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ELSEVIER Applied Mathematics and Computation 149 (2004) 351–358

APPLIED MATHEMATICS AND COMPUTATION

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Testing exponentiality against NBUFR (NWUFR)

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Abstract

In the spirit of [J. Statist. Planning Inf. 92 (2001) 121], a moment inequality for the class of new better than used failure rate NBUFR (NWUFR) of aging distributions is derived. From this inequality we constructed a new family of test statistics. We used this family in testing exponentiality against NBUFR (NWUFR). The asymptotic normality of the proposed statistic is presented. Pitman asymptotic efficacy, the power and critical values of the proposed statistic are calculated. It is shown that, the proposed statistic has a high asymptotic relative efficiency with respect to tests of other classes for commonly used alternatives.

1. Introduction

Let X be the lifetime of a device with distribution function (df) $F(x) = P(X \leqslant x)$ and survival function (sf) $\overline{F}(x) = P(X > x)$. In practice, X is often assumed (but need not be) to be absolutely continuous with probability density function (pdf) $f(x) = F'(x)$. The most commonly applied concepts of positive aging are in terms of failure rate, $r(t)$, $t \geqslant 0$, of the distribution. In this paper we provide one more criterion describing positive aging in terms of the

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0096-3003/$ - see front matter
doi:10.1016/S0096-3003(03)00143-7

failure (hazard) rate. Formally the NBUFR class is defined in the following definition, cf. Deshpande et al. [7] and Abouammoh and Ahmed [1].

Definition 1.1. An aging r.v. $X \geqslant 0$ is said to have NBUFR (NWUFR) if $r(0) \leqslant (\geqslant) r(x)$, for all $x \geqslant 0$, where $r(x) = f(x)/\overline{F}(x)$, $x \geqslant 0$, is the failure rate at $X = x$ and $f(0) > 0$.

Note that the mean μ of an aging r.v. X is called mean time to failure, cf. Zacks [11], and it is expected to be finite since otherwise studying aging has no meaning. Thus throughout this paper it is assumed that $\mu < \infty$.

One of the oldest problem in aging distributions is testing exponentiality against the known classes of aging distributions such as IFR, IFRA, NBU and others, for example see Proschan and Pyke [10], Hollander and Proschan [6], Ahmad [2,3] among others. But, with respect to the NBUFR class, we think that in the literature no one tested it before against exponentiality. So, we will compare the results of the NBUFR class with the NBU. Further, NBUFR is wider than NBU, since every NBU (NWU) distribution is NBUFR (NWUFR). It is known that any k-out-of-n, $1 \leqslant k < n$, system has the NBUFR property, cf. Abouammoh and Ahmed [1]. That encourage us to test exponentiality against NBUFR (NWUFR).

In this paper we test H_0: F is $\exp(\mu)$ against the alternative H_1: F is NBUFR (NWUFR) and is not exponential.

The moment inequality is derived in Section 2. In Section 3, we use this inequality to construct a family of test statistics that depends on a non-negative integer valued parameter. The choice of this parameter is also addressed in the context.

2. A moment inequality

The following theorem gives the moment inequality deals with the NBUFR (NWUFR) class.

Theorem 2.1. *If F is NBUFR (NWUFR), then for all integers $r \geqslant 0$*

$$\mu_r \geqslant (\leqslant) \frac{f(0)\mu_{r+1}}{r+1}, \quad f(0) > 0, \tag{2.1}$$

where $\mu_s = E(X^s)$, $s \geqslant 0$.

