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Quantile based tests for exponentiality against DMRQ and NBUE alternatives

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ABSTRACT

Quantile-based reliability analysis has received much attention recently. We propose new quantile-based tests for exponentiality against decreasing mean residual quantile function (DMRQ) and new better than used in expectation (NBUE) classes of alternatives. The exact null distribution of the test statistic is derived when the alternative class is DMRQ. The asymptotic properties of both the test statistics are studied. The performance of the proposed tests with other existing tests in the literature is evaluated through simulation study. Finally, we illustrate our test procedure using real data sets.

1. Introduction

The exponential distribution is a very appealing model, particularly in life testing and reliability studies ever since the path-breaking paper by Epstein and Sobel (1953). For properties and applications of exponential distribution we refer to Galambos and Kotz (1978), Johnson, Kotz, and Balakrishnan (1994) and Marshall and Olkin (2007). The importance of the distribution is partly due to the remarkable property of constant hazard rate or equivalently constant mean residual life function and hence it is often used as a base line for evaluating families with non constant failure rates or mean residual life times.

Let X be non-negative continuous random variable with distribution function $F(x) = P(X \leq x)$. There are mainly two kinds of tests regarding the ageing criteria in reliability theory. Tests can be constructed to measure the departure from an ageing class to a different one or to check whether F follows exponential distribution against the alternative that F belongs to a particular ageing class. For a review on it, one can refer to Doksum and Yandell (1984, Ch. 26) and Henze and Meintanis (2005).

The works on problems related to tests for exponentiality against decreasing mean residual life (DMRL) and NBUE alternatives started from Hollander and Proschan (1975). Later, several researchers studied the same problem with different approaches. See Aly (1990), Anis (2013), Anis and Mitra (2011), Bergman and Klefsjö (1989), Bhattacharjee and Sen (1995), Jammalamadaka and Taufer (2002), Koul (1978), Lorenzo, Malla, and Mukerjee (2013) and Sankaran and Midhu (2016).

Modelling lifetime data using quantile functions has received much attention in recent years. Nair, Sankaran, and Vineshkumar (2012) explored the importance of Govindarajulu distribution as a lifetime model. Nair, Sankaran, and

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Balakrishnan (2013) studied the reliability properties of various models including Power-Pareto distribution, generalized lambda distribution of Ramberg and Schmeiser and four-parameter distribution of Van Staden and Loots. The main feature of above listed models is that their corresponding distributions are not in closed form. In such cases, where the quantile functions are only available to represent a distribution, quantile based criteria for ageing are needed and are listed in Nair and Vineshkumar (2011). Also while it comes to the testing exponentiality problems related to quantile function model, our usual distribution function based approach is not applicable. Strengthened by this reason, the present paper aims at proposing quantile-based test for exponentiality against DMRQ and NBUE alternatives.

In this context, problem of testing exponentiality against different alternatives in the quantile based framework has great significance. Janssen, Swanepoel, and Veraverbeke (2009) proposed a new test for exponentiality against new better than used in pth quantile of residual lifetime distribution. Fernández-Ponce and Rodríguez-Griñolo (2015) provided a test for exponentiality against NBUE class of distributions based on the excess wealth function and discussed its applications in environmental extremes. Sankaran and Midhu (2016) obtained a test for exponentiality against DMRQ or increasing mean residual quantile function (IMRQ) alternatives. However, we noted that the test proposed by Sankaran and Midhu (2016) is actually applicable to NBUE alternatives rather than DMRQ or IMRQ alternatives (see Remark 1). Motivated by this, we develop a test for exponentiality against DMRQ alternatives using mean residual quantile function. A test for exponentiality against NBUE alternatives using a characterization based on Gini mean difference is also proposed.

The article is organized as given below. In Section 2, we propose a Quantile-based test for exponentiality against DMRQ alternatives. The exact null distribution of the test statistic is derived. The asymptotic properties of the test statistics are also studied. In Section 3, we develop a quantile based test for exponentiality against NBUE alternatives using a characterization based on Gini mean difference. The asymptotic properties of the test statistic are also presented. The result of the simulation study and real data analysis are reported in Section 4. Finally in Section 5, we give conclusions of our study.

2. Test for exponentiality against DMRQ

For right continuous function F, the corresponding quantile function is defined as

$$Q(u) = \inf\{x \;:\; F(x) \geq u\}, \quad 0 \leq u \leq 1.$$

If F is continuous, then we have a unique inverse $Q(u) = F^{-1}(u)$. Let $F_n(.)$ be the empirical distribution function defined as $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$, where I denotes the indicator function and $X_1, X_2, \ldots, X_n$ are independent random sample of size n; from F. The non-parametric estimator of $Q(u)$ is given by

$$\widehat{Q}(u) = \inf\{x \;:\; F_n(x) \geq u\}, \quad 0 \leq u \leq 1.$$

Analysis based on remaining lifetime of an object is intuitively more appealing than the popular hazard function. Expected value of the remaining lifetime is extensively used to study the ageing behaviour of a component or a system. The mean residual life function is defined as

$$m(x) = E(X - x|X > x) = \frac{1}{(1 - F(x))}\int_x^{\infty} y(1 - F(y))dy.$$

Based on the random sample $X_1, X_2, \ldots, X_n$; from F, a non-parametric estimator of mean residual life function at x is given by

$$\widehat{m}(x) = \frac{\sum_{i=1}^{n}(X_i - x)I(X_i > x)}{\sum_{i=1}^{n} I(X_i > x)},$$

where $I(X_i > x) = 1$, if $X_i > x$ and 0 otherwise. Next we give the definition of mean residual quantile function.

Definition 1. Let X be non-negative random variable with continuous distribution function $F(.)$ and quantile function $Q(.)$, the mean residual quantile function is defined as (Nair et al., 2013)

$$M(u) = \frac{1}{1 - u}\int_u^1 (Q(p) - Q(u))\, dp. \tag{1}$$

The $M(u)$ can be interpreted as the average remaining life beyond the $100(1 - u)\%$ of the distribution.

A non-parametric estimator of $M(u)$ is given by (Sankaran & Midhu, 2016)

$$\widehat{M}(u) = \frac{1}{1 - u}\int_u^1 \left(\widehat{Q}(p) - \widehat{Q}(u)\right) dp.$$

Based on the random sample $X_1, X_2, \ldots, X_n$; from F, using the definition of $\widehat{Q}(u)$ we have

$$\widehat{M}(i/n) = \frac{1}{(n - i)}\sum_{j=i+1}^{n}\left(X_{(j)} - X_{(i)}\right), \quad i = 1, 2, \ldots, n - 1. \tag{2}$$

The mean residual quantile function $M(u)$ uniquely determines the distribution through the relationship

$$Q(u) = \int_0^u \frac{M(p)}{1-p}dp - M(u) + \mu,$$

where $\mu = \int_0^1 Q(u)du$. Moreover, $M(u)$ is constant when X has exponential distribution. This enables us to develop test for exponentiality against different ageing classes. For a detailed discussion of reliability concepts in the quantile function framework we refer to Nair et al. (2013).

