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Some new applications of the total time on test transforms

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ABSTRACT

The concept of total time on test transforms (TTT) is well known for its applications in different fields of scientific study. In this article we present four applications of TTT in reliability theory. First we characterize ageing criteria such as IFRA and NBU in terms of TTT. Then we utilize an iterated version to construct bathtub shaped hazard quantile functions and corresponding lifetime models. Further, an index is developed for numerically measuring the extent of IFR-ness of a life distribution. Finally we demonstrate how the distributional properties such as kurtosis and skewness can be derived from the TTT.

1. Introduction

The concept of the total time on test transform (TTT) was introduced and developed in the early seventies; see for example [3,2]. When several units are simultaneously put under test to ascertain their life lengths, some units may fail during the test while others may survive it. The sum of all the completed and incomplete life lengths constitutes the total time on test statistic, and the limit of this statistic as the number of units increases indefinitely is called the TTT. For a non-negative continuous random variable with distribution function $F(x)$, the TTT is defined as

$$T(u) = \int_0^{Q(u)} [1 - F(t)]dt \tag{1.1}$$

where

$$Q(u) = \inf[x|F(x) \geq u], \quad 0 \leq u \leq 1$$

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is the quantile function of X. When $F(x)$ is continuous and strictly increasing, $Q(u) = x$ is obtained as the solution of $F(x) = u$. In terms of the quantile function, (1.1) becomes

$$T(u) = \int_0^u (1-p)q(p)dp \tag{1.2}$$

where $q(p)$ is the derivative of $Q(p)$, called the quantile density function. Initially, the work on TTT concentrated on its applications in reliability analysis and engineering problems as discussed in [17,5] and their references. However, recently the topic has been capturing attention in other areas of study like economics [27], optimal energy sales [6], risk assessment [31], analysis of censored data [30], repairable limits [9], spacings [10], maintenance scheduling [16] estimation in stationary observations [7] and stochastic modelling [29].

In the rest of the present paper we focus attention on some new applications of TTT that have relevance in reliability analysis. The potential of TTT in characterizing ageing criteria considered in the literature is limited to only four (among a large number) of concepts, namely IFR, DMRL, NBUE and HNBUE. Our first problem is to find appropriate analogues for other important concepts like IFRA, NBU etc. Thus by studying the TTT alone, we can evaluate the behaviour of the unit as it ages, in the sense of various criteria, without treating each criterion separately under different definitions. As defined in (1.1), the transform $T(u)$ is again a quantile function generated from $Q(u)$. Nair et al. [22] have discussed the properties of a hierarchy of transformed models generated from $Q(u)$. A method of constructing bathtub shaped hazard quantile function models from such a scheme is the problem that is addressed in this work. Then we discuss how the descriptive characteristics of a life distribution can be computed using the TTT. Among the various ageing classes discussed, the IFR is a popular choice in theory and practice. When two or more distributions are compared for modelling purposes, which one of them is more IFR than the other(s) is vital information in the choice of the appropriate model. An index for measuring the IFR-ness of a distribution is also proposed as part of the present work.

2. Characterization of ageing criteria

In the following, the distribution function $F(x)$ of X is assumed to be strictly increasing. When X has a density $f(x)$, the hazard rate is $h(x) = \frac{f(x)}{(1-F(x))}$. Setting $x = Q(u)$ in this definition, the equivalent of the hazard rate is the hazard quantile function

$$H(u) = h(Q(u)) = [(1-u)q(u)]^{-1}. \tag{2.1}$$

In terms of TTT the following characterizations of ageing concepts are discussed in the literature.

(i) A lifetime X with finite mean μ is IFR (DFR) if and only if $\phi(u)$ is concave (convex) for $0 \le u \le 1$ [2], where $\phi(u) = \frac{T(u)}{\mu}$ is often called the scaled TTT.
(ii) X is NBUE (NWUE) if and only if $\phi(u) \ge (\le)\, u$ [4].
(iii) X is (a) DMRL (IMRL) if and only if $\frac{1-\phi(u)}{1-u}$ is decreasing (increasing) in u, and (b) HNBUE (HNWUE) if and only if $\phi(u) \le (\ge)\, 1 - \exp[-\frac{Q(u)}{\mu}]$ [14].

