Przykladowyegzamin

placeholder

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Contents

1 [8/9] Algebra

1.1 DONE Zad 1

$$\begin{aligned}\Im\left(\frac{1+3i}{3-2i}+i^3+5\right) &= \Im\left(\frac{1+3i}{3-2i}+\frac{i^3(3-2i)}{3-2i}+\frac{5(3-2i)}{3-2i}\right)\\ &= \Im\left(\frac{1+3i+3i^3-2i^4+15-10i}{3-2i}\right)\\ &= \Im\left(\frac{16-7i+3i^3-2i^4}{3-2i}\right)\\ &= \Im\left(\frac{14-10i}{3-2i}\right)\\ &= \Im\left(\frac{14-10i}{3-2i}\cdot\frac{3+2i}{3+2i}\right)\\ &= \Im\left(\frac{42+28i-30i+20}{9+4}\right)\\ &= \Im\left(\frac{62-2i}{13}\right)\\ &= \frac{-2}{13}\end{aligned}$$

1.2 DONE Zad 2

$$\frac{(3-3i)^{14}}{(-1+i\sqrt{3})^{11}} = \frac{z^{14}}{w^{11}}$$

1.2.1 z

$$\sin(\varphi_z) = \frac{-3}{3\sqrt{2}} = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2} \to \varphi_z = \frac{7}{4}\pi$$

$$\begin{aligned}z^{14} &= (3-3i)^{14}\\ &= (3-3i)^{14}\\ &= (3\sqrt{2})^{14}(\cos 14\varphi + i\sin 14\varphi)\\ &= (3\sqrt{2})^{14}\left(\cos\left(14\cdot\frac{7}{4}\pi\right)+i\sin\left(14\cdot\frac{7}{4}\pi\right)\right)\\ &= (3\sqrt{2})^{14}\left(\cos\left(\frac{49}{2}\pi\right)+i\sin\left(\frac{49}{2}\pi\right)\right)\\ &= (3\sqrt{2})^{14}\left(\cos\left(\frac{1}{2}\pi\right)+i\sin\left(\frac{1}{2}\pi\right)\right)\\ &= (3\sqrt{2})^{14}(0+i1)\\ &= (3\sqrt{2})^{14}i\end{aligned}$$

1.2.2 w

$$\sin(\varphi_w) = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2} \to \varphi_w = \frac{2}{3}\pi$$

$$
\begin{aligned}
w^{11} &= 2^{11}\left(\cos\left(11\cdot\frac{2}{3}\pi\right) + i\sin\left(11\cdot\frac{2}{3}\pi\right)\right) \\
&= 2^{11}\left(-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) \\
&= 2^{11}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\
&= 2^{10}\left(-1 - i\sqrt{3}\right)
\end{aligned}
$$

1.2.3 Podstawiamy

$$
\begin{aligned}
\frac{(3-3i)^{14}}{(-1+i\sqrt{3})^{11}} &= \frac{z^{14}}{w^{11}} \\
&= \frac{(3\sqrt{2})^{14} i}{2^{10}(-1-i\sqrt{3})} \\
&= \frac{((3\sqrt{2})^{14} i)(-1+i\sqrt{3})}{2^{10}(-1-i\sqrt{3})(-1+i\sqrt{3})} \\
&= \frac{((3\sqrt{2})^{14} i)(-1+i\sqrt{3})}{2^{10}(-2)} \\
&= \frac{((3\sqrt{2})^{14} i)(-1+i\sqrt{3})}{-2^{11}}
\end{aligned}
$$

1.3 DONE Zad 3

Wyznacznik macierzy głownej = 20.

