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Assignments 3

Exercise 4

Given the function $F : (State \to State_\perp) \longrightarrow (State \to State_\perp)$ defined as follows:

$$F(f) = cond(\mathcal{B}[\![n > 0]\!], f \circ \mathcal{S}[\![x := 2 * x; n := n - 1]\!], id)$$

(a) Give an explicit definition for $F(\lambda\sigma.\ \perp)$, $F^2(\lambda\sigma.\ \perp)$ and $F^3(\lambda\sigma.\ \perp)$.

(b) From the results above, conjecture a general definition for $F^i(\lambda\sigma.\ \perp)$ where $i \geq 1$. [Optional] Prove by induction on i that your conjecture is correct.

(c) Give an explicit definition for $\bigsqcup_i F^i(\lambda\sigma.\ \perp)$.

(d) Which is the least fixed point of F? Justify your answer.

(e) Given the above, compute the state resulting from the execution of the following program

$$x := 1; \texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1)$$

under the initial stage $\sigma = [n \to 4]$.

a) An explicit definition for $F(\lambda\sigma.\ \perp)$, $F^2(\lambda\sigma.\ \perp)$ and $F^3(\lambda\sigma.\ \perp)$

Let $F : (State \to State_\perp) \longrightarrow (State \to State_\perp)$ be a function defined as follow

$$F(f) = cond(\mathcal{B}[\![n > 0]\!], f \circ \mathcal{S}[\![S]\!], id)$$

where $S = (x := 2 * x; n := n - 1)$.

Let us consider $f_0 : State \longrightarrow State_\perp$, $f_0 = \lambda\sigma.\ \perp$ for all σ. We will define explicitly $f_1 = F(f_0)$, $f_1 = F^2(f_0)$ and $f_3 = F^3(f_0)$:

Definition of $F(f_0)$

Let us consider a function $f_1 : State \longrightarrow State_\perp$ defined as follow

$$f_1 := F(f_0) = cond(\mathcal{B}[\![n > 0]\!], f_0 \circ \mathcal{S}[\![S]\!], id)$$

composition

$$= \lambda\sigma. \begin{cases} f_0 \circ \mathcal{S}[\![S]\!]\ \sigma & \mathcal{B}[\![n > 0]\!]\sigma \\ id\ \sigma & \mathcal{B}[\![n \leq 0]\!]\sigma \end{cases}$$

$$= \lambda\sigma. \begin{cases} f_0 \circ \sigma[x \to 2 \cdot \sigma(x), n \to \sigma(n) - 1] & \sigma(n) > 0 \\ \sigma & \sigma(n) \leq 0 \end{cases}$$

apply state σ

is not the same as composition.

but since $f_0 = \lambda\sigma.\perp\ \forall\sigma$, then $f_0 \circ \sigma[x \to 2\cdot\sigma(x), n \to \sigma(n)-1] = \perp$ for all σ. Therefore

$$f_1 = F(f_0) = \lambda\sigma. \begin{cases} \perp & \sigma(n) > 0 \\ \\ \sigma & \sigma(n) \leq 0 \end{cases} \tag{1}$$

Definition of $F^2(f_0)$

Let us consider a function $f_2 : \mathit{State} \longrightarrow \mathit{State}_\perp$ defined as follow

$$f_2 := F^2(f_0) = F(f_1) = \mathit{cond}(\mathcal{B}[\![n>0]\!], f_1 \circ \mathcal{S}[\![S]\!], \mathit{id})$$

$$= \lambda\sigma. \begin{cases} f_1 \circ \mathcal{S}[\![S]\!]\ \sigma & \mathcal{B}[\![n>0]\!]\sigma \\ \\ \mathit{id}\ \sigma & \mathcal{B}[\![n\leq 0]\!]\sigma \end{cases}$$

$$= \lambda\sigma. \begin{cases} f_1 \circ \sigma[x \to 2\cdot\sigma(x), n \to \sigma(n)-1] & \sigma(n) > 0 \\ \\ \sigma & \sigma(n) \leq 0 \end{cases}$$

we know now that (1) $f_1 = \lambda\sigma_1.\perp$ if $\sigma_1(n) > 0$ and $f_1 = \lambda\sigma_1.\sigma_1$ (the identity) if $\sigma_1(n) \leq 0$, being more precise $f_1 = \lambda\sigma_1.\sigma_1$ either $\sigma_1(n) = 0$ or $\sigma_1(n) < 0$, i.e.:

$$f_1 = \lambda\sigma_1. \begin{cases} \perp & \sigma_1(n) > 0 \\ \\ \sigma_1 & \sigma_1(n) = 0 \\ \\ \sigma_1 & \sigma_1(n) < 0 \end{cases}$$

So, in order to compose f_1 with $\sigma[x \to 2\cdot\sigma(x), n \to \sigma(n)-1]$, we have to distinguish two cases when $\sigma_1(n) > 0$ and $\sigma_1(n) = 0$, i.e. when $\sigma(n) - 1 > 0$, and $\sigma(n) - 1 = 0$, respectively.

