A Kernel-Type Estimator of a Quantile Function From Right-Censored Data

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Based on right-censored data from a lifetime distribution F_0, a kernel-type estimator of the quantile function $Q^o(p) = \inf\{t: F_0(t) \geqq p\}$, $0 \leqq p \leqq 1$, is proposed. The estimator is defined by $Q_n(p) = h_n^{-1} \int_0^1 \hat{Q}_n(t)K((t - p)/h_n)\,dt$, which is smoother than the usual product-limit quantile function $\hat{Q}_n(p) = \inf\{t: \hat{F}_n(t) \geqq p\}$, where $\hat{F}_n$ denotes the product-limit estimator of F_0 from the censored sample. Under the random censorship model and general conditions on h_n, K, and F_0, it is shown that $Q_n(p)$ is strongly consistent. In addition, an approximation to Q_n is shown to be asymptotically equivalent to Q_n with probability one. A small Monte Carlo simulation study shows that for several values of the bandwidth h_n, Q_n performs better than $\hat{Q}_n$ in the sense of estimated mean squared errors. An optimal bandwidth h_n may be estimated by bootstrap methods in some cases. The procedure is illustrated by an application to data from a mechanical-switch life test.

KEY WORDS: Random censorship; Product-limit quantile function; Kernel estimation; Median survival time estimation; Nonparametric quantile estimation.

1. INTRODUCTION

Arbitrarily right-censored data arise naturally in industrial life testing and medical follow-up studies. In these situations it is important to be able to obtain nonparametric estimates of various characteristics of the survival function S. Based on such right-censored data, Kaplan and Meier (1958) gave the nonparametric maximum likelihood estimator of S itself, called the product-limit estimator. Since that time many authors have studied nonparametric estimation of other characteristics of S under right censorship. Among them, Reid (1981) proposed two methods for setting confidence intervals for the median survival time, and more recently, Nair (1984) gave a comparison of confidence bands for S that had been proposed by Aalen (1976), Hall and Wellner (1980), and Csörgő and Horváth (1982). The nonparametric estimation of a density for S has been studied by Blum and Susarla (1980); Földes, Rejtö, and Winter (1981); Burke (1983); Burke and Horváth (1984); Yandell (1983); and McNichols and Padgett (1984), among others. Padgett and McNichols (1984) reviewed the methods for nonparametric density estimation from censored data.

One characteristic of the survival distribution that is of interest is the *quantile function*, which is useful in reliability and medical studies. For any probability distribution function G, the quantile function is defined by $Q(p) \equiv G^{-1}(p) \equiv \xi_p = \inf\{x: G(x) \geqq p\}$, $0 \leqq p \leqq 1$. In particular, $\xi_{.5}$ is a median of G. For a random (uncensored) sample $Y_1, \ldots, Y_n$ from G, the sample quantile function $G_n^{-1}(p) = \inf\{x: G_n(x) \geqq p\}$, $0 \leqq p \leqq 1$, has been used to estimate ξ_p, where $G_n(x)$ denotes the sample distribution function. Note that $G_n^{-1}(p) = Y_{([np])}$, the $[np]$th order statistic among $Y_1, \ldots, Y_n$, where $[\cdot]$ denotes the greatest integer function. Csörgő (1983) gave many of the known results concerning $G_n^{-1}(p)$. In addition, Falk (1984) has recently studied the relative deficiency of the sample quantile with respect to kernel-type estimators.

Other nonparametric estimators of the quantile function from uncensored data have been proposed that are smoother than the sample quantile function. For example, Kaigh and Lachenbruch (1982) considered a "generalized sample quantile" obtained by averaging an appropriate subsample quantile over all subsamples of a given size. Also recently, Yang (1985) has studied the properties of kernel-type estimators of ξ_p that smooth the sample quantile function. Parzen (1979) had mentioned kernel estimators as a possible class of quantile estimators but did not investigate their properties.

For arbitrarily right-censored data, Sander (1975) proposed estimation of ξ_p by the quantile function of the product-limit estimator of S, and she and Cheng (1984) obtained some asymptotic properties of that estimator. Aly, Csörgő, and Horváth (1985) proved strong approximation results for this estimator (cf. also Csörgő 1983, chap. 8).

The quantile function of the product-limit estimator is a step function with jumps corresponding to the uncensored observations. The purpose of this article is to present a smoothed nonparametric estimator of the quantile function from arbitrarily right-censored data based on the kernel method. It will be shown that under general conditions this estimator, mentioned briefly by Parzen (1979, p. 119), is strongly consistent, and based on the results of a small Monte Carlo simulation study, it performs better than the quantile function of the product-limit estimator in the sense of smaller mean squared error. In particular, better estimates of the median survival time are obtainable. In addition, an approximation to the kernel estimator will be shown to be almost surely asymptotically equivalent to it under certain conditions.

In kernel estimation, one problem is the optimal choice (in some sense) of the bandwidth. Since no results on the exact mean squared error of the proposed estimator are currently available, the bandwidth values that minimize the mean squared error cannot be obtained. Bootstrap methods for randomly right-censored data (Efron 1981) may be used in some cases, however, to estimate the optimal bandwidth from the data. The procedure for estimating the quantile function using the estimated bandwidth from right-censored data will be illustrated in Section 5, using the data of Nair (1984).

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Journal of the American Statistical Association
March 1986, Vol. 81, No. 393, Theory and Methods

2. ARBITRARILY RIGHT-CENSORED DATA

Let $X_1^o, \ldots, X_n^o$ denote the true survival times of n items or individuals that are censored on the right by a sequence $U_1, U_2, \ldots, U_n$, which in general may be either constants or random variables. It is assumed that the X_i^o's are nonnegative independent identically distributed random variables with common unknown distribution function F_0 and unknown quantile function $Q^o(p) \equiv \xi_p^o \equiv \inf\{t: F_0(t) \geqq p\}, 0 \leqq p \leqq 1$. In addition, $Q^o(p)$ is sometimes denoted by $F_0^{-1}(p)$.

