# Renormalization Group Flows, Line Defects and, the $g-$theorem

Rafael F. Cordoba. L.
Advisor: Miguel F. Paulos.

April 6, 2024

**Abstract**

An important application of the study of Renormalization Group flows is in the presence of a line defect [1]. Particularly useful are the existence of charges that do not renormalize along the flow or that are monotonically decreasing as they allow us to study both strongly and weekly interacting theories. In this work, we describe those that are monotonic as they bring beautiful physical consequences such as screening, flow irreversibility, entropy, etc. The existence of such charges, initially explored by Zamolodchikov [2] in the celebrated $c-$theorem and later extended to higher dimensions by Komagordoski et. al. [3], has lately been extended to theories in the presence of non-local observables. In this case, in the presence of a line defect, a $g-$theorem [4, 5] was presented by Cuomo, Komargodski, and Raviv-Moshe. We focus on how RG flows are useful to study these theories and provide a proof of the $c-$and $g-$theorems.

## Contents

# 1 Introduction

The Ising model is a simple but challenging model of statistical physics. In two dimensions, it has an exact solution due to Onsager, but computing the partition function is still computationally expensive. In higher dimensions, no analytical solutions are known. However, near the critical temperature $T_c$, the Ising model exhibits universal features that are shared by many other systems with different physical realizations. The Wilsonian renormalization group (RG) approach, developed by Kenneth Wilson, explains this universality and answers fundamental questions such as how we can do physics without knowing the details of the long and short scales, and how we can describe the intermediate scales accurately.

The RG approach generates flows in the space of quantum field theories (QFTs). Special points in this space are the fixed points, where the theories are scale-invariant and, in most cases, conformally invariant. This means that the QFTs at the fixed points, called conformal field theories (CFTs), have an enhanced space-time symmetry group that includes the Poincaré group, dilations, and special conformal transformations. This symmetry enhancement is a powerful tool to solve CFTs, as it allows us to determine the $1-$, $2-$, and $3-$point functions of the theory using symmetry arguments. CFTs are of great interest because they model physically relevant phenomena (such as critical points of phase transitions) and they are well-behaved in the UV and IR. Moreover, the CFT bootstrap program has led to many advances in the study of CFTs and has revealed many interesting aspects of QFTs, see [6] for details.

On the other hand, recent developments in QFTs had recognised the importance of extended operators. Line operators (and boundaries) are the easiest to study and they provide, already, many examples of interesting phenomena. For instance, they describe point-like impurities in space, topological line operators can be interpreted as (potentially non-invertible) symmetry generators [7], Wilson lines are natural observables in gauge theories [8] and, in the context of algebraic quantum field theory, where observables and states are separated constructions, extended operators comprise a large class of observables [9]. Therefore, line operators and, in general, extended operators are believed to be essential in the understanding of quantum field theory.

In this work, we aim to understand how to construct monotonic charges under an RG flow ($c-$theorem) and how the introduction of a defect line affects the degrees of freedom and the evolution of the theory along the flow ($g-$theorem). To do this, we review Wilson's renormalization ideas and we see how triggering flows and taking care of the symmetries of the theory allow us to find the corresponding RG charges.

## 2 Renormalization Group Flows

In this section we introduce the concept of Renormalization Group (RG) in Quantum Field Theory, we closely follow [10] and [8]. Consider a generic quantum field theory governed by the action

$$S_{\Lambda_0}[\varphi] = \int d^dx \left[\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + \sum_i \left(\frac{\Lambda_0}{\Lambda}\right)^{d-d_i} g_{i0}\mathcal{O}_i(x)\right] \tag{1}$$

where we put explicitly the dimensionless couplings $\{g_i\}$ and the scale of the theory. We define the regularized partition function:

$$\mathcal{Z}_{\Lambda_0}(g_{i0}) = \int_{C^\infty(M)\leq\Lambda_0} \mathcal{D}\varphi e^{-S_{\Lambda_0}[\varphi]/\hbar}$$

over the measure of all fields $\varphi$ with momentum of atmost $\Lambda_0$. This cut-off explicitly regularizes the theory making it absent of UV divergences. If we set an scale $\Lambda < \Lambda_0$ we can split the field $\varphi = \phi + \chi$ by the low $\phi$ $(p < \Lambda)$ and high energy $\chi$ $(p > \Lambda)$ fields that separate the new scale $\Lambda$. We want to see how the theory behaves in the regime of this new low energy i.e. we are probing a theory where the details of the high energy modes can be disregarded.

To do so we integrate the high energy modes in the partition function. Since the measure is assumed to factorize in low and high energy modes, we derive the effective action:

$$S_\Lambda^{eff}[\phi] = -\hbar\log\left[\int_{C^\infty(M)_{(\Lambda,\Lambda_0]}} \mathcal{D}\chi e^{S_{\Lambda_0}[\phi+\chi]/\hbar}\right]. \tag{2}$$

This action describes the new scale physics where the window of "validity" of our theory is $0 \leq p \leq \Lambda < \Lambda_0$. Thus, we have done a *change in the scale of the theory*, we integrated out high energy frequency modes. An important observation is that the initial (normalized) partition function $\mathcal{Z}_\Lambda(g_i(\Lambda))$ remains the same, we have just integrated out high energy modes to give an explicit form of the low energy action and thus,

$$\Lambda\frac{d\mathcal{Z}_\Lambda(g)}{d\Lambda} = 0.$$

Notice that this tells us that changing the scale by integrating out modes implies that the couplings in the effective action must vary to account for this change in degrees of freedom over which we take the path integral so that the partition function is independent of the scale we define the theory.

We deduce that the effective interaction thus will be of the general form

$$S_\Lambda^{eff}[\phi] = \int d^dx \left[\frac{Z_\Lambda}{2}(\partial\phi)^2 + \sum_i \Lambda^{d-d_i} Z_\Lambda^{n_i/2} g_i(\Lambda)\mathcal{O}_i(x)\right]$$

where we define the normalized fields $\varphi = Z_\Lambda^{1/2}\phi$ such that we work in a canonically normalized theory. We define also the $\beta-$functions of the couplings $g_i$ by

$$\beta_i = \Lambda\frac{\partial g_i}{\partial\Lambda}.$$

Likewise, we define the anomalous dimension $\gamma_\phi = -\frac{1}{2}\Lambda\frac{\partial \ln Z_\Lambda}{\partial\Lambda}$ that is just the $\beta-$function for the coupling of the kinetic term.

As a result of this analysis, the net contribution of integrating out high energy modes is to modify the couplings $\{g_i\}$ running from the initial theory to the low-energy (long distances) IR theory. This running of couplings generates a flow on the moduli space of QFTs (each one parametrized by the couplings $\{g_i\}$ ) which is called the Renormalization Group flow and it provides a tool to study theories as we probe different scales.

## 2.1 Long and short distance probes

Consider an $n-$point function in this theory,

$$\langle\phi_1...\phi_n\rangle = Z_\Lambda^{-n/2}\langle\varphi_1...\varphi_n\rangle := Z_\Lambda^{-n/2}\Gamma_\Lambda(x_1, ..., x_n, g_i(\Lambda)).$$

We want to see how energies are probed as we go to the high and low energies. Since we could take a lower scale $s\Lambda < \Lambda$ and the partition function remains the same, the correlation $\Gamma$ should obey

$$Z_{s\Lambda}^{-n/2}\Gamma_{s\Lambda}(x_1, ..., x_n, g_i(\Lambda)) = Z_\Lambda^{-n/2}\Gamma_\Lambda(x_1, ..., x_n, g_i(\Lambda)) \tag{3}$$

known as the Callan–Symanzik equation.

A crucial observation is that we can equivalently probe energy scales by going to an energy $s\Lambda$ or by changing coordinates from $x^\mu \to x^{'\mu} = sx^\mu$. In doing this transformation we see that the kinetic term of action (1) is invariant provided $\phi(sx) = s^{(2-d)/2}\phi(x)$ and the reaming terms are unchanged provided we scale the energies $\Lambda \to \Lambda/s$, as expected. This transformation makes the theory to the initial energy $\Lambda$, however, this does not integrate degrees of freedom in the path integral, they are just scalings i.e. we are probing the theory in the scale $s\Lambda$. To see the net effect of integrating out high-energy modes we should also run the couplings $g_i$ to the desired scale.

Indeed, from Equation (3) we can immediately deduce that the correlations follow

$$\Gamma_\Lambda(x_1/s, ..., x_n/s, g_i(\Lambda)) = \left[s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}^{n/2}\right]\Gamma_\Lambda(x_1, ..., x_n, g_i(s\Lambda))$$

and thus, as correlators depend (on a Poincare invariant theory) by $|x_i - x_j|/s$, the $s \to 0$ probes the long distance (IR properties) of the theory. We see that such correlators can be computed by studying the separations constant but equivalently changing the couplings $g_i(s\Lambda)$. This is just that the IR properties are governed by the low-energy modes that survive as we integrate out high-energy degrees of freedom.

## 2.2 Counterterms and the continuum limit

Renormalization Group flows describe trajectories in the moduli space of theories parametrized by the couplings $\{g_i\}$. Rather remarkably there exists fixed points of this trajectory, points where the $\beta$ functions vanish, that leaves the couplings constant as we prove all energy levels in this theory. Qualitatively, theories at this point comprise a scale-invariant theory and, more precisely, they imply that the expected value of trace of the energy-momentum tensor $T$ vanishes as the metric is a scale transformation,

$$0 = \delta g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}(x)}\ln\mathcal{Z} = -\delta g^{\mu\nu}(x)\langle T_{\mu\nu}(x)\rangle \propto \langle T\rangle.$$

Examples of this are Conformal Field Theories (CFTs) which are QFTs that extend the Poincarè global symmetry to the conformal group. Among other things, CFTs are often QFTs that are "well" behaved and better understood as, for example, they have well-defined limits in the UV and IR i.e. they don't require counterterms for the theory to be well-defined, as we will see below, and the bigger symmetry group allows to work with them rather axiomatically without the need of an explicit action, see [6] for more details. Fixed points of RG flows are thus a great source for constructing CFTs.