Definition 2. A distribution $F(.)$ is said to belong to decreasing mean residual quantile function (DMRQ) class if (Nair et al., 2013)

$$M(u) \le M(v); \quad 0 \le v \le u \le 1.$$

We use this Quantile-based definition to develop test against DMRQ alternatives.

2.1. Test statistic

We are interested to test the null hypothesis

$$H_0 : F \quad \text{is exponential}$$

against the alternative

$$H_1 : F \quad \text{is DMRQ, but not exponential,}$$

on the basis of a random sample $X_1, X_2, \ldots, X_n$; from F. To construct the test, we propose a measure of deviation from H_0 towards H_1 as follows.

$$\delta_1(u, v) = M(u) - M(v) \quad 0 \le u \le v \le 1.$$

From Definition 1, it is clear that $\delta_1(u, v) = 0$ for all $u < v$ if and only if H_0 is true. Hence we develop the test based on

$$\Delta_1 = \int_0^1 \int_0^v (1-u)(1-v)\,[M(u) - M(v)]\,du dv, \tag{3}$$

for $u < v$ and $0 \le u, v \le 1$. The measure of departure Δ_1 is zero when X follows exponential distribution and $\Delta_1 > 0$ when $F(.)$ belongs to DMRQ class. Substituting Eq. (1) in (3), Δ_1 can be written as

$$\Delta_1 = A - B - C + D,$$

where

$$A = \int_0^1 \int_0^v (1-v) \int_u^1 Q(p)dpdudv,$$

$$B = \int_0^1 \int_0^v (1-v)(1-u)Q(u)dudv,$$

$$C = \int_0^1 \int_0^v (1-u) \int_v^1 Q(p)dpdudv$$

and

$$D = \int_0^1 \int_0^v (1-v)(1-u)Q(v)dudv.$$

Thus by changing order of integration in A, B, C and D, the expression in (3) can be simplified as

$$\Delta_1 = \int_0^1 \left(\frac{4}{3}u^3 - 4u^2 + 3u - \frac{1}{2}\right) Q(u)du. \tag{4}$$

Hence the test statistic is given by

$$\widehat{\Delta}_1 = \int_0^1 \left(\frac{4}{3}u^3 - 4u^2 + 3u - \frac{1}{2}\right) \widehat{Q}(u)du, \tag{5}$$

where $\widehat{Q}(u)$ is the non-parametric estimator of $Q(u)$ given by $\widehat{Q}(u) = \inf\{x : F_n(x) \ge u\}$, with $F_n(.)$ as the empirical distribution function.

We consider a scale invariant measure given by

$$\Delta_1^* = \frac{\Delta_1}{\mu}.$$

Hence the scale invariant test statistic is given by

$$\widehat{\Delta}_1^* = \frac{\widehat{\Delta}_1}{\bar{X}},$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Therefore the test procedure is to reject the null hypothesis H_0 against H_1 for large values of $\widehat{\Delta}_1^*$.

2.2. Exact distribution

In this subsection we discuss the exact null distribution of $\widehat{\Delta}_1^*$. The following theorem is proved based on a result from Box (1954).

Theorem 1. *Let X be an exponential random variable with rate $\frac{1}{2}$. Then for fixed n*

$$P\left(\widehat{\Delta}_1^* > x\right) = \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(\frac{d_{i,n} - x}{d_{i,n} - d_{j,n}}\right) I(x, d_{i,n}),$$

provided $d_{i,n} \neq d_{j,n}$ when $i \neq j$, where

$$I(x, y) = \begin{cases} 1 & \text{if} \quad x \leq y \\ 0 & \text{if} \quad x > y \end{cases}$$

and

$$d_{i,n} = \frac{1}{3n^3}\Big(i^3 - (3n+1)i^2 + (\frac{3n^2}{2} + 2n)i - n^2\Big).$$

Proof. Under the null hypothesis H_0, from (5), we obtain

$$\widehat{\Delta}_1 = \frac{1}{n}\sum_{i=1}^{n} \left(\frac{4i^3}{3n^3} - \frac{4i^2}{n^2} + \frac{3i}{n} - \frac{1}{2}\right) X_{(i)}, \tag{6}$$

where $X_{(i)}$, $i = 1, 2, \ldots, n$ are ith order statistics based on the identical and independent random sample $X_1, X_2, \ldots, X_n$; from F. By simple algebraic manipulation, $\widehat{\Delta}_1$ can be written as

$$\widehat{\Delta}_1 = \frac{1}{3n^4}\sum_{i=1}^{n} (A(i,n) + B(i,n) + C(i,n) + D(i,n))\, X_{(i)}.$$

where

$$A(i,n) = -(n-i+1)^4 + (n-i)^4 + 1 + 4n^3 + 6n^2 + 4n,$$

$$B(i,n) = 2\big((n-i+1)^3 - (n-i)^3 - 3n^2 - 3n - 1 - \frac{3n^3}{4}\big),$$

$$C(i,n) = -(n-i+1)^2 + (n-i)^2 + 1 + 2n$$

and

$$D(i,n) = \frac{3n^2}{2}\big((n-i+1)^2 - (n-i)^2 - 2n - 1\big).$$

In terms of normalized spacings, $S_i = (n-i+1)(X_{(i)} - X_{(i-1)})$ with $X_0 = 0$, we can represent the test statistic as

$$\widehat{\Delta}_1^* = \frac{\sum_{i=1}^{n} d_{i,n} S_i}{\sum_{i=1}^{n} S_i},$$

where $d_{i,n}$'s are given in the theorem. Note that the exponential random variable with distribution function $F(x) = 1 - e^{-\frac{x}{2}}$ is distributed same as the random variable following χ^2 distribution with d.f. 2. Therefore using Theorem 2.4 of Box (1954) with $g_i = 1$, $v_i = 2$ and $s = 1$ we have

$$\begin{aligned} P\left(\widehat{\Delta}_1^* > x\right) &= P\left(\frac{\sum_{i=1}^{n} d_{i,n} S_i}{\sum_{i=1}^{n} S_i} > x\right) \\ &= P\left(\sum_{i=1}^{n} (d_{i,n} - x) S_i > 0\right) \\ &= \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(\frac{d_{i,n} - x}{d_{i,n} - d_{j,n}}\right) I(x, d_{i,n}). \qquad \square \end{aligned}$$

2.3. Asymptotic properties

In this section, we study the asymptotic properties of the proposed test statistic. The following theorem explains the consistency properties of $\widehat{\Delta}_1$.

