel}{g}}] = 0.$$

<u>Killing applications:</u>

- we want a metric that is <u>static</u> symmetrys in time (time translations) $\longrightarrow$ Killing associated is $\frac{\partial}{\partial t} = \boxed{\partial_t}$
- spherically symmetric $\leadsto$ Rotations of $\mathbb{R}^3$ leave inv. $\leadsto$ $so(3)$ reps tell us that generators are $[K_i, K_j] = \epsilon_{ijk} K_k$.

$\hookrightarrow$ $\boxed{\partial_t, \underset{L_z}{\overset{\shortparallel}{\partial_\varphi}}, \partial_\pm} = \partial_\theta + i\cot\theta\, \partial_\varphi$

$\nwarrow$ Killings $\Rightarrow$ metric doesn't depend on θ, φ, t $\leadsto$ parameterized by r + some conditions

$$ds^2 = -B^2(r)dt^2 + A^2(r)dr^2 + C(r)drdt + r^2 d\Omega^2$$

$$\boxed{ds^2 = -e^{2\phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2}$$

Tetrads: How to compute the Christoffel, Riemann, Ricci, Einstein fast?

- ∇_μ ← can be viewed as a 1-form

→ Pick an orthonormal basis of e_μ.

→ $\boxed{\nabla_\mu = \omega_\mu{}^\lambda e_\lambda}$ ~~#~~, where $\boxed{\omega^\lambda_\mu = \Gamma^\lambda_{\mu\nu} \omega^\nu}$

ω^ν dual basis of e_λ

i.e. $\omega^\nu(e_\lambda)$.

Hence,

$$\nabla_\mu e_\sigma = \Gamma^\lambda_{\mu\nu} \omega^\nu(e_\sigma) e_\lambda$$

$$= \Gamma^\lambda_{\mu\nu} \delta^\nu_\sigma e_\lambda = \Gamma^\lambda_{\mu\sigma} e_\lambda \checkmark$$

And $\nabla^2 V = \dots = V^\nu e_\mu \underbrace{\left(d\omega^\mu{}_\nu + \omega^\mu{}_\rho \omega^\rho{}_\nu \right)}_{R^\mu{}_\nu}$

→ $\boxed{R^\mu{}_\nu = d\omega^\mu{}_\nu + \omega \wedge \omega}$

Curvature 2-form.

why? use

$$\langle ds, u \wedge v \rangle = \nabla_u \langle s, v \rangle - \nabla_v \langle s, u \rangle - \langle s, [u, v] \rangle$$

$\rightsquigarrow$ take $s = d\omega$

$\Rightarrow$

$$\langle d^2\omega, u \wedge v\rangle = \nabla_u \nabla_v \omega - \nabla_v \nabla_u \omega$$

$$- \nabla_{[u,v]}\omega = R(u,v)\omega$$

$$= \langle R, u \wedge v\rangle \omega$$

$$\rightsquigarrow \boxed{R^\mu{}_\nu = R^\mu{}_{\nu\rho\sigma}\, \omega^\rho \wedge \omega^\sigma}$$

How to find ~~use~~ g from ω (or ~~other~~ the other way around)

Consider $\underline{I}$ = identity tensor = $e_\sigma \omega^\sigma$

$\Rightarrow \langle d\underline{I}, u \wedge v\rangle = R(u,v)\underline{I} \overset{\text{torsion free}}{=} 0$

and on the other hand,

$$d\underline{I} = d(e_\mu \omega^\mu) = \dots = e_\mu(\omega^\mu{}_\nu \omega^\nu + d\omega^\mu)$$

$$= 0$$

$$\Rightarrow \boxed{d\omega^\mu + \omega^\mu{}_\nu \wedge \omega^\nu = 0}$$

and $dg_{\mu\nu} = g(de_\mu, e_\nu) + g(e_\mu, de_\nu) = \dots = \omega_{\mu\nu} + \omega_{\nu\mu}$

$$\boxed{dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}}$$

Philosophy:

Compute the curvature

- $\Gamma^{\mu}_{\nu\theta} \to \omega^{\mu}{}_{\nu} \leftarrow$ 1 form
- $R^{\mu}{}_{\nu\rho\sigma} \to R^{\mu}{}_{\nu} \leftarrow$ 2 form
- $e_{\mu} \to \omega^{\mu} \leftarrow$ dual of e_{μ}.

- $R^{\mu}{}_{\nu} = R^{\mu}{}_{\nu\rho\sigma}\, \omega^{\rho} \wedge \omega^{\sigma}$
- $d\omega^{\mu} \cancel{+} \omega^{\mu}{}_{\nu} \wedge \omega^{\nu} = 0$
- $dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$
- $R^{\mu}{}_{\nu} = d\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\rho}\, \omega^{\rho}{}_{\nu}$

Ex: $ds^2 = -e^{2\psi(r)} dt^2 + e^{2\Lambda} dr^2 - r^2 d\Omega^2$

$r^2 d\theta^2 + r^2 \cos^2\theta\, d\varphi^2$

ON: basis:

$$\begin{cases} \omega^t = e^{\psi(r)} dt \\ \omega^r = e^{\Lambda} dr \\ \omega^\theta = r\, d\theta \\ \omega^\varphi = r\cos\theta\, d\varphi \end{cases} \to$$

find $\omega^{\mu}{}_{\nu}$ with the help of $\boxed{d\omega^{\mu} + \omega^{\mu}{}_{\nu} \wedge \omega^{\nu} = 0}$

$\rightsquigarrow$ Compute $R^{\mu}{}_{\nu} \leftarrow \boxed{R^{\mu}{}_{\nu} = d\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\rho}\, \omega^{\rho}{}_{\nu}}$

Identify $R^{\alpha}{}_{\beta\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma}\, \omega^{\rho} \wedge \omega^{\sigma}$.

~~Solving EFE~~

Notice that $R^{\mu}{}_{\nu}$ depends on $\Psi, \Lambda \rightsquigarrow$ EFE's to see the dependence $\rightsquigarrow$

$$\boxed{\frac{8\pi G}{c^4} T_{\mu\nu} = G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}$$

$\downarrow$ (t, t) component. $T_{tt} \sim \rho \leftarrow$ density of mass

Schwarzschild

$$\rightsquigarrow \quad 8\pi\rho = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r(1-e^{-2\Lambda})\right)$$

notice that $\int \rho \, d^3r = 4\pi \int \rho(r) r^2 dr = M = \int \frac{dm}{dr} dr$

$$\Rightarrow \frac{1}{r^2}\frac{dm}{dr} = 4\pi\rho \quad \Rightarrow \quad r(1-e^{-2\Lambda}) = 2m(r)$$

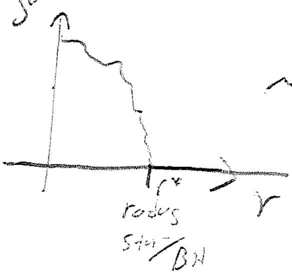
$$e^{-2\Lambda} = 1 - \frac{2m(r)}{r}$$

and other field eq. $\rightsquigarrow$

$$\Rightarrow \boxed{ds^2 = -\left(\frac{1}{1-\frac{2m}{r}}\right)dt^2 + \left(1-\frac{2m}{r}\right)dr^2 + r^2 d\Omega^2}$$

Schwarzschild eq. metric.

However, out of the star/BH. ρ vanishes



$\leadsto$ M is indep of r $\leadsto$ by the ADM mass is the coefficient of $\Phi(r) = \frac{M}{r} + O(r^2)$ where

$g_{tt} = 1 + \Phi + \dots$ ← Newtonian limit.

Now take (t,r) part $\leadsto$

$$G^t{}_t = 8\pi T^t{}_t \overset{r>0}{\leadsto}$$

$$\| $$

$$-\frac{1}{r^2} + \frac{e^{-2\Lambda}}{r^2} + \frac{2}{r} e^{-2\Lambda} \psi' = 0 \Rightarrow \psi' = \frac{r}{2m} - \frac{1}{2r}$$

$$\Rightarrow \psi(r) = \frac{1}{2} \ln\left(1 - \frac{2m}{r}\right)$$

$$\Rightarrow \boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2}$$

where we identify

$\Phi(r) = -\frac{2M}{r}$ $\leadsto$ newtonian limit.

Study of Schwarzschild geometry:

- **Red shift:**

Consider the proper time of a particle
(in rest frame)

$$-d\tau^2 = -\left(1 - \frac{2M}{r}\right) dt^2$$

$$\leadsto d\tau = \sqrt{1 - \frac{2M}{r}}\, dt$$

For an observer far away the mass

$\leadsto r \to \infty$. $d\tau_\infty \approx dt$ $\leadsto$

So the time (and therefore wave length) in the rest frame at radius r is shorter than the time at infinity!

$$\frac{d\tau}{d\tau_\infty} = \sqrt{1 - \frac{2M}{r}} < 1$$

$\leadsto$ red shift is observed $E = h\nu$

$$\frac{\frac{1}{\tau}}{\frac{1}{\tau_\infty}} = \frac{\nu_{obs}}{\nu_{emit}} < 1 (?) \quad \frac{E_{obs}}{E_{emit}} =$$

$$\frac{E_{obs}}{E_{emt}} = \sqrt{1-\frac{2M}{r}} \rightsquigarrow E_{obs} = \sqrt{1-\frac{2M}{r}}\, E_{emit}$$

$\tau_\infty \cong \cancel{\sqrt{\ }}\, dt \leftarrow$ observer.

$$\tau_{emt} = \sqrt{1-\frac{2M}{r}}\, dt$$

$$\rightsquigarrow \quad \frac{\tau_{emit}}{\tau_{obs}} = \sqrt{1-\frac{2M}{r}} = \frac{\frac{1}{\tau_{obs}}}{\frac{1}{\tau_{emt}}} = \frac{\nu_{obs}}{\nu_{emit}}$$

$E = h\nu$

$\frac{E_{obs}}{E_{emit}} \rightsquigarrow$ if r is emitted near ∞

$\rightsquigarrow$ nothing <u>shifts</u>

If r is near $r \rightsquigarrow E_{obs} < E_{emt} \rightsquigarrow$ looses <u>energy</u> and ~~it~~ is red shifted

$E_{obs} < E_{emit}$.

Geodesic equation

$$S = -m\int_{\tau_1}^{\tau_2}\sqrt{-ds^2} = -m\int_{\tau_1}^{\tau_2}\sqrt{-\dot{x}^\nu \dot{x}_\nu}\,d\tau$$

$$\mathcal{L} = -m\sqrt{-\dot{x}^\mu \dot{x}_\mu}$$ ← (can depend on x^ν inside the metric)

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial \dot{x}^\nu} = \frac{\partial \mathcal{L}}{\partial x} = 0 = \frac{d}{d\tau}\left(\frac{m\,\cancel{g^{\nu\mu}}}{2\sqrt{-\dot{x}^\mu\dot{x}_\mu}}\left(\dot{x}_\nu + \dot{x}_\nu \cancel{g}\right)\right)$$

$$\rightsquigarrow \frac{m(\dot{x}_\nu)}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} = C, \quad P_\mu = \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} =$$

$$\rightsquigarrow \frac{d}{d\tau}P_\mu = C \rightsquigarrow \quad P_\mu P^\mu = \frac{m^2\,\dot{x}_\nu \cancel{\dot{x}^\nu}}{-(\cancel{\dot{x}_\nu \dot{x}^\nu})}$$

$$\rightsquigarrow \boxed{P_\mu P^\mu + m^2 = 0}$$ ← is more useful, doesn't depend on x^μ.

$$\mathcal{L} = -m\sqrt{-\left(-\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{1}{\left(1-\frac{2M}{r}\right)}\dot{r}^2 + r^2\dot{\varphi}^2\right)}$$

$$= -m\sqrt{\left(1-\frac{2M}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2M}{r}} - r^2\dot{\varphi}^2}$$

$$\rightsquigarrow \frac{\partial\mathcal{L}}{\partial t} = 0 = \frac{d}{d\tau}P^0 \rightsquigarrow P^0 = \text{Constant} = -E$$

$$\frac{\partial\mathcal{L}}{\partial\varphi} = 0 = \frac{d}{d\tau}P^\varphi \rightsquigarrow P^\varphi = \text{const}$$

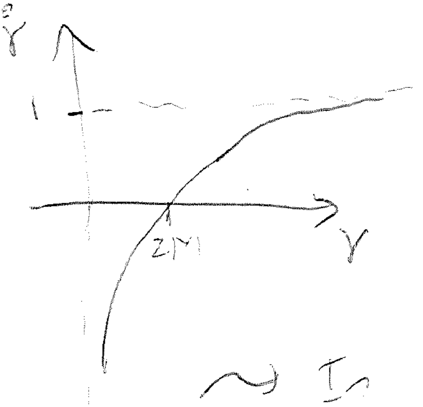
Now consider a photon (radially moving)

$\rightsquigarrow$ $$ds^2 = 0 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{\left(1-\frac{2M}{r}\right)}dr^2$$

$\rightsquigarrow$ $$\frac{dr}{dt} = \dot{r} = \pm\left(1-\frac{2M}{r}\right)$$

$\rightsquigarrow$ velocity of ~~tra~~jectory is < 1 and

far away $\rightsquigarrow$ $\dot{r} = 1$.



photon doesn't come out for $r = 2M$ since $\boxed{\dot{r} = 0}$

$\rightsquigarrow$ In Kruskal (without singularity)

at $r = 2M \rightsquigarrow r < 2M \rightsquigarrow$ photons will be pulled to $r = 0$ $\leftarrow$ Black hole

" we cannot see the colapse, it will take an infinite amount of time to never get inside the BH. So we will see matter around the horizon + stuff already in there when the star collapsed.

t: $\cancel{0 = \frac{\partial \mathcal{L}}{\partial t} = \frac{d}{dt}\left(\frac{m\,\dot{x}_0}{\sqrt{-\dot{x}_\nu \dot{x}^\nu}}\right)}$

$$\cancel{= \frac{d}{d\tau}\left(\frac{m\,\dot{x}_0}{\sqrt{-\dot{x}_\nu \dot{x}^\nu}}\right) = \frac{d}{d\tau}\left(\frac{-m\,\dot{t}}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}\left(1-\frac{2M}{r}\right)^{-1}\right)}$$

$t = x^0$

$$p^t = \text{const} = \frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{\partial \mathcal{L}}{\partial \dot{x}^0} = \frac{m(\dot{x}_\nu \ldots)}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{\partial \mathcal{L}}{\partial \dot{x}^0} = \pm\frac{m}{2}\,\frac{1}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}\,\frac{d}{d\dot{t}}\left(\dot{x}^\nu \dot{x}_\nu\right)$$

$$\left(\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}\right) = 2\dot{x}^0 g_{00} = \dot{t}\left(-\left(1-\frac{2M}{r}\right)\right)$$

$$= -\frac{m}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}\left(1-\frac{2M}{r}\right)\dot{t}$$

we set 4-vector normalized -1

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -m\left(1-\frac{2M}{r}\right)\dot{t} = \boxed{-E = \text{const}}$$

$$\Rightarrow \boxed{E = m\left(1-\frac{2M}{r}\right)\dot{t}}$$

φ: $\frac{\partial \mathcal{L}}{\partial \varphi} = 0 \Rightarrow p^\varphi = \frac{\partial \mathcal{L}}{\partial \dot{x}^\varphi} = \frac{m\,\dot{x}^\varphi}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}\,g_{\varphi\varphi} = \boxed{m\,\dot{\varphi}\,r^2 = L_z}$

Since

$$-1 = \dot{x}^{\mu}\dot{x}_{\mu} = -\left(1-\frac{2M}{r}\right)\dot{t}^{2} + \frac{1}{1-\frac{2M}{r}}\dot{r}^{2} + r^{2}\dot{\varphi}^{2}$$

using
$$\left.\begin{aligned} \dot{t} &= \frac{E}{m\left(1-\frac{2M}{r}\right)} \\ \dot{\varphi} &= \frac{L_z}{m r^{2}} \end{aligned}\right\}$$

$$-1 = -\frac{E^{2}}{m^{2}\left(1-\frac{2M}{r}\right)} + \frac{\dot{r}^{2}}{\left(1-\frac{2M}{r}\right)} + \frac{L_z^{2}}{m^{2}r^{2}}$$

$$\Rightarrow \frac{E^{2}}{m^{2}} - 1 = \dot{r}^{2} - \frac{2M}{r} + \cancel{\frac{L_z^{2}}{m^{2}r^{2}}}$$
$$+ \frac{L_z^{2}}{m^{2}}\left(\frac{1}{r^{2}} - \frac{2M}{r^{3}}\right)$$

$$\frac{E^{2}}{m^{2}} - 1 = \dot{r}^{2} + V_{eff}$$

$$V_{eff} = -\frac{2M}{r} + \cancel{\frac{L_z^{2}}{m^{2}r^{2}} - \frac{L_z^{2}}{m^{2}}\frac{2M}{r^{3}}}$$
$$+ \frac{L_z^{2}}{m^{2}}\left(\frac{1}{r^{2}} - \frac{2M}{r^{3}}\right)$$

$$E^{2} - 1 = \dot{r}^{2} - \frac{2M}{r} - \frac{L^{2}}{r^{2}} + \frac{2ML^{2}}{r^{3}}$$

← relativistic corrections

rest mass

newtonian potential

charge potential

If V_{eff} has a minimum the particle can oscillate around with energy moving back and forth between the kinetic term $\dot{r}^2$ and the effective potential.

