Lecture Notes for 3/18/2025

5.2 Subspaces (continued)

$\mathbb{R}_{m\times n}$, $\mathcal{P}_n$

5.3 Coordinatization

Remaining topic for Section 5.2: if a subspace is the span of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$, how do we find a basis (and the dimension) for this subspace?

$\mathbb{R}^3$: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

Standard basis for a general vector space such as $\mathbb{R}_{m\times n}$ or $\mathcal{P}_n$.

Example 1. A standard basis for $\mathcal{P}_2$ is $\boxed{\{1, x, x^2\}}$, or $\{x^2, x, 1\}$. The coordinate vector of $-3 + 4x - 7x^2$ under the basis $\{1, x, x^2\}$ is $\begin{bmatrix}-3\\4\\-7\end{bmatrix}$, but under the basis $\{x^2, x, 1\}$ is $\begin{bmatrix}-7\\4\\-3\end{bmatrix}$.

$\begin{bmatrix}2\\3\\-5\end{bmatrix} \longrightarrow 2+3x-5x^2$

The coordinate vector under a given basis.

Example 2. A standard basis for $\mathbb{R}^3$ is $\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\}$ of $\mathbb{R}^3$. The coordinate vector of $\mathbf{v} = \begin{bmatrix}2\\-4\\-2\end{bmatrix}$ is the same as itself under the standard basis since

$$2\begin{bmatrix}1\\0\\0\end{bmatrix} - 4\begin{bmatrix}0\\1\\0\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\-4\\-2\end{bmatrix}$$

$\mathbb{R}_{2\times 3}$ $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

Example 3. A standard basis for $\mathbb{R}_{2\times 2}$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. What is the coordinate vector of $\begin{bmatrix} -2 & 0 \\ 3 & 8 \end{bmatrix}$ under this basis?

$$\begin{bmatrix} -2 & 0 \\ 3 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 \\ 0 \\ 3 \\ 8 \end{bmatrix} : \quad -2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 8\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 3 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 1 \\ -7 \\ 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & 1 \\ -7 & 3 \end{bmatrix}$$

What if we choose $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \mathcal{B}$ to be the ~~standard~~ basis?

$$\begin{bmatrix} 2 & 5 \\ 7 & -8 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 7 \\ 5 \\ -8 \end{bmatrix}_{\mathcal{B}} \Rightarrow 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 7\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 8\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Quiz Question 1. What is the coordinate vector of $2 - x + 4x^2 - 7x^3$ under the standard basis $\{x^3, x^2, x, 1\}$ of $\mathcal{P}_3$?

A. $\begin{bmatrix} 2 \\ -1 \\ 4 \\ -7 \end{bmatrix}$; B. $\begin{bmatrix} -7 \\ 4 \\ -1 \\ 2 \end{bmatrix}$; C. $\begin{bmatrix} -7 & 4 & -1 & 2 \end{bmatrix}$; D. $\begin{bmatrix} -7x^3 \\ 4x^2 \\ -x \\ 2 \end{bmatrix}$.

Quiz Question 2. What is the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v} = \begin{bmatrix} 2 & -3 \\ 4 & 7 \end{bmatrix}$ under the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$[v]_{B}$ ✓

A. $\begin{bmatrix} 2 & -3 \\ 4 & 7 \end{bmatrix}$; B. $\begin{bmatrix} -3 \\ 2 \\ 4 \\ 7 \end{bmatrix}$; C. $\begin{bmatrix} 2 \\ -3 \\ 7 \\ 4 \end{bmatrix}$; D. $\begin{bmatrix} 2 \\ -3 \\ 4 \\ 7 \end{bmatrix}$.

Being able to write out the coordinate vector of a vector under a standard basis can help us to solve the remaining question of Section 5.2: if a subspace is the span of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$, how do we find a basis (and the dimension) for this subspace?

Example 4. Let W be the subspace of $\mathcal{P}_3$ with a spanning set consisting of $1-3x+x^2$, $2-6x+2x^2$, $x+2x^2+x^3$ and $1-x+4x^2+2x^3$. Find a basis for W and determine the dimension of W.

