Example 3

July 18, 2021

M.F. Atiyah, I.G. MacDonald *Introduction to Commutative Algebra*
1. RINGS AND IDEALS

Fact. *In a PID, a non-zero prime ideal is maximal.*

Let $y \notin (x)$. Consider the ideal $(x,y) = (z)$. There is $x = az$, $y = bz$, and $cx + dy = z$; it follows that $acz + bdz = z$, and by cancellation, $ac + bd = 1$. Now $cx = z - dy = z - bdz = (1 - bd)z \in (x)$, so it must be one of $z \in (x)$ or $1 - bd \in (x)$; with the first, it would be $y \in (x)$, which is not true, so we stay with the second: $1 - bd \in (x)$. There is $x = a(cx + dy) = acx + ady$, then $ady \in (x)$, then $ad \in (x)$ or $y \in (x)$, but as the second is not true, $ad \in (x)$, whence $a \in (x)$ or $d \in (x)$. If it were $d \in (x)$, then we would have $bd \in (x)$, but $1 + (x) = bd + (x)$; so d cannot be in (x). We are left only with $a \in (x)$, now $a = a'x$, whence $x = az = aa'z$ and cancelling, $1 = aa'z$, then z is a unit, and $(x,y) = (1)$.