0.0.1 Related result [Section 4.1.12]

Here is a related result:

$$\mathfrak{p} \text{ is the contraction of a prime ideal if and only if } A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \otimes_A B \neq 0. \tag{1}$$

This will follow from Claim 1 and Claim 2 below.

Claim 1. Let $C_1, \dots, C_5$ be the five B-algebras

$$A_{\mathfrak{p}} \otimes_A B/\mathfrak{p}B, \quad (B/\mathfrak{p}B)_{\mathfrak{p}}, \quad B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}, \quad \frac{A_{\mathfrak{p}}}{\mathfrak{p}_{\mathfrak{p}}} \otimes_A B, \quad (A/\mathfrak{p})_{\mathfrak{p}} \otimes_A B.$$

Then for any $1 \le i, j \le 5$ there is a unique B-algebra morphism $C_i \to C_j$, and this morphism is bijective. ???

Proof of Claim 1. Note that C_i has a unique $B_{\mathfrak{p}}$-algebra structure which extends its natural B-algebra structure. In other words, the image in C_i of any $s \in A \setminus \mathfrak{p}$ is a unit. Moreover, any element of C_i is equal to $\frac{b}{s} \cdot 1$ for some b in B and s in $A \setminus \mathfrak{p}$. This implies that there is at most one morphism of B-algebras $\phi_{ji} : C_i \to C_j$. To prove that such morphisms exist, we proceed as follows. If U is an additive group and V a subgroup, we denote the class in U/V of $u \in U$ by $\overline{u}$ when V is clear from the context. For $i = 1, \dots, 5$ define the set theoretical map $f_i : A \times (A \setminus \mathfrak{p}) \times B \to C_i$ by

$$\big(f_1(a,s,b), \dots, f_5(a,s,b)\big) := \left(\frac{a}{s} \otimes \overline{b}\,,\ \frac{\overline{a \cdot b}}{s}\ ,\left(\frac{a \cdot b}{s}\right)^{-}, \left(\frac{a}{s}\right)^{-} \otimes b\ ,\ \frac{\overline{a}}{s} \otimes b\right).$$

We leave it to the reader to check that, for $i, j = 1, \dots, 5$, there is a B-algebra morphism $\phi_{ji} : C_i \to C_j$ such that $\phi_{ji}(f_i(a,s,b)) = f_j(a,s,b)$ for all a, s, b.

Note that C_i is an initial object in the category of those B-algebras C such that the image of $a \in A$ in C is zero if $a \in \mathfrak{p}$ and is a unit if $a \notin \mathfrak{p}$.

Claim 2. We have $\big(B/\mathfrak{p}^{\mathrm{e}}\big)_{\mathfrak{p}} = 0 \iff \mathfrak{p}^{\mathrm{ec}} \neq \mathfrak{p}$.

Proof of Claim 2.

$$\big(B/\mathfrak{p}^{\mathrm{e}}\big)_{\mathfrak{p}} = 0 \iff \frac{1}{1} = \frac{0}{1} \text{ in } \big(B/\mathfrak{p}^{\mathrm{e}}\big)_{\mathfrak{p}} \iff \exists\, s \in A \setminus \mathfrak{p} \mid f(s) \in \mathfrak{p}^{\mathrm{e}} \iff \exists\, s \in \mathfrak{p}^{\mathrm{ec}} \setminus \mathfrak{p} \iff \mathfrak{p}^{\mathrm{ec}} \neq \mathfrak{p}.$$

Statement (1) follows also from Exercise 3.21iv p. 47 of the book [see Section 0.0.2 p. 2 below].

[Reminder.]

Proposition 1 (Proposition 3.11 p. 41 of the book)**.** (i) *Every ideal in $S^{-1}A$ is an extended ideal.*

(ii) *If $\mathfrak{a}$ is an ideal in A, then $\mathfrak{a}^{\mathrm{ec}} = \bigcup_{s \in S} (\mathfrak{a} : s)$. Hence $\mathfrak{a}^{\mathrm{e}} = (1)$ if and only if $\mathfrak{a}$ meets S.*

(iii) $\mathfrak{a} \in C \iff$ *no element of S is a zero-divisor in $A/\mathfrak{a}$.*

(iv) *The prime ideals of $S^{-1}A$ are in one-to-one correspondence ($\mathfrak{p} \leftrightarrow S^{-1}\mathfrak{p}$) with the prime ideals of A which don't meet S.*

(v) *The operation S^{-1} commutes with formation of finite sums, products, intersections and radicals.*

0.0.2 Page 46, Exercise 3.21 [Section 4.2.21]

Statement. (i) Let A be a ring, S a multiplicatively closed subset of A, and $\phi : A \to S^{-1}A$ the canonical homomorphism. Show that $\phi^* : \operatorname{Spec}(S^{-1}A) \to \operatorname{Spec}(A)$ is a homeomorphism of $\operatorname{Spec}(S^{-1}A)$ onto its image in $X = \operatorname{Spec}(A)$. Let this image be denoted by $S^{-1}X$.

In particular, if $f \in A$, the image of $\operatorname{Spec}(A_f)$ in X is the basic open set X_f (Chapter 1, Exercise 17).

(ii) Let $f : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$, and let $f^* : Y \to X$ be the mapping associated with f. Identifying $\operatorname{Spec}(S^{-1}A)$ with its canonical image $S^{-1}X$ in X, and $\operatorname{Spec}(S^{-1}B)(= \operatorname{Spec}(f(S)^{-1}B))$ with its canonical image $S^{-1}Y$ in Y, show that $S^{-1}f^* : \operatorname{Spec}(S^{-1}B) \to \operatorname{Spec}(S^{-1}A)$ is the restriction of f^* to $S^{-1}Y$, and that $S^{-1}Y = f^{*-1}(S^{-1}X)$.

