Isomorphisms 1 to 2 in Claim 1

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Fact. The isomorphism 1 to 2.

$$A_{\mathfrak{p}} \otimes_A B/\mathfrak{p}B \cong (B/\mathfrak{p}B)_{\mathfrak{p}}$$

Step 1. *How is $A_{\mathfrak{p}} \otimes B/\mathfrak{p}B$ a B-algebra?*

We need a ring structure and a homomorphism from B. The first is defined in [ItCA] TENSOR PRODUCT OF ALGEBRAS pp 30-31.

$$\left(\frac{a}{s} \otimes (b + \mathfrak{p}B)\right)\left(\frac{a'}{s'} \otimes (b' + \mathfrak{p}B)\right) = \frac{aa'}{ss'} \otimes (bb' + \mathfrak{p}B)$$

but first the $B/\mathfrak{p}B$has to be an algebra. That is, to be a ring. We have this in Facts, $\mathfrak{p}B = \mathfrak{p}^e$, an ideal of B that does not have to be prime, still, $B/\mathfrak{p}B$ is a ring.

The $B/\mathfrak{p}B$ is a ring and a B-module. How is it a B-algebra? A homomorphism of rings? One proposition

$$B \longrightarrow B/\mathfrak{p}B$$

$$b \mapsto b + \mathfrak{p}$$

In the tensor product of algebras, $A \xrightarrow{f} B$, $A \xrightarrow{g} C$, the mapping from A is $a \mapsto f(a) \otimes g(a)$, here the mapping from B to $A_{\mathfrak{p}} \otimes_A B/\mathfrak{p}B$ is

$$b \mapsto \frac{b}{1} \otimes (b + \mathfrak{p}B$$

But are they both, the tensorands, B-algebras?

What is $(B/\mathfrak{p}B)_{\mathfrak{p}}$?
$\mathfrak{p}B$ is an ideal of B .
$B/\mathfrak{p}B$ is a ring and an A-module.
We can take the $(B/\mathfrak{p}B)_{\mathfrak{p}}$ $A\mathfrak{p}$-module.

What a ring structure do we have on $(B/\mathfrak{p}B)_{\mathfrak{p}}$? General element is $(b + \mathfrak{p}B)/s$. The obvious proposition is

$$\frac{b + \mathfrak{p}B}{s} \cdot \frac{b' + \mathfrak{p}B}{s'} = \frac{bb' + \mathfrak{p}B}{ss'}$$

Well-definition. Show that if

$$\frac{b_1 + \mathfrak{p}B}{s_1} = \frac{b_2 + \mathfrak{p}B}{s_2}, \quad \frac{b'_1 + \mathfrak{p}B}{s'_!} = \frac{b'_2 + \mathfrak{p}B}{s'_2} \tag{1}$$

then

$$\frac{b_1 b'_1 + \mathfrak{p}B}{s_1 s'_1} = \frac{b_2 b'_2 + \mathfrak{p}B}{s_2 s'_2} \tag{2}$$

that is,

$$v s_2 s'_2 b_1 b'_1 - v s_1 s'_1 b_2 b'_2 \in \mathfrak{p}B \tag{3}$$

The (1) means

$$u(s_2(b_1 + \mathfrak{p}B) - s_1(b_2 + \mathfrak{p}B)) = 0,$$

$$u'(s'_2(b'_1 + \mathfrak{p}B) - s'_1(b'_2 + \mathfrak{p}B)) = 0$$

That is,

$$u s_2 b_1 - u s_1 b_2 \in \mathfrak{p}B,$$

$$u's_2'b_1' - u's_1'b_2' \in \mathfrak{p}B,$$

Then

$$\overline{us_2b_1} = \overline{us_1b_2}, \quad \overline{u's_2'b_1'} = \overline{u's_1'b_2'}$$

$$\overline{us_2b_1u's_2'b_1'} = \overline{us_1b_2u's_1'b_2'}$$

$$uu's_2s_2'b_1b_1' - uu's_1s_1'b_2b_2' \in \mathfrak{p}B$$

And it is enough to take $v = uu'$ in (3).

Is a localization $B_\mathfrak{p}$ of an A-algebra B in a prime ideal $\mathfrak{p}$ of A a B-algebra? It is an $A_\mathfrak{p}$-module. As the multiplication we try

$$\frac{b}{s} \cdot \frac{b'}{s'} = \frac{bb'}{ss'}$$

Well-definition.

$$\frac{b_1}{s_1} = \frac{b_2}{s_2}, \quad \frac{b_1'}{s_1'} = \frac{b_2'}{s_2'} \implies \frac{b_1b_1'}{s_1s_1'} = \frac{b_2b_2'}{s_2s_2'} \ ?$$

The left

$$u(s_2b_1 - s_1b_2) = 0, \quad u'(s_2'b_1' - s_1'b_2') = 0$$

$$us_2b_1 = us_1b_2, \quad u's_2'b_1' = u's_1'b_2'$$

The right

$$v(s_2s_2'b_1b_1' - s_1s_1'b_2b_2') = 0$$

$$vs_2s_2'b_1b_1' = vs_1s_1'b_2b_2'$$

Take $v = uu'$.

