Specifying, programming, and verifying in Maude

Narciso Martí-Oliet

Departamento de Sistemas Informáticos y Computación
Universidad Complutense de Madrid
narciso@esi.ucm.es

Summary

Maude is a high-level language and high-performance system supporting both equational and rewriting computation for a wide range of applications. In this course we provide an introduction to equational specification and rule-based programming in Maude 2, showing the difference between equations and rules. We use typical data structures (stacks, queues, lists, binary trees) and well-known mathematical games and puzzles to illustrate the features and expressive power of Maude, emphasizing the full generality with which membership equational logic and rewriting logic are supported.

Summary

Indeed, the expressive version of equational logic in which Maude is based, namely membership equational logic, allows the faithful specification of types (like sorted lists or search trees) whose data are defined not only by means of constructors, but also by the satisfaction of additional properties. Moreover, exploiting the fact that rewriting logic is reflective, a key distinguishing feature of Maude is its systematic and efficient use of reflection, a feature that makes Maude remarkably extensible and powerful, and that allows many advanced metaprogramming and metalanguage applications.

Summary

- Maude follows a long tradition of algebraic specification languages in the OBJ family, including
  - OBJ3,
  - CafeOBJ,
  - Elan.
- Computation = Deduction in an appropriate logic.
- Functional modules = (Admissible) specifications in membership equational logic.
- System modules = (Admissible) specifications in rewriting logic.
- Operational semantics based on matching and rewriting.

Introduction

http://maude.cs.uiuc.edu

- Maude is a high-level language and high-performance system.
- It supports both equational and rewriting logic computation.
- Membership equational logic improves order-sorted algebra.
- Rewriting logic is a logic of concurrent change.

Introduction

http://maude.cs.uiuc.edu

- Maude is a high-level language and high-performance system.
- It supports both equational and rewriting logic computation.
- Membership equational logic improves order-sorted algebra.
- Rewriting logic is a logic of concurrent change.
- It is a flexible and general semantic framework for giving semantics to a wide range of languages and models of concurrency.
- It is also a good logical framework, i.e., a metalogic in which many other logics can be naturally represented and implemented.

Introduction

http://maude.cs.uiuc.edu

- Maude is a high-level language and high-performance system.
- It supports both equational and rewriting logic computation.
- Membership equational logic improves order-sorted algebra.
- Rewriting logic is a logic of concurrent change.
- It is a flexible and general semantic framework for giving semantics to a wide range of languages and models of concurrency.
- It is also a good logical framework, i.e., a metalogic in which many other logics can be naturally represented and implemented.
- Moreover, rewriting logic is reflective.
- This makes possible many advanced metaprogramming and metalanguage applications.

Contents

Contents

Many-sorted equational specifications

- Algebraic specifications are used to declare different kinds of data together with the operations that act upon them.
- It is useful to distinguish two kinds of operations:
    - constructors, used to construct or generate the data, and
    - the remaining operations, which in turn can also be classified as modifiers or observers.
- The behavior of operations is described by means of (possibly conditional) equations.
- We start with the simplest many-sorted equational specifications and incrementally add more sophisticated features.

Signatures

- The first thing a specification needs to declare are the types (or sorts) of the data being defined and the corresponding operations.
- A many-sorted signature (S, Σ) consists of
  - a sort set S, and
  - an $S^* \times S$-sorted family

    $$\Sigma = \{\Sigma_{\overline{s},s} \mid \overline{s} \in S^*, s \in S\}$$

    of sets of operation symbols $f : s_1 \ldots s_n \to s$.
- When $f \in \Sigma_{\overline{s},s}$, we write $f : \overline{s} \to s$ and say that f has rank $\langle \overline{s}, s \rangle$, arity (or argument sorts) $\overline{s}$, and coarity (or value sort, or range sort) s.
- The symbol ε denotes the empty sequence in S^*.

Terms

- With the declared operations we can construct terms to denote the data being specified.
- Terms are typed and can have variables.
- Given a many-sorted signature (S, Σ) and an S-sorted family $X = \{X_s \mid s \in S\}$ of variables, the S-sorted set of terms

$$\mathcal{T}_\Sigma(X) = \{\mathcal{T}_{\Sigma,s}(X) \mid s \in S\}$$

is inductively defined by the following conditions:

1. $X_s \subseteq \mathcal{T}_{\Sigma,s}(X)$ for $s \in S$;
2. $\Sigma_{\varepsilon,s} \subseteq \mathcal{T}_{\Sigma,s}(X)$ for $s \in S$;
3. If $f \in \Sigma_{\overline{s},s}$ and $t_i \in \mathcal{T}_{\Sigma,s_i}(X)$ $(i = 1, \ldots, n)$, where $\overline{s} = s_1 \ldots s_n \neq \varepsilon$, then $f(t_1, \ldots, t_n) \in \mathcal{T}_{\Sigma,s}(X)$.

Equations

- A Σ-equation is an expression

$$(\overline{x} : \overline{s})\ l = r$$

  where
  - $\overline{x} : \overline{s}$ is a (finite) set of variables, and
  - l and r are terms in $\mathcal{T}_{\Sigma,s}(\overline{x} : \overline{s})$ for some sort s.
- A conditional Σ-equation is an expression

$$(\overline{x} : \overline{s})\ l = r \ \text{ if } \ u_1 = v_1 \ \wedge \ldots \wedge \ u_n = v_n$$

  where $(\overline{x} : \overline{s})\ l = r$ and $(\overline{x} : \overline{s})\ u_i = v_i$ $(i = 1, \ldots, n)$ are Σ-equations.
- A many-sorted specification (S, Σ, E) consists of:
  - a signature (S, Σ), and
  - a set E of (conditional) Σ-equations.

Equational logic deduction rules

1. **Reflexivity**. $\dfrac{}{(\forall X)\ t = t}$
2. **Symmetry**. $\dfrac{(\forall X)\ t_1 = t_2}{(\forall X)\ t_2 = t_1}$
3. **Transitivity**. $\dfrac{(\forall X)\ t_1 = t_2 \quad (\forall X)\ t_2 = t_3}{(\forall X)\ t_1 = t_3}$
4. **Congruence**. For each $f \in \Sigma$, $\dfrac{(\forall X)\ t_1 = t'_1 \quad \ldots \quad (\forall X)\ t_n = t'_n}{(\forall X)\ f(t_1, \ldots, t_n) = f(t'_1, \ldots, t'_n)}$
5. **Modus ponens**. For each sentence

$$(\overline{x} : \overline{s})\ l = r \ \text{ if } \ u_1 = v_1 \ \wedge \ldots \wedge \ u_m = v_m,$$

$$\frac{(\forall Y)\ \theta(u_1) = \theta(v_1) \quad \ldots \quad (\forall Y)\ \theta(u_m) = \theta(v_m)}{(\forall Y)\ \theta(l) = \theta(r)}$$

Semantics

- A many-sorted (S, Σ)-algebra $\mathbf{A}$ consists of:
  - a carrier set A_s for each sort $s \in S$, and
  - a function $A_f^{\overline{s},s} : A_{\overline{s}} \to A_s$ for each operation symbol $f \in \Sigma_{\overline{s},s}$.
- The meaning $[\![t]\!]_{\mathbf{A}}$ of a term t in an algebra $\mathbf{A}$ is inductively defined.
- An algebra $\mathbf{A}$ satisfies an equation $(\overline{x} : \overline{s})\; l = r$ when both terms have the same meaning: $[\![l]\!]_{\mathbf{A}} = [\![r]\!]_{\mathbf{A}}$.
- An algebra $\mathbf{A}$ satisfies a conditional equation

$$(\overline{x} : \overline{s})\; l = r \;\text{ if }\; u_1 = v_1 \;\wedge \ldots \wedge\; u_n = v_n$$

when satisfaction of all the conditions $(\overline{x} : \overline{s})\; u_i = v_i \;(i = 1, \ldots, n)$ implies satisfaction of $(\overline{x} : \overline{s})\; l = r$.

Semantics

- The loose semantics of a many-sorted specification (S, Σ, E) is defined as the set of all (S, Σ)-algebras that satisfy all the (conditional) equations in E.
- But we are usually interested in the so-called initial semantics given by a particular algebra in this class (up to isomorphism).
- A concrete representation $\mathcal{T}_{\Sigma,E}$ of such an initial algebra is obtained by imposing a congruence relation on the term algebra $\mathcal{T}_{\Sigma}$ whose carrier sets are the sets of ground terms, that is, terms without variables.
- Two terms are identified by this congruence if and only if they have the same meaning in all algebras in the loose semantics (equivalently, if and only if the equation $(\forall\emptyset)\ t = t'$ can be deduced from E).

Initiality = No junk + No confusion (Burstall & Goguen)

- No junk: Every data item can be constructed using only the constants and operations in the signature.
- A data item that cannot be so constructed is junk.
- No confusion: Two data items are equivalent if and only if they can be proved equal using the equations.
- Two data items that are equivalent but cannot be proved so from the given equations are said to be confused.

Maude functional modules

```
fmod BOOLEAN is
  sort Bool .

  op true : -> Bool [ctor] .
  op false : -> Bool [ctor] .

  op not_ : Bool -> Bool .
  op _and_ : Bool Bool -> Bool .
  op _or_ : Bool Bool -> Bool .
```

Maude functional modules

```
fmod BOOLEAN is
  sort Bool .

  op true : -> Bool [ctor] .
  op false : -> Bool [ctor] .

  op not_ : Bool -> Bool .
  op _and_ : Bool Bool -> Bool .
  op _or_ : Bool Bool -> Bool .

  var A : Bool .

  eq not true = false .
  eq not false = true .
  eq true and A = A .
  eq false and A = false .
  eq true or A = true .
  eq false or A = A .
endfm
```

Operational semantics: Matching

- Given an S-sorted family of variables X for a signature (S, Σ), a (ground) substitution is a sort-preserving map

$$\sigma : X \to \mathcal{T}_\Sigma$$

- Such a map extends uniquely to terms

$$\sigma : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma$$

- Given a term $t \in \mathcal{T}_\Sigma(X)$, the pattern, and a subject ground term $u \in \mathcal{T}_\Sigma$, we say that t matches u if there is a substitution σ such that $\sigma(t) \equiv u$, that is, $\sigma(t)$ and u are syntactically equal terms.

Rewriting and equational simplification

- In a Σ-equation $(\overline{x} : \overline{s})\ l = r$ all variables in the righthand side r must appear among the variables of the lefthand side l.
- A term t rewrites to a term t' using such an equation in E if
  1. there is a subterm $t|_p$ of t at a given position p of t such that l matches $t|_p$ via a substitution σ, and
  2. t' is obtained from t by replacing the subterm $t|_p \equiv \sigma(l)$ with the term $\sigma(r)$.
- We denote this step of equational simplification by $t \rightarrow_E t'$.
- It can be proved that if $t \rightarrow_E t'$ then $[\![t]\!]_{\mathbf{A}} = [\![t']\!]_{\mathbf{A}}$ for any algebra $\mathbf{A}$ satisfying E.
- We write $t \rightarrow^*_E t'$ to mean either $t = t'$ (0 steps) or $t \rightarrow_E t_1 \rightarrow_E t_2 \rightarrow_E \cdots \rightarrow_E t_n \rightarrow_E t'$ with $n \geq 0$ ($n + 1$ steps).

Confluence and termination

- A set of equations E is confluent (or Church-Rosser) when any two rewritings of a term can always be unified by further rewriting: if $t \rightarrow_E^* t_1$ and $t \rightarrow_E^* t_2$, then there exists a term t' such that $t_1 \rightarrow_E^* t'$ and $t_2 \rightarrow_E^* t'$.

$$\begin{tikzcd}
& t \arrow[dl, "E"', "*"] \arrow[dr, "*", "E"'] & \\
t_1 \arrow[dr, dotted, "*", "E"'] & & t_2 \arrow[dl, dotted, "E"', "*"] \\
& t' &
\end{tikzcd}$$

- A set of equations E is terminating when there is no infinite sequence of rewriting steps $t_0 \rightarrow_E t_1 \rightarrow_E t_2 \rightarrow_E \dots$

Confluence and termination

- If E is both confluent and terminating, a term t can be reduced to a unique canonical form $t{\downarrow_E}$, that is, to a term that can no longer be rewritten.
- Therefore, in order to check semantic equality of two terms $t = t'$, it is enough to check that their respective canonical forms are equal, $t{\downarrow_E} = t'{\downarrow_E}$, but, since canonical forms cannot be rewritten anymore, the last equality is just syntactic coincidence: $t{\downarrow_E} \equiv t'{\downarrow_E}$.
- Functional modules in Maude are assumed to be confluent and terminating, and their operational semantics is equational simplification, that is, rewriting of terms until a canonical form is obtained.

Natural numbers

```
fmod UNARY-NAT is
  sort Nat .

  op 0 : -> Nat [ctor] .
  op s : Nat -> Nat [ctor] .
  op _+_ : Nat Nat -> Nat .

  vars N M : Nat .

  eq 0 + N = N .
  eq s(M) + N = s(M + N) .
endfm
```

- Can we add the equation

  ```
  eq M + N = N + M .
  ```

  expressing commutativity of addition?

Modularization

- `protecting M .`
  Importing a module M into M' in `protecting` mode intuitively means that no junk and no confusion are added to M when we include it in M'.
- `extending M .`
  The idea is to allow junk, but to rule out confusion.
- `including M .`
  No requirements are made in an `including` importation: there can now be junk and/or confusion.

Modularization

- `protecting M .`
  Importing a module M into M' in `protecting` mode intuitively means that no junk and no confusion are added to M when we include it in M'.
- `extending M .`
  The idea is to allow junk, but to rule out confusion.
- `including M .`
  No requirements are made in an `including` importation: there can now be junk and/or confusion.

```
fmod NAT3 is
  including UNARY-NAT .
  var N : Nat .
  eq s(s(s(N))) = N .
endfm
```

Operations on natural numbers

```
fmod NAT+OPS is
  protecting BOOLEAN .
  protecting UNARY-NAT .

  ops _*_ _-_ : Nat Nat -> Nat .
  ops _<=_ _>_ : Nat Nat -> Bool .

  vars N M : Nat .
  eq 0 * N = 0 .
  eq s(M) * N = (M * N) + N .
  eq 0 - N = 0 .
  eq s(M) - 0 = s(M) .
  eq s(M) - s(N) = M - N .
  eq 0 <= N = true .
  eq s(M) <= 0 = false .
  eq s(M) <= s(N) = M <= N .
  eq M > N = not (M <= N) .
endfm
```

Conditional equations

- Equational conditions in conditional equations are made up of individual equations $t = t'$; satisfaction is checked by syntactic equality $t{\downarrow_E} \equiv t'{\downarrow_E}$.
- A condition can be either a single equation or a conjunction of equations using the binary conjunction connective /\ which is assumed associative.
- Furthermore, the concrete syntax of equations in conditions has three variants:
  - ordinary equations `t = t'`,
  - matching equations `t := t'`, and
  - abbreviated Boolean equations of the form `t`, with `t` a term of sort `Bool`, abbreviating the equation `t = true`.
- The Boolean terms appearing most often in abbreviated Boolean equations are terms using the built-in equality _==_ and inequality _=/=_ predicates (from predefined module `BOOL`).

Integers

```
fmod INTEGERS is
  sort Int .

  op 0 : -> Int [ctor] .
  op s : Int -> Int [ctor] .
  op p : Int -> Int [ctor] .

  op _+_ : Int Int -> Int .
  op _-_ : Int Int -> Int .
  op _*_ : Int Int -> Int .
  op -_ : Int -> Int .
  op _<=_ : Int Int -> Bool .

  vars N M : Int .

  eq s(p(N)) = N .
  eq p(s(N)) = N .
```

Integers

```
    eq 0 + N = N .
    eq s(M) + N = s(M + N) .
    eq p(M) + N = p(M + N) .
    eq N - 0 = N .
    eq M - s(N) = p(M - N) .
    eq M - p(N) = s(M - N) .
    eq - N = 0 - N .
    eq 0 * N = 0 .
    eq s(M) * N = (M * N) + N .
    eq p(M) * N = (M * N) - N .

    eq s(M) <= N = M <= p(N) .
    eq p(M) <= N = M <= s(N) .
    eq 0 <= 0 = true .
    eq 0 <= p(0) = false .
    ceq 0 <= s(M) = true if 0 <= M .
    ceq 0 <= p(M) = false if not(0 <= M) .
  endfm
```

Cardinality of finite sets

```
fmod NAT< is
  sort Nat .
  op 0 : -> Nat [ctor] .
  op s_ : Nat -> Nat [ctor] .
  op _<_ : Nat Nat -> Bool .

  vars N N' : Nat .

  eq N < 0 = false .
  eq 0 < s N = true .
  eq s N < s N' = N < N' .
endfm
```

Cardinality of finite sets

```
fmod NAT< is
  sort Nat .
  op 0 : -> Nat [ctor] .
  op s_ : Nat -> Nat [ctor] .
  op _<_ : Nat Nat -> Bool .

  vars N N' : Nat .

  eq N < 0 = false .
  eq 0 < s N = true .
  eq s N < s N' = N < N' .
endfm

fmod CARDINAL is
  protecting NAT< .
  sort Set .
  op emptySet : -> Set [ctor] .
  op _._ : Nat Set -> Set [ctor] .   *** add element
  op # : Set -> Nat .                *** count elements
```

Cardinality of finite sets

```
    vars N N' : Nat .
    var  S : Set .

    eq N . N . S = N . S .                            *** idempotency
    ceq N . N' . S = N' . N . S if N' < N .  *** commutativity

    eq #(emptySet) = 0 .
    eq #(N . emptySet) = s 0 .
    eq #(0 . s N . emptySet) = s s 0 .
    eq #(s N . s N' . emptySet) = #(N . N' . emptySet) .
    ceq #(N . N' . S) = s s #(S)
     if #(N . N' . emptySet) = s s 0
     /\ #(N . S) = s #(S)
     /\ #(N' . S) = s #(S) .
  endfm
```

Order-sorted equational specifications

- There are operations that are not defined for some values, like division on natural numbers.
- We can often avoid the possibility of considering partial functions by extending many-sorted equational logic to order-sorted equational logic.
- We can define subsorts corresponding to the domain of definition of a function, whenever such subsorts can be specified by means of constructors.

Order-sorted equational specifications

- There are operations that are not defined for some values, like division on natural numbers.
- We can often avoid the possibility of considering partial functions by extending many-sorted equational logic to order-sorted equational logic.
- We can define subsorts corresponding to the domain of definition of a function, whenever such subsorts can be specified by means of constructors.
- An order-sorted signature adds a partial order relation to the set of sorts S, such that $s \leq s'$ is interpreted semantically by the subset inclusion $A_s \subseteq A_{s'}$ between the corresponding carrier sets in the algebras.
- Moreover, operations can be overloaded:
  - subsort overloading: addition both on natural numbers and on integers,
  - ad-hoc overloading: the same symbol can be used in unrelated sorts.

Preregularity and sort-decreasingness

- A term can have several different sorts.
- Preregularity requires each term to have a least sort that can be assigned to it.
- Maude assumes that modules are preregular, and generates warnings when a module is loaded if the property does not hold.

Preregularity and sort-decreasingness

- A term can have several different sorts.
- Preregularity requires each term to have a least sort that can be assigned to it.
- Maude assumes that modules are preregular, and generates warnings when a module is loaded if the property does not hold.
- Another important property is sort-decreasingness.
- Assuming E is confluent and terminating, the canonical form $t\downarrow_E$ of a term t by the equations E should have the least sort possible among the sorts of all the terms equivalent to it by the equations E; and it should be possible to compute this least sort from the canonical form itself, using only the operator declarations.
- By a Church-Rosser and terminating theory (Σ, E) we precisely mean one that is confluent, terminating, and sort-decreasing.

Natural numbers division

```
fmod NAT-DIV is
  sorts Nat NzNat .
  subsort NzNat < Nat .

  op 0 : -> Nat [ctor] .
  op s : Nat -> NzNat [ctor] .
  op _+_ : Nat Nat -> Nat .
  op _*_ : Nat Nat -> Nat .
  op _-_ : Nat Nat -> Nat .
  op _<=_ : Nat Nat -> Bool .
  op _>_ : Nat Nat -> Bool .
  op _div_ : Nat NzNat -> Nat .
  op _mod_ : Nat NzNat -> Nat .

  vars M N : Nat .
  var P : NzNat .
```

Natural numbers division

```
    eq 0 + N = N .
    eq s(M) + N = s(M + N) .
    eq 0 * N = 0 .
    eq s(M) * N = (M * N) + N .
    eq 0 - N = 0 .
    eq s(M) - 0 = s(M) .
    eq s(M) - s(N) = M - N .
    eq 0 <= N = true .
    eq s(M) <= 0 = false .
    eq s(M) <= s(N) = M <= N .
    eq N > M = not (N <= M) .
    ceq N div P = 0 if P > N .
    ceq N div P = s((N - P) div P) if P <= N .
    ceq N mod P = N if P > N .
    ceq N mod P = (N - P) mod P if P <= N .
  endfm
```

Predefined modules

$$\begin{tikzcd}
\boxed{\texttt{QID}} & \boxed{\texttt{CONVERSION}} & \\
\boxed{\texttt{STRING}} \arrow[u, Rrightarrow] \arrow[ur, Rrightarrow] & \boxed{\texttt{RAT}} \arrow[u, Rrightarrow] & \boxed{\texttt{FLOAT}} \arrow[ul, Rrightarrow] \\
\boxed{\texttt{COUNTER}} & \boxed{\texttt{INT}} \arrow[u, Rrightarrow] & \\
\boxed{\texttt{RANDOM}} & \boxed{\texttt{NAT}} \arrow[u, Rrightarrow] \arrow[uul, Rrightarrow] \arrow[ul, Rrightarrow] \arrow[l, Rrightarrow] & \boxed{\texttt{EXT-BOOL}} \\
 & \boxed{\texttt{BOOL}} \arrow[u, Rrightarrow] \arrow[uuur, Rrightarrow] \arrow[ur, Rrightarrow] & \\
 & \boxed{\texttt{TRUTH}} \arrow[u, Rrightarrow] & \\
 & \boxed{\texttt{TRUTH-VALUE}} \arrow[u, Rrightarrow] &
\end{tikzcd}$$

Lists of natural numbers

```
fmod NAT-LIST-CONS is
   protecting NAT .

   sorts NeList List .
   subsort NeList < List .

   op [] : -> List [ctor] .                  *** empty list
   op _:_ : Nat List -> NeList [ctor] .      *** cons
   op tail : NeList -> List .
   op head : NeList -> Nat .
   op _++_ : List List -> List .             *** concatenation
   op length : List -> Nat .
   op reverse : List -> List .
   op take_from_ : Nat List -> List .
   op throw_from_ : Nat List -> List .

   vars N M : Nat .
   vars L L' : List .
```

Lists of natural numbers

```
    eq tail(N : L) = L .
    eq head(N : L) = N .
    eq [] ++ L = L .
    eq (N : L) ++ L' = N : (L ++ L') .
    eq length([]) = 0 .
    eq length(N : L) = 1 + length(L) .
    eq reverse([]) = [] .
    eq reverse(N : L) = reverse(L) ++ (N : []) .

    eq take 0 from L = [] .
    eq take N from [] = [] .
    eq take s(N) from (M : L) = M : take N from L .
    eq throw 0 from L = L .
    eq throw N from [] = [] .
    eq throw s(N) from (M : L) = throw N from L .
endfm
```

Equational attributes

- Equational attributes are a means of declaring certain kinds of equational axioms in a way that allows Maude to use these equations efficiently in a built-in way.
- Currently Maude supports the following equational attributes:
  - `assoc` (associativity),
  - `comm` (commutativity),
  - `idem` (idempotency),
  - `id:` ⟨*Term*⟩ (identity, with the corresponding term for the identity element),
  - `left id:` ⟨*Term*⟩ (left identity, with the corresponding term for the left identity element), and
  - `right id:` ⟨*Term*⟩ (right identity, with the corresponding term for the right identity element).
- These attributes are only allowed for binary operators satisfying some appropriate requirements that depend on the attributes.

Matching and simplification modulo

- In the Maude implementation, rewriting modulo A is accomplished by using a matching modulo A algorithm.
- More precisely, given an equational theory A, a term t (corresponding to the lefthand side of an equation) and a subject term u, we say that t matches u modulo A if there is a substitution σ such that $\sigma(t) =_A u$, that is, $\sigma(t)$ and u are equal modulo the equational theory A.
- Given an equational theory $A = \cup_i A_{f_i}$ corresponding to all the attributes declared in different binary operators, Maude synthesizes a combined matching algorithm for the theory A, and does equational simplification modulo the axioms A.

A hierarchy of data types

- nonempty binary trees, with elements only in their leaves, built with a free binary constructor, that is, a constructor with no equational axioms,
- nonempty lists, built with an associative constructor,
- lists, built with an associative constructor and an identity,
- multisets (or bags), built with an associative and commutative constructor and an identity,
- sets, built with an associative, commutative, and idempotent constructor and an identity.

Basic natural numbers

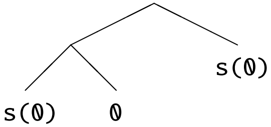
```
fmod BASIC-NAT is
  sort Nat .
  op 0 : -> Nat [ctor] .
  op s : Nat -> Nat [ctor] .
  op _+_ : Nat Nat -> Nat .
  op max : Nat Nat -> Nat .

  vars N M : Nat .
  eq 0 + N = N .
  eq s(M) + N = s(M + N) .
  eq max(0, M) = M .
  eq max(N, 0) = N .
  eq max(s(N), s(M)) = s(max(N, M)) .
endfm
```

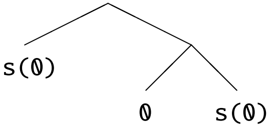
Nonempty binary trees

```
fmod NAT-TREES is
  protecting BASIC-NAT .

  sorts Tree .
  subsort Nat < Tree .
  op __ : Tree Tree -> Tree [ctor] .
  op depth : Tree -> Nat .
  op width : Tree -> Nat .

  var N : Nat .
  vars T T' : Tree .

  eq depth(N) = s(0) .
  eq depth(T T') =  s(max(depth(T), depth(T'))) .
  eq width(N) = s(0) .
  eq width(T T') = width(T) + width(T') .
endfm
```

Nonempty binary trees

- An expression such as `s(0) 0 s(0)` is ambiguous because it can be parsed in two different ways, and parentheses are necessary to disambiguate `(s(0) 0) s(0)` from `s(0) (0 s(0))`.
- These two different terms correspond to the following two different trees:





Nonempty lists

```
fmod NAT-NE-LISTS is
  protecting BASIC-NAT .

  sort NeList .
  subsort Nat < NeList .
  op __ : NeList NeList -> NeList [ctor assoc] .
  op length : NeList -> Nat .
  op reverse : NeList -> NeList .

  var N : Nat .
  var L L' : NeList .

  eq length(N) = s(0) .
  eq length(L L') = length(L) + length(L') .
  eq reverse(N) = N .
  eq reverse(L L') = reverse(L') reverse(L) .
endfm
```

Lists

```
fmod NAT-LISTS is
  protecting BASIC-NAT .

  sorts NeList List .
  subsorts Nat < NeList < List .
  op nil : -> List [ctor] .
  op __ : List List -> List [ctor assoc id: nil] .
  op __ : NeList NeList -> NeList [ctor assoc id: nil] .
  op tail : NeList -> List .
  op head : NeList -> Nat .
  op length : List -> Nat .
  op reverse : List -> List .

  var N : Nat .
  var L : List .

  eq tail(N L) = L .
  eq head(N L) = N .
```

Lists

```
  eq length(nil) = 0 .
  eq length(N L) = s(0) + length(L) .
  eq reverse(nil) = nil .
  eq reverse(N L) = reverse(L) N .
endfm
```

- The alternative equation `length(L L') = length(L) + length(L')` (with `L` and `L'` variables of sort `List`) causes problems of nontermination.
- Consider the instantiation with `L'` $\mapsto$ `nil` that gives

```
  length(L nil) = length(L) + length(nil)
                = length(L nil) + length(nil)
                = (length(L) + length(nil)) + length(nil)
                = ...
```

  because of the identification `L = L nil`.

Multisets

```
fmod NAT-MSETS is
  protecting BASIC-NAT .
  sort Mset .
  subsorts Nat < Mset .
  op empty-mset : -> Mset [ctor] .
  op __ : Mset Mset -> Mset [ctor assoc comm id: empty-mset] .
  op size : Mset -> Nat .
  op mult : Nat Mset -> Nat .
  op _in_ : Nat Mset -> Bool .

  vars N N' : Nat .
  var  S : Mset .

  eq size(empty-mset) = 0 .
  eq size(N S) = s(0) + size(S) .
  eq mult(N, empty-mset) = 0 .
  eq mult(N, N S) = s(0) + mult(N, S) .
  ceq mult(N, N' S) = mult(N, S) if N =/= N' .
  eq N in S = (mult(N, S) =/= 0) .
endfm
```

Multisets with otherwise

```
fmod NAT-MSETS-OWISE is
  protecting BASIC-NAT .
  sort Mset .
  subsorts Nat < Mset .
  op empty-mset : -> Mset [ctor] .
  op __ : Mset Mset -> Mset [ctor assoc comm id: empty-mset] .
  op size : Mset -> Nat .
  op mult : Nat Mset -> Nat .
  op _in_ : Nat Mset -> Bool .

  vars N N' : Nat .
  var  S : Mset .

  eq size(empty-mset) = 0 .
  eq size(N S) = s(0) + size(S) .
  eq N in N S = true .
  eq N in S = false [owise] .
  eq mult(N, N S) = s(0) + mult(N, S) .
  eq mult(N, S) =  0 [owise] .
endfm
```

Sets

```
fmod NAT-SETS is
  protecting BASIC-NAT .
  sort Set .
  subsorts Nat < Set .
  op empty-set : -> Set [ctor] .
  op __ : Set Set -> Set [ctor assoc comm id: empty-set] .

  vars N N' : Nat .
  vars S S' : Set .

  eq N N = N .
```

The idempotency equation is stated only for singleton sets, because stating it for arbitrary sets in the form `S S = S` would cause nontermination due to the identity attribute:

$$\texttt{empty-set} = \texttt{empty-set empty-set} \rightarrow \texttt{empty-set} \ldots$$

Sets

```
    op _in_ : Nat Set -> Bool .
    op delete : Nat Set -> Set .
    op card : Set -> Nat .

    eq N in empty-set = false .
    eq N in (N' S) = (N == N') or (N in S) .
    eq delete(N, empty-set) = empty-set .
    eq delete(N, N S) = delete(N, S) .
    ceq delete(N, N' S) = N' delete(N, S) if N =/= N' .
    eq card(empty-set) = 0 .
    eq card(N S) = s(0) + card(delete(N,S)) .
  endfm
```

- The equations for `delete` and `card` make sure that further occurrences of `N` in `S` on the righthand side are also deleted or not counted, respectively, because we cannot rely on the order in which equations are applied.
- The operations `_in_` and `delete` can also be defined with `owise`.