The observed right-censored data are denoted by the pairs (X_i, Δ_i), $i = 1, \ldots, n$, where

$$X_i = \min\{X_i^o, U_i\}, \qquad \Delta_i = 1 \quad \text{if } X_i^o \leqq U_i$$
$$= 0 \quad \text{if } X_i^o > U_i.$$

Thus, it is known which observations are times of failure or death and which ones are censored or loss times. The nature of the censoring depends on the U_i's. (a) If $U_1, \ldots, U_n$ are fixed constants, the observations are time-truncated. If all U_i's are equal to the same constant, then the case of Type I censoring results. (b) If all $U_i = X_{(r)}^o$, the rth order statistic of $X_1^o, \ldots, X_n^o$, then the situation is that of Type II censoring. (c) If $U_1, \ldots, U_n$ constitute a random sample from a distribution H (usually unknown) and are independent of $X_1^o, \ldots, X_n^o$, then (X_i, Δ_i), $i = 1, 2, \ldots, n$, is called a *randomly right-censored sample*.

The random censorship model (c) is assumed for the asymptotic results of Section 4 and for the simulations reported in Section 5. For this model, $\Delta_1, \ldots, \Delta_n$ are independent Bernoulli random variables, and the distribution function F of each X_i $(i = 1, \ldots, n)$ is given by $F = 1 - (1 - F_0)(1 - H)$.

Based on the censored sample (X_i, Δ_i), $i = 1, 2, \ldots, n$, a popular estimator of the survival function $1 - F_0(t)$ at $t \geqq 0$ is the product-limit estimator, proposed by Kaplan and Meier (1958) as the "nonparametric maximum likelihood estimator." Efron (1967) showed that this estimator, defined next, is "self-consistent." Let (Z_i, Δ_i'), $i = 1, \ldots, n$, denote the ordered X_i's along with their corresponding Δ_i's. A value of the censored sample will be denoted by the corresponding lower-case letters (x_i, δ_i) and (z_i, δ_i') for the unordered and ordered sample, respectively. Then the product-limit estimator of $1 - F_0(t)$ is defined by

$$\begin{aligned} \hat{P}_n(t) &= 1, & & 0 \leqq t \leqq Z_1, \\ &= \prod_{i=1}^{k-1} \left(\frac{n-i}{n-i+1}\right)^{\Delta_i'}, & & Z_{k-1} < t \leqq Z_k, \\ & & & k = 2, \ldots, n \\ &= 0, & & Z_n < t. \end{aligned}$$

Denote the product-limit estimator of $F_0(t)$ by $\hat{F}_n(t) = 1 - \hat{P}_n(t)$, and let s_j denote the jump of $\hat{P}_n$ (or $\hat{F}_n$) at Z_j; that is,

$$\begin{aligned} s_j &= 1 - \hat{P}_n(Z_2), & & j = 1 \\ &= \hat{P}_n(Z_j) - \hat{P}_n(Z_{j+1}), & & j = 2, \ldots, n-1 \\ &= \hat{P}_n(Z_n), & & j = n. \end{aligned}$$

Note that $s_j = 0$ if and only if $\Delta_j' = 0$, $j < n$—that is, if Z_j is a censored observation. Moreover, denote $S_i \equiv \hat{F}_n(Z_{i+1}) = \sum_{j=1}^{i} s_j$ $(i = 1, 2, \ldots, n-1)$, $S_n \equiv 1$.

The product-limit estimator has played a central role in the analysis of censored survival data (Miller 1981). Its properties have been studied by many authors—for example, Breslow and Crowley (1974); Földes and Rejtö (1981); Földes, Rejtö, and Winter (1980); and Gill (1983).

Based on randomly right-censored data, it is natural to estimate the quantile function $Q^o(p)$ by the product-limit (PL) quantile function $\hat{Q}_n(p) \equiv \hat{\xi}_p \equiv \inf\{t: \hat{F}_n(t) \geqq p\}$. Cheng (1984) obtained asymptotic normality results for $\hat{\xi}_p$, and Aly et al. (1985) presented strong approximation theorems for the PL quantile process $\hat{Q}_n$.

3. THE QUANTILE ESTIMATOR

In this section the kernel estimator of $Q^o(p)$, $0 \leqq p \leqq 1$, from the randomly right-censored observations (X_i, Δ_i), $i = 1, \ldots, n$, will be defined. Analogous to Yang's (1985) estimators for the uncensored case, an approximation that is often easier to compute will be given. First, some assumptions and notation concerning the kernel, the bandwidth sequence, and the lifetime and censoring distributions will be listed.

Let $\{h_n\}$ be a bandwidth sequence of positive numbers such that

1. $h_n \to 0$ as $n \to \infty$.

Let K be a real-valued function defined on $(-\infty, \infty)$ such that

2. $K(x) \geqq 0$, all real numbers x
3. $\int_{-\infty}^{\infty} K(x)\, dx = 1$
4. K has finite support; that is, $K(x) = 0$ for $|x| > c$ for some $c > 0$
5. K is symmetric about zero
6. K satisfies a Lipschitz condition; that is, there exists a constant Γ such that for all x, y,

$$|K(x) - K(y)| \leqq \Gamma|x - y|.$$

Notice that conditions 2 and 3 simply say that K must be a probability density function. Moreover, assume that the lifetime distribution F_0 is such that

7. F_0 is continuous with density function f_0
8. f_0 is continuous at $\xi_p^o = Q^o(p)$ and $f_0(\xi_p^o) > 0$, $0 < p < 1$
9. F_0 has a finite mean.

It should be noted that conditions 7–9 are not prohibitive and are similar to those assumed by Cheng (1984). Conditions 1, 4–6, and 8–9 are required for the asymptotic results of Section 4.

Now, for $0 \leqq p \leqq 1$, define the kernel-type quantile function estimator

$$\begin{aligned} Q_n(p) &= h_n^{-1} \int_0^1 \hat{Q}_n(t) K((t - p)/h_n)\, dt \\ &= h_n^{-1} \sum_{i=1}^{n} Z_i \int_{S_{i-1}}^{S_i} K((t - p)/h_n)\, dt, \end{aligned} \tag{3.1}$$

where $S_0 \equiv 0$. It should be noted that only those Z_i that are uncensored appear in the sum (3.1), since

$$\int_{S_{i-1}}^{S_i} K((t - p)/h_n)\,dt = 0, \quad \text{if } Z_i \text{ is censored}$$

$$= h_n[K^*((S_i - p)/h_n) - K^*((S_{i-1} - p)/h_n)] \quad \text{if } Z_i \text{ is uncensored,}$$

where K^* denotes the cumulative distribution function of K.

