The Total Time on Test Concept and Its Use in Reliability Theory

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More than 30 years ago, Epstein and Sobel introduced the Total Time on Test (TTT-) concept. During the last 10 years different generalizations of this concept have been defined and studied, e.g. the TTT-transform and the TTT-plot; a basic paper was presented by Barlow and Campo in 1975. Many of these generalizations have proven to be very useful in different areas of reliability, both from a theoretical and a practical point of view. The purpose of this expository paper is to present some of these generalizations and illustrate their value.

LET $0 = x_{(0)} \le x_{(1)} \le \cdots \le x_{(n)}$ denote an ordered sample from a life distribution F (i.e., a distribution function with $F(0\text{-}) = 0$), with survival function $\bar{F} = 1 - F$, and with a finite mean $\mu_F = \int_0^\infty \bar{F}(x)dx$. The successive *Total Time on Test (TTT-) statistics* are then defined as $T_i = \sum_{j=1}^{i} (n - j + 1)(x_{(j)} - x_{(j-1)})$ for $i = 1, 2, \cdots, n$, and *the scaled TTT-statistics* as $u_i = T_i/T_n$ for $i = 1, 2, \cdots, n$. TTT-statistics were first used by Epstein and Sobel [1953] to make inference about the exponential distribution. Originating with a paper by Barlow and Campo [1975], researchers have studied different generalizations of the original TTT-concept that have proven to be very useful in a great number of applications; e.g. for model identifications, as a basis for the characterization of life distribution classes, for hypothesis testing, and to determine optimal age replacement intervals.

The purpose of this expository paper is to summarize some applications of the TTT-concept in reliability theory. We want to point out that in recent papers, Chandra and Singpurwalla [1981] and Klefsjö [1984] have discussed the close relationship between the TTT-concept and the Lorenz curve that is frequently used in econometrics. Thus, we believe that many aspects of the TTT-concept are also of interest in other applications.

Subject classification: 721 the TTT-concept and its use in reliability.

1. THE TTT-CONCEPT

If F is the exponential distribution then $u_i = T_i/T_n$ for $i = 1, 2, \cdots, n-1$, are distributed as the order statistics from a uniform distribution on [0, 1]. By plotting u_i against i/n for $i = 0, 1, \cdots, n$ and then connecting the points by straight lines, we obtain a curve, named the *TTT-plot*, which, at least for large values of n, is close to the diagonal in the unit square. The importance of the TTT-concept in reliability theory is due to the fact that this result can be generalized to distributions other than the exponential.

By using the notations $\bar{G} = 1 - G$ and $G^{-1}(t) = \inf\{x:G(x) \geq t\}$ for any life distribution G, we can write T_i (cf. Barlow and Campo) as

$$T_i = \int_0^{F_n^{-1}(i/n)} \bar{F}_n(x)dx;$$

here F_n is the empirical distribution function determined from the ordered sample $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$. This result combined with the Glivenko-Cantelli Lemma and the Strong Law of Large Numbers (cf. Barlow et al. [1972, p. 237]) implies that with probability one

$$T_i \rightarrow \int_0^{F^{-1}(t)} \bar{F}(x)dx \quad \text{uniformly in } [0, 1], \tag{1}$$

when $n \rightarrow \infty$ and $i/n \rightarrow t$ if F is strictly increasing. (Langberg et al. [1980] proved a somewhat more general result.)

The function

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(x)dx \quad \text{for} \quad 0 \leq t \leq 1,$$

which appears in (1), is called the *TTT-transform of F*. The mean of F is given by $\mu_F = \int_0^{F^{-1}(1)} \bar{F}(x)dx$ and the transform $\varphi_F(t) = H_F^{-1}(t)/\mu_F$ for $0 \leq t \leq 1$, is scale invariant and is called the *scaled TTT-tranform of F*. When F is the exponential distribution, it is easy to see that $\varphi_F(t) = t$ for $0 \leq t \leq 1$.

