Semantics of Programming Languages
Assignment 1

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Exercise 1. *Let us assume $e, e' \in$ **AExp** and $x \in$ **Var**. The notation $e[x/e']$ denotes the result of replacing all occurrences of x in e by e'. For example: $(x+y)[x/(3*z)] = (3*z)+y$.*

1. *Define $e[x/e']$ in a compositional way.*
2. *Prove the following substitution lemma: for all $e, e' \in$ **AExp**, $x \in$ **Var**, $\sigma \in$ **State***

$$\mathcal{A}[\![e[x/e']]\!]\ \sigma = \mathcal{A}[\![e]\!]\ \sigma[x \mapsto \mathcal{A}[\![e']\!]\ \sigma]$$

Solution. We can define $e[x/e']$ as follows:

$$\begin{array}{rcl} n[x/e'] & := & n, \quad n \in \mathbb{N} \\ y[x/e'] & := & \begin{cases} e' & if\ y = x \\ y & otherwise \end{cases} \qquad y \in \mathbf{Var} \\ (e_1 + e_2)[x/e'] & := & e_1[x/e'] + e_2[x/e'] \\ (e_1 - e_2)[x/e'] & := & e_1[x/e'] - e_2[x/e'] \\ (e_1 * e_2)[x/e'] & := & e_1[x/e'] * e_2[x/e'] \end{array}$$

Then we can prove the substitution lemma compositionally. Let $e, e' \in$ **AExp**, $x \in$ **Var**, $\sigma \in$ **State**, we have several cases:

- $e = n$ such that $n \in \mathbb{N}$. Then

$$\mathcal{A}[\![n[x/e']]\!]\ \sigma = n = \mathcal{A}[\![n]\!]\ \sigma[x \mapsto \mathcal{A}[\![e']\!]\ \sigma]$$

  directly.

- If $e = y$ with $y \in$ **Var**, supposing $y \neq x$ we have

$$\begin{array}{rcl} \mathcal{A}[\![y[x/e']]\!]\ \sigma & \overset{\mathcal{A}-def}{=} & \sigma(y) \\ & \overset{\sigma-def}{=} & \sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma](y) \\ & \overset{\mathcal{A}-def}{=} & \mathcal{A}[\![y]\!]\ \sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] \end{array}$$

  because $\sigma(y) = \sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma](y)$ since $y \neq x$. Otherwise if $x = y$ then

$$\begin{array}{rcl} \mathcal{A}[\![y]\!]\ \sigma[x \mapsto \mathcal{A}[\![e']\!]\ \sigma] & \overset{\mathcal{A}-def}{=} & \sigma[x \mapsto \mathcal{A}[\![e']\!]\ \sigma](y) \\ & \overset{\sigma-def}{=} & \mathcal{A}[\![e']\!]\ \sigma \\ & \overset{subs-def}{=} & \mathcal{A}[\![y[x/e']]\!]\ \sigma \end{array}$$

- Suppose that $e_1, e_2 \in \mathbf{AExp}$ verifies the lemma and $\Box \in \{+, -, *\}$, then

$$\begin{aligned}
\mathcal{A}[\![(e_1 \Box e_2)[x/e']]\!]\ \sigma &\overset{subs-def}{=} \mathcal{A}[\![e_1[x/e']\ \Box e_2[x/e']]\!]\ \sigma \\
&\overset{\mathcal{A}-def}{=} \mathcal{A}[\![e_1[x/e']]\!]\ \sigma\ \Box\ \mathcal{A}[\![e_2[x/e']]\!]\ \sigma \\
&\overset{hip}{=} \mathcal{A}[\![e_1]\!]\ \sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma]\ \Box\ \mathcal{A}[\![e_2]\!]\ \sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] \\
&\overset{\mathcal{A}-def}{=} \mathcal{A}[\![e_1 \Box e_2]\!]\ \sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma]
\end{aligned}$$

as desired.