How to Identify a Bathtub Hazard Rate

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Key Words—**Lifetime data, Hazard rate, Bathtub curve, Total time on test**

Reader Aids—
Purpose: Widen state of art
Special math needed for explanations: Probability and elementary statistics
Special math needed to use results: Same
Results useful to: Reliability analysts and theoreticians

Abstract—**The Total Time on Test (TTT) concept is a useful tool in several reliability contexts. This note presents a new test statistic, based on the TTT plot, for testing if a random sample is generated from a life distribution with constant versus bathtub-shaped hazard rate.**

1. INTRODUCTION

The hazard rate is a basic concept in reliability theory. If the life distribution is absolutely continuous, which very often can be assumed, the hazard rate uniquely determines the life distribution. An important class of life distributions arises when the hazard rate is bathtub-shaped (section 2).

Probability plotting methods are widely used in applied statistics. This paper studies the Total Time on Test (TTT) plot [4], and shows that the asymptotic distribution of a test statistic based on the TTT plot under exponentiality has a well known (among statisticians) distribution. This statistic is one of the very few statistics that is specially derived for testing exponentiality (constant hazard rate) against bathtub distributions [1].

Notation

T_i	random sample from a life distribution, $i = 1, \ldots, N$.
$T_{N:i}$	ordered sample from a life distribution, $i = 1, \ldots, N$.
F	absolutely continuous Cdf of a life distribution.
F_N	empirical Cdf of a life distribution.
h	hazard rate
$-$	implies the complement
$F^{-1}(s)$	$\inf\{u : F(u) \geq s\}$.
$F_N^{-1}(s)$	$\inf\{u : F_N(u) \geq s\}$.

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

Nomenclature

TTT transform: $H_F^{-1}(t) \equiv \int_0^{F^{-1}(t)} \bar{F}(u)du$.
Scaled TTT transform: $\phi_F(t) \equiv H_F^{-1}(t)/H_F^{-1}(1)$.
Empirical TTT transform: $H_N^{-1}(r/N) \equiv \int_0^{F_N^{-1}(r/N)} \bar{F}_N(u)du$.
Scaled empirical TTT transform: $\phi_N(r/N)$

$$\equiv H_N^{-1}(r/N)/H_N^{-1}(1) = \left[\left(\sum_{i=1}^{r} T_{N:i}\right) + (N - r)\, T_{N:r}\right] \Big/ \sum_{i=1}^{N} T_i.$$

For $0 \leq t \leq 1$, $\phi_N(t)$ is defined by linear interpolation.

TTT plot: The plot of $(r/N, \phi_N(r/N))$ $(r = 0, 1, \ldots, N)$, where consecutive points are connected by straight lines.

TTT process: $T_N(t) \equiv \sqrt{N}\,\{\phi_N(r/N) - \phi_F(t)\}$, for $(r-1)/N < t \leq r/N$, $1 \leq r \leq N$ and $T_N(0) \equiv 0$.

Minimax rule: The minimax rule minimizes the maximum probability of error.

2. TOTAL TIME ON TEST

The TTT concept was introduced by Barlow et al. [3]. They proved that if F is strictly increasing, $\phi_N(r/N) \to \phi_F(t)$ uniformly on (0, 1) with probability one as $r/N \to t$ and $N \to \infty$. Barlow & Campo [4] therefore suggested a comparison of the TTT plot to graphs of TTT transforms for model identification. Some properties of the TTT transform are listed by Bergman & Klefsjö [7].

An important feature of the TTT transform is that it gives immediate information about the shape of the hazard rate. If F is strictly increasing, then —

$$\frac{d}{dt} H_F^{-1}(t) = [h(F^{-1}(t))]^{-1} \tag{1}$$

for almost all $t \in (0, 1)$. Based on (1), Barlow & Campo [4] proved that a life distribution has increasing (decreasing) hazard rate if and only if the scaled TTT transform is concave (convex) for $0 \leq t \leq 1$.

A class of life distributions can be constructed by assuming that the hazard rate: 1) decreases during the infant mortality phase, 2) is constant during the so-called useful life phase and, 3) increases during the wear-out phase. In reliability literature such hazard rate functions are said to have a bathtub shape. Eq (1) shows that the TTT transform of a life distribution with bathtub-shaped hazard rate is illustrated in figure 1.

3. A TEST BASED ON THE TTT PLOT

Many TTT-based procedures for testing exponentiality against different classes of life distributions have been

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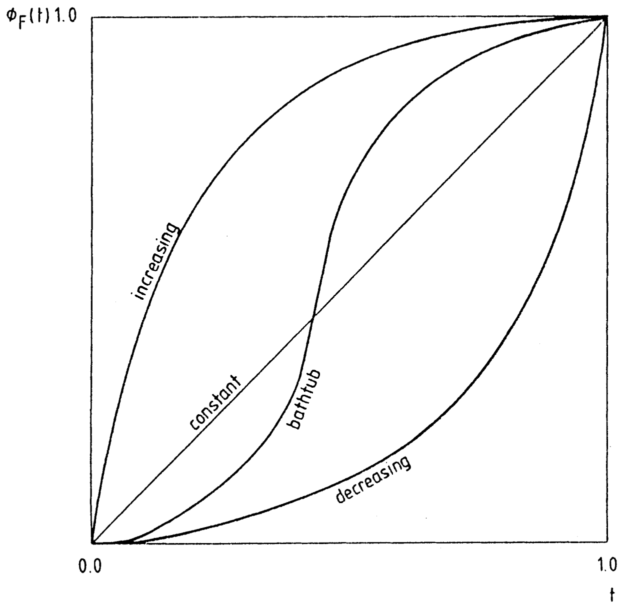

Fig. 1. Scaled TTT transform from distributions with form of the hazard rate as shown on each curve.

suggested [1, 4, 7]. Barlow & Doksum [5] proved that a test which rejects exponentiality in favor of a distribution with increasing hazard rate when the signed area between the TTT plot and the diagonal is large, asymptotically is minimax. A test based on the signed area is poor for discovering a bathtub distribution. But a test based on the (strictly positive) area might be more powerful. With this in mind, and wishing to benefit from the literature on goodness-of-fit tests, I propose the test statistic:

$$R_N = \int_0^1 T_N^2(t)dt.$$

The null hypothesis is rejected when R_N is large.

