Bisimulation Semantics

- behavioural equivalences
- trace equivalence and black box experiments
- strong bisimilarity and how to show it
- and how to disprove it: bisimulation games
- properties of strong bisimilarity

Behavioural Equivalence

Implementation

$$CM \stackrel{\mathrm{def}}{=} coin.\overline{coffee}.CM$$

$$CS \stackrel{\mathrm{def}}{=} \overline{work}.\overline{coin}.coffee.CS$$

$$Uni \stackrel{\mathrm{def}}{=} (CM \mid CS) \smallsetminus \{coin, coffee\}$$

Specification

$$Spec \stackrel{\mathrm{def}}{=} \overline{work}.Spec$$

Question

Are the processes *Uni* and *Spec* behaviorally equivalent?

$$Uni \equiv Spec\ ?$$

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Goals

What should a reasonable behavioural equivalence satisfy?

- abstract from states (consider only the behaviour – actions)
- abstract from nondeterminism
- abstract from internal behaviour

What else should a reasonable behavioural equivalence satisfy?

- reflexivity $P \equiv P$ for any process P
- transitivity $Spec_0 \equiv Spec_1 \equiv Spec_2 \equiv \cdots \equiv Impl$ gives that
$$Spec_0 \equiv Impl$$
- symmetry $P \equiv Q$ iff $Q \equiv P$

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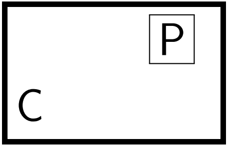
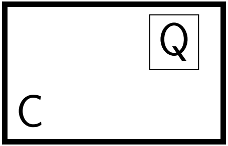
Congruence



$C(P)$



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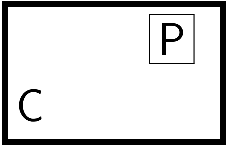

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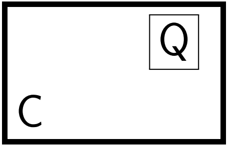

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Trace Equivalence

Let $(Proc, Act, \{\xrightarrow{a} \mid a \in Act\})$ be an LTS.

Trace Set for $s \in Proc$

$$Traces(s) = \{w \in Act^* \mid \exists s' \in Proc.\ s \xrightarrow{w} s'\}$$

Let $s \in Proc$ and $t \in Proc$.

Trace Equivalence

We say that s and t are trace equivalent ($s \equiv_t t$) if and only if

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Black-Box Experiments

Experiment in A	Experiment in B	Experiment in B
coin *tea* *coffee*	*coin* *tea* *coffee*	*coin* *tea* *coffee*
press *coin*	press *coin*	press *coin*
coin *tea* *coffee*	*coin* *tea* *coffee*	*coin* *tea* *coffee*

Main Idea

Two processes are behaviorally equivalent if and only if an external observer cannot tell them apart.

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Strong Bisimilarity

Let $(Proc, Act, \{\xrightarrow{a} \mid a \in Act\})$ be an LTS.

Strong Bisimulation

A binary relation $R \subseteq Proc \times Proc$ is a strong bisimulation iff whenever $(s, t) \in R$ then for each $a \in Act$:

- if $s \xrightarrow{a} s'$ then $t \xrightarrow{a} t'$ for some t' such that $(s', t') \in R$
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Two processes $p_1, p_2 \in Proc$ are strongly bisimilar ($p_1 \sim p_2$) if and only if there exists a strong bisimulation R such that $(p_1, p_2) \in R$.

