Facts about Rings of Fractions

1 Introduction

Fact 1.1. *For saturated S, if $f(a)$ is a unit in $S^{-1}A$, then $a \in S$.*

Proof.

$$\frac{a}{1} \cdot \frac{b}{t} = \frac{1}{1}$$

$$\frac{ab}{t} = \frac{1}{1}$$

$$(ab, t) \equiv (1, 1)$$

$$(ab - t)u = 0$$

$$abu = tu$$

$$abu \in S$$

As S is saturated, $a \in S$. □

Fact 1.2. *For A a field, and $S = \{-1, 1\}$, $S^{-1}A \cong A$.*

Proof. It is easily verified that the standard isomorphism from A to $S^{-1}A$ is 1-1 and onto. □

Fact 1.3. *For A a field, and S a multiplicatively closed subset of A not containing zero, $S^{-1}A \cong A$.*

Proof. The standard homomorphism $f : a \mapsto a/1$ of A into $S^{-1}A$ is injectve: if $a/1 = a'/1$ then $a \cdot 1 = a1 \cdot 1$, then $a = a'$. It is surjective: $f(as^{-1}) = f(a)f(s^{-1}) = (a/1)(s^{-1}/1) = \ldots$, but $s^{-1}/1 = 1/s$ as $s^{-1}s = 1 \cdot 1$; continuing, $\ldots = (a/1)(1/s) = a/s$. □

Fact 1.4. *For A a field, and S a multiplicatively closed subset of A not containing zero, $S^{-1}A \cong A$.*

Proof. The standard homomorphism $f : a \mapsto a/1$ of A into $S^{-1}A$ is injectve: if $a/1 = a'/1$ then $a \cdot 1 = a1 \cdot 1$, then $a = a'$. It is surjective: $f(as^{-1}) = f(a)f(s^{-1}) = (a/1)(s^{-1}/1) = \ldots$, but $s^{-1}/1 = 1/s$ as $s^{-1}s = 1 \cdot 1$; continuing, $\ldots$

Example 1.5. *Some example.*