Lecture 1

In this course, we will learn methods of geometry in modern physics. Historically, mathematics and physics influenced each other in their developments. We know, for example, that Newton invented calculus and classical mechanics simultaneously. However, in the mid 20th century, there was a period when interactions between physicists, in particular high energy physicists, and mathematicians almost stopped. There were two main reasons. One was on the mathematics side. The Bourbaki movement.

Another was on the physics side. Quantum field theory was conceived by Heisenberg and Pauli in 1929 and Feynman, Schwinger, and Tomonaga established the renormalization procedure to make sense of QED. However, there was not mathematical formulation of quantum field theory. In fact, there is still no rigorous mathematical foundation for it, except for special cases, some of which we will see later in this course. Making sense of quantum Yang-Mills theory is posed as one of the seven Millennium problems by the Clay Mathematics Institute.

The situtation has dramatically changed since the 1980's. It is because of the use of supersymmetry. Supersymmetry brought mathematics and physics closer since it is close to the language mathematicians use to describe geoemtric concepts.

Why supersymmetry?

Differential forms play important roles. Differential forms are composed of objects which do not commute with each other but anti-commute, just like fermions. In fact, as we will learn in the next couple of lectures, at each point on a manifold, the space of differential forms can be regarded as the Fock space of fermons. On the other hand, coordinates of a manifold commute with each other and behave like bosons. Combining the fermions and the bosons together, the total space of differential forms will become a Hilbert space of a supersymmetric system.

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In this first lecture, we will study **exterior product algebra**. This is going to be a space of differential operators at a given point on a manifold.

Consider an n-dimensional vector space V with basis vectors $\{e_i\}$ $(i = 1, ..., n)$. Each element v of V can be expanded as

$$v = \sum_{i=1}^{n} v^i e_i = v^i e_i,$$

where I used the *Einstein convension*: a repeated index should be summed over. The coefficient v^i are called components of the vector v.

For a given vector space V, we can consider its dual space V^*. It is a space of linear functions on V,

$$\omega \in V^* : \quad V \to \mathbf{R}, \quad \text{(linear)}.$$

The dimensions of V and V^* are the same.

For a given set of basis vectors $\{e_i\}$, we can define the dual basis $\{e^i\}$ by

$$e^i : \quad e_j \to e^i(e_j) = \delta^i_j.$$

Any element ω of V^* can be expanded as

$$\omega = \omega_i e^i,$$

so that

$$\omega(v) = \omega_i v^i.$$

Question 1: Show $V^{**} = V$.

We can consider a tensor product of V^* to define a space of multi-linear functions, $V^* \otimes V^* \otimes \cdots \otimes V^*$. We can expand its element as

$$\omega = \omega_{i_1 i_2 \cdots i_k} e^{i_1} \otimes e^{i_2} \otimes \cdots \otimes e^{i_k}.$$

We can also consider mixed tensors in $\otimes^k V^* \otimes^l V$.

forms

Let $\mathcal{S}_k$ be the symmetric group of k object,

$$\sigma \in \mathcal{S}_k : \ (1, 2, ..., k) \to (\sigma(1), \sigma(2), ..., \sigma(k)).$$

We call $\omega \in \otimes^k V^*$ as a k-form iff (this should read as "if and only if"),

$$\omega(v_{\sigma(1)}, v_{\sigma(2)}, ..., v_{\sigma(k)}) = (-1)^\sigma \omega(v_1, v_2, ..., v_k).$$

Here $(-1)^\sigma$ is equal to $+1$ or -1 depending on whether σ can be expressed as a product of an even or odd number of permutations (exchange of two elements).

wedge product

We can take a product of a k-form and an l-form to make a $(k+l)$-form. Let us start with a product of 1-forms. If α^a's are 1-forms (i.e., $\alpha^a \in V^*$), their wedge product are defined by

$$\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^k(v_1, v_2, ..., v_k) = \det(\alpha^a(v_b)).$$

More generally, if α is a k-form and β is an l-form,

$$(\alpha \wedge \beta)(v_1, ..., v_{k+l}) = \frac{1}{k!l!} \epsilon^{i_1 \cdots i_{k+l}} \alpha(v_{i_1}, ..., v_{i_k}) \beta(v_{i_{k+1}}, ..., v_{i_{k+l}}).$$

We can choose a basis of the space of k-forms, $\wedge^k V^*$, as

$$e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

Any k-form α can be expanded as

$$\alpha = \frac{1}{k!} \alpha_{i_1, ..., i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

Question 2: When $\alpha \in V^*$ and $\beta \in V^* \wedge V^*$, express $\alpha \wedge \beta$ in components.

Question 3: When $\alpha \in \wedge^k V^*$ and $\beta \in \wedge^l V^*$, show

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha.$$

There are transformations that map a k-form to a $(k+1)$-form and a $(k-1)$-form. The former is easy,

$$\alpha \in \wedge^k V^* \to e^i \wedge \alpha \in \wedge^{k+1} V^*.$$

The latter is defined as follows.

interior product

For $\alpha \in \wedge^k V^*$ and $u \in V$, we define $i(u)\omega \in \wedge^{k-1} V^*$ as,

$$(i(u)\alpha)(v_1, ..., v_{k-1}) = \alpha(u, v_1, ..., v_{k-1}).$$

Question 4-1: Show $i(v_1)i(v_2) = -i(v_2)i(v_1)$.

Question 4-2: When α is a k-form and β is an l-form, show

$$i(v)(\alpha \wedge \beta) = (i(v)\alpha) \wedge \beta + (-1)^k \alpha \wedge (i(v)\beta).$$

metric

We can do more if there is a metric $g \in V^* \otimes V^*$, which is symmetric, $g(v_1, v_2) = g(v_2, v_2)$ and non-degenerate, namely $g(v, u) = 0$ for any v implies $u = 0$. The metric g can be expanded as

$$g = g_{ij} e^i \otimes e^j, \qquad g_{ij} = g(e_i, e_j).$$

When a metric g is given, we can identify V with V^*, namely there is a map $\tilde{g} : V \to V^*$,

$$\tilde{g}(v) : u \to g(v, u).$$

Question 5-1: Show $\tilde{g}(e_i) = g_{ij} e^j$.

Question 5-2: What is the inverse of the map $\tilde{g}$?

Since g_{ij} is symmetric, we can always diagonalize it. In fact, we can choose a basis $\{\mathcal{O}^i\}$, so that

$$g(\mathcal{O}i, \mathcal{O}j) = \pm\delta_{ij}.$$

In this basis,

$$g = \mathcal{O}^1 \otimes \mathcal{O}^1 + \cdots + \mathcal{O}^k \otimes \mathcal{O}^k - \mathcal{O}^{k+1} \otimes \mathcal{O}^{k+1} - \cdots - \mathcal{O}^n \otimes \mathcal{O}^n.$$

In this case, we call that g is of signature $(k, n-k)$.

volume element

is defined by

$$\eta = \mathcal{O}^1 \wedge \mathcal{O}^2 \wedge \cdots \wedge \mathcal{O}^k.$$

Question 6: Show that, for a generic basis $\{e_i\}$,

$$\eta = \pm\sqrt{|\det g(e_i, e_j)|}\, e^1 \wedge \cdots \wedge e^k,$$

where the sign is determined by the relative orientation of $\{\mathcal{O}_i\}$ and $\{e_i\}$.

Hodge $*$ operator

The Hodge $*$ operator is a map $\wedge^k V^* \to \wedge^{n-k} V^*$ defined by

$$*\alpha(v_{k+1}, \cdots, v_n)\eta = \alpha \wedge \tilde{g}(v_{k+1}) \wedge \cdots \tilde{g}(v_n).$$

Show that, in components, the Hodge $*$ operator can be expressed as

$$(*\alpha)_{i_{k+1}\cdots i_n} = \pm \frac{1}{k!} \alpha_{j_1 \cdots j_k} \eta^{j_1 \cdots j_n} g_{i_{k+1} j_{k+1}} \cdots g_{i_n j_n},$$

where the sign in the right-hand side is determined by the signature of the metric g. When the mertic is positive definite,

$$** \omega = (-1)^{k(n-k)} \omega.$$

fermions

Consider free fermions obeying the anti-commutation relations,

$$\{\psi_i, \psi_j\} = 0, \quad \{\bar{\psi}^i, \bar{\psi}^j\} = 0, \quad \{\psi_i, \bar{\psi}^j\} = \delta_i^j \qquad (i, j = 1, ..., n).$$

The Fock space is built, starting with the vacuum state $|0\rangle$ annihilated by all ψ_i.

We can identify the Fock space with the space of forms $\sum_{k=0}^n \wedge^k V^*$, with the identification,

$$|0\rangle \leftrightarrow 1, \quad \bar{\psi}^i \leftrightarrow e^i \wedge, \quad \psi_i \leftrightarrow i(e_i).$$

When we have a metric g on V, we can use it to define an inner product in the Fock space. It is defined so that

$$(\psi_i)^\dagger = g_{ij} \bar{\psi}^j,$$

and

$$(\bar{\psi}^i|0\rangle, \bar{\psi}^j|0\rangle) = \langle 0|(\bar{\psi}^i)^\dagger \bar{\psi}^j|0\rangle = g^{ik}\langle 0|\psi_k \bar{\psi}^j|0\rangle = g^{ij}.$$

Lecture 2

tangent space, differential forms, Riemannian manifolds

differentiable manifolds

A manifold is a set that locally look like $\mathbf{R}^n$. For example, a two-dimensional sphere S^2 can be covered by two subspaces, one can be the northen hemisphere extended slightly below the equator and another can be the southern hemisphere extended slightly above the equator. Each patch can be mapped smoothly into an open set of $\mathbf{R}^2$.

In general, a manifold M consists of a family of open sets U_i which covers M, *i.e.* $\cup_i U_i = M$, and, for each U_i, there is a continuous invertible map $\varphi_i : U_i \to \mathbf{R}^n$. To be precise, to define what we mean by a continuous map, we has to define M as a topological space first. This requires a certain set of properties for open sets of M. We will discuss this in a couple of weeks. For now, we assume we know what continuous maps mean for M. If you need to know now, look at one of the standard textbooks (e.g., Nakahara).

Each (U_i, φ_i) is called a coordinate chart. Their collection $\{(U_i, \varphi_i)\}$ is called an atlas.

The map has to be one-to-one, so that there is an inverse map from the image $\varphi_i(U_i)$ to U_i. If U_i and U_j intersects, we can define a map $\varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j))$ to $\varphi_i(U_i \cap U_j)$. Since $\varphi_j(U_i \cap U_j))$ to $\varphi_i(U_i \cap U_j)$ are both subspaces of $\mathbf{R}^n$, we express the map in terms of n functions and ask if they are differentiable. If the map is differentiable for every intersecting pair of coordinate charts, namely if every change of coordinates is differentiable, then we call M a differentiable manifold.

An important point of this definition of differential manifolds is the following. Suppose there is a function $f : M \to \mathbf{R}$. Consider its restriction on $U_i \cap U_j$. We can express the function in two different sets of coordiantes, $f_i = f \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \mathbf{R}$ and $f_j = f \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \mathbf{R}$. If f_i is differentiable, f_j is also differentiable, and vice versa, since $\varphi_i \circ \varphi_j^{-1}$ and its inverse are both differnetiable. Thus, on a differentiable manifold, we can tell whether a given function is differentiable or not.

We can also give an invariant meaning to differentials of a function as follows.

tangent vectors

A function $f : M \to \mathbf{R}$ is called differentiable if $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbf{R}$ is differentiable for every U_i. Let us denote the space of such differentiable functions by $C^0(M)$.

A tangent vector field v at M is defined as a linear map $C^0(M) \to C^0(M)$ obeying the rule,

$$v(fg) = fv(g) + gv(f).$$

Namely, v behaves like a differential operator on $C^0(M)$.

Note that, when u and v are tangent vector fields, $f \to u(v(f))$ does not give a tangent vector field. On the other hand, their commutator $[u, v] : f \to u(v(f)) - v(u(f))$ is a tangent vector field.

A tangent vector v_p at a point $p \in M$ is a linear map $C^0(M) \to \mathbf{R}$ obeying

$$v_p(fg) = f(p)v_p(g) + g(p)v_p(f).$$

A set of tangent vectors at p is called a tangent space and is denoted by T_pM.

There is another way to think about tangent vectors. Consider two diffentiable curves $c_1, c_2 : \mathbf{R} \to M$. We say that the two curves are tangent at $t = 0$ if $c_1(t = 0) = c_2(t = 0) = p$ and

$$\frac{d}{dt}\varphi(c_1(t))_{|t=0} = \frac{d}{dt}\varphi(c_2(t))_{|t=0}$$

for some coordinate chart containing p. For each curve $c(t)$ with $c(t = 0) = p$, we can define a tangent vector v_p at p by

$$v_p(f) = \frac{d}{dt}f(c(t))_{|t=0}.$$

If c_1 and c_2 are tangent at p, they define the same tangent vector at p. Conversely, any tangent vector can be constructed in this way.

Let us try to express tangent vectors using coordinates. Consider a chart (U, φ), so that $q \in M$ is mapped to $\varphi(q) = (x^1, ..., x^n) \in \mathbf{R}^n$. We can then define a curve $c(t) = \varphi^{-1}(x^1, ..., x^{i-1}, t, x^{i+1}, ..., x^n)$ for fixed x's and a tangent vector e_i at $\varphi^{-1}(x)$ by

$$e_i(f) = \frac{d}{dt}f(c(t))_{|t=x^i} = \frac{\partial}{\partial x^i}f(\varphi^{-1}(x^1, ..., x^n)),$$

or $e_i = \partial_i$ for short. Any tangent vector v_p at $p = \varphi^{-1}(x^1, ..., x^n)$ can then be expressed as

$$v_p = v(p)^i \left(\frac{\partial}{\partial x^i}\right)_p.$$

When v_p is defined in terms of a curve $c(t)$, its component $v(p)^i$ can be obtained by

$$v(p)^i = \frac{d}{dt}\varphi(c(t))^i_{|t=0}.$$

differential forms

Remember what we did in Lecture 1. For each vector space V, we can consider its dual space V^* and their wedge product $\wedge^k V^*$ to define a space of k-forms. We can apply this to the case when $V = T_pM$. Its dual space is $V^* = T^*_pM$ and is called the space of co-tangent vectors.

For a given coordinate chart, a natual basis of T_pM is $e_i = \partial/\partial x^i$. Its dual basis is denoted by $e^i = dx^i$, so that

$$dx^i \left(\frac{\partial}{\partial x^j}\right) = \delta^i_j.$$

We can express a k-form ω at $p \in M$ as

$$\omega(p) = \frac{1}{k!}\omega_{i_1 \cdots i_k}(p) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

If the coefficients $\omega_{i_1 \cdots i_k}(p)$ are differentiable functions of p, we call ω as differentiable. Note that this definition of differentiability is independent of a choice of a coordinate chart. The space of differentiable k-forms is denoted by $C^k(M)$ or $\wedge^k T^*M$.

exterior derivative d

The exterior derivative operator d is a map from $C^k(M)$ to $C^{k+1}(M)$. When acting on $f \in C^0$,

$$df(p) = \frac{\partial f}{\partial x^i}(p) dx^i.$$

This is a good notation since d of $x^i : p \to x^i(p)$ gives dx^i. For other forms, d is defined by the requirements,

$$(1)\ d^2 = 0, \quad (2)\ d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^\alpha \alpha \wedge \beta.$$

Here $(-1)^\alpha$ is equal to $+1$ or -1 depending on whether α is an even or odd form.

Question 1: Usin the above definition, show that the exterior derivative of a k-form ω can be expressed in terms of components as

$$d\omega = \frac{1}{k!} \partial_j \omega_{i_1 \cdots i_k} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

So, we can write,

$$d\omega = dx^i \wedge \left(\frac{\partial}{\partial x^i} \omega \right).$$

metric

A metric on M is an element of $T_p^* M \otimes T_p^* M$ at each $p \in M$. It is symmetric and non-degenerate, as the metric on $V \otimes V$ discussed in Lecture 1. It components are given by $g_{ij} = g(\partial_i, \partial_j)$. If g_{ij} is positive definite, (M, g) is called a Riemannian manifold. We can also write this as,

$$ds^2 = g_{ij} dx^i \otimes dx^j.$$

vielbeins, volume form, Hodge * operator

For simplicity, we will assume that the metric g_{ij} is positive definite. For a metric with more general signature, we just have to introduce appropriate sign factors to some of the formulae below.

Since the metric g_{ij} is symmetric, we can find a basis $\{e_i^a\}_{a=1,\cdots,n}$ so that

$$g_{ij} = \sum_{a=1}^{n} e_i^a e_j^a.$$

In the last lecture, we used the symbol $\{e^a\}$ to denote general basis (and used $\mathcal{O}^a$ for orthonormal basis). From now on, we reserve this symbol for the orthonormal frame. For a given metric, the frame is defined modulo $O(n)$.

This can be done at each point p on M. e^a's are called vielbeins. *viel* means many in German, and *bein* is a leg. (In 4 dimensions, they are also called vierbeins or tetrads. In dimensions other than 4, words like fünfbein, etc. have been used. Vielbein covers all dimensions.)

Using the vierbeins, the volume form vol is defined by

$$\text{vol} = e^1 \wedge e^2 \wedge \cdots \wedge e^n.$$

Note that it may not be possible to define vol globally on M since it is invariant under $SO(n)$ but not under $O(n)$. It may not be possible to choose a sign factor for vol (associated to

$Z_2 = O(n)/SO(n)$) consistently over M. The volume form is well-defined if and only if M is orientable.

Using coordinates, we can express the volume form as

$$\text{vol} = \sqrt{g} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

where $g = \det g$ (we are assuming that the metric is positive definite). Or,

$$(\text{vol})_{i_1 i_2 \cdots i_n} = \sqrt{g} \epsilon_{i_1 i_2 \cdots i_n}.$$

For a k-form ω, the Hodge * operator is defined as

$$(*\omega)_{i_{k+1} \cdots i_n} = \frac{1}{k!} \frac{\epsilon^{j_1 \cdots j_k j_{k+1} \cdots j_n}}{\sqrt{g}} \omega_{j_1 \cdot j_k} g_{j_{k+1} i_{k+1}} \cdots g_{j_n i_n}.$$

Here I used the totally anti-symmetric tensor $\epsilon_{i_1 \cdots i_n}$ and $\epsilon^{i_1 \cdots i_n}$ normalized as

$$\epsilon_{12 \cdots n} = \epsilon^{12 \cdots n} = 1.$$

Under coordinate transformations, $\epsilon_{i_1 \cdots i_n}$ does not transform as a tensor. However, we can remedy this by multiplying $\sqrt{g}$ to make it into the volume form. The volume form transforms as a tensor if coordinate transformations preserve the orientation. If we change the orientation, we get an extra (-1).

Question 2: Show

$$** \omega = (-1)^{k(n-k)} \omega.$$

co-differential δ

The co-differential δ on C^k is defined by

$$\delta \omega = (-1)^{nk+n+1} * d * \omega.$$

The sign is chosen so that δ is *hermitian conjugate* to the exterior derivative d, we we will see later. If $\dim = n$ is even,

$$\delta = - * d * .$$

If n is odd,

$$\delta = (-1)^k * d * .$$

We can easily verify the following properties,

$$\delta^2 = 0, \quad * \delta d = d \delta *, \quad d * \delta = \delta * d = 0.$$

We can use d and δ to define the Laplace-Beltrami operator $\Delta : C^k(M) \to C^k(M)$ as

$$\Delta = \delta d + d \delta.$$

Question 3: Show that, for $f \in C^0(M)$, the Laplace-Beltrami operator is

$$\Delta f = -\frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f \right),$$

where $g = \det g$.

integral

We can integrate an n-form ω over an oriented n-dimensional manifold M. Since ω is a top form, it has only one component,

$$\omega = a(x) dx^1 \wedge \cdots \wedge dx^n.$$

The integral is defined by

$$\int_M \omega = \int a(x) dx^1 \cdots dx^n.$$

When M is covered by several coordinate charts, we devide M into segments and use the above in each segment.

Using the integral, we can define an inner product between α and $\beta \in C^k(M)$ by

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta.$$

Question 4: Show that it is symmetric, $(\alpha, \beta) = (\beta, \alpha)$.

If the manifold M has a boundary ∂M, and if $\alpha \in C^n(M)$ is of the form $\omega = d\alpha$ for some $\alpha \in C^{n-1}(M)$, the Stokes theorem holds,

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

In particular, if M has no boundary,

$$\int_M d\alpha = 0.$$

Question 5: Suppose that M has no boundary. For $\alpha \in C^{k+1}(M)$ and $\beta \in C^k(M)$, show

$$(\alpha, \delta\beta) = (d\alpha, \beta).$$

(This is the reason for the choice of sign in the definition of δ.)

supersymmetry

Let us try to compare the mathematics we discussed in this lecture with the fermion picture. As in the previous lecture, we identify the fermion creation operator $\bar\psi^i$ with the multiplication of the 1-form

$$\bar\psi^i \leftrightarrow dx^i \wedge .$$

This maps a k-form to a $(k+1)$-form. The fermion annihilation operator should map a k-form to a $(k-1)$-form. Thus, we define

$$\psi^i \leftrightarrow (-1)^{nk+k+1} * dx^i * .$$

The sign is chosen so that

$$\{\psi^i, \psi^j\} = 0, \quad \{\bar\psi^i \bar\psi^j\} = 0, \quad \{\psi^i, \bar\psi^j\} = g^{ij},$$

and

$$(\psi^i \alpha, \beta) = (\alpha, \bar{\psi}^i \beta).$$

We also identify, according to the quantum mechanics, a bosonic momentum operator p_i with the derivative $\partial/\partial x^i$.

$$p_i \leftrightarrow -i\frac{\partial}{\partial x^i}.$$

We can then write the exterior derivative operator d and the co-differential δ as,

$$d = i\bar{\psi}^i p_i, \quad \delta = -i\psi^i p_i.$$

Thus,

$$[d, x^i] = \bar{\psi}^i, \quad [\delta, x^i] = -\psi^i,$$

and

$$\{d, \bar{\psi}^i\} = 0, \quad \{d, \psi^i\} = ig^{ij}p_j,$$
$$\{\delta, \bar{\psi}^i\} = -ig^{ij}p_j, \quad \{\delta, \bar{\psi}^i\} = 0.$$

Namely, d and δ generate exchanges between the bosonic oprators p_i and the fermionic operators $\psi^i, \bar{\psi}^i$.

We can also think of the Laplace-Beltrami operator $\Delta = \{d, \delta\}$ as the *Hamiltonian*. On a Riemannian manifold, it is interesting to find the spectrum of Δ. Since d and δ commute with Δ, we can think of d and δ as generating some symmetry. In fact, it is supersymmetry since they exchange bosons and fermions.

We can think of $\sum_{k=0}^n C^k(M)$ as the Hilbert space of the quantum system. The innter product is defined by (α^*, β), so we should restrict the space to be those with normalizable differential forms. With respect to this metric, δ is hermitian conjugate of d, and Δ is hermitian.

In a later lecture, we will see that there is a dynamical system associated to M which has supersymmetry and whose quantization gives $\sum_{k=0}^n C^k(M)$ as the Hilbert space.

Lecture 3

Cohomologies, curvatures

Maxwell equations

The Maxwell equations for electromagnetic fields are expressed as

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{H}}{\partial t}, \quad \vec{\nabla} \cdot \vec{H} = 0,$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \quad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{E}}{\partial t} = 4\pi\vec{j}.$$

These equations can be simplified if we use the 4-dimensional notation as

$$\partial_\mu F_{\nu\rho} - \partial_\nu F_{\mu\rho} = 0, \quad \partial^\mu F_{\mu\nu} = 4\pi J_\nu,$$

where an anti-symmetric tensor field $F_{\mu\nu}$ and a vecfor field J_μ are given by by

$$F_{0i} = E_i, \quad F_{ij} = -\epsilon_{ijk} H_k,$$

$$J_0 = \rho, \quad J_i = -j_i.$$

It is natural to regard them as differential forms, $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ and $J = J_\mu dx^\mu$. The Maxwell equations can then be expressed as

$$dF = 0, \quad \delta F = -4\pi J.$$

The equation $dF = 0$ can be solved if we set $F = dA$. On $\mathbf{R}^4$, one can show conversely that any solution to $dF = 0$ is given by $F = dA$ for some A. This is an example of the Poincaré lemmna. In the terminology of electromagnetism, $\phi = A_0$ is the scalar potential and $-A_i$ is the vector potential.

The choice of A is not unique. If $F = dA$, we can replace A by $A' = A + d\lambda$ and we still have $F = dA'$. The Maxwell equations have gauge symmetry.

Question 1: We can restrict the gauge degrees of freedom by imposing the condition $\delta A = 0$. For any F, can we choose an appropriate gauge to satisfy this condition? Is there any remaining gauge symmetry after imposing this condition? Discuss this for both a positive definite metric and a Lorentzian signature metrics.

co-homologies

We can generalize the relation between the field strength 2-form F and the gauge potential 1-form A to other differential forms. If a k-form ω satisfies $d\omega = 0$, it is called closed. If it can be written as $\omega = d\lambda$ for some $(k-1)$-form λ, we say ω is exact. Clearly, if an exact form is closed. On $\mathbf{R}^n$, the converse is also true.

[Poincaré Lenma]

On $\mathbf{R}^n$, if a k-form ω is closed, it is also exact.

This also holds when M is contractible, namley when one can smoothly shrink M to a point. This is not true on a general manifold. However, it is true on each coordinate chart. The issue is how these charts are patched together globally.

It is interesting to ask what are closed differential forms that are not exact. This leads us to introduce the concept of cohomology. We first consider a space of closed k-forms,

$$Z^k(M) = \{\omega \in C^k(M) : \ d\omega = 0\}.$$

Clearly, if ω is closed, so is $\omega + d\lambda$. So, we can introduce an equivalence relation in Z^k: $\omega \sim \omega'$ if and only if their difference is exact. The k-th de Rham cohomology $H^k(M)$ is defined as a quotient of Z^k by the space of exact forms,

$$B^k(M) = \{d\lambda : \ \lambda \in C^{k-1}(M)\},$$

as

$$H^k(M) = Z^k(M)/B^k(M).$$

The dimensions of H^k is called the Betti number,

$$b_k = \dim H^k(M).$$

The Betti numbers are topological invariant of M. In particular, their alternative sum is the Euler characteristic,

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b_k.$$

Question 2: Show that $b_0 = 1$ if M is connected. What if M has m disjoint components?

Question 3: Compute the first cohomology H^1 of a circle S^1.

$H^k(M)$ is called co-homology since it is related to an object called homology. That will be discussed later.

representatives of cohomologies

If M is endowed with a Riemannian (positive definite) metric g, we can define the codifferential δ and a positive definite inner product (α, β) as discussed in Lecture 2.

Question 4: Express the inner product (α, β) of k-forms in their components.

It is easy to see that the Laplace-Beltrami operaotr $\Delta = \delta d + d\delta$ commutes with d. Thus, we can choose a solution to $d\omega = 0$ to be an eigenstate of Δ.

$$(d\delta + \delta d)\omega = \epsilon\omega.$$

If $\epsilon \neq 0$, since $d\omega = 0$, we have $\omega = d(\delta\omega/\epsilon)$, which would mean that ω is exact. So, if we want a closed but not exact form, it should be a zero eigenstate of Δ, i.e., a harmonic form. Thus, we can choose $\omega \in H^k(M)$ as a harmonic form.

In fact, this fixes the gauge degrees of freedom. To see that, we write

$$(\omega, \Delta\omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega).$$

and observe that the right-hand side is a sum of non-negative numbers. Therefore, ω is harmonic if and only if ω is closed ($d\omega = 0$) and co-closed ($\delta\omega = 0$). Now, suppose ω and $\omega + d\lambda$ are both harmonic. This means in particular that $\delta d\lambda = 0$. However, $(d\lambda, d\lambda) = (\lambda, \delta d\lambda) = 0$. Thus, $d\lambda = 0$. Namely, if we require ω to be harmonic, there is no more gauge degrees of freedom. Since $\Delta\omega = 0$ is equivalent to $d\omega = 0$ and $\delta\omega = 0$, this means that we can choose representatives ω of $H^k(M)$ to be co-closed $\delta\omega = 0$.

The condition $\delta\omega = 0$ is a generalization of the Lorentz gauge condition in the Maxwell theory.

It is also useful to know that any k-form on a compact orientable Riemannian manifold can be uniquely expressed as a sum of an exact form, a co-exact form, and a harmonic form. This is called the Hodge decomposition.

As we saw in the last lecture, the operators d and δ are analogous to the supersymmetry operators (supercharges) acting on the space of differential forms $\oplus_k C^k(M)$. Differential forms with k even are bosons and those with k odd are fermions. The supercharges map bosons to fermions and vice versa. The Laplace-Beltrami operator $\Delta = \{d, \delta\}$ is analogous to the Hamiltonian. If ω is a supersymmetric state, $d\omega = 0, \delta\omega = 0$. This means that $\Delta\omega = 0$. Namely, any supersymmetric state has zero energy. Some people thinks that this may help us explain why the dark energy (cosmological constant) of the Universe is so small.

covariant derivatives, curvature

The exterior derivative is a special derivative operator defined for differential forms. It is also just one derivative operator. On the other hand, to derive differential equations on M, we may want more differential operators. Unfortunately, on a general differnetiable manifold M without futhre structure, it is not possible to define a partial derivative $\partial/\partial x^i$ on a general tensor field, as it depends on a choice of coordinates.

