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Years of research and development have led to this moment: the release of dark mode (beta). Enable it now i Profile" tab. Happy April 1st from the zyBooks team :)

6.1 Linear transformations between Euclidean spaces

Section summary

In general, a function maps an element of one vector space to an element of another space. A mapping is also referred to as a transformation and can be used to manipulate an image. If an image needs to be enlarged or reduced, each vector in the original image can be scaled. An image can be rotated using a similar principle. These transformations can often be represented in terms of matrix multiplication. Some linear transformations $T(\mathbf{x})$ on vectors can be represented by matrix multiplication. A transformation represented by matrix multiplication has both properties of a linear transformation. The *standard matrix* for a linear transformation can be found using the standard basis of the domain.

In this section, students will:

- *Identify terminology associated with transformations.*
- *Determine if a transformation is linear.*
- *Use matrices to express linear transformations.*
- *Identify the standard matrix for a linear transformation.*
- *Use properties of matrix transformations.*
- *Use the standard basis to find a standard matrix.*

Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

A transformation generalizes the concept of a function. A ***transformation*** T from $\mathbb{R}^n$ to $\mathbb{R}^m$, denoted by $T : \mathbb{R}^n \to \mathbb{R}^m$, is a rule that assigns an element of a vector space $\mathbb{R}^n$ to an element of another vector space $\mathbb{R}^m$. The ***domain*** of a transformation is the set of all inputs. The ***codomain*** of a transformation is the vector space that contains the set of all outputs, in this case $\mathbb{R}^m$. Given a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbb{R}^n$ is the domain and $\mathbb{R}^m$ is the codomain. A transformation T is called an ***operator*** if the domain and codomain are the same.

Transformations are usually defined by vector-valued functions as shown below.

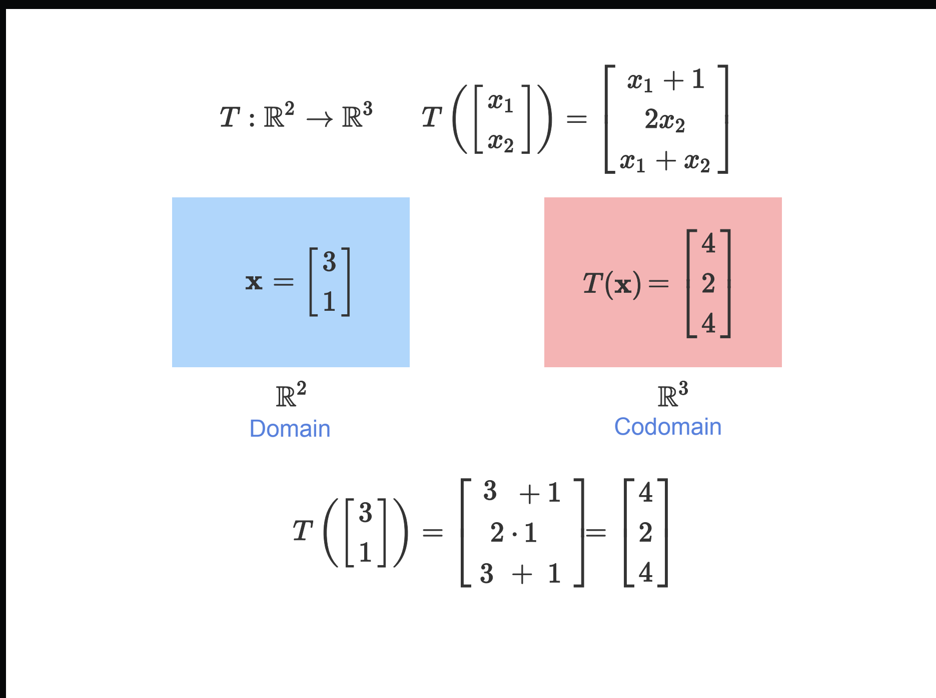
$$T(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

or

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$$

The vector $T(\mathbf{x}) \in \mathbb{R}^m$ for a given $\mathbf{x} \in \mathbb{R}^n$ in the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the ***image*** of $\mathbf{x}$ under T. The ***pre-image*** of $\mathbf{v} \in \mathbb{R}^m$ under the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of all elements $\mathbf{u} \in \mathbb{R}^n$ such that $T(\mathbf{u}) = \mathbf{v}$. The set of all elements of the form $T(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$ for a given transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the ***range*** of T.