Theorem 2. *The test statistic, $\widehat{\Delta}_1$ is a strongly consistent estimator of Δ_1.*

Proof. From (4) and (5), it follows that

$$(\widehat{\Delta}_1 - \Delta_1) = \int_0^1 \left(\frac{4}{3}u^3 - 4u^2 + 3u - \frac{1}{2}\right)\left(\widehat{Q}(u) - Q(u)\right) du.$$

The proof of the theorem is immediate from the fact that (Andersen, Gill, & Keiding, 1993) $Sup_u|\widehat{Q}(u) - Q(u)| \to 0$ almost surely as $n \to \infty$. □

Corollary 1. *The $\widehat{\Delta}_1^*$ is a strongly consistent estimator of Δ_1^*.*

Proof. We know that $\bar{X}$ is a strongly consistent estimator of μ. Since we can express

$$\frac{\widehat{\Delta}_1^*}{\Delta_1^*} = \frac{\widehat{\Delta}_1}{\Delta_1}\frac{\mu}{\bar{X}},$$

we have the result. □

Theorem 3. *As $n \to \infty$, the distribution of $\sqrt{n}\left(\widehat{\Delta}_1 - \Delta_1\right)$ is Gaussian with mean zero and variance σ_1^2, where σ_1^2 is given by*

$$\begin{aligned}\sigma_1^2 \; = & \int_0^1 \Big(\big(\frac{4}{3}u^3 - \frac{u^4}{4} - \frac{3u^2}{2} + \frac{u}{2}\big)Q'(u) \\ & - \int_0^1 \big(\frac{4}{3}v^3 - \frac{v^4}{4} - \frac{3v^2}{2} + \frac{v}{2}\big)Q'(v)dv\Big)^2 du,\end{aligned} \tag{7}$$

provided $\lim_{u\to 1} R(u)Q(u) = 0$, *where* $R(u) = \frac{4}{3}u^3 - \frac{u^4}{3} - \frac{3u^2}{2} + \frac{u}{2}$.

Proof. Applying integration by parts on the right side of (4), we obtain

$$\begin{aligned}\Delta_1 &= \lim_{u\to 1} R(u)Q(u) + \int_0^1 \left(\frac{4}{3}u^3 - \frac{u^4}{4} - \frac{3u^2}{2} + \frac{u}{2}\right) Q'(u)du \\ &= \int_0^1 \left(\frac{4}{3}u^3 - \frac{u^4}{4} - \frac{3u^2}{2} + \frac{u}{2}\right) Q'(u)du.\end{aligned}$$

By non-parametric delta method (Wasserman, 2006), we have the asymptotic normality of the test statistics where the asymptotic variance is as specified in the theorem. □

Next we obtain the null distribution of the test statistic.

Corollary 2. *Let X be non-negative continuous random variable with $F(x) = 1 - exp(-x/\lambda)$; $\lambda > 0$, as $n \to \infty$, the distribution of $\sqrt{n}\widehat{\Delta}_1$, is Gaussian with mean zero and variance $\sigma_{10}^2 = \frac{\lambda^2}{210}$.*

Proof. The quantile function of the exponential distribution with mean λ is specified by

$$Q(u) = -\lambda log(1 - u),$$

which is a logarithmic function in u. Since $R(u)$ is a polynomial of degree 4, the limit condition stated in Theorem 3 holds true. Under the null hypothesis, from (7) we obtain

$$\sigma_{10}^2 = \int_0^1 \left(\frac{4}{3}u^3 - \frac{u^4}{4} - \frac{3u^2}{2} + \frac{u}{2}\right)^2 (Q'(u))^2 du,$$

which yields

$$\sigma_{10}^2 = \frac{\lambda^2}{210}. \quad \square$$

Using Slutsky's theorem we have the following result.

Corollary 3. *Let X be non-negative continuous random variable with $F(x) = 1 - exp(-x/\lambda)$, as $n \to \infty$, the distribution of $\sqrt{n}\widehat{\Delta}_1^*$, is Gaussian with mean zero and variance $\frac{1}{210}$.*

Hence, the asymptotic test is to reject the null hypothesis H_0 in favour of H_1, if

$$\sqrt{210n}\widehat{\Delta}_1^* > Z_\alpha,$$

where Z_α is the upper α-percentile of $N(0, 1)$. One can also look at the problem of testing exponentiality against the dual concept IMRQ class. We reject the null hypothesis H_0 in favour of IMRQ class, if

$$\sqrt{210n}\widehat{\Delta}_1^* < -Z_\alpha.$$

3. Testing exponentiality against NBUE

Developing a quantile based test for testing exponentiality against NBUE alternatives based on property of Gini mean difference is of specific interest in this section. For a sample of n positive variates $X_1, X_2, \ldots, X_n$, the Gini statistic is given by

$$G_n = (2n(n-1)\bar{X})^{-1} \sum_{i,j=1}^{n} |X_i - X_j|.$$

Gail and Gastwirth (1978) proved the asymptotic normality of G_n and obtained exact distribution of G_n under exponentiality. Sudheesh and Deemat (2015) proposed a non-parametric test based on Gini index for testing exponentiality against NBUE alternatives. Recently Fernández-Ponce and Rodríguez-Griñolo (2015) developed a test for exponentiality against NBUE alternatives using a quantile dispersion measure. However this test does not perform well for small sample sizes (see Tables 5 and 6). Hence we propose a quantile-based test for H_0 against NBUE alternatives. The proposed test is developed using a result of the NBUE class based on the Gini mean difference of residual life in terms of quantiles (Sreelakshmi, Asha, & Nair, 2015). We compare our test with that of Fernández-Ponce and Rodríguez-Griñolo (2015) and show that it performs well even for small sample sizes.

3.1. Test statistics

We recall few definitions which will be used in the sequel.

Definition 3. A random variable X is said to have new better than used in expectation (NBUE) if (Nair et al., 2013)

$$M(u) \leq \mu.$$

Remark 1. Sankaran and Midhu (2016) considered a Quantile-based test for testing exponentiality against DMRQ (IMRQ) alternatives. However, we observed that the test is applicable to NBUE alternatives as they considered the departure measure based on $\mu \geq M(u)$, which is the defining inequality of NBUE class (Hollander & Proschan, 1975).