Another say to point out the problem is as follows. If we have a smooth function $f : M \to \mathbf{R}$, we can define its partial derivatives with respect to coordinates x^i as

$$\frac{\partial f}{\partial x^i} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(f(x^1, ..., x^i + \epsilon, ..., x^n) - f(x^1, ..., x^i, ..., x^n) \right).$$

This requires comparing values of f at x and $x + \epsilon$. Since f is taking values in $\mathbf{R}$, there is no problem with taking the difference of the two numbers, at x and $x + \epsilon$. This does not work for a tangent vector v, for example. At x it takes value in $T_x M$. At $x + \epsilon$, it takes value in $T_{x+\epsilon} M$. A priori, there is not natural identification of the two vector spaces, so we cannot compare the values of v at the two points.

The situation is better if we have a metric g_{ij}. With it, we can define the affine connection (Christoffel symbol) as,

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}).$$

A covariant derivative ∇_i of a rank-p tensor field $T_{i_1 \cdots i_p}$ can be defined as

$$\nabla_i T_{i_1 \cdots i_p} = \partial_i T_{i_1 \cdots i_p} - \Gamma^j_{i i_1} T_{j i_2 \cdots i_p} - \cdots - \Gamma^j_{i i_p} T_{i_1 \cdots i_{p-1} j}.$$

A covariant derivative of a tensor with upper indices, e.g., T^{ij}, can be defined similarly, with the + sign instead of the − sign in front of the affine connection Γ. In a later lecture, we will

discuss why this procedure resolves the issue of comparing T_xM and $T_{x+\epsilon}M$. For now, I just point out that, if T transforms as a tensor of rank (r, s), then ∇T transforms as a tensor of rank $(r + 1, s)$.

The affine connection is defined in such a way that the metric is covariantly constant,

$$\nabla_i g_{jk} = 0,$$

and the torsion tensor defined as an antisymmetric part of the Christoffel symbol is zero,

$$T^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) = 0.$$

The Riemann curvatuve $R^i{}_{jkl}$ is defined as a failure of the covariant derivatives to commute with each other,

$$[\nabla_i, \nabla_j]T_k = R_{ijk}{}^l T_l.$$

Explicitly,

$$R_{ijk}{}^l = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{jk}\Gamma^l_{mi}.$$

The Riemann curvature satisfies the following properties,

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}.$$

The first half of the equation follows simply by the definition and the second half is a consequance of the fact that the metric is covariantly constant.

$$R_{[ijk]l} = 0,$$

where $[ijk]$ means the antisymmetrization of the 3 indices. This can be proven by writing $d^2\omega = 0$ in terms of the covariant derivatives.

$$\nabla_{[i} R_{jk]lm} = 0.$$

The last equation is called the Bianchi identity. It is an important identity, but is not easy to verify this directly using the component notation.

curvature as seen by differential forms

The above expressions are simplified if we use the language of differential forms. First remember the vielbein e^a_i satisfying $g_{ij} = \sum_a e^a_i e^a_j$. Use this to devine a basis of $C^1(M)$ as,

$$e^a = e^a_i dx^i.$$

The affine spin connection $\omega^a{}_b$ is a 1-form defined by

$$(1) \quad de^a + \omega^a{}_b \wedge e^b = 0.$$

(The left-hand side is the torsion 2-form. We are setting the torsion to be zero.)

The curvature 2-form is defined by

$$(1) \quad R^c{}_d = \frac{1}{2} R_{ab}{}^c{}_d e^a \wedge e^b = d\omega^c{}_d + \omega^c{}_f \wedge \omega^f{}_d.$$

Question 5: Show that the curvature defined in this way is related to the Riemann curvature discussed earlier by

$$R_{ij\ l}^{\ k} = R_{ab\ d}^{\ \ c} e_i^a e_j^b e_c^k e_l^d.$$

By taking the exterior derivative of (1), we find

$$R^a_{\ b} \wedge e^b = 0,$$

which in components is equivalent to $R_{[ijk]l} = 0$.

Question 6: Show that the exterior derivative of (2) gives,

$$dR^a_{\ b} + \omega^a_{\ c} \wedge R^c_{\ b} - R^a_{\ c} \wedge \omega^c_{\ b} = DR^a_{\ b} = 0,$$

and that it is equivalent to the Bianchi identity.

Lecture 4

complex manifolds, Kähler manifold

almost complex structure

Suppose dim $M = n$ is even, $n = 2m$. We may consider combining $2m$ coordinates into complex combinations,

$$z^i = x^{2i-1} + \sqrt{-1}x^{2i}, \quad (i = 1, ..., m).$$

But, what is a motivation for us to do this? More importantly, this would depend on a choice of coordinates. The manifold M must have some structure which compells us to introduce such complex combinations.

A natural structure would be a tensor field $J^\mu_{\ \nu}$ ($\mu, \nu = 1, ..., n = 2m$), which has the property,

$$J^2 = -1.$$

(In today's lecture, I use Roman characters $i, j, ...$ for complex coordinates,and Greek characters $\mu, \nu, ...$ for real coordinates. So, $i, j... = 1, ..., n$ and $\mu, \nu, ... = 1, ..., m$.)

In a slightly more sophisticated way of saying, at each point $p \in M$, we have a linear map $J : T_pM \to T_pM$ obeying $J^2 = -1$. If there is such J, may consider diagonalizing it. However, since eigenvalues of J must be $\pm\sqrt{-1}$, we cannot do so in a vector space with real coefficients like T_pM. To do that, we need to allow vectors in T_pM have complex-valued coefficients. The complexified tangent space can be decompoesed into a holomorphic part T_pM^+ and an anti-holomorphic part T_pM^-, both m-dimensional over $\mathbf{C}$, and J has eigenvalues $\pm\sqrt{-1}$ on $T_pM^\pm$.

Note that it is not always possible to have such a tensor field J. For example, it is known that one cannot have such a tensor field on the 4-sphere S^4.

If we can define J on M satisfying $J^2 = -1$, we say that (M, J) is an *almost* complex manifold. The tensor J is called almost complex structure.

We say *almost* since we do not yet have a fully complex manifold. To be called a complex manifold, on each coordinate patch U, we need to be able to define complex coordinates z^i ($i = 1, ..., m$) so that $\{\partial/\partial z^i\}$ gives basis for the holomorphic part T_pM^+ of the tangent space at each point p on U. In order for this to be possible, the tensor field J has to satisfy the following differential equations,

$$J^\nu_\mu \partial_\rho J^\mu_\sigma - J^\nu_\mu \partial_\sigma J^\mu_\rho - J^\mu_\sigma \partial_\mu J^\nu_\rho + J^\mu_\sigma \partial_\rho J^\nu_\mu = 0.$$

The left-hand side of this equation is known as the Nijenhuis tensor.

If the Nijenhuis tensor vanishes, we can find holomorphic coordinates z^i so that $J(\partial/\partial z^i) = \sqrt{-1}\partial/\partial z^i$. Since J is defined globally on M, it follows that the manifold can be covered smoothly with complex coordinate charts. In holomorphic coordinates z^i, the tensor J is of the form,

$$J^i_j = \sqrt{-1}\delta^i_j, \quad J^{\bar{i}}_{\bar{j}} = -\sqrt{-1}\delta^{\bar{i}}_{\bar{j}},$$
$$J^i_{\bar{j}} = 0, \quad J^{\bar{i}}_j = 0.$$

Question 1: Show that, when two coordinatepatches U and V overlap, the transition function between the two complex coordinates obeying the above set of conditions are holomorphic.

Not all almost complex manifolds can be made to complex manifolds. For example, we can define an almost complex structure on the 6-sphere S^6, but its Nijenhuis tensor does not vanish.

Kähler manifolds

So far, we did not assume that M is equipped with a metric $g_{\mu\nu}$. When it exists, it is natural to impose the following compatibility conditions,

$$g_{\mu\nu}J^\mu_\rho J^\nu_\sigma = g_{\rho\sigma}, \quad \nabla_\mu J^\nu_\rho = 0.$$

These imply that the Nijenhuise tensor vanishes and therefore (M, J) is a complex manifold. But, it implies more. The first condition implies that, in complex coordinates z^i, the only non-zero components of the metric is $g_{i\bar{j}}$. Namely,

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu = 2ig_{i\bar{j}}dz^i \otimes d\bar{z}^{\bar{j}}.$$

To understand what the second equation means, we introduce the Kähler form,

$$k = \frac{1}{2}g_{\mu\nu}J^\mu_\rho dx^\rho \wedge dx^\nu = ig_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}.$$

The second condition then implies that k is closed, $dk = 0$. In components, it means

$$\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}, \quad \partial_{\bar{j}} g_{i\bar{k}} = \partial_{\bar{k}} g_{i\bar{j}}.$$

Locally, we can always integrate these equations as,

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(z, \bar{z}),$$

for some function $K(z, \bar{z})$, which is known as the Kähler potential. It is unique up to Kähler transformation, $K(z, \bar{z}) \to K(z, \bar{z}) + f(z) + \bar{f}(\bar{z})$ for any holomorphic function $f(z)$. We should note that the Kähler potential cannot be a globally defined smooth function on a compact Riemannian manifold M without boundary. This is because $J^m/m!$ is equal to the volume form vol (dim $M = 2m$). If K were globally defined, J would be an exact form and so is vol $= J^m/m!$. If the volume form were exact, its integral over M would vanish, which would be inconsistent with the assumption that the metric is non-degenerate.

Given the metric in the above form, we can compute the affine connection (Christoffel symbol).

Question 2: Show that the only non-zero components are

$$\Gamma^i_{jk} = g^{i\bar{l}}\partial_j g_{k\bar{l}},$$

and its complex conjugate. Components with mixed indices (mixed in i and $\bar{j}$) all vanish.

Thus, the only non-zero components of the curvature tensor are,

$$R_{\bar{i}j\ \ l}^{\ \ k} = \partial_{\bar{i}}\Gamma^k_{jl}.$$

The connection and the curvature describe how tangent vectors are parallel transported on the manifold M, in such a way that the metric structure is respected, $\nabla_\mu g_{\nu\rho} = 0$. For a general

Riemannian manifold without complex structure, the anti-symmetry $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$ means that the transport is done by an element of the orthogonal group $SO(n)$ (n =dim$M$). For the Kähler manifold, we can use complex coordinates z^i, and the curvature takes the form $R_{\bar{i}j}{}^k{}_l$. This means that the transport is done by an element of the unitary group $U(m) \in SO(n = 2m)$.

When we start at $p \in M$, pick a tangent vector, and transport a tangent vector by using the affine connection. We move around the manifold M and come back to the same point p. We may not get the same vector as the one we started with. On a general Riemannian manifold, the vectors before and after are related by an element of $SO(n)$. For each closed path on M, we can associate an element of $SO(n)$. It is called a holonomy for the path. If we consider holonomies of all paths starting and ending at $p \in M$, it makes a group – the holonomy group. We could start with another point on M. As far as M is connected, the resulting holonomy group are equivalent. When M is Kähler, the holonomy group is a subgroup of $U(m) \in SO(2m)$.

Question 3: Show that the Ricci tensor for a Kähler manifold takes a particularly simple form,

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log\det(g).$$

This expression will be useful when we discuss Calabi-Yau manifolds later.

Hodge-de Rham cohomology

On a Kähler manifold, we can consider forms that contains p dz's and q $d\bar{z}$'s. The space of k-forms $C^k(M)$ can then be decomposed into,

$$C^k(M) = \oplus_{p+q=k} C^{p,q}(M),$$

where elements of $\omega \in C^{p,q}(M)$, called (p,q)-forms, are of the form,

$$\omega = \frac{1}{p!q!} \omega_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \cdots \wedge d\bar{z}^{\bar{j}_q}.$$

Similarly, the exterior derivative and its adjoint $\delta = - * d *$ (note: the sign factor is (-1) since M is even dimensional) can be split as

$$d = \partial + \bar{\partial}, \quad \delta = \partial^\dagger + \bar{\partial}^\dagger,$$

where

$$\partial\omega = dz^i \wedge \left(\frac{\partial}{\partial z^i}\omega\right), \quad \bar{\partial}\omega = d\bar{z}^{\bar{j}} \wedge \left(\frac{\partial}{\partial \bar{z}^{\bar{j}}}\omega\right),$$

and

$$\partial^\dagger = - * \partial *, \quad \bar{\partial}^\dagger = - * \bar{\partial} * .$$

These operators $\partial, \bar{\partial}, \partial^\dagger, \bar{\partial}^\dagger$ map (p,q)-forms into $(p+1,q)$, $(p,q+1)$, $(p-1,q)$, and $(p,q-1)$-forms, respectively.

Question 4: Show that the Laplace-Beltrami operator is also decomposed as

$$\Delta = d\delta + \delta d = 2(\partial\partial^\dagger + \partial^\dagger\partial) = 2(\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}),$$

and

$$\partial\bar{\partial}^\dagger + \bar{\partial}^\dagger\partial = 0, \quad \bar{\partial}\partial^\dagger + \partial^\dagger\bar{\partial} = 0.$$

We can consider cohomology of (p,q)-forms with respect to $d = \partial + \bar{\partial}$. It is called the Hodge-de Rham cohomology and denoted by $H^{p,q}(M)$. Since ∂ and $\bar{\partial}$ change degrees of forms differently, elements of $H^{p,q}(M)$ are annihilated by both ∂ and $\bar{\partial}$, modulo images of ∂ and $\bar{\partial}$. This gives the decomposition,

$$H^k(M) = \oplus_{p+q=k} H^{p,q}(M).$$

Since the de Rham cohomology $H^k(M)$ is generated by harmonic forms, so is each $H^{p,q}(M)$ in the above decomposition.

On a compact Kähler manifold without boundary, the Kähler form $k = ig_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ always generate a non-trivial element of $H^{1,1}(M)$ since J^m is proportional to the volume form (dim $M = 2m$). Thus, dim$H^{1,1}(M) \geq 1$. The $(1,1)$-cohomology defined by k is called the Kähler class.

Another potential generator of $H^{1,1}(M)$ is the Ricci form, $R_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$. Later in this course, we will learn that it is related to the first Chern class of M. On a Calabi-Yau manifold, the Ricci form is zero. It was conjectured by Calabi that, if the Ricci form is an exact form (i.e., trivial as an element of $H^{1,1}(M)$), we can always choose a metric on M with the same complex structure and the Kähler class such that the Ricci curvautre is identically equal to zero. This conjecture was proven by Yau, and a Kähler manifold with zero Ricci curvature is called a Calabi-Yau manifold.

hyper-Kähler manifolds

Suppose there are more than one J's that satisfy the compatibility conditions with the metric,

$$g_{\mu\nu} J^{(a)\mu}_{\rho} J^{(a)\nu}_{\sigma} = g_{\rho\sigma}, \quad \nabla_\mu J^{(a)\nu}_{\rho} = 0, \quad (a = 1, ..., N-1).$$

The complex structure we considered in the above corresponds to the case of $N = 2$. (The reason we set the number of J's to be $(N-1)$ will become clear later.) Since $J^{(a)}$ is covariantly constant, it is invariant under parallel transport. Therefore, when we start at a point $p \in M$, transport $J^{(a)}$ along a closed path, and come back to the same p, then $J^{(a)}$ does not change. This means that $J^{(a)}$ commutes with the holonomy group of M. By Shur's lemman in group theory, one can show that $J^{(a)}$ must for a division algebra over real numbers. Namely, they make either the complex numbers, in which case $N = 2$ and there is a single complex structure J, or the quaternions, in which case $N = 2$ and thre are three possible imaginary units $J^{(1)}, J^{(2)}, J^{(3)}$. As we saw in the above, the holonomy group for $N = 2$ case is a subgroup of $U(m)$. The second case with $N = 4$ requires that dim $M = 4r$ and the holonomy group is a subgroup of $Sp(r) \in U(2r) \in SO(4r)$. In this case, M is called hyper-Kähler. The Ricci curvature of a hyper-Kähler manifold is zero.

There are manifolds with other types of holonomies. For simply connected Riemannian manifolds which are not locally a product space and are not a symmetric space (space with continous group symmetry preserving its metric), possible holonomy groups are:

$SO(n)$, $U(n)$, $SU(n)$, $Sp(n) \cdot Sp(1)$, $Sp(n)$, G_2, and Spin (7). A manifold with $SU(n)$ holonomy is a Calabi-Yau manifold, and a manifold with $Sp(n) \cdot Sp(1)$ holonomy is known as a quaternionic-Kähler manifold.

extended supersymmetry

On a Kähler manifold, differential forms are generated by dz^i and $d\bar{z}^{\bar{i}}$. Correspondingly, we can consider fermions,

$$\bar{\psi}^i \leftrightarrow dz^i\wedge, \quad \bar{\psi}^{\bar{i}} \leftrightarrow d\bar{z}^{\bar{i}}\wedge,$$

and

$$\psi^i \leftrightarrow (-1)^{nk+k+1} * dz^i *, \quad \psi^{\bar{i}} \leftrightarrow (-1)^{nk+k+1} * d\bar{z}^{\bar{i}} * .$$

They obey the anti-commutation relations,

$$\{\psi^i, \bar{\psi}^{\bar{j}}\} = \{\bar{\psi}^i, \bar{\psi}^{\bar{j}}\} = g^{i\bar{j}}, \quad \{(\text{others})\} = 0.$$

If we identify p_i and $p_{\bar{j}}$ by $-i\partial/\partial z^i$ and $-i\partial/\partial \bar{z}^{\bar{j}}$ according to quantum mechanics, we can write

$$\partial = i\bar{\psi}^i p_i, \quad \bar{\partial} = i\bar{\psi}^{\bar{j}} p_{\bar{j}}.$$

These, together with their hermitian conjugates, generate extended supersymmetry. We have twice as many supersymmetry generators (supercharges), so the resulting symmetry is called $N = 2$ supersymmetry.

Similarly, on a hyper-Kähler manifold, we have 4 times more supercharges generating $N = 4$ supersymmetry.

Lecture 5

vector bundles, gauge theory

tangent bundle

In Lecture 2, we defined the tangent space T_pM at each point p on M. Let us consider a collection of tangent bundles over every point on M,

$$TM = \cup_{p\in M} T_pM.$$

It is naturally a manifold. For a given coordinate chart (U_i, ϕ_i), we can define coordinates on $\cup_{p\in U_i} T_pM$ as (x^μ, v^μ), where (x^μ) are coordinates on U_i and we parametrize a tangent vector as

$$v = v^\mu \frac{\partial}{\partial x^\mu}.$$

This defines differential structure on TM (namely, TM is a differential manifold). TM is called a tangend bundle.

A smooth vector field is $v : p \in M \to v(p) \in T_pM$ such that its components v^μ expressed in coordinates x^μ are smooth functions of the coordinates on each U_i. We also call it a smooth *section* of TM. The reason for this name is as follows. The tangent bundle TM is locally a product space, $U_i \times \mathbf{R}^n$. Imagine that U_i is stretched in a horizontal direction and $\mathbf{R}^n$ in a vertical direction. The vector field v is then a graph over U_i, which lifts U_i in $U_i \times \mathbf{R}^n$. It cuts TM along the direction of M, which is why it is called a section.

When we change coordinates, $x^\mu \to \tilde{x}^\mu(x)$, the tangent space coordinates change as

$$v^\mu \to \tilde{v}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} v^\nu,$$

so that $v = v^\mu \partial/\partial x^\mu = \tilde{v}^\mu \partial/\partial \tilde{x}^\mu$ is independent of coordinates.

vector bundle

Vector bundles generalize the notion of the tangent bundle TM. On each coordinate chart $(U_i\phi_i)$, it should be of the form $U_i \times V$ for some vector space V. (dim V does not have to be the same as dim M.) To define a vector bundle more abstractly, mathematicians say that a differential manifold E is a vector bundle if

(1) there is a projection map π,

$$\pi : E \to M,$$

so that, for each point $p \in M$, its inverse image $\pi^{-1}(p)$ is isomorphic to V. For the tangent bundle TM, $\pi^{-1}(p) = T_pM$.

(2) we can choose atlases of E and M so that, for each local coordinate chart U of M, there is a smooth map $\varphi : \pi^{-1}(U) \to U \times V$. The map φ is called local trivialization of the vector bundle E over U.

The vector space V, which sits on the top of each $p \in M$, is called a fiber. When V is a vector space over $\mathbf{R}$, which we will consider in this lecture, we say that E is a real vector bundle. We

can also consider a complex vector bundle. In particular, when V is 1-dimensional over $\mathbf{C}$, we say that E is a line bundle.

Suppose two coordinate charts U_i and U_j of M overlap with each other. Over $U_i \cap U_j$, there are two local trivializations φ_i and φ_j. Their composition, $\varphi_i \circ \varphi_j^{-1}$, maps $U_i \cap U_j \times V$ to itself as

$$\varphi_j \circ \varphi_i^{-1}(p, v) \to (p, g_{j \leftarrow i}(p)v),$$

where $p \in M$, $v \in V$ and $g(p)$ is an invertible linear map on V. This $g_{j \leftarrow i}(p) \in GL(V, \mathbf{R})$ is called a transition function.

If there is a triple intersection of three charts U_i, U_j and U_k, the transition function must satisfy the consistency condition,

$$g_{k \leftarrow j}(p) g_{j \leftarrow i}(p) = g_{k \leftarrow i}(p),$$

on $p \in U_i \cap U_j \cap U_k$. This is called a cocycle condition.

Conversely, if we have a differential manifold M, and if we have a transition function $g_{j \leftarrow i}(p) \in GL(V, \mathbf{R})$ for $p \in U_i \cap U_j$ satisfyin the cocycle condition, then there is a unique vector bundle E over M.

For the tangent bundle TM, we considered a tangent vector field v, which we may consider as a map $p \in M \to v(p) \in T_p$. Similarly, for a general vector bundle E, we may consider a map $s : p \in M \to s(p) \in \pi^{-1}(p)$. An example of s is the zero section where $s(p) = 0 \in V$ for all p.

fiber bundle

We can consider a more general class of manifolds E called fiber bundles, where there is a projection $\pi : E \to M$, but the fiber $F \sim \pi^{-1}(p)$ for $p \in M$ is not necessarily a vector space. For example, one can consider the case where the fiber is a group G. Over a coordinate chart U of the base manifold M, E looks like $U \times G$. When two charts U_i and U_j overlap the transition function is given by $(p, g \in G) \to (p, g(p)\rho \in G)$, where $g(p) \in G$.

When E is a vector bundle, we can consider its associated principal bundle whose transition functions are given by those of E.

example: magnetic mnopole bundle and Hopf fibration

Consider the 2-sphere S^2 and a $U(1)$ principal bundle E over S^2. As a manifold, the group $U(1)$ can be regarded as a circle S^1; the angle coordinate $\theta \in [0, 2\pi)$ of S^1 gives an element $e^{i\theta} \in U(1)$. Thus, we are considering an S^1 bundle over S^2.

As we discussed in Lecture 1, S^2 can be covered by 2 coordinate charts, U_+ and U_-. They can be chosen so that U_+ (U_-) contains the northen (southern) pole and that they overlap in a region near the equator of S^2. We can choose their coordinates as $(r_\pm, \phi)$, where $t_\pm$ is a distance from the northen (southen) pole and ϕ is the longitude of S^2.

We can then choose two coordinate charts of E. Over $U_\pm$, we can use $(r_\pm, \phi; \theta_\pm)$, where $(r_\pm, \phi)$ are coordinates of $U_\pm$ and $\theta_\pm$ parametrizes the S^1 fiber.

Let us consider the transition function,

$$e^{i\theta_-} = e^{in\phi} e^{i\theta_+},$$

for some integer n. This represents the configuration of the electro-magnetic field in the presence of a magnetic monopole of charge n.

When $n = 0$, the principal bundle is a trivial product, $E_{n=0} = S^2 \times S^1$.

When $n = 1$, the total space of the principal bundle makes the 3-sphere,

$$E_{n=1} = S^3.$$

This is known as the Hopf fibration. To exhibit the fibration structure, let us present S^3 as a subspace of $\mathbf{R}^4$ subject to the condition,

$$a^2 + b^2 + c^2 + d^2 = 1.$$

This is to be identified with the total space of the bundle E. This bundle is suppose to have S^2 has a base manifold, so we need to exhibit the projection map $\pi : S^3 \to S^2$. This, according to Heiz Hopf, is given by

$$x = a^2 + b^2 - c^2 - d^2,\ y = 2(ad + bc),\ z = 2(bd - ac).$$

It is elementary to verify that $z^2 + y^2 + z^2 = 1$.

Question 1: Show that $\pi^{-1}(p) \sim S^1$ for each $p \in S^2$.

(Hint: Introducing $u = a + ib$ and $v = c - id$, we can express the equation for S^3 as

$$u\bar{u} + v\bar{v} = 1.$$

The projection map $\pi : S^3 \to S^2$ is

$$x = u\bar{u} - v\bar{v},\quad z + iy = 2u\bar{v}.$$

If we fix (x, y, z), what are the remaining degrees of freedom on S^3?)

connection and curvature

As we discuss in Lecture 3, the problem with defining partial derivatives of a tangent vector field on M is that, a priori, there is no indentification of T_pM and $T_{p'}M$ even when p and $p' \in M$ are closed to each other. To define a derivative, we need a way to perform *paralell transport* of a vector v along a smooth path $c(t)$ on M. Consider a smooth path $c(t)$ and an arbitrary vector $u \in T_pM$ at $p = c(t = 0)$. A parallel transport means that we can define $\Omega(t) \cdot u \in T_{c(t)}M$. Since the tangent space is a linear space, we are writing the parallel transport as a linear map $u \to \Omega(t) \cdot u$. Then, we can define a covariant derivative ∇_t of a vector field $v(x)$ at $p \in M$ as

$$\nabla_t v = \frac{d}{dt}\big[v(x(t)) - \Omega(t) \cdot v(x(t = 0))\big].$$

Since we can choose $c(t)$ to be tangent to any direction at p, this defines a covariant derivative. (For example, if we want to compute $\nabla_i v$ in the x^i direction, we can just choose $c(t)$ to be $(x^1, ..., x^i + t, ..., x^n)$.

For a Riemannian manifold, $\Omega(t)$ is uniquely determined by requiring that the covariant derivative of the metric, which is a section of $T^*M \otimes T^*M$, is zero, and that the torsion tensor of the connection is zero.

This can be done for any vector bundle. For each point $p \in M$, there is a vector space $\pi^{-1}(p)$. To define a covariant derivative, we introduce a parallel transport, which is a linear map $\Omega(t) : \pi^{-1}(c(t=0)) \to \pi^{-1}(c(t))$ along any smooth curve $c(t)$. In fact, all we need is an infinitesimal limit of this since we just need to take one derivative with respect to t. The infinitesimal version of the parallen transport should give a one-form, valued in matrix on V, where V is the fiber over p, since it should give a linear map on $V = \pi^{-1}(p)$ to any direction along the tangent space T_pM. This one-form is called a *connection form*.

Pick a coordinate chart $(U_i, \mathbf{R}^n)$ of a vector bundle E. The covariant derivative of a section of E, expressed in the coordinates as $(v(x)^\alpha)_{\alpha=1,\ldots,n}$, can be written as

$$\nabla_\mu v^\alpha(x) = \partial_\mu v^\alpha(x) + A_\mu{}^\alpha{}_\beta v^\beta(x),$$

where $A^\alpha{}_\beta = A_\mu{}^\alpha{}_\beta dx^\mu$ is a matrix valued connection of E.

When two coordinate charts U_i and U_j overlap, coordinates on the fiber over U_i and over U_j are related by a linear map as $v \to g(p)v$ for some $g \in GL(V, \mathbf{R})$. To be compatible with the derivative operation, the connection form should transform as

$$A \to g^{-1}Ag + g^{-1}dg.$$

Question 2: Show that the spin connection $\omega_\mu^a{}_b$ defined in Lecture 3 transforms as a connection.

The curvature F for the connection is a matrix-valued 2-form defined by

$$F = dA + A \wedge A.$$

In components, one can show that

$$F_{\mu\nu} v(p) = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) v(p),$$

for any smooth section $v(p)$ of E. Under the change of coordinates of the fiber, $v \to g(p)v$, the connection 2-form gtransforms as

$$F \to g^{-1}Fg.$$

holonomy

Pick any point $p \in M$ and move around M along a closed path γ and come back to the same point p. We can parallel transport a vector v in the fiber $\pi^{-1}(p)$ along the path. When we come back to p, the vector v is rotated to $g(\gamma)v$ by some element $g(\gamma) \in GL(V, \mathbf{R})$. It is called a holonomy along γ. If we have two such paths γ_1 and γ_2, we can combine them (start at p, go around γ_1 to come back to p, then start at p again and go around γ_2) to make another path γ_3. It is easy to show that $g(\gamma_3) = g(\gamma_2)g(\gamma_1)$. Thus holonomies along closed paths starting and ending at p makes a subgroup of $GL(V, \mathbf{R})$. It is called a holonomy group.

Question 3: Suppose any two points on M can be connected by a path on M. Show that holonomy groups at two different points p and q are isomorphic. (Two groups G_1 and G_2 are called isomophic if there is a map $f : G_1 \to G_2$ that is one-to-one and onto and if the map respect the group operations, $f(gg') = f(g)f(g')$ for $g, g' \in G_1$.