An approximation to the estimator (3.1) can be obtained by noticing that the derivative of K^* at $(S_i - p)/h_n$ is approximated by

$$(h_n/s_i)[K^*((S_i - p)/h_n) - K^*((S_{i-1} - p)/h_n)] \approx K((S_i - p)/h_n).$$

Hence, when $S_i - S_{i-1}$ is small, (3.1) is approximately equal to

$$Q_n^*(p) = h_n^{-1} \sum_{i=1}^{n} Z_i s_i K((S_i - p)/h_n). \tag{3.2}$$

Again, since $s_i = 0$ when $\Delta_i' = 0$ $(i < n)$, only the uncensored observations explicitly appear in the sum (3.2). Note that when heavy censoring is present, $S_n - S_{n-1}$ will be large, so $Q_n^*(p)$ may not provide a good approximation in this situation, as indicated by the simulation results given in Section 5. When no censoring is present, $S_i - S_{i-1} = 1/n$ for all i.

In the case of no censoring, (3.1) and (3.2) reduce to the kernel estimators of Yang (1985). He has shown that his estimators are asymptotically equivalent in mean square and obtained rates of convergence for the variance and bias. Due to the censoring, similar results for the variance, bias, and mean squared consistency of (3.1) and (3.2) seem to be difficult to obtain under general conditions on F_0 and H. Some asymptotic results, however, have been obtained under reasonable conditions and are discussed in the next section.

4. SOME ASYMPTOTIC RESULTS

Next, conditions that imply the almost sure consistency of the kernel estimator $Q_n(p)$ are stated. The proofs will be presented in the Appendix.

For a distribution function G, let $T_G \equiv \sup\{t: G(t) < 1\}$.

Theorem 1. Suppose conditions 1, 2–6, and 7–9 hold. (a) If $T_{F_0} < T_H \leqq \infty$ and $h_n^{-1}(\log\log n/n)^{3/4} \to 0$ as $n \to \infty$, then for each $0 \leqq p \leqq 1$, $Q_n(p) \to Q^o(p)$ as $n \to \infty$ with probability one. (b) Let $T_{F_0} \leqq T_H \leqq \infty$ and assume that H is continuous in an arbitrarily short open left neighborhood (T, T_{F_0}) of T_{F_0}. Let ϕ be a function on the interval $[0, 1 - F(T)]$ so that $\phi(x) \geqq x$, $\phi(x) \to 0$ as $x \to 0^+$, and $1 - F_0(t) \leqq \phi(1 - F(t))$ for $t \in (T, T_{F_0})$. If, for any $c > 1$, $\lim_n\sup \phi(c(\log\log n/(2n))^{1/2})h_n^{-1} < \infty$, then for each $0 \leqq p \leqq 1$, $Q_n(p) \to Q^o(p)$ as $n \to \infty$ with probability one.

The conditions of Theorem 1(a) do not allow both F_0 and H

Table 1. Ratios of Mean Squared Errors With Triangular Kernel: 50% Censoring

		h_n													
p		.01	.03	.05	.07	.09	.11	.13	.15	.19	.21	.25	.31	.35	.41
								$n = 100$							
.10	a	1.07	1.16	1.18	1.31	1.34	1.43	1.40	1.51	1.40	1.27	1.17	.84	.65	.38
	b	1.17	1.29	1.37	1.52	1.55	1.66	1.68	1.85	1.75	1.66	1.58	1.15	.88	.50
.25	a	1.03	1.04	1.07	1.09	1.12	1.16	1.21	1.16	1.23	1.22	1.17	1.15	1.08	.93
	b	.58	1.17	1.20	1.22	1.27	1.31	1.35	1.35	1.40	1.44	1.43	1.50	1.47	1.40
.50	a	1.01	1.04	1.07	1.10	1.12	1.14	1.17	1.14	1.16	1.15	1.10	1.03	.89	.73
	b	.08	1.14	1.24	1.34	1.34	1.39	1.47	1.46	1.50	1.61	1.72	1.74	1.85	1.83
.75	a	1.03	1.07	1.19	1.23	1.41	1.41	1.31	1.20	1.43	1.34	1.31	1.60	1.85	2.64
	b	.04	.33	.63	1.16	1.53	1.51	1.69	1.27	1.47	1.31	1.30	2.40	2.66	2.65
.90	a	1.04	1.11	1.13	1.23	1.30	1.40	1.51	1.45	1.14	.94	.92	.50	.51	.37
	b	.03	.10	.16	.21	.22	.37	.64	.79	.76	.70	.76	.45	.48	.36
.95	a	1.02	1.02	1.02	.88	.69	.57	.49	.41	.34	.29	.28	.22	.22	.19
	b	.06	.09	.10	.37	.55	.57	.54	.49	.40	.33	.31	.24	.24	.20
								$n = 50$							
.10	a	1.05	1.08	1.12	1.22	1.26	1.29	1.38	1.41	1.47	1.34	1.24	.91	.87	.59
	b	.49	1.48	1.52	1.58	1.75	1.79	1.92	2.06	2.17	2.04	1.97	1.54	1.43	.98
.25	a	1.02	1.04	1.09	1.11	1.13	1.16	1.16	1.19	1.20	1.22	1.21	1.21	1.17	.97
	b	.10	1.27	1.32	1.32	1.48	1.45	1.53	1.57	1.57	1.71	1.71	1.90	1.88	1.80
.50	a	1.03	1.09	1.07	1.07	1.10	1.15	1.17	1.26	1.21	1.10	1.21	1.04	.96	.97
	b	.04	.34	1.08	1.34	1.57	1.73	1.92	2.09	1.89	2.00	2.40	2.14	2.11	2.25
.75	a	1.05	1.13	1.10	1.19	1.19	1.33	1.29	1.41	1.56	1.42	1.62	1.92	2.35	2.65
	b	.04	.17	.36	.61	.76	1.07	.93	.95	1.29	.98	1.22	2.16	2.74	2.20
.90	a	1.02	1.06	1.11	1.13	1.15	1.16	1.16	1.08	.90	.77	.69	.51	.48	.34
	b	.03	.11	.13	.17	.16	.25	.45	.60	.70	.63	.63	.50	.48	.34
.95	a	.99	1.00	.99	.88	.75	.63	.55	.51	.41	.39	.36	.31	.31	.29
	b	.11	.12	.13	.47	.66	.67	.68	.68	.55	.53	.46	.38	.36	.32

