Exercise 3.21

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Exercise 3.21.i. Show that ϕ^* : Spec $(S^{-1}A) \to \text{Spec}(A)$ is a homeomorphism of $\operatorname{Spec}(S^{-1}A)$ onto its image in $X = \operatorname{Spec}(A)$.

We want to prove that for D closed in $\operatorname{Spec}(S^{-1}A)$ there is C closed in $\operatorname{Spec}(A)$ and reverse, that the equation holds

$$C \cup \phi^*(\operatorname{Spec}(S^{-1}A)) = \phi^*(D)$$

We take

$$C = V(\mathfrak{a}), \quad D = V(S^{-1}\mathfrak{a})$$

That is, we are going to prove that

$$V(\mathfrak{a}) \cap \phi^*(\operatorname{Spec}(S^{-1}\mathfrak{a})) = \phi^*(V(S^{-1}\mathfrak{a}))$$

Recalling that $S^{-1}\mathfrak{p} \stackrel{\phi^*}{\longmapsto} \mathfrak{p}$, the right are all prime ideals not meeting S and holding $S^{-1}\mathfrak{p} \supseteq$ $S^{-1}\mathfrak{a}$. By the fact that this happens iff $\mathfrak{p}\supseteq\mathfrak{a}$, they are all prime ideals not meeting S and containing a. Which is precisely the left.

Now ϕ^* maps closed files to closed files in any direction.

Exercise 3.21.ii. Let $f: A \to B$ be a ring homomorphism...

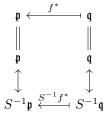
$$X = \operatorname{Spec}(A) \longleftarrow f^* \operatorname{Spec}(B) = Y$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$S^{-1}X = \phi^*(\operatorname{Spec}(S^{-1}A)) \qquad \psi^*(\operatorname{Spec}(S^{-1}B)) = S^{-1}Y$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$X = \operatorname{Spec}(S^{-1}A) \longleftarrow \operatorname{Spec}(S^{-1}B) = Y$$



That $(S^{-1}f)^*: \operatorname{Spec}(S^{-1}B) \to \operatorname{Spec}(S^{-1}A)$ is the restriction of f^* to $S^{-1}Y$ we have already proved in Facts, showing the action of $(S^{-1}f)^*$:

$$(S^{-1}f)^*: S^{-1}\mathfrak{q} \mapsto S^{-1}\mathfrak{p}$$

What is $S^{-1}X = \phi^*(\operatorname{Spec}(S^{-1}A))$? All prime ideals of A not meeting S. What is $f^{*-1}(S^{-1}X) = f^{*-1}(\phi^*(\operatorname{Spec}(S^{-1}A)))$? All prime ideals of B whose preimages in A do not meet S.

What is $S^{-1}X = \psi^*(\operatorname{Spec}(S^{-1}B))$? All prime ideals of B not meeting f(S).

We show that the last two sets are equal.

 \supseteq : If \mathfrak{q} does not meet f(S), may its preimage $f^{-1}(\mathfrak{q})$ meet S? Let $s \in f^{-1}(\mathfrak{q})$; $f(s) \in \mathfrak{q}$; now \mathfrak{q} meets f(S), a contradiction. So it cannot.

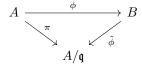
 \subseteq : Let's not let the preimage $f^{-1}(\mathfrak{q})$ of a prime ideal \mathfrak{q} of B meet S. May \mathfrak{q} meet f(B)? $f(s) \in \mathfrak{q}$; $s \in f^{-1}(\mathfrak{q})$; now the preimage $f^{-1}(\mathfrak{q})$ in A meets S, a contradiction. So it cannot.

Exercise 3.21.iii. Let \mathfrak{a} be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B...

Let \mathfrak{a} be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B. What is the homomorphism?

$$\tilde{f}:A/\mathfrak{a}\to B/\mathfrak{b}$$

Recall what does a homomorphism need to factor through a quotient?



The map $\tilde{\phi}$ has to be defined on representatives and cannot differ between them.

$$\tilde{\phi}(a+\mathfrak{a}) = \tilde{\phi}(a'+\mathfrak{a})$$

if $a + \mathfrak{s} = a' + \mathfrak{s}$ iff $a - a' \in \mathfrak{s}$.

