① Direct Proof:

Statement: The $\underbrace{\text{sum of two even numbers}}_{P}$ $\underbrace{\text{is even}}_{Q}$

Step 1: Assume P (Hypotheses):

"Let $a=2K$ and $b=2m$"

Step 2: Use logical steps to derive Q (the conclusion):

"Then, $a+b=2K+2m=2(K+m)$.
Since $(k+m)$ is an integer, $a+b$ is even"

> **Answer:**
> Let $a=2K$ and $b=2m$, where K and m are integers.
> Then, $a+b=2K+2m=2(K+m)$.
> Since $(K+m)$ is an integer, $a+b$ is even.
> Therefore, the sum of two even numbers is even

② Proof by Contrapositive:

Statement: If $\underbrace{n^2 \text{ is even}}_{P}$, $\underbrace{\text{then } n \text{ is even}}_{Q}$

Step 1: Formulate contrapositive statement:

"If $\underbrace{n \text{ is odd}}_{\neg Q}$, $\underbrace{\text{then } n^2 \text{ is odd}}_{\neg P}$"

Step 2: Assume $\neg Q$ (the negation of the conclusion)

"Assume n is odd, so $n=2K+1$ for some integer k"

Step 3: Deduce $\neg P$ (the negation of the hypothesis:

"Then, $n^2=(2K+1)^2=4K^2+4K+1=2(2K^2+2K)+1$, which is odd

Step 4: Conclude that $P \rightarrow Q$ is true since $\neg Q \rightarrow \neg P$ is established

"Since the contrapositive is true, the original statement is true."

> **Answer:**
> Assume n is odd, so $n=2K+1$ for some integer K.
> Then, $n^2=(2K+1)^2=4K^2+4K+1=2(2K^2+2K)+1$, which is odd.
> Since the contrapositive is true, the orignal statement is true.

③ Proof by Contradiction:

Statement: $\sqrt{2}$ is irrational

Step 1: Assume $\neg S$ (the negation of the statement to be proven; assume the contrary):

"Suppose $\sqrt{2}$ is rational"

Step 2: Derive a Contradiction:

"Then $\sqrt{2}$ can be expressed as a fraction $\frac{a}{b}$ in lowest terms (where a and b are integers with no common factors.

Squaring both sides: $2=\frac{a^2}{b^2}$, which implies $a^2=2b^2$.

Hence, a^2 is even, so b is even. Let $a=2K$ for some integer K.

Substitute back:
$(2K)^2=2b^2 \rightarrow 4K^2=2b^2 \rightarrow b^2=2K^2$.

Thus, b^2 is even, so b is even.

But if both a and b are even, they have a common factor of 2, contradicting the assumption that $\frac{a}{b}$ is in lowest terms

Step 3: Conclude that S must be true:

"Since the assumption that $\sqrt{2}$ is rational leads to a contradiction, $\sqrt{2}$ is irrational"

> **Answer:**
> Suppose $\sqrt{2}$ is rational.
> Then, $\sqrt{2}$ can be expressed as a fraction $\frac{a}{b}$ in lowest terms (where a and b are integers with no common factors).
> Squaring both sides: $2=\frac{a^2}{b^2}$, which implies $a^2=2b^2$.
> Hence, a^2 is even, so a must be even. Let $a=2K$ for some integer K.
> Substitute back: $(2K)^2=2b^2 \rightarrow 4K^2=2b^2 \rightarrow b^2=2K^2$.
> Thus, b^2 is even, so b is even.
> But if both a and b are even, they have a common factor of 2, contradicting the assumption that $\frac{a}{b}$ is in lowest terms.
> Since the assumption that $\sqrt{2}$ is rational leads to a contradiction, $\sqrt{2}$ is irrational

④ Proof by Cases:

Statement: For any integer n, n^2 is either of the form $4K$ or $4k+1$ for some integer k

Step 1: Identify all possible cases that exhaust the possibilities (Case 1, Case 2, etc.):

1. Case 1: n is even
2. Case 2: n is odd

Step 2: Prove the statement for each individual case:

Case 1:

"Let $n=2m$.
Then, $n^2=(2m)^2=4m^2$, which is of the form $4K$ (with $K=m^2$)"

Case 2:

"Let $n=2m+1$.
Then, $n^2=(2m+1)^2=4m^2+4m+1=4(m^2+m)+1$, which is of the form $4K+1$ (with $K=m^2+m$)"

Step 3: Conclude that the statement is universally true:

"In both cases, n^2 fits one of the forms, Proving the statement"

> **Answer:**
> Case 1: n is even:
> Let $n=2m$.
> Then, $n^2=(2m)^2=4m^2$, which is of form $4K$ (with $k=m^2$)
> Case 2: n is odd:
> Let $n=2m+1$
> Then, $n^2=(2m+1)^2=4m^2+4m+1=4(m^2+m)+1$, which is of form $4K+1$ (with $K=m^2+m$).
> In both cases, n^2 fits one of the forms, proving the statement.