NOTE: a = (MSE $\hat{F}_n^{-1}$)/(MSE Q_n); b = (MSE $\hat{F}_n^{-1}$)/(MSE Q_n^*).

to have infinite support as do the conditions of Theorem 1(b). For example, both F_0 and H could be exponential distributions under (b). The condition on h_n in (a) however, may be somewhat less restrictive than that of (b). By example (1) of Csörgő and Horváth (1983, p. 416), ϕ may be chosen so that $\phi(x) = (x/k)^{1/(1+\gamma)}$ for $0 < k \leqq 1$ and $\gamma \geqq 0$. With this choice of ϕ, the condition of Theorem 1(b) becomes $\lim_n\sup(\log\log n/n)^{1/2(1+\gamma)}h_n^{-1} < \infty$. Then $h_n \approx Dn^{-1/2}$, for some $D > 0$, satisfies the condition in (a), but not this condition with $\gamma = 0$.

The two estimators Q_n and Q_n^* can be shown to be asymptotically almost surely equivalent (uniformly in p) under general conditions. First, define $\hat{\mu}(t) \equiv \int_0^t \hat{P}_n(x)\,dx$, $t \geqq 0$.

Theorem 2. Let conditions 1, 2, 3, 6, and 9 hold. Under the conditions of Theorem 1(b) if $\lim_n\sup \hat{\mu}(Z_n) < \infty$ with probability one, then

$$P[\lim_n\sup|Q_n^*(p) - Q_n(p)| \leqq M^* \lim_n\sup \phi(c(\log\log n/2n)^{1/2})h_n^{-2}] = 1,$$

where $M^* < \infty$ and $c > 1$ is some constant.

Thus, if $\phi(c(\log\log n/2n)^{1/2})h_n^{-2} \to 0$, then under the conditions of Theorem 2, Q_n^* and Q_n are asymptotically equivalent (uniformly in p) with probability one. As before, if $\phi(x) = (x/k)^{1/(1+\gamma)}$ for some $0 < k \leqq 1$ and $\gamma \geqq 0$ is chosen, then the condition above becomes (for $\gamma = 0$ and $k = 1$) $(\log\log n/n)^{1/2}h_n^{-2} \to 0$, which is satisfied by $h_n \approx Dn^{-b}$ for $0 < b < \frac{1}{4}$ and some positive constant D, for example. In addition, it should be remarked that $\lim_n\sup \hat{\mu}(Z_n) < \infty$ almost surely under the conditions given by Susarla and Van Ryzin (1980).

Lio, Padgett, and Yu (1985) have proven the asymptotic normality of $Q_n(p)$ for $0 < p < T$, where $T < 1$, utilizing results for generalized Kiefer processes (cf. Csörgő 1983, chap. 8), corollary 1 of Cheng (1984), and the proof of theorem 1 of Yang (1985). In particular, assume that conditions 1, 2–5, and 7 hold; $H(T_{F_0}) \leqq 1$; and the derivative f_0' is continuous at ξ_p^o and $f_0(\xi_p^o) > 0$. If $n^{1/4}h_n \to 0$ as $n \to \infty$, then for $0 < p < T$, with $T < 1$, $\sqrt{n}[Q_n(p) - Q^o(p)] \to W$ in distribution, where W has a normal distribution with mean zero and variance

$$\sigma_p^2 = (1 - p)^2 \int_0^{\xi_p^o} [1 - F(u)]^{-2} f_0^{-2}(\xi_p^o)\,dF_0^*(u).$$

Here F_0^* is the subdistribution function of the uncensored observations. Notice that the sequence $h_n \approx Dn^{-b}$ for $b > \frac{1}{4}$ is valid for the asymptotic normality but might not work for Theorem 2. Lio et al. (1985) also considered the asymptotic mean squared equivalence of Q_n and Q_n^*.

It seems to be difficult to obtain the exact mean squared error of Q_n or Q_n^* and to then be able to choose an exact value of h_n to minimize this mean squared error or to choose an exact optimal value of h_n in some other sense. It is possible, however, to estimate optimal values of h_n to use in practice with Q_n for a given right-censored sample by using bootstrap methods to estimate mean squared errors of $Q_n(p)$. This will be discussed

Table 2. Ratios of Mean Squared Errors With Triangular Kernel: 30% Censoring

		h_n													
p		.01	.03	.05	.07	.09	.11	.13	.15	.19	.21	.25	.31	.35	.41
								$n = 100$							
.10	a	1.01	1.10	1.15	1.24	1.26	1.32	1.34	1.41	1.37	1.19	1.11	.80	.62	.35
	b	1.14	1.22	1.30	1.39	1.44	1.50	1.55	1.66	1.64	1.49	1.42	1.02	.78	.43
.25	a	1.03	1.04	1.08	1.09	1.13	1.14	1.17	1.17	1.20	1.21	1.22	1.22	1.09	.94
	b	1.01	1.09	1.16	1.17	1.22	1.22	1.25	1.28	1.29	1.35	1.38	1.43	1.31	1.19
.50	a	1.01	1.04	1.05	1.07	1.09	1.10	1.11	1.12	1.16	1.09	1.11	1.06	.94	.74
	b	.42	1.10	1.11	1.15	1.19	1.20	1.21	1.22	1.26	1.24	1.29	1.29	1.22	1.05
.75	a	1.01	1.02	1.15	1.10	1.10	1.08	1.11	1.12	1.05	1.00	.79	.82	.98	1.41
	b	.08	1.02	1.38	1.27	1.33	1.35	1.40	1.43	1.53	1.57	1.50	1.39	1.46	1.68
.90	a	1.08	1.11	1.23	1.34	1.21	1.23	1.67	1.71	1.73	1.21	1.05	.55	.52	.36
	b	.06	.69	.97	1.25	1.15	1.34	2.07	1.72	1.61	1.10	.98	.53	.50	.35
.95	a	1.09	1.23	1.27	1.61	1.66	1.18	.94	.69	.51	.40	.34	.27	.26	.21
	b	.05	.32	.38	1.18	1.50	1.08	.94	.70	.53	.42	.36	.28	.27	.22
								$n = 50$							
.10	a	.98	.95	1.00	1.06	1.12	1.15	1.21	1.20	1.34	1.17	1.17	.83	.77	.55
	b	.37	1.17	1.26	1.29	1.44	1.46	1.54	1.59	1.80	1.61	1.66	1.23	1.10	.77
.25	a	1.03	1.07	1.07	1.08	1.12	1.14	1.10	1.17	1.17	1.21	1.18	1.23	1.23	1.12
	b	.14	1.21	1.21	1.18	1.28	1.29	1.27	1.35	1.37	1.45	1.42	1.59	1.60	1.55
.50	a	1.03	1.05	1.04	1.08	1.09	1.11	1.08	1.12	1.18	1.08	1.13	1.10	1.06	.93
	b	.06	1.13	1.12	1.21	1.24	1.28	1.25	1.33	1.40	1.42	1.42	1.48	1.55	1.46
.75	a	1.06	1.08	1.07	1.12	1.14	1.13	1.15	1.12	1.04	1.25	.92	1.11	1.14	1.79
	b	.04	.62	1.16	1.24	1.42	1.50	1.54	1.74	1.65	2.11	1.64	2.01	1.81	2.15
.90	a	1.04	1.10	1.24	1.39	1.23	1.42	1.69	1.83	1.84	1.59	1.22	.94	.67	.51
	b	.05	.22	.53	.76	.65	1.08	1.53	1.78	1.68	1.44	1.17	.92	.66	.50
.95	a	1.08	1.13	1.24	1.21	1.14	.96	.83	.65	.49	.42	.39	.36	.28	.24
	b	.06	.19	.21	.71	1.04	.98	.91	.72	.56	.47	.42	.39	.30	.25

