The (simply-typed) λ-calculus

Theory for Programming Languages
MASTER IN FORMAL METHODS FOR SW ENGINEERING
UCM / UPM / UAM

December 13, 2021

Exercise 1. Give a proof in TA_λ for the following type assignment for function composition:

$$\vdash \lambda f.\lambda g.\lambda x.f(g\ x) : (b \to c) \to (a \to b) \to (a \to c)$$

Exercise 2. Provide a type for the combinator $\mathsf{S} = \lambda x.\lambda y.\lambda z.(x\ z)(y\ z)$.

Exercise 3. Provide a type for the term $\mathsf{II} = (\lambda x.x)(\lambda x.x)$.

Exercise 4. Provide types for the terms in exercise 6 of the previous sheet (Church booleans) along with their corresponding derivation.

Exercise 5. Prove that if $\Gamma \vdash M : \sigma$ then $\Gamma[\rho \mapsto \tau] \vdash M : \sigma[\rho \mapsto \tau]$.

Exercise 6. Prove that if $\Gamma, x : \sigma \vdash M : \tau$ and $\Gamma \vdash N : \sigma$ then $\Gamma \vdash M[x \mapsto N] : \tau$.

Exercise 7. Prove that if $M \rightsquigarrow_\beta^* M'$ and $\Gamma \vdash M : \sigma$ then $\Gamma \vdash M' : \sigma$.

Ex 1.

$$\frac{[x : a] \qquad [g : a \to b]}{x : a,\ g : a \to b \vdash g\,x : b}\ \text{App}$$

$$\frac{x : a,\ g : a \to b \vdash g\,x : b \qquad [f : b \to c]}{x : a,\ g : a \to b,\ f : b \to c \vdash f\,(g\,x) : c}\ \text{App}$$

$$\frac{x : a,\ g : a \to b,\ f : b \to c \vdash f\,(g\,x) : c}{x : a,\ g : a \to b,\ f : b \to c \vdash \lambda x.\ f\,(g\,x) : a \to c}\ \text{Abs}$$

$$\frac{x : a,\ g : a \to b,\ f : b \to c \vdash \lambda x.\ f\,(g\,x) : a \to c}{x : a,\ g : a \to b,\ f : b \to c \vdash \lambda g\ \lambda x.\ f\,(g\,x) : (a \to b) \to (a \to c)}\ \text{Abs}$$

$$\frac{x : a,\ g : a \to b,\ f : b \to c \vdash \lambda g\ \lambda x.\ f\,(g\,x) : (a \to b) \to (a \to c)}{x : a,\ g : a \to b,\ f : b \to c \vdash \lambda f.\ \lambda g\ \lambda x.\ f\,(g x) : (b \to c) \to (a \to b) \to (a \to c)}\ \text{Abs}$$

Ex 2.

$$S = \lambda x.\lambda y.\lambda z.(xz)(yz)$$

$$\dfrac{\dfrac{[z:a] \qquad [x:a \to b \to c]}{z:a,\ x:a\to b\to c \vdash xz : b\to c}\,\text{App} \qquad \dfrac{[z:a] \qquad [y:a\to b]}{z:a,\ y:a\to b \vdash yz : b}\,\text{App}}{z:a,\ x:a\to b\to c,\ y:a\to b \vdash (xz)(yz):c}\,\text{App}$$

$$\dfrac{\dfrac{\dfrac{z:a,\ x:a\to b\to c,\ y:a\to b \vdash (xz)(yz):c}{z:a,\ x:a\to b\to c,\ y:a\to b \vdash \lambda z.(xz)(yz) : a\to c}\,\text{Abs}}{z:a,\ x:a\to b\to c,\ y:a\to b \vdash \lambda y.\lambda z.(xz)(yz) : (a\to b)\to(a\to c)}\,\text{Abs}}{z:a,\ x:a\to b\to c,\ y:a\to b \vdash \lambda x.\lambda y.\lambda z(xz)(yz) : (a\to b\to c)\to(a\to b)\to(a\to c)}\,\text{Abs}$$

Ex 3.

$$(\lambda x.x)(\lambda x.x)$$

First of all, we must do α-conversion.

$$(\lambda x.x)(\lambda x.x) \rightsquigarrow_\alpha (\lambda x.x)(\lambda y.y)$$

$$\dfrac{\dfrac{[x:a\to a] \quad [x:a\to a]}{x:a\to a \vdash \lambda x.x : (a\to a)\to(a\to a)}\,\text{Abs} \qquad \dfrac{[y:a] \qquad [y:a]}{y:a \vdash \lambda y.y : a\to a}\,\text{Abs}}{x:a\to a,\ y:a \vdash (\lambda x.x)(\lambda y.y) : a\to a}\,\text{App}$$

Ex 4.

