Facts about Rings of Fractions

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1 Introduction

Fact 1.1. *If $0 \in S$, then $S^{-1}A$ is a trivial ring.*

Proof. Any $(a, s), (a', s')$ are related because $(as' - a's) \cdot 0 = 0$ with $0 \in S$. □

Fact 1.2. *A a PID, the equivalence relation in $A \times S$ is: $(a, s) \equiv (a', s')$ iff $as' = a's$.* □

Fact 1.3. *For A a field, and $S = \{-1, 1\}$, $S^{-1}A \cong A$.*

Proof. It is easily verified that the standard isomorphism from A to $S^{-1}A$ is 1-1 and onto. □

Fact 1.4. *For A a field, and S a multiplicatively closed subset of A not containing zero, $S^{-1}A \cong A$.*

Proof. The standard homomorphism $f : a \mapsto a/1$ of A into $S^{-1}A$ is injectve: if $a/1 = a'/1$ then $a \cdot 1 = a1 \cdot 1$, then $a = a'$. It is surjective: $f(as^{-1}) = f(a)f(s^{-1}) = (a/1)(s^{-1}/1) = \ldots$, but $s^{-1}/1 = 1/s$ as $s^{-1}s = 1 \cdot 1$; continuing, $\ldots = (a/1)(1/s) = a/s$. □

Fact 1.5. *For A a field, the ring of fractions and the field of fractions are isomorphic.*

Proof. For isomorphism of A with its field of fractions, see Math Exchange 79188. About the isomorphism with its ring of fractions, is the fact above. □

Example 1.6. *Some example.*

Fact 1.7. *The quotient ring A/I can be viewed as an A-module, and then the ring of fractions $T^{-1}(A/I)$, where T is the image of S in A/I, equals the module of fractions $S^{-1}(A/I)$.*

Proof. On the left, the relation is in $(A/I) \times T$: $([a], [s]) \equiv ([a'], [s'])$ iff (ring notation) $([a][s'] - [a'][s])[s''] = [0]$ iff $[as's'' - a'ss''] = [0]$. On the right, the relation works in $(A/I) \times S$: $([a], s) \equiv ([a'], s')$ iff (module notation) $s''(s'[a] - s[a']) = [0]$ iff $[as's'' - a'ss''] = [0]$. The conditions are identical so the classes must be in bijective correspondence. However, they are not identical as sets, so saying *equals* is too much. □

Fact 1.8. *What is $S^{-1}\mathfrak{a}$?*

It can be either an $S^{-1}A$-module $S^{-1}\mathfrak{a}$, because $\mathfrak{a}$ is an A-module, or the extension $S^{-1}\mathfrak{a} = \mathfrak{a}\, S^{-1}A$ in $S^{-1}A$ of the ideal $\mathfrak{a}$ in A via the canonical $A \to S^{-1}A : a \mapsto a/s$. In both cases elements of $S^{-1}\mathfrak{a}$ are written as a/s with $a \in \mathfrak{a}$, $s \in S$, but they come from different sets. In the first, module case, a/s is in the quotient of $\mathfrak{a} \times S$, in the second, extension ideal case, a/s is in the quotient of $A \times S$. We are talking of $S^{-1}A$-modules, not rings, so there can only be an $S^{-1}A$-module isomorphism, which is obvious:

$$\mathfrak{a} \times S/\sim_{\mathfrak{a}} \;\ni\; a/s \mapsto a/s \;\in\; A \times S/\sim_A$$

□

Fact 1.9. *Case $\mathfrak{a} = \mathfrak{p}$, a prime ideal. What is $S^{-1}\mathfrak{p}$?*