Lets see effective potential:

$$V_{eff} = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$$

- For $r \to \infty$ $V_{eff} = 1 \rightsquigarrow \dot{r}^2 = V^2 = \tilde{E} - 1$

~~$\dot{r}^2 >$~~ unbounded orbits have $\boxed{\tilde{E} \geq 1}$ since in this case $\dot{r} > 0$ ← ever increasing or decreasing.

find extremals:

$$0 < \frac{\partial V}{\partial r} = \frac{2M}{r^2}\left(1 + L^2/r^2\right) + \left(1 - \frac{2M}{r}\right)\left(-\frac{2L^2}{r^3}\right) = 0$$

$$\rightsquigarrow M\left(1 + L^2/r^2\right) = \left(1 - \frac{2M}{r}\right)\frac{L^2}{r}$$

$$Mr^2 + ML^2 = rL^2 - 2ML^2$$

$$Mr^2 - L^2 r + 3ML^2 = 0$$

$$r^2 - \frac{L^2}{M} r + 3L^2 = 0 \rightsquigarrow r_\pm = \frac{L^2}{M} \pm \sqrt{\frac{L^4}{M^2} - 12L^2}$$

$$= \frac{L^2}{2M} \pm L\sqrt{\frac{L^2}{4M^2} - 3}$$

$$\to r_\pm = \frac{L^2}{2M} \pm L^2 \sqrt{\frac{L^2}{4M^2} - 3}$$

Note that at $\frac{L^2}{4M^2} = 3 \rightsquigarrow \boxed{r_+ = r_-} = r^*$

$\rightsquigarrow L^* = 2\sqrt{3}M$

In this case $r^* = \frac{L^2}{2M}$

$$\rightsquigarrow r^* = 12\frac{M}{2M} = \boxed{6M = r^*}$$

So minimum radius of stable (circular orbit. root is positive) is $r^* = r$.

If $r < r^*$ they are sucked into the BH.

Orbits of Photons:

$$\frac{ds^2}{d\tau^2} = 0 = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{(1)}{1-\frac{2M}{r}}\dot{r}^2 + r^2\dot{\varphi}^2$$

~~and using~~ $\cancel{E = \dot{t}\left(1-\frac{2M}{r}\right)}$

using the conserved quantities
$$\begin{cases} E = \dot{t}\left(1-\frac{2M}{r}\right) \\ \ell = \dot{\varphi}\, r^2 \end{cases}$$

$$\rightsquigarrow \quad 0 = -\left(1-\frac{2M}{r}\right)\frac{E^2}{\left(1-\frac{2M}{r}\right)^2} + \frac{\dot{r}^2}{\left(1-\frac{2M}{r}\right)} + \frac{\ell^2}{r^2}$$

$$\rightsquigarrow \quad \dot{r}^2 = E^2 - \frac{\ell^2}{r^2}\left(1-\frac{2M}{r}\right)$$

$$V_{eff} = \frac{\ell^2}{r^2}\left(1-\frac{2M}{r}\right)$$

$$\rightsquigarrow \quad \frac{\partial V_{eff}}{\partial r} = 0 = -\frac{2\ell^2}{r^3} + \frac{6\ell^2 M}{r^4} \rightsquigarrow$$

$$\boxed{r = 3M}$$

↑ maximum

and fall off from there in either direction.

⇝ no stable orbits. (unstable at $r = 3M$)

~~small < r~~ $\left.\begin{array}{l} r < 3M \text{ sucked in} \\ r > 3M \text{ deflected} \end{array}\right\}$ photon

using $\dot{r}^2 = \mathcal{E}^2 - \frac{\ell^2}{r^2} + \frac{\ell^2 2M}{r^3}$

$$\rightsquigarrow 2\dot{r}\ddot{r} = +2\frac{\ell^2}{r^3}\dot{r} - \frac{6\ell^2 M}{r^4}\dot{r}$$

$$\rightsquigarrow \ddot{r} - \frac{\ell^2}{r^3} + \frac{3\ell^2 M}{r^4} = 0$$

using $\dot{r} = \frac{dr}{d\sigma} = \frac{dr}{d\varphi}\dot{\varphi} = \frac{\ell}{r^2}\frac{dr}{d\varphi}$

$$\rightsquigarrow \begin{cases} u'' + u - \frac{3r_s}{2}u^2 = 0 \\ u = \frac{1}{r}, \quad u' = \frac{du}{d\varphi}. \end{cases}$$

- If $M = 0 \rightsquigarrow u'' + u = 0 \rightsquigarrow \boxed{u = \frac{1}{r_0}\cos\varphi}$

Straight line written in polar coordinates.

put $u = \frac{1}{r_0}\cos\varphi + D$ in the whole ~~eq~~. $r_s = 2M$

$$\rightsquigarrow D'' + D - \frac{3r_s}{2}\left(\frac{1}{r_0^2}\cos^2\varphi\right) \simeq 0$$

In full we get

$$u(\varphi) = \frac{\cos\varphi}{r_0} - \frac{r_s}{2r_0^2}\cos^2\varphi + \frac{r_s}{r_0^2}$$

we can compute the deflection of light. at large r, $u \to 0$ and small $\frac{r_s}{r_0}$ we expect

$\varphi \approx \frac{\pi}{2}$.

$$\rightsquigarrow \quad u(\varphi) = \frac{\cos\varphi}{r_0} - \frac{r_s}{2r_0^2}\cos^2\varphi + \frac{r_s}{r_0^2} = 0$$

$$= \frac{\cos\left(\frac{\pi}{2}+\delta\right)}{r_0} + \frac{r_s}{r_0^2} = 0$$

$\rightsquigarrow \boxed{\delta = \frac{r_s}{r_0}}$ this is half the deflection of light since the trajectory is symmetric

and starts at $-\frac{\pi}{2}+\delta$. the total deflection of light is 2δ

$$\Delta = 2\frac{r_s}{r_0} \leftarrow$$ 1.75 arc seconds for light if $M = M_{sun}$ measured during an eclipse

Evaporation takes a very important role in the discussion.

BH ~~evaporates~~ Strong gravitational ~~effect~~ field causes radiation to be emitted outside the Schwarzschild radius that escapes to infinity.

$\rightsquigarrow$ BH evaporate (if not accreting matter)

here light BH evaporate quickly an large BH takes more time since matter is falling into them.

- The BH evaporates at a finite amount of time (as viewed for large r)
- It takes an infinite amount of time for light to be emitted at $r_s = 2M$ (Schwarzschild radius)

Hence the observer at large r, the BH collapses down to nearly Schwarzschild radius, then slows to zero and emits radiation eventually evaporating.

So as seen from the outside, the BH never collapses to a point. It maintains its info near r_s

NO causal

curve can come out

from $r < 2M$ $\rightsquigarrow$ BH

study $v = t - r^*$

r^* s.t.

$$ds^2 = +\left(1-\frac{2M}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 d\Omega^2$$

Causal curves

↳ white and B holes.

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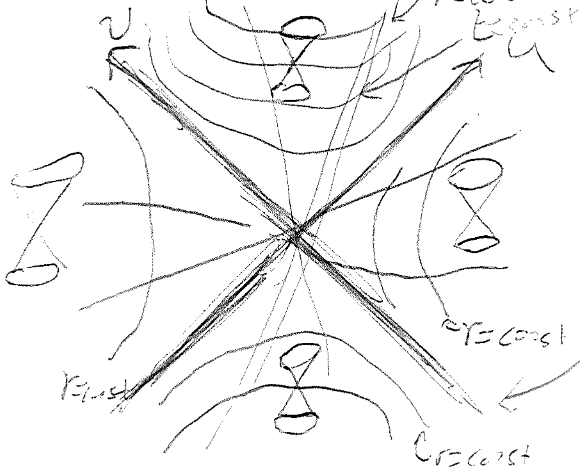
but there is nothing special about $r=2M$!!

It follows that



// straight lines through (0,0) t=const
r=const by hyperbolae

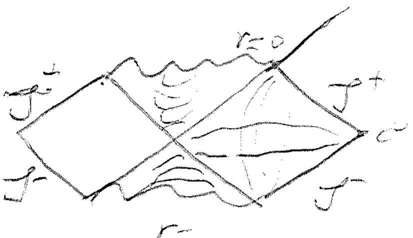
$uv=0 \Leftrightarrow r=2M$

$\Leftrightarrow$ Horizon

Changes from spacelike to timelight.

- South quadrant: everything must go out
  $\rightsquigarrow$ white hole
- North quadrant: ev... ... ... in
  $\rightsquigarrow$ BH. $\rightsquigarrow$ end in the singularity
- $r>2M$ is the right quadrant and left is another asymptotic flat space.

$\hookrightarrow$ compactificate. $u=\tan\tilde{u}$, $v=\tan\tilde{v}$

$\rightsquigarrow$ 

- Chronological future/past $\mathcal{J}^\pm(U)$ (inextendable timelike paths ...)

Rec Schwarzschild sol:

$$ds^2 = (1-\frac{2M}{r})(-dt + dr^{*2}) + r^2 d\Omega^2$$

choose $v = t + r^*$ ← light rays.
$u = t - r^*$

→ $$ds^2 = -(1-\frac{2M}{r})dv^2 + 2\,dv\,dr^* + r^2 d\Omega^2$$
← regular at $r = 2M$

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{(1)}{1-\frac{2M}{r}}dr^2 + r^2 d\Omega^2$$

→ only for $\boxed{r > 2M}$

→ extend with a chart that is analytical (instead of only C^∞).

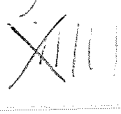
→ Kruskal coordinates

$$ds^2 = -32\frac{M^3}{r} e^{-r/2M} dU\,dV + r^2 d\Omega^2$$

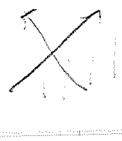
$U = -e^{-u/4m}$, $u = t - x'$, $x' = \log x$

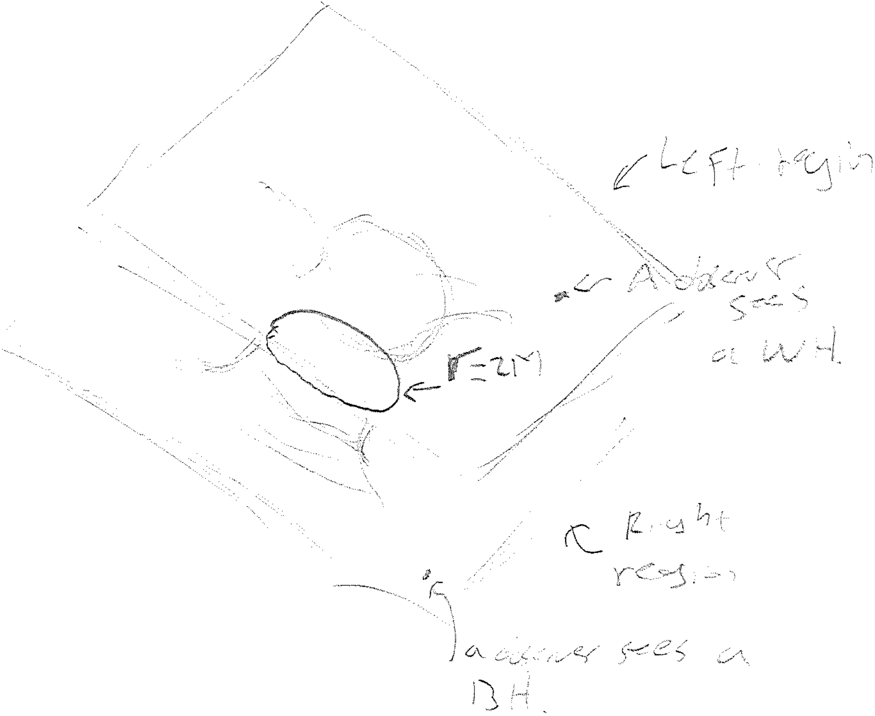
$V = e^{v/4m}$, $v = t + x'$

Really cover

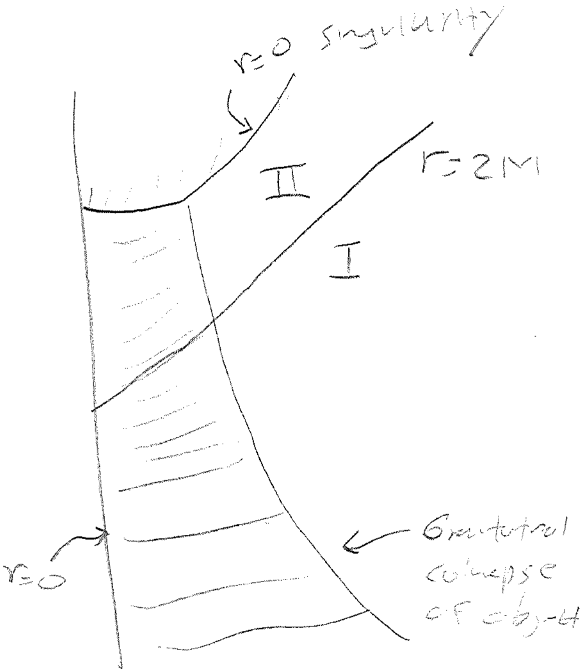


Really cover





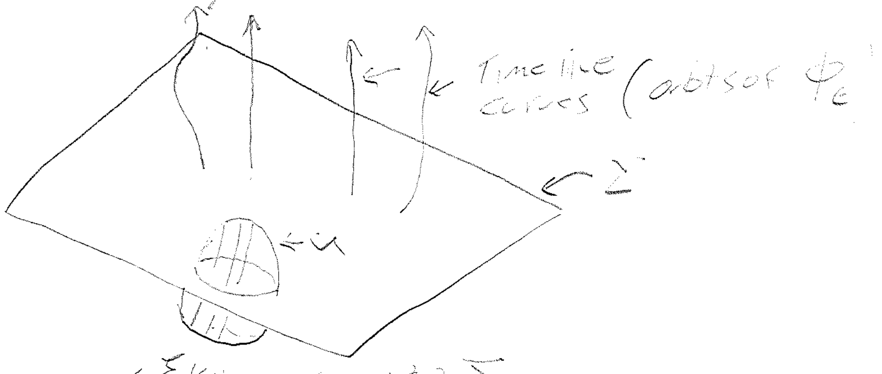
WH = BH with time reversal.



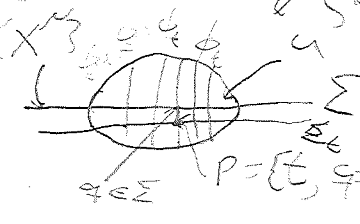
Remarks on Schwarzschild

Def: A ~~stationary spacetime is a~~ spacetime is **stationary** if ∃ ϕ_t ← group of isometries whose orbits are time like curves. This expresses the "time translation symmetry". ← "Stationary if it possesses a T.L. Killing."

A ST is **static** if it's stationary and ∃ Σ hypersurface orthogonal to the orbits of the isometry.



ξ Killing orthogonal to Σ.



Since ∂_t Killing $\rightsquigarrow$ $\partial_t g_{\mu\nu} = 0$

$\rightsquigarrow$ $g_{\mu\nu}$ indep of t

$\Sigma_t = \phi_t(\Sigma)$

also orthogonal to ξ

$\rightsquigarrow \quad ds^2 = -V^2(x^1 \dots)\, dt^2 + h_{\mu\nu}(x^1 \dots x^3)\, dx^\mu dx^\nu$

$\cdot \; \|\partial_t\| = \cancel{g} = -\xi_a \xi^a = V^2$

Killing

- Absence of $dt\, dx^\mu \rightsquigarrow \xi \perp \Sigma$.

Here a stationary but not static must have $dt\, dx^\mu$ components in *any* coordinate which uses Killing parameter as one coordinate ($\rightsquigarrow \xi$ ... t).

<u>Remark</u> ~~ϕ_t are~~ Isometries generated by Killing are connected to $\mathbb{1}$ identity. Symmetry ... , for example is not ... (White ~~Hole~~ Hole.)
$\rightarrow$ more isometries

Birkhoff's theorem $\rightsquigarrow$ ~~stationary~~ static is not a relevant only stationary. (Analogous to EM theorem: no monopoles in GR)

Kruskal coordinates (or how to not trust a ~~[illegible]~~ coordinate system)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2 d\Omega^2 \qquad r > 2M$$

Is it the coordinate metric the problem or the manifold structure

+ Clearly infinite the ... time (as seen of a for any observer) to get stuck up into the BH horizon or ... inside?

AWe require a Chart which is C^ω instead of C^∞

→ null geodesics (radially)

$$0 = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2$$

$$\rightsquigarrow \frac{dr^*(t)}{dt} = \pm\left(1-\frac{2M}{r}\right) \rightsquigarrow \pm\left(\frac{r-2M}{r}\right)$$

$$\rightsquigarrow \frac{dt}{dr} = \cancel{\ldots} = \frac{r-2M+2M}{r-2M}$$

$$\rightsquigarrow r^* = r + 2M\log(r-2M) \rightsquigarrow$$

$t = \pm\left(r + 2M \ln\left(\frac{r}{2M} - 1\right)\right) + c$

↳ null geodesics

so define $\left.\begin{aligned} u &= t - r_* \\ v &= t + r_* \end{aligned}\right\}$ $du\,dv = -dt^2 + dr_*^2 = -dt^2 + \frac{1}{(1-\frac{2M}{r})^2} dr^2$

if $u = 0$ or $v = 0$ we have geodesics.

and so

$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{1}{1-\frac{2M}{r}} dr^2 \rightsquigarrow ds^2 = -(1-\frac{2M}{r})\,du\,dv$

$= -\frac{2M}{r} e^{-r/2M} e^{2u/4M}\,du\,dv$

non singular at $r \to 2M$ but the schwarzschild

$U = e^{-\frac{u}{4M}}, V = e^{\frac{v}{4M}}$

$ds^2 = \pm\frac{32M^3}{r} e^{-r/2M}\,dU\,dV$ this extends

Note there is no more a singularity at $r = 2M$ but there is still at $r = 0$.

this is ~~not~~ physical as the $R \sim \frac{1}{r}$ also. (indep of coordinates)

& kruscal

$ds^2 = \frac{32}{r} M^3 e^{-r/2M}\left(-dT^2 + dX^2\right) + r^2 d\Omega^2$

$T = (U+V)/2$

~~F~~ similar to Minkowski

$X = (V-U)/2$

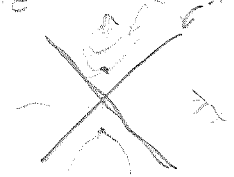
Penrose diagrams ~~At $r=2M$ ∂_t, ∂_r ...~~

- ~~...~~

	∂_t	∂_r
$r>2M$	timelike	spacelike
$r<2M$	spacelike	timelike

$$0=-dt^2+dx^2$$

$\rightsquigarrow$ ∂_t timelike



$$-1=-dt^2+dx^2$$

"Step reveals the ... $r=0$ ... timelike ... we cannot ... beyond".

