Exercise 1.06

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6. A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and the Jacobson radical of A are equal.

Assume that the ring has this property and has the Jacobson radical strictly larger than the nilradical. Now the J-radical must have nonzero idempotent: $e^2 = e \neq 0$. By Proposition 1.9 characterizing J-radicals, $1 - ey$ is a unit for any element y of the ring, especially $1 - e$ is a unit. Now we have $e - e^2 = 0$, then $e(1 - e) = 0$. Multiplying by $(1 - e)^{-1}$ whe get $e = 0$, which is a contradiction.