Lossy Petri nets

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A *Lossy* Petri net is a variant which can loose tokens at any time.

In this document we define them and study the decidability of its boundedness problem.

Definition 1 (Lossy Petri net)**.** *Given a Petri net model with its enabling and firing rules, its lossy variant is syntactically identical but with the following semantics:*

A lossy transition $M_1 \xrightarrow{t}_l M_2$ *can happen whenever there are markings* M'_1 *and* M'_2 *such that:*

- $M_1 \geq M'_1$
- $M'_2 \geq M_2$
- $M'_1 \xrightarrow{t} M'_2$ *according to the usual enabling and firing rules of its counterpart model.*

As we have used the symbol $\rightarrow_l$ *to differentiate the lossy semantics from the ones of its associated model, we also use* $[\rangle_l$ *to differentiate the set of reachable markings with this lossy semantics.*

Lemma 1. *Given a Petri net model and a marking* M*, its reachable markings are also reachable for its lossy variant:*

$$[M\rangle \subseteq [M\rangle_l$$

Proof. Let $M_2 \in [M\rangle$. As it is reachable, we have $M_1 \xrightarrow{t} M_2$ for some transition t and $M_1 \in [M\rangle$. These markings trivially satisfy $M_1 \geq M_1$ and $M_2 \geq M_2$, so we also have $M_1 \xrightarrow{t}_l M_2$. □

Lemma 2. *Given a Petri net model which satisfies at least the weak monotonicity lemma and a marking* M*, and let* $M' \in [M\rangle_l$ *be a lossy-reachable marking from* M*, then* $\exists \overline{M'} \in [M\rangle$ *such that* $\overline{M'} \geq M'$*.*

Proof. As M' is lossy-reachable from M, we have $M \rightarrow^k_l M'$. We are going to prove the property by natural induction over k.

If $k = 0$, then $M = M'$ and, as $M \in [M\rangle$, we trivially have $M' \in [M\rangle$ with $M' \geq M'$.

Proceeding with the inductive step, if $k > 0$ we have

$$M \rightarrow_l^{k-1} M'' \xrightarrow{t}_l M'$$

for some transition t and some lossy-reachable marking $M'' \in [M\rangle_l$.

On the one hand, this last lossy transition of the sequence requires that $M_1 \xrightarrow{t} M_2$ for some markings M_1 and M_2 with $M'' \geq M_1$ and $M_2 \geq M'$.

On the other hand, by the induction hypothesis, there exists a reachable marking $\overline{M} \in [M\rangle$ such that $\overline{M} \geq M''$.

As this Petri net satisfies the weak monotonicity lemma, $\overline{M} \geq M'' \geq M_1$ and $M_1 \xrightarrow{t} M_2$, we have $\overline{M} \xrightarrow{t} \overline{M_2}$ with $\overline{M_2} \geq M_2 \geq M'$ and $\overline{M_2} \in [M\rangle$. $\square$

Theorem 1. *Given a Petri net (N, M_0) from a model which satisfies at least the weak monotonicity lemma, it is bounded if and only if it is bounded with lossy semantics.*

Proof. For the left to right implication, let s be a place of N. If the net is bounded with its regular model semantics, then there exists a number $b \geq 0$ such that $M(s) \leq b \; \forall M \in [M_0\rangle$.

By the Lemma 2, for any lossy-reachable marking $M' \in [M_0\rangle_l$, there exists a reachable marking $\overline{M'} \in [M\rangle$ such that $\overline{M'} \geq M'$, so $b \geq \overline{M'}(s) \geq M'(s)$ and therefore the net is lossy-bounded.

For the right to left implication, let s be a place of N. If the net is lossy-bounded, then there exists a number $b \geq 0$ such that $M(s) \leq b \; \forall M \in [M_0\rangle_l$.

By the Lemma 1, any reachable marking $M \in [M\rangle$ is also lossy-reachable (i.e. $M \in [M\rangle_l$), thus $M(s) \leq b$ and the net is bounded. $\square$

Corollary 1. *By the Theorem 1, as regular Petri nets satisfy the weak monotonicity lemma and boundedness is decidable for them, boundedness is also decidable for regular Petri nets with lossy semantics.*

For studying its boundedness, it suffices to study the property without taking into account the lossy transitions by using the procedure that we know for regular Petri nets.

Corollary 2. *By the Theorem 1, as Petri nets with reset arcs satisfy the weak monotonicity lemma and boundedness is undecidable for them, boundedness is also undecidable for lossy Petri nets with reset arcs.*

This theorem cannot be applied to Petri nets with inhibitor arcs because they do not satisfy any version of the monotonicity lemma. However, we can find a close relation between lossy Petri nets with reset arcs and lossy Petri nets with inhibitor arcs.

Lemma 3. *Given a Petri net with reset arcs (R, M_0) and the Petri net with inhibitor arcs (I, M_0) which consists of replacing the reset arcs of R by inhibitor arcs:*

$$M_1 \xrightarrow{t}_l M_2 \textit{ for } R \iff M_1 \xrightarrow{t}_l M_2 \textit{ for } I$$

As they have the same lossy transitions and initial marking, the reachability graph of both Petri nets with lossy semantics is isomorphic.

Proof. First of all, note that the set of transitions T is the same for both nets. Let $t \in T$ be a transition and M_1 be a marking.

Due to the definition of I, there are reset arcs relating some place and t for R if and only if there are inhibitor arcs relating some place and t for I.

If t does not involve any reset arc, then it is trivial that M_1 lossy-enables t leading to a marking M_2 for R if an only if that is also the case for I, as the preconditions and postconditions of t are the same for both nets.

If there is any reset arc (s, t) for some place s, the enabling of t for R does 3.1
not depend on $M_1(s)$ and, as it is an inhibitor arc for I, the enabling of t also does not depend on $M_1(s)$ with lossy semantics. For both nets, $M_2(s)$ will be 1 or 0 depending on whether the arc (t, s) exists or not. $\square$

Corollary 3. *Boundedness is undecidable for lossy Petri nets with inhibitor arcs.*

This is because, if it were decidable, then we could reduce the boundedness problem for lossy Petri nets with reset arcs to the boundedness problem for lossy Petri nets with inhibitor arcs by using the transformation from the Lemma 3, which preserves the reachability graph, and it would be a contradiction due to the Corollary 2.

Índice de comentarios

1.1 Siendo estrictos habría que hacer inducción sobre la longitud de esa traza

2.1 cuidado: M_0 en vez de M

3.1 Aquí quizás vendría bien extender un poco las explicaciones: ¿por qué en I se puede activar una transición cuando hay lugares inhibidos?