$A = \begin{bmatrix} 3 & -2 & 1 & 0 \\ 2 & -1 & 3 & 1 \\ 2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 3 \end{bmatrix}$

1.3.1 DONE A_4

$$
\begin{aligned}
A_4 = \begin{vmatrix} 3 & -2 & 1 & 4 \\ 2 & -1 & 3 & 1 \\ 2 & -1 & 3 & -2 \\ 0 & 1 & 3 & 3 \end{vmatrix} &\xrightarrow[k_3=k_3-k4]{k_4=k_4-3k_2} \begin{vmatrix} 3 & -2 & -3 & 10 \\ 2 & -1 & 2 & 4 \\ 2 & -1 & 5 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix} \\
&= 1\cdot(-1)^6\cdot\begin{vmatrix} 3 & -3 & 10 \\ 2 & 2 & 4 \\ 2 & 5 & 1 \end{vmatrix} \\
&= 1\cdot(6+100-24)-(40+60-6) \\
&= 82-94 \\
&= -12
\end{aligned}
$$

1.3.2 DONE Podstawianie

$$
x_4 = \frac{-12}{20} = \frac{-3}{5}
$$

1.4 DONE Zad 4

$$\left[\begin{array}{cccc|c} 3 & -2 & 1 & 0 & 4 \\ 2 & -1 & 3 & 1 & 1 \\ 2 & -1 & 3 & 4 & -2 \\ x_1 & x_2 & x_3 & x_4 & y \end{array}\right] \xrightarrow[w_1=w_1-w_2]{} \left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & 3 \\ 2 & -1 & 3 & 1 & 1 \\ 2 & -1 & 3 & 4 & -2 \\ x_1 & x_2 & x_3 & x_4 & y \end{array}\right]$$

$$\xrightarrow[w_2=w_2-2w_1]{w_3=w_3-2w_1} \left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & 3 \\ 0 & 1 & 7 & 3 & -5 \\ 0 & 1 & 7 & 6 & -8 \\ x_1 & x_2 & x_3 & x_4 & y \end{array}\right]$$

$$\xrightarrow[w_3=w_3-w_2]{w_1=w_1+w_2} \left[\begin{array}{cccc|c} 1 & 0 & 5 & 2 & -2 \\ 0 & 1 & 7 & 3 & -5 \\ 0 & 0 & 0 & 3 & -3 \\ x_1 & x_2 & x_3 & x_4 & y \end{array}\right]$$

$$\xrightarrow[k_4=k_3]{k_3=k_4} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 5 & -2 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 0 & 3 & 0 & -3 \\ x_1 & x_2 & x_4 & x_3 & y \end{array}\right]$$

$$\xrightarrow[w_3=w_3\cdot\frac{1}{3}]{} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 5 & -2 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 0 & 1 & 0 & -1 \\ x_1 & x_2 & x_4 & x_3 & y \end{array}\right]$$

$$\xrightarrow[w_2=w_2-3\cdot w_3]{w_1=w_1-2\cdot w_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 7 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ x_1 & x_2 & x_4 & x_3 & y \end{array}\right]$$

$x_1 = -5x_3$
$x_2 = -7x_3 - 2$
$x_3 \in \mathbb{R}$
$x_4 = -1$

1.5 DONE Zad 5

$$XA = B \xrightarrow[\text{prawostronnie } A^{-1}]{} XAA^{-1} = BA^{-1} \to X = BA^{-1}$$

$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 0 & -2 \\ -1 & -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 & -5 \\ 3 & 0 & -2 \end{bmatrix}$

$\det A = (36 + 4) - (8 + 30) = 40 - 38 = 2$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} \begin{vmatrix} 0 & -2 \\ -4 & 5 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ -1 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ -1 & -4 \end{vmatrix} \\ -\begin{vmatrix} 2 & -3 \\ -4 & 5 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix} \\ \begin{vmatrix} 2 & -3 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \end{bmatrix}^T$$
$$= \frac{1}{2} \begin{bmatrix} -8 & -13 & -12 \\ 2 & 2 & 2 \\ -4 & -7 & -6 \end{bmatrix}^T$$
$$= \frac{1}{2} \begin{bmatrix} -8 & 2 & -4 \\ -13 & 2 & -7 \\ -12 & 2 & -6 \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 1 & -2 \\ -\frac{13}{2} & 1 & -\frac{7}{2} \\ -6 & 1 & -3 \end{bmatrix}$$

