Lecture Notes for 2/20/2025

3.5 Invertibility and determinants

3.6 Cramer's rule

Theorem 3.5.1. Let A be a square matrix. Then A is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then $\det(A^{-1}) = 1/\det(A)$.

Theorem 3.5.4. Let A be an $n \times n$ matrix, then the follow statements are equivalent:

1. A is invertible;
2. The reduced row echelon form of A is the identity matrix I_n;
3. A has n pivot columns (or n pivots, or n pivot positions);
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$;
5. There exists an $n \times n$ matrix C such that $CA = I_n$;
6. There exists an $n \times n$ matrix D such that $AD = I_n$;
7. The transpose A^T of A is invertible;
8. $\det(A) \neq 0$.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & +7 \end{bmatrix} \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A \quad I] \rightsquigarrow [I \quad A^{-1}]$$

We will be using this result later (many times in fact) so try to read it a few times to keep it in your memory for a while, it will be handy!

$\det(AB) = \det(A)\det(B)$ ✓

$\det(kA) = k^n \det(A)$ ✓

$\det(A+B) \overset{?}{=} \det(A) + \det(B)$? Not true.

$\underset{1}{\det(A+B)}$ $\quad$ $\underset{0}{\det(A)} + \underset{0}{\det(B)}$

$$\underset{A}{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}} + \underset{B}{\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}} = \underset{A+B}{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}$$

One more important property of determinants (this is covered in WebWork and will be covered in the next test):

Let A and B be two $n \times n$ matrices that are identical except for one row (or one column). Say the first row of A and B may be different, that is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

If C is the matrix obtained from A by adding the first row of B to the first row of A:

$$C = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \neq A+B$$

then $\det(C) = \det(A) + \det(B)$.

$3d + -4 \quad 3e + -6 \quad 3f + 2$

Example. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$, $\begin{vmatrix} a & b & c \\ 2 & 3 & -1 \\ g & h & i \end{vmatrix} = -7$, find $\begin{vmatrix} a & b & c \\ 3d-4 & 3e-6 & 3f+2 \\ g & h & i \end{vmatrix}$. $= 26$

$$\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ -4 & -6 & 2 \\ g & h & i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - 2\begin{vmatrix} a & b & c \\ 2 & 3 & -1 \\ g & h & i \end{vmatrix}$$

$$= 3 \cdot 4 - 2(-7) = 12 + 14 = 26$$

Example. If $\begin{vmatrix} a & b & c \\ d & e & f \\ 3 & 3 & 3 \end{vmatrix} = 12$, $\begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 2 & 3 \end{vmatrix} = -5$, find $\begin{vmatrix} a & b & c \\ d & e & f \\ -3 & -4 & -5 \end{vmatrix}$. $\overset{?}{=} k_1 \begin{vmatrix} a & b & c \\ d & e & f \\ 3 & 3 & 3 \end{vmatrix} + k_2 \begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 2 & 3 \end{vmatrix}$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 3k_1+k_2 & 3k_1+2k_2 & 3k_1+3k_3 \end{vmatrix} = k_1 \begin{vmatrix} & & \\ & & \\ 3k_1 & 3k_1 & 3k_1 \end{vmatrix} + k_2 \begin{vmatrix} & & \\ & & \\ k_1 & 2k_2 & 3k_3 \end{vmatrix}$$

$(-3 \quad -4 \quad -5) = k_1(3 \quad 3 \quad 3) + k_2(1 \quad 2 \quad 3)$

$3k_1 \quad 3k_1 \quad 3k_1$

$-3 = \boxed{3k_1 + k_2} = 3k_1 - 1 \qquad 3k_1 = -2 \qquad k_1 = -\frac{2}{3}$

$-4 = \boxed{3k_1 + k_2} + k_2 = -3 + k_2 \qquad k_2 = -1$

$-5 = 3k_1 + 3k_2$

$-\frac{2}{3} \cdot 12 - 1 \cdot (-5)$

$= -8 + 5$

$= -3$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 2 & -1 & 5 \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ 5 & 3 & 3 \end{vmatrix}$$

$\qquad = 3 \qquad\qquad = -8 \qquad\qquad \overset{?}{=} -5$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 6 & -3 & 15 \end{vmatrix} \overset{?}{=} 3 \begin{vmatrix} a & b & c \\ d & e & f \\ 2 & -1 & 5 \end{vmatrix} = 3 \cdot 3 = 9.$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ -6 & -8 & 4 \end{vmatrix} = -2 \begin{vmatrix} a & b & c \\ d & e & f \\ 3 & 4 & -2 \end{vmatrix} = 16 \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ 0 & -11 & 19 \end{vmatrix} = 3 \begin{vmatrix} a & b & c \\ d & e & f \\ 2 & -1 & 5 \end{vmatrix} - 2 \begin{vmatrix} a & b & c \\ d & e & f \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 9 + 16 = 25$$

Quiz Question 1. Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 1 & 4 \end{vmatrix} = -3 \text{ and } \left[\begin{matrix} a & b & c \\ d & e & f \\ -2 & 3 & 1 \end{matrix}\right| = 4, \text{ find } \begin{vmatrix} a & b & c \\ d & e & f \\ 4 & -5 & 2 \end{vmatrix}.$$

