Exercise 3.21

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Exercise 3.21.i. Show that $\phi^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is a homeomorphism of $\mathrm{Spec}(S^{-1}A)$ onto its image in $X = \mathrm{Spec}(A)$.

We want to prove that for D closed in $\mathrm{Spec}(S^{-1}A)$ there is C closed in $\mathrm{Spec}(A)$ and reverse, that the equation holds

$$C \cup \phi^*(\mathrm{Spec}(S^{-1}A)) = \phi^*(D)$$

We take

$$C = V(\mathfrak{a}), \quad D = V(S^{-1}\mathfrak{a})$$

That is, we are going to prove that

$$V(\mathfrak{a}) \cap \phi^*(\mathrm{Spec}(S^{-1}\mathfrak{a})) = \phi^*(V(S^{-1}\mathfrak{a}))$$

Recalling that $S^{-1}\mathfrak{p} \overset{\phi^*}{\longmapsto} \mathfrak{p}$, the right are all prime ideals not meeting S and holding $S^{-1}\mathfrak{p} \supseteq S^{-1}\mathfrak{a}$. By the fact that this happens iff $\mathfrak{p} \supseteq \mathfrak{a}$, they are all prime ideals not meeting S and containing $\mathfrak{a}$. Which is precisely the left.
Now ϕ^* maps closed files to closed files in any direction. $\square$

Exercise 3.21.ii. Let $f : A \to B$ be a ring homomorphism...

$$\begin{tikzcd}
X \arrow[r, equal] & \mathrm{Spec}(A) & & \mathrm{Spec}(B) \arrow[ll, "f^*"'] \arrow[r, equal] & Y \\
S^{-1}X \arrow[r, equal] & \phi^*(\mathrm{Spec}(S^{-1}A)) \arrow[u, hook] & & \psi^*(\mathrm{Spec}(f(S)^{-1}B)) \arrow[u, hook] \arrow[r, equal] & S^{-1}Y \\
& \mathrm{Spec}(S^{-1}A) \arrow[u, leftrightarrow] & & \mathrm{Spec}(f(S)^{-1}B) \arrow[ll, "S^{-1}f^*"'] \arrow[u, leftrightarrow] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{p} \arrow[d, equal] & \mathfrak{q} \arrow[l, mapsto, "f^*"'] \arrow[d, equal] \\
\mathfrak{p} \arrow[d, leftrightarrow] & \mathfrak{q} \arrow[d, leftrightarrow] \\
S^{-1}\mathfrak{p} & f(S)^{-1}\mathfrak{q} \arrow[l, mapsto, "S^{-1}f^*"']
\end{tikzcd}$$

That $(S^{-1}f)^* : \mathrm{Spec}(S^{-1}B) \to \mathrm{Spec}(S^{-1}A)$ is the restriction of f^* to $S^{-1}Y$ we have already proved in Facts, showing the action of $(S^{-1}f)^*$:

$$(S^{-1}f)^* : S^{-1}\mathfrak{q} \mapsto S^{-1}\mathfrak{p}$$

What is $S^{-1}X = \phi^*(\mathrm{Spec}(S^{-1}A))$? All prime ideals of A not meeting S.
What is $f^{*-1}(S^{-1}X) = f^{*-1}(\phi^*(\mathrm{Spec}(S^{-1}A)))$? All prime ideals of B whose preimages in A do not meet S.
What is $S^{-1}Y = \psi^*(\mathrm{Spec}(S^{-1}B))$? All prime ideals of B not meeting $f(S)$.
We show that the the last two sets are equal.
$\supseteq$: If $\mathfrak{q}$ does not meet $f(S)$, may its preimage $f^{-1}(\mathfrak{q})$ meet S? Let $s \in f^{-1}(\mathfrak{q})$; $f(s) \in \mathfrak{q}$; now

$\mathfrak{q}$ meets $f(S)$, a contradiction. So it cannot.
$\subseteq$: Let's not let the preimage $f^{-1}(\mathfrak{q})$ of a prime ideal $\mathfrak{q}$ of B meet S. May $\mathfrak{q}$ meet $f(B)$? $f(s) \in \mathfrak{q}$; $s \in f^{-1}(\mathfrak{q}$; now the preimage $f^{-1}(\mathfrak{q})$ in A meets S, a contradiction. So it cannot.

