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Dana Lacey[a] Anh Nguyen[b]

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Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
http://www.rose-hulman.edu/mathjournal

[a]North Central College
[b]Texas Christian University

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Abstract. There are a number of bathtub and unimodal hazard shape parametric lifetime distributions available in the literature. Therefore, it is important to classify these distributions based on their hazard flexibility to facilitate their use in applications. For this purpose we use the Total Time on Test (TTT) transform plot with two different criteria: I. measure the slope at the inflection point on the scaled TTT transform curve; II. measure the slope at selected points from the constant hazard line on the scaled TTT transform curve. We confine our research to classify the flexibility of Weibull extensions and generalizations and also select one-shape and two-shape parameter lifetime distributions to exemplify the two criteria process.

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1 Introduction

Bathtub and Unimodal hazard shapes (illustrated in Figure 1) are widely used in industrial and medical applications to determine the useful lifetime (ending in failure or death) and the peak-time failure of a sample or a population. Parametric lifetime distributions can generate both of these hazard function shapes, and in this article we classify several common distributions in terms of their hazard shapes.

Mathematically speaking, a *lifetime distribution* is the cumulative distribution function $F(t)$ of a random variable T representing the time until failure, so that

$$F(t) = P(T \leq t) = 1 - S(t), \tag{1}$$

where $S(t) = P(T > t)$ is called a *survival function*. We will refer to $F(t)$ as a "parametric lifetime distribution functions" since its shape will depend on the values of several parameters inherent in its definition. The associated *hazard function* is defined as

$$h(t) = f(t)/S(t), \tag{2}$$

where $f(t)$ is probability distribution function of T.

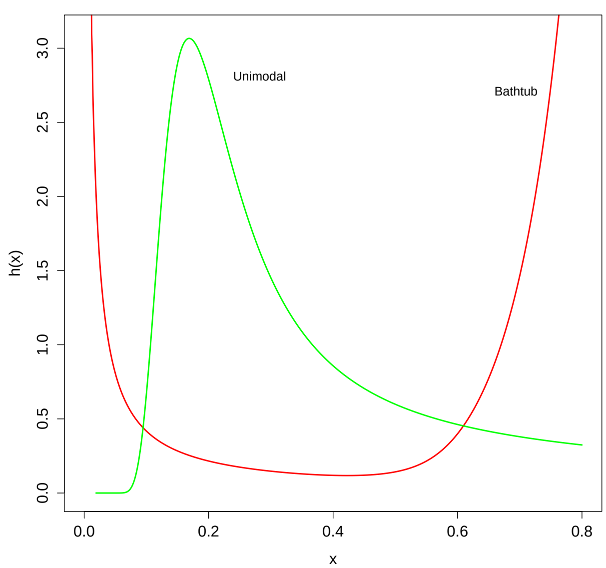

Figure 1: Bathtub and Unimodal Hazard Curves

The shape of the hazard function (shown in Figure 1) allows us to see changes in risk over time for data modeled with lifetime distributions. For example, in the typical bathtub shape, the flat line at the bottom represents the useful lifetime. The longer that flat line is, the longer lifetime the distribution can model. A V-shaped bathtub with a very short flat line is more suitable to model shorter lifetimes, such as the lifetime of a mosquito, but a wider U-shaped bathtub with a longer flat line would be more suitable for modeling the lifetime of a human. On the other hand, parametric lifetime distributions with unimodal hazard shapes can be used to model situations such as survivability after surgery, where risk quickly increases due to the chances of complications such as infection, and then decreases as the patient recovers.

Flexibility is a loose term referring to the range of shapes that a hazard function can assume, depending upon values of its parameters. To our knowledge, there are no clear comparative criteria for the flexibility of hazard functions currently available. However, the flexibility of hazard functions has been compared for specific data sets. In this article, we focus on finding criteria to classify the hazard flexibility of various lifetime distributions. This classification is important since knowing the hazard flexibility of a lifetime distribution is important when attempting to fit a distribution to a data set. For example, when we classify the flexibility of unimodal hazard shapes, we can see the range of peak-time failure length that a given parametric distribution can model.

We establish criteria for this classification based on the Total Time on Test (TTT) transform, as defined by Barlow and Campo [1]. The TTT transform of F(t) is defined as

$$H_F^{-1}(u) = \int_0^{F^{-1}(u)} (1 - F(x))dx, \quad 0 \leq u \leq 1 \tag{3}$$

and is related to the hazard function by

$$\frac{d}{du} H_F^{-1}(u)|_{u=F(t)} = \frac{1}{h(t)}. \tag{4}$$

Barlow and Campo have used the TTT transform to identify the non-monotonic hazard shapes: increasing, decreasing, and constant. Later this relationship was explored further by Aarset [3] and Mudholkar et al. [4] to identify the bathtub and unimodal hazard shapes, respectively. However, for our study we will apply a scaled TTT transform, $\phi_F(u)$, to identify the hazard shape for a given distribution, where

$$\phi_F(u) = \frac{H_F^{-1}(u)}{H_F^{-1}(1)}. \tag{5}$$

We use the scaled TTT transform because it allows for easy comparison of hazard function flexibility as plots of TTT transforms are confined to the unit square.

Because of the relationship between the TTT transform and the hazard function, we can use the TTT transform to observe the hazard shapes of certain distributions; for example, a linear TTT curve corresponds with a constant hazard function, a convex TTT curve corresponds with a decreasing hazard, and a concave TTT curve corresponds with an increasing hazard [1]. Similarly, a convex then concave TTT curve indicates a bathtub hazard shape [3] and a concave then convex TTT curve indicates a unimodal hazard shape [4]. Thus, the TTT plot not only displays as much information as the hazard curve, but as mentioned, the scaled TTT plot also allows us to conveniently compare different hazard curves because it is confined to the unit square.

TTT plots for both bathtub and unimodal hazard functions have a single inflection point, as shown in the following figure:

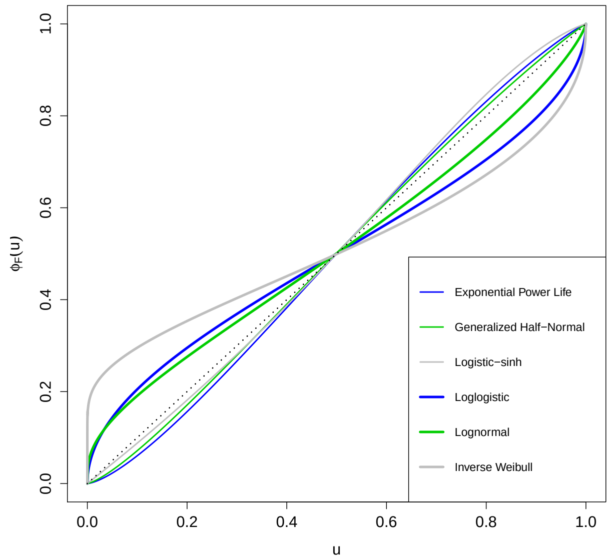

Figure 2: TTT plot of bathtub and unimodal hazard shapes

The shape flexibility of both the bathtub and unimodal hazard curves are dependent on the behavior of the slope at the inflection point. In the unimodal case for example, the closer the slope at the inflection point is to zero, the further away the TTT curve is from the constant hazard line $y = x$. We create our criteria to classify the flexibility bathtub and unimodal hazard shapes based off of this idea.

The rest of the paper is organized as follows: in Section 2 we explain the criteria from which we compare and classify the hazard flexibility. Also in that section, we look at examples of one-shape parameter lifetime distributions and examine their hazard shapes using the TTT transform and our criteria. In Section 3 we examine the hazard flexibility of two-shape parameter Weibull extensions: four regular Weibull extensions and one non-regular Weibull extension. Finally, in Section 4 we conclude our study by discussing the implications of the results.

2 Single-shape parameter lifetime distributions

As mentioned, TTT plots of bathtub and unimodal hazard curves each have a single inflection point. Thus, examining the slope at the inflection point allows us to have a better understanding of the distribution's hazard flexibility. For unimodal hazard shapes, the smaller the slope at the inflection point is, the further away the hazard curve is from the constant hazard line. Thus that unimodal hazard curve can model a longer peak-time failure. For bathtub hazard shapes, the larger the slope at the inflection point is, the further away the hazard curve is from the constant hazard line. Also, that bathtub hazard curve can model a longer useful lifetime. To more easily compare the slopes at inflection points, we converted each slope to the angle of the tangent line at the inflection point, where the angle must be between 0° and 45° for the unimodal hazard shape and between 45° and 90° for the bathtub hazard shape.