Membership equational logic specifications

- In order-sorted equational specifications, subsorts must be defined by means of constructors, but it is not possible to have a subsort of sorted lists, for example, defined by a property over lists.
- There is also a different problem of a more syntactic character. In the example of natural numbers division, the term

```
s(s(s(0))) div (s(s(0)) - s(0))
```

  is not even well formed.
- The subterm `s(s(0)) - s(0)` has least sort `Nat`, while the `div` operation expects its second argument to be of sort `NzNat < Nat`.

Membership equational logic specifications

- In order-sorted equational specifications, subsorts must be defined by means of constructors, but it is not possible to have a subsort of sorted lists, for example, defined by a property over lists.
- There is also a different problem of a more syntactic character. In the example of natural numbers division, the term

  ```
  s(s(s(0))) div (s(s(0)) - s(0))
  ```

  is not even well formed.
- The subterm `s(s(0)) - s(0)` has least sort `Nat`, while the `div` operation expects its second argument to be of sort `NzNat < Nat`.
- This is too restrictive and makes most (really) order-sorted specifications useless, unless there is a mechanism that gives at parsing time the benefit of the doubt to this kind of terms.
- Membership equational logic solves both problems, by introducing sorts as predicates and allowing subsort definition by means of conditions involving equations and/or sort predicates.

Membership equational logic

- A signature in membership equational logic is a triple $\Omega = (K, \Sigma, S)$ where K is a set of kinds, (K, Σ) is a many-kinded signature, and $S = \{S_k\}_{k \in K}$ is a K-kinded set of sorts.
- An Ω-algebra is then a (K, Σ)-algebra $\mathbf{A}$ together with the assignment to each sort $s \in S_k$ of a subset $A_s \subseteq A_k$.
- Atomic formulas are either Σ-equations, or membership assertions of the form $t : s$, where the term t has kind k and $s \in S_k$.
- General sentences are Horn clauses on these atomic formulas, quantified by finite sets of K-kinded variables.

$$(\forall X)\ \ t = t' \ \text{ if } \ (\bigwedge_i u_i = v_i) \wedge (\bigwedge_j w_j : s_j)$$

$$(\forall X)\ \ t : s \ \text{ if } \ (\bigwedge_i u_i = v_i) \wedge (\bigwedge_j w_j : s_j).$$

Membership equational logic deduction rules

1. **Reflexivity**. $\dfrac{}{(\forall X)\ t = t}$
2. **Symmetry**. $\dfrac{(\forall X)\ t_1 = t_2}{(\forall X)\ t_2 = t_1}$
3. **Transitivity**. $\dfrac{(\forall X)\ t_1 = t_2 \quad (\forall X)\ t_2 = t_3}{(\forall X)\ t_1 = t_3}$
4. **Congruence**. For each $f \in \Sigma$, $\dfrac{(\forall X)\ t_1 = t_1' \ \ \ldots \ \ (\forall X)\ t_n = t_n'}{(\forall X)\ f(t_1, \ldots, t_n) = f(t_1', \ldots, t_n')}$
5. **Membership**. $\dfrac{(\forall X)\ t_1 = t_2 \quad (\forall X)\ t_1 : s}{(\forall X)\ t_2 : s}$

Membership equational logic deduction rules

6. **Modus ponens 1**. For each sentence

$$(\forall X)\ \ l = r\ \ \text{if}\ \ (\bigwedge_i u_i = v_i) \wedge (\bigwedge_j w_j : s_j),$$

$$\frac{(\forall Y)\,\theta(u_i) = \theta(v_i)\ \text{for all}\ i, \quad (\forall Y)\,\theta(w_j) : s_j\ \text{for all}\ j}{(\forall Y)\,\theta(l) = \theta(r)}$$

7. **Modus ponens 2**. For each sentence

$$(\forall X)\ \ t : s\ \ \text{if}\ \ (\bigwedge_i u_i = v_i) \wedge (\bigwedge_j w_j : s_j),$$

$$\frac{(\forall Y)\,\theta(u_i) = \theta(v_i)\ \text{for all}\ i, \quad (\forall Y)\,\theta(w_j) : s_j\ \text{for all}\ j}{(\forall Y)\,\theta(t) : s}$$

Membership equational logic in Maude

- Maude functional modules are membership equational specifications and their semantics is given by the corresponding initial membership algebra in the class of algebras satisfying the specification.
- Maude does automatic kind inference from the sorts declared by the user and their subsort relations.
- Kinds are not declared explicitly, and correspond to the connected components of the subsort relation.
- The kind corresponding to a sort `s` is denoted `[s]`.
- If `NzNat` < `Nat`, then `[NzNat]` = `[Nat]`.

Membership equational logic in Maude

- An operator declaration like

```
op _div_ : Nat NzNat -> Nat .
```

  can be understood as a declaration at the kind level

```
op _div_ : [Nat] [Nat] -> [Nat] .
```

  together with the conditional membership axiom

```
cmb N div M : Nat if N : Nat and M : NzNat .
```

- A subsort declaration `NzNat < Nat` can be understood as the conditional membership axiom

```
cmb N : Nat if N : NzNat .
```

Multiples of 3

```
fmod 3*NAT is
  sorts Zero Nat .
  subsort Zero < Nat .

  op 0 : -> Zero [ctor] .
  op s_ : Nat -> Nat [ctor] .

  sort 3*Nat .
  subsorts Zero < 3*Nat < Nat .

  var M3 : 3*Nat .

  mb (s s s M3) : 3*Nat  .
endfm
```

Palindromes

```
fmod PALINDROME is
  sorts Letter Word Pal .
  subsort Letter < Pal < Word .

  ops a b c d e f g h i j k l m
      n o p q r s t u v w x y z : -> Letter [ctor] .
  op nil : -> Pal [ctor] .
  op __ : Word Word -> Word [ctor assoc id: nil] .

  var A : Letter .
  var P : Pal .

  mb A P A : Pal  .
endfm
```

Sorted lists

```
fmod NAT-SORTED-LIST is
  protecting NAT-LIST-CONS .

  sorts SortedList NeSortedList .
  subsort NeSortedList < SortedList NeList < List .

  op insertion-sort : List -> SortedList .
  op insert-list : SortedList Nat -> SortedList .

  op mergesort : List -> SortedList .
  op merge : SortedList SortedList -> SortedList [comm] .

  op quicksort : List -> SortedList .
  op leq-elems : List Nat -> List .
  op gr-elems : List Nat -> List .

  vars N M : Nat .
  vars L L' : List .
  vars OL OL' : SortedList .
  var NEOL : NeSortedList .
```

Sorted lists

```
mb [] : SortedList .
mb N : [] : NeSortedList .
cmb N : NEOL : NeSortedList if N <= head(NEOL) .

eq insertion-sort([]) = [] .
eq insertion-sort(N : L) = insert-list(insertion-sort(L), N) .

eq insert-list([], M) = M : [] .
ceq insert-list(N : OL, M) = M : N : OL if M <= N .
ceq insert-list(N : OL, M) = N : insert-list(OL, M) if M > N .

eq mergesort([]) = [] .
eq mergesort(N : []) = N : [] .
ceq mergesort(L) =
    merge(mergesort(take (length(L) quo 2) from L),
          mergesort(throw (length(L) quo 2) from L))
 if length(L) > s(0) .
```

Sorted lists

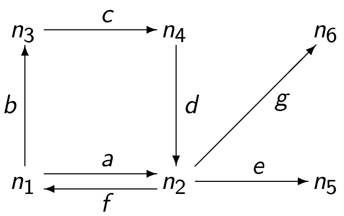
```
  eq merge(OL, []) = OL .
  ceq merge(N : OL, M : OL') = N : merge(OL, M : OL') if N <= M .

  eq quicksort([]) = [] .
  eq quicksort(N : L)
    = quicksort(leq-elems(L,N)) ++ (N : quicksort(gr-elems(L,N))) .

  eq leq-elems([], M) = [] .
  ceq leq-elems(N : L, M) = N : leq-elems(L, M) if N <= M .
  ceq leq-elems(N : L, M) = leq-elems(L, M) if N > M .
  eq gr-elems([], M) = [] .
  ceq gr-elems(N : L, M) = gr-elems(L, M) if N <= M .
  ceq gr-elems(N : L, M) = N : gr-elems(L, M) if N > M .
endfm
```

Partial constructors: powerlists

- A powerlist must be of length 2^n for some $n \in \mathbb{N}$.

```
fmod POWERLIST is
  protecting NAT .

  sort Pow .

  op [_] : Nat -> Pow [ctor] .
  op _|_ : [Pow] [Pow] -> [Pow] [assoc] .
  op len : Pow -> Nat .

  var I : Nat .
  vars P Q : Pow .

  cmb P | Q : Pow if len(P) = len(Q) .

  eq len([I]) = 1 .
  eq len(P | Q) = len(P) + len(Q) .
endfm
```

Partial constructors: paths


$$\begin{tikzcd}
n_3 \arrow[r, "c"] & n_4 \arrow[d, "d"] & n_6 \\
n_1 \arrow[u, "b"] \arrow[r, shift left, "a"] & n_2 \arrow[l, shift left, "f"] \arrow[r, "e"] \arrow[ur, "g"] & n_5
\end{tikzcd}$$

```
fmod A-GRAPH is
  sorts Edge Node .
  ops n1 n2 n3 n4 n5 n6 : -> Node [ctor] .
  ops a b c d e f g : -> Edge [ctor] .
  ops source target : Edge -> Node .
  eq source(a) = n1 .  eq target(a) = n2 .
  eq source(b) = n1 .  eq target(b) = n3 .
  eq source(c) = n3 .  eq target(c) = n4 .
  eq source(d) = n4 .  eq target(d) = n2 .
  eq source(e) = n2 .  eq target(e) = n5 .
  eq source(f) = n2 .  eq target(f) = n1 .
  eq source(g) = n2 .  eq target(f) = n6 .
endfm
```

Paths

```
fmod PATH is
  protecting NAT .
  protecting A-GRAPH .
  sorts Path .
  subsorts Edge < Path .
  op _;_ : [Path] [Path] -> [Path] [assoc] .
  ops source target : Path -> Node .
  op length : Path -> Nat .

  var E : Edge .
  var P : Path .
  cmb (E ; P) : Path if target(E) == source(P) .

  eq source(E ; P) = source(E) .
  eq target(P ; E) = target(E) .
  eq length(E) = 1 .
  eq length(E ; P) = 1 + length(P) .
endfm
```

Parameterization: theories

- Parameterized datatypes use theories to specify the requirements that the parameter must satisfy.
- A (functional) theory is a membership equational specification whose semantics is loose.
- Equations in a theory are not used for rewriting or equational simplication and, thus, they need not be confluent or terminating.
- Simplest theory only requires existence of a sort:

```
fth TRIV is
  sort Elt .
endfth
```

Order theories

- Theory requiring a strict total order over a given sort:

```
fth STOSET is
  protecting BOOL .
  sort Elt .
  op _<_ : Elt Elt -> Bool .
  vars X Y Z : Elt .
  eq X < X = false [nonexec label irreflexive] .
  ceq X < Z = true if X < Y /\ Y < Z [nonexec label transitive] .
  ceq X = Y if X < Y /\ Y < X [nonexec label antisymmetric] .
  ceq X = Y if X < Y = false /\ Y < X = false [nonexec label total] .
endfth
```

Order theories

- Theory requiring a nonstrict total order over a given sort:

```
fth TOSET is
  including STOSET .
  op _<=_ : Elt Elt -> Bool .
  vars X Y : Elt .
  eq X <= X = true [nonexec] .
  ceq X <= Y = true if X < Y [nonexec] .
  ceq X = Y if X <= Y /\ X < Y = false [nonexec] .
endfth
```

Parameterization: views

- Theories are used in a parameterized module expression such as

  ```
  fmod LIST{X :: TRIV} is  ...  endfm
  ```

  to make explicit the requirements over the argument module.
- A view shows how a particular module satisfies a theory, by mapping sorts and operations in the theory to sorts and operations in the target module, in such a way that the induced translations on equations and membership axioms are provable in the module.
- Each view declaration has an associated set of proof obligations, namely, for each axiom in the source theory it should be the case that the axiom's translation by the view holds true in the target. This may in general require inductive proof techniques.
- In many simple cases it is completely obvious:

  ```
  view Nat from TRIV to NAT is
    sort Elt to Nat .
  endv
  ```

Parameterization: instantiation

- A module expression such as LIST{Nat} denotes the instantiation of the parameterized module LIST{X :: TRIV} by means of the previous view Nat.

$$\begin{tikzcd}
\texttt{TRIV} \arrow[r, "\texttt{Nat}"] \arrow[d] & \texttt{NAT} \arrow[d] \\
\texttt{LIST\{X :: TRIV\}} \arrow[r] & \texttt{LIST\{Nat\}}
\end{tikzcd}$$

- Views can also go from theories to theories, meaning an instantiation that is still parameterized.

```
view Toset from TRIV to TOSET is
  sort Elt to Elt .
endv
```

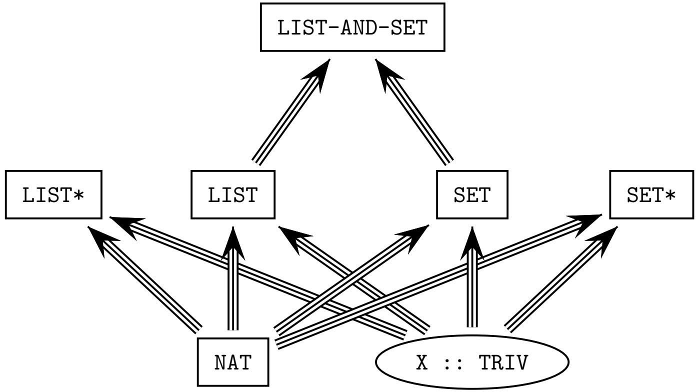
- It is possible to have more than one view from a theory to a module or to another theory.

Parameterized modules: Simple example

```
fmod MAYBE{X :: TRIV} is
  sort Maybe{X} .
  subsort X$Elt < Maybe{X} .
  op maybe : -> Maybe{X} [ctor] .
endfm
```

- The sort `Maybe{X}` is declared as a supersort of the sort `Elt` coming from the parameter theory.
- We add a "new" constant `maybe` to this sort `Maybe{X}`.
- This technique is useful to declare a partial function as a total function, so that `maybe` represents the undefined value.

Predefined parameterized modules



Stacks

```
fmod STACK{X :: TRIV} is
  sorts NeStack{X} Stack{X} .
  subsort NeStack{X} < Stack{X} .
  op empty : -> Stack{X} [ctor] .
  op push : X$Elt Stack{X} -> NeStack{X} [ctor] .
  op pop : NeStack{X} -> Stack{X} .
  op top : NeStack{X} -> X$Elt .
  op isEmpty : Stack{X} -> Bool .

  var S : Stack{X} .
  var E : X$Elt .

  eq pop(push(E,S)) = S .
  eq top(push(E,S)) = E .
  eq isEmpty(empty) = true .
  eq isEmpty(push(E,S)) = false .
endfm
```

Stacks

```
view Int from TRIV to INT is
  sort Elt to Int .
endv

fmod STACK-TEST is
  protecting STACK{Int} .
endfm

Maude> red top(push(4,push(5,empty))) .
result NzNat : 4
```

Queues

```
fmod QUEUE{X :: TRIV} is
  sort NeQueue{X} Queue{X} .
  subsort NeQueue{X} < Queue{X} .
  op empty : -> Queue{X} [ctor] .
  op enqueue : Queue{X} X$Elt -> NeQueue{X} [ctor] .
  op dequeue : NeQueue{X} -> Queue{X} .
  op first : NeQueue{X} -> X$Elt .
  op isEmpty : Queue{X} -> Bool .

  var Q : Queue{X} .
  var E : X$Elt .

  eq dequeue(enqueue(empty,E)) = empty .
  ceq dequeue(enqueue(Q,E)) = enqueue(dequeue(Q),E) if Q =/= empty .
  eq first(enqueue(empty,E)) = E .
  ceq first(enqueue(Q,E)) = first(Q) if Q =/= empty .
  eq isEmpty(empty) = true .
  eq isEmpty(enqueue(Q,E)) = false .
endfm
```

Priority queues

```
fmod PRIORITY-QUEUE{X :: TOSET} is
  sort NePQueue{X} PQueue{X} .
  subsort NePQueue{X} < PQueue{X} .
  op empty : -> PQueue{X} .
  op insert : PQueue{X} X$Elt -> NePQueue{X} .
  op deleteMin : NePQueue{X} -> PQueue{X} .
  op findMin : NePQueue{X} -> X$Elt .
  op isEmpty : PQueue{X} -> Bool .

  var PQ : PQueue{X} .
  vars E F : X$Elt .
```

Priority queues

```
    eq insert(insert(PQ,E),F) = insert(insert(PQ,F),E) [nonexec] .
    eq  deleteMin(insert(empty,E)) = empty .
    ceq deleteMin(insert(insert(PQ,E),F)) =
        insert(deleteMin(insert(PQ,E)),F)
     if findMin(insert(PQ,E)) <= F .
    ceq deleteMin(insert(insert(PQ,E),F)) = insert(PQ,E)
     if F < findMin(insert(PQ,E)) .
    eq  findMin(insert(empty,E)) = E .
    ceq findMin(insert(insert(PQ,E),F)) = findMin(insert(PQ,E))
     if findMin(insert(PQ,E)) <= F .
    ceq findMin(insert(insert(PQ,E),F)) = F
     if F < findMin(insert(PQ,E)) .
    eq isEmpty(empty) = true .
    eq isEmpty(insert(PQ,E)) = false .
  endfm
```

Priority queues

```
view IntAsToset from TOSET to INT is
  sort Elt to Int .
endv

fmod PRIORITY-QUEUE-TEST is
  protecting PRIORITY-QUEUE{IntAsToset} .
endfm

Maude> red findMin(insert(insert(empty,4),5)) .
result NzNat: 4
```

Parameterized lists

```
fmod LIST-CONS{X :: TRIV} is
  protecting NAT .

  sorts NeList{X} List{X} .
  subsort NeList{X} < List{X} .

  op [] : -> List{X} [ctor] .
  op _:_ : X$Elt List{X} -> NeList{X} [ctor] .
  op tail : NeList{X} -> List{X} .
  op head : NeList{X} -> X$Elt .

  var E : X$Elt .
  var N : Nat .
  vars L L' : List{X} .

  eq tail(E : L) = L .
  eq head(E : L) = E .
```

Parameterized lists

```
    op _++_ : List{X} List{X} -> List{X} .
    op length : List{X} -> Nat .
    op reverse : List{X} -> List{X} .
    op take_from_ : Nat List{X} -> List{X} .
    op throw_from_ : Nat List{X} -> List{X} .

    eq [] ++ L = L .
    eq (E : L) ++ L' = E : (L ++ L') .
    eq length([]) = 0 .
    eq length(E : L) = 1 + length(L) .
    eq reverse([]) = [] .
    eq reverse(E : L) = reverse(L) ++ (E : []) .
    eq take 0 from L = [] .
    eq take N from [] = [] .
    eq take s(N) from (E : L) = E : take N from L .
    eq throw 0 from L = L .
    eq throw N from [] = [] .
    eq throw s(N) from (E : L) = throw N from L .
  endfm
```

Parameterized sorted lists

```
view Toset from TRIV to TOSET is
  sort Elt to Elt .
endv

fmod SORTED-LIST{X :: TOSET} is
  protecting LIST-CONS{Toset}{X} .

  sorts SortedList{X} NeSortedList{X} .
  subsorts NeSortedList{X} < SortedList{X} < List{Toset}{X} .
  subsort NeSortedList{X} < NeList{Toset}{X} .

  vars N M : X$Elt .
  vars L L' : List{Toset}{X} .
  vars OL OL' : SortedList{X} .
  var NEOL : NeSortedList{X} .
```

Parameterized sorted lists

```
    mb [] : SortedList{X} .
    mb (N : []) : NeSortedList{X} .
    cmb (N : NEOL) : NeSortedList{X} if N <= head(NEOL) .

    op insertion-sort : List{Toset}{X} -> SortedList{X} .
    op insert-list : SortedList{X} X$Elt -> SortedList{X} .

    op mergesort : List{Toset}{X} -> SortedList{X} .
    op merge : SortedList{X} SortedList{X} -> SortedList{X} [comm] .

    op quicksort : List{Toset}{X} -> SortedList{X} .
    op leq-elems : List{Toset}{X} X$Elt -> List{Toset}{X} .
    op gr-elems : List{Toset}{X} X$Elt -> List{Toset}{X} .

    *** equations as before
endfm
```

Parameterized sorted lists

```
view NatAsToset from TOSET to NAT is
  sort Elt to Nat .
endv

fmod SORTED-LIST-TEST is
  protecting SORTED-LIST{NatAsToset} .
endfm

Maude> red insertion-sort(5 : 4 : 3 : 2 : 1 : 0 : []) .
result NeSortedList{NatAsToset}: 0 : 1 : 2 : 3 : 4 : 5 : []

Maude> red mergesort(5 : 3 : 1 : 0 : 2 : 4 : []) .
result NeSortedList{NatAsToset}: 0 : 1 : 2 : 3 : 4 : 5 : []

Maude> red quicksort(0 : 1 : 2 : 5 : 4 : 3 : []) .
result NeSortedList{NatAsToset}: 0 : 1 : 2 : 3 : 4 : 5 : []
```

Parameterized multisets

```
fmod MULTISET{X :: TRIV} is
  protecting NAT .
  sort Mset{X} .
  subsort X$Elt < Mset{X} .
  op empty : -> Mset{X} [ctor] .
  op __ : Mset{X} Mset{X} -> Mset{X} [ctor assoc comm id: empty] .

  vars E E' : X$Elt .
  vars S S' : Mset{X} .

  op isEmpty : Mset{X} -> Bool .
  eq isEmpty(empty) = true .
  eq isEmpty(E S) = false .
  op size : Mset{X} -> Nat .
  eq size(empty) = 0 .
  eq size(E S) = 1 + size(S) .
  op mult : X$Elt Mset{X} -> Nat .
  eq mult(E, E S) = 1 + mult(E, S) .
  eq mult(E, S) = 0 [owise] .
```

Parameterized multisets

```
    op isIn : X$Elt Mset{X} -> Bool .
    eq isIn(E, E S) = true .
    eq isIn(E, S) = false [owise] .

    op delete : X$Elt Mset{X} -> Mset{X} .
    eq delete(E, E S) = delete(E, S) .
    eq delete(E, S) = S [owise] .
    op delete1 : X$Elt Mset{X} -> Mset{X} .
    eq delete1(E, E S) = S .
    eq delete1(E, S) = S [owise] .

    op intersection : Mset{X} Mset{X} -> Mset{X} .
    eq intersection(E S, E S') = E intersection(S, S') .
    eq intersection(S, S') = empty [owise] .
    op difference : Mset{X} Mset{X} -> Mset{X} .
    eq difference(E S, E S') = difference(S, S') .
    eq difference(S, S') = S [owise] .
  endfm
```

Binary trees

```
fmod BIN-TREE{X :: TRIV} is
  protecting LIST-CONS{X} .

  sorts NeBinTree{X} BinTree{X} .
  subsort NeBinTree{X} < BinTree{X} .

  op empty : -> BinTree{X} [ctor] .
  op _[_]_ : BinTree{X} X$Elt BinTree{X} -> NeBinTree{X} [ctor] .
  ops left right : NeBinTree{X} -> BinTree{X} .
  op root : NeBinTree{X} -> X$Elt .

  var E : X$Elt .
  vars L R : BinTree{X} .
  vars NEL NER : NeBinTree{X} .

  eq left(L [E] R) = L .
  eq right(L [E] R) = R .
  eq root(L [E] R) = E .
```

Binary trees

```
    op depth : BinTree{X} -> Nat .
    ops leaves preorder inorder postorder : BinTree{X} -> List{X} .

    eq depth(empty) = 0 .
    eq depth(L [E] R) = 1 + max(depth(L), depth(R)) .
    eq leaves(empty) = [] .
    eq leaves(empty [E] empty) = E : [] .
    eq leaves(NEL [E] R) = leaves(NEL) ++ leaves(R) .
    eq leaves(L [E] NER) = leaves(L) ++ leaves(NER) .

    eq preorder(empty) = [] .
    eq preorder(L [E] R) = E : (preorder(L) ++ preorder(R)) .
    eq inorder(empty) = [] .
    eq inorder(L [E] R) = inorder(L) ++ (E : inorder(R)) .
    eq postorder(empty) = [] .
    eq postorder(L [E] R)
      = postorder(L) ++ (postorder(R) ++ (E : [])) .
  endfm
```

Binary search trees

- Search trees, like dictionaries, contain in the nodes pairs formed by a key and its associated contents.
- The search tree structure is with respect to a strict total order on keys, but contents can be over an arbitrary sort.
- When we insert a pair $\langle K, C\rangle$ and the key K already appears in the tree in a pair $\langle K, C'\rangle$, insertion takes place by combining the contents C' and C.
- As part of the requirement parameter theory for the contents we will have an associative binary operator `combine` on the sort `Contents`.

```
fth CONTENTS is
  sort Contents .
  op combine : Contents Contents -> Contents [assoc] .
endfth
```

Binary search trees

- The module for search trees will be imported in the specifications of AVL and red-black trees.
- It is important to be able to add new information in the nodes: for AVL trees, the depth of the tree hanging in each node, and for red-black trees the appropriate node color.

Binary search trees

- The module for search trees will be imported in the specifications of AVL and red-black trees.
- It is important to be able to add new information in the nodes: for AVL trees, the depth of the tree hanging in each node, and for red-black trees the appropriate node color.
- A record is defined as a collection (with an associative and commutative union operator denoted by _,_) of pairs consisting of a field name and an associated value.

```
fmod RECORD is
  sorts Record .
  op null : -> Record [ctor] .
  op _,_ : Record Record -> Record [ctor assoc comm id: null] .
endfm

view Record from TRIV to RECORD is
  sort Elt to Record .
endv
```

Binary search trees

```
fmod SEARCH-TREE{X :: STOSET, Y :: CONTENTS} is
  extending BIN-TREE{Record} .
  protecting MAYBE{Contents}{Y} * (op maybe to not-found) .

  sort SearchRecord{X, Y} .
  subsort SearchRecord{X, Y} < Record .

  sorts SearchTree{X, Y} NeSearchTree{X, Y} .
  subsorts NeSearchTree{X, Y} < SearchTree{X, Y} < BinTree{Record} .
  subsort NeSearchTree{X, Y} < NeBinTree{Record} .

  --- Search records, used as nodes in search trees.
  var Rec : [Record] .
  var K : X$Elt .
  var C : Y$Contents .
```

Binary search trees

```
op key:_ : X$Elt -> Record [ctor] .
op key : Record ~> X$Elt .
op numKeys : Record -> Nat .
eq numKeys(key: K, Rec) = 1 + numKeys(Rec) .
eq numKeys(Rec) = 0 [owise] .
ceq key(Rec, key: K) = K if numKeys(Rec, key: K) = 1 .

op contents:_ : Y$Contents -> Record [ctor] .
op numContents : Record -> Nat .
op contents : Record ~> Y$Contents .
eq numContents(contents: C, Rec) = 1 + numContents(Rec) .
eq numContents(Rec) = 0 [owise] .
ceq contents(Rec, contents: C) = C
  if numContents(Rec, contents: C) = 1 .

cmb Rec : SearchRecord{X, Y}
  if numContents(Rec) = 1 /\ numKeys(Rec) = 1 .
```

Binary search trees

```
--- Definition of binary search trees.
ops min max : NeSearchTree{X, Y} -> SearchRecord{X, Y} .
var  SRec : SearchRecord{X, Y} .
vars L R : SearchTree{X, Y} .
vars L' R' : NeSearchTree{X, Y} .
var  C' : Y$Contents .
eq min(empty [SRec] R) = SRec .
eq min(L' [SRec] R) = min(L') .
eq max(L [SRec] empty) = SRec .
eq max(L [SRec] R') = max(R') .

mb empty : SearchTree{X, Y} .
mb empty [SRec] empty : NeSearchTree{X, Y} .
cmb L' [SRec] empty : NeSearchTree{X, Y}
  if key(max(L')) < key(SRec) .
cmb empty [SRec] R' : NeSearchTree{X, Y}
  if key(SRec) < key(min(R')) .
cmb L' [SRec] R' : NeSearchTree{X, Y}
  if key(max(L')) < key(SRec) /\ key(SRec) < key(min(R')) .
```

Binary search trees

```
--- Operations for binary search trees.
op insert : SearchTree{X, Y} X$Elt Y$Contents -> SearchTree{X, Y} .
op lookup : SearchTree{X, Y} X$Elt -> Maybe{Contents}{Y} .
op delete : SearchTree{X, Y} X$Elt -> SearchTree{X, Y} .
op find : SearchTree{X, Y} X$Elt -> Bool .

eq insert(empty, K, C) = empty [key: K, contents: C] empty .
ceq insert(L [Rec, key: K, contents: C] R, K, C')
  = L [Rec, key: K, contents: combine(C, C')] R
  if numKeys(Rec) = 0 /\ numContents(Rec) = 0 .
ceq insert(L [SRec] R, K, C) = insert(L, K, C) [SRec] R
  if K < key(SRec) .
ceq insert(L [SRec] R, K, C) = L [SRec] insert(R, K, C)
  if key(SRec) < K .

eq lookup(empty, K) = not-found .
ceq lookup(L [SRec] R, K) = C
  if key(SRec) = K /\ C := contents(SRec) .
ceq lookup(L [SRec] R, K) = lookup(L, K) if K < key(SRec) .
ceq lookup(L [SRec] R, K) = lookup(R, K) if key(SRec) < K .
```

Binary search trees

```
  eq delete(empty, K) = empty .
  ceq delete(L [SRec] R, K) = delete(L, K) [SRec] R
    if K < key(SRec) .
  ceq delete(L [SRec] R, K) = L [SRec] delete(R, K)
    if key(SRec) < K .
  ceq delete(empty [SRec] R, K) = R if key(SRec) = K .
  ceq delete(L [SRec] empty, K) = L if key(SRec) = K .
  ceq delete(L' [SRec] R', K)
    = L' [min(R')] delete(R', key(min(R')))
    if key(SRec) = K .

  eq find(empty, K) = false .
  ceq find(L [SRec] R, K) = true if key(SRec) = K .
  ceq find(L [SRec] R, K) = find(L, K) if K < key(SRec) .
  ceq find(L [SRec] R, K) = find(R, K) if key(SRec) < K .
endfm
```

Binary search trees

```
view StringAsContents from CONTENTS to STRING is
  sort Contents to String .
  op combine to _+_ .
endv

view IntAsStoset from STOSET to INT is
  sort Elt to Int .
endv
```

Binary search trees

```
view StringAsContents from CONTENTS to STRING is
  sort Contents to String .
  op combine to _+_ .
endv

view IntAsStoset from STOSET to INT is
  sort Elt to Int .
endv

fmod SEARCH-TREE-TEST is
  protecting SEARCH-TREE{IntAsStoset, StringAsContents} .
endfm

Maude> red insert(insert(empty, 1, "a"), 2, "b") .
result NeSearchTree{IntAsStoset,StringAsContents}:
       empty[key: 1,contents: "a"](empty[key: 2,contents:"b"]empty)

Maude> red lookup(insert(insert(insert(empty, 1, "a"),
                                       2, "b"), 1, "c"), 1) .
result String: "ac"
```

AVL trees

- AVL trees are binary search trees satisfying the additional constraint in each node that the difference between the depth of both children is at most one.
- Then the depth of the tree is always logarithmic with respect to the number of nodes.
- We obtain a logarithmic cost for the operations of search, lookup, insertion, and deletion, assuming that the last two are implemented in such a way that they keep the properties of the balanced tree.
- In our presentation we add a `depth` field to the search records used to hold the information in the nodes of binary search trees.