Exercise 3.21.iii. Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B...

Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B. What is the homomorphism?

$$\tilde{f} : A/\mathfrak{a} \to B/\mathfrak{b}$$

Recall what does a homomorphism need to factor through a quotient?

$$\begin{tikzcd}
A \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & B \arrow[dl, "\tilde{\phi}"] \\
& A/\mathfrak{q} &
\end{tikzcd}$$

The map $\tilde{\phi}$ has to be defined on representatives and cannot differ between them.

$$\tilde{\phi}(a + \mathfrak{a}) = \tilde{\phi}(a' + \mathfrak{a})$$

if $a + \mathfrak{a} = a' + \mathfrak{a}$ iff $a - a' \in \mathfrak{a}$.
We define $\tilde{\phi}$ by $\tilde{\phi}(a + \mathfrak{a}) = \phi(a)$ so it has to be

$$\phi(a) = \phi(a') \text{ if } a - a' \in \mathfrak{a}$$

$$\phi(a - a') = 0 \text{ if } a - a' \in \mathfrak{a}$$

$$\phi(a) = 0 \text{ if } a \in \mathfrak{a}$$

$$\ker \phi \supseteq \mathfrak{a}$$

To factor through the quotient by an ideal, the homomorphism's kernel must contain this ideal.
A homomorphism factors through any ideal contained in its kernel.

If $\mathfrak{a} \subseteq \phi^{-1}(0)$ then $\phi(a) = 0$ for $a \in \mathfrak{a}$ then $\phi(a - a') = 0$ for $a - a' \in \mathfrak{a}$ then $\phi(a) = \phi(a')$ for $a + \mathfrak{a} = a' + \mathfrak{a}$ and we can say $\tilde{\phi}(a + \mathfrak{a}) = \phi(a)$.

We return to $f : A \to B$, $\mathfrak{a} = f^{-1}(\mathfrak{b})$, $\mathfrak{b}$ an ideal of B.

$$A \xrightarrow{f} B \xrightarrow{\rho} B/\mathfrak{b}$$

Does the kernel contain $\mathfrak{a}$? If $a \in \mathfrak{a}$ then $a \mapsto f(a) \mapsto f(a) + \mathfrak{b}$, but $f(a) \in \mathfrak{b}$ so a maps to zero and is in the kernel of this composition homomorphism, which then factors through the quotient:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, two heads] & B \arrow[r, "\rho", two heads] & B/\mathfrak{b} \\
& A/\mathfrak{a} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[r, mapsto, "f"] \arrow[dr, mapsto] & f(a) \arrow[r, mapsto, "\rho"] & f(a) + \mathfrak{b} \\
& a + \mathfrak{a} \arrow[ur, mapsto] &
\end{tikzcd}$$

How does $\mathrm{Spec}(A/\mathfrak{a})$ have it canonical image $V(\mathfrak{a})$ in $\mathrm{Spec}(A)$?

$$A \xrightarrow{\pi} A/\mathfrak{a}$$

$$a \mapsto a + \mathfrak{a}$$

That there is a 1-1 correspondence between ideals of $A/\mathfrak{a}$ and ideals of A containing $\mathfrak{a}$, we are told in the text on page 9. And that prime ideals correspond to prime ideals. So we have a bijection between Spec(A/$\mathfrak{a}$) and prime ideals of A containing $\mathfrak{a}$, which comprise the set $V(\mathfrak{a})$. We are not required to prove a homeomorphism here.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "\pi"] & B \arrow[d, two heads, "\rho"] \\
A/\mathfrak{a} \arrow[r, "\tilde{f}"] & B/\mathfrak{b}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Spec}(A) & \mathrm{Spec}(B) \arrow[l, "f^*"'] \\
\pi^*(\mathrm{Spec}(A/\mathfrak{a})) \arrow[u, hook] \arrow[d, leftrightarrow] & \rho^*(\mathrm{Spec}(B/\mathfrak{b})) \arrow[l] \arrow[u, hook] \arrow[d, leftrightarrow] \\
\mathrm{Spec}(A/\mathfrak{a}) \arrow[uu, bend left, "\pi^*"] & \mathrm{Spec}(B/\mathfrak{b}), \arrow[l, "\tilde{f}^*"'] \arrow[uu, bend right, "\rho^*"']
\end{tikzcd}
$$