Proof. Since F is NBUFR (NWUFR), then from Definition (1.1), we get

$$\overline{F}(x) f(0) \leqslant (\geqslant) f(x).$$

Hence

$$\int_0^\infty x^r f(0)\overline{F}(x)\,\mathrm{d}x \leqslant (\geqslant) \int_0^\infty x^r f(x)\,\mathrm{d}x = \mu_r. \tag{2.2}$$

But

$$\int_0^\infty x^r\overline{F}(x)\,\mathrm{d}x = E\int_0^\infty x^r I(X > x)\,\mathrm{d}x = \frac{E(X^{r+1})}{r+1} = \frac{\mu_{r+1}}{r+1}.$$

Thus the proof of the theorem is completed. □

3. Testing against NBUFR (NWUFR) alternatives

This section is divided into two main subsections. The first one is concerned with the construction of the proposed statistic as a classical U-statistic, discussing its asymptotic normality and explains how one can use it as an application of testing of hypotheses. In the second subsection, the simulated upper percentile values for 90, 95, and 99 of the proposed statistic are presented.

3.1. The U-statistic test procedure

The main aim in this section is to test

$\mathrm{H}_0 : F$ is exponential with mean μ

against

$\mathrm{H}_1 : F$ is NBUFR (NWUFR) and is not exponential.

From Theorem 2.1 with $r \geqslant 0$, a measure of departure from H_0 in favor H_1 can be defined by

$$\delta_r = \mu_r - \frac{f(0)\mu_{r+1}}{r+1}. \tag{3.1}$$

To make the test scale invariant we use $\Delta_r = \delta_r/\mu^r$, which can be estimated by

$$\hat{\Delta}_r = \frac{\hat{\delta}_r}{\bar{X}^r},\ \hat{\delta}_r = \frac{2}{n(n-1)}\sum_{i<j}\left[X_i^r - \frac{X_i^{r+1}}{(r+1)h}K\left(-\frac{X_j}{h}\right)\right], \tag{3.2}$$

where $\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^n K(\frac{x-X_i}{h})$ is an estimate of the pdf $f(x)$ based on the kernel method with the band width

$$h = \left(\frac{4}{3n}\right)^{1/5}\hat{\sigma}, \quad \hat{\sigma} = \sqrt{\left(\frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{X})^2\right)} \quad \text{and} \quad K(u) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2},$$

cf. Härdle [5]. Further, it is known that, cf. Härdle [5],

$$E\hat{f}(0) = f(0) + \mathrm{o}(h),$$
$$\mathrm{var}(\hat{f}(0)) = n^{-1}(f(0) + \mathrm{o}(h))\{h^{-1}\|K\|_2^2 - (f(0) + \mathrm{o}(h))\}, \quad \text{as } h \to 0.$$

Note that, with

$$\phi_r(X_1, X_2) = X_1^r - \frac{X_1^{r+1}}{(r+1)h} K\left(-\frac{X_2}{h}\right) \tag{3.3}$$

$\hat{\delta}_r$ is the classical U-statistic, cf. Lee [8]. The following theorem deals with the asymptotic normality of the proposed test.

Theorem 3.1. *If $n \to \infty$, then $\sqrt{n}(\hat{\Delta}_r - \Delta_r)$ is asymptotically normal with mean* 0 *and variance given in* (3.4). *Under H_0, $\Delta_r = 0$ and the variance is given by* (3.5).

Proof. Since $\hat{\Delta}_r$ and $\hat{\delta}_r/\mu^r$ have the same limiting distribution, we concentrate on $\sqrt{n}(\hat{\delta}_r - \delta_r)$. This is asymptotically normal with mean 0 and variance

$$\sigma^2 = \frac{1}{\mu^{2r}} \lim_{h\to 0} E\left\{X_1^r - \frac{X_1^{r+1}}{r+1}\left[\int K(s)(f(0) + \mathrm{o}(h))\,\mathrm{d}s\right]\right.$$
$$\left. + \mu_r - \frac{\mu_{r+1}}{(r+1)h} K\left(-\frac{X_1}{h}\right)\right\}^2 \tag{3.4}$$

Under H_0

$$\sigma_0^2 = (2r)! \tag{3.5}$$

We will take the value $r = 1$ because it maximizes the Pitman asymptotic efficacy (PAE) values for the proposed test as will be shown later in Table 2.

