Classical Mechanics

- System → described (locally) by a collection of continuous parameters

↓

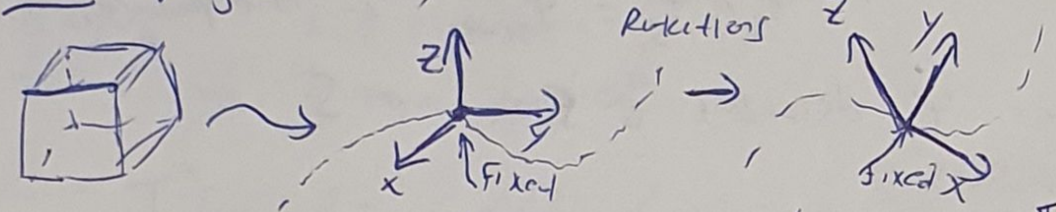
"Set" of all positions is a differential manifold $\boxed{M}$

"configuration space"

Ex. 3 free particles → $M' = \mathbb{E}^3 \times \mathbb{E}^3 \times \mathbb{E}^3$ $(\mathbb{E}^3)^3$

If not able to occupy same position → $M = M' - \{\langle v_1, v_2, v_3\rangle : v_i = v_j \ i, j\}$

Ex Rigid body



M = All oriented orthonormal bases of $\mathbb{E}^3$

3-dim

positions ~ $\langle v_1, v_2, v_3\rangle$ ⟵ $v_i = Ae_i$

Rotation

$M \cong O^+(3)$

orthogonal linear transf with positive determinant

- Configuration ~ $C(t) \in \boxed{M}$

any time gives the config.

"instantaneous information → configuration + things"

Config + things = "State of the system" $\boxed{S}$

$\pi : S \to M$ diff

$\pi(s) = C(t)$ ~~$\pi(s)$~~.

we then want to predict the state at a time t knowing at a t_0. i.e

- $\varphi_{t,t_0} : S \to S$, $\varphi_{t,t_0}(s)$ ← state at time t

- $\varphi_{t,t_1} \circ \varphi_{t_1,t_0} = \varphi_{t,t_0}$ ← Reasonable

$$\varphi_{t-t_1} \circ \varphi_{t_1-t_0} = \varphi_{t-t_0}$$

$$\underset{s_2}{\parallel} \quad \underset{s_1}{\parallel} \quad \underset{s}{\parallel}$$

$$\varphi_{s_2} \circ \varphi_{s_1} = \varphi_{s_1+s_2}$$ ← one parameter group

NOT necessarily semi (only positive)

Here φ describes a flow on S.

Ingredients

$$M, S, \varphi, \pi$$

Here $c(t) = \pi \varphi_t(x)$

<u>Reminder</u> Classical ⤳ configuration = position

+ things = momentum (locally)

$p = mv$ ← "infinitesimal resistance to change in instantaneous velocity" is a <u>cotangent</u> vector

<u>Heuristic</u>

$x \in M$ ⤳ $v = \dot{x} \in T_x M$.

At any given v, "a resistance to change in v" will be some function defined near v and vanishing at v.

⤳ linear function ~~on~~ on $T_v T_x M$ ← identify with $T_x M$ for all v ⟹

all "momenta" are identified as elements of $T_x^* M$

$TM = \bigcup_{x\in M} T_xM \xleftarrow{\text{diff structure}}$ Haz tangente "Tangent bundle"

$\pi: TM \to M$

$\boxed{TM \to M}$

(U,α) chart

$T(\alpha)\xi = \langle \alpha(x), \xi_\alpha \rangle$ if $\xi \in T_xM$, $x \in U$.

$\cancel{T(\alpha)(\xi)A}$

~~$T(\alpha)\xi = \langle \alpha(x), \xi_\alpha \rangle$ if $\xi \in T_xM$, $x \in U$~~

$T(\alpha)(\xi) = \langle q^1(\xi), \dots, q^n(\xi), \dot{q}^1(\xi), \dots, \dot{q}^n(\xi) \rangle$

T^*M haz cotangente "cotangent bundle"

$T^*(\alpha)\ell = \langle \alpha(x), \ell_\alpha \rangle = \langle q^1(\ell), \dots, q^n(\ell), p^1(\ell), \dots, p^n(\ell) \rangle$

$q^i = x^i \circ \pi$, $\ell = \sum p^i(\ell)\, dx^i_x$

Equations of variation

Let M_1, M_2 and $\varphi: M_1 \to M_2$ map,

$\varphi_{*x}(\xi) = T(\varphi)\xi$ ($D\varphi$), if φ is diff

- $T(\varphi \circ \psi) = T(\varphi) \circ T(\psi)$
- $\pi \circ T(\varphi) = \varphi \circ \pi$ $\rightsquigarrow$

$$\begin{tikzcd} TM_1 \arrow[r, "T(\varphi) = \varphi_*"] \arrow[d, "\pi"'] & TM_2 \arrow[d, "\pi"] \\ M_1 \arrow[r, "\varphi"'] & M_2 \end{tikzcd}$$

In particular $\pi \circ T(\varphi_t)\xi = \varphi_t(\pi(\xi))$

Flow φ is generated by X (infinitesimal generator) $\star$ $(\varphi' = X(\varphi(t)))$

$\Rightarrow \pi_{*\xi}(T(X)_\xi) = X_\xi$

Let us see $T(X)$ in terms of (U, α).