The general prime ideal of $B/\mathfrak{b}$ is $\rho(\mathfrak{q})$ where $\mathfrak{q}$ is a prime ideal of B containing $\mathfrak{b}$.

$$\tilde{f}^* : \rho(\mathfrak{q}) \mapsto \tilde{f}^{-1}(\rho(\mathfrak{q}))$$

$$\tilde{f}^{-1}(\rho(\mathfrak{q})) = \{a + \mathfrak{a} : f(a) + \mathfrak{b} \in \rho(\mathfrak{q})\} = \dots$$

Property. $b + \mathfrak{b} \in \rho(\mathfrak{q}) \iff b \in \mathfrak{q}$.

Probably general for surjective homomorphism and an ideal, or even a set, containing the kernel.
If $b + \mathfrak{b} \in \rho(\mathfrak{q})$ then $b + \mathfrak{b} = b' + \mathfrak{b}$ for some $b' \in \mathfrak{q}$, then $b - b' \in \mathfrak{q}$ and $b' \in \mathfrak{q}$, then $b \in \mathfrak{q}$.
If $b \in \mathfrak{q}$ then $\rho(b) \in \rho(\mathfrak{q})$, meaning $b + \mathfrak{b} \in \mathfrak{q}$. □

$$
\begin{aligned}
\dots &= \{a + \mathfrak{a} : f(a) \in \mathfrak{q}\} \\
&= f^{-1}(\mathfrak{q}) + \mathfrak{a} \\
&= \pi(f^{-1}(\mathfrak{q}))
\end{aligned}
$$

Now π^* maps this to $\pi^{-1}(\pi(f^{-1}(\mathfrak{q})))$. As π is surjective, this set is $f^{-1}(\mathfrak{q}) = f^*(\mathfrak{q})$.
The up-left path: $\rho(\mathfrak{q})$ is identified in $\mathrm{Spec}(B)$ with $\mathfrak{q}$ then this is mapped by f^* to $f^{-1}(\mathfrak{q}) = f^*(\mathfrak{q})$.

$$
\begin{tikzcd}
\mathfrak{p} \arrow[d, equal] & \mathfrak{q} \arrow[l, mapsto, "f^*"'] \arrow[d, equal] \\
\mathfrak{p}0 \arrow[d, leftrightarrow] & \mathfrak{q} \arrow[d, leftrightarrow] \\
\mathfrak{p} + \mathfrak{a} & \mathfrak{q} + \mathfrak{b} \arrow[l, mapsto, "\tilde{f}^*"']
\end{tikzcd}
$$

We took the risk to write $\pi(\mathfrak{p})$ as $\mathfrak{p} + \mathfrak{a}$ and $\rho(\mathfrak{q})$ as $\mathfrak{q} + \mathfrak{b}$.

Exercise 3.21.iv. Let $\mathfrak{p}$ be a prime ideal of A...
We take $S = A \setminus \mathfrak{p}$ in (ii)

$$
\begin{tikzcd}
\mathrm{Spec}(A) & \mathrm{Spec}(B) \arrow[l, "f^*"'] \\
\phi^*(\mathrm{Spec}(A_\mathfrak{p})) \arrow[u, hook] \arrow[d, leftrightarrow] & \psi^*(\mathrm{Spec}(B_\mathfrak{p})) \arrow[l] \arrow[u, hook] \arrow[d, leftrightarrow] \\
\mathrm{Spec}(A_\mathfrak{p}) \arrow[uu, bend left, "\phi^*"] & \mathrm{Spec}(B_\mathfrak{p}\ ) \arrow[l, "f_\mathfrak{p}^*"'] \arrow[uu, bend right, "\psi^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{p} \arrow[d, equal] & \mathfrak{q} \arrow[l, mapsto, "f^*"'] \arrow[d, equal] \\
\mathfrak{p} \arrow[d, leftrightarrow] & \mathfrak{q} \arrow[d, leftrightarrow] \\
S^{-1}\mathfrak{p} & S^{-1}\mathfrak{q} \arrow[l, mapsto, "S^{-1}f^*"']
\end{tikzcd}
$$

What a ring can be reduced modulo $S^{-1}\mathfrak{p} = (A \setminus \mathfrak{p})\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$? Only a ring of which $S^{-1}\mathfrak{p}$ is an ideal, that is, the ring $A_\mathfrak{p}$. In (iii), the ring B to the right is reduced by the extension of the ideal $\mathfrak{a}$ of A. What is the extension of the ideal $\mathfrak{p}A_\mathfrak{p}$ in B? We know from Fact that this is $\mathfrak{p}B_\mathfrak{p}$.

