Facts about Rings of Fractions

Rafal Michalski

1 Introduction

Fact 1.1. *If $0 \in S$, then $S^{-1}A$ is a trivial ring.*

Proof. Any $(a, s), (a', s')$ are related because $(as' - a's) \cdot 0 = 0$ with $0 \in S$.

Fact 1.2. *A a PID, the equivalence relation in $A \times S$ is: $(a, s) \equiv (a', s')$ iff $as' = a's$.* □

Fact 1.3. *For A a field, and $S = \{-1, 1\}$, $S^{-1}A \cong A$.*

Proof. It is easily verified that the standard isomorphism from A to $S^{-1}A$ is 1-1 and onto. □

Fact 1.4. *For A a field, and S a multiplicatively closed subset of A not containing zero, $S^{-1}A \cong A$.*

Proof. The standard homomorphism $f : a \mapsto a/1$ of A into $S^{-1}A$ is injectve: if $a/1 = a'/1$ then $a \cdot 1 = a1 \cdot 1$, then $a = a'$. It is surjective: $f(as^{-1}) = f(a)f(s^{-1}) = (a/1)(s^{-1}/1) = \ldots$, but $s^{-1}/1 = 1/s$ as $s^{-1}s = 1 \cdot 1$; continuing, $\ldots = (a/1)(1/s) = a/s$. □

Fact 1.5. *For A a field, the ring of fractions and the field of fractions are isomorphic.*

Proof. For isomorphism of A with its field of fractions, see Math Exchange 79188. About the isomorphism with its ring of fractions, is the fact above. □

Example 1.6. *Some example.*

Fact 1.7. *The quotient ring A/I can be viewed as an A-module, and then the ring of fractions $T^{-1}(A/I)$, where T is the image of S in A/I, equals the module of fractions $S^{-1}(A/I)$.*

Proof. On the left, the relation is in $(A/I) \times T$: $([a], [s]) \equiv ([a'], [s'])$ iff (ring notation) $([a][s'] - [a'][s])[s''] = [0]$ iff $[as's'' - a'ss''] = [0]$. On the right, the relation works in $(A/I) \times S$: $([a], s) \equiv ([a'], s')$ iff (module notation) $s''(s'[a] - s[a']) = [0]$ iff $[as's'' - a'ss''] = [0]$. The conditions are identical so the classes must be in bijective. However, they are not identical as sets, so saying *equals* is too much. □

Fact 1.8. *What is $S^{-1}\mathfrak{a}$?*

It can be either an $S^{-1}A$-module $S^{-1}\mathfrak{a}$, because $\mathfrak{a}$ is an A-module, or the extension $S^{-1}\mathfrak{a} = \mathfrak{a}\, S^{-1}A$ in $S^{-1}A$ of the ideal $\mathfrak{a}$ in A. In both cases elements of $S^{-1}\mathfrak{a}$ are written as a/s with $a \in \mathfrak{a}$, $s \in S$, but they come from different sets. In the first, module case, a/s is in the quotient of $\mathfrak{a} \times S$, in the second, extension ideal case, a/s is in the quotient of $A \times S$. We are talking of $S^{-1}A$-modules, not rings, so there can only be A-module and $S^{-1}A$-module isomorphism:

$$\mathfrak{a} \times S/ \sim_{\mathfrak{a}} \;\ni\; a/s \mapsto a/s \;\in\; A \times S/ \sim_A$$

□

Fact 1.9. *What is $S^{-1}\mathfrak{a}$ in case $S = A \setminus \mathfrak{p}$?*

It can be either an $A_{\mathfrak{p}}$-module $\mathfrak{a}_{\mathfrak{p}}$, because $\mathfrak{a}$ is an A-module, or the extension $S^{-1}\mathfrak{a} = \mathfrak{a}\, A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}$ of the ideal $\mathfrak{a}$ in A. In both cases elements of $S^{-1}\mathfrak{a}$ are written as a/s with $a \in \mathfrak{a}$, $s \notin \mathfrak{p}$, but they come from different sets. In the first, module case, a/s is in the quotient of $\mathfrak{a} \times (A \setminus \mathfrak{p})$, in the second, extension ideal case, a/s is in the quotient of $A \times (A \setminus \mathfrak{p})$. We are talking of $A_{\mathfrak{p}}$-modules, not rings, so there can only be an A-module and $A_{\mathfrak{p}}$-module isomorphism:

$$\mathfrak{a} \times (A \setminus \mathfrak{p})/ \sim_{\mathfrak{a}} \ \ni \ a/s \mapsto a/s \ \in \ A \times (A \setminus \mathfrak{p})/ \sim_A$$

□

Fact 1.10. *Case $\mathfrak{a} = \mathfrak{p}$, a prime ideal. What is $S^{-1}\mathfrak{p}$?*

It can be either the $A_{\mathfrak{p}}$-module $\mathfrak{p}_{\mathfrak{p}}$, because $\mathfrak{p}$ is an A-module, or the extension $\mathfrak{p}A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}$ of the ideal $\mathfrak{p}$ in A, via the canonical $A \to A_{\mathfrak{p}} : a \mapsto a/s$. Looks like we don't have the $\cdot_{\mathfrak{p}}$-instead-of-$S^{-1}\cdot$ notation in the ideal extension case, but then, the quotient notation $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ is used, which makes sense only if $\mathfrak{p}_{\mathfrak{p}}$ is an ideal in $A_{\mathfrak{p}}$

$$\mathfrak{p}_{\mathfrak{p}} = \mathfrak{p}\, A_{\mathfrak{p}}$$

□

Fact 1.11. *The contraction of $S^{-1}\mathfrak{p}$ is $\mathfrak{p}$.*

$$\begin{aligned}\{a : a/1 \in S^{-1}\mathfrak{p}\} &= \{a : a/1 = a'/s' \text{ for some } a' \in \mathfrak{p}, s' \notin \mathfrak{p}\}\\ &= \{a : \exists u \in S\ (as' - a')u = 0\}\\ &= \{a : \exists u \in S, s' \in S, a' \in \mathfrak{p} \quad as'u = a'u\}\end{aligned}$$