$\cancel{T(\alpha) \circ T(\varphi_t) \circ T(\alpha)^{-1} \langle v, w \rangle}$

$\cancel{= \langle (\alpha \circ \varphi_t \circ \alpha^{-1}) v, J_{\alpha \circ \varphi_t \circ \alpha^{-1}}(\varphi) \rangle}$

$T(X)_{T(\alpha)} \langle v, w \rangle = \langle X_\alpha(v), dX_{\alpha(v)} w \rangle$

$= \langle X^1(v), \dots, X^n(v), \sum \frac{\partial X^1}{\partial x^j} w^j, \dots, \sum \frac{\partial X^n}{\partial x^j} w^j \rangle$

Hence with

$\langle q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n \rangle = \langle v, w \rangle$

$$\begin{cases} \dfrac{dq^i}{dt} = X^i \\ \dfrac{d\dot{q}^i}{dt} = \dfrac{\partial X^i}{\partial x^1} \dot{q}^1 + \dots + \dfrac{\partial X^i}{\partial x^n} \dot{q}^n \end{cases}$$

~~Ex: $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$~~

~~$\dot{\varphi}^1(t) = \frac{\partial}{\partial x} \varphi^1 + \frac{\partial}{\partial y} \varphi^1$~~

~~$\dot{\varphi}^2(t) = \frac{\partial}{\partial x} \varphi^2 + \frac{\partial}{\partial y} \varphi^2$~~

Ex. $X = y \frac{\partial}{\partial x} \pm x \frac{\partial}{\partial y}$

$\dot{\varphi}^1 = \varphi^2$

$\dot{\varphi}^2 = \pm \varphi^1$

$\frac{d\varphi^1}{dt} = \varphi^2$

$\frac{d\varphi^2}{dt} = \pm \varphi^1$

$\frac{d\varphi^1}{d\varphi^2} = \pm \frac{\varphi^2}{\varphi^1}$

$\frac{\varphi^{1\,2}}{2} = \pm \frac{\varphi^{2\,2}}{2} + C$

$\Rightarrow \frac{\varphi^{1\,2}}{2} \mp \frac{\varphi^{2\,2}}{2} = C$

i.e. can regard $\frac{\partial X^i}{\partial x^j} = \frac{\partial X^i}{\partial x^j}(x^1(t), \dots, x^n(t))$

↑ only as t

$$\Rightarrow \quad \frac{d\dot\varphi^i}{dt} = \sum_j \frac{\partial X^i}{\partial x^j}\dot\varphi^j \quad \text{is equivalent}$$

$$\frac{dw}{dt} = A(t)\, w \quad \Big\}\ \text{equation of variation of } X \text{ along } \varphi$$

Similar to the cotangent bundle

if $\varphi : M_1 \to M_2 \Rightarrow \varphi^*_x : T^*_{\varphi(x)} M_2 \to T^*_{\cancel{\varphi x}\, x} M_1$

↑ wrong direction!

Hence

$$T^*(\varphi)\, l = (\varphi^{-1})^* l = (\varphi^*_{\varphi(x)})^{-1} l, \quad l \in T^*_x M_1$$

↑ since φ is a diffeo. ✓

$$\begin{tikzcd}
T^*M_1 \arrow[r, "T^*(\varphi)"] \arrow[d, "\pi"'] & T^*M_2 \arrow[d, "\pi"] \\
M_1 \arrow[r, "\varphi"'] & M_2
\end{tikzcd}$$

If φ_t is a flow with vector $X \Rightarrow$

- $\pi \circ T^*(\varphi_t)\, l = \varphi_t(\pi(l))$
- $\pi_* T^*(X)_t = X_x$ for $l \in T^*_x M$

3. The fund linear form

$M \;,\; TM \;,\; T^*M$

$q \qquad\qquad p$

$z \in T^*M \to \xi \in T_z M$

$\langle \xi_z, \theta_z \rangle \equiv \langle \pi_* \xi, z \rangle$

$\theta_z = p_i \, dq^i$

-

Let $\varphi: M_1 \to M_2$ diff.

θ_1 form fund $\quad \theta_2$ form fund

$\pi_* T^*(\varphi)_* \xi = \varphi_* \pi_* \xi$

$$\boxed{(T^*(\varphi))^* \theta_2 = \theta_1}$$

In particular

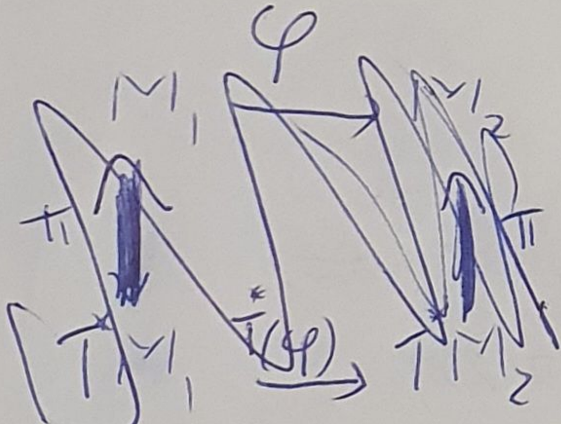
$$(T^*(\varphi_t))^* \theta = \theta$$

~~Let $\xi \in T_z(T^*M_1)$, $\pi(z) = x \in M_1$~~

~~$\Rightarrow \pi_* T^*(\varphi)_* \xi = \varphi_* \pi_* \xi$~~

using $(\pi_* T^*(X)_z = X_x)$

$$\boxed{D_{T^*(X)} \theta = 0}$$



note that $f_X(z) = \langle X_x, z \rangle$

$, \; x = \pi(z)$

$\Rightarrow f_X = \langle T^*(X), \theta \rangle$

since $\langle T^*(X)_z, \theta_z \rangle = \langle \pi_* T^*(X)_z, z \rangle$

$= \langle X_x, z \rangle$.