Then, when $r = 1, \delta_1 = \mu - (f(0)\mu^2/2)$.

In this case $\sigma_0^2 = 2$ and the test statistic is

$$\frac{2}{n(n-1)X} \sum_{i<j} \left\{X_i - \frac{X_i^2}{2h} K\left(-\frac{X_j}{h}\right)\right\}.$$

One can use the proposed test to calculate $\sqrt{n}\hat{\Delta}_r/\sigma_0$ and reject H_0 if $\sqrt{n}\hat{\Delta}_r/\sigma_0 \geqslant Z_\alpha(\leqslant -Z_\alpha)$, where Z_α is the upper α-quantile of the standard normal distribution.

3.2. Monte Carlo null distribution critical values for $\hat{\Delta}_1$ test

In practice, simulated percentiles for small samples are commonly used by applied statisticians and reliability analyst commonly used by applied statisticians and reliability analyst. We have simulated the upper percentile values for 90, 95 and 99. Table 1 presents these percentile values of the statistic $\hat{\Delta}_1$ and

Table 1
The upper percentile of $\hat{\Delta}_1$

n	90	95	99
5	0.88754	0.92227	0.99852
6	0.86498	0.88618	0.97330
7	0.85368	0.87827	0.95717
8	0.83905	0.86481	0.93144
9	0.82835	0.85098	0.91818
10	0.82511	0.84500	0.90788
11	0.81560	0.83764	0.90393
12	0.80914	0.82751	0.88353
13	0.80483	0.82480	0.87802
14	0.79645	0.81552	0.86569
15	0.79540	0.81213	0.87390
16	0.79083	0.81012	0.86193
17	0.78748	0.80687	0.86226
18	0.78424	0.80028	0.85083
19	0.78106	0.80011	0.85228
20	0.77619	0.79331	0.83887
21	0.77422	0.79015	0.84393
22	0.76953	0.78661	0.83478
23	0.76776	0.78513	0.83628
24	0.76725	0.78451	0.82589
25	0.76484	0.78115	0.82637
26	0.76252	0.77997	0.82275
27	0.76192	0.77830	0.82125
28	0.75926	0.77482	0.81936
29	0.75717	0.77187	0.82009
30	0.75238	0.76703	0.81064
35	0.74587	0.76268	0.80508
40	0.74233	0.75543	0.79946
45	0.73288	0.74636	0.78598
50	0.72621	0.73966	0.77672
55	0.72375	0.73730	0.77616
60	0.71897	0.73191	0.76640
65	0.71498	0.72803	0.76337
70	0.71010	0.72378	0.76237

the calculations are based on 5000 simulated samples of sizes $n = 5(1)30(5)70$. It is clear from Table 1 that the percentile values decrease slowly as n increases.

4. An application of the $\hat{\Delta}_1$ test

To illustrate this test we use the special case $r = 1$ and calculate $\sqrt{(n/2)}\hat{\Delta}_1$ for the data in Bryson and Siddiqui [4] which gives the value 0.08207. This value is smaller than $Z_\alpha(-Z_\alpha)$ for all $\alpha < 0.44$. Hence H_0 is not rejected as expected and this agrees with the conclusion of Bryson and Siddiqui [4] and others.

5. Asymptotic efficiency

To study the efficiency of the proposed test, we employ the concept of PAE of the proposed test, which is defined as (cf. Pitman [9]),

$$\mathrm{PAE}(\Delta_r(\theta)) = \left\{ \frac{\mathrm{d}}{\mathrm{d}\theta}\Delta_r(\theta)|_{\theta=\theta_0} \right\}^2 \Big/ \sigma_0^2. \tag{5.1}$$