Besides this, Barlow and Campo [2] have also shown that if X is IFRA then $\frac{\phi(u)}{u}$ is decreasing in u, which is not a sufficient condition. This result was further refined in [19], showing that X is "new better than renewal used" in the reversed hazard rate order, that is $\frac{F(t)}{\int_0^t \bar{F}(u)du}$ is increasing in $t \ge 0$, if and only if $\phi(t)$ is anti-star shaped.

Proposition 2.1. *A necessary and sufficient condition for X to be IFRA (DFRA) is that*

$$t(u) \le (\ge) -\frac{Q(u)}{\log(1-u)} \tag{2.2}$$

where $t(u) = T'(u) = (1-u)q(u)$.

Proof.

$$\begin{aligned} X \text{ is IFRA } &\Leftrightarrow -\frac{\log \bar{F}(x)}{x} \text{ is increasing} \\ &\Leftrightarrow -\frac{\log(1-u)}{Q(u)} \text{ is increasing} \\ &\Leftrightarrow H(u) \geq -\frac{\log(1-u)}{Q(u)} \\ &\Leftrightarrow t(u) \leq -\frac{Q(u)}{\log(1-u)}. \end{aligned}$$

Three other ageing classes based on the hazard rate are NBUFRA, NBUFR [20] and IFRA $*t_0$ [18]. Recall that (a) X is NBUFR iff $h(0) \leq h(x)$ or $H(0) \leq H(u)$, (b) X is NBUFRA if $h(0) \leq x^{-1}\int_0^x h(t)dt$ or $H(0) \leq \frac{-\log(1-u)}{Q(u)}$ and (c) X is IFRA $*t_0$ iff $\bar{F}(bx) \geq \bar{F}^b(x)$ for $x \geq t_0 > 0$ and $\frac{t_0}{x} \leq b < 1$, when $x = Q(u)$ and $t_0 = Q(u_0)$. We have the following proposition, proved along the lines of Proposition 2.1. □

Proposition 2.2. *A lifetime random variable X is:*

(a) *NBUFR iff $t(u) \leq t(0)$.*
(b) *NBUFRA iff $-\frac{\log(1-u)}{Q(u)} \leq t(0)$.*
(c) *IFRA $*t_0$ iff for all $u \geq u_0$,*

$$\int_0^u \frac{t(p)dp}{(1-p)} \geq \frac{Q(u_0)\log(1-u)}{\log(1-u_0)}, \quad 0 \leq u_0 \leq 1.$$

The dual classes NWUFR and NWUFRA are obtained by reversing the above inequalities.

The mean residual life of X is given by

$$m(x) = E(X - x|X > x) = \frac{1}{\bar{F}(x)}\int_0^\infty \bar{F}(t)dt$$

and accordingly in the quantile form, the mean residual quantile function is [21]

$$M(u) = m(Q(u)) = (1-u)^{-1}\int_u^1 (1-p)q(p)dp.$$

We recall that X is said to be DMRLHA (IMRLHA) [8] iff $[\frac{1}{x}\int_0^x \frac{dt}{m(t)}]^{-1}$ is decreasing (increasing) in x or $\int_0^u \frac{q(p)}{M(p)}dp / \int_0^u q(p)dp$ is decreasing (increasing) in u.

Proposition 2.3. *A necessary and sufficient condition for X to be DMRLHA (IMRLHA) is that $-\frac{1}{Q(u)}\log(1-\phi(u))$ is decreasing (increasing) in u.*

Proof.

$$\begin{aligned} X \text{ is DMRLHA } &\Leftrightarrow \left[\frac{1}{x}\int_0^x \frac{dt}{m(t)}\right]^{-1} \text{ is decreasing in } x \\ &\Leftrightarrow \frac{1}{Q(u)}\int_0^u \frac{q(p)}{M(p)}dp \text{ is increasing in } u \\ &\Leftrightarrow \frac{1}{Q(u)}\int_0^u \frac{(1-p)q(p)}{(1-p)M(p)}dp \text{ is increasing} \\ &\Leftrightarrow \frac{1}{Q(u)}\int_0^u -\frac{d\log}{dp}\left(\int_p^1 (1-t)q(t)dt\right) dp \text{ is increasing.} \end{aligned}$$

$$\Leftrightarrow \frac{1}{Q(u)} \int_0^u \left(-\frac{d \log(1-\phi(p))}{dp}\right) dp \text{ is decreasing.}$$
$$\Leftrightarrow -\frac{1}{Q(u)} \log(1-\phi(u)) \text{ is decreasing.} \quad \square$$

Remark 2.1. The appearance of $Q(u)$ in the above results does not require an additional input as $Q(u) = \int_0^u \frac{t(p)dp}{1-p}$.

