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Assignments 4

1. Define an abstract function and a concretization function between **Interval** and $\mathbf{Z}_{\perp}^{\top}$

$$\alpha : \textbf{Interval} \longrightarrow \mathbb{Z}_{\perp}^{\top}$$
$$\gamma : \mathbb{Z}_{\perp}^{\top} \longrightarrow \textbf{Interval}$$

   and prove that $(\textbf{Interval}, \alpha, \gamma, \mathbb{Z}_{\perp}^{\top})$ is a Galois connection.

Solution

In order to prove that $(\textbf{Interval}, \alpha, \gamma, \mathbb{Z}_{\perp}^{\top})$ is a Galois connection, we will define two function α and σ, such that both are monotonically increasing and hold $\sigma \circ \alpha \sqsupseteq id$ and $\alpha \circ \sigma \sqsubseteq id$.

Definition of α and γ

Let $int \in \textbf{Interval}$, $a, b \in \mathbf{Z} \cup \{-\infty, +\infty\}$ and $z \in \mathbb{Z}_{\perp}^{\top}$ be. Let us consider $\alpha : \textbf{Interval} \longrightarrow \mathbb{Z}_{\perp}^{\top}$ and $\gamma : \mathbb{Z}_{\perp}^{\top} \longrightarrow \textbf{Interval}$ two functions. We define the abstract and the concretization function as follow:

$$\alpha = \lambda\, int. \begin{cases} a & \text{if } int := [a, b] \text{ with } a = b \wedge a \neq -\infty \wedge b \neq +\infty \\ \perp & \text{if } int = \perp \\ \top & \text{otherwise} \end{cases}$$

and

$$\gamma = \lambda\, z. \begin{cases} [z, z] & \text{if } z \in \mathbb{Z} \\ \perp & \text{if } z = \perp \\ [-\infty, +\infty] & \text{if } z = \top \end{cases}$$

Proof of α is monotonically increasing

In order to prove that α is monotonically increasing, we will see for two given intervals in **Interval** which are related by $\sqsubseteq$ relationship, to apply α on both keep that relationship in the same way.

Let us consider $int_1, int_2 \in \textbf{Interval}$ such that $int_1 \sqsubseteq int_2$. By definition, that mean there exist $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ such that $int_1 = [a_1, b_1]$ and $int_2 = [a_2, b_2]$ with $a_2 \leq a_1$ and $b_1 \leq b_2$.

Case $a_1 = b_1$

We can see if $a_1 = b_1$ then $\alpha(int_1) = a_1$. Let's see what append with int_2.

- If $a_2 = b_2$, then $\alpha(int_2) = a_2$. We know that $int_1 \sqsubseteq int_2$, i.e. $a_2 \leq a_1$ and $b_1 \leq b_2$ with $a_1 = b_1$ and $a_2 = b_2$, then $a_1 = a_2$. Therefore $a_1 = a_2 \iff \mathrm{a}_1 \sqsubseteq a_2 \iff \alpha(int_1) \sqsubseteq \alpha(int_2)$.
- If $a_2 < b_2$ or $a_1 = -\infty$ or $b_1 = +\infty$, then $\alpha(int_2) = \top$. Therefore, $a_1 \sqsubseteq \top \iff \alpha(int_1) \sqsubseteq \alpha(int_2)$.

Case $a_1 < b_1$

We can see if $a_1 < b_1$ then $\alpha(int_1) = \top$ and we know that $int_1 \sqsubseteq int_2$, so a_2, b_2 are necessarily different. Therefore $\alpha(int_2) = \top$, then $\top \sqsubseteq \top \iff \alpha(int_1) \sqsubseteq \alpha(int_2)$

Case $int_1 = \bot$

The case when $int_1 = \bot$ is trivial, because $\alpha(\bot) = \bot$ and $\bot \sqsubseteq z$ for all $z \in \mathbb{Z}^{\top}_{\bot}$. In particular, $\top \sqsubseteq \alpha(int_2) \iff \alpha(int_1) \sqsubseteq \alpha(int_2)$.

Proof of γ is monotonically increasing

In order to prove that γ is monotonically increasing, we will do in a similar way than the proof of α. We will see for two given elements in $\mathbb{Z}^{\top}_{\bot}$ which are related by $\sqsubseteq$ relationship, to apply γ on both keep that relationship in the same way.