Distributivity over addition.

$$\frac{b}{s}\left(\frac{b'}{s'} + \frac{b''}{s''}\right) = \frac{b}{s}\frac{s''b' + s'b''}{s's''} = \frac{s''bb' + s'bb''}{ss's''}$$

$$\frac{b}{s}\frac{b'}{s'} + \frac{b}{s}\frac{b''}{s''} = \frac{bb'}{ss'} + \frac{bb''}{ss''} = \frac{ss''bb' + ss'bb''}{s^2s's''} = \frac{s(s''bb' + s'bb'')}{s^2s's''} = \frac{s''bb'' + s'bb''}{ss's''}$$

We now use this fact to get the multiplication in the A-algebra $B/\mathfrak{p}B$ localized at an ideal $\mathfrak{p}$ of A, that is, $(B/\mathfrak{p}B)_\mathfrak{p}$.

$$\frac{b + \mathfrak{p}B}{s} \cdot \frac{b' + \mathfrak{p}B}{s'} = \frac{(b + \mathfrak{p}B)(b' + \mathfrak{p}B)}{ss'} = \frac{bb' + \mathfrak{p}B}{ss'}$$

which is an easier way to define this multiplication, avoiding lots of calculations.

We now move to to a proposition of mutually reverse maps between

$$A_\mathfrak{p} \otimes_A B/\mathfrak{p}B \quad \text{and} \quad (B/\mathfrak{p}B)_\mathfrak{p}$$

Both are B-algebras. And B-algebras' isomorphisms must be commutative with homomorphisms from B.

$$\begin{tikzcd}
 & A_\mathfrak{p} \otimes_A B/\mathfrak{p}B \arrow[dd] \\
B \arrow[ur] \arrow[dr] & \\
 & (B/\mathfrak{p}B)_\mathfrak{p}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1/1 \otimes (b + \mathfrak{p}B) \arrow[dd, mapsto] \\
b \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & (b + \mathfrak{p}B)/1
\end{tikzcd}$$

For the top-down, the proposition is

$$a/s \otimes (b + \mathfrak{p}B) \mapsto (ab + \mathfrak{p}B)/s \tag{4}$$

For the bottom-up, the proposition is

$$(b + \mathfrak{p}B)/s \mapsto 1/s \otimes (b + \mathfrak{p}B) \tag{5}$$

We start with A-bilinear map $A_{\mathfrak{p}} \times B/\mathfrak{p}B \to (B/\mathfrak{p}B)_{\mathfrak{p}}$

$$(a/s, b + \mathfrak{p}B) \mapsto (ab + \mathfrak{p}B)/s \tag{6}$$

First we verify well-defibition then bilinearity.

$$\frac{a}{s} = \frac{a'}{s'},\ b + \mathfrak{p}B = b' + \mathfrak{p}B \overset{?}{\Rightarrow} \frac{ab + \mathfrak{p}B}{s} = \frac{a'b' + \mathfrak{p}B}{s'}$$

$$u(s'a - sa') = 0,\ b - b' \in \mathfrak{p}B \overset{?}{\Rightarrow} v(s'(ab + \mathfrak{p}B) - s(a'b' + \mathfrak{p}B)) = 0$$

$$us'a = usa',\ b = b' + b'', b'' \in \mathfrak{p}B \overset{?}{\Rightarrow} vs'ab - vsa'b' \in \mathfrak{p}B$$

By the multiplication property of equality

$$us'ab = usa'b' + usa'b'',$$

the second summand being in the ideal $\mathfrak{p}B$,

$$us'ab - usa'b' \in \mathfrak{p}B$$

Taking $v = u$ we see our map well-defined. We now verify the bilinearity. Additivity in the first.

$$\begin{aligned}
\left(\frac{a}{s} + \frac{a'}{s'}, b + \mathfrak{p}B\right) &= \left(\frac{s'a + sa'}{ss'}, b + \mathfrak{p}B\right) \\
&\mapsto \frac{(s'a + sa')b + \mathfrak{p}B}{ss'} \\
&= \frac{s'ab + sa'b + \mathfrak{p}B}{ss'} \\
&= \frac{s'ab + \mathfrak{p}B}{ss'} + \frac{sa'b + \mathfrak{p}B}{ss'} \\
&= \frac{s'(ab + \mathfrak{p}B)}{ss'} + \frac{s(a'b + \mathfrak{p}B)}{ss'} \\
&= \frac{ab + \mathfrak{p}B}{s} + \frac{a'b + \mathfrak{p}B}{s'} \\
&= \leftarrow\!\shortmid \left(\frac{a}{s}, b + \mathfrak{p}B\right) + \leftarrow\!\shortmid \left(\frac{a'}{s'}, b + \mathfrak{p}B\right)
\end{aligned}$$

Scaling in the first.