The set finally is $\mathfrak{p}$:
$\Longrightarrow$: $as'u \in \mathfrak{p}$, so $a \in \mathfrak{p}$ or $s'u \in \mathfrak{p}$ but $\mathfrak{p} \cap S = \emptyset$; must be $a \in \mathfrak{p}$.
$\Longleftarrow$: $a \in \mathfrak{p}$; $(a \cdot 1 - a) \cdot 1 = 0$, so a is in the set.

□

Fact 1.12. *What is $\mathfrak{a}_{\mathfrak{p}}$?*

It is the A-module $\mathfrak{a}$ localized at $\mathfrak{p}$. It is an $A_{\mathfrak{p}}$-module. We also use this notation for the ideal $S^{-1}\mathfrak{a}$ of $S^{-1}A$, where $S = A \setminus \mathfrak{p}$. How are they isomorphic? $a/s \mapsto a/s$ with $a \in \mathfrak{a}$, $s \notin$. Of what it is an isomorphism? Of A-modules, of $A_{\mathfrak{p}}$-modules. They are not rings. □

Fact 1.13. *What is $\mathfrak{p}_{\mathfrak{p}}$?*

It is the A-module $\mathfrak{p}$ localized at $\mathfrak{p}$. We also use this notation for the ideal $S^{-1}\mathfrak{p}$ of $S^{-1}A$, where $S = A \setminus \mathfrak{p}$, that is, the ideal $\mathfrak{p}A_{\mathfrak{p}}$. □

Fact 1.14. *What is $\mathfrak{p}A_{\mathfrak{p}}$?*

As $A_{\mathfrak{p}}$ is an A-module, we can multiply it by a prime ideal in A in a standard way

$$\sum a_i' \frac{a_i}{s_i} = \sum \frac{a' a_i}{s_i}$$

After bringing to a common denominator, this is

$$\frac{a}{s}$$

with $a \in \mathfrak{p}$, so $\mathfrak{p}A_{\mathfrak{p}}$ is the single maximal ideal of the local ring $A_{\mathfrak{p}}$. □

Fact 1.15. *What are $\mathfrak{p}A_{\mathfrak{p}}$ and $\mathfrak{a}A_{\mathfrak{p}}$?*

In the Solutions by Y. P. Gaillard, the residue field is $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, so the single maximal ideal of $A_{\mathfrak{p}}$ from the Example 1, p. 38 of ItCA must be just $\mathfrak{p}A_{\mathfrak{p}}$

$$\mathfrak{p}A_{\mathfrak{p}} = \{a/s : a \in \mathfrak{p}, s \notin \mathfrak{p}\}$$

Then $\mathfrak{a}A_{\mathfrak{p}}$ must be the generalization n □

Fact 1.16. *When $S = A \setminus \mathfrak{p}$, as $A_\mathfrak{p}$-modules*

$$S^{-1}\mathfrak{a} = \mathfrak{a}A_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}$$

$$S^{-1}\mathfrak{p} = \mathfrak{p}A_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}$$

$$S^{-1}\mathfrak{p}' = \mathfrak{p}'A_\mathfrak{p}$$

for prime ideal $\mathfrak{p}' \subseteq \mathfrak{p}$.

□

Fact 1.17. *Same notation for general S*

$$S^{-1}\mathfrak{a} = \mathfrak{a}S^{-1}A$$

□

Fact 1.18. *How is $A_\mathfrak{p}$ an A-module ?*

The canonical map $\phi : A \to A_\mathfrak{p} : a \mapsto \frac{a}{1}$ gives the multiplication by scalars from A

$$a'\frac{a}{s} = \phi(a')\frac{a}{s} = \frac{a'}{1}\frac{a}{s} = \frac{a'a}{s}$$

□

Fact 1.19. *How is $B_\mathfrak{q}$ an $A_\mathfrak{p}$-module?*

Let $g = \psi \circ f$ be the composition $A \to B \to T^{-1}B : a \to f(a) \to f(a)/1$. This composition sends $s \in S$ to a unit in $T^{-1}B$, as $(f(s)/1)(1/f(s)) = 1/1$, where $f(s) \in f(S) = f(A \setminus \mathfrak{p}) \subseteq B \setminus \mathfrak{q} = T$. Why the inclusion? If $a \notin \mathfrak{p} = f^{-1}(\mathfrak{q})$ then $f(a) \notin \mathfrak{q}$. By the universal property of the ring of fractions, g factorizes

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f"] \arrow[dr, "g"] & S^{-1}A \arrow[d, "h"] \\ B \arrow[r, "\psi"] & T^{-1}B \end{tikzcd}$$

where the recipe for h is given in **Proposition 3.1** of ItCA as $a/s \mapsto g(a)g(s)^{-1} = (f(a)/1)(1/f(s)) = f(a)/f(s)$. □

Fact 1.20. *What is $B_\mathfrak{p}$?*

(For $f : A \mapsto B$ and $\mathfrak{p}$ a prime ideal of A).
The ring B is an A-module by the restriction of scalars. We can localize it in the prime ideal $\mathfrak{p}$ of A. The cartesian product is $B \times (A \setminus \mathfrak{p})$, the relation is

$$(b, s) \sim (b', s') \iff \exists t \notin \mathfrak{p} \;\; t(sb' - s'b) = 0$$

