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A change point estimation problem related to age replacement policies

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ABSTRACT

The mean time to failure (MTTF) function plays a vital role in the theory of age replacement policies. The point at which the MTTF function changes trend has important implications in the context of cost optimization in such policies. We develop a general methodology for change point estimation in this scenario and also establish the strong consistency of the proposed estimator. We also examine the performance of our estimator by applying it to simulated and real life data sets.

1. Introduction

In the modern era, industrial systems are typically complex and multi-functional. Consequently, failures associated with sophisticated industrial equipment result in a great deal of expenditure due to cost of repair and replacement. Moreover, in-service failures or failures when the system is operational, not only have economic consequences but can also pose serious hazards both to human life and the surrounding environment. Under the circumstances, it makes sense to reduce such costs and risks by following appropriate maintenance policies. Preventive maintenance policies are primarily of two types: (i) *age replacement policy* (a component is replaced by a new one on failure or at a pre-specified time t whichever occurs earlier) and (ii) *block replacement policy* (components are replaced at fixed time epochs of $t, 2t, 3t \ldots$ and also at failure) (see [4] and [3]). Suppose X is the lifetime of a new component with *cumulative distribution function* (cdf) $F(x)$ and *survival function* $\bar{F}(x) := 1 - F(x)$. Let $X_{[t]}$ denote the time to the first in-service failure of a component under age replacement policy with age replacement time t. In the context of age replacement, Barlow and Proschan [4] introduced the notion of the *mean time to failure* (MTTF) function given by

$$M_F(t) := \mathbb{E}(X_{[t]}) = \frac{\int_0^t \bar{F}(x)dx}{F(t)}, \quad F(t) > 0. \tag{1}$$

Since then, the MTTF function has found extensive usage in the investigation of expected life cycles of a system and also in evaluating the performance and effectiveness of age replacement policies. Due to its widespread application in reliability analysis, the MTTF function has received considerable attention in the literature; see [13], [18], [11], [10] and others. A distribution function F is said to be DMTTF (IMTTF) if $M_F(t)$ is *non-increasing* (*non-decreasing*) in $t \in (0, \infty)$ which is indicative of a positive (negative) ageing pattern. The work of Barlow and Proschan [4] established that the DMTTF class is sandwiched between the *increasing failure rate* (IFR) and the *new better than used in expectation* (NBUE) families of life distributions. Later Klefsjö [13] proved that the DMTTF class also contains all increasing failure rate average (IFRA) distributions and investigated the relationship between the DMTTF family and various well-known classes of life distributions. Knopik [15] established preservation of the DMTTF class under some reliability operations while closure under weak convergence and convolutions were studied in [16]. In a different context, Li and Xu [18] introduced the *new better than renewal used in the reversed hazard order* (NBRUrh) family which turns out to be equivalent to the DMTTF class and studied some of its properties. They also considered the problem of testing exponentiality against DMTTF alternatives using a U-statistic approach. Asha and Nair [2] defined a new stochastic ordering in terms of MTTF to compare life distributions and connected it with some known ageing orderings. Some preservation properties of the MTTF order and test for exponentiality against DMTTF class were considered by Kayid et al. [11].

Though the ageing patterns in all of the above mentioned classes are monotonic, the ageing profile of many real life systems as well as biological entities happen to be non-monotonic. To model such scenarios, there are several nonparametric classes of distributions which exhibit ageing that is non-monotonic

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(for example, bathtub failure rate (BFR) [8], upside-down bathtub failure rate (UBFR), new worse then better than used in expectation (NWBUE) [19], increasing initially then decreasing mean residual life (IDMRL) [9], increasing initially, then decreasing residual life in the Laplace order ($\mathrm{IDRL_{Lt}}$) [6], NBUE-NWUE [14] etc.). In the same spirit, Izadi et al. [10] introduced the following important non-monotonic class of distributions (and its dual) based on the MTTF function.

Definition 1.1. A life distribution F is said to be an increasing then decreasing mean time to failure (IDMTTF) distribution if there exists a change point $t_0 \geq 0$ such that $M_F(t)$ is non-decreasing on $[0, t_0)$ and non-increasing on $[t_0, \infty)$.

In this case we write 'F is IDMTTF(t_0)'. By changing the order of monotonicity in the above definition, the dual class of distributions (DIMTTF) can be defined in an analogous manner.