$$0 = \mathcal{L}_{T^*(X)}\theta = d\langle T^*(X), \theta\rangle + T^*(X) \lrcorner d\theta$$

$$\Rightarrow \quad df_X = - T^*(X) \lrcorner d\theta$$

$\langle T^*(X), \theta\rangle = f_X$ ← "momentum function associated to the vector field X"

4. The fundamental exterior 2-form on T^*M $\qquad X \lrcorner \omega = i_X \omega$

$\Omega = d\theta$ ← fundamental

① $d^2\theta = d\Omega = 0 \rightarrow$ ✓

② Ω is non singular on $T_z(T^*M)$

i.e. is $\xi \in T_z(T^*M)$ is such that

$$\xi \lrcorner \Omega = i_\xi \Omega = 0 \Rightarrow \xi = 0$$

<u>Proof</u>

$$\theta = \sum p^i dq_i = p_i dq^i$$

$$\Omega = dp_i \wedge dq^i \qquad X = A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial p^i}$$

$$i_X \Omega = B^i dq^i - A^i dp_i = 0 \Leftrightarrow A = B = 0$$

$\Rightarrow X \rightarrow X \lrcorner \Omega$. one to one correspondence between diff forms and vector fields.

$$\omega_X \equiv X \lrcorner \Omega \quad , \quad \omega \equiv X_\omega \lrcorner \Omega$$

$\uparrow$ given $\qquad\qquad \uparrow$ given

$$d\omega = 0 \iff \mathcal{O}_{X_\omega}\Omega = 0$$
$$\| $$
$$d(X_\omega \lrcorner \Omega) = d\omega$$

In particular there is a distinguished class of vector fields T^*M (corresponding to functions), vector fields of the form X_{dF} where F is a function on T^*M. These vector fields are called Hamiltonian vector fields.

- if $\mathcal{O}_X\Omega = 0$, locally X is of the form X_{dF}

since $d\omega = 0 \rightsquigarrow \omega = dF$ locally.

If we make the assumption that T^*M has first cohomology group vanishing $\Rightarrow$ $\mathcal{O}_X\Omega = 0$ is equivalent to X being Hamiltonian.

Restriction to the nature of the configuration space

we don't take here this restriction and thus not define $\mathcal{O}_X\Omega = 0$ as X being a Hamiltonian vector field

- Notes: 1) $\underbrace{aX+bY}_{\text{Hamiltonian}} = \underbrace{X_{d(aF+bG)}}_{\text{Hamiltonian}}$

• $[X, Y]$ Hamiltonian

$$\mathcal{D}_X(Y_{dG} \lrcorner \Omega) = \mathcal{D}_X Y \lrcorner \Omega + Y \lrcorner \underbrace{\mathcal{D}_X \Omega}_{=0}$$

$$= \mathcal{D}_X Y \lrcorner \Omega$$

$$= [X, Y] \lrcorner \Omega$$

$$= \mathcal{D}_X (dG) = d\underbrace{\mathcal{D}_X G}$$

$$\Rightarrow [X_{dF}, Y_{dG}] = X_{d(X_{dF} G)}$$

$$\cancel{[X, Y] = X_{\mathcal{D}_X G}} = X$$

$$\cancel{X_{dG} \lrcorner \Omega = dG}$$

Here $[\cdot, \cdot]$ closed Lie Bracket on Hamiltonian vector fields. Poisson brackets denoted by $\{F, G\}$.

$$\{F, G\} = X_{dF} G$$

$$\Rightarrow [X_{dF}, X_{dG}] = X_{d\{F, G\}}$$

<u>note</u>

$$X_{dF} G = \langle X_{dF}, dG \rangle = \langle X_{dF}, X_{dG} \lrcorner \Omega \rangle$$

$$= \langle X_{dG} \wedge X_{dF}, \Omega \rangle$$

$$\Rightarrow \{F, G\} = -\{G, F\}.$$

In coordinates $\langle q^1, \ldots, q^n, p^1, \ldots, p^n \rangle \Rightarrow$

$$dF = \sum \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial p^i} dp_i$$

$$\Rightarrow X_{dF} = \sum \frac{\partial F}{\partial q^i} \frac{\partial}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial}{\partial q^i}$$

$$\Rightarrow \{F, G\} = \sum \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial G}{\partial q^i}$$

<u>Prop</u> if F, G are s.t.

$$X_{dF} G = 0 \Rightarrow X_{dG} F = 0.$$

" if G is constant along the solution curves of X_{dF} $\Rightarrow$ F is constant along X_{dG} ".

$\Rightarrow$ All conservation laws of mechanics.

Let Y be ~~vector~~ vector field on M. The momentum function of Y is a function f_Y on T^*M.

$$\Rightarrow -T^*(Y) = X_{df_Y}$$ (using $df_X = -T^*(X) \rfloor d\theta$)

3. Hamiltonian mechanics

Assumptions

•) Evolution of the system is determined by a flow on T^*M

•) The infinitesimal generator of the flow is a Hamiltonian vector field.

// there is a function H (energy) on T^*M s.t. X_{-dH} is the infinitesimal generator of the flow on T^*M "

Let us see:

$$X_{dH} = \sum_i \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right)$$

$\Rightarrow$ $\langle q^i(t), p^i(t) \rangle$ is an integral curve of the flow.

$$\Rightarrow \quad \frac{\partial H}{\partial p^i} = \frac{dq^i(t)}{dt} \quad \& \quad -\frac{\partial H}{\partial q^i} = \frac{dp^i(t)}{dt}.$$

• Trivial consequence of $\{F, G\} = -\{G, F\}$ $\Rightarrow$

$$X_{-dH} H = 0 \quad \leftarrow \text{"Conservation law"}$$

proposition.