On the other hand, flows near but not at the critical point have trajectories, to first order, characterized by a non-trivial flow

$$\Lambda \frac{\partial g_i}{\partial \Lambda}\Big|_{g_j^* + \delta g_j} = B_{ij}\delta g_j + O(\delta g^2)$$

where $B_{ij}$ is an infinite dimensional matrix i.e. we linearized the beta functions. Working in the eigenbasis $\sigma_i$ of $B_{ij}$, whose scaling dimension is $\Delta_i$ and eigenvalue $\Delta_i - d$ , we can write:

$$\Lambda \frac{\partial \sigma_i}{\partial \Lambda} = (\Delta_i - d)\sigma_i + O(\sigma^2) \implies \sigma_i = \left(\frac{\Lambda}{\Lambda_0}\right)^{\Delta_i - d} \sigma_i(\Lambda_0)$$

in perturbation to this order. Next orders are the effects of integrating out degrees of freedom which modifies the scaling. This parametrization of the flow allow us to identify the possible relevant and irrelevant contributions to the theory.

- Starting near a critical point and turning on a coupling to any operator with $\Delta_i > d$ then implies that the coupling becomes smaller as $\Lambda$ is lowered in the IR. We say the corresponding operator is *irrelevant*, and the RG flow will go back to the critical $g^*$.
- If $\Delta_i < d$ we say the operator is *relevant*. In the presence of these, the RG flow will drive away from the critical surface as we head to IR. Starting from a critical point and turning on a relevant coupling is known as a normalized trajectory.
- *Marginal* operators are the remaining possibility, they can go either way considering quantum corrections.

Classically we have infinite irrelevant operators, derivatives, and field insertions raise the dimensions so the point $g^*$ lies on an infinite dimensional surface $\mathcal{C}$ such that all couplings in the surface drive us to the critical point. In the neighborhood of the critical point, irrelevant operators form a basis of this surface. On the other hand, the critical surface has finite codimension as only finite many operators are less than $d$.Marginal operators can turn, after quantum corrections in either irrelevant, relevant or exactly marginal.

Sending $\Lambda_0 \to \infty$, without affecting what the theory predicts for low–energy phenomena, is called *the continuum limit* of our theory. If our initial couplings $g_{i0}$ lie on the critical surface C, within the domain of attraction of $g_i^*$. Then as we raise the cut–off $\Lambda_0$, the theory we obtain at any fixed scale $\Lambda$ will be driven to the critical point $g_i^*$ as all the irrelevant operators become arbitrarily suppressed by positive powers of $\Lambda/\Lambda_0$. The critical point is a fixed point of the RG flow and is scale-invariant, so the limit

$$S_\Lambda^{eff}[\phi]) = -\hbar \log \lim_{\Lambda_0 \to \infty} \left[\int_{C^\infty(M)_{(\Lambda,\Lambda_0]}} \mathcal{D}\chi e^{S_{\Lambda_0}[\phi+\chi]/\hbar}\right] \tag{4}$$

exists provided we take this limit after computing the path integral. The resulting effective theory will be a CFT, independent of $\Lambda$. Since C has only finite codimension, we only have to tune finitely many coefficients (those of all the relevant operators) to ensure that $g_{i0} \in C$.

On the other hand, a theory whose initial conditions are near, but not on C eventually will diverge away. Let $\mu$ denote the energy scale at which this theory passes closest to $g_i^*$. Since the RG flow is determined by the initial conditions, $\mu$ depends only on the theory we started with. On dimensional grounds we must have

$$\mu = \Lambda_0 f(g_{i0})$$

and $f = 0$ at the critical surface. To obtain a theory with relevant or marginal operators, we tune the initial couplings $g_{i0}$ so that $\mu$ remains finite as we take $\Lambda_0 \to \infty$. If $codim(C) = r$ then the theory we end up with thus depends on $(r-1)$ parameters, together with the scale $\mu$.

We achieve this tuning by introducing new counterterms $S_{CT}[\phi, \Lambda_0]$ modifying the initial action to

$$S_{\Lambda_0}[\varphi] \to S_{\Lambda_0}[\varphi] + S_{CT}[\varphi, \Lambda_0].$$

The effective action we considered before already contained all possible monomials in fields and their derivatives, so in this sense, the counterterms add nothing new. However, the values of the counterterm couplings are to be chosen by hand, varying these couplings thus changes which initial high–energy theory we're considering, as opposed to running a set of couplings under RG flow, which just describes how the same theory appears at different scales. The counterterms are tuned so that the limit (4) exists. Sending $\Lambda_0 \to \infty$ defines a continuum QFT with finite (or renormalized) relevant couplings at scale $\Lambda$.

### 2.2.1 Gaussian critical point

The simplest type of critical point is the Gaussian fixed–point where $g_{2k} = 0 \, \forall k > 1$, corresponding to a free theory. If we start from a theory where the couplings to each of these vertices are precisely set to zero, then no interactions can ever be generated. Indeed, the first couplings can be found to evolve according to (see [10] for details):

$$\begin{aligned}
\Lambda \frac{dg_2}{d\Lambda} &= -2g_2 - \frac{ag_4}{1+g_2}, \\
\Lambda \frac{dg_4}{d\Lambda} &= (d-4)g_4 - \frac{ag_6}{1+g_2} + \frac{3ag_4^2}{(1+g_2)^2}, \\
\Lambda \frac{dg_8}{d\Lambda} &= (2d-6)g_6 - \frac{ag_8}{1+g_2} + \frac{15ag_4g_6}{(1+g_2)^2} - \frac{30ag_4^3}{(1+g_2)^3},
\end{aligned} \tag{5}$$

which shows that the Gaussian point is indeed a fixed–point of the RG flow, with the mass term $\beta-$function $\beta_2 = -2g_2$ simply compensating for the scaling of the explicit power of $\Lambda$ introduced to make the coupling dimensionless.

Now, consider a $\phi^4$ theory. To linear order in the couplings, only the first two terms on Equation (5) are present and, in general [10],

$$\beta_{2k} = (k(d-2)-d)g_{2k} - ag_{2k+2}.$$

Thus, in $4$ dimensions $k \geq 3$ are irrelevant operators and the $\phi^4$ is marginal but quantum corrections may change its nature. The mass term $g_2$ is relevant and thus turning masses will take us away from the massless theory as the cut-off $\Lambda$ is lowered. Once $g_2$ is large, perturbation theory is not valid anymore but since $\beta_2 = -2g_2$ quantum corrections are eventually suppressed as the mass becomes large in units of the cut-off.

The coupling $g_4$, to second order is given $\beta_4 = \frac{3}{4\pi^2}g_4^2 + O(g_4^2 g_2)$ so we find

$$g_4 = \frac{16\pi^2}{3\log(\mu/\Lambda)},$$

for some scale $\mu$. Since $g_4$ is the highest coefficient in the potential then we require $g_4 > 0$ for the action to be bounded at $\phi = \pm\infty$ so $\mu > \Lambda$. Dimensional transmutation! we see that $g_4(\Lambda)$ decreases as $\Lambda \to 0$, ultimately being driven to zero. However, the scale dependence of $g_4$ is rather mild; instead of power–law behavior we have only logarithmic dependence on the cut–off. Thus the $\phi^4$ coupling, which was marginal at the classical level, is marginally irrelevant once quantum effects are taken into account. In the deep IR, we see only a free massive theory.

### 2.2.2 Wilson-Fisher critical point

$\phi_4$ theory does not have a continuum limit in $d = 4$ as the theory breaks down for cut-off higher than $\mu$. Since the only critical point is the Gaussian free theory we reach at low energies, four-dimensional scalar theory, is known as a trivial theory. Wilson and Fisher had the idea of introducing a parameter $\varepsilon := 4 - d \geq 0$ which is treated so that one is near four dimensions.

From the local approximation of Equation (5), Wilson and Fisher showed that there is a critical point $g_i^{WF}$ where

$$g_2^{WF} = -\frac{1}{6}\varepsilon + O(\varepsilon^2), \quad g_4^{WF} = \frac{1}{3a}\varepsilon + O(\varepsilon^2)$$

and $g_{2k}^{WF} \sim \varepsilon^k$ for $k > 2$. ($\varepsilon > 0$ so the potential is bounded and the theory stable). The beta function around $g_{2k}^{WF}$ is linearized by truncating the subspace spanned by $\{g_2, g_4\}$:

$$\Lambda\frac{\partial}{\partial}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix} = \begin{pmatrix}\varepsilon/3 - 2 & -a(1+\varepsilon/6)\\ 0 & \varepsilon\end{pmatrix}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix}$$

whose eignevalues are $\varepsilon/3 - 2$ and $\varepsilon$ with eigenvectors:

$$\sigma_2 = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad \sigma_4 = \begin{pmatrix}-a(3+\varepsilon/2)\\ 2(3+\varepsilon)\end{pmatrix}$$

respectively. In $d = 4 - \varepsilon$ we have $a = \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}|_{d=4-\varepsilon} = \frac{1}{16\pi^2} + \frac{\varepsilon}{32\pi^2}(1 - \gamma + \ln 4\pi) + O(\varepsilon^2)$ and, since $\varepsilon$ is small, then the first eigenvalue is negative so mass term is relevant perturbation of Wilson-Fisher fixed point. On the other hand, the operator $-a(3+\varepsilon/2)\phi^2 + 2(3+\varepsilon)\phi^4$ corresponding to $\sigma_4$ is an irrelevant operator.

Both, the Gaussian and, the Wilson–Fisher fixed–points, lie on the critical surface. A particular RG trajectory, with initial Gaussian model and turning on the operator $\sigma_4$ ends at the Wilson–Fisher fixed point in the IR as shown in Figure 1. Theories on the line heading vertically out of the Gaussian fixed–point correspond to massive free theories, while theories in the region $I$ are massless and free in the deep UV but become interacting and massive in the IR.

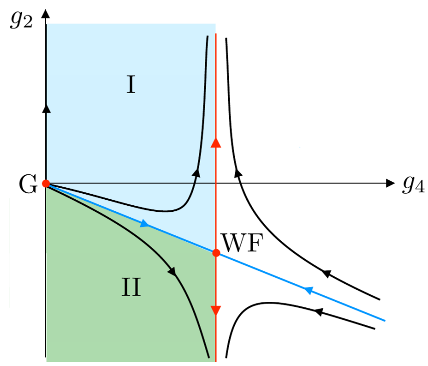


Figure 1: $\phi^4$ RG flow. These theories are parametrized by the scalar mass and by the strength of the interaction at any given energy scale. The RG trajectory obtained by deforming the Wilson–Fisher fixed point by a mass term is shown in red. All couplings in any theory to the right of this line diverge as we try to follow the RG back to the UV; these theories do not have well–defined continuum limits. Figure retrieved from [10].

