Lecture Notes for 2/27/2025

4.2 Spanning sets

4.3 Linear independence and dependence

Definition of linear combination. Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ be any k vectors of $\mathbb{R}^n$, then for any scalars a_1, a_2, ..., a_k, the vector $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$ is called a *linear combination* of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$. It is very important for us to understand the difference between **a linear combination** and **all linear combinations** of a given set of vectors.

Examples. $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

$$2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-4)\begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} + 0\cdot\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

solutions of $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 3 & -4 & 0 \end{array}\right]$

$$\begin{aligned} x_1 &= x_3 - 2x_4 \\ x_2 &= -3x_3 + 4x_4 \\ x_3 &= 1\cdot x_3 + 0\cdot x_4 \\ x_4 &= 0\cdot x_3 + 1\cdot x_4 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3\underline{\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}} + x_4\underline{\begin{bmatrix} -2 \\ 4 \\ 0 \\ 1 \end{bmatrix}}$$

Quiz Question 1. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two given vectors in $\mathbb{R}^4$. Determine which of the following vectors is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

(a) $\mathbf{v}_1 - 3\mathbf{v}_2$; (b) $10\mathbf{v}_2$; (c) $\mathbf{0}$ (the zero vector in $\mathbb{R}^4$)

A. (a) only; B. (a) and (b) only; (C) all of them; D. None of them

One important take away from the examples is that we can view a linear equation system as the question of determining whether a given vector is a linear combination of other vectors. (Here is a question for you: How many ways can we view a linear equation system up to this point?)

For example, the equation system

$$\begin{aligned} 2x_1 + 3x_2 \quad\quad + 5x_4 &= -1, \\ -7x_1 + x_2 - 3x_3 + 2x_4 &= 8, \\ x_1 - 6x_2 + 2x_3 - 2x_4 &= 0, \end{aligned}$$

can be rewritten as

$$x_1 \begin{bmatrix} 2 \\ -7 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 0 \end{bmatrix},$$

and can also be rewritten as a matrix equation:

$$\begin{bmatrix} 2 & 3 & 0 & 5 \\ -7 & 1 & -3 & 2 \\ 1 & -6 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 0 \end{bmatrix},$$

and we also use an augmented matrix to solve this equation system:

$$\left[\begin{array}{cccc|c} 2 & 3 & 0 & 5 & -1 \\ -7 & 1 & -3 & 2 & 8 \\ 1 & -6 & 2 & -2 & 0 \end{array}\right].$$

Quiz Question 2. If we write the equation system

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= -1, \\ 7x_1 + 8x_2 + 9x_3 &= 0, \\ -x_1 - 2x_2 - 3x_3 &= 1, \\ -2x_1 - 3x_2 - 4x_3 &= 2 \end{aligned}$$

in the form of a linear combination equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b},$$

then which one of the following is NOT correct?

A. $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 7 \\ -1 \\ -2 \end{bmatrix}$; B. $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix}$; C. $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 9 \\ -3 \\ -4 \end{bmatrix}$; D. $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

$\text{span}\{v_1, v_2, v_3\}$ $S = \{v_1, v_2, \cdots, v_k\}$

$\text{span}(S)$

Definition of span and spanning set. Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ be any k vectors of $\mathbb{R}^n$, then the set of all possible linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ is called the *span* of the vector set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ (denoted by span(S)). By Theorem 4.2.2, span(S) is a subspace of $\mathbb{R}^n$. For any subspace W of $\mathbb{R}^n$, if $W = \text{span(S)}$ for some vector set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$, then S is called the *spanning set* of W.

Theorem 4.2.2: Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ be any k given vectors of $\mathbb{R}^n$, then $\text{span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n)$ is a subspace of $\mathbb{R}^n$. (In fact, any subspace in $\mathbb{R}^n$ is the span of some subset of vectors in $\mathbb{R}^n$.)

Example 1. The span of the vectors $\underbrace{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-3\\2\end{bmatrix} \text{ and } \begin{bmatrix}1\\-1\end{bmatrix}}_{S}$ would be the set containing all linear combination of these vectors, namely vectors that can be written as

$$W = \text{span}(S) = \left\{ x_1 \begin{bmatrix}1\\1\end{bmatrix} + x_2 \begin{bmatrix}-3\\2\end{bmatrix} + x_3 \begin{bmatrix}1\\-1\end{bmatrix} \right\}, \text{ is a subspace}$$

for some scalars x_1, x_2 and x_3. Determining whether the vector $\begin{bmatrix}-5\\-5\end{bmatrix}$ is in the span of these vectors is the same as the question whether the equation system

$$x_1 \begin{bmatrix}1\\1\end{bmatrix} + x_2 \begin{bmatrix}-3\\2\end{bmatrix} + x_3 \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-5\\-5\end{bmatrix}$$

is consistent (that is, it has at least one solution). Can you see whether this equation system is consistent?

Note the above equation system can also be written as a matrix equation:

$$\begin{bmatrix}1 & -3 & 1\\1 & 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-5\\-5\end{bmatrix}$$

and the augmented matrix of the equation system is

$$\left[\begin{array}{ccc|c}1 & -3 & 1 & -5\\1 & 2 & -1 & -5\end{array}\right]. \xrightarrow{-1\cdot R_1 + R_2 \to R_2}$$

$$\begin{bmatrix}1 & -3 & 1 & -5\\0 & 5 & & \end{bmatrix} \checkmark$$

Example 2. The span of the vectors $\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}$ would be the set containing all vectors that can be written as

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}$$

for some scalars c_1, c_2.