AVL trees

```
fmod AVL{X :: STOSET, Y :: CONTENTS} is
  extending SEARCH-TREE{X, Y} .

  --- Add depth to search records.
  var N : Nat .
  var Rec : Record .

  sort AVLRecord{X, Y} .
  subsort AVLRecord{X, Y} < SearchRecord{X, Y} .
  op depth:_ : Nat -> Record [ctor] .
  op numDepths : Record -> Nat .
  op depth : Record ~> Nat .
  eq numDepths(depth: N, Rec) = 1 + numDepths(Rec) .
  eq numDepths(Rec) = 0 [owise] .
  ceq depth(Rec,depth: N) = N if numDepths(Rec) = 0 .

  var SRec : SearchRecord{X, Y} .
  cmb SRec : AVLRecord{X, Y} if numDepths(SRec) = 1 .
```

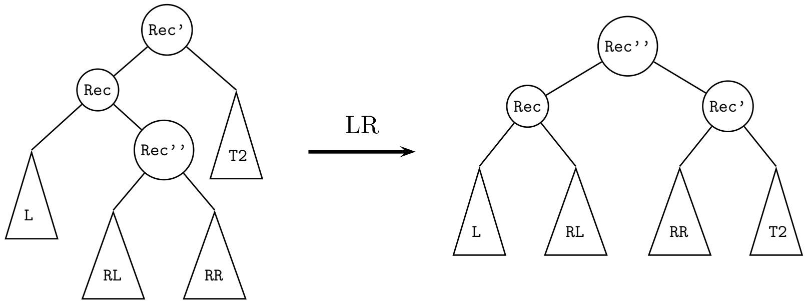
AVL trees

```
sorts NeAVL{X, Y} AVL{X, Y} .
subsorts NeAVL{X, Y} < AVL{X, Y} < SearchTree{X, Y} .
subsorts NeAVL{X, Y} < NeSearchTree{X, Y} .
vars AVLRec AVLRec' AVLRec'' : AVLRecord{X, Y} .
vars L R L' R' RL RR LR LL T1 T2 : AVL{X, Y} .
var  ST : NeSearchTree{X, Y} .

mb empty : AVL{X, Y} .
cmb ST : NeAVL{X, Y}
  if L [AVLRec] R := ST /\ sd(depth(L), depth(R)) <= 1
     /\ 1 + max(depth(L), depth(R)) = depth(AVLRec) .
```

AVL trees

```
  op insertAVL : AVL{X, Y} X$Elt Y$Contents -> NeAVL{X, Y} .
  op deleteAVL : X$Elt AVL{X, Y} -> AVL{X, Y} .
  op depthAVL : AVL{X, Y} -> Nat .
  op buildAVL : AVL{X, Y} Record AVL{X, Y} ~> AVL{X, Y} .
  op join : AVL{X, Y} Record AVL{X, Y} ~> AVL{X, Y} .
  op lRotate : AVL{X, Y} AVLRecord{X, Y} AVL{X, Y} ~> AVL{X, Y} .
  op rRotate : AVL{X, Y} AVLRecord{X, Y} AVL{X, Y} ~> AVL{X, Y} .

  vars K K' : X$Elt .
  vars C C' : Y$Contents .

 *** EQUATIONS NOT SHOWN
endfm
```

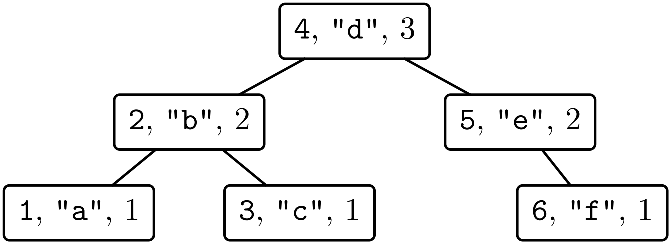
AVL trees



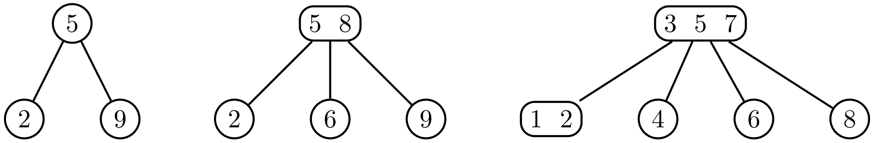
AVL trees

```
fmod AVL-TEST is
  protecting AVL{IntAsStoset, StringAsContents} .
endfm

Maude> red insertAVL(insertAVL(insertAVL(insertAVL(
              insertAVL(insertAVL(empty, 1, "a"), 2, "b"), 3, "c"),
                4, "d"), 5, "e"), 6, "f") .
result NeAVL{IntAsStoset,StringAsContents}:
   ((empty[key: 1,contents: "a",depth: 1]empty)
    [key: 2,contents: "b",depth: 2]
    (empty[key: 3,contents: "c",depth: 1]empty))
  [key: 4,contents: "d",depth:3]
   (empty
    [key: 5,contents: "e",depth: 2]
   (empty[key: 6,contents: "f",depth: 1]empty))
```

AVL trees



2-3-4 trees



2-3-4 trees

```
op empty234 : -> 234Tree{T} [ctor] .
op _[_]_ : 234Tree?{T} T$Elt 234Tree?{T} -> Ne234Tree?{T} [ctor] .
op _<_>_<_>_ : 234Tree?{T} T$Elt 234Tree?{T} T$Elt 234Tree?{T}
             -> Ne234Tree?{T} [ctor] .
op _{_}_{_}_{_}_ : 234Tree?{T} T$Elt 234Tree?{T} T$Elt
           234Tree?{T} T$Elt 234Tree?{T} -> Ne234Tree?{T} [ctor] .

cmb TL [ N ] TR : Ne234Tree{T}
  if greaterKey(N, TL) /\ smallerKey(N, TR)
     /\ depth(TL) = depth(TR) .
cmb TL < N1 > TC < N2 > TR : Ne234Tree{T}
  if N1 < N2
     /\ greaterKey(N1, TL) /\ smallerKey(N1, TC)
     /\ greaterKey(N2, TC) /\ smallerKey(N2, TR)
     /\ depth(TL) = depth(TC) /\ depth(TC) = depth(TR) .
cmb TL { N1 } TLM { N2 } TRM { N3 } TR : Ne234Tree{T}
  if N1 < N2 /\ N2 < N3
     /\ greaterKey(N1, TL) /\ smallerKey(N1, TLM)
     /\ greaterKey(N2, TLM) /\ smallerKey(N2, TRM)
     /\ greaterKey(N3, TRM) /\ smallerKey(N3, TR)
     /\ depth(TL) = depth(TLM) /\ depth(TL) = depth(TRM)
     /\ depth(TL) = depth(TR) .
```

2-3-4 trees

```
eq find(M, empty234) = false .
ceq find(M, T1 [N1] T2) = find(M, T1) if M < N1 .
eq find(M, T1 [M] T2) = true .
ceq find(M, T1 [N1] T2) = find(M, T2) if N1 < M .
ceq find(M, T1 < N1 > T2 < N2 > T3) = find(M, T1) if M < N1 .
eq find(M, T1 < M > T2 < N2 > T3) = true .
ceq find(M, T1 < N1 > T2 < N2 > T3) = find(M, T2)
  if N1 < M /\ M < N2 .
eq find(M, T1 < N1 > T2 < M > T3) = true .
ceq find(M, T1 < N1 > T2 < N2 > T3) = find(M, T3) if N2 < M .
ceq find(M, T1 { N1 } T2 { N2 } T3 { N3 } T4) = find(M, T1)
  if M < N1 .
eq find(M, T1 { M } T2 { N2 } T3 { N3 } T4) = true .
ceq find(M, T1 { N1 } T2 { N2 } T3 { N3 } T4) = find(M, T2)
  if N1 < M /\ M < N2 .
eq find(M, T1 { N1 } T2 { M } T3 { N3 } T4) = true .
ceq find(M, T1 { N1 } T2 { N2 } T3 { N3 } T4) = find(M, T3)
  if N2 < M /\ M < N3 .
eq find(M, T1 { N1 } T2 { N2 } T3 { M } T4) = true .
ceq find(M, T1 { N1 } T2 { N2 } T3 { N3 } T4) = find(M, T4)
  if N3 < M .
```

Red-black trees

```
sorts NeRBTree{X, Y} RBTree{X, Y} .
subsort NeRBTree{X, Y} < RBTree{X, Y} < SearchTree{X, Y} .
subsort NeRBTree{X, Y} < NeSearchTree{X, Y} .

var  RBRec : RBRecord{X, Y} .
vars ST RBTL? RBTR? : SearchTree{X, Y} .

mb empty : RBTree{X, Y} .
cmb ST : NeRBTree{X, Y}
  if RBTL? [RBRec] RBTR? := ST /\ color(RBRec) = b
     /\ blackDepth(RBTR?) = blackDepth(RBTR?)
     /\ blackBalance(RBTL?) /\ blackBalance(RBTR?)
     /\ not twoRed(RBTL?) /\ not twoRed(RBTR?) .

--- Auxiliary operations
op blackDepth : BinTree{Record} -> Nat .
op blackBalance : BinTree{Record} -> Bool .
op twoRed : BinTree{Record} -> Bool .
```

Rewriting logic

- We arrive at the main idea behind rewriting logic by dropping symmetry and the equational interpretation of rules.
- Rewriting logic was introduced by J. Meseguer in 1990 as a unifying framework for concurrency.
- We interpret a rule $t \to t'$ computationally as a local concurrent transition of a system, and logically as an inference step from formulas of type t to formulas of type t'.
- Rewriting logic is a logic of becoming or change, that allows us to specify the dynamic aspects of systems.

Rewriting logic

- We arrive at the main idea behind rewriting logic by dropping symmetry and the equational interpretation of rules.
- Rewriting logic was introduced by J. Meseguer in 1990 as a unifying framework for concurrency.
- We interpret a rule $t \rightarrow t'$ computationally as a local concurrent transition of a system, and logically as an inference step from formulas of type t to formulas of type t'.
- Rewriting logic is a logic of becoming or change, that allows us to specify the dynamic aspects of systems.
- Representation of systems in rewriting logic:
  - The static part is specified as an equational theory.
  - The dynamics is specified by means of possibly conditional rules that rewrite terms, representing parts of the system, into others.
  - The rules need only specify the part of the system that actually changes: the frame problem is avoided.

Rewriting logic

- A rewriting logic signature is an equational specification (Ω, E) that makes explicit the set of equations in order to emphasize that rewriting will operate on congruence classes of terms modulo E.
- Sentences are rewrites of the form $[t]_E \longrightarrow [t']_E$.
- A rewriting logic specification $\mathcal{R} = (\Omega, E, L, R)$ consists of:
  - a signature (Ω, E),
  - a set L of labels, and
  - a set R of labelled rewrite rules $\quad r : [t]_E \longrightarrow [t']_E$ where r is a label and $[t]_E, [t']_E$ are congruence classes of terms in $\mathcal{T}_{\Omega,E}(X)$.
- The most general form of a rewrite rule is conditional:

$$r : t \to t' \;\; \mathit{if} \;\; (\bigwedge_i u_i = v_i) \wedge (\bigwedge_j w_j : s_j) \wedge (\bigwedge_k p_k \to q_k)$$

The meaning of rewriting logic

$$\begin{array}{ccccc}
\textit{State} & \longleftrightarrow & \textit{Term} & \longleftrightarrow & \textit{Proposition} \\
\textit{Transition} & \longleftrightarrow & \textit{Rewriting} & \longleftrightarrow & \textit{Deduction} \\
\textit{Distributed} & \longleftrightarrow & \textit{Algebraic} & \longleftrightarrow & \textit{Propositional} \\
\textit{structure} & & \textit{structure} & & \textit{structure}
\end{array}$$

Rewriting logic deduction rules (unconditional, unsorted)

1. **Reflexivity**. $\dfrac{}{[t] \longrightarrow [t]}$
2. **Transitivity**. $\dfrac{[t_1] \longrightarrow [t_2] \qquad [t_2] \longrightarrow [t_3]}{[t_1] \longrightarrow [t_3]}$
3. **Congruence**. For each $f \in \Sigma$, $\dfrac{[t_1] \longrightarrow [t_1'] \quad \dots \quad [t_n] \longrightarrow [t_n']}{[f(t_1, \dots, t_n)] \longrightarrow [f(t_1', \dots, t_n')]}$
4. **Replacement**. For each rewrite rule

$$r : [t(x_1, \dots, x_n)] \longrightarrow [t'(x_1, \dots, x_n)],$$

$$\frac{[w_1] \longrightarrow [w_1'] \quad \dots \quad [w_n] \longrightarrow [w_n']}{[t(\overline{w}/\overline{x})] \longrightarrow [t'(\overline{w'}/\overline{x})]}$$

Rewriting logic semantics

- A rewrite theory is a static description of what a system can do. The meaning should be given by a model of its actual behavior.
- A rewrite theory $\mathcal{R} = (\Omega, E, L, R)$ has initial and free models that capture computationally the intuitive idea of a "rewrite system" whose states are E-equivalence classes of terms and whose transitions are concurrent rewritings using the rules in R.
- Logically, we can view such models as "logical systems" in which formulas are validly rewritten to other formulas by concurrent rewritings which correspond to proofs.
- The free model is a category $\mathcal{T}_{\mathcal{R}}(X)$ whose objects are equivalence classes of terms $[t] \in T_{\Omega,E}(X)$ and whose morphisms are equivalence classes of "proof terms" representing proofs in rewriting deduction, i.e., concurrent $\mathcal{R}$-rewrites.

Rewriting logic semantics: proof term generation

1. **Identities**. $$\frac{}{[t] : [t] \longrightarrow [t]}$$
2. **Composition**. $$\frac{\alpha : [t_1] \longrightarrow [t_2] \qquad \beta : [t_2] \longrightarrow [t_3]}{\alpha;\beta : [t_1] \longrightarrow [t_3]}$$
3. **Σ-structure**. For each $f \in \Sigma$,
$$\frac{\alpha_1 : [t_1] \longrightarrow [t'_1] \quad \ldots \quad \alpha_n : [t_n] \longrightarrow [t'_n]}{f(\alpha_1, \ldots, \alpha_n) : [f(t_1, \ldots, t_n)] \longrightarrow [f(t'_1, \ldots, t'_n)]}$$
4. **Replacement**. For each rewrite rule
$$r : [t(x_1, \ldots, x_n)] \longrightarrow [t'(x_1, \ldots, x_n)],$$
$$\frac{\alpha_1 : [w_1] \longrightarrow [w'_1] \quad \ldots \quad \alpha_n : [w_n] \longrightarrow [w'_n]}{r(\alpha_1, \ldots, \alpha_n) : [t(\overline{w}/\overline{x})] \longrightarrow [t'(\overline{w'}/\overline{x})]}$$

Rewriting logic semantics: proof term equalities

1. **Category**.
   - *Associativity*. For all α, β, γ

$$(\alpha; \beta); \gamma = \alpha; (\beta; \gamma)$$

   - *Identities*. For each $\alpha : [t] \longrightarrow [t']$

$$\alpha; [t'] = \alpha \qquad\qquad [t]; \alpha = \alpha$$

2. **Functoriality of the Σ-algebraic structure**. For each $f \in \Sigma$,
   - *Preservation of composition*. For all α_i, β_i,

$$f(\alpha_1; \beta_1, \ldots, \alpha_n; \beta_n) = f(\alpha_1, \ldots, \alpha_n); f(\beta_1, \ldots, \beta_n)$$

   - *Preservation of identities*.

$$f([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$$

Rewriting logic semantics: proof term equalities

3. **Axioms in** E.
   For each equation $t(x_1,\ldots,x_n) = t'(x_1,\ldots,x_n)$ in E, for all α_i,
   $$t(\alpha_1,\ldots,\alpha_n) = t'(\alpha_1,\ldots,\alpha_n)$$

4. **Exchange**.
   For each $r : [t(x_1,\ldots,x_n)] \longrightarrow [t'(x_1,\ldots,x_n)]$ in R,
   $$\frac{\alpha_1 : [w_1] \longrightarrow [w'_1] \quad \ldots \quad \alpha_n : [w_n] \longrightarrow [w'_n]}{r(\overline{\alpha}) = r(\overline{[w]}); t'(\overline{\alpha}) = t(\overline{\alpha}); r(\overline{[w']})}$$

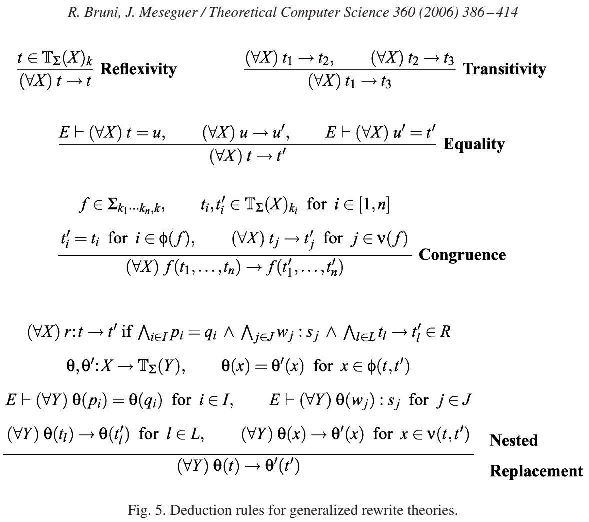
$\mathcal{R}$-Systems

- The category $\mathcal{T}_{\mathcal{R}}(X)$ is one among many models that can be assigned to the rewrite theory $\mathcal{R}$.
- Given a rewrite theory $\mathcal{R} = (\Omega, E, L, R)$, an $\mathcal{R}$-system $\mathcal{S}$ is a category $\mathcal{S}$ together with:
  - a (Σ, E)-algebra structure given by a family of functors

    $$\{f_{\mathcal{S}} : \mathcal{S}^n \longrightarrow \mathcal{S} \mid f \in \Sigma_n\}$$

    satisfying E, i.e., for any equation $t(x_1, \ldots, x_n) = t'(x_1, \ldots, x_n)$ in E we have an identity of functors $t_{\mathcal{S}} = t'_{\mathcal{S}}$, where the functor $t_{\mathcal{S}}$ is defined inductively from the functors $f_{\mathcal{S}}$.
  - for each rewrite rule $r : [t(\overline{x})] \longrightarrow [t'(\overline{x})]$ in R a natural transformation $r_{\mathcal{S}} : t_{\mathcal{S}} \Rightarrow t'_{\mathcal{S}}$.

A more detailed deduction system



R. Bruni, J. Meseguer / Theoretical Computer Science 360 (2006) 386–414

$$\frac{t \in \mathbb{T}_\Sigma(X)_k}{(\forall X)\ t \rightarrow t}\ \textbf{Reflexivity} \qquad \frac{(\forall X)\ t_1 \rightarrow t_2, \qquad (\forall X)\ t_2 \rightarrow t_3}{(\forall X)\ t_1 \rightarrow t_3}\ \textbf{Transitivity}$$

$$\frac{E \vdash (\forall X)\ t = u, \qquad (\forall X)\ u \rightarrow u', \qquad E \vdash (\forall X)\ u' = t'}{(\forall X)\ t \rightarrow t'}\ \textbf{Equality}$$

$$\frac{\begin{array}{c} f \in \Sigma_{k_1 \cdots k_n, k}, \qquad t_i, t_i' \in \mathbb{T}_\Sigma(X)_{k_i} \text{ for } i \in [1,n] \\ t_i' = t_i \text{ for } i \in \phi(f), \qquad (\forall X)\ t_j \rightarrow t_j' \text{ for } j \in \nu(f) \end{array}}{(\forall X)\ f(t_1, \ldots, t_n) \rightarrow f(t_1', \ldots, t_n')}\ \textbf{Congruence}$$

$$\frac{\begin{array}{c} (\forall X)\ r{:}t \rightarrow t' \text{ if } \bigwedge_{i\in I} p_i = q_i \wedge \bigwedge_{j\in J} w_j : s_j \wedge \bigwedge_{l\in L} t_l \rightarrow t_l' \in R \\ \theta, \theta'{:}X \rightarrow \mathbb{T}_\Sigma(Y), \qquad \theta(x) = \theta'(x) \text{ for } x \in \phi(t,t') \\ E \vdash (\forall Y)\ \theta(p_i) = \theta(q_i) \text{ for } i \in I, \qquad E \vdash (\forall Y)\ \theta(w_j) : s_j \text{ for } j \in J \\ (\forall Y)\ \theta(t_l) \rightarrow \theta(t_l') \text{ for } l \in L, \qquad (\forall Y)\ \theta(x) \rightarrow \theta'(x) \text{ for } x \in \nu(t,t') \end{array}}{(\forall Y)\ \theta(t) \rightarrow \theta'(t')}\ \begin{array}{l}\textbf{Nested} \\ \textbf{Replacement}\end{array}$$

Fig. 5. Deduction rules for generalized rewrite theories.

System modules

- System modules in Maude correspond to rewrite theories in rewriting logic.
- A rewrite theory has both rules and equations, so that rewriting is performed modulo such equations.
- The equations are divided into
  - a set A of structural axioms, for which matching algorithms exist in Maude, and
  - a set E of equations that are Church-Rosser and terminating modulo A;

  that is, the equational part must be equivalent to a functional module.

System modules

- The rules R in the module must be coherent with the equations E modulo A, allowing us to intermix rewriting with rules and rewriting with equations without losing rewrite computations by failing to perform a rewrite that would have been possible before an equational deduction step was taken.

$$\begin{tikzcd}
t \arrow[rr, "1", "R/A"'] \arrow[dd, "E/A"', "!"] & & t' \arrow[dr, dotted, "!", "E/A"'] & \\
& & & w \\
u \arrow[rr, dotted, "1", "R/A"'] & & u' \arrow[ur, dotted, "!", "E/A"'] &
\end{tikzcd}$$

- A simple strategy available in these circumstances is to always reduce to canonical form using E before applying any rule in R.
- In this way, we get the effect of rewriting modulo $E \cup A$ with just a matching algorithm for A.

Transition systems

$$\begin{tikzcd}
n_3 \arrow[r, "c"] & n_4 \arrow[d, "d"] & n_6 \\
n_1 \arrow[u, "b"] \arrow[r, shift left, "a"] & n_2 \arrow[l, shift left, "f"] \arrow[r, "e"] \arrow[ur, "g"'] & n_5
\end{tikzcd}$$

```
mod A-TRANSITION-SYSTEM is
  sort State .
  ops n1 n2 n3 n4 n5 n6 : -> State [ctor] .
  rl [a] : n1 => n2 .     rl [b] : n1 => n3 .
  rl [c] : n3 => n4 .     rl [d] : n4 => n2 .
  rl [e] : n2 => n5 .     rl [f] : n2 => n1 .
  rl [g] : n2 => n6 .
endm
```

- not confluent: there are, for example, two transitions out of `n2` that are not joinable
- not terminating: there are cycles creating infinite computations

A vending machine

```
fmod VENDING-MACHINE-SIGNATURE is
  sorts Coin Item State .
  subsorts Coin Item < State .
  op __ : State State -> State [assoc comm] .
  op $ : -> Coin [format (r! o)] .
  op q : -> Coin [format (r! o)] .
  op a : -> Item [format (b! o)] .
  op c : -> Item [format (b! o)] .
endfm

mod VENDING-MACHINE is
  including VENDING-MACHINE-SIGNATURE .
  var M : State .
  rl [add-q] : M => M q .
  rl [add-$] : M => M $ .
  rl [buy-c] : $ => c .
  rl [buy-a] : $ => a q .
  rl [change]: q q q q => $ .
endm
```

A vending machine

- The top-down rule-fair rewrite command (abbreviated rew) applies rules on the top operator (__ in this case) in a fair way; in this example, it tries to apply add-q, add-$, and change in this order.

```
Maude> rew [20] $ $ q q .
rewrite [20] in VENDING-MACHINE : $ $ q q .
rewrites: 20 in 0ms cpu (0ms real) (~ rewrites/second)
result State: $ $ $ $ $ $ $ $ $ $ $ $ $ q q q
```

A vending machine

- The top-down rule-fair rewrite command (abbreviated `rew`) applies rules on the top operator (`__` in this case) in a fair way; in this example, it tries to apply `add-q`, `add-$`, and `change` in this order.

```
Maude> rew [20] $ $ q q .
rewrite [20] in VENDING-MACHINE : $ $ q q .
rewrites: 20 in 0ms cpu (0ms real) (~ rewrites/second)
result State: $ $ $ $ $ $ $ $ $ $ $ $ $ q q q
```

- The frewrite command (abbreviated `frew`) uses a depth-first position-fair strategy that makes it possible for some rules to be applied that could be "starved" using the `rewrite` command.

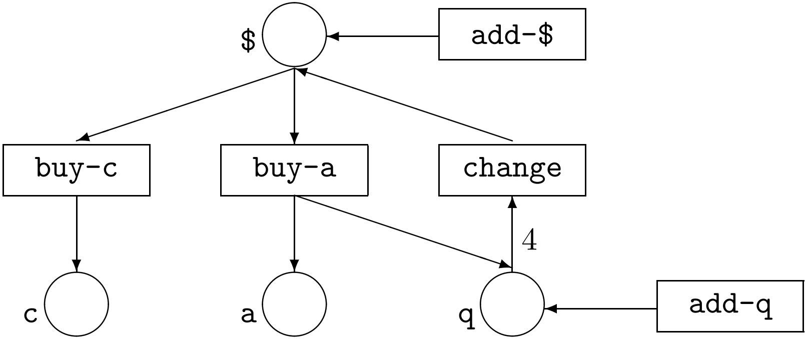
```
Maude> frew [20] $ $ q q .
frewrite [20] in VENDING-MACHINE : $ $ q q .
rewrites: 20 in 0ms cpu (1ms real) (~ rewrites/second)
result (sort not calculated):
    c (q a) ($ q) ($ $) c $ q q q q q q q q q q a c
```

A vending machine

- With the search command we can look for different ways to use a dollar and three quarters to buy an apple and two cakes (in a depth-first manner). First we ask for one solution, and then use the bounded continue command to see another solution.

```
Maude> search [1] in VENDING-MACHINE : $ q q q =>+ a c c M .

Solution 1 (state 108)
states: 109  rewrites: 1857 in 20ms cpu (57ms real)
     (92850 rewrites/second)
M --> q q q q

Maude> cont 1 .

Solution 2 (state 160)
states: 161  rewrites: 1471 in 20ms cpu (76ms real)
    (73550 rewrites/second)
M --> q q q q q
```

A vending machine

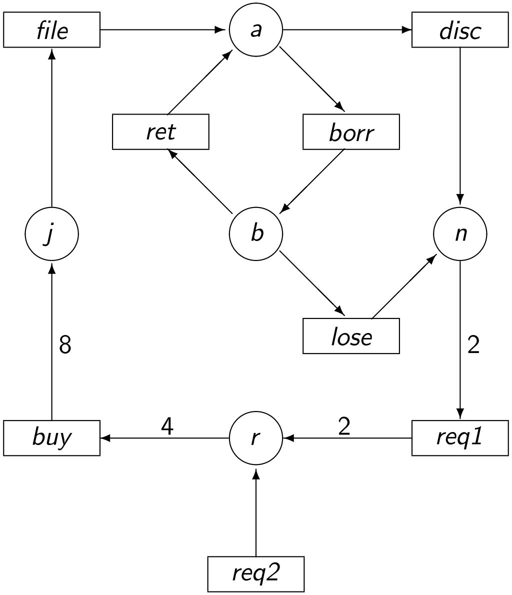


Petri nets

Consider a Petri net modelling a small library, where a token represents a book, that can be in several different states: just bought (j), available (a), borrowed (b), requested (r), and not available (n).
The possible transitions are the following:

- *buy*: When there are four accumulated requests, the library places an order to buy two copies of each requested book (here we do not distinguish among different books or copies of the same book).
- *file*: A book just bought is filed, making it available.
- *borr*: An available book can be borrowed.
- *ret*: A borrowed book can be returned.
- *lose*: A borrowed book can become lost, and thus no longer available.
- *disc*: An available book is discarded because of its bad condition, and thus it is no longer available either.
- *req1*: A user may place a request to buy a non available book, but only when there are two accumulated requests these are honored.
- *req2*: The library may decide to buy a new book, thus creating a new token in the system.