It can be either the $A_{\mathfrak{p}}$-module $\mathfrak{p}_{\mathfrak{p}}$, because $\mathfrak{p}$ is an A-module, or the extension $\mathfrak{p}A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}$ of the ideal $\mathfrak{p}$ in A, via the canonical $A \to A_{\mathfrak{p}} : a \mapsto a/s$. Looks like we don't have the $\cdot_{\mathfrak{p}}$-instead-of-$S^{-1}\cdot$ notation in the ideal extension case, but then, the quotient notation $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ is used, which makes sense only if $\mathfrak{p}_{\mathfrak{p}}$ is an ideal in $A_{\mathfrak{p}}$

$$\mathfrak{p}_{\mathfrak{p}} = \mathfrak{p}\, A_{\mathfrak{p}}$$

□

Fact 1.10. *How is $B_{\mathfrak{q}}$ an $A_{\mathfrak{p}}$-module?*

Let $g = \psi \circ f$ be the composition $A \to B \to T^{-1}B : a \to f(a) \to f(a)/1$. This composition sends $s \in S$ to a unit in $T^{-1}B$, as $(f(s)/1)(1/f(s)) = 1/1$, where $f(s) \in f(S) = f(A \setminus \mathfrak{p}) \subseteq B \setminus \mathfrak{q} = T$. By the universal property of the ring of fractions, g factorizes

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f"] \arrow[dr, "\eta"] & S^{-1}A \arrow[d, "h"] \\ B \arrow[r, "\psi"] & T^{-1}B \end{tikzcd}$$

where the recipe for h is given in **Proposition 3.1** of [ItCA] as $a/s \mapsto g(a)g(s)^{-1} = (f(a)/1)(1/f(s)) = f(a)/f(s)$. □

Fact 1.11. *How is $B_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}}$ an $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$-module?*

The kernel of the composition $A_{\mathfrak{p}} \to B_{\mathfrak{q}} \to B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} : a/s \mapsto f(a)/f(s) + \mathfrak{q}B_{\mathfrak{q}}$ contains $\mathfrak{p}A_{\mathfrak{p}}$ (because $\mathfrak{p} = f^{-1}(\mathfrak{q})$) so the composition factors through $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$: $a/s + \mathfrak{p}A_{\mathfrak{p}} \mapsto f(a)/f(s) + \mathfrak{q}B_{\mathfrak{q}}$. This is a ring homomorphism that makes $B_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}}$ an $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$-module. □

Fact 1.12. *What is $\mathfrak{p}M_{\mathfrak{p}}$?*

When $M_{\mathfrak{p}}$ is seen as an A-module, $\mathfrak{p}M_{\mathfrak{p}} = \{am/s : a \in \mathfrak{p}, m \in M, s \notin \mathfrak{p}\}$. When $M_{\mathfrak{p}}$ is seen as an $A_{\mathfrak{p}}$-module, $\mathfrak{p}$ is not even an ideal in $A_{\mathfrak{p}}$, but its extension, $\mathfrak{p}A_{\mathfrak{p}}$ is, and $(\mathfrak{p}A_{\mathfrak{p}})M_{\mathfrak{p}} = \{(a/s')(m/s) : a \in \mathfrak{p}, m \in M, s, s' \notin \mathfrak{p}\} = \{am/s : a \in \mathfrak{p}, m \in M, s \notin \mathfrak{p}\}$, the same set, which we write $\mathfrak{p}M_{\mathfrak{p}}$ for:

$$\mathfrak{p}M_{\mathfrak{p}} = (\mathfrak{p}A_{\mathfrak{p}})M_{\mathfrak{p}}$$

□

Fact 1.13. *How*

$$\frac{(B \otimes_A M)_{\mathfrak{q}}}{\mathfrak{q}(B \otimes_A M)_{\mathfrak{q}}} \cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_B B \otimes_A M$$

?