Similar to the above example, determining whether the vector $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ is in the span of these vectors is the same as the question whether the equation system

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 5 & -4 & 2 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{cc|c} & & \\ - & - & - \\ 0 & 0 & 2 \end{array}\right]$$

Quiz Question 3. See if you can determine quickly which vector $\mathbf{v}$ is NOT in the span of the corresponding S. (Hint: write down the matrix form of the equation $A\mathbf{x} = \mathbf{b}$ and using determinant to determine whether the coefficient matrix A is invertible, if it is, then you know the equation has a unique solution.)

A. $\mathbf{v} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$

B. $\mathbf{v} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

C. $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$

D. $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} \right\}$

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 6 \\ -2 & 2 & 1 \end{array}\right] \xrightarrow{2R_1 + R_2} \left[\begin{array}{cc|c} 1 & -1 & 6 \\ 0 & 0 & 13 \end{array}\right]$$

Definition of linear independence and dependence.

Let us consider linear *dependence* first. If a vector $\mathbf{u}$ is a linear combination of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_m$, then it makes sense to say that $\mathbf{u}$ is *dependent* on $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_m$. Generalizing this idea, it then makes sense for us to say that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are *linearly dependent* if one of them can be written as linear combination of the rest. And, of course, we would say that $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are linearly **independent** when **none** of them can be written as a linear combination of the rest. This is in fact what linear independence and dependence mean, however the formal definition in the book is not like it. Instead, here is how it is defined:

Definition. The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are linearly independent if the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution $a_1 = a_2 = \cdots = a_k = 0$. Otherwise $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are linearly dependent.

Why? The intuitive definition is not very practical. Say, if we have 100 vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_{100}$ and we want to use our intuitive definition to prove that they are linearly independent, then we will need to show that NONE of these vectors can be written as a linear combination of the rest. That means we need to set up 100 equations and show that each one does not have a solution. For example, to show that $\mathbf{v}_{100}$ is not a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_{99}$ we need to show that the equation

$$\mathbf{v}_{100} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_{99}\mathbf{v}_{99}$$

has no solution.

Whereas in the definition given in the book, we only need to solve one equation, namely

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_{99}\mathbf{v}_{99} + a_{100}\mathbf{v}_{100} = \mathbf{0}$$

But does this definition mean the same thing as our intuitive definition? The answer is YES. Why?

$$a_1 V_1 + a_2 V_2 + a_3 V_3 = \vec{0}$$

suppose $a_1 = 2$, $a_2 = 0$, $a_3 = -5$ is a solution

$$2V_1 + 0\cdot V_2 + (-5)V_3 = \vec{0}$$

$$2V_1 = 0\cdot V_2 + 5V_3 \qquad V_1 = 0\cdot V_2 + \frac{5}{2}\cdot V_3$$

if only trivial solution.

say $V_3 = c_1 V_1 + c_2 V_2$ $c_1 V_1 + c_2 V_2 + (-1) V_3 = \vec{0}$

If the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_k\mathbf{v}_k = \mathbf{0}$$

has a non-trivial solution $a_1 = c_1$, ..., $a_k = c_k$ for some constants c_1, ..., c_k that are not all zero, say $c_1 \neq 0$, then we have

$$c_1\mathbf{v}_1 = -c_2\mathbf{v}_2 - \cdots - c_k\mathbf{v}_k$$

hence

$$\mathbf{v}_1 = (-c_2/c_1)\mathbf{v}_2 + (-c_3/c_1)\mathbf{v}_3 + \cdots + (-c_k/c_1)\mathbf{v}_k.$$

So $\mathbf{v}_1$ is a linear combination of the rest hence these vectors are dependent by our "intuitive" definition.

On the other hand, if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution $a_1 = a_2 = \cdots = a_k = 0$, then the vectors must be independent by our "intuitive" definition. Because if this is not true, that is, these vectors are in fact dependent by our "intuitive" definition, then one of them is a linear combination of the rest. Say $\mathbf{v}_1$ is a linear combination of the rest, that is, there exist constants c_2, .., c_k such that

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k$$

But this means

$$\mathbf{v}_1 - c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - \cdots - c_k\mathbf{v}_k = \mathbf{0}$$

That is, $a_1 = 1$, $a_2 = -c_2$, $a_3 = -c_3$, ..., $a_k = -c_k$ is a nontrivial solution of the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_k\mathbf{v}_k = \mathbf{0}$$

which is a contradiction.

Example 1. For any vectors $\mathbf{v}_1$, ..., $\mathbf{v}_k$ in $\mathbb{R}^n$, if the zero vector is one of $\mathbf{v}_1$, ..., $\mathbf{v}_k$, then they are always linearly dependent.

$$\vec{0},\ \vec{v}_2,\ \vec{v}_3$$

$$a_1\vec{0} + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$$

$$\underline{1}\cdot\vec{0} + \underline{0}\cdot\vec{v}_2 + \underline{0}\cdot\vec{v}_3 = \vec{0}$$

Example 2. The vectors $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ are linearly independent. Why?

$$a_1\begin{bmatrix} -2 \\ 3 \end{bmatrix} + a_2\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 3. The vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ in $\mathbb{R}^n$ are obviously linearly independent. Why?

$\mathbb{R}^2$: $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\mathbb{R}^3$: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix},\ v_3 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix},\ v_4 = \begin{bmatrix} 1 \\ 7 \\ 8 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 5 & 1 & 0 \\ 1 & -1 & 1 & 7 & 0 \\ 2 & -2 & 4 & 8 & 0 \end{array}\right]$$

Quiz Question 4. If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are vectors in $\mathbb{R}^3$ and they are linearly dependent, then which statement below is true?

A. The equation $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$ has no solutions;

B. The equation $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$ has only the trivial solution;

C. The equation $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$ has non-trivial solutions;

D. None of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ can be written as a linear combination of the rest.