The curvature $F_{\mu\nu}$ is a holonomy for an infinitesimal loop.

When the curvature vanishes, the holonomy for a look γ is invariant under continuous deformation of γ. In that case the holonomy depends only on topological (global) data of γ. It is called monodoromy in that case.

The holonomy group of an n-dimensional Riemannian manifold M is a subgroup of $SO(n)$. If M is a Kähler manifold and $n = 2m$, its holonomy group is a subgroup of $U(m)$. If M is a Calabi-Yau manifold, its holonomy group is a subgroup of $SU(m)$.

gauge theory

Consider a vector bundle with a complex 1-dimensional fiber. A section s is a complex-valued function in each coordinate patch and transforms as $s(p) \to g(p)s(p)$ under a change of coordinates, where $g(p)$ is a non-vanishing complex-valued function. The covariant derivative takes the form, $\nabla_\mu s(p) = (\partial_\mu + iA_\mu)s(p)$ for a complex-valued connection form A_μ. (I included the imaginary unit in front of A_μ for a later convenience.) Under the change of coordinate, the connection transforms as

$$A \to A - id \log g(p).$$

Since everything commutes over complex number, $g^{-1}A_\mu g = A_\mu$.

If we restrict the transition function $g(p)$ to be in $U(1)$ and write $g(p) = e^{i\lambda(p)}$ for some real valued function λ,

$$A_\mu \to A_\mu + \partial_\mu \lambda,$$

and the curvature 2-form $F = dA$ is given in components by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

If we identify A_μ as the vector potential of the Maxwell theory of electro-magnetism, these are the gauge transformation rule and the definition of the field strength.

For a vector bundle with a higher dimensional fiber, the connection form A_μ is matrix-valued, and the curvature is given in components by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu],$$

where I made the substitution $A \to iA$, in comparison with the convention in the previous section. This gives a non-Abelian generalization of the Maxwell theory, known as the Yang-Mills theory.

Lecture 6

topological space, homology

ϵ-δ definition

In the earlier lecture, we assume that we know when a function $f : M \to \mathbf{R}$ is continuous. To define a continuous function, we need to define a topological space. Here we will give such a definition.

Before we discuss a general notion of topological spaces, let us remind ourselves on the ϵ-δ definition of limit of a function $f : \mathbf{R} \to \mathbf{R}$. The formula,

$$\lim_{x \to x_0} f(x) = y,$$

means that "for any real $\epsilon > 0$, there is always $\delta > 0$ such that, for every x satisfying $|x - x_0| < 0$, we have $|f(x) - y| < \epsilon$."

This method is also used to define continuous functions (often attributed to Karl Weierstrass). We say that $f(x)$ for $x \in \mathbf{R}$ is continuous if, for any $x \in \mathbf{R}$ and $\epsilon > 0$, there is always $\delta > 0$ such that for every y satisfying $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$.

A closely related notion is a Cauchy sequence. An infinite sequence of numbers $\{x_1, x_2, ...\}$ is called Cauchy if and only if, "for any $\epsilon > 0$, there is always an integer $N > 0$ such that, for any pair of integers $n, m > N$ such that $|x_n - x_m| < \epsilon$." In the set of real numbers, every Cauchy sequence has a limit. This is not the case for the set of rational numbers. We say that the real numbers are complete, but the rational numbers are not.

One way to define the real numbers is to use a set of all possible Cauchy sequences in the rational numbers. We can add and multiply two Cauchy sequences. We say that two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ define the same real number if, for any $\epsilon > 0$ there is always an integer $N > 0$ such that $|x_n - y_n| < \epsilon$ for any $n > N$. The set of all Cauchy sequences with this equivalence relation defined the real numbers.

topological space

We note that the inequality symbol $>$ plays a prominent role in these definitions. In fact, we can define a continuous function on $\mathbf{R}$ by using open intevals $(a, b) = \{x : a < x < b\}$ alone. Thus, to define continuous functions on a general space M, we will introduce on M a generalization of the notion of open intervals. They are called open sets of M. To do so, we start with the following list of basic properties of open intervals of $\mathbf{R}$:

Let $\mathcal{U}$ be a set of unions of open intervals on $\mathbf{R}$. This means that

(0) $\mathcal{U}$ contains (a, b) $(a < b)$, and

(1) If $\{U_a \in \mathcal{A} : a \in S\}$ is a (possibly infite) subset of $\mathcal{U}$ parametrized by A, then $\cup_{a \in A} U_a \in \mathcal{U}$.

(2) We also assume $\phi, \mathbf{R} \in \mathcal{U}$. (Here, ϕ means the empty set.)

There is one more property of open intervals:

(3) If $U_1, U_2, ..., U_k$ (k: positive integer) are in $\mathcal{U}$, so is $\cap_{i=1}^{k} U_i$.

Question 1: Why do we consider only finite subsets of $\mathcal{U}$ in (3) while we allow infinite subsets in (2)?

To definte open sets for M, we generalize the above properties of open intervals of $\mathbf{R}$.

Definition of topological space: Suppose M is a set. It is called a topological space if there is a set $\mathcal{U}$ of subsets of M satisfying the following properties.

(1) $\phi, M \in \mathcal{U}$.

(2) If $\{U_a \in \mathcal{U} : a \in A\}$ is any subset of $\mathcal{U}$, then $\cup_{a\in A} U_a \in \mathcal{U}$.

(3) If f $U_1, U_2, ..., U_k$ are in $\mathcal{U}$, so is $\cap_{i=1}^{k} U_i$.

In this case, we call $U \in \mathcal{U}$ an open set of M. The set $\mathcal{U}$ of open sets *defines* the topology of M. For a point p, an open set $U \in \mathcal{U}$ containing p is called an open neighborhood of p.

removing pathological cases

The definition of topological spaces in the above is the best attempt by mathematicians to generalize the notion of continuity. However, this allows a few pathological cases. For example, for any set M, we can define a set $\mathcal{U} = \{\phi, M\}$. This satisfies all the properties (1)-(3) in the above. This is a funny topology. All the points in M are in the same neighborhood. To avoid such pathological cases, Hausdorff introduced an additional assumption. We say that a topological space M is Hausdorff iff, for any $p, q \in M$ $(p \neq q)$, we can find open neighborhood U, V $(p \in U, q \in V)$ such that $U \cap V = \phi$. This is to say that topology can separate two different points.

Often, we emploly one more assumption, the second countability assumption. It states that we can choose a countably many elements $U_{ii\in\mathbf{Z}}$ so that their unions and intersetions make all open sets in $\mathcal{U}$.

Question 1: Generalize the Weierstrass definition of continuous functions on $\mathbf{R}$

When we can choose $\{U_i \in \mathcal{U}\}_i$ so that there is a continuous one-to-one map $f_i : U_i \to \mathbf{R}^n$, we call M a topological manifold.

triangulation

A simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimension. A p-simplex is a p-dimensional polytope which is the convex hull of its $(p + 1)$ vertices. A simplicial complex K is a topological space of a particular kind, constructed by gluing them together. For a topological space M, if we can find a topologically equivalent simplicial complex K (namely, when there is a continuous map $K \to M$ that is one-to-one and onto), we call it a triangulation of M. We can also consider a smooth (C^∞) triangulation. That is a triangulation so that, for each simplex in K, the map gives its C^∞ embedding in M. (C^∞ means that it is differentiable infinite times.)

In order to study geometric objects, it is often convenient to use their triangulations. It is known that every C^∞ manifolds has a smooth triangulation.

homologies

A simplicial complex K contains simplexes of dimensions $0, 1, ..., n$. Each p-simplex σ has $(p+1)$ vertices $v_0, ..., v_p$. An orientation of the simplex is determined by an ordring of the vetices, modulo even permutations. So let us denote such an oriented simplex by $\langle\sigma\rangle = \langle v_0, ..., v_p\rangle$. We can consider a vector space with integer coefficients (free Abelian group) whose basis vectors are p-simplexes of K, and denote it by $C_p(K)$. If σ' is the same simplex as σ but with the opposite orientation, we write $\langle\sigma'\rangle = -\langle\sigma\rangle$. An element of $C_p(M)$ is a sum of simplexes with integer coefficients, and we call them p-chais. We can formally add or subtract p-chains in $C_p(K)$.

The boundary operator ∂ is a map from $C_p(K)$ to $C_{p-1}(K)$. On each simplex, it acts as

$$\partial\langle v_0, ..., v_p\rangle = \sum_{i=0}^{p}(-1)^i\langle v_0, ..., \hat{v}_i, ..., v_p\rangle, \tag{1}$$

where $\hat{v}_i$ means we remove v_i in the right-hand side. For example, the boundary of a triangle $\langle v_0, v_1, v_2\rangle$ is a sum of its 3 edges, $\langle v_1, v_2\rangle, \langle v_2, v_0\rangle, \langle v0, v_1\rangle$. Similarly, the boundary of a tetrahedron $\langle v_0, v_1, v_2, v_3\rangle$ is a sum of its four faces, $\langle v_1, v_2, v_3\rangle, ...$ We can define the action of ∂ on $C_p(K)$ by extending this linearly.

Question 2: Prove that the boundary operator is nilpotent, $\partial^2 = 0$.

Given this, we can define homologies just as we defined the de Rham cohomology. We introduce sets of p-dimensional cycles Z_k and boundaries B_k as follows,.

$$Z_p(K) = \{c \in C_p(K) : \partial c = 0\} \quad B_p(K) = \{\partial c : c \in C_{p+1}(K)\}. \tag{2}$$

Cycles are chains without boundaries. Homologies are defined as their quotients,

$$H_p(K) = Z_p(K)/B_p(K). \tag{3}$$

When M is topologically equivalent to K, we define $H_p(M) = H_p(K)$. The homologies we defined in the above is with integer coefficients, so often denoted by $H_p(M, \mathbf{Z})$. We can also consider $H_p(M, \mathbf{R})$ and $H_p(M, \mathbf{C})$. There are simple relations such as $H_p(M, \mathbf{R}) = H_p(M, \mathbf{Z}) \otimes \mathbf{R}$. Thus, these homologies are essentially the same, except for finite groups possibily in $H_p(M, \mathbf{Z})$. These finite groups, if exist, are called torsions.

Question 3: Show that $H_0(M, \mathbf{R}) = \mathbf{R}$ for any connected manifold M.

For the torus T^2, which we can think of a surface of a doughnut, we have $H_1(T^2, \mathbf{R}) = \mathbf{R} \oplus \mathbf{R}$ and $H_2(T^2, \mathbf{R}) = \mathbf{R}$.

As an example of a torsion, we can consider the group $SO(3)$. As a manifold, $SO(3)$ is equivalent to S^3 with antipodal points identified. This can be used to show $H_1(SO(3), \mathbf{Z}) = \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$, while $H_1(SO(3), \mathbf{R}) = 0$.

simplicial cohomologies, de Rham cohomologies

Simplicial cohomologies are dual spaces of homologies $H_p(M, \mathbf{R})$ of M. The fundamental theorems of de Rham implies that it is isomorphic to the de Rham cohomologies $H^p(M)$. To explain the theorems, we note that there is a natural inner product between a p-cycle c in $Z_p(M)$ and closed p-forms ω in $Z^p(M)$ as follows.

$$(c, \omega) = \int_c \omega. \tag{4}$$

Suppose γ is a p-chain, but is not necessarily a cycle (∂c may not be zero). The Stokes theorem says,

$$\int_\gamma d\lambda = \int_{\partial\gamma} \lambda. \tag{5}$$

Therefore, if $c \in Z_p(M)$ and $\omega \in Z^p(M)$, then for any $(p+1)$-chain γ and any $(p-1)$-form λ,

$$\begin{aligned} (c, \omega + d\lambda) &= (c, \omega) + \int_{\partial c} \lambda = (c, \omega), \quad (\text{since } \partial c = 0) \\ (c + \partial\gamma, \omega) &= (c, \omega) + \int_\gamma d\omega = (c, \omega) \quad (\text{since } d\omega = 0). \end{aligned} \tag{6}$$

Therefore, (c, ω) is independent of the choice of representatives of the equivalence classes $H_p(M)$ and $H^p(M)$ and defines a pairing between them. The pairing (c, ω) is also called a period integral.

Now, we are ready to state the theorems of de Rham when M is a compact manifold without boundary. Let us choose a basis $\{c^i\}$ $(i = 1, ..., \dim H_p)$ of $H_p(M)$.

Theorem 1: For any set of integers ν_i, there is a closed p-form ω such that

$$\int_{c^i} \omega = \nu_i. \tag{7}$$

Theorem 2: If all the period of a p-form ω are zero,

$$\int_{c^i} \omega = 0, \tag{8}$$

then ω must be exact (*i.e.* $= 0$ in $H^p(M)$). This means that, if $\{\omega_i\}$ is a basis of $H^p(M)$, the period matrix π_{ij} defined,

$$\pi_{ij} = \int_{c^i} \omega_j, \tag{9}$$

is invertible.

In particular, $\dim H_p(M) = \dim H^p(M) = b_p$, the Betti numbers.

Poincaré duality

More generally, Poincaré duality states that $H^p(M)$ and $H^{n-p}(M)$ are dual to each.

Poincaré duality is easy to demonstrate if M is a Riemannian manifold and is compact, orientable, and without boundary. To show this, remember that representatives of $H^p(M)$ can be chosen as p-forms ω which satisfies

$$d\omega = 0, \quad \delta\omega = 0. \tag{10}$$

We have the Hodge dual operator $*$ which maps a p-form to an $(n-p)$-form. Since $\delta \sim *d*$ and since $** \sim 1$, we find that the Hodge operator maps $H^p(M)$ to $H^{n-p}(M)$ and vice versa. In particular, $H^n(M) = \mathbf{R}$ $(n = \dim M)$ since any $\omega \in H^n(M)$ can be written as being proportional to the volume form.

Lecture 7 *characteristic classes*

In the previous lectures, we have seen cases where fiber bundles are characterized by integers. For example, monopole bundles on S^2 are classified by the monopole number n which tells us how the $U(1)$ fibers over the upper hemisphere and the lower hemisphere are glued together. Generally speaking, for a vector bundle on a manifold M, a characteristic class associates a cohomology class of M.

Invariant Polynomials

Characteristic classes are constructed as polynomials of the curvature $F = dA + A \wedge A$. Under gauge transformation, F transforms as $F \to \Omega^{-1} F \Omega$, where Ω is a map from the manifold M to the gauge group (structure group) G. In the following, we consider the cases where $G = U(k)$ and $SO(2r)$. To construct characteristic classes, we need to introduce invariant polynomials of matrixes. We look for a function $P(X)$ of a matrix X that is invariant under the conjugation, $P(\Omega^{-1} X \Omega) = P(X)$. We consider two cases:

(1) X is a $k \times k$ hermitial matrix and $\Omega \in U(k)$. This will be used when E is a complex vector bundle.

(2) X being a $2r \times 2r$ real anti-symmetric matrix and $\Omega \in SO(2r)$. This will be used when E is a real vector bundle.

Examples of invariant polynomials are $\mathrm{tr} X^m$ $(m = 1, 2, ...)$ and $\det X$. In fact we can use these to construct a nice basis. The following two are particularly useuful (we are using the notation in the case of $G = U(k)$):

(1) $\sigma_i(X)$ defined by

$$\det(1 + tX) = 1 + t\sigma_1(X) + t^2 \sigma_2(X) + \cdots + t^k \sigma_k(X).$$

(2) $s_i(X)$ defined by

$$s_i(X) = \mathrm{tr} X^i, \quad (i = 1, ..., k).$$

They are related to each other by Newton's formula,

$$s_1 = \sigma_1, \ s_2 = \sigma_1^2 - \sigma_2, \ s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3, ...$$

We can also express the invariant polynomials in terms of eigenvalues. If X is a hermitian matrix, we can diagonalize it with eigenvalues $x_1, ..., x_k$. Then,

$$\prod_{i=1}^{k} (1 + t x_i) = 1 + t\sigma_1(x) + \cdots + t^k \sigma_k(x).$$

Similarly,

$$s_i(x) = \sum_{j=1}^{k} (x_j)^i.$$

Characteristic Classes

If $P_i(X)$ is an invariant polynomial of degree i, we can use the curvature 2-form F to define a $2i$-form $P_i(F)$. By construction, it is invariant under the gauge transformation, $F \to \Omega^{-1}F\Omega$. It is also a closed form. To see this, we first note that F satisfies the Bianchi identity,

$$dF + [A, F] = 0.$$

Question 1: Verfity the above identity. To show this, one has to use the Jacobi identity

$$[a, [b, c]] + \text{cyclic perm.} = 0.$$

Using this, we find

$$\begin{aligned} d\text{tr}F^i &= \text{tr}\left\{dFF^{i-1} + FdFF^{i-1} + \cdots F^{i-1}dF\right\} \\ &= \text{tr}\left\{(dF + [A, F])F^{i-1} + \cdots + F^{i-1}(dF + [A, F])\right\} \quad (1) \\ &= 0 \quad (2) \end{aligned}$$

Thus, $P_i(F)$ is potentially a non-trivial element of $H^{2i}(M)$.

Moreover, $P_i(F)$ is invariant under continuos deformation of the gauge field A as an element of $H^{2i}(M)$. Suppsoe we change $A \to A + \eta$ with η being an infinitesimal one-form. Note that, although A transforms inhomogeneously as $A \to \Omega^{-1}A\Omega + \Omega^{-1}d\Omega$, the one-form transforms homogeneously, $\eta \to \Omega^{-1}\eta\Omega$.

Under this deformation, F changes by $\delta F = d\eta + [A, \eta]$. Therefore,

$$\begin{aligned} \delta\text{tr}F^i &= \text{tr}\left\{(d\eta + [A, \eta])F^{i-1} + \cdots + F^{i-1}(d\eta + [A, \eta])\right\} \\ &= i\text{tr}\left\{(d\eta + [A, \eta])F^{i-1}\right\} \\ &= i\text{tr}\left\{d\eta F^{i-1} + \eta dFF^{i-2} \cdots + \eta F^{i-2}dF\right\} \quad (3) \\ &= id\text{tr}\left(\eta F^{i-1}\right). \quad (4) \end{aligned}$$

Since both η and F transform homogeneously under the gauge transformation, $\text{tr}(\eta F^{i-1})$ is a well-defiend $(2i-1)$-form. Thus, under any infinitesimal deformation, $P_i(F)$ changes by an exact form. Thus, $P_i(F)$ depends only on the type of the bundle E and not on a specific type of the connection A on E. For this reason, we sometime write a characteristic class as a function of E.

Chern classes, Chern characters, etc

Among characteristic classes for an $U(n)$ bundle, we have the Chern classes and the Chern characters.

The **Chern classes**, $c_i \in H^{2i}(M)$ $(i = 0, 1, 2..., k)$, are defined by

$$\det\left(1 + \frac{\sqrt{-1}}{2\pi}F\right) = c_0 + c_1 + c_2 + \cdots.$$

For example,

$$c_0 = 1,\ c_1 = \frac{\sqrt{-1}}{2\pi}\text{tr}F,\ c_2 = -\frac{1}{8\pi^2}\left(\text{tr}F \wedge \text{tr}F - \text{tr}F \wedge F\right),\ \cdots.$$

If the holonomy is in $SU(k) \in U(k)$, we have a trivial first Chern class $c_1 = 0$.

The sum $c = c_0 + c_1 + c_2 + \cdots$ is called the total Chern class. One of the important properties of the Chern classes is that it behaves nicely when we take a direct sum of vector bundles $E_1, E_2 \to E_1 \oplus E_2$ as,

$$c(E_1 \oplus E_2) = c(E_1) \wedge c(E_2).$$

On the other hand, it does not behave nicely under the direct product $E_1 \otimes E_2$.

The **Chern characters**, $ch_i(F) \in H^{2i}(M)$ $(i = 0, 1, 2, \cdots)$, are defined by

$$ch_i(F) = \frac{1}{i!}\mathrm{tr}\left(\frac{\sqrt{-1}F}{2\pi}\right)^i.$$

We can also write it as

$$ch(F) = ch_0 + ch_1 + \cdots = \mathrm{tr}\exp\left(\frac{\sqrt{-1}F}{2\pi}\right).$$

The Chern characters behave nicely under both the direct sum and direct product as,

$$\begin{aligned} ch(E_1 \oplus E_2) &= ch(E_1) + ch(E_2), \\ ch(E_1 \otimes E_2) &= ch(E_1) \wedge ch(E_2). \end{aligned} \tag{5}$$

Sometime we encounter other characteristic classes, such as Todd classes, Hirzebuch L-polynomials, and $\hat{A}$ polynomials. They correspond to different basis' of invariant polynomials. To desribe these, use eigenvalues $x_1, ..., x_k$ of $\frac{\sqrt{-1}}{2\pi}F$. For example, the total Chern classes can be expressed as

$$c(F) = \det\left(1 + \frac{\sqrt{-1}}{2\pi}F\right) = \prod_{i=1}^{k}(1 + x_i).$$

(Tangentially, it is interesting to note that the right-hand side takes the form $\prod_i c(L_i)$, where L_i is a line bundle with a curvatuve given by x_i and $c(L_i) = 1+x_i$. Thus, as far as the Chern classes are concerned, the bector bundle E behaves as a direct sum of the line bundles $L_1 \oplus L_2 \oplus \cdots \oplus L_k$. This phenomenon is called the splitting principle.)

Using this notation, the Todd classes are defined by,

$$td(E) = \prod_i \frac{x_i}{1 - e^{-x_i}},$$

the L-polynomials are defined by,

$$L(E) = \prod_i \frac{x_i}{\tanh x_i},$$

and the $\hat{A}$-polynomials are defined by,

$$\hat{A}(E) = \prod_i \frac{x_i/2}{\sinh(x_i/2)}.$$

Chern number

Remarkably, the Chern classes and the Chern characters are integral. That means that if we integrate, say, $c_i(E)$ over any $2i$-cycle in M with integer coefficients, we find an integer that is independent of the choice of the connection of E. If $2k \geq n$, we can integrate $c_n(F)$ over the entire manifold M and get the Chern number. Let us compute Chern numbers in some examples.

(1) Consider the monopole bundle over S^2. It has the $U(1)$ gauge field A. Let us denote the northen and southern hemispheres of S^2 as $H_\pm$, and the gauge fields on them as $A_\pm$. For the monopole bundle with n monopole charge, the gauge field transforms as

$$A_- = A_+ + nd\phi$$

across the equator, where ϕ is the longitude of S^2. We can then evaluate the Chern number as

$$\begin{aligned} C_1 &= \int_{S^2} c_1 \\ &= \frac{-1}{2\pi}\left(\int_{H_+} A_+ + \int_{H_-} A_-\right) \\ &= \frac{-1}{2\pi}\int_{S^1}(A_+ - A_-) = \frac{1}{2\pi}\int_0^{2\pi} nd\phi = n. \end{aligned} \tag{6}$$

Thus, the monopole number is the first Chern number in this case.

(2) Consider an $SU(2)$ bundle over S^4. We can then consider the second Chern number,

$$C_2 = \int_{S^4} c_2 = \frac{1}{8\pi^2}\int_{S^4} \mathrm{tr} F \wedge F.$$

We again split S^4 into $H_\pm$ such that $H_+ \cap H_- = S^3$. Over S^3, the gauge field transforms as

$$A_- = \Omega^{-1} A_+ \Omega + \Omega^{-1} d\Omega.$$

When we integrate $\mathrm{tr} F_\pm \wedge F_\pm$ over $H_\pm$, we note that the integrand can be written as

$$\mathrm{tr} F \wedge F = d\mathrm{tr}\left(AdA + \frac{2}{3}A^3\right).$$

Note that this does not mean that $\mathrm{tr} F \wedge F$ is an exact form since the right-hand side, called the Chern-Simons form, is not necessarily globally defined over S^4. Thus,

$$\begin{aligned} C_2 &= \int_{S^4} c_2 \\ &= \frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}\left(A_+ dA_+ + \frac{2}{3}A_+^3 - A_- dA_- - \frac{2}{3}A_-^3\right) \\ &= \frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}(\Omega^{-1}d\Omega)^3. \end{aligned} \tag{7}$$

The gauge transformation matrix Ω is a map from S^3 to $SU(2)$. Since the group $SU(2)$ is diffeomorphic to S^3 as a manifold, we can think of it as a map from S^3 to itself. Such a map

can be classified by its winding number, which turns out to be the same as the second Chern number in the above.

Euler class, Poltryagn classes

Let us turn to real vector bundles. Suppose X is a $2r \times 2r$ real and anti-symmetrix matrix. In this case, in addtion to tr and det, we can consider one more way to construct an invariant polynomial. That is the Pfaffian,

$$Pf(X) = \frac{(-1)^r}{2^r r!} \epsilon^{i_1 j_1 i_2 j_2 \cdots i_r j_r} X_{i_1 j_1} X_{i_2 j_2} \cdots X_{i_r j_r}.$$

Note that, for antisymmetric matrices, the Pfaffian is a square root of the determinant,

$$\det X = Pf(X)^2.$$

If X is real and anti-symmetric, we can block-diagonarize it by $SO(2r)$ as

$$X = \begin{pmatrix} 0 & x_1 & 0 & 0 & \cdots & 0 & 0 \\ -x_1 & 0 & 0 & 0 & \cdots & \cdot & 0 \\ 0 & 0 & 0 & x_2 & & & \\ 0 & 0 & -x_2 & 0 & & & \\ \cdot & \cdot & & & \cdot & & \\ 0 & \cdot & & & & 0 & x_r \\ 0 & 0 & & & & -x_r & 0 \end{pmatrix}$$

We can then write the Pfaffian as

$$Pf(X) = (-1)^r \prod_{i=1}^{r} x_i.$$

Under the conjugation $X \to \Omega^t X \Omega$, the Pfaffian transforms as

$$Pf(\Omega^t X \Omega) = \det\Omega \cdot Pf(X).$$

Thus, if $\Omega \in SO(2r)$, the Pfaffian is invariant. (Note that if $\Omega \in O(2r)$, the Pfaffian may change its sign.)

We can now define the Euler class by $e(F) = Pf(F)$.

In particular, the tangent bundle TM of an orientable Riemannian manifold M of dimensions $n = 2r$ is an $SO(2r)$ bundle. For example,

$$\begin{aligned} n = 2: \quad e(TM) &= \frac{1}{2\pi} R_{12}, \\ n = 4: \quad e(TM) &= \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} R^{\mu\nu} \wedge R^{\rho\sigma}, \end{aligned} \tag{8}$$

where the Riemann curvature is regarded as the 2-form as,

$$R^a{}_b = \frac{1}{2} R_{cd}{}^a{}_b e^c \wedge e^d = \frac{1}{2} R_{\mu\nu}{}^a{}_b dx^\mu \wedge dx^\nu.$$

The Gauss-Bonnet theorem for an even-dimensional manifold M relates the Euler characteristic $\chi(M) = \sum_p (-1)^p b_p$ to the Euler class by

$$\chi(M) = \int_M e(TM).$$

The Pontryagin classes are defined similarly to the Chern classes as

$$p(E) = 1 + p_1(E) + p_2(E) + \cdots = \det\left(1 - \frac{1}{2\pi}F\right).$$

Since F is anti-symmetric, we only have nontrivial polynomials with even degrees in F. Thus, we choose $p_i(E)$ to be a $4i$-form. The highest Pontryagin class is at $i = r$ where $2r$ is the dimension of the fiber (unless $2r > n = \dim M$). At this highest degree, it is the square of the Euler class,

$$p_r(E) = e(E)^2.$$

Lecture 8

supersymmetry and index theorems

bosonic sigma model

Let us consider a dynamical system describing a motion of a particle in a Riemannian manifold M. The motion is a map $\phi : \mathbf{R} \to M$. By choosing coordinates $(\phi^\mu)_{\mu=1,\dots,n}$ on M, we can write the Lagrangian,

$$L = \frac{1}{2} g_{\mu\nu}(\phi)\dot{\phi}^\mu \dot{\phi}^\nu,$$

where $g_{\mu\nu}$ is the metric on M and $\dot{\phi} = d\phi/dt$. Note that this is independent of the choice of coordinates. The momentum conjugate to ϕ^μ is given by

$$p_\mu = \frac{\partial L}{\partial \dot{\phi}^\mu} = g_{\mu\nu}\dot{\phi}^\nu.$$

The Hamiltonian is then,

$$H = p_\mu \dot{\phi}^\mu - L = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu.$$

To quantize this system, we consider the Hilbert space consisting of square integrable functions $\Psi(\phi)$ over M. We can then define the inner product,

$$(\Psi_1, \Psi_2) = \int d^n\phi \sqrt{g} \bar{\Psi}_1 \Psi_2.$$

The multiplication of the coordinates ϕ^μ defines hermitian operators. We can also define the momentum components p_μ as hermitian operators as

$$p_\mu = -\sqrt{-1} g^{-\frac{1}{4}} \frac{\partial}{\partial \phi^\mu} g^{\frac{1}{4}},$$

so that

$$[\phi^\mu, p_\nu] = \sqrt{-1}\delta^\mu_\nu.$$

Similarly, there is some operator ordering ambiguity in defining the quantum version of the H. Requiring invariance under coordiante change, we use

$$H = -\frac{1}{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \phi^\mu}\left(g^{\mu\nu}\sqrt{g}\frac{\partial}{\partial \phi^\nu}\right).$$

This is nothing but the (minus of) Laplace operator on M.