Let us consider $z_1, z_2 \in \mathbb{Z}^{\top}_{\bot}$ such that $z_1 \sqsubseteq z_2$. By definition, that mean $z1 = \bot$ or $z_1 = z_2$ or $z_2 = \top$.

- If $z_1 = \bot$, then $\gamma(z_1) = \bot$. We know that $\bot \sqsubseteq int$ for all $int \in$ **Interval**, in particular $\bot \sqsubseteq \gamma(z_2) \iff \gamma(z_1) \sqsubseteq \gamma(z_2)$.
- If $z_1 = \top$, then $z_2 = \top$. That mean that $\gamma(z_1) = [-\infty, +\infty]$ and $\gamma(z_2) = [-\infty, +\infty]$. Therefore, $[-\infty, +\infty] \sqsubseteq [-\infty, +\infty] \iff \gamma(z_1) \sqsubseteq \gamma(z_2)$.
- If $z_1 \in \mathbb{Z}$ such that $z_1 \sqsubseteq z_2$, then $z_2 = z_1$ or $z_2 = \top$.
  In the first case, $z_2 = z_1$ then $\gamma(z_1) = [z_1, z_1] = [z_2, z_2] = \gamma(z_2) \iff \gamma(z_1) \sqsubseteq \gamma(z_2)$.
  In the second one, $\gamma(z_2) = [-\infty, +\infty]$. We know that $int \sqsubseteq [-\infty, +\infty]$ for all $int \in$ **Interval**, in particular $\gamma(z_1) \sqsubseteq [-\infty, +\infty] \iff \gamma(z_1) \sqsubseteq \gamma(z_2)$.

Proof of $\gamma \circ \alpha \sqsupseteq id$

Let $int \in$ **Interval** and $a, b \in \mathbb{Z}$ be. We know that

$$\alpha(int) = \begin{cases} a & \text{if } int := [a,b] \text{ with } a = b \wedge a \neq -\infty \wedge b \neq +\infty \\ \bot & \text{if } int = \bot \\ \top & \text{otherwise} \end{cases} \Rightarrow \gamma(\alpha(int)) = \begin{cases} [a,a] & \text{if } int := [a,b] \text{ with } a = b \wedge a \neq -\infty \wedge b \neq +\infty \\ \bot & \text{if } int = \bot \\ [-\infty, +\infty] & \text{otherwise} \end{cases}$$

- If $int = [a, a]$ with $a \in \mathbb{Z}$, then $id(int) = int = [a, a]$. On the other hand, $\gamma(\alpha(int)) = [a, a]$. Therefore, $\gamma(\alpha(int)) = [a, a] = id(int)$, i.e. $\gamma \circ \alpha(int) \sqsupseteq id(int)$.
- If $int = \bot$, then $id(int) = int = \bot$. On the other hand, $\gamma(\alpha(int)) = \bot$. Therefore, $\gamma(\alpha(int)) = \bot = id(int)$, i.e. $\gamma \circ \alpha(int) \sqsupseteq id(int)$.
- If $int = [a, b]$ with $a, b \in \mathbb{Z}$ and $a \neq b$ or $a \neq -\infty$ or $b \neq +\infty$, then $id(int) = id([a, b]) = [a, b]$. On the other hand, $\gamma(\alpha(int)) = [-\infty, +\infty]$. Therefore, $\gamma \circ \alpha(int) = [-\infty, +\infty] \sqsupseteq [a, b] = id(int)$, i.e. $\gamma \circ \alpha(int) \sqsupseteq id(int)$

Proof of $\alpha \circ \gamma \sqsubseteq id$

Let $z \in \mathbb{Z}^{\top}_{\bot}$ be. We know that

$$\gamma(z) = \begin{cases} [z, z] & \text{if } z \in \mathbb{Z} \\ \bot & \text{if } z = \bot \\ [-\infty, +\infty] & \text{if } z = \top \end{cases} \quad \Rightarrow \quad \alpha(\gamma(z)) = \begin{cases} z & \text{if } z \in \mathbb{Z} \\ \bot & \text{if } z = \bot \\ \top & \text{if } z = \top \end{cases}$$