Izadi et al. [10] also proved the following chain of implications:

$$\text{BFR} \implies \text{IDMTTF} \implies \text{NWBUE}$$

$$\text{and} \quad \text{UBFR} \implies \text{DIMTTF} \implies \text{NBWUE}.$$

Let c_1 and c_2 be the cost of replacement at failure and the cost of a planned replacement respectively; clearly $c_1 > c_2$. Then the expected cost rate (see [21, p. 72]) is given by

$$C_F(t) = \frac{c_1 F(t) + c_2 \bar{F}(t)}{\int_0^t \bar{F}(x)dx} \tag{2}$$

One of the most familiar criteria to determine the optimal replacement time is minimizing the expected cost rate $C_F(t)$. Now (1) and (2) yield the relationship between the expected cost rate function and the MTTF function which is given by

$$C_F(t) = \frac{c_1 + c_2(\bar{F}(t)/F(t))}{M_F(t)} \tag{3}$$

Let T be the optimal age replacement time that minimizes $C_F(t)$ and F be an IDMTTF distribution with a change point t_0. From the relationship given in (3), Izadi et al. [10] showed that t_0 is a lower bound for T and for large t_0, T is approximated by t_0 under the natural assumption that c_2/c_1 is relatively small. Thus the estimation of the change point of the IDMTTF distribution is of great interest.

Earlier efforts in estimating change points were limited to some parametric and semiparametric models, see for example [5,22,24,26] and [23]. The problem of estimating the change point in a purely nonparametric setup was first addressed by [17]. They dealt with the problem in the context of BFR distributions. Later Mitra and Basu [20] considered the change point estimation problem for the NWBUE, IDMRL and BFR classes of life distributions and proposed consistent estimators of the change point in each case.

In this paper we consider the problem of estimating the change point of IDMTTF distributions. In the next section we develop a general methodology for consistent estimation of the change point. In the last section we analyse a real life data set for illustrative purposes.

2. Change point estimation

Given a random sample $X_1, X_2, \ldots, X_n$ of size n from an unknown IDMTTF(t_0) distribution, our aim is to estimate the unknown change point t_0. We make the following two assumptions:

(A1) The finite change point t_0 of the continuous distribution function F is *unique*.

(A2) There exists an upper bound δ of the unknown change point t_0 i.e. $t_0 \leq \delta < \infty$ and $F(\delta) < 1$.

It is worth noting that (A2) is a weak assumption because in many practical scenarios one can get some idea about δ on the basis of prior knowledge and experience. Assumption (A1) suggests that t_0 is the unique maximizer of $M_F(t)$. Now we estimate $M_F(t)$ by $M_{F_n}(t)$ where F_n is the empirical cdf given by

$$F_n(x) = \begin{cases} 0, & 0 \leq x < X_{(1)} \\ \frac{i}{n} & X_{(i)} \leq x < X_{(i+1)}, \quad i = 1, 2, \ldots, n-1 \\ 1 & x \geq X_{(n)} \end{cases}$$

where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the order statistics based on the random sample $X_1, X_2, \ldots, X_n$.

Direct calculation yields the empirical MTTF function as

$$M_{F_n}(t) = \frac{1}{k}\sum_{i=1}^{k} X_{(i)} + (\frac{n}{k} - 1)t \qquad \text{for } t \in \left[X_{(k)}, X_{(k+1)}\right)$$

and $k = 1, 2, \ldots, n-1$. For $t \geq X_{(n)}$, $M_{F_n}(t) = \bar{X}$ and is undefined otherwise.

Lemma 2.1. *With probability* 1, *$M_{F_n}(t)$ converges uniformly to $M_F(t)$.*

Proof. Using the Glivenko–Cantelli theorem we get,

$$\sup_{t\in(0,\infty)} |F_n(t) - F(t)| \to 0 \quad \text{almost surely} \quad \text{as } n \to \infty. \tag{4}$$

Now $\bar{F}_n(u)I_{(0,t)}(u)$ is bounded by an integrable function $g(u)$ which is given by

$$g(u) = \begin{cases} 1 & \text{when } u \in (0, t) \\ 0 & \text{otherwise} \end{cases}$$

where I denotes an indicator function. Now by Lebesgue's dominated convergence theorem and (4) we get that, with probability 1,

$$\int_0^t \bar{F}_n(u)du \to \int_0^t \bar{F}(u)du \qquad \text{uniformly as } n \to \infty. \tag{5}$$