// H is constant along trajectories of the system.

Proposition: Let X_{-dH} ← infinitesimal generator of the system with energy H. Let F such that

$$X_{dF} H = 0 \quad \Rightarrow$$

F is constant on the trajectories, (solution curves) of the flow.

Prototype of "momentum conservation"

The kinetic energy K is a form on T^*M associated with the Riemann metric $(\ ,\ )$ on M.

$\rightarrow$ Scalar product $(\ ,\ )_x$ on T_xM $\rightarrow$ isomorphism of T_xM with T_x^*M $\rightarrow$ scalar product on T_x^*M induced by $(\ ,\ )$

$$\Rightarrow \quad K = \frac{1}{2}(\ell, \ell).$$

Ex:

particle of mass m in $\mathbb{E}^3$. $\boxed{p = m\dot{q}}$ can be formulated as:

$$\langle q_x, q_y, q_z, \dot{q}_x, \dot{q}_y, \dot{q}_z \rangle \mapsto \|(\dot{q}, \dot{q}_y, \dot{q}_z)\|^2 = m\dot{q}_x^2 + m\dot{q}_y^2 + m\dot{q}_z^2$$

The map $T_x\mathbb{E}^3 \rightarrow T_x^*\mathbb{E}^3$ sends

$$\langle q_\alpha, \dot{q}_\alpha \rangle \rightarrow \langle q_\alpha, p_\alpha \rangle$$

$$\{ p_\alpha = m\dot{q}_\alpha$$

$$\Rightarrow \quad K(q_\alpha, p_\alpha) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

"the passage from ~~[illegible]~~ velocity to momentum depends on the choice of a Riemann metric, which defines a map from $TM \rightarrow T^*M$.

"Generalized choice of mass".

The fuctiy U c is assumd to be $U = \bar{U} \circ \pi$
where $\bar{U}$ is a fuctio on M. the form $\vec{F} = -d\bar{U}$
is calld the <u>force field</u> with potty $\bar{U}$.

$$\Rightarrow \langle \xi, F \rangle = -\langle \xi, d\bar{U} \rangle$$

using U and K $\rightarrow$ $H = K + U = \frac{1}{2m} \Sigma(p_\alpha^2) + U$

$$\frac{\partial H}{\partial p_i} = \boxed{\frac{p_i}{m} = \frac{dq_i}{dt}}$$

$$-\frac{\partial H}{\partial q^i} = \boxed{+F = \frac{dp}{dt} = \dot{p}}$$

Dirac manifolds

A symplectic structure $\begin{cases} \to \text{non-degenerate } \underbrace{\text{closed 2-form}}_{\text{Presymplectic structure}} \\ \to \text{non degenerate } \underbrace{\text{Poisson bivector field}}_{\text{Poisson structures.}} \end{cases}$

M a manifold $\rightsquigarrow$ a 2-form $\omega \in \Omega^2(M)$ is called symplectic if it is non degenerate and $d\omega = 0$.

Nondegeneracy:

$$\omega^{\#} : TM \to T^{*}M$$

- $X \to i_X \omega$

Isomorphism $\leftarrow$ Isomorphism theorems $(\ker \omega^{\#} = 0)$

- or, $\omega = \frac{1}{2}\boxed{\omega_{ij}}\, dx^i \wedge dx^j$

matrix is invertible (pointwise)

$(M, \omega) \leftarrow$ $\underbrace{\text{symplectic}}$ manifold

word coined by Weyl $\underbrace{\text{Com}}_{\text{Latin}}$plex

$\underbrace{\text{Sym}}_{\text{Greek}}$plex

Hamilton formalism on a sympt manifold are:

1) For any $f \in C^\infty(M)$ there is a $X_f \in \mathfrak{X}(M)$ defined by $i_{X_f}\omega = df$

i.e. $X_f = (\omega^{\#})^{-1}(df)$.

2) There is a induced op.

$\{\cdot,\cdot\} : C^\infty(M) \dots$

Poisson bracket

desc. by $\{f,g\} := \omega(X_g, X_f) = \mathcal{L}_{X_f} g$

Results

- $\{f,g\} = -\{f,g\}$
- Jac Poiss $= 0 = d\omega$.

$(C^\infty(M), \{\cdot,\cdot\}) \leftarrow$ Poisson Algebra

Lie Algebra + $\{,\}$ is compat with the associative, commutative product via Leibniz

$\Downarrow$

$\{f, gh\} = \{f,g\}h + \{f,h\}g$

$\mathcal{L}_{X_f}(gh) = (\mathcal{L}_{X_f} g)h + (\mathcal{L}_{X_f} h)g$.

The Poisson bracket is defined by a bivector field

$\pi \in \Gamma(\wedge^2 TM) \rightsquigarrow \boxed{\pi(df, dg) = \{f,g\} = \omega(X_g, X_f)}$

locality ⇝ $\Pi = \frac{1}{2}\pi^{ij}\,\partial_i \wedge \partial_j$

Π defines a bundle map

$$\Pi^\# : T^*M \longrightarrow TM$$

$$\alpha \longrightarrow \iota_\alpha \Pi$$

in such a way that $X_f = \Pi^\#(df)$.

compare

$$X_f = (\omega^\#)^{-1}(df)$$

$$\Rightarrow \quad \omega_{ij} = (\Pi^{ij})^{-1}$$

Here one can either choose Π or ω.