# 3 The Stress-Energy Tensor in Quantum Field Theory

The Wilsonian renormalization has provided us with an understanding of how degrees of freedom evolve in the theory, now we develop some tools for getting the hand of them.

Consider a quantum field theory (QFT) and let $\{\mathcal{O}_i\}_{i\in I}$ be the set of observables. The energy-momentum tensor $T$ is defined to parametrize the responses of the system to changes in metric i.e. it measures the variation of the action with respect to the metric $g$ as we replaced $\eta \to g$,

$$\delta S = -\frac{1}{2}\int d^dx\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}.$$

The existence of a global tensor $T$ means that energy is smeared out in space-time where the degrees of freedom can be extended. An immediate consequence is thus the existence of theories that may not have an energy-momentum tensor, the so-called topological theories. In these types of theories degrees of freedom can not be localized, instead, observables are extended operators that depend on the global structure (topological invariants) i.e. degrees of freedom do not couple to local geometrical quantities (such as the metric $g$, frame fields, curvature, etc.), and consequently, the energy-momentum tensor vanishes.

A hallmark of the study of the stress-energy tensor, and that we are going to study in detail, is the celebrated $c-$theorem for 2d and many subsequent theorems describing the existence of monotonic charges along RG flows[1]. If such charges exist one can prove that the flows are irreversible and so they constitute a *gradient* flow in the space of QFTs.

[1]One can also study charges that are conserved along the RG flow, these are observables in the UV (strongly coupled) that remain constant in the IR (weakly coupled) theory and therefore it provides results that under perturbation theory are virtually inaccessible.

### 3.1 The dilaton field $\Phi$ and the Zamolodchikov's $C-$theorem

Consider a $2d$ CFT. The introduction of a mass term in the action,

$$S_{CFT_{UV}} \to S_{CFT_{UV}} + \int M\mathcal{O}(x)d^2x,$$

breaks conformal symmetry explicitly by both trace anomalies and operational violation of $T^\mu_\mu = 0$. We introduce a dilation field $\Phi$ to compensate for the operational violation of the traceless condition by replacing every scale $M$ in the Lagrangian by $M \to M_0 e^{-\Phi(\sigma)}$,

$$S_{CFT_{UV}} \to S_{CFT_{UV}} + \int M_0\mathcal{O}(x)e^{(\Delta_{\mathcal{O}}-2)\Phi}d^2x.$$

Conformal symmetry is then restored[2] if Weyl scalings $x \to e^\sigma x$, $\mathcal{O} \to e^{-b\Delta_{\mathcal{O}}}\mathcal{O}$ are accompanied by $\Phi \to \Phi + \sigma$ as only then the new action remains invariant under rescalings.

Notice that the new action has triggered an RG flow from the CFT at short distances, $CFT_{UV} = QFT_{UV} + QFT_\Phi$ towards the CFT at long distances, $CFT_{IR} = QFT_{IR} + QFT_\Phi$. As we flow along the RG we integrate out high energy modes and, since we do not integrate out massless particles, the dependence on $\Phi$ remains regular and local and thus remains a well-defined local field. Moreover, since in even dimensions the conformal group has trace anomalies, the conformal field theory at long distances (IR), contributes to the trace anomalies but to match the defining $UV$ theory, the $\Phi$ functional has to compensate precisely for the difference between the anomalies of the conformal field theory at short distances, $CFT_{UV}$, and the conformal field theory at long distances, $CFT_{IR}$.

To see this more precisely, let us see how $\Phi$ can be coupled in this theory. We introduce a fiducial metric $g_{\mu\nu}$ into the system and Weyl transformation are performed by $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$ and $\Phi \to \Phi + \sigma$. Since we must keep the theory to be a CFT we need to find all actions $S_\Phi$ that are $diff \times Weyl$ invariant for the dilation and metric background fields. Moreover, since the stress tensor must couple linearly to the dilaton field by

$$\int \Phi T^\mu_\mu d^2x$$

we must match the generated effective interaction to the action $S_\Phi$ of the dilaton field. Indeed, the interacting terms are given by

$$\begin{aligned}
\langle e^{\int \Phi T^\mu_\mu d^2x}\rangle &= 1 + \frac{1}{2}\int d^2x_1 d^2x_2 \Phi(x_1)\Phi(x_2)\langle T(x_1)T(x_2)\rangle \\
&= 1 + \frac{1}{2}\int d^2x_1 d^2x_2 \Phi(x_1)\Big(\Phi(x_1) + \partial_\alpha\Phi(x)\,|_{x_1}\,(x_1-x_2)^\alpha \\
&\qquad + \frac{1}{2}(\partial_\alpha\partial_\beta\Phi(x)\,|_{x_1})(x_1-x_2)^\alpha(x_1-x_2)^\beta + O(3)\Big)\langle T(x_1)T(x_2)\rangle \\
&= 1 + \frac{1}{4}\int dx_1\Phi(x_1)(\partial_\alpha\partial_\beta\Phi(x)\,|_{x_1})\int dx_2(x_1-x_2)^\alpha(x_1-x_2)^\beta\langle T(x_1)T(x_2)\rangle + O(3)
\end{aligned}$$

[2]In two dimensional QFT Weyl invariance implies conformal invariance, this was motivated by Zamolodchikov and proved by Polchinski in [11].

where we performed a gradient expansion on $\Phi(x_2)$. Using translation invariance the last integral gives $\int dx_2(x_1-x_2)^\alpha(x_1-x_2)^\beta\langle T(x_1)T(x_2)\rangle = \frac{1}{2}\eta^{\alpha\beta}\int d^2y y^2\langle T(y)T(0)\rangle$ so we have an effective coupling

$$\int d^2x\Phi(x)\Box\Phi\Big(\frac{1}{8}\int dyy^2\langle T(y)T(0)\rangle\Big). \tag{6}$$

Using the unitarity of the theory, the term in the parenthesis is definite positive and, therefore, among all the possible actions of the dilaton, we search for all possible actions $S_\Phi$ that reproduce this kinetic term.

Notice that $e^{-2\phi}g$ is Weyl invariant and therefore the curvature $\hat{R}=R[e^{-2\Phi}g]$ is also a Weyl invariant that contains second order derivatives. At the level of two derivatives, this is the only diffeomorphism and Weyl invariant term, $\int\sqrt{g}\hat{R}$, but this term is topological in 2d. We conclude that even with an invariant theory, upon setting $g=\eta$, $\int d^2x(\partial\Phi)^2$ is absent because there is no appropriate local term that could generate it. The key realization is to recall that unitary-two dim theories have trace anomaly,

$$T=-\frac{c}{24\pi}R.$$

One therefore should allow the transformations to break Weyl in such a way that they recreate the trace anomaly. Putting the action in curved space-time, the action

$$\int M_0\mathcal{O}(x)e^{(\Delta_{\mathcal{O}}-2)\Phi}\sqrt{g}d^2x,$$

is scale-invariant as $g_{\mu\nu}\to e^{2\sigma}g_{\mu\nu}$ and $\Phi\to\Phi+\sigma$, but we allow additionally an anomaly term that reproduces $T=-\frac{\Delta c}{24\pi}R$. Since Weyl transformations give a variation $\delta g_{\mu\nu}=2\sigma g_{\mu\nu}$ then, plugging in the definition of the stress tensor, we require an additional anomaly-matching term whose variation is

$$\delta_\sigma S_{anom}=\frac{\Delta c}{24\pi}\int d^2x\sqrt{g}R\sigma.$$

The term that does the job is

$$S_{anom}=\frac{\Delta c}{24\pi}\int\sqrt{g}\Big(\Phi R+(\partial\Phi)^2\Big)d^2x$$

because $\delta R=2\partial^2\sigma+O(2)$ and therefore

$$\begin{aligned}\delta S_{anom}&=\frac{\Delta c}{24\pi}\int(\delta\sqrt{g})\Big(\Phi R+(\partial\Phi)^2\Big)+\sqrt{g}((\delta\Phi)R+\Phi\delta R+2\partial\Phi\partial\delta\Phi)\\&=\frac{\Delta c}{24\pi}\int d^2x\sqrt{g}(\sigma R+2\Phi\partial\sigma+2\partial\Phi\partial\delta\sigma)\\&=\frac{\Delta c}{24\pi}\int d^2x\sqrt{g}\sigma R,\end{aligned}$$

where we disregard the first variation as it contains higher powers of $\sigma$ and integrated by parts in the third equality. Going back to flat space the anomaly vanishes but, rather remarkably, the kinetic term for the dilaton field survives and is fixed by the coefficient $\Delta c$,

$$S_{anom}\,|_{g=\eta}=\frac{\Delta c}{24\pi}\int(\partial\Phi)^2d^2x. \tag{7}$$

This, on the other hand, is precisely the second-order term of the dilaton coupling $\Phi$ of Equation (6) and, by matching the coefficients we conclude :

$$\Delta c = 3\pi \int d^2 y y^2 \langle T(y) T(0) \rangle \geq 0. \tag{8}$$

Finally, as we said previously, the dilaton action $S_{anom}$ must compensate for the difference of the anomalies such that the CFT in the UV has an anomaly of the form

$$\delta_\sigma S = \frac{c_{UV}}{24\pi} \int d^2 x \sqrt{g} R \sigma \tag{9}$$

and, as we evolve to the CFT in the IR, we have:

$$\delta_\sigma S = \frac{c_{UV}}{24\pi} \int d^2 x \sqrt{g} R \sigma \quad \text{and} \quad \delta_\sigma S_{CFT} = \delta_\sigma (S - S_{anom}) = \frac{c_{IR}}{24\pi} \int d^2 x \sqrt{g} R \sigma. \tag{10}$$

We conclude that $\Delta c = c_{UV} - c_{IR}$ precisely recovers Equations (9) and (10). The constant $\Delta c \geq 0$ implies $c_{IR} > c_{UV}$, a result known as the *Zamolodchikov's $C-$theorem*.

