$$\begin{bmatrix} 2 & * & * & * \\ 0 & -3 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\det = 2(-3)\cdot 1(-5)$$
$$= 30$$

Lecture Notes for 2/18/2025

3.4 Properties of determinants

Property 3.4.1: How elementary row operations on a square matrix affect the determinant of the matrix.

Let A be an $n \times n$ square matrix and k be any scalar, then the following statements are true.

(1) If B is obtained from A by performing a row operation of the form $R_i \longleftrightarrow R_j$, then $\det(B) = -\det(A)$.

(2) If B is obtained from A by a row operation of the form $kR_j \longrightarrow R_j$, then $\det(B) = k\det(A)$.

(3) If B is obtained from A by performing a row operation of the form $kR_j + R_i \longrightarrow R_i$, then $\det(B) = \det(A)$.

Note 1: The above statements hold true if the row operations are replaced by column operations. The reason being that the cofactor expansion formula also applies to columns. The proof outlined next can be easily modified by replacing rows with columns.

Note 2: We know that the determinant of a triangular matrix is simply the product of its entries on the diagonal line, so we can use the above results to calculate the determinant of a matrix by using row operations to bring the matrix to an echelon form (which would be triangular since the matrix is square).

Example. Compute the determinant of the following matrix.

$$\begin{array}{c} \begin{matrix}2 & -8 & 4 & 0\end{matrix} \\ \begin{bmatrix}1 & -4 & 2 & 0\\ 0 & 0 & -2 & 1\\ 0 & 2 & -5 & -2\\ -2 & 8 & 2 & 1\end{bmatrix}\end{array} \xrightarrow{2R_1+R_4\to R_4} \begin{vmatrix}1 & -4 & 2 & 0\\ 0 & 0 & -2 & 1\\ 0 & 2 & -5 & -2\\ 0 & 0 & 6 & 1\end{vmatrix}$$

$$\xrightarrow{R_2\leftrightarrow R_3} -\begin{vmatrix}1 & -4 & 2 & 0\\ 0 & 2 & -5 & -2\\ 0 & 0 & -2 & 1\\ 0 & 0 & 6 & 1\end{vmatrix} \xrightarrow{3R_3+R_4\to R_4} -\begin{vmatrix}1 & -4 & 2 & 0\\ 0 & 2 & -5 & -2\\ 0 & 0 & -2 & 1\\ 0 & 0 & 0 & 4\end{vmatrix}$$

$$= -1\cdot 2\cdot(-2)\cdot 4 = 16$$

$$\begin{bmatrix}2 & 3 & 0 & -1\\ 0 & 1 & -2 & 1\\ 4 & 2 & 3 & -2\\ -2 & 0 & 2 & 1\end{bmatrix} \xrightarrow[1\cdot R_1+R_4\to R_4]{-2R_1+R_3\to R_3} \begin{vmatrix}2 & 3 & 0 & -1\\ 0 & 1 & -2 & 1\\ 0 & -4 & 3 & 0\\ 0 & 3 & 2 & 0\end{vmatrix}$$

$$\xrightarrow{C_2\leftrightarrow C_4} -\begin{vmatrix}2 & -1 & 0 & 3\\ 0 & 1 & -2 & 1\\ 0 & 0 & 3 & -4\\ 0 & 0 & 2 & 3\end{vmatrix}\begin{matrix}\\ \\ 2\\ -3\end{matrix} \xrightarrow[-3R_4\to R_4]{2R_3\to R_3} -\frac{1}{(-6)}\begin{vmatrix}2 & -1 & 0 & 3\\ 0 & 1 & -2 & 1\\ 0 & 0 & 6 & -8\\ 0 & 0 & -6 & -9\end{vmatrix}$$

$\cancel{-6\times}$

$$\xrightarrow{R_3+R_4\to R4} +\frac{1}{6}\begin{vmatrix}2 & -1 & 0 & 3\\ 0 & 1 & -2 & 1\\ 0 & 0 & 6 & -8\\ 0 & 0 & 0 & -17\end{vmatrix} = \frac{1}{\cancel{6}}\cdot 2\cdot 1\cdot \cancel{6}\cdot(-17)$$

$$= -34$$

$$\det(A) \xrightarrow{2R_3\to R_3} 2\det(A)$$

Quiz Question 1. Find the determinant of the matrix

$$\begin{bmatrix} 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & -1 \\ -2 & 6 & 1 & 0 \\ 1 & -3 & 3 & -9 \end{bmatrix}$$