- Given $\Pi \in \Gamma(\Lambda^2 TM)$, Π is <u>nondegenerate</u>
  (bivector field)

  if the bundle map $\Pi^\#$ is an isomorphism

  (or (π^{ij}) are invertible)

- Π is <u>Poisson</u> if $\{f,g\} = \pi(df, dg)$

  satisfies the Jacobi identity

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$$

Remarks we asked for

- Lie Algebra ✓
- $\{,\}$ compatible with the product via Leibniz

↑ ge time gatis

- Skew symmetric $\pi \in \Gamma(\Lambda^2 TM)$.

The relation $\pi(df, dg) = \omega(X_g, X_f)$ is a 1-1 correspondence between

bivector fields $\Longleftrightarrow$ non degenerate 2-forms

in such a way that π ar poisson iff ω is closed.

non-deg π non-deg ω

- $Jac_\pi = 0 \Longleftrightarrow d\omega = 0$
- $X_f = \pi^\#(df) \Longleftrightarrow X_f = (\omega^\#)^{-1}(df)$

$$\Updownarrow$$

$$\iota_{X_f}\omega = df$$

- $\{f,g\}$ $\pi(df, dg) = \{f,g\} = \omega(X_g, X_f)$

(Optional)

if $\alpha \in \Omega^1(X)$ 1-forma en X ⇝

$$f_\alpha : X \to T^*X$$
$$x \mapsto (x, \alpha_x)$$

$f_\alpha(X) = X_\alpha$ ⇝

theorem $\alpha \in \Omega^1(X) \Rightarrow X_\alpha$ is a lagrangian submanifold of (T^*X, ω_{can}) if and only if α is closed

2 example of degeneracy

Lie Group G acting on (M, ω) by symmetries (preserving the symplectic structure)

consider M/G ← assume smooth

It is more convenient to consider Π the bivector field

$\varphi : M \to M/G$ $\quad \varphi_* \Pi$ ← push forward (or projection)

$\varphi_*\Pi$ always satisfies $Jac_{\varphi_*\Pi} = 0$ but in general it is ~~not~~ could be degenerated.

<u>presymplectic manifolds</u> (degenerate)

For a vector bundle $E \to M$, a $\boxed{\text{distribution}}$ D in E assigns $x \in M$ a vector subspace $D_x \subseteq E_x$. If $\dim(D_x)$, called rank of D at x, is independent of x, D is <u>regular</u>

Degenerate

$\iota : C \hookrightarrow M$

(submanifold)

~~Input~~

$\iota^* \omega$

symplectic ω.

Optional

$\Rightarrow$ ω can be pulled back

$\iota : TM \to M$

$\iota^* \omega \leftarrow$ symp on C.

closed ✓

degenerated?

Ex:

M is necessary even-dim (in order to the (non degeneracy condition to be possible satisfied)

If X is smooth and $f : X \to M$ an embedding

X is called a Lagrangian submanifold if

- $f^* \omega = 0$ and • $\dim(X) = \frac{1}{2} \dim(M)$

↑ pullback dc bel,

$\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ diffeomorp between symplectic manifolds

symplectomorphism

if $\varphi^* \omega_2 = \omega_1$.

If X is smooth, $T^* X$ has a natural symplectic form

$\omega_{can} = -d\lambda$, λ is 1-form on $T^* X$

defined $\lambda = d\pi^* \xi$

where $\pi : T^* X \to X$ is the bundle projection. $\langle x, \xi \rangle$ $\langle x, \xi \rangle$

A ~~distribution~~ D in E is smooth is $\forall x \in M$ and $v_0 \in D_x$, there exist a smooth local section v of E s.t. $v(y) \in D_y$ and $v(x) = v_0$.

A distribution that is smooth and regular is ~~a subbundle~~ a subbundle.

3)

For a vector bundle map $\phi: E \to A$ covering the identity. $\phi(E)$ is a smooth distribution of A; $\ker(\phi)$ is a distribution of E.

A smooth distribution D in TM is integrable if any $x \in M$, is contained in an integral submanifold i.e. a connected immersed submanifold O so that $D|_O = TO$.

A integrable distribution defines a decomposition of M into leaves (maximal integral submanifolds) & this decomposition is " singular foliation. or foliation.

When D is smooth and has constant rank, the classical Frobenius theorem asserts that D is integrable iff it is involutive. The resulting foliation is referred as regular.

With out the non-degeneracy assumption there might be functions admitting no Hamiltonian vector fields.

Hence we say that $f \in C^\infty(M)$ is **admissible** if there exist a vector field X_f s.t.

$$\boxed{\iota_{X_f} \omega = df}$$

- X_f is not uniquely defined

$$X_f + \underset{\text{null}}{K}$$

- but the Poisson bracket formula is well defined (indep of X_f) when f and g are admissible.

$$\underset{\text{Poisson algebra}}{C^\infty_{adm}(M)} \subseteq C^\infty(M)$$

Poisson manifolds

(Conversely ~~to~~ symplectic) if (M, Π) is a Poisson manifold, every $f \in C^\infty(M)$ defines a unique Hamiltonian $X_f = \Pi^\sharp(df) \Rightarrow C^\infty(M)$ is Poisson Algebra with bracket

$$(C^\infty(M), \{f, g\} = \Pi(df, dg))$$

Dirac Structures

Courant, Dirac manifolds ('90)
Courant, Weinstein Beyond Poisson structures ('88)

"A way to treat both types of degenerate" symplectic structures" in a unified manner.