**Remark 1.** *Recall that a $\phi^4$ theory evolves towards a free scalar theory. An immediate consequence of the $c-$theorem is that if we start the other way around, with a free field theory together with some $\phi$ degrees of freedom, there exists no RG flow that recombines the free field and the $\phi$ degrees of freedom to reproduce the $\phi^4$ theory in the IR as it would violate the existence of the monotonic charge.*

## 3.2 Defect Conformal Field Theories

The energy-momentum tensor also allows us to explicitly construct the symmetry generators of the theory. Indeed, in Conformal Field Theories local operators at any fixed point in $\mathbb{R}^d$ furnish a representation of the charge algebra $\{Q_\varepsilon\}$ generated by the conformal Killing vectors $\varepsilon$, i.e. vector fields satisfying the conformal Killing equation

$$\partial_{(\mu}\varepsilon_{\nu)} = \frac{\partial \cdot \varepsilon}{d} \eta_{\mu\nu},$$

and the relations[3] $[Q_{\varepsilon_1}, Q_{\varepsilon_2}] = Q_{[\varepsilon_1, \varepsilon_2]}$. If the energy-momentum exists, we can explicitly write

$$Q_\varepsilon[\Sigma] = \int_\Sigma dS_\mu \varepsilon_\mu T^{\mu\nu} \tag{11}$$

as the charge associated with the vector field. The charge is conserved provided the energy-momentum tensor is traceless and conserved. Notice that the conserved charge (11) dependence is topological meaning that we can deform the contour enclosed by the hypersurface it bounds without changing the symmetry generator.

### 3.2.1 Line defects

Defects are ubiquitous in quantum field theory both for high-energy physics and condensed matter. Magnetic impurities in metals, worldlines of heavy charged particles, lattice systems (spin chain) with

[3] This relation is not always guaranteed. In two dimensional CFTs, you have an infinite dimensional algebra due to analytic generators of Killing. In this case, the algebra of charges is the central extension of the algebra of vector fields.

boundaries, D-Branes in worldsheet string theory are some examples of defect realizations that constitute important models of extended operators. The canonical example of a defect line are the Wilson lines on gauge theories, observables of the form

$$W[x_i, x_f] = \mathcal{P}\exp\left(i\int_{x_i}^{x_f} A_\mu dx^\mu\right),$$

where $A$ is the gauge field, $\mathcal{P}$ is the path ordering and we are integrating along a trajectory from $x_i$ to $x_f$.

In general extended operators are characterized by the symmetries they break or preserve and correlation functions with both local and non-local (defects) operators. In this work, we consider theories in the presence of conformal line operators, these are extended line operators where the conformal group symmetry acting on the line is broken into the subgroup $SL(2,\mathbb{R})$.

Indeed, starting from the bulk $SO(d+1,1)$ conformal symmetry, a conformal Wilson line, extending in time and centered at $\vec{\mathbf{v}}=0$, would preserve the maximal allowed subgroup that leaves $\vec{x}=0$ invariant. The allowed transformations, therefore, consist of dilations, and translations in time, and out of the special conformal transformations we must only allow those with $\vec{b}=0$ and hence, denoting $b^0=\beta$ we have the transformation (evaluated at $\vec{x}=0$),

$$\vec{x}=0, \quad t' = \frac{t}{1-\beta t}.$$

The three transformations we have found comprise an $SL(2,\mathbb{R})$ subgroup which at $\vec{x}=0$ acts as

$$t' = \frac{at+b}{ct+d}, \quad ad-bc=1.$$

Therefore, our definition of conformal line operators is any line operator that preserves the $SL(2,\mathbb{R})$ subgroup[4]. Any CFT theory that admits conformal line operators is said to be a defect Conformal Field Theory (dCFT).

Consider a CFT with stress-momentum tensor $T$ and operators $\mathcal{O}$. The introduction of a defect line, in general, does not admit a stress tensor on the line, examples of this are t'Hooft lines or boundaries. This is due to the degrees of freedom on the defect do not couple with the metric so there is no possibility of localization of energy on the line defect and energy always ends up being smeared into the bulk. In the case there exist local degrees of freedom localized in the defect, the energy-momentum tensor parallel to the defect is conserved but energy can flow between bulk and defect, momentum normal to the defect is not conserved. This is a particular consequence of the Ward identity derived below.

### 3.2.2 Ward Identity of the defect theory

Similarly to the derivation of $T^{\mu\nu}$, we consider an auxiliary metric $g$ and vary the action along all the geometric quantities in this manifold. Since we now have a defect of dimension $p$ ($p=1$) extending on the manifold, we write $X^\mu(\sigma^a)$ to be the embedding of the defect on the manifold where $\{\sigma^a\}$,

[4] In addition, the transverse rotations in $so(d-1)$ leave $\vec{x}=0$ invariant, but there are $sl(2,R)$ invariant line operators that break this symmetry. So we will not require $so(d-1)$ invariance as part of our definition of DCFT.

$a = 1, ..., p$ are the coordinates that parameterized the defect. The path integral defines the generating functional

$$e^{W[g,X]} = \int D\phi e^{-S[g,X,\phi,\partial\phi,...]}$$

and, by requiring the functional to be invariant under a symmetry transformation, we have

$$S[g + \delta_\xi g, x + \delta_\xi X, \phi + \delta_\xi \phi, \partial\phi + \delta_\xi \partial\phi, \cdots] = S[g, X, \phi, \partial\phi, \cdots].$$

Notice that the symmetry is non-trivial if and only if our action couples the defect degrees of freedom to local geometric quantities and so, for the sake of completeness, we proceed to recall these quantities.

The defect embedding defines the induced metric

$$\gamma_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}, \quad e^\mu_a = \frac{\partial X^\mu}{\partial \sigma^a} \tag{12}$$

where we raise and lower latin indices with $\gamma$ and, additionally, we construct $(d-p)-$unit vector fields $n^\mu_I$ normal to the defect by requiring

$$n^I_\mu e^\mu_a = 0, \quad n^I_\mu n^{\mu J} = \delta^{IJ}. \tag{13}$$

This allows us to write the metric $g$, restricted to the defect, as a decomposition on the normal and tangent frames by

$$g_{\mu\nu}\,|_{\text{Defect}}(X) = e^a_\mu e_{\nu a} + n^I_\mu n_{\nu I}$$

and thus, the geometric quantities that we can construct are composed of the metric $g_{\mu\nu}$ and the vector fields $e^\mu_a$ and $n^\mu_I$. Notice, however, that only $g_{\mu\nu}$ and $e^\mu_a$ are completely independent due to (13). More precisely, the geometrical quantities are: the induced metric $\gamma_{ab}$, the $(d-p)-$tuple of normal unit vectors $n^\mu_A$, the extrinsic curvatures $K^A_{ab}$ of the defect, the spin connection on the normal bundle $\mu_{aIJ}$ and components composed of the bulk Riemann tensor evaluated at the defect, etc.

To derive the Ward identities we consider the expectation value of an observable quantity $\mathcal{X}$ and vary along all of the possible local geometrical objects:

$$\begin{aligned}
\delta_\xi \langle \mathcal{X} \rangle &= \int D\phi \mathcal{X} e^{-S} \delta_\xi S \\
&= \int D\phi \mathcal{X} e^{-S} \left( \int_M \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \right) \\
&\quad + \int D\phi \mathcal{X} e^{-S} \left( \int_D \frac{\delta S}{\delta \gamma_{\mu\nu}} \delta\gamma_{ab} + \frac{\delta S}{\delta n^\mu_a} \delta n^\mu_a + \frac{\delta S}{\delta K^A_{ab}} \delta K^A_{ab} + \frac{\delta S}{\delta \mu_{aIJ}} \delta\mu_{aIJ} + \frac{\delta S}{\delta e^\mu_a} \delta e^\mu_a + \frac{\delta S}{\delta X^\mu} \delta X^\mu + \dots \right) \\
&= -\int_M \frac{1}{2} \langle T^{\mu\nu}_b \mathcal{X} \rangle \delta g_{\mu\nu} \\
&\quad + \int_D \left\langle \mathcal{X} \left( -\frac{1}{2} T^{ab}_D \delta\gamma_{ab} + \lambda^I_\mu \delta n^\mu_I + \frac{1}{2} C^{ab}_I \delta K^I_{ab} + j^{aIJ} \delta\mu_{aIJ} + \eta^a_\mu \delta e^\mu_a + D_\mu \delta X^\mu \right) \right\rangle + \dots
\end{aligned} \tag{14}$$

where we vary along all the geometric quantities in the theory, replaced the definitions of the energy-momentum tensor in the bulk $T_b$ and defect $T_D$ (associated to the variation of the metric $\gamma$) and, defined the *conjugate* operators $\lambda^I_\mu$, $C^{ab}_I$, $j^{aIJ}$, $D_\mu$ associated to the variations of the $(d-p)-$tuple of normal unit vectors $n^\mu_A$, the extrinsic curvatures $K^A_{ab}$, the spin connection on the normal bundle $\mu_{aIJ}$ and the embedding $X^\mu$ respectively. A particularly important operator that appears here is the displacement operator $D^\mu$ as the operator that parametrizes the responses of the system to changes in the location of the defect. Notice that this operator is always present as long as the variation of the defect embedding $X^\mu$ is not vanishing.

Either diffeomorphisms, reparametrizations, or Weyl rescalings act as symmetries of the theory making the variation (14) vanishing. In the remainder of the section we derive the variations for each of these and express them in terms of $\delta g_{\mu\nu}$ and $\delta e^\mu_a$.

