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ON THE ASYMPTOTIC PROPERTIES OF A KERNEL TYPE QUANTILE ESTIMATOR FROM CENSORED SAMPLES*

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Abstract: Some asymptotic results for a kernel type estimator of the quantile function from right-censored data are obtained. The estimator is defined by $Q_n(p)=h_n^{-1}\int_0^1 \hat{Q}_n(t)K((t-p)/h_n)\,dt$, which is smoother than the usual product-limit quantile function $\hat{Q}_n(p)=\inf\{t: \hat{F}_n(t)\geq p\}$, where $\hat{F}_n$ denotes the product-limit estimator of the lifetime distribution F_0. Under the random censorship model and general conditions on h_n, K, F_0, the asymptotic normality of $Q_n(p)$ is proven. In addition, an approximation to Q_n is shown to be asymptotically uniformly equivalent to Q_n in mean square.

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1. Introduction

In reliability and medical studies, it is often of interest to estimate various quantiles of the unknown lifetime distribution. In particular, the median lifetime and extreme quantiles are of interest to the experimenter in such studies. In many life testing and medical follow-up experiments, however, arbitrarily right-censored data arise, and it is important to be able to estimate the quantiles of interest based on the censored data. For such data, some kernel-type quantile estimators are considered in this paper which give smoother estimates than the usual product-limit quantile function.

For any probability distribution function G, denote the quantile function by $Q(p)\equiv G^{-1}(p)=\inf\{x: G(x)\geq p\}$, $0\leq p\leq 1$. For a random (uncensored) sample $Y_1,\dots,Y_n$ from G, the sample quantile function $G_n^{-1}(p)=\inf\{x: G_n(x)\geq p\}$, $0\leq p\leq 1$, has been used to estimate $Q(p)$, where G_n denotes the sample distribution

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function. Csörgő (1983) gave many of the known results concerning $G_n^{-1}(p)$. Also, Falk (1984) studied the relative deficiency of the sample quantile with respect to kernel-type estimators, and Falk (1985) obtained asymptotic normality for kernel estimators. Yang (1985) has obtained some convergence properties of kernel estimators of $Q(p)$ and gave some simulation results comparing kernel-type estimators with other estimators. For arbitrarily right-censored data, Sander (1975) proposed estimation of $Q(p)$ by the quantile function of the product-limit estimator, and she and Cheng (1981) derived some asymptotic properties of that estimator. Also, Csörgő (1983) presented strong approximation results for that estimator.

Recently, Padgett (1985) studied a smoothed nonparametric estimator of $Q(p)$ from arbitrarily right-censored data based on the kernel method. It was shown that his estimator, mentioned briefly by Parzen (1979, p. 119), was strongly consistent, and a small Monte Carlo study was performed to compare the estimator with the product-limit estimator. In addition, a simple approximation to this kernel estimator was shown to be almost surely asymptotically equivalent to it.

The purpose of this paper is to further study the asymptotic properties of the estimators proposed by Padgett (1985). In particular, the asymptotic normality will be proven, and the asymptotic equivalence in mean square of the estimator and its approximation will be shown under general conditions on the kernel function, bandwidth sequence, lifetime distribution, and censoring mechanism.

2. Notation and preliminaries

Let $X_1^0, X_2^0, \ldots, X_n^0$ denote the true survival times of n items or individuals which are censored on the right by a sequence $U_1, U_2, \ldots, U_n$, which in general may be either constants or random variables. It is assumed that the X_i^0's are nonnegative independent identically distributed random variables with common unknown distribution function F_0 and unknown quantile function

$$Q^0(p) \equiv \xi_p^0 = \inf\{t: F_0(t) \geq p\}, \quad 0 \leq p \leq 1.$$

The observed right-censored data are denoted by the pairs (X_i, Δ_i), $i = 1, \ldots, n$, where

$$X_i = \min\{X_i^0, U_i\}, \qquad \Delta_i = \begin{cases} 1 & \text{if } X_i^0 \leq U_i, \\ 0 & \text{if } X_i^0 > U_i. \end{cases}$$

For the asymptotic results of this paper, the random right-censorship model will be assumed, that is, $U_1, \ldots, U_n$ constitute a random sample from a distribution H (usually unknown) and are independent of $X_1^0, \ldots, X_n^0$. The distribution function of each X_i, $i = 1, \ldots, n$, is then $F = 1 - (1 - F_0)(1 - H)$.