Petri nets



Petri nets as system modules

- A marking on a Petri net is a multiset over its set of places, denoting the available resources in each place.
- A transition goes from the marking representing the resources it consumes to the marking representing the resources it produces.

Petri nets as system modules

- A marking on a Petri net is a multiset over its set of places, denoting the available resources in each place.
- A transition goes from the marking representing the resources it consumes to the marking representing the resources it produces.

```
mod LIBRARY-PETRI-NET is
  sorts Place Marking .
  subsort Place < Marking .
  op 1 : -> Marking [ctor] .
  op __ : Marking Marking -> Marking  [ctor assoc comm id: 1] .
  ops a b n r j : -> Place [ctor] .
  var M : Marking .
  rl [buy] : r r r r => j j j j j j j j .
  rl [file] : j => a .        rl [borr] : a => b .
  rl [ret] : b => a .         rl [lose] : b => n .
  rl [disc] : a => n .        rl [req1] : n n => r r .
  rl [req2] : M => M r .
endm
```

Petri nets as system modules

- Computations may be nonterminating, for example, because a book can be forever in the borrow-return cycle.
- This particular net happens to be confluent by chance.

Petri nets as system modules

- Computations may be nonterminating, for example, because a book can be forever in the borrow-return cycle.
- This particular net happens to be confluent by chance.
- Starting in the empty marking and using the `rewrite` command, the system keeps adding requests until there are enough to buy a book, and then repeats the same process, like in the following sequence of 10 rewrites.

```
Maude> rew [10] 1 .
rewrite [10] in LIBRARY-PETRI-NET : 1 .
rewrites: 10 in 20ms cpu (54ms real) (500 rewrites/second)
result Marking: j j j j j j j j j j j j j j j j
```

Petri nets as system modules

- Computations may be nonterminating, for example, because a book can be forever in the borrow-return cycle.
- This particular net happens to be confluent by chance.
- Starting in the empty marking and using the `rewrite` command, the system keeps adding requests until there are enough to buy a book, and then repeats the same process, like in the following sequence of 10 rewrites.

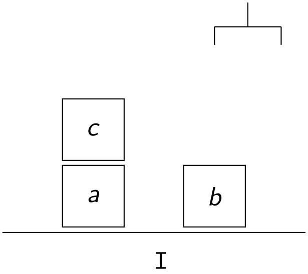
```
Maude> rew [10] 1 .
rewrite [10] in LIBRARY-PETRI-NET : 1 .
rewrites: 10 in 20ms cpu (54ms real) (500 rewrites/second)
result Marking: j j j j j j j j j j j j j j j j
```

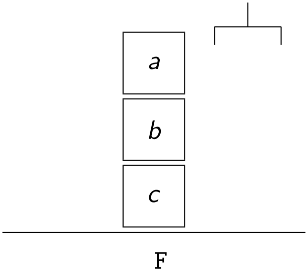
- The `frewrite` command uses a fair strategy where other rules are applied.

```
Maude> frewrite  [10] 1 .
frewrite [10] in LIBRARY-PETRI-NET : 1 .
rewrites: 10 in 0ms cpu (19ms real) (˜ rewrites/second)
result (sort not calculated): a (r j) a (r j) a j j j
```

Blocks world

- A generalization of Petri net computations is provided by conjunctive planning problems, where the states are described by means of some kind of conjunction of propositions describing basic facts.
- A typical example is the blocks world; here we have a table on top of which there are blocks, which can be moved only by means of a robot arm.





Blocks world

```
mod BLOCKS-WORLD is
  protecting QID .
  sorts BlockId Prop State .
  subsort Qid < BlockId .
  subsort Prop < State .

  op table : BlockId -> Prop [ctor] .       *** block is on the table
  op on : BlockId BlockId -> Prop [ctor] .  *** block A is on block B
  op clear : BlockId -> Prop [ctor] .       *** block is clear
  op hold : BlockId -> Prop [ctor] .        *** robot arm holds block
  op empty : -> Prop [ctor] .               *** robot arm is empty

  op 1 : -> State [ctor] .
  op _&_ : State State -> State [ctor assoc comm id: 1] .
```

Blocks world

```
    vars X Y : BlockId .

    rl [pickup] : empty & clear(X) & table(X) => hold(X) .
    rl [putdown] : hold(X) => empty & clear(X) & table(X) .
    rl [unstack] : empty & clear(X) & on(X,Y) => hold(X) & clear(Y) .
    rl [stack] : hold(X) & clear(Y) => empty & clear(X) & on(X,Y) .
  endm
```

Blocks world

```
    vars X Y : BlockId .

    rl [pickup] : empty & clear(X) & table(X) => hold(X) .
    rl [putdown] : hold(X) => empty & clear(X) & table(X) .
    rl [unstack] : empty & clear(X) & on(X,Y) => hold(X) & clear(Y) .
    rl [stack] : hold(X) & clear(Y) => empty & clear(X) & on(X,Y) .
  endm
```

If we just ask for a sequence of rewrites starting from the initial state *I*, it gets very boring as the robot arm, after picking up the block b keeps stacking and unstacking it on c; we show a sequence of 5 rewrites:

```
  Maude> rew [5] empty & clear('c) & clear('b) & table('a) &
                 table('b) & on('c,'a) .
  rewrite [5] in BLOCKS-WORLD : empty & clear('c) & clear('b) &
                                table('a) & table('b) & on('c,'a) .
  rewrites: 5 in 0ms cpu (42ms real) (~ rewrites/second)
  result State: table('a) & clear('c) & hold('b) & on('c, 'a)
```

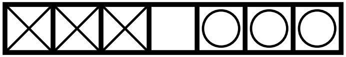
Blocks world

To see that it is possible to go from state I to state F we can use the search command as follows:

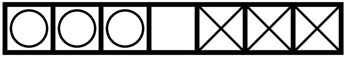
```
Maude> search empty & clear('c) & clear('b) & table('a) &
              table('b) & on('c,'a)
       =>* empty & clear('a) & table('c) & on('a,'b) & on('b,'c) .

Solution 1 (state 21)
empty substitution

No more solutions.

Maude> show path labels 21 .
unstack
putdown
pickup
stack
pickup
stack
```

Hopping rabbits

- Initial configuration (for 3 rabbits in each team):



- Final configuration:



- X-rabbits move to the right.
- O-rabbits move to the left.
- A rabbit is allowed to advance one position if that position is empty.
- A rabbit can jump over a rival if the position behind it is free.

Hopping rabbits

```
mod RABBIT-HOP is

  ***  each rabbit is represented as a constant
  ***  a special rabbit for the empty position

  sort Rabbit .
  ops x o free : -> Rabbit .

  *** a game state is represented
  *** as a nonempty list of rabbits

  sort RabbitList .
  subsort Rabbit < RabbitList .
  op __ : RabbitList RabbitList -> RabbitList [assoc] .
```

Hopping rabbits

```
  *** rules (transitions) for game moves

  rl [xAdvances] : x free => free x .
  rl [xJumps] : x o free => free o x .
  rl [oAdvances] : free o => o free .
  rl [oJumps] : free x o => o x free .

  *** auxiliary operation to build initial states

  protecting NAT .

  op initial : Nat -> RabbitList .
  var  N : Nat .
  eq initial(0) = free .
  eq initial(s(N)) = x initial(N) o .
endm
```

Hopping rabbits

```
Maude> search initial(3) =>* o o o free x x x .

Solution 1 (state 71)
empty substitution

No more solutions.

Maude> show path labels 71 .
xAdvances     oJumps     oAdvances
xJumps        xJumps     xAdvances
oJumps        oJumps     oJumps
xAdvances     xJumps     xJumps
oAdvances     oJumps     xAdvances

Maude> show path 71 .
state 0, RabbitList: x x x free o o o
===[ rl x free => free x [label xAdvances] . ]===>
state 1, RabbitList: x x free x o o o
...
```

The Josephus problem

- Flavius Josephus was a famous Jewish historian who, during the Jewish-Roman war in the first century, was trapped in a cave with a group of 40 Jewish soldiers surrounded by Romans.
- Legend has it that, preferring death to being captured, the Jews decided to gather in a circle and rotate a dagger around so that every third remaining person would commit suicide.
- Apparently, Josephus was too keen to live and quickly found out the safe position.

The Josephus problem

- Flavius Josephus was a famous Jewish historian who, during the Jewish-Roman war in the first century, was trapped in a cave with a group of 40 Jewish soldiers surrounded by Romans.
- Legend has it that, preferring death to being captured, the Jews decided to gather in a circle and rotate a dagger around so that every third remaining person would commit suicide.
- Apparently, Josephus was too keen to live and quickly found out the safe position.
- The circle representation becomes a (circular) list once the beginning position is chosen, with the dagger implicitly at position 1.
- The idea then consists in continually taking the first two elements in the list and moving them to the end of it while “killing” the third one.
- The dagger remains always implicitly located at the beginning of the list.

The Josephus problem

```
mod JOSEPHUS is
  protecting NAT .
  sorts Morituri Circle .
  subsort NzNat < Morituri .
  op __ : Morituri Morituri -> Morituri [assoc] .
  op {_} : Morituri  -> Circle .
  op initial : NzNat -> Morituri .

  var M : Morituri .
  vars I1 I2 I3 N : NzNat .

  eq initial(1) = 1 .
  eq initial(s(N)) = initial(N) s(N) .

  rl [kill>3] : { I1 I2 I3 M } => { M I1 I2 } .
  rl [kill3]  : { I1 I2 I3 } => { I1 I2 } .
    --- Rule kill3 is necessary because M cannot be empty
  rl [kill2]  : { I1 I2 } => { I2 } .
endm
```

The Josephus problem

- In this problem, transitions are deterministic: only one of the three can be applied each time, depending on the numbers of elements remaining in the list.
- Search can be used, but it is not necessary.
- The command `rewrite` provides directly the solution:

```
Maude> rewrite { initial(41) } .
result Circle: {31}
```

The generalized Josephus problem

- It is easy to generalize the program so that every *i*-th person commits suicide, where *i* is a parameter.
- The idea is the same, but because of the parameter now it is necessary to explicitly represent the dagger.
- For that, we use the constructor

  ```
  dagger : NzNat NzNat -> Morituri .
  ```

  whose second argument stores the value of *i* while the first one acts as a counter.
- Each time an element is moved from the beginning of the list to the end, the first argument is decreased by one; once it reaches 1, the element that is currently the head of the list is "killed," i.e., removed from the list.

The generalized Josephus problem

```
mod JOSEPHUS-GENERALIZED is
  protecting NAT .
  sorts Morituri Circle .
  subsort NzNat < Morituri .
  op dagger : NzNat NzNat -> Morituri .
  op __ : Morituri Morituri -> Morituri [assoc] .
  op {_} : Morituri  -> Circle .
  op initial : NzNat NzNat -> Morituri .

  var  M : Morituri .
  vars I I1 I2 N : NzNat .
  eq initial(1, I)  = dagger(I, I) 1 .
  eq initial(s(N), I) = initial(N, I) s(N) .

  rl [kill] : { dagger(1, I) I1 M } => { dagger(I, I) M } .
  rl [next] : { dagger(s(N), I) I1 M } => { dagger(N, I) M I1 } .
  rl [last] : { dagger(N, I) I1 } => { I1 } .
          --- The last one throws the dagger away!
endm
```

The three basins puzzle

- We have three basins with capacities of 3, 5, and 8 gallons.
- There is an unlimited supply of water.
- The goal is to get 4 gallons in any of the basins.
- Practical application: in the movie *Die Hard: With a Vengeance*, McClane and Zeus have to deactivate a bomb with this system.

The three basins puzzle

- We have three basins with capacities of 3, 5, and 8 gallons.
- There is an unlimited supply of water.
- The goal is to get 4 gallons in any of the basins.
- Practical application: in the movie *Die Hard: With a Vengeance*, McClane and Zeus have to deactivate a bomb with this system.
- A basin is represented with the constructor `basin`, having two natural numbers as arguments: the first one is the basin capacity and the second one is how much it is filled.
- We can think of a basin as an object with two attributes.
- This leads to an object-based style of programming, where objects change their attributes as result of interacting with other objects.
- Interactions are represented as rules on configurations that are nonempty multisets of objects.

The three basins puzzle

```
mod DIE-HARD is
  protecting NAT .

  *** objects
  sort Basin .
  op basin : Nat Nat -> Basin .    *** capacity / content

  *** configurations / multisets of objects
  sort BasinSet .
  subsort Basin < BasinSet .
  op __ : BasinSet BasinSet -> BasinSet [assoc comm] .

  *** auxiliary operation to represent initial state
  op initial : -> BasinSet .
  eq initial = basin(3, 0) basin(5, 0) basin(8,0) .
```

The three basins puzzle

```
    *** possible moves as four rules
    vars M1 N1 M2 N2 : Nat .

    rl [empty] : basin(M1, N1) => basin(M1, 0) .

    rl [fill] : basin(M1, N1) => basin(M1, M1) .

    crl [transfer1] : basin(M1, N1) basin(M2, N2)
      => basin(M1, 0) basin(M2, N1 + N2)
     if N1 + N2 <= M2 .

    crl [transfer2] : basin(M1, N1) basin(M2, N2)
      => basin(M1, sd(N1 + N2, M2)) basin(M2, M2)
     if N1 + N2 > M2 .

     *** sd is symmetric difference in predefined NAT
  endm
```

The three basins puzzle

```
Maude> search [1] initial =>* basin(N:Nat, 4) B:BasinSet .

Solution 1 (state 75)
B:BasinSet --> basin(3, 3) basin(8, 3)
N:Nat --> 5

Maude> show path 75 .
state 0, BasinSet: basin(3, 0) basin(5, 0) basin(8, 0)
===[ rl ... fill ]===>
state 2, BasinSet: basin(3, 0) basin(5, 5) basin(8, 0)
===[ crl ... transfer2 ]===>
state 9, BasinSet: basin(3, 3) basin(5, 2) basin(8, 0)
===[ crl ... transfer1 ]===>
state 20, BasinSet: basin(3, 0) basin(5, 2) basin(8, 3)
===[ crl ... transfer1 ]===>
state 37, BasinSet: basin(3, 2) basin(5, 0) basin(8, 3)
===[ rl ... fill ]===>
state 55, BasinSet: basin(3, 2) basin(5, 5) basin(8, 3)
===[ crl ... transfer2 ]===>
state 75, BasinSet: basin(3, 3) basin(5, 4) basin(8, 3)
```

Crossing the bridge

- The four components of U2 are in a tight situation. Their concert starts in 17 minutes and in order to get to the stage they must first cross an old bridge through which only a maximum of two persons can walk over at the same time.
- It is already dark and, because of the bad condition of the bridge, to avoid falling into the darkness it is necessary to cross it with the help of a flashlight. Unfortunately, they only have one.
- Knowing that Bono, Edge, Adam, and Larry take 1, 2, 5, and 10 minutes, respectively, to cross the bridge, is there a way that they can make it to the concert on time?

Crossing the bridge

- The current state of the group can be represented by a multiset (a term of sort `Group` below) consisting of performers, the flashlight, and a watch to keep record of the time.
- The flashlight and the performers have a `Place` associated to them, indicating whether their current position is to the left or to the right of the bridge.
- Each performer, in addition, also carries the time it takes him to cross the bridge.
- In order to change the position from `left` to `right` and vice versa, we use an auxiliary operation `changePos`.
- The traversing of the bridge is modeled by two rewrite rules: the first one for the case in which a single person crosses it, and the second one for when there are two.

Crossing the bridge

```
mod U2 is
  protecting NAT .

  sorts Performer Object Group Place .
  subsorts Performer Object < Group .

  ops left right : -> Place .
  op flashlight : Place -> Object .
  op watch : Nat -> Object .
  op performer : Nat Place -> Performer .
  op __ : Group Group -> Group [assoc comm] .

  op changePos : Place -> Place .

  eq changePos(left) = right .
  eq changePos(right) = left .
```

Crossing the bridge

```
    op initial : -> Group .
    eq initial
      = watch(0) flashlight(left) performer(1, left)
        performer(2, left) performer(5, left) performer(10, left) .

    var  P : Place .
    vars M N N1 N2 : Nat .

    rl [one-crosses] :
      watch(M) flashlight(P) performer(N, P)
      => watch(M + N) flashlight(changePos(P))
         performer(N, changePos(P)) .

    crl [two-cross] :
      watch(M) flashlight(P) performer(N1, P) performer(N2, P)
      => watch(M + N1) flashlight(changePos(P))
         performer(N1, changePos(P))
         performer(N2, changePos(P))
      if N1 > N2 .
  endm
```

Crossing the bridge

- A solution can be found by looking for a state in which all performers and the flashlight are to the `right` of the bridge.
- The search command is invoked with a such that clause that allows to introduce a condition that solutions have to fulfill, in our example, that the total time is less than or equal to 17 minutes:

```
Maude> search [1] initial
          =>* flashlight(right) watch(N:Nat)
              performer(1, right) performer(2, right)
              performer(5, right) performer(10, right)
          such that N:Nat <= 17 .

Solution 1 (state 402)
N --> 17
```

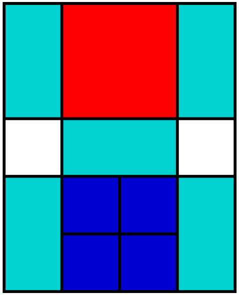
Crossing the bridge

- The solution takes exactly 17 minutes (a happy ending after all!) and the complete sequence of appropriate actions can be shown with the command

```
Maude> show path 402 .
```

- After sorting out the information, it becomes clear that Bono and Edge have to be the first to cross. Then Bono returns with the flashlight, which gives to Adam and Larry. Finally, Edge takes the flashlight back to Bono and they cross the bridge together for the last time.
- Note that, in order for the `search` command to stop, we need to tell Maude to look only for one solution. Otherwise, it will continue exploring all possible combinations, increasingly taking a larger amount of time, and it will never end.

The Khun Phan puzzle



- Can we move the big square to where the small ones are?
- Can we reach a completely symmetric configuration?

The Khun Phan puzzle

```
mod KHUN-PHAN is
  protecting NAT .
  sorts Piece Board .
  subsort Piece < Board .

  *** each piece carries the coordinates of its upper left corner
  ops empty bigsq smallsq hrect vrect : Nat Nat -> Piece .

  *** board is nonempty multiset of pieces
  op __ : Board Board -> Board [assoc comm] .

  op initial : -> Board .

  eq initial
    = vrect(1, 1)       bigsq(2, 1)         vrect(4, 1)
      empty(1, 3)       hrect(2, 3)         empty(4, 3)
      vrect(1, 4) smallsq(2, 4) smallsq(3, 4) vrect(4, 4)
                  smallsq(2, 5) smallsq(3, 5) .
```

The Khun Phan puzzle

```
vars X Y : Nat .

rl [sqr] : smallsq(X, Y) empty(s(X), Y)
  => empty(X, Y) smallsq(s(X), Y) .
rl [sql] : smallsq(s(X), Y) empty(X, Y)
  => empty(s(X), Y) smallsq(X, Y) .
rl [squ] : smallsq(X, s(Y)) empty(X, Y)
  => empty(X, s(Y)) smallsq(X, Y) .
rl [sqd] : smallsq(X, Y) empty(X, s(Y))
  => empty(X, Y) smallsq(X, s(Y)) .

rl [Sqr] : bigsq(X, Y) empty(s(s(X)), Y) empty(s(s(X)), s(Y))
  => empty(X, Y) empty(X, s(Y)) bigsq(s(X), Y) .
rl [Sql] : bigsq(s(X), Y) empty(X, Y) empty(X, s(Y))
  => empty(s(s(X)), Y) empty(s(s(X)), s(Y)) bigsq(X, Y) .
rl [Squ] : bigsq(X, s(Y)) empty(X, Y) empty(s(X), Y)
  => empty(X, s(s(Y))) empty(s(X), s(s(Y))) bigsq(X, Y) .
rl [Sqd] : bigsq(X, Y) empty(X, s(s(Y))) empty(s(X), s(s(Y)))
  => empty(X, Y) empty(s(X), Y) bigsq(X, s(Y)) .
```

The Khun Phan puzzle

```
    rl [hrectr] : hrect(X, Y) empty(s(s(X)), Y)
      => empty(X, Y) hrect(s(X), Y) .
    rl [hrectl] : hrect(s(X), Y) empty(X, Y)
      => empty(s(s(X)), Y) hrect(X, Y) .
    rl [hrectu] : hrect(X, s(Y)) empty(X, Y) empty(s(X), Y)
      => empty(X, s(Y)) empty(s(X), s(Y)) hrect(X, Y) .
    rl [hrectd] : hrect(X, Y) empty(X, s(Y)) empty(s(X), s(Y))
      => empty(X, Y) empty(s(X), Y) hrect(X, s(Y)) .

    rl [vrectr] : vrect(X, Y) empty(s(X), Y) empty(s(X), s(Y))
      => empty(X, Y) empty(X, s(Y)) vrect(s(X), Y) .
    rl [vrectl] : vrect(s(X), Y) empty(X, Y) empty(X, s(Y))
      => empty(s(X), Y) empty(s(X), s(Y)) vrect(X, Y) .
    rl [vrectu] : vrect(X, s(Y)) empty(X, Y)
      => empty(X, s(s(Y))) vrect(X, Y) .
    rl [vrectd] : vrect(X, Y) empty(X, s(s(Y)))
      => empty(X, Y) vrect(X, s(Y)) .
  endm
```

The Khun Phan puzzle

- With the following command we get all possible 964 final configurations for the game:

```
Maude> search initial =>* B:Board bigsq(2, 4) .
```

- The final state used, `B:Board bigsq(2,4)`, represents any final situation such that the upper left corner of the big square is at coordinates $(2, 4)$.
- The `search` command does not enumerate the different ways of reaching the same configuration.
- The shortest path leading to the final configuration, due to the breadth-first search, reveals that it consists of 112 moves:

```
Maude> show path labels 23721 .
```

The Khun Phan puzzle

- The following command shows that it is not possible to reach a position symmetric to the initial one.

```
Maude> search initial
       =>* vrect(1, 1) smallsq(2, 1) smallsq(3, 1) vrect(4, 1)
                       smallsq(2, 2) smallsq(3, 2)
           empty(1, 3)         hrect(2, 3)          empty(4, 3)
           vrect(1, 4)         bigsq(2, 4)          vrect(4, 4) .

No solution.
```

Black or white

- In an 8×8 board the four corners are colored white and all the others are black.
- Is it possible to make all the squares white by recoloring rows and columns?
- Recoloring is the operation of changing the colors of all the squares in a row or column.

Black or white

- In an 8×8 board the four corners are colored white and all the others are black.
- Is it possible to make all the squares white by recoloring rows and columns?
- Recoloring is the operation of changing the colors of all the squares in a row or column.
- The board is represented as a multiset of squares.
- Each square has three arguments: the first two are its column and row, and the last one its color, either black (`b`) or white (`w`).
- A predicate `allWhite?` checks whether all the squares are white.

Black or white

```
mod RECOLORING is
  protecting NAT .
  sorts Place Board Color .
  subsort Place < Board .
  ops w b : -> Color .
  op sq : Nat Nat Color -> Place .
  op __ : Board Board -> Board [assoc comm] .

  op recolor : Color -> Color .
  op allWhite? : Board -> Bool .

  vars I J : Nat .
  vars C1 C2 C3 C4 C5 C6 C7 C8 : Color .
  var B : Board .

  eq recolor(b) = w .
  eq recolor(w) = b .
  eq allWhite?(sq(I,J,b) B) = false .
  eq allWhite?(B) = true [owise] .
```

Black or white

```
    op initial : -> Board .
    eq initial = sq(1,1,w) sq(1,2,b) ...... sq(8,7,b) sq(8,8,w) .

    rl [recolor-column] : sq(I,1,C1) sq(I,2,C2) sq(I,3,C3) sq(I,4,C4)
         sq(I,5,C5) sq(I,6,C6) sq(I,7,C7) sq(I,8,C8)
      => sq(I,1,recolor(C1)) sq(I,2,recolor(C2))
         sq(I,3,recolor(C3)) sq(I,4,recolor(C4))
         sq(I,5,recolor(C5)) sq(I,6,recolor(C6))
         sq(I,7,recolor(C7)) sq(I,8,recolor(C8)) .

    rl [recolor-row] : sq(1,J,C1) sq(2,J,C2) sq(3,J,C3) sq(4,J,C4)
         sq(5,J,C5) sq(6,J,C6) sq(7,J,C7) sq(8,J,C8)
      => sq(1,J,recolor(C1)) sq(2,J,recolor(C2))
         sq(3,J,recolor(C3)) sq(4,J,recolor(C4))
         sq(5,J,recolor(C5)) sq(6,J,recolor(C6))
         sq(7,J,recolor(C7)) sq(8,J,recolor(C8)) .
  endm
```

Black or white

The command

```
Maude> search initial =>* B:Board
       such that allWhite?(B:Board) .

No solution.
```

proves that it is not possible to get a configuration with all the squares colored white from the initial configuration.

The game is not over

- Mr. Smith and his wife invited four other couples to have dinner at their house. When they arrived, some people shook hands with some others (of course, nobody shook hands with their spouse or with the same person twice), after which Mr. Smith asked everyone how many times they had shaken hands. The answers, it turned out, were different in all cases. How many people did Mrs. Smith shake hands with?

The game is not over

- The numbers 25 and 36 are written on a blackboard. At each turn, a player writes on the board the (positive) difference between two numbers already on the blackboard—if this number does not already appear on it. The loser is the player who cannot write a number. Prove that the second player will always win.
- A 3×3 table is filled with numbers. It is allowed to increase simultaneously all the numbers in any 2×2 square by 1. Is it possible, using these operations, to obtain the table $(4, 9, 5), (10, 18, 12), (6, 13, 7)$ from a table initially filled with zeros?

Untyped lambda calculus

```
fth VAR is
  protecting BOOL .
  sorts Var VarSet .
  subsort Var < VarSet .                     *** singleton sets
  op empty-set : -> VarSet .                 *** empty set
  op _U_ : VarSet VarSet -> VarSet [assoc comm id: empty-set] .
                                             *** set union
  op _in_ : Var VarSet -> Bool .             *** membership test
  op _\_ : VarSet VarSet -> VarSet .         *** set difference
  op new : VarSet -> Var .                   *** new variable

  vars E E' : Var .
  vars S S' : VarSet .

  eq E U E = E .
  eq E in empty-set = false .
  eq E in E' U S = (E == E') or (E in S) .
  eq empty-set \ S = empty-set .
  eq (E U S) \ S' = if E in S' then S \ S' else E U (S \ S') fi .
  eq new(S) in S = false [nonexec] .
endfth
```

Untyped lambda calculus

```
fmod LAMBDA{X :: VAR} is
  sort Lambda{X} .
  subsort X$Var < Lambda{X} .                     *** variables
  op \_._ : X$Var Lambda{X} -> Lambda{X} [ctor] .
                                                  *** lambda abstraction
  op __ : Lambda{X} Lambda{X} -> Lambda{X} [ctor] .
                                                  *** application
  op _[_/_] : Lambda{X} Lambda{X} X$Var -> Lambda{X} .
                                                  *** substitution
  op fv : Lambda{X} -> X$VarSet .                 *** free variables

  vars X Y : X$Var .
  vars M N P : Lambda{X} .

  *** Free variables
  eq fv(X) = X .
  eq fv(\ X . M) = fv(M) \ X .
  eq fv(M N) = fv(M) U fv(N) .
  eq fv(M [N / X]) = (fv(M) \ X) U fv(N) .
```

Untyped lambda calculus

```
  *** Substitution equations
  eq X [N / X] = N .
  ceq Y [N / X] = Y if X =/= Y .
  eq (M N)[P / X] = (M [P / X])(N [P / X]) .
  eq (\ X . M)[N / X] = \ X . M .
  ceq (\ Y . M)[N / X] = \ Y . (M [N / X])
    if X =/= Y and (not(Y in fv(N)) or not(X in fv(M))) .
  ceq (\ Y . M)[N / X]
    = \ (new(fv(M N))) . ((M [new(fv(M N)) / Y])[N / X])
    if X =/= Y /\ (Y in fv(N)) /\ (X in fv(M)) .

  *** Alpha conversion
  ceq \ X . M = \ Y . (M [Y / X]) if not(Y in fv(M)) [nonexec] .
endfm
```

Untyped lambda calculus

```
mod BETA-ETA{X :: VAR} is
  including LAMBDA{X} .
  var  X : X$Var .
  vars M N : Lambda{X} .
  rl [beta] : (\ X . M) N => M [N / X] .
  crl [eta] : \ X . (M X) => M if not(X in fv(M)) .
endm
```

```
view Nat from TRIV to NAT is
  sort Elt to Nat .
endv
```

Untyped lambda calculus

```
fmod NAT-SET-MAX is
  protecting (SET * (op _,_ to _U_)){Nat} .
  op max : Set{Nat} -> Nat .
  var N : Nat .
  var S : Set{Nat} .
  eq max(empty) = 0 .
  eq max(N U S) = if N > max(S) then N else max(S) fi .
endfm

view VarNat from VAR to NAT-SET-MAX is
  sort Var to Nat .
  sort VarSet to Set{Nat} .
  var S : VarSet .
  op empty-set to empty .
  op new(S) to term max(S) + 1 .
endv

mod UNTYPED-LAMBDA-CALCULUS is
  protecting BETA-ETA{VarNat} .
endm
```

Untyped lambda calculus

```
Maude> set trace on .
Maude> set trace eqs off .
Maude> rew [2] (\ 1 . (1 1))(\ 1 . (1 1)) .
rewrite [2] in UNTYPED-LAMBDA-CALCULUS : (\ 1 . (1 1)) \ 1 . (1 1) .
*********** rule
rl (\ X:Nat . M:Lambda{VarNat}) N:Lambda{VarNat}
  => M:Lambda{VarNat}[N:Lambda{VarNat} / X:Nat] [label beta] .
X:Nat --> 1
M:Lambda{VarNat} --> 1 1
N:Lambda{VarNat} --> \ 1 . (1 1)
(\ 1 . (1 1)) \ 1 . (1 1)
--->
(1 1)[\ 1 . (1 1) / 1]
*********** rule
rl (\ X:Nat . M:Lambda{VarNat}) N:Lambda{VarNat}
  => M:Lambda{VarNat}[N:Lambda{VarNat} / X:Nat] [label beta] .
X:Nat --> 1
M:Lambda{VarNat} --> 1 1
N:Lambda{VarNat} --> \ 1 . (1 1)
(\ 1 . (1 1)) \ 1 . (1 1)
--->
(1 1)[\ 1 . (1 1) / 1]
rewrites: 8 in 10ms cpu (31ms real) (800 rewrites/second)
result Lambda{VarNat}: (\ 1 . (1 1)) \ 1 . (1 1)
```

Untyped lambda calculus

```
Maude> set trace on .
Maude> set trace eqs off .
Maude> rew (\ 1 . (\ 2 . 2))((\ 1 . (1 1))(\ 1 .(1 1))) .

rewrite in UNTYPED-LAMBDA-CALCULUS :
  (\ 1 . \ 2 . 2) ((\ 1 . (1 1)) \ 1 . (1 1)) .
*********** rule
rl (\ X:Nat . M:Lambda{VarNat}) N:Lambda{VarNat}
  => M:Lambda{VarNat}[N:Lambda{VarNat} / X:Nat] [label beta] .
X:Nat --> 1
M:Lambda{VarNat} --> \ 2 . 2
N:Lambda{VarNat} --> (\ 1 . (1 1)) \ 1 . (1 1)
(\ 1 . \ 2 . 2) ((\ 1 . (1 1)) \ 1 . (1 1))
--->
(\ 2 . 2)[(\ 1 . (1 1)) \ 1 . (1 1) / 1]
rewrites: 36 in 0ms cpu (0ms real) (~ rewrites/second)
result Lambda{VarNat}: \ 2 . 2
```

Full Maude

- The systematic and efficient use of reflection (more on this later) through its predefined `META-LEVEL` module makes Maude remarkably extensible and powerful.
- Full Maude is an extension of Maude, written in Maude itself, that endows the language with an even more powerful and extensible module algebra of parameterized modules and module composition operations, including parameterized views.
- Full Maude also provides special syntax for object-oriented modules supporting object-oriented concepts such as objects, messages, classes, and multiple class inheritance.

