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# The TTT transformation and a new bathtub distribution model

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**Abstract**

In this paper we use the total time on test transformation to establish a method for construction of parametric models of lifetime distributions having bathtub-shaped failure rate. We study a particular model which is simple compared to the other existing models. We derive expressions for moments and quantiles and treat estimation methods. Particularly, the maximum likelihood method is studied. Consistency proofs are given.

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## 1. Introduction

Parametric models of lifetime distributions are often used in reliability. These models have to fulfil two criteria:

– simplicity,

– adequate parametric description of the lifetime distribution.

Most of the commonly used models have increasing or decreasing failure rate (IFR or DFR). Considering reliability behaviour of technical objects, however, one has to deal with bathtub-shaped failure rates: Starting with a decreasing phase, the so-called infant mortality phase, the failure rate becomes almost constant in the second phase. The final phase is characterized by wear-out processes and aging, reflected by an increasing failure rate. Unfortunately, frequently used and easy models, as e.g. the Weibull distribution, are able to model only monotonic behaviour of the failure rate. Using these distributions the phases with decreasing or increasing failure rate are

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neglected. Alidirsi et al. (1991) treated a sample with underlying bathtub failure rate using different models with monotonic behaviour of the failure rate to demonstrate this effect. On the other hand, most models with bathtub failure rates are rather complex. Estimation is possible by extensive iteration procedures and the main characteristics of the distribution as moments, quantiles, etc. are not available in closed form, see e.g. Dhillon (1981), Lai and Mukherjee (1986), Kunitz (1989), Gazer (1979) or Kogan (1988). A review of bathtub-shaped failure rate distributions (BTFRD) and further references is given by S. Rajarshi and M.B. Rajarshi (1988). Even the simplest model of a BTFRD known by now suffers from these drawbacks, see Siddiqui and Kumar (1991). These difficulties frighten practicers from using bathtub models.

However, erroneous use of models with monotone failure rate can lead to serious misleadings, see Haupt and Schäbe (1992). In most cases samples of lifetimes are censored, i.e. we do not observe the failure times of all objects under study. If censoring is heavy in the right tail of the lifetime distribution then failures coming from the increasing phase of the failure rate will almost not be observed. Fitting a Weibull distribution to these data yields a decreasing failure rate and a mean considerably larger than the true value.

The aim of the present paper is to consider known methods to construct bathtub-shaped failure rate distributions and propose a new method based on the total time on test (TTT) transformation. The new method is used to construct a model that is simpler than the well-known ones.

The plan of the paper is as follows. Section 2 presents a generalized definition for BTFRD. Section 3 gives an overview on construction techniques for BTFRD. The new construction technique is presented in Section 4. In Section 5 we study a new simple model with BTFRD. Parameter estimation methods have been comprised into an additional section. BTFRDs are lifetime distributions, consequently $F(0) = 0$. This proposition is assumed throughout the paper.

## 2. Definition and basic results for BT distributions

Glaser (1980) has given a definition of BTFRDs having a density.

**Definition 2.1** (Glaser (1980)). Let $F$ be a differentiable lifetime distribution with the density $f$ and the failure rate $\lambda$. Then $F$ is of BTFRD-type, provided there exists a $t_0$, $0 < t_0 < \infty$ such that

(a) $\lambda(t)$ is decreasing for $t \leqslant t_0$,

(b) $\lambda(t)$ is increasing for $t_0 \leqslant t$.

This definition is restricted to absolutely continuous distributions. A generalization of Definition 2.1 is given by Definition 2.2.

**Definition 2.2.** Let $F$ be a lifetime distribution. Then $F$ is of BTFRD-type if there exists a $t_0$, $0 < t_0 < \infty$, such that for

$$\bar{F}(x \mid t) = \bar{F}(t + x)/\bar{F}(t)$$

(a) $\bar{F}(x \mid t)$ is increasing in $t$ for $0 \leqslant t \leqslant t_0$, $0 \leqslant x \leqslant t_0 - t$,
(b) $\bar{F}(x \mid t)$ is decreasing in $t$ for $t_0 \leqslant t \leqslant t_\infty$, $x \geqslant 0$.

