## Exercise 3.21

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## M.F. Atiyah, I.G. MacDonald Introduction to Commutative Algebra

**Exercise 3.21.i.** Show that  $\phi^* : \operatorname{Spec}(S^{-1}A) \to \operatorname{Spec}(A)$  is a homeomorphism of  $\operatorname{Spec}(S^{-1}A)$  onto its image in  $X = \operatorname{Spec}(A)$ .

We want to prove that for D closed in  $\operatorname{Spec}(S^{-1}A)$  there is C closed in  $\operatorname{Spec}(A)$  and reverse, that the equation holds

$$C \cup \phi^*(\operatorname{Spec}(S^{-1}A)) = \phi^*(D)$$

We take

$$C = V(\mathfrak{a}), \quad D = V(S^{-1}\mathfrak{a})$$

That is, we are going to prove that

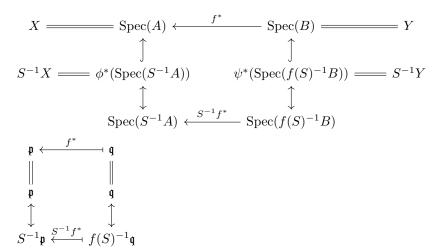
$$V(\mathfrak{a}) \cap \phi^*(\operatorname{Spec}(S^{-1}\mathfrak{a})) = \phi^*(V(S^{-1}\mathfrak{a}))$$

Recalling that  $S^{-1}\mathfrak{p} \xrightarrow{\phi^*} \mathfrak{p}$ , the right are all prime ideals not meeting S and holding  $S^{-1}\mathfrak{p} \supseteq S^{-1}\mathfrak{a}$ . By the fact that this happens iff  $\mathfrak{p} \supseteq \mathfrak{a}$ , they are all prime ideals not meeting S and containing  $\mathfrak{a}$ . Which is precisely the left.

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Now  $\phi^*$  maps closed files to closed files in any direction.

**Exercise 3.21.ii.** Let  $f: A \to B$  be a ring homomorphism...



That  $(S^{-1}f)^*$ : Spec $(S^{-1}B) \to \text{Spec}(S^{-1}A)$  is the restriction of  $f^*$  to  $S^{-1}Y$  we have already proved in Facts, showing the action of  $(S^{-1}f)^*$ :

$$(S^{-1}f)^*: S^{-1}\mathfrak{g} \mapsto S^{-1}\mathfrak{p}$$

What is  $S^{-1}X = \phi^*(\operatorname{Spec}(S^{-1}A))$ ? All prime ideals of A not meeting S.

What is  $f^{*-1}(S^{-1}X) = f^{*-1}(\phi^*(\operatorname{Spec}(S^{-1}A)))$ ? All prime ideals of B whose preimages in A do not meet S.

What is  $S^{-1}Y = \psi^*(\operatorname{Spec}(S^{-1}B))$ ? All prime ideals of B not meeting f(S).

We show that the last two sets are equal.

 $\supseteq$ : If  $\mathfrak{q}$  does not meet f(S), may its preimage  $f^{-1}(\mathfrak{q})$  meet S? Let  $s \in f^{-1}(\mathfrak{q})$ ;  $f(s) \in \mathfrak{q}$ ; now

 $\mathfrak{q}$  meets f(S), a contradiction. So it cannot.

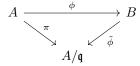
 $\subseteq$ : Let's not let the preimage  $f^{-1}(\mathfrak{q})$  of a prime ideal  $\mathfrak{q}$  of B meet S. May  $\mathfrak{q}$  meet f(B)?  $f(s) \in \mathfrak{q}$ ;  $s \in f^{-1}(\mathfrak{q})$ ; now the preimage  $f^{-1}(\mathfrak{q})$  in A meets S, a contradiction. So it cannot.

**Exercise 3.21.iii.** Let  $\mathfrak{a}$  be an ideal of A and let  $\mathfrak{b} = \mathfrak{a}^e$  be its extension in B...

