Proposition 1.11

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Let $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ $(1 \leq i \leq n)$ and the hypothesis be true for $n-1$. For each $i \leq n$, $\mathfrak{a} \not\subseteq \mathfrak{p}$ for $j \neq i$; there are $n-1$ such j's, so by the induction hypothesis, $\mathfrak{a} \not\subseteq \bigcup_{\substack{j=1 \\ j\neq i}}^{n} \mathfrak{p}_j$; then there is some x_i in $\mathfrak{a}$ that is not in any $\mathfrak{p}_j$ for $j \neq i$.

Now let

$$y = \sum_{i=1}^{n} x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$$

Can the i'th term be in $\mathfrak{p}_i$? No, since then, as the ideal is prime, we would have $x_j \in \mathfrak{p}_i$ for some $j \neq i$. The factor x_i is in every term except $x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$; factoring out, for some z

$$x_i z + x_1 \cdots x_{i-1} x_{i+1} \cdots x_n = y$$

Can y be in $\mathfrak{p}_i$? Now the first summand is in the ideal, as is the right side, ten one of factors must be in the ideal, a contradiction to what we observed.