Obviously, Definitions 2.1 and 2.2 are identical for all absolutely continuous lifetime distributions.

**Theorem 2.3.** *A BTFRD has finite moments of all order provided the density exists.*

The proof is straightforward.

## 3. Construction techniques for BTFRD

S. Rajarshi and M.B. Rajarshi (1988) have presented several methods to construct BTFRDs.

(i) *Glasers technique.* Glacer choose a function $\eta(t)$ which fulfils the following criteria:

(a) $\eta(t) = -f'(t)/f(t)$ and $f(t)$ is a density function;

(b) there exists a $t_0 > 0$ such that $\eta'(t) < 0$ for all $t \in (0, t_0)$, $\eta'(t_0) = 0$ and $\eta'(t) > 0$ for all $t > t_0$;

(c) there exists a $y_0 > 0$ such that $\int_{y_0}^{\infty} [f(y)/f(y_0)]\eta(y_0)\,\mathrm{d}y - 1 = 0$.

Then $f(t)$ is the density of a BTFRD.

(ii) *Convex function.* With the definition of BTFRD we can define a BTFRD if we choose a positive convex function $\lambda(t)$ over $(0, \infty)$ with $\int_0^{\infty} \lambda(t)\,\mathrm{d}t = \infty$.

(iii) *Function of random variables.* This procedure is due to Griffith (1982).

(iv) *Reliability and stochastic mechanisms*:

- series systems,
- stochastic failure rate models,
- DMRL and UBTRL classes,
- stochastic differential equation models and population abundance distributions,
- mixtures.

S. Rajarshi and M.B. Rajarshi (1988) presented 16 models of BTFRD's. Most of them are very uncomfortable, e.g. only for three models there exist explicit equations for the quantiles, and only in one case the moments are explicitly given. The maximum likelihood master equations can be found in the literature only for six models; only for nine model, they are explicitly for the parameters.

As an example consider the hazard rate

$$\lambda(t) = \alpha + \beta t + \gamma t^2, \quad t \geqslant 0, \beta < 0, \gamma > 0.$$

This model seems to be easy, but the moments can be obtained only by numerical methods. The quantiles are solutions of an algebraic equation of third order.

The most simple model is presented by Mukherjee and Islam (1983). This model, however, has also several drawbacks.

## 4. Using TTT to construct BTFRD

The scaled TTT transformation and TTT plot were introduced by Barlow and Campo (1975). Further developments were made by Gupta and Michalek (1985). In the sequel the concepts have proved to be useful in statistical analysis of failure date.

We define the TTT transformation $H_F^{-1}$ of a lifetime distribution $F$ by

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} (1 - F(s))\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1,$$

where $F^{-1}(t) = \inf\{x\colon F(x) \geqslant t\}$.

The scaled TTT transformation is $\Phi_F(t) = H_F^{-1}(t)/H_F^{-1}(1)$.

An obvious property of the scaled TTT transformation simple to see is that $\Phi(0) = 0$, $\Phi(1) = 1$, and $0 \leqslant \Phi(t) \leqslant 1$ for all $t$.

The graph of the scaled TTT transformation of an exponential distribution is the diagonal line in the unit square. The graph of any IFR (DFR) distribution is a concave (convex) curve in the unique square connecting the points $(0, 0)$ and $(1, 1)$. This suggests that a distribution with scaled TTT transformation which is convex in the beginning and then concave could be a BTFRD.

