Semantics of Programming Languages
Assignment 2 & 3

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Exercise 1. *Assume we extend the syntax of While statetments with a new construct:* $\texttt{repeat}\ S\ \texttt{until}\ b$*. This statement is executed as follows:*

1. *Execute S.*
2. *Check whether b is false. In this case, step back to 1). Otherwise, finish.*

Define the big-step and small-step semantic rules for this new construct. You cannout rely on the rules of `while` *to define the rules of* `repeat`*. Finally, prove that* $\texttt{repeat}\ S\ \texttt{until}\ b$ *is equivalent to* $(S; \texttt{while}\ \neg b\ \texttt{do}\ S)$

Solution. **Big-step semantic.**

$$[\text{repeatT}_{\text{BS}}]\quad \frac{\langle S, s\rangle \Downarrow \sigma' \qquad \mathcal{B}[\![b]\!]\ \sigma' = \textbf{true}}{\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle \Downarrow \sigma'}$$

$$[\text{repeatF}_{\text{BS}}]\quad \frac{\langle S, s\rangle \Downarrow \sigma' \qquad \langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma'\rangle \Downarrow \sigma'' \qquad \mathcal{B}[\![b]\!]\ \sigma' = \textbf{false}}{\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle \Downarrow \sigma''}$$

Small-step semantic.

$$[\text{repeat}_{\text{SS}}]\quad \frac{}{\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle \rightarrow \langle S; \texttt{if}\ b\ \texttt{then skip else}\ (\texttt{repeat}\ S\ \texttt{until}\ b), \sigma\rangle}$$

Equivalence with $S; \texttt{while}\ \neg b\ \texttt{do}\ S$ **(small-step).**

We will prove that

$$\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle \rightarrow^* \sigma' \Longleftrightarrow \langle S; \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \rightarrow^* \sigma'$$

$\Rightarrow$) We suppose

$$\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle \rightarrow^* \sigma'$$

The implication is trivial for the case

$$\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle \rightarrow \sigma'$$

(one step) because is impossible to be true.

Otherwise

$$\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle \rightarrow^n \sigma'$$

for $n > 1$ and we suppose implication is true for $k < n$. Then exists a chain of deduction that holds

$$\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle \rightarrow \langle S_1, \sigma_1\rangle \rightarrow \ldots \rightarrow \sigma'$$

Only we can apply $[\text{repeat}_{\text{SS}}]$ rule, so

$$\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle \rightarrow \langle S; \texttt{if}\ b\ \texttt{then skip else}\ (\texttt{repeat}\ S\ \texttt{until}\ b), \sigma\rangle \rightarrow \ldots \rightarrow \sigma'$$

As consequence of *lemma* (2) we have $\sigma'' \in \textbf{State}$ such that

$$\langle S, \sigma\rangle \rightarrow^{k_1} \sigma''$$

and

$$\langle \texttt{if}\ b\ \texttt{then skip else}\ (\texttt{repeat}\ S\ \texttt{until}\ b), \sigma''\rangle \rightarrow^{k_2} \sigma' \quad (*)$$

We have two cases (depending on whether it has been deducted with $[\text{If1}_{\text{SS}}]$ or $[\text{If2}_{\text{SS}}]$)

- [If1$_{\text{SS}}$], then $\mathcal{B}[\![b]\!]\ \sigma'' = \mathbf{true}$ and

$$\begin{aligned}\langle \texttt{if}\ b\ \texttt{then skip else (repeat}\ S\ \texttt{until}\ b), \sigma''\rangle &\xrightarrow{\text{If1}_{\text{SS}}} \langle \texttt{skip}, \sigma''\rangle \\ &\xrightarrow{\text{skip}_{\text{SS}}} \sigma''\end{aligned}$$

  As we can only apply [skip$_{\text{SS}}$] we can deduce that $\sigma' = \sigma''$,

$$\langle S, \sigma\rangle \rightarrow^{k_1} \sigma'$$

  and $\mathcal{B}[\![b]\!]\ \sigma' = \mathbf{true}$. Now applying *lemma* (1)

$$\begin{aligned}\langle S; \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle &\xrightarrow{lemma2} \langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma'\rangle \\ &\xrightarrow{\text{WhileF}_{\text{SS}}} \sigma'\end{aligned}$$

  since $\mathcal{B}[\![\neg b]\!]\ \sigma' = \neg\mathcal{B}[\![b]\!]\ \sigma' = \neg\mathbf{true} = \mathbf{false}$.