The condition reads

$$f(t)(f(s)b' - f(s')b) = 0$$

The obvious addition

$$\frac{b}{s} + \frac{b'}{s'} = \frac{s'b + sb'}{ss'} = \frac{f(s')b + f(s)b'}{ss'}$$

The obvious scalar multiplication

$$\frac{a}{s'} \cdot \frac{b}{s} = \frac{ab}{s's} = \frac{f(a)b}{s's}$$

□

Fact 1.21. *How is $B_\mathfrak{p}$ an $A_\mathfrak{p}$-module?*

And f is an homomorphism of A modules:

$$f(a'a) = f(a')f(a) = a' \cdot f(a)$$

This gives rise to an $A_\mathfrak{p}$-module homomorphism $S^{-1}f : A_\mathfrak{p} \to B_\mathfrak{p}$

$$a/s \mapsto f(a)/s$$

See how it is different from the map $A_\mathfrak{p} \to B_\mathfrak{q}$

$$a/s \mapsto f(a)/f(s)$$

By the restriction of scalars, $B_\mathfrak{p}$ is an $A_\mathfrak{p}$-module. $\square$

Fact 1.22. *$B_\mathfrak{p}$ is a ring.*

The multiplication

$$\frac{b}{s} \cdot \frac{b'}{s'} = \frac{bb'}{ss'}$$

is distributive over the addition.

$$\begin{aligned}
\frac{b''}{s''}\left(\frac{b'}{s'} + \frac{b}{s}\right) &= \frac{b''}{s''}\frac{sb' + s'b}{s's} \\
&= \frac{b''(sb' + s'b)}{s''s's'} \\
&= \frac{sb''b' + s'b''b}{s''s's} \\
&= \frac{b''b'}{s''s'} + \frac{b''b}{s''s}
\end{aligned}$$

$$\begin{aligned}
\frac{b''}{s''}\frac{b'}{s'} + \frac{b''}{s''}\frac{b}{s} &= \frac{s''sb''b' + s''s'b''b}{s''s's''s} \\
&= \frac{f(s'')f(s)b''b' + f(s'')f(s')b''b}{s''s's''s} \\
&= \frac{f(s'')f(s)b''b'}{s''s's''s} + \frac{f(s'')f(s')b''b}{s''s's''s}
\end{aligned}$$

How can we cancel here? In a general $S^{-1}A$-module $S^{-1}M$

$$\frac{f(s)m}{s} = \frac{s \cdot m}{s} = \frac{s}{s} \cdot \frac{m}{1} = \frac{1}{1} \cdot \frac{m}{1} = \frac{m}{1}$$

With this cancellation rule, both sides of the distributivity become equal. $\square$

Fact 1.23. *$S^{-1}B$ (an $S^{-1}A$-module) is a ring.*

By argument identical to that for the $B_\mathfrak{p}$ ring. $\square$

Fact 1.24. *A ring homomorphism $S^{-1}A \to S^{-1}B$.*

It is

$$\frac{a}{s} \mapsto \frac{f(a)}{s}$$

Preservation of the multiplication is immediately verified. $\square$

Fact 1.25. *A ring homomorphis $A_\mathfrak{p} \to B_\mathfrak{p}$.*

It is

$$\frac{a}{s} \mapsto \frac{f(a)}{s}$$

Preservation of the multiplication is immediately verified. $\square$

Fact 1.26. *The ring $f(S)^{-1}B$.*

The subset $f(S)$ of the ring B is multiplicatively closed, and we can take the ring of fractions. The construction starts from $B \times f(S)$,

$$(b, f(s)) \sim (b', f(s')) \iff \exists u \in S \ f(u)(f(s')b - f(s)b') = 0$$

$\square$

Fact 1.27. *The rings $S^{-1}B$ and $f(S)^{-1}B$ are isomorphic via $b/s \mapsto b/f(s)$*

The well-definition and injectivity are easily verified and the surjectivity is obvious. $\square$

Fact 1.28. *The homomorphisms $B \to S^{-1}B : b \mapsto b/s$ and $B \to f(S)^{-1}B : b \mapsto b/f(s)$.*

8 The second is natural as $f(S)$ is a multiplicatively closed subset of B. The first can arise from the isomorphism of both rings, making the diagram

$$\begin{tikzcd}
 & S^{-1}B \arrow[dd, "\cong"] \\
B \arrow[ur] \arrow[dr] & \\
 & f(S)^{-1}B
\end{tikzcd}$$

$$\begin{tikzcd}
 & b/1 \arrow[dd, mapsto] \\
b \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & b/f(1) = b/1
\end{tikzcd}$$

commutative, or can be verified directly, and the diagram after it. The top homomorphism

$\square$

Fact 1.29. *What are general ideals of $f(S)^{-1}B$ and $S^{-1}B$?*

Every ideal of $f(S)^{-1}B$ is an extended ideal $f(S)^{-1}\mathfrak{b} = \mathfrak{b}B_{\mathfrak{p}} = \{b/f(s) : b \in \mathfrak{b}, s \in S\}$. The isomorphic set in $S^{-1}B$ is $\mathfrak{b}S^{-1}B = \{b/s : b \in \mathfrak{b}, s \in S\}$. $\square$

Fact 1.30. *What are general prime ideals of $f(S)^{-1}B$ and $S^{-1}B$?*

Prime ideals of $f(S)^{-1}B$ are in 1-1 correspondence with prime ideals of B not meeting $f(S)$.

$$\mathfrak{q} \longleftrightarrow f(S)^{-1}\mathfrak{q} = \mathfrak{q}f(S)^{-1}B$$