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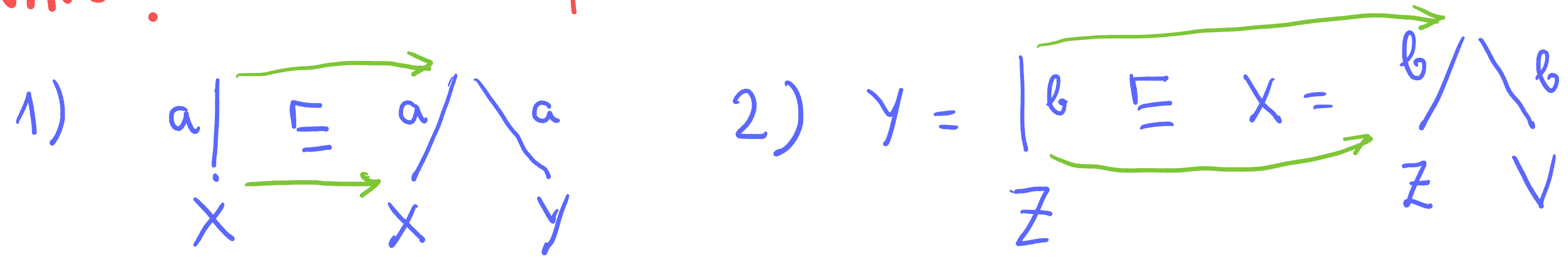
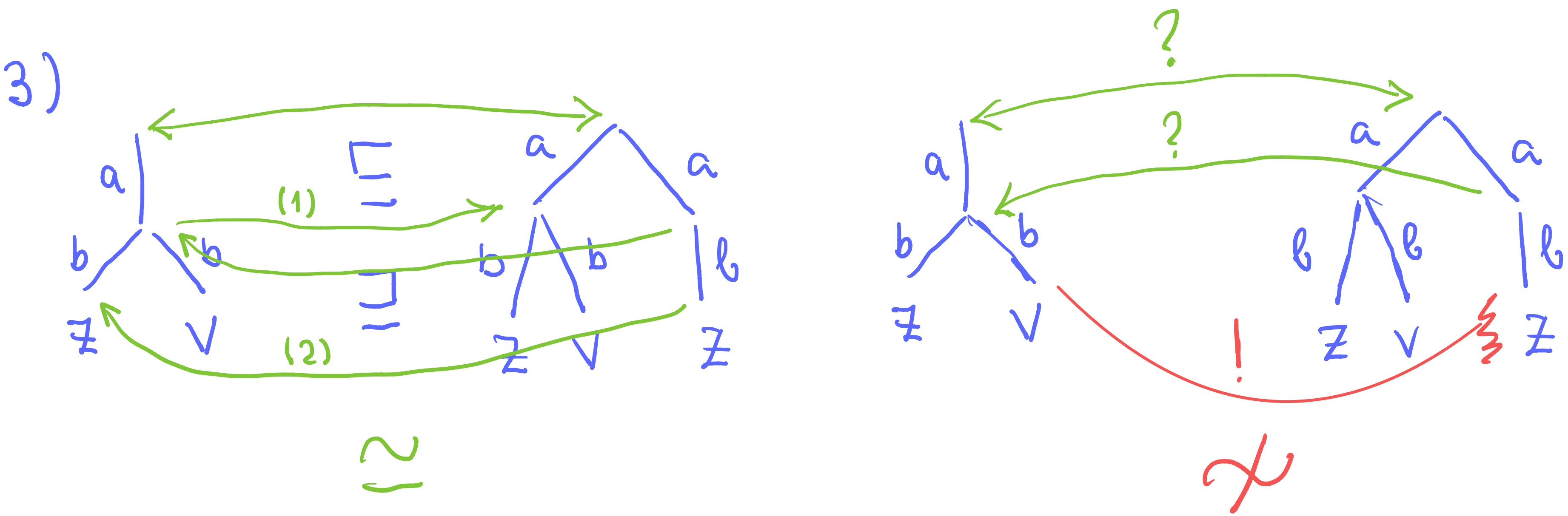
Exercise 3.17 (Simulation) *Let us say that a binary relation $\mathcal{R}$ over the set of states of an LTS is a simulation iff whenever $s_1 \mathcal{R} s_2$ and α is an action:*

- *if $s_1 \xrightarrow{\alpha} s_1'$, then there is a transition $s_2 \xrightarrow{\alpha} s_2'$ such that $s_1' \mathcal{R} s_2'$.*

We say that s' simulates s, written $s \sqsubseteq s'$, iff there is a simulation $\mathcal{R}$ with $s \mathcal{R} s'$. Two states s and s' are simulation equivalent, written $s \simeq s'$, iff $s \sqsubseteq s'$ and $s' \sqsubseteq s$ both hold.

