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Assignments 1

1. Let us assume $e, e' \in AExp$ and $x \in Var$. The notation $e[x/e']$ denotes the result of replacing all occurrences of x in e by e'. For example: $(x + y)[x/(3 * z)] = (3 * z) + y$.

   (a) Define $e[x/e']$ in a compositional way.

   (b) Prove the following substitution lemma: for all $e, e' \in AExp, x \in Var, \sigma \in State$:

$$\mathcal{A}[\![e[x/e']]\!]\sigma = \mathcal{A}[\![e]\!]\sigma[x \to \mathcal{A}[\![e']\!]\sigma]$$

Proof

(a) Let us assume that $e_1, e_2, e' \in AExp$ and $x, y \in Var$ such that $x \neq y$. We can define $e[x/e']$ in terms of $AExp$ as follow:

$$\begin{aligned}
n[x/e'] &= n \quad \forall n \in \mathbb{Z} \\
x[x/e'] &= e' \quad \forall x \in Var \\
y[x/e'] &= y \quad \forall y \in Var \quad \boxed{x \neq y} \\
(e_1 + e_2)[x/e'] &= (e_1)[x/e'] + (e_2)[x/e'] \\
(e_1 - e_2)[x/e'] &= (e_1)[x/e'] - (e_2)[x/e'] \\
(e_1 * e_2)[x/e'] &= (e_1)[x/e'] \cdot (e_2)[x/e']
\end{aligned}$$

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(b) In order to prove the following equality, we need to apply structural induction for every term in $AExp$. Now we can consider the semantics function

$$\mathcal{A}[\![\_]\!] : AExp \longrightarrow State \longrightarrow \mathbb{Z}$$

We can assume the base case when e be either n or a variable $x \in Var$. Once we proved that (it will be our structural induction hipothesis, SIH) we will assume this for any expression e and we will try to prove one of the compositioned expression, for instance the sum expression (the rest will be similiar).

Therefore, let's first consider the case $\boldsymbol{e := n}$.

$$\mathcal{A}[\![n[x/e']]\!]\sigma =^{(1)} \mathcal{A}[\![n]\!]\sigma = n$$

note that $\mathcal{A}[\![e']\!]\sigma$ might be different than n

for all $\sigma : State \longrightarrow \mathbb{Z}$, in particular for $\sigma[x \to n]$. Thus, we obtain

$$\begin{aligned}
\mathcal{A}[\![n[x/e']]\!]\sigma &=^{(1)} \mathcal{A}[\![n]\!]\sigma \\
&= n \\
&= \mathcal{A}[\![n]\!]\sigma[x \to n]
\end{aligned}$$

Let us see the case when $\boldsymbol{e := x}$ with $x \in Var$. Let $e' \in AExp$ and $m \in \mathbb{Z}$ be such that $\mathcal{A}[\![e']\!]\sigma = m \ \forall \sigma \in State$, in particular $\sigma[x \to m]$:

$$\begin{aligned}
\mathcal{A}[\![x[x/e']]\!]\sigma &=^{(1)} \mathcal{A}[\![e']\!]\sigma \\
&= m \\
&= \mathcal{A}[\![x]\!]\sigma[x \to m] \\
&= \mathcal{A}[\![x]\!]\sigma[x \to \mathcal{A}[\![e']\!]\sigma]
\end{aligned}$$

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We can obtain analogously $\boldsymbol{e := y}$ because the fact of mapping a variable y by $\sigma[x \to m]$ doesn't affect.

Let $e' \in AExp$ and $m, m' \in \mathbb{Z}$ be such that $\mathcal{A}[\![e']\!]\sigma = m$ and $\mathcal{A}[\![y]\!]\sigma = m'$ $\forall \sigma \in State$. In particular, for $\sigma[x \to m]$:

$$\mathcal{A}[\![y]\!]\sigma[x \to m] = m' \tag{2}$$

$$\begin{aligned}
\mathcal{A}[\![y[x/e']]\!]\sigma &=^{(1)} \mathcal{A}[\![y]\!]\sigma \\
&= m' \\
&=^{(2)} \mathcal{A}[\![y]\!]\sigma[x \to m] \\
&= \mathcal{A}[\![y]\!]\sigma[x \to \mathcal{A}[\![e']\!]\sigma]
\end{aligned}$$

Finally, let us consider some expressions $e_1, e_2 \in AExp$. We can continue with the sum compositioned expression proof as follow

$$\begin{aligned}
\mathcal{A}[\![(e_1 + e_2)[x/e']]\!]\sigma &=^{(1)} \mathcal{A}[\![(e_1)[x/e']]\!]\sigma + \mathcal{A}[\![(e_2)[x/e']]\!]\sigma \\
&=^{(SIH)} \mathcal{A}[\![e_1]\!]\sigma[x \to \mathcal{A}[\![e']\!]\sigma] + \mathcal{A}[\![(e_2)[x/e']]\!]\sigma \\
&=^{(SIH)} \mathcal{A}[\![e_1]\!]\sigma[x \to \mathcal{A}[\![e']\!]\sigma] + \mathcal{A}[\![e_2]\!]\sigma[x \to \mathcal{A}[\![e']\!]\sigma] \\
&=^{(def)} \mathcal{A}[\![(e_1 + e_2)]\!]\sigma[x \to \mathcal{A}[\![(e')]\!]\sigma]
\end{aligned}$$

[1]

$\blacksquare$

[1] By the ~~last section~~ definition of replacement