Idea View the presymplectic and Poisson structures as subbundles of

$$\mathbb{T}M = TM \oplus T^*M \leftarrow \text{Tangent generalized}$$

defined by the graphs of the bundle maps $\pi^\sharp, \omega^\sharp$ + additional geometric structures.

Consider $\mathbb{T}M$ equipped with

- $pr_T : \mathbb{T}M \to TM$
- $pr_{T^*} : \mathbb{T}M \to T^*M$

- A non-degenerate, symmetric fiberwise bilinear form $\langle \cdot , \cdot \rangle$ on $\mathbb{T}M$, given at each $x \in M$ by

$$\langle (X,\alpha), (Y,\beta) \rangle = \beta(X) + \alpha(Y)$$

$, X, Y \in T_xM$
$\alpha, \beta \in T^*_xM.$

~~A Courant bracket $[\![\cdot,\cdot]\!] : \Gamma(\mathbb{T}M)$~~

• Courant bracket $[\![\cdot,\cdot]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$

$$[\![(X,\alpha),(Y,\beta)]\!] = \left([X,Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \tfrac{1}{2}d(\alpha(Y) - \beta(X))\right)$$

A <u>Dirac structure</u> on M is a vector subbundle (subbundle) $L \subset \mathbb{T}M$ satisfying:

(i) $L = L^{\perp}$, respect to $\langle\cdot,\cdot\rangle$.

(ii) $[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L)$

i.e. L is involutive with respect to $[\![\cdot,\cdot]\!]$.

Remarks

• (ia) equivalent to $\langle\cdot,\cdot\rangle|_L = 0$ and $\mathrm{Rank}(L) = \dim(M)$

• The Courant bracket satisfies

$$\mathrm{Jac}_{[\![,]\!]} = [\![[\![a_1,a_2]\!],a_3]\!] + \text{cyclic permutations}$$
$$= \frac{1}{3} d\left(\langle [\![a_1,a_2]\!], a_3\rangle + \text{cyclic permutations}\right)$$

for $a_1, a_2, a_3 \in \Gamma(\mathbb{T}M)$.

IT DOES $\boxed{\text{NOT}}$ satisfies the <u>Jacobi identity</u>, <u>Not</u> a Lie bracket!

- A subbundle $L \subset \mathbb{T}M$ satisfying (i) is sometimes called Lagrangian subbundle of $\mathbb{T}M$.
- (ii) is referred as integrability condition. Can be equivalently written by (using (i))

$$\langle [\![a_1, a_2]\!], a_3 \rangle \equiv 0, \quad \forall a_1, a_2, a_3 \in \Gamma(L);$$

- For any Lagrangian subbundle $L \subset \mathbb{T}M$,

$$\Upsilon_L (a_1, a_2, a_3) := \langle [\![a_1, a_2]\!], a_3 \rangle$$

defines an element $\Upsilon_L \in \Gamma(\wedge^3 L^*)$ that we call the <u>Courant tensor</u> of L.

condition ii) is $\Upsilon_L \equiv 0$

<u>Ex</u> Any bivector field $\pi \in \Gamma(\wedge^2 TM)$ defines a Lagrangian subbundle of $\mathbb{T}M$ given by

$$L_\pi = \{ (\pi^\#(\alpha), \alpha) \mid \alpha \in T^*M \}$$

Hence, $a_i = (\pi^\#(df_i), df_i)$, $i = 1, 2, 3$,

$$\Upsilon_{L_\pi} (a_1, a_2, a_3) = \langle [\![a_1, a_2]\!], a_3 \rangle =$$

$(\pi^\#(df_3), df_3)$

$\langle [\![a_1, a_2]\!], a_3 \rangle =$

$\left([\pi^\#(df_1), \pi^\#(df_2)], \mathcal{L}_{\pi^\#(df_1)} df_2 - \mathcal{L}_{\pi^\#(df_2)} df_1 + \frac{1}{2} d\left(df_1(\pi^\#(df_2)) - df_2(\pi^\#(df_1))\right) \right)$

$\cancel{-f\, df_3([\pi^\#(df_1), \pi^\#(df_2)])}$

$$[\![(X_{f_1}, df_1), (X_{f_2}, df_2)]\!] = ([X_{f_1}, X_{f_2}],$$

$$(\mathcal{L}_{X_{f_1}} df_2 - \mathcal{L}_{X_{f_2}} df_1 - \tfrac{1}{2} d(\iota_{X_{f_1}} df_2 - \iota_{X_{f_2}} df_1)))$$

$$= (\mathcal{L}_{X_f} X_g,\ 2d\{f_1, f_2\} - d\{f_1, f_2\}) = [X_{f_1}, X_{f_2}] \oplus d\{f_1, f_2\}$$

$$\omega(X_{f_1}, X_{f_2})$$

$$\mathcal{L}_{X_{f_1}} df_2 - \mathcal{L}_{X_{f_2}} df_1 = d(\mathcal{L}_{X_{f_1}} f_2 - \mathcal{L}_{X_{f_2}} f_1) = d(\{f_1, f_2\} - \{f_2, f_1\})$$

$$= 2d\{f_1, f_2\}$$

$$\tfrac{1}{2} d(\iota_{X_{f_1}} df_2 - \iota_{X_{f_2}} df_1) = \tfrac{1}{2} d(\mathcal{L}_{X_{f_1}} f_2 - \mathcal{L}_{X_{f_2}} f_1)$$