Combining (4) and (5) we get the desired result. □

Now $\lim_{t\to X_{(k+1)}^-} M_{F_n}(t) = \frac{1}{k}\sum_{i=1}^{k} X_{(i)} + (\frac{n}{k} - 1)X_{(k+1)}$. The right hand limit and the value of the empirical MTTF function $M_{F_n}(t)$ at $t = X_{(k+1)}$ is $\frac{1}{k+1}\sum_{i=1}^{k+1} X_{(i)} + (\frac{n}{k+1} - 1)X_{(k+1)} = V$, say. After some simplification, we get

$$\begin{aligned} &\lim_{t\to X_{(k+1)}^-} M_{F_n}(t) - V \\ &= \frac{1}{k}\sum_{i=1}^{k} X_{(i)} + (\frac{n}{k} - 1)X_{(k+1)} - \frac{1}{k+1}\sum_{i=1}^{k+1} X_{(i)} - (\frac{n}{k+1} - 1)X_{(k+1)} \\ &= \frac{1}{k}\sum_{i=1}^{k} X_{(i)} - \frac{1}{k+1}\left(\sum_{i=1}^{k} X_{(i)} + X_{k+1}\right) + \left(\frac{n}{k} - \frac{n}{k+1}\right)X_{k+1} \\ &= \frac{1}{k(k+1)}\sum_{i=1}^{k} X_{(i)} + \frac{n-k}{k(k+1)}X_{(k+1)} > 0 \end{aligned} \tag{6}$$

So, $M_{F_n}(t)$ is a right continuous function having finite left limits and is piecewise linearly increasing in the intervals $\left[X_{(1)}, X_{(2)}\right)$, $\left[X_{(2)}, X_{(3)}\right)$, ..., $\left[X_{(n-1)}, X_{(n)}\right)$. Also note from (6) that $M_{F_n}(t)$ has downward jumps at each $t = X_{(i)}$, $i = 2, 3, \ldots n$.

Define,

$$\Lambda_n = \left\{0 < t \leq \delta \;\middle|\; \lim_{h\to 0^+} M_{F_n}(t-h) \text{ is maximum}\right\} \tag{7}$$

From the structure of $M_{F_n}(t)$, it is clear that Λ_n is non-empty with probability 1 and the maximum value of $\lim_{h\to 0^+} M_{F_n}(t-h)$ is attained at one or more of the order statistic points other than $X_{(1)}$. Now, define,

$$t_{0n} = \inf \Lambda_n = \min \Lambda_n$$

Our proposed estimator for t_0 is t_{0n} and the result contained in the following theorem provides a justification of its merit.

Theorem 2.2. *The estimator t_{0n} is strongly consistent for t_0.*

Proof. Fix $\omega \in \Omega$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space on which the X_i's are defined. Note that by (A2), $\{t_{0n}\}$ is bounded. Hence applying Bolzano–Weierstrass theorem we get a subsequence $\{t_{0n_k}\}$ of $\{t_{0n}\}$ converging to some t_0^* (finite) as $k \to \infty$. Now Λ_n can have at most $(n-1)$ elements so that $\inf \Lambda_n = \min \Lambda_n$ and as such $t_{0n_k} \in \Lambda_{n_k}$. As $\lim_{h\to 0^+} M_{F_{n_k}}(t-h)$ is maximum at $t = t_{0n_k}$, we have,

$$\lim_{h\to 0^+} M_{F_{n_k}}(t_0 - h) \le \lim_{h\to 0^+} M_{F_{n_k}}(t_{0n_k} - h) \tag{8}$$

Taking limit of both sides of (8) as $k \to \infty$ yields

$$\lim_{k\to\infty} \lim_{h\to 0^+} M_{F_{n_k}}(t_0 - h) \le \lim_{k\to\infty} \lim_{h\to 0^+} M_{F_{n_k}}(t_{0n_k} - h)$$