Let  $\mathfrak{a}$  be an ideal of A and let  $\mathfrak{b} = \mathfrak{a}^e$  be its extension in B. What is the homomorphism?

$$\tilde{f}:A/\mathfrak{a}\to B/\mathfrak{b}$$

Recall what does a homomorphism need to factor through a quotient?



The map  $\tilde{\phi}$  has to be defined on representatives and cannot differ between them.

$$\tilde{\phi}(a+\mathfrak{a}) = \tilde{\phi}(a'+\mathfrak{a})$$

if  $a + \mathfrak{s} = a' + \mathfrak{s}$  iff  $a - a' \in \mathfrak{s}$ .

We define  $\tilde{\phi}$  by  $\tilde{\phi}(a+\mathfrak{a})=\phi(a)$  so it has to be

$$\phi(a) = \phi(a') \text{ if } a - a' \in \mathfrak{a}$$
 
$$\phi(a - a') = 0 \text{ if } a - a' \in \mathfrak{a}$$
 
$$\phi(a) = 0 \text{ if } a \in \mathfrak{a}$$

 $\ker \phi \supseteq \mathfrak{a}$ 

To factor through the quotient by an ideal, the homomorphism's kernel must contain this ideal.

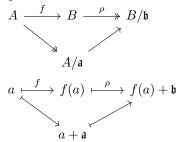
A homomorphism factors through any ideal contained in its kernel.

If  $\mathbf{a} \subseteq \phi^{-1}(0)$  then  $\phi(a) = 0$  for  $a \in \mathbf{a}$  then  $\phi(a - a') = 0$  for  $a - a' \in \mathbf{a}$  then  $\phi(a) = \phi(a')$  for  $a + \mathbf{a} = a' + \mathbf{a}$  and we can say  $\tilde{\phi}(a + \mathbf{a}) = \phi(a)$ .

We return to  $f: A \to B$ ,  $\mathfrak{a} = f^{-1}(\mathfrak{b})$ ,  $\mathfrak{b}$  an ideal of B.

$$A \xrightarrow{f} B \xrightarrow{\rho} B/\mathfrak{b}$$

Does the kernel contain  $\mathfrak{a}$ ? If  $a \in \mathfrak{a}$  then  $a \mapsto f(a) \mapsto f(a) + \mathfrak{b}$ , but  $f(a) \in \mathfrak{b}$  so a maps to zero and is in the kernel of this composition homomorphism, which then factors through the quotient:



How does  $\operatorname{Spec}(A/\mathfrak{a})$  have it canonical image  $V(\mathfrak{a})$  in  $\operatorname{Spec}(A)$ ?

$$A \xrightarrow{\pi} A/\mathfrak{a}$$

$$a\mapsto a+\mathfrak{s}$$

That there is a 1-1 correspondence between ideals of  $A/\mathfrak{a}$  and ideals of A containing  $\mathfrak{a}$ , we are told in the text on page 9. And that prime ideals correspond to prime ideals. So we have a bijection between  $\operatorname{Spec}(A/\mathfrak{a})$  and prime ideals of A containing  $\mathfrak{a}$ , which comprise the set  $V(\mathfrak{a})$ . We are not required to prove a homeomorphism here.

The general prime ideal of  $B/\mathfrak{b}$  is  $\rho(\mathfrak{q})$  where  $\mathfrak{q}$  is a prime ideal of B containing  $\mathfrak{b}$ .

$$\begin{split} \tilde{f}^*: \rho(\mathfrak{q}) \mapsto \tilde{f}^{-1}(\rho(\mathfrak{q})) \\ \tilde{f}^{-1}(\rho(\mathfrak{q})) = \{a + \mathfrak{a}: f(a) + \mathfrak{b} \in \rho(\mathfrak{q})\} = \dots \end{split}$$

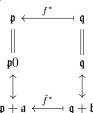
Property.  $b + \mathfrak{b} \in \rho(\mathfrak{q}) \iff b \in \mathfrak{q}$ .

Probably general for surjective homomorphism and an ideal, or even a set, containing the kernel.