Contraction of the right on the left, extension of the left on the right. $\square$

Fact 1.31. *What are general prime ideals of $B_{\mathfrak{p}}$?*

$B_{\mathfrak{p}} = S^{-1}B \cong f(S)^{-1}B$ for $S = A \setminus \mathfrak{p}$. Prime ideals of $f(A \setminus \mathfrak{p})^{-1}B$ are in 1-1 correspondence with prime ideals of B not meeting $f(A \setminus \mathfrak{p})$. We have no better option than using $f(A \setminus \mathfrak{p})$ here.

$$\mathfrak{q} \longleftrightarrow f(A \setminus \mathfrak{p})^{-1}\mathfrak{q} = \mathfrak{q}B_{\mathfrak{p}}$$

Contraction of the right on the left, extension of the left on the right. $\square$

Fact 1.32. *How does $(S^{-1}f)^* : \mathrm{Spec}(S^{-1}B) \to Spec(S^{-1}B)$ work?*

We show that $(S^{-1}f)^*(S^{-1}\mathfrak{q}) = \mathfrak{p}$ where $\mathfrak{p} = f^{-1}(\mathfrak{q})$.

$$\begin{aligned}(S^{-1}f)^*(S^{-1}\mathfrak{q}) &= (S^{-1}f)^*(\{b/s : b \in \mathfrak{q}, s \in S\}) \\ &= \{a/s : (S^{-1}f)(a/s) \in S^{-1}\mathfrak{b}\} \\ &= \{a/s : f(a)/s \in \mathfrak{b}\}\end{aligned}$$

We can show that this is $S^{-1}\mathfrak{p}$.

The $\subseteq$: $f(a)/s = b/s'$ for some $b \in \mathfrak{q}, s \in S$; we move to $f(S)^{-1}B$; $f(a)/f(s) = b/f(s')$; $(f(a)f(s') - bf(s))f(u) = 0$ for some $u \in S$; $f(a)f(s')f(u) = bf(s)f(u) \in \mathfrak{b}$; $f(a) \in \mathfrak{q}$; $a \in f^{-1}(\mathfrak{q}) = \mathfrak{p}$; $a/s \in S^{-1}\mathfrak{p}$.

What if we did not move to $f(S)^{-1}B$? $u(s'f(a) - sb) = 0$ in the $S^{-1}A$-module $S^{-1}B$; $us'f(a) = usb$. But what is the multiplication by scalar from A? It is multiplication by f of it. $f(u)f(s')f(a) = f(u)f(s)b \in \mathfrak{q}$ ... We proceed the same way.

The $\supseteq$: $a/s \in S^{-1}\mathfrak{p}$; $a \in \mathfrak{p}$; $f(a) \in \mathfrak{q}$; $f(a)/s \in S^{-1}\mathfrak{q}$.

Now we know that $(S^{-1}f)^*oll$ is the restriction of f^* to $\psi^*(\mathrm{Spec}(S^{-1}B)) = S^{-1}Y$ □

Fact 1.33. *How is $B_\mathfrak{q}/\mathfrak{q}_\mathfrak{q} = B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$ an $A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$-module?*

We know the map $A_\mathfrak{p} \to B_\mathfrak{q} : a/s \mapsto f(a)/f(s)$ from 1.19. The kernel of the composition $A_\mathfrak{p} \to B_\mathfrak{q} \to B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} : a/s \mapsto f(a)/f(s) + \mathfrak{q}B_\mathfrak{q}$ contains $\mathfrak{p}A_\mathfrak{p}$: element of $\mathfrak{p}A_\mathfrak{p}$ is a/s where $a \in \mathfrak{p}, s \notin \mathfrak{p}$; it follow that $f(s) \notin \mathfrak{q}$ (otherwise $s \in f^{-1}(\mathfrak{q}) = \mathfrak{p}$); so the image in the first map of a/s is in $\mathfrak{q}B_\mathfrak{q}$, the kernel of the second map, then a/s is in the kernel of the composition. The composition then factors through $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$: $a/s + \mathfrak{p}A_\mathfrak{p} \mapsto f(a)/f(s) + \mathfrak{q}B_\mathfrak{q}$. This is a ring homomorphism that makes $B_\mathfrak{q}/\mathfrak{q}_\mathfrak{q}$ an $A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$-module. □

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Fact 1.34. *What is $\mathfrak{p}M_\mathfrak{p}$?*

When $M_\mathfrak{p}$ is seen as an A-module, $\mathfrak{p}M_\mathfrak{p} = \{am/s : a \in \mathfrak{p}, m \in M, s \notin \mathfrak{p}\}$. When $M_\mathfrak{p}$ is seen as an $A_\mathfrak{p}$-module, $\mathfrak{p}$ is not even an ideal in $A_\mathfrak{p}$, but its extension, $\mathfrak{p}A_\mathfrak{p}$ is, and $(\mathfrak{p}A_\mathfrak{p})M_\mathfrak{p} = \{(a/s')(m/s) : a \in \mathfrak{p}, m \in M, s, s' \notin \mathfrak{p}\} = \{am/s : a \in \mathfrak{p}, m \in M, s \notin \mathfrak{p}\}$, the same set, which we write $\mathfrak{p}M_\mathfrak{p}$ for:

$$\mathfrak{p}M_\mathfrak{p} = (\mathfrak{p}A_\mathfrak{p})M_\mathfrak{p}$$

□

Fact 1.35. *What is $\mathfrak{p}B$?*

For $f : A \to B$, we can think in two ways. As we identify $ab = f(a)b$, $\mathfrak{p}B = \{\sum a_ib_i = \sum f(a_i)b_i : a_i \in \mathfrak{p}, b_i \in B\}$ is the extension $f(\mathfrak{p})B$ of the ideal $\mathfrak{p}$. The second way is that B is an A-module, and $\mathfrak{p}$ a prime ideal in A, so we can form $\mathfrak{p}B = \{\sum a_ib_i = \sum f(a_i)b_i\}$ with $a_i \in \mathfrak{p}$, $b_i \in B$, getting the same set.