The Hamiltonian H can be used to write the Schrödinger equation,

$$i\frac{\partial}{\partial t}\Psi = H\Psi,$$

which can be formally integrated as

$$\Psi(t) = e^{-itH}\Psi(t=0).$$

For our purpose, it is often convenient to analytically continue the time variable t to pure imaginary value $t = -i\tau$ with $\tau \in \mathbf{R}$. Thus,

$$\Psi(\phi, \tau) = e^{-\tau H}\Psi(\phi, \tau = 0).$$

We can introduce the heat kernel $G(\phi_1, \phi_2; \tau)$ to express this as

$$\Psi(\phi_1, \tau) = \int d^n\phi_2 G(\phi_1, \phi_2; \tau)\Psi(\phi_2).$$

The heat kernel obeys the Schrödinger equation (or more precisely the diffusion equation),

$$\frac{\partial}{\partial\tau}G = -HG$$

with the initial condition,

$$G(\phi_1, \phi_2; \tau = 0) = \delta(\phi_1 - \phi_2).$$

To understand the energy spectrum $\{\epsilon_0, \epsilon_1, ...\}$ of the Hamiltonian, it is useful if we can evaluate $\mathrm{tr}e^{-\tau H}$ as it gives,

$$\mathrm{tr}e^{-\tau H} = \sum_{i=0}^{\infty} e^{-\tau\epsilon_i},$$

where tr is over the Hilbert space (*i.e.* the space of square integrable functions over M). Using the heat kernel, we can write it as

$$\mathrm{tr}e^{-\tau H} = \int d^n\phi G(\phi, \phi; \tau).$$

In the Feynman path integral formulation of quantum mechanics, the heat kernel $G(\phi, \phi'; \tau)$ is expressed as

$$G(\phi_1, \phi_2; \tau) = \int_{\phi(\tau)=\phi_1;\phi(0)=\phi_2} \mathcal{D}\phi \exp\left(-\int_0^\tau d\tau' L(\dot{\phi}(\tau'), \phi(\tau'))\right).$$

Combining this with the above expression for $\mathrm{tr}e^{-\tau H}$, we find that

$$\mathrm{tr}e^{-\tau H} = \int_{\phi(\tau)=\phi(0)} \mathcal{D}\phi e^{-\int_0^\tau d\tau' L}.$$

supersymmetric sigma-model

We introduce fermionic coordinates ψ^μ for the tangent space $T_\phi M$ at ϕ. The Lagrangian

$$\begin{aligned} L &= \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{\sqrt{-1}}{2}g_{\mu\nu}\bar{\psi}^\mu\left(\frac{d}{dt}\psi^\nu + \Gamma^\nu_{\rho\sigma}\dot{\phi}^\rho\psi^\sigma\right) \\ &\quad + \frac{1}{4}R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma. \end{aligned} \tag{1}$$

As I advertised earlier, the Hilbert space is the space of differential forms $\Omega(M)$ (we require them to be integrable with respect to the inner product on $\Omega(M)$). The fermions obey the anti-commutation relations,

$$\{\psi^\mu, \psi^\nu\} = 0,\ \{\bar{\psi}^\mu, \bar{\psi}^\nu\} = 0,\ \{\psi^\mu, \bar{\psi}^\nu\} = g^{\mu\nu}.$$

The vacuum $|0\rangle$ in the ferimion Fock space is annihilated by all the ψ's. We identify it as the 0-form. Other forms are generated by multiplying $\bar{\psi}$'s, as $\bar{\psi}^\mu|0\rangle, \bar{\psi}^\mu\bar{\psi}^\nu|0\rangle, ...$

The supercharges $Q, \bar{Q}$ are identified with the exterior derivative operator d and its conjugate $\delta \sim *d*$, so that $H = \{Q, \bar{Q}\}$ gives the Laplace-Beltrami operator.

Witten index

Let F be the operator that counts the degree of forms. We can also call it as the fermion number. We can use it to define the fermion number parity $(-1)^F$, which gives $+1$ for bosonic states (even forms) and -1 for fermoinic states (odd forms). Note that $(-1)^F Q = -Q(-1)^F$ and similarly for $\bar{Q}$. Since Q and $\bar{Q}$ commute with H, it maps a bosonic energy eigenstate to a fermionic energy eigenstate and vice versa. Moreover, since $H = \{Q, \bar{Q}\}$, the map is bijection (one-to-one and onto) for states with non-zero energies. The situation is different for states with $H = 0$; the numbers of bosonic and fermionic states may be different. The difference is called the Witten index. The Witten index is a convenient quantity since it is invariant under continous deformations of the system, provided that the spectrum does not contain continuous bands.

For the supersymmetric sigma model we are discussing, the Witten index is given by

$$\mathrm{tr}(-1)^F e^{-\beta H} = \sum_{p=0}^{n} (-1)^p b_p = \chi(M).$$

Grassmannian integral

Let us evaluate $\chi(M)$ using the Feynman path integral. To do so, we need to formulate the path integral for the fermionic variables $\psi, \bar{\psi}$. This is done by introducing Grassmannian numbers.

A Grassmannian number θ is nilpotent $\theta^2 = 0$, and its integral is defined like a differentiation,

$$\int d\theta 1 = 0, \quad \int d\theta\theta = 1.$$

More generally, if we have a function $f(\theta) = a + b\theta$, wehre a and b are ordinary numbers (since $\theta^2 = 0$, the most general function is a linear function), we have

$$\int d\theta f(\theta) = b.$$

A nice thing about this definition is that we can perform the integration by parts,

$$\int d\theta \frac{\partial}{\partial\theta} f(\theta) = \int d\theta b = 0.$$

Similarly, we an show that

$$\int d\theta f(\theta) \frac{\partial}{\partial\theta} g(\theta) = \pm \int d\theta \left[\frac{\partial}{\partial\theta} f(\theta)\right] g(\theta),$$

where the sign on the right-hand side depends on whether $b\theta = \theta b$ (choose $+$) or $b\theta = -\theta b$ (choose $-$).

Question 1: For a bosonic variable x, achange of variables $x \to y$ with $x = f(y)$ transforms an integral as

$$\int dx G(x) = \int dy |f'(y)| G(f(y)).$$

Show that, for a Grassmannian variable θ, a change of variables $\theta \to \rho$ with $\theta = f(\rho)$ gives

$$\int d\theta G(\theta) = \int d\rho \frac{1}{f'(\rho)} G(f(\rho)).$$

This definition fits well with the interpretation of the supersymmetric sigma-model in terms of differential forms. Consider the top form $\omega \in \Omega^n(M)$. We can write the integral of ω over M as,

$$\int_M \omega = \int d^n\phi d^n\bar{\psi} \omega_{\mu_1\mu_2\ldots\mu_n}(\phi)\bar{\psi}^{\mu_1}\bar{\psi}^{\mu_2}\cdots\bar{\psi}^{\mu_n}.$$

Question 2: Use the result of Question 1 to show that the right-hand side of the above is independent of the choice of coordinates ϕ^μ.

Gauss-Bonnet theorem

Now we are ready to evaluate the Witten index $\mathrm{tr}(-1)^F e^{-\beta H}$ using the Feynman path integral. As in the computation $\mathrm{tr}e^{-\beta H}$ in the case of the bosonic sigma-model, we consider a sum over periodic paths, but now in TM not in M.

It is important to note that we require periodic boundary condtion for ψ and $\bar{\psi}$ as $\tau \to \tau + \beta$ because of the insertion of $(-1)^F$. To see this, let us look at the following quantity, $\mathrm{tr}(-1)^F e^{-\beta H}\mathcal{O}\psi(0)$ for some operator $\mathcal{O}$. By using the cyclicity of the trace, we see that this is equal to $-\mathrm{tr}(-1)^F e^{-\beta H}\psi(\beta)\mathcal{O}$, where the minus sign comes from the exchange of ψ with $(-1)^F$. However, we should also note that $\mathcal{O}$ must be a fermionic operator – otherwise the trace would vanish. Thus, we get an extra minus sign when we exchange ψ and $\mathcal{O}$. The end result is that $\psi(0) = \psi(\beta)$ in the trace.

Now we are ready to evaluate the path integral, with periodic boundary conditions for both ϕ and $\psi, \bar{\psi}$. The main idea is use of the fact that the answer is independent of β and look at the limit of $\beta \to 0+$. Define $t = \beta\tau$ keeping the periodicity of τ to be 1. Rescale $\psi \to \beta^{-1/4}\psi$, and we obtain,

$$L = \frac{1}{2\beta^2} g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{\sqrt{-1}}{2\beta^{\frac{3}{2}}} g_{\mu\nu}\bar{\psi}^\mu\left(\frac{d}{dt}\psi^\nu + \Gamma^\nu_{\rho\sigma}\dot{\phi}^\rho\psi^\sigma\right) + \frac{1}{4\beta} R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma.$$

The action is then

$$\begin{aligned} S &= \beta\int_0^1 d\tau L \\ &= \iint_0^1 d\tau\Big[\frac{1}{2\beta} g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{\sqrt{-1}}{2\beta^{\frac{1}{2}}} g_{\mu\nu}\bar{\psi}^\mu\left(\frac{d}{dt}\psi^\nu + \Gamma^\nu_{\rho\sigma}\dot{\phi}^\rho\psi^\sigma\right) + \frac{1}{4} R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma\Big]. \quad (2) \end{aligned}$$

In the limit of $\beta \to 0$, the only configurations that can contribute to the path integral are are those with ϕ and ψ being constant since any non-constant configuration would make the action

infinite in the limit. Thus, the path integral reduces to an integral over constant ϕ and ψ as,

$$\mathrm{tr}(-1)^f e^{-\beta H} = \int \frac{d^n\phi d^n\psi d^n\bar{\psi}}{(2\pi)^{n/2}} e^{-\frac{1}{4}R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma}.$$

This would be 0 if $n = \dim M$ is odd. If $n = 2m$, we can evaluate the Grassmannian integral to find

$$= \frac{(-1)^m}{2^{2m}m!\pi^m}\int d^n\phi\sqrt{g}\epsilon^{\mu_1\nu_1\cdots\mu_m\nu_n}\epsilon^{\rho_1\sigma_1\cdots\rho_m\sigma_m}R_{\mu_1\nu_1\rho_1\sigma_1}\cdots R_{\mu_m\nu_m\rho_m\sigma_m}.$$

This gives a path integral proof of the Gauss-Bonnet theorem.

Morse theory

We can also turn on a potential $h(\phi)$ in a supersymmetric way,

$$\Delta L = -\frac{1}{2}g^{\mu\nu}\frac{\partial h}{\partial\phi^\mu}\frac{\partial h}{\partial\phi^\nu} - \frac{1}{2}\frac{\partial^2 h}{\partial\phi^\mu\partial\phi^\nu}\psi^\mu\psi^\nu.$$

In this case, the path integral localizes at teh minima of $|\partial h/\partial\phi|^2$, namely at $\partial h/\partial\phi^\mu = 0$. By using a scaling argument similar to the one in the above, we find

$$\mathrm{tr}(-1)^F e^{-\beta H} = \sum_{\phi_0:\partial h(\phi_0)=0}(-1)^{\#(\text{negative eigenvalues of }\partial^2 h)}.$$

In fact, to the leading order in the $\hbar$ expansion, we find

$$b_p = \dim H^p(M) = \#\left\{\phi_0 : \frac{\partial h}{\partial\phi^\mu} = 0,\ \#(\text{negative eigenvalues}) = p\right\}.$$

This can be refined by incorporating instanton corrections.

Dirac operator

In the supersymmetric sigma-model discussed in the above, there are two supercharges Q and $\bar{Q}$. We can reduce the amount of supersymmetry by half, by setting $\bar{\psi} = \psi$. In this case, we have only $Q = \bar{Q}$. The Lagrangian is simplified as

$$L = \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{\sqrt{-1}}{2}g_{\mu\nu}\psi^\mu D_t\psi^\nu.$$

The quantization gives the anti-commutatition relation,

$$\{\psi^\mu, \psi^\nu\} = g^{\mu\nu}.$$

This is the Clifford algebra. It is known that the Clifford algebra has a unique non-trivial irreducible representation and that is the representation in terms of the Dirac matrices γ^μ. Thus, the Hilbert space of this model is the space of spinor fields on M.

In this case, $Q = \gamma^\mu D_\mu$.

The Witten index in this case is called the Dirac genus. The sigma-model path integral shows that it is given by the $\hat{A}$ polynomial,

$$\mathrm{ind}(\gamma^\mu D_\mu) = \int_M \hat{A},$$

where

$$\hat{A} = \prod_{i=1}^{m} \frac{x_i/2}{\sinh(x_i/2)},$$

and x_i are eigenvalues of the Riemann curvature ($i = 1, ..., m; \dim M = 2m$).

Dolbeault index

If M is a Kähler manifold, we can double the number of supercharges, $\partial, \bar{\partial}, \partial^\dagger, \bar{\partial}^\dagger$. The arithmetic genus of the manifold is defined by

$$\text{index}(\bar{\partial}) = \sum_q (-1)^q \dim H_{\bar{\partial}}^{(0,q)}(M).$$

The path integral computation is exactly the same as that for the Dirac genus, and it relates the index to the Todd class,

$$td(M) = \prod_i \frac{x_i}{1 - e^{-x_i}}.$$

These index theorems are interesting in that they relate global quantitizes such as the Euler characteristic to integrals of curvatures (local quantities).

Lecture 9

Riemann surfaces, elliptic functions

Laplace equation

Consider a Riemannian metric $g_{\mu\nu}$ in two dimensions. In two dimensions it is always possible to choose coordinates (x, y) to diagonalize it as $ds^2 = \Omega(x, y)(dx^2 + dy^2)$. We can then combine them into a complex combination $z = x+iy$ to write this as $ds^2 = \Omega dz d\bar{z}$. It is actually a Kähler metric since the condition $\partial_{[i,}g_{j]\bar{k}} = 0$ is trivial if $i, j = 1$. Thus, an orientable Riemannian manifold in two dimensions is always Kähler.

In the diagonalized form of the metric, the Laplace operator is of the form,

$$\Delta = 4\Omega^{-1}\partial_z\bar{\partial}_{\bar{z}}.$$

Thus, any solution to the Laplace equation $\Delta\phi = 0$ can be expressed as a sum of a holomorphic and an anti-holomorphid function.

$$\Delta\phi = 0 \ \rightarrow \ \phi = f(z) + \bar{f}(\bar{z}).$$

In the following, we assume $\Omega = 1$ so that the metric is $ds^2 = dzd\bar{z}$. It is not difficult to generalize our results for non-constant Ω.

Now, we would like to prove the following formula,

$$\bar{\partial}\frac{1}{z} = -\pi\delta(z),$$

where $\delta(z) = \delta(x)\delta(y)$. Since $1/z$ is holomorphic except at $z = 0$, the left-hand side should vanish except at $z = 0$. On the other hand, by the Stokes theorem, the integral of the left-hand side on a disk of radius r gives,

$$\int_{x^2+y^2\leq r^2} dxdy\ \bar{\partial}\frac{1}{z} = \frac{i}{2}\oint_{|z|=r}\frac{dz}{z} = -\pi.$$

This proves the formula. Thus, the Green function $G(z, w)$ obeying

$$\Delta_z G(z, w) = 4\pi\delta(z - w),$$

should behave as

$$G(z, w) = -\log|z - w|^2 = -\log(z - w) - \log(\bar{z} - \bar{w}),$$

near $z = w$. (In this lecture, log = ln.) Note that the right-hand side is a sum of a holonorphic function and an anti-holomrophic function.

Rational, trigonometric, elliptic

Let us start with the simplest topology in two dimensions – the plane. On the Euclidean plane, $G(z, w) = \log|z - w|^2$ is a solution with the delta-function source at $z = w$. We can add a purely holomorphic or anti-holomorphic function to this, and it still solves the same equation. Compared to three or higher dimensions, none of solutions decay at inifinity. For example, the

corresponding solution in three dimensions decays at $1/r$. In n-dimensions, it would be like $1/r^{n-2}$. The limit $n \to 2$ of this gives the logarithmis singularity $\log r$. This is a deep fact and implies, for example, Coleman's theorem that there is no spontaneous breaking of continuous symmetry in quantum field theory in two dimensions.

The next simplest case is the cylinder. Let us periodically identify the x direction with the period 1. We then look for the Green function with the periodicity, $G(z+1,w) = G(z,w+1) = G(z,w)$. This can be satisfied by

$$G(z,w) = -\log|\sin 2\pi(z-w)|^2 .$$

If we periodically identify the cylinde in the y direction, we obtain the torus – the surface of a doughnut. More generally we may twist the cylinder before identification. So, we can impose $(x,y) \sim (x+\theta, y+\beta)$ in addition to $(x,y) \sim (x+1,y)$. In terms of the complex coordinate z, we have

$$z \sim z+1 \sim z+\tau, \quad \tau = \theta + i\beta.$$

Without loss of generality, we assume that β is positive. We can think of the torus as the parallelogram with four vertices $0, 1, \tau, \tau+1$ on the complex planes with its parallel edges identified pair-wisely. The area of the torus is $\beta = \mathrm{Im}\ \tau$.

To find the Green function on the torus, we may consider using the method of images and sum the Green function $G(z,w) = -\log|\sin[2\pi(z-w)]|^2$ over $w \to w + n\tau$ ($n \in \mathbf{Z}$). However, the sum is not convergent. In fact, we can anticipate the problem by noting that the equation,

$$\Delta G(z,w) = \delta(z-w),$$

cannot be compatible with the fact that the torus is compact and without boundary. Integrating both hand side over the torus, the right-hand side gives 1 but the left-hand side would vanish by integration by parts. In order for the Laplace equation to have a solution on a compact space without boundary, the total charge must be zero since the electric flux generated at the source has to go somewhere. To remedy this, we can add a constant negative charge density to cancel the positive charge at $z = w$,

$$\Delta G(z,w) = \delta(z-w) - \frac{1}{\mathrm{Im}\tau}.$$

A solution to this equation is unique up to an additional constant, and it is given by,

$$G(z,w) = -\frac{1}{4\pi}\log\left|\frac{\vartheta_1(z-w|\tau)}{\eta(\tau)}\right| + \frac{1}{2}\frac{(\mathrm{Im}(z-w))^2}{\mathrm{Im}\tau}.$$

Here ϑ_1 is one of the four elliptic theta functions and is given by

$$\begin{aligned}\vartheta_1(z|w) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} e^{2\pi i\left(n-\frac{1}{2}\right)z} \\ &= 2\sum_{n=1}^{\infty}(-1)^{n-1} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}\sin(2n-1)\pi z, \end{aligned} \tag{1}$$

where

$$q = e^{2\pi i\tau}.$$

By construction, it satisfies the quasi-periodicity,

$$\vartheta(z+1|\tau) = -\vartheta_1(z|\tau), \quad \vartheta_1(z+\tau|\tau) = -q^{-\frac{1}{8}}e^{-2\pi iz}\vartheta_1(z|\tau).$$

To see that it has a series of zeros at $z = n + m\tau$ $(n, m \in \mathbf{Z})$, it is useful to use the product formula,

$$\vartheta_1(z|\tau) = -iq^{\frac{1}{12}}e^{\pi iz}\eta(\tau)\prod_{n=1}^{\infty}(1-q^n e^{2\pi iz})(1-q^{n-1}e^{-2\pi iz}),$$

where $\eta(\tau)$ is the Dedekint eta-function,

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n).$$

There are three other theta functions,

$$\begin{aligned}
\vartheta_0(z|\tau) &= q^{-\frac{1}{24}}\eta(\tau)\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}e^{2\pi iz})(1-q^{n-\frac{1}{2}}e^{-2\pi iz}),\\
\vartheta_2(z|\tau) &= q^{\frac{1}{12}}e^{\pi iz}\eta(\tau)\prod_{n=1}^{\infty}(1+q^n e^{2\pi iz})(1+q^{n-1}e^{-2\pi iz}),\\
\vartheta_3(z|\tau) &= q^{-\frac{1}{24}}\eta(\tau)\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}e^{2\pi iz})(1+q^{n-\frac{1}{2}}e^{-2\pi iz}). \qquad (2)
\end{aligned}$$

In particular, ϑ_3 is called the Jacobi theta function. The other three theta functions are obtained from the Jacobi theta function by shifting z by $1/2$ and $\tau/2$.

Modular invariance

We have regarded the torus as the quotient of $\mathbf{C}$ with the metric $ds^2 = dzd\bar{z}$ by the lattice $\{n + m\tau : n, m \in \mathbf{Z}\}$. Equivalently, we can consider a fixed lattice and modify the metric as a function of τ. Consider coordinates (u, v) on $\mathbf{R}^2$ with the periodicity $(u, v) \sim (u + n, v + m)$ $(n, m \in \mathbf{Z})$. Namely, we are considering the square lattice. To reproduce the periodicity $z \sim z + 1 \sim z + \tau$, we can write $z = u + \tau v$. The metric in terms of the coordinates (u, v) then becomes

$$ds^2 = |du + \tau dv|^2.$$

Physics should be independent of the choice of coordinates we make. We can change coordinates as

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

This does not change the lattice structure if the matrix is in $SL(2, \mathbf{Z})$. Consider two examples of $SL(2, \mathbf{Z})$ transformations:

(1) $(u, v) \to (u + v, v)$: This changes the metric as

$$ds^2 = |du + (\tau + 1)dv|^2.$$

This means that the metrics with τ and $(\tau + 1)$ are related to each other by the coordinate transformation.

(2) $(u, v) \to (v, -u)$: This changes the metric

$$ds^2 = |\tau|^2 \left| du - \frac{1}{\tau} dv \right|^2 .$$

This means that the metrics with τ and $-1/\tau$ are related to each other by the coordinate transformation adn the overall rescaling. The Laplace equation is invariant under the rescaling of the metric, so its solution must have this symmetry too.

Question 1: Show that, in two dimensions, the Laplace equation with the unit source, $\Delta G(z, w) = \delta(z - w)$ is invariant under an arbitrary rescaling of the metric $g_{\mu\nu} \to \Omega(z, \bar{z}) g_{\mu\nu}$. Here the delta-function is normalized with respect to the volume form,

$$\int \sqrt{g}\ \delta(z - w) = 1.$$

It is known that the two transformation $(u, v) \to (u + v, v)$ and $(v, -u)$ generate the whole $SL(2, \mathbf{Z})$ group. Its action on τ can be seen as

$$\tau \to \frac{a\tau + b}{c\tau + d}.$$

This is called the modular transformation. As expected, the ingredients of the Green function on the torus transforms nicely under the modular transformation as,

$$\eta(\tau + 1) = e^{\frac{2\pi i}{24}} \eta(\tau), \quad \eta\left(-1/\tau\right) = \sqrt{-i\tau}\eta(\tau),$$

and

$$\vartheta_1(z|\tau + 1) = e^{\frac{2\pi i}{8}} \vartheta_1(z|\tau), \quad \vartheta_1\left(z/\tau| - 1/\tau\right) = e^{-\frac{2\pi i}{8} + \frac{2\pi i z^2}{2\tau}} \tau^{-\frac{1}{2}} \vartheta_1(z|\tau).$$

The parameter τ specifies the complex structure of torus as it determines its complex coordinate $z = u + \tau v$ with respect to the fixed real coordinates (u, v). Two different τ's related to each other under the modular transformation describe the same complex structure. The space of complex structures on the torus is called the moduli space of the torus, and it can be identifies as the upper half-plane of the τ-space, modulo $SL(2, \mathbf{Z})$.

Elliptic integral

With the elliptic theta function, we can define Weirstrass' elliptic function as

$$\mathcal{P}(z) = -\frac{\partial^2}{\partial z^2} \log \vartheta_1(z|\tau) - 2\eta_1(\tau),$$

where

$$\eta_1(\tau) = 2\pi i \frac{\partial}{\partial \tau} \log \eta(\tau).$$

This function has a double pole at $z = 0$ and is doubly-periodic,

$$\mathcal{P}(z) = \mathcal{P}(z + 1) = \mathcal{P}(z + \tau).$$

Its derivative $\mathcal{P}'(z)$ has a triple pole at $z = 0$ and is also doubly-periodic. A doubly-periodic function on the torus is called an elliptic function. It can be shown that any elliptic function is a rational function of $\mathcal{P}(z)$ and $\mathcal{P}'s(z)$.

They satisfies the relation,

$$(\mathcal{P}'(z))^2 = 4(\mathcal{P}(z) - e_1)(\mathcal{P}(z) - e_2)(\mathcal{P}(z) - e_3),$$

where e_1, e_2, e_3 are some functions of τ, given by

$$e_1 = \mathcal{P}\left(\frac{1}{2}\right), \quad e_2 = \mathcal{P}\left(\frac{\tau}{2}\right), \quad e_3 = \mathcal{P}\left(\frac{1}{2} + \frac{\tau}{2}\right).$$

The above relation between $\mathcal{P}'$ and $\mathcal{P}$ means that $\mathcal{P}'$ is a function of $\mathcal{P}$ with branch points at $\mathcal{P} = e_1, e_2, e_3$. We can have two branch cuts, one connecting e_1 and e_2, for example, and another going from e_3 to the infinity. One can see that the covering space is topologically equivalent to the torus.

The equation can also be expressed as,

$$z = \int^{\mathcal{P}} \frac{d\mathcal{P}}{2\sqrt{(\mathcal{P}(z) - e_1)(\mathcal{P}(z) - e_2)(\mathcal{P}(z) - e_3)}} + \text{const.}$$

The integral on the right-hand side is of the form known as elliptic integral. Thus, the Weierstrass $\mathcal{P}$-function can be regarded as the inverse of the elliptic integral. It is because of this historical origin that doubly-periodic functions are called elliptic functions. Historically, elliptic integrals were studied to compute the arc length of an ellipse. It occured later to Abel and Jacobi that its inverse has the double-periodicity $z \sim z + 1 \sim z + \tau$.

Riemann surfaces of higher genera

The torus can be regarded as a set of solution to the equation,

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3).$$

This can be solved by setting $y = \mathcal{P}'(z)$ and $x = \mathcal{P}(z)$, with z being the doubly-periodic coordinate. This can be generalized. For example, one can consider the equation,

$$y^2 = \prod_{i=1}^{2g+1} (x - e_i).$$

We can think of y as a function of x with $(g+1)$ branch cuts, one of which extends to the infinity in the x-plane. The covering space is then a surface with g handles. It is a special example of Riemann surfaces of genus g, called the hyper-elliptic surface. The torus has genus $g = 1$.

The torus as a complex manifold is parametrized by the modulus τ. For a general Riemann surface Σ_g of genus $g > 1$, the moduli space $\mathcal{M}_g$ is a complex manifold of dimensions $(3g - 3)$. To understand $\mathcal{M}_g$, it is useful to consider period integrals. It is easy to see that $b_1(\Sigma_g) = 2g$. We can choose a basis $H_1(\Sigma_g)$ so that for each handle, we have two generators α_i, β^i $(i = 1, ..., g)$ and that they intersect with each other as,

$$\alpha_i \cap \alpha_j = \beta^i \cap \beta^j = 0, \quad \alpha_i \cap \beta^j = \delta_i^j.$$

With respect to this basis, we can choose a basis of $H^{1,0}(\Sigma_g)$ as $\{\omega_i\}$ so that

$$\int_{\alpha_i} \omega_j = \delta_j^i.$$

The period matrix is defined as

$$\Omega_{ij} = \int_{\beta^i} \omega_j.$$

There is a one-to-one map from the moduli space $\mathcal{M}_g$ to Ω_{ij}, namely we can use Ω_{ij} to distinguish complex structures of Σ_g. Unfortunately, the space of Ω_{ij} is too big; it is a symmetric matrix, so it has $\frac{1}{2}g(g+1)$ components while $\dim\mathcal{M}_g = 3g-3$. This raises the question on how to characterize the image of $\mathcal{M}_g$ in the space of $g \times g$ symmetric matrices. This so-called Schottkey problem was found by using the theory of integrable systems.

There is an analogue of the theta functions at higher genera, called the Riemann theta function,

$$\Theta(\vec{z}|\Omega) = \sum_{\vec{n}\in\mathbf{Z}^g} \exp\left(2\pi i\left(\frac{1}{2}\vec{n}^t\Omega\vec{n} + \vec{n}\cdot\vec{z}\right)\right).$$

We can build the theory of holomorphic functions and sections of various line bundles over Σ_g using the Riemann theta function.

Lecture 10

homotopy

Consider continuous maps from a topological space X to another topological space Y. Two such maps are called homotopic if one can continuously deform one to another. This provides a useful way to define topological invariants. In particular, when X is the n-sphere S^n, the space of maps (modulo homotopy equivalence) becomes a group — the homotopy group — of Y, and the group is denoted by $\pi_n(Y)$.

Homotopy group should be contrasted with homology groups we studied earlier. For one thing, homotopy groups can be non-abelian, while homology groups are always abelian. In fact, $H_1(Y)$ is an abelian reduction of $\pi_1(Y)$. Thus, $\pi_1(Y)$ can contain more informaion on Y than $H_1(Y)$ does.