We realize that for a general life distribution F, the TTT-plot will approach the graph of the scaled TTT-transform φ_F as n, the number of observations, increases to infinity; cf. Figure 1.

A form of TTT-transform first appeared in Marshall and Proschan [1965]. Later Barlow and Doksum [1972], see also Barlow et al., used the TTT-transform when studying the cumulative TTT-statistic (see Section 4 of this paper). Barlow and Campo introduced the scaled TTT-transform and the TTT-plot. For different properties of the TTT-transform, we refer to Barlow et al. [Chapters 5 and 6], Barlow and Campo, Barlow [1979] and Langberg et al.

2. CHARACTERIZATIONS OF AGING PROPERTIES

The scaled TTT-transform can be used to illustrate and characterize different aging properties. In the following discussion, we summarize some of these situations. For definitions of the aging properties, see, for example, Barlow and Proschan [1975] or Klefsjö [1982].

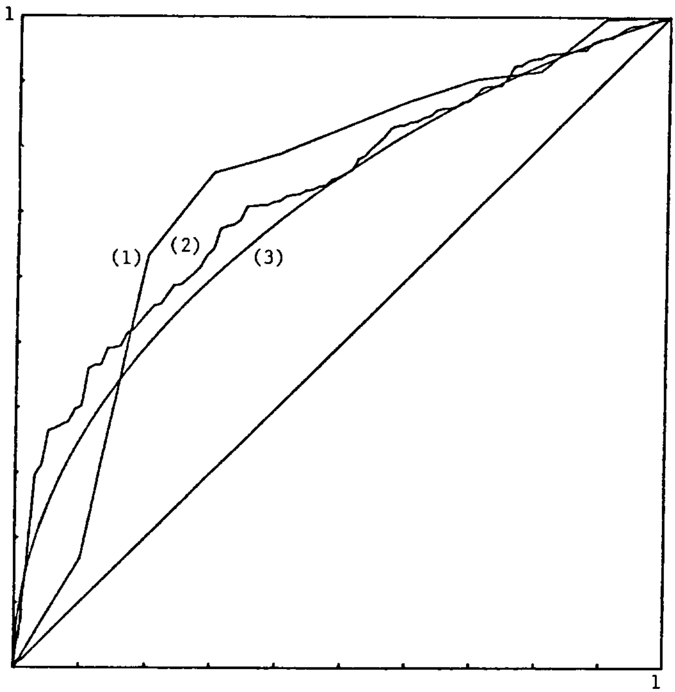

Figure 1. TTT-plots based on simulated Weibull data with $\beta = 2.0$, $n = 10$ in (*1*) and $\beta = 2.0$, $n = 100$ in (*2*) and the scaled TTT-transform of a Weibull distribution with $\beta = 2.0$ in (*3*).

The life distribution F is IFR if and only if the scaled TTT-transform φ_F is concave on [0, 1]; cf. Barlow and Campo and Langberg et al. This characterization is illustrated by the scaled TTT-transform in Figure 1.

If F is IFRA then $\varphi_F(t)/t$ is decreasing in $0 < t < 1$ (cf. Barlow [1979]. However, the reversed implication does not hold. Klefsjö [1982] observed that the class of life distributions for which $\varphi_F(t)/t$ is decreasing in $0 < t < 1$ is a class, studied by Marshall and Proschan [1972], which naturally arises in replacement theory.

A life distribution F is DMRL if and only if $(1 - \varphi_F(t))/(1 - t)$ is decreasing in $0 \le t < 1$; cf. Klefsjö [1982].

A life distribution F is NBUE if and only if $\varphi_F(t) \geq t$, $0 \leq t \leq 1$; cf. Bergman [1979]. This property assures that the scaled TTT-transform of a NBUE life distribution never crosses the diagonal in the unit square.