$$X = BA^{-1} = \begin{bmatrix} 4 & 2 & -5 \\ 3 & 0 & -2 \end{bmatrix} \begin{bmatrix} -4 & 1 & -2 \\ -\frac{13}{2} & 1 & -\frac{7}{2} \\ -6 & 1 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} -16 - 13 + 30 & 4 + 2 - 5 & -8 - 7 + 15 \\ -12 + 0 + 12 & 3 + 0 - 2 & -6 + 0 + 6 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

1.6 DONE Zad 6

Macierz obrotu o kąt $\frac{\pi}{3}$ w kierunku dodatnim $A = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}$.

1.6.1 DONE Związek między współrzędnymi x_1, x_2 w bazie $\vec{e}_1, \vec{e}_2$ a współrzędnymi $\hat{x}_1, \hat{x}_2$ w bazie $\vec{u}, \vec{v}$.

$$\vec{u} = A \cdot \vec{e}_1$$
$$= \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

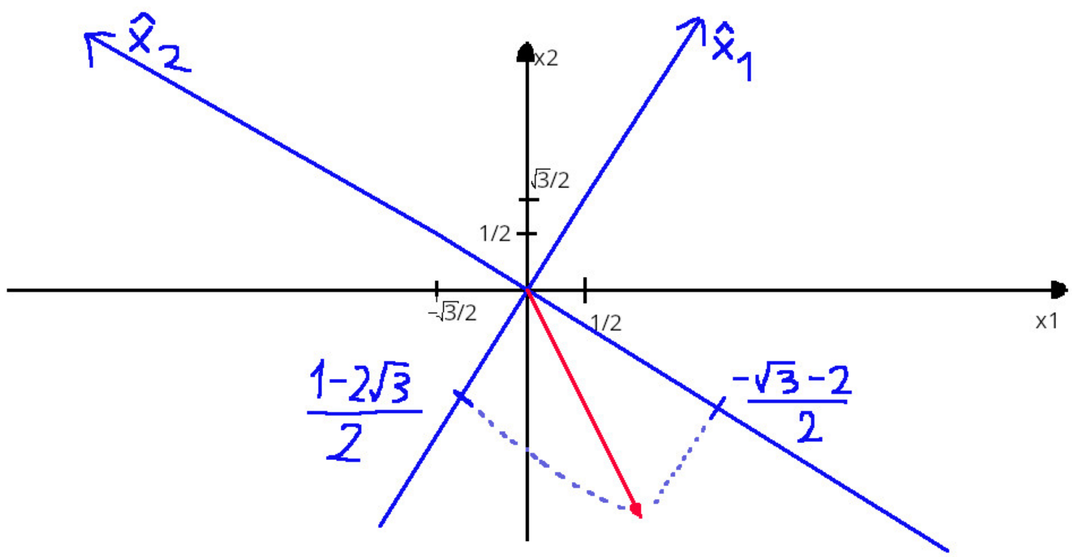
$$\vec{v} = A \cdot \vec{e}_2$$
$$= \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

1.6.2 DONE Znalesć współrzędne wektora $\vec{x}$ w bazie $\vec{u}, \vec{v}$, jeżeli jego współrzędne w bazie $\vec{e}_1, \vec{e}_2$ wynoszą: $1, -2$.