Proposition 3.5 states, in the language of subscript-$\mathfrak{p}$, that $M_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A M$ over $A_{\mathfrak{p}}$. Here $(B \otimes_A M)_{\mathfrak{q}} \cong B_{\mathfrak{q}} \otimes_B (B \otimes_A M)$. Then

$$\begin{aligned} \frac{B_{\mathfrak{q}} \otimes_B (B \otimes_A M)}{(\mathfrak{q}B_{\mathfrak{q}})(B_{\mathfrak{q}} \otimes_B (B \otimes_A M))} &\cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}B_{\mathfrak{q}}} \otimes_{B_{\mathfrak{q}}} (B_{\mathfrak{q}} \otimes_B (B \otimes_A M)) \\ &\cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_B B \otimes_A M \end{aligned}$$

□

In P. Y. Gaillard solution to ItCA Exercise 3.19 (viii).

Fact 1.14. *The diagram*

$$\begin{tikzcd} A_{\mathfrak{p}} \arrow[r, "\phi"] \arrow[d, "f"] \arrow[dr, "\eta"] & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \arrow[d, "h"] \\ B_{\mathfrak{q}} \arrow[r, "\psi"] & B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \end{tikzcd}$$

$$\begin{tikzcd} a/s \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "f"] \arrow[dr, mapsto, "\eta"] & a/s + \mathfrak{p}A_{\mathfrak{p}} \arrow[d, mapsto, "h"] \\ f(a)/f(s) \arrow[r, mapsto, "\psi"] & f(a)/f(s) + \mathfrak{q}B_{\mathfrak{q}} \end{tikzcd}$$

is commutative.

All calculated on the diagram. $\square$

Now $\kappa_{\mathfrak{q}} = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ is an $A_{\mathfrak{p}}$-module by $A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ (with the formula as on the bottom diagram) and we may tensor over $A_{\mathfrak{p}}$.

If a field K is an A-module for some ring A, can it be a zero A-module?

$$1_A 1_K = 1_k \neq 0_K$$

It cannot.

Now that $\kappa_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = 0$, both tensorands finitely generated, and $\kappa_{\mathfrak{q}} \neq 0$, it must be $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = 0$ by ItCA Exercise 2.3.

In solution of ItCA 3.19 (viii) by J. D. Taylor.

Fact 1.15. *What is $\mathfrak{p}B$?*

For $f : A \to B$, we can think in two ways. As we identify $ab = f(a)b$, $\mathfrak{p}B = \{ab = f(a)b : a \in \mathfrak{p}, b \in B\}$ is the extension $f(\mathfrak{p})B$ of the ideal $\mathfrak{p}$. The second way is that B is an A-module, and $\mathfrak{p}$ a prime ideal in A, so we can form $\mathfrak{p}B = \{\sum a_i b_i = \sum f(a_i)b_i\}$ with $a_i \in \mathfrak{p}$, $b_i \in B$, getting the same set.

$\square$

Fact 1.16. *What is $B_{\mathfrak{p}}$?*

Since B is an A-module, $B_{\mathfrak{p}}$ consists of all elements b/s where $b \in B, s \in A \setminus \mathfrak{p}$. This is the standard construction of $S^{-1}A$-module $S^{-1}M$ in the text. It is:

- An A-module.
- An $A_{\mathfrak{p}}$-module: the standard construction.
- A B-module.
- A ring.

$\square$

Fact 1.17. *What is $\mathfrak{p}B_{\mathfrak{p}}$?*

$B_{\mathfrak{p}}$ is an A - module, $\mathfrak{p}$ is a prime ideal of A, so $\mathfrak{p}B_{\mathfrak{p}}$ makes sense and consists of finite sums $\sum a_i(b_i/s)$ where $a_i \in \mathfrak{p}$, $b_i \in B$, and $s_i \in A \setminus \mathfrak{p}$. After bringing to common denominator, the sum becomes ab/s where $a \in \mathfrak{p}$, $b \in B$ and $s_i \in A \setminus \mathfrak{p}$ that is, b/s where $b \in \mathfrak{p}B$ and $s_i \in A \setminus \mathfrak{p}$. $\square$

Fact 1.18. *$\mathfrak{p}B_{\mathfrak{p}}$ is an ideal in $B_{\mathfrak{p}}$.*

As a module, it is an abelian group, then the multiplication property is easily verified. $\square$

The ideal $\mathfrak{p}A_{\mathfrak{p}}$ was the single maximal ideal in $A_{\mathfrak{p}}$. We do not know this for $\mathfrak{p}B_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$.

