Exact Sequences and Fractions
Remark after Corollary 3.13

Rafal Michalski

An exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M' \longrightarrow 0$$

induces isomorphism $M/f(M') \cong M''$. It works like this:

$$m + f(M') \mapsto g(m)$$

The case of a quotient

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\pi} M/N \longrightarrow 0$$

$$m + N \mapsto m + N$$

as $\iota(N) = N$ here, then $M/\iota(N) = M/N$.

Apply the exact functor S^{-1} now:

$$0 \longrightarrow S^{-1}N \xrightarrow{S^{-1}\iota} S^{-1}M \xrightarrow{S^{-1}\pi} M/N \longrightarrow 0$$

where $S^{-1}\iota$ is

$$y/s \mapsto y/s$$

the left in $S^{-1}N$, the right in $S^{-1}M$, and $S^{-1}\pi$ is

$$x/s \mapsto \pi(x)/s$$

$$x/s \mapsto \frac{x+N}{s}$$

As sets, $N \times S \subseteq M \times S$. And $S^{-1}\iota$ is an injection. Is it an embedding? Is a class in $N \times S$ also a class in $M \times S$? Any class in $N \times S$ is contained in some class in $M \times S$: if $(y,s) \sim (y',s')$ in $N \times S$, then $u(s'y - sy')$ for some $u \in S$, which is also true in $M \times S$ so the pairs are also related in this larger product. The map $S^{-1}\iota : y/s \mapsto y/s$ assigns to a class in the smaller the containing class in the larger. Can the larger class contain more smaller classes? This would contradict the injectivity of the map, so it cannot. Can the larger class be a strictly larger set? Yes, two cases may happen. First, $(y,s) \sim (x,s')$ for some $x \notin N$, second, $(x,s') \sim (x',s'')$ for some another $x' \notin N$. The map $S^{-1}\iota$ is an injection, but not an embedding.

The co-kernel isomorphism $S^{-1}M/S^{-1}\iota(S^{-1}N) \to S^{-1}(M/N)$ is

$$\frac{S^{-1}M}{S^{-1}\iota(S^{-1}N)} \cong S^{-1}(M/N)$$

$$\frac{x}{s} + S^{-1}\iota(S^{-1}N) \mapsto S^{-1}\pi(\frac{x}{s})$$

$$\frac{x}{s} + S^{-1}\iota(S^{-1}N) \mapsto \frac{x+N}{s}$$

If we decide to identify $S^{-1}\iota(S^{-1}N)$ with $S^{-1}N$, since both are $\{y/s : y \in N\}$, although in the image the fraction is in $S^{-1}M$ not $S^{-1}N$, we can write the cokernel isomorphism as

$$\frac{x}{s} + S^{-1}N \mapsto \frac{x+N}{s}$$

Way 1: localizing at $\mathfrak{p}$ then taking the quotient mod $\mathfrak{q}$. But $\mathfrak{q}$ is not an ideal of $A_\mathfrak{p}$, only an ideal of A. By Corollary 3.13, ideals of $A_\mathfrak{p}$ are in 1-1 correspondence with the prime ideals of A contained in $\mathfrak{p}$

$$\mathfrak{q} \mapsto S^{-1}\mathfrak{q} = \{a/s : a \in \mathfrak{q}, s \notin \mathfrak{p}\}$$

As $S^{-1}\mathfrak{q}$ is an ideal in $A_\mathfrak{p}$, we can take the quotient $A_\mathfrak{p}/S^{-1}\mathfrak{q}$. Its elements look like

$$\frac{a}{s} + S^{-1}\mathfrak{q}, a \in \mathfrak{q}, s \notin \mathfrak{p}$$

Way 2: taking the quotient by $\mathfrak{q}$ then localizing in the image of $\mathfrak{p}$ which is $\{a + \mathfrak{q} : a \in \mathfrak{p}\}$ that we can write ambiguously as $\mathfrak{p} + \mathfrak{q}$. The localization now has symbol $(A/\mathfrak{q})_{\mathfrak{p}+\mathfrak{q}}$. The general element of the localization is
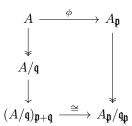
$$\frac{a + \mathfrak{q}}{s + \mathfrak{q}}, \ s + \mathfrak{q} \notin \mathfrak{p} + \mathfrak{q}$$

Fact 1. $s + \mathfrak{q} \notin \mathfrak{p} + \mathfrak{q} \iff s \notin \mathfrak{p}$.

Proof. The $\implies$ direction: Let $s + \mathfrak{q} \in \mathfrak{p} + \mathfrak{q}$; $s + \mathfrak{q} = a + \mathfrak{q}$ for some $a \in \mathfrak{p}$; $s - a \in \mathfrak{q}$, but since $\mathfrak{q} \subseteq \mathfrak{p}$, $s - a \in \mathfrak{p}$; now $s \in \mathfrak{p}$. The $\impliedby$ direction is obvious.

The general element of the localization $(A/\mathfrak{q})_{\mathfrak{p}+\mathfrak{q}}$ is now

$$\frac{a + \mathfrak{q}}{s + \mathfrak{q}}, \ s + \mathfrak{q} \notin \mathfrak{p} + \mathfrak{q}$$

Where $\mathfrak{q}_\mathfrak{p} = S^{-1}\mathfrak{q}$ with $S = A - \mathfrak{p}$. Is this notation used anywhere?