We say that X is "used better than aged", UBA (UWA) if $\mu = E(X) < \infty$, $m(\infty) < \infty$ and

$$\bar{F}(x+t) \geq (\leq) \bar{F}(t) \exp\left(-\frac{x}{m(\infty)}\right), \quad x, t \geq 0 \tag{2.3}$$

or $Q(v + (1-v)u) - Q(v) \leq -M(1) \log(1-u)$, $0 \leq u, v < 1$, $0 < M(1) < \infty$, where $t = Q(v)$. Further X is UBAE if $m(x) \geq m(\infty)$ [1] or $M(u) \geq M(1)$, $0 < M(1) < \infty$. These two concepts are translated to forms in terms of TTT as follows.

Proposition 2.4. (a) *X is UBA* $\Leftrightarrow \int_v^{v+(1-v)u} \frac{t(p)}{1-p} dp \leq -\frac{1}{M(1)} \log(1-u)$, *for* $0 \leq u, v < 1$.
(b) *X is UBAE* $\Leftrightarrow T(u) \leq \mu - (1-u)M(1)$.

Proof. Part (a) follows from the fact that X is UBA iff

$$Q(v + (1-v)u) - Q(v) \geq -\frac{1}{M(1)} \log(1-u)$$

and (b) from the identity

$$T(u) = \mu - (1-u)M(u). \tag{2.4}$$

Another ageing criterion is based on the variance residual life

$$\sigma^2(x) = \frac{2}{\bar{F}(x)} \int_x^\infty \int_u^\infty \bar{F}(t) dt du - m^2(x).$$

From [21],

$$V(u) = \sigma^2(Q(u)) = \frac{1}{1-u} \int_u^1 M^2(p) dp$$

and the monotonicity of $\sigma^2(x)$ in x is the same as the monotonicity of $V(u)$ in u. Accordingly, we say that X is DVRL (decreasing variance residual life) or IVRL (increasing variance residual life) iff $V(u)$ is decreasing (increasing) in u. See [11,25]. $\square$

Proposition 2.5. *X is DVRL (IVRL)*

$$\Leftrightarrow \int_u^1 \left(\frac{1-\phi(p)}{1-p}\right)^2 dp \leq (\geq) \frac{(1-\phi(u))^2}{1-u}.$$

Proof.

$$\begin{aligned} X \text{ is DVRL } &\Leftrightarrow (1-u)^{-1} \int_u^1 M^2(p) dp \text{ is decreasing in } u \\ &\Leftrightarrow \int_u^1 M^2(p) dp \leq (1-u) M^2(u) \\ &\Leftrightarrow \int_u^1 \left(\frac{\mu - T(p)}{1-p}\right)^2 dp \leq (1-u) \frac{(\mu - T(u))^2}{(1-u)^2} \\ &\Leftrightarrow \int_u^1 \left(\frac{1-\phi(p)}{1-p}\right)^2 dp \leq \frac{(1-\phi(u))^2}{1-u} dp. \quad \square \end{aligned}$$

The NBU and its variants can also be expressed in terms of TTT. Recall the following definitions.

(a) X is NBU (NWU) if $\bar{F}(x+t) \le (\ge) \bar{F}(x)\bar{F}(t)$ for all $x, t > 0$.
(b) X is NBU-t_0 (NWU-t_0) if $\bar{F}(x+t_0) \le (\ge) \bar{F}(x)\bar{F}(t_0)$ for all x and some $t_0 > 0$ [12].
(c) X is NBU$*t_0$ (NWU$*t_0$) if $\bar{F}(x+y) \le (\ge) \bar{F}(x)\bar{F}(y)$ for all x and $y \ge t_0 > 0$ [18].

Converting (a) in terms of quantile functions, the condition for X to be NBU can be seen as

$$Q(u+v-uv) \le Q(u) + Q(v), \quad 0 \le u < v < 1.$$

This and similar considerations in cases (b) and (c) lead to the next proposition.

Proposition 2.6. (a)

$$X \text{ is NBU (NWU)} \Leftrightarrow \int_{v}^{u+v-uv} \frac{t(p)dp}{1-p} \le (\ge) \int_0^u \frac{t(p)}{1-p}dp + \int_0^v \frac{t(p)}{1-p}dp. \tag{2.5}$$

(b) *X is NBU-u_0 $\Leftrightarrow$ inequality (2.5) with $u = u_0$.*
(c) *X is NBU $*u_0$ $\Leftrightarrow$ inequality (2.5) for some $u \ge u_0$ and all u.*

The need for the above characterization is illustrated in the following example.