Fundamental Group

Pick a point x_0 in a topological space M. A continous map $\alpha : [0,1] \to M$ such that $\alpha(0) = \alpha(1) = x_0$ is called a loop or a closed path with x_0 as its base point. Suppose there are two such loops, α and β. They are called homotope, $\alpha \sim \beta$ if there is a continuous map $H : [0,1] \times [0,1] \to M$ such that,

$$\begin{aligned} H(t,0) = \alpha(t), \quad H(t,1) = \beta(t) \\ H(0,s) = H(0,t) = x_0 \end{aligned} \tag{1}$$

Namely, for each $s \in [0,1]$, $H(t,s)$ defines a loop with x_0 as its base point. As we vary s, H interpolates between α and β continuously.

Question 1: Show that $\alpha \sim \beta$ defines an equivalence relation, namely (1) $\alpha \sim \alpha$, (2) $\alpha \sim \beta$ implies $\beta \sim \alpha$, (3) $\alpha \sim \beta$, $\beta \sim \gamma$ implies $\alpha \sim \beta$.

The set of homotopy classes $[\alpha]$ of loops at x_0 is denoted by $\pi_1(M, x_0)$. It naturally is a group since we can combine two loops α and β to make another loop $\alpha * \beta$. The class containing the constant map $c : [0,1] \to x_0$ gives the identity. The group is called the fundamental group or the first homotopy group. For example, if we view S^1 as $\mathbf{R}/\mathbf{Z}$, we can choose representatives of the fundamental group as

$$\alpha_n : \theta \in [0,1] \to x_0 + n\theta \in \mathbf{R}.$$

Question 2: Show that $[\alpha_n] * [\alpha_m] = [\alpha_{n+m}]$ and that $\pi_1(S^1, x_0) = \mathbf{Z}$.

On the other hand, $\pi_1(S^2, x_0)$ is trivial since any loop on S^2 can be contractible to a point.

A topological space is called arcwise connected if any pair of points x_0 and y_0 in M can be connected by a continuous path, namely there is a continuous map $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$ and $\gamma(1) = y_0$.

Question 3: Show that $\pi_1(M, x_0) \simeq \pi(M, y_0)$ for any x_0, y_0 in M if M is archwise connected.

When M is archwisely connected, we can denote the fundamental group as $\pi_1(M)$ without refering to the base point.

Free Group and Relations

Homology groups are always abelian, but the fundamental group can be non-abelian.

It is a good place to introduce the notion of free group. Start with a finite set of letters, $X = \{a, b, c, ...\}$. Words are ordered lists of letters of the form, $\omega = x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N}$ with $n_1, ..., n_N \in \mathbf{Z}$ and $x_1, ..., x_N \in X$. If we have $x^n x^m$, we can replace it with x^{n+m}. For example, $a^{-2}bc^3c^{-2}ae = a^{-2}bcae$. This is called reduction. In a reduced word, $x_i \neq x_{i+1}$ and $n_1, ..., n_N \neq 0$. The emply word gives the identity. We can take a produce of words, buy multiplying them together and then reducing the result. This defines the free group $F[X]$ finitely generated by X.

Consider the space $M = \mathbf{R}^2 - \{z_1, z_2\}$. The fundamental group $\pi_1(M)$ has two generators, one going around z_1, call a, and another going around z_2, call b. In fact, the fundamental group is a free group generated by a and b. Note that $aba^{-1}b^{-1}$ is not trivial. In general, $\pi_1(\mathbf{R}^2 - \{z_1, ..., z_N\})$ is a free group generated by n letters.

We can impose relations on a free group to construct another group. For example, $\mathbf{Z}_N = \mathbf{Z}/n\mathbf{Z}$ is x (regard $\mathbf{Z} = \{1, x^{\pm 1}, x^{\pm 2}, ...\}$) with the relation $x^N = 1$. The fundamental group of T^2 is generated by a and b with the relation $aba^{-1}b^{-1} = 1$. Thus, $\pi_1(T^2) = \mathbf{Z} \oplus \mathbf{Z}$.

Suppose G is an arbitrary finite group generated by $a_1, ..., a_N$. Consider the free group with the same number of generators, $F = F[\{A_1, ..., A_N\}]$. We can define a homomorphism, $\varphi : F \to G$ by $\varphi(A_{i_1}^{n_1} \cdots A_{i_N}^{n_N}) = a_{i_1}^{n_1} \cdots a_{i_N}^{n_N}$. By construction, G is surjective, $G = \mathrm{Im}\varphi$. By the homomorphism theorem,

$$G = \mathrm{Im}\varphi = F/\mathrm{ker}\varphi.$$

We see that $\mathrm{ker}\varphi$ gives a set of relations to define G from the free group F. This shows that any finitely generated discrete group can be constructed as a free group with relations. In particular, the fundamental group can also been seen in this way.

The first homology group of a genus-g Riemann surface Σ_g is $H_1(\Sigma_g; \mathbf{Z}) = \mathbf{Z}^{\oplus 2g}$. On the other hand, the fundamental group $\pi_1(\Sigma_g)$ is isomorphic to the quotient group of the free group on the generators $a_1, b_1, a_2, b_2, ..., a_g, b_g$ by the normal subgroup generated by $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. Except for $\pi_1(\Sigma_{g=0}) = 0$ and $\pi_1(\Sigma_{g=1}) = \mathbf{Z} \oplus \mathbf{Z}$, the group is non-abelian.

For a group G, its commutator subgroup F is generated by elements of the form $aba^{-1}b^{-1}$. It is a normal subgroup of G and the quotient group G/F is abelian. In fact, F is the smallest normal subgroup of G such that the quotient is abelian. One can show that the first homology group $H_1(M)$ is isomorphic to the fundamental group $\pi_1(M)$ divided by its commutator subgroup.

As an aside, I would like to mention another important group associated to topology in three dimensions, the knot group. Consider two sets of points $\{A_1, A_2, ..., A_N\}$ and $\{B_1, B_2, ..., B_N\}$ in three dimensions, and connect them pairwisely by N ropes. We allow homotopy transformations of configurations of the ropes. Configurations of ropes natually make a group by multiplication, the knot group. There is the identity connecting A_1 to B_1, etc in the straight fashion. σ_i connects A_i to B_{i+1} and A_{i+1} to B_i. σ_i^{-1} also does so but with opposite orientation. The knot group is generated $\{\sigma_i\}_{i=1,...,N-1}$ with the two set of relations,

$$\sigma_i \sigma_k = \sigma_k \sigma_i \ (i + 1 < k), \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

Higher Homotopy Groups

A natural generalization of a loop is the n-sphere. So, we try to classify continuous maps from S^n to M. A continuous map α from the n-cube $I_n = [0,1]\times[0,1]\times\cdots\times[0,1]$ to M such that $\alpha : \partial I_n \to x_0$ is called an n-loop with base x_0. We say that two n-loops, α and β, are homotopic if there is a continuous family of n-loops $H(s)$ such that $H(s=0) = \alpha$ and $H(s=1) = \beta$. The set of homotopically equivalent classes $[\alpha]$ of n-loops defines the n-th homotopy group $\pi_n(M, x_0)$. Again, if M is archwise connected, the homotopy group can be defined without referring to the base point x_0. Higher homotopy groups are all abelian.

Homotopy Type

Homotopy of maps

A pair of continuous maps, $f, g : X \to Y$, are homotopic if there is a continuous map $H : [0,1] \times X \to Y$ such that $H(s=0) = f$ and $H(s=1) = g$.

Homotopy of spaces

Two topological spaces are of the same homotopy type iff there are continuous maps, $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is *homotopic* to the identity map on Y and $g \circ f$ is *homotopic* to the identity on X. For example, $\mathbf{R}$ is homotopic to a set consisting of one point $p \in \mathbf{R}$.

A subspace R of M is called a deformation retract of M if there is a continuous map $H : [0,1] \times M \to M$ such that $H(s=0)$ is the identity map on M, and $H(s=1)$ maps M to R and acts as the identity on R. In this case, M and R are of the same homotopy type. (R is called a strong deformation retract if $H(s)$ leaves points in R fixed for all $s \in [0,1]$.) In particular, if $p \in M$ is a deformation retract of M, then M is called contractible.

If two topological spaces are of the same homotopy type, they have the same homotopy groups.

Question 4: Show that the fundamental group of a contractible space is trivial.

Homotopy Groups of Spheres

When $0 < i < n$, any continuous map from S^i to S^n is homotopic to a constant map, and $\pi_i(S^n)$ is trivial. As for $\pi_n(S^n)$, there is the identity map $f : S^n \to S^n$, and we can then consider $[f] * [f] * \cdots * [f]$, which wrap S^n multiple times. In fact, $\pi_n(S^n) = \mathbf{Z}$, parametrized by the winding number.

More interesting is $\pi_i(S^n)$ with $i > n$. The first non-trivial example is $\pi_3(S^2)$, namely classification of maps from $S^3 \to S^2$. We have encountered this earlier in this course — the Hopf fibration. There, we regarded S^3 as the total space of fiber bundle over S^2. For a fiber bundle, there must be a projection map from the total space to the base. The Hopf fibration is characterized by an integer n, which we identified as the monopole number. In fact, $\pi_3(S^2) = \mathbf{Z}$.

Homology groups $H_i(S^n)$ for $i > n$ are all trivial. In contrast, $\pi_i(S^n)$ for $i > n$ are complex and difficult to compute.

Stable region

$$\pi_4(S^3) = \pi_5(S^4) = \pi_6(S^5) = \cdots = \mathbf{Z}_2,$$
$$\pi_6(S^4) = \pi_7(S^5) = \pi_8(S^6) = \cdots = \mathbf{Z}_2,$$
$$\pi_8(S^5) = \pi_9(S^6) = \pi_{10}(S^7) = \cdots = \mathbf{Z}_{24},$$
$$\pi_{10}(S^6) = \pi_{11}(S^7) = \pi_{12}(S^8) = \cdots = 0,$$
$$\cdots . \tag{2}$$

In general, $\pi_{2k+n}(S^{k+1+n})$ $(k = 2, 3, 4, ...; n = 0, 1, 2, 3...)$ are called stable homotopy groups of spheres, and they depend only on k.

Homotopy Groups of Groups

We will discuss topology of Lie groups in the next week. Here we will summarize some basic facts about their homotopy groups.

Suppose M is a topological space with $\pi_0(M) = \pi_1(M) = 0$. Such a space is called simply connected. Suppose that a group G acts on M, *i.e.* each $g \in G$ gives a homeomorphism on M, and that the quotient M/G is smooth. The fundamental group of the quotient is given by,

$$\pi_1(M/G) = \pi_0(G).$$

In particular, if G is a discrete group, $\pi_0(G) = G$ and thus $\pi_1(M/G) = G$. For example, for $M = \mathbf{R}$ and $G = \mathbf{Z}$, we have $\pi_1(S^1) = \pi_1(\mathbf{R}/\mathbf{Z}) = \mathbf{Z}$.

Bott periodicity

For the unitary group,

$$\pi_k(U(n)) = \pi_k(SU(N)) = 0 \ (k : \text{even}), \quad \mathbf{Z} \ (k : \text{odd}),$$

when $k > 1$ and $n \geq (n + 1)/2$. This proporty was discovered by Raoul Bott and is called the Bott periodicity. It is closely related to the behavior of the homotopy groups of spheres in the stable region. There are similar periodicities for the orthogonal and symplectic groups.

It is not possible to end a lecture on homotopy without referring to the Poincare conjecture in three dimensions. Poincare initially conjectured that any three-dimensional manifold with the same betti number as those of the 3-sphere must be homeomorphic to the 3-sphere. However, he later found a counter-example, the first example of what are now called homology spheres. He then revised his conjecture as: Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

The generalized Poincare conjecture has been proven in higher dimensions. In 1961, Stephen Smale proved the conjecture for dimensions greater than four. The four-dimensional conjecture was proven by Michael Freedman.

It should be noted that the conjecture is about homeomorphism and not about diffeomorphism. For example, Milnor's exotic spheres show that the smooth Poincare' conjecture is false in dimension seven.

Finally, in 2002-2003, Grigori Perelman proved the original three-dimensional conjecture, by refinding the technique of the Ricci flow conceived by Richard Hamilton.

Lecture 11

geometry of continuous groups

topological groups

If G is both a group and a topological space, and if the group operators (the multiplication: $G \times G \to G$ and the inverse: $G \to G$) are continuous, G is a topological group. If G is a topological group, for each $a \in G$, the maps, $L_a : g \to a \cdot g$ and $R_a : g \to g \cdot a$ for $g \in G$ define homeomorphism, called left and right translations.

For a topological group G and its subgroup H, one can define the left and right quotients as follows. For each $g \in G$, define $gH = \{gh : \ h \in H\}$ and call it the left-coset for g. The left quotient G/H is a set of left-cosets. The right quotient is defined by revesing the order of G and H. The quotients are topological spaces. In particular, if H is an invariant subgroup (normal subgroup) of G, namely if, for any $h \in H$ and $g \in G$, ghg^{-1} is also in H, then G/H is also a topological group.

Lie groups

If G is both a group and a differential manifold, and if the group operations are differentiable maps, G is called a Lie group.

Consider the unitary grou $U(N)$ as an example of Lie groups. First we note that any element g of the unitary group can be diagonalized by conjugation by another unitary matrix u as,

$$g = u \cdot \mathrm{diag}(e^{i\theta_a})_{a=1,\ldots,N} \cdot u^{-1}.$$

$gg^\dagger = 1$ requires that $\theta_a \in \mathbf{R}$. Since the exponential function has the Taylor expansion,

$$e^{i\theta_a} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \theta_a^n,$$

we can express g as,

$$\begin{aligned} g &= \sum_{n=0}^{\infty} \frac{i^n}{n!} u \cdot \mathrm{diag}(\theta_a^n) \cdot u^{-1} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(u \cdot \mathrm{diag}(\theta_a) \cdot u^{-1}\right)^n \\ &= \exp\left(iu \cdot \mathrm{diag}(\theta_a) \cdot u^{-1}\right). \end{aligned} \tag{1}$$

Note that $H = u \cdot \mathrm{diag}(\theta_a) \cdot u^{-1}$ is a hermitian matrix. Conversely, if H is a hermitian matrix, $\exp(iH)$ is a unitary matrix. Therefore, any untary matrix can be expressed as an exponential of i times a hermitian matrix.

One important feature of the set of hermitian matrices is that it is closed by the commutator. If H_1 and H_2 are hermitian, then $[H_1, H_2]$ is equal to i times another hermitian matrix. A set of matrices that makes a linear space (linear combinations also belong to the set) and is closed under the commutator in this way is called a Lie algebra.

The need for the closure under the commutator can be explained as floows. Suppose that X and Y are matrices and that we want e^{iX} and e^{iY} to be in the Lie group G. The Baker-Campbell-Hausdorff formula says,

$$e^{iX}e^{iY} = e^{iW},$$

where

$$W = X + Y + \frac{i}{2}[X,Y] - \frac{1}{12}[X-Y,[X,Y]] + \cdots,$$

and $\cdots$ in the above is expressed as a sum of commutators of X and Y. Thus, if X and Y are in the Lie algebra, so is W. If e^{iX} and e^{iY} belong to G, so is $e^{iX}e^{iY}$.

Pull-back, Push-forward, Lie derivative

These concepts are defined for any differentiable manifold, but they are useful to study geometry of Lie groups.

Suppose there is a differentiable map, $\varphi : M \to N$, between manifolds M and N. We do not have to assume φ is injective. This map induces the pull-back, $\varphi^* : C^\infty(N) \to C^\infty(M)$. For any function $f(q)$ $(q \in N)$, we can defined a function on M as $[\varphi^* f](p) = f(\varphi(p))$ $(p \in M)$.

We can also define the push-forward tangent vector fields on M and N. Pick a tangent vector v on M. For any differentiable function f on M, it gives another function $v(f)(p)$. What we want is a way to find a function on N for each function g on N. This can be done as follows. First pull-back g to define a function $[\varphi^* g](p) = g(\varphi(p))$ on M. Now we can evaluate it with v to define another function $v(\varphi^* g)(p) = v(g(\varphi(p)))$. Thus, the tangent vector $v(p)$ at T_pM is mapped to $[\varphi_* v](q)$ at T_qN, where $q = \varphi(p)$.

Question 1: This may sounds a bit abstract, so let us express it in terms of coordinates. Choose coordinates x^μ $(\mu = 1, ..., m)$ on M and y^i $(i = 1, ..., n)$ on N. Note that M and N may have different dimensions! Consider the tangent vector field $v = v^\mu(x)\partial/\partial x^\mu$. For $\varphi : x^\mu \to y^i(x)$, show

$$\varphi_* v = v^\mu \frac{\partial y^i}{\partial x^\mu}\frac{\partial}{\partial y^i}.$$

In the GR speak, the tangent vector v transforms as a contravariant vector.

We can repeat this for one-forms. Since a one-form is a linear function on tangent vectors, we can naturally define a pull-back φ_*.

Question 2: Define a pull-back for a one-form ω. Show that, in coordinates,

$$\omega = \omega_i dy^i \to \varphi^*\omega = \omega_i(\varphi(x))\frac{\partial y^i}{\partial x^\mu}dx^\mu.$$

Now we define a Lie derivative for a given tangent vector field v. For a given vector field v on M, we can define the exponential map, $\varphi_t : M \to M$, defined by the property that

$$\frac{d}{dt}f(\varphi_t(p)) = [v(f)](\varphi_t(p)).$$

We often write it as $\varphi_t(p) = \exp(tv)(p)$. Infinitesimally (which is all we need for now), and in terms of coordinates x^μ,

$$\varphi_t(x)^\mu = x^\mu + tv^\mu(x) + \cdots.$$

Now, we can define a push-forward map, $\varphi_{-t\ *} : T_{\varphi_t(p)}M \to T_pM$. Using this, the Lie derivative $\mathcal{L}_v$ is defined as a map from a tangent vector field u on M to another tangent vector field $\mathcal{L}_v u$,

$$\mathcal{L}_v u(p) = \lim_{t\to 0} \frac{1}{t} \left(\varphi_{-t\ *} v(\varphi_t(p)) - v(p)\right).$$

Question 3: Writing u and v as differential operators $u = u^\mu \partial_\mu$, $v = v^\mu \partial_\mu$, show,

$$\mathcal{L}_v u = [v, u].$$

Using coordinates, show that $[v, u]$ transforms as a tangent vector field.

Lie Algebra as Space of Left-Invariant Vector Fields

Cnsidering the Lie group G as a differential manifold, the left and right translations of $a \in G$,

$$L_a : g \to ag, \qquad R_a : g \to ga,$$

define diffeomorphisms from G to itself. Thus, we can define push-forwards, L_{a*} and R_{a*}. We say that a tangent vector field v is left (right) invariant iff $L_{a*}v = v$ ($R_{a*}v = v$). Let us denote the space of left invariant vector fields of G by $\mathcal{G}$. If v is a left invariant vector field, $v(g) = L_{a*}v(a^{-1}g) = L_{g*}v(1)$. Thus, it is determined by a tangent vector at $g = 1$. This means that $\mathcal{G}$ as a linear space is isomorphic to $T_{g=1}G$. In particular, $\dim \mathcal{G} = \dim G$.

If u and v are left invariant vector fields, so is $[u, v]$. This way, the space $\mathcal{G}$ of left invariant vector fields can be identified with the Lie algebra of G.

If we think of the group G as a matrix group, with each element $g \in G$ represented by a matrix g_{ij}, left invariant vector fields can also be written as

$$\sum_k g_{ki} \frac{\partial}{\partial g_{kj}}.$$

For each (i, j), the above vector is invariant under $g \to ag$.

Maurer-Cartan Forms

The Maurer-Cartan forms are left invariant one-forms, defined by

$$\Phi = g^{-1} dg.$$

It satisfies the Maurer-Cartan equation,

$$d\Phi + \Phi \wedge \Phi = 0.$$

The left invariant vector field $g^T \partial/\partial g$ is dual to the Maurer-Cartan forms. In fact, the Maurer-Cartan form is locally determined by this equation since we can think of the equation as saying that the gauge connection one-form given by Φ has vanishing curvature and is locally gauge equivalent to the trivial configuration.

Connected and Simpy Connected

Two elements of a Lie Group G are called connected if there is a continuous path in the group connecting them. Being connected is an equivalence relation, and thus the group can

be divided into its equivalence classes, called connected components. Let us call the connected component containing the identity e as G_0. Since any element of the unitary group $U(N)$ can be expressed as $\exp(iX)$ by some hermitian matrix X, the unitary group is connected.

It is easy to see that G_0 is a sugroup of G. It important to note that G_0 is an invariant subgroup (normal subgroup). Namely, for any $g \in G$ and $h \in G_0$, ghg^{-1} is also in G_0. To see this, suppose $h(t)$ is a continuous path connecting e to h as $h(0) = e, h(1) = h$. It then follows that $gh(t)g^{-1}$ gives a path connecting e to ghg^{-1}.

If $\pi_1(G_0)$ is trivial, G_0 is called simply connected.

examples

$O(3)$ is not connected. To see that, we note that any $g \in O(3)$ satisfies $gg^T = 1$ and thus $(\det g)^2 = 1$. Thus, elements of $O(3)$ are divided into those with $\det g = 1$ and -1, and these two classes of elements are not continuously connected. On the other hand, $SO(3)$ is connected. However, it is not simply conneced. As explained below, $SU(2)$ is locally the same as $SO(3)$, but it is both connected and simply connected.

Universal Covering Group

The fundamental theorem of the Lie group (initiated by S. Lie and completed by E. Cartan) states that, for any n-dimensional Lie algebra, there is a unique simply connected Lie group. This simply connected group is called the universal coversing group and is denoted by UG. Any other Lie group with the same Lie algebra is of the form UG/Γ, where Γ is a discrete invariant subgroup.

In fact, if Γ is a discrete invariant subgroup, each of its elements must commutes with elements of UG. To see this, note that Γ being invariant means that, for any $h \in \Gamma$ and for any $g \in UG$, there is $h' \in \Gamma$ such that $ghg^{-1} = h'$. However, g can be continuously connected to the identity e in UG. Since Γ is discrete, h and h' cannot change continuously as we change g to e. Thus, $ehe^{-1} = h = h'$, and $gh = hg$.

If we regard G as a matrix group, then by Shur's lemma, any group element that commutes with all the elements of G must be proportional to the identity. Therefore, Γ should consist of elements of the form $\lambda \cdot \mathrm{id}$ for some $\lambda \in \mathbf{C}$.

examples

Let us enumerate all discrete invariant subroups of $SU(2)$. The group $SU(2)$ consists of matrices of the form,

$$g = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}.$$

The condition that $\det g = 1$ means

$$|\alpha|^2 + |\beta|^2 = 1.$$

Namely, the group $SU(2)$ is diffeomorphic to the 3-sphere in $\mathbf{R}^4$. If Γ is an invariant subgroup, its element must be of the form $\lambda \cdot \mathrm{id}$. For this to belong to $SU(2)$, $\lambda = \pm 1$.

Thus, there are two possibilities: $\Gamma = \{\mathrm{id}\}$ or $= \{\pm \mathrm{id}\}$. In the former case, we have $SU(2)$ itself. In the latter case, we have $SU(2)/\mathbf{Z}_2$. The $\mathbf{Z}_2$ action identifies $\alpha \sim -\alpha$, $\beta \sim -\beta$. This gives the group $SO(3)$. To see that, we note that any rotation in three dimensions can be

parametrized by the vector $\vec{\Omega} = \theta\vec{n}$, where $\vec{n}$ is a unit vector representing the axis of rotation and θ is the amount of rotation. We can choose, for example, $0 \leq \theta \leq \pi$. Thus, the space of $(\theta, \vec{n})$ is the disk, with $\theta = \pi$ representating the boundary of the disk. However, we need to identify $(\pi, \vec{n})$ with $(\pi, -\vec{n})$. Namely, we make the antipodal identification of the boundary of the disk. If we view the disk as the upper hemisphere of the S^3, it is the same as $SU(2)/\mathbf{Z}_2$.

In general, the universal covering group of $SO(N)$ is called $spin(N)$. For example, $spin(3) \sim SU(2)$, $spin(4) \sim SU(2) \times SU(2)$, $spin(5) \sim USp(4)$, $spin(6) \sim SU(4)$.

Homotopy of Lie Groups

If $\pi_1(G)$ trivial, $\pi_1(G/H) = \pi_0(H)/\pi_0(G)$. Thus, for example, we see that $\pi_1(SO(3)) = \pi_1(SU(2)/Z_2) = \pi_0(Z_2) = Z_2$.

If $\pi_2(G)$ is trivial, $\pi_2(G/H) = \pi_1(H)/\pi_1(G)$. For example, for any compact connected Lie group, $\pi_2(G)$ is trivial. In a gauge theory with the Higgs mechanism to break the gauge symmetry $G \to H$, in the symmetry breaking phase, the Higgs field Ψ takes value in G/H. Suppose we are in $\mathbf{R}^3$ and there is a point in the space where the gauge symmetry is restored. There, the Higgs field must vanish. So, we remove the neighborhood of the point from $\mathbf{R}^3$. The resulting space is homotopic to S^2. Thus, the Higgs field configuration is classified by $S^2 \to G/H$. This gives the classification of the 't Hooft-Polyakov magnetic monopole.

In the Standard Model of Particle Physics , the gauge group $SU(3) \times SU(2) \times U(1)$ is spontaneously broken to $SU(3) \times U(1)_{\rm EM}$, where $U(1)_{\rm EM}$ is a combination of the $U(1)$ and an $U(1)$ subgroup of the $SU(2)$ in the original gauge group. Thus, we should consider $\pi_2(SU(2) \times U(1)/U(1)_{\rm EM}) = \pi_1(U(1)_{\rm EM})/\pi_1(U(1))$ and it is trivial, where we used the fact that the $SU(2)$ is simply connected. This shows that there is no magnetic monopole with the Standard Model.

However, in the grand unified theory with gauge groups $SU(5)$ or $SO(10)$, we can have $\mathbf{Z}$ valued magnetic monopoles since $\pi_1(SU(5))$ is trivial and $\pi_1(SO(n)) = \mathbf{Z}_2$ $(n \geq 3)$. This is a prediction of these grand unified theories.

To prove these relations, we can use the exact sequence,

$$\begin{aligned} &\to \pi_3(G) \to \pi_3(G/H) \to \pi_2(H) \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \\ &\quad \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(G) \to \pi_0(G) \to \pi_0(G/H) \to 0. \end{aligned} \tag{2}$$

That this is the exact sequence means that the image of each map is the kernel of the next-right map. For example, the image of $\pi_1(G)$ in $\pi_1(G/H)$ is the kernel of the map from $\pi_1(G/H)$ into $\pi_0(G)$. Thus, if $\pi_1(G)$ is trivial, there is no kernel for the map from $\pi_1(G/H)$ to $\pi_0(G)$, i.e., it is injective. Thus, $\pi_1(G/H)$ must be the same as its image in $\pi_0(G)$. But, by the exact sequence, the image must be the same as the kernel of the map from $\pi_0(G)$ to $\pi_0(H)$. The latter is the same as the quotient $\pi_0(H)/\pi_0(G)$. This shows that $\pi_1(G/H) = \pi_0(H)/\pi_0(G)$. Similar relations can be derived for $\pi_n(G/H)$.

Lecture 12

geometry of gauge theory

Gauge Fixing

The Yang-Mills Lagrangian density takes the form,

$$\mathcal{L} = \frac{1}{4}\mathrm{tr}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)^2.$$

We can choose a basis $\{T^a\}$ for the Lie algebra of the gauge group G and write

$$A_\mu = \sum_a A_\mu^a T^a.$$

This action does not contain the time derivative of the variable A_0, which is the source of fun. Let us denote spacelike directions by the indix i. The action can then be expressed as,

$$\mathcal{L} = \mathrm{tr}\left(\frac{1}{2}E_i\dot{A}_i - \frac{1}{4}(E_iE_i + B_iB_i) + \frac{1}{2}A_0C\right),$$

where

$$E_i = F_{0i}, \quad B_i = \epsilon_{ijk}F_{jk},$$

and

$$C = \partial_i E_i + [A_i, E_i].$$

If we view the Yang-Mills theory as a classical mechanical system and apply the standard Lagrangian formalism, we see that E_i and A_i are canonically conjugate to each otehr with the Poisson bracket,

$$\{E_i^a(t,\vec{x}), A_j^b(t,\vec{y})\} = \delta_{ij}\delta^{ab}\delta(\vec{x}-\vec{y}),$$

where a, b refer to gauge group indices and we assumed that the basis of the Lie algebra is chosen so that $\mathrm{tr}T^aT^b = \delta^{ab}$. The time component A_0 does not have its conjugate variable, and it serves as a Lagrange multiplier to impose the constraint, $C = 0$. We see that the Poisson bracket for E_i and A_i imply

$$\{C(\vec{x})^a, C(\vec{y})^b\} = f^{abc}C^c(\vec{x})\delta(\vec{x}-\vec{y}),$$

where we suppressed the time coordinate t, which are common to all, and f^{abc} is the structure constant defined by $[T^a, T^b] = if^{abc}T^c$. Thus, it is consistent to impose the condition $C = 0$.