In our case, $\mathrm{PAE}(\Delta_r(\theta))$ becomes

$$\mathrm{PAE}(\Delta_r(\theta)) = \left\{ \mu_r'(\theta_0). - \frac{f_{\theta_0(0)}}{r+1}\mu_{r+1}'(\theta_0) - \frac{f_{\theta_0(0)}}{r+1}\mu_{r+1}'(\theta_0) \right\}^2 \Big/ (2r)!, \tag{5.2}$$

where $\mu_r(\theta) = r\int_0^\infty x^{r-1}\bar{F}_\theta(x)\,\mathrm{d}x$.Two of the most commonly used alternatives (cf. Hollander and Proschan [6]) are:

(i) The linear failure rate family: $\overline{F}_1(x) = \exp\left\{-x - \frac{\theta}{2}x^2\right\}$, $x \geqslant 0$, $\theta \geqslant 0$.
(ii) The Makeham family: $\overline{F}_2(x) = \exp\{-x - \theta(x + \exp -x - 1)\}$, $x \geqslant 0$, $\theta \geqslant 0$.

Note that $f(0) = 1$ for the above two alternatives. the null hypothesis is at $\theta = 0$ for these two alternatives. The PAE's of these alternatives of our procedure are, respectively,

$$\mathrm{PAE}(\Delta_r(\theta), \mathrm{Mekeham}) = \frac{\left[r!\left(1 - \left(\frac{1}{2}\right)^{r+1}\right)\right]^2}{(2r)!} \tag{5.3}$$

and

$$\mathrm{PAE}(\Delta_r(\theta), \mathrm{LFR}) = \frac{[(r+1)!]^2}{(2r)!}. \tag{5.4}$$

Table 2 summarizes the PAE values of the proposed test for some values of r with the above two alternatives.

Table 2
The PAE values

Distribution	r			
	0	1	2	3
LFR	1	2	1.5	0.8
Makeham	0.25	0.28	0.13	0.044

Table 3
The PAEs values

The compared NBU tests	Alternatives	
	LFR	Makeham
Hollander and Proschan [6]	0.58095 (3.44)	0.25582 (1.09)
Distribution Ahmad [3]	0.8065 (2.48)	0.28544 (0.98)
Ahmad (2001)	1(2)	0.0625 (4.48)

Based on the data presented in Table 2, we could conclude that PAE value of our proposed test for both F_1 and F_2 takes its maximum when $r = 1$.

As far as we know, no other tests have as yet been proposed for testing against NBUFR (NWUFR) alternatives. Thus we will compare the proposed NBUFR (NWUFR) test with tests designed for a smaller class of alternatives, the NBU class. When the underlying distribution is actually NBU, it is to be expected that an NBU test will in general perform better than NBUFR (NWUFR) test. Switching the comparison to grounds where the NBUFR (NWUFR) test should excel, we exhibit a class of NBUFR (NWUFR) distributions for which the NBUFR (NWUFR) test performs distinctly better than the NBU tests.

Table 3 presents the PAEs of three of famous tests for the NBU class up to now, with respect to the alternatives LFR and Makeham. Also, in this Table we present the asymptotic relative efficiencies, AREs, of the proposed test for the NBUFR (NWUFR) class to the three tests for the NBU class. It is shown that the proposed test has an AREs more than one in most of the cases.

6. The power of the proposed test

The power of the proposed test at a significance level α with respect to the alternatives F_1 and F_2 is calculated based on simulation data. In such simulation, 10 000 samples were generated with sizes $n = 10$ and $n = 20$ from the alternatives. Table 4 shows the power of test at different values of θ and the significance level $\alpha = 0.05$.

Table 4
The power of the test

θ	LFR		Makeham	
	$n = 10$	$n = 20$	$n = 10$	$n = 20$
0.5	0.138	0.215	0.103	0.133
1.0	0.193	0.332	0.136	0.203
1.5	0.230	0.414	0.165	0.272

From Table 4, it is noted that the power of the test increases by increasing the value of the parameter θ, as it was expected.

Acknowledgement

The authors would like to thank professor I.A. Ahmad for his comments which led to substantial improvements of the earlier version of this paper.

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