If  $b + \mathfrak{b} \in \rho(\mathfrak{q})$  then  $b + \mathfrak{b} = b' + \mathfrak{b}$  for some  $b' \in \mathfrak{q}$ , then  $b - b' \in \mathfrak{q}$  and  $b' \in \mathfrak{q}$ , then  $b \in \mathfrak{q}$ .  $\Box$ 

$$\dots = \{a + \mathfrak{a} : f(a) \in \mathfrak{q}\}$$
$$= f^{-1}(\mathfrak{q}) + \mathfrak{a}$$
$$= \pi (f^{-1}(\mathfrak{q}))$$

Now  $\pi^*$  maps this to  $\pi^{-1}(\pi(f^{-1}(\mathfrak{q})))$ . As  $\pi$  is surjective, this set is  $f^{-1}(\mathfrak{q}) = f^*(\mathfrak{q})$ . The up-left path:  $\rho(\mathfrak{q})$  is identified in  $\operatorname{Spec}(B)$  with  $\mathfrak{q}$  then this is mapped by  $f^*$  to  $f^{-1}(\mathfrak{q}) = f^*(\mathfrak{q})$ .

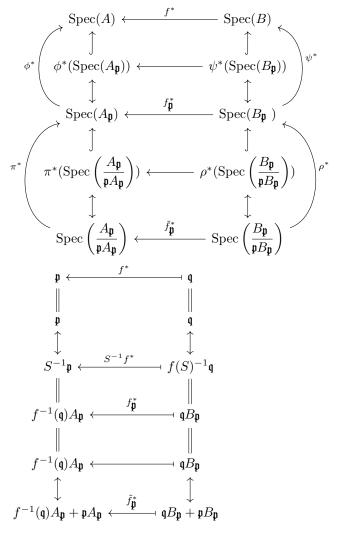


We took the risk to write  $\pi(\mathfrak{p})$  as  $\mathfrak{p} + \mathfrak{s}$  and  $\rho(\mathfrak{q})$  as  $\mathfrak{q} + \mathfrak{b}$ .

**Exercise 3.21.iv.** Let  $\mathfrak{p}$  be a prime ideal of A... We take  $S = A \setminus \mathfrak{p}$  in (ii)

What a ring can be reduced modulo  $S^{-1}\mathfrak{p}=(A\setminus\mathfrak{p})\mathfrak{p}=\mathfrak{p}A_{\mathfrak{p}}$ ? Only a ring of which  $S^{-1}\mathfrak{p}$  is an ideal, that is, the ring  $A_{\mathfrak{p}}$ . In (iii), the ring B to the right is reduced by the extension of the ideal  $\mathfrak{p}A_{\mathfrak{p}}$  in B? We know from Fact that this is  $\mathfrak{p}B_{\mathfrak{p}}$ .

This is how we apply (iii):  $A := A_{\mathfrak{p}}, \mathfrak{a} := \mathfrak{p}A_{\mathfrak{p}}, \mathfrak{b} := \mathfrak{p}B_{\mathfrak{p}}, f := f_{\mathfrak{p}} := S^{-1}f : A_{\mathfrak{p}} \to B_{\mathfrak{p}}.$ 



What is  $f^{*-1}(\mathfrak{p})$ ? It is the set  $\{\mathfrak{q}: f^*(\mathfrak{q}) = \mathfrak{p}\} = \{\mathfrak{q}: \mathfrak{p} = f^{-1}(\mathfrak{q})\}$ . We know that  $f^{*-1}(S^{-1}X) = S^{-1}Y$  and  $\mathfrak{p} \in S^{-1}X$  so  $f^{*-1}(\mathfrak{p}) \subseteq S^{-1}Y$ . Any ideal  $\mathfrak{q} \in f^{*-1}(\mathfrak{p})$  does not meet  $f(A \setminus \mathfrak{p})$ :  $a \in A \setminus \mathfrak{p}$ ;  $f(a) \in \mathfrak{q}$ ;  $a \in f^{-1}(\mathfrak{q})$ ;  $a \in \mathfrak{p}$ , a contradiction. So it is in  $\psi^*(\operatorname{Spec}(B_{\mathfrak{p}}))$ . To fall into  $\rho^*(\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}))$ ,  $\mathfrak{q}B_{\mathfrak{p}}$  should contain  $\mathfrak{p}B_{\mathfrak{p}}$ .