Full Maude

- Full Maude itself can be used as a basis for further extensions, by adding new functionality.
- Full Maude becomes a common infrastructure on top of which one can build other tools:
  - Church-Rosser and coherence checkers for Maude,
  - declarative debuggers for Maude, for wrong and missing answers,
  - Real-Time Maude tool for specifying and analyzing real-time systems,
  - MSOS tool for modular structural operational semantics,
  - Maude-NPA for analyzing cryptographic protocols,
  - strategy language prototype.

Object-oriented systems

- An object in a given state is represented as a term

  ```
  < O : C | a1 : v1,..., an : vn >
  ```

  where O is the object's name, belonging to a set `Oid` of object identifiers, C is its class, the `ai`'s are the names of the object's attributes, and the `vi`'s are their corresponding values.
- Messages are defined by the user for each application.
- In a concurrent object-oriented system the concurrent state, which is called a configuration, has the structure of a multiset made up of objects and messages that evolves by concurrent rewriting (modulo the multiset structural axioms) using rules that describe the effects of communication events between some objects and messages.
- We can regard the special syntax reserved for object-oriented modules as syntactic sugar, because each object-oriented module can be translated into a corresponding system module.

Object-oriented rules

- General form of rules in object-oriented systems:

$$
\begin{aligned}
& M_1 \ldots M_n \; \langle O_1 : F_1 \mid atts_1 \rangle \ldots \langle O_m : F_m \mid atts_m \rangle \\
& \longrightarrow \langle O_{i_1} : F'_{i_1} \mid atts'_{i_1} \rangle \ldots \langle O_{i_k} : F'_{i_k} \mid atts'_{i_k} \rangle \\
& \qquad \langle Q_1 : D_1 \mid atts''_1 \rangle \ldots \langle Q_p : D_p \mid atts''_p \rangle \\
& \qquad M'_1 \ldots M'_q \\
& if \;\; C
\end{aligned}
$$

- By convention, the only object attributes made explicit in a rule are those relevant for that rule:
  - the attributes mentioned only in the lefthand side of the rule are preserved unchanged,
  - the original values of attributes mentioned only in the righthand side of the rule do not matter, and
  - all attributes not explicitly mentioned are left unchanged.

Another puzzle

7	1	2
6		8
5	4	3

7	8	1
6		2
5	4	3

Another puzzle

7	1	2
6		8
5	4	3

7	8	1
6		2
5	4	3

```
eq initial
  = < (1, 1) : Tile | value : 7 >     < (1, 2) : Tile | value : 1 >
    < (1, 3) : Tile | value : 2 >     < (2, 1) : Tile | value : 6 >
    < (2, 2) : Tile | value : blank > < (2, 3) : Tile | value : 8 >
    < (3, 1) : Tile | value : 5 >     < (3, 2) : Tile | value : 4 >
    < (3, 3) : Tile | value : 3 > .

eq final
  = < (1, 1) : Tile | value : 7 >     < (1, 2) : Tile | value : 8 >
    < (1, 3) : Tile | value : 1 >     < (2, 1) : Tile | value : 6 >
    < (2, 2) : Tile | value : blank > < (2, 3) : Tile | value : 2 >
    < (3, 1) : Tile | value : 5 >     < (3, 2) : Tile | value : 4 >
    < (3, 3) : Tile | value : 3 > .
```

Another puzzle

```
(omod 8-PUZZLE is
   sorts NumValue Value Coordinate .
   subsort NumValue < Value .
   ops 1 2 3 4 : -> Coordinate [ctor] .
   ops 1 2 3 4 5 6 7 8 : -> NumValue [ctor] .
   op blank : -> Value [ctor] .

   op s_ : Coordinate -> Coordinate .
   eq s 1 = 2 .
   eq s 2 = 3 .
   eq s 3 = 4 .
   eq s 4 = 4 .

   op ‘(_‘,_‘) : Coordinate Coordinate -> Oid .
   class Tile | value : Value .
   msgs left right up down : -> Msg .

   vars N M R : Coordinate .
   var P : NumValue .
```

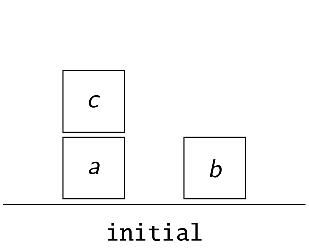
Another puzzle

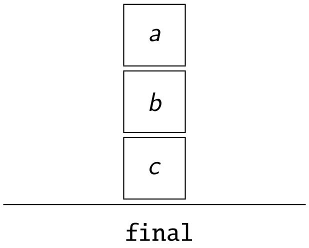
```
crl [r] :  right
     < (N, R) : Tile | value : blank >  < (N, M) : Tile | value : P >
  => < (N, R) : Tile | value : P >  < (N, M) : Tile | value : blank >
  if R == s M .

crl [l] :  left
     < (N, R) : Tile | value : blank >  < (N, M) : Tile | value : P >
  => < (N, R) : Tile | value : P >  < (N, M) : Tile | value : blank >
  if s R == M .

crl [u] :  up
     < (R, M) : Tile | value : blank >  < (N, M) : Tile | value : P >
  => < (R, M) : Tile | value : P >  < (N, M) : Tile | value : blank >
  if s R == N .

crl [d] :  down
    < (R, M) : Tile | value : blank >  < (N, M) : Tile | value : P >
  => < (R, M) : Tile | value : P >  < (N, M) : Tile | value : blank >
  if R == s N .

ops initial final : -> Configuration .
eq initial =  ...  .
eq final = ...  .
endom)
```

Another puzzle

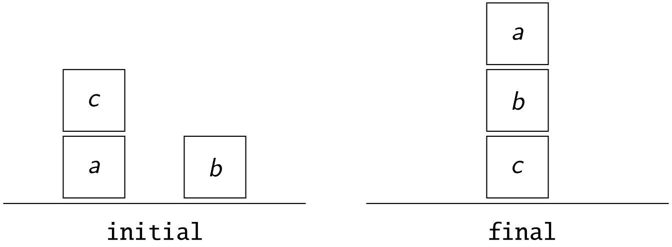
```
Maude > (search up left down right initial
           =>* C:Configuration
           such that C:Configuration == final .)
rewrites: 974 in 50ms cpu (68ms real) (19480 rewrites/second)

Solution 1
C:Configuration -->
  < (1, 1) : Tile | value : 7 > < (1, 2) : Tile | value : 8 >
  < (1, 3) : Tile | value : 1 > < (2, 1) : Tile | value : 6 >
  < (2, 2) : Tile | value : blank > < (2, 3) : Tile | value : 2 >
  < (3, 1) : Tile | value : 5 > < (3, 2) : Tile | value : 4 >
  < (3, 3) : Tile | value : 3 >

No more solutions.
```

An object-oriented blocks world





An object-oriented blocks world



```
eq initial
  = < 'a : Block | under : 'c, on : table >
    < 'c : Block | under : clear, on : 'a >
    < 'b : Block | under : clear, on : table > .

eq final
  = < 'a : Block | under : clear, on : 'b >
    < 'b : Block | under : 'a, on : 'c >
    < 'c : Block | under : 'b, on : table > .
```

An object-oriented blocks world

```
(omod OO-BLOCKS-WORLD is
   protecting QID .

   sorts BlockId Up Down .
   subsorts Qid < BlockId < Oid .
   subsorts BlockId < Up Down .

   op clear : -> Up [ctor] .
   op table : -> Down [ctor] .

   class Block | under : Up, on : Down .
   msg move : Oid Oid Oid -> Msg .
   msgs unstack stack : Oid Oid -> Msg .

   vars X Y Z : BlockId .
```

An object-oriented blocks world

```
    rl [move] : move(X, Z, Y)
      < X : Block | under : clear, on : Z >
      < Z : Block | under : X > < Y : Block | under : clear >
      => < X : Block | on : Y >
         < Z : Block | under : clear >  < Y : Block | under : X > .

    rl [unstack] :  unstack(X,Z)
      < X : Block | under : clear, on : Z >
      < Z : Block | under : X >
      => < X : Block | on : table >  < Z : Block | under : clear > .

    rl [stack] :  stack(X, Z)
      < X : Block | under : clear, on : table >
      < Z : Block | under : clear >
      => < X : Block | on : Z >  < Z : Block | under : X > .

    ops initial final : -> Configuration .
    eq initial =  ...  .
    eq final = ...  .
  endom)
```

An object-oriented blocks world

```
Maude > (search unstack('c,'a) stack('b,'c) stack('a,'b)
                initial
           =>* C:Configuration
           such that C:Configuration == final .)
rewrites: 1318 in 20ms cpu (107ms real) (65900 rewrites/second)

Solution 1
C:Configuration --> < 'a : Block | on : 'b, under : clear >
                    < 'b : Block | on : 'c, under : 'a >
                    < 'c : Block | on : table, under : 'b >

No more solutions.
```

Colored blocks

```
(omod OO-BLOCKS-WORLD+COLOR is
   including OO-BLOCKS-WORLD .
   sort Color .
   ops red blue yellow : -> Color [ctor] .
   class ColoredBlock | color : Color .
   subclass ColoredBlock < Block .
 endom)
```

Colored blocks

```
(omod OO-BLOCKS-WORLD+COLOR is
   including OO-BLOCKS-WORLD .
   sort Color .
   ops red blue yellow : -> Color [ctor] .
   class ColoredBlock | color : Color .
   subclass ColoredBlock < Block .
 endom)

 Maude> (rewrite unstack('c, 'a)
   < 'a : ColoredBlock | color : red, under : 'c, on : table >
   < 'b : ColoredBlock | color : yellow, under : clear, on : table >
   < 'c : ColoredBlock | color : blue, under : clear, on : 'a > .)
 Result Configuration :
   < 'a : ColoredBlock | on : table , under : clear , color : red >
   < 'b : ColoredBlock | on : table , under : clear , color : yellow >
   < 'c : ColoredBlock | on : table , under : clear , color : blue >
```

Readers and writers

```
mod READERS-WRITERS is
  sort Nat .
  op 0 : -> Nat [ctor] .
  op s : Nat -> Nat [ctor iter] .
  sort Config .
  op <_,_> : Nat Nat -> Config [ctor] .  --- readers/writers
  vars R W : Nat .
  rl < 0, 0 > => < 0, s(0) > .
  rl < R, s(W) > => < R, W > .
  rl < R, 0 > => < s(R), 0 > .
  rl < s(R), W > => < R, W > .
endm
```

- mutual exclusion: readers and writers never access the resource simultaneously: only readers or only writers can do so at any given time.
- one writer: at most one writer will be able to access the resource at any given time.

Invariant checking through bounded search

- We represent the invariants implicitly by representing their negations through patterns.
- The negation of the first invariant corresponds to the simultaneous presence of both readers and writers: `< s(N:Nat), s(M:Nat) >`.
- The negation of the fact that zero or at most one writer should be present at any given time is captured by `< N:Nat, s(s(M:Nat)) >`.

```
Maude> search [1, 100000] in READERS-WRITERS :
         < 0, 0 > =>* < s(N:Nat), s(M:Nat) > .
No solution.
states: 100002  rewrites: 200001 in 610ms cpu (1298ms real)
  (327870 rews/sec)

Maude> search [1, 100000] in READERS-WRITERS :
         < 0, 0 > =>* < N:Nat, s(s(M:Nat)) > .
No solution.
states: 100002  rewrites: 200001 in 400ms cpu (1191ms real)
  (500002 rews/sec)
```

Abstraction by adding equations

```
mod READERS-WRITERS-PREDS is
  protecting READERS-WRITERS .
  sort NewBool .
  ops tt ff : -> NewBool [ctor] .
  ops mutex one-writer : Config -> NewBool [frozen] .
  eq mutex(< s(N:Nat), s(M:Nat) >) = ff .
  eq mutex(< 0, N:Nat >) = tt .
  eq mutex(< N:Nat, 0 >) = tt .
  eq one-writer(< N:Nat, s(s(M:Nat)) >) = ff .
  eq one-writer(< N:Nat, 0 >) = tt .
  eq one-writer(< N:Nat, s(0) >) = tt .
endm

mod READERS-WRITERS-ABS is
  including READERS-WRITERS-PREDS .
  including READERS-WRITERS .
  eq < s(s(N:Nat)), 0 > = < s(0), 0 > .
endm
```

Abstraction by adding equations

In order to check both the executability and the invariant-preservation properties of this abstraction, since we have no equations with either `tt` or `ff` in their lefthand side, we need to check:

1. that the equations in both `READERS-WRITERS-PREDS` and `READERS-WRITERS-ABS` are ground confluent, sort-decreasing, and terminating;
2. that the equations in both `READERS-WRITERS-PREDS` and `READERS-WRITERS-ABS` are sufficiently complete (this is equivalent to requiring that properties are preserved); and
3. that the rules in both `READERS-WRITERS-PREDS` and `READERS-WRITERS-ABS` are ground coherent with respect to their equations.

Church-Rosser Checker

```
Maude> (check Church-Rosser READERS-WRITERS-PREDS .)
Church-Rosser checking of READERS-WRITERS-PREDS
Checking solution:
  All critical pairs have been joined. The specification is
    locally-confluent.
The specification is sort-decreasing.

Maude> (check Church-Rosser READERS-WRITERS-ABS .)
Church-Rosser checking of READERS-WRITERS-ABS
Checking solution:
  All critical pairs have been joined. The specification is
    locally-confluent.
The specification is sort-decreasing.
```

Sufficient Completeness Checker

```
Maude> (scc READERS-WRITERS-PREDS .)
Checking sufficient completeness of READERS-WRITERS-PREDS ...
Success: READERS-WRITERS-PREDS is sufficiently complete under the
  assumption that it is weakly-normalizing, confluent, and
  sort-decreasing.

Maude> (scc READERS-WRITERS-ABS .)
Checking sufficient completeness of READERS-WRITERS-ABS ...
Success: READERS-WRITERS-ABS is sufficiently complete under the
  assumption that it is weakly-normalizing, confluent, and
  sort-decreasing.
```

Coherence Checker

```
Maude> (check coherence READERS-WRITERS-PREDS .)
Coherence checking of READERS-WRITERS-PREDS
Coherence checking solution:
  All critical pairs have been rewritten and all equations
    are non-constructor.
  The specification is coherent.

Maude> (check coherence READERS-WRITERS-ABS .)
Coherence checking of READERS-WRITERS-ABS
Coherence checking solution:
  The following critical pairs cannot be rewritten:
  cp < s(0), 0 > => < s(N*@:Nat), 0 > .
```

Finite-state invariant checking

```
Maude> search in READERS-WRITERS-ABS :
         < 0, 0 > =>* C:Config
         such that mutex(C:Config) = ff .

No solution.
states: 3  rewrites: 9 in 0ms cpu (0ms real) (~ rews/sec)

Maude> search in READERS-WRITERS-ABS :
         < 0, 0 > =>* C:Config
         such that one-writer(C:Config) = ff .

No solution.
states: 3  rewrites: 9 in 0ms cpu (0ms real) (~ rews/sec)
```

Model checking

- Two levels of specification:
  - a system specification level, provided by the rewrite theory specified by that system module, and
  - a property specification level, given by some properties that we want to state and prove about our module.

Model checking

- Two levels of specification:
    - a system specification level, provided by the rewrite theory specified by that system module, and
    - a property specification level, given by some properties that we want to state and prove about our module.
- Temporal logic allows specification of properties such as safety properties (ensuring that something bad never happens) and liveness properties (ensuring that something good eventually happens), related to the infinite behavior of a system.
- Maude 2 includes a model checker to prove properties expressed in linear temporal logic (LTL).

Linear temporal logic

- Main connectives:
  - True: $\top \in \mathrm{LTL}(AP)$.
  - Atomic propositions: If $p \in AP$, then $p \in \mathrm{LTL}(AP)$.
  - Next operator: If $\varphi \in \mathrm{LTL}(AP)$, then $\bigcirc\varphi \in \mathrm{LTL}(AP)$.
  - Until operator: If $\varphi, \psi \in \mathrm{LTL}(AP)$, then $\varphi\, \mathcal{U}\, \psi \in \mathrm{LTL}(AP)$.
  - Boolean connectives: If $\varphi, \psi \in \mathrm{LTL}(AP)$, then the formulae $\neg\varphi$, and $\varphi \vee \psi$ are in LTL(AP).
- Other Boolean connectives:
  - False: $\bot = \neg\top$
  - Conjunction: $\varphi \wedge \psi = \neg((\neg\varphi) \vee (\neg\psi))$
  - Implication: $\varphi \rightarrow \psi = (\neg\varphi) \vee \psi$.

Linear temporal logic

- Other temporal operators:
  - Eventually: $\Diamond\varphi = \top\,\mathcal{U}\,\varphi$
  - Henceforth: $\Box\varphi = \neg\Diamond\neg\varphi$
  - Release: $\varphi\,\mathcal{R}\,\psi = \neg((\neg\varphi)\,\mathcal{U}\,(\neg\psi))$
  - Unless: $\varphi\,\mathcal{W}\,\psi = (\varphi\,\mathcal{U}\,\psi) \vee (\Box\varphi)$
  - Leads-to: $\varphi \rightsquigarrow \psi = \Box(\varphi \rightarrow (\Diamond\psi))$
  - Strong implication: $\varphi \Rightarrow \psi = \Box(\varphi \rightarrow \psi)$
  - Strong equivalence: $\varphi \Leftrightarrow \psi = \Box(\varphi \leftrightarrow \psi)$.

Linear temporal logic

- Before considering their formal meaning, let us note that the intuition behind the main temporal connectives is the following:
  - $\top$ is a formula that always holds at the current state.
  - $\bigcirc\varphi$ holds at the current state if φ holds at the state that follows.
  - $\varphi\,\mathcal{U}\,\psi$ holds at the current state if ψ is eventually satisfied at a future state and, until that moment, φ holds at all intermediate states.
  - $\Box\varphi$ holds if φ holds at every state from now on.
  - $\Diamond\varphi$ holds if φ holds at some state in the future.

Kripke structures

- A Kripke structure is a triple $\mathcal{A} = (A, \rightarrow_{\mathcal{A}}, L)$ such that
  - A is a set, called the set of states,
  - $\rightarrow_{\mathcal{A}}$ is a total binary relation on A, called the transition relation, and
  - $L : A \longrightarrow \mathcal{P}(AP)$ is a function, called the labeling function, associating to each state $a \in A$ the set $L(a)$ of those atomic propositions in AP that hold in the state a.
- A path in a Kripke structure $\mathcal{A}$ is a function $\pi : \mathbb{N} \longrightarrow A$ with $\pi(i) \rightarrow_{\mathcal{A}} \pi(i+1)$ for every i.
- We use π^i to refer to the suffix of π starting at $\pi(i)$.

Kripke structures: semantics

- The semantics of the temporal logic LTL is defined by means of a satisfaction relation between a Kripke structure $\mathcal{A}$, a state $a \in A$, and an LTL formula $\varphi \in \mathrm{LTL}(AP)$:

$$\mathcal{A}, a \models \varphi \iff \mathcal{A}, \pi \models \varphi \quad \text{for all paths } \pi \text{ with } \pi(0) = a.$$

- The satisfaction relation $\mathcal{A}, \pi \models \varphi$ is defined by structural induction on φ:

$$\begin{array}{lcl}
\mathcal{A}, \pi \models p & \iff & p \in L(\pi(0)) \\
\mathcal{A}, \pi \models \top & \iff & \mathit{true} \\
\mathcal{A}, \pi \models \varphi \vee \psi & \iff & \mathcal{A}, \pi \models \varphi \text{ or } \mathcal{A}, \pi \models \psi \\
\mathcal{A}, \pi \models \neg\varphi & \iff & \mathcal{A}, \pi \not\models \varphi \\
\mathcal{A}, \pi \models \bigcirc\varphi & \iff & \mathcal{A}, \pi^1 \models \varphi \\
\mathcal{A}, \pi \models \varphi \,\mathcal{U}\, \psi & \iff & \text{there exists } n \in \mathbb{N} \text{ such that } \mathcal{A}, \pi^n \models \psi \text{ and,} \\
 & & \text{for all } m < n, \mathcal{A}, \pi^m \models \varphi
\end{array}$$

The semantics of the remaining Boolean and temporal operators (e.g., $\bot$, $\wedge$, $\rightarrow$, $\Box$, $\Diamond$, $\mathcal{R}$, and $\leadsto$) can be derived from these.

Kripke structures associated to rewrite theories

- Given a system module M specifying a rewrite theory $\mathcal{R} = (\Sigma, E, R)$, we
  - choose a kind k in M as our kind of states;
  - define some state predicates Π and their semantics in a module, say M-PREDS, protecting M by means of the operation

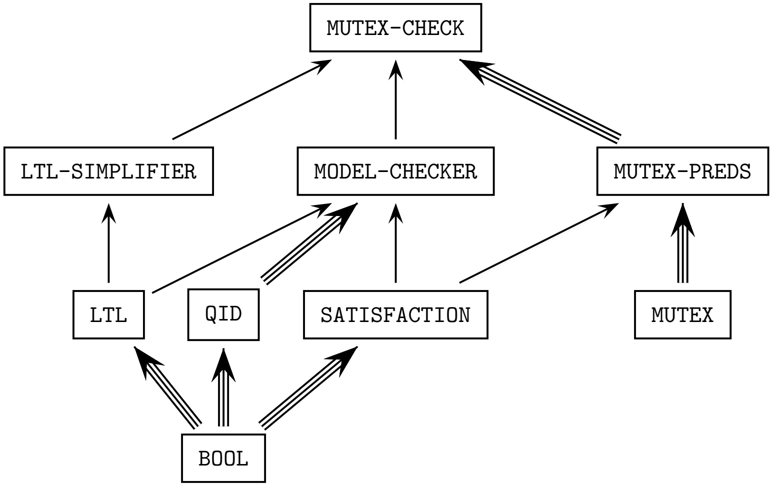
    ```
    op _|=_ : State Prop -> Bool .
    ```

    coming from the predefined SATISFACTION module.
- Then we get a Kripke structure

$$\mathcal{K}(\mathcal{R}, k)_\Pi = (T_{\Sigma/E,k}, (\rightarrow^1_{\mathcal{R}})^\bullet, L_\Pi).$$

- Under some assumptions on M and M-PREDS, including that the set of states reachable from $[t]$ is finite, the relation $\mathcal{K}(\mathcal{R}, k)_\Pi, [t] \models \varphi$ becomes decidable.

Model-checking modules



Mutual exclusion: processes

```
mod MUTEX is
  sorts Name Mode Proc Token Conf .
  subsorts Token Proc < Conf .
  op none : -> Conf [ctor] .
  op __ : Conf Conf -> Conf [ctor assoc comm id: none] .

  ops a b : -> Name [ctor] .
  ops wait critical : -> Mode [ctor] .
  op [_,_] : Name Mode -> Proc [ctor] .
  ops * $ : -> Token [ctor] .

  rl [a-enter] : $ [a, wait] => [a, critical] .
  rl [b-enter] : * [b, wait] => [b, critical] .
  rl [a-exit] : [a, critical] => [a, wait] * .
  rl [b-exit] : [b, critical] => [b, wait] $ .
endm
```

Mutual exclusion: basic properties

```
mod MUTEX-PREDS is
  protecting MUTEX .
  including SATISFACTION .
  subsort Conf < State .

  op crit : Name -> Prop .
  op wait : Name -> Prop .

  var N : Name .
  var C : Conf .
  var P : Prop .

  eq [N, critical] C |= crit(N) = true .
  eq [N, wait] C |= wait(N) = true .
  eq C |= P = false [owise] .
endm
```

Model checking mutual exclusion

```
mod MUTEX-CHECK is
  protecting MUTEX-PREDS .
  including MODEL-CHECKER .
  including LTL-SIMPLIFIER .
  ops initial1 initial2 : -> Conf .
  eq initial1 = $ [a, wait] [b, wait] .
  eq initial2 = * [a, wait] [b, wait] .
endm
```

Model checking mutual exclusion

```
mod MUTEX-CHECK is
  protecting MUTEX-PREDS .
  including MODEL-CHECKER .
  including LTL-SIMPLIFIER .
  ops initial1 initial2 : -> Conf .
  eq initial1 = $ [a, wait] [b, wait] .
  eq initial2 = * [a, wait] [b, wait] .
endm

Maude> red modelCheck(initial1, [] ~(crit(a) /\ crit(b))) .
ModelChecker: Property automaton has 2 states.
ModelCheckerSymbol: Examined 4 system states.
result Bool: true

Maude> red modelCheck(initial2, [] ~(crit(a) /\ crit(b))) .
ModelChecker: Property automaton has 2 states.
ModelCheckerSymbol: Examined 4 system states.
result Bool: true
```

A strong liveness property

If a process waits infinitely often, then it is in its critical section infinitely often.

```
Maude> red modelCheck(initial1, ([]<> wait(a)) -> ([]<> crit(a))) .
ModelChecker: Property automaton has 3 states.
ModelCheckerSymbol: Examined 4 system states.
result Bool: true

Maude> red modelCheck(initial1, ([]<> wait(b)) -> ([]<> crit(b))) .
ModelChecker: Property automaton has 3 states.
ModelCheckerSymbol: Examined 4 system states.
result Bool: true

Maude> red modelCheck(initial2, ([]<> wait(a)) -> ([]<> crit(a))) .
ModelChecker: Property automaton has 3 states.
ModelCheckerSymbol: Examined 4 system states.
result Bool: true

Maude> red modelCheck(initial2, ([]<> wait(b)) -> ([]<> crit(b))) .
ModelChecker: Property automaton has 3 states.
ModelCheckerSymbol: Examined 4 system states.
result Bool: true
```

Counterexamples

- A counterexample is a pair consisting of two lists of transitions, where the first corresponds to a finite path beginning in the initial state, and the second describes a loop.
- If we check whether, beginning in the state `initial1`, process b will always be waiting, we get a counterexample:

```
Maude> red modelCheck(initial1, [] wait(b)) .
ModelChecker: Property automaton has 2 states.
ModelCheckerSymbol: Examined 4 system states.

result ModelCheckResult:
  counterexample({$ [a, wait] [b, wait], 'a-enter}
                 {[a, critical] [b, wait], 'a-exit}
                 {* [a, wait] [b, wait], 'b-enter} ,
                 {[a, wait] [b, critical], 'b-exit}
                 {$ [a, wait] [b, wait], 'a-enter}
                 {[a, critical] [b, wait], 'a-exit}
                 {* [a, wait] [b, wait], 'b-enter})
```

Crossing the river

- A shepherd needs to transport to the other side of a river
  - a wolf,
  - a lamb, and
  - a cabbage.
- He has only a boat with room for the shepherd himself and another item.
- The problem is that in the absence of the shepherd
  - the wolf would eat the lamb, and
  - the lamb would eat the cabbage.

Crossing the river

- The shepherd and his belongings are represented as objects with an attribute indicating the side of the river in which each is located.
- Constants `left` and `right` represent the two sides of the river.
- Operation `change` is used to modify the corresponding attributes.
- Rules represent the ways of crossing the river that are allowed by the capacity of the boat.

Crossing the river

- The shepherd and his belongings are represented as objects with an attribute indicating the side of the river in which each is located.
- Constants `left` and `right` represent the two sides of the river.
- Operation `change` is used to modify the corresponding attributes.
- Rules represent the ways of crossing the river that are allowed by the capacity of the boat.
- Properties define the good and bad states:
  - `success` characterizes the state in which the shepherd and his belongings are in the other side,
  - `disaster` characterizes the states in which some eating takes place.

