

# Facts about Rings of Fractions

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## 1 Introduction

**Fact 1.1.** *If  $0 \in S$ , then  $S^{-1}A$  is a trivial ring.*

*Proof.* Any  $(a, s), (a', s')$  are related because  $(as' - a's) \cdot 0 = 0$  with  $0 \in S$ . □

**Fact 1.2.** *A a PID, the equivalence relation in  $A \times S$  is:  $(a, s) \equiv (a', s')$  iff  $as' = a's$ .* □

**Fact 1.3.** *For  $A$  a field, and  $S = \{-1, 1\}$ ,  $S^{-1}A \cong A$ .*

*Proof.* It is easily verified that the standard isomorphism from  $A$  to  $S^{-1}A$  is 1-1 and onto. □

**Fact 1.4.** *For  $A$  a field, and  $S$  a multiplicatively closed subset of  $A$  not containing zero,  $S^{-1}A \cong A$ .*

*Proof.* The standard homomorphism  $f : a \mapsto a/1$  of  $A$  into  $S^{-1}A$  is injective: if  $a/1 = a'/1$  then  $a \cdot 1 = a' \cdot 1$ , then  $a = a'$ . It is surjective:  $f(as^{-1}) = f(a)f(s^{-1}) = (a/1)(s^{-1}/1) = \dots$ , but  $s^{-1}/1 = 1/s$  as  $s^{-1}s = 1 \cdot 1$ ; continuing,  $\dots = (a/1)(1/s) = a/s$ . □

**Fact 1.5.** *For  $A$  a field, the ring of fractions and the field of fractions are isomorphic.*

*Proof.* For isomorphism of  $A$  with its field of fractions, see Math Exchange 79188. About the isomorphism with its ring of fractions, is the fact above. □

**Example 1.6.** *Some example.*

**Fact 1.7.** *The quotient ring  $A/I$  can be viewed as an  $A$ -module, and then the ring of fractions  $T^{-1}(A/I)$ , where  $T$  is the image of  $S$  in  $A/I$ , equals the module of fractions  $S^{-1}(A/I)$ .*

*Proof.* On the left, the relation is in  $(A/I) \times T$ :  $([a], [s]) \equiv ([a'], [s'])$  iff (ring notation)  $([a][s'] - [a'][s])[s''] = [0]$  iff  $[as's'' - a'ss''] = [0]$ . On the right, the relation works in  $(A/I) \times S$ :  $([a], s) \equiv ([a'], s')$  iff (module notation)  $s''(s'[a] - s[a']) = [0]$  iff  $[as's'' - a'ss''] = [0]$ . The conditions are identical so the classes must be in bijective correspondence. However, they are not identical as sets, so saying *equals* is too much. □

**Fact 1.8.** *What are  $\mathfrak{p}A_{\mathfrak{p}}$  and  $\mathfrak{a}A_{\mathfrak{p}}$ ?*

In the Solutions by Y. P. Gaillard, the residue field is  $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ , so the single maximal ideal of  $A_{\mathfrak{p}}$  from the Example 1, p. 38 of ItCA must be just  $\mathfrak{p}A_{\mathfrak{p}}$

$$\mathfrak{p}A_{\mathfrak{p}} = \{a/s : a \in \mathfrak{p}, s \notin \mathfrak{p}\}$$

Then  $\mathfrak{a}A_{\mathfrak{p}}$  must be the generalization

$$\mathfrak{a}A_{\mathfrak{p}} = \{a/s : a \in \mathfrak{a}, s \notin \mathfrak{p}\}$$