□

Fact 1.36. *What is $\mathfrak{p}B_\mathfrak{p}$?*

$B_\mathfrak{p}$ is an A - module, $\mathfrak{p}$ is a prime ideal of A, so $\mathfrak{p}B_\mathfrak{p}$ makes sense and consists of finite sums $\sum a_i(b_i/s_i) = \sum(a_ib_i)/s_i$ where $a_i \in \mathfrak{p}$, $b_i \in B$, and $s_i \in A \setminus \mathfrak{p}$. After bringing to common denominator, the sum becomes ab/s where $a \in \mathfrak{p}$, $b \in B$ and $s \in A \setminus \mathfrak{p}$. We observe that $b \in \mathfrak{p}B+$. □

Fact 1.37. *What is $(\mathfrak{p}A_\mathfrak{p})B_\mathfrak{p}$?*

As $B_\mathfrak{p}$ is an $A_\mathfrak{p}$-module, and $\mathfrak{p}A_\mathfrak{p}$ is the single maximal ideal of the local ring $A_\mathfrak{p}$, Any element is, from the definition of the ideal-by-module and from the general element of $\mathfrak{p}A_\mathfrak{p}$ $(a \in \mathfrak{p})$

$$\sum_i \frac{a_i}{s'_i}\frac{b_i}{s_i} = \sum \frac{ab}{s's}$$

After bringing to a common denominator, this becomes

$$ab/s = f(a)b/s$$

where $a \in \mathfrak{p}$. Notice we got the general element of $\mathfrak{p}B_{\mathfrak{p}}$, so

$$(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$$

$\square$

Fact 1.38. *The extension in $B_{\mathfrak{p}}$ of the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ is $\mathfrak{p}B_{\mathfrak{p}}$*

$$\begin{aligned}
B_{\mathfrak{p}}(S^{-1}f)(\mathfrak{p}A_{\mathfrak{p}}) &= B_{\mathfrak{p}}(S^{-1}f)\left\{\frac{a}{s} : a \in \mathfrak{p}, s \notin \mathfrak{p}\right\} \\
&= B_{\mathfrak{p}}\left\{\frac{f(a)}{s} : a \in \mathfrak{p}, s \notin \mathfrak{p}\right\} \\
&= \left\{\frac{bf(a)}{s} : a \in \mathfrak{p}, b \in B, s \notin \mathfrak{p}\right\} \\
&= \left\{\frac{ab}{s} : a \in \mathfrak{p}, b \in B, s \notin \mathfrak{p}\right\}
\end{aligned}$$

We know from Facts that this is $\mathfrak{p}B_{\mathfrak{p}}$. $\square$

Fact 1.39. *How*

$$\frac{(B \otimes_A M)_{\mathfrak{q}}}{\mathfrak{q}(B \otimes_A M)_{\mathfrak{q}}} \cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_B B \otimes_A M$$

?

Proposition 3.5 states, in the language of subscript-$\mathfrak{p}$, that $M_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A M$ over $A_{\mathfrak{p}}$. Here $(B \otimes_A M)_{\mathfrak{q}} \cong B_{\mathfrak{q}} \otimes_B (B \otimes_A M)$ over $B_{\mathfrak{p}}$. Then

$$\begin{aligned}
\frac{B_{\mathfrak{q}} \otimes_B (B \otimes_A M)}{(\mathfrak{q}B_{\mathfrak{q}})(B_{\mathfrak{q}} \otimes_B (B \otimes_A M))} &\cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}B_{\mathfrak{q}}} \otimes_{B_{\mathfrak{q}}} (B_{\mathfrak{q}} \otimes_B (B \otimes_A M)) \\
&\cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_B B \otimes_A M
\end{aligned}$$

$\square$

The first equality is from Exercise 2.2: $M/\mathfrak{a}M \cong A/\mathfrak{a} \otimes_A M$. In P. Y. Gaillard solution to ItCA Exercise 3.19 (viii).

Fact 1.40. *How*

$$\frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}} \otimes_A M \cong \frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}}$$

?

$$\frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}} \otimes_A M \cong \frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}} \otimes_{A_{\mathfrak{p}}} A =_{\mathfrak{p}} \otimes_A M \cong \frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \cong \frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}}$$

The second by Proposition 3.5, the third by Exercise 2.2.
In Y. P. Gaillard solution of ItCA Exercise 3.19 (viii). $\square$

Fact 1.41. *How $(B \otimes_A M)_{\mathfrak{q}} = B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$?*

$$\begin{aligned}(B \otimes_A M)_{\mathfrak{q}} &= B_{\mathfrak{q}} \otimes_B (B \otimes_A M)\\ &= B_{\mathfrak{q}} \otimes_A M\\ &= (B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}}) \otimes_A M\\ &= B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A M)\\ &= B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}\end{aligned}$$

The first and the last equalities are applications of Proposition 3.5:

$$S^{-1}A \otimes_A M \cong S^{-1}M$$

$$A_{\mathfrak{p}} \otimes_A M \cong M_{\mathfrak{p}}$$
$$\frac{a}{s} \otimes m \mapsto \frac{am}{s}$$

In Y. P. Gaillard solution to ItCA Exercise 3.19 (iii). □

Fact 1.42. *The diagram*

$$\begin{tikzcd} A_{\mathfrak{p}} \arrow[r, "\phi"] \arrow[d, "f"] \arrow[dr, "\eta"] & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \arrow[d, "h"] \\ B_{\mathfrak{q}} \arrow[r, "\psi"] & B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \end{tikzcd}$$

$$\begin{tikzcd} a/s \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "f"] \arrow[dr, mapsto, "\eta"] & a/s + \mathfrak{p}A_{\mathfrak{p}} \arrow[d, mapsto, "h"] \\ f(a)/f(s) \arrow[r, mapsto, "\psi"] & f(a)/f(s) + \mathfrak{q}B_{\mathfrak{q}} \end{tikzcd}$$

is commutative.