Using Equation (12) the variation of the induced metric gives

$$\delta\gamma_{ab} = 2e_{\mu(a}\delta e^\mu_{b)} + e^\mu_a e^\nu_b \delta g_{\mu\nu} \tag{15}$$

and, since a infinitesimal local rotation in the normal bundle, parametrized by an antisymmetric matrix $\omega$, acts as

$$\delta_\omega n^\mu_I = \omega_{IJ} n^{\mu J}, \quad \delta_\omega K_{Iab} = \omega_{IJ} K^J_{ab}, \quad \delta_\omega \mu_{aIJ} = -\nabla_a \omega_{IJ} \tag{16}$$

we can use Equations (13) and (16) to deduce the variation of the normal bundle to be

$$\delta n^\mu_I = \left(-\frac{1}{2} n^\rho_I n^\sigma_J \delta g_{\rho\sigma} + \omega_{IJ}\right) n^{\mu J} - \left(n^\rho_I e^\sigma_a \delta g_{\rho\sigma} + n_{I\rho}\delta e^\rho_a\right) e^{a\mu}. \tag{17}$$

Extrinsic curvature is a little bit more involved. The relation of the extrinsic curvature and the frame bundles are given by

$$\nabla_a e^\mu_b = \partial_a e_b{}^\mu - \hat\Gamma^c_{ab} e_c{}^\mu + \Gamma^\mu_{\lambda\sigma} e^\lambda_a e_b{}^\sigma = n^\mu_I K^I_{ab}, \quad K^I_{ab} = K^I_{(ab)} \tag{18}$$

where we denoted by $\hat\Gamma$ the Christoffel symbols of the induced metric. The variation is then found by using Equation (18) together with (16),

$$\delta K^I_{ab} = \nabla_{(a}\delta e^\mu_{b)} n^I_\mu + n^I_\mu \Gamma^\mu_{\lambda\sigma}\delta e^\lambda_{(a} e^\sigma_{b)} + n^I_\mu \delta\Gamma^\mu_{\lambda\sigma} e^\lambda_a e^\sigma_b + n^\mu_J K^J_{ab} \delta n^I_\mu \omega^I_J K^J_{ab}. \tag{19}$$

Finally, the spin connection $\mu$ is related by the *Weingarten decomposition*:

$$\nabla_a n^\mu_I = \partial_a n^\mu_I + \Gamma^\mu_{\lambda\nu} e^\lambda_a n^\nu_I - \mu_a{}^J n^\mu_J = -K^I_{ab} e^{\mu b}, \quad \mu_{aIJ} = \mu_{a[IJ]}. \tag{20}$$

The variation of $\mu$ is then deduced from Equation (20) and (16) to be:

$$\delta\mu^{IJ}_a = -\nabla_a \delta n^{[I\mu} n^{J]}_\mu - n^{[J}_\mu \Gamma^\mu_{\lambda\nu}\delta e^\lambda_a n^{I]\nu} - n^{[J}_\mu \delta\Gamma^\mu_{\lambda\nu} e^\lambda_a n^{I]\nu} + K^{[I}_{ab} e^{b\mu}\delta n^{J]}_\mu - \nabla_a \omega^{IJ}. \tag{21}$$

Further geometrical quantities are derived from the bulk and defect Riemann tensors and, the extrinsic curvatures are related by the Gauss-Codazzi equations:

$$\begin{aligned} \hat R_{abcd} &= K^I_{ac} K^I_{bd} - K^I_{ad} K^I_{bc} + R_{abcd} \\ \nabla_c K^I_{ab} - \nabla_b K^I_{ac} &= R^I_{abc}, \end{aligned}$$

where the Riemann $R_{abcd}$ is with respect to the $e^\mu_a$ frame.

It remains to see how the $g_{\mu\nu}$ and $e_a^\mu$ transform under diffeomorphisms, reparametrizations and Weyl rescalings. Since we have a sub-manifold embedding the total variation on $X$ of $g$ follows

$$\delta g_{\mu\nu} = \delta_g g_{\mu\nu} + \delta_X g_{\mu\nu}, \quad \text{with } \delta_X g_{\mu\nu} = \delta X^\lambda \partial_\lambda g_{\mu\nu}$$

which is simply stating the variation $g'(X+\delta X) - g(X)$. The Christoffel symbols $\Gamma$ and any bulk quantity evaluated on $D$ follow the same total variations. On the other hand, variation of tangent vectors only changes due to the embedding $\delta e_a^\mu = \delta_X e_a^\mu$.

**Diffeomorphism**

Under an infinitesimal $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ the embedding of the defect change by $\delta_\xi X^\mu = \xi^\mu$ and thus

$$\delta_\xi e_a^\mu = \frac{\partial \delta X^\mu}{\partial \sigma^a} = \partial_a \xi^\mu.$$

The metric, according to the above discussion, changes accordingly to

$$\delta_\xi g_{\mu\nu} = \delta_{\xi,g} g_{\mu\nu} + \delta X^\lambda \partial_\lambda g_{\mu\nu} = -2\nabla_{(\mu}\xi_{\nu)} + \xi^\lambda \partial_\lambda g_{\mu\nu} = -2 g_{\rho(\mu}\partial_{\nu)}\xi^\rho$$

and similarly for $\Gamma$,

$$\delta_\xi \Gamma^\mu_{\nu\rho} = \partial_\sigma \xi^\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \xi^\sigma \Gamma^\mu_{\sigma\rho} - \partial_\rho \xi^\sigma \Gamma^\mu_{\sigma\nu} - \partial^2_{\nu\rho}\xi^\mu.$$

Plugging this in Equations (15), (19),(21) and (17) we find:

$$\begin{aligned}
\delta_\xi \gamma_{ab} &= 0 \\
\delta_\xi n_I^\mu &= n_I^\lambda \partial_\lambda \xi^\mu + \omega'_{IJ} n^{\mu J} \\
\delta_\xi K^I_{ab} &= \omega'_{IJ} K^J_{ab}, \\
\delta_\xi \mu_{aIJ} &= -\nabla_a \omega'_{IJ},
\end{aligned} \tag{22}$$

with $\omega'_{IJ}$ an arbitrary antisymmetric matrix. Up to transverse rotations these are the expected tensorial transformations, $n$ is a vector while the others are scalars.

**Reparametrizations**

Reparametrizations are diffeomorphisms of the defect and thus they are the ones that generate the variation for the energy-momentum tensor in the defect $T_D$. Under this $\sigma^a \to \sigma^a + \zeta^a(\sigma)$, an infinitesimal parametrization of the defect, the embedding and metric changes by

$$\delta_\zeta X^\mu = -e_a^\mu \zeta^a \tag{23}$$

$$\delta_\zeta g_{\mu\nu} = \delta X^\lambda \partial_\lambda g_{\mu\nu} = -e_a^\mu \zeta^a \partial_\lambda g_{\mu\nu}. \tag{24}$$

Hence, the tangent frame transforms by

$$\delta_\zeta e_a^\mu = \frac{\partial \delta_\zeta X^\mu}{\partial \sigma^a} = -\partial_a(e_b^\mu \zeta^b) = -\nabla_a \zeta^b - \Gamma^b_{ac}\zeta^c = -\nabla_a(\zeta^b e_b^\mu) - K^I_{ab}\zeta^b n_I^\mu + \Gamma^\mu_{b\lambda} e_a^\lambda \zeta^b$$

where we used the fact $\nabla_a$ is covariant both under diffeomorphisms in D and local orthogonal transformations in the normal bundle ($\nabla_a \zeta^b = \partial_a(e_b^\mu \zeta^b) + \Gamma^b_{ac}\zeta^c$) and Equation (18).

Inserting the expressions in their respective variations we get:

$$\begin{aligned}\delta_\zeta \gamma_{ab} &= -2\nabla_{(a}\zeta_{b)} \\ \delta_\zeta n_I^\mu &= K^I_{ab} e^{\mu a}\zeta^b + \Gamma^\mu_{\lambda b} n_I^\lambda \zeta^b + \omega''_{IJ} n^{\mu J} \\ \delta_\zeta K^I_{ab} &= -2\nabla_{(a}\zeta^c K^I_{b)c} - \zeta^c \nabla_c K^I_{ab} + \omega''_{IJ} K^J_{ab}, \\ \delta_\zeta \mu_{aIJ} &= K_a^{[Ib} K_{bc}^{J]}\zeta^c - \frac{1}{2}\zeta^c R_{caIJ} - \zeta^c R_{c[IaJ]} - \nabla_a \omega''_{IJ}.\end{aligned} \tag{25}$$

### Weyl scalings

Finally, for a Weyl rescaling $\sigma$ only the metric has a non vanishing variations giving

$$\delta_\sigma g_{\mu\nu}(X) = 2\sigma(X) g_{\mu\nu}(X).$$

We find:

$$\begin{aligned}\delta_\zeta \gamma_{ab} &= 2\sigma\gamma_{ab} \\ \delta_\zeta n_I^\mu &= -\sigma n_I^\mu + \omega_{IJ} n^{\mu J} \\ \delta_\zeta K^I_{ab} &= \sigma K^I_{ab} - \gamma_{ab} n^{\mu I}\partial_\mu \sigma + \omega^{IJ} K_{Jab}, \\ \delta_\zeta \mu_{aIJ} &= -\nabla_a \omega_{IJ}.\end{aligned} \tag{26}$$

**Remark 2.** *We conclude that all of these quantities transform as tensors under a diffeomorphism in M, up to a local rotation in the normal bundle. We find therefore convenient to ask for symmetry under local rotations in the normal bundle. Then the defect action may be constructed as the integral of a scalar function over D.*

Equation (14) expresses relations between the different operators. In particular, for a symmetry transformation, taking the differential form we can find the conservation of the bulk energy-momentum tensor $T_b$ which will be affected by the defect localization of all the quantities that are defined in the defect $\mathcal{D}$. Crucially, we see that the energy-momentum tensor in the defect $T_D$ and the displacement operator $D^\mu$ must always exist and therefore we consider only the case where defect degrees of freedom only couple to these. Under this assumption Equation (14) becomes:

$$0 = -\int_M \frac{1}{2}\langle T_b^{\mu\nu}\mathcal{X}\rangle \delta g_{\mu\nu} + \int_D \left\langle \mathcal{X}\left(-\frac{1}{2}T_D^{ab}\delta\gamma_{ab} + D_\mu \delta X^\mu\right)\right\rangle. \tag{27}$$

In this case, the bulk variation, since $\delta_\xi g_{\mu\nu} = -2\nabla_{(\mu}\xi_{\nu)}$, allow us to write

$$\int_M T_b^{\mu\nu}\delta g_{\mu\nu} = 2\int_M (\nabla_\mu T^{\mu\nu})\xi_\nu$$

and similarly for the defect,

$$\int_D T_D^{ab}\delta\gamma_{ab} = 2\int_D (\nabla_a T^{ab})\zeta_b.$$

Further, for reparametrizations of the defect (which are the only ones that contribute for the stress-energy tensor of the defect due to (22)), according to (23), gives $\delta X^\mu = -e^\mu_a \zeta^a$ which is proportional to the tangent $e_a^\mu$ and therefore $D_\mu$ is only non-vanishing for perpendicular coordinates i.e. $D^\mu = n_i^\mu D^i$. Putting all back together gives:

$$0 = \int_M \langle (\nabla_\mu T^{\mu\nu})\xi_\nu \mathcal{X}\rangle + \int_D \langle (\nabla_a T_D^{ab})\zeta_b \mathcal{X}\rangle + \langle \mathcal{X} n_i^\mu D^i \xi^\mu \rangle.$$

Since this holds for any $\xi$, $\zeta$ and $\mathcal{X}$ we finally conclude the Ward identity:

$$\langle \nabla_\mu T_b^{\mu\nu}\rangle = -\delta_D^{d-1}\langle \nabla_\mu T_d^{\mu\nu}\rangle - \langle \delta_D^{d-1} n_i^\nu D^i\rangle, \tag{28}$$

where $n_i$ are the unit vectors normal to the defect and $D^i$ is as before the displacement operator which parameterized the breaking of translations in the directions normal to the defect. Finally, in a defect theory with a line defect $(p = 1)$ we have the explicit relation $\nabla_a T_d^{a\nu}\xi_\nu = \dot{X}^\nu \dot{T}_D \xi_\nu$ which gives the conservation equation:

$$\langle \nabla_\mu T_b^{\mu\nu}\rangle = -\delta_D^{d-1}\langle \dot{X}^\nu \dot{T}_D + n_i^\nu D^i\rangle. \tag{29}$$

Crucially we conclude that the stress-energy tensor in the presence of a defect will not be conserved but exchange of energy between line and bulk will be present.