□

**Fact 1.9.** *What is  $S^{-1}\mathfrak{a}$ ?*

It can be either an  $S^{-1}A$ -module  $S^{-1}\mathfrak{a}$ , because  $\mathfrak{a}$  is an  $A$ -module, or the extension  $S^{-1}\mathfrak{a} = \mathfrak{a}S^{-1}A$  in  $S^{-1}A$  of the ideal  $\mathfrak{a}$  in  $A$ . In both cases elements of  $S^{-1}\mathfrak{a}$  are written as  $a/s$  with  $a \in \mathfrak{a}$ ,  $s \in S$ , but they come from different sets. In the first, module case,  $a/s$  is in the quotient of  $\mathfrak{a} \times S$ , in the second, extension ideal case,  $a/s$  is in the quotient of  $A \times S$ . We are talking of  $S^{-1}A$ -modules, not rings, so there can only be  $A$ -module and  $S^{-1}A$ -module isomorphism:

$$\mathfrak{a} \times S / \sim_{\mathfrak{a}} \ni a/s \mapsto a/s \in A \times S / \sim_A$$

□

**Fact 1.10.** *What is  $\mathfrak{a}_{\mathfrak{p}}$ ?*

It is the  $A$ -module  $\mathfrak{a}$  localized at  $\mathfrak{p}$ . It is an  $A_{\mathfrak{p}}$ -module. We also use this notation for the ideal  $S^{-1}\mathfrak{a}$  of  $S^{-1}A$ , where  $S = A \setminus \mathfrak{p}$ . How are they isomorphic?  $a/s \mapsto a/s$  with  $a \in \mathfrak{a}$ ,  $s \notin \mathfrak{p}$ . Of what it is an isomorphism? Of  $A$ -modules, of  $A_{\mathfrak{p}}$ -modules. They are not rings.

**Fact 1.11.** *What is  $S^{-1}\mathfrak{a}$  in case  $S = A \setminus \mathfrak{p}$ ?*

It can be either an  $A_{\mathfrak{p}}$ -module  $\mathfrak{a}_{\mathfrak{p}}$ , because  $\mathfrak{a}$  is an  $A$ -module, or the extension  $S^{-1}\mathfrak{a} = \mathfrak{a}A_{\mathfrak{p}}$  in  $A_{\mathfrak{p}}$  of the ideal  $\mathfrak{a}$  in  $A$ . In both cases elements of  $S^{-1}\mathfrak{a}$  are written as  $a/s$  with  $a \in \mathfrak{a}$ ,  $s \notin \mathfrak{p}$ , but they come from different sets. In the first, module case,  $a/s$  is in the quotient of  $\mathfrak{a} \times (A \setminus \mathfrak{p})$ , in the second, extension ideal case,  $a/s$  is in the quotient of  $A \times (A \setminus \mathfrak{p})$ . We are talking of  $A_{\mathfrak{p}}$ -modules, not rings, so there can only be an  $A$ -module and  $A_{\mathfrak{p}}$ -module isomorphism:

$$\mathfrak{a} \times (A \setminus \mathfrak{p}) / \sim_{\mathfrak{a}} \ni a/s \mapsto a/s \in A \times (A \setminus \mathfrak{p}) / \sim_A$$

□

**Fact 1.12.** *What is  $\mathfrak{p}_{\mathfrak{p}}$ ?*

It is the  $A$ -module  $\mathfrak{p}$  localized at  $\mathfrak{p}$ . We also use this notation for the ideal  $S^{-1}\mathfrak{p}$  of  $S^{-1}A$ , where  $S = A \setminus \mathfrak{p}$ .

**Fact 1.13.** *Case  $\mathfrak{a} = \mathfrak{p}$ , a prime ideal. What is  $S^{-1}\mathfrak{p}$ ?*

It can be either the  $A_{\mathfrak{p}}$ -module  $\mathfrak{p}_{\mathfrak{p}}$ , because  $\mathfrak{p}$  is an  $A$ -module, or the extension  $\mathfrak{p}A_{\mathfrak{p}}$  in  $A_{\mathfrak{p}}$  of the ideal  $\mathfrak{p}$  in  $A$ , via the canonical  $A \rightarrow A_{\mathfrak{p}} : a \mapsto a/s$ . Looks like we don't have the  $\cdot \mathfrak{p}$ -instead-of- $S^{-1}$  notation in the ideal extension case, but then, the quotient notation  $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$  is used, which makes sense only if  $\mathfrak{p}_{\mathfrak{p}}$  is an ideal in  $A_{\mathfrak{p}}$