- At $r=2M$ $g_{tt}=0$ $\rightsquigarrow$ $\int_{r=2M} dV = 0$

$\rightsquigarrow$ The surface $r=2M$ which appears to be 3-dim is really 2-dim

$$\int_{r=2M} \ldots = 4\pi(2M)^2$$

- g_{rr} at $2M$ diverges ... distances can be compared to $2M$!

$$d = \int_{2M}^{r>2M} g_{rr}^{1/2}\, dr < \infty$$ ... in finite amount of time.

In-going

Recall $u = t - r_*$, $v = t + r_*$ } parametrise by

ingoing rays and outgoing
(u) ← velocity is ↑ v, velocity
needs ... to the ... Penrose

useful to see light cones

Use $ds^2 = -(1-2M/r)\,dv^2 + 2\,dv\,dr + r^2 d\Omega^2$

Radial light cone → $ds^2 = 0$, as expected

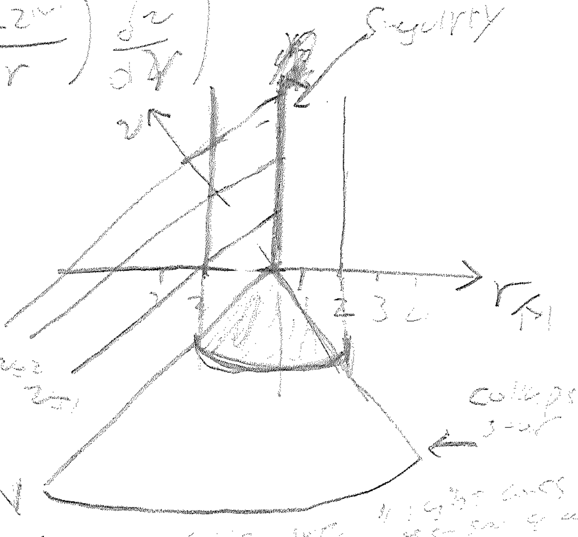
$$0 = -(1-2M/r)\left(\frac{dv}{dr}\right)^2 + 2\frac{dv}{dr}$$

$$= \frac{dv}{dr}\left(2 - \left(\frac{r-2M}{r}\right)\frac{dv}{dr}\right)$$

$$\Rightarrow \frac{dv}{dr} = 0$$

$$\frac{dv}{dr} = \frac{2r}{r-2M}$$

$$= \frac{2}{1-2M/r}$$



So ⟶ r left-leaning light cones ...

$$\frac{du}{dr} = \frac{\partial u}{\partial r} = -\frac{2}{1-2M/r}$$

Theorem (Birkoff I) A [illegible] spacetime

1) Spherically symmetric

2) Solution of EFE in vacuum

[illegible] geometry is that of a Schwarzschild

geometry.

(Birkoff II)

3) EM field source

[illegible] space of Reissner–Nordström eq.

[illegible]

Theorem: A BH is fully characterized

[illegible] O.S (mass, charge, angular momentum)

[illegible] these quantities should be,

[illegible] these quantities

[illegible] off the

[illegible] some

[illegible] posed however

[illegible] we can find that local

[illegible] / [illegible] momentum, stress etc.

[illegible] inside the BH

[illegible] no contribution to

[illegible] causal [illegible] S.T.

A proof of the "BH has no hair" ([illegible] 3) theorem

[illegible] external gravitational field and electromagnetic fields of a stationary BH are determined by M, Q and S (mass charge and intrinsic angular momentum of the BH)

i.e. A BH can have no hair.

Kerr-Newman Solution Solutions

Implications and insights:

$a = S/M$

+ Kerr-Newman has an horizon

$\rightsquigarrow$ BH iff $M^2 \geq Q^2 + a^2$

[illegible] repulsion or centrifugal forces [illegible] stop the gravitational collapse

+ $Q = 0 \rightsquigarrow$ Kerr

$S = 0 \rightsquigarrow$ Reissner-Nordström

$Q = S = 0 \rightsquigarrow$ Schwarzschild

$M^2 = Q^2 + a^2 \rightsquigarrow$ "Extreme Kerr-Newman"

Here:

$$ds^2 = -\frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2\cos^2\theta}\left[dt - a\sin^2\theta\, d\phi\right]^2$$

$$+ \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}\left[(r^2+a^2)\,d\phi - a\,dt\right]^2$$

$$+ \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2 + Q^2}\,dr^2 + (r^2 + a^2\cos^2\theta)\,d\theta^2$$

$$\boxed{ds^2 = -\frac{\Delta}{\rho^2}\left[dt - a\sin^2\theta\, d\phi\right]^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)\,d\phi - a\,dt\right]^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2}$$

Corresponding EM field tensor is:

$$F = \frac{Q}{\rho^4}(r^2 - a^2\cos^2\theta)\,dr \wedge \left[dt - a\sin^2\theta\, d\phi\right]$$

$$- \frac{2Q}{\rho^4}\, a r \cos\theta \sin\theta\, d\theta \wedge \left[(r^2+a^2)\,d\phi - a\,dt\right]$$

In a generalization of Schwarzschild coordinates

g is independent of t and ϕ $\leadsto$ $\partial_t, \partial_\phi$ Killing

spacetime is 'stationary' and axially symmetric

Boyer [illegible] (Generalization of [illegible])

$$ds^2 = -\frac{\Delta}{\rho^2}\left[dt - a\sin^2\theta\, d\phi\right]^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)\,d\phi - a\,dt\right]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$F = \frac{Q}{\rho^4}(r^2 - a^2\cos^2\theta)\,dr\wedge\left[dt - a\sin^2\theta\, d\phi\right] + \frac{2Q}{\rho^4}\,a r\cos\theta\sin\theta\, d\theta\wedge\left[(r^2+a^2)\,d\phi - a\,dt\right]$$

Kerr coordinates: (Generalization of Eddington–Finkelstein u, v)

$$ds^2 = -\left[1 - \frac{2Mr - Q^2}{\rho^2}\right]dv^2 + 2\,dr\,dv + \rho^2 d\theta^2 + \left[(r^2+a^2)^2 - \Delta a^2\sin^2\theta\right]\sin^2\theta\, d\tilde{\phi}^2$$

$$-2a\sin^2\theta\, d\tilde{\phi}\, dr - \frac{2a}{\rho^2}(2Mr - Q^2)\sin^2\theta\, d\tilde{\phi}\, dv.$$

$$F = \frac{Q}{\rho^4}\Big[(r^2 - a^2\cos^2\theta)\,dr\wedge dv - 2a^2 r\cos\theta\sin\theta\, d\theta\wedge dv - a\sin^2\theta(r^2 - a^2\cos^2\theta)\,dr\wedge d\tilde{\phi}$$

$$+ 2ar(r^2+a^2)\cos\theta\sin\theta\, d\theta\wedge d\tilde{\phi}$$

where $dv = dt + (r^2+a^2)\dfrac{dr}{\Delta}$, $d\tilde{\phi} = d\phi + \dfrac{a\,dr}{\Delta}$.

$\Delta = r^2 - 2Mr + a^2 + Q^2$, $\rho^2 = r^2 + a^2\cos^2\theta$.

The metric is indep of v and $\tilde{\phi}$ $\Rightarrow$ ∂_v, $\partial_{\tilde{\phi}}$ are Killing

associated to $\partial_t = \partial_v$, $\partial_\phi = \partial_{\tilde{\phi}}$.

+ The dragging of inertial frames by BH's produces a process of [illegible] relative to distant stars $\sim$ one detects [illegible] the angular momentum [illegible] (of the BH) [illegible] process.

— The dragging occurs more extreme the nearer one approaches the horizon.

Before the horizon is reached, [illegible] defined by $r = r_0(\theta)$

$$r = M + \sqrt{M^2 - Q^2 - a^2\cos^2\theta}$$

the dragging is so extreme that no observer can possibly remain at rest (i.e. be static) relative to the distant stars.

At and inside this surface all observers [illegible] r and θ must [illegible] in the same direction in which the hole rotates

$$\Omega = \frac{d\varphi}{dt} > \frac{[illegible]\,\theta - \sqrt{\Delta}}{[illegible]\,\theta - \sqrt{\Delta}\,a\sin^2\theta}$$

- no matter the engine he is using, he can never halt his angular motion

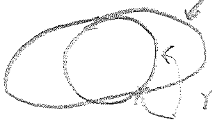
(motion of (r,θ) occurs)

This occurs ~~off~~ [illegible] for [illegible] = speed of light, therefore no observer can remain at rest.

+ Horizon at $r = r_+ = M + \sqrt{M^2 - Q^2 - a^2}$

Ergosphere : region between horizon and static limit.

$r = M + \sqrt{M^2 - Q^2 - a^2\cos^2\theta}$



$r = M + \sqrt{M^2 - Q^2 - a^2}$

+ Singularity ~~of the~~ [illegible] have Schwarzschild singularities on [illegible] ~> infinite [illegible] for any point [illegible] cannot be [illegible] (as seen [illegible] by an external observer).

- To remove this use new coordinates to compress [illegible] by Eddington–Finkelstein analogy
~> Kerr coordinate

If rotating then also [illegible] ϕ [illegible]

+ $E_r \sim \frac{Q}{r^2}$, $D_r \sim \frac{2\mu}{r^3}\cos\theta$, $D_\theta \sim \frac{\mu}{r^3}\sin\theta$, [illegible]

Killing Horizons:

- Recall K^a killing if

$$0 = \nabla_\mu K_\nu + \nabla_\nu K_\mu = \nabla_{\{\mu} K_{\nu\}}$$

$$\nabla_\mu \nabla_\nu K_\rho = -R_{\nu\rho\mu\sigma} K^\sigma$$

K^a generate isometries $\rightsquigarrow$

Thm F, G are isometries s.t. $F(p) = G(p)$ and $DF(p) = DG(p)$ for some $p \in M$

$\Rightarrow F = G.$

Proof: Geodesics $\to$ Geodesics

determined at p and its derivatives $\Rightarrow$ Both.

Corollary: K^a is determined by $K(p)$ and $\nabla_\mu K_\nu|_p$

Killing horizon: null hypersurface h s.t. $\|K\| = 0$ on h

$\rightsquigarrow h = \{ p \in M \mid \|p\| = 0, \|K(p)\| = 0, T \in T_p h \}$

$f \stackrel{\text{def}}{=} \|K(p)\| \rightsquigarrow df = n^\mu \leftarrow$ normal

$\rightsquigarrow n^\mu = \cancel{\nabla(K\cdot K)} = 2K\cdot\nabla K$. Let $T \in T_p h$, $T\cdot K =$

Let's see that K is normal to h.

K is null to h iff $K \cdot T = 0$ $\forall$ T tangent to h.

$\leadsto$ we have
- $T^2 = 0$
- $K^2 = 0$
- $K \cdot T = 0$

$\Leftrightarrow (K + \lambda T)^2 = K^2 + 2\lambda K T + T^2 = 0$

$\Leftrightarrow$ $\boxed{T = \sigma K}$ constant

~~Let's see that~~

~~Suppose T_1, T_2 two indep~~

~~Since we have the desired result.~~

To see that K is normal we have to see that $T = \sigma K$ for all T.

Let Σ be a Killing horizon wrt K.

Prop: K is normal to Σ.

Proof: we want to see that $T\cdot K = 0$ for all $T \in T\Sigma$.
which in turn is equivalent to
$(T+\lambda K)^2 = 0$ for all T (because $\cancel{T^2} + 2\lambda T\cdot K + \cancel{K^2} = 0$, $T\cdot K = 0$)

but only holds if $T = \sigma K$ hence
we want to prove that $\boxed{T = \sigma K \text{ for all } T.}$

To notice that $n_\mu = \nabla_\mu K^\nu K_\nu$ and
$$K\cdot n = K^\mu n_\mu = K^\mu K^\nu \nabla_\mu K_\nu + K^\mu K^\nu \underbrace{\nabla_\mu K_\nu}_{-\nabla_\nu K_\mu} = 0.$$
$\Rightarrow K\cdot n = 0 \Rightarrow \boxed{K \text{ is tangent}}$

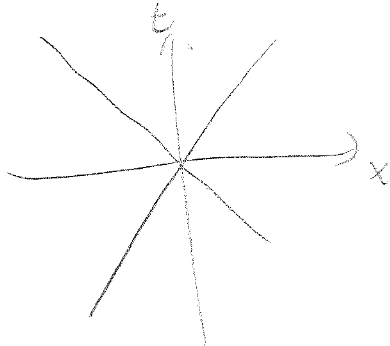
Hence, at least for 1 T, $T = \sigma K$ is satisfied.
now lets prove that there are no more
indep tangents. Assume T_1, T_2 are indep.
$\Rightarrow T_1 \cdot T_2 \neq 0$ as since all tangents are
normal at the same (in a null hypersurface)
$\Rightarrow \underbrace{T_1}_{\text{tangent}} \cdot \underbrace{T_2}_{\text{normal}} = 0$, contradiction $\Rightarrow T = K\sigma$.

Ex: choose $ds^2 = -dt^2 + (dx^{\mu})^2$

$K = x\partial_t + t\partial_x$

$\hookrightarrow \|K\|^2 = K^\mu K_\mu = g_{\mu\nu} K^\mu K^\nu = -x^2 + t^2 = (t-x)(t+x)$

$\rightarrow$ define $\Sigma = \{ p \in M \mid f(p) = 0 \}$, $f(t,x) = (t-x)(t+x)$



Tangents?

$\gamma = (t, -t)$

$\gamma = (t, t)$

$\hookrightarrow \dot\gamma = (1,-1) \rightsquigarrow$

$\dot\gamma = (1,1) \rightsquigarrow \|T\|^2 = -1 + 1 = 0$

Now, recall $n^\mu = \nabla^\mu K^\nu K_\nu$ is a normal to Σ

$n = \nabla K^2$

and ... tangent $\Rightarrow n^\mu = \sigma K \rightsquigarrow$

$\sigma K^\mu = \nabla^\mu K^\nu K_\nu \stackrel{!}{=} K^\nu \nabla^\mu K_\nu + K_\nu \nabla^\mu K^\nu$

"surface gravity" $\quad \stackrel{!}{=} 2 K^\nu \nabla^\mu K_\nu$

$\rightsquigarrow 0 \stackrel{!}{=} \mathcal{L}_K (\nabla K^2) = \mathcal{L}_K (\sigma K) = (\mathcal{L}_K \sigma) K + \sigma \cancelto{0}{[K, K]}$

$\rightsquigarrow$ σ is constant along the orbits of K.

[Since $\mathcal{L}_K \sigma(x) = \lim_{t\to 0} \frac{\sigma(x) - \sigma(\phi_t(x))}{t}$ flow generated by K

$\rightsquigarrow 0 = \mathcal{L}_K \sigma(x) \Rightarrow \sigma$ is constant along ϕ_t curves of K]

Now, if S is tangent to Σ $\rightsquigarrow$
Killing

$\nabla_S \sigma^2 = \nabla_S\left(-\frac{1}{2}(\nabla_\mu K_\nu)(\nabla^\mu K^\nu)\right) = +\frac{1}{2}\left(S^\rho R_{\mu\nu\rho\sigma} K^\sigma \nabla^\mu K^\nu + (S^\rho R^\mu{}_{\rho\sigma} K^\sigma) \times \nabla_\mu K_\nu\right)$

$\nabla\nabla\xi = -R_{\nu\rho\mu\alpha} K^\alpha$

including $\mu\leftrightarrow\nu$ and using
- Killing antisymmetry
- Riemann symmetry $\Rightarrow$

$$\boxed{\nabla_S \sigma^2 = 0}$$ $\rightsquigarrow$ σ^2 is constant on Σ.

Now, notice that the equation

$\sigma K^\nu = \nabla^\mu K^\nu K_\mu$ is $-2\sigma K = \nabla_K K$

which is geodesic equation with non-affine parameter. $\rightsquigarrow$ put $V = e^{\sigma \tau}$ $\leftarrow$ affine parameter more like given us earlier.

$\rightsquigarrow \nabla_K K = 0 \leftarrow$ geodesic equation.

Proof of Key//could

$\Sigma = \{ x \in M \mid \|K(x)\| = 0 \}$

$\rightsquigarrow K \perp T\Sigma \quad , \quad K \parallel T\Sigma .$

Let $T, T' \in T\Sigma \rightsquigarrow [T, T'] \in T\Sigma \rightsquigarrow [T, T'] = \sigma K$

$\rightsquigarrow \overset{\sigma K^2}{0 =} K \cdot [T, T'] = K^b ([T, T'])$

$= \cancel{K^c (T^a \partial_a T' - T^b \partial_b T^a)}$

$= K_\mu (T^\nu \nabla_\nu T'^\mu - T'^\nu \nabla_\nu T^\mu)$

$[\nabla_\nu (K_\mu T^\mu) = K \nabla_\nu T_\mu + T^\mu \nabla_\nu K_\mu$

$\rightsquigarrow T^\mu = \sigma K^\mu \rightsquigarrow K_\mu T^\mu = \sigma K^2 = 0 \rightsquigarrow$

$\hookrightarrow K^\mu \nabla_\nu T_\mu = - T^\mu \nabla_\nu K_\mu]$

$= - T^\nu T'^\mu \nabla_\nu K_\mu + T'^\nu T^\mu \nabla_\nu K_\mu$

$= T^{[\mu} T'^{\nu]} \nabla_\nu K_\mu = T^\mu T'^\nu \nabla_{[\nu} K_{\mu]}$

$$\cancel{0 = (\nabla^\mu \kappa^\rho)\,\nu}$$

$$\cancel{0 = (\nabla^\mu K^\rho)\, K_\nu\, (\nabla_\mu K_\rho)}$$

$$K_{[\sigma} \nabla_\nu K_{\rho]} = K^\flat \wedge dK^\flat$$

$$=$$

$$\cancel{= \nabla^\mu K^\rho \; K_\nu \; \nabla_\mu K_\rho}$$

$$K_{[\mu} \nabla_\nu K_{\rho]}$$

$$0 \overset{!}{=} K_\mu \nabla_\nu K_\rho + K_\nu \nabla_\rho K_\mu + K_\rho \nabla_\mu K_\nu$$

$$= - K_\mu \nabla_\rho K_\nu + K_\nu \nabla_\rho K_\mu + K_\rho \nabla_\mu K_\nu$$

$$= 2 K_{[\rho} \nabla_{\mu]} K_\nu + \cancel{K_\nu \nabla_\rho K_\mu}$$

$$- K_\nu \nabla_\mu K_\rho$$

$$\Rightarrow \quad K_\nu \nabla_\mu K_\rho = 2 K_{[\rho} \nabla_{\mu]} K_\nu$$

$$(\nabla^\mu K^\rho)\, \nu\, (\nabla_\mu K_\rho) = (\nabla^\mu K^\rho)\, 2 K_{[\rho} \nabla_{\mu]} K_\nu$$

$$= -2\, \nabla^\mu K^\rho\, K_{[\mu} \nabla_{\rho]} K_\nu$$

$$= -2 \left(K_\mu \nabla^\mu K^\rho \nabla_\rho K_\nu - \tfrac{1}{2} \underbrace{K_\rho \nabla^\mu K^\rho}_{=0} \ldots \right.$$

$$= -2\, \sigma K^\rho \nabla_\rho K_\nu = -2\sigma^2 K_\nu$$

$$\Rightarrow \quad \boxed{(\nabla^\mu K^\rho)(\nabla_\mu K_\rho) = -2\sigma^2}$$

Furthermore:

$$3\, K^{[\mu} \nabla^{\nu} K^{\rho]}\, K_{[\mu} \nabla_{\nu} K_{\rho]}$$

$$= K^{\mu} \nabla^{\nu} K^{\rho} \left(\quad \right)$$

$$= \|K\|^2 (\nabla^{\nu} K^{\rho})(\nabla_{\nu} K_{\rho}) - 2 (K^{\mu} \nabla^{\nu} K^{\rho})(K_{\rho} \nabla_{\nu} K_{\mu})$$

In $\underline{\overline{\Delta}}$ LHS vanishes

Surface gravity example (Schwarzschild).