A popular estimator of the survival function $S_0(t) = 1 - F_0(t)$ based on (X_i, Δ_i), $i = 1, \ldots, n$, is the product-limit estimator, proposed by Kaplan and Meier (1958) as the 'nonparametric maximum likelihood estimator'. Let (Z_i, Δ_i'), $i = 1, \ldots, n$, denote the ordered X_i's along with their corresponding Δ_i's. The product-limit estimator of $S_0(t)$, shown to be 'self-consistent' by Efron (1967), is defined by

$$\hat{P}_n(t)=\begin{cases}1, & 0\le t\le Z_1,\\ \prod_{i=1}^{k-1}\left(\dfrac{n-i}{n-i+1}\right)^{\Delta_i'}, & Z_{k-1}<t\le Z_k,\ k=2,\dots,n,\\ 0, & t>Z_n.\end{cases}$$

Denote the product-limit estimator of $F_0(t)$ by $\hat{F}_n(t)=1-\hat{P}_n(t)$, and let s_j denote the jump of $\hat{P}_n$ at Z_j, that is

$$s_j=\begin{cases}1-\hat{P}_n(Z_2), & j=1,\\ \hat{P}_n(Z_j)-\hat{P}_n(Z_{j+1}), & j=2,\dots,n-1,\\ \hat{P}_n(Z_n), & j=n.\end{cases}$$

Note that $s_j=0$ if and only if $\Delta_j'=0$, $j<n$, i.e. whenever Z_j is a censored observation. Also, denote $S_i\equiv\hat{F}_n(Z_{i+1})=\sum_{j=1}^{i}s_j$, $i=1,\dots,n$, with $S_0\equiv 0$, $Z_0\equiv 0$, and $Z_{n+1}\equiv Z_n+\varepsilon$, for some positive constant ε.

It is natural to estimate ξ_p^0 by the product-limit quantile function $\hat{Q}_n(p)\equiv\hat{\xi}_p^0=\inf\{t: \hat{F}_n(t)\ge p\}$. Sander (1975) and Cheng (1981) obtained asymptotic normality results for $\hat{\xi}_p^0$. Padgett (1985) smoothed $\hat{Q}_n$ by the kernel method to obtain the estimator

$$\begin{aligned}Q_n(p)&=h_n^{-1}\int_0^1 \hat{Q}_n(t)K((t-p)/h_n)\,\mathrm{d}t\\ &=h_n^{-1}\sum_{i=1}^{n} Z_i\int_{S_{i-1}}^{S_i} K((t-p)/h_n)\,\mathrm{d}t,\end{aligned} \tag{2.1}$$

where K is an appropriate kernel function and $\{h_n\}$ is a bandwidth sequence. Also, a simpler kernel-type estimator which is an approximation to (2.1) was defined by

$$Q_n^*(p)=h_n^{-1}\sum_{i=1}^{n} Z_i s_i K((S_i-p)/h_n). \tag{2.2}$$

Note that only the uncensored observations actually appear in the sums of (2.1) and (2.2).

In the next section, the asymptotic normality of $Q_n(p)$ and $Q_n^*(p)$ will be obtained. The following general conditions on the kernel function, the bandwidth sequence, and the lifetime and censoring distributions will be assumed:

(h.1) $h_n\to 0$ as $n\to\infty$.

(K.1) $K(x)$ is a bounded probability density function which has finite support, i.e. $K(x)=0$ for $|x|>c$ for some $c>0$.

(K.2) K is symmetric about zero.

(K.3) K satisfies a Lipschitz condition, i.e. there exist a constant Γ such that for all x, y,

$$|K(x)-K(y)| \leq \Gamma |x-y|.$$

(F.1) F_0 is continuous with density function f_0.

(F.2) $H(T_{F_0}) \leq 1$, where $T_{F_0} = \sup\{t: F_0(t) < 1\}$.