Definition 4. Let X_1 and X_2 be two i.i.d. random variables. Then Gini mean difference (GMD) is defined as (Xu, 2007)

$$G = E|X_1 - X_2|.$$

Another useful expression of GMD is given by

$$G = \int_0^\infty 2x(2F(x) - 1)dF(x).$$

For the truncated random variable $X(t) = (X|X > t)$ with survival function

$$\bar{F}_t(x) = \frac{\bar{F}(x)}{\bar{F}(t)}; \quad x > t,$$

the GMD of residual life $X(t)$ is given by

$$G(t) = 2\int_t^\infty F_t(x)\bar{F}_t(x)dx.$$

In terms of quantile functions GMD of $X(t)$ can be expressed as (Nair et al., 2013)

$$\eta(u) = G(Q(u)) = 2\int_u^1 \frac{(1-p)(p-u)}{(1-u)^2} Q'(p)dp, \tag{8}$$

where prime denotes differentiation. A relationship between $\eta(u)$ and $M(u)$ is given by (Nair & Vineshkumar, 2010)

$$\eta(u) = \frac{2}{(1-u)^2}\int_u^1 (1-p)M(p)dp. \tag{9}$$

Using (2), we propose a non-parametric estimator of $\eta(u)$ as

$$\widehat{\eta}(u) = \frac{2}{(1-u)^2}\int_u^1 (1-p)\widehat{M}(p)dp. \tag{10}$$

Also a necessary condition for a distribution F to be NBUE is (Sreelakshmi et al., 2015),

$$\eta(u) \le \mu, \quad 0 < u < 1. \tag{11}$$

From (11) it is clear that when the difference between GMD of $X(t)$ and total mean income is negligible, the corresponding income distribution is exponential. Accordingly this inequality can be used to develop a test for testing exponentiality against NBUE alternatives.

Let $X_1, X_2, \ldots, X_n$ be independent random sample of size n from F. Consider the testing problem

$H_0 : F$ is exponential

against

$H_2 : F$ is NBUE, but not exponential.

We define the measure of departure from H_0 towards H_2 as

$$\Delta_2 = \int_0^1 (\mu - \eta(u))\, du. \tag{12}$$

Using Eq. (9), the expression given in (12) becomes

$$\Delta_2 = \int_0^1 \left(\mu - \frac{2}{(1-u)^2}\int_u^1 (1-p)M(p)dp\right) du.$$

By changing order of integration in above equation, we obtain

$$\Delta_2 = \int_0^1 (\mu - 2uM(u))\, du. \tag{13}$$

Hence the test statistic is given by

$$\widehat{\Delta}_2 = \int_0^1 \left(\bar{X} - 2u\widehat{M}(u)\right) du.$$

To make the test scale invariant we consider the measure defined by

$$\Delta_2^* = \frac{\Delta_2}{\mu}.$$

And the scale invariant test is given by

$$\widehat{\Delta}_2^* = \frac{1}{\bar{X}}\int_0^1 \left(\bar{X} - 2u\widehat{M}(u)\right) du.$$

Given the random sample $X_1, X_2, \ldots, X_n$; from F, using the expression for $\widehat{M}(u)$ given in Eq. (2) we can express $\widehat{\Delta}_2^*$ as

$$\widehat{\Delta}_2^* = \frac{1}{n\bar{X}}\left\{\sum_{i=1}^{n} X_{(i)} - \frac{1}{n}\sum_{i=1}^{n-1}\frac{2i}{n-i}\sum_{j=i+1}^{n}(X_{(j)} - X_{(i)})\right\}. \tag{14}$$

The test procedure is to reject H_0 against H_2 for large values of $\widehat{\Delta}_2^*$.

3.2. Asymptotic properties

The asymptotic properties of the test statistic are studied in this subsection. First we discuss the consistency of the test statistic.

Theorem 4. *Let $\widehat{\eta}(u)$ be non-parametric estimator of $\eta(u)$ give in* Eq. (9), *then $Sup_u|\widehat{\eta}(u) - \eta(u)| \to 0$ almost surely, as $n \to \infty$.*

Proof. Using (9), we have

$$\widehat{\eta}(u) - \eta(u) = \frac{2}{(1-u)^2}\int_u^1 (1-p)(\widehat{M}(p) - M(p))dp \tag{15}$$

Note that $Sup_u|\widehat{M}(u) - M(u)| \to 0$ almost surely, as $n \to \infty$ (Sankaran & Midhu, 2016). Hence from (15) we have, $Sup_u|\widehat{\eta}(u) - \eta(u)| \to 0$ almost surely, as $n \to \infty$. □

Hence the following result is immediate.

Corollary 4. *The test statistic* $\widehat{\Delta}_2$ *is a strongly consistent estimator of* Δ_2.

The proof of the following result is similar to that of Corollary 1.

Corollary 5. *The* $\widehat{\Delta}_2^*$ *is a strongly consistent estimator of* Δ_2^*.

Theorem 5. *As* $n \to \infty$, *the distribution of* $\sqrt{n}\left(\widehat{\Delta}_2 - \Delta_2\right)$ *is Gaussian with mean zero and variance* σ_2^2 *given by*

$$\sigma_2{}^2 = \int_0^1 \left(\mu - 2uM(u) - \int_0^1 (\mu - 2vM(v))dv\right)^2 du.$$

Next we obtain the null distribution of Δ_2.

Corollary 6. *Let* X *be non-negative continuous random variable with* $F(x) = 1 - exp(-x/\lambda)$, *as* $n \to \infty$, *the distribution of* $\sqrt{n}\widehat{\Delta}_2$ *is Gaussian with mean zero and variance* $\sigma_{20}^2 = \frac{\lambda^2}{3}$.

Proof. Using Eqs. (1) and (9), the test statistic given in Eq. (12) can be written as

$$\begin{aligned}\Delta_2 &= \int_0^1 \left(Q(u) - \frac{2}{(1-u)^2}\int_u^1 (1-p)\left[\frac{1}{(1-p)}\int_p^1 Q(t)dt - Q(p)\right]dp\right)du \\ &= \int_0^1 \left(Q(u) - \frac{2}{(1-u)^2}\int_u^1 \int_p^1 Q(t)dtdp + \frac{2}{(1-u)^2}\int_u^1 (1-p)Q(p)dp\right)du.\end{aligned}$$

Using Fubini's theorem, changing the order of integration in second and third terms of the above equation we have

$$\Delta_2 = \int_0^1 Q(u)du + 2\int_0^1 Q(u)\left(u + log(1-u)\right)du + 2\int_0^1 uQ(u)du,$$

which yields

$$\Delta_2 = \int_0^1 (1 + 4u + 2log(1-u))Q(u)du.$$

Using the quantile function of exponential distribution and by integration by parts we obtain

$$\Delta_2 = \int_0^1 (1-u)(1+2u-2Q(u))Q'(u)du.$$

By non-parametric delta method we have the asymptotic normality of Δ_2 where the asymptotic variance is given by

$$\sigma_{20}{}^2 = \int_0^1 \left((1-u)(1+2u-2Q(u))Q'(u)\right)^2 du = \frac{\lambda^2}{3}. \quad \square$$

Using Slutsky's theorem we have the following result.