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + \cdots$$

Killing $K^a = \frac{\partial}{\partial t} = (1,0,0,0) \leftarrow (t,r,\theta,\varphi)$

$\rightsquigarrow \left\|\frac{\partial}{\partial t}\right\|^2 = 0 \rightsquigarrow 1-\frac{2M}{r} = 0 \Rightarrow$ $r = 2M$ is the hypersurface Σ

At Σ let's see the surface gravity:

Taking Eddington Finkelstein (not singularity at $r=2M$)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2 + \cdots$$

and $\frac{\partial}{\partial t} = \frac{\partial v}{\partial t}\frac{\partial}{\partial v}$, $v = t + r + 2M\ln|r-2M|$

$$= \frac{\partial}{\partial v} \rightsquigarrow K^{a'} = (1,0,0,0) = \delta^{a'}_{v}$$

and $K_{a'} = g_{\alpha a'}K^{\alpha'} = (g_{vv}\cdot 1, g_{vr}\cdot 1, 0, 0)$

$$= \left(-1+\frac{2M}{r}, 1, 0, 0\right)$$

$\rightsquigarrow$ Killing surface gravity

$$K^{a'}\nabla_{a'}K^{b'} = \alpha K^{b'} \rightsquigarrow$$

$$K^{a'} \nabla_{a'} K^{b'} = K^{a'} \left(K^{b'}_{\ ,a'} + \Gamma^{b'}_{a'c'} K^{a'} K^{c'} \right) = \sigma K^{b'}$$

original eqn.

$K^{b'} = v \rightsquigarrow$

$$\sigma = 1 \cdot \left(\cancel{1_{,a'}}^{0} + \Gamma^{v}_{a'c'} \delta^{a'}_{v} \delta^{c'}_{v} \right)$$

$$= \Gamma^{v}_{vv}$$

$$\rightsquigarrow \quad \sigma \overset{\text{def christoffel}}{=} -\frac{1}{2} \left(\partial_v g_{vv} \right) = -\frac{1}{2} \partial_r \left(-1 + \frac{2M}{r} \right)$$

$$\partial_v = \frac{\partial r}{\partial v} \frac{\partial}{\partial r} + \cancel{\frac{\partial \theta}{\partial v}} \to 0$$

$$\Rightarrow \sigma = -\frac{1}{2} \left(-\frac{2M}{r^2} \right) = \frac{M}{r^2}, \text{ at the}$$

Killing horizon, $r = 2M \rightsquigarrow$

$$\boxed{\sigma = \frac{1}{4M}}$$. Recall that $\boxed{\nabla_k K = a K}$

non affine parametization of geodesic $= a$

hence "Surface gravity σ is the acceleration, as exerted at infinity, needed to keep an object at the horizon".

HW Chapter 3.4, Reissner–Nordström sol.

→ horizons at r_+ and r_-

Cauchy horizon

$$\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)$$ ← electric field

Cosmic censorship consistent with $m \geq Q$

• otherwise naked singularity

• when $m = Q$ ⟶ extremal black hole

$r_+ = r_-$

i.e. mass comes from charge (field energy)

2 extremal black holes with same charge: they attract gravitationally & repel electrically ⟶ net force = 0

Let's generalize this (in D dimensions)

$$S = \int d^D x \sqrt{-g} \left(R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2n!} e^{a\phi} F^2 \right)$$

R ← E-H, ϕ ← scalar field, $e^{a\phi}$ ← coupling, F ← n-form field strength

This action generalizes Einstein–Maxwell + massless scalar action (Dilaton found in string theory)

Coupling of p-brane with charge Q to electric field

$$S_{\text{coupl.}} = \int^{p}$$

world-volume

generalize to a $(n-1)$ form A to be integrated

over $(n-1)$ dim world volume

dim	object	p
1	particle	0
2	string	1
3	membrane	2
$n-1$	p-brane, $p=n-2$	p

$= \#$ spatial dim

eg $p=1 \to n=3$, "electric charge"

Gauss Law

$D-n$ form

$$Q = \int_{S^{D-n}} *F$$

p Form

$= S^{D-p-2}$

$D=4,\ p=1$

$S^{4-1-2} = S^1$

also magnetic coupling

$$Q_m = \int_{S^n} F$$

Equations of motion

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu \phi \partial_\nu \phi + S_{\mu\nu}$$

$$= \frac{1}{2(n-1)!} e^{a\phi}\left(F_{\mu\rho_1 \ldots \rho_{n-1}} F_\nu{}^{\rho_1 \ldots \rho_{n-1}} - \frac{n-1}{n(D-2)} F^2 g_{\mu\nu}\right)$$

$$\nabla_{\mu_1}\left(e^{a\phi} F^{\mu_1 \ldots \mu_n}\right) = 0$$

$$\Box \phi = \frac{a}{2n!} e^{a\phi} F^2$$

$\mu = 1, \ldots, D$.

Now, make (split) the coords into $x^\mu = \mu = 0, \ldots, n-1$

along the p-brane $y^m,\ m = n, \ldots, D-1$

transverse the p-brane

$$ds^2 = e^{2A(r)}\, dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)}\, dy^m dy^n \delta_{mn}$$

for the case being studied in HW) A and B only depend on $r=\sqrt{y^2}$

- Electric brane $A_{\mu_1 \dots \mu_n} = \varepsilon_{\mu_1 \dots \mu_n}\, e^{C(r)}$

$$F_{m \mu_1 \dots \mu_n} = \begin{cases} \varepsilon_{\mu_1 \dots \mu_n} \partial_m C\, e^{C} \\ \text{others } 0 \end{cases}$$

- Magnetic Brane

$$F_{m_1 \dots m_n} = \lambda\, \varepsilon_{m_1 \dots m_n p} \frac{y^p}{r^{n+1}}$$

$\Rightarrow \dots \Rightarrow$

$$R_{\mu\nu} = -\eta_{\mu\nu}\, e^{2(A-B)} \left(A'' + D (A')^2 + \tilde d A' B' + \frac{\tilde d + 1}{r} A' \right)$$

$D-n-2$

$$R_{mn} = -\delta_{mn} \left(B'' - D A' B' + \tilde d (B')^2 + \frac{2\tilde d + 1}{r} B' + \frac{1}{r} A' \right) - \frac{y^m y^n}{r^2} \left(\tilde d B'' + D A'' - 2 D A' B' + D (A')^2 - \tilde d (B')^2 - \frac{\tilde d}{r} B' - \frac{D}{r} A' \right)$$

$D = n$

Finally the EoM

- $A'' - n (A')^2 + \tilde d A' B' + \frac{\tilde d + 1}{r} A' = \frac{\tilde d}{2(D-2)} S^2$
- $B'' - n A' B' + \tilde d (B')^2 + \frac{2\tilde d + 1}{r} B' + \frac{n}{r} A' = -\frac{n}{2(D-2)} S^2$

$$\tilde{d}\, B' + n A'' - 2n\, A' B' + n A'^2 - \tilde{d}\,(B')^2 - \frac{\tilde{d}}{r} B' - \frac{n}{r} A' + \frac{1}{2}(\phi')^2 = \frac{1}{2} S^2.$$

$$\phi'' + n A' \phi' - \tilde{d}\, B'\phi' + \frac{\tilde{d}+1}{r}\, \phi' = \frac{1}{2}\, \rho\, a\, S^2$$

$$\rho = \pm 1$$

$$S = \begin{cases} e^{\frac{1}{2} a\phi} \text{ magnetic} \\ \lambda \left(e^{\frac{1}{2} a\phi - \tilde{d} B}\right) r^{-\tilde{d}-1} \text{ electric} \end{cases}$$

All non-linearities are multiples of $\boxed{n A' + \tilde{d} B'}$. If it is zero they become linear!

Assume that.

Notice $\nabla^2\phi = \phi'' - \frac{\tilde{d}+1}{r}\phi' \rightarrow$

- $\nabla^2\phi = -\frac{1}{2}\rho a S^2$
- $\nabla^2 A = \frac{\tilde{d}}{2(D-2)} S^2$
- $n(D-2)(A')^2 + \tilde{d}(\phi')^2 = \frac{1}{2}\tilde{d} S^2$

This suggest an assumption $\boxed{\phi' = -\rho \frac{a(D-2)}{\tilde{d}} A'}$

$$\Delta = a^2 + \frac{2n\tilde{d}}{D-2}, \quad S^2 = \frac{\Delta(\phi')^2}{a^2}$$

$$\Rightarrow \quad \nabla^2 e^{\frac{\rho\Delta}{2a}\phi} = 0$$

$$e^{\frac{a\Delta}{2a}\phi} = H(y) = 1 + \sum \frac{k_i}{|y-y_i|^{\tilde d}}$$

summation for charges k_i and points y_i.

It remains to find C.

$$\frac{\partial}{\partial r} e^{C} = -\frac{\sqrt{\Delta}}{a} e^{-\frac{a}{2}\phi} \ldots A_\phi'$$

In the end,

$$ds^2 = H^{-\frac{4\tilde d}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}} dy^m dy^n \delta_{mn}$$

$$e^{\phi} = H^{\frac{2a}{\epsilon\Delta}}$$

$$e^{C} = \frac{2}{\sqrt{\Delta}} H^{-1} \quad \text{e.e.m.c.}$$

$$k_i = \frac{\sqrt{\Delta}}{2\tilde d} \lambda$$

This solution generalizes the extremal Reissner-Nordström solution. $M = Q$ is connected to the assumptions.

Supersymmetry implies this bound eq. and fermionic fields: gravitino + gaugino to your theory and search solutions where fermi fields vanish and preserve some SUSY.

BH Laws

$\boxed{0^{th}}$ law: the horizon has constant surface gravity for a stationary BH.

$\boxed{1^{st}}$ Statement of conservation of Energy, momentum, angular momentum and charge.

$\boxed{2^{nd}}$ When anything falls down a BH or several BHs coalesce, the sum of the surface areas (or "irreducible masses") of all BH can never decrease.

Reversible and irreversible transf

$$A = 4\pi(r_+^2+a^2) = 4\pi\left[\left(M+\sqrt{M^2-Q^2-a^2}\right)^2+a^2\right]$$

reversible process: change M, a, Q but fix A

irreversible process: change A (only increase)

+ If a [illegible] angular momentum [illegible]

+ [illegible] the "irreducible mass"

$M_{irr} = \left(\frac{A}{16\pi}\right)^{1/2}$ = mass of Schwarzschild BH of surface area A.

↳ $A = (2M_{irr})^2 4\pi = 16\pi (M_{irr})^2$

[illegible] terms,

$$M^2 = \left(M_{irr} + \frac{Q^2}{4M_{irr}}\right)^2 + \frac{S^2}{4M_{irr}^2}$$

(analogous to $E^2 = m^2 + p^2$)

[illegible] nearly added

Recall that BH's can only exist if

$Q^2 + a^2 \leq M^2$.

Penrose process

If [illegible] infalling ~~mass~~ body is [illegible] by [illegible] negative gravitational [illegible] energy [illegible] rotation [illegible] things happen

so [illegible] things happen [illegible] small?

But [illegible] a [illegible]

we can neglect the [illegible]

assuming the size object size ≪ BH

and mass ≪ mass of hole.

Apply BH dynamics laws!

$\Delta M = E$ ("energy at infinity of the object")

$\Delta J = \ell$ ([illegible] of the object)

$\Delta S = \Delta|S| = L_z =$ (component of angular momentum of object onto the BH rotation axis).

Since $S = (0, 0, S)$ only L_z component changes magnitude (significantly) but others can change as

$$\Delta S_x = L_x = -\sin\phi_\infty \sqrt{\Theta_\infty} - \cot\theta_\infty \cos\phi_\infty L_z$$

$$\Delta S_y = L_y = \cos\phi_\infty \sqrt{\Theta_\infty} - \cot\theta_\infty \sin\phi_\infty L_z$$

$\Delta S_\perp$ [illegible]

Now: Penrose process

1) Shoot a object A into BH with energy/charge E_A, e_A and L_{zA}.

2) When it reaches (almost) horizon, let it explode/divide into B and C with $E_{B/C}$, $e_{B/C}$, $L_{zB/C}$.

3) Design the explosion so that B falls into the horizon but C escapes.

ΔM = Total energy going in (E_A) − total energy re-emerge (E_C).

$$= E_A - E_C = E_B$$

$$\Delta Q = e_A - e_C = e_B$$

$$\Delta S = L_{zA} - L_{zC} = L_{zB}$$

× using charge conservation

× using 4-momentum conservation ~~local~~ charge conservation

$$p_A = P_A - e_A A = P_B + P_C + (e_B + e_C) A$$

$= p_B + p_C$

$\Rightarrow$ Conservation of components of generalized momentum along $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$

$\rightsquigarrow E_A = -p_{tA} = -p_{tB} - p_{tC} = E_B + E_C$

$L_{zA} = p_{\phi B} + p_{\phi C} = L_{zB} + L_{zC}$

$\Rightarrow \Delta M = E_B,\ \Delta Q = e_b,\ \Delta S = L_{zB}$

consequence.

! By injecting matter in a carefully way one can decrease total mass-energy of the BH, one can extract energy from the hole!

Proof (using Ergosphere)

outside $E = -p \cdot \xi_{(t)} = -p^0_{\text{proper}}$ ($\xi_{(t)} = \frac{\partial}{\partial t}$)

but inside the ergosphere, the $\frac{\partial}{\partial t}$ changes to space-like, so E could be negative.

but restricted inside an orbit inside the ergosphere

For a charged object electromagnetic forces alter the region of E negative

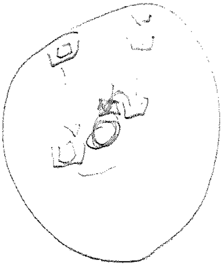
since
$$E = -\underbrace{(p - eA)}_{\text{generalized momentum}} \cdot \xi_{(t)} = -p \cdot \xi_{(t)} + \frac{eQr}{\rho^2}$$

So $E<0$ exist in a region larger than the ergosphere if $eQ<0$.

If $eQ>0$ then the region is smaller.

The region where $E<0$ for given e, Q and rest mass μ is called the "effective ergosphere".

$E_{\text{vehicle out}} = E_{\text{vehicle+garbage down}} - E_{\text{garbage gone}}$
$> E_{\text{vehicle+garbage down}}$



$E_{\text{per trip}} = E_{\text{vehicle out}} - \text{rest mass of vehicle}$
$= (\text{rest mass vehicle} + \text{rest mass garbage}) - \Delta M - \text{rest mass vehicle}$

$= \text{rest mass garbage} - \Delta M$

Energy produced for a rotating {positive} mass BH and the equivalent {charge} {electrical}

The restriction to every power having r_+ is

$$M^2 \geq Q^2 + a^2$$

$$\rightsquigarrow \delta M \geq \frac{a\,\delta S + M Q\,\delta Q}{M^2 + a^2}$$

and if the hole is on a extremal state

$M = r_+ \rightsquigarrow$ $$\delta M \geq \frac{a\,\delta S + r_+ Q\,\delta Q}{r_+^2 + a^2}$$

$$\rightsquigarrow \boxed{\delta M_{irr} \geq 0}$$

$$M_{irr} = \frac{1}{2}\left(\left(M + \sqrt{M^2 - Q^2 - a^2}\right)^2 + a^2\right)^{1/2}$$

$$= \frac{1}{2}\sqrt{r_+^2 + a^2}$$

Few remarks and further analysis

How s affects the rotation Ω of a body? (the "linear rotation" of the horizon)

$$\Omega = \frac{d\varphi}{dt} = \frac{d\varphi/d\tau}{dt/d\tau} = \frac{u^\varphi}{u^t}$$

$$u = \frac{\partial_t + \Omega\partial_\varphi}{\|\partial_t + \Omega\partial_\varphi\|} = \frac{\partial_t + \Omega\partial_\varphi}{\sqrt{-(g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi})}}$$

u has to be time like, (<0)
boundary at $(=0)$

$\rightsquigarrow \quad \|\Omega\|=0= g_{tt}+2\Omega_{min} g_{t\varphi}+\Omega_{min}^2 g_{\varphi\varphi}$

$$\rightsquigarrow \quad \Omega_{min} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} \pm \sqrt{\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}$$

<u>EOM Penrose process</u>

$$H = \frac{1}{2} g^{\mu\nu}(\Pi_\mu - eA_\mu)(\Pi_\nu - eA_\nu)$$

$$\nearrow \quad \frac{d\Pi_\alpha}{d\tau} = -\frac{\partial H}{\partial x^\alpha} \quad \nearrow \quad \left\{\begin{array}{l} E = -\Pi_t = P_t - eA_t \\ L_z = \Pi_\varphi = P_\varphi - eA_\varphi \\ \mu = \|p\| = \sqrt{-g^{\mu\nu} P_\mu P_\nu} \end{array}\right.$$

constants of motion

$\boxed{\text{Analy}}$ in Kerr-Newman other constants

$$Q = P_\theta^2 + \cos^2\theta\left(a^2(\mu^2 - E^2) + \frac{L_z^2}{\sin^2\theta}\right)$$

$$P = E(r^2+a^2) - L_z a - eQr$$

EOM

ρ^2

Equations of motion test particles:

Let a test point of charge e and test mass μ move in the external fields of a BH.