By (3.4.iii), the bottom arrow should be an isomorphism.

1. Map between sets: $\frac{a+\mathfrak{q}}{s} \mapsto \frac{a+\mathfrak{q}}{s+\mathfrak{q}}$.

2. Show this is an A-module isomorphism.

3. Show that $(A/\mathfrak{q})_{\mathfrak{p}+\mathfrak{q}}$ is an $A_\mathfrak{p}$-module, as $A_\mathfrak{p}/\mathfrak{q}_\mathfrak{p}$ already is. This step is probably optional as we target a ring isomorphism.

4. Prove our map being a ring isomorphism. So why did we need $A_\mathfrak{p}$-modules? Why even A-modules? For commutative group maybe.

The map

$$(A/\mathfrak{q})_{\mathfrak{p}+\mathfrak{q}} \ni \frac{a + \mathfrak{q}}{s} \mapsto \frac{a + \mathfrak{q}}{s + \mathfrak{q}} \in A_\mathfrak{p}/\mathfrak{q}_\mathfrak{p}$$

Well-definition. Let

$$\frac{a + \mathfrak{q}}{s} = \frac{a' + \mathfrak{q}}{s'}$$
$$u(s'(a + \mathfrak{q}) - s(a' + \mathfrak{q})) = 0$$
$$us'a - usa' \in \mathfrak{q}$$
$$u(s'a - sa') \in \mathfrak{q}$$

but $u \notin \mathfrak{p}$ then $u \notin \mathfrak{p}$ then $s'a - sa' \in \mathfrak{q}$. Now when

$$\frac{a + \mathfrak{q}}{s + \mathfrak{q}} = \frac{a' + \mathfrak{q}}{s' + \mathfrak{q}}$$

in the ring $(A/\mathfrak{q})_{\mathfrak{p}+\mathfrak{q}}$? The multipliset is $A/\mathfrak{q} - (\mathfrak{p} + \mathfrak{q})$.

$$(u + \mathfrak{q}((s' + \mathfrak{q})(a + \mathfrak{q}) - (s + \mathfrak{q})(a' + \mathfrak{q})) = 0 + \mathfrak{q}$$

for some $u + \mathfrak{q} \notin \mathfrak{p} + \mathfrak{q} \iff u \notin \mathfrak{p}$.

$$u(s'a - sa') \in \mathfrak{q}$$

but $u \notin \mathfrak{q}$

$$s'a - sa' \in \mathfrak{q}$$

We have got the same condition as above, so our map is well defined.

Our map looks surjective. Is it injective? Let

$$\frac{a + \mathfrak{q}}{s + \mathfrak{q}} = \frac{a' + \mathfrak{q}}{s' + \mathfrak{q}}$$

We already know the condition for this:

$$s'a - sa' \in \mathfrak{q} \implies \frac{a + \mathfrak{q}}{s} = \frac{a' + \mathfrak{q}}{s}$$

The map is finally bijective.

It remains to verify the preservation of multiplication

$$\frac{a + \mathfrak{q}}{s} \cdot \frac{a' + \mathfrak{q}}{s'} = \frac{aa' + \mathfrak{q}}{ss'} \mapsto \frac{aa' + \mathfrak{q}}{ss' + \mathfrak{q}} = \frac{a + \mathfrak{q}}{s + \mathfrak{q}} \cdot \frac{a' + \mathfrak{q}}{s' + \mathfrak{q}}$$

the image of multiplication is the multiplication of images, thus the multiplication is preserved.

Let's consider the localized ring $(A/\mathfrak{q})_{\mathfrak{p}+\mathfrak{q}}$ as an $A_\mathfrak{p}$-module. The scalar multiplication will be

$$\frac{a}{s} \cdot \frac{a' + \mathfrak{q}}{s' + \mathfrak{q}} = \frac{aa' + \mathfrak{q}}{ss' + \mathfrak{q}}$$

Well-definition? Let

$$\frac{a_1}{s_1} = \frac{a_2}{s_2} \quad \text{and} \quad \frac{a_1' + \mathfrak{q}}{s_1' + \mathfrak{q}} = \frac{a_2' + \mathfrak{q}}{s_2' + \mathfrak{q}}$$

$$u(a_1 s_2 - a_2 s_1) = 0, u \notin \mathfrak{p}; \quad s_2' a_1' - s_1' a_2' \in \mathfrak{q}$$

$$a_1 s_2 - a_2 s_1 = 0$$

Now the condition for

$$\frac{a_1 a_1' + \mathfrak{q}}{s_1 s_1' + \mathfrak{q}} = \frac{a_2 a_2' + \mathfrak{q}}{s_2 s_2' + \mathfrak{q}}$$

is

$$s_2 s_2' a_1 a_1' - s_1 s_1' a_2 a_2' \in \mathfrak{q}$$

Let

$$c = a_1 s_2 - a_2 s_1$$

$$c(s_2' a_1' - s_1' a_2') \in \mathfrak{q}$$

because the parentheses are in $\mathfrak{q}$. The scalar multiplication is well-defined, because this condition is satisfied.