It is important to note that

$$\{C_\mu, A_i(\vec{x})\} = D_i\mu = \partial_i\mu + [A_i, \mu],$$

where

$$C_\mu = \int d\vec{y}\ \mathrm{tr}\left(\mu^a(\vec{y})C^a(\vec{y})\right).$$

Namely, $C^a(\vec{x})$ is the generator of the gauge transformation, $A_i \to A_i + D_i\mu$. In particular, the Hamiltonian

$$H = \int d\vec{x}\mathrm{tr}\left(E_iE_i + B_iB_i\right),$$

which is gauge invariant, commutes with C.

If we have constraints that are closed under the Poisson bracket, they are called the first class.

Symplectic Reduction

To understand the phase space of the Yang-Mills theory, it is useful to review the notion of the symplectic reduction. A phase space M is an even dimensional space (say $2m$ dimensional) with a non-degenerate Poisson bracket. Locally, we can choose canonical coordinates (q^i, p_i) $(i = 1, ..., m)$ so that $\{q\ i, p_j\} = \delta^i_j$. Suppose we have a Hamiltonian H and a set of constraints $\{\varphi^a\}$ $(a = 1, ..., n)$, satisfying,

$$\{\varphi^a, \varphi^b\} = \sum_c c^{ab}{}_c \varphi^c, \quad \{H, \varphi^a\} = \sum_b d^a_b \varphi^b,$$

for some functions $c^{ab}{}_c, d^a_b$ on M. These are of the first class.

To impose the constraints, the Lagrangian is defined as

$$L = p_i \dot{q}^i - H - \sum_a \lambda_a \varphi^a,$$

where λ_a's are Lagrange multipliers.

Question 1: Derive the equations of motion for (q, p) with keeping the Lagrange multipliers as arbitrary functions of t, and show that trajectories stay within the constrainted subspace in M.

Trajectories depend on λ's, but we would like time evolutions of physical observables be independent of them. Namely, observable functions f should satisfy

$$\{f, \varphi^a\} = 0 \quad (\mathrm{mod}\varphi).$$

This means that we should not only evaluate f on the $(2m - n)$-dimensional subspace with $\varphi^a = 0$, but f should be independent of n more directions generated by φ's.

To paramatrize the additional n directions, let us choose functions $\chi_a(q, p)$ $(a = 1, ..., n)$ so that

$$\det\{\chi_a, \varphi^b\} \neq 0.$$

For simplicity, let us also assume $\{\chi_a, \chi_b\} = 0$, though we can relax this condition. Since φ_a's do not Poisson commute with the constraints φ^a, they are not physical.

Consider a new subspace M^* defined by the two sets of constraints, $\chi_a = 0, \varphi^a = 0$. They do not Poisson commute and they are called the second class constraints.

Since χ_a commute with each other, we can choose canonical coordinates on M such that the first n momenta are $p_a = \chi_a$ $(a = 1, ..., n)$. Their conjugate coordinates are q^a. Let us denote the rest of canonical coordinates by (q^s_*, p_{*s}) $(s = 1, ..., m - n)$. Since χ_a and φ^a have non-degenerate Poisson bracket,

$$\det\left(\frac{\partial \varphi^a}{\partial q^b}\right) = \det\{\varphi^a, \chi_b\} \neq 0.$$

This means that we can solve the constraints $\varphi^a = 0$ on M^* by choosing q^a appropriately. Namely, we can characterize M^* as a subspace of M obeying,

$$p_a = 0, \quad q^a = q^a(q_*, p_*).$$

In particular, (q_*, p_*) are natural canonical coordinates on M^* and we can use them to define the Poisson bracket on $M*$. This procedure, to derive the new phase space $M*$ from the old phase space M subject to the constraints φ^a, is called the symplectic reduction.

This can be done whenever a group G is acting on a phase space G as a canonical transformation. Then there is a generator (or a set of generators) φ for the G action. Define $M_0 = \varphi^{-1}(0)$, namely the subspace of M where $\varphi = 0$. There is a G action on M_0 and that the quotient M_0/G is naturally a phase space (i.e., with non-degenerate symplectic form). This is the same as M^* discussed here. The reduced phase is also denoted as $M//G$. It is also called a symplectic quotient or a Marsden-Weinstein quotient.

Faddeev-Popov Determinant

When one quantize a system using the path integral, the integral should be over trajectories on the physical phase space M^*, so that the resulting quantum amplitudes obey the unitarity conditions. On the other hand, it is often convenient to write the integral using canonical coordinates on M. The Jacobian for the change of variables is called the Faddeev-Popov determinant.

Consider the natural measure ω_* on M^*,

$$\omega_* = \prod_{s=1}^{m-n} dq_*^s dp_{*s},$$

and compare it with

$$\omega = \prod_{i=1}^{m} dq^i dp_i.$$

Since $M*$ is a subspace of M with the conditions, $p_a = 0, q^a = q^a(q_*, p_*)$, the two measures are related as

$$\omega_* = \prod_{a=1}^{n} \delta(q^a = q^a(q_*, p_*))\delta(p_a)\omega_*.$$

Since $p_a = \chi_a$ and $q^a = q^a(q_*, p_*)$ solve $\varphi(q, p) = 0$, we can write,

$$\prod_a \delta(q^a)\delta(p_a) = \prod_a \delta(\chi_a)\delta(\varphi^a)\det\left(\frac{\partial\varphi^a}{\partial q\ b}\right) = \prod_a \delta(\chi_a)\delta(\varphi^a)\det\{\varphi^a, \chi_b\}.$$

If the Lagrange multiplier term $\lambda_a\varphi^a$ is added to the Lagrangian, we can replace the constraint φ^a by an integral over λ_a. Thus, the measure ω_* on the physical phase space can be replaced by,

$$\det\{\varphi^a, \chi_b\} \prod_a \delta(\chi_a) d\lambda_a \prod_i dq^i dp_i.$$

The result of the integral is independent of the choice of χ_a as far as $\det\{\chi_a, \varphi^b\} \neq 0$.

Going back to the Yang-Mills theory, we see that the constraints are $C(\vec{x}) = 0$ and the Lagrange multipliers are $A_0(\vec{x})$. As the second set of constraints χ, we should choose ones that

do not Poisson commute with $C(\vec{x})$, namely gauge non-invariant conditions. They are gauge fixing conditions. For example, in the Coulomb gauge, we choose $\partial_i A_i = 0$ as such conditions. In this case, the Faddeev-Popov determinant is $\det \partial_i D_i$.

Geometry of Gauge Field Configurations

So far, we have looked at local geometry of gauge fixing in the phase space picture. Let us turn to global structure of the physical phase space of the Yang-Mills theory. Consider the four-dimensional Yang-Mills theory on S^4. Call the gauge field configuration space as $\mathcal{A}$ and the group consisting of gauge transformations by $\mathcal{G}$, which is a subset of $\Omega^4(G) = \{g(x) : S^4 \to G\}$. The quotient $\mathcal{A}/\mathcal{G}$ is a set of physically inequivalent gauge field configrations. One can show that this is homotope to $\Omega^3(G)$, i.e. space of maps from S^3 to G. This comes from separating S^4 into the northern and southern hemispheres and gluing them by gauge transformations across the equator $\sim S^3$.

Let us study the topology of this infinite dimensional space, $\Omega^3(G)$. To do so, we note that $\Omega^n(\Omega^m(G))$ is homotope to $\Omega^{n+m}(G)$. This in particular means that $\pi_n(\Omega^m(G)) = \pi_{n+m}(G)$. This simplifies our task since we know how to compute the homotopy of the group G itself.

For example, if $G = SU(N)$, $\pi_0(\Omega^3(G)) = \pi_3(G) = \mathbf{Z}$ if $N \geq 2$. This means that the gauge field configuration space $\mathcal{A}/\mathcal{G}$ consists of infinitely many disjoint components parametrized by $\mathbf{Z}$. In fact, this integer parameter is nothing but the second Chern number (instanton number),

$$C_2 = \frac{1}{8\pi^2} \int_{S^4} \mathrm{tr} F \wedge F.$$

The index theorem discussed in Lecture 8 tells you that this number is related to the number of zero modes of the Dirac operator coupled to the gauge field.

We have $\pi_1(\Omega^3(G)) = \pi_4(G) = 0$ for $N \geq 3$, so the space of gauge transformations is simply connected for $SU(N \geq 3)$. However, $\pi_1(\Omega^3(SU(2)) = \pi_4(SU(2)) = \mathbf{Z}_2$. Since $\pi_0(\Omega^4(G))$ is also equal to $\pi_4(G)$, this means that, for $G = SU(2)$, there is topologically nontrivial gauge transformation on S^4 that cannot be continuously deformed to the identity. It was shown by E. Witten that, if there is an odd number of chiral fermions that are in the doublet representation of $SU(2)$, its Dirac determinant changes the sign under the $\mathbf{Z}_2$ action of the gauge symmetry. This makes it impossible to quantize such a model consistently with the gauge symmetry. This is called the Witten anomaly.

For $N \geq 3$, we also have $\pi_2(\Omega^3(G)) = \pi_5(G) = \mathbf{Z}$ (it is $\mathbf{Z}_2$ for $SU(2)$). This is related the so-called non-abelian anomaly, a violation of the gauge symmetry and non-conservation of the Noether current for the gauge symmetry.

Question 2: Repeat this analysis in other spacetime dimensions.

Calabi-Yau manifolds

What is CY?

- complex : one can define holomorphic coordinates x^i $(i=1,\cdots,n)$
- Kähler : $g_{ij}=0$, $g_{\bar{i}\bar{j}}=0$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$$

- Ricci flat : $R_{i\bar{j}} = 0$

For a Kähler manifold, $R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \det g$.

$\Rightarrow R_{ij}=0$ means $\det g = \Omega\bar{\Omega}$

Ω : holomorphic on M
transforms as $(n,0)$-form

M : CY $\Rightarrow$ $\exists$ no-where vanishing $(n,0)$-form
$(\Leftrightarrow c_1 \neq 0)$

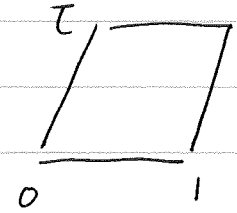
conjectured by Calabi
proven by Yau

example of CY

1d (complex 1d)

- $\mathbb{C}$
- $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ ~ cylinder

- $T^2 = \mathbb{C} / \{\mathbb{Z} + \tau\mathbb{Z}\}$

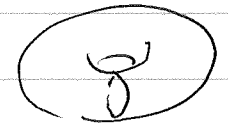


$x = x^1 + \tau x^2$

$0 \le x^1, x^2 \le 1.$

τ : complex structure

$dx = dx^1 + \tau dx^2$; (1,0) form

$1 = \int_0^1 dx$ 

$\tau = \int_0^\tau dx$  period integral.

$$(h^{p,q}) \qquad \begin{matrix} & h^{00} & \\ h^{10} & & h^{01} \\ & h^{11} & \end{matrix} = \begin{matrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{matrix}$$

r : area of T^2.

metric $g_{ij}\, dx^i dx^j = r\, dx\, d\bar{x} / \mathrm{Im}\,\tau$

$= r\,(dx^1 dx^1 + 2\,\mathrm{Re}\,\tau\, dx^1 dx^2 + |\tau|^2 dx^2 dx^2) / \mathrm{Im}\,\tau.$

Kähler form

$$k = \frac{i}{2} g_{i\bar{j}}\, dx^i \wedge dx^{\bar{j}} = r\, dx^1 \wedge dx^2.$$

r : Kähler class

Yau's theorem

M : $c_1 = 0$, given complex str and Kähler class

$\Rightarrow$

$\exists$ unique CY metric

String theory r : complexify $t = i r + \theta$ Kähler moduli

$(\tau, t) \in O(2,2)/O(2)\times O(2)$

$O(2,2;\mathbb{Z})$

2 complex dim

There are only 2 classes of compact CY's

- T^4

$$\begin{array}{ccccc} & & h^{00} & & \\ & h^{10} & & h^{01} & \\ h^{20} & & h^{11} & & h^{02} \\ & h^{21} & & h^{12} & \\ & & h^{22} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

complex moduli space $4\mathbb{C}$.

Kähler class $4 \leftarrow h^{1,1} = 4$.

$\hookrightarrow$ string theory $4\mathbb{C}$ $8\mathbb{C}$.

They combine to make $O(4,4)/O(4)\times O(4)$

$O(4,4;\mathbb{Z})$.

In general, the moduli space for string theory on T^D is

$O(D,D;\mathbb{Z}) \backslash O(D,D)/O(D)\times O(D)$

K_3 : only non-flat compact CY_2

$(h^{p,q}) =$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\chi = 24$

$h^{1,0} = 0 \Rightarrow$ no isometry

moduli space of metric = locally $O(19,3)/O(19)\times O(3)$

A Todorov.

~~Before~~ Complex moduli $\Leftrightarrow$ $h^{1,1} = 20$ complex dim

stringy Kähler moduli $\Leftrightarrow$ $h^{1,1} = 20$.

$O(20,4;\mathbb{Z}) \backslash O(20,4)/O(20)\times O(4)$

40 complex dim

examples

start with $\mathbb{CP}^3$

i.e. $(z_1 \cdots z_4) \sim \lambda(z_1 \cdots z_4)$, $\lambda \in \mathbb{C}^\times$

Consider homogeneous funct $P(z)$
of degree d

$P(z_1 - z_4) = 0$.

In general, a hypersurface X in $\mathbb{CP}^{k-1}$

defined by $P(z_1 \cdots z_k) = 0$
$\uparrow$ deg d

has $c_1 \sim (d-k)\, c_1(\mathbb{CP}^{k-1})$

So, we need $d = k$ for CY.

in our case $d = 4$

e.g. $(X_1)^4 + (X_2)^4 + (X_3)^4 + (X_4)^4 = 0$

in $\mathbb{CP}^4$

• CY_3

$h^{p,q} =$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & h^{11} & & 0 \\ 1 & h^{12} & & h^{21} & 1 \\ 0 & & h^{22} & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$h^{11} = h^{22}$, $h^{21} = h^{12}$ duality.

$h^{1,0} = 0$ no isometry

($h^{2,0} = h^{0,2} = h^{0,1} = 0$ follows by complex conjugate + duality)

If $h^{1,0} \neq 0 \Rightarrow$ torus

$\chi = 2(h^{1,1} - h^{2,1})$

complex structure deformation $\Leftrightarrow h^{2,1}$

Kähler deformation $\Leftrightarrow h^{1,1}$

(In general complex deformation $\Leftrightarrow h^{d-1,1}$)

Complex structure moduli space $\mathcal{M}_c$

holomorphic $(3,0)$-form Ω

Ω defines a line bundle over $\mathcal{M}_c$
(subbundle of the Hodge bundle)

with a metric $\|\Omega\|^2 = i\int_M \Omega \wedge \bar{\Omega}$

Define $K = -\log \|\Omega\|^2$

Then $\mathcal{M}_c$ becomes a Kähler mfd.

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K.$$

- flat coordinates on $\mathcal{M}_c$.

Choose a basis $\{\alpha_I, \beta^I\}_{I=0,1,\cdots,h^{2,1}}$

of $H_3(M, \mathbb{Z})$

$b_3 = 2 + 2h^{2,1}$

Define the periods:

$$X^I = \int_{\alpha_I} \Omega \quad , \quad F_I = \int_{\beta^I} \Omega$$

$F_I = F_I(X)$ homogeneous, degree 1

$$\|\Omega\|^2 = X^I \bar{F}_I - \bar{X}^I F_I$$

$\exists F$, $F_I = \partial_I F(X)$

↖ pre-potential ($\Rightarrow$ Seiberg-Witten theory)

$$t^a = \frac{X^a}{X^0} \qquad a = 1, \dots, h^{2,1}$$

One can show

$$K = -\log\left(4F - 4\bar{F} + \bar{t}^a \partial_a F - t^a \partial_a \bar{F}\right)$$

$$R_{a\bar{b}c\bar{d}} = G_{a\bar{b}} G_{c\bar{d}} + G_{a\bar{d}} G_{c\bar{b}} - e^{2K} C_{ace} \bar{C}_{\bar{b}\bar{d}\bar{f}} G^{e\bar{f}}$$

where $C_{abc} = \dfrac{\partial^3 F}{\partial t^a \partial t^b \partial t^c}$

$\Updownarrow$

$$[\nabla - C, \nabla - C] = 0$$

non-compact examples

1. local $\mathbb{C}P^2$

$\mathbb{C}P^2$ is not CY, Consider a line bundle over $\mathbb{C}P^1$.
so that the 1st chern class of the fiber cancels that of the base.

$\Rightarrow$ start with $(x, z_1, z_2, z_3) \in \mathbb{C}^4 \setminus \{0\}$

$$(x, z_1 z_2 z_3) \sim (\lambda^{-3} x, \lambda z_1 \lambda z_2 \lambda z_3)$$

Weighted projective space

It is a total space of $\mathcal{O}(-3) \to \mathbb{C}P^2$.

2. local $\mathbb{C}P^1$

$$(x_1, x_2, z_1, z_2) \sim (\lambda^{-1} x_1 \lambda^{-1} x_2 \lambda z_1 \lambda z_2)$$

$$\Rightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$$

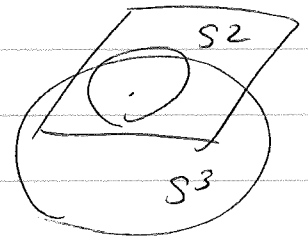
3. conifold

$(x, y, w, z) \in \mathbb{C}^4$

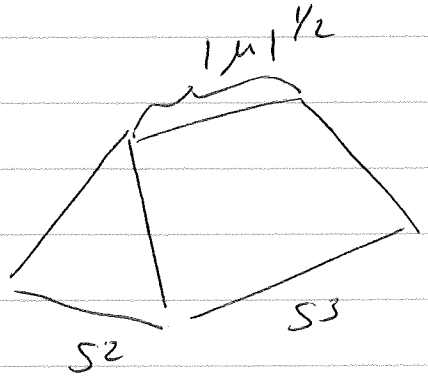
$xy - wz = \mu$. μ : complex moduli

This is the same as T^*S^3

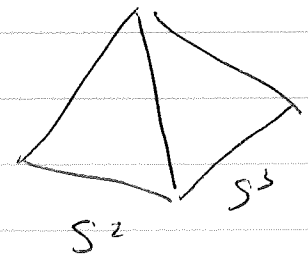
(radius $S^3 \sim \sqrt{|\mu|}$)



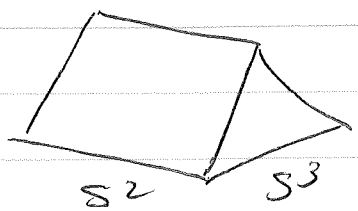
$\Rightarrow$ the infinity of T^*S^3 looks like $S^2 \times S^3$



$\mu \to 0$: conifold singularity



$\Longrightarrow$ resolution



toric CTs

Example local $\mathbb{CP}^2$

$(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \setminus \{0\}$

$(Q_0, Q_1, Q_2, Q_3) \equiv (-3, 1, 1, 1)$ "charge"

Consider $z_i \to e^{iQ_i\theta} z_i \qquad i = 0, 1, 2, 3$

If $\mathbb{C}^4$ is given with the Kähler fm

$$k = i \sum_{i=0}^{3} dz_i \wedge d\bar{z}_i$$

the $U(1)$ symmetry is generated by

$$\varphi \equiv -3|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2$$

local $\mathbb{CP}^2 = \varphi^{-1}(r) / U(1)$

Define $z_i = \sqrt{p_i}\, e^{i\phi_i}$ $\qquad p_i \geq 0$

$$k = i\sum_i dz_i \wedge d\bar{z}_i = i\sum_i dp_i \wedge d\phi_i$$

$$\varphi = -3p_0 + p_1 + p_2 + p_3 = r.$$

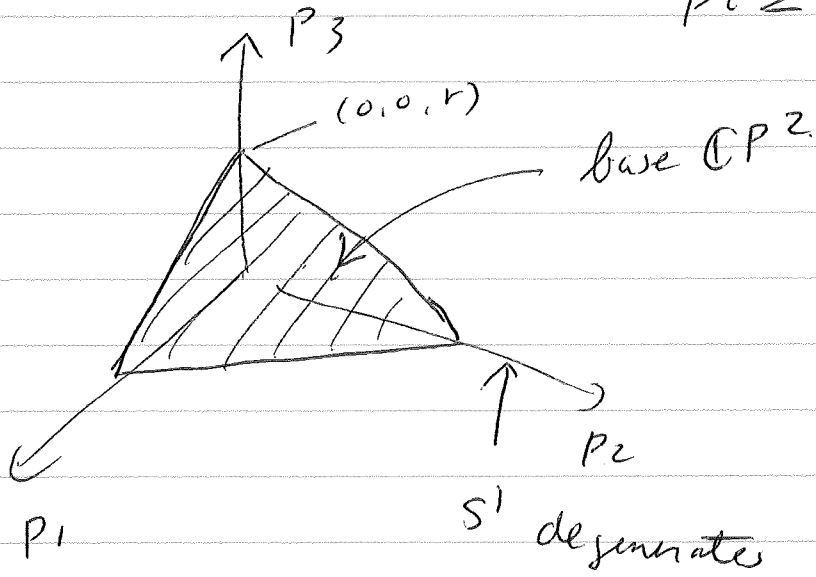
$$(\phi_0, \phi_1, \phi_2, \phi_3) \sim (\phi_0 - 3\theta, \phi_1 + \theta, \phi_2 + \theta, \phi_3 + \theta)$$

We can eliminate (p_0, ϕ_0)

$\Rightarrow$ T^3 (ϕ_1, ϕ_2, ϕ_3) fibered over

(p_1, p_2, p_3) $\qquad p_1 + p_2 + p_3 \geq r$

$p_i \geq 0$



In general,

- Start with $(z_1, \cdots, z_{N+3}) \in \mathbb{C}^{N+3} \setminus \{0\}$
- Divide by $U(1)^N$

$$Q^1 = (Q_1^1 \cdots Q_{N+3}^1)$$

$$\vdots$$

$$Q^N = (Q_1^1 \cdots Q_{N+3}^N)$$

- $\varphi_a = \sum_{i=1}^{N+3} Q_i^a |z_i|^2$

$\varphi_a = t^a$: Kähler moduli

toric $CY_3 = \varphi^{-1}(t) / U(1)^N$

---

local $\mathbb{CP}^1$.

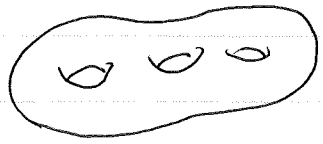
$$Q = (-1, -1, 1, 1)$$

Question: Draw its toric diagram

Topological String Theory

- string theory perturbative amplitudes

Start with 2d conformal field theory (CFT)
(↑ for Caltech students, more in the next week.)

2d surface Σ  with a metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

CFT : local scale invariance $g_{\mu\nu} \to \Omega g_{\mu\nu}$

$\Updownarrow$

$T^\mu{}_\mu$ = c-number

Locality + scaling $\Rightarrow$ $T^\mu{}_\mu = c R$ (R: scalar curvature)

c : central charge.

In this case, CFT amplitudes depend only on conformal equivalence class of $g_{\mu\nu}$.

$\{ g_{\mu\nu} \} / _{g_{\mu\nu} \sim \Omega g_{\mu\nu}} = \mathcal{M}$: moduli space of complex structure of Σ.

g-loop string amplitudes

$$= \int_{\mathcal{M}} (\text{CFT amplitudes on } \Sigma \text{ with genus } g)$$

$$(\dim \mathcal{M} = 6g-6)$$

This makes sense only when $\underline{c=0}$.
(There are other conditions to define the measure.)

- sigma-model --- basic example of CFT

M : n-dim Riemannian manifold, G_{IJ} : metric

sigma-model variable X^I : $\Sigma \to M$

$$\mathcal{L} = \frac{1}{2}\sqrt{g}\, g^{\mu\nu}\, G_{IJ}(X)\, \partial_\mu X^I \partial_\nu X^J$$

↑ metric on Σ ↖ metric on M

If we choose complex coordinate z on Σ,

$$ds^2 = 2 g_{z\bar{z}}\, dz\, d\bar{z}$$

$$\mathcal{L} = \frac{1}{2} G_{IJ}(X)\, \partial_z X^I \partial_{\bar{z}} X^J$$

$\mathcal{L}$: $(1,1)$-form on Σ.

$S = \int_\Sigma \mathcal{L}$ is (classically) scale invariant.

There is another scale invariant term one can add.

$$B_{IJ}\, dX^I \wedge dX^J \in \Omega^2(M) \qquad \text{(antisym 2 tensor)}$$

$$\epsilon^{\mu\nu} B_{IJ}(X)\, \partial_\mu X^I\, \partial_\nu X^J$$

$$= i\, B_{IJ}\, \partial_z X^I\, \partial_{\bar z} X^J$$

$$\mathcal{L} = \frac{1}{2}\underbrace{(G_{IJ} + i\, B_{IJ})}\, \partial_z X^I\, \partial_{\bar z} X^J$$

complexification of the metric.

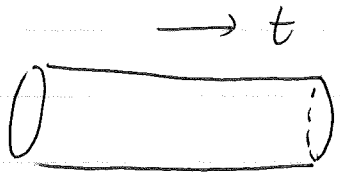
Question 1: Show that, when B_{IJ} is shifted by $\partial_I \Lambda_J - \partial_J \Lambda_I$, $\mathcal{L}$ changes by a total derivative on Σ.

- Sigma-model on a torus

Consider $M = S^1$, radius R.

$$G_{11} = R^2, \quad B_{11} = 0$$

Consider a Lorentzian metric $(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ on Σ



cylindrical worldsheet

$$ds^2 = -dt^2 + d\theta^2$$

$$S = \int_0^{2\pi} d\theta \, \frac{R^2}{4\pi} \left((\partial_t X)^2 - (\partial_\theta X)^2 \right)$$

$$0 \le X(t,\theta) \le 2\pi$$

$X(t,\theta)$ represents ∞-many degrees of freedom

momentum conjugate to X : $P = R^2 \partial_t X$

$$H = \int_0^{2\pi} \frac{d\theta}{4\pi} \left(\frac{1}{R^2} P^2 + R^2 (\partial_\theta X)^2 \right)$$

- Since X is periodic, the center of mass momentum is quantized.

$$P = n + \text{oscillators}$$

(non-zero Fourier modes)

- As θ goes from 0 to 2π, X can wind S^1 several times.

$$X = m\theta + \text{oscillators}$$

$$n, m \in \mathbb{Z}$$

$$H = \frac{1}{2}\left(\left(\frac{n}{R}\right)^2 + (mR)^2 \right) + \text{harmonic oscillators}$$

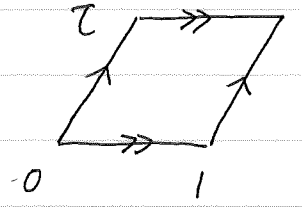
n : momentum , m : winding number

This is invariant under $R \to \frac{1}{R}$.

In fact, this is symmetry of this CFT
--- T-duality

Moduli space of S^1 :

2d torus T^2

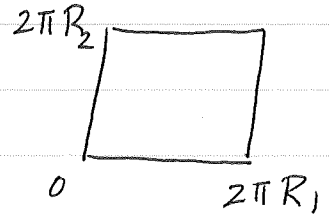


τ : complex structure moduli

$t = i(\text{area of } T^2) + B_{12}$

Kähler moduli.

For simplicity, consider · τ = pure imaginary
· $B_{12} = 0$



$$\tau = i R_2/R_1$$
$$t = i R_1 R_2$$

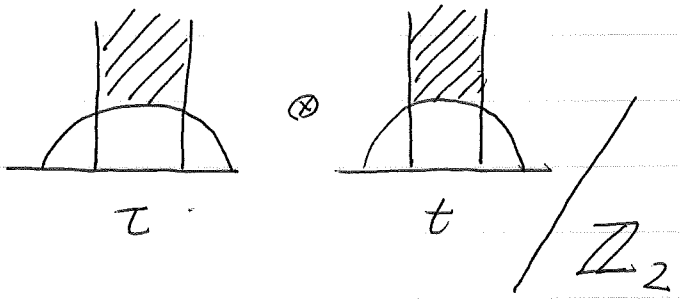
$R_1 \to 1/R_1 \Leftrightarrow t \to \tau$
$\tau \to t$

This holds even if τ, t : complex

$\tau \leftrightarrow t$: mirror symmetry

Since the moduli space of τ is $H/SL(2,\mathbb{Z})$, so is that of t.

Moduli space of T^2



- How about curved target space ?

Quantum effects can break conformal invariance because of UV divergences.

e.g. QED with massless electron(s) is classically scale invariant, but the renormalization introduces a scale.

2d sigma-model with metric G_{IJ} is one-loop scale invariant if $R_{IJ} = 0$. (more general condition for $B_{IJ} \neq 0$)

Asymptotic free if $R_{IJ} > 0$.

- supersymmetric sigma-model.

c.f. supersymmetric quantum mechanics

$$X^I : \mathbb{R} \to M$$
$$\psi, \bar{\psi}^I : \mathbb{R} \to T_{X(t)} M$$

$\Rightarrow$ differential forms, superchanges $= d, d^\dagger$.

