# Exercise 3.21

October 18, 2024

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**Exercise 3.21.i**. Show that $\phi^* : \operatorname{Spec}(S^{-1}A) \to \operatorname{Spec}(A)$ is a homeomorphism of $\operatorname{Spec}(S^{-1}A)$ onto its image in $X = \operatorname{Spec}(A)$.

We want to prove that for $D$ closed in $\operatorname{Spec}(S^{-1}A)$ there is $C$ closed in $\operatorname{Spec}(A)$ and reverse, that the equation holds

$$C \cup \phi^*(\operatorname{Spec}(S^{-1}A)) = \phi^*(D)$$

We take

$$C = V(\mathfrak{a}), \quad D = V(S^{-1}\mathfrak{a})$$

That is, we are going to prove that

$$V(\mathfrak{a}) \cap \phi^*(\operatorname{Spec}(S^{-1}\mathfrak{a})) = \phi^*(V(S^{-1}\mathfrak{a}))$$

Recalling that $S^{-1}\mathfrak{p} \xmapsto{\phi^*} \mathfrak{p}$, the right are all prime ideals not meeting $S$ and holding $S^{-1}\mathfrak{p} \supseteq S^{-1}\mathfrak{a}$. By the fact that this happens iff $\mathfrak{p} \supseteq \mathfrak{a}$, they are all prime ideals not meeting $S$ and containing $\mathfrak{a}$. Which is precisely the left.
Now $\phi^*$ maps closed files to closed files in any direction. $\square$

**Exercise 3.21.ii**. Let $f : A \to B$ be a ring homomorphism...

$$\begin{tikzcd}
X \arrow[r, equal] & \operatorname{Spec}(A) & & \operatorname{Spec}(B) \arrow[ll, "f^*"'] \arrow[r, equal] & Y \\
S^{-1}X \arrow[r, equal] & \phi^*(\operatorname{Spec}(S^{-1}A)) \arrow[u, hook] & & \psi^*(\operatorname{Spec}(S^{-1}B)) \arrow[u, hook] \arrow[r, equal] & S^{-1}Y \\
X \arrow[r, equal] & \operatorname{Spec}(S^{-1}A) \arrow[u, leftrightarrow] & & \operatorname{Spec}(S^{-1}B) \arrow[ll, "S^{-1}f^*"'] \arrow[u, leftrightarrow] \arrow[r, equal] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{p} & \mathfrak{q} \arrow[l, mapsto, "f^*"'] \\
\mathfrak{p} \arrow[u, equal] \arrow[d, leftrightarrow] & \mathfrak{q} \arrow[u, equal] \arrow[d, leftrightarrow] \\
S^{-1}\mathfrak{p} & S^{-1}\mathfrak{q} \arrow[l, mapsto, "S^{-1}f^*"']
\end{tikzcd}$$

That $(S^{-1}f)^* : \operatorname{Spec}(S^{-1}B) \to \operatorname{Spec}(S^{-1}A)$ is the restriction of $f^*$ to $S^{-1}Y$ we have already proved in Facts, showing the action of $(S^{-1}f)^*$:

$$(S^{-1}f)^* : S^{-1}\mathfrak{q} \mapsto S^{-1}\mathfrak{p}$$

What is $S^{-1}X = \phi^*(\operatorname{Spec}(S^{-1}A))$? All prime ideals of $A$ not meeting $S$.
What is $f^{*-1}(S^{-1}X) = f^{*-1}(\phi^*(\operatorname{Spec}(S^{-1}A)))$? All prime ideals of $B$ whose preimages in $A$ do not meet $S$.
What is $S^{-1}X = \psi^*(\operatorname{Spec}(S^{-1}B))$? All prime ideals of $B$ not meeting $f(S)$.
We show that the the last two sets are equal.

