Lecture Notes for 3/11/2025

4.4 Basis and dimension

First, let us do a brief review.

Definition of a span and a spanning set: Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ be k given vectors in $\mathbb{R}^n$. The span of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$, $\mathrm{Span}(\underline{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_\mathrm{k}})$, is the set that contains all possible linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$. $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_\mathrm{k})$ is a subspace of $\mathbb{R}^n$ and $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ is called a spanning set of this subspace.

$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$

$\mathbb{R}^2$

Example 1. The span of $\underline{\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$ is what? What about the span of $\underline{\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}}$?

$\mathrm{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\uparrow \text{ spanning set}} \right\}$

any vector in $\mathbb{R}^2$ looks like

$\begin{bmatrix} x \\ y \end{bmatrix} = \underline{x} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underline{y} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix}$

$\mathrm{span}\left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \end{bmatrix} \right\} \overset{?}{=} \mathbb{R}^2$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{x}{3} \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \frac{y}{-4} \begin{bmatrix} 0 \\ -4 \end{bmatrix}$

Example 2. The span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is the same as the span of $\mathbf{v}_1$ since $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$. So $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \neq \mathbb{R}^2$. Why?

$\mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\} \overset{?}{=} \mathbb{R}^2$ $\qquad$ $\begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -2 \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2y \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (x+2y) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Quiz Question 1. Which of the following statement is correct?

A. The span of $\overset{e_1}{\mathbf{v}_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\overset{e_2}{\mathbf{v}_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\overset{e_3}{\mathbf{v}_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ is $\mathbb{R}^2$; $\begin{bmatrix}x\\y\\z\end{bmatrix}$

B. The span of $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ is $\mathbb{R}^3$;

C. The span of $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}0\\0\\0\end{bmatrix}$ is $\mathbb{R}^3$;

D. The span of $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ is not $\mathbb{R}^3$.

Definition of linear independence and dependence: The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are (called) linearly independent if the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution $a_1 = a_2 = \cdots = a_k = 0$. If on the other hand, this equation has non-trivial solutions, then $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are (said to be) linearly dependent.

Example 1. Determine whether

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

dependent

are linearly ~~independent~~.

$$a_1\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + a_3\begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 5 \\ -1 & -1 & -2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 5 & 0 \\ -1 & -1 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\det = 2 + 0 - 5 - 0 - (-4) - 1$$
$$= -3 + 3 = 0$$

Example 2. Determine the values of c so that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} c \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

are linearly independent.

$$\Rightarrow \quad \det\begin{bmatrix} c & 1 & 0 \\ 1 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \neq 0$$

$$\parallel$$

$$-2c + 0 + 0 - 0 - 3c - 2 = -5c - 2 \neq 0$$

$$c \neq -\frac{2}{5}$$

$$-5c \neq 2$$

Quiz Question 2. Determine the value of c so that the vectors

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}3\\-1\\1\end{bmatrix}, \quad \begin{bmatrix}c\\5\\-2\end{bmatrix}$$

are linearly DEPENDENT.

A. 1 B. 12 C. -12 D. 22

$$\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}4\\4\\7\end{bmatrix}, \begin{bmatrix}-9\\0\\8\end{bmatrix} \quad ?$$

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ for $\mathbb{R}^2 \longrightarrow \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\-4\end{bmatrix}$

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ for $\mathbb{R}^3$ $\qquad \begin{bmatrix}x\\y\\z\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix}$

Definition of a basis. A spanning set of a vector space is called a basis of the vector space if it is linearly independent. Note a vector space can be a subspace in $\mathbb{R}^n$, not necessary the entire $\mathbb{R}^n$.

Example: $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_n$ is a basis of $\mathbb{R}^n$ (called the standard basis).

span$\{v_1, v_2, \cdots, v_k\}$

The following theorem is quite important in the context of this section.

Theorem. If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ and $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m$ are both bases of the same vector space V, then $m = n$.

Because of this theorem, we can define the dimension of a vector space V as the number of vectors in any of its bases.

It is also important to note the following theorem.

Theorem. If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ is a basis of the vector space V, then any vector $v \in V$ can be written a linear combination of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ and this linear combination is unique. Say

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

We say that $c_1, c_2, ..., c_n$ are the *coordinates* of $\mathbf{v}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$.

If our vector space is $\mathbb{R}^n$ then obviously its dimension is n since we know its standard basis contains n vectors. But what if our vector space is a subspace of $\mathbb{R}^n$ which is spanned by some vectors?

Answer: using Gaussian elimination method to find a basis within a known spanning set of the subspace.

Example. Find the dimension of the subspace W of $\mathbb{R}^3$ that is the span of

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \cancel{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -11 \end{bmatrix}} . \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ -3 & 2 & 1 & -11 \end{bmatrix} \xrightarrow{3R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

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$$\xrightarrow{-2R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Quiz Question 3. Find the dimension of the subspace in $\mathbb{R}^4$ whose spanning set contains the following vectors:

$$\begin{bmatrix} 2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 3 \\ 4 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}.$$

A. 1 B. 2 C. 3 D. 4

$V = \text{span}\{\underline{V_1, V_2, \dots, V_k}\}$

$A\vec{x} = \vec{b} \neq \vec{0}$

Theorem. Let A be a matrix of size $\underline{m \times n}$. The set of all solutions to a homogeneous linear equation system (written in matrix form) $\boxed{A\mathbf{x} = \mathbf{0}}$ is a subspace of $\mathbb{R}^n$.

$A\vec{x}_1 = \vec{0}$

$A\vec{x}_2 = \vec{0}$

$\vec{x}_1$, $\vec{x}_2$ solutions, is $\vec{x}_1 + \vec{x}_2$ a solution?

Why?

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 4 & 1 \\ 1 & 3 & -5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\underline{3 \times 4}}$

$A(\vec{x}_1 + \vec{x}_2) = \vec{0}$?

$A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$

$A(c\vec{x}_1) = c A\vec{x}_1 = c \cdot \vec{0} = \vec{0}$

$A\vec{x} = \vec{0}$

Let W be the subspace given by the above theorem. How do we determine its dimension (or find a basis of it)?

Example.

$$\begin{bmatrix} | & | & | & | \end{bmatrix}$$

$$A \rightsquigarrow A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_2 + 3x_4$

$x_2 = 1 \cdot x_2 + 0 \cdot x_4$

$x_3 = 0 \cdot x_2 + 1 \cdot x_4$

$x_4 = 0 \cdot x_2 + 1 \cdot x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + x_4 \underbrace{\begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\vec{v}_2} = \text{span}\{\vec{v}_1, \vec{v}_2\}$$

Quiz Question 4. Let W be the subspace of $\mathbb{R}^7$ that consists of the solutions to the equation system $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & 4 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

span {

v_1 - - v_7

Find the dimension of W.

A. 1 B. 2 C. 3 D. 4