A life distribution F is light-tailed (i.e. $\lim_{t\to\infty} (\int_t^\infty \bar{F}(x)dx/\bar{F}(t)) = 0$; cf. Vännman [1975]) if and only if $\lim_{t\to 1-} (\varphi_F(t) - \varphi_F(1))/(t-1) = 0$; cf. Klefsjö [1982].

Each one of the aging properties above has a dual property (DFR, DFRA, IMRL, NWUE and heavy-tailed, respectively) defined in an obvious way. For these dual classes, we have results corresponding to those just mentioned.

3. MODEL IDENTIFICATION

As mentioned in Section 1, the TTT-plot based on a sample from F will (under general conditions) approach the graph of the scaled TTT-transform φ_F when $n \to \infty$. This property prompted Barlow and Campo to suggest this plotting procedure for model identification. A TTT-plot can easily be compared to graphs on transparencies of scaled TTT-transforms corresponding to different life distributions.

Our opinion is that this technique is a suitable complement to other methods used in practice (e.g. probability papers) since the plot is very easy to make, it is invariant of scale, and requires only one plot.

In practice data is often truncated or censored in one sense or another. One important situation arises when the data are randomly right censored. Let $X_1, X_2, \cdots, X_n$ denote the true life times of n units and let the censoring times $Y_1, Y_2, \cdots, Y_n$ be independent (and independent of $X_1, X_2, \cdots, X_n$) with life distribution G. If we must make model identification on the basis of $Z_i = \min(X_i, Y_i)$ for $i = 1, 2, \cdots, n$, and knowledge whether the different observations are censored or not, a natural generalization of the mentioned plotting technique is to plot

$$u_i = \int_0^{K_n^{-1}(X_{(i)})} \bar{K}_n(x)dx \Big/ \int_0^{K_n^{-1}(1)} \bar{K}_n(x)dx \quad \text{against} \quad K_n(X_{(i)}),$$

for uncensored $X_{(i)}$'s, where K_n is the Kaplan-Meier estimator of the true life distribution F, based on $Z_1, Z_2, \cdots, Z_n$ (cf. Kaplan and Meier [1958]). Figure 2 illustrates this plotting procedure using data in Koul and Susarla [1980]. Similar plotting procedures may be based on other estimators of F. Barlow and Campo also discussed TTT-plotting for different forms of censored data.

4. TESTS AGAINST AGING

For testing exponentiality against the IFR (or DFR) property, Barlow and Doksum suggested a test based on the cumulative TTT-statistic

$V = \sum_{j=1}^{n-1} u_j$. They proved that this test possesses a certain minimax property; see also Barlow et al. [Chapter 6]. Having been suggested by Laplace in 1733 the analogue of the cumulative TTT-statistic for tests for trend in a series of events has been described as the oldest known statistical test.

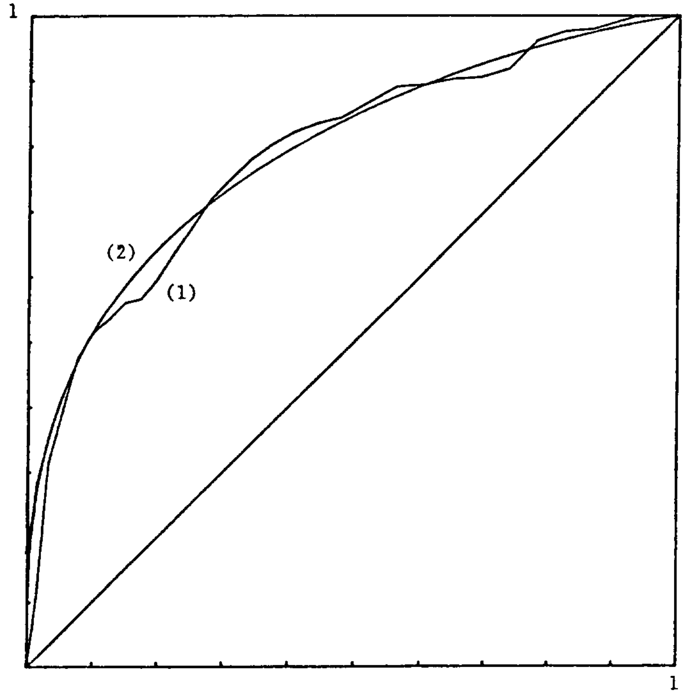

Figure 2. The smooth curve (2) is the scaled TTT-transform of a Weibull distribution ($\beta = 3.0$). The curve (*1*) is the generalized TTT-plot based on 69 observations (29 uncensored and 40 censored) from a melanoma study at the University of Wisconsin-Madison (cf. Koul and Susarla [1980]).