PARTICIPATION ACTIVITY 6.1.1: Domain, codomain, and image of a transformation.



Animation content:

Static figure: The domain and codomain of a linear transformation.
Step 1: A transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ maps an element of $\mathbb{R}^2$ to an element of $\mathbb{R}^3$.
$T : \mathbb{R}^2 \to \mathbb{R}^3$. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 1 \\ 2x_2 \\ x_1 + x_2 \end{bmatrix}$. $\mathbf{x}$ is in $\mathbb{R}^2$. $T(\mathbf{x})$ is in $\mathbb{R}^3$.
Step 2: $\mathbb{R}^2$ is the domain and $\mathbb{R}^3$ is the codomain of T.
Step 3: To find the image of $\mathbf{x}$ under T, $\mathbf{x}$ is substituted into the vector-valued function.
The image of $\mathbf{x}$ under T is in the codomain of T. $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} (3) + 1 \\ 2(1) \\ (3) + (1) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}.$$

Animation captions:

1. A transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ maps an element of $\mathbb{R}^2$ to an element of $\mathbb{R}^3$.

2. $\mathbb{R}^2$ is the domain and $\mathbb{R}^3$ is the codomain of T.
3. To find the image of $\mathbf{x}$ under T, $\mathbf{x}$ is substituted into the vector-valued function. The image of $\mathbf{x}$ under T is in the codomain of T.

PARTICIPATION ACTIVITY | 6.1.2: Transformation of a vector.

Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be a transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 - x_4 \end{bmatrix}$$

Match each item with the correct description.

How to use this tool

$\mathbb{R}^4$ | $\mathbb{R}^2$ | $\begin{bmatrix} x_1 + x_2 \\ x_3 - x_4 \end{bmatrix}$ | $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

	Domain of T
	Codomain of T
	Range of T
	Image of $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ under T

Reset

PARTICIPATION ACTIVITY | 6.1.3: Evaluating transformations.

Evaluate each transformation.

1)

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T(\mathbf{x}) = \begin{bmatrix} x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \text{a} \\ 2 \end{bmatrix}, \text{where}$$

a =

Check Show answer

2)

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

$$T(\mathbf{x}) = \begin{bmatrix} 2x_1 + x_2 + x_3 \\ x_1 - x_2 - x_3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \text{b} \\ -1 \end{bmatrix}, \text{where}$$

b =

Check Show answer

3)

$$T : \mathbb{R}^2 \to \mathbb{R}^3$$

$$T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ x_2 - 3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ \text{c} \\ -2 \end{bmatrix}, \text{where}$$

c =

Check Show answer

CHALLENGE ACTIVITY | 6.1.1: Domain, codomain, and image of a linear transformation.

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Start

T: R [Ex: 1] $\to$ R [Ex: 1]

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + 5x_2 \\ -4x_2 + 4x_1 \\ -3x_1 + 5x_2 \end{bmatrix}$$

1 2 3

Check Next

Linear transformations

A ***linear transformation*** is a transformation that preserves the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ in the domain of T and all scalars r.

Property 6.1.1: Properties of linear transformations.

a. Additivity property:
$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
b. Homogeneity property: $T(r\mathbf{v}) = rT(\mathbf{v})$

The properties of vector algebra can be used to determine if a transformation is or is not a linear transformation.

Example 6.1.1: Determining whether a transformation is a linear transformation.

Use the properties of linear transformations to determine whether each of the following transformations is a linear transformation.

a. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 \end{bmatrix}$
b. $S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 x_2 \end{bmatrix}$

Solution

PARTICIPATION ACTIVITY | 6.1.4: Determining whether a transformation is linear.

Let T and S be transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $T(\mathbf{x}) = \begin{bmatrix} \sqrt[3]{x_1^3 - x_2^3} \\ 2x_1 \end{bmatrix}$ and $S(\mathbf{x}) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$. Answer true or false.

