Proposition 1.10

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M.F. Atiyah, I.G. MacDonald *Introduction to Commutative Algebra*
1 RINGS AND IDEALS

Fact. *If* $z_1 \equiv 1 \pmod{\mathfrak{a}}$ *and* $z_2 \equiv 1 \pmod{\mathfrak{a}}$, *then* $z_1 z_2 \equiv 1 \pmod{\mathfrak{a}}$

$z_1 - 1 = a_1 \in \mathfrak{a}$, $z_1 - 1 = a_1 \in \mathfrak{a}$; $(z_1 - 1)(z_2 - 1) = a_1 a_3 \in \mathfrak{a}$; $z_1 z_2 - z_1 - z_2 + 1 \in \mathfrak{a}$; but $-z_2 + 1 \in \mathfrak{a}$; now $z_1 z_2 - z_1 \in \mathfrak{a}$; $z_1 z_2 \equiv z_1 \equiv 1 \pmod{\mathfrak{a}}$; and the relation is transitive...
This fact is also because $A/\mathfrak{a}$ is a ring: $z_1 z_2 + \mathfrak{a} = (z_1 + \mathfrak{a})(z_2 + \mathfrak{a}) = (1 + \mathfrak{a})(1 + \mathfrak{a}) = 1 + \mathfrak{a}$.

Back in the proof of Proposition 1.10, $x_i = 1 - y_i \equiv 1 \pmod{\mathfrak{a}_n}$; $\prod_{i=1}^{n-1} x_i \equiv 1 \pmod{\mathfrak{a}_n}$

Fact. *If for* $y \in \mathfrak{b}$, $y \equiv 1 \pmod{\mathfrak{a}}$, *then* $\mathfrak{a} + \mathfrak{b} = (1)$.

$y - 1 \in \mathfrak{a}$; $y - 1 = x\ in \mathfrak{a}$; $x + y = 1$. Now $\mathfrak{a}, \mathfrak{b}$ are coprime by the remark from the first paragraph on page 7.

(ii) $\Rightarrow$: $\phi(x) = (1, 0, \ldots, 0) = (1 + \mathfrak{a}_1, 0 + \mathfrak{a}_2, \ldots, 0)$; $x + \mathfrak{a}_1 = 1 + \mathfrak{a}_1$ and $x + \mathfrak{a}_1 = 0 + \mathfrak{a}_1$; $x \equiv 1 \pmod{\mathfrak{a}_1}$ and $x \equiv 0 \pmod{\mathfrak{a}_2}$.
(ii) $\Leftarrow$: Same technique as in (i) to show that $x = \prod(1 - u_i) \equiv 1 \pmod{\mathfrak{a}_1}$. As $x \in \mathfrak{a}_1$ for $i > 1$, $x \equiv 0 \pmod{\mathfrak{a}_1}$.