Interchanging limits in view of Lemma 2.1 we get

$$\lim_{h\to 0^+} M_F(t_0 - h) \le \lim_{h\to 0^+} M_F(t_0^* - h) \tag{9}$$

Now combining (9) and the fact that $\lim_{h\to 0^+} M_F(t-h)$ is maximum at $t = t_0$ we get

$$\lim_{h\to 0^+} M_F(t_0 - h) = \lim_{h\to 0^+} M_F(t_0^* - h).$$

Hence continuity of $M_F(t)$ yields $M_F(t_0) = M_F(t_0^*)$ which implies $t_0^* = t_0$ as change point t_0 is unique (A1). Therefore $t_{0n} \to t_0$ almost surely as $n \to \infty$ which completes the proof. □

3. Computation of change point and confidence interval

It is interesting to note that while the estimator proposed for the change point is the infimum of the set Λ_n defined by (7), the process for computing it is far from obvious. In what follows, we develop a neat algorithm to circumvent this difficulty. We also outline a method for obtaining confidence interval for the unknown change point.

Since $M_{F_n}(t)$ is right continuous having finite left limits and is piecewise linearly increasing in each of the segments $[X_{(i)}, X_{(i+1)})$ and jumps down at each $t = X_{(i)}$, $i = 1, \ldots, n-1$, its maximum value is the maximum of these finite left limits. Consequently, a natural estimator of the change point is the unique or first such order statistic whose left limit is the largest.

For a given random sample $X_1, X_2, \ldots, X_n$, we estimate the change point using the following algorithm.

1. Find the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ based on the random sample $X_1, X_2, \ldots, X_n$.
2. Calculate $\frac{1}{k}\sum_{i=1}^{k} X_{(i)} + (\frac{n}{k} - 1)X_{(k+1)}$ $\forall\, k = 1, 2, \ldots, n-1$.
3. Find the smallest k, call it k^*, such that $k^* = \arg\max_k \left(\frac{1}{k}\sum_{i=1}^{k} X_{(i)} + (\frac{n}{k} - 1)X_{(k+1)}\right)$.
4. Estimate the change point by $X_{(k^*+1)}$.

In practical situations, it is of great importance to obtain confidence intervals for the unknown change point. In connection with nonparametric estimation problems, the bootstrap method (see [7]) is a popular technique to construct confidence interval for the unknown 'parameter' when analytical methods fail or the estimator is complicated.

We now proceed to construct confidence intervals based on nonparametric percentile bootstrap method. For the sake of completeness, an algorithm for constructing $100(1-\alpha)\%$ confidence interval is outlined as follows:

1. For a given random sample $X_1, X_2, \ldots, X_n$ calculate $t_{0n} = X_{(k^*+1)}$.
2. Draw a bootstrap sample $X_1^*, X_2^*, \ldots, X_n^*$ and calculate $t_{0n}^* = X_{(k^*+1)}^*$ as in step 1.
3. Repeat step 2, N times.
4. Let $\hat{H}(x) := P(t_{0n}^* \le x)$ be the cdf of t_{0n}^*. Then define $t_{0n}^{\text{Boot-}}\text{p}(x) = \hat{H}^{-1}(x)$ for a given x. The approximate $100(1-\alpha)\%$ confidence interval for t_0 is now given by $\left(t_{0n}^{Boot-p}(\frac{\alpha}{2}),\ t_{0n}^{\text{Boot-}}\text{p}(1-\frac{\alpha}{2})\right)$.

Table 1
A simulated data.

0.5853	1.0403	1.6765	1.8040	2.9743	5.8484
9.7380	11.8790	14.1116	19.1553	19.9501	22.3294
22.6520	33.5220	35.6394	36.5797	41.0354	41.7409
42.5455	46.6129	50.2489	55.1265	58.9307	60.2697
63.4661	63.5706	76.4850	77.1538	101.1656	108.0788

Table 2
Lifetimes of 50 devices.

0.1	0.2	1	1	1	1	1	2	3	6	7	11
12	18	18	18	18	18	21	32	36	40	45	46
47	50	55	60	63	63	67	67	67	67	72	75
79	82	82	83	84	84	84	85	85	85	85	85
86	86										

Simulated data set: Here we analyse a simulated data set from Modified Weibull Extension (MWE) distribution introduced by Xie et al. [25]. The cumulative distribution function (cdf) of MWE is given by $F(x) = 1 - \exp\left[-\lambda\beta\left\{e^{(t/\beta)^\alpha} - 1\right\}\right]$, $t \ge 0$, $\alpha, \beta, \lambda > 0$. A random sample of size 30 from MWE with true parameter values $\alpha = 0.8408, \beta = 110.0909, \lambda = 0.0141$ is generated and the resulting observations are given in Table 1. In this case, the true change point is 34.4614. Using the above algorithm, the estimated change point is 33.5220 and 95% confidence interval is given by $(1.6765, 58.9307)$. The plot of the empirical MTTF function for this data set is given in Fig. 1.