Corollary 7. *Let* X *be non-negative continuous random variable with* $F(x) = 1 - exp(-x/\lambda)$, *as* $n \to \infty$, *the distribution of* $\sqrt{n}\widehat{\Delta}_2^*$ *is Gaussian with mean zero and variance* $\sigma_{20}^2 = \frac{1}{3}$.

Therefore for large values of n, we reject H_0 if

$$\sqrt{3n}\widehat{\Delta}_2^* > Z_\alpha.$$

If the alternative hypothesis we consider is new better than worse in expectation (NBWE) class, then we reject the null hypothesis H_0 in favour of NBWE class, if

$$\sqrt{3n}\widehat{\Delta}_2^* < -Z_\alpha.$$

4. Simulation and data analysis

In this section, we report the result of the simulation study done to evaluate the performance of our tests developed in Sections 2 and 3. The simulation is done using *R* software. We also explained the proposed testing methods using three real data sets.

Table 1
Empirical type 1 error of $\widehat{\Delta}_1^*$.

n	5% level	1% level
10	0.210	0.071
20	0.162	0.070
50	0.100	0.050
75	0.090	0.032
100	0.083	0.028
200	0.076	0.021
500	0.051	0.012

4.1. Empirical study: DMRQ alternatives

First we compute the empirical type 1 error of $\widehat{\Delta}_1^*$ by generating a random sample of size n from standard exponential distribution. The simulation is repeated for thousand times. The empirical type 1 error calculated for different sample sizes is reported in Table 1.

For finding the empirical power, we simulate observations from Weibull, Rescaled beta and Gompertz distributions where the form of the quantile functions is specified below:

- Weibull distribution:
$$Q(u) = (-log(1-u))^{\frac{1}{\beta}};\ \ \beta > 0$$
- Rescaled beta distribution:
$$Q(u) = R\big(1-(1-u)^{\frac{1}{c}}\big);\ \ c,\ R > 0,\ \ 0 \leq Q(u) < R$$
- Gompertz distribution:
$$Q(u) = \frac{1}{c}log(1-\frac{\lambda}{c}log(1-u));\ \ \lambda,\ c > 0.$$

The empirical powers of above given alternatives are compared with the tests proposed by Abu-Youssef (2002), Ahmad (1992) and Li, Cao, and Feng (2006). Next we report their test statistics and its asymptotic null distribution.

Ahmad (1992) developed a test for exponentiality against DMRQ alternative and is given by

$$T_1 = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1,j<i}^{n}(\phi(X_i, X_j) + \phi(X_j, X_i)),$$

where $\phi(X_1, X_2) = (3X_1 - X_2)I(X_2 - X_1)$. The asymptotic null distribution of $\sqrt{n}T_1$ is Gaussian with mean 0 and variance 1/3.

Abu-Youssef (2002) proposed a test for exponentiality against DMRQ alternative based on the inequality

$$E(\max(X_1, X_2)) \geq \frac{(E(X_1))^2}{2},$$

where X_1 and X_2 are independently and identically distributed as that of X. The test statistic is given by

$$T_2 = \frac{1}{\bar{X}^2}\frac{2}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1,j<i}^{n}\left(\min(X_i^2, X_j^2) - \frac{X_iX_j}{2}\right).$$

Under H_0, $\sqrt{n}T_2$ is asymptotically distributed as Gaussian with mean zero and variance 2/27.

For the same testing problem, Li et al. (2006) proposed a test given by

$$T_3 = \frac{1}{n(n-1)(n-2)}\sum_{i\neq j\neq k} h\left(X_i, X_j, X_k\right),$$

where $h\left(X_1, X_2, X_3\right) = -\frac{1}{6}X_1 + \frac{2}{3}\min\left(X_1, X_2\right) - \frac{1}{2}\min\left(X_1, X_2, X_3\right)$. Asymptotic null distribution of $\sqrt{n}T_3$ is Gaussian with mean 0 and variance $\frac{\mu^2}{270}$, where $\mu = E(X)$.

We compared the power of our test with T_1, T_2 and T_3. The empirical powers calculated for rescaled beta, Gompertz and Weibull distributions are listed in Tables 2, 3 and 4, respectively. The plot of $M(u)$ corresponding to rescaled beta ($R = 20$, $c = 4$) and Gompertz ($c = 0.3$, $\lambda = 0.05$) distributions is given in Figs. 1 and 2. In Figs. 3 and 4 we plot $M(u)$ of Weibull distributions with shape parameters $\beta = 4$ and $\beta = 2$, respectively. From Tables 2 and 3 it is clear that our test performs better than the other tests when the data is generated from rescaled beta and Gompertz distribution. Moreover as the sample size increases, the power of the test approaches one. We noted that empirical power of our test is comparable with the other tests for large sample sizes when the data is simulated from Weibull distribution (Table 4). In this case, for small sample sizes, the test proposed by Abu-Youssef (2002) and Ahmad (1992) performs better than the other two tests. However, one can notice that our test can be implemented easily in the computational part.

Table 2
Empirical power: Rescaled beta distribution ($R = 20, c = 4$).

n	$\widehat{\Delta}_1^*$		T_1		T_2		T_3	
	5% level	1% level	5% level	1% level	5% level	1% level	5% level	1% level
10	0.572	0.532	0.175	0.061	0.101	0.015	0.11	0.006
20	0.616	0.587	0.223	0.06	0.141	0.038	0.121	0.008
50	0.678	0.646	0.405	0.173	0.367	0.107	0.338	0.061
75	0.749	0.721	0.505	0.247	0.528	0.201	0.471	0.148
100	0.781	0.756	0.607	0.349	0.681	0.308	0.645	0.244

Table 3
Empirical power: Gompertz distribution ($c = 0.3, \lambda = 0.05$).

n	$\widehat{\Delta}_1^*$		T_1		T_2		T_3	
	5% level	1% level	5% level	1% level	5% level	1% level	5% level	1% level
10	0.725	0.706	0.109	0.036	0.07	0.015	0.074	0.006
20	0.829	0.806	0.146	0.026	0.082	0.016	0.103	0.008
50	0.948	0.94	0.218	0.062	0.152	0.027	0.14	0.025
75	0.981	0.979	0.235	0.077	0.184	0.043	0.174	0.032
100	0.988	0.985	0.292	0.102	0.25	0.074	0.244	0.052

