Recursive Formulas: Managing Fixed Points

- from Hennessy-Milner logic to Recursive Formulas
- lattice theory, Tarski's fixed point theorem

But Is Hennessy-Milner Logic Really Powerful Enough?

Theorem (for Image-Finite LTS)

It holds that $p \sim q$ if and only if p and q satisfy exactly the same Hennessy-Milner formulae.

Modal depth (nesting degree) for Hennessy-Milner formulae:

- $md(tt) = md(ff) = 0$
- $md(F \wedge G) = md(F \vee G) = \max\{md(F), md(G)\}$
- $md([a]F) = md(\langle a \rangle F) = md(F) + 1$

Idea: a formula F can "see" only upto depth $md(F)$.

Theorem (let F be a HM formula and $k = md(F)$)

If the defender has a defending strategy in the strong bisimulation game from s and t upto k rounds then $s \models F$ if and only if $t \models F$.

Temporal Properties not Expressible in HM Logic

$s \models Inv(F)$ iff all states reachable from s satisfy F
$s \models Pos(F)$ iff there is a reachable state which satisfies F

Fact

Properties $Inv(F)$ and $Pos(F)$ are not expressible in HM logic.

Let $Act = \{a_1, a_2, \ldots, a_n\}$ be a finite set of actions. We define

- $\langle Act \rangle F \stackrel{\text{def}}{=} \langle a_1 \rangle F \vee \langle a_2 \rangle F \vee \ldots \vee \langle a_n \rangle F$
- $[Act]F \stackrel{\text{def}}{=} [a_1]F \wedge [a_2]F \wedge \ldots \wedge [a_n]F$

$Inv(F) \equiv F \wedge [Act]F \wedge [Act][Act]F \wedge [Act][Act][Act]F \wedge \ldots$
$Pos(F) \equiv F \vee \langle Act \rangle F \vee \langle Act \rangle \langle Act \rangle F \vee \langle Act \rangle \langle Act \rangle \langle Act \rangle F \vee \ldots$

Infinite Conjunctions and Disjunctions vs. Recursion

Problems

- infinite formulae are not allowed in HM logic
- infinite formulae are difficult to handle

Why not to use recursive formulas?

- $Inv(F)$ expressed by $X \stackrel{\text{def}}{=} F \wedge [Act]X$
- $Pos(F)$ expressed by $X \stackrel{\text{def}}{=} F \vee \langle Act \rangle X$

Question: How to define the semantics of such equations?

Solving Equations (in General Frameworks) can be Tricky

Equations over Natural Numbers ($n \in \mathbb{N}$)

$n = 2 * n$	one solution $n = 0$
$n = n + 1$	no solution
$n = 1 * n$	many solutions (every $n \in \mathbb{N}$ is a solution)

Equations over Sets of Integers ($M \in 2^{\mathbb{N}}$)

$M = (\{7\} \cap M) \cup \{7\}$	one solution $M = \{7\}$
$M = \mathbb{N} \setminus M$	no solution
$M = \{3\} \cup M$	many solutions (every $M \supseteq \{3\}$)

What about Equations over Sets of Processes?

$$X \stackrel{\text{def}}{=} [a]ff \vee \langle a \rangle X \quad \Rightarrow \quad \text{find } S \subseteq 2^{Proc} \text{ s.t. } S = [\cdot a \cdot]\emptyset \cup \langle \cdot a \cdot \rangle S$$

General Approach – Lattice Theory

Problem

For a set D and a function $f : D \to D$, for which elements $x \in D$ we have

$$x = f(x) \text{ ?}$$

Such x's are called fixed points.

Partially Ordered Set

Partially ordered set (or simply a partial order) is a pair $(D, \sqsubseteq)$ s.t.

- D is a set
- $\sqsubseteq \subseteq D \times D$ is a binary relation on D which is
  - reflexive: $\forall d \in D.\ d \sqsubseteq d$
  - antisymmetric: $\forall d, e \in D.\ d \sqsubseteq e \ \wedge \ e \sqsubseteq d \ \Rightarrow \ d = e$
  - transitive: $\forall d, e, f \in D.\ d \sqsubseteq e \ \wedge \ e \sqsubseteq f \ \Rightarrow \ d \sqsubseteq f$

Supremum and Infimum

Upper/Lower Bounds (Let $X \subseteq D$)

- $d \in D$ is an upper bound for X (written $X \sqsubseteq d$) iff $x \sqsubseteq d$ for all $x \in X$
- $d \in D$ is a lower bound for X (written $d \sqsubseteq X$) iff $d \sqsubseteq x$ for all $x \in X$

Least Upper Bound and Greatest Lower Bound (Let $X \subseteq D$)

- $d \in D$ is the least upper bound (supremum) for X ($\sqcup X$) iff
  1. $X \sqsubseteq d$
  2. $\forall d' \in D.\ X \sqsubseteq d' \ \Rightarrow\ d \sqsubseteq d'$
- $d \in D$ is the greatest lower bound (infimum) for X ($\sqcap X$) iff
  1. $d \sqsubseteq X$
  2. $\forall d' \in D.\ d' \sqsubseteq X \ \Rightarrow\ d' \sqsubseteq d$

Complete Lattices and Monotonic Functions

Complete Lattice

A partially ordered set $(D, \sqsubseteq)$ is called complete lattice iff $\sqcup X$ and $\sqcap X$ exist for any $X \subseteq D$.

We define the top and bottom by $\top \stackrel{\text{def}}{=} \sqcup D$ and $\bot \stackrel{\text{def}}{=} \sqcap D$.

Monotonic Function and Fixed Points

A function $f : D \to D$ is called monotonic iff

$$d \sqsubseteq e \quad \Rightarrow \quad f(d) \sqsubseteq f(e)$$

for all $d, e \in D$.

Element $d \in D$ is called fixed point iff $d = f(d)$.

Tarski's Fixed Point Theorem

Theorem (Tarski)

Let $(D, \sqsubseteq)$ be a complete lattice and let $f : D \to D$ be a monotonic function.

Then f has a unique largest fixed point z_{max} and a unique least fixed point z_{min} given by:

$$z_{max} \stackrel{\text{def}}{=} \sqcup\{x \in D \mid x \sqsubseteq f(x)\}$$

$$z_{min} \stackrel{\text{def}}{=} \sqcap\{x \in D \mid f(x) \sqsubseteq x\}$$

Computing Min and Max Fixed Points on Finite Lattices

Let $(D, \sqsubseteq)$ be a complete lattice and $f : D \to D$ monotonic.
Let $f^1(x) \stackrel{\text{def}}{=} f(x)$ and $f^n(x) \stackrel{\text{def}}{=} f(f^{n-1}(x))$ for $n > 1$, i.e.,

$$f^n(x) = \underbrace{f(f(\dots f}_{n \text{ times}}(x)\dots)).$$

Theorem

If D is a finite set then there exist integers $M, m > 0$ such that

- $z_{max} = f^M(\top)$
- $z_{min} = f^m(\bot)$

Idea (for z_{min}): The following sequence stabilizes for any finite D

$$\bot \sqsubseteq f(\bot) \sqsubseteq f(f(\bot)) \sqsubseteq f(f(f(\bot))) \sqsubseteq \cdots$$