It should be noted that these conditions are not prohibitive and (F.1) and (F.2) are similar to conditions required by Cheng (1981). Also, (F.2) insures that observations will be available from the entire support of F_0, a common condition in random right-censorship models.

3. The main results

In this section, the main results are summarized in Theorems 3.1 and 3.2. The proofs will be presented in the next section. Theorem 3.1 gives conditions for the asymptotic normality of $Q_n(p)$. The asymptotic uniform mean-squared equivalence of Q_n and Q_n^* will be shown in Theorem 3.2.

Theorem 3.1. *Assume, in addition to conditions* (h.1), (K.1)–(K.2), (F.1), *and* (F.2), *that the derivative f_0' is continuous at ξ_p^0 and $f_0(\xi_p^0) > 0$. Suppose $\{h_n\}$ is such that $n^{1/4} h_n \to 0$ as $n \to \infty$. Then for $0 < p < T$, where $T < 1$, as $n \to \infty$, $\sqrt{n}[Q_n(p) - Q^0(p)] \to Z$ in distribution, where Z is a normally distributed random variable with mean zero and variance*

$$\sigma_p^2 = (1-p)^2 \int_0^{\xi_p^0} [1-F(u)]^{-2} \frac{\mathrm{d}F_0^*(u)}{f_0^2(\xi_p^0)}$$

with $1-F(u) = [1-F_0(u)][1-H(u)]$ and $F_0^*(u) = P(X_i \leq u, \Delta_i = 1)$, the subdistribution function of the uncensored observations.

Note that an example of a bandwidth sequence that satisfies the conditions of Theorem 3.1 is $h_n = cn^{-\delta}$ with $\delta > \frac{1}{4}$.

The next theorem gives some conditions for which Q_n^* and Q_n are asymptotically uniformly equivalent in mean square.

Theorem 3.2. *Suppose F_0 and H are continuous and* (h.1), (K.1), *and* (K.3) *hold. Assume $E(X_1^{2q}) < \infty$ for some $q > 1$, where $X_1 = \min\{X_1^0, U_1\}$. Let η be such that $[1-F_0(\eta)][1-H(\eta)] > 0$ and let $T^* = F_0(\eta)$. Then for all $T \in [0, T^*)$,*

$$\lim_{n \to \infty} E\left[\sup_{0 \leq p \leq T} |Q_n^*(p) - Q_n(p)|^2\right] = 0,$$

provided $n^{1/2} h_n^4 \to \infty$ as $n \to \infty$.

4. Proofs of theorems

The following two lemmas will be needed in the proof of Theorem 3.1.

In this section, $\{K(s,t): 0\le s\le t,\ t\ge 0\}$ will denote the generalized Kiefer process as stated by Csörgő (1983, Ch. 8).

Lemma 1. *For $0<p<T$, where $T<1$, and $\delta<\min\{T-p,p\}$, as $n\to\infty$,*

$$\sup_{|h|<\delta} |n^{-1/2}[K(p+h,n)-K(p,n)]|\to 0 \quad \textit{in probability.}$$

The proof of Lemma 1 follows the same argument as the proof of Theorem 8.2.1 of Csörgő (1983).

Lemma 2. *Suppose the derivative f_0' is continuous at ξ_p^0 and $f_0(\xi_p^0)>0$. Under assumptions* (h.1), (K.1), (K.2), (F.1) *and* (F.2), *for* $0<p<1$,

$$\left|\int_0^1 [q_n(t)-q_n(p)]h_n^{-1}K\left(\frac{t-p}{h_n}\right)\mathrm{d}t\right|\to 0 \quad \textit{in probability}$$

as $n\to\infty$, where $q_n(t)=n^{1/2}[\hat{Q}_n(t)-Q^0(t)]$ denotes the product-limit quantile process.