Crossing the river

```
mod RIVER-CROSSING is
  sorts Side Group .

  ops left right : -> Side [ctor] .
  op change : Side -> Side .
  eq change(left) = right .
  eq change(right) = left .

  ops s w l c : Side -> Group [ctor] .
  op __ : Group Group -> Group [ctor assoc comm] .

  var S : Side .

  rl [shepherd] : s(S) => s(change(S)) .
  rl [wolf] : s(S) w(S) => s(change(S)) w(change(S)) .
  rl [lamb] : s(S) l(S) => s(change(S)) l(change(S)) .
  rl [cabbage] : s(S) c(S) => s(change(S)) c(change(S)) .
endm
```

Crossing the river

```
mod RIVER-CROSSING-PROP is
  protecting RIVER-CROSSING .
  including MODEL-CHECKER .
  subsort Group < State .

  op initial : -> Group .
  eq initial = s(left) w(left) l(left) c(left) .

  ops disaster success : -> Prop .

  vars S S' S'' : Side .

  ceq (w(S) l(S) s(S') c(S'') |= disaster) = true if S =/= S' .
  ceq (w(S'') l(S) s(S') c(S) |= disaster) = true if S =/= S' .
  eq (s(right) w(right) l(right) c(right) |= success) = true .
  eq G:Group |= P:Prop = false [owise] .
endm
```

Crossing the river

- The model checker only returns paths that are counterexamples of properties.
- To find a safe path we need to find a formula that somehow expresses the negation of the property we are interested in: a counterexample will then witness a safe path for the shepherd.
- If no safe path exists, then it is true that whenever `success` is reached a disastrous state has been traversed before:

  ```
  <> success -> (<> disaster /\ ((˜ success) U disaster))
  ```

  Note that this formula is equivalent to the simpler one

  ```
  <> success -> ((˜ success) U disaster)
  ```

- A counterexample to this formula is a safe path, completed so as to have a cycle.

Crossing the river

```
Maude> red modelCheck(initial,
         <> success -> (<> disaster /\ ((˜ success) U disaster))) .

result ModelCheckResult: counterexample(
    {s(left) w(left) l(left) c(left),'lamb}
    {s(right) w(left) l(right) c(left),'shepherd}
    {s(left) w(left) l(right) c(left),'wolf}
    {s(right) w(right) l(right) c(left),'lamb}
    {s(left) w(right) l(left) c(left),'cabbage}
    {s(right) w(right) l(left) c(right),'shepherd}
    {s(left) w(right) l(left) c(right),'lamb}
    {s(right) w(right) l(right) c(right),'lamb}
    {s(left) w(right) l(left) c(right),'shepherd}
    {s(right) w(right) l(left) c(right),'wolf}
    {s(left) w(left) l(left) c(right),'lamb}
    {s(right) w(left) l(right) c(right),'cabbage}
    {s(left) w(left) l(right) c(left),'wolf},
    {s(right) w(right) l(right) c(left),'lamb}
    {s(left) w(right) l(left) c(left),'lamb})
```

Crossing the river through search

```
mod RIVER-CROSSING is
  sorts Side Group State .
  ops left right : -> Side [ctor] .
  op change : Side -> Side .
  eq change(left) = right .
  eq change(right) = left .
  ops s w l c : Side -> Group [ctor] .
  op __ : Group Group -> Group [ctor assoc comm] .

  vars S S' : Side .      var  G : Group .

  ceq w(S) l(S) s(S') = w(S) s(S') if S =/= S' .
      --- wolf eats lamb
  ceq c(S) l(S) w(S') s(S') = l(S) w(S') s(S') if S =/= S' .
      --- lamb eats cabbage
```

- Problem: lack of coherence!
- Solution: encapsulation and rule refinement

Crossing the river through search

```
    op {_} : Group -> State [ctor] .
    op toBeEaten : Group -> Bool .

    ceq toBeEaten(w(S) l(S) s(S') G) = true if S =/= S' .
    ceq toBeEaten(c(S) l(S) s(S') G) = true if S =/= S' .
    eq toBeEaten(G) = false [owise] .

    crl [shepherd-alone] : { s(S) G } => { s(change(S)) G }
      if not(toBeEaten(s(S) G)) .
    crl [wolf] : { s(S) w(S) G } => { s(change(S)) w(change(S)) G }
      if not(toBeEaten(s(S) w(S) G)) .
    rl [lamb] : { s(S) l(S) G } => { s(change(S)) l(change(S)) G }.
    crl [cabbage] : { s(S) c(S) G } => { s(change(S)) c(change(S)) G }
      if not(toBeEaten(s(S) c(S) G)) .

    op initial : -> State .
    eq initial = { s(left) w(left) l(left) c(left) } .
  endm
```

Crossing the river through search

```
Maude> search initial
         =>* { w(right) s(right) l(right) c(right) } .
Solution 1 (state 27)
empty substitution

No more solutions.

Maude> show path labels 27 .
lamb
shepherd-alone
wolf
lamb
cabbage
shepherd-alone
lamb
```

Reflection

- Rewriting logic is reflective, because there is a finitely presented rewrite theory $\mathcal{U}$ that is universal in the sense that:
  - we can represent any finitely presented rewrite theory $\mathcal{R}$ and any terms t, t' in $\mathcal{R}$ as terms $\overline{\mathcal{R}}$ and $\overline{t}, \overline{t'}$ in $\mathcal{U}$,
  - then we have the following equivalence

$$\mathcal{R} \vdash t \longrightarrow t' \iff \mathcal{U} \vdash \langle \overline{\mathcal{R}}, \overline{t} \rangle \longrightarrow \langle \overline{\mathcal{R}}, \overline{t'} \rangle.$$

- Since $\mathcal{U}$ is representable in itself, we get a reflective tower

$$\begin{array}{c} \mathcal{R} \vdash t \to t' \\ \Updownarrow \\ \mathcal{U} \vdash \langle \overline{\mathcal{R}}, \overline{t} \rangle \to \langle \overline{\mathcal{R}}, \overline{t'} \rangle \\ \Updownarrow \\ \mathcal{U} \vdash \langle \overline{\mathcal{U}}, \overline{\langle \overline{\mathcal{R}}, \overline{t} \rangle} \rangle \to \langle \overline{\mathcal{U}}, \overline{\langle \overline{\mathcal{R}}, \overline{t'} \rangle} \rangle \\ \vdots \end{array}$$

Maude's metalevel

$$\begin{tikzcd}
 & \boxed{\texttt{META-LEVEL}} & & \\
 & \boxed{\texttt{META-MODULE}} \arrow[u, Rrightarrow] & & \\
\boxed{\texttt{QID-LIST}} \arrow[ur, Rrightarrow] & \boxed{\texttt{QID-SET*}(\beta)} \arrow[u, Rrightarrow] & \boxed{\texttt{META-TERM}} \arrow[ul, Rrightarrow] & \boxed{\texttt{NAT-LIST}} \arrow[ull, Rrightarrow] \\
 & \boxed{\texttt{QID-SET}} \arrow[u, dotted, "\texttt{\_*}(\beta)"] & \boxed{\texttt{QID}} \arrow[u, Rrightarrow] &
\end{tikzcd}$$

Maude's metalevel

In Maude, key functionality of the universal theory $\mathcal{U}$ has been efficiently implemented in the functional module `META-LEVEL`:

- Maude terms are reified as elements of a data type `Term` in the module `META-TERM`;
- Maude modules are reified as terms in a data type `Module` in the module `META-MODULE`;
- operations `upModule`, `upTerm`, `downTerm`, and others allow moving between reflection levels;
- the process of reducing a term to canonical form using Maude's `reduce` command is metarepresented by a built-in function `metaReduce`;
- the processes of rewriting a term in a system module using Maude's `rewrite` and `frewrite` commands are metarepresented by built-in functions `metaRewrite` and `metaFrewrite`;

Maude's metalevel

- the process of applying a rule of a system module at the top of a term is metarepresented by a built-in function `metaApply`;
- the process of applying a rule of a system module at any position of a term is metarepresented by a built-in function `metaXapply`;
- the process of matching two terms is reified by built-in functions `metaMatch` and `metaXmatch`;
- the process of searching for a term satisfying some conditions starting in an initial term is reified by built-in functions `metaSearch` and `metaSearchPath`; and
- parsing and pretty-printing of a term in a module, as well as key sort operations such as comparing sorts in the subsort ordering of a signature, are also metarepresented by corresponding built-in functions.

Representing terms

```
sorts Constant Variable Term .
subsorts Constant Variable < Qid Term .

op <Qids> : -> Constant [special (...)] .
op <Qids> : -> Variable [special (...)] .

sort TermList .
subsort Term < TermList .
op _,_ : TermList TermList -> TermList
    [ctor assoc gather (e E) prec 120] .

op _[_] : Qid TermList -> Term [ctor] .
```

Representing terms: Example

- Usual term: `c (q M:State)`
- Metarepresentation

```
'__['c.Item, '__['q.Coin, 'M:State]]
```

Representing terms: Example

- Usual term: `c (q M:State)`
- Metarepresentation

```
'__['c.Item, '__['q.Coin, 'M:State]]
```

- Meta-metarepresentation

```
'_`[_`]['’__.Qid,
        '_`,_[''c.Item.Constant,
              '_`[_`][''__.Qid,
                      '_`,_[''q.Coin.Constant,
                            ''M:State.Variable]]]]
```

Representing modules

```
sorts FModule SModule FTheory STheory Module .
subsorts FModule < SModule < Module .
subsorts FTheory < STheory < Module .
sort Header .
subsort Qid < Header .
op _{_}  : Qid ParameterDeclList -> Header [ctor] .
op fmod_is_sorts_.____endfm : Header ImportList SortSet
     SubsortDeclSet OpDeclSet MembAxSet EquationSet -> FModule
     [ctor gather (& & & & & & &)] .
op mod_is_sorts_._____endm : Header ImportList SortSet
     SubsortDeclSet OpDeclSet MembAxSet EquationSet RuleSet
     -> SModule [ctor gather (& & & & & & & &)] .
op fth_is_sorts_.____endfth : Qid ImportList SortSet SubsortDeclSet
     OpDeclSet MembAxSet EquationSet -> FTheory
     [ctor gather (& & & & & & &)] .
op th_is_sorts_._____endth : Qid ImportList SortSet SubsortDeclSet
     OpDeclSet MembAxSet EquationSet RuleSet -> STheory
     [ctor gather (& & & & & & & &)] .
```

Representing modules: Example at the object level

```
fmod VENDING-MACHINE-SIGNATURE is
  sorts Coin Item State .
  subsorts Coin Item < State .
  op __ : State State -> State [assoc comm] .
  op $ : -> Coin [format (r! o)] .
  op q : -> Coin [format (r! o)] .
  op a : -> Item [format (b! o)] .
  op c : -> Item [format (b! o)] .
endfm
```

Representing modules: Example at the metalevel

```
fmod 'VENDING-MACHINE-SIGNATURE is
  nil
  sorts 'Coin ; 'Item ; 'State .
  subsort 'Coin < 'State .
  subsort 'Item < 'State .
  op '__ : 'State 'State -> 'State [assoc comm] .
  op 'a : nil -> 'Item [format('b! 'o)] .
  op 'c : nil -> 'Item [format('b! 'o)] .
  op '$ : nil -> 'Coin [format('r! 'o)] .
  op 'q : nil -> 'Coin [format('r! 'o)] .
  none
  none
endfm
```

Representing modules: Example at the object level

```
mod VENDING-MACHINE is
  including VENDING-MACHINE-SIGNATURE .
  var M : State .
  rl [add-q] : M => M q .
  rl [add-$] : M => M $ .
  rl [buy-c] : $ => c .
  rl [buy-a] : $ => a q .
  rl [change] : q q q q => $ .
endm
```

Representing modules: Example at the metalevel

```
mod 'VENDING-MACHINE is
  including 'VENDING-MACHINE-SIGNATURE .
  sorts none .
  none
  none
  none
  none
  rl 'M:State => '__['M:State, 'q.Coin] [label('add-q)] .
  rl 'M:State => '__['M:State, '$.Coin] [label('add-$)] .
  rl '$.Coin => 'c.Item [label('buy-c)] .
  rl '$.Coin => '__['a.Item, 'q.Coin] [label('buy-a)] .
  rl '__['q.Coin,'q.Coin,'q.Coin,'q.Coin]
    => '$.Coin [label('change)] .
endm
```

Moving between levels

```
op upModule : Qid Bool ~> Module [special (...)] .
op upSorts : Qid Bool ~> SortSet [special (...)] .
op upSubsortDecls : Qid Bool ~> SubsortDeclSet [special (...)] .
op upOpDecls : Qid Bool ~> OpDeclSet [special (...)] .
op upMbs : Qid Bool ~> MembAxSet [special (...)] .
op upEqs : Qid Bool ~> EquationSet [special (...)] .
op upRls : Qid Bool ~> RuleSet [special (...)] .
```

In all these (partial) operations

- The first argument is expected to be a module name.
- The second argument is a Boolean, indicating whether we are interested also in the imported modules or not.

Moving between levels: Example

```
Maude> reduce in META-LEVEL : upEqs('VENDING-MACHINE, true) .
result EquationSet:
  eq '_and_['true.Bool, 'A:Bool] = 'A:Bool [none] .
  eq '_and_['A:Bool, 'A:Bool] = 'A:Bool [none] .
  eq '_and_['A:Bool, '_xor_['B:Bool, 'C:Bool]]
    = '_xor_['_and_['A:Bool, 'B:Bool], '_and_['A:Bool, 'C:Bool]]
    [none] .
  eq '_and_['false.Bool, 'A:Bool] = 'false.Bool [none] .
  eq '_or_['A:Bool,'B:Bool]
    = '_xor_['_and_['A:Bool, 'B:Bool],'_xor_['A:Bool, 'B:Bool]]
    [none] .
  eq '_xor_['A:Bool, 'A:Bool] = 'false.Bool [none] .
  eq '_xor_['false.Bool, 'A:Bool] = 'A:Bool [none] .
  eq 'not_['A:Bool] = '_xor_['true.Bool, 'A:Bool] [none] .
  eq '_implies_['A:Bool, 'B:Bool]
    = 'not_['_xor_['A:Bool, '_and_['A:Bool, 'B:Bool]]] [none] .
```

Moving between levels: Example

```
Maude> reduce in META-LEVEL : upEqs('VENDING-MACHINE, false) .
result EquationSet: (none).EquationSet
```

Moving between levels: Example

```
Maude> reduce in META-LEVEL : upEqs('VENDING-MACHINE, false) .
result EquationSet: (none).EquationSet
```

```
Maude> reduce in META-LEVEL : upRls('VENDING-MACHINE, true) .
result RuleSet:
  rl '$.Coin => 'c.Item [label('buy-c)] .
  rl '$.Coin => '__['q.Coin,'a.Item] [label('buy-a)] .
  rl 'M:State => '__['$.Coin,'M:State] [label('add-$)] .
  rl 'M:State => '__['q.Coin,'M:State] [label('add-q)] .
  rl '__['q.Coin,'q.Coin,'q.Coin,'q.Coin] => '$.Coin
    [label('change)] .
```

Moving between levels: Terms

```
op upTerm : Universal -> Term .
op downTerm : Term Universal -> Universal .
```

- `upTerm` takes a term t and returns the metarepresentation of its canonical form.
- `downTerm` takes the metarepresentation of a term t and a term t', and returns the canonical form of t, if t is a term in the same kind as t'; otherwise, it returns the canonical form of t'.

Moving between levels: Terms

```
 fmod UP-DOWN-TEST is protecting META-LEVEL .
  sort Foo .
  ops a b c d : -> Foo .
  op f : Foo Foo -> Foo .
  op error : -> [Foo] .
  eq c = d .
endfm

Maude> reduce in UP-DOWN-TEST : upTerm(f(a, f(b, c))) .
result GroundTerm: 'f['a.Foo,'f['b.Foo,'d.Foo]]

Maude> reduce downTerm('f['a.Foo,'f['b.Foo,'c.Foo]], error) .
result Foo: f(a, f(b, c))

Maude> reduce downTerm('f['a.Foo,'f['b.Foo,'e.Foo]], error) .
Advisory: could not find a constant e of
          sort Foo in meta-module UP-DOWN-TEST.
result [Foo]: error
```

metaReduce

- Its first argument is the representation in META-LEVEL of a module $\mathcal{R}$ and its second argument is the representation in META-LEVEL of a term t.
- It returns the metarepresentation of the canonical form of t, using the equations in $\mathcal{R}$, together with the metarepresentation of its corresponding sort or kind.
- The reduction strategy used by metaReduce coincides with that of the reduce command.

```
Maude> reduce in META-LEVEL :
         metaReduce(upModule('NAT-SORTED-LIST, false),
           'quicksort['_:_['s_^3['0.Zero],'_:_['s_^18['0.Zero],
                      '_:_['s_^1['0.Zero],'_:_['s_^9['0.Zero],
                      '`[`].List]]]]]) .
result ResultPair:
  {'_:_['s_['0.Zero],'_:_['s_^3['0.Zero],'_:_['s_^9['0.Zero],
        '_:_['s_^18['0.Zero],'`[`].SortedList]]]],
   'NeSortedList}
```

metaRewrite

- Its first two arguments are the representations in `META-LEVEL` of a module $\mathcal{R}$ and of a term t, and its third argument is a natural n.
- Its result is the representation of the term obtained from t after at most n applications of the rules in $\mathcal{R}$ using the strategy of Maude's command `rewrite`, together with the metarepresentation of its corresponding sort or kind.

```
Maude> reduce in META-LEVEL :
        metaRewrite(upModule('VENDING-MACHINE, false),
          '__['$.Coin, '__['$.Coin, '__['q.Coin, 'q.Coin]]], 1) .
result ResultPair:
  {'__['$.Coin, '$.Coin, '$.Coin, 'q.Coin, 'q.Coin], 'State}

Maude> reduce in META-LEVEL :
        metaRewrite(upModule('VENDING-MACHINE, false),
          '__['$.Coin, '__['$.Coin, '__['q.Coin, 'q.Coin]]], 2) .
result ResultPair:
  {'__['$.Coin, '$.Coin, '$.Coin, 'q.Coin, 'q.Coin, 'q.Coin], 'State}
```

metaApply

- The first three arguments are representations in `META-LEVEL` of a module $\mathcal{R}$, a term t in $\mathcal{R}$, and a label l of some rules in $\mathcal{R}$.
- The fourth argument is a set of assignments (possibly empty) defining a partial substitution σ for the variables in the rules.
- The last argument is a natural number n, used to enumerate (starting from `0`) all the possible solutions of the intended rule application.
- It returns a triple of sort `ResultTriple` consisting of the metarepresentation of a term, the metarepresentation of its corresponding sort or kind, and the metarepresentation of a substitution.

metaApply

The operation `metaApply` is evaluated as follows:

1. the term t is first fully reduced using the equations in $\mathcal{R}$;
2. the resulting term is matched at the top against all rules with label l in $\mathcal{R}$ partially instantiated with σ, with matches that fail to satisfy the condition of their rule discarded;
3. the first n successful matches are discarded; if there is an $(n+1)$th match, its rule is applied using that match and the steps 4 and 5 below are taken; otherwise `failure` is returned;
4. the term resulting from applying the given rule with the $(n+1)$th match is fully reduced using the equations in $\mathcal{R}$;
5. the triple formed by the metarepresentation of the resulting fully reduced term, the metarepresentation of its corresponding sort or kind, and the metarepresentation of the substitution used in the reduction is returned.

metaApply

```
Maude> reduce in META-LEVEL :
         metaApply(upModule('VENDING-MACHINE, false),
           '$.Coin, 'buy-c, none, 0) .
result ResultTriple: {'c.Item,'Item,(none).Substitution}

Maude> reduce in META-LEVEL :
         metaApply(upModule('VENDING-MACHINE, false),
           '$.Coin, 'add-$, none, 0) .
result ResultTriple:
  {'__['$.Coin, '$.Coin], 'State, 'M:State <- '$.Coin}

Maude> reduce in META-LEVEL :
         metaApply(upModule('VENDING-MACHINE, false),
           '__['q.Coin, '$.Coin], 'buy-c, none, 0) .
result ResultTriple?: (failure).ResultTriple?
```

metaXapply

- The operation `metaXapply(`$\overline{\mathcal{R}}$`, `$\bar{t}$`, `$\bar{l}$`, `σ`, `n`, `b`, `m`)` is evaluated as the function `metaApply` but using extension and in any possible position, not only at the top.
- The arguments n and b can be used to localize the part of the term where the rule application can take place.
- n is the lower bound on depth in terms of nested operators, and should be set to `0` to start searching from the top, while
- The `Bound` argument b indicates the upper bound, and should be set to `unbounded` to have no cut off.
- m is used to enumerate (starting from 0) all the possible solutions of the intended rule application.
- The result of `metaXapply` has an additional component, giving the context (a term with a single "hole", represented `[]`) inside the given term t, where the rewriting has taken place.

metaXapply

```
Maude> reduce in META-LEVEL :
         metaXapply(upModule('VENDING-MACHINE, false),
           '__['q.Coin, '$.Coin], 'buy-c, none, 0, unbounded, 0) .
result Result4Tuple:
  {'__['q.Coin, 'c.Item], 'State, none, '__['q.Coin, []]}
```

```
Maude> reduce in META-LEVEL :
         metaXapply(upModule('VENDING-MACHINE, false),
           '__['q.Coin, '$.Coin], 'buy-c, none, 0, unbounded, 1) .
result Result4Tuple?: (failure).Result4Tuple?
```

metaMatch and metaXmatch

- The operation metaMatch($\overline{\mathcal{R}}$, $\overline{t}$, $\overline{t'}$, *Cond*, *n*) tries to match at the top the terms t (the pattern) and t' in the module $\mathcal{R}$ in such a way that the resulting substitution satisfies the condition *Cond*.
- The last argument is used to enumerate possible matches.
- If the matching attempt is successful, the result is the corresponding substitution; otherwise, noMatch is returned.
- The generalization to metaXmatch follows exactly the same ideas as for metaXapply.
- metaMatch provides the metalevel counterpart of the object-level command match.
- metaXmatch provides a generalization of the object-level command xmatch. The object-level behavior of the xmatch command is obtained by setting both min and max depth to 0.

metaMatch and metaXmatch

```
Maude> reduce in META-LEVEL :
         metaMatch(upModule('VENDING-MACHINE, false),
           '__['M:State, '$.Coin],
           '__['$.Coin, 'q.Coin, 'a.Item, 'c.Item], nil, 0) .
result Assignment: 'M:State <- '__['q.Coin, 'a.Item, 'c.Item]

Maude> reduce metaXmatch(upModule('VENDING-MACHINE, false),
                '__['M:State, '$.Coin],
                '__['$.Coin, 'q.Coin, 'a.Item, 'c.Item],
                nil, 0, unbounded, 0) .
result MatchPair: {'M:State <- '__['q.Coin, 'a.Item, 'c.Item], []}

Maude> reduce metaXmatch(upModule('VENDING-MACHINE, false),
                '__['M:State,'$.Coin],
                '__['$.Coin,'q.Coin,'a.Item,'c.Item],
                nil, 0, unbounded, 1) .
result MatchPair: {'M:State <- '__['a.Item,'c.Item],'__['q.Coin,[]]}
```

metaSearch

- `metaSearch` takes as arguments
  - the metarepresentation of a module,
  - the metarepresentation of the starting term for search,
  - the metarepresentation of the pattern to search for,
  - the metarepresentation of a condition to be satisfied,
  - the metarepresentation of the type of search to carry on,
  - a `Bound` value that indicates the maximum depth of the search, and
  - a natural number, to enumerate solutions.
- The searching strategy used by `metaSearch` coincides with that of the object level `search` command.
- The possible types of search are:
  - `'*` for a search involving zero or more rewrites (corresponding to =>* in the `search` command),
  - `'+` for a search consisting in one or more rewrites (=>+),
  - `'!` for a search that only matches canonical forms (=>!).
- The result has the same form as the result of `metaApply`.

metaSearch

```
Maude> reduce in META-LEVEL :
         metaSearch(upModule('VENDING-MACHINE, false),
           '__['$.Coin, 'q.Coin, 'q.Coin,'q.Coin],
           '__['c.Item, 'a.Item, 'c.Item, 'M:State],
           nil, '+, unbounded, 0) .
result ResultTriple:
  {'__['q.Coin,'q.Coin,'q.Coin,'q.Coin,'a.Item,'c.Item,'c.Item],
   'State,
   'M:State <- '__['q.Coin, 'q.Coin, 'q.Coin, 'q.Coin]}

Maude> reduce in META-LEVEL :
         metaSearch(upModule('VENDING-MACHINE, false),
           '__['$.Coin, 'q.Coin, 'q.Coin, 'q.Coin],
           '__['c.Item, 'a.Item, 'c.Item, 'M:State],
           nil, '+, unbounded, 1) .
result ResultTriple:
  {'__['$.Coin,'q.Coin,'q.Coin,'q.Coin,'q.Coin,'a.Item,'c.Item,'c.Item],
   'State,
   'M:State <- '__['$.Coin,'q.Coin,'q.Coin,'q.Coin,'q.Coin]}
```

metaSearchPath

```
Maude> reduce in META-LEVEL :
         metaSearchPath(upModule('VENDING-MACHINE, false),
           '__['$.Coin, 'q.Coin, 'q.Coin,'q.Coin],
           '__['c.Item, 'a.Item, 'c.Item, 'M:State],
           nil, '+, unbounded, 0) .
result Trace:
  {'__['$.Coin,'q.Coin,'q.Coin,'q.Coin],
   'State,
   rl 'M:State => '__['$.Coin,'M:State] [label('add-$)] .}
  {'__['$.Coin,'$.Coin,'q.Coin,'q.Coin,'q.Coin],
   'State,
   rl 'M:State => '__['$.Coin,'M:State] [label('add-$)] .}
  {'__['$.Coin,'$.Coin,'$.Coin,'q.Coin,'q.Coin,'q.Coin],
   'State,
   rl '$.Coin => 'c.Item [label('buy-c)] .}
  {'__['$.Coin,'$.Coin,'q.Coin,'q.Coin,'q.Coin,'c.Item],
   'State,
   rl '$.Coin => 'c.Item [label('buy-c)] .}
  {'__['$.Coin,'q.Coin,'q.Coin,'q.Coin,'c.Item,'c.Item],
   'State,
   rl '$.Coin => '__['q.Coin,'a.Item] [label('buy-a)] .}
```

Internal strategies

- System modules in Maude are rewrite theories that do not need to be Church-Rosser and terminating.
- Therefore, we need to have good ways of controlling the rewriting inference process—which in principle could not terminate or go in many undesired directions—by means of adequate strategies.
- In Maude, thanks to its reflective capabilities, strategies can be made internal to the system.
- That is, they can be defined using statements in a normal module in Maude, and can be reasoned about as with statements in any other module.
- In general, internal strategies are defined in extensions of the `META-LEVEL` module by using `metaReduce`, `metaApply`, `metaXapply`, etc., as building blocks.

Internal strategies

We illustrate some of these possibilities by implementing the following strategies for controlling the execution of the rules in the `VENDING-MACHINE` module:

- `insertCoin`: insert either a dollar or a quarter in the vending machine;
- `onlyCakes`: only buy cakes, and buy as many cakes as possible, with the coins already inserted;
- `onlyNitems`: only buy either cakes or apples, and buy at most *n* of them, with the coins already inserted;
- `cakesAndApples`: buy the same number of apples and cakes, and buy as many as possible, with the coins already inserted.

Internal strategies: insertCoin

```
var T : Term .
var Q : Qid .
var  N : Nat .
vars BuyItem? BuyCake? Change? : [Result4Tuple].

op insertCoin : Qid Term -> Term .

ceq insertCoin(Q, T)
  = if BuyItem? :: Result4Tuple
    then getTerm(BuyItem?)
    else T
    fi
  if (Q == 'add-q or Q == 'add-$)
     /\ BuyItem? := metaXapply(upModule('VENDING-MACHINE, false),
                               T, Q, none, 0, unbounded, 0) .

eq insertCoin(Q, T) = T [owise] .
```

Internal strategies: onlyCakes

```
op onlyCakes : Term -> Term .

ceq onlyCakes(T)
  = if BuyCake? :: Result4Tuple
    then onlyCakes(getTerm(BuyCake?))
    else (if Change? :: Result4Tuple
          then onlyCakes(getTerm(Change?))
          else T
          fi)
    fi
  if BuyCake? := metaXapply(upModule('VENDING-MACHINE, false),
                    T, 'buy-c, none, 0, unbounded, 0)
   /\ Change? := metaXapply(upModule('VENDING-MACHINE, false),
                    T, 'change, none, 0, unbounded, 0) .
```

Internal strategies: onlyNitems

```
op onlyNitems : Term Qid Nat -> Term .

ceq onlyNitems(T, Q, N)
  = if N == 0
    then T
    else (if BuyItem? :: Result4Tuple
          then onlyNitems(getTerm(BuyItem?), Q, sd(N, 1))
          else (if Change? :: Result4Tuple
                then onlyNitems(getTerm(Change?), Q, N)
                else T
                fi)
          fi)
    fi
  if (Q == 'buy-c or Q == 'buy-a)
     /\ BuyItem? := metaXapply(upModule('VENDING-MACHINE, false),
                        T, Q, none, 0, unbounded, 0)
     /\ Change? := metaXapply(upModule('VENDING-MACHINE, false),
                        T, 'change, none, 0, unbounded, 0) .

eq onlyNitems(T, Q, N) = T [owise] .
```

Internal strategies: cakesAndApples

```
op buyItem? : Term Qid -> Bool .

  ceq buyItem?(T, Q)
    = if BuyItem? :: Result4Tuple
      then true
      else (if Change? :: Result4Tuple
            then buyItem?(getTerm(Change?), Q)
            else false
            fi)
      fi
    if (Q == 'buy-c or Q == 'buy-a)
       /\ BuyItem? := metaXapply(upModule('VENDING-MACHINE, false),
                                 T, Q, none, 0, unbounded, 0)
       /\ Change? := metaXapply(upModule('VENDING-MACHINE, false),
                                T, 'change, none, 0, unbounded, 0) .

  eq buyItem?(T, Q) = false [owise] .
```

Internal strategies: cakesAndApples

```
op cakesAndApples : Term -> Term .

eq cakesAndApples(T)
  = if buyItem?(T, 'buy-c)
    then (if buyItem?(onlyNitems(T, 'buy-c, 1), 'buy-a)
          then cakesAndApples(onlyNitems(onlyNitems(T, 'buy-c, 1),
                                         'buy-a, 1))
          else T
          fi)
    else T
    fi .
```

Internal strategies: Examples

```
Maude> reduce in BUYING-STRATS :
  insertCoin('q.Coin, insertCoin( '$.Coin,
                        '__['$.Coin,'$.Coin,'$.Coin, 'q.Coin])) .

Maude> reduce in BUYING-STRATS :
  onlyCakes('__['$.Coin,'$.Coin,'$.Coin, 'q.Coin]) .

Maude> reduce in BUYING-STRATS :
  onlyNitems('__['$.Coin,'$.Coin,'$.Coin, 'q.Coin], 'buy-a, 3) .

Maude> reduce in BUYING-STRATS :
  cakesAndApples('__['$.Coin,'$.Coin,'$.Coin, 'q.Coin]) .
```

Metaprogramming

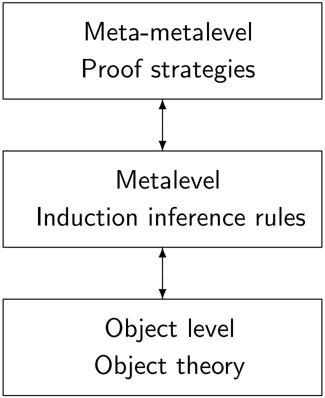
- Programming at the metalevel: the metalevel equations and rewrite rules operate on representations of lower-level rewrite theories.
- Reflection makes possible many advanced metaprogramming applications, including
  - user-definable strategy languages,
  - language extensions by new module composition operations,
  - development of theorem proving tools, and
  - reifications of other languages and logics within rewriting logic.