This is how we apply (iii): $A := A_\mathfrak{p}$, $\mathfrak{a} := \mathfrak{p}A_\mathfrak{p}$, $\mathfrak{b} := \mathfrak{p}B_\mathfrak{p}$, $f := f_\mathfrak{p} := S^{-1}f : A_\mathfrak{p} \to B_\mathfrak{p}$.

$$
\begin{tikzcd}
\operatorname{Spec}(A) & \operatorname{Spec}(B) \arrow[l, "f^*"'] \\
\phi^*(\operatorname{Spec}(A_{\mathfrak{p}})) \arrow[u, hook] & \psi^*(\operatorname{Spec}(B_{\mathfrak{p}})) \arrow[l] \arrow[u, hook] \\
\operatorname{Spec}(A_{\mathfrak{p}}) \arrow[u, leftrightarrow] \arrow[uu, bend left=60, "\phi^*"] & \operatorname{Spec}(B_{\mathfrak{p}}) \arrow[l, "f_{\mathfrak{p}}^*"'] \arrow[u, leftrightarrow] \arrow[uu, bend right=60, "\psi^*"'] \\
\pi^*\left(\operatorname{Spec}\left(\dfrac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}}\right)\right) \arrow[u, hook] & \rho^*\left(\operatorname{Spec}\left(\dfrac{B_{\mathfrak{p}}}{\mathfrak{p}B_{\mathfrak{p}}}\right)\right) \arrow[l] \arrow[u, hook] \\
\operatorname{Spec}\left(\dfrac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}}\right) \arrow[u, leftrightarrow] \arrow[uu, bend left=60, "\pi^*"] & \operatorname{Spec}\left(\dfrac{B_{\mathfrak{p}}}{\mathfrak{p}B_{\mathfrak{p}}}\right) \arrow[l, "\tilde{f}_{\mathfrak{p}}^*"'] \arrow[u, leftrightarrow] \arrow[uu, bend right=60, "\rho^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathfrak{p} & \mathfrak{q} \arrow[l, mapsto, "f^*"'] \\
\mathfrak{p} \arrow[u, equal] & \mathfrak{q} \arrow[u, equal] \\
S^{-1}\mathfrak{p} \arrow[u, leftrightarrow] & f(S)^{-1}\mathfrak{q} \arrow[l, mapsto, "S^{-1}f^*"'] \arrow[u, leftrightarrow] \\
f^{-1}(\mathfrak{q})A_{\mathfrak{p}} \arrow[u, equal] & \mathfrak{q}B_{\mathfrak{p}} \arrow[l, mapsto, "f_{\mathfrak{p}}^*"'] \arrow[u, equal] \\
f^{-1}(\mathfrak{q})A_{\mathfrak{p}} \arrow[u, equal] & \mathfrak{q}B_{\mathfrak{p}} \arrow[l, mapsto] \arrow[u, equal] \\
f^{-1}(\mathfrak{q})A_{\mathfrak{p}} + \mathfrak{p}A_{\mathfrak{p}} \arrow[u, leftrightarrow] & \mathfrak{q}B_{\mathfrak{p}} + \mathfrak{p}B_{\mathfrak{p}} \arrow[l, mapsto, "\tilde{f}_{\mathfrak{p}}^*"'] \arrow[u, leftrightarrow]
\end{tikzcd}
$$