$$\mathfrak{p}_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$$

□

**Fact 1.14.** *How is  $B_{\mathfrak{q}}$  an  $A_{\mathfrak{p}}$ -module?*

Let  $g = \psi \circ f$  be the composition  $A \rightarrow B \rightarrow T^{-1}B : a \rightarrow f(a) \rightarrow f(a)/1$ . This composition sends  $s \in S$  to a unit in  $T^{-1}B$ , as  $(f(s)/1)(1/f(s)) = 1/1$ , where  $f(s) \in f(S) = f(A \setminus \mathfrak{p}) \subseteq B \setminus \mathfrak{q} = T$ . By the universal property of the ring of fractions,  $g$  factorizes

$$\begin{array}{ccc} A & \xrightarrow{\phi} & S^{-1}A \\ & \searrow \eta & \downarrow h \\ B & \xrightarrow{\psi} & T^{-1}B \end{array}$$

where the recipe for  $h$  is given in **Proposition 3.1** of [ItCA] as  $a/s \mapsto g(a)g(s)^{-1} = (f(a)/1)(1/f(s)) = f(a)/f(s)$ . □

**Fact 1.15.** *How is  $B_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}}$  an  $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ -module?*

The kernel of the composition  $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}} \rightarrow B_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}} : a/s \mapsto f(a)/f(s) + \mathfrak{q}_{\mathfrak{q}}$  contains  $\mathfrak{p}A_{\mathfrak{p}}$  (because  $\mathfrak{p} = f^{-1}(\mathfrak{q})$ ) so the composition factors through  $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}}$ :  $a/s + \mathfrak{p}_{\mathfrak{p}} \mapsto f(a)/f(s) + \mathfrak{q}_{\mathfrak{q}}$ . This is a ring homomorphism that makes  $B_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}}$  an  $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ -module. □

**Fact 1.16.** *What is  $\mathfrak{p}M_{\mathfrak{p}}$ ?*

When  $M_{\mathfrak{p}}$  is seen as an  $A$ -module,  $\mathfrak{p}M_{\mathfrak{p}} = \{am/s : a \in \mathfrak{p}, m \in M, s \notin \mathfrak{p}\}$ . When  $M_{\mathfrak{p}}$  is seen as an  $A_{\mathfrak{p}}$ -module,  $\mathfrak{p}$  is not even an ideal in  $A_{\mathfrak{p}}$ , but its extension,  $\mathfrak{p}A_{\mathfrak{p}}$  is, and  $(\mathfrak{p}A_{\mathfrak{p}})M_{\mathfrak{p}} = \{(a/s')(m/s) : a \in \mathfrak{p}, m \in M, s, s' \notin \mathfrak{p}\} = \{am/s : a \in \mathfrak{p}, m \in M, s \notin \mathfrak{p}\}$ , the same set, which we write  $\mathfrak{p}M_{\mathfrak{p}}$  for:

$$\mathfrak{p}M_{\mathfrak{p}} = (\mathfrak{p}A_{\mathfrak{p}})M_{\mathfrak{p}}$$

□

**Fact 1.17.** *How*

$$\frac{(B \otimes_A M)_{\mathfrak{q}}}{\mathfrak{q}(B \otimes_A M)_{\mathfrak{q}}} \cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_B B \otimes_A M$$

?

Proposition 3.5 states, in the language of subscript- $\mathfrak{p}$ , that  $M_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A M$  over  $A_{\mathfrak{p}}$ . Here  $(B \otimes_A M)_{\mathfrak{q}} \cong B_{\mathfrak{q}} \otimes_B (B \otimes_A M)$ . Then

$$\begin{aligned} \frac{B_{\mathfrak{q}} \otimes_B (B \otimes_A M)}{(\mathfrak{q}B_{\mathfrak{q}})(B_{\mathfrak{q}} \otimes_B (B \otimes_A M))} &\cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}B_{\mathfrak{q}}} \otimes_{B_{\mathfrak{q}}} (B_{\mathfrak{q}} \otimes_B (B \otimes_A M)) \\ &\cong \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_B B \otimes_A M \end{aligned}$$

□

In P. Y. Gaillard solution to ItCA Exercise 3.19 (viii).