Barlow and Campo proposed another test for testing exponentiality against the IFR (or DFR) property that is based on the number of crossings between the TTT-plot and the diagonal in the unit square. Bergman [1977a] presented properties, exact and asymptotic, of this test statistic.

For testing exponentiality against alternatives with an initially decreasing failure rate (indicating, for example, a mixture of quality classes; cf. Proschan [1963]) a test based on $\inf\{i{:}u_i \geq i/n\}$ seems natural. Bergman [1979] studied this test and some other test statistics based on the TTT-plot; see also Aarset [1983].

Klefsjö [1983] used the characterizations mentioned in Section 2 to develop ideas for test statistics when testing exponentiality against different aging properties. These ideas lead to some new statistics when testing against IFR, IFRA and light-tailedness. In the NBUE and DMRL cases, he obtained the cumulative TTT-statistic and a statistic K^*, respectively. Hollander and Proschan [1975, 1980] obtained the same test statistics using different arguments.

5. AGE REPLACEMENT

Replacement theory, i.e. the study of strategies for replacement of units in active use in order to minimize costs due to failures and replacements, is an important subject in reliability theory; see e.g. Barlow and Proschan [1965, Chapter 4] and Pierskalla and Voelker [1976]. Bergman [1977b] found (see also Barlow [1978] and Bergman [1979]) that the TTT-concept is very useful when studying the age replacement problem.

Assume that a certain type of unit is needed in a continuous production process and that a unit always can be replaced by a new one at a cost c. At failure, the unit is replaced and an additional consequence cost K is incurred. Let F be the life distribution of the time to failure of the prescribed type of unit. Assume that F is continuous and strictly increasing and that the units can be regarded as independent. The ordinary age replacement problem is to find an optimal age replacement interval $T = T^*$ that minimizes $C(T)$, the average long run cost per unit time, which is given by (cf. Barlow and Proschan [1965, Chapter 4])

$$C(T) = (c + KF(T))/\int_0^T \bar{F}(x)dx. \tag{2}$$

The optimal age replacement policy is then to "replace at T^* or at failure, whichever occurs first." The definition of the scaled TTT-transform implies that T^* minimizes (2) exactly when $t^* = F(T^*)$ maximizes the expression $\varphi_F(t)/(t + (c/K))$. Hence, we can find the optimal age replacement interval $T^* = F^{-1}(t^*)$ using the graphical procedure illustrated in Figure 3.

This graphical procedure has a great educational value. Many properties of the optimal age replacement interval are easier to understand by using this graphical interpretation; it is, for example, easy to find and interpret criteria for a unique optimal replacement interval to exist and for the optimal age replacement interval to be finite (cf. Bergman and Klefsjö [1982]). The procedure is also convenient for performing sensitivity analyses when different values are assigned to c and K (cf. Bergman [1980]).

If F is unknown, but some observations are available, φ_F in Figure 3 is replaced by the TTT-plot based on these observations and we get an estimator of T^* in a natural way; see Figure 4. Bergman [1979] studied this estimator.