Table 4
Empirical power: Weibull distribution ($\beta = 4$).

n	$\widehat{\Delta}_1^*$		T_1		T_2		T_3	
	5% level	1% level	5% level	1% level	5% level	1% level	5% level	1% level
10	0.020	0.002	0.888	0.559	0.975	0.838	0.036	0.002
20	0.042	0.009	0.999	0.968	1.000	0.998	0.125	0.003
50	0.702	0.253	1.000	1.000	1.000	1.000	0.716	0.085
75	0.954	0.356	1.000	1.000	1.000	1.000	0.948	0.387
100	1.000	0.779	1.000	1.000	1.000	1.000	0.998	0.771

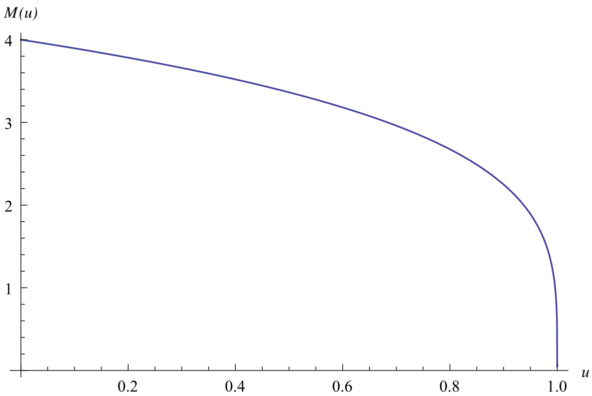

Fig. 1. Plot of $M(u)$: Rescaled beta distribution ($R = 20, c = 4$).

4.2. Empirical study: NBUE alternatives

In this subsection, we check the performance of $\widehat{\Delta}_2^*$. The empirical type 1 error of $\widehat{\Delta}_2^*$ is given in Table 5. We compare our test with some other tests available in literature. Next we briefly discuss about the test statistics that are used for comparison. For more details about these tests and Monte Carlo comparison we refer to Anis and Basu (2014).

Mugdadi and Ahmed (2005) considered a test statistic given by

$$T_4 = \frac{1}{\bar{X}}\left(\frac{2}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\frac{X_i}{2} - \min(X_i, X_j)\right)\right),$$

to test exponentiality against NBUE alternatives. Under the null, using U-statistics theory they proved that $\sqrt{n}T_4$ is asymptotically distributed as Gaussian with mean 0 and variance 1/12. For the same testing problem, Anis and Mitra (2011)

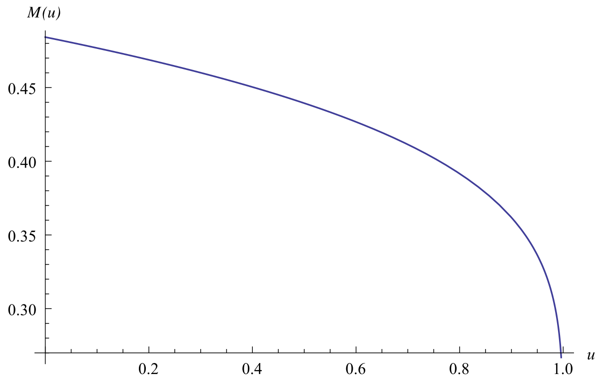

Fig. 2. Plot of $M(u)$: Gompertz distribution ($c = 0.3$, $\lambda = 0.05$).

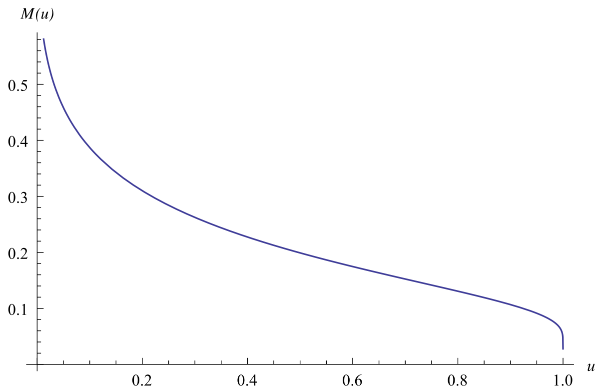

Fig. 3. Plot of $M(u)$: Weibull distribution ($\beta = 4$).

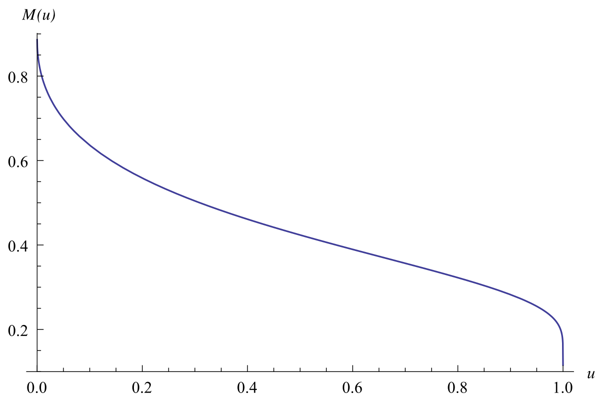

Fig. 4. Plot of $M(u)$: Weibull distribution ($\beta = 2$).

proposed a family of tests given by

$$T_5 = \frac{1}{\bar{X}}\frac{1}{j}\sum_{i=1}^{n}\left(\left(\frac{n-i+1}{n}\right)^{j+1} - \left(\frac{n-i}{n}\right)^{j+1} - \frac{1}{n(j+1)}\right)X_{(i)}.$$

We compare our test with T_5 for $j = 1$. When $j = 1$, under the null, T_5 is asymptotically distributed as normal having variance $1/12$ with convergence rate $n^{-1/2}$.