$$= d\{f_1, f_2\}$$

Cartan: $\mathcal{L}_X \omega = \iota_X d\omega + d(\iota_X \omega)$

$$\Rightarrow \iota_{X_{f_1}} df_2 - \iota_{X_{f_2}} df_1 = \mathcal{L}_{X_{f_1}} f_2 - d(\iota_{X_{f_1}} f_2)$$

$$- \mathcal{L}_{X_{f_2}} f_1 + d(\iota_{X_{f_2}} f_1)$$

$$\gamma_L(a_1, a_2, a_3) = \langle ([X_{f_1}, X_{f_2}], d\{f_1, f_2\}), a_3 \rangle$$

$$a_3 = (\Pi^\sharp(df_3), df_3)$$

$$= df_3([X_{f_1}, X_{f_2}]) + d\{f_1, f_2\}(\Pi^\sharp(df_3))$$

$$\cancel{df_3(\mathcal{L}_{X_{f_1}} X_{f_2})} + d\{f_1, f_2\}(X_{f_3})$$

$$= [X_{f_1}, X_{f_2}] f_3 + X_{f_3}\{f_1, f_2\}$$

$$= \cancel{\{f_3} \qquad + \{f_3, \{f_1, f_2\}\}$$

$$X_{f_1}\{f_2, f_3\} - X_{f_2}\{f_1, f_3\}$$

$$= \{f_1, \{f_2, f_3\}\} - \{f_2, \{f_1, f_3\}\} + \{f_3, \{f_1, f_2\}\}$$

$$= \{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} \checkmark$$

So

$\langle [\![a_1, a_2]\!], a_3 \rangle = 0$ iff π is Poisson

i.e $\boxed{J_{\cancel{\pi}}\, C_\pi = 0}$

i.e L_π is a Dirac structure.

Similarly $L_\omega = \text{graph}(\omega^\#) = \{ (X, \omega^\#(X)) \mid X \in TM \} \subset \mathbb{T}M$

$\Rightarrow \Upsilon_{L_\omega}(a_1, a_2, a_3) = \langle [\![a_1, a_2]\!], a_3 \rangle = d\omega(X_1, X_2, X_3)$

$\Rightarrow L_\omega$ is a Dirac structure iff ω is presymplectic.

Lie Algebroid

A Lie algebroid is a vector bundle $A \to M$ equipped with a Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$ and a bundle map (called the anchor) $\rho : A \to TM$ so that

$$[u, fv] = (\mathcal{L}_{\rho(u)} f) v + f[u, v]$$

$u, v \in \Gamma(A)$

$f \in C^\infty(M)$

Hamiltonian vector fields and Poisson algebra

Let L be a Dirac structure on M.

A function $f \in C^\infty(M)$ is called admissible if there is a vector field X s.t.

$$(X, df) \in L$$

In this case X is called hamiltonian relative to f.

- As presymplectic manifolds, X is not uniquely determined.
- C^∞_{adm} is always a Poisson algebra

---

morphisms: consider M_i. $\varphi : M_1 \to M_2$ can preserve symplectic structures by

$$\varphi^* \omega_2 = \omega_1 \quad \boxed{\text{or}} \quad \varphi_* \Pi_1 = \Pi_2$$

$$(\Pi_2)_{\varphi(x)}(\alpha,\beta) = (\Pi_1)_x(\varphi^*\alpha, \varphi^*\beta) \quad \forall \alpha,\beta \in T^*_x M_2.$$

<u>Ex</u> conditions are not equivalent.

$$\omega_{\mathbb{R}^2} = dq^1 \wedge dp_1 \quad , \quad \Pi_{\mathbb{R}^2} = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q^1}$$

$$\omega_{\mathbb{R}^4} = dq^1 \wedge dp_1 + dq^2 \wedge dp_2 \quad , \quad \Pi_{\mathbb{R}^4} = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q^1} + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q^2}$$

- $\cancel{\varphi} : \mathbb{R}^4 \to \mathbb{R}^2$

$$(q^1, p_1, q^2, p_2) \to (q^1, p_1)$$

Satisfies $\varphi_* \Pi_1 = \Pi_2$ but not $\varphi^* \omega_2 = \omega_1$

$\cancel{\varphi_* \Pi_1 = P_2}$

$\cancel{k}$ $\iota : \mathbb{R}^2 \to \mathbb{R}^4$

$$(q^1, p_1) \to (q^1, p_1, 0, 0)$$

Satisfies $\varphi^* \omega_2 = \omega_1$ but not $\varphi_* \Pi_1 = \Pi_2$

Estructuras de Dirac y Sistemas de Ligaduras.

La evolución de un sistema está dado por

$$\frac{df}{dt} = \{f, H\} \qquad \overset{\text{sistema físico}}{(M, \{\cdot,\cdot\}, H)}$$

observable físico. (espacio de fases simpléctico)

Una propiedad que exhibe son las <u>Ligaduras</u>

determinan el conjunto de funciones admisibles
⇒ Subespacios del espacio de fase.