We define $\tilde{\phi}$ by $\tilde{\phi}(a+\mathfrak{a}) = \phi(a)$ so it has to be

$$\phi(a) = \phi(a') \text{ if } a - a' \in \mathfrak{a}$$

$$\phi(a - a') = 0 \text{ if } a - a' \in \mathfrak{a}$$

$$\phi(a) = 0 \text{ if } a \in \mathfrak{a}$$

$$\ker \phi \supseteq \mathfrak{a}$$

To factor through the quotient by an ideal, the homomorphism's kernel must contain this ideal.

A homomorphism factors through any ideal contained in its kernel.

If $\mathbf{a} \subseteq \phi^{-1}(0)$ then $\phi(a) = 0$ for $a \in \mathbf{a}$ then $\phi(a - a') = 0$ for $a - a' \in \mathbf{a}$ then $\phi(a) = \phi(a')$ for $a + \mathbf{a} = a' + \mathbf{a}$ and we can say $\tilde{\phi}(a + \mathbf{a}) = \phi(a)$.

We return to $f: A \to B$, $\mathfrak{a} = f^{-1}(\mathfrak{b})$, \mathfrak{b} an ideal of B.

$$A \xrightarrow{f} B \xrightarrow{\rho} B/\mathfrak{b}$$

Does the kernel contain \mathfrak{a} ? If $a \in \mathfrak{a}$ then $a \mapsto f(a) \mapsto f(a) + \mathfrak{b}$, but $f(a) \in \mathfrak{b}$ so a maps to zero and is in the kernel of this composition homomorphism, which then factors through the quotient:

$$A \xrightarrow{f} B \xrightarrow{\rho} B/\mathfrak{b}$$

$$A/\mathfrak{a}$$

$$a \xrightarrow{f} f(a) \xrightarrow{\rho} f(a) + \mathfrak{b}$$

$$a + \mathfrak{a}$$

How does $\operatorname{Spec}(A/\mathfrak{a})$ have it canonical image $V(\mathfrak{a})$ in $\operatorname{Spec}(A)$?

$$A \xrightarrow{\pi} A/\mathfrak{a}$$
$$a \mapsto a + \mathfrak{a}$$

That there is a 1-1 correspondence between ideals of A/\mathfrak{a} and ideals of A containing \mathfrak{a} , we are told in the text on page 9. And that prime ideals correspond to prime ideals. So we have a bijection between $\operatorname{Spec}(A/\mathfrak{a})$ and prime ideals of A containing \mathfrak{a} , which comprise the set $V(\mathfrak{a})$. We are not required to prove a homeomorphism here.

The general prime ideal of B/\mathfrak{b} is $\rho(\mathfrak{q})$ where \mathfrak{q} is a prime ideal of B containing \mathfrak{b} .

$$\begin{split} \tilde{f}^*: \rho(\mathfrak{q}) \mapsto \tilde{f}^{-1}(\rho(\mathfrak{q})) \\ \tilde{f}^{-1}(\rho(\mathfrak{q})) = \{a+\mathfrak{a}: f(a)+\mathfrak{b} \in \rho(\mathfrak{q})\} = \dots \end{split}$$

Property. $b + \mathfrak{b} \in \rho(\mathfrak{q}) \iff b \in \mathfrak{q}$.

Probably general for surjective homomorphism and an ideal, or even a set, containing the kernel.

If $b + \mathfrak{b} \in \rho(\mathfrak{q})$ then $b + \mathfrak{b} = b' + \mathfrak{b}$ for some $b' \in \mathfrak{q}$, then $b - b' \in \mathfrak{q}$ and $b' \in \mathfrak{q}$, then $b \in \mathfrak{q}$. \Box

$$\dots = \{a + \mathfrak{a} : f(a) \in \mathfrak{q}\}$$
$$= f^{-1}(\mathfrak{q}) + \mathfrak{a}$$
$$= \pi (f^{-1}(\mathfrak{q}))$$

Now π^* maps this to $\pi^{-1}(\pi(f^{-1}(\mathfrak{q})))$. As π is surjective, this set is $f^{-1}(\mathfrak{q}) = f^*(\mathfrak{q})$. The up-left path: $\rho(\mathfrak{q})$ is identified in $\operatorname{Spec}(B)$ with \mathfrak{q} then this is mapped by f^* to $f^{-1}(\mathfrak{q}) = f^*(\mathfrak{q})$.