**Fact 1.18.** *The diagram*

$$\begin{array}{ccc} A_{\mathfrak{p}} & \xrightarrow{\phi} & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \\ \downarrow f & \searrow \eta & \downarrow h \\ B_{\mathfrak{q}} & \xrightarrow{\psi} & B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \end{array}$$
  

$$\begin{array}{ccc} a/s & \xrightarrow{\phi} & a/s + \mathfrak{p}A_{\mathfrak{p}} \\ \downarrow f & \searrow \eta & \downarrow h \\ f(a)/f(s) & \xrightarrow{\psi} & f(a)/f(s) + \mathfrak{q}B_{\mathfrak{q}} \end{array}$$

is commutative.

All calculated on the diagram. □

Now  $\kappa_{\mathfrak{q}} = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$  is an  $A_{\mathfrak{p}}$ -module by  $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$  (with the formula as on the bottom diagram) and we may tensor over  $A_{\mathfrak{p}}$ .

If a field  $K$  is an  $A$ -module for some ring  $A$ , can it be a zero  $A$ -module?

$$1_A 1_K = 1_K \neq 0_K$$

It cannot.

Now that  $\kappa_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = 0$ , both tensorands finitely generated, and  $\kappa_{\mathfrak{q}} \neq 0$ , it must be  $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = 0$  by ItCA Exercise 2.3.

In solution of ItCA 3.19 (viii) by J. D. Taylor.

**Fact 1.19.** *What is  $\mathfrak{p}B$ ?*

For  $f : A \rightarrow B$ , we can think in two ways. As we identify  $ab = f(a)b$ ,  $\mathfrak{p}B = \{ab = f(a)b : a \in \mathfrak{p}, b \in B\}$  is the extension  $f(\mathfrak{p})B$  of the ideal  $\mathfrak{p}$ . The second way is that  $B$  is an  $A$ -module, and  $\mathfrak{p}$  a prime ideal in  $A$ , so we can form  $\mathfrak{p}B = \{\sum a_i b_i = \sum f(a_i) b_i\}$  with  $a_i \in \mathfrak{p}$ ,  $b_i \in B$ , getting the same set. □

**Fact 1.20.** *What is  $B_{\mathfrak{p}}$ ?*

Since  $B$  is an  $A$ -module,  $B_{\mathfrak{p}}$  consists of all elements  $b/s$  where  $b \in B$ ,  $s \in A \setminus \mathfrak{p}$ . This is the standard construction of  $S^{-1}A$ -module  $S^{-1}M$  in the text. It is:

- An  $A$ -module.
- An  $A_{\mathfrak{p}}$ -module: the standard construction.
- A  $B$ -module.
- A ring.

□

**Fact 1.21.** *What is  $\mathfrak{p}B_{\mathfrak{p}}$ ?*

$B_{\mathfrak{p}}$  is an  $A$ -module,  $\mathfrak{p}$  is a prime ideal of  $A$ , so  $\mathfrak{p}B_{\mathfrak{p}}$  makes sense and consists of finite sums  $\sum a_i (b_i/s_i)$  where  $a_i \in \mathfrak{p}$ ,  $b_i \in B$ , and  $s_i \in A \setminus \mathfrak{p}$ . After bringing to common denominator, the sum becomes  $ab/s$  where  $a \in \mathfrak{p}$ ,  $b \in B$  and  $s_i \in A \setminus \mathfrak{p}$  that is,  $b/s$  where  $b \in \mathfrak{p}B$  and  $s_i \in A \setminus \mathfrak{p}$ . □

**Fact 1.22.**  $\mathfrak{p}B_{\mathfrak{p}}$  is an ideal in  $B_{\mathfrak{p}}$ .

As a module, it is an abelian group, then the multiplication property is easily verified.  $\square$

The ideal  $\mathfrak{p}A_{\mathfrak{p}}$  was the single maximal ideal in  $A_{\mathfrak{p}}$ . We do not know this for  $\mathfrak{p}B_{\mathfrak{p}}$  in  $B_{\mathfrak{p}}$ .