Generalize this to 2d.

$$\Omega^{(m,n)}(\Sigma) \ni \omega (dz)^m (d\bar{z})^n$$

spinors : $\Omega^{(\frac{1}{2},0)}$, $\Omega^{(0,\frac{1}{2})}$ (Weyl basis)

$$\begin{cases} \psi_L{}^I : \Sigma \to T_x M \otimes \Omega^{(\frac{1}{2},0)} \\ \psi_R{}^I : \Sigma \to T_x M \otimes \Omega^{(0,\frac{1}{2})} \end{cases}$$

$$\mathcal{L} = \frac{1}{2} G_{IJ} \partial_z X^I \partial_{\bar{z}} X^J + \frac{i}{2} G_{IJ} \bar{\psi}^I \gamma^\mu D_\mu \psi^J \sqrt{g}$$
$$+ \frac{1}{12} R_{IJKL} \bar{\psi}^I \psi^J \bar{\psi}^K \psi^L$$

$$D_\mu \psi^I = \partial_\mu \psi^I + \Gamma^I_{JK} \partial_\mu X^J \psi^K$$

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi$$
$$\psi_R = \frac{1}{2}(1+\gamma^5)\psi$$

Theorem : If M is a Calabi-Yau manifold,

i.e.

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \,, \quad R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \det G = 0 \,,$$

then $G_{i\bar{j}}$ is invariant under renormalization

modulo $K \to K +$ globally defined function

In particular, complex structure and Kähler class are not renormalized.

Namely, the supersymmetric sigma-model can be made a CFT by appropriately adjusting K.

$$\begin{cases} X : \Sigma \to M \\ \psi_L^{\,i} : \Sigma \to T^{1,0}_X M \otimes \Omega^{(\frac{1}{2},0)} \\ \psi_L^{\,\bar{i}} : \Sigma \to T^{0,1}_X M \otimes \Omega^{(\frac{1}{2},0)} \\ \psi_R^{\,i} \,, \ \psi_R^{\,\bar{i}} \end{cases}$$

can be changed to

so that $g_{i\bar{j}}\, \psi^{\bar{j}} \bar{\partial} \psi^{i}$ is $(1,1)$ form

$$\begin{cases} \otimes \Omega^{(p,0)} \\ \otimes \Omega^{(1-p,0)} \end{cases}$$

The central charges $c_L = 12 p_L (1-p_L)$

$c_R = 12 p_R (1-p_R)$

2 interesting cases :

A-model : $\psi_L^i : \Omega^{(0,0)}$, $\psi_L^{\bar{i}} : \Omega^{(1,0)}$

$\psi_R^i : \Omega^{(0,1)}$, $\psi_R^{\bar{i}} : \Omega^{(0,0)}$

B-model : $\psi_L^i : \Omega^{(1,0)}$, $\psi_L^{\bar{i}} : \Omega^{(0,0)}$

$\psi_R^i : \Omega^{(0,1)}$, $\psi_R^{\bar{i}} : \Omega^{(0,0)}$

In each model, there are 2 supersymmetries with scalar parameters $(\varepsilon, \bar{\varepsilon})$.

A-model $\delta X^i = \varepsilon \psi_L^i$ $\quad \delta X^{\bar{i}} = \bar{\varepsilon} \psi_R^{\bar{i}}$

$\delta \psi_L^{\bar{i}} = \varepsilon \partial X^{\bar{i}}$ $\quad \delta \psi_R^i = \bar{\varepsilon} \bar{\partial} X^i$

SUSY configuration : $\bar{\partial} X^i = 0$

--- holomorphic map.

B-model $\delta X^i = 0$

$\delta X^{\bar{i}} = \varepsilon \psi_L^{\bar{i}} + \bar{\varepsilon} \psi_R^{\bar{i}}$

$\delta \psi_L^i = \varepsilon \partial X^i$, $\quad \delta \psi_R^i = \bar{\varepsilon} \bar{\partial} X^i$

SUSY configuration : $\partial X^i = 0$, $\bar{\partial} X^i = 0$

--- constant map

In A-model, supersymmetric amplitudes depend only on Kähler moduli.

In fact, $k \equiv (i G_{i\bar{j}} + B_{i\bar{j}}) dX^i \wedge dX^{\bar{j}}$

$$S\Big|_{\substack{\bar{\partial} X^i = 0 \\ \psi_\cdot = 0}} = \int_\Sigma X^* k$$

In B-model, supersymmetric amplitudes depend only on complex structure.

M, $\tilde{M}$: mirror pair

$\Leftrightarrow$ A-model on M = B-model on $\tilde{M}$

e.g.

$T^2(\tau, t)$ and $T^2(t, \tau)$ make a mirror pair.

D branes

- brief review of CFT$_2$.

energy-momentum tensor $T_{\mu\nu} = \frac{2\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}$ ← action

$$T_{z\bar{z}} = 0, \quad \partial_{\bar{z}} T_{zz} = 0, \quad \partial_z T_{\bar{z}\bar{z}} = 0.$$

$$T(z)\,T(w) \sim \frac{c/2}{(z-w)^4} + \left(\frac{2}{(z-w)^2} + \frac{1}{z-w}\partial_w \right) T(w) + \cdots$$

c : central charge.

$\phi(z,\bar{z})\,(dz)^h (d\bar{z})^{\bar{h}}$: primary field.

$$\Rightarrow \quad T(z)\,\phi(w,\bar{w}) \sim \left(\frac{h}{(z-w)^2} + \frac{1}{z-w}\partial_w \right) \phi(w,\bar{w}) + \cdots$$

Hilbert space $= \bigoplus_{h,\bar{h}} N_{h,\bar{h}}\ Vir^{(h)} \otimes Vir^{(\bar{h})}$

$Vir^{(h)}$: highest weight rep of $L_n = \oint_0 \frac{dz}{2\pi i} z^{n+1} T(z)$

$$\left(T(z) = \sum_n L_n z^{-n-2} \right)$$

$L_0 |h\rangle = h\,|h\rangle$

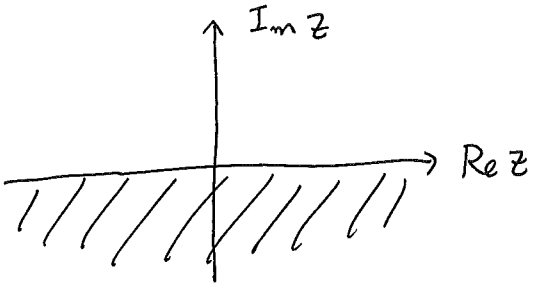
$L_n |h\rangle = 0 \ (n \geq 1)$

$Vir^{(h)} = \{ L_{-n_1} \cdots L_{-n_k} |h\rangle \}$

state-operator correspondence :

$$\phi(z,\bar{z}) \quad \longleftrightarrow \quad |h,\bar{h}\rangle = \lim_{z\to 0} \phi(z,\bar{z})\,|0\rangle$$

• CFT$_2$ with boundary.



boundary = real axis's

Conformal transf : $z \to z + \varepsilon_n z^n$

$\varepsilon_n \Leftrightarrow L_n$

$\bar{\varepsilon}_n \Leftrightarrow \bar{L}_n$

If we want to maintain conformal transf that keep the real axis, $\varepsilon_n \in \mathbb{R}$.

$\Rightarrow$ For the boundary condition compatible with this

$$T(z) = \bar{T}(\bar{z}) \text{ on the real axis's.}$$

method of images $\Rightarrow$ $T(z)$ can be extended over the entire $\mathbb{C}$.

<u>example</u> free massless scalar ϕ.

$$T = \frac{1}{2}(\partial\phi)^2, \quad \bar{T} = \frac{1}{2}(\bar{\partial}\phi)^2$$

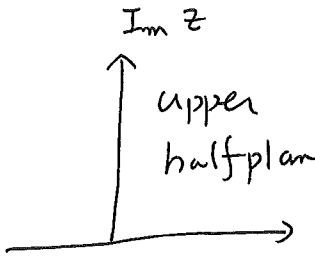
$T = \bar{T}$ on $z \in \mathbb{R} \Leftrightarrow$ $\partial\phi = \bar{\partial}\phi$: Neumann

$\partial\phi = -\bar{\partial}\phi$: Dirichelet

($\Leftrightarrow \phi$: const on $\mathbb{R}$)

D branes

Consider X^I :



$\longrightarrow \mathbb{R}^N$

$I = 1, \cdots, N$.

Some X^μ may obey Neumann condition,

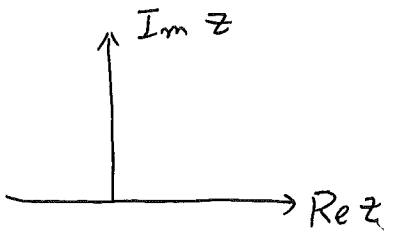
other X^i may obey Dirichelet condition.

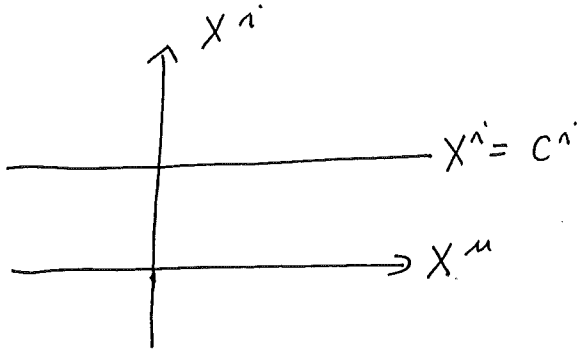
(There may also be some mix's of them.)

Suppose $\partial_\perp X^\mu = 0 \quad \mu = 1, \cdots, n$

$X^i = C^i \quad i = n+1, \cdots, N$ on $z \in \mathbb{R}$.

X^I :





The upper half plane

is mapped to $\mathbb{R}^N$ in such a way

that the Real Axis is on $\underline{X^i = C^i}$.

$\Uparrow$

D brane.

geometric way to

think about the boundary condition

Ishibashi State / Cardy State

Consider periodically identifying the boundary

boundary condition $\Rightarrow$ a state in CFT_2. $|B\rangle\!\rangle$

$T(z) = \bar{T}(\bar{z}) \quad \Rightarrow \quad L_n |B\rangle\!\rangle = \bar{L}_{-n} |B\rangle\!\rangle$

There is a general solution to this.

Start with a highest weight state $|h\rangle$.

Choose an orthonormal basis $\{ |h, N : j\rangle \}$

for $\{ L_{-n_1} \cdots L_{-n_k} |h\rangle \}_{n_1 + \cdots + n_k = N}$.

Then $|h\rangle\!\rangle = \sum_N \sum_j |h, N : j\rangle \otimes \overline{|h, N; j\rangle}$

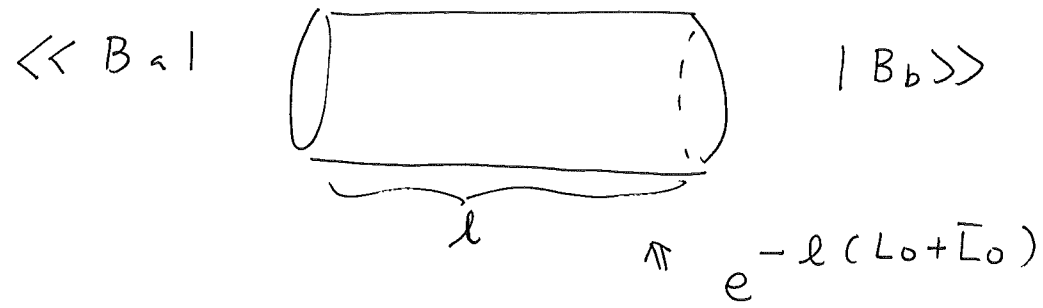
solves $L_n |h\rangle\!\rangle = \bar{L}_{-n} |h\rangle\!\rangle$

In general, $|B\rangle\!\rangle$ is a linear combination of $|h\rangle\!\rangle$ (Ishibashi states).

Cardy condition.

Not all linear combinations of $|h\rangle\!\rangle$ represent consistent boundary conditions.

Suppose we have $|B_a\rangle\!\rangle$, $|B_b\rangle\!\rangle$



One should be able to express this as $\mathrm{tr}\left(e^{-\frac{1}{l}H_{\mathrm{open}}}\right)$ for CFT$_2$ on the line segment $\overset{}{\longleftarrow}$ (l) with boundary conditions a and b at the two ends.

In particular.

$\langle\langle B_a | e^{-l(L_0+\bar{L}_0)} | B_b \rangle\rangle$ should be expanded by $e^{-\frac{\varepsilon}{l}}$ with <u>integer</u> coefficients

<u>Cardy states</u> $\Leftarrow$ For the minimal model, one can construct such $|B\rangle\!\rangle$ as linear combinations of $|h\rangle\!\rangle$ and classify them.

D branes for SUSY sigma model on CY_n

$$\mathcal{L} = \frac{1}{2} g_{i\bar{j}} (\partial X^i \bar{\partial} X^{\bar{j}} + \partial X^{\bar{j}} \bar{\partial} X^i)$$

$$+ \frac{i}{2} \bar{\psi}_L^{\bar{j}} \bar{D} \psi_L^j \, g_{i\bar{j}} + \frac{i}{2} \bar{\psi}_R^{\bar{j}} D \psi_R^i \, g_{i\bar{j}} + \cdots$$

$$T = \frac{1}{2} g_{i\bar{j}} \partial X^i \partial X^{\bar{j}} + \text{fermions}$$

$$G_L^+ = g_{i\bar{j}} \psi_L^i \partial X^{\bar{j}}, \quad G_L^- = g_{i\bar{j}} \psi_L^{\bar{j}} \partial X^i$$

$$G_R^+, \; G_R^-$$

There are two types of D branes:

$$T = \bar{T}$$

A branes : $G_L^+ = \pm G_R^+$, $G_L^- = \pm G_R^-$

B branes : $G_L^+ = \pm G_R^-$, $G_L^- = \pm G_R^+$

- A brane = γ in CY_n such that ... Lagrangian
  - $\dim_{\mathbb{R}} \gamma = n$
  - $k|_\gamma = 0$ $\quad k = i g_{i\bar{j}} \, dx^i \wedge dx^{\bar{j}}$
- B brane = γ in CY_n

  such that $\gamma \subset CY_n$

  is a holomorphic submanifold.

Large N duality

$F_{g;n_1 \cdots n_k}$: open string amplitudes

$$F_g(t_1, \cdots, t_k) = \sum_{n_1 \cdots n_k} F_{g;n_1 \cdots n_k} (t_1)^{n_1} \cdots (t_k)^{n_k}$$

Can we interpret $F_g(t)$ as

a closed string amplitude on genus g

for some CFT$_2$ with parameters $t_1 \cdots t_k$?

Example 1 AdS / CFT correspondence

$F_{g,n}$ $\Leftarrow$ $N=4$ super Yang-Mills theory on $\mathbb{R}^4$

with gauge group $SU(N)$ ($t = g_{YM}^2 N$)

$F_g(t)$ $\Leftarrow$ Type IIB superstring on $AdS_5 \times S^5$

$t \sim$ curvature radius of AdS_5 and S^5.

example 2

$F_{g,n} \Leftarrow$ Chern-Simons gauge theory on S^3 with gauge group $SU(N)$

$$S_{CS} = \frac{k}{4\pi} \int_{S^3} \mathrm{tr}\left(A dA + \frac{2}{3} A^3\right)$$

partition functi

$$Z_1 = \frac{e^{i\frac{\pi}{8}N(N-1)}}{(k+N)^{N/2}} \sqrt{\frac{k+N}{N}} \prod_{s=1}^{N-1} \left(2\sin\left(\frac{s\pi}{k+N}\right)\right)^{N-s}$$

$$= \exp\left(-\sum_g \sum_n F_{g,n}\, \lambda^{2g-2}\, t^n\right)$$

$$\lambda = \frac{2\pi}{k+N}, \qquad t = \frac{2\pi i N}{k+N}$$

Then

$$F_0(t) = \frac{i}{12} t^3 + \sum_{m=1}^{\infty} m^{-3} e^{-mt}$$

$$F_1(t) = \frac{1}{24} t + \frac{1}{12} \log\left(1 - e^{-t}\right)$$

$$F_{g\geq 2}(t) = \frac{2 B_g \zeta(2g-2)}{(2\pi)^{2g-2} 2g(2g-2)} - \frac{(-1)^{g-1}}{2g(2g-2)!} B_g \sum_m m^{2g-3} e^{-mt}$$

Close topological strin on the cunifold

Lecture A1

matrix model

matrix integral

Consider an $N \times N$ hermitian matrix M and its potential of the form $\mathrm{tr}W(M)$, where $W(M)$ is a polynomial of M. It is invariant under conjugation, $\mathrm{tr}W(\Omega M \Omega^{-1}) = \mathrm{tr}W(M)$, where $\Omega \in U(N)$. The partition function Z of the matrix model is defined by the integral,

$$Z = \int dM e^{-\mathrm{tr}\ W(M)},$$

where the measure is $dM = 2^{\frac{1}{2}N(N-1)} \prod_i dM_{ii} \prod_{i<j} d\mathrm{Re}M_{ij} d\mathrm{Im}M_{ij}$ and is invariant under the $U(N)$ action. The matrix model has widespread applications in theoretical physics. It is a toy model of the functional integral of the gauge theory (it can be thought of as a zero-dimensional quantum field theory). It was introduced to understand spectra of atomic nuclei, it was used to understand non-Abelian gauge theory in the limit of a large gauge group, it described dynamics of D branes in string theory in certain situations, it has close connections to quantum geometry of Calabi-Yau manifolds, *etc.*

Feynman diagrams

Let us start with the simplest case of $N = 1$ and when the integral is Gaussian, $W(M) = \frac{a}{2}M^2$,

$$Z_0 = \int_{-\infty}^{\infty} dM e^{-\frac{a}{2}M^2} = \sqrt{\frac{2\pi}{a}}.$$

We can also compute *correlation functions*,

$$\langle M^k \rangle_0 = \frac{1}{Z_0} \int_{-\infty}^{\infty} dM\ M^k\ e^{-\frac{a}{2}M^2} = \frac{1}{Z_0} \left(-2\frac{d}{da}\right)^k Z_0 = (2k-1)!! a^{-k},$$

where $(2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$. This can be used to evaluate a matrix integral for a more complicated potential as,

$$\begin{aligned} Z(a,g) &= \int_{-\infty}^{\infty} dM e^{-\frac{a}{2}M^2 - \frac{g}{4!}M^4} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{g}{4!}\right)^n \langle M^{4n} \rangle_0 Z_0(a) \\ &= \sum_{n=0}^{\infty} \frac{(4n-1)!!}{n!(4!)^n} \left(-\frac{g}{a^2}\right)^n Z_0(a). \end{aligned} \tag{1}$$

Note that the sum is an asymptotic expansion in g, and it is not convergent. The combinatorial factor $(4n-1)!!/n!(4!)^n$ is equal to the number of Feynman diagrams with n vertices with 4 legs each. Namely, it is equal to the number of ways to connect the n M^4 vertices using $2n$ lines (propagators). When a diagram has a symmetry, we divide its contribution by the order of the symmetry group. For example, the figure-8 shape diagram for $n = 1$ has $\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2}$ symmetry, and its contribution to the above sum is $1/8$ times $-g/a^2$.

't Hooft counting

Now we move on to $N > 1$. Consider the potential,

$$W(M) = \frac{1}{2}M^2 + \sum_{p=3}^{\infty} \frac{g_p}{p} M^p.$$

Since both the potential and the measure dM is $U(N)$ invariant, it is reasonable to define the partition function by divising the matrix integral by the volume of the $U(N)$ group given by

$$\text{vol } U(N) = \frac{(2\pi)^{\frac{1}{2}N(N+1)}}{G_2(N+1)},$$

where $G_2(z)$ is the Barnes double-Γ function satisfying,

$$G_2(z+1) = \Gamma(z)G_2(z), \quad G_2(1) = 1.$$

The matrix model partition function is then given by,

$$Z = \frac{1}{\text{vol } U(N)} \int dM e^{-\frac{1}{\lambda}\text{tr}W(M)}.$$

Here we introduced the parameter λ to keep track of the perturbative exapansion. Later we will indentify it with the string coupling constant.

Let us evaluate the integral Z in powers of λ. As in the case of $N = 1$, we should first understand the Gaussian integral with $g_p = 0$ ($p = 3, 4, ...$). We can show, for example,

$$\langle M_{ij} M_{kl} \rangle_{Gaussian} = \lambda \delta_{il} \delta_{jk}.$$

Expectation values of higher powers of M are given as sums of products of this two-point function $\langle M_{ij} M_{kl} \rangle_{Gaussian}$. This property of the Gaussian integral is known as the Wick theorem. Since the actio of the $U(N)$ group is given by $M \to \Omega M \Omega^{-1}$, we can think of M_{ij} as in the tensor product of the fundamental representation i and the anti-fundamental representation j of $U(N)$. We can visualize the two-point function $\langle M_{ij} M_{kl} \rangle_{Gaussian}$ by connecting an arrow from i to l and another arrow from k to j, namely from one of the fundamental representation to one of the anti-fundamental representation.

We can compute $\langle \text{tr} M^2 \rangle_{Gaussian}$ by visualizing a pair of loops, one going clockwisely and another going counter-clockwisely. We should also remember that each double-line (a pair of arrows) is weighted by λ. Thus, we see that it should be equal to λN^2.

Question 1: Compute $\langle \text{tr} M^2 \ \text{tr} M^2 \rangle_{Gaussian}$.

To compute $\langle \text{tr} M^3 \ \text{tr} M^3 \rangle_{Gaussian}$, we can visualize 2 cubic-vertices. From each of the cubic vertices, 3 double-ines are emenating, and we need to tie them together, paying attention to the orientations of the arrows. It turns out that there are 2 topologically distinct ways of doing it. One involves 3 loops of arrows, giving $12N^3\lambda^3$ (we have N^3 since there are 3 loops, and we have λ^3 since there are 3 double-lines). Another one involves just 1 loop, giving $3N\lambda^3$ (we have N since there is only 1 loop). These two Feynman diagrams are topologically distinct. For example, the former can be drawn on a plane sheet of paper without any lines crossing each other. It

is not possible to do so for the latter. Another way to see the difference is to pay attention to each loop in the diagrams and to identify a disk bounded by the loop so that the loop goes clockwisely as seen from the top of the disk. By attaching such disks, each Feynman diagram can be turned into a closed surface. In our example, this generates two distinct surfaces. For the first type of Feynman diagrams, the resulting surface is the 2-sphere. For the second type, it is a torus.

In general, the perturbative expansion of the partition function Z generates a sum of possibly disjoint diagrams. We can generate a sum of connected diagrams if we take the logarithm $\log Z$. We can then express is as a sum over connected surfaces. It is interesting to note that surfaces generated in this way are always orientable. This is because we started with arrows with definite orientations and is ultimately because we started with the integral over hermitial matrices. If we had started with an integral over anti-symmetric matrices, we would have generated surfaces without orientations (*e.g.* including the Klein bottle).

Each Feynman diagram is weighted as follows. Let us call the number of p-valent vertices as V_p with the total number of vertices $V = \sum_p V_p$, the number of double-lines (propagators) as E, the number of disks (faces) as F. The weight can be counted as,

$$\prod_p \left(-\frac{g_p}{\lambda}\right)^{V_p} \lambda^E N^F = \lambda^{E-V} N^F \prod_p (-g_p)^{V_p} = \lambda^{-V+E-F} (\lambda N)^F \prod_p (-g_p)^{V_p}.$$

It is interesting to note that the combination $(V - E + F)$ is the Euler characteristic of the surface one gets by attaching disks to the Feynman diagram. For a closed surface,

$$V - E + F = 2 - 2g,$$

where g is the number of handles attached to the surface ($g = 0$ for the 2-sphere, $g = 1$ for the torus, *etc*).

The combination $t = \lambda N$ is called the 't Hooft coupling. When we use the perturbative expansion in λ, we assume that $\lambda \ll 1$. The above observation shows that, if we take this limit while keeping the 't Hoof coupling finite, we can express $\log Z$ as a sum of connected surfaces weighted by λ^{2g-2},

$$F = \log Z = \sum_{g=0}^{\infty} F_g(g_p, t) \lambda^{2g-2}.$$

This is called the 't Hooft expansion or the large N expansion (since $\lambda \ll 1$ with $t = \lambda N$ finite means $N \gg 1$). Gerard 't Hooft speculated that the resulting $F_g(g_p, t)$ should have a nice interpretation in terms of a closed string theory. This turned out to be the case for the matrix model.

eigenvalue integral

Since the matrix integral is invariant under $M \to \Omega M \Omega^{-1}$, we can express it as an integral over eigenvalues of M.

$$Z = \frac{1}{\mathrm{vol}U(N)} \int dM \; e^{-\frac{1}{\lambda}\mathrm{tr}W(M)} = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{d\lambda_i}{2\pi} e^{-\frac{1}{\lambda}W(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2.$$

This follows from the following identity,

$$\Delta(M)^2 \cdot \int_{U(N)} d\Omega \prod_{i<j} \delta\Big((\Omega M \Omega^{-1})_{ij}\Big) = 1,$$

where $\Delta(M)$ defined so that,

(1) It is an invariant function, $\Delta(\Omega M \Omega^{-1}) = \Delta(M)$.

(2) It is equal to $\prod_{i<j}(\lambda_i - \lambda_j)$ when M is diagonal.

One can think of $\Delta(M)^2$ as the Faddeev-Popov determinant for the gauge condition, $M_{ij} = 0$ $(i < j)$, namely M being diagonal. Using this,

$$\begin{aligned}
\int dM\ e^{-\frac{1}{\lambda}\mathrm{tr}W(M)} &= \int dM\ e^{-\frac{1}{\lambda}\mathrm{tr}W(M)} \Delta(M)^2 \int_{U(N)} d\Omega \prod_{i<j} \delta\Big((\Omega M \Omega^{-1})_{ij}\Big) \\
&= \int dM'\ e^{-\frac{1}{\lambda}\mathrm{tr}W(M')} \Delta(M')^2 \prod_{i<j} \delta(M'_{ij}) \int_{U(N)} d\Omega, \quad \text{(we set } M' = \Omega M \Omega^{-1}\text{)} \\
&= \frac{\mathrm{vol}(U(N))}{(2\pi)^N N!} \int \prod_{i=1}^{N} d\lambda_i e^{\frac{1}{\lambda}W(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2. \qquad (2)
\end{aligned}$$

This proves the eigenvalue integral expression for Z. One can think of the factor $1/N!$ as taking care of the residual gauge symmetry of exchanging the eigenvalues, which remains after imposing the gauge condition $M_{ij} = 0$ $(i < j)$.

eigenvalue distribution

We can write the eigenvalue integral as,

$$Z = \frac{1}{N!} \int \prod_i \frac{d\lambda_i}{2\pi} e^{-N^2 V(\lambda)},$$

where

$$V(\lambda) = \frac{1}{N} \sum_i t^{-1} W(\lambda_i) - \frac{1}{N^2} \sum_{i<j} \log(\lambda_i - \lambda_j)^2.$$

One can think of this as the potential energy of N particles in one dimensions with coordinate λ in the potential $t^{-1}W(\lambda)$ and with the repulsive potential $-\log(\lambda_i - \lambda_j)^2$.

In the large N limit, the sum over i gives a factor of N, and $V(\lambda)$ is then of the order 1. In fact, by introducing the eigenvalue distribution function,

$$\rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i),$$

which is normalized as

$$\int d\lambda \rho(\lambda) = 1,$$

we can write the potential as a functional of ρ,

$$V(\rho) = \frac{1}{t} \int d\lambda \rho(\lambda) W(\lambda) - \int d\lambda d\lambda' \rho(\lambda)\rho(\lambda') log|\lambda - \lambda'|.$$

Since the integrand for the eigenvalue integral is $\exp(-N^2 V)$, the eigenvalues will try to settle in the configuration to minimize the potential V. A variation of V with respect to ρ gives,

$$\frac{1}{2t}W'(\lambda) = \mathrm{P}\int \frac{\rho(\lambda')}{\lambda - \lambda'}d\lambda'.$$

Here P in the right-hand side means that we take the principal value of the integral. To be precise, the above is obtained by taking a derivative of the equation $\delta V/\delta\rho = 0$ with respect to λ. This turns out to be more convenient than the orignal equation.

resolvent

To solve the eigenvalue equation, it is convenient to introduce the resolvent,

$$\omega(p) = \frac{1}{N}\langle \mathrm{tr}\frac{1}{p - M}\rangle.$$

In the large N limit, it becomes

$$\omega_0(p) = \int d\lambda \frac{\rho(\lambda)}{p - \lambda}.$$

Suppose that the eigenvalue distribution $\rho(\lambda)$ has a finite support C on the real line $\mathbf{R}$. The resolvent $\omega_0(p)$ has a branch cut on C. The resolvent has several properties, which can be derived from its definition:

(1) It is analytic, except on C.

(2) $\omega_0(p) \sim 1/p$ for $p \to \infty$.

(3) $\omega_0(\lambda + i\epsilon) - \omega_0(\lambda - i\epsilon) = -2\pi i\rho(\lambda)$ for $\lambda \in C$ and $0 < \epsilon \ll 1$.

The eigenvalue equation then gives,

(4) $\omega_0(\lambda + i\epsilon) + \omega_0(\lambda - i\epsilon) = -\frac{1}{t}W'(\lambda)$.

Assuming the eigenvalue distribution generates a single cut, $C = [b, a]$, there is an explicit solution to these conditions of the form,

$$\omega_0(p) = \frac{1}{2t}\oint \frac{dz}{2\pi i}\frac{W'(z)}{p - z}\sqrt{\frac{(p-a)(p-b)}{(z-a)(z-b)}}.$$

The end points of the cut are determined by the conditions (1) and (2).