R_N can be used to measure discrepancy between observations and an hypothesis in general, but here I derive its asymptotic distribution under exponentiality only. Observe that R_N is large if the data seem to support a distribution with, for instance, first increasing hazard rate, then constant, and finally decreasing hazard rate (the "opposite" of a bathtub curve). If the observations are claims an insurance company gets after fires, it often seems realistic to assume that the underlying distribution has such a hazard rate. If the null hypothesis is rejected, we therefore study which alternative the TTT plot indicates. Are the observations generated from a life distribution with increasing hazard rate, decreasing hazard rate, bathtub hazard rate, or something else?

4. THE ASYMPTOTIC DISTRIBUTION OF R_N.

Notation

$\xrightarrow{D}$ convergence in distribution

$\{U(t); 0 \leqslant t \leqslant 1\}$ Brownian Bridge on (0, 1).

$\{W(t); t \geqslant 0\}$ Wiener process.

Nomenclature

Wiener process: A stochastic process $\{W(t); t \geqslant 0\}$ where

(i) $W(0) = 0$

(ii) $\{W(t); t \geqslant 0\}$ has stationary independent increments

(iii) $W(t)$ is normally distributed with mean t and variance t for all $t \geqslant 0$.

Brownian Bridge on (0, 1): A stochastic process $\{U(t); 0 \leqslant t \leqslant 1\}$ where $U(t) = W(t) - tW(t)$, $0 \leqslant t \leqslant 1$.

Under exponentiality $T_N(t) \xrightarrow{D} U(t)$ when $N \to \infty$ [4]. According to the invariance principle [9], the R_N therefore has the same asymptotic distribution as:

$$W^2 = \int_0^1 U^2(t)dt.$$

Because the Cramèr-von Mises statistic also has the property —

$$W_N^2 = N \int_{-\infty}^{\infty} [F_N(t) - F(t)]^2 \, dF(t) \xrightarrow{D} W^2,$$

the W^2 has been extensively studied. Durbin [10] gives its Cdf

$$\Pr\{W^2 \leqslant x\} = 1 - (1/\P) \sum_{j=1}^{\infty} (-1)^{j-1} \int_{(2j-1)^2\pi^2}^{(2j)^2\pi^2}$$

$$y^{-1}((-\sqrt{y})/\sin\sqrt{y})^{1/2} e^{-xy/2} dy; \; x \geqslant 0,$$

a distribution tabulated by Anderson & Darling [2].

Under exponentiality

$$R_N = \sum_{r=1}^{N} \phi_N(r/N)(\phi_N(r/N) - (2r - 1)/N) + N/3.$$

5. EXAMPLE

Table 1 contains the times to failure of 50 devices put on life test at time 0. Now, $R_{50} = 1.3$, and according to Anderson & Darling [2, p 203, table 1] — $\Pr\{R_{50} \geqslant 1.3\} < 0.001$. That is, the null hypothesis of exponentiality is rejected at all levels of statistical significance greater than 0.1%. (Remember this is an asymptotic result.) Furthermore, the TTT plot indicates a bathtub-shaped hazard rate (figure 2).

TABLE 1
Lifetimes of 50 devices

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

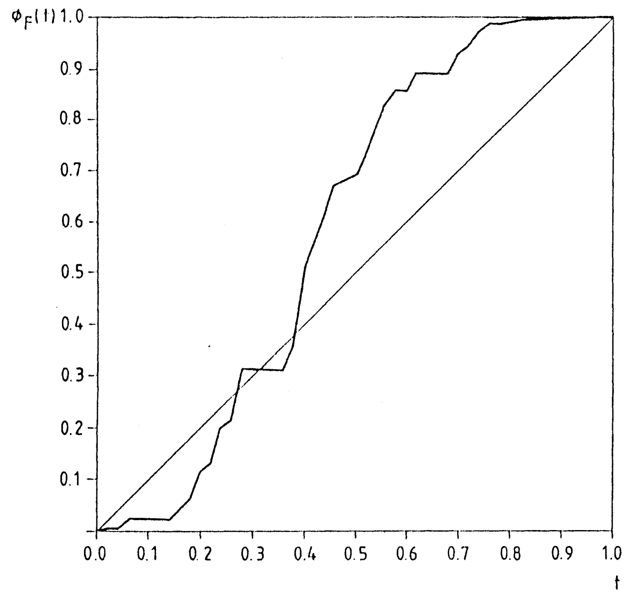

Fig. 2. TTT plot based on the 50 observations in table 1.

ACKNOWLEDGMENT

I am pleased to thank N. L. Hjort and A. Höyland for their helpful advice and suggestions.

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Manuscript received 1985 September 26; revised 1986 July 26; revised 1986 November 17. ★★★

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