NOTE: $a = (\text{MSE } \hat{F}_n^{-1})/(\text{MSE } Q_n)$; $b = (\text{MSE } \hat{F}_n^{-1})/(\text{MSE } Q_n^*)$.

Table 3. Ratios of Mean Squared Errors With Uniform Kernel: 50% Censoring (n = 100)

		h_n											
p		.03	.05	.07	.09	.10	.13	.15	.20	.25	.30	.35	.40
.10	a	1.12	1.13	1.25	1.26	1.29	1.30	1.40	1.52	1.53	1.50	1.19	.99
	b	.83	1.10	1.35	1.41	1.42	1.50	1.64	1.81	1.88	1.90	1.57	1.35
.25	a	1.02	1.05	1.06	1.08	1.10	1.15	1.13	1.17	1.24	1.24	1.29	1.15
	b	.66	.83	1.03	1.08	1.12	1.17	1.23	1.30	1.43	1.42	1.47	1.40
.50	a	1.03	1.06	1.09	1.10	1.06	1.13	1.13	1.12	1.17	1.16	1.18	1.03
	b	.26	.54	.83	.95	.96	1.07	1.17	1.31	1.32	1.40	1.60	1.50
.75	a	1.06	1.14	1.19	1.35	1.13	1.25	1.14	1.32	1.52	1.48	1.31	1.16
	b	.13	.27	.54	.74	.73	1.02	.84	1.15	1.39	1.28	1.21	.88
.90	a	1.08	1.08	1.17	1.21	1.26	1.35	1.47	1.46	1.22	.93	.71	.50
	b	.06	.10	.17	.20	.22	.23	.23	.20	1.32	1.12	.83	.56
.95	a	1.02	1.01	1.02	1.01	1.01	.68	.53	.36	.29	.26	.24	.22
	b	.08	.10	.09	.10	.10	.53	.73	.53	.38	.31	.27	.24

NOTE: $a = (\text{MSE } \hat{F}_n^{-1})/(\text{MSE } Q_n)$; $b = (\text{MSE } F_n^{-1})/(\text{MSE } Q_n^*)$.

further later. In addition, simulation results presented in the next section indicate a range of possible values of h_n for which the mean squared errors of $Q_n(p)$ (and $Q_n^*(p)$) are less than those of the PL quantile estimator for each p.

5. SOME SIMULATION RESULTS AND AN EXAMPLE

A small Monte Carlo study was performed in order to provide some small-sample comparisons of Q_n and Q_n^* with the PL quantile estimator, and with each other, in the sense of mean squared errors. The study also provides some insight into the choice of reasonable values for h_n that might be used in practice to estimate Q^o with smaller mean squared error than $\hat{Q}_n$. The random censorship model with $F_0(t) = 1 - \exp(-t)$ and $H(t) = 1 - \exp(-\lambda t)$ was used with λ chosen to give 50% censoring ($\lambda = 1$) or 30% censoring ($\lambda = \frac{3}{7}$) as in Reid (1981). The ratios of the mean squared error of $\hat{Q}_n(p)$ to the mean squared errors of the smoothed estimators $Q_n(p)$ and $Q_n^*(p)$ were computed for various $0 < p < 1$ and sample sizes $n = 50$ and 100. For each case, 1,000 censored samples were generated using the uniform random number generator GGUBS in the International Mathematical and Statistical Libraries (1982) on a DEC VAX 11-750 computer. The computed standard errors of the estimated mean squared errors from the simulations ranged from 10^{-1} to 10^{-4}.

Tables 1 and 2 show some of the results for the triangular kernel $K(x) = 1 - |x|$, $|x| \leqq 1$, which satisfies conditions 2–6 of Section 3. The simulations were run for values of $h_n = .01(.02).61$. For the estimator $Q_n(p)$, for each value of p listed there is an h_n that gives smaller estimated mean squared error than the PL quantile estimator. In particular, this is true for several h_n values for the *median* estimators $Q_n(.5)$ and $\hat{Q}_n(.5)$. The approximation $Q_n^*(p)$ performs well for several h_n values when $p \leqq .5$ but not so well for larger p. As would be expected for more severe censoring, the performance of either estimator at large values of p is not as good as for values near .5. Notice that h_n values from .09 to .13 appear to be best for $Q_n(p)$ over most values of p in Table 1 with $n = 100$, whereas for $Q_n^*(p)$ the h_n should be somewhat larger (.15–.21) for a good estimator for most values of p. Generally, in Tables 1 and 2 the best h_n for $Q_n^*(p)$ is larger than that for $Q_n(p)$, indicating that Q_n^* required more smoothing than Q_n.

The results of Tables 3 and 4 are for the uniform kernel $K(x) = 1$, $|x| \leqq \frac{1}{2}$. This kernel does not satisfy condition 6, but the simulation results are quite similar to the results in Tables 1 and 2.