Such an s-shaped curve need not cross the diagonal line. An example can be given by the following function:

$$H_F^{-1}(u) = \begin{cases} \frac{1}{4}u & \text{for } 0 \leqslant u \leqslant \frac{1}{4}, \\ \frac{1}{16}(12u - 2) & \text{for } \frac{1}{4} \leqslant u \leqslant \frac{1}{2}, \\ \frac{1}{16}(28u - 10) & \text{for } \frac{1}{2} \leqslant u \leqslant \frac{3}{4}, \\ \frac{1}{16}(20u - 4) & \text{for } \frac{3}{4} \leqslant u \leqslant 1. \end{cases}$$

**Theorem 4.1.** *Let $F(t)$ be a lifetime distribution with absolutely continuous density $f(t)$. The following statements are equivalent*:

(i) *$F(t)$ is BTFRD.*

(ii) *The scaled TTT transformation $\phi_F(u)$ has one inflection point $u_0$ such that $0 < u_0 < 1$ and $\phi_F(u)$ is convex on $[0, u_0]$ and concave on $[u_0, 1]$.*

**Proof.** The scaled TTT transformation

$$\Phi_F(s) = \int_0^{F^{-1}(s)} \bar{F}(t)\,\mathrm{d}t \bigg/ \int_0^{F^{-1}(1)} \bar{F}(t)\,\mathrm{d}t$$

has second derivative given by

$$\frac{\mathrm{d}^2\Phi(u)}{\mathrm{d}u^2} = -\frac{\lambda'(F^{-1}(u))(1-u)}{\lambda^3(F^{-1}(u))} \quad \text{for } u \in \operatorname{supp}(F),$$

where $\operatorname{supp}(F)$ denotes the support of $F$.

Since $1 - u \geqslant 0$, $\mathrm{d}^2\Phi(u)/\mathrm{d}u^2$ is positive when $\lambda'$ is negative and vice versa. Consequently, $\phi$ is convex (concave) if $\lambda$ is decreasing (increasing). Moreover, $\mathrm{d}^2\Phi(u)/\mathrm{d}u^2$ is zero inside the interval (0, 1) if and only if

$$\lambda'(t_0) = 0 \text{ with } F(t_0) = u_0.$$

Finally, we have to assume that $\lambda$ is bounded inside the support of $F$. Since $f$ is absolutely continuous, unboundedness of $\lambda(u')$ implies unboundedness of $\Lambda(u')$ which is equivalent to $F(u') = 1$. However, the point $u'$ cannot lie inside the interval $[0, 1]$. □

Theorem 4.1 can be used to construct the following:
Choose a twice differentiable function $\Phi(u)$, fulfilling the properties

(1) $\Phi(0) = 0$, $\Phi(1) = 1$, $0 \leqslant \phi(u) \leqslant 1$:

(2) the solution $F(t)$ of the differential equation

$$\frac{\theta\Phi(F(t))\,\mathrm{d}F(t)}{1 - F(t)} = \mathrm{d}t, \quad \text{with } \theta := H_F^{-1}(1) > 0 \tag{1}$$

is a lifetime distribution;

(3) $\Phi$ satisfies statement (ii) of Theorem 4.1.

In the sequel we demonstrate the method with the help of an example.

**Example.** It is suitable, to choose $\Phi(t)$ such that in (1) the denominator disappears. We take

$$\Phi'(u) = \alpha(1 - u)(\beta + u)$$

and get

$$\Phi(u) = -\tfrac{1}{3}\alpha u^3 + \tfrac{1}{2}(\alpha - \alpha\beta)u^2 + \alpha\beta u.$$

$\Phi(u)$ is a polynomial of third degree.

Property (1) is fulfilled for $\alpha = 2/(\beta + \frac{1}{2})$.

To get a $\Phi(t)$ which fulfils property (2) we have to ensure that there exists a $u_0$ with $\Phi''(u_0) = 0$ and $0 < u_0 < 1$.