$$\mathfrak{p}B\mathfrak{p} = \left\{ \frac{f(a)}{s} : a \in \mathfrak{p}, b \in B, s \notin \mathfrak{p} \right\}$$
 
$$\mathfrak{q}B\mathfrak{p} = \left\{ \frac{b}{s} : b \in \mathfrak{q}, s \notin \mathfrak{p} \right\}$$

But  $f(a) \in \mathfrak{q}$ , then  $f(a)b \in \mathfrak{q}$ . So  $\mathfrak{q}B_{\mathfrak{p}} \supseteq \mathfrak{p}B_{\mathfrak{p}}$ . We have shown that

$$f^{*-1}(\mathfrak{p}) \subseteq \psi^*(\rho^*(B\mathfrak{p}/\mathfrak{p}B\mathfrak{p}))$$

The reverse inclusion is obvious from both diagrams combined (the large diagram). Also the diagram shows a bijection.

A closed set in Spec $(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$  is  $V(\rho(\mathfrak{b}B_{\mathfrak{p}})) = \rho(V(\mathfrak{b}B_{\mathfrak{p}}))$ . We want to show that the  $\psi^* \circ \rho^*$  image of this set is a closed set in  $\operatorname{Spec}(B)$  i.e. the set of prime ideals containing some ideal of B, intersected with  $f^{*-1}(\mathfrak{p})$ .

Note that all arrows of the combined diagram work on prime, not ordinary ideals.

Our contained ideal is the surjection of some ideal  $\mathfrak{b}B_{\mathfrak{p}}$  and the same is true for containing prime ideals, they are surjections of prime ideals of  $B_{\mathfrak{p}}$  containing  $\mathfrak{b}B_{\mathfrak{p}}$ .

Now there is a fact:  $\mathfrak{b} \subseteq \mathfrak{q} \iff \mathfrak{b}B_{\mathfrak{p}} \subseteq \mathfrak{q}B_{\mathfrak{p}}$ :  $\mathfrak{b} \subseteq \mathfrak{b}^{ec} \subseteq \mathfrak{q}^{ec} = \mathfrak{q}$ . The inclusion relations have moved from  $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$  to  $\operatorname{Spec}(B)$  through  $\psi^* \circ \rho^*$ :

$$(\psi^* \circ \rho^*)(V(\mathfrak{b}B\mathfrak{p} + \mathfrak{p}B\mathfrak{p})) = \psi^*(V(\mathfrak{b}B\mathfrak{p})) = V(\mathfrak{b})$$

a closed set in Spec(B). This set happens to be contained in  $f^{*-1}(\mathfrak{p})$  due to the bijection proven above.

Now take a closed set in  $f^{*-1}(\mathfrak{p})$ . It is the intersection of a closed set in  $\operatorname{Spec}(B)$  with the set  $f^{*-1}(\mathfrak{p})$  itself. A closed set in  $\operatorname{Spec}(B)$  is  $V(\mathfrak{b})$ .  $f^{*-1}(\mathfrak{p})$  are prime ideals of B such that  $f^*(\mathfrak{q}) = \mathfrak{p}$  that is  $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ . The intersection are prime ideals  $\mathfrak{q}$  of B such that  $f^{-1}(\mathfrak{q}) = \mathfrak{p}$  and  $\mathfrak{q} \supseteq \mathfrak{p}$ . We have already proved that they do not meet  $f(A \setminus \mathfrak{p})$ , so they are all in  $\psi^*(\operatorname{Spec}(B_{\mathfrak{p}}))$ . In  $\operatorname{Spec}(B_{\mathfrak{p}})$  each of them is mapped to  $\mathfrak{q}B_{\mathfrak{p}} \supseteq \mathfrak{b}B_{\mathfrak{p}}$ ; it is in  $V(\mathfrak{b}B_{\mathfrak{p}})$ . After surjection onto  $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ , they all fall into V of the surjection of  $\mathfrak{b}B_{\mathfrak{p}}$ 

$$\rho(V(\mathfrak{b}B_{\mathfrak{p}})) = V(\rho(\mathfrak{b}B_{\mathfrak{p}}))$$