Since the TTT plot is confined into the unit square by definition, it is natural to examine the inflection point in the center of the plot first, when $u = 1/2$. To have a better idea of the curve's behavior throughout the entire plot, we also look at the inflection point when $u = 1/4$ and $u = 3/4$. This is our first criterion. However, while this criterion allows us to accurately examine the flexibility of hazard shapes of a distribution at different points on the TTT plot, it does not allow us to graphically compare the hazard flexibility of one distribution to another because the inflection points occur at different locations for a given value of u. Thus, for comparison, we create the second criterion where we examine the slope of the scaled TTT plot on the constant hazard line at the points $(1/2, 1/2), (1/4, 1/4)$, and $(3/4, 3/4)$. Using the second criterion, we graphically compare and observe the hazard flexibility.

Our criteria are formalized as follows:

Criterion I: Measure the slope at the inflection point when $u = 1/4$, $1/2$, $3/4$ on the scaled TTT transform curve.

Criterion II: Measure the slope of the scaled TTT transform curve at (1/2, 1/2), (1/4, 1/4), and (3/4, 3/4).

To begin with, we use the TTT transform to examine the bathtub and unimodal hazard shapes of six commonly used one-shape parametric lifetime distributions. We choose the lognormal, inverse Weibull, and loglogistic distributions as examples of unimodal hazard curves from one-shape parameter distributions, and choose the logistic-sinh [6], exponential power life [7], and generalized half-normal [8] as the examples of bathtub hazard curves from one-shape parameter distributions.

For the one-shape parameter case, there is no hazard flexibility to examine because given a single point on the TTT plot there is only one parameter value that will give a curve going through that point. However, we can use the TTT transform not only to exemplify our two criteria, but also to observe the behavior of the hazard functions and determine which of our distributions has the bathtub hazard shape that can model the longest useful lifetime and which has the unimodal hazard shape that can model the longest peak-time failure.

According to Table 1, for the unimodal hazard shape in the second criterion, when $u = 1/2$, the inverse Weibull's angle (26.18°) is lower than the other two distributions'. In addition, the loglogistic has the same angle (32.48°) for both of the criteria. This is explained further in Subsection (i) of Section 3. Meanwhile, for the bathtub hazard shape in the second criterion, when $u = 1/2$, the exponential power life distribution's angle (49.51°) is higher than the other two distributions'. As we can see in Figure 1, for the unimodal hazard shape, the inverse Weibull has a longer peak-time failure, since its slope is lower than the other

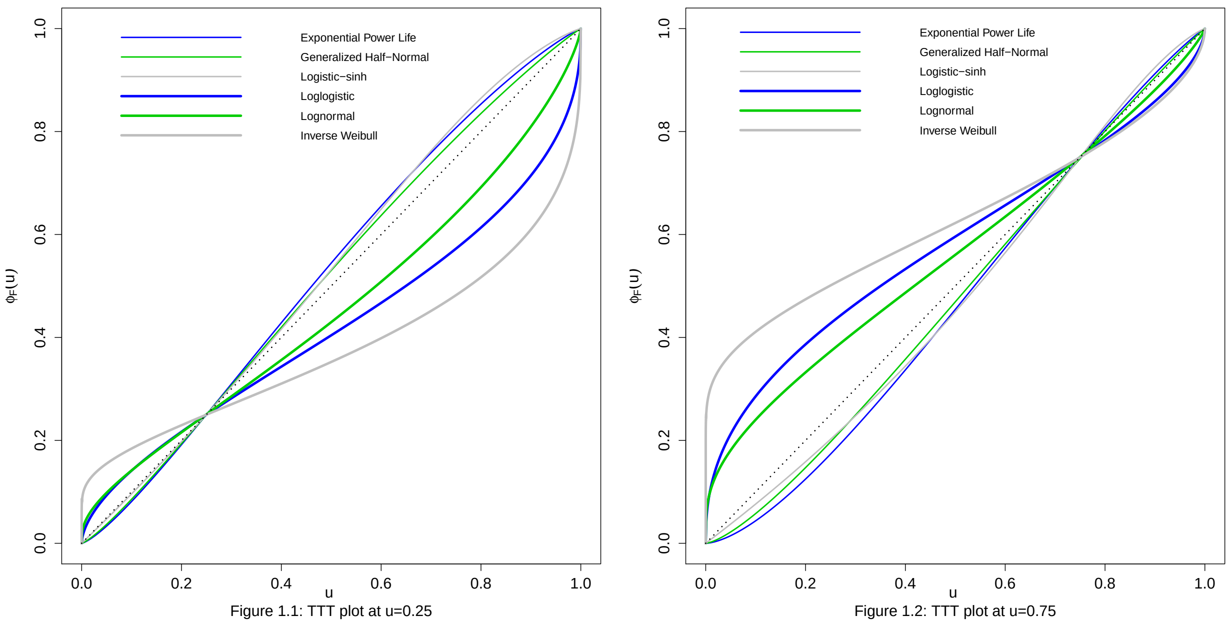

Figure 3: TTT plot of one-shape parameter lifetime distributions at $u = 1/4,\ 3/4$

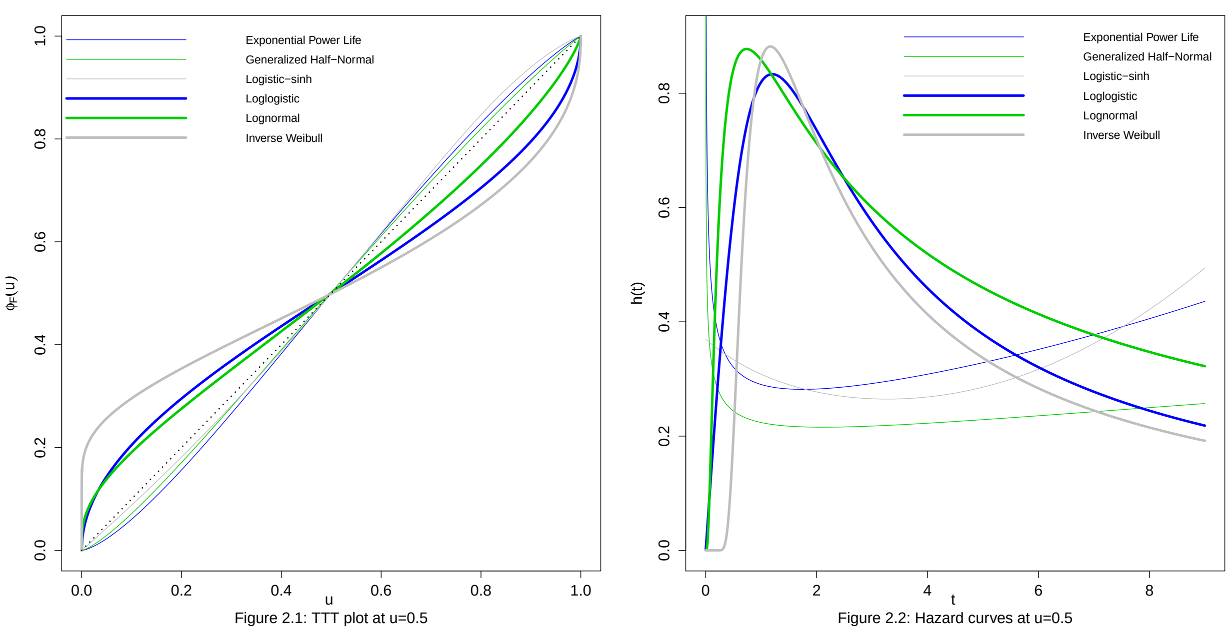

Figure 4: TTT plot and hazard curves of one-shape parameter lifetime distributions at $u = 1/2$