A. -11; B. 5; C. 1; D. -2.

$$k_1 \begin{pmatrix} 0 & 1 & 4 \end{pmatrix} + k_2 \begin{pmatrix} -2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \end{pmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$C_{23} = -M_{23}$ $\quad$ $C_{42} = M_{42}$

$2+3=5$ odd

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ & \cdots & & \\ C_{41} & C_{42} & C_{42} & C_{44} \end{bmatrix}$$

Cofactor matrix and adjoint matrix of A:

Let A be an $n \times n$ matrix, with C_{ij} being the cofactor of the entry a_{ij} in A. Then

C^T

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

$C^T = \boxed{adj(A)}$

is called the cofactor matrix of A and the transpose of it is called the adjoint matrix of A, denoted by $\mathrm{adj}(A)$:

$k(AB)$
$=(kA)\cdot B$
$= A\cdot(kB)$

$$\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$A \cdot adj(A) = det(A) I$

$adj(A) \cdot A = det(A) \cdot I_n$

$\neq 0$

Theorem 3.5.2. Let A be a square matrix and $adj(A)$ the adjoint matrix of A, then $A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = \det(A) I_n$. Thus if $\det(A) \neq 0$, then $\boxed{A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A).}$

The proof of this theorem is given in the book.

$\frac{1}{det(A)} A \cdot adj(A) = I$

If A is a 4×4 matrix such that

$A \cdot \underbrace{\frac{1}{det(A)} \cdot adj(A)}_{= A^{-1}} = I$

$$A^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

(so $\det(A) = 1/\det(A^{-1}) = 1/2$), find the cofactor matrix of A and use it to find the cofactors C_{23} and C_{44}.

$\boxed{A^{-1} = \frac{1}{det(A)} adj(A)}$ or $\quad adj(A) = det(A) \cdot A^{-1}$

$C^T = det(A) \cdot A^{-1}$

$\boxed{C = det(A) \cdot (A^{-1})^T}$

$$\rightarrow C = \frac{1}{2} \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ 0 & 5 & 3 & -2 \\ 1 & 4 & -2 & 1 \end{bmatrix}}_{(A^{-1})^T}$$

$C_{11} = 1 \quad C_{12} = C_{13} = C_{14} = 0 \quad C_{23} = 0 \quad C_{44} = \frac{1}{2}$

det(A) = 0 ? NOT so easy to find C.

Quiz Question 2. Continue from the last example, where 4×4 matrix such that

$$A^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

(so $\det(A) = 1/\det(A^{-1}) = 1/2$), find the cofactor C_{14}.

A. $-1/2$; B. 0; C. $1/2$; D. 1.

$C_{34} = ?$

$$\begin{cases} 2x_1 + 3x_2 = 1 \\ -x_1 + 5x_2 = 4 \end{cases} \Rightarrow \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

3.6 Cramer's rule

Theorem 3.6.1. Consider the linear equation system $A\mathbf{x} = \mathbf{b}$ with n equations and n variables, so that the coefficient matrix A is of size $n \times n$. Let A_j be the matrix obtained from A by replacing its j-th column with $\mathbf{b}$, then if $\det(A) \neq 0$, the equation has a unique solution which is given by $x_j = \boxed{\frac{\det(A_j)}{\det(A)}}$ for $1 \leq j \leq n$.

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)} \quad \cdots$$

Note that under this setting A is an $n \times n$ matrix, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Example. Find the solution of the equation system

$$\left[\begin{array}{cc|c} 2 & 5 & a \\ -1 & 3 & b \end{array}\right]$$

$$2x_1 + 5x_2 = a, \quad -x_1 + 3x_2 = b$$

where a and b are some constant numbers. Express your answer in terms of a and b.

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix} \qquad \det(A) = 6 - 5(-1) = 6 + 5 = 11$$

$$A_1 = \begin{bmatrix} a & 5 \\ b & 3 \end{bmatrix} \qquad \det(A_1) = 3a - 5b \qquad x_1 = \frac{3a - 5b}{11}$$

$$x_2 = \frac{a + 2b}{11}$$

$$A_2 = \begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix} \qquad \det(A_2) = 2b - a(-1) = a + 2b$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\det(A) = 6$$

$$\det(A_1) = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 3 \end{vmatrix} = 6$$

$$\det(A_2) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 0 & 3 \end{vmatrix} = -3 - 6 = -9$$

$$\Rightarrow \quad x_1 = \frac{6}{6} = 1, \quad x_2 = \frac{-9}{6} = -\frac{3}{2}, \quad x_3 = \frac{0}{6} = 0$$

$$\det(A_3) = \underline{\det(A)} = 6$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$A_{n\times n}$

Quiz Question 3. Consider the linear equation system $A\mathbf{x} = \mathbf{b}$ with n equations and n unknowns. If $\det(A) = 0$, then which of the following statement is always true?

a. We can still apply Cramer's rule to solve the equation;

b. The equation is definitely inconsistent;

c. The equation is consistent and has a unique solution;

d. The equation is either inconsistent, or is consistent with infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \cancel{2}\ 0 \end{array}\right]$$

inconsistent

Quiz Question 4. Consider the linear equation system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 4 & 3 & 1 \\ 1 & -5 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -17 \end{bmatrix}.$$

Given that $\det(A) = -19$, find x_2.

a. The solution does not exist; b. 2; c. -2; d. $2/19$.

$$x_2 = \frac{\det(A_2)}{-19}$$