The first one is actually easy, $\sigma_1(n) > 0 \iff \sigma(n) - 1 > 0 \iff \sigma(n) > 1$ then

$$f_2 = \lambda\sigma.\quad \perp, \quad \sigma(n) > 1$$

In the second one $\sigma_1(n) = 0 \iff \sigma(n) - 1 = 0 \iff \sigma(n) = 1$, then

$$f_2 = \lambda\sigma.\quad \sigma[x \to 2\cdot\sigma(x), n \to 0], \quad \sigma(n) = 1$$

And finally in the case $\sigma(n) \leq 0$, $f_2 = \lambda\sigma.\sigma$. Therefore

$$f_2 = \lambda\sigma. \begin{cases} \perp & \sigma(n) > 1 \\ \\ \sigma[x \to 2\cdot\sigma(x), n \to 0] & \sigma(n) = 1 \\ \\ \sigma & \sigma(n) \leq 0 \end{cases} \tag{2}$$

Definition of $F^3(f_0)$

Let us consider a function $f_3 : State \longrightarrow State_\perp$ defined as follow

$$f_3 := F^3(f_0) = F^2(f_1) = F(f_2) = cond(\mathcal{B}[\![n > 0]\!], f_2 \circ \mathcal{S}[\![S]\!], id)$$

$$= \lambda\sigma. \begin{cases} f_2 \circ \mathcal{S}[\![S]\!]\ \sigma & \mathcal{B}[\![n > 0]\!]\sigma \\ \\ id\ \sigma & \mathcal{B}[\![n \leq 0]\!]\sigma \end{cases}$$

$$= \lambda\sigma. \begin{cases} f_2 \circ \sigma[x \to 2 \cdot \sigma(x), n \to \sigma(n) - 1] & \sigma(n) > 0 \\ \\ \sigma & \sigma(n) \leq 0 \end{cases}$$

In order to proceed with the definition of f_3 we will think pretty similar to the way when we defined f_2, we know that (2) $f_2 = \lambda\sigma_2.\ \perp$ if $\sigma_2(n) > 1$, $f_2 = \lambda\sigma_2.\sigma_2[x \to 2 \cdot \sigma_2(x), n \to 0]$ if $\sigma_2(n) = 1$ and $f_2 = \lambda\sigma_2.\sigma_2$ (the identity) if $\sigma_2(n) \leq 0$, being more precise $f_2 = \lambda\sigma_2.\sigma_2$ either $\sigma_2(n) = 0$ or $\sigma_2(n) < 0$, i.e.:

$$f_2 = \lambda\sigma_2. \begin{cases} \perp & \sigma_2(n) > 1 \\ \\ \sigma_2[x \to 2 \cdot \sigma_2(x), n \to 0] & \sigma_2(n) = 1 \\ \\ \sigma_2 & \sigma_2(n) = 0 \\ \\ \sigma_2 & \sigma_2(n) < 0 \end{cases}$$

Again, when $\sigma(n) > 0$, we will compose f_2 with $\sigma[x \to 2 \cdot \sigma(x), n \to \sigma(n) - 1]$. Then, we have to distinguish three cases when $\sigma_2(n) > 1$, $\sigma_2(n) = 1$ and $\sigma_2(n) = 0$, i.e. when $\sigma(n) - 1 > 1$, $\sigma(n) - 1 = 1$ and $\sigma(n) - 1 = 0$, respectively.

The first one is actually easy, $\sigma_2(n) > 1 \iff \sigma(n) - 1 > 1 \iff \sigma(n) > 2$ then

$$f_3 = \lambda\sigma.\ \ \perp, \quad \sigma(n) > 2$$

The third one $\sigma_2(n) = 0 \iff \sigma(n) - 1 = 0 \iff \sigma(n) = 1$ then

$$f_3 = \lambda\sigma.\ \ \sigma[x \to 2 \cdot \sigma(x), n \to 0], \quad \sigma(n) = 1$$

But in the case when $\sigma_2(n) = 1$, we note that in addition to map n to 2 by σ, because $\sigma_2(n) = 1 \iff \sigma(n) - 1 = 1 \iff \sigma(n) = 2$, we have to compose

$$(\lambda\sigma_2.\sigma_2[x \to 2 \cdot \sigma_2(x), n \to 0])\ \circ\ (\lambda\sigma.\sigma[x \to 2 \cdot \sigma(x), n \to 2])$$