$$\begin{aligned}
\left(a' \cdot \frac{a}{s}, b + \mathfrak{p}B\right) &= \left(\frac{aa'}{s}, b + \mathfrak{p}B\right) \\
&\mapsto \frac{a'ab + \mathfrak{p}B}{s} \\
&= a' \cdot \frac{ab + \mathfrak{p}B}{s} \\
&= a' \cdot \leftarrow\!\shortmid \left(\frac{a}{s}, b + \mathfrak{p}B\right)
\end{aligned}$$

Additivity in the second.

$$\begin{aligned}
\left(\frac{a}{s}, b + \mathfrak{p}B + b' + \mathfrak{p}B\right) &= \left(\frac{a}{s}, b + b' + \mathfrak{p}B\right) \\
&\mapsto \frac{a(b + b') + \mathfrak{p}B}{s} \\
&= \frac{ab + ab' + \mathfrak{p}B}{s} \\
&= \frac{ab + \mathfrak{p}B + ab' + \mathfrak{p}B}{s} \\
&= \frac{ab + \mathfrak{p}B}{s} + \frac{ab' + \mathfrak{p}B}{s} \\
&= \leftarrow\!\shortmid \left(\frac{a}{s}, b + \mathfrak{p}B\right) + \leftarrow\!\shortmid \left(\frac{a}{s}, b' + \mathfrak{p}B\right)
\end{aligned}$$

Scaling in the second.

$$\begin{aligned}\left(\frac{a}{s}, a'(b+\mathfrak{p}B)\right) &= \left(\frac{a}{s}, a'b+\mathfrak{p}B\right)\\ &\mapsto \frac{aa'b+\mathfrak{p}B}{s}\\ &= a'\cdot\frac{ab+\mathfrak{p}B}{s}\\ &= a'\cdot \hookleftarrow \left(\frac{a}{s}, b+\mathfrak{p}B\right)\end{aligned}$$

The map (6) is now proved bilinear. By the universal property it factors.

$$\begin{tikzcd} A_\mathfrak{p}\times B/\mathfrak{p}B \arrow[r] \arrow[dr] & A_\mathfrak{p}\otimes_A B/\mathfrak{p}B \arrow[d] \\ & (B/\mathfrak{p}B)_\mathfrak{p} \end{tikzcd} \qquad \begin{tikzcd} (a/s, b+\mathfrak{p}B) \arrow[r, mapsto] \arrow[dr, mapsto] & a/s\otimes b+\mathfrak{p}B \arrow[d, mapsto] \\ & (ab+\mathfrak{p}B)/s \end{tikzcd}$$

The top-down arrow is our map (4). We move to the well-definition of the map (5).

$$\frac{b+\mathfrak{p}B}{s} = \frac{b'+\mathfrak{p}B}{s'} \overset{?}{\Longrightarrow} 1/s\otimes(b+\mathfrak{p}B) = 1/s'\otimes(b'+\mathfrak{p}B)$$

The left.

$$u(s'(b+\mathfrak{p}B) - s(b'+\mathfrak{p}B)) = 0$$
$$us'b - usb' \in \mathfrak{p}B$$

The right.

$$\begin{aligned}\frac{1}{s'}\otimes(b'+\mathfrak{p}B) &= \frac{us}{uss'}\otimes(b'+\mathfrak{p}B)\\ &= \frac{1}{uss'}\otimes(usb'+\mathfrak{p}B)\\ &= \frac{1}{uss'}\otimes(us'b+\mathfrak{p}B)\\ &= \frac{us'}{uss'}\otimes(b+\mathfrak{p}B)\\ &= \frac{1}{s}\otimes(b+\mathfrak{p}B)\end{aligned}$$

The map (5) is now well-defined. Additivity.

$$\begin{aligned}b+\mathfrak{p}B+b'+\mathfrak{p}B &= b+b'+\mathfrak{p}B\\ &\mapsto 1/1\otimes(b+b'+\mathfrak{p}B)\\ &= 1/1\otimes(b+\mathfrak{p}B+b'+\mathfrak{p}B)\\ &= 1/1\otimes(b+\mathfrak{p}B)+1/1\otimes(b'+\mathfrak{p}B)\\ &= \hookleftarrow(b+\mathfrak{p}B)+\hookleftarrow(b'+\mathfrak{p}B)\end{aligned}$$

Multiplicativity.

$$\begin{aligned}(b+\mathfrak{p}B)(b'+\mathfrak{p}B) &= bb'+\mathfrak{p}B\\ &\mapsto 1/1\otimes(bb'+\mathfrak{p}B)\\ &= (1/1)(1/1)\otimes(b+\mathfrak{p}B)(b'+\mathfrak{p}B)\\ &= (1/1\otimes(b+\mathfrak{p}B))(1/1\otimes(b'+\mathfrak{p}B))\\ &= \hookleftarrow(b+\mathfrak{p}B)\hookleftarrow(b'+\mathfrak{p}B)\end{aligned}$$