- If $z \in \mathbb{Z}$, then $id(z) = z$. On the other hand, $\alpha(\gamma(z)) = z$. Therefore, $\alpha(\gamma(z)) = z = id(z)$, i.e. $\alpha \circ \gamma(z) \sqsubseteq id(z)$.
- If $z = \bot$, then $id(z) = \bot$. On the other hand, $\alpha(\gamma(z)) = \bot$. Therefore, $\alpha(\gamma(z)) = \bot = id(z)$, i.e. $\alpha \circ \gamma(z) \sqsubseteq id(z)$.
- If $z = \top$, then $id(z) = \top$. On the other hand, $\alpha(\gamma(z)) = \top$. Therefore, $\alpha \circ \gamma(z) = \top = id(int)$, i.e. $\alpha \circ \gamma(z) \sqsupseteq id(z)$

2. Extend this Galois connection to mappings on program variables

$$(\mathbf{Var} \longrightarrow \mathbf{Interval}, \alpha', \gamma', \mathbf{Var} \longrightarrow \mathbb{Z}^{\top}_{\bot})$$

and define α' and γ' in the standard way.

Solution

By the last exercise, we know that $(\mathbf{Interval}, \alpha, \gamma, \mathbb{Z}^{\top}_{\bot})$ is a Galois connection.

Let us consider the following sets $\mathbf{State} = \mathbf{Var} \longrightarrow \mathbf{Interval}$ and $\mathbf{State}^{\sharp} = \mathbf{Var} \longrightarrow \mathbb{Z}^{\top}_{\bot}$. We can extend $(\mathbf{Interval}, \alpha, \gamma, \mathbb{Z}^{\top}_{\bot})$ to $(\mathbf{State}, \alpha', \gamma', \mathbf{State}^{\sharp})$ where α' and γ' are two function defined as follow:

$$\begin{array}{rccc} \alpha' : & \mathbf{State} & \longrightarrow & \mathbb{Z}^{\top}_{\bot} \\ & f & \longrightarrow & \alpha \circ f \end{array}$$

$$\begin{array}{rccc} \gamma' : & \mathbf{State}^{\sharp} & \longrightarrow & \mathbf{Interval} \\ & g & \longrightarrow & \gamma \circ g \end{array}$$

3. Define an abstract interpreter $[\![e]\!]^\sharp : \mathbf{State}^\sharp \longrightarrow \mathbb{Z}_\bot^\top$ that determines whether the result of an expression must be constant at runtime.

Solution

Let $n, z_1, z_2 \in \mathbb{Z}_\bot^\top$, $\in \mathbf{Var}$ and $e_1, e_2 \in AExp$ be. Let us consider $\oplus^\sharp, \ominus^\sharp, \otimes^\sharp : (\mathbb{Z}_\bot^\top \times \mathbb{Z}_\bot^\top) \longrightarrow \mathbb{Z}_\bot^\top$ defined as follow:

$$\oplus^\sharp : (\mathbb{Z}_\bot^\top \times \mathbb{Z}_\bot^\top) \longrightarrow \mathbb{Z}_\bot^\top$$

$$(z_1, z_2) \longrightarrow \begin{cases} z_1 +_{\mathbb{Z}} z_2 & \text{if } z_1 \notin \{\top, \bot\} \text{ and } z_2 \notin \{\top, \bot\} \\ \top & \text{otherwise} \end{cases}$$

Mais preciso si
$\bot \oplus \bot = \bot$
$\bot \oplus \top = \bot$

$$\ominus^\sharp : (\mathbb{Z}_\bot^\top \times \mathbb{Z}_\bot^\top) \longrightarrow \mathbb{Z}_\bot^\top$$

$$(z_1, z_2) \longrightarrow \begin{cases} z_1 -_{\mathbb{Z}} z_2 & \text{if } z_1 \notin \{\top, \bot\} \text{ and } z_2 \notin \{\top, \bot\} \\ \top & \text{otherwise} \end{cases}$$

$$\otimes^\sharp : (\mathbb{Z}_\bot^\top \times \mathbb{Z}_\bot^\top) \longrightarrow \mathbb{Z}_\bot^\top$$

$$(z_1, z_2) \longrightarrow \begin{cases} z_1 *_{\mathbb{Z}} z_2 & \text{if } z_1 \notin \{\top, \bot\} \text{ and } z_2 \notin \{\top, \bot\} \\ \top & \text{otherwise} \end{cases}$$