Table 1: One-Shape Parameter Distribution Results

Maximum Slope for Bathtub Hazard Curves										
		u = 1/2			u=1/4			u=3/4		
Distribution	Crit.	Shape Parameter	Slope	Angle	Shape Parameter	Slope	Angle	Shape Parameter	Slope	Angle
Logistic-sinh	I	$\lambda = 2.0080$	1.1759	$\theta = 49.62°$	$\lambda = 1.2510$	1.2922	$\theta = 52.26°$	$\lambda = 3.2697$	1.2264	$\theta = 50.81°$
	II	$\lambda = 2.5860$	1.1531	$\theta = 49.07°$	$\lambda = 2.1600$	1.0677	$\theta = 46.87°$	$\lambda = 3.4900$	1.0454	$\theta = 46.28°$
Exponential Power Life	I	$\alpha = 0.6551$	1.1830	$\theta = 49.79°$	$\alpha = 0.7982$	1.2173	$\theta = 50.60°$	$\alpha = 0.5348$	1.3031	$\theta = 52.50°$
	II	$\alpha = 0.6875$	1.1711	$\theta = 49.51°$	$\alpha = 0.7580$	1.1829	$\theta = 49.79°$	$\alpha = 0.6246$	1.1426	$\theta = 48.81°$
Generalized Half Normal	I	$\alpha = 0.6168$	1.1264	$\theta = 48.40°$	$\alpha = 0.8081$	1.1404	$\theta = 48.75°$	$\alpha = 0.2987$	1.4440	$\theta = 55.40°$
	II	$\alpha = 0.7500$	1.1072	$\theta = 47.91°$	$\alpha = 0.7977$	1.1314	$\theta = 48.53°$	$\alpha = 0.7010$	1.0784	$\theta = 47.16°$
Maximum Slope for Unimodal Hazard Curves										
		u = 1/2			u=1/4			u=3/4		
Distribution	Crit.	Shape Parameter	Slope	Angle	Shape Parameter	Slope	Angle	Shape Parameter	Slope	Angle
Lognormal	I	$\sigma = 7.9790$	0.7274	$\theta = 36.03°$	$\sigma = 1.0479$	0.7044	$\theta = 35.16°$	$\sigma = 0.7724$	0.7592	$\theta = 37.21°$
	II	$\sigma =0.9300$	0.7564	$\theta = 37.10°$	$\sigma = 1.0600$	0.6978	$\theta = 34.91°$	$\sigma = 0.8250$	0.8056	$\theta = 38.86°$
Inverse Weibull	I	$\tau = 2.5887$	0.4402	$\theta = 23.76°$	$\tau = 1.1787$	0.2262	$\theta = 12.75°$	$\tau = 6.6346$	0.1893	$\theta = 10.72°$
	II	$\tau = 1.7500$	0.4916	$\theta = 26.18°$	$\tau = 1.4315$	0.4043	$\theta = 22.01°$	$\tau = 2.1830$	0.5729	$\theta = 29.81°$
Loglogistic	I	$\gamma = 2.0000$	0.6366	$\theta = 32.48°$	$\gamma = 1.3333$	0.3949	$\theta = 21.55°$	$\gamma = 4.0000$	0.3949	$\theta = 21.55°$
	II	$\gamma = 2.0000$	0.6366	$\theta = 32.48°$	$\gamma = 1.7165$	0.6489	$\theta = 32.98°$	$\gamma = 2.3953$	0.6489	$\theta = 32.98°$

two distributions'. Furthermore, for the bathtub hazard shape, the exponential power life distribution has a longer useful lifetime, as its slope is higher than others'. Nonetheless, in Figure 1.1, we see that the exponential power life and the logistic-sinh distribution seem to overlap each other, as the difference between their angles is relatively small (Table 1). Also, this can be observed in the graph of the corresponding hazard curves, as seen in Figure 1.2.

Similarly, when $u = 1/4$, for the unimodal case, the inverse Weibull's angle is still lower than that of the lognormal's and loglogistic's in the second criterion. For the bathtub case, the exponential power life's angle is still higher than the logisitic-sinh's and the generalized half-normal's. Hence, on the TTT plot (Figure 2.1), the inverse Weibull has a longer peak-time failure for the unimodal hazard shape, while the exponential power life has longer useful lifetime for the bathtub shape. Similar results occur for when $u = 3/4$ (Figure 2.2).

When $u = 1/4$ and $u = 3/4$, according to the first criterion, we observe that the loglogistic distribution has the same slope for its unimodal hazard curve. See Appendix 11 for justification of this claim.

3 Two-shape parameter Weibull extensions

As discussed above, while the single shape parametric lifetime distributions have no hazard flexibility, the two shape parameter distributions give us hazard flexibility. As a result, we will now examine the hazard flexibility of two-shape parameter distributions.

To begin with, we choose these four regular Weibull extensions: exponentiated Weibull [4], generalized power Weibull [9], and odd Weibull [10]. Since these four distributions are Weibull exensions, these distributions produce both unimodal and bathtub hazard shapes, rather than simply one hazard shape as in the one-shape parameter case. Additionally, we

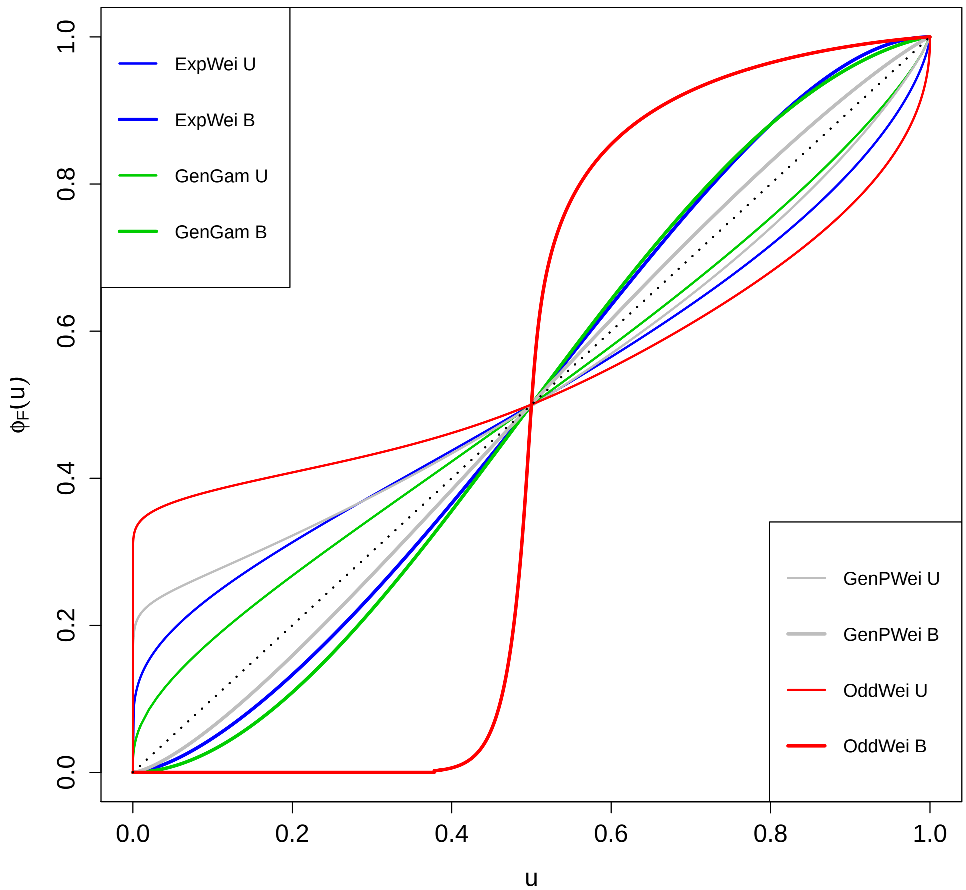

Figure 5: TTT plot of two-shape parameter lifetime distributions at $u = 1/2$

choose Weibull extensions because their hazard range includes the constant hazard line on the TTT plot. These Weibull extensions can model any hazard curve on the TTT plot from the constant hazard line to the bathtub or unimodal hazard curve with the maximum distance from constant hazard line. Thus, for both criteria, we will find the maximum slope for the bathtub hazard shape and the minimum slope for the unimodal hazard shape. Again, the first criterion allows us to individually classify hazard flexibility, while the second criterion allows us to graphically compare the hazard flexibility of these four distributions.