**Fact 1.23.** How is  $A_{\mathfrak{p}}$  an  $A$ -module ?

The canonical map  $\phi : A \rightarrow A_{\mathfrak{p}} : a \mapsto \frac{a}{1}$  gives the multiplication by scalars from  $A$

$$a' \frac{a}{s} = \phi(a') \frac{a}{s} = \frac{a'}{1} \frac{a}{s} = \frac{a'a}{s}$$

$\square$

**Fact 1.24.** What is  $\mathfrak{p}A_{\mathfrak{p}}$  ?

As  $A_{\mathfrak{p}}$  is an  $A$ -module, we can multiply it by a prime ideal in  $A$  in a standard way

$$\sum a'_i \frac{a_i}{s_i} = \sum \frac{a'_i a_i}{s_i}$$

After bringing to a common denominator, this is

$$\frac{a}{s}$$

with  $a \in \mathfrak{p}$ , so  $\mathfrak{p}A_{\mathfrak{p}}$  is the single maximal ideal of the local ring  $A_{\mathfrak{p}}$ .  $\square$

**Fact 1.25.** How is  $B_{\mathfrak{p}}$  an  $A_{\mathfrak{p}}$ -module ?

Definition of the  $S^{-1}M$  as  $S^{-1}A$ -module in the text. The multiplication by a scalar is

$$\frac{a}{s'} \frac{b}{s} = \frac{ab}{s's} = \frac{f(a)b}{s's}$$

$\square$

**Fact 1.26.** What is  $(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}}$  ?

As  $B_{\mathfrak{p}}$  is an  $A_{\mathfrak{p}}$ -module, and  $\mathfrak{p}A_{\mathfrak{p}}$  is the single maximal ideal of the local ring  $A_{\mathfrak{p}}$ , Any element is, from the definition of the ideal-by-module and from the general element of  $\mathfrak{p}A_{\mathfrak{p}}$  ( $a \in \mathfrak{p}$ )

$$\sum_i \frac{a_i}{s'_i} \frac{b_i}{s_i} = \sum \frac{ab}{s's}$$

After bringing to a common denominator, this becomes

$$ab/s = f(a)b/s$$

where  $a \in \mathfrak{p}$ . Notice we got the general element of  $\mathfrak{p}B_{\mathfrak{p}}$ , so

$$(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$$

$\square$

**Fact 1.27.** How  $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}$  ?

Apply Exercise 2.2

$$A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$$

to  $M := B_{\mathfrak{p}}$ ,  $A := A_{\mathfrak{p}}$ ,  $\mathfrak{a} := \mathfrak{p}A_{\mathfrak{p}}$

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} = B_{\mathfrak{p}}/(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}}$$

now apply  $(\mathfrak{p}A_{\mathfrak{p}})B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$ .  $\square$

**Fact 1.28.** How  $A_{\mathfrak{p}} \otimes_A B \cong B_{\mathfrak{p}}$  ?

Apply Proposition 3.5:  $S^{-1}A \otimes_A M \cong S^{-1}M$ .

□

We now understand the isomorphisms in the solution of ItCA's 3.21(iv) by J D. Taylor.

$$\begin{aligned} B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} &= A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} \\ &= K_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}} \otimes_A B \\ &= K_{\mathfrak{p}} \otimes_A B \end{aligned}$$

## 2 Saturated

**Fact 2.1.** *For saturated  $S$ , if  $f(a)$  is a unit in  $S^{-1}A$ , then  $a \in S$ .*

*Proof.*

$$\frac{a}{1} \cdot \frac{b}{t} = \frac{1}{1}$$

$$\frac{ab}{t} = \frac{1}{1}$$

$$(ab, t) \equiv (1, 1)$$

$$(ab - t)u = 0$$

$$abu = tu$$

$$abu \in S$$

As  $S$  is saturated,  $a \in S$ .

□