From $\Phi''(u_0) = 0$ we have $u_0 = \frac{1}{2}(1 - \beta)$. For $-1 < \beta < 1$ $u_0$ is in the interval (0, 1). For $\beta \geqslant 1$ or $\beta \leqslant -1$ the failure rate, provided it exists, is monotonic.

We solve Eq. (1) and get

$$\alpha \int_0^u \frac{(1-u)(\beta+u)}{1-u}\,\mathrm{d}u = \frac{t}{\theta}.$$

Substituting $\theta := \alpha\theta > 0$ we have a quadratic equation

$$u^2 + 2\beta u - \frac{2t}{\theta'} = 0.$$

Taking $F(t) = u$ we have

$$F(t) = -\beta \pm \sqrt{\beta^2 + (2t/\theta')}. \tag{2}$$

For simplicity, we introduce a new parameter $T$ by $T := \theta' \frac{1}{2}(1+2\beta)$. Then (2) becomes

$$F(t) = -\beta \pm \sqrt{\beta^2 + (1+2\beta)t/T}. \tag{3}$$

Since $F(t)$ is a distribution function we have the restriction that $F(t)$ is nondecreasing. Then the two solutions comprised in (3) read

$$F(t) = -\beta + \sqrt{\beta^2 + (1+2\beta)t/T}, \quad 0 \leqslant t \leqslant T,\ \beta > -\tfrac{1}{2} \tag{4}$$

and

$$F(t) = \gamma - \sqrt{\gamma^2 - (2\gamma-1)t/T}, \quad 0 \leqslant t \leqslant T,\ \gamma > \tfrac{1}{2}. \tag{5}$$

In the sequel we will study both solutions. Solution (5) can be proved to have monotone failure rate.

The density of distribution (5) is

$$f(t) = \frac{2\gamma - 1}{2T\sqrt{\gamma^2 - (2\gamma-1)t/T}}, \quad \gamma > \tfrac{1}{2}.$$

If $\gamma \neq 1$ the density is increasing from the value $(2\gamma-1)/(2T\gamma)$ at $t=0$ up to $(2\gamma-1)/(2T|\gamma-1|)$ at $t=T$. The same tendency holds for $\gamma = 1$, approaches infinity at $t = T$.

To investigate the failure rate we study the behaviour of $\lambda'(t)$. We introduce the notation

$$g := \gamma^2 - (2\gamma-1)t/T.$$

$$\lambda'(t) = \frac{(2\gamma-1)^2}{4T^2}\,\frac{2g^{1/2} + 1 - \gamma}{[(1-\gamma) + g^{1/2}]^2 g^{3/2}}.$$

There exist no $t^0 \in (0,T)$ with $\lambda'(t^0) = 0$. This means $\lambda(t)$ is monotone for all $\gamma > \frac{1}{2}$. $\lambda(t)$ is increasing from $(2\gamma-1)/(2T\gamma)$ at $t=0$ up to $(2\gamma-1)/(4T(1-\gamma)^2)$ at $t=T$ for $\gamma < 1$ and for $\gamma \geqslant 1$ to infinity at $t = T$.

## 5. The bathtub distribution

This section is dedicated to the distribution function defined by

$$F(t) = \begin{cases} 1, & t \geqslant T, \\ -\beta + \sqrt{\beta^2 + (1+2\beta)t/T}, & 0 \leqslant t < T, \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

The new model will be called the BT distribution. Obviously, it has finite origin, i.e. with probability 1 all items fail until $T$. This property is natural since a technical object usually has a finite life length.

The density of the BT distribution is given by

$$f(t) = \begin{cases} \dfrac{1+2\beta}{2T\sqrt{\beta^2 + (1+2\beta)t/T}}, & 0 \leqslant t \leqslant T, \\ 0, & \text{otherwise}. \end{cases}$$

If $\beta > 0$ the density is decreasing from the value $(1+2\beta)/(2T\beta)$ at $t = 0$ down to $(1+2\beta)/(2T(1+\beta))$ at $t = T$. The same tendency holds for $\beta = 0$, approaches infinity value at $t = 0$. For $\beta < 0$ the density is increasing from $(1+2\beta)/(2T|\beta|)$ up to $(1+2\beta)/(2T(1+\beta))$.