### 3.2.3 The relation between the $\beta-$function and the energy-momentum tensor

Notice that we have two notions of conformal invariance, as a group-theoretic property, which translates to the vanishing of the trace, and as a property of quantum theories, which translates as fixed points of some RG flow. To conclude this section we show a neat way in which RG flows and the stress tensor are explicitly related.

Once again the Ward identities provide this connection. Consider a QFT with couplings $\{g_i\}$. If we perform an infinitesimal scale transformation $x^\nu \to (1+\epsilon)x^\nu$ i.e. $\Lambda \to \Lambda/(1+\varepsilon) = \Lambda - \varepsilon\Lambda$, then the couplings $\{g_i\}$ transform by:

$$g_i(\Lambda) \to g_i(\mu - \epsilon\mu) = g_i(\mu) - \epsilon\beta_i,$$

where we used the definition of the $\beta-$function. This transformation gives an extra contribution to the variation of the action:

$$\delta S = \delta S_{\text{classic}} + \delta S_{\text{quantum}} = \int d^d x\, \partial_\nu T_\mu^\nu \varepsilon x^\mu + \int d^d x \frac{\partial \mathcal{L}}{\partial g_i}\delta g_i = \int d^d x \partial_\nu T_\mu^\nu\, \varepsilon x^\mu - \int d^d x \frac{\partial \mathcal{L}}{\partial g_i}\varepsilon\beta_i,$$

where we identify $J^\mu \equiv T_\nu^\mu x^\nu$, the conserved current for a scale transformation (c.f. to Equation (11)) and, we used the variation of $g_i$. Setting $\delta S = 0$ (i.e. a symmetry transformation), we find:

$$\partial_\nu J^\nu = \frac{\partial \mathcal{L}}{\partial g_i}\,\beta_i,$$

where this equation should be interpreted as an operator equation valid under correlation functions. Using that the stress-tensor is conserved gives[5]:

$$T_\mu^\mu = \sum_i \beta_i \mathcal{O}_i. \tag{30}$$

[5] Here we assumed $g$ dimension full coupings i.e. $\mathcal{L} \sim g_i\mathcal{O}_i$ instead of the previous, dimensionless, notation $\mathcal{L} \sim \Lambda^{\Delta_i} g_i \mathcal{O}_i$.

**Remark 3.** *Consider an scalar theory with an interaction term $\lambda\phi^4$. The energy-momentum tensor can be easily seen to be: $T^{\mu}_{\nu} = \partial_\nu\phi\partial^\mu\phi - \delta^\mu_\nu(\frac{1}{2}(\partial\phi)^2 - \lambda\phi^4)$. Since we allow total derivatives, we can find the symmetric "improved" stress-tensor:*

$$T^{\mu\nu} = \frac{1}{2(d-1)}\left[d\partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\partial_\rho\phi\partial^\rho\phi - (d-2)\phi\partial^\mu\partial^\nu\phi + (d-2)\eta^{\mu\nu}\phi\Box\phi\right] + \lambda\eta^{\mu\nu}\phi^4. \tag{31}$$

*Naively, the trace of the energy-momentum tensor is not of the form of Equation* (30)*, it's only under the equations of motion that we find*

$$T = \frac{d-2}{2}\phi(-4g\phi^3) + dg\phi^4 = \beta_\lambda(\Lambda)\phi^4.$$

# 4 g-theorem for Line Defects

We consider a circular defect as it is conformal equivalent[6] to a line by noticing that a special conformal transformation sends lines to circles and circles to lines. It is therefore desirable to consider instead circular defects as they have the advantage of finite distance operators, periodic parametrizations and, no boundary conditions. We locate the circle of radius $R$ centered around the origin of the $x_1 - x_2$ plane $(x_3 = \cdots = 0)$.

Under this assumption, the symmetry group of the DCFT is the $SL(2,\mathbb{R})$ whose Killing vector fields preserved by the circle are:

$$\xi_{(1)} = \frac{1}{2}\left[\left(R + \frac{x^2}{R}\right)\partial_1 - \frac{2}{R}x_1x^\mu\partial_\mu\right] = \frac{1}{2}[RP_1 + \frac{1}{R}K_1], \tag{32}$$

$$\xi_{(2)} = \frac{1}{2}\left[\left(R + \frac{x^2}{R}\right)\partial_2 - \frac{2}{R}x_2x^\mu\partial_\mu\right] = \frac{1}{2}[RP_2 + \frac{1}{R}K_2], \tag{33}$$

$$\xi_{(\phi)} = x_1\partial_2 - x_2\partial_1. \tag{34}$$

Indeed, considering polar coordinates in the $(x_1 - x_2)$ plane, i.e. the plane of the circle, we want a linear combination of the Killing fields that have only $\partial_\theta$ components. Clearly, rotations of this plane (generated by the orthogonal axis) are a symmetry which immediately proves Equation (34). More generally, let $\xi = \alpha^\mu K_\mu + \beta^\mu P_\mu + \gamma D + \delta^{\mu\nu}M_{\mu\nu}$ be a Killing vector field for some coefficients $\alpha$, $\beta$, $\gamma$ and $\delta$. Going to polar coordinates and looking at the $\partial_r$ coordinate, we require $0 = \xi_r = \gamma r + (\beta_1 - \alpha_1 r^2)\sin(\theta) + (\beta_2 - \alpha_2 r^2)\cos(\theta)$ and therefore, we conclude $\gamma = 0, \beta_a = \alpha_a r^2$ and no restriction for $\delta$. Since for coordinates in the defect $r = R$, we have the result.

With the symmetry generators of the defect circle, we can define a charge $Q_\xi$ of each one of the associated killing vectors of Equations (32), (33) and (34), analogously to Equation (11), by

$$Q_\xi(D) = \int_{T(\varepsilon)} d^{d-1}\Sigma^\mu\langle T^b_{\mu\nu}\rangle\xi^\nu = \int_D d\sigma(\nabla_\mu\langle T^b_{\mu\nu}\rangle)\xi^\nu,$$

where we consider a surface wrapping the circular defect whose distance is $\varepsilon << 1$ to the circular defect points and, in the second equality we applied Stokes theorem and the fact that $\xi$ follows the Killing

[6] Meaning that we have $\langle\rangle_{Circ} = e^a\rangle\langle_{Flat}$

equation. This charge vanishes as they generate a symmetry of the circular defect provided $\varepsilon$ is small enough so that the surface does not intersect other fields insertions. We conclude,

$$Q_\xi(D) = 0. \tag{35}$$

## 4.1 Defect Renormalization Group Flow of line defects in $d > 2$

We have now all the ingredients to work on the DCFT Renormalization Group (RG) flow. In the defect case, we can start from a bulk CFT, introduce a line defect that breaks the symmetry, and evolve this line along the RG flow generated by the introduction of a scale towards a fixed point. One would expect the resulting theory to be a DCFT. Furthermore, away from the critical points in the RG flow, we expect the existence of a stress tensor $T_d$ of the defect which allows us to use Equation (29) to study the theory similarly as for the $c-$theorem in Section 3.1.

If there exists a relevant defect[7] operator on the bulk $\mathcal{O}$, a Defect Renormalization Group (DRG) flow may be triggered by perturbing the DCFT as,

$$S_{DCFT} \to S_{DCFT} + M_0 \int_D d\sigma \mathcal{O}(\sigma). \tag{36}$$

$SL(2,\mathbb{R})$ (conformal) invariance is now explicitely broken by the introduction of the physical scale $M_0$ to translations on the defect. Notice, however, that the line defect becomes trivial in the UV. This is called the *pinning field* or external field defect which is the continuous realization of turning a localized magnetic field on a lattice.

Due to the locality of the bulk CFT, the bulk stress tensor remains conserved and traceless (up to possible bulk trace anomalies in curved space) away from the line, however, as we saw in Section 3.2, energy can now be stored on the defect and $T_D$ always exists away from the fixed points of the defect. We introduce a dimensionless background dilaton field $\Phi$ to parameterize the mass scale by $M(\sigma) = M_0 e^{-\Phi(\sigma)}$ . The dilaton field acts as a source for the theory and, since $T_D = \beta_{M_0}\mathcal{O}(\sigma) \propto M_0$, according to Equation (30), this, in turn, modifies the conservation Equation (29) to:

$$\langle \nabla_\mu T_b^{\mu\nu} \rangle_\Phi = -\langle \delta_D^{d-1} \dot{X}^\nu \Big( \dot{T}_D - \dot{\Phi} T_D \Big) \rangle_\Phi - \langle \delta_D^{d-1} n_i^\nu D^i \rangle_\Phi. \tag{37}$$

Notice that Equation (35), the existence of a vanishing charge wrapping the defect, together with Equation (37) gives a constrain on the dilation field given by

$$\begin{aligned} 0 =& Q_\xi(D) = \int_D d\sigma (\nabla_\mu \langle T^b_{\mu\nu} \rangle) \xi^\nu = \int_D d\sigma (-\xi_D) \langle \dot{T}_D \rangle_\Phi + \xi_D \dot{\Phi} \langle T_D \rangle_\Phi & (38) \\ =& \int_D d\sigma (\dot{\xi}_D + \xi_D \dot{\Phi}) \langle T_D \rangle_\Phi, & (39) \end{aligned}$$

with $\xi_D$ the projection of the Killing along the defect ($\xi_D = \xi^\mu \dot{X}_\mu$). Since the dilaton field and the stress tensor must couple linearly, this equation may be interpreted as a part of the effective action of the dilaton and stress-tensor coupling and thus it states that fields $\Phi$ and the one generated by the infinitesimal $\Phi(\sigma) + \alpha(\dot{\xi}(\sigma) + \xi_D(\sigma)\dot{\Phi}(\sigma))$ couple the same to the stress tensor.