Real data Set: In this example the data set presented in Table 2 contains lifetimes of 50 devices. This data set was analysed in [1] using the TTT-transform. Using above algorithm, the estimated change point turns out to be 60 and 95% confidence interval is given by $(18, 79)$. The plot of the empirical MTTF function for this data set is given in Fig. 2.

Finally, we feel that a couple of comments regarding the confidence intervals in Section 3 would be in order. It appears that for both simulated and real data sets, the confidence intervals are rather wide. The reason for this in the context of the simulated data set might be that the sample size 30 is not large enough — either increasing the sample size or using a parametric bootstrap would lead to improved confidence intervals. On the other hand, for the real data set, a significant proportion (more than 65%) of the data are clustered in the regions $(0, 18)$ and $(82, 86)$, far away from the change point indicated in Fig. 2.

4. Discussion and future work

In this work, our goal has been to propose an estimator for the change point of an IDMTTF distribution. While we have been able to establish the strong consistency of our proposed estimator, no

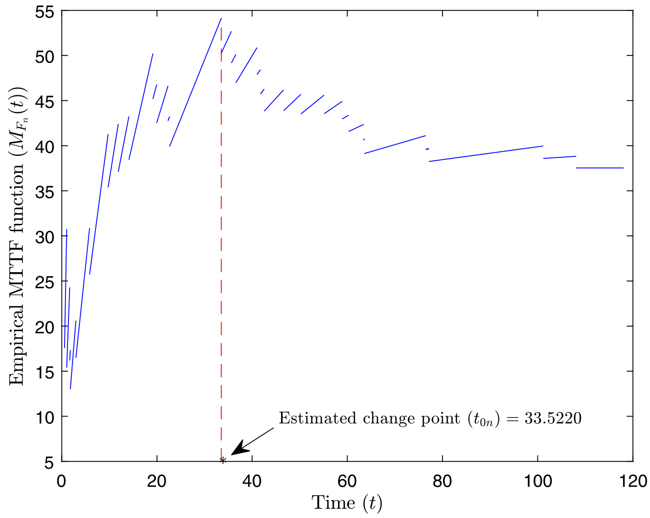

Fig. 1. Empirical MTTF function for the data set given in Table 1.

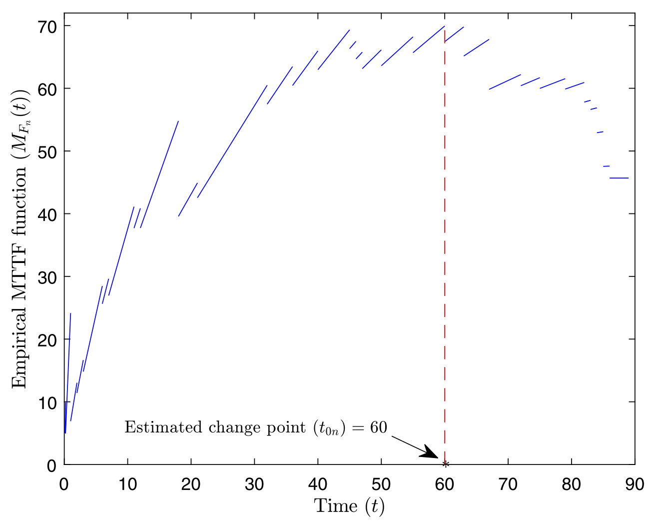

Fig. 2. Empirical MTTF function for the data set given in Table 2.

result on the rate of convergence is available. Since the structure of the estimator does not make it amenable to standard techniques like Taylor expansion, obtaining results in this direction is far from simple. Results involving convergence rate of the proposed estimator would thus be a significant contribution from a theoretical standpoint. From a practical viewpoint, this would enable us to give a theoretical assessment of the accuracy of the estimator.

Another direction for pursuing future work might be the determination of sharp bounds on the survival function $\bar{F}(t)$ for a given t, assuming that the life distribution is IDMTTF (or, more generally, a member of some class based on the MTTF function), perhaps with certain additional specifications (see for example, [12] and the references therein).

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