Example 2

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M.F. Atiyah, I.G. MacDonald *Introduction to Commutative Algebra*
1 RINGS AND IDEALS

$A = k[x_1, \ldots, x_n]$, $\mathfrak{a} = (x_1, \ldots, x_n)$. How is $\mathfrak{a}^m$ the set of all polynomials with no terms of degree less than m?

In general, $\mathfrak{a}^m$ is the set of sums of products $a_1 \cdots a_n$ where $a_i \in \mathfrak{a}$. First, $\mathfrak{a} = (x_1, \ldots, x_m)$ is the set of finite sums

$$f_1(x_1, \ldots, x_n)x_1 + \cdots + f_n(x_1, \ldots, x_n)x_n$$

Then, $\mathfrak{a}^m$ is is the set of finite sums of products

$$\begin{gathered}(f_{11}(x_1, \ldots, x_n)x_1 + \cdots + f_{1n}(x_1, \ldots, x_n)x_n)\cdot \\ \cdot \cdots \cdot \\ \cdot(f_{m1}(x_1, \ldots, x_n)x_1 + \cdots + f_{mn}(x_1, \ldots, x_n)x_n)\end{gathered}$$

Each such product has degree not less than m.

Now consider any polynomial of degree m. Each term is $x_1^{i_1} \cdots x_n^{i_n}$ with $i_1 + \cdots + i_n \geq m$. Now set $f_{11} = \cdots = f_{i_1 1} = 1$ and $f_{1j} = \cdots = f_{i_1 j} = 0$ for $j \neq 1$, then ...