Facts about Rings of Fractions

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1 Introduction

Fact 1.1. *If $0 \in S$, then $S^{-1}A$ is a trivial ring.*

Proof. Any $(a, s), (a', s')$ are related because $(as' - a's) \cdot 0 = 0$ with $0 \in S$. □

Fact 1.2. *A a PID, the equivalence relation in $A \times S$ is: $(a, s) \equiv (a', s')$ iff $as' = a's$.* □

Fact 1.3. *For A a field, and $S = \{-1, 1\}$, $S^{-1}A \cong A$.*

Proof. It is easily verified that the standard isomorphism from A to $S^{-1}A$ is 1-1 and onto. □

Fact 1.4. *For A a field, and S a multiplicatively closed subset of A not containing zero, $S^{-1}A \cong A$.*

Proof. The standard homomorphism $f : a \mapsto a/1$ of A into $S^{-1}A$ is injectve: if $a/1 = a'/1$ then $a \cdot 1 = a1 \cdot 1$, then $a = a'$. It is surjective: $f(as^{-1}) = f(a)f(s^{-1}) = (a/1)(s^{-1}/1) = \ldots$, but $s^{-1}/1 = 1/s$ as $s^{-1}s = 1 \cdot 1$; continuing, $\ldots = (a/1)(1/s) = a/s$. □

Fact 1.5. *For A a field, the ring of fractions and the field of fractions are isomorphic.*

Proof. For isomorphism of A with its field of fractions, see Math Exchange 79188. About the isomorphism with its ring of fractions, is the fact above. □

Example 1.6. *Some example.*

Fact 1.7. *The quotient ring A/I can be viewed as an A-module, and then the ring of fractions $T^{-1}(A/I)$, where T is the image of S in A/I, equals the module of fractions $S^{-1}(A/I)$.*

Proof. On the left, the relation is in $(A/I) \times T$: $([a], [s]) \equiv ([a'], [s'])$ iff (ring notation) $([a][s'] - [a'][s])[s''] = [0]$ iff $[as's'' - a'ss''] = [0]$. On the right, the relation works in $(A/I) \times S$: $([a], s) \equiv ([a'], s')$ iff (module notation) $s''(s'[a] - s[a']) = [0]$ iff $[as's'' - a'ss''] = [0]$. The conditions are identical so the classes must be in bijective correspondence. However, they are not identical as sets, so saying *equals* is too much. □

Fact 1.8. *What are $\mathfrak{p}A_\mathfrak{p}$ and $\mathfrak{a}A_\mathfrak{p}$?*

In the Solutions by Y. P. Gaillard, the residue field is $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, so the single maximal ideal of $A_\mathfrak{p}$ from the Example 1, p. 38 of ItCA must be just $\mathfrak{p}A_\mathfrak{p}$

$$\mathfrak{p}A_\mathfrak{p} = \{a/s : a \in \mathfrak{p}, s \notin \mathfrak{p}\}$$

Then $\mathfrak{a}A_\mathfrak{p}$ must be the generalization

$$\mathfrak{a}A_\mathfrak{p} = \{a/s : a \in \mathfrak{a}, s \notin \mathfrak{p}\}$$

□

Fact 1.9. *What is $S^{-1}\mathfrak{a}$?*

It can be either an $S^{-1}A$-module $S^{-1}\mathfrak{a}$, because $\mathfrak{a}$ is an A-module, or the extension $S^{-1}\mathfrak{a} = \mathfrak{a}\, S^{-1}A$ in $S^{-1}A$ of the ideal $\mathfrak{a}$ in A. In both cases elements of $S^{-1}\mathfrak{a}$ are written as a/s with $a \in \mathfrak{a}$, $s \in S$, but they come from different sets. In the first, module case, a/s is in the quotient of $\mathfrak{a} \times S$, in the second, extension ideal case, a/s is in the quotient of $A \times S$. We are talking of $S^{-1}A$-modules, not rings, so there can only be A-module and $S^{-1}A$-module isomorphism:

$$\mathfrak{a} \times S/\sim_\mathfrak{a} \;\ni\; a/s \mapsto a/s \;\in\; A \times S/\sim_A$$

□

Fact 1.10. *What is $\mathfrak{a}_\mathfrak{p}$?*

It is the A-module $\mathfrak{a}$ localized at $\mathfrak{p}$. It is an $A_\mathfrak{p}$-module. We also use this notation for the ideal $S^{-1}\mathfrak{a}$ of $S^{-1}A$, where $S = A \setminus \mathfrak{p}$. How are they isomorphic? $a/s \mapsto a/s$ with $a \in \mathfrak{a}, s \notin$. Of what it is an isomorphism? Of A-modules, of $A_\mathfrak{p}$-modules. They are not rings.

