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Assignments 2

Exercise 2

Assume we extend the syntax of *While* statements with a new construct: $\texttt{repeat}\ S\ \texttt{until}\ b$. This statement is executed as follows:

(1) Execute S.

(2) Check whether b is false. In this case, step back to (1). Otherwise, finish.

Define the big-step and small-step semantic rules for this new construct. You cannot rely on the rules of `while` to define the rules of `repeat`. Finally, prove that $\texttt{repeat}\ S\ \texttt{until}\ b$ is equivalent to (S; $\texttt{while}\ \neg b\ \texttt{do}\ S$)

Definition of big-step and small-step semantic rules

Let $b \in BExp$ and $S \in Stm$ be. We can define the big-step and small-step semantic rules for $\texttt{repeat}\ S\ \texttt{until}\ b$ constructor as follow:

(a) **Big-step rules**.

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma' \qquad \mathcal{B}[\![b]\!]\sigma' = true}{\langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma\rangle \Downarrow \sigma'}\ [\text{UntilT}_{BS}]$$

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$$\frac{\langle S, \sigma\rangle \Downarrow \sigma'' \qquad \mathcal{B}[\![b]\!]\sigma'' = false \qquad \langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma''\rangle \Downarrow \sigma'}{\langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma\rangle \Downarrow \sigma'}\ [\text{UntilF}_{BS}]$$

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(b) **Small-step rules**. The small step semantics is defined by rewriting steps.

$$\frac{}{\langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma\rangle \longrightarrow \langle S;\ \texttt{if}\ b\ \texttt{then skip else repeat}\ S\ \texttt{until}\ b\ , \sigma\rangle}\ [\text{Until}_{SS}]$$

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Proof the equivalence

In order to proof that both expression in the extension of while semantic are equivalent, we need to see

$$\langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma\rangle \Downarrow \sigma' \iff \langle S;\ \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'$$

For this purpose, we will prove a base case. Then we will continue by applying rule-induction.

Proof. $\Rightarrow$

Base case.

Let's consider a statement $S \in Stm$, a boolean expression $b \in BExp$ and any states $\sigma, \sigma' \in State$. The base case is when the semantic denotational of b is true for any σ'. We assume that

$$(S;\ \texttt{while}\ \neg b\ \texttt{do}\ S)$$

?

you meant repeat S until b?

holds. Therefore we know that there is a derivation tree for it:

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma' \qquad \mathcal{B}[\![b]\!]\sigma' = true}{\langle \texttt{repeat } S \texttt{ until } b\ , \sigma\rangle \Downarrow \sigma'}\ [\text{UntilT}_{BS}] \tag{1}$$

We can rewrite $\mathcal{B}[\![b]\!]\sigma' = true$ to $\mathcal{B}[\![\neg b]\!]\sigma' = false$ and reshape the $WhileF_{BS}$ rule as follow:

$$\frac{\mathcal{B}[\![\neg b]\!]\sigma = false}{\langle \texttt{while } \neg b \texttt{ do } S, \sigma\rangle \Downarrow \sigma}\ [\text{WhileF}_{BS}] \tag{2}$$

Therefore, using the same assertions in (1) and appliying the Seq_{BS} rule and (2), we obtain:

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma' \qquad \dfrac{\mathcal{B}[\![\neg b]\!]\sigma' = false}{\langle \texttt{while } \neg b \texttt{ do } S, \sigma'\rangle \Downarrow \sigma'}\ [\text{WhileF}_{BS}]}{\langle S\texttt{; while } \neg b \texttt{ do } S, \sigma\rangle \Downarrow \sigma'}\ [\text{Seq}_{BS}]$$

What? The base case?

That will be our induction hypothesis (IH). Now, we have to prove the inductive case, i.e when the semantic denotational of the boolean expression b is false. In this case and being rigorous, we can consider that exist a knowed $k \in \mathbb{N}$ such that $\mathcal{B}[\![b]\!]\sigma_k = true$ and before that $\mathcal{B}[\![b]\!]\sigma_i = false$ for all $i \in \mathbb{N}$ with $i \leq k$.

Inductive case.

Let $S \in Stm$, $b \in BExp$, $\sigma, \sigma' \in State$ be and let $k \in \mathbb{N}$ be a natural number such that $\mathcal{B}[\![b]\!]\sigma_k = true$ and $\mathcal{B}[\![b]\!]\sigma_i = false$ for all $i \in \mathbb{N}$ with $i \leq k$.