According to Table 2, of these four regular Weibull extensions, in the second criterion, when $u = 1/2$, the odd Weibull has both the smallest angle ($23.97°$) for the unimodal hazard shape and the largest angle ($86.34°$) for the bathtub hazard shape. Therefore, if we take a look at their hazard flexibility on the TTT plot when $u = 1/2$ (Figure 3), odd Weibull, in terms of both unimodal and bathtub hazard shapes, is more flexible than the other three

Table 2: Two-Shape Parameter Distribution Results

Two-shape parameter results when u = 1/2									
Unimodal Hazard Results					Bathtub Hazard Results				
Distribution	Crit.	Shape Parameters		Angle	Distribution	Crit.	Shape Parameters		Angle
Exponentiated	I	$\alpha \to 0$	$\beta \to \infty$	$\theta \to 0°$	Exponentiated	I	$\alpha = 25.000$	$\beta = 0.0200$	$\theta = 59.14°$
Weibull	II	$\alpha = 0.2100$	$\beta = 603.40$	$\theta = 32.33°$	Weibull	II	$\alpha = 29.191$	$\beta = 0.0200$	$\theta = 58.45°$
Generalized	I	$\alpha = 17543.7$	$\beta =$ 5.7x10ˆ-5	$\theta = 26.57°$	Generalized	I	$\alpha = 0.6550$	$\beta = 600.30$	$\theta = 49.79°$
Power Weibull	II	$\alpha = 17.0700$	$\beta =$ 0 .0315	$\theta = 34.05°$	Power Weibull	II	$\alpha = 0.7000$	$\beta = 24.000$	$\theta = 49.30°$
Generalized	I	$\alpha \to -\infty$	$\beta \to -\infty$	$\theta \to 0°$	Generalized	I	$\alpha = 8.3790$	$\beta = 0.0030$	$\theta = 81.31°$
Weibull	II	$\alpha =$ -5.0000	$\beta = 7.8700$	$\theta = 20.55°$	Weibull	II	$\alpha = 2.0500$	$\beta = 0.2500$	$\theta = 64.39°$
Odd Weibull	I	$\alpha \to \infty$	$\beta =$ -0.4427	$\theta \to 0°$	Odd Weibull	I	$\alpha = 3.2589$	$\beta \to 0$	$\theta \to 90°$
	II	$\alpha =$ -6.3000	$\beta =$ -0.2900	$\theta = 23.97°$		II	$\alpha = 6.4800$	$\beta = 0.0136$	$\theta = 86.34°$
Two-shape parameter results when u = 1/4									
Unimodal Hazard Results					Bathtub Hazard Results				
Distribution	Crit.	Shape Parameters		Angle	Distribution	Crit.	Shape Parameters		Angle
Exponentiated	I	$\alpha \to 0$	$\beta \to \infty$	$\theta \to 0°$	Exponentiated	I	$\alpha = 22.802$	$\beta = 0.0408$	$\theta = 57.9°$
Weibull	II	$\alpha = 0.0392$	$\beta =$ 7x10ˆ14	$\theta = 23.45°$	Weibull	II	$\alpha = 29.500$	$\beta = 0.0400$	$\theta = 55.23°$
Generalized	I	$\alpha = 5000.0$	$\beta =$ 0 .0002	$\theta = 26.57°$	Generalized	I	$\alpha = 0.7982$	$\beta = 90100.0$	$\theta = 50.60°$
Power Weibull	II	$\alpha = 5.0000$	$\beta =$ 0 .0891	$\theta = 26.54°$	Power Weibull	II	$\alpha = 0.7577$	$\beta =$ 10ˆ6	$\theta = 49.78°$
Generalized	I	$\alpha \to -\infty$	$\beta \to -\infty$	$\theta \to 0°$	Generalized	I	$\alpha = 19.050$	$\beta = 0.0040$	$\theta = 84.07°$
Weibull	II	$\alpha =$ -19.000	$\beta = 23.9002$	$\theta = 11.63°$	Weibull	II	$\alpha = 6.6900$	$\beta = 0.1000$	$\theta = 74.46°$
Odd Weibull	I	$\alpha =$ -27556.8	$\beta =$ -0.3431	$\theta = 0.002°$	Odd Weibull	I	$\alpha = 29.677$	$\beta = 0.0336$	$\theta = 61.77°$
	II	$\alpha =$ -1.8420	$\beta =$ -0.7800	$\theta = 20.05°$		II	$\alpha = 14.980$	$\beta = 0.0500$	$\theta = 58.32°$
Two-shape parameter results when u = 3/4									
Unimodal Hazard Results					Bathtub Hazard Results				
Distribution	Crit.	Shape Parameters		Angle	Distribution	Crit.	Shape Parameters		Angle
Exponentiated	I	$\alpha \to 0$	$\beta \to \infty$	$\theta \to 0°$	Exponentiated	I	$\alpha = 24.409$	$\beta = 0.0081$	$\theta = 70.65°$
Weibull	II	$\alpha = 0.3000$	$\beta = 147.50$	$\theta = 36.20°$	Weibull	II	$\alpha = 46.200$	$\beta = 0.0100$	$\theta = 60.44°$
Generalized	I	$\alpha = 111.12$	$\beta =$ 0 .0090	$\theta = 26.57°$	Generalized	I	$\alpha = 0.5348$	$\beta = 45000.0$	$\theta = 52.50°$
Power Weibull	II	$\alpha = 8.0000$	$\beta =$ 0 .0773	$\theta = 39.09°$	Power Weibull	II	$\alpha = 0.6510$	$\beta = 12.400$	$\theta = 48.45°$
Generalized	I	$\alpha \to -\infty$	$\beta \to -\infty$	$\theta \to 0°$	Generalized	I	$\alpha = 4.444$	$\beta = 0.002$	$\theta = 82.07°$
Weibull	II	$\alpha =$ -6.998	$\beta = 14.060$	$\theta = 26.67°$	Weibull	II	$\alpha = 3.0500$	$\beta = 0.0300$	$\theta = 73.02°$
Odd Weibull	I	$\alpha =$ -8.4774	$\beta =$ -0.8200	$\theta = 10.46°$	Odd Weibull	I	$\alpha = 1.2107$	$\beta = 0.7200$	$\theta = 51.61°$
	II	$\alpha =$ -2.2000	$\beta =$ -0.9930	$\theta = 29.81°$		II	$\alpha = 1.5950$	$\beta = 0.0200$	$\theta = 55.01°$

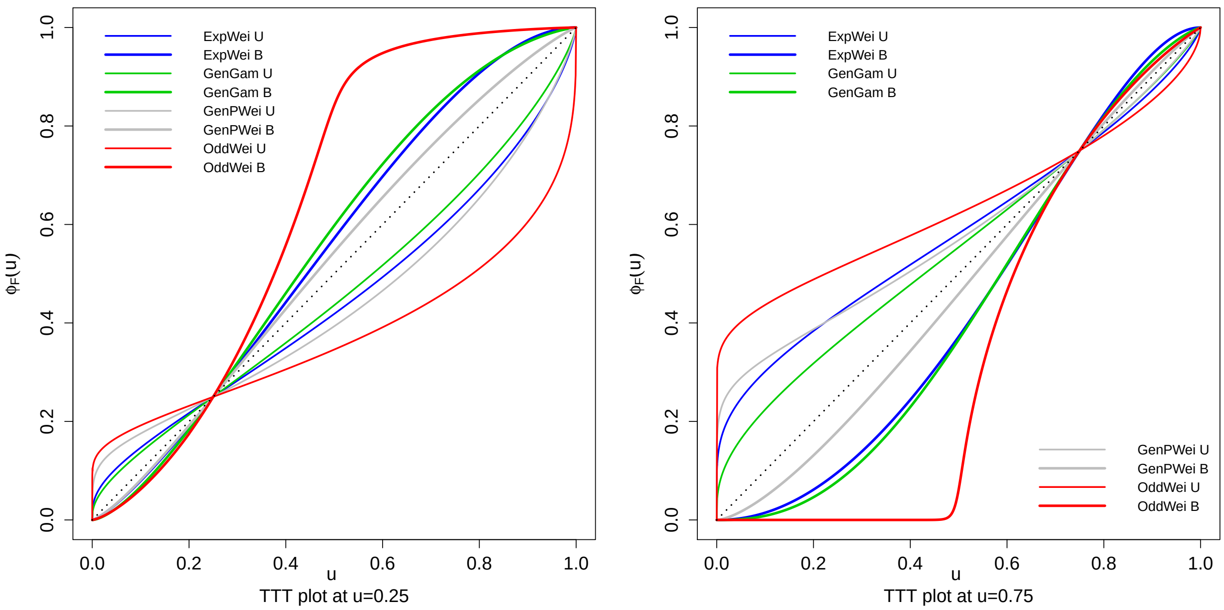

Figure 6: TTT plot of two-shape parameter lifetime distributions at $u = 1/4, 3/4$

regular distributions.

Furthermore, when $u = 1/4$, considering the second criterion, we see that it is also the same case. From Table 2, of the four regular Weibull extensions in the second criterion, the odd Weibull has the smallest angle (20.05°) for the unimodal case and largest angle (58.32°). Hence, as seen in Figure 4.1, the odd Weibull is still more flexible than others with respect to unimodal and bathtub shapes.