1) T preserves the additivity property.
- True
- False

2) T preserves the homogeneity property.
- True
- False

3) T is a linear transformation.
- True
- False

4) S preserves the additivity property.
- True
- False

5) S preserves the homogeneity property.
- True
- False

6) S is a linear transformation.
- True
- False

Matrix transformation

When the functions $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ that define a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ are linear, $T(\mathbf{x}) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$ can be written as

$$T(\mathbf{x}) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$

which can also be expressed using matrix multiplication as

$$T(\mathbf{x}) = A\mathbf{x}$$
$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Thus, a system of linear equations can be seen as a linear transformation that maps a column vector $\mathbf{x}$ in $\mathbb{R}^n$ to a column vector in $\mathbb{R}^m$ by multiplying $\mathbf{x}$ by an $m \times n$ matrix A. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ that can be defined as $T(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix is called a ***matrix transformation***. The matrix A that defines the transformation T is called the ***standard matrix*** for T.

PARTICIPATION ACTIVITY 6.1.5: Finding the standard matrix.

Find the standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(\mathbf{x}) = \begin{bmatrix} 4x_1 + 6x_2 \\ 2x_1 - 7x_2 \\ -3x_1 + 5x_2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4 & 6 \\ 2 & -7 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} 4 & 6 \\ 2 & -7 \\ -3 & 5 \end{bmatrix}$$

$$T(\mathbf{x}) = A\mathbf{x}$$

$$T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 4 & 6 \\ 2 & -7 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -10 \\ 25 \\ -21 \end{bmatrix}$$

Animation content:

Static figure: Finding the standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \begin{bmatrix} 4x_1 + 6x_2 \\ 2x_1 - 7x_2 \\ -3x_1 + 5x_2 \end{bmatrix}$.

Step 1: $T(\mathbf{x})$ is expressed as a matrix product. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4 & 6 \\ 2 & -7 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

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Step 2: A is the standard matrix of the linear transformation T. $T(\mathbf{x}) = A\mathbf{x}$. $A = \begin{bmatrix} 4 & 6 \\ 2 & -7 \\ -3 & 5 \end{bmatrix}$.

Step 3: The standard matrix is used to find images. $T\left(\begin{bmatrix}2\\-3\end{bmatrix}\right) = \begin{bmatrix}4 & 6\\2 & -7\\-3 & 5\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}-10\\25\\-21\end{bmatrix}$

.

Animation captions:

1. $T(\mathbf{x})$ is expressed as a matrix product.
2. A is the standard matrix of the linear transformation T.
3. The standard matrix is used to find images.

PARTICIPATION ACTIVITY | 6.1.6: Finding the standard matrix.

Match each transformation with the correct standard matrix for the transformation.

How to use this tool

$\begin{bmatrix}1 & -2 & 4\\2 & 1 & -1\end{bmatrix}$ $\begin{bmatrix}1 & 1\\3 & -3\\-2 & 1\end{bmatrix}$ $\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$ $\begin{bmatrix}3 & -1\\1 & 2\end{bmatrix}$

$T : \mathbb{R}^2 \to \mathbb{R}^2, T(\mathbf{x}) = \begin{bmatrix}2x_1\\3x_2\end{bmatrix}$

$T : \mathbb{R}^2 \to \mathbb{R}^2, T(\mathbf{x}) = \begin{bmatrix}3x_1 - x_2\\x_1 + 2x_2\end{bmatrix}$

$T : \mathbb{R}^2 \to \mathbb{R}^3, T(\mathbf{x}) = \begin{bmatrix}x_1 + x_2\\3x_1 - 3x_2\\-2x_1 + x_2\end{bmatrix}$

$T : \mathbb{R}^3 \to \mathbb{R}^2, T(\mathbf{x}) = \begin{bmatrix}x_1 - 2x_2 + 4x_3\\2x_1 + x_2 - x_3\end{bmatrix}$

Reset

The properties of matrix transformations are similar to the properties of a linear transformation.

Property 6.1.2: Properties of matrix transformations.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix transformation. Then for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalars $a \in \mathbb{R}$.

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(a\mathbf{v}) = aT(\mathbf{v})$
3. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$

Proof

PARTICIPATION ACTIVITY | 6.1.7: Properties of matrix transformations.

Determine whether each statement is true or false.

1) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $T(\mathbf{0}) = \mathbf{0}$, then T is a matrix transformation.
- True
- False

2) If T is a matrix transformation, then T is a linear transformation.
- True
- False

3) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ x_1 + x_3 \end{bmatrix}$ is a matrix transformation.
- True
- False

Finding a standard matrix using the standard basis

The standard matrix is uniquely determined by the images of the standard basis vectors under a linear transformation T. The image of each standard basis vector forms the columns of the standard matrix as stated in the theorem below.

Theorem 6.1.1: Standard matrix of a linear transformation.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then a matrix A exists such that $T(\mathbf{x}) = A\mathbf{x}$ where the columns of A are formed by the image of each standard basis vector in $\mathbb{R}^n$,

$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$.

PARTICIPATION ACTIVITY | 6.1.8: Proof: The standard matrix of a linear transformation.

Full screen

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then a matrix A exists such that $T(\mathbf{x}) = A\mathbf{x}$ where the colu are formed by the image of each standard basis vector in $\mathbb{R}^n$, $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$.

How to use this tool

Unused blocks

Thus, $T(\mathbf{x}) = A\mathbf{x}$, where $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \dots \quad T(\mathbf{e}_n)]$.

$= A\mathbf{x}$

Let $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.

The proof will show that $T(\mathbf{x}) = A\mathbf{x}$.

$$= \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n)$

Since the basis vector $\mathbf{e}_i$ has 0 in every entry except the i^{th} entry of

Proof

Move blocks here

0 of 7 blocks correct

CHALLENGE ACTIVITY | 6.1.2: Proof: The uniqueness of a standard matrix.

Full screen

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Theorem: The standard matrix for any linear transformation T is unique.

How to use this tool

Unused blocks

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Assume two standard matrices A and B for T exist, so that $T(\mathbf{x}) = A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. The proof will show that $A = B$.

Since $A\mathbf{e}_i = B\mathbf{e}_i$ for $i = 1, 2, \dots n$:

$$\begin{aligned}[A\mathbf{e}_1 \quad A\mathbf{e}_2 \quad \cdots \quad A\mathbf{e}_n] &= [B\mathbf{e}_1 \quad B\mathbf{e}_2 \quad \cdots \quad B\mathbf{e}_n] \\ &= B[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n] \\ &= B \cdot I_n\end{aligned}$$

Matrix A can be written in terms of the identity matrix and standard basis vectors as follows.

$A = A \cdot I_n = A[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n] = [A\mathbf{e}_1 \quad A\mathbf{e}_2 \quad \cdots \quad A\mathbf{e}_n]$

The assumption that $A\mathbf{x} = B\mathbf{x}$ must be true for standard basis vectors. That is, $A\mathbf{e}_i = B\mathbf{e}_i$ for all $i = 1, 2, \dots, n$.

Proof

Move blocks here

Since $T(\mathbf{x}) = A\mathbf{x} = B\mathbf{x}$ implies $A = B$, the standard matrix for

Check

PARTICIPATION ACTIVITY

6.1.9: Finding the standard matrix using the standard basis.

Find the standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T(\mathbf{x}) = \begin{bmatrix} 4x_1 + 6x_2 \\ 2x_1 - 7x_2 \\ -3x_1 + 5x_2 \end{bmatrix}$$

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4(1) + 6(0) \\ 2(1) - 7(0) \\ -3(1) + 5(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$$

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4(0) + 6(1) \\ 2(0) - 7(1) \\ -3(0) + 5(1) \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 5 \end{bmatrix}$$

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)] = \begin{bmatrix} 4 & 6 \\ 2 & -7 \\ -3 & 5 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 4 & 6 \\ 2 & -7 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \\ -14 \end{bmatrix}$$

Animation content:

Static figure: Finding the standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \begin{bmatrix} 4x_1 + 6x_2 \\ 2x_1 - 7x_2 \\ -3x_1 + 5x_2 \end{bmatrix}$ using standard basis vectors.

Step 1: The standard matrix can also be obtained by determining the image of each standard basis vector.

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4(1) + 6(0) \\ 2(1) - 7(0) \\ -3(1) + 5(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}.$$

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4(0) + 6(1) \\ 2(0) - 7(1) \\ -3(0) + 5(1) \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 5 \end{bmatrix}.$$

Step 2: The image of each standard basis vector forms the columns of the standard matrix.