Fact 1.11. *What is $S^{-1}\mathfrak{a}$ in case $S = A \setminus \mathfrak{p}$?*

It can be either an $A_\mathfrak{p}$-module $\mathfrak{a}_\mathfrak{p}$, because $\mathfrak{a}$ is an A-module, or the extension $S^{-1}\mathfrak{a} = \mathfrak{a}\, A_\mathfrak{p}$ in $A_\mathfrak{p}$ of the ideal $\mathfrak{a}$ in A. In both cases elements of $S^{-1}\mathfrak{a}$ are written as a/s with $a \in \mathfrak{a}$, $s \notin \mathfrak{p}$, but they come from different sets. In the first, module case, a/s is in the quotient of $\mathfrak{a} \times (A \setminus \mathfrak{p})$, in the second, extension ideal case, a/s is in the quotient of $A \times (A \setminus \mathfrak{p})$. We are talking of $A_\mathfrak{p}$-modules, not rings, so there can only be an A-module and $A_\mathfrak{p}$-module isomorphism:

$$\mathfrak{a} \times (A \setminus \mathfrak{p})/ \sim_\mathfrak{a} \;\ni\; a/s \mapsto a/s \;\in\; A \times (A \setminus \mathfrak{p})/ \sim_A$$

□

Fact 1.12. *What is $\mathfrak{p}_\mathfrak{p}$?*

It is the A-module $\mathfrak{p}$ localized at $\mathfrak{p}$. We also use this notation for the ideal $S^{-1}\mathfrak{p}$ of $S^{-1}A$, where $S = A \setminus \mathfrak{p}$, that is, the ideal $\mathfrak{p}A_\mathfrak{p}$.

Fact 1.13. *Case $\mathfrak{a} = \mathfrak{p}$, a prime ideal. What is $S^{-1}\mathfrak{p}$?*

It can be either the $A_\mathfrak{p}$-module $\mathfrak{p}_\mathfrak{p}$, because $\mathfrak{p}$ is an A-module, or the extension $\mathfrak{p}A_\mathfrak{p}$ in $A_\mathfrak{p}$ of the ideal $\mathfrak{p}$ in A, via the canonical $A \to A_\mathfrak{p} : a \mapsto a/s$. Looks like we don't have the $\cdot_\mathfrak{p}$-instead-of-$S^{-1}\cdot$ notation in the ideal extension case, but then, the quotient notation $A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is used, which makes sense only if $\mathfrak{p}_\mathfrak{p}$ is an ideal in $A_\mathfrak{p}$

$$\mathfrak{p}_\mathfrak{p} = \mathfrak{p}\, A_\mathfrak{p}$$

□

Fact 1.14. *When $S = A \setminus \mathfrak{p}$, as $A_\mathfrak{p}$-modules*

$$S^{-1}\mathfrak{a} = \mathfrak{a}A_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}$$

$$S^{-1}\mathfrak{p} = \mathfrak{p}A_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}$$

□

Fact 1.15. *How is $B_\mathfrak{q}$ an $A_\mathfrak{p}$-module?*

Let $g = \psi \circ f$ be the composition $A \to B \to T^{-1}B : a \to f(a) \to f(a)/1$. This composition sends $s \in S$ to a unit in $T^{-1}B$, as $(f(s)/1)(1/f(s)) = 1/1$, where $f(s) \in f(S) = f(A \setminus \mathfrak{p}) \subseteq B \setminus \mathfrak{q} = T$. Why the inclusion? If $a \notin \mathfrak{p} = f^{-1}(\mathfrak{q})$ then $f(a) \notin \mathfrak{q}$. By the universal property of the ring of fractions, g factorizes

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f"] \arrow[dr, "g"] & S^{-1}A \arrow[d, "h"] \\ B \arrow[r, "\psi"] & T^{-1}B \end{tikzcd}$$

where the recipe for h is given in **Proposition 3.1** of ItCA as $a/s \mapsto g(a)g(s)^{-1} = (f(a)/1)(1/f(s)) = f(a)/f(s)$. □