Proof. For any given $\delta>0$, there exists N such that when $n\ge N$,

$$\begin{aligned}\left|\int_0^1 [q_n(t)-q_n(p)]h_n^{-1}K\left(\frac{t-p}{h_n}\right)\mathrm{d}t\right| &= \left|\int_{A(\delta)} [q_n(t)-q_n(p)]h_n^{-1}K\left(\frac{t-p}{h_n}\right)\mathrm{d}t\right|\\ &\le \sup_{t\in A(\delta)} |q_n(t)-q_n(p)|, \end{aligned}\tag{4.1}$$

where $A(\delta)=[p-\delta,p+\delta]$. By the conditions on f_0, for δ sufficiently small, $f_0(Q^0(t))>0$ for all $t\in A(\delta)$. Hence, the right-hand side of (4.1) is less than or equal to

$$\sup_{t\in A(\delta)} |\tilde{\varrho}_n(t)-\tilde{\varrho}_n(p)|\cdot\left|\frac{1}{f_0(Q^0(t))}\right| + \sup_{t\in A(\delta)}\left|\tilde{\varrho}_n(p)\left[\frac{1}{f_0(Q^0(t))}-\frac{1}{f_0(Q^0(p))}\right]\right|,$$

where $\tilde{\varrho}_n(t)=f_0(Q^0(t))q_n(t)$.

Let

$$a=\sup_{t\in A(\delta)}\left|\frac{1}{f_0(Q^0(t))}\right|,\qquad b=\sup_{t\in A(\delta)}\left|\frac{1}{f_0(Q^0(t))}-\frac{1}{f_0(Q^0(p))}\right|.$$

From Corollary 1 of Cheng (1981), since f_0 is continuous at ξ_p^0, $\tilde{\varrho}_n(p)\to Z$ in distribution as $n\to\infty$, where Z is a normally distributed random variable with mean zero and variance

$$\sigma^2=(1-p)^2\int_0^{\xi_p^0}[1-F(u)]^{-2}\mathrm{d}F_0^*(u),$$

with $1-F(u)=[1-F_0(u)][1-H(u)]$ and $F_0^*(u)=P(X_i\le u, \Delta_i=1)$. Therefore,

$$\sup_{t\in A(\delta)}\left|\tilde{\varrho}_n(p)\left[\frac{1}{f_0(Q^0(t))}-\frac{1}{f_0(Q^0(p))}\right]\right|\le b|\tilde{\varrho}_n(p)| \tag{4.2}$$

and for given $\varepsilon>0$,

$$\limsup_{n\to\infty} P(|\tilde{\varrho}_n(p)|>\varepsilon/b)\le P(|Z|>\varepsilon/b)\le b\sigma^2/\varepsilon^2. \tag{4.3}$$

Now,

$$\begin{aligned}\sup_{t\in A(\delta)}|\tilde{\varrho}_n(t)-\tilde{\varrho}_n(p)|\cdot\left|\frac{1}{f_0(Q^0(t))}\right| &\le a\sup_{t\in A(\delta)}|\tilde{\varrho}_n(t)-\tilde{\varrho}_n(p)|\\ &\le a\left\{\sup_{t\in A(\delta)}|\tilde{\varrho}_n(t)-n^{-1/2}K(t,n)|\right.\\ &\quad+\sup_{t\in A(\delta)}|\tilde{\varrho}_n(p)-n^{-1/2}K(p,n)|\\ &\quad\left.+\sup_{t\in A(\delta)}|n^{-1/2}[K(p,n)-K(t,n)]|\right\}.\end{aligned} \tag{4.4}$$

For small enough δ, $p+\delta<T<1$ and $p-\delta\ge 0$, so that by Corollary 8.3.3 of Csörgő (1983) as $n\to\infty$,

$$\sup_{t\in A(\delta)}|\tilde{\varrho}_n(t)-n^{-1/2}K(t,n)|\to 0\quad\text{in probability}$$

and

$$\sup_{t\in A(\delta)}|\tilde{\varrho}_n(p)-n^{-1/2}K(p,n)|\to 0\quad\text{in probability.}$$

By Lemma 1, the third term on the right-hand side of inequality (4.4) converges to zero in probability for sufficiently small δ. Therefore, (4.4) converges to zero in probability as $n\to\infty$.