Let $\sigma^\sharp \in \mathbf{State}^\sharp$ be. We can define an abstract interpreter $[\![e]\!]^\sharp : \mathbf{State}^\sharp \longrightarrow \mathbb{Z}_\bot^\top$ as follow:

$$\begin{aligned} [\![n]\!]^\sharp &= \lambda\sigma^\sharp.\ n \\ [\![x]\!]^\sharp &= \lambda\sigma^\sharp.\ \sigma^\sharp(x) \\ [\![e_1 + e_2]\!]^\sharp &= \lambda\sigma^\sharp.\ [\![e_1]\!]^\sharp\sigma^\sharp \oplus^\sharp [\![e_2]\!]^\sharp\sigma^\sharp \\ [\![e_1 - e_2]\!]^\sharp &= \lambda\sigma^\sharp.\ [\![e_1]\!]^\sharp\sigma^\sharp \ominus^\sharp [\![e_2]\!]^\sharp\sigma^\sharp \\ [\![e_1 * e_2]\!]^\sharp &= \lambda\sigma^\sharp.\ [\![e_1]\!]^\sharp\sigma^\sharp \otimes^\sharp [\![e_2]\!]^\sharp\sigma^\sharp \end{aligned}$$

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4. Finally, show that the interpreter is correct by proving that $[\![e]\!]^\sharp \sqsupseteq \alpha \circ [\![e]\!] \circ \gamma'$, where $[\![e]\!]$ is the abstract interpretation in the lattice of intervals.

Solution

Case $[\![n]\!]^\sharp$.

On the one hand, we know that $[\![n]\!]^\sharp = \lambda\sigma^\sharp.\ n$ for all $\sigma^\sharp \in \mathbf{State}^\sharp$.
On the other hand, $\alpha[\![n]\!]\gamma' = \lambda\sigma^\sharp.\ \alpha[\![n]\!]\left(\gamma \circ \sigma^\sharp\right)$ for all $\sigma^\sharp \in \mathbf{State}^\sharp \iff \alpha[\![n]\!]\gamma' = \lambda\sigma^\sharp.\ \alpha\left([n,n]\right)$ for all $\sigma^\sharp \in \mathbf{State}^\sharp \iff \alpha[\![n]\!]\gamma' = \lambda\sigma^\sharp.\ n$ for all $\sigma^\sharp \in \mathbf{State}^\sharp$. Therefore,

$$[\![n]\!]^\sharp = \lambda\sigma^\sharp.\ n = \alpha[\![n]\!]\gamma'$$

for all $\sigma^\sharp \in \mathbf{State}^\sharp$. Therefore, $[\![n]\!]^\sharp \sqsupseteq \alpha[\![n]\!]\gamma'$.

Case $[\![x]\!]^\sharp$.

On the one hand, we know that $[\![x]\!]^\sharp = \lambda\sigma^\sharp.\ \sigma^\sharp(x)$ for all $\sigma^\sharp \in \mathbf{State}^\sharp$.
On the other hand, $\alpha[\![x]\!]\gamma' = \lambda\sigma^\sharp.\ \alpha[\![x]\!]\left(\gamma \circ \sigma^\sharp\right)$ for all $\sigma^\sharp \in \mathbf{State}^\sharp \iff \alpha[\![x]\!]\gamma' = \lambda\sigma^\sharp.\ \alpha\left(\gamma \circ \sigma^\sharp\right)(x)$ for all $\sigma^\sharp \in \mathbf{State}^\sharp \iff \alpha[\![x]\!]\gamma' = \lambda\sigma^\sharp.\ (\alpha \circ \gamma) \circ \sigma^\sharp(x)$ for all $\sigma^\sharp \in \mathbf{State}^\sharp$.

By exercise 1, we know that $\alpha \circ \gamma \sqsubseteq id$, in particular:

$$[\![x]\!]^\sharp = \lambda\sigma^\sharp.\ \sigma^\sharp(x) = \lambda\sigma^\sharp.\ id(\sigma^\sharp(x)) \sqsupseteq \lambda\sigma^\sharp.\ (\alpha \circ \gamma) \circ \sigma^\sharp(x) = \alpha[\![x]\!]\gamma'$$

for all $\sigma^\sharp \in \mathbf{State}^\sharp$. Therefore, $[\![x]\!]^\sharp \sqsupseteq \alpha[\![x]\!]\gamma'$.