$\supseteq$: If $\mathfrak{q}$ does not meet $f(S)$, may its preimage $f^{-1}(\mathfrak{q})$ meet $S$? Let $s \in f^{-1}(\mathfrak{q})$; $f(s) \in \mathfrak{q}$; now $\mathfrak{q}$ meets $f(S)$, a contradiction. So it cannot.
$\subseteq$: Let's not let the preimage $f^{-1}(\mathfrak{q})$ of a prime ideal $\mathfrak{q}$ of $B$ meet $S$. May $\mathfrak{q}$ meet $f(B)$? $f(s) \in \mathfrak{q}$; $s \in f^{-1}(\mathfrak{q}$; now the preimage $f^{-1}(\mathfrak{q})$ in $A$ meets $S$, a contradiction. So it cannot.

**Exercise 3.21.iii**. Let $\mathfrak{a}$ be an ideal of $A$ and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in $B$...

Let $\mathfrak{a}$ be an ideal of $A$ and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in $B$. What is the homomorphism?

$$\tilde{f} : A/\mathfrak{a} \to B/\mathfrak{b}$$

Recall what does a homomorphism need to factor through a quotient?

$$\begin{tikzcd}
A \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & B \arrow[dl, "\tilde{\phi}"] \\
& A/\mathfrak{q} &
\end{tikzcd}$$

The map $\tilde{\phi}$ has to be defined on representatives and cannot differ between them.

$$\tilde{\phi}(a + \mathfrak{a}) = \tilde{\phi}(a' + \mathfrak{a})$$

if $a + \mathfrak{a} = a' + \mathfrak{a}$ iff $a - a' \in \mathfrak{a}$.
We define $\tilde{\phi}$ by $\tilde{\phi}(a + \mathfrak{a}) = \phi(a)$ so it has to be

$$\phi(a) = \phi(a') \text{ if } a - a' \in \mathfrak{a}$$

$$\phi(a - a') = 0 \text{ if } a - a' \in \mathfrak{a}$$

$$\phi(a) = 0 \text{ if } a \in \mathfrak{a}$$

$$\ker \phi \supseteq \mathfrak{a}$$

To factor through the quotient by an ideal, the homomorphism's kernel must contain this ideal.
A homomorphism factors through any ideal contained in its kernel.

If $\mathfrak{a} \subseteq \phi^{-1}(0)$ then $\phi(a) = 0$ for $a \in \mathfrak{a}$ then $\phi(a - a') = 0$ for $a - a' \in \mathfrak{a}$ then $\phi(a) = \phi(a')$ for $a + \mathfrak{a} = a' + \mathfrak{a}$ and we can say $\tilde{\phi}(a + \mathfrak{a}) = \phi(a)$.

We return to $f : A \to B$, $\mathfrak{a} = f^{-1}(\mathfrak{b})$, $\mathfrak{b}$ an ideal of $B$.

$$A \xrightarrow{f} B \xrightarrow{\rho} B/\mathfrak{b}$$

Does the kernel contain $\mathfrak{a}$? If $a \in \mathfrak{a}$ then $a \mapsto f(a) \mapsto f(a) + \mathfrak{b}$, but $f(a) \in \mathfrak{b}$ so $a$ maps to zero and is in the kernel of this composition homomorphism, which then factors through the quotient:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, two heads] & B \arrow[r, two heads, "\rho"] & B/\mathfrak{b} \\
& A/\mathfrak{a} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[r, maps to, "f"] \arrow[dr, maps to] & f(a) \arrow[r, maps to, "\rho"] & f(a) + \mathfrak{b} \\
& a + \mathfrak{a} \arrow[ur, maps to] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "\pi"] & B \arrow[d, two heads, "\rho"] \\
A/\mathfrak{a} \arrow[r, "\tilde{f}"] & B/\mathfrak{b}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Spec}(A) & & \mathrm{Spec}(B) \arrow[ll, "f^*"'] \\
\pi^*(\mathrm{Spec}(A/\mathfrak{a})) \arrow[u] \arrow[d, leftrightarrow] & & \rho^*(\mathrm{Spec}(B/\mathfrak{b})) \arrow[ll] \arrow[u] \arrow[d, leftrightarrow] \\
\mathrm{Spec}(A/\mathfrak{a}) \arrow[uu, bend left=60, "\pi^*"] & & \mathrm{Spec}(B/\mathfrak{b}), \arrow[ll, "\tilde{f}^*"'] \arrow[uu, bend right=60, "\rho^*"']
\end{tikzcd}$$