If BH's $Q=0 \rightsquigarrow$ equations of electrodynamic free giving
momentum, 4-velocity

$$\mu a = eF \cdot u$$

. uncharged case $(e=0) \rightsquigarrow$ geodesic equation

$a=0$ is equivalent to

$$\frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu}$$

where λ is an affine parameter so that

$$\frac{\partial}{\partial \lambda} = p = \text{4-momentum}.$$

and $H =$ 'Super-Hamiltonian' $= \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$

. charged case $H = \frac{1}{2} g^{\mu\nu} (\pi_\mu - eA_\mu)(\pi_\nu - eA_\nu)$

π_μ is generalized momentum

$$\boxed{\pi^\mu = \mu u^\mu + eA^\mu}$$

and $\frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial \pi_\mu}, \quad \frac{d\pi_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu}$

Solution:

1^{st}: $$p^\mu = \frac{dx^\mu}{d\lambda} = \pi^\mu - eA^\mu$$

2^{nd}: $$\frac{d\pi_\mu}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^\mu} = -\frac{\partial}{\partial x^\mu}\left(\frac{1}{2} g^{\mu\nu} P_\mu P_\nu\right)$$

$d(P_\mu + eA_\mu)$

$$\hookrightarrow \frac{dp^\mu}{d\lambda} + \frac{deA^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta$$

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = e F^{\mu\nu} P_\nu$$

$\uparrow P_\nu$ not u_ν

Kerr-Newman in Boyer-Lindquist coords thus $\lambda = \tau/\mu$

$$\hookrightarrow A = -\frac{Qr}{\rho^2}\left(dt - a\sin^2\theta\, d\phi\right)$$

$$\hookrightarrow dA = \frac{1}{2}\left(A_{\beta,\alpha} - A_{\alpha,\beta}\right) dx^\alpha \wedge dx^\beta$$

(Exact 2-form)

- Why didn't we used
$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = eF^\alpha{}_\beta \frac{dx^\beta}{d\lambda} ?$$

(this system)

$\amalg$ allows to find constants of motion.

Since $g_{\mu\nu}$ and A_μ are indep of t, ϕ

$\rightsquigarrow$ $\mathcal{H}$ is indep of t, ϕ $\Rightarrow$

$\boxed{\Pi_t \text{ and } \Pi_\phi \text{ are constants of motion}}$

Far from the BH (almost Minkowski)

$$\rightsquigarrow \quad ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

$$\Rightarrow \quad \Pi_t = p_t = -\text{Energy} = -p^{\hat{t}}$$

$$\Pi_\phi = p_\phi = g_{\phi\phi} p^\phi = r p^{\hat{\phi}} = \text{projection of angular momentum along BH's rotation axis}$$

$$\rightsquigarrow \quad \boxed{E} \equiv (\text{"energy at infinity"}) = -\Pi_t$$

$$= -(p_t + eA_t)$$

$$\boxed{L_z} \equiv \Pi_\phi = p_\phi + eA_\phi$$

And $\boxed{\mu} = |p| = (-g^{\alpha\beta} p_\alpha p_\beta)^{1/2}$

← particle's rest mass

We need # constants = # dofs to determine orbits

Carter's fourth constant of motion is:

$$\boxed{Q} = p_\theta^2 + \cos^2\theta \left[a^2(\mu^2 - E^2) + \sin^{-2}\theta \, L_z^2 \right]$$

and define $K = Q + (L_z - aE)^2 \geq 0.$

Hence we can express the 4-momentum p^α in terms of these constants

↳ resulting in

$$\rho^2 \frac{d\theta}{d\lambda} = \sqrt{\Theta}$$

$$\rho^2 \frac{dr}{d\lambda} = \sqrt{R}$$

$$\rho^2 \frac{d\phi}{d\lambda} = -\left(aE - \frac{L_z}{\sin^2\theta}\right) + \frac{a}{\Delta} P$$

$$\rho^2 \frac{dt}{d\lambda} = -a\left(aE\sin^2\theta - L_z\right) + \frac{(r^2+a^2)}{\Delta} P$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and

$$\Theta = Q - \cos^2\theta \left[a^2(\mu^2 - E^2) + L_z^2/\sin^2\theta\right]$$

$$P = E(r^2 + a^2) - L_z a - eQr$$

$$R = P^2 - \Delta\left[\mu^2 r^2 + (L_z - aE)^2 + Q\right]$$

Hence as

$$L_x = \sin\phi \frac{\partial}{\partial\theta} + \cot\theta\cos\phi \frac{\partial}{\partial\phi}$$

$$L_y = -\cos\phi \frac{\partial}{\partial\theta} + \cot\theta\sin\phi \frac{\partial}{\partial\phi}$$

$$L_z = \frac{\partial}{\partial\phi} = L_z$$

p_θ, L^2

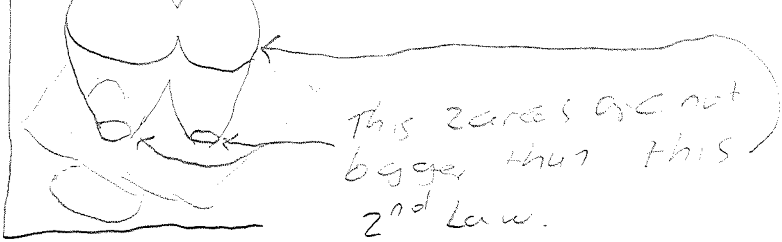
Penrose (Theorem):

$\dot{J}^-(\dot{J}^+)$ is generated by null geodesics that have no future endpoints.

- The generators are null geodesics that lie in $\dot{J}^-(\dot{J}^+)$ for atleast a finite amount of affine parameter.
- when followed to the past a generator may leave $\dot{J}^-(\dot{J}^+)$ but only at a caustic.

horizon

- when they leave, they leave to $J^-(J^+)$
- Once a generator enters a caustic it can never leave again or intersect another generator, thus through any non-caustic event there passes exactly one generator.



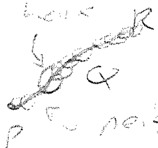
Remark 1:  If c_1 is future directed causal curve from p to Q, c_2 ... Q to R

and $p \not\ll R \Rightarrow c_1$ and c_2 are null geodesics with same tangent at Q.

Corollary: $c_1 \circ c_2$ is a null geodesic.

Proof: Assume that they are not null geodesics.

- If there is a timelike segment

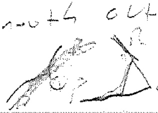
⇝ open condition of $p \ll q$ ⇝  neighborhoods

there ... $\Rightarrow p \ll R$, a contradiction.

So c_1 and c_2 has to be lightlike everywhere.

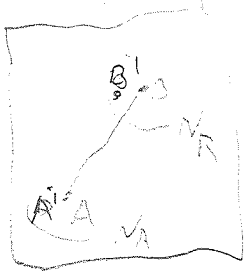
- If ... not geodesic at some point $\Rightarrow$ ... length is not extremal ... since it's 0 (the curves are null)

$\Rightarrow$ there is a neighborhood ~~where~~ of curves with positive and negative proper length. I.e. there exist a timelike curve. From p to $R \Rightarrow p \ll R$. A contradiction.

- If tangents in Q don't agree then one could smooth out the corner by introducing a timelike segment ... the whole curve ... will not be  a null geodesic. A contradiction.

Lemma 2: If $A, B \in \dot{J}^-(\mathcal{J}^+)$ then $A \not\ll B$

Proof: Assume $A \ll B$, since preceed is an open property $\exists N_B$ s.t. there is a timelike from A to $B' \in N_B$, $A \ll C$ and symmetry for $A' \in N_A$ $D \ll C_B$



Since $A, B \in \dot{J}^-(\mathcal{J}^+)$ ~~MANY~~ both neighborhoods contain points in and out of ~~the bo..~~ $J^-(\mathcal{J}^+)$. Take B' inside $J^-(\mathcal{J}^+)$ and A' outside.

B' is causally to $J^-(\mathcal{J}^+)$ but A' not.

And since $A' \ll B'$, $\exists$ a A' to a causal curve to $\mathcal{J}^+ \Rightarrow A' \in J^-(\mathcal{J}^+)$

A contradiction

Lemma 3: Let $c(\lambda)$ be a causal curve that intersects $\dot{J}^-(\mathcal{J}^+)$ at some event B, then, before B, $c(\lambda)$ is in $J^-(\mathcal{J}^+) \cup \dot{J}^-(\mathcal{J}^+)$

Proof: Assume A on $c(\lambda)$ that is not in $J^-(\mathcal{J}^+)$ then, there $\exists$ N_A nbhd of A. In N_A there $\exists$ $A' \ll B$

Similarly N_B such that $A' \ll N_B$. Since N_B is a neighborhood of a point in the Boundary

$\exists\, B' \in N_B \cap J^-(\mathscr{I}^+) \rightsquigarrow A' \in J^-(\mathscr{I}^+)$

$A' \ll B' \in J^-(\mathscr{I}^+)$

A contradiction.

<u>Proof of theorem</u> (Penrose)

1) Pick $p \in \dot{J}^-(\mathscr{I}^+)$, lets show there is a null geodesic through p that lies inside of $\dot{J}^-(\mathscr{I}^+)$.

- There's a sequence $\{P_i\}_{i=1}^{\infty} \subseteq J^-(\mathscr{I}^+)^N$ with $P_i \xrightarrow{i\to\infty} p$, pick one of the C_i that connects P_i to $\mathscr{I}^+$.

Then, pick a ball B_p of p s.t. ~~all~~ all but finite $P_i \in B_p$. For those, denote $Q_i = C_i \cap \partial B_p$. Since ∂B is compact, Q_i has an accumulation point Q. Since $P_i \ll Q_i \xrightarrow{\text{limit}}$ causal curve from p to Q

i.e. $Q \ll p$.

Notice that $Q \notin \bar{J}^-(\mathscr{I}^+)$ as otherwise there would be ~~a~~ nbhd of Q s.t. outside the horizon. ~~In this~~

which means that $R \gg Q$ ~>

the curve $c \cup c'$ where c is curve from P to Q and c' is timelike from Q to R $\Rightarrow$

and therefore we can connect to events to $\mathscr{I}^+$.

Now) a nbhd around P will have some $\dot{J}^-(\mathscr{I}^+)$

and the ... component will be ... S

succeeds $\mathscr{I}^+$. A contradiction $\Rightarrow Q \in \dot{J}^-(\mathscr{I}^+)$

Now, by lemma 2, $P, Q \in \dot{J}^-(\mathscr{I}^+)$ and

~~$Q \ll P$ ~>~~ $\Rightarrow Q \not\ll P$. but since

there's a causal curve from Q to P $\Rightarrow$

this curve is null geodesic by lemma 1.

By lemma 3, since its causal and ~~...~~

~~...~~/connects $P, Q \in \dot{J}^-(\mathscr{I}^+) \Rightarrow c$ lies

in $\dot{J}^-(\mathscr{I}^+)$

$\Rightarrow$ Through every $P \in \dot{J}^-(\mathscr{I}^+)$ there passes a null geodesic c which, when followed into the future from P, lies in $\dot{J}^-(\mathscr{I}^+)$.

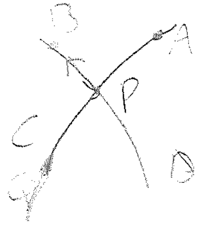
This is called a "generator" of $\dot{J}^-(\mathcal{J}^+)$
(it cannot go out.)

2) Extend C by geodesic equation but can't really leave! suppose till p'.
[illegible] the construction [illegible], by [illegible]!
[illegible] curves.

<u>Hence</u> once a generator, being followed into the future, enters $\dot{J}^-(\mathcal{J}^+)$ — can never leave $\dot{J}^-(\mathcal{J}^+)$

3) Geodesics can enter $\dot{J}^-(\mathcal{J}^+)$ at caustics but cannot cross away later.
Lemma 3 guarantees that it enters from $J^-(\mathcal{J}^+)$

4)



Let's assume they cross.
to the causal future of P,
both generators lie in $\dot{J}^-(\mathcal{J}^+)$. i.e.
A and B [illegible] $\dot{J}^-(\mathcal{J}^+)$.

Since P is in the causal future of the caustic where ~~they enter~~ C enters $\dot{J}^-(\mathcal{J}^+)$
$\to \exists\, Q \in \dot{J}^-(\mathcal{J}^+) \cap C$. to causal past of P.
Since $Q, B \in \dot{J}^-(\mathcal{J}^+)$ $Q \not\ll B$ by Lemma 2.

and, by lemma 1, the curves C and D connecting $C - p - B$ $\Rightarrow$ tangents are equal $\Rightarrow$ C and D are the same curve C.

<u>Hence</u> — once a generator has entered $\mathring{J}^-(\gamma^+)$

it can never thereafter cross any other generator.

Second law of ~~thermodynamics~~ BH dynamics.

How to convert penrose theorem into areas?

Penrose states that $\dot{J}^-(\mathscr{J}^+)$ ← Horizon is generated by null geodesics, this have no future end points. The generators lie in $\dot{J}^-(\mathscr{J}^+)$ for a finite ~~time~~ amount of ... [illegible], this generators may leave ... when followed to the past and, in $\dot{J}^-(\mathscr{J}^+)$ they never cross (other generators)

We use the "focusing inequality"

$$\frac{d^2 a}{d\lambda^2} \leq 0 \quad \leftarrow a \text{ area density.}$$

Thus, if $\frac{da}{d\lambda} < 0 \Rightarrow \lambda' > \lambda \rightsquigarrow \left.\frac{da}{d\lambda}\right|_{\lambda'} \leq \left.\frac{da}{d\lambda}\right|_{\lambda}$

$\rightsquigarrow a(\lambda)$ after some time must be negative at some point (ever decreasing), a contradiction.

$$\Rightarrow \left\{\frac{d^2a}{d\lambda^2} \leq 0 \text{ and } \frac{da}{d\lambda} \geq 0\right\} \Rightarrow a \text{ increases or stays the same.}$$

The fact that after affine parameter $\frac{a(\lambda)}{\frac{da}{d\lambda}}$

It has to become negative implies that horizon intersects singularities at finite affine param (i.e. there exist a 'naked' singularity).

$$\text{If } \frac{da}{d\lambda} < 0 \rightsquigarrow \frac{da(\lambda')}{d\lambda}\Big|_{\lambda'} \le \frac{da(\lambda)}{d\lambda}\Big|_{\lambda}, \ \lambda' > \lambda$$

(focusing) Hence, using an affine parameter we can set a bound where a goes to zero by

$$\frac{a(\lambda') - 0}{\Delta \lambda'} \approx -\frac{da(\lambda')}{d\lambda}$$

$$\rightsquigarrow \Delta \lambda' \le \frac{a(\lambda')}{\underbrace{-\frac{da(\lambda')}{d\lambda}}_{\text{magnitude}}} \leftarrow \text{finite}$$

So we can reach $a = 0$ in a finite affine parameter and adjacent null geodesics cross each other in $\dot{J}(\mathcal{J}^+)$ so we know one geodesic pokes in ... that $\rightsquigarrow$ Penrose th is ...

so either $\frac{da}{d\lambda} > 0$ or if $\frac{da}{d\lambda} < 0$

and before the generators cross to each other (before $\Delta\lambda$ as above given) the generators hit a singularity and cease to exist.

Hawking: assume space time is "strongly asymptotically predictable" ← i.e. no naked singularities

Critics: no singularities is lift by the horizon and then by $\frac{da}{d\lambda} \geq 0$. (so assume no naked singularities.)

Proof of Focusing inequality:

Preliminaries: Geometric optics.

valid asymptotically as $\lambda \ll \mathcal{L}$ ← length scale on which scale changes amplitude

Phase θ: $R \ll \frac{1}{\lambda^2}$ (curvature) $A_\mu = a_\mu e^{i\theta/\varepsilon}$

$k = \nabla\theta$ wave vector ($\theta = k_\mu x^\mu$)

$a = \sqrt{a \cdot a}$ scalar amplitude

$f = \frac{a}{a}$ polarization vector

$A_\mu = \mathrm{Re}\left[(a_\mu + \varepsilon b_\mu + \dots) e^{i\theta/\varepsilon}\right]$

Lorentz Gauge:

$O(\frac{1}{\varepsilon})$: $K \cdot a = 0 = \vec{K} \cdot f$

$\rightsquigarrow$ wave equation

$$0 = (\Box A)^\mu = -A^{\mu;\nu}{}_\nu + R^\mu{}_\nu A^\nu$$

wave eq acting on $e^{i\theta/\varepsilon} \rightsquigarrow$

at $O(\frac{1}{\varepsilon^2})$

$$K^\nu K_\nu a^\mu = 0 \Leftarrow \|K\| = 0$$

$$0 = \underbrace{K^\nu K_\nu}_{=0} b^\mu - 2i\left(K^\nu a^\mu{}_{;\nu} + \frac{1}{2} K^\nu{}_{;\nu} a^\mu\right)$$

$\hookrightarrow$ $\boxed{\nabla_K a = -\frac{1}{2}(\nabla \cdot K)\, a}$

$K = \nabla\theta$ (i.e. $K_\nu = \theta_{;\nu}$)

$$0 = \|K\|^2 \Rightarrow 0 = 2K^\nu K_{\nu;\mu} = 2K^\nu \theta_{;\nu\mu}$$

$$= 2K^\nu \left[\theta_{;\mu\nu} + R\,\cancel{\theta}\right]$$

$$= 2K^\nu \nabla_\nu K_\mu = \boxed{\nabla_K K = 0}$$

^ K is parallel transported along the light ray.