Example. Let X be distributed according to the power–Pareto law specified by the quantile function

$$Q(u) = \frac{Cu^2}{(1-u)^3}, \quad C > 0,\ 0 \le u < 1.$$

In this case $\bar{F}(x)$ cannot be expressed in algebraic form by inverting

$$x = \frac{Cu^2}{(1-u)^3}$$

and $u = F(x)$ has to be found by numerical solution of the above equation for chosen values of x. Hence the conventional definition of NWU,

$$\bar{F}(x+t) \ge \bar{F}(x)\bar{F}(t)$$

for all $x, t > 0$, is difficult to verify. On the other hand X is NWU according to (2.5) if and only if

$$\int_0^{u+v-uv} \frac{t(p)dp}{(1-p)} - \int_0^u \frac{t(p)}{1-p}dp - \int_o^v \frac{t(p)}{1-p}dp \ge 0.$$

The expression on the left side for the power–Pareto law simplifies to

$$\frac{C(u+v-uv)^2}{(1-u)^2(1-v)^2} - \frac{Cu^2}{(1-u)^3} - \frac{Cv^2}{(1-v)^3}$$

and to

$$t^2(1-s) + s^2(1-t) + t(1-s) + s + 1$$

where $t = 1-u$ and $v = 1-s$. For all $0 \le u, v < 1$, the above expression in non-negative and, therefore, X is NWU. It may be noticed that the role of the general power–Pareto model in lifetime data analysis is demonstrated in [21,26]. Further, when the quantile function does not give an analytically tractable survival function, Propositions 2.1 to 2.6 become handy for studying the ageing behaviour.

The other ageing concepts can also be characterized in a similar manner. Expressions for the ageing concepts in terms of quantile functions given above, the interrelationships and examples are discussed in [25].

3. Ordering IFR distributions

The scaled TTT, $\phi(u)$, is a strictly increasing function in the unit square with $\phi(0) = 0$ and $\phi(1) = 1$. When X is IFR (DFR), $\phi(u) \geq (\leq u)$, and in the exponential case, $\phi(u) = u$. This shows that

$$I = \int_0^1 \phi(u)du = 1 - \frac{L_2}{L_1} = 1 - \tau_2 \tag{3.1}$$

where τ is the L-coefficient of variation. See Section 5 for a detailed discussion on the measure I in terms of L-moments. From (3.1) and $0 \leq I < 1$, we have the following result.

Proposition 3.1. *If X is IFR (DFR), then $I \geq (\leq) \frac{1}{2}$. The exponential distribution corresponds to $I = \frac{1}{2}$.*

Interpreting geometrically, the distribution becomes more and more IFR when the $\phi(u)$ curve moves closer to the upper side of the unit square. Thus the relative IFR-ness of two distributions can be ascertained graphically by looking at their $\phi(u)$ curves or ascertained numerically by comparing their I values. In practice the empirical counterpart of $\phi(u)$ is the scaled total time on test statistic

$$\phi_{r:n} = \frac{\sum_{j=1}^{r}(n-j+1)(x_{j:n} - x_{j-1:n})}{\sum_{j=1}^{n}(n-j+1)(x_{j:n} - x_{j-1:n})}.$$

It gives approximately an idea about the magnitude of I. In the above expression, n is the number of units put under test, $X_{j:n}$ are order statistics of the time to failure and we were looking at the total time up to the rth failure.

A reflexive, antisymmetric and transitive relation $\leq$ on a set is called a partial order. If in addition, for any two elements a and b in the set, either $a \leq b$ or $b \leq a$ holds, then the partial order is said to be a total order. Let X and Y be lifetime random variables with finite expectations and indices I_X and I_Y. Then we have the following total ordering of X and Y.

Definition 3.1. We say that X is more IFR (less DFR) than Y in the weak sense if $I_X \geq I_Y$.