Fact 1.16. *What is $B_\mathfrak{p}$?*

(For $f : A \mapsto B$ and $\mathfrak{p}$ a prime ideal of A).
The ring B is an A-module by the restriction of scalars. We can localize it in the prime ideal $\mathfrak{p}$ of A. The cartesian product is $B \times (A \setminus \mathfrak{p})$, the relation is

$$(b, s) \sim (b', s') \iff \exists t \notin \mathfrak{p}\;\; t(sb' - s'b) = 0$$

The condition reads

$$f(t)(f(s)b' - f(s')b) = 0$$

The obvious addition

$$\frac{b}{s} + \frac{b'}{s'} = \frac{s'b + sb'}{ss'} = \frac{f(s')b + f(s)b'}{ss'}$$

The obvious scalar multiplication

$$\frac{a}{s'} \cdot \frac{b}{s} = \frac{ab}{s's} = \frac{f(a)b}{s's}$$

And f is an A-module homomorphism:

$$f(a'a) = f(a')f(a) = a' \cdot f(a)$$

It gives rise to an $A_\mathfrak{p}$-module homomorphism $S^{-1}f : A_\mathfrak{p} \to B_\mathfrak{p}$

$$a/s \mapsto f(a)/s$$

See how it is different from the map $A_\mathfrak{p} \to B_\mathfrak{q}$

$$a/s \mapsto f(a)/f(s)$$

□

Fact 1.17. *How is $B_\mathfrak{p}$ an $A_\mathfrak{p}$-module ?*

Definition of the $S^{-1}M$ as $S^{-1}A$-module in the text. The multiplication by a scalar is

$$\frac{a}{s'}\frac{b}{s} = \frac{ab}{s's} = \frac{f(a)b}{s's}$$

□

Fact 1.18. *How $(B \otimes_A M)_\mathfrak{q} = B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p}$?*

$$\begin{aligned}(B \otimes_A M)_\mathfrak{q} &= B_\mathfrak{q} \otimes_B (B \otimes_A M)\\ &= B_\mathfrak{q} \otimes_A M\\ &= (B_\mathfrak{q} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}) \otimes_A M\\ &= B_\mathfrak{q} \otimes_{A_\mathfrak{p}} (A_\mathfrak{p} \otimes_A M)\\ &= B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p}\end{aligned}$$

The first and the last equalities are applications of Proposition 3.5:

$$S^{-1}A \otimes_A M \cong S^{-1}M$$

$$A_\mathfrak{p} \otimes_A M \cong M_\mathfrak{p}$$
$$\frac{a}{s} \otimes m \mapsto \frac{am}{s}$$

□

Fact 1.19. *$B_\mathfrak{p}$ is not a ring.*

The multiplication

$$\frac{b}{s} \cdot \frac{b'}{s'} = \frac{bb'}{ss'}$$

fails to be distributive over the addition.

$$\begin{aligned}\frac{b''}{s''}\left(\frac{b'}{s'} + \frac{b'}{s'}\right) &= \frac{b''}{s''}\frac{sb' + s'b}{ss'}\\ &= \frac{b''(sb' + s'b)}{ss'}\\ &= \frac{sb''b' + s'b''b}{ss'}\\ &= \frac{sb''b'}{ss'} + \frac{s'b''b}{ss'}\end{aligned}$$

$$\begin{aligned}\frac{b''}{s''}\frac{b'}{s'}+\frac{b''}{s''}\frac{b}{s} &= \frac{b''b'}{s''s'}+\frac{s'b''b}{s''s} \\ &= \frac{s'sb''b'+s''s'b''b}{s''s's''s} \\ &= \frac{f(s')f(s)b''b'+f(s'')f(s')b''b}{s''s's''s} \\ &= \frac{f(s')f(s)b''b'}{s''s's''s}+\frac{f(s'')f(s')b''b}{s''s's''s}\end{aligned}$$

and we cannot cancel here to make them equal. □

Fact 1.20. *$B_{\mathfrak{p}} \cong B_{\mathfrak{q}}$ as $A_{\mathfrak{p}}$-modules.*

By Exercise 4: $T = f(S)$ then $S^{-1}B \cong T^{-1}B$ since $f(\mathfrak{p}) = \mathfrak{q}$. □

Fact 1.21. *How is $B_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}} = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ an $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$-module?*