A. 35; B. -35; C. -14; D. 14.

$$\xrightarrow{R_1 \leftrightarrow R_4} -\begin{vmatrix} 1 & -3 & 3 & -9 \\ 0 & 2 & -2 & -1 \\ -2 & 6 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{vmatrix} \xrightarrow{2R_1 + R_3 \to R_3} -\begin{vmatrix} 1 & -3 & 3 & -9 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 7 & -18 \\ 0 & 1 & -1 & 2 \end{vmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{vmatrix} 1 & -3 & 3 & -9 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 7 & -18 \\ 0 & 2 & -2 & -1 \end{vmatrix} \longrightarrow$$

$$\begin{bmatrix} a_{11}=0 & a_{12}=0 & a_{13}=0 & a_{14}\neq 0 \\ & * & & \end{bmatrix} \qquad \det(A) = \underset{0}{a_{11}} C_{11} + \underset{0}{a_{12}} C_{12} + \underset{0}{a_{13}} C_{13} + \underset{0}{a_{14}} C_{14}$$

Some consequences of the properties (1)–(3) and the cofactor expansion formula.

• If A has a row of 0s or a column of 0s, then $\det(A) = 0$;

• If A is a triangular matrix, then $\det(A)$ is the multiplication of its entries on the diagonal line;

• If A has two identical rows or columns, then $\det(A) = 0$;

• If A has two rows or two columns that are scalar multiples of each other, then $\det(A) = 0$;

• $\det(AB) = \det(A)\det(B)$ (this actually requires additional proof by using the elementary matrices);

• The above can be extended to the multiplication of more than two matrices: $\det(A_1A_2\cdots A_k) = \det(A_1)\det(A_2)\cdots\det(A_k)$;

• $\det(\underline{kA}) = \underline{k^n\det(A)}$; $\boxed{\text{(NOT } \det(kA) = k\det(A)\text{!!!)}}$

• If A is invertible then $\det(A) \neq 0$ (and $\det(A^{-1}) = \frac{1}{\det(A)}$);

• If $\det(A) \neq 0$ then A is invertible.

$$AB \overset{?}{\neq} BA$$

$$\underline{\det(AB) = \det(A)\det(B)} = \det(BA)$$

$$\det(A \cdot A^{-1}) = \det(I) = 1$$

$$\det(A)\cdot \underline{\det(A^{-1})} = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

Examples.

1. If we know the determinant of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is 5, what is the determinant of the matrix $\begin{bmatrix} -2a & -2b & -2c \\ d & e & f \\ 3g & 3h & 3i \end{bmatrix}$? What about $\det(-2A)$?

$$\det(-2A) = (-2)^3 \cdot 5 = -40$$

$-2 \times R_1 \to R_1$
$3 \cdot R_3 \to R_3$

$$-2 \cdot 3 \cdot \det(A) = -2 \cdot 3 \cdot 5 = -30$$

2. If A has size 2×2 and $\det(A) = -3$, what is $\det(-4A)$?

$$\det(-4A) = (-4)^2 \det(A) = 16 \cdot (-3) = -48$$

3. If $\det(A) = 5$, $\det(B) = 20$, what is $\det(A^{-2}B^2)$?

$$A^{-2} = A^{-1} \cdot A^{-1}$$
$$B^2 = B \cdot B$$

$$\det(A^{-2}B^2) = \det(A^{-1})\det(A^{-1})\det(B)\det(B) = \frac{1}{5} \cdot \frac{1}{5} \cdot 20 \cdot 20 = 16$$

Quiz Question ~~2~~ 3. If $\det(A) = -5$ and A has size 3×3, then $\det(-2A) =$

A. 10; B. -10; C. -40; D. 40.

Quiz Question ~~3~~2. Which of the following statements is correct (A and B are both $n \times n$ matrices in these statements)?

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A. If $AB = 0$ (the $n \times n$ zero matrix), then $A = 0$ or $B = 0$;

B. If $\det(A) = \det(B)$, then $A = B$;

C. If $\det(AB) = 0$, then $\det(A) = 0$ or $\det(B) = 0$; $\det(A) \cdot \det(B)$

D. It is possible that $\det(A) = 0$ when A is invertible.

DO NOT ANSWER

The proof of the statements (1)–(3).