We can assume that exist a derivation tree with root

$$\langle \texttt{repeat } S \texttt{ until } b\ , \sigma\rangle \Downarrow \sigma'$$

such that is obtained by derivating and apliying $UntilF_{BS}$ in all T_i subtree (for each i-big-step) and finally we obtain:

$$\cfrac{\cfrac{\langle S, \sigma\rangle \Downarrow \sigma_1 \qquad \mathcal{B}[\![b]\!]\sigma_1 = false \qquad \langle \texttt{repeat } S \texttt{ until } b\ , \sigma_1\rangle \Downarrow \sigma_2}{\langle \texttt{repeat } S \texttt{ until } b\ , \sigma\rangle \Downarrow \sigma_2}\ [\text{UntilF}_{BS}]}{\vdots}$$

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma_k \qquad \mathcal{B}[\![b]\!]\sigma_k = true \qquad \langle \texttt{repeat } S \texttt{ until } b\ , \sigma_k\rangle \Downarrow \sigma'}{}$$

How we have obtained both $\langle S, \sigma\rangle \Downarrow \sigma_k$ and $\mathcal{B}[\![\neg b]\!]\sigma_k = false$ ($\mathcal{B}[\![b]\!]\sigma_k = true$) assertions. By IH results:

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma_k \qquad \mathcal{B}[\![\neg b]\!]\sigma_k = false}{\langle S\texttt{; while } \neg b \texttt{ do } S, \sigma\rangle \Downarrow \sigma'}\ [\text{ IH }]$$

Therefore

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma_k \qquad \mathcal{B}[\![\neg b]\!]\sigma_k = false \qquad \langle \texttt{repeat } S \texttt{ until } b\ , \sigma_k\rangle \Downarrow \sigma'}{\langle S\texttt{; while } \neg b \texttt{ do } S, \sigma\rangle \Downarrow \sigma'}\ [\text{ IH }]$$

and finally

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma_k \qquad \mathcal{B}[\![\neg b]\!]\sigma_k = false \qquad \langle S\texttt{; while } \neg b \texttt{ do } S, \sigma_k\rangle \Downarrow \sigma'}{\langle S\texttt{; while } \neg b \texttt{ do } S, \sigma\rangle \Downarrow \sigma'}\ [\text{ IH }]$$

Proof. $\Leftarrow$

In order to prove this sense, we can proceed very similar as before. Assuming that $\langle S;\ \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'$ holds, we have to deconstruct it in its assertions that is derivated in and to considerate the case base when the boolean expression is false.

Base case.

Let a statement $S \in Stm$, a boolean expression $b \in BExp$ and any states $\sigma, \sigma' \in State$ be. We assume $\langle S;\ \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'$ holds. So we have the following derivation tree:

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma' \qquad \langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'}{\langle S;\ \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'}\ [\text{Seq}_{BS}] \tag{3}$$

Furthermore, since we assume that $\langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'$ holds, then there is a derivation subtree for that expression from which it is derived.

$$\frac{\mathcal{B}[\![\neg b]\!]\sigma' = false}{\langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'}\ [\text{WhileF}_{BS}]$$

And by rewriting the boolean expression

$$\frac{\mathcal{B}[\![b]\!]\sigma' = true}{\langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'}\ [\text{WhileF}_{BS}] \tag{4}$$

Thus, combining (3) and (4) we finally obtain

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma' \qquad \dfrac{\mathcal{B}[\![b]\!]\sigma' = true}{\langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'}\ [\text{WhileF}_{BS}]}{\langle S;\ \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'}\ [\text{Seq}_{BS}] \tag{5}$$

On the other hand, using the same assertions in the equation (5), we obtain by definition:

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma' \qquad \mathcal{B}[\![b]\!]\sigma' = true}{\langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma\rangle \Downarrow \sigma'}\ [\text{UntilT}_{BS}]$$

Inductive case.

We can prove the inductive case analogously ~~at the right sense of the proof.~~ to the left-right implication

Let $S \in Stm$, $b \in BExp$, $\sigma, \sigma' \in State$ be and let $k \in \mathbb{N}$ be a natural number such that $\mathcal{B}[\![\neg b]\!]\sigma_k = false$ and $\mathcal{B}[\![\neg b]\!]\sigma_i = true$ for all $i \in \mathbb{N}$ with $i \leq k$.

