Lecture Notes for 3/13/2025

5.1 General vector spaces

5.2 Subspaces

Field: a system similar to the set of real numbers with two operations (addition and multiplications) defined, and behaves like the real numbers. (Do not overburden yourself with this: we will only be using real numbers in this course.)

If we take out the "geometric meaning" of a vector in $\mathbb{R}^n$, then what we are looking at is just a matrix of size $n \times 1$. The vector addition and scalar multiplication in $\mathbb{R}^n$ are just the matrix addition and matrix scalar multiplication. So if we replace the vectors in $\mathbb{R}^n$ by matrices of the same size, say by the set of all 2×2 matrices, then Conditions (1) to (10) that we used to define the vector space $\mathbb{R}^n$ would still hold. That is, the set of all 2×2 matrices with real number entries behaves just like the vector space $\mathbb{R}^4$ under the matrix addition and scalar multiplication, so there is no reason why we cannot think it as a vector space. We just have to ignore its geometric meaning. We would call such a vector space a *general vector space* to distinguish it from a Euclidean vector space $\mathbb{R}^n$. (Or we can call it an "abstract vector space" like in some literature.)

Definition 5.1.2: General vector spaces.

A ***vector space*** V over a field F is a set that satisfies a list of properties under two binary operations, vector addition and scalar multiplication. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars a and b in F. Then,

- Closure under vector addition: $\mathbf{u} + \mathbf{v} \in V$ ✓
- Closure under scalar multiplication: $a\mathbf{u} \in V$ ✓
- Commutativity of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Associativity of addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- Additive identity: $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- Additive inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- Associativity of scalar multiplication: $a(b\mathbf{u}) = (ab)\mathbf{u}$
- Scalar identity: $1 \cdot \mathbf{u} = \mathbf{u}$
- Distributivity of scalars over vector addition: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- Distributivity of vectors over scalar addition: $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

$A + B = B + A$

$V = \mathbb{R}_{2\times 3}$

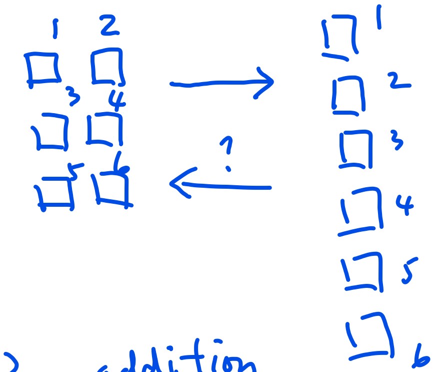
$\mathbb{R}^4 = \mathbb{R}_{4\times 1}$

$A_{2\times 3} + B_{2\times 3}$

More examples of "general vector spaces". $\mathbb{R}_{2\times 2}$

$\mathbb{R}_{3\times 2}$: the set of all 3×2 matrices with real number entries. What about the set that contains all 3×2 AND all 2×2 matrices with real entries? Or the set of all 2×2 matrices with integer entries?

$\mathbb{R}_{3\times 2}$ $\mathbb{R}_{4\times 6}$



$\mathbb{R}^6$

$\mathbb{R}_{3\times 1} = \mathbb{R}^3$

$\mathbb{R}_{3\times 2} \cup \mathbb{R}_{2\times 2}$ NO, addition not defined.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W, \text{ but } \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \notin W$$

so W is not a subspace.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 5 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 4 & 2 \end{bmatrix}$$

Similar to the case of the Euclidean vector space $\mathbb{R}^n$, a non-empty subset W of a general vector space U is a subspace (that is, it is also a vector space by itself) if and only if it is closed under addition and scalar multiplication.

$\to \mathbb{R}_{1\times 5}$

Quiz Question 1. Let U be the set that contains all 1×5 matrices with real entries, V be the set that contains all 3×3 matrices with real entries, and W be $\mathbb{R}^4$, which of the following statement is NOT true?

$V = \mathbb{R}_{3\times 3}$

A. U is a vector space; ✓

B. V is a vector space; ✓

$\underline{\mathbb{R}_{m\times n}}$

C. The union of U, V and W is a vector space.

D. Each of U, V and W is a vector space. ✓

Are there any other general vector spaces other than $\mathbb{R}_{m\times n}$? The answer is yes, in fact there are many. We shall only introduce you to one of these.