The failure rate is given by

$$\lambda(t) = \begin{cases} \dfrac{1+2\beta}{2T\sqrt{\beta^2 + (1+2\beta)t/T}\,(1+\beta-\sqrt{\beta^2+(1+2\beta)t/T})}, & 0 \leqslant t \leqslant T, \\ 0, & \text{otherwise}. \end{cases}$$

To investigate for which values of $\beta$ the failure rate is bathtub or monotone we study the behaviour of $\lambda'(t)$.

We introduce the notation $z := \beta^2 + (1+2\beta)t/T$. Then

$$\lambda'(t) = \frac{(1+2\beta)^2(2z^{1/2} - (1+\beta))}{4T^2(1+\beta-z^{1/2})^2 z^{3/2}}, \quad 0 \leqslant t \leqslant T.$$

There exists a unique $t^0$ with $\lambda'(t^0) = 0$, defined by

$$t^0 = \frac{T}{4(1+2\beta)}(1 + 2\beta - 3\beta^2),$$

$$\lambda(t^0) = \frac{2(1+2\beta)}{(1+\beta)^2 T}.$$

For $-\frac{1}{3} < \beta < 1$ $t^0$ lies within the origin and $\lambda(t^0)$ is a minimum. Since

$$\lambda'(0) = \frac{(1+2\beta)^2(2|\beta| - (1+\beta))}{4T^2} = \begin{cases} > 0, & \beta > 1, \\ < 0, & -\frac{1}{3} < \beta < 1, \\ > 0, & -\frac{1}{2} < \beta < -\frac{1}{3}, \end{cases}$$

and

$$\lim_{t \to t_0} \lambda'(t) = +\infty,$$

$\lambda$ has bathtub shape for $-\frac{1}{3} < \beta < 1$. For $\beta > 1$, $\lambda'(t)$ is strictly positive over the origin and hence $\lambda(t)$ must be increasing. For $-\frac{1}{2} < \beta < -\frac{1}{3}$ $\lambda'(t)$ is also positive over the origin and hence the failure rate must be increasing.

Fig. 1 shows the failure rate for $\beta = 0.2$ and $T = 1$.

To calculate the moments of the distribution we introduce a random variable by

$$U(X) := \beta^2 + X(1+2\beta)/T.$$

The moments of $U$ are

$$\mathbb{E}U^k = \int_{\beta^2}^{(1+\beta)^2} u^k \, \mathrm{d}u^{1/2} = \int_{\beta^2}^{(1+\beta)^2} \tfrac{1}{2} u^{k-1/2} \, \mathrm{d}u$$
$$= \frac{(1+\beta)^{2k+1} - \beta^{2k+1}}{2k+1}.$$