All calculated on the diagram. □

Now $\kappa_{\mathfrak{q}} = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ is an $A_{\mathfrak{p}}$-module by $A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ (with the formula as on the bottom diagram) and we may tensor over $A_{\mathfrak{p}}$.

If a field K is an A-module for some ring A, can it be a zero A-module?

$$1_A 1_K = 1_k \neq 0_K$$

It cannot.

Now that $\kappa_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = 0$, both tensorands finitely generated, and $\kappa_{\mathfrak{q}} \neq 0$, it must be $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = 0$ by ItCA Exercise 2.3.

In solution of ItCA 3.19 (viii) by J. D. Taylor.

Fact 1.43. *How* $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}$ *?*

Apply Exercise 2.2

$$A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$$

to $M := B_{\mathfrak{p}}$, $A := A_{\mathfrak{p}}$, $\mathfrak{a} := \mathfrak{p}A_{\mathfrak{p}}$

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} = B_{\mathfrak{p}}/(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}}$$

now apply $(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$. □

Fact 1.44. *How* $A_{\mathfrak{p}} \otimes_A B \cong B_{\mathfrak{p}}$ *?*

Apply Proposition 3.5: $S^{-1}A \otimes_A M \cong S^{-1}M$. □

We now understand the isomorphisms in the solution of ItCA's 3.21(iv) by J D. Taylor.

$$\begin{aligned}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} &= A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}\\ &= K_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}} \otimes_A B\\ &= K_{\mathfrak{p}} \otimes_A B\end{aligned}$$

Fact 1.45. $\mathfrak{p} \supseteq \mathfrak{a} \iff S^{-1}\mathfrak{p} \supseteq S^{-1}\mathfrak{a}$

The $\implies$ direction is universal for ideal extensions. For the $\impliedby$ direction, $(S^{-1}\mathfrak{p})^c \supseteq (S^{-1}\mathfrak{a})^c$ meaning $\mathfrak{p} \supseteq \mathfrak{a}^{ec} \supseteq \mathfrak{a}$ □

Fact 1.46. *If* $\mathfrak{p} \supseteq \mathfrak{a}$ *then* $\mathfrak{p} \supseteq \mathfrak{a}^{ec} \supseteq \bigcup_{s\in S} (\mathfrak{a} : s)$

If $x \in (\mathfrak{a} : s)$ then $xs \in \mathfrak{a} \subseteq \mathfrak{p}$ then $xs \in \mathfrak{p}$ then $x \in \mathfrak{p}$ or $s \in \mathfrak{p}$ but $\mathfrak{p} \cap S = O$ so $x \in \mathfrak{p}$. □

Fact 1.47. $S^{-1}(\mathfrak{a}M) = S^{-1}\mathfrak{a}S^{-1}M = \mathfrak{a}S^{-1}M$

What is $S^{-1}(\mathfrak{a}M)$? $\mathfrak{a}M$ is a submodule of the A-module M that is, it is an A-module. $S^{-1}(\mathfrak{a}M)$. $S^{-1}(\mathfrak{a}M)$ is the module of fractions, with respect to S. Its construction starts from $\mathfrak{a}M \times S$. Its elements are am/s, with $a \in \mathfrak{a}$, classes in the quotient of $\mathfrak{a}M \times S$.

What is $S^{-1}\mathfrak{a} \cdot S^{-1}M$? $S^{-1}\mathfrak{a}$ is the extension of $\mathfrak{a}$ in $S^{-1}A$. Its elements are a/s with $a \in \mathfrak{a}$. $S^{-1}M$ is the module of fractions of M with respect to S. Its elements are m/s. It is an $S^{-1}A$-module so we can multiply it by the ideal $S^{-1}\mathfrak{a}$ of $S^{-1}A$. The elements of $S^{-1}\mathfrak{a} \cdot S^{-1}M$ are $(a/s)(m/s')$ where $a \in \mathfrak{a}$. Any element can be written as am/s with $a \in \mathfrak{a}$. But the construction of $S^{-1}M$ started from $M \times S$. Any am/s is a class in the quotient of $M \times S$.

What is $\mathfrak{a} \cdot S^{-1}M$? $S^{-1}M$ is an $S^{-1}A$-module, but $\mathfrak{a}$ is an ideal in A, not $S^{-1}A$. Still $S^{-1}M$ is also an A-module through the restriction of scalars

$$A \xrightarrow{\phi} S^{-1}A$$

$$a \mapsto a/1$$

The scaling by an element of A is

$$a \cdot \frac{m}{s} = \phi(a) \cdot \frac{m}{s} = \frac{a}{1}\frac{m}{s} = \frac{am}{s}$$

Now $\mathfrak{a} \cdot S^{-1}M$ are am/s with $a \in \mathfrak{a}$. Any of them is a class in the quotient of $M \times S$.