When $u = 3/4$, of the four distributions, although the odd Weibull still has the smallest angle for the unimodal shape (Table 2), it does not have the largest angle for the bathtub shape. In fact, the exponeniated Weibull forms the larger angle (60.44°). Thus, as seen from Figure 4.2, the odd Weibull is more flexible in terms of unimodal hazard shape, while the exponentiated Weibull is more flexible with regards to bathtub hazard shape. In addition, if we look at the first criterion for the unimodal case of the exponentiated Weibull at $u = 1/4, 1/2$ and $3/4$, we see that when α approaches 0 and β approaches ∞, then the slope approaches 0. Moreover, at $u = 1/2$, for the odd Weibull distribution, when α approaches ∞ and $\beta = 0.4427$, then the slope approaches 0. Also in this distribution, when $\alpha = 3.2589$ and β approaches 0, then the slope approaches ∞. These will be clearly justified in sections (i) and (ii) respectively.

(i) Limiting behavior of exponentiated Weibull on the TTT plot

Under the first criterion, the most flexible unimodal hazard curve of the exponentiated Weibull distribution does not practically exist. In other words, when β approaches ∞ and α approaches 0 the slope of the scaled TTT transform approaches 0, meaning that finite parameter values are not available to give a slope of 0. Furthermore, the boundary line ($\alpha\beta = 1$) separates the parameter region of the exponentiated Weibull for decreasing hazard, $\alpha < 1$ and $\alpha\beta < 1$, and for unimodal hazard, $\alpha < 1$ and $\alpha\beta > 1$, and so we see that when when β approaches ∞ and α approaches 0 the boundary line approaches the y-axis. Thus, although these two parameter values seem to give us a unimodal hazard curve, they actually produce a decreasing hazard curve. Therefore, these results do not represent a unimodal hazard function with a positive mode and, hence, it is not effective to use the first criterion in this case to examine hazard flexibility. This justifies the benefit of including the second criterion to classify the hazard flexibility.

(ii) Odd Weibull when $u = 1/2$

To justify the results for the first criterion when $u = 1/2$, first observe that α in terms of β (Appendix 8.e) simplifies to

$$\alpha = \{1 + (-\ln(2))\,(1-\beta)\}^{-1}.$$

When α approaches ∞, $\beta = 1 - \frac{1}{\ln(2)} = -0.442695$ and the slope is

$$\frac{\partial \phi_F(u)}{\partial u} = \lim_{\alpha\to\infty} \frac{1}{4\alpha\beta} \frac{\ln^{1/\alpha-1}(2)}{\int_0^1 \ln^{1/\alpha}\left(\left(\frac{t}{1-t}\right)^{1/\beta}+1\right)dt} = 0\frac{\ln^{-1}(2)}{\int_0^1 1dt} = 0.$$

Also, when β approaches 0, $\alpha = (1-\ln(2))^{-1} = 3.258891$, and we can compute the slope using L'Hôpital's rule:

$$\begin{aligned}
\lim_{\beta\to 0} \frac{\frac{1}{4\alpha\beta}\ln^{1/\alpha-1}(2)}{\int_0^1 \ln^{1/\alpha}\left(\left(\frac{t}{1-t}\right)^{1/\beta}+1\right)} dt &= \lim_{\beta\to 0} \frac{-\frac{1}{\beta^2}\frac{1}{4\alpha}\ln^{1/\alpha-1}(2)}{\int_0^1 \frac{-\frac{1}{\alpha}\ln^{1/\alpha-1}\left(1+\left(\frac{t}{1-t}\right)^{1/\beta}\right)\ln\left(\frac{t}{1-t}\right)}{\beta^2\left(\left(\frac{1-t}{t}\right)^{1/\beta}+1\right)}dt} \\
&= \lim_{\beta\to 0} \frac{\frac{1}{4(3.258891)}\ln^{1/3.258891-1}(2)}{\int_0^1 \frac{\frac{1}{3.258891}\ln\left(\frac{t}{1-t}\right)}{\ln^{0.693147}\left(1+\left(\frac{t}{1-t}\right)^{1/\beta}\right)\left(\left(\frac{1-t}{t}\right)^{1/\beta}+1\right)}dt}.
\end{aligned}$$

And as β approaches 0, the slope approaches ∞.

Therefore, when α approaches ∞, $\beta = 1 - \frac{1}{ln(2)} = -0.442695$ and the slope approaches 0 and when β approaches 0, $\alpha = (1 - ln(2))^{-1} = 3.258891$ the slope approaches ∞.

(iii) Hazard flexibility classification of generalized Weibull

After examining the flexibility of the aforementioned regular Weibull extensions, we next study the flexibility of the non-regular Weibull extensions and thus compare their flexibility with that of the regular Weibull extensions. To begin with, we use the generalized Weibull distribution [11] given in Appendix 10, to illustrate the case. We choose the generalized Weibull, because the regular Weibull is a limiting case of it; that is, when α approaches 0, β approaches 1 then the slope approaches 1. In other words, even though the generalized Weibull distribution cannot produce the constant hazard line on the TTT plot, it can approach this line. This enables us to classify and compare the flexibility of non-regular Weibull extensions, specifically the generalized Weibull, with that of the regular Weibull extensions.

For the generalized Weibull, we use the same two criteria to evaluate its bathtub and unimodal flexibility. Since the odd Weibull is more flexible than the other three regular Weibull extensions, we use it to compare with the generalized Weibull. The generalized Weibull can obtain four hazard shapes: increasing, decreasing, bathtub and unimodal [11]. However, because of the main purpose of this paper, we only focus on the bathtub and unimodal hazard shapes. When $\alpha > 0$ and $\beta < 1$, the generalized Weibull produce a bathtub hazard shape. Meanwhile, when $\alpha < 0$ and $\beta > 1$, the generalized Weibull produces the unimodal shape.

Table 2 indicates the results and comparisons between generalized Weibull and odd Weibull at $u = 1/2$. Considering the second criterion, we see that the generalized Weibull does not give as much flexibility for the bathtub shape as the odd Weibull does (Figure 5). However, for the unimodal shape, the generalized Weibull is more flexible than the odd Weibull.

Table 2 also indicates the unimodal flexibility of the two distributions at $u = 1/4$ and 3/4. For the unimodal shape, when we apply the second criterion, generalized Weibull allows more flexibility than the odd Weibull does. Nevertheless, this is not the case for the bathtub shape. While the generalized Weibull maintains more flexibility than the odd Weibull does at $u = 1/2$, it no longer does at $u = 1/4$ and 3/4 (Figure 6).

In order to justify this, we examine the CDF of the Burr distribution [12]:

$$F(x) = 1 - 1/(1 + (\tfrac{x}{\lambda})^{\beta})^{\gamma};\ x, \lambda, \gamma > 0.$$

Let $\gamma = \frac{1}{\alpha}$, and we have

$$F(x) = 1 - 1/(1 + (\frac{x}{\lambda})^{\beta})^{1/\alpha}.$$

Then, let $\alpha < 0$ and $\frac{1}{\lambda^{\beta}} = \frac{\alpha}{\theta^{\beta}}$, for some $\theta > 0$. We can easily obtain the generalized Weibull CDF [9] as follows:

$$\begin{aligned} F(x) &= 1 - 1/(1 - \alpha \left(\frac{x}{\theta}\right)^{\beta})^{-1/\alpha} \\ &= 1 - (1 - \alpha \left(\frac{x}{\theta}\right)^{\beta})^{1/\alpha}. \end{aligned}$$

Thus, the generalized Weibull is an embedded form of the Burr distribution, meaning that it is more flexible. Consequently, it is also more flexible than the odd Weibull with