[7] Here relevant operators are taken to be with respect to the defect dimension e.g. for a line defect operators $\mathcal{O}$ with $\Delta_{\mathcal{O}} < 1$ are relevant operators.

We can thus expand Equation (38) expected value around $\Phi = 0$,

$$\begin{aligned}
0 = Q_\xi(D) &= \int_D d\sigma(\dot{\xi}_D + \xi_D\dot{\Phi})\Big(\langle T_D\rangle_{\Phi=0} + \int d\sigma\Phi(\sigma_2)\langle T_D(\sigma)T_D(\sigma_2)\rangle_{\Phi=0} + O(\Phi^2)\Big) \\
&= \int d\sigma\dot{\xi}_D(\sigma)\langle T_D(\sigma)\rangle_{\Phi=0} + \int d\sigma\xi_D(\sigma)\dot{\Phi}(\sigma)\langle T_D(\sigma)\rangle_{\Phi=0} \\
&\qquad + \int d\sigma_1 d\sigma_2\dot{\xi}_D(\sigma_1)\Phi(\sigma_2)\langle T_D(\sigma_1)T_D(\sigma_2)\rangle_{\Phi=0} + O(\Phi^2).
\end{aligned} \tag{40}$$

This generates an infinite amount of relations. Particularly, to first order in $\Phi$ we conclude:

$$\int d\sigma\dot{\xi}_D(\sigma)\langle T_D(\sigma)\rangle_{\Phi=0} = -\int d\sigma_1 d\sigma_2\dot{\xi}_D(\sigma_1)\Phi(\sigma_2)\langle T_D(\sigma_1)T_D(\sigma_2)\rangle. \tag{41}$$

Equation (41) relates one and two-point functions of the defect stress scalar for our Killing vector fields of the defect. Explicitly, if we parametrizing the circle by $x^1 = R\cos\phi$, $x^2 = R\sin\phi$ and, $\sigma = R\phi$ then $\dot{X} = (-\frac{\sin\phi}{R}, \frac{\cos\phi}{R}, \vec{0})$. The projection of the Killing fields (Equations (32), (33) and (34)) on the defect gives:

$$\begin{aligned}
\xi_{(\phi)\,D} &= \xi_{(\phi)}\cdot\dot{X} = x_1\dot{X}_2 - x_2\dot{X}_1 = -\cos^2\phi - \sin^2\phi = -1, \\
\xi_{(1)\,D} &= \frac{1}{2}\left((R+\frac{R^2}{R}) - \frac{2}{R}x_1^2\right)\dot{X}_1 + \frac{1}{2}\left(-\frac{2}{R}x_1x_2\right)\dot{X}_2 = R\sin^2\phi(-\frac{\sin\phi}{R}) - \frac{2}{2}\cos^2\phi\sin\phi = -\sin\phi, \\
\xi_{(2)\,D} &= \frac{1}{2}\left((R+\frac{R^2}{R}) - \frac{2}{R}x_2^2\right)\dot{X}_2 + \frac{1}{2}\left(-\frac{2}{R}x_1x_2\right)\dot{X}_1 = \cos^2\phi(\cos\phi) + \sin^2\phi\cos\phi = \cos\phi.
\end{aligned}$$

Now, for each one of these Killings the Ward Identity of Equation (41) gives a constrain on the two and one-point functions, however, due to translation invariance we can set $\phi \to \phi - \pi/2$ to see that the constrain generated by $\xi_{(1)\,D}$ is equivalent to the one of $\xi_{(2)\,D}$. We have then two independent relations. The first, generated by $\xi_{(\phi)}$, does not add information about the relation between one- and two-point functions as the left-hand side of the equation vanishes. The second, picking the one generated by $\xi_{(1)}$ and, making a judicious choice of the dilaton profile $\Phi = C\cos\phi$ for some constant $C$[8], plugged in Equation (41), gives[9]:

$$R\int_D d\phi\left(\frac{C}{R}\sin^2\phi\right)\langle T_D(\phi)\rangle = R^2\int_D d\phi_1 d\phi_2(\cos\phi_1)(\frac{C}{R}\cos\phi_2)\langle T_D(\phi_1)T_D(\phi_2)\rangle. \tag{42}$$

Using translation invariance $\phi \to \phi + \pi/2$, $\phi_1 \to \phi_1 + \pi/2$ and, $\phi_2 \to \phi_2 + \pi/2$, we can equivalently write Equation (42) as

$$R\int_D d\phi\left(\frac{C}{R}\cos^2\phi\right)\langle T_D(\phi)\rangle = R^2\int_D d\phi_1 d\phi_2(\sin\phi_1)(\frac{C}{R}\sin\phi_2)\langle T_D(\phi_1)T_D(\phi_2)\rangle, \tag{43}$$

which, adding up both Equations (42) and (43), allows us to simplify the trigonometric functions to find the relation:

$$R\int d\phi\langle T_D(\sigma)\rangle = R^2\int d\phi_1 d\phi_2\cos(\phi_1 - \phi_2)\langle T_D(\sigma_1)T_D(\sigma_2)\rangle. \tag{44}$$

[8] The dilaton profile must be a $2\pi$-periodic function such that the action defined in (1) is well defined.

[9] Note that here $\xi_D$ is either $0$ or $\cos\phi$ and therefore, the zeroth-order term of Equation (41) vanishes identically or, by translation invariance of the stress tensor respectively. As it should be due to $Q_\xi = 0$.

### 4.1.1 The defect entropy

Finally we can define an observable for lines called the *defect entropy* $s$ in terms of the partition function in the presence of the defect,

$$\log g(M_0 R) = \log Z_M - \log Z_M^{(CFT)}.$$

This function depends on the dimensionless product $M_0 R$, however, there is an ambiguity of $\log g$ due to renormalization counter terms. Let us emphasize that the set of counterterms needed to make the functional integral finite should be diffeomorphism invariant as well. Of course, we also assume the action to be invariant under reparametrizations of the defect and, since the mass renormalization can only be made of operators of dimension 1, the only operator that can appear on a renormalization scheme is the shifted energy difference

$$\alpha \int d\sigma M_0 \sim M_0 R,$$

where $R$ is the radius of the circular defect. All other non-geometrical invariant terms that are analytic around the flat metric are of dimension larger than one and cannot appear as counter-terms. This implies that there are no other ambiguities in $d > 2$. Therefore, this renders the scheme-independent quantity

$$s(M_0 R) = (1 - R\partial_R) \log g(M_0 R), \tag{45}$$

called the *defect entropy*. In general, this is an operator with $R$ dependence but at the fixed points this gives a constant number as the theory becomes scale-invariant. Consequently, since the extremal points define the UV and IR CFTs (could be empty), one has

$$s(M_0 R) \to \begin{cases} \log g_{UV}, & R \to 0 \\ \log g_{IR}, & R \to \infty \end{cases} \tag{46}$$

where we used the fact that the renormalization flow is implemented by changing scale $R$ as we saw in Section 2.1.

**Theorem 4** ($g-$theorem for DCFTs)**.** *Under the above assumptions, the defect entropy $s(M_0 R)$ is a scheme-independent monotone function. In particular, $\log g$ is a number on the fixed points and we have*

$$g_{UV} > g_{IR}.$$

*Proof.* Using Equation (46), it remains to probe that $s$ is monotonically decreasing along the RG flow. Since

$$M_0 \frac{\partial}{\partial M_0} \log g(M_0 R) = M_0 R \frac{1}{g} g' = -\frac{d}{d\phi} \log g(M_0 R e^{-\Phi}) \,|_{\Phi=0} \,,$$

then we compute the change of entropy in terms of the $\Phi$ expansion,

$$\log Z\Big|_{\phi+\delta\phi} = \log Z\Big|_{\phi} + \int_D d\sigma \delta\phi(\sigma) \langle T_D(\sigma) \rangle_\Phi + \frac{1}{2} \int_D d\sigma_1 d\sigma_2 \delta\phi(\sigma_1) \delta\phi(\sigma_2) \langle T_D(\sigma_1) T_D(\sigma_2) \rangle_\Phi + \ldots \tag{47}$$

Using Equation (47), we have:

$$
\begin{aligned}
M_0 \frac{\partial}{\partial M_0} s(M_0 R) &= -\left[ \left(\frac{d}{d\Phi} - \frac{d^2}{d\Phi^2}\right) \log g(R M_0 e^{-\Phi}) \right]_{\Phi=0} \\
&= R \int_D d\phi \langle T_D(\phi) \rangle - R^2 \int_D d\phi_1 d\phi_2 \langle T_D(\phi_1) T_D(\phi_2) \rangle, \\
&= -R^2 \int_D d\phi_1 d\phi_2 \langle T_D(\phi_1) T_D(\phi_2) \rangle [1 - \cos(\phi_1 - \phi_2)],
\end{aligned} \tag{48}
$$

where we used the parametrization $\sigma = R\phi$ and the Ward Identity of Equation (44) in the third line. This quantity is a negative definite function (for unitary theories) and thus we conclude $s$ is monotonically decreasing along the defect RG flow which, in turn, implies

$$g_{UV} > g_{IR}.$$

As we will see in the example below, this means that the defect impurity can only push out bulk degrees of freedom! ■

### 4.1.2 Example

The simplest example of a nontrivial defect is done by perturbing the massless scalar Free Field (FF) by

$$S_{FF} \to \int d^d x \frac{1}{2} (\partial\phi)^2 + h \int_\gamma d\sigma \phi, \tag{49}$$

where $\gamma$ is some path in the space and $h$ has dimension $\frac{d-4}{2}$ (the dimension of the free field). The line term operator, as the free field scaling dimension is $2 - \frac{d}{2}$, is relevant for $d < 4$, exactly[10] marginal at $d = 4$ and, irrelevant otherwise. In $d > 2$, the variation of the action (49) gives the classical equation of motion:

$$\Box\phi(x) = h \int_\gamma d\sigma \delta^d(x - \gamma(\sigma)).$$

Thus, noticing that this is nothing but the Green function of the d'Alambertian operator integrated along the curve $\gamma$, we find:

$$\phi_{cl}(x) = \frac{2h}{(d-2)\Omega_{d-1}} \int_0^{2\pi} d\sigma \frac{1}{|x - \gamma(\sigma)|^{d-2}}, \quad \Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}. \tag{50}$$

Notice that, since the theory is free, the fluctuations around $\phi_{cl}(x)$ are insensitive to the existence of the line defect and therefore, we can compute the defect partition function, normalized by the partition function of the theory without the defect, as a function of $R$ by simply plugging the classical solution back into the action.

