QFT2 Exercises - 2022/23

1 Formulae for Skyrmions

Check that equations (1.76) and (1.83) of the lecture notes reproduce the general formulas for the action and winding number of a nonlinear sigma model when one uses (1.77-79). Check the additivity property of the winding number, equation (1.84).

2 The $\mathbb{CP}^{N-1}$ models

The real projective space $\mathbb{RP}^{N-1}$ is the space of lines through the origin in $\mathbb{R}^N$. Since every line crosses the unit sphere in exactly two points, it can also be thought of as $S^{N-1}/\mathbb{Z}_2$. Thinking of the sphere as the coset $O(N)/O(N-1)$ and $\mathbb{Z}_2 = O(1)$,

$$\mathbb{RP}^{N-1} = \frac{O(N)}{O(N-1)\times O(1)} \ .$$

In a similar way one defines the complex projective space $\mathbb{CP}^{N-1}$ to be the space of complex lines in $\mathbb{C}^N$. By a similar reasoning as above, it is the quotient

$$\mathbb{CP}^{N-1} = \frac{U(N)}{U(N-1)\times U(1)} = S^{2N-1}/U(1) \ .$$

It is a complex manifold of complex dimension $N-1$. Show that $\mathbb{CP}^1 = S^2$.

Let $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, where z_i are complex scalar fields. We impose the constraint

$$z^\dagger z = 1 \ ,$$

so z defines a map into S^{2N-1}. The group $U(1)$ acts on z multiplying all fields by the same phase. The group $U(N)$ also acts on z in the usual way. Define the covariant derivative

$$D_\mu z = \partial_\mu z + iA_\mu z$$

where A_μ is an auxiliary $U(1)$ gauge field. Consider the action

$$S = f^2 \int d^n x D_\mu z^\dagger D^\mu z \ .$$

The equation of motion for A expresses A as function of z. Inserting back the solution, show that the action becomes that of a nonlinear sigma model with hermitian metric (the barred index multiplies a dz^*)

$$g_{a\bar{b}} = \delta_{a\bar{b}} - z^*_a z_{\bar{b}} \ .$$

Show global $U(N)$- and local $U(1)$-invariance of the action in the form where A has been eliminated.

The next step is optional. The sphere S^{2n-1} is a fibration over $\mathbb{CP}^{N-1}$ with fiber S^1. Use the long exact sequence of homotopy groups, equation (3) in the appendix to the lecture notes, to compute the lowest ($n = 1, 2, 3$) homotopy groups of $\mathbb{CP}^{N-1}$ (for this you only need to use the exactness of the sequence and the homotopy groups of spheres). In what dimensions could the models have topological solitons?

3 Path integral of the harmonic oscillator

Consider a harmonic oscillator with Hamiltonian $H = \frac{1}{2}(\dot{q}^2 + \omega^2 q^2)$. Knowing the energy levels, evaluate the (Euclidean) partition function:

$$Z_E = \mathrm{tr}\, e^{-\beta H} = \sum_n e^{-\beta E_n} = \frac{1}{2\sinh(\beta\hbar\,\omega/2)}$$

Conversely, one can calculate the partition function using the path integral and derive from it the energy eigenvalues. The transition amplitude between two position eigenstates $|q_0\rangle$ and $|q_1\rangle$ in time T can be written as a matrix element of the time evolution operator

$$K(q_1, T; q_0, 0) = \langle q_1 | e^{-\frac{i}{\hbar}HT} | q_0 \rangle$$

or as a path integral

$$K(q_1, T; q_0, 0) = \int (dq(t)) e^{\frac{i}{\hbar}S(q(t))}$$

over paths with the boundary conditions $q(0) = q_0$ and $q(T) = q_1$. We are interestes in the trace

$$Z_M = \mathrm{tr} e^{-\frac{i}{\hbar}HT} = \int dq_0 \langle q_0 | e^{-\frac{i}{\hbar}TH} | q_0 \rangle$$

which is just the Minkowski version of the partition function, with the time T identified as $\hbar\beta$. Use the WKB approximation to calculate the path integral that corresponds to Z_M. In the WKB approximation one writes $q(t) = q_{cl}(t) + \eta(t)$, where q_{cl} is the classical solution with the given boundary conditions and η is the quantum fluctuation. The action has to be expanded to second order in η:

$$S(q) = S(q_{cl}) + \frac{1}{2}\int dt\, \eta L \eta + O(\eta^3)$$

where

$$L = -\frac{d^2}{dt^2} + \omega^2$$

and the linear term is absent. The boundary conditions on η at $t = 0, T$ follow from the boundary conditions on q. Then

$$Z_M = e^{\frac{i}{\hbar}S(q_{cl})}(\det L)^{-1/2} \ .$$

Show that

$$S(q_{cl}) = -\omega q_0^2 \tan(\omega T/2)$$

and

$$(\det L)^{-1/2} = K(T)\sqrt{\frac{\omega T}{\sin(\omega T)}}$$

where K is a normalization factor that does not depend on ω. From here one finds that Z_M is just the Wick-rotated form of Z_E, and can be reexpressed as

$$Z_M = \sum_n e^{-i(n+1/2)\omega T}$$

provided we identify K suitably. In this way the eigenvalues of H have been extracted from the path integral.

As a side result, show that when $T \to \infty$, the Euclidean determinant becomes

$$\det L \to \left(\frac{\omega}{\pi\hbar}\right) e^{-\frac{1}{2}\omega T}$$

4 Instantons for the double well potential

Consider a particle in a potential

$$V(q) = \frac{\lambda}{4}(q^2 - f^2)^2 \ .$$

Derive the energy splitting between the two ground states using a dilute instanton gas calculation.

5 Spin of the NLSM soliton

Write the Hopf invariant as a local piece of the action for the spherical non-linear sigma model, viewed as the $\mathbb{CP}^1$ model. Show that for a soliton subjected to an adiabatic 2π rotation, the Hopf invariant is equal to one. (See e.g. Wilczek and Zee, PRL **25**, 2250 (1983))

6 The descent equations

The gauge anomaly in d dimensions can be obtained from the axial anomaly in $d+2$ dimensions by a procedure of dimensional descent. This is described in section 5.5 of the lecture notes. It relates several quantities encountered in class. Derive the gauge anomalies in 2 and 4 dimensions starting from the topological invariants c_2 and c_3 in 4 and 6 dimensions respectively. Then verify the WZ consistency conditions. (These are just equation (5.87), whose right hand side vanishes on a d-dimensional manifold without boundary.)