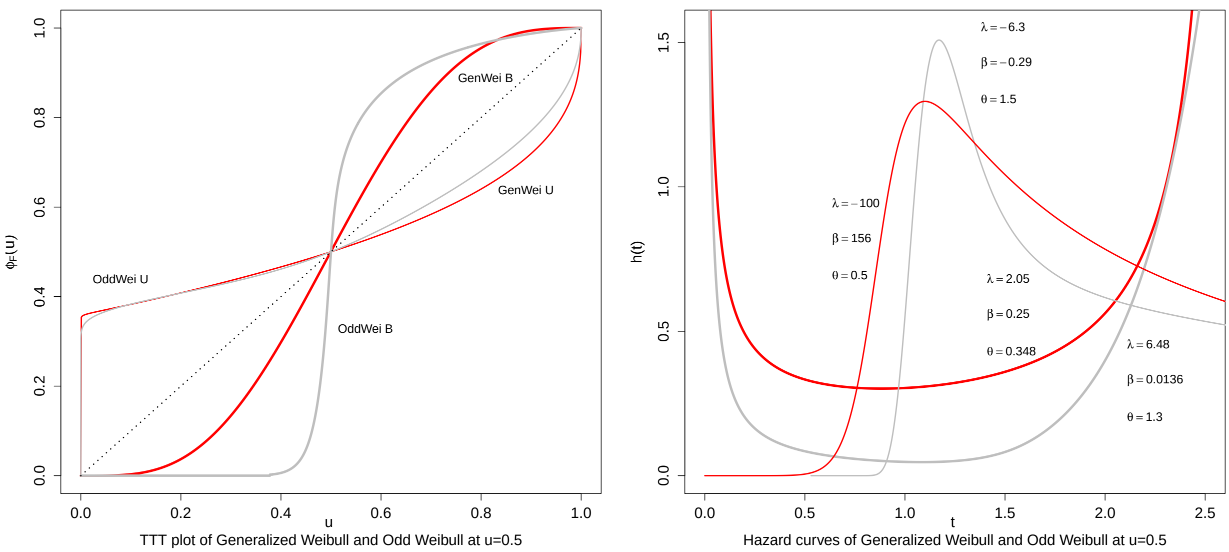

Figure 7: generalized Weibull and odd Weibull TTT plot and their corresponding hazard curves when $u = 1/2$

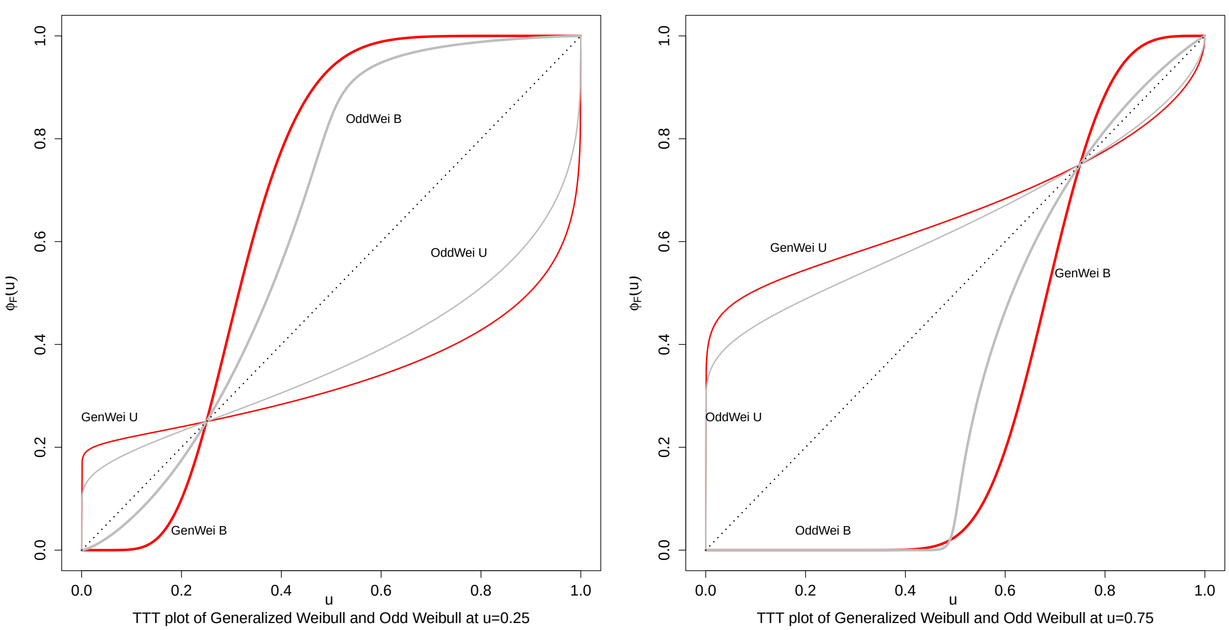

Figure 8: generalized Weibull and odd Weibull TTT plot when $u = 1/4, 3/4$

respect to unimodal shape.

Furthermore, since the generalized Weibull, as previously mentioned, is a non-regular

Weibull extension, the support of the distribution depends on the value of its parameters ($0 < x < \alpha^{-\beta}\theta$). In other words, the number of values of x is limited and hence, finite. Therefore, for the bathtub hazard shape, at $u = 1/2$, the generalized Weibull is not as flexible as the odd Weibull. Given this situation, users may encounter difficulties in modeling sets of data that fit the generalized Weibull bathtub shape.

Also, considering the first criterion with regards to the unimodal shape at all three locations of u on the TTT plot (Table 2), we see that when α approaches $-\infty$ and β approaches ∞, then the slope approaches 0. However, this does not necessarily mean that the slope will equal 0.

Assume for contradiction that $\frac{\partial \phi_F(u)}{\partial u} = 0$, then,

$$\begin{aligned}[\frac{1-(1-u)^\alpha}{\alpha}]^{1/\beta-1} &= 0\\ \frac{1-(1-u)^\alpha}{\alpha} &= 0.\end{aligned}$$

Because $\lambda < 0$ for the unimodal hazard shape, then

$$\begin{aligned}1-(1-u)^\alpha &= 0\\ \alpha &= 0.\end{aligned}$$

This contradicts with the fact that $\alpha < 0$ for the unimodal hazard shape. Hence, for the generalized Weibull, $\frac{\partial \phi_F(u)}{\partial u} \neq 0$.

4 Conclusion

From this study, we can see how the two established criteria can be used to classify hazard flexibility of parametric lifetime distributions with bathtub and unimodal hazard shapes. Because of the limitations of one-shape parameter distributions, those distributions do not have any flexibility to classify. However, we are able to use the criteria to compare the lengths of useful lifetime and peak-time failure each distribution can model.

For the two-shape parameter regular Weibull extensions we are able to classify the flexibility of the bathtub and unimodal hazard shape each can produce. We also apply these criteria to examine the generalized Weibull as an example of a non-regular Weibull extension and compare its flexibility to that of the odd Weibull. From the four regular Weibull extensions we cannot conclude which one is the most flexible overall, for the flexibility varies at different locations on the TTT plot. For example, the odd Weibull has the most flexible bathtub hazard shape of the regular Weibull extensions at $u = 1/2$, $1/4$, but the exponentiated Weibull has the most flexible bathtub hazard shape at $u = 3/4$. However, the purpose of this study is not to find the most flexible distribution, but to create a method to better understand the behavior and flexibility of a distribution at any location on the TTT plot.

We hope to continue classifying the flexibility of more distributions using these criteria, so that practitioners will have more resources and comparative tools to use when deciding which distribution is the most appropriate to model a given set of data.

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Appendix

To apply the two-criteria test on a given lifetime distribution, we start with the cumulative distribution function $F(x)$, which we use to derive the quantile function $F^{-1}(x)$. Then we apply the definition of the scaled TTT transform to obtain the equation for $\phi_F(u)$. Taking the first partial derivative of $\phi_F(u)$ with respect to u ($\frac{\partial \phi_F(u)}{\partial u}$) allows us to derive an equation for the slope of the TTT curve. Taking the second partial derivative with respect to u and equating to 0 allows us to solve for the parameter value at an inflection point. $\frac{\partial^2 \phi_F(u)}{\partial u^2} = 0$ is necessary for the first criterion, while $\phi_F(u)$ is necessary for the second criterion. In addition,

$\frac{\partial \phi_F(u)}{\partial u}$ is used for both criteria to calculate the slope at given points and thus classify the hazard flexibility.

These equations are listed here for each of the distributions used in this research.