The coordinate vectors of the spanning set under the standard basis $\{1, x, x^2, x^3\}$ are

$$\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -6 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \begin{matrix} 1 \\ -1 \\ 4 \\ 2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ -3 & -6 & 1 & -1 \\ 1 & 2 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[-R_1+R_3 \to R_3]{3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Example 5. Let W be the subspace of $\mathbb{R}_{2\times 2}$ that is spanned by the vectors $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. Find a basis of W and the dimension of W.

$$\therefore \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 1 & 2 & 3 & -1 \end{bmatrix} \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 1 & -1 \end{bmatrix}$$

basis $= \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \right\}$ dim $= 3$

Quiz Question 3. Let W be the subspace of $\mathcal{P}_2$ with a spanning set consisting of $1-3x+x^2$, $2-6x+2x^2$, $-1+x+2x^2$, $1-x+4x^2$ and $3+x^2$. In order to find a basis for W and to determine the dimension of W, we need to perform the Gaussian elimination method on which of the following matrices?

A. Don't choose this one: anyone choosing this one is not attending class and is cheating!

B. $\begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ -3 & -6 & 1 & -1 & 0 \\ 1 & 2 & 2 & 4 & 1 \end{bmatrix}$;

C. $\begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ -3x & -6x & 1x & -1x & 0 \\ x^2 & 2x^2 & 2x^2 & 4x^2 & x^2 \end{bmatrix}$;

D. $\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ -1 & 1 & 2 \\ 1 & -1 & 4 \\ 3 & 0 & 1 \end{bmatrix}$;

Coming back to the coordinate vector question. We did that using a standard basis, and that was pretty easy (and useful). But what if we do not have a standard basis (remember life is not always easy)? For example, we know that

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is also a basis for $\mathbb{R}^3$, so we can also ask what is the coordinate vector of $\mathbf{v} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$ under this basis $\mathcal{B}$?

Answer:

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}_{\mathcal{B}}$$

Since

$$2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} . = v$$

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$M_B \cdot$

How do you find $[\mathbf{v}]_{\mathcal{B}}$? If $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, it means

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}.$$

$Ax = b$

$A^{-1}Ax = A^{-1}b$

$x = A^{-1}b$

$M_B \cdot [v]_B = v$

So we are solving the equation system $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$, which can be solved as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

$[v]_B = M_B^{-1} \cdot v$

$\{1, x, x^2\}$

Similarly, if we know that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1 - x, x + x^2, 2x - x^2\}$ is a basis for $\mathcal{P}_2$, then how do we find the coordinate vector of $\mathbf{v} = 1 + 9x + x^2$ under this basis?

Again, keep in mind that the coordinate vector $[\mathbf{v}]_\mathcal{B} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ means

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = a_1(1 - x) + a_2(x + x^2) + a_3(2x - x^2) = 1 + 9x + x^2.$$

It is most helpful if we translate the above equation using the coordinate vectors under a standard basis, for example $\{1, x, x^2\}$.

$$a_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix}.$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}}_{M_B} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix}$$

$$[v]_B = M_B^{-1} \cdot v = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

Solving it you should get

$$[\mathbf{v}]_\mathcal{B} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

You can verify that $1(1 - x) + 4(x + x^2) + 3(2x - x^2) = 1 + 9x + x^2$.

In general, finding $[\mathbf{v}]_\mathcal{B}$ is just solving the equation

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$$

where $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$. Once we replace each vector in the equation by their corresponding coordinate vector under a standard basis, it becomes a standard linear equation system.

Quiz Question 4. Given that

$$\mathcal{B} = \{1 - 2x + 3x^2 - x^3, 2x^2 + x^3, 3 + x^2, x + 4x^3\}$$

$\{1, x, x^2, x^3\}$

is a basis for $\mathcal{P}_3$, in order to find $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ for $\mathbf{v} = 1 - x + 2x^2 + 5x^3$, which equation below will lead to the correct answer?

A. Huh? I don't believe there is such an equation.

B. $a_1 \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix}$

C. $a_1 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix}$

D. $a_1 \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix}$