In fact, as was noted by Bergman and Klefsjö [1982], every problem that can be transformed into a problem of maximizing $(\alpha + \beta \int_0^T \bar{F}(x)dx)/(\gamma + \delta F(T))$, where β, $\delta > 0$, can be analyzed in a similar way. This is the case for the age replacement problem with imperfect repair and several other maintenance models. The age replacement problem utilizing discounted cost models can also be studied by using

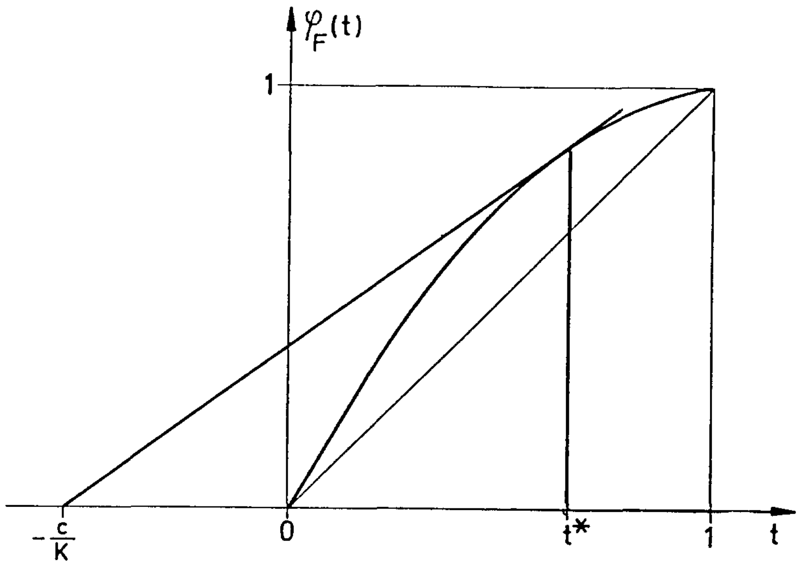

Figure 3. Illustration of the graphical determination of the optimal age replacement interval.

TTT-transforms; see Bergman and Klefsjö [1983]. A forthcoming paper by the authors will show that optimal burn-in procedures for nonrepairable electronic devices can also be found in a similar way.

6. GENERALIZATIONS

This section will indicate some further generalizations of the TTT-concept. We are, however, convinced that there are many other generalizations of the TTT-concept that have not yet been utilized.

Barlow and Davies [1979] have studied bivariate generalizations of the TTT-plot. By using modern computer graphics, this way of displaying bivariate life distributions and bivariate life data becomes very illuminating. Barlow and Proschan [1977a, 1977b] also studied bivariate TTT-processes.

The TTT-plot also is a valuable tool in point process contexts. Barlow [1978] utilized this approach to study the intensity function of a non-homogeneous Poisson process of which a number of replicates have been observed; see also Barlow and Davies.

For discrete life distributions with support in {0, 1, 2, · · ·}, it is possible, in a rather analogous way, to define discrete TTT-transforms. However, it is important to know whether or not zero belongs to the support. The discrete TTT-concept will be studied in a forthcoming paper.