One-shape parameter unimodal hazard functions

1. Inverse Weibull

   (a) $F(x) = \exp(-(\frac{\theta}{x})^{\tau});\;\; x > 0, \theta > 0, \tau > 0$

   (b) $F^{-1}(u) = \theta(-\ln u)^{1/\tau}$

   (c) $\phi_F(u) = \frac{(1-u)(-\ln u)^{-1/\tau} + \int_0^u (-\ln u)^{-1/\tau} dt}{\int_0^1 (-\ln u)^{-1/\tau} dt}$

   (d) $\frac{\partial \phi_F(u)}{\partial u} = \frac{(1-u)(-\ln u)^{-1/\tau - 1}}{u\tau \int_0^1 (-\ln u)^{1/\tau} dt}$

   (e) $\frac{\partial^2 \phi_F(u)}{\partial u^2} = 0$
   $\implies \frac{\left[(1-u)(\frac{1}{\tau}+1)(-\ln u)^{-1/\tau-2} - (-\ln u)^{-1/\tau-1}\right]}{u^2\tau \int_0^1 (-\ln u)^{1/\tau} dt} = 0$

2. Loglogistic

   (a) $F(x) = \left(\left(\frac{x}{\theta}\right)^{\gamma}\right) / \left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right);\;\; x > 0, \theta > 0, \gamma > 0$

   (b) $F^{-1}(u) = \sigma \left(\frac{u}{1-u}\right)^{1/\gamma}$

   (c) $\phi_F(u) = \frac{\left(\frac{u}{1-u}\right)^{1/\gamma}(1-u) + \int_0^u \left(\frac{t}{1-t}\right)^{1/\gamma} dt}{\int_0^1 \left(\frac{t}{1-t}\right)^{1/\gamma} dt}$

   (d) $\frac{\partial \phi_F(u)}{\partial u} = \frac{(1-u)\left(\frac{u}{1-u}\right)^{1/\gamma+1}}{u^2\gamma \int_0^1 \left(\frac{t}{1-t}\right)^{1/\gamma} dt}$

   (e) $\frac{\partial^2 \phi_F(u)}{\partial u^2} = 0$
   $\implies \frac{\left(\frac{u}{1-u}\right)^{1/\gamma+1}\left[(u-2) + \frac{1-u}{u}\left(\frac{1}{\gamma}+1\right)\frac{u}{1-u}\right]}{u^3\gamma \int_0^1 \left(\frac{t}{1-t}\right)^{1/\gamma} dt} = 0$

3. Lognormal

   (a) $F(x) = \Phi(\frac{\ln x - \mu}{\sigma});\;\; x > 0, -\infty < \mu < \infty, \sigma > 0.$
   where, $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt, -\infty < z < \infty.$

   (b) $F^{-1}(u) = \exp\left(\mu + \sigma\Phi^{-1}(u)\right)$

(c) $\phi_F(u) = \frac{H_F^{-1}(u)}{H_F^{-1}(1)} = (1-u)\exp\left(\sigma\Phi^{-1}(u) - \frac{1}{2}\sigma^2\right) + \Phi(\Phi^{-1}(u) - \sigma)$

(d) $\frac{\partial\phi_F(u)}{\partial u} = \sqrt{2\pi}\sigma(1-u)\exp\left(\frac{1}{2}(\Phi^{-1}(u))^2 + \sigma\Phi^{-1}(u) - \frac{1}{2}\sigma^2\right)$

(e) $\frac{\partial^2\phi_F(u)}{\partial u^2} = 0$
$\implies (2\pi)\sigma(1-u)\left(\Phi^{-1}(u)+\sigma^2\right)\exp\left((\Phi^{-1}(u))^2 + \sigma\Phi^{-1}(u) - \frac{1}{2}\sigma^2\right)$
$-\sqrt{2\pi}\sigma\exp\left(\frac{1}{2}(\Phi^{-1}(u))^2 + \sigma\Phi^{-1}(u) - \frac{1}{2}\sigma^2\right) = 0$

One-shape parameter bathtub hazard functions

4. Exponential power life [7]

(a) $F(x) = 1 - \exp\left(1 - \exp\left(\left(\frac{x}{\theta}\right)^\alpha\right)\right);\ \ x > 0, \theta > 0, \alpha > 0$

(b) $F^{-1}(x) = \theta\left[\ln\left(1 - \ln\left(1-u\right)\right)\right]^{1/\alpha}$

(c) $\phi_F(u) = \frac{(1-u)[\ln(1-\ln(1-u))]^{1/\alpha} + \int_0^u [\ln(1-\ln(1-t))]^{1/\alpha}dt}{\int_0^1 [\ln(1-\ln(1-t))]^{1/\alpha}dt}$

(d) $\frac{\partial\phi_F(u)}{\partial u} = \frac{[\ln(1-\ln(1-u))]^{1/\alpha-1}}{\alpha(1-\ln(1-u))\int_0^1[\ln(1-\ln(1-t))]^{1/\alpha}dt}$

(e) $\frac{\partial^2\phi_F(u)}{\partial u^2} = 0$
$\implies \frac{-[\ln(1-\ln(1-u))]^{1/\alpha-1} + [\ln(1-\ln(1-u))]^{1/\alpha-2}(1/\alpha-1)}{(1-\ln(1-u))^2\alpha(1-u)\int_0^1[\ln(1-\ln(1-t))]^{1/\alpha}dt} = 0$

5. Generalized half-normal [8]

(a) $F(x) = 2\Phi\left(\left(\frac{x}{\theta}\right)^\alpha\right) - 1;\ \ x > 0, \theta > 0, \alpha > 0$

(b) $F^{-1}(x) = \theta\left(k(u)\right)^{1/\alpha}$, where $k(u) = \Phi^{-1}\left(\frac{u+1}{2}\right)$

(c) $\phi_F(u) = \frac{(1-u)(k(u))^{1/\alpha} + \int_0^u (k(u))^{1/\alpha}dt}{\int_0^1 (k(u))^{1/\alpha}dt}$

(d) $\frac{\partial\phi_F(u)}{\partial u} = \frac{\sqrt{2\pi}(1-u)(k(u))^{1/\alpha-1}\exp\left(\frac{1}{2}(k(u))^2\right)}{2\alpha\int_0^1(k(u))^{1/\alpha}dt}$

(e) $\frac{\partial^2\phi_F(u)}{\partial u^2} = 0$
$\implies \frac{\sqrt{2\pi}}{2\alpha}\left(-\exp\left(\frac{1}{2}\left(k(u)\right)^2\right)\left(k(u)\right)^{1/\alpha-1}\right)$
$+ (1-u)\exp\left((k(u))^2\right)\frac{2\pi}{4\alpha}\left(k(u)\right)^{1/\alpha-2}\left((1/\alpha - 1) + k(u)^2\right) = 0$

6. Logistic-sinh [6]

   (a) $F(x) = 1 - (1 + \lambda \sinh(\exp(\frac{x}{\theta}) - 1))^{-1};\ x > 0, \theta > 0, \lambda > 0$
   (b) $F^{-1}(u) = \theta \ln(1 + \text{arcsinh}\ (\frac{u}{\lambda(1-u)}))$
   (c) $\phi_F(u) = \frac{(1-u)\ln\left(1+\text{arcsinh}\ \left(\frac{u}{\lambda(1-u)}\right)\right)+\int_0^u \ln\left(1+\text{arcsinh}\ \left(\frac{t}{\lambda(1-t)}\right)\right)dt}{\int_0^1 \ln\left(1+\text{arcsinh}\ \left(\frac{t}{\lambda(1-t)}\right)\right)dt}$
   (d) $\frac{\partial \phi_F(u)}{\partial u} = \frac{1/\left(1+\text{arcsinh}\ \left(\frac{u}{\lambda(1-u)}\right)\left(u^2+\lambda^2(1-u)^2\right)^{1/2}\right)}{\int_0^1 \ln\left(1+\text{arcsinh}\ \left(\frac{t}{\lambda(1-t)}\right)\right)dt}$
   (e) $\frac{\partial^2 \phi_F(u)}{\partial u^2} = 0$
   $\implies\ -\frac{1}{1-u} - \frac{\left(1+\text{arcsinh}\ \left(\frac{u}{\lambda(1-u)}\right)\right)\left(u-\lambda^2(1-u\right)}{\left(u^2+\lambda^2(1-u)^2\right)^{1/2}} = 0$

Two-shape parameter Weibull extensions functions

7. Exponentiated Weibull [4]

   (a) $F(x) = [1 - \exp(-(\frac{x}{\theta})^\alpha)]^\beta;\ x > 0, \theta > 0, \alpha > 0, \beta > 0$
   (b) $F^{-1}(u) = \sigma(-\ln(1 - u^{1/\beta}))^{1/\lambda}$
   (c) $\phi_F(u) = \frac{(1-u)\ln^{1/\alpha}(\frac{1}{1-u^{1/\beta}})+\int_0^u \ln^{1/\alpha}(\frac{1}{1-t^{1/\beta}})dt}{\int_0^1 \ln^{1/\alpha}\left(\frac{1}{1-t^{1/\beta}}\right)dt}$
   (d) $\frac{\partial \phi_F(u)}{\partial u} = \frac{\frac{1-u}{\lambda\beta}(-\ln^{1/\lambda-1}(1-u^{1/\beta}))[\frac{u^{1/\beta-1}}{1-u^{1/\beta}}]}{\int_0^1 \ln^{1/\alpha}(\frac{1}{1-t^{1/\beta}})dt}$
   (e) $\frac{\partial^2 \phi_F(u)}{\partial u^2} = 0 \implies$
   $$\alpha = \left(1 - \frac{\left(u\left(1-u^{1/b}\right)\left(\frac{(1-u)u^{\frac{1}{\beta}-1}}{\beta} - u^{1/\beta}\right) - (1-u)u^{1/\beta}\left(\left(1-u^{1/\beta}\right) - \frac{u^{\frac{1}{\beta}}}{\beta}\right)\right)\left(\frac{1}{1-u^{1/\beta}}\right)}{\frac{(1-u)u^{2/\beta}}{\beta}}\right)^{-1}$$
   (f) The hazard function has a bathtub shape when $\alpha > 1$ and $\alpha\beta < 1$ and has a unimodal shape when $\alpha < 1$ and $\alpha\beta > 1$.