What is $f^{*-1}(\mathfrak{p})$? It is the set $\{\mathfrak{q} : f^*(\mathfrak{q}) = \mathfrak{p}\} = \{\mathfrak{q} : \mathfrak{p} = f^{-1}(\mathfrak{q})\}$. We know that $f^{*-1}(S^{-1}X) = S^{-1}Y$ and $\mathfrak{p} \in S^{-1}X$ so $f^{*-1}(\mathfrak{p}) \subseteq S^{-1}Y$. Any ideal $\mathfrak{q} \in f^{*-1}(\mathfrak{p})$ does not meet $f(A\backslash\mathfrak{p})$: $a \in A\backslash\mathfrak{p}$; $f(a) \in \mathfrak{q}$; $a \in f^{-1}(\mathfrak{q})$; $a \in \mathfrak{p}$, a contradiction. So it is in $\psi^*(\operatorname{Spec}(B_{\mathfrak{p}}))$. To fall into $\rho^*(\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}))$, $\mathfrak{q}B_{\mathfrak{p}}$ should contain $\mathfrak{p}B_{\mathfrak{p}}$.

$$\mathfrak{p}B_{\mathfrak{p}} = \left\{\frac{f(a)}{s} : a \in \mathfrak{p}, b \in B, s \notin \mathfrak{p}\right\}$$

$$\mathfrak{q}B_{\mathfrak{p}} = \left\{\frac{b}{s} : b \in \mathfrak{q}, s \notin \mathfrak{p}\right\}$$

But $f(a) \in \mathfrak{q}$, then $f(a)b \in \mathfrak{q}$. So $\mathfrak{q}B_{\mathfrak{p}} \supseteq \mathfrak{p}B_{\mathfrak{p}}$. We have shown that

$$f^{*-1}(\mathfrak{p}) \subseteq \psi^*(\rho^*(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}))$$

The reverse inclusion is obvious from both diagrams combined (the large diagram). Also the diagram shows a bijection.

A closed set in $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$ is $V(\rho(\mathfrak{b}B_{\mathfrak{p}})) = \rho(V(\mathfrak{b}B_{\mathfrak{p}}))$. We want to show that the $\psi^* \circ \rho^*$ image of this set is a closed set in $\operatorname{Spec}(B)$ i.e. the set of prime ideals containing some ideal of B, intersected with $f^{*-1}(\mathfrak{p})$.

Note that all arrows of the combined diagram work on prime, not ordinary ideals.

Our contained ideal is the surjection of some ideal $\mathfrak{b}B_{\mathfrak{p}}$ and the same is true for containing prime ideals, they are surjections of prime ideals of $B_{\mathfrak{p}}$ containing $\mathfrak{b}B_{\mathfrak{p}}$.

Now there is a fact: $\mathfrak{b} \subseteq \mathfrak{q} \iff \mathfrak{b}B_{\mathfrak{p}} \subseteq \mathfrak{q}B_{\mathfrak{p}}$: $\mathfrak{b} \subseteq \mathfrak{b}^{ec} \subseteq \mathfrak{q}^{ec} = \mathfrak{q}$.

The inclusion relations have moved from $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$ to $\operatorname{Spec}(B)$ through $\psi^* \circ \rho^*$:

$$(\psi^* \circ \rho^*)(V(\mathfrak{b}B_{\mathfrak{p}} + \mathfrak{p}B_{\mathfrak{p}})) = \psi^*(V(\mathfrak{b}B_{\mathfrak{p}})) = V(\mathfrak{b})$$

a closed set in $\operatorname{Spec}(B)$. This set happens to be contained in $f^{*-1}(\mathfrak{p})$ due to the bijection proven above.

Now take a closed set in $f^{*-1}(\mathfrak{p})$. It is the intersection of a closed set in $\mathrm{Spec}(B)$ with the set $f^{*-1}(\mathfrak{p})$ itself. A closed set in $\mathrm{Spec}(B)$ is $V(\mathfrak{b})$. $f^{*-1}(\mathfrak{p})$ are prime ideals of B such that $f^*(\mathfrak{q}) = \mathfrak{p}$ that is $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. The intersection are prime ideals $\mathfrak{q}$ of B such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ and $\mathfrak{q} \supseteq \mathfrak{p}$. We have already proved that they do not meet $f(A \setminus \mathfrak{p})$, so they are all in $\psi^*(\mathrm{Spec}(B_\mathfrak{p})$. In $\mathrm{Spec}(B_\mathfrak{p})$ each of them is mapped to $\mathfrak{q}B_\mathfrak{p} \supseteq \mathfrak{b}B_\mathfrak{p}$; it is in $V(\mathfrak{b}B_\mathfrak{p})$. After surjection onto $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$, they all fall into V of the surjection of $\mathfrak{b}B_\mathfrak{p}$

$$\rho(V(\mathfrak{b}B_\mathfrak{p})) = V(\rho(\mathfrak{b}B_\mathfrak{p}))$$

We move to the final isomorphism $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.
The first way is by J. D. Taylor.