Table 5
Empirical type 1 error of $\widehat{\Delta}_2^*$.

n	5% level	1% level
10	0.064	0.012
20	0.059	0.011
50	0.052	0.011
75	0.051	0.010
100	0.050	0.010

Table 6
Empirical power: Weibull distribution ($\beta = 2$).

n	$\widehat{\Delta}_1^*$		T_4		T_5		T_6	
	5% level	1% level	5% level	1% level	5% level	1% level	5% level	1% level
10	1.000	0.975	0.799	0.467	0.951	0.838	0.899	0.657
20	1.000	1.000	0.987	0.882	0.995	0.945	0.996	0.945
50	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
75	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

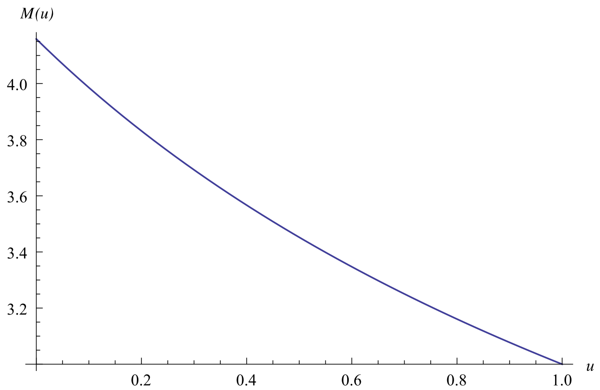

Fig. 5. Plot of $M(u)$: Half logistic distribution ($\sigma = 3$).

Fernández-Ponce and Rodríguez-Griñolo (2015) proposed a test for exponentiality against NBUE alternatives based on spacing. They developed a test using the quantile based inequality

$$M(u) \le M(0)(1-u); \quad 0 \le u \le 1$$

and the test statistic is given by

$$T_6 = 0.5 - \frac{1}{n}\frac{\sum_{j=2}^{n}(j-1)S_j}{\sum_{j=1}^{n} S_j}.$$

Under the null hypothesis, $\sqrt{n}T_6$ is asymptotically distributed as Gaussian with mean zero and variance $1/12$.

Now we compare the performance of our test $\widehat{\Delta}_2^*$ with the tests T_4, T_5 and T_6 in terms of empirical power. For finding the empirical power, we simulate observations from Weibull, half logistic, re-scaled beta and mixture of Weibull distributions. The quantile function that corresponds to half logistic distribution is given as

$$Q(u) = \sigma \log\left(\frac{1+u}{1-u}\right); \ \sigma > 0.$$

We consider mixture of Weibull distribution with survival function given by

$$\bar{F}(x) = pe^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}} + (1-p)e^{-\left(\frac{x}{\alpha_2}\right)^{\beta_2}}; \ \alpha_1, \alpha_2, \beta_1, \beta_2 > 0, \ 0 < p < 1.$$

Note that mixed Weibull distribution with scale parameters $\alpha_1 = 3$ and $\alpha_2 = 2$, shape parameters $\beta_1 = 1.5$ and $\beta_2 = 5$ and proportion $p = 0.5$ belongs to NBUE class, not DMRQ. The plot of $M(u)$ of half logistic ($\sigma = 3$) distribution and mixed Weibull distribution is given in Figs. 5 and 6, respectively.

Tables 6–9 provide the power comparisons of $\widehat{\Delta}_2^*$ with T_4, T_5 and T_6. From these tables we can see that empirical powers of the test approach one when n takes large values. Also from Tables 6–8, it is clear that our test performs well compared to the

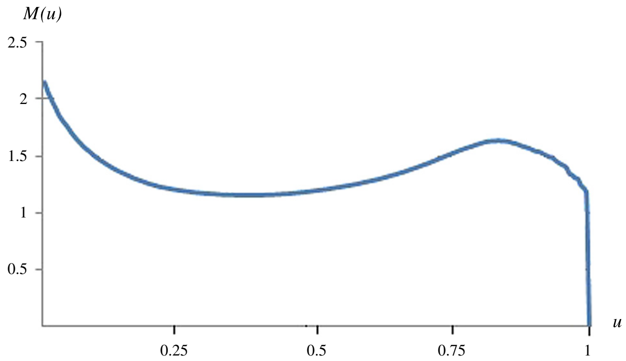

Fig. 6. Plot of $M(u)$: Mixed Weibull distribution ($\alpha_1 = 3, \alpha_2 = 2, \beta_1 = 1.5, \beta_2 = 5, p = 0.5$).

Table 7
Empirical power: Half logistic distribution ($\sigma = 3$).

n	$\widehat{\Delta}_2^*$		T_4		T_5		T_6	
	5% level	1% level	5% level	1% level	5% level	1% level	5% level	1% level
10	0.171	0.012	0.250	0.068	0.125	0.032	0.286	0.086
20	0.422	0.084	0.344	0.114	0.223	0.066	0.336	0.113
50	0.875	0.563	0.487	0.220	0.395	0.154	0.520	0.231
75	0.990	0.866	0.589	0.317	0.503	0.264	0.600	0.331
100	1.000	0.971	0.710	0.428	0.648	0.376	0.719	0.446

Table 8
Empirical power: Rescaled beta distribution ($R = 2, c = 2$).

n	$\widehat{\Delta}_2^*$		T_4		T_5		T_6	
	5% level	1% level	5% level	1% level	5% level	1% level	5% level	1% level
10	0.881	0.598	0.074	0.018	0.149	0.036	0.444	0.155
20	0.996	0.962	0.085	0.018	0.137	0.033	0.546	0.248
50	1.000	1.000	0.072	0.017	0.108	0.027	0.875	0.631
75	1.000	1.000	0.058	0.011	0.078	0.021	0.949	0.811
100	1.000	1.000	0.068	0.014	0.091	0.021	0.992	0.921

Table 9
Empirical power: Mixed Weibull distribution ($\alpha_1 = 3, \alpha_2 = 2, \beta_1 = 1.5, \beta_2 = 5, p = 0.5$).

n	$\widehat{\Delta}_1^*$		T_4		T_5		T_6	
	5% level	1% level	5% level	1% level	5% level	1% level	5% level	1% level
10	0.797	0.617	0.702	0.426	0.837	0.574	0.732	0.490
20	0.879	0.769	0.931	0.808	0.973	0.867	0.865	0.760
50	0.992	0.957	1.000	0.997	1.000	0.998	0.988	0.948
75	1.000	0.989	1.000	1.000	1.000	1.000	1.000	1.000
100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 10
Jet airplanes data: ordered time intervals between successive failures (in hours).

1	4	11	16	18	18	18	24	31
39	46	51	54	63	68	77	80	82
97	106	111	141	142	163	191	206	216

other tests. We also noticed that the test proposed by Fernández-Ponce and Rodríguez-Griñolo (2015) performs better than those of Anis and Mitra (2011) and Mugdadi and Ahmed (2005). For mixed Weibull distribution, our test performs better than those of Fernández-Ponce and Rodríguez-Griñolo (2015) and Mugdadi and Ahmed (2005) for small sample sizes. In this case, the test developed by Anis and Mitra (2011) performs better than our test. However the performances of all four tests are comparable for large sample sizes.