8. Generalized Power Weibull [9]

   (a) $F(x) = 1 - \exp(1 - [1 + (\frac{x}{\theta})^\alpha]^\beta)\ x > 0, \theta > 0, \alpha > 0, \beta > 0$

(b) $F^{-1}(x) = \left((1 - \ln(1-u))^{1/\beta} - 1\right)^{1/\alpha}$

(c) $\phi_F(u) = \frac{(1-u)\left((1-\ln(1-u))^{1/\beta}-1\right)^{1/\alpha}+\int_0^u\left((1-\ln(1-u))^{1/\beta}-1\right)^{1/\alpha}dt}{\int_0^1\left((1-\ln(1-u))^{1/\beta}-1\right)^{1/\alpha}dt}$

(d) $\frac{\partial\phi_F(u)}{\partial u} = \frac{(1-\ln(1-u))^{1/\beta-1}\left((1-\ln(1-u))^{\frac{1}{\beta}}-1\right)^{1/\alpha-1}}{\alpha\beta\int_0^1\left((1-\ln(1-t))^{a/\beta-1}\right)^{1/\alpha}dt}$

(e) $\frac{\partial^2\phi_F(u)}{\partial u^2} = 0$
$\implies \alpha = \left(1 - \frac{(1-\beta)\left((1-\ln(1-u))^{1/\beta}-1\right)}{(1-\ln(1-u))^{1/\beta}}\right)^{-1}$

(f) The hazard function has a bathtub shape when $0 < 1/\beta < \alpha < 1$ and has a unimodal shape when $1/\beta > \alpha > 1$.

9. Generalized Weibull [11]

(a) $F(x) = 1 - \left[1 - \alpha\left(\frac{x}{\theta}\right)^{\beta}\right]^{1/\alpha}$

(b) $F^{-1}(x) = \theta\left(\frac{1-(1-u)^{\alpha}}{\alpha}\right)^{1/\beta}$

(c) $\phi_F(u) = \frac{(1-u)\left(\frac{1-(1-u)^{\alpha}}{\alpha}\right)^{1/\beta}+\int_0^u\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{1/\beta}dt}{\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{1/\beta}dt}$

(d) $\frac{\partial\phi_F(u)}{\partial u} = \frac{(1-u)^{\alpha}\left(\frac{1-(1-u)^{\alpha}}{\alpha}\right)^{\frac{1}{\beta}-1}}{\beta\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{1/\beta}dt}$

(e) $\frac{\partial^2\phi_F(u)}{\partial u^2} = 0$
$\implies \alpha = \frac{1-(1-u)^{\beta}}{(1-u)^{\beta}} + 1$

(f) The hazard function has a bathtub shape when $\beta < 1$ and $\alpha > 0$ and has a unimodal shape when $\beta > 1$ and $\alpha < 0$.

10. Odd Weibull [10]

(a) $F(x) = 1 - 1/(1 + [\exp((\frac{x}{\theta})^{\alpha}) - 1]^{\beta});\ x > 0, \alpha\beta > 0, \theta > 0$

(b) $F^{-1}(x) = \left(\ln\left(\left(\frac{u}{1-u}\right)^{1/\beta} + 1\right)\right)^{1/\alpha}$

(c) $\phi_F(u) = \frac{(1-u)\left(\ln\left(\left(\frac{u}{1-u}\right)^{1/\beta}+1\right)\right)^{1/\alpha}+\int_0^u\left(\ln\left(\left(\frac{u}{1-u}\right)^{1/\beta}+1\right)\right)^{1/\alpha}dt}{\int_0^1\left(\ln\left(\left(\frac{u}{1-u}\right)^{1/\beta}+1\right)\right)^{1/\alpha}dt}$

(d) $\frac{\partial\phi_F(u)}{\partial u} = \frac{\ln^{\frac{1}{\alpha}-1}\left(\left(\frac{u}{1-u}\right)^{1/\beta}+1\right)}{\alpha\beta u\left(\left(\frac{1-u}{u}\right)^{1/\beta}+1\right)\int_0^1\left(\ln\left(\left(\frac{u}{1-u}\right)^{1/\beta}+1\right)\right)^{1/\alpha}dt}$

(e) $\frac{\partial^2\phi_F(u)}{\partial u^2} = 0 \implies$

$$\alpha = \left(\frac{(1-u)\left(-u^{1/\beta}+(1-\beta)\left(u^{1/\beta}+(1-u)^{1/\beta}\right)+u(1-u)^{\frac{1}{\beta}-1}\right)\left(-\ln\left(\left(\frac{u}{1-u}\right)^{1/\beta}+1\right)\right)}{u^{1/\beta}}+1\right)^{-1}$$

(f) The hazard function has a bathtub shape when $\alpha > 1$ and $\alpha\beta \le 1$ and has a unimodal shape when either $\beta, \alpha < 0$ or $\alpha < 1$ and $\alpha\beta \ge 1$.

11. Loglogistic at $u = 1/4$ and $3/4$

When $u = 1/4$ and $u = 3/4$, according to the first criterion, we observe that the loglogistic distribution has the same slope for its unimodal hazard curve. To justify this we look at the equation of the slope at $u = 1/4$, $3/4$:
First, when $u = 1/4$ the slope is

$$\frac{\partial\phi_F(u)}{\partial u}|_{u=1/4} = \frac{\frac{3/4}{(1/4)^2\gamma}(1/3)^{1/\gamma+1}}{\int_0^1(t)^{1/\gamma}(1-t)^{-1/\gamma}dt}.$$

Next, set the second partial derivative (Appendix 3.e) equal to 0 and solve for the parameter value at the inflection point:

$$\gamma = (1-u)^{-1}$$

.

Thus, when $u = 1/4$, $\gamma = 4/3$.
Then,

$$\frac{\partial\phi_F(u)}{\partial u}|_{u=1/4,\gamma=4/3} = \frac{3^{1/4}}{\int_0^1 t^{3/4}(1-t)^{-3/4}dt}.$$

Since the density of the beta function is $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ and $\Gamma(\alpha+1) =$

$\alpha\Gamma(\alpha)$,

$$\begin{aligned}
\int_0^1 (t)^{3/4}(1-t)^{-3/4}dt &= \int_0^1 t^{7/4-1}(1-t)^{1/4-1}dt \\
&= \frac{\Gamma(7/4)\Gamma(1/4)}{\Gamma(7/4+1/4)} \int_0^1 \frac{\Gamma(7/4+1/4)}{\Gamma(7/4)\Gamma(1/4)} t^{7/4-1}(1-t)^{1/4-1}dt \\
&= \frac{\Gamma(7/4)\Gamma(1/4)}{\Gamma(2)} \\
&= (3/4)\Gamma(3/4)\Gamma(1/4).
\end{aligned}$$

Then,

$$\frac{\partial \phi_F(u)}{\partial u}|_{u=1/4,\gamma=4/3} = \frac{3^{1/4}}{(3/4)\Gamma(3/4)\Gamma(1/4)} = \frac{3^{-3/4}}{(1/4)\Gamma(3/4)\Gamma(1/4)}.$$

Similarly, it can be shown that when $u = 3/4$, the parameter $\gamma = 4$ and the slope will also be

$$\frac{3^{-3/4}}{(1/4)\Gamma(3/4)\Gamma(1/4)}.$$

Therefore, $\frac{\partial \phi_F(u)}{\partial u}|_{u=1/4} = \frac{\partial \phi_F(u)}{\partial u}|_{u=3/4}$.