Fact 1.48. *Case* $S = A \setminus \mathfrak{p}$

$$(\mathfrak{a}M)_\mathfrak{p} = \mathfrak{a}_\mathfrak{p} M_\mathfrak{p} = \mathfrak{a}M_\mathfrak{p}$$

□

Fact 1.49. *Case* $\mathfrak{a} = \mathfrak{p}$

$$(\mathfrak{p}M)_\mathfrak{p} = \mathfrak{p}_\mathfrak{p} M_\mathfrak{p} = \mathfrak{p}M_\mathfrak{p}$$

□

Fact 1.50. *Case* $M = A$

$$\mathfrak{p}_\mathfrak{p} = \mathfrak{p}_\mathfrak{p} A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$$

□

Fact 1.51. *For all A-linear map $g : M \to N$ from M to an S^{-1}-module N such that $sm = 0$ for some $s \in S$ and some $m \in M$ implies $g(m) = 0$, there is a unique $S^{-1}A$-linear map $h : S^{-1}M \to N$ such that $g = h \circ f$:*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"] & N \\ S^{-1}M \arrow[ur, dashed, "h"] & \end{tikzcd}$$

What if $sm = 0$ but $g(m) \neq 0$?

$$g(sm) = g(0) = 0$$

but

$$sg(m) = \frac{s}{1}g(m)$$

(restriction of scalars!) is a nonzero vector scaled by a unit, which cannot be zero. The map becomes non-A-linear.

Existence. Let $h(m/s) = (1/s)g(m) = s^{-1}g(m)$. Then h will clearly be an A-module isomorphism provided it is well-defined. Suppose that $m/s = m'/s'$; then there exists $t \in S$ such that $t(s'm - sm') = 0$; taking g on this, $t(s'g(m) - sg(m')) = 0$; multiplying by $1/tss'$, we get $(1/s')g(m) = (1/s)g(m')$. Thus h is well-defined and we get the existence proved.

Uniqueness If h satisfies the condition then $h(m/1) = g(m)/1$ for all $m \in M$; hence, if $s \in S$, $h(m/s) = h((1/s)(m/s)) = (1/s)h(m/s) = (1/s)(g(m)/1) = g(m)/s$ so that h is uniquely determined by g. □

This is the first time we encounter a module over a localized ring that is not itself a localization (in the same multipliset). But maybe it is? By the restriction of scalars, N is also an A-module with scaling

$$an = \phi(a)n = \frac{a}{1}n$$

What if we localize it at S? In $N \times S$, the relation is

$$(n, s) \sim (n', s') \iff \exists u \in S \ u(s'n - sn') = 0$$

Under the quantifier there is

$$\phi(u)(\phi(s')n - \phi(s)n') = 0$$

$$\frac{u}{1}(\frac{s'}{1}n - \frac{s}{1}n') = 0$$

We can use the universal property after verification of A-linearity of the horizontal map g. On the left, N is an A-module, on the right, it is an $S^{-1}A$-module.

$$\begin{tikzcd} N \arrow[r, "g"] \arrow[d, "f"'] & N \\ S^{-1}N \arrow[ur, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} n \arrow[r, mapsto, "g"] \arrow[d, "f"'] & n \\ n/1 \arrow[ur, mapsto, "h"'] & \end{tikzcd}$$

If $sn = 0$ then in the restriction of scalars this means $(1/s)n = 0$ and since $1/s$ is a unit, $n = 0$, then $g(n) = 0$. Now we can use the universal property. The map h on general element is

$$n/s \mapsto \frac{1}{s}n$$

We verify injectivity. Let

$$\frac{1}{s}n = \frac{1}{s'}n'$$

Multiplying by $ss'/1$

$$\frac{s'}{1}n = \frac{s}{1}n'$$

$$s'n = sn'$$

in the A-module N. Then, in $S^{-1}N$,

$$\frac{n}{s} = \frac{n'}{s'}$$

and we have the injectivity proved. Clearly the map is surjective. Then it is an isomorphism

$$S^{-1}N \cong N$$

of $S^{-1}A$-modules.

Fact 1.52. *Any module over a ring of fractions with respect to a multipliset, is a module of fractions with respect to this multipliset.*

□

Fact 1.53. $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \cong (M/\mathfrak{p}M)_\mathfrak{p}$

We start from the exact sequence

$$0 \to \mathfrak{p}M \to M \to M/\mathfrak{p}M \to 0$$

By the exactness of S^{-1} (Proposition 3.3 of the Book), the sequence

$$0 \to (\mathfrak{p}M)_{\mathfrak{p}} \to M_{\mathfrak{p}} \to (M/\mathfrak{p}M)_{\mathfrak{p}} \to 0$$

is exact. As $(\mathfrak{p}M)_{\mathfrak{p}} = \mathfrak{p}M_{\mathfrak{p}}$, the sequence

$$0 \to \mathfrak{p}M_{\mathfrak{p}} \to M_{\mathfrak{p}} \to (M/\mathfrak{p}M)_{\mathfrak{p}} \to 0$$

is exact, with ordinary inclusion on the second left. □

Fact 1.54.

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A M \cong (A/\mathfrak{p})_{\mathfrak{p}} \otimes_A M$$

Follows from the preceding Fact for $M \coloneqq A$. □

Fact 1.55. *Exercise 2.2 of the Book*

$$A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$$

on elements.