Simulations were also performed with F_0 equal to a Weibull distribution function with shape parameter α and scale parameter equal to one and H an exponential distribution with scale

Table 4. Ratios of Mean Squared Errors With Uniform Kernel: 30% Censoring (n = 100)

		h_n											
p		.03	.05	.07	.09	.10	.13	.15	.20	.25	.30	.35	.40
.10	a	1.08	1.07	1.18	1.19	1.22	1.24	1.34	1.46	1.46	1.40	1.14	.92
	b	1.14	1.28	1.39	1.41	1.17	1.40	1.56	1.73	1.73	1.72	1.43	1.18
.25	a	1.03	1.07	1.07	1.09	1.09	1.12	1.15	1.17	1.26	1.22	1.26	1.19
	b	.61	.90	1.03	1.07	1.09	1.16	1.19	1.26	1.35	1.34	1.41	1.35
.50	a	1.04	1.04	1.05	1.07	1.08	1.08	1.10	1.13	1.14	1.13	1.13	1.08
	b	.47	.74	.90	.94	1.03	1.09	1.09	1.17	1.21	1.26	1.32	1.22
.75	a	1.01	1.02	1.08	1.08	1.04	1.11	1.18	1.10	1.03	.91	.80	.71
	b	.24	.56	.76	.93	.87	1.11	1.26	1.20	1.29	1.27	1.21	1.28
.90	a	1.07	1.23	1.28	1.19	1.21	1.35	1.24	1.34	1.77	1.07	.78	.43
	b	.23	.44	.69	.80	.77	.99	.94	.75	1.82	1.21	.85	.46
.95	a	1.16	1.27	1.35	1.53	1.51	1.55	.96	.59	.46	.32	.28	.24
	b	.18	.31	.37	.38	.29	1.63	1.19	.72	.51	.35	.30	.25

NOTE: $a = (\text{MSE } \hat{F}_n^{-1})/(\text{MSE } Q_n)$; $b = (\text{MSE } \hat{F}_n^{-1})/(\text{MSE } Q_n^*)$.

Table 5. Ratios of MSE's With Triangular Kernel for Weibull Life Distribution and Exponential Censoring Distribution: Shape Parameter .5, n = 100 (49.86% censoring)

		h_n											
p		*.03*	*.05*	*.07*	*.09*	*.11*	*.13*	*.15*	*.21*	*.25*	*.31*	*.35*	*.41*
.10	a	.90	.87	.85	.78	.64	.57	.46	.21	.15	.10	.06	.03
	b	1.10	1.09	1.08	1.00	.83	.73	.59	.26	.18	.12	.07	.03
.25	a	1.02	1.05	1.05	1.05	.99	1.00	.88	.73	.57	.37	.25	.15
	b	1.15	1.20	1.20	1.21	1.15	1.16	1.04	.88	.70	.44	.31	.18
.50	a	1.00	1.04	1.06	1.03	1.03	1.00	.96	.80	.67	.47	.32	.17
	b	1.21	1.22	1.24	1.23	1.24	1.22	1.19	1.07	.98	.77	.63	.42
.75	a	1.08	1.12	.98	1.11	1.37	1.06	1.08	.75	.94	1.08	1.27	1.53
	b	1.94	2.38	2.14	3.27	4.61	4.08	3.41	2.84	3.44	5.13	5.18	4.55
.90	a	1.14	1.18	1.30	1.30	1.48	1.52	1.54	1.20	1.19	.75	.71	.58
	b	.23	.33	.37	.32	.48	.85	.93	.85	.93	.64	.63	.53
.95	a	1.00	1.00	.88	.75	.66	.58	.53	.41	.38	.33	.33	.29
	b	.19	.21	.48	.59	.59	.56	.53	.42	.39	.34	.33	.29

NOTE: a = (MSE $\hat{F}_n^{-1}$)/(MSE Q_n); b = (MSE $\hat{F}_n^{-1}$)/(MSE Q_n^*).

one for the triangular kernel. Some of the simulation results are shown in Tables 5 and 6 when the Weibull shape parameter is $\alpha = .5$ (which gives 49.68% censoring) and $\alpha = 2$ (yielding 50.14% censoring). Comparison of Tables 1 and 5 suggests that Q_n performs better for exponential lifetime distributions than for Weibull distributions with shape $\alpha = .5$ for $p \leqq .5$, whereas Tables 2 and 6 indicate that Q_n performs better for Weibull distributions with $\alpha = 2$ for the same range of p. In particular, for $p = .1$, Q_n performs rather poorly for the Weibull lifetime distributions with $\alpha = .5$.

It should be noted that based on the simulations the approximation to Q_n, Q_n^*, does not perform very well relative to the PL quantile estimator for small h_n values (with fixed sample size n) and seems to be slightly worse for higher censoring percentages (at least for $p \geqq .5$). By definition of $Q_n^*(p)$, suppose for a given sample size n, $p \approx S_k \equiv \hat{F}_n(Z_{k+1})$. Then for sufficiently small h_n, $Q_n^*(p) = 0$ if Z_k is censored and $Q_n^*(p) \approx Z_k s_k / h_n$, which may be very large, if Z_k is uncensored. Hence, under random right-censorship, $Q_n^*(p)$ can have a large mean squared error for very small h_n values. However, when $p \approx S_k$ and h_n is sufficiently small, $Q_n(p)$ yields a value near Z_k for uncensored Z_k, which is near the value of $\hat{Q}_n(p)$. This is evident in Tables 1 and 2 when $h_n = .01$.