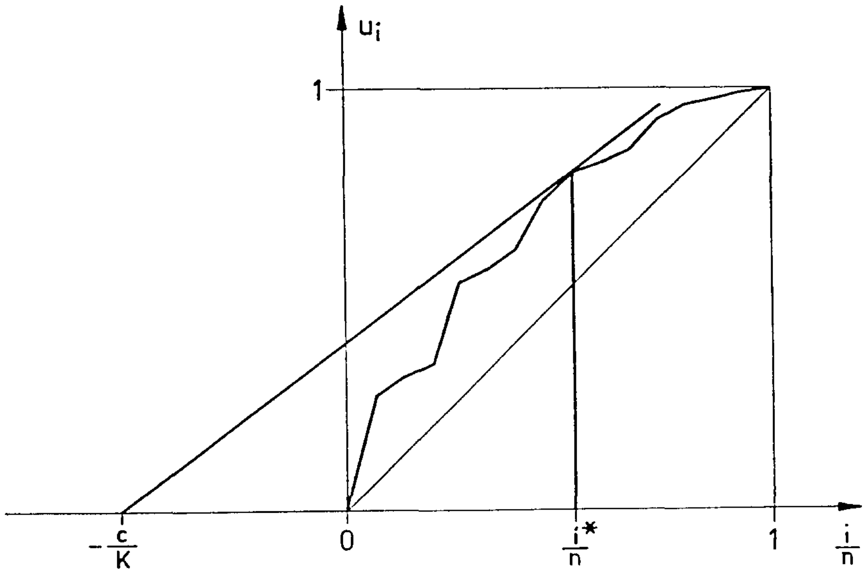

Figure 4. Graphical illustration of the estimation of the optimal age replacement interval; the estimator is determined as $T^* = x_{(i^*)}$.

Bergman [1981] has shown that the TTT-concept is useful when, for the multinomial distribution with parameters $(n, \bar{p})$, we want to test $H_0{:}\bar{p} = \bar{p}_0$ against certain ordered alternatives.

Very often, the failure rate of an aging device is dependent on the age of the device only indirectly through some parameter indicating the wear or the amount of deterioration of the unit. Sometimes this parameter is observable. Bergman [1978] studied optimal replacement strategies for such situations. Bergman [1977b] indicated that procedures similar to the estimation method using TTT-plots described in Section 5 can be used in this case as well. Hypothesis testing concerning the shape of the parameter dependent failure rate can also be studied by using the generalized TTT-concept introduced by Bergman [1977b]

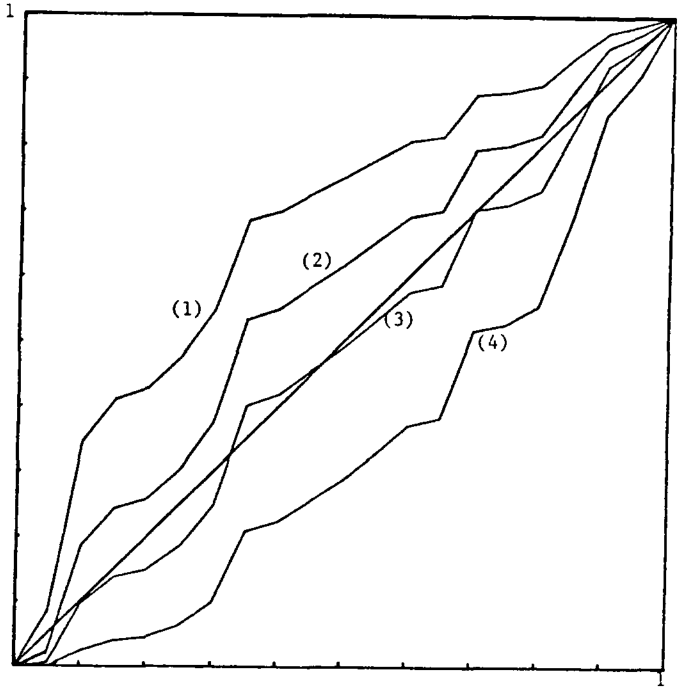

Figure 5. A TQT-plot is obtained by plotting Q_i/Q_n, where $Q_i = \sum_{j=1}^{i} (n - j + 1)(Q(x_{(j)}) - Q(x_{(j-1)}))$, against i/n, $i = 0, 1, \cdots, n$ and then connecting the points by straight lines. This figure shows TQT-plots from simulated Weibull data ($\beta = 2.0$, $n = 20$) with $Q(x) = x^\alpha$, where $\alpha = 1.0$ in (*1*), $\alpha = 1.5$ in (*2*), $\alpha = 2.0$ in (*3*) and $\alpha = 3.0$ in (*4*). The corresponding scaled TQT-transform coincides in this case with the diagonal when $Q(x) = x^2$.

Jewell [1977] introduced a generalization of the scaled TTT-transform, called the *scaled TQT-transform*. By using this transform, we get, under suitable conditions, the diagonal in the unit square as the transform of F. This property may be advantageous in certain situations, as in model identification. Figure 5 illustrates a TQT-plot based on a simulated sample.

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