Assignment 1

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Let $e, e' \in AExp$, $x \in Var$.

i) Define $e[x/e']$ in a compositional way.

Let $e' \in AExp$ and $x \in Var$. We can define $e[x/e'] \in AExp$ in a compositional way depending on the structure of $e \in AExp$ as follows:

$$n[x/e'] = n \text{, for } n \in \mathbb{Z}$$

$$y[x/e'] = \begin{cases} y & \text{if } x \neq y \\ e' & \text{if } x = y \end{cases} \text{, for } y \in Var$$

$$(e_1 + e_2)[x/e'] = e_1[x/e'] + e_2[x/e'] \text{, for } e_1, e_2 \in AExp$$

$$(e_1 - e_2)[x/e'] = e_1[x/e'] - e_2[x/e'] \text{, for } e_1, e_2 \in AExp$$

$$(e_1 * e_2)[x/e'] = e_1[x/e'] * e_2[x/e'] \text{, for } e_1, e_2 \in AExp$$

ii) Prove the following lemma: for all $e, e' \in AExp$, $x \in Var$, $\sigma \in State$

$$\mathcal{A}[\![e[x/e']]\!] \sigma = \mathcal{A}[\![e]\!] \sigma [x \mapsto \mathcal{A}[\![e']\!] \sigma].$$

By structural induction in e:

- if $e = n$ for some $n \in \mathbb{Z}$, we have that

$$e[x/e'] = n[x/e'] = n \text{, so}$$

$$\mathcal{A}[\![e[x/e']]\!]\sigma = \mathcal{A}[\![n]\!]\sigma = n, \text{ and}$$

$$\mathcal{A}[\![e]\!]\sigma' = \mathcal{A}[\![n]\!]\sigma' = n \text{ for any } \sigma' \in \text{State}.$$

- if $e = y$ for $y \in Var$ we would have that

$$e[x/e'] = \begin{cases} y & \text{if } x \neq y \\ e' & \text{if } x = y \end{cases}, \text{ and then}$$

$$\mathcal{A}[\![e[x/e']]\!]\sigma = \begin{cases} \mathcal{A}[\![y]\!]\sigma & \text{if } x \neq y \\ \mathcal{A}[\![e']\!]\sigma & \text{if } x = y \end{cases},$$

$$\mathcal{A}[\![e]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] = \mathcal{A}[\![y]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] =$$

$$= \begin{cases} \mathcal{A}[\![y]\!]\sigma & \text{if } x \neq y \\ \mathcal{A}[\![e']\!]\sigma & \text{if } x = y \end{cases}, \text{ and both values are the same.}$$

- if $e = e_1 + e_2$ for $e_1, e_2 \in AExp$, we have

$$e[x/e'] = e_1[x/e'] + e_2[x/e'] \text{ and}$$

$$\mathcal{A}[\![e[x/e']]\!]\sigma = \mathcal{A}[\![e_1[x/e']]\!]\sigma + \mathcal{A}[\![e_2[x/e']]\!]\sigma =$$

$$\stackrel{I.H.}{=} \mathcal{A}[\![e_1]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma]$$

$$+ \mathcal{A}[\![e_2]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] =$$

$$= \mathcal{A}[\![e_1 + e_2]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma]$$

- if $e = e_1 - e_2$ for $e_1, e_2 \in AExp$,

$$e[x/e'] = e_1[x/e'] - e_2[x/e'],$$

$$\mathcal{A}[\![e[x/e']]\!]\sigma = \mathcal{A}[\![e_1[x/e']]\!]\sigma - \mathcal{A}[\![e_2[x/e']]\!]\sigma \overset{I.H.}{=}$$

$$= \mathcal{A}[\![e_1]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] - \mathcal{A}[\![e_2]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma]$$

$$= \mathcal{A}[\![e_1 - e_2]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma].$$

- if $e = e_1 * e_2$ for $e_1, e_2 \in AExp$, repeating the same argument we would have that

$$e[x/e'] = e_1[x/e'] * e_2[x/e'], \text{ so}$$

$$\mathcal{A}[\![e[x/e']]\!]\sigma = \mathcal{A}[\![e_1[x/e']]\!]\sigma * \mathcal{A}[\![e_2[x/e']]\!]\sigma \overset{I.H.}{=}$$

$$= \mathcal{A}[\![e_1]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] * \mathcal{A}[\![e_2]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] =$$

$$= \mathcal{A}[\![e_1 * e_2]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma].$$

That proves the lemma.