Example. Let X be distributed as a power distribution with $F_X(x) = x^c$, $0 \leq x \leq 1$, $c \geq 1$, and Y as a beta distribution specified by $\bar{F}(x) = (1-x)^c$, $0 \leq x \leq 1$. Then

$$I_X = \frac{2c}{1+2c} \quad \text{and} \quad I_Y = \frac{c+1}{2c+1}.$$

Since $I_X \geq I_Y$ for $c > 1$, we conclude that the power distribution is more IFR than the beta distribution. As both are IFR, the above comparison is between two IFR models. Suppose $c < 1$. Then X is bathtub shaped and Y is IFR. A comparison between them is still possible but $I_X \leq I_Y$. Notice also that when X is IFR, $I_X \geq \frac{1}{2}$. But the converse need not be true as the power distribution for c in $(\frac{1}{2}, 1)$ illustrates.

Let X and Y have continuous distribution functions with $F_X(0) = 0 = F_Y(0)$ and F_Y be strictly increasing on the interval support. From [15], X is more IFR than Y, denoted by $X \leq_c Y$, iff $F_Y^{-1}(F(x))$ is a convex function of x. It is of interest to know the relationship that this total order has with $I_X \leq I_Y$. It is known from the implications given on p. 223 and p. 224 of [28] that

$$\begin{aligned} X \leq_c Y &\Rightarrow \phi_X(u) \geq \phi_Y(u) \quad \text{for all } u \text{ in } [0, 1] \\ &\Rightarrow I_X \geq I_Y. \end{aligned}$$

Thus the I-measure is consistent with the existing notion for comparison on the basis of the IFR order.

4. Construction of bathtub models

There have been several techniques proposed for constructing models that have non-monotonic hazard rates; see [17] for various distribution functions and [25,23,24] for quantile function models. In this section, we propose and discuss such models on the basis of TTT. The TTT of order n of X is defined recursively as [22]

$$T_n(u) = \int_0^u (1-p)t_{n-1}(p)dp, \quad n = 1, 2, \ldots$$

with $T_0(u) = Q(u)$, $t_n(u) = \frac{dT_n(u)}{du}$ and $\mu_{n-1} = \int_0^1 T_{n-1}(p)dp < \infty$. We see that

$$t_n(u) = (1-u)^n q(u).$$

Denoting by X_n the random variable with quantile function $T_n(u)$, the hazard quantile function $H_n(u)$ of X_n and $H(u)$ of X are related by

$$H(u) = (1-u)^n H_n(u), \quad n = 0, 1, \ldots.$$

Thus X_n has a hazard quantile function larger than that of X_{n-1} and hence the successive TTT's specify distributions that have larger hazard quantile functions than their predecessors.

Hence if we start with a known DFR distribution it is possible that one of the successive transformed distributions possesses a bathtub shaped hazard quantile function. The procedure is illustrated by considering X to have the Weibull distribution

$$\bar{F}(x) = \exp\left[-\left(\frac{x}{\alpha}\right)^\beta\right], \quad x > 0,\ \alpha > 0,\ \beta \le 1.$$

Using the quantile function of X,

$$Q(u) = \alpha(-\log(1-u))^{\frac{1}{\beta}},$$

we have

$$q(u) = \frac{\alpha}{\beta(1-u)}(-\log(1-u))^{\frac{1}{\beta}-1}$$

and hence

$$t_n(u) = \frac{\alpha(1-u)^{n-1}}{\beta}(-\log(1-u))^{\frac{1}{\beta}-1}.$$

Differentiating the above with respect to u, we have

$$t_n'(u) = \frac{\alpha}{\beta}(1-u)^{n-2}(-\log(1-u))^{\frac{1}{\beta}-2}\left[\frac{1}{\beta} - 1 + (n-1)\log(1-u)\right].$$

Thus when $\beta \le 1$, $T_{n+1}(u)$ is convex on $[0, u_0]$ and concave on $[u_0, 1]$ where

$$u_0 = 1 - \exp\left(\frac{\beta - 1}{(n-1)\beta}\right)$$

and hence we have a bathtub shaped hazard quantile function, for each $n > 1$. Specializing to $n = 2$, we have the distribution with the quantile function

$$T_2(u) = \frac{\alpha}{\beta}\int_0^{-\log(1-u)} e^{-t}t^{\frac{1}{\beta}-1}dt$$

and with the hazard quantile function

$$H_2(u) = \frac{\beta}{\alpha}(1-u)^{-1}(-\log(1-u))^{1-\frac{1}{\beta}}$$

which has bathtub shape with the change point at $u_0 = 1 - \exp(\frac{\beta-1}{\beta})$. To study the distributional aspects of the model, we rely on the first four L-moments (see the next section):