$\mathcal{P}_n$: the set of all polynomials of x with real coefficients and powers up to n.

P_2 : 1, 0, 5, $\cdots$

$1+3x$, $\pi+2x$, $\cdots$, $-3x$, $\cdots$

$-1+2x-5x^2$, x^2, $3x-7x^2$, $\cdots$

$$3(2-7x+5x^2) = 6-21x+15x^2$$

$\mathbb{R}_{m\times n}$, $\mathcal{P}_n$, $(n \geq 0)$

$\mathcal{P}_0 = \mathbb{R}^1$

$\mathcal{P}_1$

$a_0 + a_1 x$

$a_0 + a_1 x + a_2 x^2$

$$\begin{bmatrix} 2 & -5 \\ 4 & -8 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 5 & -2 \end{bmatrix}$$

More examples of subspaces of these general vector spaces.

W is the set of all 2×2 matrices such that the sum of its entries is zero.

W is closed under + and scalar mult. so it is a subspace.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$ means $a+b+c+d=0$

$+$

$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \in W$ . . . $e+f+g+h=0$

$\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \xrightarrow{\text{in}} W$

$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

$W = \{a_1 x + a_2 x^2 : \ a_1, \ a_2 \in \mathbb{R}\}$ (which is a subset of $\mathcal{P}_2$).

$1 \cdot x + 3x^2$, x^2, $-5x$, 0, . . .

$a_0 + a_1 x + a_2 x^2$

$a_1 x + a_2 x^2 + b_1 x + b_2 x^2$

$= (a_1 + b_1) x + (a_2 + b_2) x^2 \in W$

same idea for scalar. mult.

so W is a subspace.

Examples of subsets that are not subspaces.

The set of all 2×2 matrices whose $\boxed{\text{traces}}$ are integers.

$$tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d \qquad tr\begin{bmatrix} 1/3 & 1/2 \\ 0 & 5/3 \end{bmatrix} = 2$$

$$\frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ \pi & 2/3 \end{bmatrix}$$

fails the closure condi. under scalar mult.

The set of all degree 2 (not less or equal to two!) polynomials with real number coefficients.

P_2 $\qquad 0(1+2x+5x^2)$

Quiz Question 2. Identify which of the following is a vector space.

A. The set of all 3×3 matrices with integer traces;

B. The set of all linear functions with non-zero slopes (namely the set of all functions of the form $mx + b$ with $m \neq 0$);

C. The set of all 2×2 matrices with real number entries whose determinants are zero;

D. The set of all polynomials of the form $ax^3 + b$ with a and b being any real numbers.

$$\det(A+B) \;\cancel{=}\; \underset{\underset{0}{\parallel}}{\det(A)} + \underset{\underset{0}{\parallel}}{\det(B)}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$2 - x + 5x^2 = 0$

Linear dependence/independence, basis. Examples.

Determine whether $1 - 2x + 3x^2$, $x - x^2$ and $3 - 8x + 11x^2$ are linearly independent.

$v_1 \qquad v_2 \qquad v_3$

$a_1 v_1 + a_2 v_2 + a_3 v_3 = \vec{0}$

if only solution is $a_1 = a_2 = a_3 = 0$ then independent

$\underline{a_1}(1 - 2x + 3x^2) + a_2(x - x^2) + a_3(3 - 8x + 11x^2) = 0$

$$\begin{aligned} 1 \cdot a_1 + 0 \cdot a_2 + 3 \cdot a_3 &= 0 \\ -2a_1 + 1 \cdot a_2 - 8a_3 &= 0 \\ 3a_1 - 1 \cdot a_2 + 11a_3 &= 0 \end{aligned} \rightarrow \underline{\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -8 \\ 3 & -1 & 11 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

Repeat the above for $\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$, $\begin{bmatrix} 5 & -1 \\ 2 & 10 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$.

Quiz Question 3. Determine whether the vectors $1 - 2x$, x^2 and $x + x^2$ of $\mathcal{P}_2$ are linearly independent.

A. They are dependent today, but will be independent tomorrow;

B. They are linearly independent;

C. They are linearly dependent;

D. It is not possible to determine this.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$