Let's see the scalar amplitude a:

$$2a\nabla_K a = \nabla_K a^2 = \nabla_K(\bar{a}a) = (\nabla_K \bar{a})a + \bar{a}\nabla_K a$$

Let's see how light moves using Maxwell eq!

$$\nabla_a F^{ab} = 0 \quad , \quad F_{ab} = \nabla_a A_b - \nabla_b A_a$$

$\Rightarrow$ $$\underbrace{\nabla_a \nabla^a A_b}_{\Box A_b} - \underbrace{\nabla_a \nabla^b}_{\nabla^b \nabla_a + R^{ba}} A_a = 0$$

$\Rightarrow$ $$\Box A^b - \nabla^b \underbrace{\nabla_a A^a}_{\text{choose Lorentz gauge: } (\nabla_a A^a = 0)} - R^{ba} A_a = 0$$

$\Rightarrow$ $$\boxed{\Box A^b = R^{ba} A_a}$$

$\Rightarrow$ Approximation: wave length << curvature

$\Rightarrow$ $A_a = a_a e^{i\theta}$ ← phase; a_a ← polarization

$\theta = k_\mu x^\mu$ ← direction of the ray

$\Rightarrow$ $\boxed{\nabla\theta = k}$

In flat space a_μ does not depend on position

~~but~~ ~~...~~ $\rightsquigarrow$ $A_\mu \sim e^{iK_b X^b/\varepsilon}$

$\hookrightarrow$ $\hat{P}_c = -i\hbar\partial_c \rightsquigarrow \hat{P}_c A_a = -\hbar i \cdot \frac{i}{\varepsilon} K_a A_a$

$= \underset{\underset{K_c}{\parallel}}{P_c} A_a$ ✓

wave eq $\rightsquigarrow$ $p^2 = 0$

Gauge condition $\rightsquigarrow$ $P_a a^a = 0$

In CST

$A_\mu \sim (a_a + \varepsilon b_\mu + \varepsilon^2 c_\mu + \dots) e^{i\theta/\varepsilon}$

Expansion in powers of ε ($\hbar$)

Gauge $\rightsquigarrow$ $\nabla_a A^a = 0 = (\nabla_a a^a - \varepsilon \nabla_a \dots) e^{i\theta/\varepsilon}$

$+ (a^a + \varepsilon \dots) e^{i\theta/\varepsilon} \left(\frac{i\nabla_a \theta}{\varepsilon}\right)$

order $\frac{1}{\varepsilon}$ $\longrightarrow$ $a^a \nabla_a \theta = 0$ $\rightsquigarrow$ here, $\boxed{\nabla_a \theta \equiv P_a \text{ "momentum"}}$

$0 = O(\varepsilon^2)$

order 1 $\rightsquigarrow$ $\nabla_a a^a + ib^a \nabla_a \theta = 0$

$\hookrightarrow$ $\boxed{\nabla_a a^a + i P_a b^a = 0}$

Maxwell eq's $\rightsquigarrow$

$$\Box A^a - R^{ab} A_b = 0$$

$$\nabla_b \nabla^b \left((a^a + \hbar b^a + \dots) e^{i\theta/\hbar} \right) - R^{ab} A_b = 0$$

$$= \nabla_b \left((\nabla^b a^a + \dots) e^{i\theta/\hbar} + (a^a + \dots)(\nabla^b \theta) e^{i\theta/\hbar} \left(\frac{i}{\hbar}\right) \right) - R^{ab} A_b$$

$$= (\Box a^a + \dots) e^{i\theta/\hbar} + \frac{i}{\hbar} (\nabla_b \theta)(\nabla^b a^a \dots)$$

$$+ (\nabla_b a^a + \dots)(\nabla^b \theta) e^{i\theta/\hbar} \left(\frac{i}{\hbar}\right)$$

$$+ (a^a + \dots)(\nabla_b \theta)(\nabla^b \theta) e^{i\theta/\hbar} \left(\frac{-1}{\hbar^2}\right)$$

$$+ (a^a + \dots)(\nabla_b \nabla^b \theta) e^{i\theta/\hbar} \frac{i}{\hbar}$$

$$- R^{ab} A_b = 0$$

$\boxed{\text{To order } \frac{1}{\hbar^2}}$ $\rightsquigarrow$ $(\nabla_a \theta)^2 = 0$

$p_a p^a$ $\leftarrow$ Thus the tangent to the trajectory of light rays is null

$\boxed{\text{To order } \frac{1}{\hbar}}$ $\rightsquigarrow$

$$a_a \nabla_b p^b + 2 p^b \nabla_b a_a = 0$$

$$+ \cancel{p_b} \nabla_b p^b \cdot b_a$$

$\rightsquigarrow -\frac{1}{2} a_a \nabla_b p^b = \nabla_p a_a$

$$\boxed{\nabla_k a = -\frac{1}{2}(\nabla \cdot k)\, a}$$

remark

$p_a p^a = 0$ and $p_a = \nabla_a \theta$

since $\nabla_b p_a = \nabla_a p_b$, p_a is exact.

$0 = \nabla_b p_a p^a = 2p^a \nabla_b p_a$

$= 2p^a \nabla_a p_b$

$= 2\nabla_p p \rightsquigarrow$ p obeys geodesic equation

Parallel transport of a along p (geodesic) null is the divergence of p.

a_a contains info about { polarization, amplitude

$\rightsquigarrow a_a = a\, \varepsilon_a$ polarization (unit spacelike vector), amplitude

↳ insert a and ε_a ⤳

$$\boxed{p^b \nabla_b (a \varepsilon_a) = -\frac{1}{2} a \varepsilon_a \nabla_b p^b}$$

$$p^b \nabla_b a \, \varepsilon_a + p^b a \nabla_b \varepsilon_a = -\frac{1}{2} a \varepsilon_a \nabla_b p^b$$

$\varepsilon_a \varepsilon^a = 1$, $\varepsilon^a \nabla_b \varepsilon_a = 0$

↳ contract with ε^a ⤳

$$p^b (\nabla_b a) + p^b a \, \cancel{\varepsilon^a \nabla_b \varepsilon_a} = -\frac{1}{2} a \nabla_b p^b$$

$$\nabla_k a = -\frac{1}{2} a \nabla_b p^b = -\frac{1}{2} a (\nabla \cdot p)$$

$$\boxed{\nabla_p a = -\frac{1}{2} a (\nabla \cdot p)} \quad \text{amplitud eq.}$$

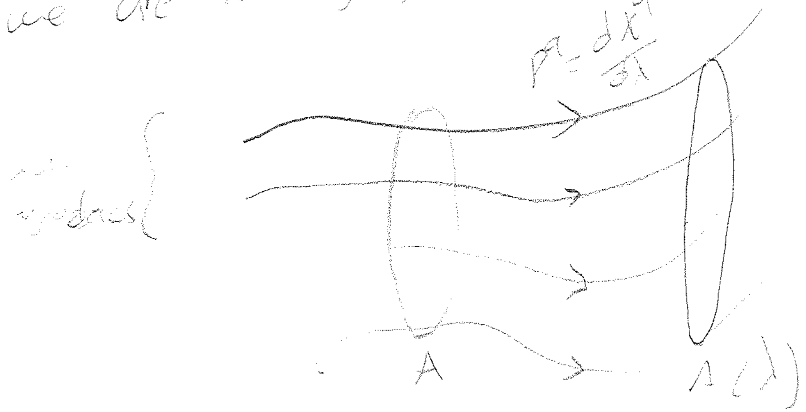
⤳

$$p^b a \nabla_b \varepsilon_a = 0$$

$$a \nabla_p \varepsilon_a = 0 \rightsquigarrow \boxed{\nabla_p \varepsilon_a = 0}$$

polarization is parallel transported along null geodesics p.

we are looking for



How are spread out?

$A(\lambda)$ = the cross-sectional area of a beam

How to see this relation?

Energy is conserved for the beam as

$$\frac{d(A a^2)}{d\lambda} = 0$$ Energy ~ radiation

equivalently

$$p^a \nabla_a (A a^2) = 0$$

$$= p^a(\nabla_a A)\, a^2 + 2\underbrace{p^a(\nabla_a a)}_{\text{we know}} A$$

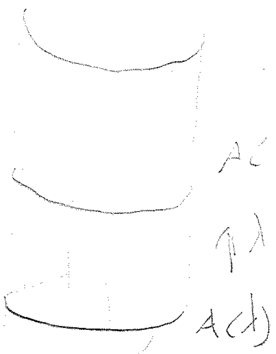
amplitude eq.

$$p^a \nabla_a A\, a^2 - a^2 \nabla_a p^a A \overset{!}{=} 0$$

$$\Leftrightarrow \boxed{p^a \nabla_a A = A \nabla_a p^a} = A(\nabla \cdot p)$$

Focusing equation:

The event horizon consist of null geodesics that propagate through spacetime.



Focusing equation: ~~Analyzes~~ Analyze things p... a perturbation (matter like BH) i.e. stationary no more.

$$\frac{d^2}{d\lambda^2} A^{\frac{1}{2}} = p^a \nabla_a \left(p^b \nabla_b A^{\frac{1}{2}}\right)$$

$$= p^a \nabla_a \left(\frac{1}{2} p^b (\nabla_b A) \cdot A^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2} p^a \nabla_a \left(A^{\frac{1}{2}} \nabla_b p^b\right)$$

$$= \frac{1}{4} p^a (\nabla_a A)(A^{-\frac{1}{2}})(\nabla_b p^b) + \frac{1}{2} A^{\frac{1}{2}} p^a \nabla_a \nabla_b p^b$$

$$= \frac{1}{4} A^{\frac{1}{2}} (\nabla_a p^a)(\nabla_b p^b) + \frac{1}{2} A^{\frac{1}{2}} p^a \nabla_b \nabla_a p^b + \frac{1}{2} A^{\frac{1}{2}} R_{a\ c}^{\ b} p^c$$

$$= \frac{1}{4} A^{1/2} (\nabla_a p^a)(\nabla_b p^b) - \frac{1}{2} A^{1/2} R_{ab} p^a p^b + \frac{1}{2} A^{1/2} p^a \nabla_b \nabla_a p^b$$

$$= \frac{1}{4} A^{1/2} (\nabla_a p^a)(\nabla_b p^b) - \frac{1}{2} A^{1/2} R_{ab} p^a p^b + \frac{1}{2} A^{1/2} \nabla_b \underbrace{\cancel{\left(p^a \nabla_a p^b\right)}}_{p \text{ geodesic}} - \frac{1}{2} A^{1/2} (\nabla_b p_a)(\nabla^a p^b)$$

(Focusing eq.

III $$\boxed{\frac{d^2}{d\lambda^2} A^{1/2} = - A^{1/2} \left(\frac{1}{2} R_{ab} p^a p^b + \sigma^2 \right)}$$

Focusing eq.

where $\sigma^2 = \frac{1}{2} (\nabla_a p_b)(\nabla^b p^a) - \frac{1}{4} (\nabla_a p^a)(\nabla_b p^b)$

Let's see if σ^2 is indeed positive.

$$\sigma^2 = \frac{1}{2} (\nabla_a p_b)(\nabla_c p_d) \left(g^{ac} g^{bd} - \frac{1}{2} g^{ab} g^{cd} \right)$$

check positivity.

$\nabla_a p_b - \nabla_b \nabla_a$ ...

$X_{ab} \cdot \nabla_a p_b = X_{ba}$.

Let L^μ be a null vector, with $L^\mu k_\mu = -1$

(e.g. $k = \frac{e_0 + e_x}{\sqrt{2}}$, $L = \frac{e_0 - e_1}{\sqrt{2}}$

$\leadsto L^\mu k_\mu = -\frac{1}{2} - \frac{(-1)}{2}$)

Tues

$$\begin{cases} X_{ab} = \nabla_a p_b \\ p^a \nabla_a p_b = 0 \\ \quad \shortparallel \\ p^a X_{ab} = 0 \end{cases} \Rightarrow X = \begin{pmatrix} \alpha & \alpha & \beta & \gamma \\ \alpha & \alpha & \beta & \gamma \\ \beta & \beta & \varepsilon & \lambda \\ \gamma & \gamma & \lambda & \xi \end{pmatrix}$$

$g^{00} = -1$, $g^{ij} = \delta^{ij}$ in a suitable local frame.

$\omega^2 = \lambda^2 - \frac{1}{4}(\varepsilon - \xi)^2 \geq 0.$

$= \frac{1}{2} \cancel{X}_{ab} X_{cd} (g^{ac} g^{bd} - \frac{1}{2} g^{ab} g^{cd})$

EFE's

$R_{ab} = \frac{1}{2} R g_{ab} = 8\pi T_{ab}$

$R_{ab} p^a p^b = 8\pi T_{ab} p^a p^b$

$T = \begin{pmatrix} \text{Energy density} & \text{Momentum flux} \\ \shortparallel & (\text{stress}) \end{pmatrix}$

$T_{ab} p^a p^b$ = Energy density + |momentum flux|

Believe this is positive

$\Rightarrow$ $\boxed{R_{ab} p^a p^b \geq 0}$ ← (Strong energy condition.)

Holds for all classical matter.

⇝ using focusing eq.,

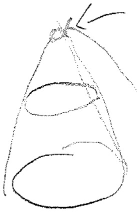
$$\boxed{\frac{d^2 A^{1/2}}{d\lambda^2} \leq 0}$$

focusing inequality

"Gravity attracts light ⇝ light converge"

Hawking singularity theorems.

Suppose A decreases to zero.



BH squeezed out to existence

the notion of spacetime breakdown

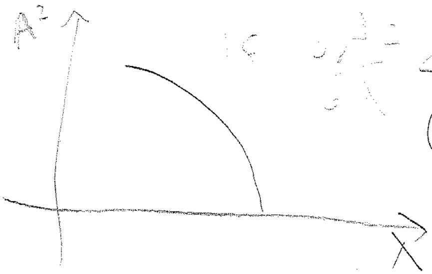
at a point. "singular"

we assume this don't happend (reasonable)

"cosmic censorship hypothesis"

Suppose $\frac{dA^{1/2}}{d\lambda} < 0$ then $A^{1/2} \to 0$ at some point in the future.

(using) $\frac{d^2A^{1/2}}{d\lambda^2} \le 0$.



if $\frac{dA^{1/2}}{d\lambda} < 0$ $\left(\frac{d^2A^{1/2}}{d\lambda^2} \le 0\right)$

Then $\frac{dA^{1/2}}{dA} > 0$ otherwise geodesic reaches singularities.

1st law $dM = \frac{\kappa\, dA}{8\pi} + \Omega\, dJ$

2nd law $dA \ge 0$

Strongly suggests $\begin{cases} A \sim S & \text{(entropy)} \\ \kappa \sim T \end{cases}$

but in reality $\frac{A}{4} \sim S$, $\frac{\kappa}{2\pi} \sim T$ $T \sim 6K$, $S \sim \frac{A}{4}$

Lets see $\nabla_K(\nabla\cdot K) = K^\mu K^\nu{}_{;\nu\mu}$

$$= K^\mu R^\nu{}_{\rho\mu\nu} K^\rho + K^\mu K^\nu{}_{;\mu\nu}$$

$$= -K^\mu K^\rho R_{\mu\nu} + \Big(\underbrace{K^\mu K^\nu{}_{;\mu}}_{\nabla_K K^\nu}\Big)_{;\nu} - K^\mu{}_{;\nu} K^\nu{}_{;\mu}$$

$$= -K^\mu K^\nu R_{\mu\nu} - K^\mu{}_{;\nu} K^\nu{}_{;\mu}$$

$$\Rightarrow \frac{d^2 a^{1/2}}{d\lambda^2} = \Big(\frac{1}{2} K^\mu K^\nu (+R_{\mu\nu}) - \frac{1}{2} K^\mu{}_{;\nu} K^\nu{}_{;\mu} + \frac{1}{4}(K^\mu{}_{;\mu})^2\Big)\sqrt{a}$$

$$= -\Big(\frac{1}{2} K^\mu K^\nu (+R_{\mu\nu}) + \underbrace{\|\sigma\|^2}_{\text{shear}}\Big)\sqrt{a}$$

Lets see if the shear is [illegible] space.

Consider $B_{\mu\nu} \equiv K_{\mu;\nu} \underset{K_\mu = \partial_\mu \theta}{=} \frac{1}{2}(K_{\mu;\nu} + K_{\nu;\mu})$

$$K^\mu B_{\mu\nu} = K^\nu B_{\mu\nu} = (\nabla_K K)_\nu = 0.$$

$$= -\frac{1}{2}(\nabla\cdot K)\,\bar{a}a - \bar{a}(\nabla\cdot K)\,a = -a^2\,\nabla\cdot K$$

$$\Rightarrow \boxed{\nabla_K a = -\frac{1}{2}(\nabla\cdot K)\,a}$$

or $\nabla\cdot(a^2K) = \nabla_K a^2 + a^2\,\nabla\cdot K$

$= -a^2\,\nabla\cdot K + a^2\,\nabla\cdot K = 0$

$$\Rightarrow \boxed{\nabla\cdot(a^2K) = 0}$$

For polarization, $a = f\,a$

$$\Rightarrow 0 = \nabla_K(af) + \frac{1}{2}(\nabla\cdot K)\,af$$

$$= a\,\nabla_K f + f\,\underbrace{\left(\nabla_K a + \frac{1}{2}(\nabla\cdot K)\,a\right)}_{0}$$

$\Rightarrow \boxed{\nabla_K f = 0}$ & f is parallel transported.