$\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}_{\vec{e}_1, \vec{e}_2}$

$$
\begin{aligned}
\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} &= A^T \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} - \sqrt{3} \\ -\frac{\sqrt{3}}{2} - 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1-2\sqrt{3}}{2} \\ \frac{-2-\sqrt{3}}{2} \end{bmatrix}
\end{aligned}
$$

1. Sporządzić rysunek



1.7 TODO Zad 7

1.8 DONE Zad 8

$$A = (0, 1, 5), \qquad B = (-2, 3, 1), \qquad C = (-2, 7, 3)$$

$|AB| = \sqrt{(-2)^2 + (3-1)^2 + (5-1)^2} = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$
$|AC| = \sqrt{4+36+4} = \sqrt{44} = 2\sqrt{11}$
$|BC| = \sqrt{0+16+4} = \sqrt{20} = 2\sqrt{5}$

1.8.1 DONE Rówanie prostej w postaci kanonicznej

$$\frac{x-0}{-2-0} = \frac{y-1}{7-1} = \frac{z-5}{3-5} \to \frac{x-0}{-2} = \frac{y-1}{6} = \frac{z-5}{-2} = t$$

1.8.2 DONE Równanie prostej w postaci paraetrycznej

$$
\begin{cases}
x - 0 = -2t \to & x = -2t \\
y - 1 = 6t \to & y = 6t + 1 \\
z - 5 = -2t \to & z = -2t + 5
\end{cases}
$$

1.8.3 DONE Oblcizanie odległości punktu od prostej

$$\begin{aligned}
\overrightarrow{BD} \times \vec{k} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 4 \\ -2 & 6 & -2 \end{vmatrix} \\
&= (4\vec{i} + 12\vec{k} - 8\vec{j}) - (4\vec{k} + 24\vec{i} - 4\vec{j}) \\
&= 4\vec{i} + 12\vec{k} - 8\vec{j} - 4\vec{k} - 24\vec{i} + 4\vec{j} \\
&= -20\vec{i} - 4\vec{j} + 8\vec{k}
\end{aligned}$$

$B = (-2, 3, 1)$
$D = (0, 1, 5)$
wektor kierunkowy: $\vec{k} = [-2, 6, -2]$
$\overrightarrow{BD} = [0 + 2, 1 - 3, 5 - 1] = [2, -2, 4]$

$$|BD| = \frac{|\overrightarrow{BD}|}{|\vec{k}|} = \frac{\sqrt{(-20)^2 + 4^2 + 8^2}}{\sqrt{(-2)^2 + 6^2 + (-2)^2}} = \frac{\sqrt{400 + 16 + 64}}{\sqrt{4 + 36 + 4}} = \frac{\sqrt{480}}{\sqrt{44}} = \sqrt{\frac{120}{11}} = \frac{\sqrt{120}}{\sqrt{11}}$$

1.8.4 DONE Obliczanie pola trójkąta

$$P = \frac{2\sqrt{11} \cdot \frac{\sqrt{120}}{\sqrt{11}}}{2} = \frac{2\sqrt{120}}{2} = \sqrt{120} = 2\sqrt{30}$$

1.9 DONE Zad 9

Płaszczyzna: $\pi : x - y + z - 2 = 0$

Postać krawędziowa prostej: $l_1 : \begin{cases} 3x + 2y - z - 4 = 0 \\ -x - 2y + z + 2 = 0 \end{cases}$

1.9.1 DONE Wyznaczyć wektor kierunkowy porstej

$$\begin{aligned}
\vec{k} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ -1 & -2 & 1 \end{vmatrix} \\
&= 2\vec{i} - 6\vec{k} + 1\vec{j} - (-2\vec{k} + 2\vec{i} + 3\vec{j}) \\
&= [0, -2, -4]
\end{aligned}$$

1.9.2 DONE Wyznaczyc punkt przebica płaszczyzny i prostej

1. Znaleźć punkt na prostej Strzlamy punkt $Q(1, 1, 1)$, bo spełnia równanie prostej.
2. Równanie paramtryczne prostej
$$l_1 : \begin{cases} x = 1 + 0t = 1 \\ y = 1 - 2t \\ z = 1 - 4t \end{cases}$$
3. Obliczyć punkt przecięcia płaszczyzny π oraz prostej l_1
   (a) Obliczyć t. Podstawiamy x, y, z z równania parametryczego do równiania płaszczyzny.
$$\begin{aligned}
0 &= 1 - 1 + 2t + 1 - 4t - 2 && \text{uprościć} \\
0 &= -2t - 1 \\
-2t &= 1 && / -2 \\
t &= -\frac{1}{2}
\end{aligned}$$