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)] = \begin{bmatrix} 4 & 6 \\ 2 & -7 \\ -3 & 5 \end{bmatrix}.$$

Step 3: The standard matrix can now be used to find images.

$$T\left(\begin{bmatrix}3\\-1\end{bmatrix}\right)=\begin{bmatrix}4&6\\2&-7\\-3&5\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix}=\begin{bmatrix}6\\13\\-14\end{bmatrix}.$$

Animation captions:

1. The standard matrix can also be obtained by determining the image of each standard basis vector.
2. The image of each standard basis vector forms the columns of the standard matrix.
3. The standard matrix can now be used to find images.

Example 6.1.2: Finding the standard matrix for a projection of a vector along a line through the origin.

Find the standard matrix for the projection transformation of a vector along a line through the origin in terms of θ, the angle the line makes with respect to the horizontal axis x_1. Use the standard matrix to find the projection of $\mathbf{v}=\begin{bmatrix}4\\8\end{bmatrix}$ onto the line that makes a $30°$ angle with the x_1-axis.

Solution

PARTICIPATION ACTIVITY | 6.1.10: Finding a standard matrix for a projection of a vector onto a line that passes through the origin.

Let T be an operator on $\mathbb{R}^2$ that projects a vector onto a line L, which makes an angle of $\theta=60°$ with respect to the horizontal axis x_1.

1) The unit vector $\mathbf{v}$ in the direction of L is $\begin{bmatrix}0.500\\ \text{a}\end{bmatrix}$, where a = _____.

Ex: 1.234

Check Show answer

2) $T(\mathbf{e}_1)=\begin{bmatrix}\text{b}\\0.433\end{bmatrix}$, where b = _____.

Ex: 1.234

Check Show answer

3) $T(\mathbf{e}_2) = \begin{bmatrix} \mathrm{c} \\ 0.750 \end{bmatrix}$, where c = _____.

Ex: 1.234

Check Show answer

4) Use the standard matrix to find the projection of $\mathbf{v} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ onto the line which makes a $60°$ angle with the horizontal axis. The projection is $\begin{bmatrix} -0.049 \\ \mathrm{d} \end{bmatrix}$, where d = _____.

Ex: 1.234

Check Show answer

Additional exercises

EXERCISE 6.1.1: Linear transformations between Euclidean spaces.

Determine whether each statement is true or false. Justify each answer or provide a counterexample when appropriate.

(a) For a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the codomain is the space $\mathbb{R}^m$ and the range the set of elements formed by $T(\mathbf{x})$.

(b) For the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$, the codomain and the range are both $\mathbb{R}^2$.

(c) A linear transformation satisfies the property $T(r(\mathbf{u} + \mathbf{v})) = rT(\mathbf{u}) + rT(\mathbf{v})$.

(d) Some matrix transformations $T(\mathbf{x}) = A\mathbf{x}$ are not linear transformations.

(e) If A is the 5×2 matrix corresponding to $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is $\mathbb{R}^2$

(f) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a transformation given by $T(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$ for some $\mathbf{x}_0 \in \mathbb{R}^3$ where $\mathbf{x}_0 \neq \mathbf{0}$. Then T is *not* a linear transformation.

EXERCISE 6.1.2: Evaluating transformations.

Consider the transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 3x_2x_4 \\ x_1 + x_2 + x_3 \\ x_5^2 - x_3^2 \end{bmatrix}.$$

(a) What is the image of $\begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ under T?

(b) What is the domain of T?

(c) What is the codomain of T?

EXERCISE 6.1.3: Linear and matrix transformations.

Determine if the transformation given is a linear transformation. If the transformation is linear, give the standard matrix for the corresponding matrix transformation.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 - x_3 \\ 2x_1 + x_3 \end{bmatrix}$

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{x_1} \\ -\sqrt{x_2} \end{bmatrix}$.

EXERCISE 6.1.4: Finding a projection of a vector onto a line that passes through the origin.

Let T be the linear operator on $\mathbb{R}^2$ that projects a vector onto the line that forms a $45°$ angle with the x_1-axis.

(a) What is the standard matrix for the projection transformation T?

(b) Find the projection of the vector $\mathbf{v} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$ onto a line that goes through the origin and makes a $45°$ angle with the x_1-axis.