Now, lets see a

$$\frac{d^2 a^{1/2}}{d\lambda^2} = \nabla_K\nabla_K\sqrt{a} = \nabla_K\left(\frac{(\nabla\cdot K)\,a}{2\sqrt{a}}\right)$$

$$= \frac{1}{2}\nabla_K\left(\nabla\cdot K\,\sqrt{a}\right) = \frac{1}{2}\left(\nabla_K(\nabla\cdot K)\sqrt{a} + \frac{1}{2}(\nabla\cdot K)^2\sqrt{a}\right.$$

Let L^μ be a null vector, with $L^\mu K_\mu = -1$

(e.g. $K = \frac{e_t + e_x}{\sqrt{2}}$, $L = \frac{e_t - e_x}{\sqrt{2}}$

$K^\mu L_\mu = -\frac{1}{2} + \frac{1\cdot(-1)}{2} = -1$)

Define "orthogonal" metric

$$h_{\mu\nu} = g_{\mu\nu} + L_\mu K_\nu + L_\nu K_\mu$$

$$K^\mu h_{\mu\nu} = K_\nu + \underbrace{L_\mu K_\nu K^\mu}_{-K_\nu} + L_\nu \underbrace{K_\mu K^\mu}_{0}$$

$$= 0 \quad (\text{same for } h_{\mu\nu} L^\mu)$$

and $h_{\mu\nu} h^{\mu\nu} = h_{\mu\nu}(g^{\mu\nu} + L^\mu K^\nu + L^\nu K^\mu)$

$$= (D + (-1) + (-1)) = D-2$$

$\Rightarrow$ split $B_{\mu\nu} = K_{\mu;\nu}$ with this metric into trace and traceless part:

$$\theta := h^{\mu\nu} B_{\mu\nu} \underset{\uparrow}{=} g^{\mu\nu} B_{\mu\nu}$$

$$B_{\mu\nu} K^\mu = 0$$

$$\sigma_{\mu\nu} = B_{\mu\nu} - \frac{1}{D-2} h_{\mu\nu} \theta$$

$L^\mu B_{\mu\nu} L^\nu = L^\mu L^\nu K_{\mu;\nu} = L^\mu K^\nu L_{\mu;\nu} = \frac{1}{2} \nabla_\nu (L^2) = 0$

[which used the fact that

$$0 = \mathcal{L}_K L = \nabla_K L - \nabla_L K = K^\nu L_{\mu;\nu} - L^\nu K_{\mu;\nu}\,]$$

Hence $B_{\mu\nu}$ lives only on $D-2$ spatial manifold projected on by h.

(orthogonal to $[L,K]$)

$\sigma^{\mu\nu}\sigma_{\mu\nu} \geq 0$ completing the assertion.

$$= K_{\mu;\nu}K^{\mu;\nu} - \frac{1}{2}\left(K^{\mu}{}_{;\mu}\right)^2$$

Notice that our here

$$B = \nabla K$$

$$\bigcirc \longrightarrow \bigcirc \sim \operatorname{Tr} B$$

$$\theta = \nabla\cdot K = \frac{d\sqrt{a}/d\lambda}{\sqrt{a}}$$

Shear (symmetric traceless part) of B

$$\sigma = B - \frac{1}{D-2} h\theta$$

$\bigcirc \longrightarrow$ ⬭

rotation (antisymmetric traceless part of ?)

$\nabla\theta = K$ ~ Antisymmetry "the derivatives can be exchanged"

$\bigcirc \longrightarrow \circlearrowleft$

In the focusing equation

$$\frac{d^2 a^{1/2}}{d\lambda^2} = -\left(|\sigma|^2 + \frac{1}{2} R_{\mu\nu} k^\mu k^\nu\right)$$

using EFE's ⇝

$$R_{\mu\nu} k^\mu k^\nu = 8\pi T_{\mu\nu} k^\mu k^\nu$$

Hence, focusing eq has a definite sign if matter ($T_{\mu\nu}$) obeys the "null energy condition"

$$\boxed{\forall \text{ light like } k : \quad \boxed{T_{\mu\nu} k^\mu k^\nu \geq 0}}$$

The null energy condition ~~is strong~~ follows from the weak energy condition which requires this for all timelike k (all observers see non-negative energy)

Ex: perfect fluid $T = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$

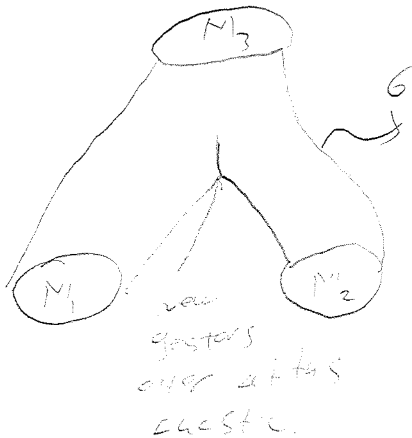
NEC: $\rho + p \geq 0$. WEC: $\rho + p \geq 0$, $\rho \geq 0$

Ruy Chaleri' eli

Consequences of 2nd law

1) Efficiency of mass / energy conversion in BH collisions.

Consider



How much mass to energy is converted?

$$\eta = \frac{\text{Mass-Energy radiated}}{\text{Mass energy initial}} = \frac{M_1 + M_2 - M_3}{M_1 + M_2} = 1 - \frac{M_3}{M_1 + M_2}$$

Assume M_1 and M_2 are initially stationary

$16\pi M_{1,2}^2 = A_{1,2}$ $\rightsquigarrow$ the area theorem says

$$A_3 \geq A_1 + A_2 = 16\pi (M_2^2 + M_1^2)$$

$\rightsquigarrow$ ~~$M_3 \geq \sqrt{M_1^2 + M_2^2}$. but,~~

but $16\pi M_3^2 \geq A_3$ with equality at late times

$$\Rightarrow M_3 \geq \sqrt{M_2^2 + M_1^2} \Rightarrow$$

$$\eta \le 1 - \frac{\sqrt{M_1^2+M_2^2}}{M_1+M_2} \le 1-\frac{1}{\sqrt{2}}$$

So the maximum efficient process will

convert $1-\frac{1}{\sqrt{2}}$ %.

This analogous as 2^{nd} law limits efficiency of heat engines.

2) BH cannot bifurcate.

suppose $M_3 \to M_1 + M_2$

$$\Rightarrow \quad M_3 \le \sqrt{M_1^2+M_2^2} < M_1+M_2 \quad \text{Always}$$

by the area theorem

but energy conservation implies that

$M_3 \ge M_1 + M_2$, a contradiction.

Penrose diagram Reissner Nordström

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)} + r^2 d\Omega^2$$

$$= \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\left(-dt^2 + \left(\frac{dr}{\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)}\right)^2\right) + r^2 d\Omega^2$$

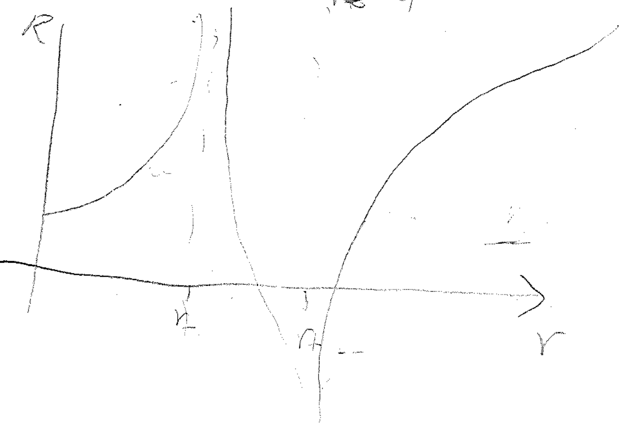
$$dR = \frac{dr}{1-\frac{2M}{r}+\frac{Q^2}{r^2}} = \frac{r^2}{r^2-2Mr+Q^2}\,dr = \frac{r^2}{(r-r_+)(r-r_-)}\,dr$$

$$1-\frac{2M}{r}+\frac{Q^2}{r^2} \qquad r^2-2Mr+Q^2 = (r-r_+)(r-r_-)$$

$$r_\pm = M \pm \sqrt{M^2-Q^2}$$

$r_+ \neq r_-$

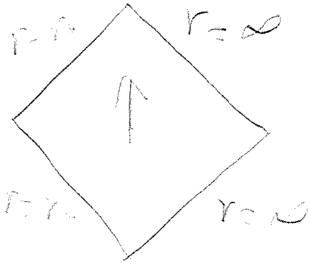
$$\hookrightarrow \quad R = r + \frac{1}{r_+-r_-}\left(r_+^2 \ln|r-r_+| - r_-^2 \ln|r-r_-|\right)$$



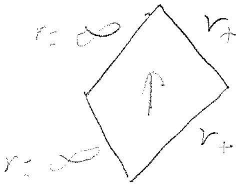
$r > r_+$:

$$\underbrace{(r-r_+)(r-r_-)}_{>0}\left(-dt^2+dR^2\right)$$

time as usual ✓



And $r\to -r$, $t\to -t$

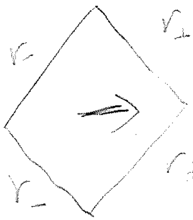


$r_- < r < r_+$:

$$\underbrace{(r-r_+)}_{-}\underbrace{(r-r_-)}_{+}$$

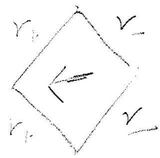
no direction

and $t\to -t$



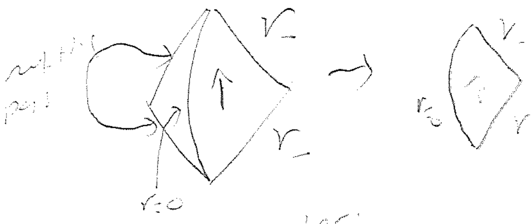
space:

$r\to -r$
$t\to -t$

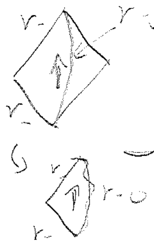


$0 < r < r_-$:

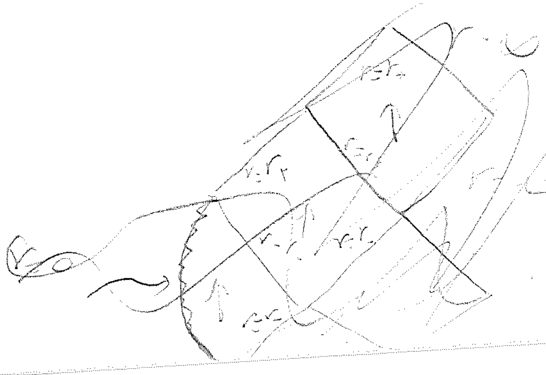
$$\underbrace{(r-r_+)}_{-}\underbrace{(r-r_-)}_{-}$$



$r\to -r$
$t\to -t$

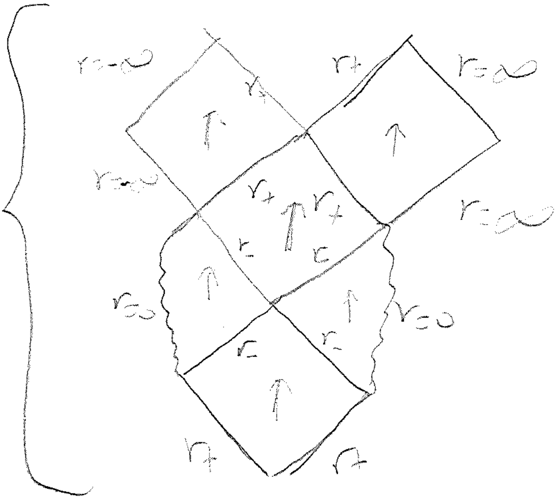


Glue together

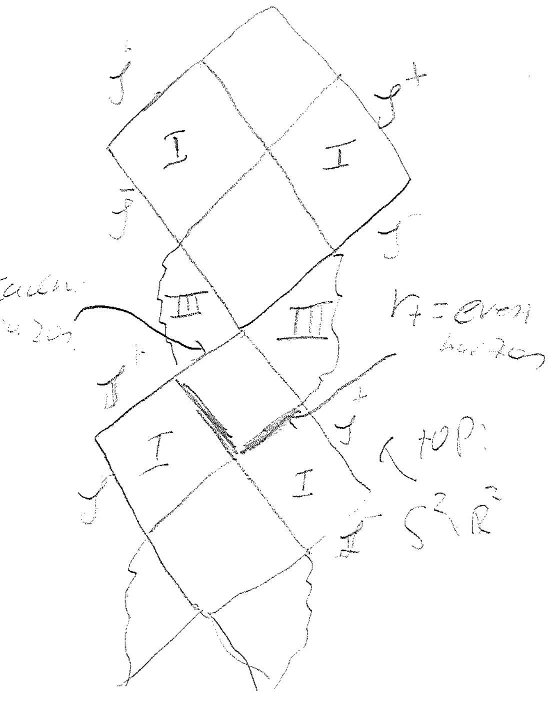


Glue together with correct time orientation?

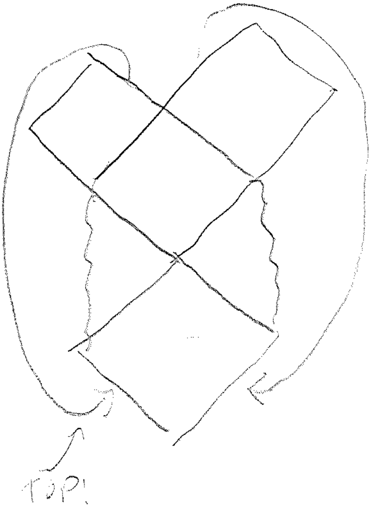
Building Block



one can either
⇒ Glue halfway: or identify



TOP!
$S^2 \times R^2$



TOP!
$S^2 \times R \times S^1$!

Event horizon $(r=r_+)$ ⇝ one cannot interact with region I any more.

Cauchy horizon $(r=r_-)$: is the region where starts to not being globally hyperbolic.

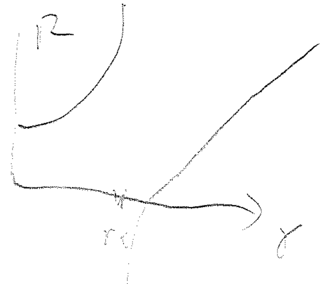
Since for example this Σ is a Cauchy hypersurface

but the whole space is not globally hyperbolic ($r=0$ and you're the things) information can get into i.e. III is not G.H.

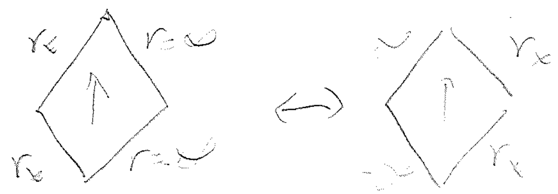
In the case of top $S^2 \times \mathbb{R} \times S^1$ there are closed timelike curves.

For $r_* = r_+ = r_-$ ⇝

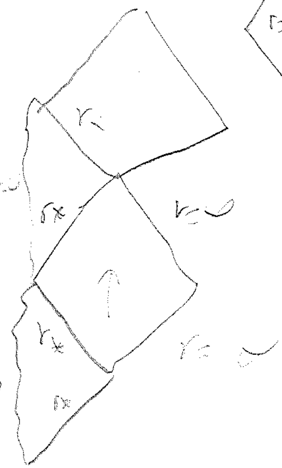
$$ds^2 = +(r-r_*)^2\left(-dt^2 + dR^2\right) + \cdots$$



• $\{r > r_*\}$ $ds^2 = (r-r_*)^2(-dt^2 + dR^2)$



• $\{0 < r < r_*\}$



or identify ends (causal curves closed)

Now, if not real root, $m^2 - q^2 < 0$.
$= -a^2$

⇝ $ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 + \cdots$

$= -\frac{1}{r^2}\left(r^2 - 2mr + \cdots\right)dt^2 + \cdots$

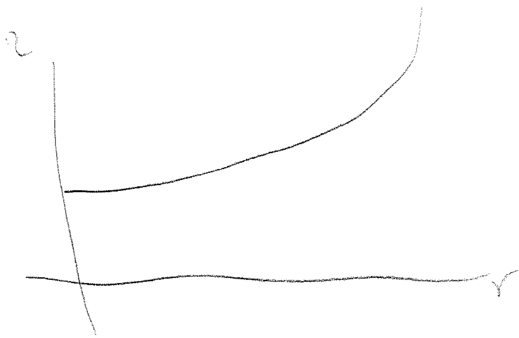
$$ds^2 = -\frac{1}{r^2}\left((r-M)^2+Q^2\right)dt^2 + \frac{r^2}{((r-M)^2+Q^2)}dr^2 + r^2 d\Omega^2$$

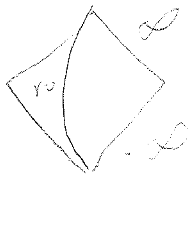
$$= +\frac{1}{r^2}\left((r-M)^2+Q^2\right)\left(-dt^2+dR^2\right)+r^2 d\Omega^2$$

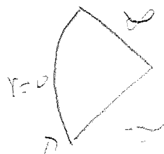
$$dR = \frac{r^2}{((r-M)^2+Q^2)}dr \rightsquigarrow$$

$$Q^2+(r-M)^2$$

$$\int dR = \int \frac{r^2}{\ldots} = R = M\ln(M^2-2Mr+Q^2+r^2) + \left(Q-\frac{M^2}{Q}\right)\tan^{-1}\left(\frac{M-r}{Q}\right)+r$$







presence of a naked singularity!

asymptotic future and -10 singularity is not separated from null

The Unruh effect + algebraic things

we want a theory $f \to \phi(f)$
solutions ↑ — operators labeled

by solutions.

↳ the map must be: • $\mathbb{R}$-linear

• Real field $(\phi(f))^\dagger = \phi(f)$

with this "quantization" how are construct the Fock space? what are $a, a^\dagger$?

• $[\phi(f), \phi(g)] = i\sigma(f,g)$

symplectic (bilinear, antisymmetric, non degenerate)

Ex: space h of solutions of K-G eq.

σ the wronskian $\left(\sigma(f,g) = \int d^{D-1}x \; f\dot{g} - \dot{f}g\right)$

Σ cauchy (for example $t=0$)

⤳ solutions of K-G

⤳ $e^{ik_\mu x^\mu}$ ⤳ $[\phi(f), \phi^\dagger(g)] = \int d^{D-1}x \; e^{ik\cdot x} \ldots$

$f = e^{ikx}$, $g = e^{ipx}$

$= i\delta(k-p)$

Ex: $h = \mathbb{C} = \mathbb{R}^2$ ⤳ $f = x$, $g = ip$

symplectic form

$\operatorname{Im}(\bar{z}w) = \sigma(z, w) = bc - da$

with $z = a + ib$, $w = c + id$

$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in$ in $h = \mathbb{R}^2$.