We can assume that exist a derivation tree with root

$$\langle S;\ \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma'$$

such that is obtained by derivating and applying $WhileF_{BS}$ in all T_i subtree (for each i-big-step):

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma_1 \qquad \mathcal{B}[\![\neg b]\!]\sigma_1 = true \qquad \langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma_1\rangle \Downarrow \sigma_2}{\underline{\langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma\rangle \Downarrow \sigma_2}}\ [\text{WhileF}_{BS}]$$

$$\vdots$$

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma_k \qquad \mathcal{B}[\![\neg b]\!]\sigma_k = false \qquad \langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma_k\rangle \Downarrow \sigma'}{} \tag{6}$$

Then, rewriting the boolean expression and applying inductive hypothesis IH:

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma_k \qquad \mathcal{B}[\![b]\!]\sigma_k = true}{\langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma\rangle \Downarrow \sigma'}\ [\text{ IH }] \tag{7}$$

Combining (6) and (7)

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma_k \qquad \mathcal{B}[\![b]\!]\sigma_k = true \qquad \langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma_k\rangle \Downarrow \sigma'}{\langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma\rangle \Downarrow \sigma'}\ [\text{ IH }]$$

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma'' \qquad \mathcal{B}[\![b]\!]\sigma'' = false \qquad \langle \texttt{while}\ \neg b\ \texttt{do}\ S, \sigma''\rangle \Downarrow \sigma'}{\langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma\rangle \Downarrow \sigma'}\ [\text{ IH }]$$

And finally we obtain $UntilF_{BS}$ axiom

$$\frac{\langle S, \sigma\rangle \Downarrow \sigma'' \qquad \mathcal{B}[\![b]\!]\sigma'' = false \qquad \langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma''\rangle \Downarrow \sigma'}{\langle \texttt{repeat}\ S\ \texttt{until}\ b\ , \sigma\rangle \Downarrow \sigma'}\ [\text{UntilF}_{BS}]$$

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Exercise 3

Add the following iterative construct to $While$: `for` $x := e_1$ `to` e_2 `do` S . Define its big-step and small-step semantic rules. You cannot rely on the while or repeat construct to do this exercise.

Definition of big-step and small-step semantic rules

Let $b \in BExp$ and $S \in Stm$ be. We can define the big-step and small-step semantic rules for `repeat` S `until` b constructor as follow:

why assign but not execute S?

(a) **Big-step rules**.

$$\frac{\mathcal{B}[\![e_1 \leq e_2]\!]\sigma = false}{\langle \texttt{for}\ x := e_1\ \texttt{to}\ e_2\ \texttt{do}\ S\ , \sigma\rangle \Downarrow \sigma[x \Rightarrow \mathcal{A}[\![e_1]\!]\sigma]}\ [\text{ForF}_{BS}]$$

$$\frac{\mathcal{B}[\![e_1 \leq e_2]\!]\sigma = true \qquad \langle S, \sigma[x \rightarrow \mathcal{A}[\![e_1]\!]\sigma]\rangle \Downarrow \sigma_1 \qquad \langle \texttt{for}\ x := e_1 + 1\ \texttt{to}\ e_2\ \texttt{do}\ S\ , \sigma_1\rangle \Downarrow \sigma'}{\langle \texttt{for}\ x := e_1\ \texttt{to}\ e_2\ \texttt{do}\ S\ , \sigma\rangle \Downarrow \sigma'}\ [\text{ForT}_{BS}]$$

(b) **Small-step rules**. The small step semantics is defined by rewriting steps.

$$\frac{}{\langle \texttt{for}\ x := e_1\ \texttt{to}\ e_2\ \texttt{do}\ S\ , \sigma\rangle \longrightarrow \langle x := e_1;\ \texttt{if}\ e_1 \leq e_2\ \texttt{then}\ S_1\ \texttt{else skip}, \sigma\rangle}\ [\text{For}_{SS}]$$

where $S_1 = (S;\ \texttt{for}\ x := e_1 + 1\ \texttt{to}\ e_2\ \texttt{do}\ S)$