$$L_1 = E(X) = \alpha\Gamma\left(\frac{1}{\beta}\right)\beta^{-1}2^{-\frac{1}{\beta}},$$

$$L_2 = \alpha\beta^{-1}[2^{-\frac{1}{\beta}} - 3^{-\frac{1}{\beta}}]\Gamma\left(\frac{1}{\beta}\right),$$

$$L_3 = \alpha\beta^{-1}[2^{-\frac{1}{\beta}} - 3(3^{-\frac{1}{\beta}}) + 2(4^{-\frac{1}{\beta}})]\Gamma\left(\frac{1}{\beta}\right),$$

and

$$L_4 = \alpha\beta^{-1}[2^{-\frac{1}{\beta}} - 6(3^{-\frac{1}{\beta}}) + 10(4^{-\frac{1}{\beta}}) - 5(5^{-\frac{1}{\beta}})]\Gamma\left(\frac{1}{\beta}\right).$$

We have simple expressions for the L-skewness and L-kurtosis:

$$\tau_3 = \frac{L_3}{L_2} = 1 - \frac{2\left(1 - \left(\frac{3}{4}\right)^{\theta}\right)}{\left(\frac{3}{2}\right)^{\theta} - 1}, \quad \theta = \beta^{-1}$$

and

$$\tau_4 = \frac{L_4}{L_2} = 1 - \frac{5 - \left(\frac{3}{4}\right)^{\theta} + 5\left(\frac{3}{5}\right)^{\theta}}{\left(\frac{3}{2}\right)^{\theta} - 1}.$$

Estimation of the parameters is easily done. Equating τ_3 with the sample L-skewness and solving, we have the estimate $\hat{\beta}$ of β. Then from L_1, α is solved from

$$\bar{X} = \alpha\hat{\beta}^{-1}2^{-\frac{1}{\hat{\beta}}}\Gamma\left(\frac{1}{\hat{\beta}}\right).$$

Choosing more flexible distributions as initial models by the above method we can have more general bathtub distributions.

5. *L*-moments

In this section, we show that the L-moments of a distribution can be derived directly from the TTT. Nair and Vineshkumar [26] have discussed the relevance and role of L-moments in reliability modelling. Recall that the rth L-moment is defined in terms of the quantile function as [13]

$$L_r = \int_0^1 \sum_{k=0}^{r-1}(-1)^{r-k}\binom{r}{k}\binom{r+k}{k}u^k Q(u)du.$$

Specializing to $r = 1, 2, 3, 4$,

$$L_1 = \int_0^1 Q(u)du = \mu,$$

$$L_2 = \int_0^1 (2u-1)Q(u)du,$$

$$L_3 = \int_0^1 (6u^2 - 6u + 1)Q(u)du,$$

and

$$L_4 = \int_0^1 (20u^3 - 30u^2 - 12u - 1)Q(u)du.$$

Using integration by parts and substituting in $t(u) = (1-u)q(u)$, we have the formulae for L-moments in terms of TTT as follows:

$$L_1 = \int_0^1 t(p)dp = T(1),$$
$$L_2 = \int_0^1 pt(p)dp = \mu - \int_0^1 T(p)dp,$$
$$L_3 = \int_0^1 p(2p-1)t(p)dp = \mu - \int_0^1 (4p-1)T(p)dp,$$
$$L_4 = \int_0^1 p(1-5p-5p^2)t(p)dp = \mu - \int_0^1 (1-10p+15p^2)T(p)dp.$$

Various descriptive measures for the model are readily available from the above four moments (see the previous section). The index I defined in the previous section becomes

$$I = \int_0^1 \frac{T(p)dp}{\mu} = 1 - \frac{L_2}{L_1} = 1 - \tau_2.$$

The ratio $\frac{L_2}{L_1}$ is called the L-coefficient of variation, which always lies between 0 and 1 is the measure analogous to the conventional coefficient of variation for relative spread. Thus the measure of relative IFR-ness is expressed in terms τ_2, and X is more IFR than Y in the weak sense of X having a smaller L-coefficient of variation than Y.

To conclude the study, we note that some simple properties of the TTT were presented that could be used in reliability analysis. When the functional form of TTT is known, all of the ageing properties can be deduced along with all the distributional characteristics of the lifetime random variable. Like the hazard rate, mean residual life and similar functions, TTT also uniquely determines the distribution. As noted above, it gives more information relevant to the analysis than the other functions in certain cases.

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