

# Introduction to Generalized Linear Models

## Part 2 of 3

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# Outline

- ~~The normal model: A recap~~
- ~~Models for binary data~~
- ~~Models for binomial data~~
- Models for multinomial data
- Models for count data
- Extensions: overdispersion models
- Extensions: zero-inflated models

# Models for Multinomial Data

# The Multinomial GLM

- Example: elephant grass grazing data



# The Multinomial GLM

- We now are looking at multivariate extensions of the Bernoulli and binomial distributions
- For the sake of simplicity we will consider the *individual* case (denominator of 1)
- A Bernoulli random variable can assume only two values, success or failure
- Recall the Bernoulli variable  $Y_i$ :

$$Y_i = \begin{cases} 1, & \text{if success} \\ 0, & \text{if failure} \end{cases}$$

- We can re-write it as a bivariate response variable  $\mathbf{Y}_i$  such that:

$$\mathbf{Y}_i = \begin{cases} (1, 0)^\top, & \text{if success} \\ (0, 1)^\top, & \text{if failure} \end{cases}$$

# The Multinomial GLM

- The vectors are of dimension 2 because there are only 2 possible categories of response
- Extending this to  $K$  possible response categories, we have:

$$Y_{ik} = \begin{cases} 1, & \text{if category } k \\ 0, & \text{otherwise} \end{cases}$$

- As an example, for  $K = 3$  categories, this translates to

$$\mathbf{Y}_i = \begin{cases} (1, 0, 0)^\top, & \text{if category 1} \\ (0, 1, 0)^\top, & \text{if category 2} \\ (0, 0, 1)^\top, & \text{if category 3} \end{cases}$$

# The Multinomial GLM

- We write  $Y_i \sim \text{Multinomial}(\pi_i)$ , where  $\pi_i = (\pi_1, \pi_2, \dots, \pi_K)^\top$  is now a vector of probabilities
- Recall that probabilities are bounded in the  $(0, 1)$  interval
- Therefore, we must make the restriction

$$\sum_{j=1}^K \pi_{ij} = 1$$

- This commonly translates as taking one of the categories as the *reference category*
- In the `nnet` package in R which implements this model as the `multinom` function, the first category is taken as reference, and therefore

$$\pi_{i1} = 1 - \sum_{j=2}^K \pi_{ij}$$

# The Multinomial GLM

- There are many variations and extensions to multinomial GLMs
- We will now see the basic building blocks of the so-called *generalized logits model*
- We have  $K - 1$  logits for a multinomial model with  $K$  categories
- Taking the first category as reference, we have

$$\eta_{ki} = \log \left( \frac{\pi_{ki}}{\pi_{1i}} \right) = \beta_{0k} + \beta_{1k}x_{1i} + \dots + \beta_{pk}x_{pi}$$

- This means that a generalized logits model estimates  $(K - 1) \times (p + 1)$  parameters



# The Multinomial GLM

- For our grazing experiment example, we have  $K = 3$  categories, and therefore, 2 logits:

$$\log \left( \frac{\pi_{2i}}{\pi_{1i}} \right) = \beta_{02} + \beta_{12} \text{month}_i$$

$$\log \left( \frac{\pi_{3i}}{\pi_{1i}} \right) = \beta_{03} + \beta_{13} \text{month}_i$$

- $\beta_{02}, \beta_{12}, \beta_{03}$ , and  $\beta_{13}$  are the four parameter estimates we see when we fit the model using `multinom`

# The Multinomial GLM

- It is easy to obtain fitted probabilities for each category based on predictors:

$$\log\left(\frac{\pi_{ki}}{\pi_{1i}}\right) = \eta_{ki} \Leftrightarrow \frac{\pi_{ki}}{\pi_{1i}} = e^{\eta_{ki}} \Leftrightarrow \pi_{ki} = \pi_{1i}e^{\eta_{ki}}$$

- We then use the restriction

$$\pi_{1i} = 1 - \sum_{j=2}^K \pi_{ji} = 1 - \sum_{j=2}^K \pi_{1i}e^{\eta_{ji}}$$

- Solving for  $\pi_{1i}$  yields

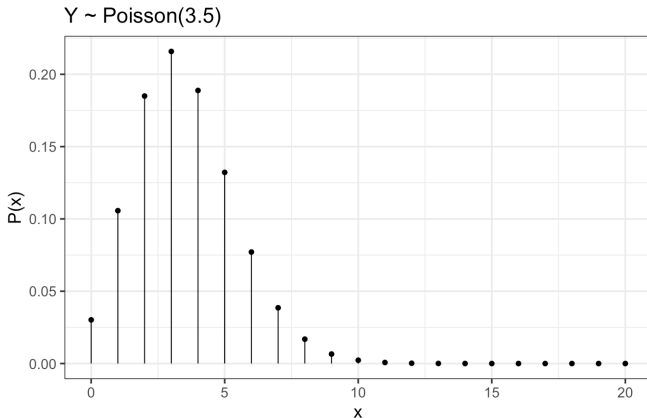
$$\pi_{1i} = \frac{1}{1 + \sum_{j=2}^K e^{\eta_{ji}}}$$

- Using the two equations in blue above we are able to recover all probabilities; we compute the reference probability, and then use it to compute the probabilities for all other categories

# Models for Count Data

# The Poisson GLM

- The Poisson distribution is a discrete probability distribution over the non-negative integers  $0, 1, 2, \dots$