It was mentioned in Section 4 that the *exact* mean squared error of Q_n (or Q_n^*) for small n has not yet been calculated due to mathematical difficulties arising from the right-censorship. Moreover, the mean squared convergence (with a rate) has not yet been obtained. Hence, to find an optimal h_n in the sense of minimum mean squared error of Q_n (or Q_n^*) seems to be quite difficult. The simulations reported in Tables 1–6, however, indicate reasonable ranges of h_n values that give smaller mean squared errors than does the PL quantile function $\hat{Q}_n$ under the assumed models and censoring percentages. In addition, as mentioned earlier, the bootstrap method for randomly right-censored samples (Efron 1981) can sometimes be used to estimate values of h_n giving the smallest bootstrap mean squared error over a reasonable range of h_n values for a given sample. Efron (1981) showed that for *random* right-censorship, a bootstrap sample from (x_i, δ_i), $i = 1, \ldots, n$, is obtained by randomly choosing n pairs (x_j^*, δ_j^*) with replacement (with equal probabilities) from the given n censored observations. As an example, a randomly right-censored sample of size $n = 50$

Table 6. Ratios of MSE's With Triangular Kernel for Weibull Life Distribution and Exponential Censoring Distribution: Shape Parameter 2.0, n = 50 (50.14% censoring)

		h_n											
p		*.03*	*.05*	*.07*	*.09*	*.11*	*.13*	*.15*	*.21*	*.25*	*.31*	*.35*	*.41*
.10	a	1.16	1.24	1.32	1.36	1.47	1.59	1.72	1.88	2.03	2.22	2.54	2.82
	b	1.13	1.26	1.29	1.39	1.39	1.45	1.52	1.40	1.58	1.60	1.96	2.16
.25	a	1.07	1.11	1.09	1.13	1.15	1.13	1.19	1.34	1.37	1.50	1.54	1.60
	b	1.05	1.11	1.08	1.15	1.16	1.16	1.21	1.34	1.33	1.35	1.34	1.27
.50	a	1.03	1.04	1.08	1.11	1.08	1.13	1.17	1.19	1.24	1.30	1.36	1.41
	b	.86	.99	1.04	1.11	1.11	1.15	1.20	1.24	1.29	1.37	1.47	1.51
.75	a	1.11	1.08	1.12	1.15	1.14	1.17	1.19	1.24	1.19	1.35	1.49	1.03
	b	.42	.88	.96	1.15	1.11	1.16	1.20	1.40	1.27	1.45	1.41	.94
.90	a	1.12	1.19	1.22	1.26	1.36	1.49	1.34	.55	.33	.22	.16	.13
	b	.17	.57	.66	.75	.91	1.16	1.20	.60	.37	.24	.17	.14
.95	a	1.17	1.25	1.25	.78	.51	.36	.30	.16	.14	.11	.10	.08
	b	.14	.22	.70	.86	.69	.50	.40	.21	.16	.13	.11	.09

NOTE: a = (MSE $\hat{F}_n^{-1}$)/(MSE Q_n); b = (MSE $\hat{F}_n^{-1}$)/(MSE Q_n^*).

Table 7. Failure Times (in millions of operations) of Switches

Z_i	δ_i'	Z_i	δ_i'	Z_i	δ_i'	Z_i	δ_i'
1.151	0	1.667	1	2.119	0	2.547	1
1.170	0	1.695	1	2.135	1	2.548	1
1.248	0	1.710	1	2.197	1	2.738	0
1.331	0	1.955	0	2.199	0	2.794	1
1.381	0	1.965	1	2.227	1	2.883	0
1.499	1	2.012	0	2.250	0	2.883	0
1.508	0	2.051	0	2.254	1	2.910	1
1.543	0	2.076	0	2.261	0	3.015	1
1.577	0	2.109	1	2.349	0	3.017	1
1.584	0	2.116	0	2.369	1	3.793	0

from the exponential distributions with 30% censoring used in Table 2 was generated. At each of several different values of p and h_n, the bootstrap mean squared errors for $Q_n(p)$ were calculated based on 300 bootstrap samples from the given censored sample. The estimates of the optimal h_n values for $Q_n(p)$ at $p = .10, .25, .50, .75, .90$, and $.95$ were found to be approximately $h_n^* = .27, .09, .19, .35, .35$, and $.35$, respectively. The results of Table 2 for $n = 50$ indicate that the values of h_n yielding the smallest *ratio* of mean squared errors of $\hat{Q}_n(p)$ and $Q_n(p)$ for the same values of p as above were $h_n = .19, .11, .19, .41, .19$, and $.05$, respectively. These values are very similar to the h_n^* for $p \leqq .75$.

As an example of the quantile estimators, the life test data for $n = 40$ mechanical switches reported by Nair (1984) are used. Two failure modes, A and B, were recorded and Nair (1984) estimated the survival function of mode A assuming the random right-censorship model. Table 7 shows the 40 observations with the corresponding δ_i values ($\delta_i = 1$ indicates failure mode A and $\delta_i = 0$ denotes a censored value). There are 17 uncensored observations, which is 57.5% censoring in this particular sample. Again using 300 bootstrap samples of size 40 at each value of p and h_n, the bootstrap mean squared errors of Q_n and Q_n^* were calculated. For Q_n, the bootstrap estimates of h_n at $p = .10, .25, .50, .75, .90$, and $.95$ were approximately $h_n^* = .30, .26, .34, .34, .40$, and $.47$, respectively. The bootstrap estimates of optimal h_n for Q_n^* were virtually the same. Due to the rather heavy right-censoring in this data and small sample size, more smoothing is required for

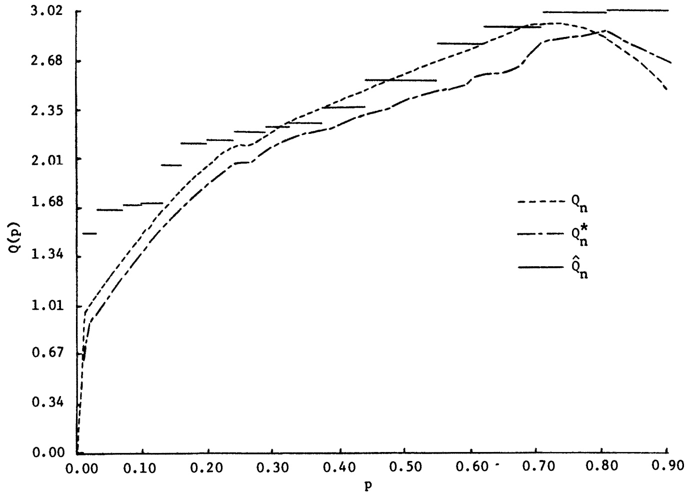

Figure 1. Quantile Estimators for Switch Life Data.

larger quantiles; hence h_n should be larger as the bootstrap estimates indicate. Thus, based on these results, a value of h_n of .28 was chosen to calculate $Q_n(p)$ for $0 < p \leqq .25$, the h_n value of .34 was used for $.25 < p < .90$, and .40 was used for $.90 \leqq p < 1$. Figure 1 shows the estimates $Q_n(p)$ and $Q_n^*(p)$ for this example, calculated using the triangular kernel, along with the PL quantile function $\hat{Q}_n(p)$. The estimates of median lifetime are $Q_n(.5) = 2.5874$, $Q_n^*(.5) = 2.4020$, and $\hat{Q}_n(.5) = 2.5480$.