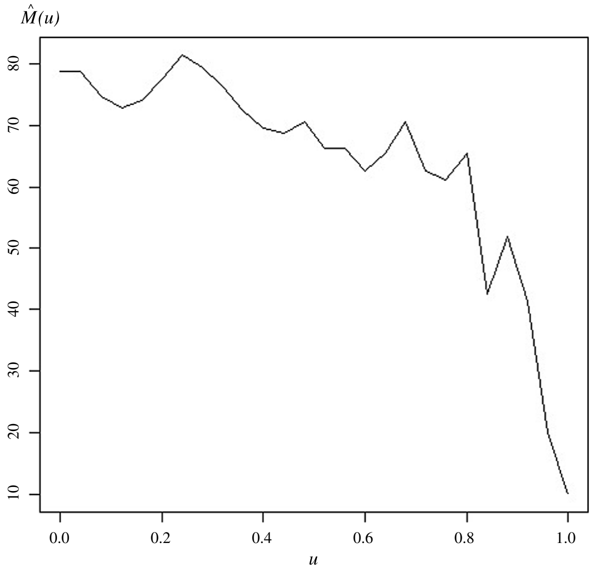

Fig. 7. Jet airplane data: Plot of $\widehat{M}(u)$.

Table 11
Ball bearings data.

17.88	28.92	33.00	41.52	42.12	45.60	48.40	51.84
51.96	54.12	55.56	67.80	68.64	68.64	68.88	84.12
93.12	98.64	105.12	105.84	127.92	128.04	173.40	

Table 12
Blood cancer data.

115	181	255	418	441	461	516	739	743	789
807	865	924	983	1024	1062	1063	1165	1191	1222
1222	1251	1277	1290	1357	1369	1408	1455	1478	1549
1578	1578	1599	1603	1605	1696	1735	1799	1815	1852

4.3. Data analysis

The tests for exponentiality explained in Sections 2 and 3 are applied to three sets of real data. First data set is taken from Proschan (1963) and is reported in Table 10. It consists of 27 observations of the intervals between successive failures (in hours) of the air conditioning systems of 7913 jet airplanes. The plot of $\widehat{M}(u)$ obtained using this data is given in Fig. 7. The value of $\sqrt{210n}\widehat{\Delta}_1^*$ is calculated as 0.353. Therefore at 95% level, we fail to reject the null hypothesis of exponentiality against DMRQ alternative.

But we applied the same data to test the exponentiality against NBUE alternatives based on Gini mean difference of truncated random variable. The value of $\sqrt{3n}\widehat{\Delta}_2^*$ is calculated as 2.672. Therefore at 95% level, we reject the null hypothesis of exponentiality against NBUE alternative.

Second data set is taken from Pavur, Edgeman, and Scott (1992) which gives the number of revolutions (in millions) to failure of 23 ball bearings in a life test study. The data is reported in Table 11. The plot of $\widehat{M}(u)$ obtained for ball bearings data is given in Fig. 8. The value of $\sqrt{210n}\widehat{\Delta}_1^*$ is calculated as 0.488. Therefore at 95% level, we fail to reject the null hypothesis of exponentiality against DMRQ alternative. Also we apply the same data to test the exponentiality against NBUE alternatives and we obtain the value of $\sqrt{3n}\widehat{\Delta}_2^*$ as 4.285. Therefore at 95% level, we reject the null hypothesis of exponentiality against NBUE alternative.

Next we consider the data given in Abouammoh, Abdulghani, and Qamber (1994) which represents the ordered life times (in days) of 40 patients suffering from blood cancer collected from one of the Health Ministry Hospital in Saudi Arabia (see Table 12). The plot of $\widehat{M}(u)$ for the data is given in Fig. 9. The value of $\sqrt{210n}\widehat{\Delta}_1^*$ and $\sqrt{3n}\widehat{\Delta}_2^*$ is 2.956 and 8.565, respectively. Therefore at 95% level, we reject the null hypothesis of exponentiality against both DMRQ and NBUE alternatives.

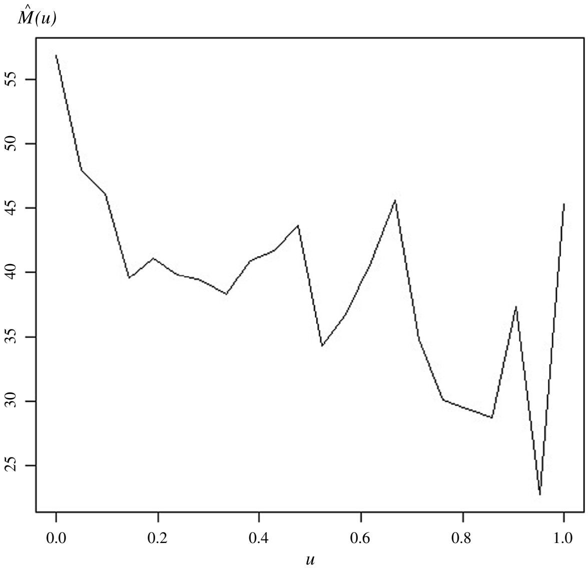

Fig. 8. Ball bearings data: Plot of $\widehat{M}(u)$.

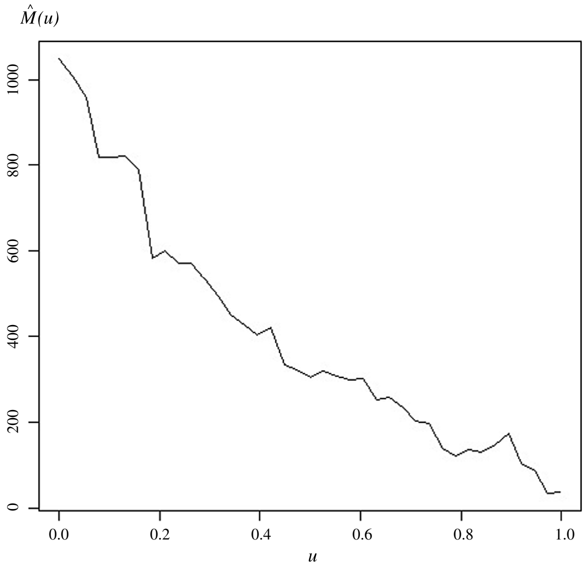

Fig. 9. Blood cancer data: Plot of $\widehat{M}(u)$.

5. Conclusion

In this paper, we proposed quantile based tests for exponentiality against two alternative classes, decreasing mean residual quantile function and new better than used in expectation. The test statistic are simple to devise and can easily apply in practice. The exact null distribution is derived when the alternative is DMRQ. The asymptotic properties of the tests are studied. The simulation study shows that our tests perform well against various alternatives. Also we compared our tests with some existing tests and showed that our tests perform better in terms of empirical power.

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