$$\begin{aligned} B/\mathfrak{p}B_\mathfrak{p} &\cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{p} \\ &\cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p} \otimes_A B \\ &\cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_A B \end{aligned}$$

The first isomorphism is an application of Exercise 2.2:i9i

$$A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$$

to $M \coloneqq B_\mathfrak{p}, A \coloneqq A_\mathfrak{p}, \mathfrak{a} \coloneqq \mathfrak{p}A_\mathfrak{p}$, then application of $(\mathfrak{p}A_\mathfrak{p})B_\mathfrak{p} = \mathfrak{p}B_\mathfrak{p}$.
The second isomprphism is an application of Proposition 3.5:

$$\begin{gathered} S^{-1}A \otimes_A M \cong S^{-1}M \\ A_\mathfrak{p} \otimes_A M \cong M_\mathfrak{p} \\ A_\mathfrak{p} \otimes_A B \cong B_\mathfrak{p} \end{gathered}$$

The second way is inspired by Y. P. Gaillard. We take take the exact sequence

$$0 \to \mathfrak{p}A_\mathfrak{p} \to A_\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to 0$$

Then tensor it with M over A. The sequence

$$\mathfrak{p}A_\mathfrak{p} \otimes_A M \to A_\mathfrak{p} \otimes_A M \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_A M \to 0$$

is exact. This sequence on elements

$$\begin{aligned} \frac{a}{s} \otimes m \mapsto \frac{a}{s} \otimes m &\mapsto \left[\frac{a}{s}\right] \otimes m \\ \frac{a}{s} \otimes m &\mapsto \left[\frac{a}{s}\right] \otimes m \end{aligned}$$

To its second module, we apply Proposition 3.5

$$\begin{gathered} A_\mathfrak{p} \otimes_A M \cong M_\mathfrak{p} \\ \frac{a}{s} \otimes m \mapsto \frac{am}{s} \\ \frac{1}{s} \otimes m \mapsfrom \frac{m}{s} \end{gathered}$$

getting the third exact sequence

$$\mathfrak{p}A_\mathfrak{p} \otimes_A M \to M_\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_A M \to 0$$

The first homomorphism of this sequence is

$$\mathfrak{p}A_\mathfrak{p} \otimes_A M \to A_\mathfrak{p} \otimes_A M \to M_\mathfrak{p}$$

$$\frac{a}{s} \otimes m \mapsto \frac{a}{s} \otimes m \mapsto \frac{am}{s}$$

The second isomorphism is

$$M_\mathfrak{p} \to A_\mathfrak{p} \otimes_A M \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_A M$$

$$\frac{m}{s} \mapsto \frac{1}{s} \otimes m \mapsto \left[\frac{1}{s}\right] \otimes m$$

What is the image of $\mathfrak{p}A_\mathfrak{p} \otimes_A M$ in M_ppp in the third sequence? It is

$$\left\{ \frac{am}{s} : a \in \mathfrak{p}, s \notin \mathfrak{p} \right\} = \mathfrak{p}M_\mathfrak{p}$$

Now we can state that

$$M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_A M$$

$$\frac{m}{s} = \mathfrak{p}M_\mathfrak{p} \mapsto \left(\frac{1}{s} + \mathfrak{p}A_\mathfrak{p} \right) \otimes m$$

What is the inverse? The general element of $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_A M$

$$\left(\frac{a}{s} + \mathfrak{p}A_\mathfrak{p} \right) \otimes m = a \left(\frac{1}{s} + \mathfrak{p}A_\mathfrak{p} \right) \otimes m = \left(\frac{1}{s} + \mathfrak{p}A_\mathfrak{p} \right) \otimes am$$

is the image of

$$\frac{am}{s} + \mathfrak{p}M_\mathfrak{p}$$

The corresponding element in $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ is its preimage. The inverse map on elements is

$$\left(\frac{a}{s} + \mathfrak{p}A_\mathfrak{p} \right) \otimes m \mapsto \frac{am}{s} + \mathfrak{p}M_\mathfrak{p}$$