TRUE $= \lambda x.\lambda y.x$

$$\dfrac{\dfrac{[x:a] \qquad [y:b]}{x:a,\ y:b \vdash \lambda y.x : b\to a}\,\text{Abs}}{x:a,\ y:b \vdash \lambda x.\lambda y.x : a\to b\to a}\,\text{Abs}$$

FALSE $= \lambda x.\lambda y.y$

$$\dfrac{\dfrac{[y:b]}{y:b \vdash \lambda y.y : b\to b}\,\text{Abs} \qquad [x:a]}{x:a,\ y:b \vdash \lambda x.\lambda y.y : a\to b\to b}\,\text{Abs}$$

COND = $\lambda b\,\lambda x\,\lambda y\ b\,x\,y$

$$\frac{[b : a \to f \to c] \qquad [x : \alpha]}{b : a \to f \to c,\ x : \alpha \vdash bx : f \to c}\ \text{App}$$

$$\frac{b : a \to f \to c,\ x : \alpha \vdash bx : f \to c \qquad [y : b]}{b : a \to f \to c,\ x : \alpha,\ y : f \vdash bx\,y : c}\ \text{App}$$

$$\frac{b : a \to f \to c,\ x : \alpha,\ y : f \vdash bx\,y : c}{b : a \to f \to c,\ x : \alpha,\ y : f \vdash \lambda y.\, bxy : f \to c}\ \text{Abs}$$

$$\frac{}{b : a \to f \to c,\ x : \alpha,\ y : f \vdash \lambda x.\lambda y.\, bxy : a \to f \to c}\ \text{Abs}$$

$$\frac{}{b : a \to f \to c,\ x : \alpha,\ y : f \vdash \lambda b.\lambda x.\lambda y.\, bxy : (a \to f \to c) \to a \to f \to c}\ \text{Abs}$$

CONJ = $\lambda b_1.\ \lambda b_2.\ b_1\ b_2\ \text{FALSE}$

For the proof tree above, we know that FALSE : $a \to b \to b$

$$\frac{[b_1 : \alpha \to (a \to b \to b) \to \gamma] \qquad [b_2 : \alpha]}{b_1 : \alpha \to (a \to b \to b) \to \gamma,\ b_2 : \alpha \vdash b_1 b_2 : (a \to b \to b) \to \gamma}\ \text{App}$$

$$\frac{b_1 : \alpha \to (a \to b \to b) \to \gamma,\ b_2 : \alpha \vdash b_1 b_2 : (a \to b \to b) \to \gamma \qquad [\text{FALSE} : a \to b \to b]}{b_1 : \alpha \to (a \to b \to b) \to \gamma,\ b_2 : \alpha,\ \text{FALSE} : a \to b \to b \vdash b_1 b_2\,\text{FALSE} : \gamma}\ \text{App}$$

$$\frac{}{b_1 : \alpha \to (a \to b \to b) \to \gamma,\ b_2 : \alpha,\ \text{FALSE} : a \to b \to b \vdash \lambda b_2.\, b_1 b_2\,\text{FALSE} : \alpha \to \gamma}\ \text{Abs}$$

$$\frac{}{b_1 : \alpha \to (a \to b \to b) \to \gamma,\ b_2 : \alpha,\ \text{FALSE} : a \to b \to b \vdash \lambda b_1.\lambda b_2.\, b_1 b_2\,\text{FALSE} : (\alpha \to (a \to b \to b) \to \gamma) \to \alpha \to \gamma}\ \text{Abs}$$

DISJ = $\lambda b_1.\ \lambda b_2\ .\ b_1\ \text{TRUE}\ b_2$

For the proof tree above, we know that TRUE : $a \to b \to a$

$$\frac{[b_1 : (a \to b \to a) \to \alpha \to \gamma] \qquad [\text{TRUE} : a \to b \to a]}{b_1 : (a \to b \to a) \to \alpha \to \gamma,\ \text{TRUE} : a \to b \to a \vdash b_1\,\text{TRUE} : \alpha \to \gamma}\ \text{App}$$

$$\frac{b_1 : (a \to b \to a) \to \alpha \to \gamma,\ \text{TRUE} : a \to b \to a \vdash b_1\,\text{TRUE} : \alpha \to \gamma \qquad [b_2 : \alpha]}{b_1 : (a \to b \to a) \to \alpha \to \gamma,\ \text{TRUE} : a \to b \to a,\ b_2 : \alpha \vdash b_1\,\text{TRUE}\,b_2 : \gamma}\ \text{App}$$

$$\frac{}{b_1 : (a \to b \to a) \to \alpha \to \gamma,\ \text{TRUE} : a \to b \to a,\ b_2 : \alpha \vdash \lambda b_2.\ b_1\,\text{TRUE}\,b_2 : \alpha \to \gamma}\ \text{Abs}$$

$$\frac{}{b_1 : (a \to b \to a) \to \alpha \to \gamma,\ \text{TRUE} : a \to b \to a,\ b_2 : \alpha \vdash \lambda b_1.\lambda b_2.\ b_1\,\text{TRUE}\,b_2 : ((a \to b \to a) \to \alpha \to \gamma) \to \alpha \to \gamma}\ \text{Abs}$$

NEG = $\lambda b\ \lambda x\,\lambda y\ .\ b\ y\ x$

$$\frac{[b : f \to a \to c] \qquad [y : f]}{b : f \to a \to c,\ y : f \vdash by : a \to c}\ \text{App}$$

$$\frac{b : f \to a \to c,\ y : f \vdash by : a \to c \qquad [x : a]}{b : f \to a \to c,\ y : f,\ x : a \vdash by\,x : c}\ \text{App}$$

$$\frac{}{b : f \to a \to c,\ y : f,\ x : a \vdash \lambda y.\, byx : f \to c}\ \text{Abs}$$

$$\frac{}{b : f \to a \to c,\ y : f,\ x : a \vdash \lambda x\,\lambda y.\, byx : a \to f \to c}\ \text{Abs}$$

$$\frac{}{b : f \to a \to c,\ y : f,\ x : a \vdash \lambda b\,\lambda x\,\lambda y.\, byx : (f \to a \to c) \to a \to f \to c}\ \text{Abs}$$