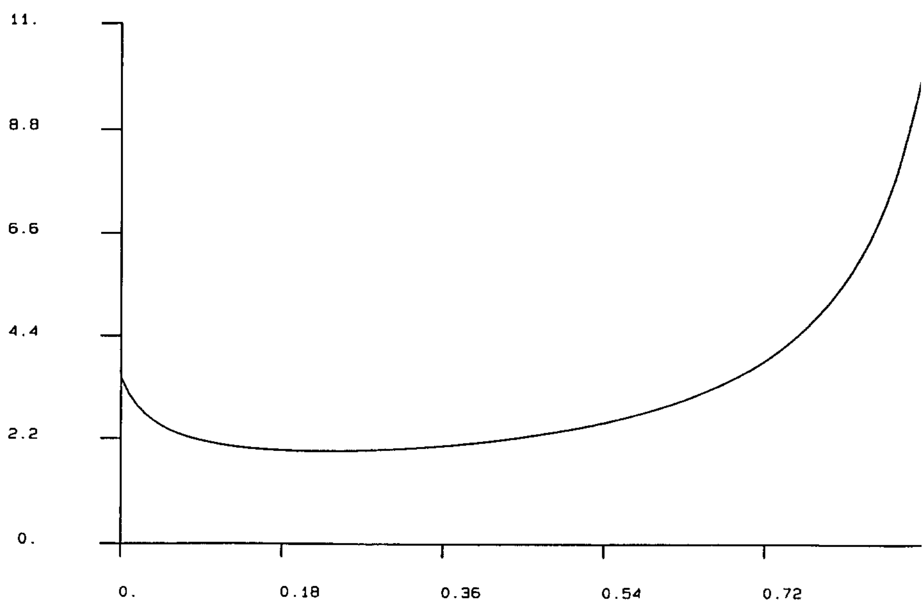


Fig. 1. Failure rate of BT distribution for $\beta = 0.2$ and $T = 1$.

The moments of $X$ can be obtained from

$$m_k = \mathbb{E}T^k = (T/(1+2\beta))^k \sum_{i=0}^{k} \binom{k}{i} \mathbb{E}U^i(-\beta^2)^{k-i}.$$

Particularly,

$$\mathbb{E}X = (\beta + \tfrac{1}{3})T/(1+2\beta), \tag{7}$$

$$\mathbb{E}X^2 = (\tfrac{4}{3}\beta^2 + \beta + \tfrac{1}{5})T^2/(1+2\beta)^2.$$

Similarly, for the central moments $\mu_k$ of $X$ we can write

$$\mu_k = (T/(1+2\beta))^k \sum_{i=0}^{k} \binom{k}{i} \mathbb{E}U^i(-(\beta^2 + \beta + \tfrac{1}{3})^{k-i},$$

$$= (T/(1+2\beta))^k \sum_{i=0}^{k} \binom{k}{i} \frac{(1+2\beta)^{2k+1} - \beta^{2k+1}}{2k+1}(-(\beta^2+\beta+\tfrac{1}{3}))^{k-i},$$

$\mathrm{Var}(X) = (T/(1+2\beta))^2(\frac{1}{3}\beta^2 + \frac{1}{3}\beta + \frac{4}{45})$.

The variation coefficient turns out to be

$$v = \sqrt{\tfrac{1}{3}\beta^2 + \tfrac{1}{3}\beta + \tfrac{4}{45}}/(\beta + \tfrac{1}{3}). \tag{8}$$

For the quantities we have

$$t_p = (p^2 + 2p\beta)T/(1+2\beta).$$

The main characteristics of the BT distribution can be given explicitly by elementary algebraic operations. This is the main effort of the new model.

## 6. Parameter estimation

A commonly used estimation technique is maximum likelihood estimation since it is asymptotically efficient and straightforward to implement. It can be shown that our BT distribution fulfils the regularity conditions provided $|\beta| \geqslant \varepsilon$, $\varepsilon$ small. For $\beta = 0$ the density function tends to infinity as $t$ tends to zero, i.e. there exists a pole.

### *6.1. Maximum likelihood*

Let us denote the failure times by $t_1, \ldots, t_n$ and assume $t_1 < \cdots < t_n$. For the BT distribution the maximum likelihood estimators are

$$\hat{T} = t_n, \tag{12}$$

$$\frac{2}{1+2\hat{\beta}} = \frac{1}{n}\sum_{i=0}^{n} \frac{\hat{\beta} + t_i/\hat{T}}{\hat{\beta}^2 + (1+2\hat{\beta})t_i/\hat{T}}. \tag{13}$$

Eq. (13) for $\hat{\beta}$ can be solved by simple bisection algorithms.