(iii) Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^{\mathrm{e}}$ be its extension in B. Let $\overline{f} : A/\mathfrak{a} \to B/\mathfrak{b}$ be the homomorphism induced by f. If $\operatorname{Spec}(A/\mathfrak{a})$ is identified with its canonical image $V(\mathfrak{a})$ in X, and $\operatorname{Spec}(B/\mathfrak{b})$ with its image $V(\mathfrak{b})$ in Y, show that $\overline{f}^*$ is the restriction of f^* to $V(\mathfrak{b})$.

(iv) Let $\mathfrak{p}$ be a prime ideal of A. Take $S = A \setminus \mathfrak{p}$ in (ii) and then reduce mod $S^{-1}\mathfrak{p}$ as in (iii). Deduce that the subspace $f^{*-1}(\mathfrak{p})$ of Y is naturally homeomorphic to $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_\mathfrak{p}$.

$\operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$ is called the fiber of f^* over $\mathfrak{p}$.

Solution. (i) Set $S^{-1}X := \{\mathfrak{p} \in X \mid \mathfrak{p} \cap S = \varnothing\}$. By Proposition 3.11iv p. 41 of the book [Proposition 1 p. 1] the maps

$$\operatorname{Spec}(S^{-1}A) \underset{S^{-1}}{\overset{\phi^*}{\rightleftarrows}} S^{-1}X$$

are inverse bijections. Let us equip $S^{-1}X \subset X$ with the induced topology. Then the closed subsets of $S^{-1}X$ are precisely the subsets of the form $V(\mathfrak{a}) \cap S^{-1}X$ where $\mathfrak{a}$ is an ideal of A. By Proposition 3.11i p. 41 of the book [Proposition 1 p. 1], the closed subsets of $S^{-1}A$ are precisely the subsets of the form $V(S^{-1}\mathfrak{a})$ where $\mathfrak{a}$ is an ideal of A. Thus it suffices to show that, given an ideal $\mathfrak{a}$ of A and a prime ideal $\mathfrak{p}$ of A, we have $S^{-1}\mathfrak{a} \subset S^{-1}\mathfrak{p} \iff \mathfrak{a} \subset \mathfrak{p}$. Implication $\Leftarrow$ is clear. Conversely $S^{-1}\mathfrak{a} \subset S^{-1}\mathfrak{p}$ implies

$$\mathfrak{a} \subset \phi^* S^{-1}\mathfrak{a} \subset \phi^* S^{-1}\mathfrak{p} = \mathfrak{p}.$$

(ii) Since the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ S^{-1}A \arrow[r, "S^{-1}f"'] & S^{-1}B \end{tikzcd}$$

commutes, so does

$$\begin{tikzcd} X & Y \arrow[l, "f^*"'] \\ S^{-1}X \arrow[u, hook] & S^{-1}Y. \arrow[l, "(S^{-1}f)^*"] \arrow[u, hook] \end{tikzcd}$$

This proves the first claim. To show $S^{-1}Y = f^{*-1}(S^{-1}X)$, note that, for $\mathfrak{q} \in Y$, we have

$$\mathfrak{q} \in S^{-1}Y \iff f(S) \cap \mathfrak{q} = \varnothing \iff S \cap f^*(\mathfrak{q}) = \varnothing \iff f^*(\mathfrak{q}) \in S^{-1}X \iff \mathfrak{q} \in f^{*-1}(S^{-1}X).$$

(iii) Since the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A/\mathfrak{a} \arrow[r, "\overline{f}"'] & B/\mathfrak{b} \end{tikzcd}$$

commutes, so does

$$\begin{tikzcd} X & Y \arrow[l, "f^*"'] \\ V(\mathfrak{a}) \arrow[u, hook] & V(\mathfrak{b}). \arrow[l, "\overline{f}^*"] \arrow[u, hook] \end{tikzcd}$$

(iv) We have the commuting diagrams

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d] & B \arrow[d] & \\ & A/\mathfrak{p} \arrow[r] \arrow[d] & B/\mathfrak{p}B \arrow[d] & \\ k(\mathfrak{p}) \arrow[r, equal] & (A/\mathfrak{p})_\mathfrak{p} \arrow[r] & (B/\mathfrak{p}B)_\mathfrak{p} \arrow[r, equal] & B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \end{tikzcd}$$

and

$$\begin{tikzcd} X & Y \arrow[l, "f^*"'] \\ V(\mathfrak{p}) \arrow[u, hook] & V(\mathfrak{p}B) \arrow[l, "\overline{f}^*"'] \arrow[u, hook] \\ \{\mathfrak{p}\} \arrow[u, hook] & Z \arrow[l, "\overline{f}_\mathfrak{p}^*"] \arrow[u, hook] \end{tikzcd}$$

with $Z := \mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$. The second claim of (ii) implies $Z = \overline{f}^{*-1}(\mathfrak{p})$. Finally note that we have

$$\overline{f}^{*-1}(\mathfrak{p}) = \{\mathfrak{q} \in Y \mid \mathfrak{q} \supset f(\mathfrak{p}),\ f^*(\mathfrak{q}) = \mathfrak{p}\} = \{\mathfrak{q} \in Y \mid f^*(\mathfrak{q}) = \mathfrak{p}\} = f^{*-1}(\mathfrak{p}),$$

that is, $\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = f^{*-1}(\mathfrak{p})$, as required.