- Case $[\![e_1 + e_2]\!]^\sharp$

Let $z_1, z_2 \in \mathbb{Z}$ be. On the one hand, we know that

$$[\![e_1 + e_2]\!]^\sharp = \lambda\sigma^\sharp.\ [\![e_1]\!]^\sharp\sigma^\sharp \oplus [\![e_2]\!]^\sharp\sigma^\sharp$$

$$= \lambda\sigma^\sharp \begin{cases} z_1 + z_2 & \text{if } z_1 \notin \{\top, \bot\} \text{ and } z_2 \notin \{\top, \bot\} \\ \top & \text{otherwise} \end{cases}$$

On the other hand we know that $\alpha \circ [\![e_1 + e_2]\!]\gamma' = \lambda\sigma^\sharp.\ \alpha \circ [\![e_1 + e_2]\!](\gamma\sigma^\sharp) =$
$\lambda\sigma^\sharp.\ \alpha\left([\![e_1]\!](\gamma\sigma^\sharp) \oplus [\![e_2]\!](\gamma\sigma^\sharp)\right)\ \forall \sigma^\sharp \in \text{State}^\sharp$

- if $\lambda\sigma^\sharp\ [\![e_1]\!](\gamma\sigma^\sharp) = \bot$ or $\lambda\sigma\ [\![e_2]\!](\gamma\sigma^\sharp) = \bot$
$\Rightarrow \lambda\sigma^\sharp.\ \left([\![e_1]\!](\gamma\sigma^\sharp) \oplus [\![e_2]\!](\gamma\sigma^\sharp)\right) = \bot$
$\Rightarrow \lambda\sigma^\sharp.\ \alpha\left([\![e_1]\!](\gamma\sigma^\sharp) \oplus [\![e_2]\!](\gamma\sigma^\sharp)\right) = \lambda\sigma^\sharp.\ \alpha(\bot) = \lambda\sigma^\sharp.\ \bot$
$\Rightarrow \alpha\ [\![e_1 + e_2]\!]\gamma' = \lambda\sigma^\sharp.\ \bot \sqsubseteq \lambda\sigma^\sharp\ z \quad \forall z \in \mathbb{Z}^\top_\bot$

In particular $\alpha\ [\![e_1 + e_2]\!] \sqsubseteq [\![e_1 + e_2]\!]^\sharp$

- if $\lambda\sigma^{\#}.\ [\![e_1]\!]^{\#}\sigma^{\#} \in \{\top, \bot\}$ or $\lambda\sigma^{\#}.\ [\![e_2]\!]^{\#}\sigma^{\#} \in \{\top, \bot\}$

$\Rightarrow z \sqsubseteq \lambda\sigma^{\#}.\ \top = \lambda\sigma^{\#}.\ [\![e_1]\!]^{\#}\sigma^{\#} \oplus^{\#} [\![e_2]\!]^{\#}\sigma^{\#} = [\![e_1 + e_2]\!]^{\#} \quad \forall z \in \mathbb{Z}^{\top}_{\bot}$

- by the exercise 1, we know that $\alpha \circ \gamma \sqsubseteq id$

in particular

$\lambda\sigma^{\#}.\ (\alpha \circ ([\![e_1]\!](\gamma\sigma^{\#})) \sqsubseteq \lambda\sigma^{\#}.\ [\![e_1]\!]^{\#}\sigma^{\#} = z_1$

$\lambda\sigma^{\#}.\ (\alpha \circ ([\![e_2]\!](\gamma\sigma^{\#})) \sqsubseteq \lambda\sigma^{\#}.\ [\![e_2]\!]^{\#}\sigma^{\#} = z_2$

$\alpha([\![e_1]\!](\gamma(\sigma^{\#}))) = (\alpha \circ [\![e_1]\!] \circ \gamma)(\sigma^{\#}) \sqsubseteq [\![e_1]\!]^{\#}\sigma^{\#}$

por H.I, que es: $\alpha \circ [\![e_1]\!] \circ \gamma \sqsubseteq [\![e_1]\!]^{\#}$

with $(\alpha \circ [\![e_1]\!](\gamma\sigma^{\#}))$ and $(\alpha \circ [\![e_2]\!](\lambda\sigma^{\#}))$ distin to $\bot$ or $\top$, then both must to be z_1 and z_2 respectly

therefore,

$\lambda\sigma^{\#}.\ \alpha([\![e_1]\!](\gamma\sigma^{\#}) \oplus [\![e_2]\!](\gamma\sigma^{\#})) = \lambda\sigma^{\#}.\ \alpha([z_1, z_1] \oplus [z_2, z_2]) =$

$= \lambda\sigma^{\#}.\ \alpha([z_1 + z_2, z_1 + z_2]) = \lambda\sigma^{\#}.\ z_1 + z_2 =$