As an example, consider the Gaussian model with $W(\lambda) = \frac{1}{2}\lambda^2$. In this case, we expect $a = -b$ by symmetry, and we have

$$\omega_0(p) = \frac{\sqrt{p^2 - a^2}}{2t}\oint \frac{dz}{2\pi i}\frac{z}{(p - z)\sqrt{z^2 - a^2}}.$$

Requiring (2), one finds $a = 2\sqrt{t}$ and

$$\omega_0(p) = \frac{1}{2t}\left(p - \sqrt{p^2 - 4t}\right).$$

By using (3), the eigenvalue distribution is given by

$$\rho(\lambda) = \frac{1}{2\pi t}\sqrt{4t - \lambda^2}.$$

This is the famous semi-circle law of Eugene Wigner.

Generally speaking, if we write

$$\omega_0(p) = \frac{1}{2t}\left(y(p) + W'(p)\right),$$

$y(p)$ obeys

$$y(p)^2 = W'(p)^2 - R(p),$$

where

$$R(p) = 4t\int d\lambda\rho(\lambda)\frac{W'(p) - W'(\lambda)}{p - \lambda}.$$

If $W(p)$ is a polynomial of p of degree k, $R(p)$ is a polynomial of degree $(k-2)$. Thus, (y, p) obeying the above equation defines an algebraic curve (Riemann surface) in $\mathbf{C}^2$. Here we considered a one-cut solution, but in general the curve can have $(k-1)$ branch cuts. This is as expected since it is the same as the number of extrema of $W(p)$, where eigenvalues can congregate.

Given the algebraic curve, we can consider a complex 3-dimensional manifold defined by

$$uv = y^2 - W'(p)^2 + R(p), \qquad (u, v, y, p) \in \mathbf{C}^4.$$

It turns out that this manifold is Calabi-Yau and we can introduce a Ricci-flat Kähler metric on it. We can consider the closed topological string theory of B-type with this Calabi-Yau manifold as its target space, and its partition function is equal to the matrix model partition function Z. In this case, the topological string theory gives the large N dual of the matrix model anticipated by 't Hooft.

Lecture A2

conformal field theory

Killing vector fields

The sphere S^n is invariant under the group $SO(n+1)$. The Minkowski space is invariant under the Poincaré group, which includes translations, rotations, and Lorentz boosts. For a general Riemannian manifold M, take a tangent vector field $\xi = \xi^\mu \partial_\mu$ and consider the infinitesimal coordinate transformation,

$$x^\mu \to x^\mu + \epsilon \xi^\mu, \quad (|\epsilon| \ll 1).$$

Question 1: Show that the metric components $g_{\mu\nu}$ transforms as

$$g_{\mu\nu} \to g_{\mu\nu} + \epsilon \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \xi^\rho \partial_\rho\right) + 0(\epsilon^2) = g_{\mu\nu} + \epsilon \left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right) + 0(\epsilon^2).$$

The metric is invariant under the infinitesimal transformation by ξ iff $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$. A tangent vector field satisfying this equation is called a Killing vector field.

Question 2: Suppose we have two tangent vector fields, $\xi_a = \xi_a^\mu \partial_\mu$ $(a = 1, 2)$. Show that their commutator

$$[\xi_1, \xi_2] = (\xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu)\partial_\mu$$

is also a tangent vector field. Explain why $\xi_1^\nu \partial_\nu \xi_2^\mu$ is not a tangent vector field in general.

Question 3: Suppose there are two Killing vector fields, ξ_a $(a = 1, 2)$. One can consider an infinitesimal transformation, g_a, corresponding to each of them. Namely,

$$g_a : x^\mu \to x^\mu + \epsilon \xi_a^\mu, \quad (a = 1, 2).$$

Show that $g_1 g_2 g_1^{-1} g_2^{-1}$ is generated by the commutator, $[\xi_1, \xi_2]$.

If the metric is invariant under infinitesimal transformations generated by ξ_1, ξ_2, it should also be invariant under their commutator, $[\xi_1, \xi_2]$. Thus, the space of Killing vector fields is closed under the commutator – it makes a Lie algebra.

Question 4: What is the Lie algebra of Killing vector fields on S^2?

conformal Killing vector fields

Let us relax the condition somewhat and allow the metric to be scaled as $g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x)$ under transformation, $x \to x + \epsilon\xi$ for some positive-definite function $\Omega(x)$. This means, $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = f(x) g_{\mu\nu}$ for some function $f(x)$. By taking the trace of both sides, one finds $f(x) = \frac{2}{n} \nabla \cdot \xi$ where $n = \mathrm{dim} M$. Thus,

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \frac{2}{n} \nabla \cdot \xi g_{\mu\nu}.$$

A tangent vector field ξ satisfying this equation is called a conformal Killing vector field. Note that $\nabla \cdot \xi = 0$ for a Killing vector field.

Question 5: Count the dimension of the space of conformal Killing vector fields for the n-dimensional Euclidean space and Minkowski space.

In two dimensions, the space of local solutions to the conformal Killing vector field equation is infinite dimensional. Consider a two-dimensional Riemannian manifold M with Euclidean signature (analysis for Minkowskian signature case is the same). In two dimensions, any Riemannian metric is Kähler and one can choose complex coordinates $(z, \bar{z})$ so that the metric becomes $ds^2 = 2g_{z\bar{z}}dzd\bar{z}$. In this metric, the conformal Killing vector equation becomes,

$$\partial_{\bar{z}}\xi^z = 0, \quad \partial_z\xi^{\bar{z}} = 0.$$

Namely, it simply means that ξ^z is holomorphic and $\xi^{\bar{z}}$ is anti-holomorphic. There are infinitely many solutions to this condition. In fact, under any holomorphic coordinate transformation, $w = w(z)$, one finds,

$$g_{w\bar{w}} = \left|\frac{\partial z}{\partial w}\right|^2 g_{z\bar{z}}.$$

Namely, it generates a scale transformation of the metric.

The space of conformal Killing vector fields also makes a Lie algebra. In two dimensions, one can choose $\xi = z^{n+1}\partial_z$ ($n \in \mathbf{Z}$) locally. Then,

$$[\xi_n, \xi_m] = (n-m)\xi_{n+m}. \qquad (1)$$

There is a similar relation for $\bar{\xi} = \bar{z}^{n+1}\partial_{\bar{z}}$.

free massless scalar field in two dimensions

Consider a massless scalar field $\phi(z, \bar{z})$ in two dimensions. Its action is of the form,

$$S = \frac{1}{2\pi}\int_M \sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$

so that the equations of motion is given by $\Delta\phi = 0$. The energy-momentum tensor $T_{\mu\nu}$ is defined by

$$T_{\mu\nu} = \frac{2\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2.$$

Note that $T_{\mu\nu}$ is symmetric. It is also traceless, $g^{\mu\nu}T_{\mu\nu} = 0$. It is a reflection of the fact that the action is invariant under the scale transformation, $g_{\mu\nu}(x) \to \Omega(x)g_{\mu\nu}(x)$.

If we use complex coordinates $(z, \bar{z})$, the trace part of the energy-momentum tensor is $T_{z\bar{z}}$. Thus, the only non-zero components are T_{zz} and $T_{\bar{z}\bar{z}}$. The conservation law, $\nabla^\mu T_{\mu\nu} = 0$, then implies that T_{zz} is holomorphic and $T_{\bar{z}\bar{z}}$ is anti-holomorphic. Since the transformation $z \to z + \epsilon z^{n+1}$ is symmetry, there must be a corresponding Nöther charge.

Suppose we start with the Minkowski signature space with coordiantes (t, θ) with the metric $ds^2 = -dt^2 + d\theta^2$. Take θ be periodic so that the space is a cylinder. Now Euclideanize the time coordinate $t = -i\tau$ so that $ds^2 = d\tau^2 + d\theta^2$. If we define $z = e^{\tau+i\theta}$, the past infinity $\tau \to -\infty$ corresponds to $z = 0$ and the fugure infinite $\tau \to +\infty$ corresponds to $z = \infty$. The constant time surface $\tau =$ const. corresponds to $|z| =$ const. In terms of this z coordinate, the Nöther charge for $z \to z + \epsilon z^{n+1}$ is given by

$$L_n = \oint_{z=0}\frac{dz}{2\pi i}z^{n+1}T_{zz}.$$

Here the contour is chosen to surround $z = 0$. Since both z^{n+1} and T_{zz} are holomorphic, L_n is invariant under continuous deformation of the integration contour. The Nöther charge is conserved.

For the free scalar field, the energy-momentum tensor is given by

$$T_{zz} = \frac{1}{2}(\partial_z \phi)^2.$$

Using the operator product expansion,

$$\partial_z \phi(z) \partial_w \phi(w) \sim \frac{1}{(z-w)^2} + 0(1),$$

one obtains

$$T_{zz} T_{ww} \sim \frac{1/2}{(z-w)^4} + \left(\frac{2}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) T_{ww} + 0(1).$$

Using this, one can derive the commutation relation,

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \qquad (2)$$

with $c = 1$.

The commutation relation of the conformal Killing vectors (1) is also of this form, with $c = 0$. One can think of c as arising from quantum effects. It is called the central charge since it commutes with all the generators L_n. The algebra generated by L_n's is called the Virasoro algebra.

minimal models

The Hilbert space of any conformal field theory (CFT) in two dimensions gives a representation of the Virasoro algebra (2). In fact, there are two copies of the Virasoro algebra, $Vir \oplus Vir$, one for T_{zz} and another for $T_{\bar{z}\bar{z}}$. In this lecture, we assume that the value of the central charge is the same for both Virasoro algebras.

If the CFT is unitary, one should be able to expand its Hilbert space as a sum of unitary representations of $Vir \oplus Vir$. The hermiticity of the energy-momentum tensor implies $L_n^\dagger = L_{-n}$. Since $L_0 + \bar{L}_0$ can be identified with the Hamiltonian for the translation of the time variable t and since

$$[L_0, Ln] = -nL_n,$$

a representation with energy bounded below must be of the highest weight type. Namely, we start with the highest weight state, $|h\rangle$, satisfying

$$L_0|h\rangle = h|h\rangle, \qquad L_n|h\rangle = 0 \qquad (n \geq 1),$$

and build the representation by acting L_{-n}'s. Such a representation is parameterized by the highest weight h. Note that the hermiticity of the energy-momentum tensor implies $L_n^\dagger = L_{-n}$.

It is natural to ask for what value of c, we can build a unitary CFT. For $c < 1$, unitary represenations appear only at discrete values,

$$c = 1 - \frac{6}{m(m+1)}, \qquad (m = 3, 4, 5, ...).$$

For such c, unitary hightest weight representations are parametrized as

$$h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \quad 1 \leq r \leq m-1,\ 1 \leq s \leq r.$$

Let us denote the corresponding highest weight representation by $Vir^{(m)}_{r,s}$. The Hilbert space $Hilb$ of the CFT should take the form,

$$Hilb = \sum_{r,s:r',s'} N_{r,s;r',s'} Vir^{(m)}_{r,s} \oplus Vir^{(m)}_{r',s'}, \qquad (3)$$

where the coefficients $N_{r,s;r',s'}$ are positive integers. Since the vacuum state must be unique, $N_{1,1;1,1} = 1$.

There is a complete classification of $c < 1$ unitary CFT's. The idea is to require the modular invariance discussed in Lecture 9. Consider the character of the Virasoro algebra defined by,

$$\chi_{r,s}(\tau) = \mathrm{tr}_{Vir^{(m)}_{r,s}} e^{2\pi i\tau(L_0 - \frac{c}{24})}.$$

(Do not confuse τ with the imaginary time discussed in the above. It is the modulus of the torus.) It turns out that $\chi_{r,s}(\tau)$'s with $1 \leq s \leq r \leq m-1$ transform into themselves under the modular group action,

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad (ad - bc = 1; a, b, c, d \in \mathbf{Z}).$$

The partition function of the Hilbert space defined by

$$Z(\tau, \bar{\tau}) = \mathrm{tr}_{Hilb} \left(e^{2\pi i\tau(L_0 - \frac{c}{24})} e^{-2\pi i\bar{\tau}(\bar{L}_0 - \frac{c}{24})} \right).$$

The structure of the Hilbert space implies

$$Z(\tau, \bar{\tau}) = \sum_{r,s:r',s'} N_{r,s;r',s'} \chi_{r,s}(\tau) \bar{\chi}_{r',s'}(\bar{\tau}).$$

There is a complete classification of modular invariant partition functions for $c < 1$ unitary conformal field theories.

primary fields

Suppose there is a field $\phi(z, \bar{z})$ which transforms as $\phi(dz)^h (d\bar{z})^{\bar{h}}$. Under infinitesimal coordinate transformation, $z \to z + \epsilon z^{n+1}$, it should transform as,

$$\phi \to \phi + \epsilon \left(z^{n+1} \partial_z + hnz^n \right) \phi(z).$$

Since this coordinate transformation is generated by L_n,

$$[L_n, \phi(z)] = \epsilon \left(z^{n+1} \partial_z + hnz^n \right) \phi(z).$$

This is equivalent to the operator product expansion,

$$T(z)\phi(w) \sim \left(\frac{h}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) \phi(w).$$

The field ϕ is called a primary field and $(h, \bar{h})$ are its (left and right) conformal weights. We see that the energy-momentum tensor $T(z)$ transforms almost as a primary field of weight 2, but there is a slight anomaly due to the term proportional to c.

state-operator correspondence

Suppose the conformal field theory is defined on the geometry $\mathbf{R} \times S^1$, where $\tau \in \mathbf{R}$ corresponds to the Euclideanized time variable and $\in S^1$ parametrizes the spatial section. The Hilbert space is defined at a given time τ. Since the theory is invariant under conformal coordinate transformation, let us introduce

$$z = \exp(\tau + i\theta).$$

In this coordinate, the past infinity is at $z = 0$.

If there is no operator at $z = 0$, the energy-momentum tensor is regular at $z = 0$. This means that

$$L_n(0)1 = \oint_{z=0} \frac{dz}{2\pi i} z^{n+1} T(z)1 = 0, \quad \text{for } n \geq -1.$$

On the other hand, if we put the primary field $\phi(z, \bar{z})$ of weight h at $z = 0$,

$$L_n(0)\phi(0) = \oint_{z=0} \frac{dz}{2\pi i} z^{n+1} T(z)\phi(0) = 0, \quad \text{for } n \geq 1.$$

Moreover,

$$L_0(0)\phi(0) = h\phi(0).$$

This is just as if we have the highest weight representation of the Virasoro algebra with the highest weight h.

Conversely one can start with a highest weight state $|h\rangle$ in the coordinates (τ, θ) and perform the coordinate transformation in the above to define a primary field $\phi(z, \bar{z})$ at $z = 0$.

This one-to-one correspondence between states and operators is very important in studying conformal field theories.

Lecture A3

Chern-Simons gauge theory

The Chern-Simons (CS) gauge theory in three dimensions is defined by the action,

$$\begin{aligned} S_{CS} &= \frac{k}{4\pi}\int \mathrm{tr}\left(AdA+\frac{2}{3}A^3\right), \\ &= \frac{k}{8\pi}\int \epsilon^{\mu\nu\rho}\mathrm{tr}\left(A_\mu(\partial_\nu A_\rho-\partial_\rho A_\nu)+\frac{2}{3}A_\mu[A_\nu,A_\rho]\right). \end{aligned} \tag{1}$$

where tr is the trace over the fundamental representation of the gauge group G and k is a parameter of the theory (inverse of the coupling constant). If G is compact and simple, k has to be an integer in order for the action to be gauge invariant. (This has to do with the fact that $\pi_3(G)=\mathbf{Z}$ for any such group, where π_3 is the 3rd homotopy group – to be introduced in Lecture 10.)

The CS theory is an example of a topological field theory since writing down its action does not require a metric. If the quantization can be carried out without introducing a metric, observables of the theory would give topological invariants.

abelian CS theory

As a warm-up exercise, let us study the case when $G=U(1)$, *i.e.* the abelian CS theory. It has the action,

$$S_{U(1)}=\frac{k}{4\pi}\int A\wedge dA,$$

with A being the $U(1)$ connection. Note that $U(1)$ is not a simple group and k is not quantized in this case.

Such an action appears, for example, as the low energy effective theory to describe the topological order in fractional quantum Hall effect states. The equation of motion for the action simply says that $F=dA=0$, which means that there is no local gauge-invariant observables. However, there are non-local observables. An important class of observables are the Wilson loops. Suppose that we are in $\mathbf{R}^3$. For each closed loop $\gamma\in\mathbf{R}^3$, we can consider the operator,

$$W(q;\gamma)=\exp\left(iq\oint_\gamma A\right).$$

Suppose we deform the loop infinitesimally. This can be done by attaching a small loop ϵ as $\gamma\to\gamma+\epsilon$. The Wilson loop changes as

$$W(q;\gamma+\epsilon)=\left(1+iq\oint_\epsilon A\right)W(q;\gamma)=\left(1+iq\int_D F\right)W(q;\gamma),$$

where D is a small disk in $\mathbf{R}^3$ such that $\partial D=\epsilon$ and we used the Stokes theorem. Since $F=0$ by the equations of motion, the expectation value of $W(q;\gamma)$ is invariant under infinitesimal deformation of γ. Namely, it is a topological invariant of γ. Since the functional integral is Gaussian, one can evaluate the expectation value of products of these operators exactly. For example,

$$\frac{\langle W(q_1;\gamma_1)W(q_2;\gamma_2)\rangle}{\langle W(q_1;\gamma_1)\rangle\langle W(q_2;\gamma_2)\rangle}=\exp\left(-q_1q_2\oint_{\gamma_1}dx^\mu\oint_{\gamma_2}dy^\nu\langle A_\mu(x)A_\mu(y)\rangle\right)=\exp\left(\frac{2\pi i}{k}q_1q_2\Phi(\gamma_1,\gamma_2)\right),$$

where

$$\Phi(\gamma_1,\gamma_2)=\frac{1}{4\pi}\oint_{\gamma_1}dx^\mu\oint_{\gamma_2}dy^\nu\epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}.$$

This is called the Gauss linkng number. It counts the number of links of the two loops (γ_1 and γ_2) by either $+1$ or -1 depending on the orientation of each link. This is one of the classical invariants of knots and links.

non-abelian CS theory

New features appear when the gauge group G is non-abelian. First of all, the action acquires the cubic term A^3, generating interactions. The defintion of the Wilson loop operator requires a care since A's do not commute with each other. Suppose the loop γ starts and end at a point x. Consider the vector bundle in a representation R of the structure group G and the connection A. Pick a vector v at the fiber over x. Now parallel transport v along the loop γ using the connection and come back to x. This gives another vector at the fiber over x. Thus, the parallen transport around γ defines a linear map on the fiber over x. The Wilson loop is defined as a trace of this linear map. More explicitly, we can also define the Wilson loop as

$$W(R;\gamma)=\mathrm{tr}_R P\exp\left(i\oint_\gamma A\right),$$

where tr_R indicates the trace over the representation space R, and the symbol P means the path-ordering.

In the simplest non-abelian case when $G=SU(2)$ and R is its fundamental representation, the expectation value of the Wilson loops gives the Jones polynomial invariants of knots and links. Below, we will see how one can compute such invariants.

In addition to invariants of knots and links, one can use the partition function (vacuum amplitude) of the CS theory to define a topological invariant of a 3-dimensional manifold M - the Reshetikhin-Turaev invariant.

canonical quantization

To compute amplitudes of the CS theory, it is often convenient to cut the manifold M into two parts across a two-dimensional surface. One reason for its usefulness is the existence of the Heegaard splitting. First define a handlebody as a 3-dimensional manifold with a boundary Σ such that cutting each of its handles across a disk produces the 3-sphere. Another way to think about a handle body is to embed a Riemann surface Σ in $\mathbf{R}^3$ and consider its interior plus the boundary Σ. Consider two such handlebodies with the same boundary Σ. Gluing them across the boundary produces a closed 3-dimensional manifold. It can be shown that any 3-dimensional differentiable manifold can be constructed in this way.

For example, consider a solid torus $T^2_{\alpha,\beta}$ with two homology cycles α,β and α is contractible. If we glue two copies of them together, we find $S^1\times S^2$. On the other hand, if we glue $T_{\alpha,\beta}$ with $T_{\beta,\alpha}$ together, we get S^3.

As a special case, consider a three manifold of the form $\mathbf{R}\times\Sigma$ and perform the canonical quantization with $\mathbf{R}$ as time. This produces a Hilbert space $\mathcal{H}_\Sigma$ associated to Σ. In particular, if M_a $(a=1,2)$ are handlebodies with $\partial M_a=(-1)^a\Sigma$ (the sign refers to the orientation), the CS functional integarls over M_a give particular state functions $\Psi_{a=1,2}$ in $\mathcal{H}_\Sigma$. The partition

function for the closed manifold with the Heegaard splitting intwo $M_1 \cup M_2$ is then given by the inner product of the two wave functions (Ψ_1, Ψ_2).

Let us denote the connection in the $\mathbf{R}$ direction and in the Σ direction by A_0 and $A_{i=1.2}$, respectively. The CS action on $\mathbf{R} \times \Sigma$ can then be written as

$$S_{CS} = \frac{k}{4\pi} \int dt \int_\Sigma \mathrm{tr} \left(\epsilon^{ij} A_i \frac{\partial}{\partial t} A_j + A_0 F_{ij} \right),$$

where $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$, the curvature of the vector bundle over Σ. In this action, A_0 serves as the Lagrange multiplier to enforce the constraint $F_{ij} = 0$ and the gauge connection A_i is canonically conjugate to $\frac{k}{2\pi}\epsilon^{ij} A_j$. Namely, we are dealing with a system with constraints.

When we have a system with constraints, there are two ways to quantize it:

(1) Start with the canonical commutations relations, in this case,

$$[A_i^a(x), A_j^b] = \frac{2\pi i}{k} \epsilon_{ij} \delta^{ab} \delta^2(x - y),$$

where a, b are gauge group indices, and impose $F_{ij} = 0$ as conditions on physical wave functions.

(2) Impose the constraint first. In our case, the resulting space is the space of flat vector bundles over Σ. On this space, one can define a non-degenerate symplectic form, which one can derive by applying the symplectic reduction of the canonical symplectic form on A_i^a, and then quantize the constrainted phase space.

These two procedures should give the same answer.

When $G = SU(2)$, the dimensions of the Hilbert space is given by the Verlinde formula,

$$\dim \mathcal{H}_\Sigma = \left(\frac{k+2}{2} \right)^{g-1} \sum_{j=0}^{k} \left(\sin \frac{(j+1)\pi}{k+2} \right)^{2-2g},$$

where g is the genus (the number of handles) of Σ. There is a generalization of this formula for compact G.

holomorphic quantization

In the undergraduate quantum mechanics, one considers the space of square integrable functions of $q \in \mathbf{R}$. The position operator is the multiplication by q, and its conjugate momentum is the derivative operator $p = -i\partial_q$. We can exchange the role of q and p, consider the space of square integrable functions of p and regard q as the differential operator, $i\partial_p$. We can also consider their hybrid. Choose complex coordinate,

$$u = q + ip, \quad \bar{u} = q - ip.$$

Use holomorphic functions of u for wave functions, and regard $\bar{u}$ as the derivative operator ∂_u. This is called the holomorphic quantization.

This approach is useful here since we can make use of the complex structure $(z, \bar{z})$ on the Riemann surface Σ. The commutation relations between A_i^a's can be writte in the complex coordinates as,

$$[A_z^a(z, \bar{z}), A_{\bar{z}}^b(w, \bar{w})] = \frac{2\pi}{k} \delta^{ab} \delta(z - w) \delta(\bar{z} - \bar{w}).$$

We can then consider the Hilbert space as the space of holomorphic functionals $\Psi(A)$ of $A_{\bar{z}}$ and regard A_z as the functional derivative,

$$A_z^a = \frac{2\pi}{k}\frac{\delta}{\delta A_{\bar{z}}^a}.$$

The constraint $F_ij = 0$ can then be expressed as conditions on wave functions as,

$$\left(\partial_{\bar{z}}\frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}}\right]\right)\Psi(A) = \frac{k}{2\pi}\partial_z A_{\bar{z}}\Psi(A).$$

The Hilbert space consists of normalizable solutions to these equations.

WZW model

It turns out that the Hilbert space $\mathcal{H}_\Sigma$ for the canonical quantization of the CS theory on $\mathbf{R}\times\Sigma$ is naturally related to the two-dimensional conformal field theory called the WZW model on Σ. For simplicity, here we will discuss the case with $G = SU(2)$.

In Lecture A2, we discussed the Virasoro generators L_n derived from the energy-momentum tensor T_{zz}. The energy-momentum tensor T_{zz} is meromorphic and obeys the product operator expansion,

$$T_{zz}T_{ww} \sim \frac{c/2}{(z-w)^4} + \left(\frac{2}{(z-w)^2} + \frac{1}{z-w}\partial_w\right)T_{ww} + 0(1).$$

The WZW model is a conformal field theory with holomorphic currents J_z^a $(a = 1, 2, 3)$ in the adjoint representation of $SU(2)$. They obey the product operator expansion,

$$J_z^a\ J_w^b \sim \frac{k/2\delta^{ab}}{(z-w)^2} + \frac{\epsilon^{abc}}{z-w}J_w^c + 0(1).$$

By using

$$\partial_{\bar{z}}\frac{1}{z-w} = -\pi\delta^2(z-w),$$

which we derived earlier, we can also write this as,

$$\partial_{\bar{z}}J_z^a\ J_w^b = \frac{k\pi}{2}\delta^{ab}\partial_z\delta^2(z-w) - \pi\epsilon^{abc}\delta^2(z-w)J_w^c.$$

Consider couling these currents to an $SU(2)$ gauge field on Σ. For now, let us just turn on the anti-holomorphic component $A_{\bar{z}}$. Conside the partition function with $A_{\bar{z}}$,

$$Z(A) = \langle\exp(\pi\int_\Sigma A_{\bar{z}}^a J_z)\rangle.$$

We see that functional derivatives with respect to A generate correlation functions of the currents. The operator product expansion we wrote in the above can then be expressed in terms of $Z(A)$ as

$$\left(\partial_{\bar{z}}\frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}}\right]\right)Z(A) = \frac{k}{2\pi}\partial_z A_{\bar{z}}Z(A).$$

This is exactly the same equation as that for wave functions for the Chern-Simons theory. In the context of the WZW model, this equation is called the Ward-Takahashi identity since it follows from symmetry of the model.

We can also turn on the A_z component of the gauge field. It is then known that the partition function of the WZW model on a Riemann surface Σ can be decomposed into a finite sum of products of holomorphic and anti-holomorphic parts as,

$$Z(A_z, A_{\bar{z}}) = \sum_\alpha \Psi_\alpha(A_{\bar{z}})\bar{\Psi}_\alpha(A_z).$$

Clearly, each Ψ_α should obey the Ward-Takahashi identify, and $\bar{\Psi}_\alpha$ its conjugate. Each of them can be regarded as a wave function for the canonical quantization of the $SU(2)$ WZW model on Σ. In the context of the WZW model, Ψ_α's are called conformal blocks. Thus, we have the correspondence, relating objects in 2-dimensional conformal field theory to the 3-dimensional CS theory,

"conformal blocks in the WZW model on Σ" $\leftrightarrow$ "wave functions of the CS theory on $\mathbf{R} \times \Sigma$".

The relation between the 2d conformal field theory and the 3d theory with coordinate invariance is similar to the AdS/CFT correspondence.

computation

The relation between the CS theory and the WZW model can be used to compute observables in the CS theory. For example, for a closed 3-dimensional manifold M, one can compute its CS partition function by performing the Heegaard splitting and by taking the inner product (Ψ_1, Ψ_2), where $\Psi_{a=1,2}$ are associated to particular handlebodies. Two different handlebodies are related to each other by modular transformations on the boundary Σ, and therefore we can compute this inner product if we know how the modular transformation acts on the Hilbert space $\mathcal{H}_\Sigma$. The description in terms of the WZW model provides this information.

For example, we know that there is only one conformal block on S^2. Therefore,

$$Z_{CS}(S^1 \times S^2) = \mathrm{tr}_{\mathcal{H}_{S^2}} 1 = 1.$$

We can use this to compute $Z_{CS}(S^3)$ since S^3 is related to $S^1 \times S^2$ by the modular transformation $T_{\alpha,\beta} \to T_{\beta,\alpha}$ of the Heegaard decomposition.

Similarly, suppose we want to compute the vacuum expectation values of n Wilson loops on S^3. The S^3 can be separated into two 3d balls, with a common boundary of S^2. Suppose the S^2 cuts each Wilson loops into half. We have S^2 with $2n$ punctures. In fact, they are conformal blocks of a correlation function of $2n$ local operators in the WZW model on S^2. If we move one local operator around another and bring it back to the same location, we should get the same result since that is what it means that the two operators being mutually local. However, conformal blocks can mix with each other under this monodromy transformation. It turns out that, if we cut the Wilson loops in half and perform some monodromy transformation on one side, then one can disentangle knots and links. Thus, if we know how conformal blocks transform into each other under monodromy transformations, one can relate the vacuum expectation values of n Wilson loops to the one of n unknots, the trivial knots. This enables us to compute such expectation values.