- [If2$_{\text{SS}}$], then $\mathcal{B}[\![b]\!]\ \sigma'' = \mathbf{false}$. As it can only be applied [repeatF$_{\text{SS}}$] then

$$\begin{aligned}\langle \texttt{if}\ b\ \texttt{then skip else (repeat}\ S\ \texttt{until}\ b), \sigma''\rangle &\xrightarrow{\text{If2}_{\text{SS}}} \langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma''\rangle \\ &\xrightarrow{k_2-1} \sigma'\end{aligned}$$

  As a result of the induction hypothesis ($k_2 - 1 < n$) we have

$$\langle S; \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma''\rangle \rightarrow^{*} \sigma'$$

  then, since $\langle S, \sigma\rangle \rightarrow^{k_1} \sigma''$

$$\begin{aligned}\langle S; \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle &\longrightarrow^{k_1} \langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle'' \\ &\xrightarrow{\text{WhileF}_{\text{SS}}} \langle S; \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma''\rangle \\ &\xrightarrow{ind.hyp.}{}^{*} \sigma'\end{aligned}$$

  as we want.

$\Leftarrow$) Again, implication is trivial for derivation sequence of lenght one (because not exists derivation). We suppose that

$$\langle S; \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \rightarrow^{n} \sigma'$$

with $n > 1$ and implication valid for all $k < n$. We know that exists $k_1, k_2 < n$ and σ'' such that

$$\langle S, \sigma\rangle \rightarrow^{k_1} \sigma''$$

and

$$\langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma''\rangle \rightarrow^{k_2} \sigma'$$

Then we have two cases:

- $\mathcal{B}[\![\neg b]\!]\ \sigma'' = \mathbf{false}$ so the applied rule can only be [While1$_{\text{SS}}$]

$$\langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma''\rangle \rightarrow \sigma''$$

  As result, $\sigma'' = \sigma'$. After that

$$\begin{aligned}\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle &\xrightarrow{\text{repeat}_{\text{SS}}} \langle S; \texttt{if}\ b\ \texttt{then skip else (repeat}\ S\ \texttt{until}\ b), \sigma\rangle \\ &\xrightarrow{lemma1} \langle \texttt{if}\ b\ \texttt{then skip else (repeat}\ S\ \texttt{until}\ b), \sigma'\rangle \\ &\xrightarrow{\text{If1}_{\text{SS}}} \langle \texttt{skip}, \sigma'\rangle \\ &\xrightarrow{\text{skip}_{\text{SS}}} \sigma'\end{aligned}$$

  hence proved.

- $\mathcal{B}[\![\neg b]\!]\ \sigma'' = \textbf{true}$ so the applied rule can only be [While2$_{\text{SS}}$]

$$\langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma''\rangle \to \langle S; \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma''\rangle$$

It follows that

$$\langle S; \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma''\rangle \to^{k_2-1} \sigma'$$

As a result of the induction hypothesis,

$$\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma''\rangle \to^{*} \sigma'$$

Then

$$\begin{aligned}
\langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma\rangle &\xrightarrow{\text{repeat}_{\text{SS}}} \langle S; \texttt{if}\ b\ \texttt{then skip else}\ (\texttt{repeat}\ S\ \texttt{until}\ b), \sigma\rangle \\
&\xrightarrow{\mathit{lemma1}} \langle \texttt{if}\ b\ \texttt{then skip else}\ (\texttt{repeat}\ S\ \texttt{until}\ b), \sigma''\rangle \\
&\xrightarrow{\text{If2}_{\text{SS}}} \langle \texttt{repeat}\ S\ \texttt{until}\ b, \sigma''\rangle \\
&\xrightarrow{\mathit{ind.hyp.}} \sigma'
\end{aligned}$$

as desired.

Exercise 2. *Add the following iterative construct to While : $\texttt{for}\ x := e_1\ \texttt{to}\ e_2\ \texttt{do}\ S$. Define its big-step and small-step semantic rules. You cannot rely on the* while *or* repeat *construct to do this exercise.*

Solution. First of all, we define $\mathcal{A}^{-1}[\![\_]\!] : \mathbb{N} \mapsto \textbf{Aexp}$ by

$$\mathcal{A}^{-1}[\![n]\!] = \text{n}$$

for all $n \in \mathbb{N}$.