WARNING! Simulation equivalence is much LESS than bisimilarity





Basic Properties of Strong Bisimilarity

Theorem

$\sim$ *is an equivalence (reflexive, symmetric and transitive)*

Theorem

$\sim$ *is the largest strong bisimulation*

Theorem

$s \sim t$ *if and only if for each* $a \in Act$:

- *if* $s \xrightarrow{a} s'$ *then* $t \xrightarrow{a} t'$ *for some* t' *such that* $s' \sim t'$
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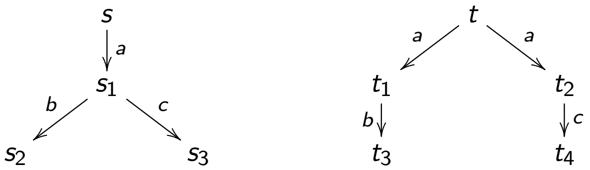
How to Show Nonbisimilarity?



To prove that $s \not\sim t$:

- Enumerate all binary relations and show that none of them at the same time contains (s, t) and is a strong bisimulation. (Expensive: $2^{|Proc|^2}$ relations.)
- Make certain observations which will enable to disqualify many bisimulation candidates in one step.
- Use game characterization of strong bisimilarity.

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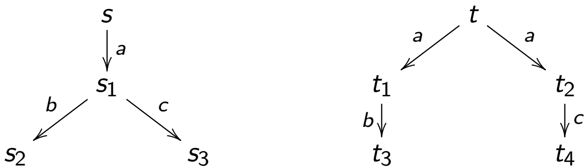

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\begin{tikzcd}[column sep=small]
 & s \arrow[d, "a"] & \\
 & s_1 \arrow[dl, "b"'] \arrow[dr, "c"] & \\
s_2 & & s_3
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & t \arrow[dl, "a"'] \arrow[dr, "a"] & \\
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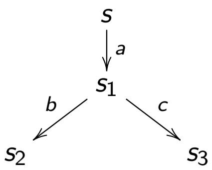

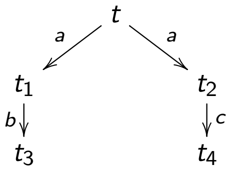

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Strong Bisimulation Game

Let $(Proc, Act, \{\xrightarrow{a} \mid a \in Act\})$ be an LTS and $s, t \in Proc$.

We define a two-player game of an 'attacker' and a 'defender' starting from s and t.

- The game is played in rounds and configurations of the game are pairs of states from $Proc \times Proc$.
- In every round exactly one configuration is called current. Initially the configuration (s, t) is the current one.

Intuition

The defender wants the show that s and t are strongly bisimilar while the attacker aims to prove the opposite.

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Rules of the Bisimulation Games

Game Rules

In each round the players change the current configuration as follows:

1. the attacker chooses one of the processes in the current configuration and makes an $\xrightarrow{a}$-move for some $a \in Act$, and
2. the defender must respond by making an $\xrightarrow{a}$-move in the other process under the same action a.

The newly reached pair of processes becomes the current configuration. The game then continues by another round.

Result of the Game

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Game Characterization of Strong Bisimilarity

Theorem

- States s and t are strongly bisimilar if and only if the defender has a universal winning strategy starting from the configuration (s, t).
- States s and t are not strongly bisimilar if and only if the attacker has a universal winning strategy starting from the configuration (s, t).

Remark

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Strong Bisimilarity is a Congruence for CCS Operations

Theorem

Let P and Q be CCS processes such that $P \sim Q$. Then

- $\alpha.P \sim \alpha.Q$ *for each action* $\alpha \in Act$
- $P + R \sim Q + R$ *and* $R + P \sim R + Q$ *for each CCS process* R
- $P \mid R \sim Q \mid R$ *and* $R \mid P \sim R \mid Q$ *for each CCS process* R
- $P[f] \sim Q[f]$ *for each relabelling function* f
- $P \setminus L \sim Q \setminus L$ *for each set of labels* L.

Other Properties of Strong Bisimilarity

Following Properties Hold for any CCS Processes P, Q and R

- $P + Q \sim Q + P$
- $P \mid Q \sim Q \mid P$
- $P + Nil \sim P$
- $P \mid Nil \sim P$
- $(P + Q) + R \sim P + (Q + R)$
- $(P \mid Q) \mid R \sim P \mid (Q \mid R)$