// Dirac encontró que es posible describir la dinámica asociada a un sistema usando un corchete de Dirac //

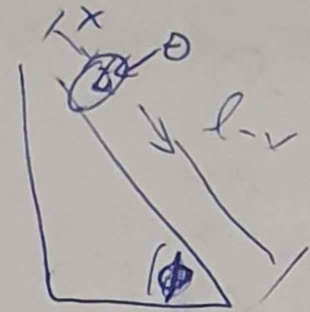
(Lectures on Quantum mechanics 1964 Dirac)

La forma en las que posición-momento-velocidad se relacionan es tal dado por Ligaduras denotadas por $\phi_r(q_i, p_i) = 0$

↑ Ligadura de primera clase

(no dependen desigualdades)

Ex



$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 - mg(l-x)\sin\phi$$

$$I = \frac{1}{2} M R^2$$

$$\phi = x - R\theta = 0$$

ligadura no desliza

E-L

$$\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = mg\sin\phi$$

mutip de lagrange

$$\Rightarrow \quad mg\sin\phi - m\ddot{x} = \lambda = \lambda \frac{\partial \phi_r}{\partial x}$$

$$\frac{\partial L}{\partial \theta} = \cancel{mg(l-x)\cos\theta} = \cancel{mg}$$
$$= 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m \ddot{\theta} \rightarrow -\frac{1}{2} M R^2 \ddot{\theta} = -\lambda R = \lambda \frac{\partial \phi_r}{\partial \theta}$$

$$\Rightarrow \quad mg\sin\phi - m\ddot{x} = +\frac{1}{2} m R \ddot{\theta}$$

$$\ddot{x} = R\ddot{\theta}$$

$$mg\sin\phi = \frac{3}{2} m \ddot{x}$$

$$\boxed{\ddot{x} = \frac{g\sin\phi \cdot 3}{2}}$$

$$\rightarrow \lambda = -m\ddot{x} + mg\sin\phi$$
$$= -\frac{mg\sin\phi\, 3}{2} + mg\sin\phi$$
$$= -\frac{mg}{2}\sin\phi$$

$$\ldots + \lambda(\nabla_i \phi_r)$$

$\Rightarrow \theta = \frac{g\sin\phi}{2R}$, $|\lambda| = Q = \frac{mg\sin\phi}{2}$

Forc constren

Def Sea C_H estructura de Dirac sobre el esp de configuraciones. una ligadura ϕ_i es indep si

$$\{ f, \sum_{i=1}^{n} \phi_i \} = 0$$

conjunto de ligaduras

Ex

$$\{\phi, H\} = \cancel{\frac{1}{m}\left(R\theta P_x + \frac{4P_\phi}{R}\right)} = \frac{P_x}{m} - \frac{4P_\theta R}{mR^2}$$

$$\cancel{\frac{1}{m}\left(m\ddot{x} + \frac{4}{2} m R \ddot{x}\right)}$$

$$\cancel{P_x = m\ddot{x} =} \qquad = \frac{m\ddot{x}}{m} - \frac{4 m\ddot{x}R^2}{4mR^2}$$

$$\cancel{P_\theta = \frac{mR^2\ddot{\theta}}{2} = \frac{m\ddot{x}R}{4}} \qquad P$$

$$= 0 \checkmark$$

$$\cancel{P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{mR^2\dot{\theta}}{2} =}$$

Dirac introdujo

$$\dot{q}_i = \sum_i \frac{\partial H}{\partial p_i} + \lambda^r \frac{\partial \phi}{\partial p_i}$$

$$\dot{p}_i = -\sum_i \left(\frac{\partial H}{\partial q_i} - \lambda^r \frac{\partial \phi}{\partial q_i}\right) \quad ; \quad \phi_r = 0$$

$$\rightarrow \dot{p}_i = \{p_i, H\} + \lambda^r \{q_i, \phi_r\} \; ; \; \dot{q}_i = \{q_i, H\} + \lambda^r \{q_i, \phi_r\}$$

y para cualquier observable físico.

$$\dot{f} = \sum \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i$$

$$= \sum \frac{\partial f}{\partial q_i}\left(\frac{\partial H}{\partial p_i} + \lambda^r \frac{\partial \phi}{\partial p_i}\right) + \frac{\partial f}{\partial p_i}\dot{p}_i\left(-\frac{\partial H}{\partial q_i} + \lambda^r\frac{\partial \phi}{\partial q_i}\right)$$

$$= \{f, H\} + \lambda^r\{f, \phi_r\}$$

Def observable debilmente nulo

$$\dot{\xi} = \{\xi_r, H\} + \lambda^s\{\xi_r, \xi_s\} \approx 0.$$

Dirac

$$(C^\infty(M,\mathbb{R}), \{\cdot,\cdot\}, H) \to (C^\infty(M,\mathbb{R}), \{\cdot,\cdot\}_D, H)$$

$$\{\cdot,\cdot\}_D : C^\infty(T^*Q,\mathbb{R}) \times C^\infty(T^*Q,\mathbb{R}) \to C^\infty(T^*Q,\mathbb{R})$$

$$\{f,g\}_D = \{f,g\} - \{f, \xi_\mu\}\Delta^{\mu\nu}\{\xi_\nu, g\}$$

matriz cuyas entradas están definidas por $\{\cdot,\cdot\}$.

ligaduras secundarias

Retomando el ejemplo, $\phi = x - R\theta = 0$

$\rightarrow$ $\omega_0 \xleftarrow{\text{Form simplectica}}$ $\rightarrow$ estructura de Dirac

$$= \sum_i dp_i \wedge dq_i = dp_x \wedge dx + dp_\theta \wedge d\theta$$

$\cancel{\omega = \phi}$ e introducimos perturbación lagrangiana ϕ

$\rightarrow$ $\omega \equiv \phi\,\omega_0 \leftarrow$ Estructura Dirac H-torcida.

Para definir el algebra de Poisson se tiene

$$L_{X_f}\omega = L_{X_f}\left((x - R\theta)(dp_x \wedge dx + dp_\theta \wedge d\theta)\right) = 0$$

$$= \{f, x - R\theta\} \checkmark$$