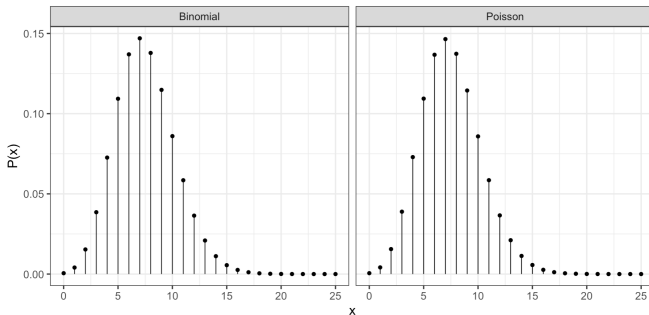


# The Poisson GLM

- The number of times an event occurs is a common form of data, e.g.
  - number of identified animal species in quadrats at different locations
  - number of eggs laid per female of an insect species
  - number of customers purchasing at a particular shop per day
  - number of people in each cell of a contingency table summarising survey responses
- Simplest assumption of the underlying process: counts arise at some average *rate*  $\lambda$ , occurring in a fixed interval of time or space
- The Poisson model offers a starting point for analysis

# The Poisson GLM

- The Poisson distribution can be seen as the limit of a binomial distribution with a small probability of success very large  $m$
- We have that under these conditions  $\text{Binomial}(m, \pi) \approx \text{Poisson}(m\pi)$
- Example:  $m = 1000$  and  $\pi = 0.0075$ :



# The Poisson GLM

- Consider events over time
- Write  $\lambda$  for the average rate per unit time
- The distribution for the number of events  $Y$  in an interval of length  $t$  is  $\text{Poisson}(\lambda t)$ , with probability mass function

$$P(Y = y) = \frac{e^{-\lambda t} (\lambda t)^y}{y!}, \quad y = 0, 1, 2, \dots$$

- We have that

$$E(Y) = \text{Var}(Y) = \lambda t = \mu$$

- This is often referred to as *equidispersion*
- In many simple applications, counts will be observed over identical time periods, areas, etc.

# The Poisson GLM

- For a random sample of  $n$  counts  $Y_i$ ,  $i = 1, \dots, n$ , with possibly different underlying rates  $\mu_i$ , we have

$$Y_i \sim \text{Poisson}(\mu_i)$$

- We typically model the rates  $\mu_i$  in terms of observed explanatory variables
- Since  $\mu_i > 0$ , the most used link function is the *log link*:

$$\log \mu_i = \beta_0 + \beta_1 x_{1i} + \dots + \beta_p x_{pi}$$



# The Poisson GLM

- Example: contamination dataset

# The Poisson GLM

- Interpreting the coefficients from a Poisson log-linear model is intuitive
- Since we use the log-link, the inverse link function is the exponential, i.e.

$$\mu_i = e^{\beta_0 + \beta_1 x_{1i} + \dots + \beta_p x_{pi}}$$

- Take a model with a single continuous predictor, i.e.  $\mu_i = e^{\beta_0 + \beta_1 x_i}$
- For a one-unit increase in the predictor we have

$$\begin{aligned}\mu_i^+ &= e^{\beta_0 + \beta_1(x_i + 1)} \\ &= e^{\beta_0 + \beta_1 x_i + \beta_1} \\ &= e^{\beta_0 + \beta_1 x_i} e^{\beta_1}\end{aligned}$$

- Therefore,  $\mu_i^+ = \mu_i e^{\beta_1}$ , i.e.  $e^{\beta_1}$  is the multiplicative effect of an increase in one unit to the predictor variable

# Offset variables

- In many situations the length of time during which events are measured varies across individuals
  - e.g. suppose we are sampling the number of pest insects in an agricultural field, and find 10, 20, and 30 insects in quadrats of  $1\text{m}^2$ ,  $1.5\text{m}^2$  and  $4\text{m}^2$
- In such cases, the notation  $Y_i \sim \text{Poisson}(\lambda_i t_i)$  is helpful
- We have:

$$\log \mu_i = \log(\lambda_i t_i) = \log \lambda_i + \log t_i$$

- Here,  $\log \lambda_i$  is modelled with covariates, and the term  $\log t_i$  is referred to as *offset*

# Offset variables

- The model is then

$$\begin{aligned} Y_i &\sim \text{Poisson}(\mu_i) \\ \log \mu_i &= \underbrace{\beta_0 + \beta_1 x_{1i} + \cdots + \beta_p x_{pi}}_{\log \lambda_i} + \underbrace{\log t_i}_{\text{offset}} \end{aligned}$$

- Note that the offset term can be seen as adding a covariate to the linear predictor associated to a  $\beta$  coefficient equal to 1

## Goodness-of-fit Based on Residual Deviance

- For the Poisson and binomial GLMs, one way of checking whether the model fits the data well is to look at the residual deviance
- For a well fitted model, the residual deviance should be approximately equal to the  $n - p$ , the number of residual d.f.
- This is because, asymptotically, the residual deviance has a  $\chi^2$  distribution with  $n - p$  d.f.
- If the residual deviance is much larger than expected, this indicates lack-of-fit, which may be due to omission of important predictors, or simply extra variability not accounted for by the model
- Example: *Sitophilus zeamais* progeny data

# Half-Normal Plots with a Simulated Envelope

- Another way of empirically checking whether a GLM (or any<sup>1</sup> statistical model, in fact), is to use a graphical technique called *half-normal plot with a simulated envelope*
- It consists in plotting ordered residuals in absolute value versus expected order statistics of the half-normal distribution, and adding a simulated envelope based on the fitted model
- The envelope serves as a guide to what expect if the observed data are a plausible realisation of the fitted model
- In R it is implemented as the `hnp` package
- If the majority of points lie within the simulated envelope, the model can be considered to be well-fitted to the data

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<sup>1</sup>as long as response variables can be simulated from it