6. CONCLUSION

The kernel-type quantile estimator given in this article (and the approximate estimator) is smoother than the PL quantile function that has been used for estimation from right-censored data in the past. Based on the small Monte Carlo simulation study, the two proposed estimators seem to perform about equally as well (when compared with $\hat{Q}_n$) except for large values of p, where Q_n seems to be better. Hence, over all p, Q_n would be the preferred small-sample estimator. The integrals involved in computing $Q_n(p)$ are easily calculated for the simple kernels used in Section 5.

Still under study is the problem of mean squared convergence (with a rate) of the estimators $Q_n(p)$ and $Q_n^*(p)$. In addition, the choice of an exact optimal bandwidth sequence $\{h_n\}$ in the sense of minimum mean squared error or minimum bias is still being investigated. A practical choice of h_n can be estimated, however, based on the bootstrap method, as illustrated in Section 5.

APPENDIX: PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. The proof of (a) follows the same lines as the proof of (b) using the law of the iterated logarithm for the product-limit estimator given by Földes and Rejtö (1981).

To prove (b), similar to the original preprint of Yang (1985), under the given assumptions, integrate $\int_0^1 [\hat{Q}_n(t) - Q^o(t)]h_n^{-1}K((t-p)/h_n)\,dt$ by parts to obtain $h_n^{-1}\int_0^1 \hat{Q}_n(t)K((t-p)/h_n)\,dt - Q^o(p) \equiv -I_1 - I_2 + I_3$, where

$$I_1 = \int_0^{T_{F_0}} \left\{ \int_{F_0(x)}^{\hat{F}_n(x)} h_n^{-1}K((t-p)/h_n)\,dt \right.$$

$$\left. - [\hat{F}_n(x) - F_0(x)]h_n^{-1}K((F_0(x)-p)/h_n) \right\} dx$$

$$I_2 = \int_0^{T_{F_0}} [\hat{F}_n(x) - [\hat{F}_0(x)]h_n^{-1}K((F_0(x)-p)/h_n)\,dx$$

$$I_3 = h_n^{-1}\int_0^1 Q^o(t)K((t-p)/h_n)\,dt - Q^o(p).$$

Let $\|\hat{F}_n - F_0\| \equiv \sup_{-\infty<x\leq T_F} |\hat{F}_n(x) - F_0(x)|$. Csörgő and Horváth's (1983) corollary 2 (ii) and (v) then yield with probability one, for a constant b,

$$\limsup_n |I_1| \leqq M^* \limsup_n [\phi^2(c(\log\log n/2n)^{1/2})h_n^{-2}]$$

$$\times \limsup_n [bh_n + \|\hat{F}_n - F_0\|], \quad \text{(A.1)}$$

the right-hand side of which is almost surely zero under the assumptions. Moreover, by corollary 2 (ii) of Csörgő and Horváth (1983),

$$|I_2| \leqq \|\hat{F}_n - F_0\| \int_0^1 h_n^{-1} K((t-p)/h_n)[f_0(Q^o(t))]^{-1}\,dt$$

$$\to 0 \quad \text{with probability one as } n \to \infty. \quad \text{(A.2)}$$

Finally, using conditions 1, 2–6, 7, and 9 and theorem 1A of Parzen (1962), it follows that $|I_3| = o(1)$, which with (A.1) and (A.2) proves the result.

Proof of Theorem 2. For $0 \leqq p \leqq 1$,

$$Q_n^*(p) - Q_n(p) = h_n^{-1} \sum_{i=1}^{n} Z_i[s_i K((S_i - p)/h_n) - \int_{S_{i-1}}^{S_i} K((t - p)/h_n)\, dt].$$

When $s_i > 0$ (i.e., Z_i is uncensored) let S_i^* be an interior point of the interval (S_{i-1}, S_i) with probability one so that

$$s_i K((S_i^* - p)/h_n) = \int_{S_{i-1}}^{S_i} K((t - p)/h_n)\, dt \quad \text{a.s.}$$

Then using condition 6,

$$\begin{aligned} |Q_n^*(p) - Q_n(p)| &\leqq h_n^{-1} \sum_{i=1}^{n} Z_i s_i |K((S_i - p)/h_n) - K((S_i^* - p)/h_n)| \\ &\leqq \Gamma\, h_n^{-2} \sum_{i=1}^{n} Z_i s_i |S_i - S_i^*| \\ &\leqq \Gamma\, h_n^{-2} \sum_{i=1}^{n} Z_i s_i^2 \quad \text{a.s.} \end{aligned} \tag{A.3}$$

Now, by the continuity of F_0 from condition 7, using the definitions of s_i and S_i, (A.3) can be written as

$$\begin{aligned} |Q_n^*(p) - Q_n(p)| &\leqq \Gamma\, h_n^{-2} \sum_{i=1}^{n} Z_i |\hat{F}_n(Z_i^+) - \hat{F}_n(Z_i)|\, d\hat{F}_n(Z_i) \\ &= \Gamma\, h_n^{-2} \int_0^{T_{F_0}} x |\hat{F}_n(x^+) - \hat{F}_n(x)|\, d\hat{F}_n(x) \\ &\leqq \Gamma\, h_n^{-2} \int_0^{T_{F_0}} x[|\hat{F}_n(x) - F_0(x)| \\ &\qquad + |F_0(x^+) - \hat{F}_n(x^+)|]\, d\hat{F}_n(x) \\ &\leqq 2\Gamma\, h_n^{-2} \|\hat{F}_n - F_0\| \int_0^{T_{F_0}} x\, d\hat{F}_n(x) \quad \text{a.s.,} \end{aligned} \tag{A.4}$$

where $g(x^+)$ denotes the limit from the right at x of the function g.

Now, since $\hat{F}_n(x) = 0$ for $x > Z_n$, $\int_0^{T_{F_0}} x\, d\hat{F}_n(x) = \hat{\mu}(Z_n)$, and by the assumptions of the theorem and corollary 2 (v) of Csörgő and Horváth (1983), the conclusion of the theorem follows from inequality (A.4).

[*Received October 1984. Revised July 1985.*]

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