[10] As there are no interactions quantum corrections are absent.

If we consider the circular line defect of radius $R$, parametrized by $\gamma(\phi) = (R\cos\phi, R\sin\phi)$, the classical action is

$$\begin{aligned} S &= \int d^dx - \frac{1}{2}\phi_{cl}\Box\phi_{cl} + h\int_\gamma d\sigma\phi_{cl} = h\int_\gamma d\sigma\phi_{cl} \\ &= \frac{2h^2}{(d-2)\Omega_{d-1}}\int d\sigma_1 d\sigma_2 \frac{1}{|\gamma(\sigma_1)-\gamma(\sigma_2)|^{d-2}} \\ &= \frac{2h^2R^{4-d}}{(d-2)\Omega_{d-1}}\int d\sigma_1 d\sigma_2 \frac{1}{|(\cos\sigma_1-\cos\sigma_2)^2+(\sin\sigma_1-\sin\sigma_2)^2|^{\frac{d-2}{2}}}. \end{aligned} \tag{51}$$

Which, by using translation invariance (as we integrate from $0$ to $2\pi$) and trigonometric identities gives:

$$\begin{aligned} S &= \frac{2h^2R^{4-d}}{(d-2)\Omega_{d-1}}2\pi\int_0^{2\pi} d\sigma_1\frac{1}{|(\cos\sigma_1-1)^2+(\sin\sigma_1)^2|^{\frac{d-2}{2}}} \\ &= \frac{4\pi h^2R^{4-d}}{(d-2)\Omega_{d-1}}\int_0^{2\pi} d\phi\frac{1}{\left[4\sin^2(\phi/2)\right]^{d/2-1}}. \end{aligned} \tag{52}$$

We therefore find:

$$\begin{aligned} g \equiv \log\left(Z_{\text{defect}}/Z_{\text{bulk}}\right) &= \frac{4\pi h^2R^{4-d}}{(d-2)\Omega_{d-1}}\int_0^{2\pi} d\phi\frac{1}{\left[4\sin^2(\phi/2)\right]^{d/2-1}} \\ &= 4\pi h^2R^{4-d}\frac{\Gamma\left(\frac{3}{2}-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{2^{d-1}\pi^{\frac{d-1}{2}}\Gamma\left(2-\frac{d}{2}\right)}, \end{aligned} \tag{53}$$

where we used Mathematica for the computation of the integral. We can extract the defect entropy $s$ using Equation (45) to be:

$$s = 4\pi h^2R^{4-d}c(d), \quad c(d) \equiv -\frac{\Gamma\left(\frac{5}{2}-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma\left(2-\frac{d}{2}\right)}. \tag{54}$$

In $d=3$ we find

$$s = -\frac{1}{2\pi^{3/2}}\pi h^2R,$$

so we see that the defect entropy in the ultraviolet ($R\to 0$) is vanishing as expected (since the line defect is trivial) and in the infrared ($R\to\infty$) $s\to-\infty$ which is consistent with the $g$-theorem. Notice that this allows us to understand the meaning of $g$, we say that $g$ counts the degrees of freedom that are being pushed out of the line as we evolve in the RG. In this case, the entropy being negative means that degrees of freedom are being pushed into the line making the IR line nontrivial.

Finally, as a consistency, we check (48). The right-hand side of Equation (48), using Equation (54), gives:

$$M_0\frac{\partial}{\partial M_0}s = (d-4)h^2\pi R^{4-d}c(d), \tag{55}$$

where we used the fact $h = M_0^{d/2-2}g$ for some $g$ dimensionless. On the other hand, to verify the left-hand side of equation (48), we need to compute $\langle T_D(\sigma_1)T_D(\sigma_2)\rangle$.

Using Equation (30), the only $\beta-$function in the defect action is $\beta_h = \Lambda\partial_\Lambda(\Lambda^{\frac{d-4}{2}}g) = \frac{d-4}{2}h$ and thus,

$$T_D(\sigma) = \beta_h\phi(\gamma(\sigma)).$$

Since the only contribution we have is classical, as we said before, plugging in the two-point function gives:

$$\langle T_D\left(\sigma_1\right) T_D\left(\sigma_2\right)\rangle_c = \frac{\beta_h^2}{(d-2)\Omega_{d-1}R^{d-4}}\frac{1}{\left[4\sin^2\left(\frac{\phi_1-\phi_2}{2}\right)\right]^{\frac{d-2}{2}}}. \tag{56}$$

Finally, with Equation (56) we can evaluate the integral on the right-hand side of Equation (48). Using Mathematica we find:

$$-\int d\sigma_1\int d\sigma_2\,\langle T_D\left(\sigma_1\right) T_D\left(\sigma_2\right)\rangle_c\left[1-\cos\left(\frac{\sigma_1-\sigma_2}{R}\right)\right] = (4-d)4\pi h^2 R^{4-d}c(d).$$

In accordance with the defect entropy variation (55).

# 5 Conclusions

In this work we showed how different monotonic charges (under RG flows) might be constructed and, why they are useful tools for studying QFTs.

In Section 2, we studied how RG flows are essential for understanding Quantum Field Theories, how CFTs are special points of the RG flows, and how they provide a connection between theories in the UV and the IR. The first application of this connection is on how CFTs in the IR and UV are characterized by a central charge that monotonically decreases, as explained by Zamolodchikov's $c-$theorem that reveals properties of the RG flow and its irreversibility. To extend these results for the introduction of defect lines in QFT, we develop general results, where properties of (quantum) quantities, such as the stress-tensor, charges, $\beta-$functions, etc., are changed due to the presence of the defect line.

In Section 4, we extended the $c-$theorem for general $d$ in the presence of a line defect, to define the charge $g$ as the difference of the partition function of the defect theory with respect to the underlying CFT. This quantity, which has the interpretation of counting degrees of freedom that are being pushed out ($g$ decreasing) of the line as we evolve to the IR, was explicitly computed for the simplest theory, mainly a $\phi^1$ scalar theory of model (49), but, of course, theories such as $O(N)$ model, $\phi^4$ and Wilson lines have been studied in the literature [4, 5]. This example serves both as a consistency check for the $g-$theorem that was just derived before and as an illustration of how the quantity $g$ should be interpreted. In this case, the dCFT in the UV was trivial (as the line perturbation was performed by an irrelevant operator and $s = 0$), and thus the $g$ theorem states that the dCFT in the infrared limit is necessarily nontrivial as for this case we have $s < 0$, a rather remarkably conclusion: We have found a nontrivial dCFT!

## References

[1] Marco Billò et al. "Defects in conformal field theory". In: *Journal of High Energy Physics* 2016.4 (Apr. 2016), pp. 1–56. ISSN: 1029-8479. DOI: 10.1007/jhep04(2016)091. URL: http://dx.doi.org/10.1007/JHEP04(2016)091.

[2] A. B. Zomolodchikov. ""Irreversibility" of the flux of the renormalization group in a 2D field theory". In: *Soviet Journal of Experimental and Theoretical Physics Letters* 43 (June 1986), p. 730.

[3] Zohar Komargodski and Adam Schwimmer. "On renormalization group flows in four dimensions". In: *Journal of High Energy Physics* 2011.12 (Dec. 2011). ISSN: 1029-8479. DOI: 10.1007/jhep12(2011)099. URL: http://dx.doi.org/10.1007/JHEP12(2011)099.

[4] Gabriel Cuomo, Zohar Komargodski, and Márk Mezei. "Localized magnetic field in the O(N) model". In: *JHEP* 02 (2022), p. 134. DOI: 10.1007/JHEP02(2022)134. arXiv: 2112.10634 [hep-th].

[5] Gabriel Cuomo, Zohar Komargodski, and Avia Raviv-Moshe. "Renormalization Group Flows on Line Defects". In: *Phys. Rev. Lett.* 128.2 (2022), p. 021603. DOI: 10.1103/PhysRevLett.128.021603. arXiv: 2108.01117 [hep-th].

[6] Slava Rychkov. *EPFL Lectures on Conformal Field Theory in D>= 3 Dimensions*. SpringerBriefs in Physics. Jan. 2016. ISBN: 978-3-319-43625-8, 978-3-319-43626-5. DOI: 10.1007/978-3-319-43626-5. arXiv: 1601.05000 [hep-th].

[7] Chi-Ming Chang et al. "Topological Defect Lines and Renormalization Group Flows in Two Dimensions". In: *JHEP* 01 (2019), p. 026. DOI: 10.1007/JHEP01(2019)026. arXiv: 1802.04445 [hep-th].

[8] S. Weinberg. *The Quantum Theory of Fields*. vol. 1. Cambridge University Press, 1995. ISBN: 9780521585552. URL: https://books.google.fr/books?id=424vlQEACAAJ.

[9] Christopher J. Fewster and Kasia Rejzner. *Algebraic Quantum Field Theory – an introduction*. 2019. arXiv: 1904.04051 [hep-th].

[10] David Skinner. *Quantum Field Theory II*. University of Cambridge. Available online at https://www.damtp.cam.ac.uk/user/dbs26/AQFT.html. 2024.

[11] Joseph Polchinski. "Scale and conformal invariance in quantum field theory". In: *Nuclear Physics B* 303.2 (1988), pp. 226–236. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(88)90179-4. URL: https://www.sciencedirect.com/science/article/pii/0550321388901794.

[12] Ofer Aharony et al. "Phases of Wilson Lines in Conformal Field Theories". In: *Phys. Rev. Lett.* 130.15 (2023), p. 151601. DOI: 10.1103/PhysRevLett.130.151601. arXiv: 2211.11775 [hep-th].

[13] Ofer Aharony et al. "Phases of Wilson Lines: Conformality and Screening". In: (Sept. 2023). arXiv: 2310.00045 [hep-th].

[14] Gabriel Cuomo et al. "Spin impurities, Wilson lines and semiclassics". In: *JHEP* 06 (2022), p. 112. DOI: 10.1007/JHEP06(2022)112. arXiv: 2202.00040 [hep-th].

[15] S. Weinberg. *The Quantum Theory of Fields*. The Quantum Theory of Fields 2 Volume Hardback Set vol. 2. Cambridge University Press, 1995. ISBN: 9780521550024. URL: https://books.google.fr/books?id=48xXMF1oHxkC.