We move to the final isomorphism  $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \cong A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ . The first way is by J. D. Taylor.

$$B/\mathfrak{p}B_{\mathfrak{p}} \cong A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}$$

$$\cong A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}} \otimes_{A} B$$

$$\cong A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A} B$$

The first isomorphism is an application of Exercise 2.2:i9i

$$A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$$

to  $M := B_{\mathfrak{p}}, A := A_{\mathfrak{p}}, \mathfrak{a} := \mathfrak{p}A_{\mathfrak{p}}$ , then application of  $(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$ . The second isomorphism is an application of Proposition 3.5:

$$S^{-1}A \otimes_A M \cong S^{-1}M$$
$$A_{\mathfrak{p}} \otimes_A M \cong M_{\mathfrak{p}}$$
$$A_{\mathfrak{p}} \otimes_A B \cong B_{\mathfrak{p}}$$

The second way is inspired by Y. P. Gaillard. We take take the exact sequence

$$0 \to \mathfrak{p}A_{\mathfrak{p}} \to A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to 0$$

Then tensor it with M over A. The sequence

$$\mathfrak{p}A_{\mathfrak{p}} \otimes_A M \to A_{\mathfrak{p}} \otimes_A M \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A M \to 0$$

is exact. This sequence on elements

$$\frac{a}{s} \otimes m \mapsto \frac{a}{s} \otimes m \mapsto \left[\frac{a}{s}\right] \otimes m$$
$$\frac{a}{s} \otimes m \mapsto \left[\frac{a}{s}\right] \otimes m$$

To its second module, we apply Proposition 3.5

$$A_{\mathfrak{p}} \otimes_{A} M \cong M_{\mathfrak{p}}$$
$$\frac{a}{s} \otimes m \mapsto \frac{am}{s}$$
$$\frac{1}{s} \otimes m \leftarrow \frac{m}{s}$$

getting the third exact sequence

$$\mathfrak{p}A_{\mathfrak{p}} \otimes_A M \to M_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A M \to 0$$

The first homomorphism of this sequence is

$$\mathfrak{p}A_{\mathfrak{p}} \otimes_A M \to A_{\mathfrak{p}} \otimes_A M \to M_{\mathfrak{p}}$$
$$\frac{a}{s} \otimes m \mapsto \frac{a}{s} \otimes m \mapsto \frac{am}{s}$$

The second isomorphism is

$$M_{\mathfrak{p}} \to A_{\mathfrak{p}} \otimes_A M \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A M$$
$$\frac{m}{s} \mapsto \frac{1}{s} \otimes m \mapsto \left[\frac{1}{s}\right] \otimes m$$

What is the image of  $\mathfrak{p}A_{\mathfrak{p}}\otimes_A M$  in M-ppp in the third sequence? It is

$$\left\{\frac{am}{s}: a \in \mathfrak{p}, s \notin \mathfrak{p}\right\} = \mathfrak{p}M_{\mathfrak{p}}$$

Now we can state that

$$M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \cong A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A M$$

$$\frac{m}{s}=\mathfrak{p}M_{\mathfrak{p}}\mapsto\left(\frac{1}{s}+\mathfrak{p}A_{\mathfrak{p}}\right)\otimes m$$

What is the inverse? The general element of  $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}\otimes_A M$ 

$$\left(\frac{a}{s} + \mathfrak{p}A_{\mathfrak{p}}\right) \otimes m = a\left(\frac{1}{s} + \mathfrak{p}A_{\mathfrak{p}}\right) \otimes m = \left(\frac{1}{s} + \mathfrak{p}A_{\mathfrak{p}}\right) \otimes am$$

is the image of

$$\frac{am}{s} + \mathfrak{p}M_{\mathfrak{p}}$$

The corresponding element in  $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$  is its preimage. The inverse map on elements is

$$\left(\frac{a}{s} + \mathfrak{p}A_{\mathfrak{p}}\right) \otimes m \mapsto \frac{am}{s} + \mathfrak{p}M_{\mathfrak{p}}$$