If we are interested in that execution of S won't generate collateral effects on the number of iterations, thanks to this definition we can define:

Big-step semantic.

$$[\text{for1}^{\text{nat}}_{\text{BS}}]\ \frac{n_1 \in \mathbb{N} \wedge n_2 \in \mathbb{N} \wedge n_1 > n_2}{\langle \texttt{for}\ x := n_1\ \texttt{to}\ n_2\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma}$$

$$[\text{for2}^{\text{nat}}_{\text{BS}}]\ \frac{n_1 \in \mathbb{N} \wedge n_2 \in \mathbb{N} \wedge n_1 \leq n_2 \qquad \langle S, \sigma[x \mapsto n_1]\rangle \Downarrow \sigma' \qquad \langle \texttt{for}\ x := \mathcal{A}^{-1}[\![n_1+1]\!]\ \texttt{to}\ \mathcal{A}^{-1}[\![n_2]\!]\ \texttt{do}\ S, \sigma'\rangle \Downarrow \sigma''}{\langle \texttt{for}\ x := n_1\ \texttt{to}\ n_2\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma''}$$

$$[\text{for}^{\text{aexp}}_{\text{BS}}]\ \frac{e_1 \notin \mathbb{N} \vee e_2 \notin \mathbb{N} \qquad \langle \texttt{for}\ x := \mathcal{A}^{-1}[\![\mathcal{A}[\![e_1]\!]\ \sigma]\!]\ \texttt{to}\ \mathcal{A}^{-1}[\![\mathcal{A}[\![e_2]\!]\ \sigma]\!]\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'}{\langle \texttt{for}\ x := e_1\ \texttt{to}\ e_2\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'}$$

Small-step semantic.

$$[\text{for1}_{\text{SS}}]\ \frac{\mathcal{A}[\![e_1]\!]\ \sigma > \mathcal{A}[\![e_2]\!]\ \sigma}{\langle \texttt{for}\ x := e_1\ \texttt{to}\ e_2\ \texttt{do}\ S, \sigma\rangle \to \sigma}$$

$$[\text{for2}_{\text{SS}}]\ \frac{\mathcal{A}[\![e_1]\!]\ \sigma \leq \mathcal{A}[\![e_2]\!]\ \sigma}{\langle \texttt{for}\ x := e_1\ \texttt{to}\ e_2\ \texttt{do}\ S, \sigma\rangle \to \langle S; \texttt{for}\ x := \mathcal{A}^{-1}[\![\mathcal{A}[\![e_1]\!]\ \sigma + 1]\!]\ \texttt{to}\ \mathcal{A}^{-1}[\![\mathcal{A}[\![e_2]\!]\ \sigma]\!]\ \texttt{do}\ S, \sigma[x \mapsto \mathcal{A}[\![e_1]\!]\sigma]\rangle}$$

On the other hand we can define (for example, in big-step semantics)

Big-step semantic.

$$[\text{for}1^{\text{nat}}_{\text{BS}}] \quad \frac{\mathcal{A}[\![e_1]\!]\ \sigma > \mathcal{A}[\![e_2]\!]\ \sigma}{\langle \texttt{for}\ x := e_1\ \texttt{to}\ e_2\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma}$$

$$[\text{for}2^{\text{nat}}_{\text{BS}}] \quad \frac{\mathcal{A}[\![e_1]\!]\ \sigma \leq \mathcal{A}[\![e_2]\!]\ \sigma \qquad \langle S, \sigma[x \mapsto \mathcal{A}[\![e_1]\!]\ \sigma]\rangle \Downarrow \sigma' \qquad \langle \texttt{for}\ x := e_1 + 1\ \texttt{to}\ e_2\ \texttt{do}\ S, \sigma'\rangle \Downarrow \sigma''}{\langle \texttt{for}\ x := e_1\ \texttt{to}\ e_2\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma''}$$

And we can see that

```
y := 1
z := 0
for x := y to 10 do
    z := z + 1
    y := 10
```

has different interpretation if we apply the first and the last model. For the first model we receive a **State** that holds $\sigma(z) = 11$, but with the second model we receive $\sigma(z) = 1$. Both models are well-defined but have different behaviours.

The first model is the usual and expected behaviour of the for statement but the second is clever.