Metaprogramming

- Programming at the metalevel: the metalevel equations and rewrite rules operate on representations of lower-level rewrite theories.
- Reflection makes possible many advanced metaprogramming applications, including
  - user-definable strategy languages,
  - language extensions by new module composition operations,
  - development of theorem proving tools, and
  - reifications of other languages and logics within rewriting logic.
- Full Maude extends Maude with special syntax for object-oriented specifications, and with a richer module algebra of parameterized modules and module composition operations
- Theorem provers and other formal tools have underlying inference systems that can be naturally specified and prototyped in rewriting logic. Furthermore, the strategy aspects of such tools and inference systems can then be specified by rewriting strategies.

Developing theorem proving tools

- Theorem-proving tools have a very simple reflective design in Maude.
- The inference system itself may perform theory transformations, so that the theories themselves must be treated as data.
- We need strategies to guide the application of the inference rules.
- Example: Inductive Theorem Prover (ITP).



Metaprogramming examples

- A metaprogram is a program that takes programs as inputs and performs some useful computation.
- It may, for example, transform one program into another.
- Or it may analyze such a program with respect to some properties, or perform other useful program-dependent computations.
- We can easily write Maude metaprograms by importing `META-LEVEL` into a module that defines such metaprograms as functions that have `Module` as one of their arguments.
- Example:
    - rule instrumentation.

Rule instrumentation

- Instrumentation is the addition of mechanisms to some application for the purpose of gathering data.
- We are interested in collecting a history of the rules being applied on a configuration of objects and messages.
- We transform a given specification so that each rule

```
(c)rl [L] : T => T' (if cond) .
```

  with T a term of sort Configuration (or some other sort in the same kind) is transformed into a rule

```
(c)rl [L] : {T, LL}  => {T', LL 'L}  (if cond) .
```

- This transformation will instrument properly object-oriented modules whose rules rewrite terms of kind [Configuration].

Rule instrumentation in Maude

The INSTRUMENTATION-INFRASTRUCTURE module contains the basic definitions needed for our intrumentation.

```
mod INSTRUMENTATION-INFRASTRUCTURE is
  pr CONFIGURATION .
  pr QID-LIST .

  sorts InstrConfig .

  var C : Configuration . var QL : QidList .

  op {_,_} : Configuration QidList -> InstrConfig .
  op getConfig : InstrConfig -> Configuration .
  op getLabels : InstrConfig -> QidList .
  eq getConfig({C, QL}) = C .
  eq getLabels({C, QL}) = QL .
endm
```

Rule instrumentation in Maude

```
sorts InstrConfig .
op {_,_} : Configuration QidList -> InstrConfig .
op getConfig : InstrConfig -> Configuration .
op getLabels : InstrConfig -> QidList .

op getLabel : AttrSet -> Qid .
op setRls : Module RuleSet -> Module .
op addImports : Module ImportList -> Module .

op instrument : Qid -> Module .
op instrument : Module -> Module .
op instrument : Module RuleSet -> RuleSet .

eq instrument(H) = instrument(upModule(H, false)) .
eq instrument(M)
  = setRls(
      addImports(M, (including 'INSTRUMENTATION-INFRASTRUCTURE .)),
      instrument(M, getRls(M))) .
```

Rule instrumentation in Maude

```
eq instrument(M, rl T => T' [AtS] . RlS)
  = if sameKind(M, leastSort(M, T), 'Configuration)
    then (rl ''{_',_'}['__[T, 'C@:Configuration], 'QL@:QidList]
            => ''{_',_'}['__[T', 'C@:Configuration],
                 '__['QL@:QidList,
                     qid("'" + string(getLabel(AtS)) + ".Qid")]]
            [AtS] .)
    else (rl T => T' [AtS] .)
    fi
    instrument(M, RlS) .
 eq instrument(M, crl T => T' if Cd [AtS] . RlS)
  = if sameKind(M, leastSort(M, T), 'Configuration)
    then (crl ''{_',_'}['__[T, 'C@:Configuration],
                        'QL@:QidList]
            => ''{_',_'}['__[T', 'C@:Configuration],
                 '__['QL@:QidList,
                     qid("'" + string(getLabel(AtS)) + ".Qid")]]
            if Cd
            [AtS] .)
    else (crl T => T' if Cd [AtS] .)
    fi
    instrument(M, RlS) .
 eq instrument(M, none) = none .
```

Rule instrumentation: Example

```
mod BANK-ACCOUNT is
  protecting INT .
  including CONFIGURATION .
  op Account : -> Cid [ctor] .
  op bal :_ : Int -> Attribute [ctor gather (&)] .
  ops credit debit : Oid Nat -> Msg [ctor] .
  op from_to_transfer_ : Oid Oid Nat -> Msg [ctor] .
  vars A B : Oid .
  vars M N N' : Nat .

  rl [credit] : < A : Account | bal : N > credit(A, M)
    => < A : Account | bal : N + M > .

  crl [debit] : < A : Account | bal : N > debit(A, M)
    => < A : Account | bal : N - M >
    if N >= M .

  crl [transfer] : (from A to B transfer M)
    < A : Account | bal : N > < B : Account | bal : N' >
    => < A : Account | bal : N - M > < B : Account | bal : N' + M >
    if N >= M .
endm
```

Rule instrumentation: Example

```
Maude> red in INSTRUMENTATION-TEST :
         downTerm(
           getTerm(
             metaRewrite(
               addOps(instrument('BANK-ACCOUNT),
                 op 'A-001 : nil -> 'Oid [ctor] .
                 op 'A-002 : nil -> 'Oid [ctor] .
                 op 'A-003 : nil -> 'Oid [ctor] .),
               ''{_',_'}[upTerm(< A-001 : Account | bal : 300 >
                                debit(A-001, 200)
                                debit(A-001, 150)
                                < A-002 : Account | bal : 250 >
                                debit(A-002, 400)
                                < A-003 : Account | bal : 1250 >
                                (from A-003 to A-002 transfer 300)),
                         'nil.QidList],
               unbounded)),
           {(none).Configuration, (nil).QidList}) .
result InstrConfig:
  { debit(A-001, 200)
    < A-001 : Account | bal : 150 >
    < A-002 : Account | bal : 150 >
    < A-003 : Account | bal : 950 >,
    'debit 'transfer 'debit }
```

Rule instrumentation: Example

```
Maude> red in INSTRUMENTATION-TEST :
         getLabels(
           downTerm(
             getTerm(
               metaSearch(
                 addOps(instrument('BANK-ACCOUNT),
                   op 'A-001 : nil -> 'Oid [ctor] .
                   op 'A-002 : nil -> 'Oid [ctor] .
                   op 'A-003 : nil -> 'Oid [ctor] .),
                 ''{_',_'}[
                   upTerm(< A-001 : Account | bal : 300 >
                          debit(A-001, 200)
                          debit(A-001, 150)
                          < A-002 : Account | bal : 250 >
                          debit(A-002, 400)
                          < A-003 : Account | bal : 1250 >
                          (from A-003 to A-002 transfer 300)),
                   'nil.QidList],
                 ''{_',_'}[
                   upTerm(C:Configuration debit(A-001, 150)),
                   'QL:QidList],
                 nil, '!, unbounded, 1)),
             {(none).Configuration, (nil).QidList})) .
result NeTypeList: 'transfer 'debit 'debit
```

Strategy language: main ideas

- But there is a very general additional possibility of being interested in the results of only some execution paths satisfying some constraints. Strategies are needed for this.
- A strategy is described as an operation that, when applied to a given term, produces a set of terms as a result, given that the process is nondeterministic in general.
- A basic strategy consists in applying a rule to a given term, but rules can be conditional with respect to some rewrite conditions which in turn can also be controlled by means of strategies.
- Those basic strategies are combined by means of several operations.
- Furthermore, strategies can also be generic.

The strategy language: Basic strategies

Idle and fail are constants representing the simplest strategies.

- Idle does nothing leaving state equal.
- Fail always fails (empty set of results).

Basic strategies Application of a rule (identified by the corresponding rule label) to a given term (possibly with variable instantiation).

$$
\begin{array}{rcl}
\langle \mathit{Substitution} \rangle & ::= & \mathrm{none} \\
 & \mid & \langle \mathit{Variable} \rangle \texttt{ <- } \langle \mathit{Term} \rangle \\
 & \mid & \langle \mathit{Substitution} \rangle \texttt{ ; } \langle \mathit{Substitution} \rangle \\
\langle \mathit{BasicStrat} \rangle & ::= & \langle \mathit{Label} \rangle \\
 & \mid & \langle \mathit{Label} \rangle \texttt{ [} \langle \mathit{Substitution} \rangle \texttt{]} \\
\langle \mathit{Strat} \rangle & ::= & \langle \mathit{BasicStrat} \rangle
\end{array}
$$

The strategy language: Basic strategies

- For conditional rules, rewrite conditions can be controlled by means of strategy expressions.

$$\begin{array}{rcl} \langle \textit{StratList} \rangle & ::= & \langle \textit{Strat} \rangle \\ & \mid & \langle \textit{Strat} \rangle\ \langle \textit{StratList} \rangle \\ \langle \textit{BasicStrat} \rangle & ::= & \langle \textit{Label} \rangle \left[\ \texttt{[}\ \langle \textit{Substitution} \rangle\ \texttt{]}\ \right]\ \texttt{\{}\ \langle \textit{StratList} \rangle\ \texttt{\}} \end{array}$$

Top For restricting the application of a rule just to the top of the term.

$$\langle \textit{Strat} \rangle \quad ::= \quad \texttt{top(}\ \langle \textit{BasicStrat} \rangle\ \texttt{)}$$

The strategy language: Tests

Tests are strategies that test some property of a given state term, based on matching.

$$
\begin{array}{rcl}
\langle \mathit{EqCondition} \rangle & ::= & \langle \mathit{BoolTerm} \rangle \\
& | & \langle \mathit{Term} \rangle \; \texttt{=} \; \langle \mathit{Term} \rangle \\
& | & \langle \mathit{Term} \rangle \; \texttt{:=} \; \langle \mathit{Term} \rangle \\
& | & \langle \mathit{EqCondition} \rangle \; \texttt{/\textbackslash} \; \langle \mathit{EqCondition} \rangle \\
\langle \mathit{Test} \rangle & ::= & \texttt{amatch} \; \langle \mathit{Pattern} \rangle \; [\; \texttt{s.t.} \; \langle \mathit{EqCondition} \rangle \;] \\
& | & \texttt{match} \; \langle \mathit{Pattern} \rangle \; [\; \texttt{s.t.} \; \langle \mathit{EqCondition} \rangle \;] \\
\langle \mathit{Strat} \rangle & ::= & \langle \mathit{Test} \rangle
\end{array}
$$

`amatch T s.t. C` is a test that, when applied to a given state term `T'`, succeeds if there is a subterm of `T'` that matches the pattern `T` and then the condition `C` is satisfied, and fails otherwise.

`match` works in the same way, but only at the top.

The strategy language: Regular expressions

Regular expressions

$\langle \mathit{Strat} \rangle$	$::=$	$\langle \mathit{Strat} \rangle$ **;** $\langle \mathit{Strat} \rangle$	concatenation
	$\mid$	$\langle \mathit{Strat} \rangle$ **|** $\langle \mathit{Strat} \rangle$	union
	$\mid$	$\langle \mathit{Strat} \rangle$ *	iteration (0 or more)
	$\mid$	$\langle \mathit{Strat} \rangle$ +	iteration (1 or more)

The strategy language: Conditional

If-then-else is a typical if-then-else, but generalized so that the first argument is also a strategy.

$$\langle \mathit{Strat} \rangle \quad ::= \quad \langle \mathit{Strat} \rangle \texttt{ ? } \langle \mathit{Strat} \rangle \texttt{ : } \langle \mathit{Strat} \rangle$$

Using the if-then-else combinator, we can define many other useful strategy combinators as derived operations.

$$\texttt{E orelse E'} = \texttt{E ? idle : E'}$$

$$\texttt{not(E)} = \texttt{E ? fail : idle}$$

$$\texttt{E !} = \texttt{E * ; not(E)}$$

$$\texttt{try(E)} = \texttt{E ? idle : idle}$$

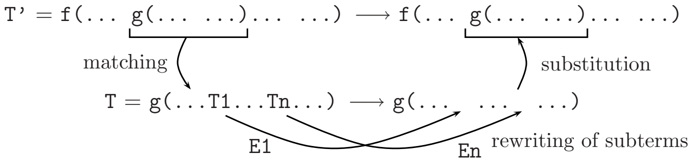
$$\texttt{test(E)} = \texttt{not(E) ? fail : idle}$$

The strategy language: Decomposition

Strategies applied to subterms with the (a)`matchrew` combinator control the way different subterms of a given state are rewritten.

$$
\begin{array}{rcl}
\langle \mathit{TermStratList} \rangle & ::= & \langle \mathit{Term} \rangle \ \texttt{using} \ \langle \mathit{Strat} \rangle \\
 & | & \langle \mathit{TermStratList} \rangle \ \texttt{,} \ \langle \mathit{TermStratList} \rangle \\
\langle \mathit{Strat} \rangle & ::= & \texttt{amatchrew} \ \langle \mathit{Pattern} \rangle \ [\ \texttt{s.t.} \ \langle \mathit{EqCondition} \rangle\ ] \ \texttt{by} \\
 & & \quad \langle \mathit{TermStratList} \rangle \\
 & | & \texttt{matchrew} \ \langle \mathit{Pattern} \rangle \ [\ \texttt{s.t.} \ \langle \mathit{EqCondition} \rangle\ ] \ \texttt{by} \\
 & & \quad \langle \mathit{TermStratList} \rangle
\end{array}
$$

`amatchrew T s.t. C by T1 using E1, ..., Tn using En`
applied to a state term `T'`:



T' = f(... g(... ...)... ...) ⟶ f(... g(... ...)... ...)
matching
substitution
T = g(...T1...Tn...) ⟶ g(... ... ...)
E1
En
rewriting of subterms

The strategy language: Recursive definitions

Recursion is achieved by giving a name to a strategy expression and using this name in the strategy expression itself or in other related strategies. For example,

```
strat solve : @ X$State .
var S : X$State .
sd solve :=   (match S s.t. isSolution(S))
            ? idle
            : (expand ;
               match S s.t. isOk(S) ;
               solve) .
```

Moreover, recursive strategies can have data arguments: values in the rewrite theory on which the strategies appy can be used as arguments.

The strategy language: Strategy modules and commands

- The user can write one or more strategy modules to define strategies for a system module M.

```
smod STRAT is
  protecting M .
  including STRAT1 .   ...   including STRATp .
  strat E1 : T11 ... T1m @ K1 .
  sd E1(P11,...,P1m) := Exp1 .
     ...
  strat En : Tn1 ... Tnp @ Kn .
  sd En(Pn1,...,Pnp) := Expn .
  csd En(Qn1,...,Qnp) := Expn' if C .
endsd
```

- The command

```
   srew T using E .
```

  rewrites the term T using a strategy expression E.

Examples: Blackboard

- A simple game: A blackboard with several natural numbers written on it. A legal move consists in selecting two numbers in the blackboard, removing them, and writing their arithmetic mean.
- The objective of the game is to get the greatest possible number written on the blackboard at the end.
- A player can choose the numbers randomly, or can follow some strategy.

```
mod BLACKBOARD is
  protecting NAT .
  sort Blackboard .
  subsort Nat < Blackboard .
  op __ : Blackboard Blackboard -> Blackboard [assoc comm] .
  vars M N : Nat .
  rl [play] : M N => (M + N) quo 2 .
endm
```

Examples: Blackboard

- Possible strategies consist in taking always the two greatest numbers, or the two smallest, or taking the greatest and the smallest.

```
mod EXT-BLACKBOARD is
  pr BLACKBOARD .
  ops max min : Blackboard -> Nat .
  op remove : Nat Blackboard -> Blackboard .
  vars M N : Nat .
  var B : Blackboard .

  eq max(N) = N .
  eq max(N M) = if (N > M) then N else M fi .
  eq max(N M B) = if (N > M) then max(N B) else max(M B) fi .
  eq min(N) = N .
  eq min(N M) = if (N < M) then N else M fi .
  eq min(N M B) = if (N < M) then min(N B) else min(M B) fi .
  eq remove(N, N B) = B .
  eq remove(N, B) = B [otherwise] .
endm
```

Examples: Blackboard

```
smod BLACKBOARD-STRAT is
  protecting EXT-BLACKBOARD .
  var B : Blackboard .     vars X Y : Nat .
  strat maxmin @ Blackboard .
  sd maxmin := (matchrew B s.t. X := max(B) /\ Y := min(B) by
                B using play[M <- X ; N <- Y] ) ! .
  strat maxmax @ Blackboard .
  sd maxmax := (matchrew B s.t. X := max(B) /\ Y := max(remove(X,B)) by
                B using play[M <- X ; N <- Y] ) ! .
  strat minmin @ Blackboard .
  sd minmin := (matchrew B s.t. X := min(B) /\ Y := min(remove(X,B)) by
                B using play[M <- X ; N <- Y] ) ! .
endsm

Maude> srew 2000 20 2 200 10 50 using maxmin .
result NzNat :  178
Maude> srew 2000 20 2 200 10 50 using minmin .
result NzNat :  1057
```

Examples: Insertion sort

- Arrays are represented as sets of pairs and there is a rule to switch the values in two positions of the array.

```
mod SORTING is
  protecting NAT .
  sorts Pair PairSet .
  subsort Pair < PairSet .
  op (_,_) : Nat Nat -> Pair .
  op empty : -> PairSet .
  op __ : PairSet PairSet -> PairSet [assoc comm id: empty] .
  op length : PairSet -> Nat .
  vars I J V W : Nat .  var PS : PairSet .
  eq length(empty) = 0 .
  eq length((I, V) PS ) = length(PS) + 1 .
  rl [switch] : (J, V) (I, W) => (J, W) (I, V) .
endm
```

Examples: Insertion sort

```
$Y := 2$
while $Y \leq N$ do
    $X := Y$
    while $X > 1 \ \wedge \ V[X-1] > V[X]$ do
        switch $V[X-1]$ and $V[X]$
        $X := X - 1$
    $Y := Y + 1$
```

Examples: Insertion sort

```
$Y := 2$
while $Y \leq N$ do
    $X := Y$
    while $X > 1 \ \wedge \ V[X-1] > V[X]$ do
        switch $V[X-1]$ and $V[X]$
        $X := X - 1$
    $Y := Y + 1$
```

```
smod INSERTION-SORT-STRAT is
  protecting SORTING .
  var PS : PairSet .   vars X Y V W : Nat .
  strat insort : Nat @ PairSet .
  sd insort(Y) := try(match PS s.t. Y <= length(PS) ;
                      insert(Y) ;  insort(Y + 1)) .

  strat insert : Nat @ PairSet .
  sd insert(1) := idle .
  csd insert(s(X)) := try(amatch (X, V) (s(X), W) s.t. V > W ;
                          switch[J <- X ; I <- s(X)] ;
                          insert(X))
                      if s(X) > 1 .
endsm
```

Examples: Generic backtracking strategy

Backtracking is a generic strategy useful for solving a problem.

```
sth BT-ELEMS is
  sort State .
  op isOk : State -> Bool .
  op isSolution : State -> Bool .
  strat expand @ State .
endsth

smod BT-STRAT{X :: BT-ELEMS} is
  var S : X$State .
  strat solve @ X$State .
  sd solve := (match S s.t. isSolution(S)) ? idle
                                           : (expand ;
                                              match S s.t. isOk(S) ;
                                              solve) .
endsm
```

Backtracking instantiation: Labyrinth

```
fmod POSITIONS is  protecting NAT .
  sort Pos .
  op [_,_] : Nat Nat -> Pos .
endfm

mod LABYRINTH is
  protecting LIST{Pos} .
  op contains : List{Pos} Pos -> Bool .
  ops isSolution isOk : List{Pos} -> Bool .
  op next : List{Pos} -> Pos .
  op wall : -> List{Pos} .
  vars X Y : Nat .    var P Q : Pos .    var L : List{Pos} .
  eq isSolution(L [8,8]) = true .
  eq isSolution(L) = false [owise] .
  eq contains(nil, P) = false .
  eq contains(Q L, P) = if P == Q then true else contains(L, P) fi .
  eq isOk(L [X,Y]) = X >= 1 and Y >= 1 and X <= 8 and Y <= 8
                       and not(contains(L, [X,Y])) and
                           not(contains(wall, [X,Y])) .
```

Backtracking instantiation: Labyrinth

```
  crl [extend] : L => L P if next(L) => P .

  rl [next] : next(L [X,Y]) => [X + 1, Y] .
  rl [next] : next(L [X,Y]) => [X, Y + 1] .
  rl [next] : next(L [X,Y]) => [sd(X, 1), Y] .
  rl [next] : next(L [X,Y]) => [X, sd(Y, 1)] .

  eq wall =       [2,1]
                  [2,2]
                  [2,3]       [4,3] [5,3] [6,3] [7,3] [8,3]

            [1,5] [2,5] [3,5] [4,5]             [7,5]
                                                [7,6]
                                                [7,7]
                                                [7,8]       .
endm
```

Backtracking instantiation: Labyrinth

```
smod LABYRINTH-STRAT is
  protecting LABYRINTH .
  strat expand @ List{Pos} .
  sd expand := top(extend{ next }) .
endsm

sview LABYRINTH-BT-ELEM from BT-ELEMS to LABYRINTH-STRAT is
  sort State to List{Pos} .
endsv

smod LABYRINTH-BT-STRAT is
  including BT-STRAT{LABYRINTH-BT-ELEM} .
endsm
```

Examples: River crossing

```
mod RIVER-CROSSING is
  sorts Side Group .
  ops left right : -> Side .
  op change : Side -> Side .
  ops s w g c : Side -> Group .
  op __ : Group Group -> Group [assoc comm] .
  op init : -> Group .
  eq change(left) = right .
  eq change(right) = left .
  eq init = s(left) w(left) g(left) c(left) .
  vars S S' : Side .

  crl [wolf-eats] : w(S) g(S) s(S') => w(S) s(S') if S =/= S' .
  crl [goat-eats] : c(S) g(S) s(S') => g(S) s(S') if S =/= S' .

  rl [shepherd-alone] : s(S) => s(change(S)) .
  rl [wolf] : s(S) w(S) => s(change(S)) w(change(S)) .
  rl [goat] : s(S) g(S) => s(change(S)) g(change(S)) .
  rl [cabbage] : s(S) c(S) => s(change(S)) c(change(S)) .
endm
```

Examples: River crossing

```
smod RIVER-CROSSING-STRAT is
  protecting RIVER-CROSSING .

  strat eating : @ Group .
  sd eating := (wolf-eats | goat-eats) ! .

  strat oneCrossing : @ Group .
  sd oneCrossing := shepherd-alone | wolf | goat | cabbage .

  strat allCE : @ Group .
  sd allCE := (eating ; oneCrossing) * .
endsm

srew init using allCE ; (match (s(right) w(right) g(right) c(right))) .
```

Rewriting semantics

Applying a strategy to a term is a task that, in the process of rewriting, can give rise to more tasks.

```
sorts Task Tasks .
subsort Task < Tasks .

op none : -> Tasks .
op __ : Tasks Tasks -> Tasks [assoc comm id: none] .
eq T:Task T:Task = T:Task .

op <_@_> : Strat Term -> Task .
```

Solved tasks are of the form sol(t), meaning that the term t is a solution.

```
op sol : Term -> Task .
```

Rewriting semantics

Sometimes, these solutions are only the intermediate results in a task that must be continued with another process; therefore, there is also some syntax to represent continuations as terms of sort Cont.

```
sort Cont .

op <_;_> : Tasks Cont -> Task .

op seq : Strat -> Cont .
op if : Strat Strat Term -> Cont .
op crl : Condition StratList Term Context -> Cont .
op mrew : Nat Variable TermStratList Term Substitution
          Context -> Cont .
```

Strategy syntax

```
sorts BasicStrat Strat .
subsorts BasicStrat < Strat .
sort  Test .
subsort Test < Strat .

op _[_] : Qid Substitution -> BasicStrat [prec 50] .
op _[_]{_} : Qid Substitution StratList -> BasicStrat [prec 50] .

op match : Term EqCondition -> Test .
op amatch : Term EqCondition -> Test .
```

Strategy syntax

```
ops idle fail : -> Strat .
op top : BasicStrat -> Strat .
op _|_ : Strat Strat -> Strat [prec 40 assoc comm] .
op _;_ : Strat Strat -> Strat [prec 50 assoc] .
op if : Strat Strat Strat -> Strat [strat(0)] .
op _orelse_ : Strat Strat -> Strat [prec 45 gather(e E)] .
op not : Strat -> Strat .
op _* : Strat -> Strat [prec 30 gather(e)] .
op _+ : Strat -> Strat [prec 30 gather(e)] .
op _! : Strat -> Strat [prec 30 gather(e)] .
op call : Term -> Strat . *** strat call
op matchrew : Term EqCondition TermStratList -> Strat .
op amatchrew : Term EqCondition TermStratList -> Strat .
```

Strategy rewrite rules

```
rl < idle @ T > => sol(T) .

rl < fail @ T > => none .

rl < L[Sb] @ T > => apply-everywhere(L, Sb, T, 0) .

ceq apply-everywhere(L, Sb, T, N)
    = sol(T') apply-everywhere(L, Sb, T, N + 1)
 if { T', Ty, Sb', CX } := metaXapply(MOD, T, L, Sb, 0, unbounded, N)

eq apply-everywhere(L, Sb, T, N) = none [owise] .

rl < top(L[Sb]) @ T > => apply-top(L, Sb, T, 0) .

ceq apply-top(L, Sb, T, N) = sol(T') apply-top(L, Sb, T, N + 1)
 if { T', Ty, Sb' } := metaApply(MOD, T, L, Sb, N) .

eq apply-top(L, Sb, T, N) = none [owise] .
```

Strategy rewrite rules

```
rl < L[Sb]{SL} @ T > => find-rules(L[Sb]{SL}, T, false) .
```

For each application of a rule
`crl [L] : LHS => RHS if T1 => T2 /\ CO .` to the term T,
`find-rules` produces a task like this one

```
    < < E @ T1 > ; crl(T1 => T2 /\ CO, E SL, RHS, CX) >
```

Solutions to the nested tasks are then handled

```
rl < sol(T) TASKS ; crl(T1 => T2 /\ CO, E SL, RHS, CX) >
=> find-matching(T, T2, CO, SL, RHS, CX, 0)
   < TASKS ; crl(T1 => T2 /\ CO, E SL, RHS, CX) > .

rl < none ; crl(T1 => T2 /\ CO, E SL, RHS, CX) > => none .
```

Strategy rewrite rules

```
rl < match(T',C) @ T > =>
   if metaMatch(MOD, T', T, C, 0) == noMatch
   then none
   else sol(T)
   fi .

rl < amatch(T',C) @ T > =>
   if metaXmatch(MOD, T', T, C, 0, unbounded, 0) == noMatch
   then none
   else sol(T)
   fi .
```

Strategy rewrite rules

```
rl  < E | E' @ T >  =>  < E @ T > < E' @ T > .

rl  < E ; E' @ T >  =>  < < E @ T > ; seq(E') > .

rl  < sol(T) TASKS ; seq(E') >  =>  < E' @ T > < TASKS ; seq(E') > .

rl  < none ; seq(E') >  =>  none .

rl  < E * @ T >  =>  sol(T) < E ; (E *) @ T > .

eq E + = E ; E *  .
```

Strategy rewrite rules

```
rl < if(E, E', E'') @ T >  =>  < < E @ T > ; if(E', E'', T) > .

rl < sol(T') TASKS ; if(E', E'', T) >
  =>  < E' @ T' > < TASKS ; seq(E') > .

rl < none ; if(E', E'', T) >  =>  < E'' @ T > .

eq E orelse E' = if(E, idle, E') .

eq not(E) = if(E, fails, idle) .

eq E ! = if(E, E !, idle) .

eq try(E) = if(E, idle, idle) .

eq test(E) = if(not(E), fails, idle) .
```

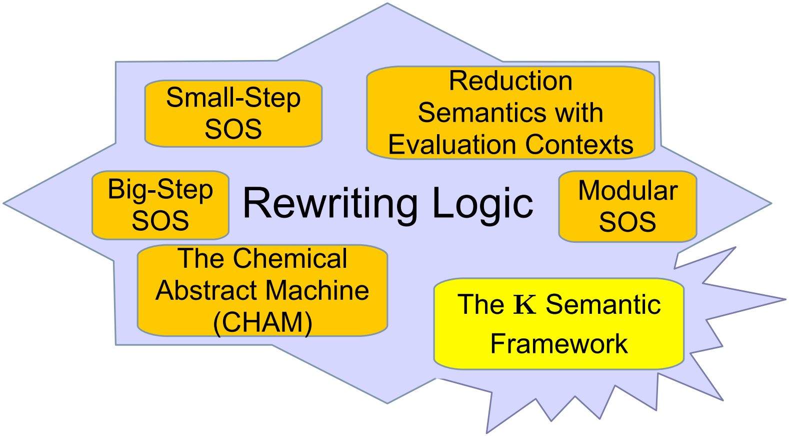
Strategy language for Maude

- The first applications of the strategy language included the operational semantics of process algebras such as CCS and the ambient calculus.
- Also the two-level operational semantics of the parallel functional programming language Eden and to modular structural operational semantics.
- It has also been used in the implementation of basic Knuth-Bendix completion algorithms and of congruence closure algorithms.
- Other applications include a sudoku solver, neural networks, membrane systems, hierarchical design graphs, and a representation of BPMN.