Finally, since b depends on δ, letting b become arbitrarily small gives from (4.2) and (4.4) that

$$\sup_{t\in A(\delta)}\left|\tilde{\varrho}_n(p)\left[\frac{1}{f_0(Q^0(t))}-\frac{1}{f_0(Q^0(p))}\right]\right|\to 0\quad\text{in probability.}$$

Thus, the result follows. □

Proof of Theorem 3.1. Analogous to the beginning of the proof of Theorem 1 of Yang (1985), write

$$\sqrt{n}[Q_n(p)-Q^0(p)]=\int_0^1[q_n(t)-q_n(p)]h_n^{-1}K\left(\frac{t-p}{h_n}\right)\mathrm{d}t$$

$$+n^{1/2}\left[\int_0^1 Q^0(t)h_n^{-1}K\left(\frac{t-p}{h_n}\right)\mathrm{d}t-Q^0(p)\right]+q_n(p), \tag{4.5}$$

where $q_n(t)=n^{1/2}[\hat{Q}_n(t)-Q^0(t)]$ as in Lemma 2.

From Lemma 2, the first term on the right-hand side of (4.5) is $o_p(1)$, which means that it converges to zero in probability as $n\to\infty$. Similar to equation (10) of Yang (1985),

$$n^{1/2}\left[\int_0^1 Q^0(t)h_n^{-1}K\left(\frac{t-p}{h_n}\right)\mathrm{d}t-Q^0(p)\right]$$

$$=o(n^{1/2}h_n^2)+n^{1/2}h_n^2Q^{0''}(p)\int_{-\infty}^{\infty}\frac{t^2}{2}K(t)\,\mathrm{d}t. \tag{4.6}$$

With the assumption that $n^{1/4}h_n\to 0$ as $n\to\infty$, (4.6) is also $o_p(1)$. Therefore, by Corollary 1 of Cheng (1981), the conclusion of the theorem follows. □

Proof of Theorem 3.2. For $0\le p\le T$, write

$$Q_n^*(p)-Q_n(p)=h_n^{-1}\sum_{i=1}^n Z_i\left[s_iK\left(\frac{S_i-p}{h_n}\right)-\int_{S_{i-1}}^{S_i}K\left(\frac{t-p}{h_n}\right)\mathrm{d}t\right].$$

When $s_i>0$, that is, Z_i is uncensored, let S_i^* be an interior point of the interval (S_{i-1},S_i) with probability one so that

$$s_iK((S_i^*-p)/h_n)=\int_{S_{i-1}}^{S_i}K((t-p)/h_n)\,\mathrm{d}t \quad \text{almost surely.}$$

Then by condition (K.3), letting I_A denote the indicator function of the set A,

$$|Q_n^*(p)-Q_n(p)|I_{[0,T]}(p)$$
$$\le h_n^{-1}\sum_{i=1}^n s_iZ_i|K((S_i-p)/h_n)-K((S_i^*-p)/h_n)|I_{[0,T]}(p)I_{[S_i^*-ch_n,1]}(p)$$
$$\le \Gamma h_n^{-2}\sum_{i=1}^n Z_is_i|S_i-S_i^*|I_{[0,T]}(p)I_{[S_i^*-ch_n,1]}(p)$$
$$\le \Gamma h_n^{-2}\sum_{i=1}^n Z_is_i^2I_{[0,T]}(p)I_{[S_i^*-ch_n,1]}(p) \quad \text{almost surely.}$$

So

$$|Q_n^*(p)-Q_n(p)|^2I_{[0,T]}(p)\le \Gamma^2h_n^{-4}\left(\sum Z_is_i^2I_{[0,T]}(p)I_{[S_i^*-ch_n,1]}(p)\right)^2. \tag{4.7}$$

Now,

$$\left(\sum Z_is_i^2I_{[0,T]}(p)I_{[S_i^*-ch_n,1]}(p)\right)^2$$
$$\le \sum Z_i^2s_i^3I_{[0,T]}(p)I_{[S_i^*-ch_n,1]}(p)$$

$$\leq \sum_{i=2}^{n+1} Z_{i-1}^2(|\hat{F}_n(Z_i)-F_0(Z_i)|+|F_0(Z_i^-)-\hat{F}_n(Z_i^-)|)^3$$

$$\cdot I_{[0,T]}(p)I_{[S_{i-2}-ch_n,1]}(p), \tag{4.8}$$

where $g(x^-)$ denotes the limit from the left at x of the function g.