$= [\![e_1 + e_2]\!]^{\#}$

$\Rightarrow \boxed{\alpha \circ [\![e_1 + e_2]\!]\gamma' \sqsubseteq [\![e_1 + e_2]\!]^{\#}}$

- the case $[\![e_1 - e_2]\!]^{\#}$ is analogous

- case $[\![e_1 * e_2]\!]$

Let $z_1, z_2 \in \mathbb{Z}$ be. On the one hand, we know that $[\![e_1 * e_2]\!]^{\#} = \lambda\sigma^{\#}.\ [\![e_1]\!]^{\#}\sigma^{\#} \otimes [\![e_2]\!]^{\#}\sigma^{\#} =$

$= \lambda\sigma^{\#}.\ \begin{cases} z_1 \cdot z_2 \text{ if } z_1 \notin \{\top, \bot\} \text{ and } z_2 \notin \{\top, \bot\} \\ \top \text{ otherwise} \end{cases}$

On the other hand, we also know that $\alpha [\![e_1 * e_2]\!]\gamma' = \lambda\sigma^{\#}.\ \alpha \circ [\![e_1 * e_2]\!](\gamma\sigma^{\#}) =$

$\lambda\sigma^{\#}.\ \alpha \circ ([\![e_1]\!](\gamma\sigma^{\#}) \otimes [\![e_2]\!](\gamma\sigma^{\#})) \quad \forall \sigma^{\#} \in state^{\#}$

- if $\lambda\sigma^{\#}.\ [\![e_1]\!](\gamma\sigma^{\#}) = \bot$ or $\lambda\sigma^{\#}.\ [\![e_2]\!](\lambda\sigma^{\#}) = \bot$ is the same the case $[\![e_1 + e_2]\!]^{\#}$ or $[\![e_1 - e_2]\!]^{\#}$

- if $\lambda\sigma^{\#}.\ [\![e_1]\!]^{\#}\sigma^{\#} \in \{\top, \bot\}$ or $\lambda\sigma^{\#}.\ [\![e_2]\!]\sigma^{\#\#} \in \{\top, \bot\}$

$\Rightarrow \lambda\sigma^{\#}.\ [\![e_1]\!]^{\#}\sigma^{\#} \otimes^{\#} [\![e_2]\!]^{\#}\sigma^{\#} = \lambda\sigma^{\#}.\ \top$

that means that $z \sqsubseteq [\![e_1 * e_2]\!]^\# \quad \forall z \in \mathbb{Z}^\top_\bot$

in particular $\alpha \circ [\![e_1 * e_2]\!] \gamma' \sqsubseteq [\![e_1 * e_2]\!]^\#$

In a similar way by the exercise 1, we know that $\alpha \circ \gamma \sqsubseteq id$ in particular

$$\lambda\sigma^\#.(\alpha \circ [\![e_1]\!](\gamma\sigma^\#)) \sqsubseteq \lambda\sigma^\#. [\![e_1]\!]^\# \sigma^\# = z_1$$

$$\lambda\sigma^\#.(\alpha \circ [\![e_2]\!](\gamma\sigma^\#)) \sqsubseteq \lambda\sigma^\#. [\![e_2]\!]^\# \sigma^\# = z_2$$

with $(\alpha \circ [\![e_1]\!](\gamma\sigma^\#))$ and $(\alpha \circ [\![e_2]\!](\lambda\sigma^\#))$ distint to $\bot$ or $\top$
then both must to be z_1 and z_2 respectly

therefore, there exists $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ such that $\min(V) = \max(V) = z_1 \cdot z_2$
where $V = \{a_1 \cdot a_2, a_1 \cdot b_2, b_1 \cdot a_2, b_1 \cdot b_2\}$ and:

$$\lambda\sigma^\#.\alpha([\![e_1]\!](\gamma\sigma^\#)) \otimes [\![e_2]\!](\gamma\sigma^\#)) = \lambda\sigma^\#.\alpha([\min(V), \max(V)]) =$$
$$= \lambda\sigma^\#.\alpha([z_1 \cdot z_2, z_1 \cdot z_2]) = \lambda\sigma^\#. z_1 \cdot z_2 =$$
$$= \lambda\sigma^\#. [\![e_1]\!]^\# \sigma^\# \otimes^\# [\![e_2]\!]^\# \sigma^\# = [\![e_1 * e_2]\!]^\#$$

$$\Rightarrow \boxed{\alpha [\![e_1 * e_2]\!] \gamma' \sqsubseteq [\![e_1 * e_2]\!]^\#}$$

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