**Theorem 6.1.1.** *The estimators* $\hat{\beta}$ *and* $\hat{T}$ *defined* (12) *and* (13) *are consistent and asymptotically normal distributed if* $|\beta| > \varepsilon$, $-\frac{1}{2} + \varepsilon < \beta < \beta_0$ *and* $T < T_0$. *Here* $\beta_0$ *and* $T_0$ *are arbitrary but fixed,* $\varepsilon$ *is an arbitrary small number.*

Verifying the conditions for the MLEs the proof is straightforward.

In practice samples of life test data are usually censored. The application to censored data is straightforward and leads to a system of two coupled equations for the two unknown parameters.

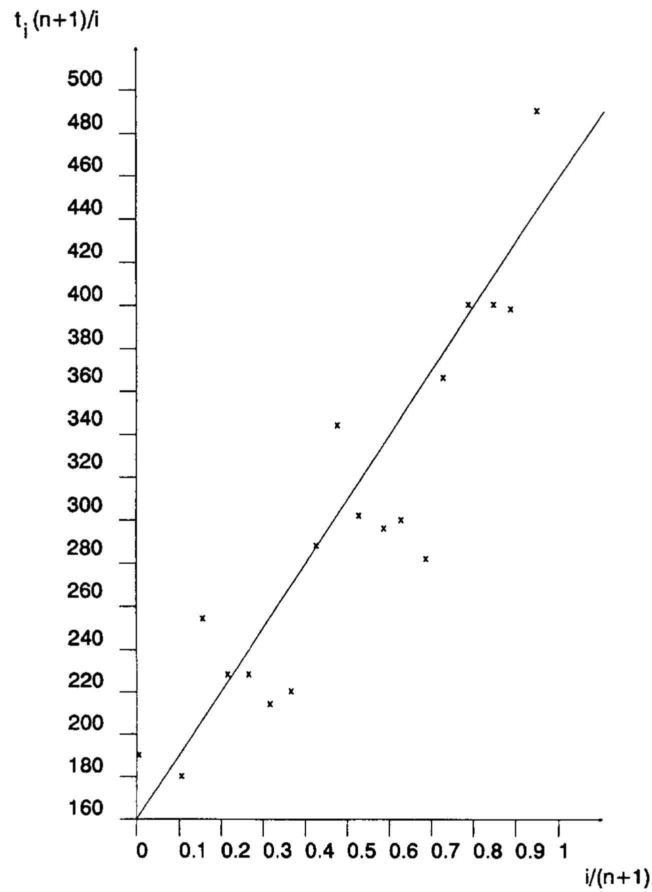


Fig. 2.

### 6.2. Probability plotting technique

Probability plotting techniques are useful for practical applications. In this case we can obtain estimators for both parameters explicitly. Eq. (4) can be rewritten as

$$\frac{t}{F(t)} = \frac{TF(t)}{1+2\beta} + \frac{2\beta T}{1+2\beta}.$$

Estimating $F(t_{(i)})$ by $i/(n+1)$ and plotting $i/(n+1)$ versus $t_i(n+1)/i$ shall give approximately a straight line with intercept

$$\alpha = 2\hat{\beta}\hat{T}/(1+2\hat{\beta})$$

and slope

$$\tan(\phi) = \hat{T}/(1+2\hat{\beta}).$$

Finally the estimators are

$$\hat{T} = \alpha + \tan\phi, \qquad \hat{\beta} = \alpha/(2\tan\phi).$$

This estimation technique can be applied even with a simple pocket calculator. The estimation by a probability plot is shown in Fig. 2. We used a sample of size 18 consisting of the failure time data 10, 19, 39, 48, 60, 68, 81, 122, 164, 170, 172, 189, 194, 293, 321, 345, 350, 465. Plotting $i/(n+1)$ versus $(n+1)t_i/i$ we obtain $\alpha = 160$ and $\tan\phi = 333$. This gives $\hat{T} = 493$ and $\hat{\beta} = 0, 24$.

The application of other techniques, as ordinary least squares or the method of moments, is possible without problems.

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