There is an N such that for all $n \geq N$, $T+ch_n < T^*$ and $T_N < F_0(\eta-\varepsilon)$ where $T_N \equiv T+ch_N$. For all such n's,

$$\sum Z_{i-1}^2(|\hat{F}_n(Z_i)-F_0(Z_i)|+|F_0(Z_i^-)-\hat{F}_n(Z_i^-)|)^3 I_{[0,T]}(p)I_{[S_{i-2}-ch_n,1]}(p)$$

$$\leq 8\sum Z_{i-1}^2 \sup_{0\leq x\leq \hat{Q}_n(T_N)} |\hat{F}_n(x)-F_0(x)|^3, \tag{4.9}$$

which is independent of p. Notice also that

$$\sup_{0\leq x\leq \hat{Q}_n(T_N)+\varepsilon} |\hat{F}_n(x)-F_0(x)|^3 \leq \sup_{0\leq x\leq \eta} |\hat{F}_n(x)-F_0(x)|^3 + I_{[\hat{Q}_n(T_N)>\eta-\varepsilon]}.$$

From the exponential bound in Theorem 2 of Földes and Retjö (1981), for any $r \geq 1$,

$$\left\{ n^{1/2} \sup_{0\leq x\leq \eta} |\hat{F}_n(x)-F_0(x)|)^r,\ n\geq 1 \right\}$$

is uniformly integrable. Also,

$$E(n^{3/2}I_{[\hat{Q}_n(T_N)>\eta-\varepsilon]})^r = n^{3r/2}P[\hat{Q}_n(T_N)>\eta-\varepsilon]$$

$$= n^{3r/2}P[\hat{Q}_n(T_N)-F_0^{-1}(T_N)>\eta-F_0^{-1}(T_N)-\varepsilon].$$

Letting $\gamma=\eta-F_0^{-1}(T_N)-\varepsilon$, where ε is chosen so that $\gamma>0$,

$$P[\hat{Q}_n(T_N)-F_0^{-1}(T_N)>\gamma] \leq P[T_N>\hat{F}_n(\eta-\varepsilon)]$$

$$= P[T_N-F_0(\eta-\varepsilon)>\hat{F}_n(\eta-\varepsilon)-F_0(\eta-\varepsilon)]$$

$$\leq P[|\hat{F}_n(\eta-\varepsilon)-F_0(\eta-\varepsilon)|>F_0(\eta-\varepsilon)-T_N].$$

By the same exponential bound as in Theorem 2 of Földes and Rejtö (1981), as $n\to\infty$,

$$E(n^{3/2}I_{[\hat{Q}_n(T_N)>\eta-\varepsilon]})^r \to 0.$$

Therefore,

$$\left\{ \left(n^{1/2} \sup_{0\leq x\leq \hat{Q}_n(T_N)+\varepsilon} |\hat{F}_n(x)-F_0(x)|)^r,\ n\geq 1 \right\}$$

is uniformly integrable. By hypothesis, $E(X_1^{2q})<\infty$ for some $q>1$, so $\{(n^{-1}\sum_{i=1}^n Z_i^2)^q,\ n\geq 1\}$ is uniformly integrable. Thus, for $1/s+1/q=1$,

$$E\left(n^{3/2} \sup_{0\leq x\leq \hat{Q}_n(T_n)+\varepsilon} |\hat{F}_n(x)-F_0(x)|^3 n^{-1}\sum_{i=1}^n Z_i^2 \right)$$

$$\leq \left(E\left(n^{3/2} \sup_{0 \leq x \leq \hat{Q}_n(T_N)+\varepsilon} |\hat{F}_n(x) - F_0(x)|^3\right)^s\right)^{1/s} E\left(\left(n^{-1} \sum_{i=1}^{n} Z_i^2\right)^q\right)^{1/q}$$

$$= \mathrm{O}(1). \tag{4.10}$$

Therefore, from (4.7)–(4.10), by the hypothesis $n^{1/2}h_n^4 \to \infty$,

$$E\left[\sup_{0 \leq p \leq T} |Q_n^*(p) - Q_n(p)|^2\right] = \mathrm{o}(1),$$

completing the proof. □

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