Exercise 3.21

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M.F. Atiyah, I.G. MacDonald *Introduction to Commutative Algebra*

Exercise 3.21.i. Show that $\phi^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is a homeomorphism of $\mathrm{Spec}(S^{-1}A)$ onto its image in $X = \mathrm{Spec}(A)$.

We want to prove that for D closed in $\mathrm{Spec}(S^{-1}A)$ there is C closed in $\mathrm{Spec}(A)$ and reverse, that the equation holds

$$C \cup \phi^*(\mathrm{Spec}(S^{-1}A)) = \phi^*(D)$$

We take

$$C = V(\mathfrak{a}), \quad D = V(S^{-1}\mathfrak{a})$$

That is, we are going to prove that

$$V(\mathfrak{a}) \cap \phi^*(\mathrm{Spec}(S^{-1}\mathfrak{a})) = \phi^*(V(S^{-1}\mathfrak{a}))$$

Recalling that $S^{-1}\mathfrak{p} \xmapsto{\phi^*} \mathfrak{p}$, the right are all prime ideals not meeting S and holding $S^{-1}\mathfrak{p} \supseteq S^{-1}\mathfrak{a}$. By the fact that this happens iff $\mathfrak{p} \supseteq \mathfrak{a}$, they are all prime ideals not meeting S and containing $\mathfrak{a}$. Which is precisely the left.
Now ϕ^* maps closed files to closed files in any direction. $\square$

Exercise 3.21.ii. Let $f : A \to B$ be a ring homomorphism...

$$\begin{tikzcd}
X \arrow[r, equal] & \mathrm{Spec}(A) & & \mathrm{Spec}(B) \arrow[ll, "f^*"'] \arrow[r, equal] & Y \\
S^{-1}X \arrow[r, equal] & \phi^*(\mathrm{Spec}(S^{-1}A)) \arrow[u, hook] & & \psi^*(\mathrm{Spec}(S^{-1}B)) \arrow[u, hook] \arrow[r, equal] & S^{-1}Y \\
X \arrow[r, equal] & \mathrm{Spec}(S^{-1}A) \arrow[u, leftrightarrow] & & \mathrm{Spec}(S^{-1}B) \arrow[ll, "S^{-1}f^*"'] \arrow[u, leftrightarrow] \arrow[r, equal] & Y
\end{tikzcd}$$