In other words,

$$(\lambda\sigma_2.\sigma_2[x \to 2 \cdot \sigma_2(x), n \to 0]) \circ (\lambda\sigma.\sigma[x \to 2 \cdot \sigma(x), n \to 2])$$

$$= \lambda\sigma.\sigma[x \to 2 \cdot \sigma(x), n \to 0][x \to 2 \cdot \sigma(x), n \to 2]$$

$$= \lambda\sigma.\sigma[x \to 2 \cdot (2 \cdot \sigma(x)), n \to 0]$$

$$= \lambda\sigma.\sigma[x \to 2 \cdot 2 \cdot \sigma(x), n \to 0]$$

$$= \lambda\sigma.\sigma[x \to 2^2 \cdot \sigma(x), n \to 0]$$

And finally, in the case $\sigma(n) \leq 0$, $f_2 = \lambda\sigma.\sigma$. Therefore

$$f_3 = \lambda\sigma. \begin{cases} \perp & \sigma(n) > 2 \\ \sigma[x \to 2^2 \cdot \sigma(x), n \to 0] & \sigma(n) = 2 \\ \sigma[x \to 2 \cdot \sigma(x), n \to 0] & \sigma(n) = 1 \\ \sigma & \sigma(n) \leq 0 \end{cases} \tag{3}$$

b) An explicit definition for $F^i(\lambda\sigma. \perp)$.

Let us consider a function $f_i : State \longrightarrow State_\perp$ defined as follow

$$f_i := F^i(\lambda\sigma. \perp) = \lambda\sigma. \begin{cases} \perp & \sigma(n) \geq i \\ \sigma[x \to 2^{\sigma(n)} \cdot \sigma(x), n \to 0] & 0 < \sigma(n) < i \\ \sigma & \sigma(n) \leq 0 \end{cases}$$

c) An explicit definition for $\bigsqcup_i F^i(\lambda\sigma. \perp)$.

$$\bigsqcup_i F^i(\lambda\sigma. \perp) = \lambda\sigma. \begin{cases} \sigma[x \to 2^{\sigma(n)} \cdot \sigma(x), n \to 0] & \sigma(n) > 0 \\ \sigma & \sigma(n) \leq 0 \end{cases}$$

d) Give the least fixed point of F and justify it.

Let $(State \to State_\perp, \sqsubseteq)$ be a pair of a set and an order relation, we know that $(State \to State_\perp, \sqsubseteq)$ is a ccpo (proposition pag. 110), and let $F : (State \to State_\perp) \longrightarrow (State \to State_\perp)$ be a function with $F(f) = cond(\mathcal{B}[\![b]\!], f \circ \mathcal{S}[\![S]\!], id)$ with $S \in Stm$ and $b \in BExp$, we also know F is monotone and continuous in $(State \to State_\perp, \sqsubseteq)$ (result in pag 116).

Applying Fixed-Point theorem we conclude that $\bigsqcup_i F^i(\lambda\sigma.\ \bot)$ is the least fixed point of the function F, i.e

$$lfp\ F = \bigsqcup_i F^i(\lambda\sigma.\ \bot) = \lambda\sigma. \begin{cases} \sigma[x \to 2^{\sigma(n)} \cdot \sigma(x), n \to 0] & \sigma(n) > 0 \\ \\ \sigma & \sigma(n) \le 0 \end{cases}$$

d) Execute the program of d) under the given initial state.

Let us consider the following statement

$$S = (x := 1; \texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1))$$

and the initial state $\sigma = [n \to 4]$.

In order to compute S under σ we have to compose σ with $\mathcal{S}[\![S]\!]$, i.e.

$$\sigma \circ \mathcal{S}[\![(x := 1; \texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1))]\!]$$

We know the semantics of a sequence $S_1; S_2$ is defined as follows:

$$\mathcal{S}[\![S_1; S_2]\!] = \mathcal{S}[\![S_2]\!] \circ \mathcal{S}[\![S_1]\!]$$

Therefore, we obtain that

$$\mathcal{S}[\![x := 1; \texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1)]\!] = \mathcal{S}[\![\texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1)]\!] \circ \mathcal{S}[\![x := 1]\!]$$

And finally

$$\begin{aligned}
\sigma \circ \mathcal{S}[\![S]\!] &= \sigma \circ \mathcal{S}[\![(x := 1; \texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1))]\!] \\
&= \sigma \circ \mathcal{S}[\![\texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1)]\!]\ \circ \mathcal{S}[\![x := 1]\!] \\
&= \sigma \circ \mathcal{S}[\![\texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1)]\!]\ [x \to 1] \\
&= [n \to 4] \circ \mathcal{S}[\![\texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1)]\!]\ [x \to 1] \\
&= \mathcal{S}[\![\texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1)]\!]\ [x \to 1][n \to 4] \\
&= \mathcal{S}[\![\texttt{while}\ n > 0\ \texttt{do}\ (x := 2 * x; n := n - 1)]\!]\ [x \to 1, n \to 4] \\
&= (lfp\ F)\ [x \to 1, n \to 4] \\
&= \left(\bigsqcup_i F^i(\lambda\sigma.\ \bot)\right)\ [x \to 1, n \to 4] \\
&= \lambda\sigma'.\sigma'[x \to 2^4, n \to 0] \\
&= \lambda\sigma'.\sigma'[x \to 16, n \to 0]
\end{aligned}$$