(b) Podstwaić t do równania parametrycznego prostej.

$$l_1 : \begin{cases} x = 1 + 0t & = 1 \\ y = 1 - 2t = 1 + 1 & = 2 \\ z = 1 - 4t = 1 + 2 & = 3 \end{cases}$$

(c) Punkt przecięcia prostej l z płaszczyzną

$$P_1 = (1, 2, 3)$$

1.9.3 DONE Obliczyć odległość punktu $P(0, 1, 0)$ od prostej l_1

$\overrightarrow{PQ} = [1 - 0, 1 - 1, 1 - 0] = [1, 0, 1]$

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{k} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_{\overrightarrow{PQ}} & y_{\overrightarrow{PQ}} & z_{\overrightarrow{PQ}} \\ x_{\vec{K}} & y_{\vec{K}} & z_{\vec{K}} \end{vmatrix} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & -2 & -4 \end{vmatrix} \\ &= -2\vec{k} - (-2\vec{i} - 4\vec{j}) \\ &= 2\vec{i} + 4\vec{j} - 2\vec{k} \\ &= [2, 4, -2] \end{aligned}$$

1. Długość odcinka PQ

$$|PQ| = \frac{|\overrightarrow{PQ}|}{|\vec{k}|} = \frac{\sqrt{2^2 + 4^2 + (-2)^2}}{\sqrt{(-2)^2 + (-4)^2}} = \sqrt{\frac{24}{20}} = \sqrt{\frac{6}{5}} = \frac{\sqrt{6} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{\sqrt{30}}{5}$$

2 [2/2] Analiza

2.1 DONE Zad 5

Obliczyć w przybliżeniu wartość $\sin(0.2)$ używając weilomianu Taylora stopnia $n = 3$.

$$f(x) = \sin x \qquad x = 0.2 \qquad n = 3$$

$$\begin{aligned} f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \end{aligned}$$

$x_0 = 0$

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x - x_0)^1 + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n-1)\prime}(x_0)}{(n-1)!}(x - x_0)^{(n-1)} + \underbrace{\frac{f^{(n)\prime}(x_0)}{n!}(x - x_0)^n}_{\text{reszta}}$$

$$f(x) \approx 0 + \frac{1}{1}x + \frac{0}{2}x + \frac{-1}{6}x$$

$$f(x) \approx 0.2 + 0 - \frac{1}{6} \cdot \left(\frac{1}{5}\right)^3 = \frac{1}{5} - \frac{1}{6} \cdot \frac{1}{125} = \frac{1}{5} - \frac{1}{750} = \frac{150}{750} - \frac{1}{750} = \frac{149}{750}$$

2.2 DONE Zad 7

2.2.1 Zad 7b

Obliczyć $\int_a^b f(x)dx$

$$a = -3 \qquad b = \frac{1}{2} \qquad f(x) = 3 + x^2$$

$$\begin{aligned}
\int_{-3}^{\frac{1}{2}} 3 + x^2 dx &= \int_{-3}^{\frac{1}{2}} 3dx + \int_{-3}^{\frac{1}{2}} x^2 dx \\
&= \left(\underbrace{3x + \frac{x^3}{3}}_{F(x)} \right) \Bigg|_{-3}^{\frac{1}{2}} \\
&= F(b) - F(a) \\
&= \left(\frac{36}{24} + \frac{1}{24} \right) - (-9 - 9) \\
&= \frac{37}{24} + 18 \\
&= \frac{37}{24} + \frac{432}{24} \\
&= \frac{469}{24}
\end{aligned}$$