We know the map $A_{\mathfrak{p}} \to B_{\mathfrak{q}} : a/s \mapsto f(a)/f(s)$ from 1.15. The kernel of the composition $A_{\mathfrak{p}} \to B_{\mathfrak{q}} \to B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} : a/s \mapsto f(a)/f(s) + \mathfrak{q}B_{\mathfrak{q}}$ contains $\mathfrak{p}A_{\mathfrak{p}}$: element of $\mathfrak{p}A_{\mathfrak{p}}$ is a/s where $a \in \mathfrak{p}, s \notin \mathfrak{p}$; it follow that $f(s) \notin \mathfrak{q}$ (otherwise $s \in f^{-1}(\mathfrak{q}) = \mathfrak{p}$); so the image in the first map of a/s is in $\mathfrak{q}B_{\mathfrak{q}}$, the kernel of the second map, then a/s is in the kernel of the composition. The composition then factors through $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$: $a/s + \mathfrak{p}A_{\mathfrak{p}} \mapsto f(a)/f(s) + \mathfrak{q}B_{\mathfrak{q}}$. This is a ring homomorphism that makes $B_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}}$ an $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$-module. □

Fact 1.22. *What is $\mathfrak{p}M_{\mathfrak{p}}$?*

When $M_{\mathfrak{p}}$ is seen as an A-module, $\mathfrak{p}M_{\mathfrak{p}} = \{am/s : a \in \mathfrak{p}, m \in M, s \notin \mathfrak{p}\}$. When $M_{\mathfrak{p}}$ is seen as an $A_{\mathfrak{p}}$-module, $\mathfrak{p}$ is not even an ideal in $A_{\mathfrak{p}}$, but its extension, $\mathfrak{p}A_{\mathfrak{p}}$ is, and $(\mathfrak{p}A_{\mathfrak{p}})M_{\mathfrak{p}} = \{(a/s')(m/s) : a \in \mathfrak{p}, m \in M, s, s' \notin \mathfrak{p}\} = \{am/s : a \in \mathfrak{p}, m \in M, s \notin \mathfrak{p}\}$, the same set, which we write $\mathfrak{p}M_{\mathfrak{p}}$ for:

$$\mathfrak{p}M_{\mathfrak{p}} = (\mathfrak{p}A_{\mathfrak{p}})M_{\mathfrak{p}}$$

□

Fact 1.23. *How*

$$\frac{(B \otimes_A M)_{\mathfrak{q}}}{\mathfrak{q}(B \otimes_A M)_{\mathfrak{q}}} \cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_B B \otimes_A M$$

?

Proposition 3.5 states, in the language of subscript-$\mathfrak{p}$, that $M_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A M$ over $A_{\mathfrak{p}}$. Here $(B \otimes_A M)_{\mathfrak{q}} \cong B_{\mathfrak{q}} \otimes_B (B \otimes_A M)$. Then

$$\begin{aligned}\frac{B_{\mathfrak{q}} \otimes_B (B \otimes_A M)}{(\mathfrak{q}B_{\mathfrak{q}})(B_{\mathfrak{q}} \otimes_B (B \otimes_A M))} &\cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}B_{\mathfrak{q}}} \otimes_{B_{\mathfrak{q}}} (B_{\mathfrak{q}} \otimes_B (B \otimes_A M)) \\ &\cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_B B \otimes_A M\end{aligned}$$

□

In P. Y. Gaillard solution to ItCA Exercise 3.19 (viii).

Fact 1.24. *The diagram*

$$\begin{tikzcd} A_{\mathfrak{p}} \arrow[r, "\phi"] \arrow[d, "f"] \arrow[dr, "\eta"] & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \arrow[d, "h"] \\ B_{\mathfrak{q}} \arrow[r, "\psi"] & B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \end{tikzcd}$$

$$\begin{tikzcd} a/s \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "f"] \arrow[dr, mapsto, "\eta"] & a/s + \mathfrak{p}A_{\mathfrak{p}} \arrow[d, mapsto, "h"] \\ f(a)/f(s) \arrow[r, mapsto, "\psi"] & f(a)/f(s) + \mathfrak{q}B_{\mathfrak{q}} \end{tikzcd}$$

is commutative.

All calculated on the diagram. $\square$

Now $\kappa_{\mathfrak{q}} = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ is an $A_{\mathfrak{p}}$-module by $A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ (with the formula as on the bottom diagram) and we may tensor over $A_{\mathfrak{p}}$.

If a field K is an A-module for some ring A, can it be a zero A-module?

$$1_A 1_K = 1_k \neq 0_K$$

It cannot.

Now that $\kappa_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = 0$, both tensorands finitely generated, and $\kappa_{\mathfrak{q}} \neq 0$, it must be $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = 0$ by ItCA Exercise 2.3.

In solution of ItCA 3.19 (viii) by J. D. Taylor.