Fact 1.19. *How is $A_{\mathfrak{p}}$ an A-module ?*

The canonical map $\phi : A \to A_{\mathfrak{p}} : a \mapsto \frac{a}{1}$ gives the multiplication by scalars from A

$$a' \frac{a}{s} = \phi(a') \frac{a}{s} = \frac{a'}{1} \frac{a}{s} = \frac{a'a}{s}$$

$\square$

Fact 1.20. *What is $\mathfrak{p}A_{\mathfrak{p}}$?*

As $A_{\mathfrak{p}}$ is an A-module, we can multiply it by a prime ideal in A in a standard way

$$\sum a'_i \frac{a_i}{s_i} = \sum \frac{a' a_i}{s_i}$$

After bringing to a common denominator, this is a/s with $a \in \mathfrak{p}$, so $\mathfrak{p}A_{\mathfrak{p}}$ is the single maximal ideal of the local ring $A_{\mathfrak{p}}$.

$\square$

Fact 1.21. *How is $B_{\mathfrak{p}}$ an $A_{\mathfrak{p}}$-module ?*

Definition of the $S^{-1}M$ as $S^{-1}A$-module in the text.

$$\frac{a}{s'}\frac{b}{s} = \frac{ab}{s's} = \frac{f(a)b}{s's}$$

□

Fact 1.22. *What is* $(\mathfrak{p}A_\mathfrak{p})B_\mathfrak{p}$ *?*

As $B_\mathfrak{p}$ is an $A_\mathfrak{p}$-module, and $\mathfrak{p}A_\mathfrak{p}$ is the single maximal ideal of the local ring $A_\mathfrak{p}$, Any element is, from the definition of the ideal-by-module and from the general element of $\mathfrak{p}A_\mathfrak{p}$ $(a \in \mathfrak{p})$

$$\sum_i \frac{a_i}{s_i'}\frac{b_i}{s_i} = \sum \frac{ab}{s's}$$

After bringing to a common denominator, this becomes ab/s where $a \in \mathfrak{p}$. Notice we got a general element of $\mathfrak{p}B_\mathfrak{p}$, so

$$(\mathfrak{p}A_\mathfrak{p})B_\mathfrak{p} = \mathfrak{p}B_\mathfrak{p}$$

□

Fact 1.23. *How* $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{p}$ *?*

Apply Exercise 2.2

$$A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$$

to $M \coloneqq B_\mathfrak{p}$, $A \coloneqq A_\mathfrak{p}$, $\mathfrak{a} \coloneqq \mathfrak{p}A_\mathfrak{p}$

$$A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{p} = B_\mathfrak{p}/(\mathfrak{p}A_\mathfrak{p})B_\mathfrak{p}$$

now apply $(\mathfrak{p}A_\mathfrak{p})B_\mathfrak{p} = \mathfrak{p}B_\mathfrak{p}$ □

Fact 1.24. *How* $A_\mathfrak{p} \otimes_A B \cong B_\mathfrak{p}$ *?*

Apply Proposition 3.5: $S^{-1}A \otimes_A M \cong S^{-1}M$. □

We now understand the isomorphisms in the solution of ItCA's 3.21(iv) by J D. Taylor.

2 Saturated

Fact 2.1. *For saturated* S*, if* $f(a)$ *is a unit in* $S^{-1}A$*, then* $a \in S$*.*

Proof.

$$\frac{a}{1} \cdot \frac{b}{t} = \frac{1}{1}$$

$$\frac{ab}{t} = \frac{1}{1}$$

$$(ab, t) \equiv (1, 1)$$

$$(ab - t)u = 0$$

$$abu = tu$$

$$abu \in S$$

As S is saturated, $a \in S$. □