The sequence

$$\begin{gathered} 0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0 \\ a \mapsto a \mapsto a + \mathfrak{a} \end{gathered}$$

is exact. We tensor it with M over A

$$\begin{aligned} \mathfrak{a} \otimes_A M \to & A \otimes_A M \to \quad A/\mathfrak{a} \otimes M \to 0 \\ a \otimes m \mapsto \quad & a \otimes m \mapsto (a + \mathfrak{a}) \otimes m \mapsto 0 \end{aligned}$$

By the isomorphism

$$\begin{gathered} A \otimes_A M \cong M \\ a \otimes m \mapsto am \\ 1 \otimes m \leftarrow m \end{gathered}$$

$$\begin{aligned} \mathfrak{a} \otimes_A M \to M \to \quad & A/\mathfrak{a} \otimes M \to 0 \\ a \otimes m \mapsto am \mapsto & (1 + \mathfrak{a}) \otimes am \mapsto 0 \end{aligned}$$

The image of the first homomorphism is precisely $\mathfrak{a}M$, and we take quotient.

$$\begin{gathered} M/\mathfrak{a}M \cong A/\mathfrak{a} \otimes_A M \\ \overline{m} \mapsto \overline{1} \otimes m \\ \overline{am} \leftarrow \overline{a} \otimes m \end{gathered}$$

The inverse map has yet to be defined. Consider the map

$$\begin{gathered} A/\mathfrak{a} \times M \longrightarrow M/\mathfrak{a}M \\ (\overline{a}, m) \longmapsto \overline{am} \end{gathered}$$

It is well-defined: if $\overline{a} = \overline{a'}$ then $a - a' \in \mathfrak{a}$ then $(a - a')m \in \mathfrak{a}M$ then $am - a'm \in \mathfrak{a}M$ then $\overline{am} = \overline{a'm}$. It is clearly A-bilinear. Then it factors through $A/\mathfrak{a} \otimes_A M$

$$\begin{tikzcd} A/\mathfrak{a} \times M \arrow[r, "\otimes"] \arrow[dr] & A/\mathfrak{a} \otimes_A M \arrow[d] \\ & M/\mathfrak{a}M \end{tikzcd} \qquad \begin{tikzcd} (\overline{a}, m) \arrow[r, mapsto, "\otimes"] \arrow[dr] & \overline{a} \otimes m \arrow[d, mapsto] \\ & \overline{am} \end{tikzcd}$$

□

Fact 1.56. *The inverse map in Proposition 3.5 of the Book is*

$$S^{-1}M \cong S^{-1}A \otimes_A M$$
$$\frac{m}{s} \mapsto \frac{1}{s} \otimes m$$

We use the universal property

$$\begin{tikzcd} M \arrow[r] \arrow[d] & S^{-1}A \otimes_A M \\ S^{-1}M \arrow[ur] & \end{tikzcd}$$

Let's define the horizontal map as

$$m \longmapsto \frac{1}{1} \otimes m$$

If $sm = 0$ for some $s \in S, m \in M$, then

$$\frac{1}{1} \otimes sm = 0; \quad s \cdot \frac{1}{1} \otimes m = 0; \quad \frac{s}{1} \otimes m = 0$$

multiplying by $1/s$,

$$\frac{1}{s}\left(\frac{s}{1} \otimes m\right) = 0; \quad \frac{1}{s} \cdot \frac{s}{1} \otimes m = 0; \quad \frac{1}{1} \otimes m = 0$$

The condition for A-bilinearity is fulfilled. Now the universal property defines the skew map as

$$\frac{m}{s} \longmapsto \frac{1}{s} \cdot \left(\frac{1}{1} \otimes m\right) = \frac{1}{s} \otimes m$$

□

Fact 1.57. *The inverse map in Proposition 3.7 of the Book is*

$$S^{-1}(M \otimes_A N) \cong \ S^{-1}M \otimes_{S^{-1}A} S^{-1}N$$
$$\frac{m \otimes n}{s} \longmapsto \frac{m}{s} \otimes \frac{n}{1}$$

We use the universal property

$$\begin{tikzcd} M \otimes_A N \arrow[r] \arrow[d] & S^{-1}M \otimes_{S^{-1}A} S^{-1}N \\ S^{-1}(M \otimes_A N) \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} m \otimes n \arrow[r, mapsto] \arrow[d, mapsto] & \frac{m}{1} \otimes \frac{n}{1} \\ \frac{m \otimes n}{1} \arrow[ur, mapsto] & \end{tikzcd}$$

which will determine the skew map as

$$\frac{m \otimes n}{s} \mapsto \frac{1}{s}\left(\frac{m}{1} \otimes \frac{n}{1}\right)$$

But first the horizontal map must be defined.

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr] & M \otimes_A N \arrow[d] \\ & S^{-1}M \otimes_{S^{-1}A} S^{-1}N \end{tikzcd} \qquad \begin{tikzcd} (m,n) \arrow[r, mapsto, "\otimes"] \arrow[dr] & m \otimes n \arrow[d, mapsto] \\ & \frac{m}{1} \otimes \frac{n}{1} \end{tikzcd}$$

The whole bilinearity is easily verified, thus the map becomes well-defined. Also, mutual inverse is easily verified.

Fact 1.58.

$$(M/\mathfrak{p}M)_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A M/\mathfrak{p}M$$

Application of Proposition 3.5

$$S^{-1}M \cong S^{-1}A \otimes_A M$$
$$m/s \mapsto 1/s \otimes m$$
$$am/s \mapsfrom a/s \otimes m$$

to $M \coloneqq M/\mathfrak{p}M$ as an A-module. □

2 Saturated

Fact 2.1. *For saturated S, if $f(a)$ is a unit in $S^{-1}A$, then $a \in S$.*

Proof.

$$\frac{a}{1} \cdot \frac{b}{t} = \frac{1}{1}$$

$$\frac{ab}{t} = \frac{1}{1}$$

$$(ab, t) \equiv (1, 1)$$

$$(ab - t)u = 0$$

$$abu = tu$$

$$abu \in S$$

As S is saturated, $a \in S$. $\square$