$\rightsquigarrow \ [\phi(x), \phi(ip)] = i\,\mathrm{Im}(-x\,ip) = -i\,xp$

$\hookrightarrow [\phi(i), \phi(1)] = i \leftarrow$ "usual" commutator relations.

we want a complex structure which is the data "added in the quantization" which is compatible with σ, i.e.

$J: h \to h, \quad J^2 = -\mathbb{1}$ compatible with σ

$$\sigma(f, Jg) = -\sigma(Jf, g)$$

Ex: $h = \mathbb{C} \rightsquigarrow \quad J\underset{a+ib}{\overset{''}{z}} = iz = -b + ia$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

and $\quad \sigma(f, Jg) = \mathrm{Im}(f\,\overline{Jg}) = \mathrm{Im}(f(-i\bar{g}))$

$$= -\mathrm{Im}(f i \bar{g})$$

$$= -\sigma(f, g)$$

or in matrices

$$(a\ b)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = (a\ b)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$$

~~$(ab)^t = b^t a^t$~~ $\quad = -\left(\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$

$$= -\left(\begin{matrix} -b \\ a \end{matrix} \right.$$

here $\quad i: x \to p$ ← see here ix behaves like p.

$\quad\quad p \to -x$

Now complexify h by taking $h \otimes \mathbb{C}$.

and define the bilinear form

$$\langle f, g \rangle \equiv \sigma(f, Jg) + i\sigma(f, g)$$

notice that $\langle g, f \rangle = \sigma(g, Jf) + i\sigma(g, f)$

$$= -\sigma(Jg, f) - i\sigma(f, g)$$

$$= \sigma(f, Jg) - i\sigma(f, g)$$

$$= \overline{\langle f, g \rangle}$$

$\rightsquigarrow$ $(h \otimes \mathbb{C}, \langle \cdot, \cdot \rangle)$ is a pre hilbert (without completness) but we can built

from $f \to \phi(f)$ a $\mathbb{C}$-linear map $f \to \underbrace{\frac{1}{\sqrt{2}}(\phi(f) - i\phi(Jf))}_{a^*(f)}$

$$a^*(Jf) = \frac{1}{\sqrt{2}}(\cancel{i}\phi(f) - i\phi(Jf))$$

$= \cancel{\frac{1}{\sqrt{2}}}\, i\, a^*(f) \rightsquigarrow$ $\quad a^*(Jf) = i a(f)$ ← $\boxed{\text{Linear}}$

$$a^*(f)^* = \frac{1}{\sqrt{2}}(\phi(f) + i\phi(Jf)) \equiv a(f).$$

↑ ϕ is selfadjoint

$\rightsquigarrow a(Jf) = -i a(f)$ ← $\boxed{\text{antilinear}}$

$$\boxed{\begin{aligned}[a^*(f), a^*(g)] &= \cdots = 0 \\ [a(f), a^*(g)] &= \langle f, g\rangle\end{aligned}}$$ CCR.

$a^*(\cancel{f + \lambda g})$

<u>Ex:</u> $h = \mathbb{C}$, $\{1\}$ basis of $\mathbb{C}$. $\rightsquigarrow$

$$a^*(1) = \frac{1}{\sqrt{2}}\big(\underset{\underset{\hat{x}}{\shortparallel}}{\phi(1)} - i\underset{\underset{\hat{p}}{\shortparallel}}{\phi(J\cdot 1)}\big)$$

$$= \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$$

<u>Fock space</u> Let $|0\rangle$ s.t. $a(f)|0\rangle = 0$.

what happens with symetrics (automorphisms)

Let $S: h \to h$ s.t. $\sigma(Sf, Sg) = \sigma(f, g)$

Define $S_{1/2} = \frac{1}{2}(S \pm JSJ)$, $S_1 + S_2 = S$.

$\rightsquigarrow \{S_1, J\} = 0 = [S_2, J]$

then $\tilde{\phi}(Sf) = \phi(Sf)$ is the corresponding and Obey the identification of quantization

Same CCR. $[\tilde{\phi}(f), \tilde{\phi}(g)] = i\underset{\sigma(Sf, Sg)}{\underset{\shortparallel}{\sigma(f,g)}}$

$\phi \to \tilde{\phi}$ is an algebra automorphism

and the creator ops are

$$\tilde{a}(f) = \frac{1}{\sqrt{2}}\left(\phi(Sf) - i\,\phi(SJf) \right)$$

$$\phi(S_1 Jf) + \phi(S_2 Jf)$$

$$= \phi(JS_1 f) + \phi(JS_2 f)$$

$$\boxed{\begin{aligned} \tilde{a}(f) &= a^*(S_2 f) + a(S_1 f) \\ &\text{and similarly:} \\ \tilde{a}^*(f) &= a(S_2 f) + a^*(S_1 f) \end{aligned}}$$

← Bogoliubov transformations

// mixes creation and annihilation operators!!

<u>Remark</u>

$$J^2 = -1 \rightsquigarrow J = \left(\begin{array}{c|c} +i & \\ \hline & -i \end{array} \right)$$

$$S = \left(\begin{array}{c|c} S_{++} & S_{+-} \\ \hline S_{-+} & S_{--} \end{array} \right) \rightsquigarrow S_1 = \begin{pmatrix} 0 & S_{+-} \\ S_{-+} & 0 \end{pmatrix}$$ ← mixes modes

$$S_2 = \begin{pmatrix} S_{++} & 0 \\ 0 & S_{--} \end{pmatrix}$$ ← transf. the lines positive→positive, negative→negative

⟹ S_2 leaves the vacuum invariant while S_1 not.

$$\tilde{a}(f)|0\rangle = \left(a(S_2 f) + a^*(S_1 f)\right)|0\rangle = a^*(S_1 f)|0\rangle$$

~~$\langle \tilde{a}(f) \rangle = 0 \ \langle 0 | a(S_2 f)\, a^*(S_1 f) \ldots \rangle \neq 0$~~

$$N(f) = a^*(f)\, a(f)$$

$$\rightsquigarrow \langle 0 | \tilde{N}(f) | 0 \rangle = \cancel{\langle 0 | a^*(f) a(f) + a^*(f) a^*(f) + a(f) a^*(f) | 0 \rangle}$$

$$\cancel{\langle 0 | \langle S_1 f \rangle | 0 \rangle + \langle 0 | a(f) a^*(S_2 f) | 0 \rangle}$$

$$= \langle 0 | \left(a(S_1 f) + a^*(S_2 f) \right) \left(a(S_2 f) + a^*(S_1 f) \right) | 0 \rangle$$

$$= \langle 0 | a(S_1 f)\, a^*(S_1 f) | 0 \rangle$$

$$= \langle 0 | [a(S_1 f), a^*(S_1 f)] | 0 \rangle$$

$$= \langle S_1 f, S_1 f \rangle \langle 0 | 0 \rangle = \langle f, S_1^* S_1 f \rangle$$

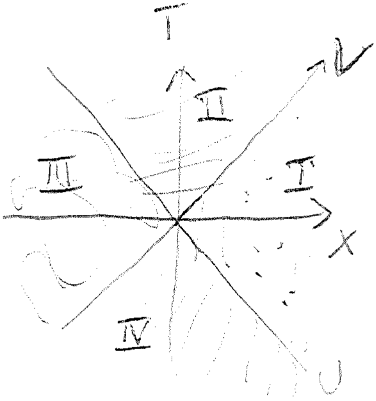
$$N_{tot} = \sum_i N(e_i) \rightsquigarrow \langle 0 | N | 0 \rangle = tr(\underbrace{S_1^* S_1}_{\rho}) = tr(\rho)$$

e_i: basis of h

For accelerated observers (Unruh effect)

$$\cancel{tr(\rho)} \quad \rho \sim e^{-\frac{\beta}{\hbar} H}, \quad \beta = \frac{1}{kT}, \quad H \leftarrow \frac{\partial}{\partial \tau}$$

• Consider an accelerated observer and an inertial observer



inertial $\leftarrow (T, X)$

accelerated $\leftarrow (t, x)$

$$\begin{cases} U = T - X \\ V = T + X \end{cases} \qquad t = \frac{1}{2}\log\left(-\frac{V}{U}\right)$$

$$x = \sqrt{-UV} = \sqrt{X^2 - T^2}$$

$$\rightsquigarrow \quad U = -x e^{-\kappa t}$$
$$V = x e^{\kappa t} \leftarrow \text{this coordinates only cover I}$$

$$\left(\begin{matrix} U \cancel{\ } < 0 \\ V > 0 \end{matrix} \right)$$

Similar for

$$ds^2 = -dT^2 + dX^2 = -(dV\,dU)$$
$$= -\left(\left(e^{\kappa t} dx + x \cancel{\kappa} e^{\kappa t} dt\right)\left(-e^{-\kappa t} dx + e^{-\kappa t} \kappa x\, dt\right)\right)$$
$$= -\left(-dx^2 + x\,dt\,dx - x\,dt\,dx + x^2 \kappa\, dt^2\right)$$
$$= -x^2 \kappa^2 dt^2 + dx^2 \leftarrow \text{Killing is } \frac{\partial}{\partial t}.$$

$$\left\| \frac{\partial}{\partial t} \right\|^2 = -x^2 \kappa^2.$$

Lets see the K-G eq.

(1) in the inertial observer
$$0 = \left(-\frac{\partial^2}{\partial T^2} + \frac{\partial^2}{\partial X^2}\right)\phi$$
$$= -\partial_U \partial_V \phi$$

$$\Rightarrow \phi(U, V) = f(U) + g(V)$$

As we know, positive frequency solutions are of the form $u_p(T,X) = C_p e^{-i(\omega T - \vec{p}\cdot\vec{X})}$, $|p| = \omega$.

$$= \begin{cases} C_p e^{-i(\omega T - px)} & p>0 \\ C_p e^{-i(\omega T + px)} & p<0 \end{cases}$$

$$= \begin{cases} C_p e^{-i\omega U} & p>0 \\ C_p e^{-i\omega V} & p<0 \end{cases}$$

only right observers $\longrightarrow$ $\phi = \begin{cases} C_p e^{-i\omega U} & \text{~~right observers~~} \\ C_p e^{-i\omega V} & \text{~~left observers~~} \end{cases}$

Let's see in Rindler coordinates:

$$0 = \Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right)$$

$$= \frac{1}{x^2}\left(x\partial_x(x\partial_x\phi) + \frac{\sigma^2}{\kappa^2}\phi\right) \overset{\text{Sol}}{\rightsquigarrow} \phi : x^{\pm\frac{\sigma}{\kappa}i} e^{-i\sigma t}$$

with $|p|\kappa = \sigma$, positive frequency.

$$\rightsquigarrow \phi = C_p e^{-i(\sigma t - p\ln x)}$$

where $\sigma t - p\ln x = \frac{\sigma}{2\kappa}\ln\left(-\frac{V}{U}\right) - \frac{p}{2}\ln(-UV)$

~~For~~

$$u_p = \begin{cases} e^{i\frac{\sigma}{\kappa}\ln(-U)} & \cancel{p>0} \\ e^{-i\frac{\sigma}{\kappa}\ln(V)} & \cancel{p<0} \end{cases}$$

only right observers. $(U<0,\ V>0)$

Similarly for left observers $(U,V)\to(-U,-V)$

$\rightsquigarrow$ (like $\rightsquigarrow$)

$$\overline{U}_L = \begin{cases} e^{\frac{i\omega}{\kappa}\ln(v)} \\ e^{-\frac{i\omega}{\kappa}\ln(v)} \end{cases}$$

Criterion of <u>positive frequency</u>

take Fourier transform $\tilde{f}(\omega) = \int_{\mathbb{R}} du\, e^{i\omega u} f(u)$

if $f(u)$ holomorphic in $\text{Im}\, u < 0$ and bounded

Choose contour ... $\leadsto$ $\tilde{f}(\omega) = 0$ $\leadsto$

f has only positive frequencies.

$\boxed{\text{i.e. Holomorphicity in } \text{Im}\, u < 0 \Rightarrow \text{positive freq}}$

Hence, notice that we can extend u_R analytically to all space as

$$\ln(-u) \equiv \begin{cases} \ln(-u) & u < 0 \\ \ln(+u) \pm i\pi & u > 0 \end{cases}$$

$\leadsto$ positive frequency solutions will have

$$V_p = u_R + \begin{cases} 0 & u < 0 \\ e^{\frac{i\omega}{\kappa}(\ln(+u) \pm i\pi)} & u > 0 \end{cases}$$

$$= u_R^p + \overline{u}_L^p e^{-\frac{\omega\pi}{\kappa}}$$ ← positive frequency solutions

rudier

$$\bar{U}_R = \begin{cases} 0 & U > 0 \\ 0 & V < 0 \\ C_p e^{\frac{i\omega}{\varkappa}\ln(-U)} & U < 0 \\ C_p e^{-\frac{i\omega}{\varkappa}\ln(V)} & V > 0 \end{cases} \qquad \bar{U}_L = \begin{cases} 0 & U < 0 \\ 0 & V > 0 \\ C_p e^{\frac{i\omega}{\varkappa}\ln(U)} & U > 0 \\ C_p e^{\frac{i\omega}{\varkappa}\ln(-V)} & V < 0 \end{cases}$$

positive frequency?

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which function does this?

$$\ln_{-1}(U) = \begin{cases} \ln(U) - i\pi 2 & U > 0 \\ \ln(-U) - i\pi & U < 0 \end{cases}$$

$\ln(U)$ is holomorphic in the $\mathrm{Im}\, U < 0$

$\Rightarrow e^{\frac{i\omega}{\varkappa}\ln(U)}$ is positive frequency modes!

$\Rightarrow$ ~~$V_p = U_p +$~~

$$\bar{U}_L = C_p e^{\frac{i\omega}{\varkappa}\ln(U)} = C_p e^{+\frac{i\omega}{\varkappa}\ln(-U)} \Rightarrow$$

$$U_R = V = U_R + C_p e^{+\frac{i\omega}{\varkappa}(\ln U - i\pi)}$$

$$= U_R + C_p e^{\frac{\omega\pi}{\varkappa}} e^{\frac{i\omega}{\varkappa}\ln U}$$

$$= U_R + \bar{U}_L e^{\frac{\omega\pi}{\varkappa}}$$

we have

$$V_p = u_R + e^{-\frac{\sigma\pi}{\kappa}} \bar{u}_L$$ ← frequency positive modes

as well as

$$V_p' = u_L + e^{-\frac{\sigma\pi}{\kappa}} \bar{u}_R$$

⇝ we can see as $V_p = a$

$V_p|0\rangle = 0$

$V_p'|0\rangle = 0$

⇝

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using Robert's denotations ⇝

$$V_p^{(1)} = u_p^R + e^{-\frac{\pi\sigma}{\kappa}} \bar{u}_p^L$$ ← positive frequency modes

$$V_p^{(2)} = u_p^L + e^{-\frac{\pi\sigma}{\kappa}} \bar{u}_p^R$$

~~we can see as annihilation ops~~

~~i.e. $V_p^{(1)}|0\rangle = 0$~~

~~$V_p^{(2)}|0\rangle = 0$~~

~~If $a_p^R|0\rangle = \ldots$~~

Change notation so it is more suggestive:

$u_p^R = a_p^R, \quad \bar{u}_p^L = u_p^{L\dagger}, \quad$ etc.

then we have

$$\underbrace{\left(a_p^R + e^{-\sigma\frac{\pi}{\kappa}}(a_p^L)^\dagger\right)}_{\text{acts as annihilation on inertial}}|0\rangle = 0$$

$a \neq$

$$\left(a_p^L + e^{-\sigma\frac{\pi}{\kappa}}(a_p^R)^\dagger\right)|0\rangle = 0$$

$$\Rightarrow \left((a_p^R)^\dagger a_p^R + (a_p^R)^\dagger e^{-\sigma\frac{\pi}{\kappa}}(a_p^L)^\dagger\right)|0\rangle$$

$$\rightsquigarrow \langle 0| a_p^{R\dagger} a_p^R |0\rangle = +e^{-\sigma\frac{\pi}{\kappa}}\langle 0| a_p^L (a_p^L)^\dagger |0\rangle$$

$$= +e^{-2\sigma\frac{\pi}{\kappa}}\left(\langle 0|(a_p^L)^\dagger a_p^L|0\rangle + \langle 0|0\rangle\right)$$

$[a^L, a^{L\dagger}] = 1$

$$\rightsquigarrow \langle 0| a_p^{R\dagger} a_p^R|0\rangle = +e^{-2\sigma\frac{\pi}{\kappa}}\langle 0|a_p^{L\dagger} a_p^L|0\rangle + e^{-2\sigma\frac{\pi}{\kappa}} \quad (1)$$

Since R and L are separate regions we expect they have the same number of particles in each wedge

$$\Rightarrow \langle 0| a_p^{R\dagger} a_p^R |0\rangle = \langle 0| a_p^{L\dagger} a_p^L |0\rangle \quad ②$$

using ② and ①

$$\Rightarrow \langle 0| a_p^{R\dagger} a_p^R |0\rangle = e^{-\frac{2\omega\pi}{\kappa}} \langle 0| a_p^{R\dagger} a_p^R |0\rangle + e^{-\frac{2\omega\pi}{\kappa}}$$

$$\Rightarrow \langle 0| a_p^{R\dagger} a_p^R |0\rangle = \frac{e^{-\frac{2\omega\pi}{\kappa}}}{1 - e^{-\frac{2\omega\pi}{\kappa}}} = \frac{e^{-\frac{\omega\pi}{\kappa}}}{e^{+\frac{\omega\pi}{\kappa}} - e^{-\frac{\omega\pi}{\kappa}}}$$

$$\langle 0| N_p^R |0\rangle = \frac{e^{-\frac{\omega\pi}{\kappa}}}{2\sinh\left(\frac{\omega\pi}{\kappa}\right)}$$

$$= \langle 0| N_p^L |0\rangle$$