For more information on the strategy language for Maude and the complete module defining its rewriting semantics, see

`http://maude.sip.ucm.es/strategies`

K framework


Small-Step SOS
Reduction Semantics with Evaluation Contexts
Big-Step SOS
Rewriting Logic
Modular SOS
The Chemical Abstract Machine (CHAM)
The $\mathbb{K}$ Semantic Framework

K framework: ambitious features

- Methodology to define languages . . .
- . . . and type checkers, abstract interpreters, domain-specific checkers, etc.
- Arbitrarily complex language features
- Modular (crucial for scalability and reuse)
- Generic (multi-language and multi-paradigm)
- Support for non-determinism and concurrency
- Efficient executability
- State-exploration capabilities (e.g., finite-state model-checking)
- Formal semantics

K basics: computations

- K is based around concepts from Rewriting Logic Semantics, with some intuitions from Chemical Abstract Machines (CHAMs) and Reduction Semantics (RS).
- Abstract computational structures contain context needed to produce a future computation (like continuations).
- Computations take place in the context of a configuration.
- Configurations are hierarchical (like in RLS), made up of K cells.
- Each cell holds specific piece of information: computation, environment, store, etc.
- Two regularly used cells:
  - $\top$ (*top*), representing entire configuration
  - k, representing current computation
- Cells can be repeated (e.g., multiple computations in a concurrent language).

K basics: equations and rules

- Cell k is made up of a list of computational tasks separated by $\curvearrowright$, like $t_1 \curvearrowright t_2 \curvearrowright \ldots \curvearrowright t_n$.
- Intuition from CHAMs: language constructs can heat (break apart into pieces for evaluation) and cool (form back together).
- Represented using $\rightleftharpoons$, like $a_1 + a_2 \rightleftharpoons a_1 \curvearrowright \square + a_2$.
- A heating/cooling pair can be seen as an equation.
- Intuition: $\square$ can be seen as similar to evaluation contexts, marking the location where evaluation can occur.
- Computations are defined used equations and rules.
- Heating/cooling rules (structural equations): manipulate term structure, non-computational, reversible, can think of as just equations.
- Rules: computational, not reversible, may be concurrent.

Example: Cink

- Cink is a kernel of C:
  - functions
  - int expressions
  - simple input/output
  - basic flow control (if, if-else, while, sequential composition)
- Used for teaching purposes.
- Many other examples can be found on Google Code site for K project:
  `http://code.google.com/p/k-framework/`
- There is an online interface of the $\mathbb{K}$ tool
  `https://fmse.info.uaic.ro/tools/K/`

$\mathbb{K}$ Syntax: BNF syntax annotated with strictness

- Consider for example the annotated syntax definition for plus:

$$Exp ::= Exp + Exp \text{ [strict]}$$

- The strict attribute means the evaluation order is strict and non-deterministic
- This is achieved by two kinds of rules (similar to CHAM):
  - heating rules:

$$E_1 + E_2 \rightarrow E_1 \curvearrowright \square + E_2$$
$$E_1 + E_2 \rightarrow E_2 \curvearrowright E_1 + \square$$

  - cooling rules:

$$I_1 \curvearrowright \square + E_2 \rightarrow I_1 + E_2$$
$$I_2 \curvearrowright E_1 + \square \rightarrow E_1 + I_2$$

$\mathbb{K}$ computations and $\mathbb{K}$ syntax

Computations

- Extend the programming language syntax with a "task sequentialization" operation
  - $t_1 \curvearrowright t_2 \curvearrowright \ldots \curvearrowright t_n$, where t_i are computational tasks
- Computational tasks: pieces of syntax (with holes), closures, ...

$\mathbb{K}$ Syntax: BNF syntax annotated with strictness

$$
\begin{array}{lrl}
\mathit{Exp} ::= \mathit{Id} & & \\
\quad \mid \ \mathit{Exp} + \mathit{Exp} & \text{[strict]} & \mathit{ERed} + \mathit{ERed} \ \rightarrow \ \mathit{ERed} \curvearrowright \square + \mathit{ERed} \\
\quad \mid \ \mathit{Exp} = \mathit{Exp} & \text{[strict(2)]} & E = \mathit{ERed} \ \rightarrow \ \mathit{ERed} \curvearrowright E = \square \\
\mathit{Stmt} ::= \mathit{Exp}\ ; & \text{[strict]} & \mathit{ERed}\ ; \ \rightarrow \ \mathit{ERed} \curvearrowright \square\ ; \\
\quad \mid \ \mathit{Stmt}\ \ \mathit{Stmt} & \text{[seqstrict]} & \mathit{SRed}\ \ S \ \rightarrow \ \mathit{SRed} \curvearrowright \square\ \ S
\end{array}
$$

Heating syntax through strictness rules

Computation

y = x+2 ; x = 7 ;

$\mathbb{K}$ Syntax: BNF syntax annotated with strictness

$\mathit{Exp} ::= \mathit{Id}$			
$\mid\ \mathit{Exp} + \mathit{Exp}$	[strict]	$\mathit{ERed} + \mathit{ERed} \rightarrow \mathit{ERed} \curvearrowright \square + \mathit{ERed}$	
$\mid\ \mathit{Exp} = \mathit{Exp}$	[strict(2)]	$E = \mathit{ERed} \rightarrow \mathit{ERed} \curvearrowright E = \square$	
$\mathit{Stmt} ::= \mathit{Exp}\ ;$	[strict]	$\mathit{ERed}\ ; \rightarrow \mathit{ERed} \curvearrowright \square\ ;$	
$\mid\ \mathit{Stmt}\ \mathit{Stmt}$	[seqstrict]	$\mathit{SRed}\ S \rightarrow \mathit{SRed} \curvearrowright \square\ S$	

Heating syntax through strictness rules

Computation

$$\mathsf{y = x{+}2\ ;} \curvearrowright \square\ \mathsf{x = 7\ ;}$$

$\mathbb{K}$ Syntax: BNF syntax annotated with strictness

$$\begin{array}{llrlcl}
\mathit{Exp} ::= \mathit{Id} & & & & & \\
\mid & \mathit{Exp} + \mathit{Exp} & [\text{strict}] & \mathit{ERed} + \mathit{ERed} & \rightarrow & \mathit{ERed} \curvearrowright \square + \mathit{ERed} \\
\mid & \mathit{Exp} = \mathit{Exp} & [\text{strict}(2)] & E = \mathit{ERed} & \rightarrow & \mathit{ERed} \curvearrowright E = \square \\
\mathit{Stmt} ::= & \mathit{Exp}\ ; & [\text{strict}] & \mathit{ERed}\ ; & \rightarrow & \mathit{ERed} \curvearrowright \square\ ; \\
\mid & \mathit{Stmt}\ \ \mathit{Stmt} & [\text{seqstrict}] & \mathit{SRed}\ \ S & \rightarrow & \mathit{SRed} \curvearrowright \square\ \ S
\end{array}$$

Heating syntax through strictness rules

Computation

$y = x{+}2 \curvearrowright \square ; \curvearrowright \square \;\; x = 7 \;;$

$\mathbb{K}$ Syntax: BNF syntax annotated with strictness

$Exp ::= Id$		
$\mid\ Exp + Exp$	[strict]	$ERed + ERed \rightarrow ERed \curvearrowright \square + ERed$
$\mid\ Exp = Exp$	[strict(2)]	$E = ERed \rightarrow ERed \curvearrowright E = \square$
$Stmt ::= Exp\ ;$	[strict]	$ERed\ ; \rightarrow ERed \curvearrowright \square\ ;$
$\mid\ Stmt\ \ Stmt$	[seqstrict]	$SRed\ \ S \rightarrow SRed \curvearrowright \square\ \ S$

Heating syntax through strictness rules

Computation

$x + 2 \curvearrowright y = \square \curvearrowright \square ; \curvearrowright \square \quad x = 7 \;;$

$\mathbb{K}$ Syntax: BNF syntax annotated with strictness

$Exp ::= Id$		
$\mid\ Exp + Exp$	[strict]	$ERed + ERed \rightarrow ERed \curvearrowright \square + ERed$
$\mid\ Exp = Exp$	[strict(2)]	$E = ERed \rightarrow ERed \curvearrowright E = \square$
$Stmt ::= Exp\ ;$	[strict]	$ERed\ ; \rightarrow ERed \curvearrowright \square\ ;$
$\mid\ Stmt\ \ Stmt$	[seqstrict]	$SRed\ \ S \rightarrow SRed \curvearrowright \square\ \ S$

Heating syntax through strictness rules

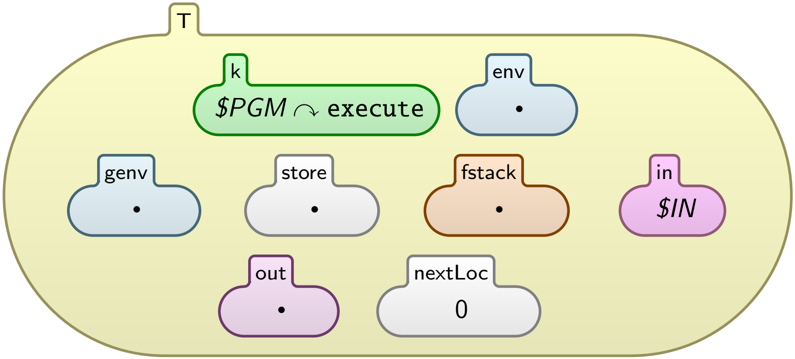
Computation

$x \curvearrowright \square + 2 \curvearrowright y = \square \curvearrowright \square ; \curvearrowright \square \;\; x = 7 \;;$

$\mathbb{K}$ Syntax: BNF syntax annotated with strictness

$Exp ::= Id$			
$\mid\; Exp + Exp$	[strict]	$ERed + ERed \rightarrow ERed \curvearrowright \square + ERed$	
$\mid\; Exp = Exp$	[strict(2)]	$E = ERed \rightarrow ERed \curvearrowright E = \square$	
$Stmt ::= Exp\ ;$	[strict]	$ERed\ ; \rightarrow ERed \curvearrowright \square\ ;$	
$\mid\; Stmt\ \ Stmt$	[seqstrict]	$SRed\ \ S \rightarrow SRed \curvearrowright \square\ \ S$	

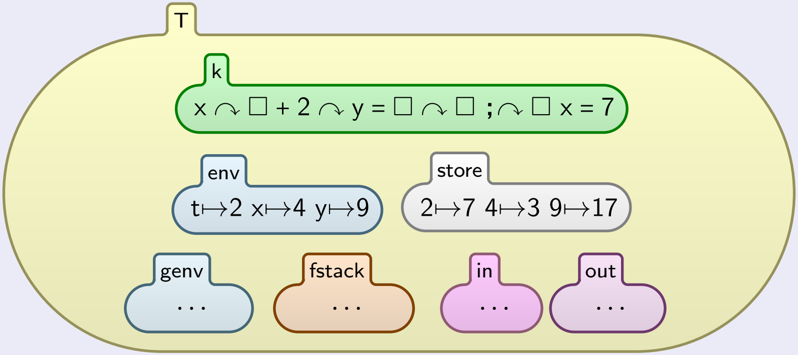
Cink Configuration



𝕂 rules: expressing natural language into rules

Reading from store via environment

If a variable X is the next thing to be processed and
if X is mapped to a location L in the environment and L to a value V in the store
then process X, replacing it by V.

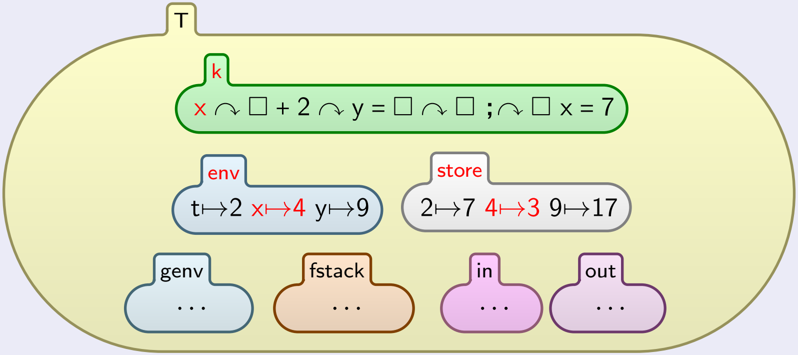


𝕂 rules: expressing natural language into rules

Focusing on the relevant part

Reading from store via environment

If a variable X is the next thing to be processed and
if X is mapped to a location L in the environment and L to a value V in the store
then process X, replacing it by V.

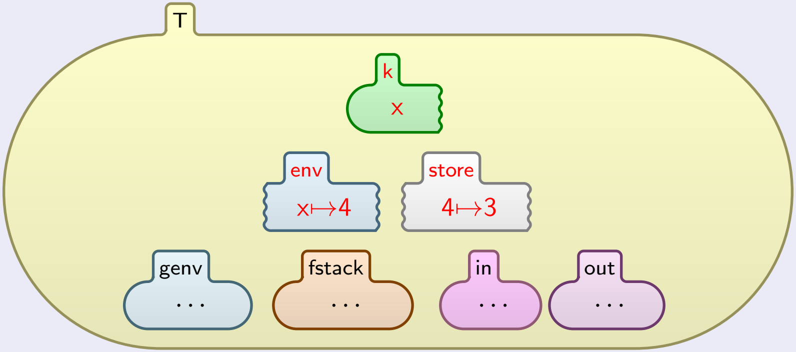


𝕂 rules: expressing natural language into rules

Unnecessary parts of the cells are abstracted away

Reading from store via environment

If a variable X is the next thing to be processed and
if X is mapped to a location L in the environment and L to a value V in the store
then process X, replacing it by V.

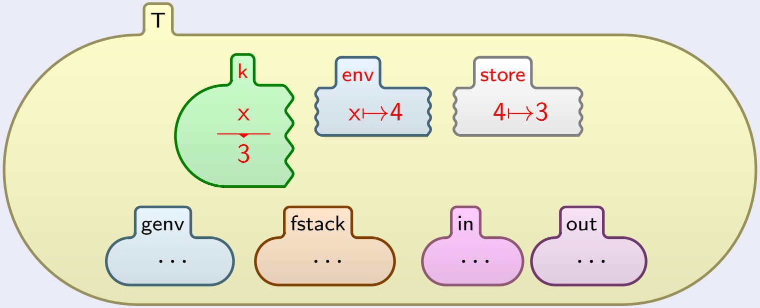


𝕂 rules: expressing natural language into rules

Underlining what to replace, writing the replacement under the line

Reading from store via environment

If a variable X is the next thing to be processed and
if X is mapped to a location L in the environment and L to a value V in the store
then process X, replacing it by V.

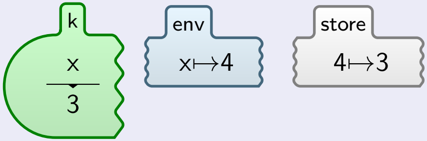


𝕂 rules: expressing natural language into rules

Configuration abstraction: Keep only the relevant cells

Reading from store via environment

If a variable X is the next thing to be processed and
if X is mapped to a location L in the environment and L to a value V in the store
then process X, replacing it by V.

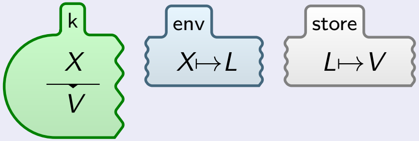


𝕂 rules: expressing natural language into rules

Generalize the concrete instance

Reading from store via environment

If a variable X is the next thing to be processed and
if X is mapped to a location L in the environment and L to a value V in the store
then process X, replacing it by V.



This is called context abstraction

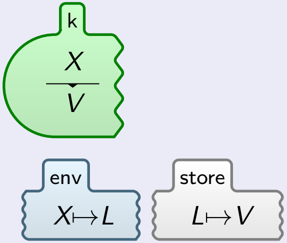
𝕂 rules: expressing natural language into rules

Voilà!

Reading from store via environment

If a variable X is the next thing to be processed and
if X is mapped to a location L in the environment and L to a value V in the store
then process X, replacing it by V.

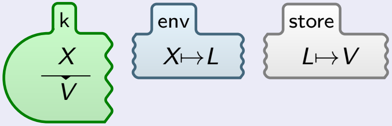
graphic

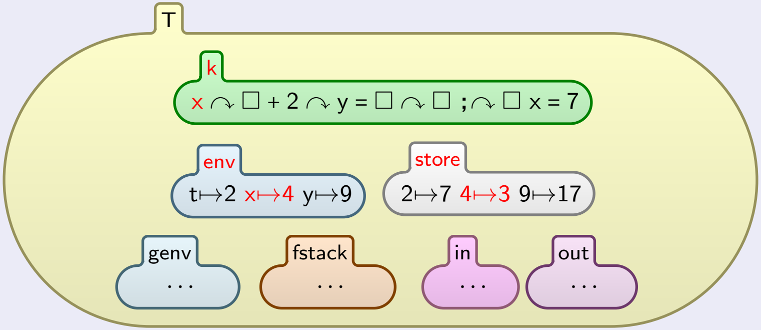


ASCII

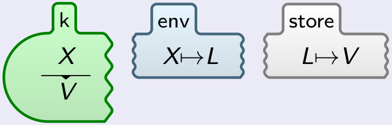
```
rule [lookup]:
     <k> X => V ...</k>
     <env>...  X |-> L ...</env>
     <store>...  L |-> V ...</store>
```

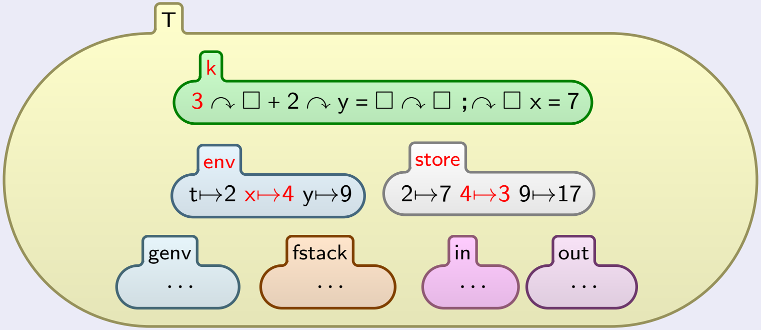
Rules at Work





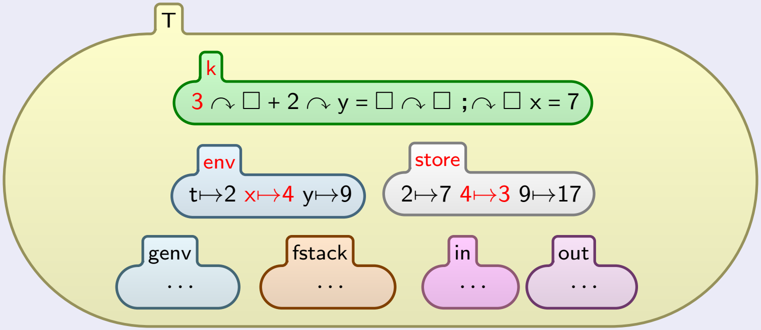
Rules at Work





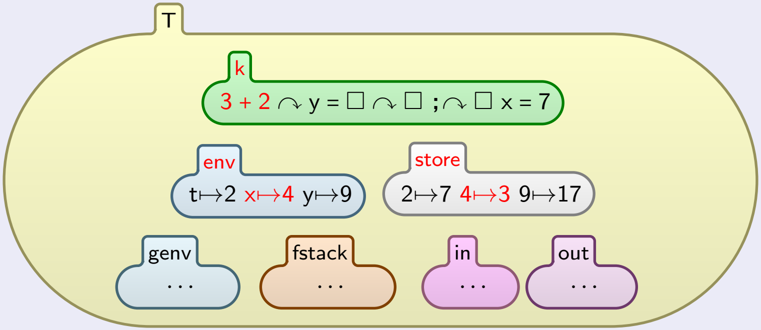
Rules at Work

$ERed \curvearrowright \square + ERed \rightarrow ERed + ERed$



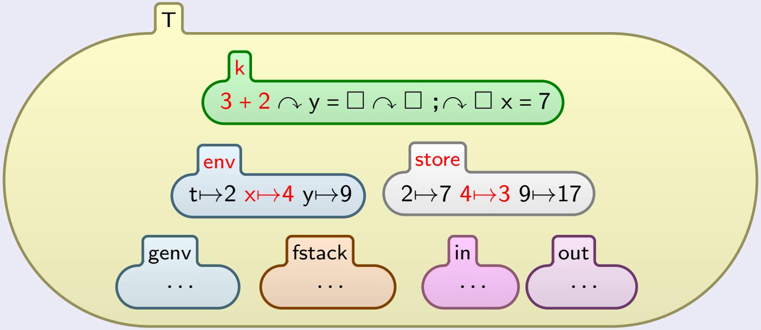
Rules at Work

$\mathit{ERed} \curvearrowright \square + \mathit{ERed} \rightarrow \mathit{ERed} + \mathit{ERed}$



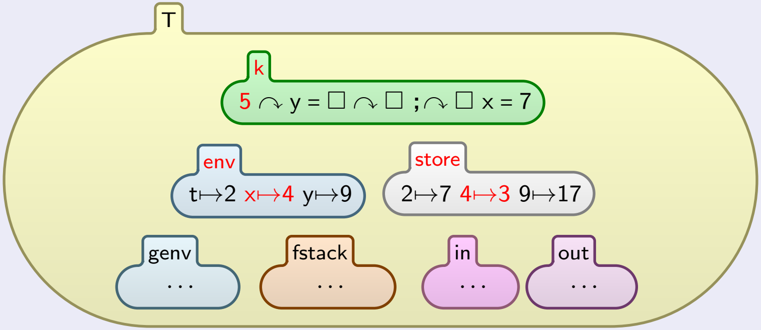
Rules at Work

$I1 + I2 \rightarrow I1 +_{Int} I2$



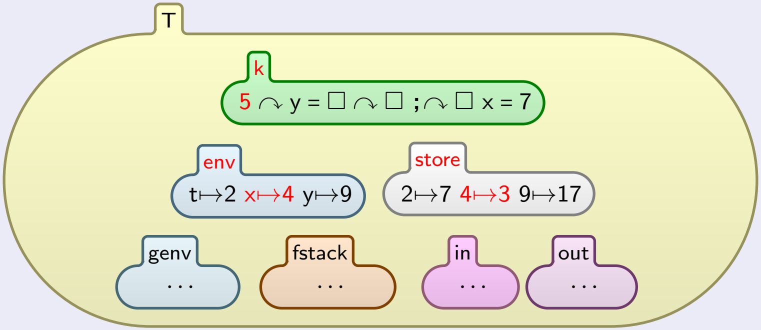
Rules at Work

$I1 + I2 \rightarrow I1 +_{Int} I2$



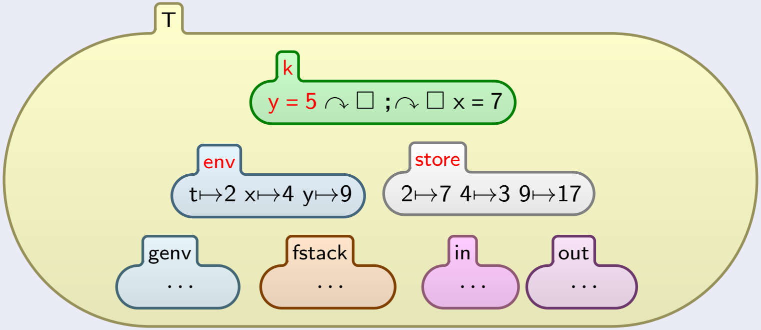
Rules at Work

$ERed \curvearrowright E = \square \rightarrow E = ERed$

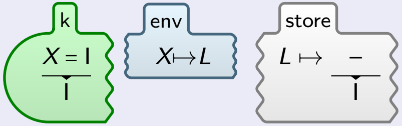


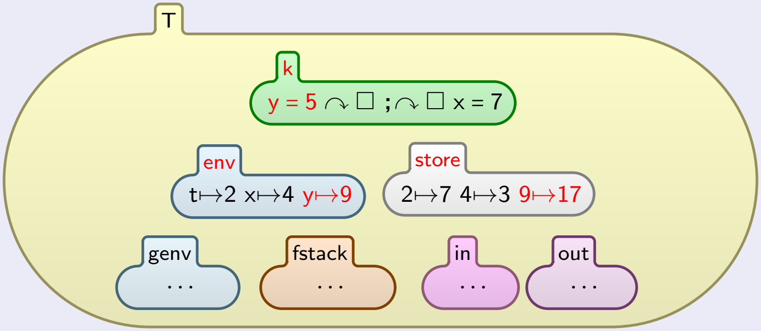
Rules at Work

$ERed \curvearrowright E = \square \to E = ERed$

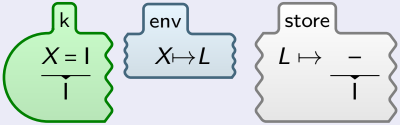


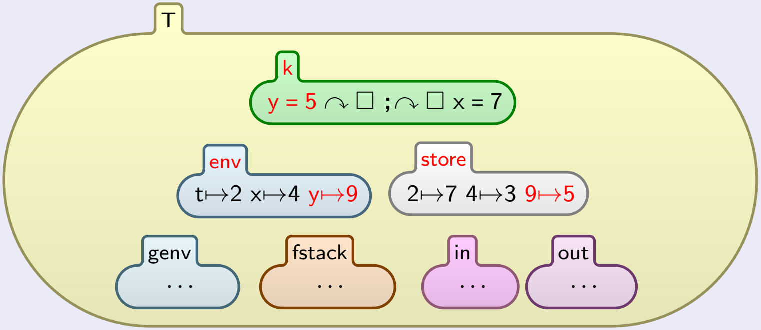
Rules at Work



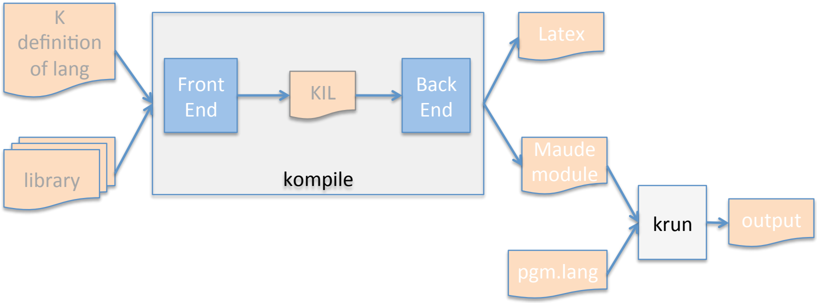


Rules at Work





K tool


K definition of lang
library
Front End
KIL
Back End
kompile
Latex
Maude module
pgm.lang
krun
output

K tool features

- Under continuous development (so still moving target!)
- Compilation of K definitions into Maude and LaTeX
- Literate programming: nicely structured printing of semantics definitions
- Efficient and interactive execution (prototype interpreters)
- Analysis through state-space exploration (from Maude):
  - search
  - model checking
- Further options to control nondeterminism

K framework: some accomplishments

- Embeddings of different operational semantics styles in K.
- Semantics of many languages, toy and real.
- Complete semantics for C (recent paper at POPL 2012).
- Implementation in Maude: K tool and K-Maude.
- Static checking of units of measurement in C.
- Runtime verification of C memory safety.
- Definition of type systems and type inference.
- Compilation of language definitions into competitive interpreters (in OCaml).

Unification

- Given terms t and u, we say that t and u are unifiable if there is a substitution σ such that $\sigma(t) \equiv \sigma(u)$.
- Given an equational theory A and terms t and u, we say that t and u are unifiable modulo A if there is a substitution σ such that $\sigma(t) \equiv_A \sigma(u)$.
- Maude 2.4 supports at the core level and at the metalevel order-sorted equational unification modulo combinations of `comm` and `assoc comm` attributes as well as free function symbols.

Narrowing

- A term t narrows to a term t' using a rule $l \Rightarrow r$ in R and a substitution σ if
  1. there is a subterm $t|_p$ of t at a nonvariable position p of t such that l and $t|_p$ are unifiable via σ, and
  2. $t' = \sigma(t[r]_p)$ is obtained from $\sigma(t)$ by replacing the subterm $\sigma(t|_p) \equiv \sigma(l)$ with the term $\sigma(r)$.
- Narrowing can also be defined modulo an equational theory A.
- Full Maude (2.4 and later) supports a version of narrowing modulo with simplification, where each narrowing step with a rule is followed by simplification to canonical form with the equations.
- There are some restrictions on the allowed rules; for example, they cannot be conditional.

Narrowing reachability analysis

Narrowing can be used as a general deductive procedure for solving reachability problems of the form

$$(\exists \vec{x})\ t_1(\vec{x}) \rightarrow t_1'(\vec{x})\ \wedge \ldots \wedge\ t_n(\vec{x}) \rightarrow t_n'(\vec{x})$$

in a given rewrite theory.

- The terms t_i and t_i' denote sets of states.
- For what subset of states denoted by t_i are the states denoted by t_i' reachable?
- No finiteness assumptions about the state space.
- Sound and complete for topmost rewrite theories.