(1) Let $n \times n$ be the size of A. If we switch two adjacent rows of A, for example the first two rows of A and let B be the resulting matrix. By the cofactor expansion and using the first row of A, we have

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

where $C_{11} = (-1)^{1+1}M_{11}$, $C_{12} = (-1)^{1+2}M_{12}$, ..., $C_{1n} = (-1)^{1+n}M_{1n}$. Now we use the cofactor expansion and the second row of B, then we have

$$\det(B) = b_{21}C'_{21} + b_{22}C'_{22} + \cdots + b_{2n}C'_{2n}.$$

But $b_{21} = a_{11}$, $b_{22} = a_{12}$, ..., $b_{2n} = a_{1n}$, and $C'_{21} = (-1)^{2+1}M_{11} = -C_{11}$, $C'_{22} = (-1)^{2+2}M_{12} = -C_{12}$, ..., $C'_{2n} = (-1)^{2+n}M_{1n} = -C_{1n}$. Thus

$$\begin{aligned} \det(B) &= b_{21}C'_{21} + b_{22}C'_{22} + \cdots + b_{2n}C'_{2n} \\ &= a_{11}(-C_{11}) + a_{12}(-C_{12}) + \cdots + a_{1n}(-C_{1n}) \\ &= -\left(a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}\right) = -\det(A). \end{aligned}$$

The same argument can be used when any two adjacent rows are switched. If the two rows R_i and R_j are not adjacent, say there are $k \geq 1$ rows between them, then we can perform $2k+1$ switches between adjacent rows to achieve the effect of switching just R_i and R_j. This causes the determinant to change signs $2k+1$ times, which will still gives us $\det(B) = -\det(A)$.

• An immediate consequent of the result (1) is that if A has two identical rows, then $\det(A) = 0$. Why?

(2) Let B be the resulting matrix after the operation $kR_j \longrightarrow R_j$ is applied to A. A and B are identical except that the entries of the j-th row of B are ka_{j1}, ka_{j2}, ..., ka_{jn}. Let us choose the j-th row for the cofactor expansion to compute $\det(A)$ and $\det(B)$. By the cofactor expansion formula, we have

$$\det(A) = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn}.$$

Because A and B are identical except the j-th row, they share the same cofactors C_{j1}, ..., C_{jn}. It follows that

$$\begin{aligned} \det(B) &= (ka_{j1})C_{j1} + (ka_{j2})C_{j2} + \cdots + (ka_{jn})C_{jn} \\ &= k\,(a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn}) = k\det(A). \end{aligned}$$

This proves (2).

(3) Let B be the resulting matrix after the operation $kR_i + R_j \longrightarrow R_j$ is applied to A. Again A and B are identical except that the entries of the j-th row of B are $ka_{i1} + a_{j1}$, $ka_{i1} + a_{j1}$, ..., $ka_{i1} + a_{j1}$. Let us choose the j-th row for the cofactor expansion to compute $\det(A)$ and $\det(B)$. By the cofactor expansion formula, we have

$$\det(A) = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn}.$$

Because A and B are identical except the j-th row, they share the same cofactors C_{j1}, ..., C_{jn}. It follows that

$$\begin{aligned} \det(B) &= (ka_{i1} + a_{j1})C_{j1} + (ka_{i2} + a_{j2})C_{j2} + \cdots + (ka_{in} + a_{jn})C_{jn} \\ &= k\left(a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}\right) \\ &+ \left(a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn}\right). \end{aligned}$$

Because $a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = 0$ (why?), it follows that $\det(B) = \det(A)$.

Quiz Question 4. Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5,$$

find

$$\begin{vmatrix} d & e & f \\ a & b & c \\ \cancel{a} - 2g & \cancel{b} - 2h & \cancel{c} - 2i \end{vmatrix}.$$

A. -10; B. -5; C. 10; D. 5

1 $R_1 \leftrightarrow R_2$ $-$

2 $-2R_3 \to R_3$ -2

3 $R_2 + R_3 \to R_3$ no change.

$$\underset{E_2}{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 4 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$