Fact 1.25. *What is $\mathfrak{p}B$?*

For $f : A \to B$, we can think in two ways. As we identify $ab = f(a)b$, $\mathfrak{p}B = \{ab = f(a)b : a \in \mathfrak{p}, b \in B\}$ is the extension $f(\mathfrak{p})B$ of the ideal $\mathfrak{p}$. The second way is that B is an A-module, and $\mathfrak{p}$ a prime ideal in A, so we can form $\mathfrak{p}B = \{\sum a_i b_i = \sum f(a_i)b_i\}$ with $a_i \in \mathfrak{p}$, $b_i \in B$, getting the same set.

$\square$

Fact 1.26. *What is $\mathfrak{p}B_{\mathfrak{p}}$?*

$B_{\mathfrak{p}}$ is an A - module, $\mathfrak{p}$ is a prime ideal of A, so $\mathfrak{p}B_{\mathfrak{p}}$ makes sense and consists of finite sums $\sum a_i(b_i/s)$ where $a_i \in \mathfrak{p}$, $b_i \in B$, and $s_i \in A \setminus \mathfrak{p}$. After bringing to common denominator, the sum becomes ab/s where $a \in \mathfrak{p}$, $b \in B$ and $s_i \in A \setminus \mathfrak{p}$ that is, b/s where $b \in \mathfrak{p}B$ and $s_i \in A \setminus \mathfrak{p}$. $\square$

Fact 1.27. *How is $A_{\mathfrak{p}}$ an A-module ?*

The canonical map $\phi : A \to A_{\mathfrak{p}} : a \mapsto \frac{a}{1}$ gives the multiplication by scalars from A

$$a'\frac{a}{s} = \phi(a')\frac{a}{s} = \frac{a'}{1}\frac{a}{s} = \frac{a'a}{s}$$

$\square$

Fact 1.28. *What is $\mathfrak{p}A_{\mathfrak{p}}$?*

As $A_{\mathfrak{p}}$ is an A-module, we can multiply it by a prime ideal in A in a standard way

$$\sum a_i' \frac{a_i}{s_i} = \sum \frac{a' a_i}{s_i}$$

After bringing to a common denominator, this is

$$\frac{a}{s}$$

with $a \in \mathfrak{p}$, so $\mathfrak{p}A_{\mathfrak{p}}$ is the single maximal ideal of the local ring $A_{\mathfrak{p}}$.

$\square$

Fact 1.29. *What is $(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}}$?*

As $B_{\mathfrak{p}}$ is an $A_{\mathfrak{p}}$-module, and $\mathfrak{p}A_{\mathfrak{p}}$ is the single maximal ideal of the local ring $A_{\mathfrak{p}}$, Any element is, from the definition of the ideal-by-module and from the general element of $\mathfrak{p}A_{\mathfrak{p}}$ $(a \in \mathfrak{p})$

$$\sum_i \frac{a_i}{s_i'}\frac{b_i}{s_i} = \sum \frac{ab}{s's}$$

After bringing to a common denominator, this becomes

$$ab/s = f(a)b/s$$

where $a \in \mathfrak{p}$. Notice we got the general element of $\mathfrak{p}B_{\mathfrak{p}}$, so

$$(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$$

$\square$

Fact 1.30. *How* $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}$ *?*

Apply Exercise 2.2

$$A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$$

to $M := B_{\mathfrak{p}}$, $A := A_{\mathfrak{p}}$, $\mathfrak{a} := \mathfrak{p}A_{\mathfrak{p}}$

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} = B_{\mathfrak{p}}/(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}}$$

now apply $(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$.

□

Fact 1.31. *How* $A_{\mathfrak{p}} \otimes_A B \cong B_{\mathfrak{p}}$ *?*

Apply Proposition 3.5: $S^{-1}A \otimes_A M \cong S^{-1}M$.

□

We now understand the isomorphisms in the solution of ItCA's 3.21(iv) by J D. Taylor.

$$\begin{aligned} B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} &= A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} \\ &= K_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}} \otimes_A B \\ &= K_{\mathfrak{p}} \otimes_A B \end{aligned}$$

2 Saturated

Fact 2.1. *For saturated* S*, if* $f(a)$ *is a unit in* $S^{-1}A$*, then* $a \in S$*.*

Proof.

$$\frac{a}{1} \cdot \frac{b}{t} = \frac{1}{1}$$

$$\frac{ab}{t} = \frac{1}{1}$$

$$(ab, t) \equiv (1, 1)$$

$$(ab - t)u = 0$$

$$abu = tu$$

$$abu \in S$$

As S is saturated, $a \in S$. □