A Lifetime Distribution With an Upside-Down Bathtub-Shaped Hazard Function

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A Lifetime Distribution With an Upside-Down Bathtub-Shaped Hazard Function

Theodora Dimitrakopoulou, Konstantinos Adamidis, and Sotirios Loukas

***Abstract*—A three-parameter lifetime distribution with increasing, decreasing, bathtub, and upside down bathtub shaped failure rates is introduced. The new model includes the Weibull distribution as a special case. A motivation is given using a competing risks interpretation when restricting its parametric space. Various statistical properties, and reliability aspects are explored; and the estimation of parameters is studied using the standard maximum likelihood procedures. Applications of the model to real data are also included.**

***Index Terms*—Bathtub curve, competing risks, hazard function, lifetime distribution, maximum likelihood estimation, survival function, upside-down bathtub curve.**

NOTATION

θ	(α, β, λ)—parameters of the distribution.
$f(t;\theta)$	pdf of T at t, depending on $theta$.
$S(t;\theta)$	survival function of T at t, depending on θ.
$F(t)$	cdf of T at t.
$F^{-1}(t)$	quantile function of T at t.
$h(t;\theta)$	hazard function of T at t, depending on θ.
$P(\cdot)$	probability function.
$I(\theta)$	observed information matrix.

I. INTRODUCTION

ALTHOUGH the most popular lifetime models are those with monotone hazard rates (e.g. gamma, Weibull), reflecting a wear out or a work hardening behavior of the population under study, there are several situations where the failure pattern is somehow different. For instance, when studying the life cycle of an industrial product, or the entire life span of a biological entity, it is very likely that a three-phase behavior of the failure rate will be observed. For example, consider a high failure rate in infancy which decreases to a certain level, where it remains essentially constant for some time, and then increases from a point onwards due to wear out or aging (Gaver & Acar [1]). Thus, in this case, a model with a bathtub or 'U' shaped failure rate would be appropriate to describe the population's survival capacity. A systematic account of such distributions can be found in Rajarshi & Rajarshi [2], and Lai *et al.* [3]. Other situations are those who call for a model with unimodal failure rate, often modeled by the lognormal, or the inverse Gaussian distributions (Johnson *et al.* [4]).

This paper aims to provide a new lifetime model with a minimum number of parameters, at least as flexible as the Weibull distribution, yet adequate to describe more complex failure patterns; two shape, and one scale parameters are included to accommodate for increasing, decreasing, bathtub shaped, and upside-down bathtub-shaped failure rates. The model belongs to the class proposed by Gurvich *et al.* [5], for generalizing the Weibull distribution, and it is further discussed by Nadarajah & Kotz [6] (see also Lai *et al.* [7], Lie & Murthy [8], and Xie *et al.* [9]). A motivation is given using a competing risks interpretation when restricting one of its parameters to be a positive integer. Several statistical properties are explored, and the estimation of parameters is studied by the method of maximum likelihood; the fit of the distribution to two sets of real data is also examined.

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II. THE MODEL

The pdf of the distribution is given by

$$f(t;\theta) = \alpha\beta\lambda t^{\beta-1}(1+\lambda t^{\beta})^{\alpha-1}\exp\left\{1-(1+\lambda t^{\beta})^{\alpha}\right\}, \quad (1)$$

for $t > 0$ with $\theta = (a, \beta, \lambda)$, where $\alpha, \beta > 0$ are shape parameters, and $\lambda > 0$ is a scale parameter. It can be shown that, for $\beta < 1$, the pdf is monotone decreasing with $\lim_{t\to 0^+} f(t;\theta) = \infty$, $\lim_{t\to\infty} f(t;\theta) = 0$; for $\beta = 1$, the same shape is exhibited with $\lim_{t\to 0^+} f(t;\theta) = \alpha\lambda$, and $\lim_{t\to\infty} f(t;\theta) = 0$. However, for $\beta > 1$, the pdf assumes the limiting value of zero at the origin, increases to a maximum, and then decreases, approaching the value of zero at infinity; the different shapes of the pdf are illustrated in Fig. 1, for selected values of the parameters.

III. STATISTICAL PROPERTIES, AND RELIABILITY ASPECTS

A. Relations With Other Distributions

When $\alpha = 1$, the proposed model reduces to the Weibull distribution with shape, and scale parameters β, and λ, respectively. Also, by setting $G(t) = (1+\lambda t^{\beta})^{\alpha} - 1$ in the Nadarajah & Kotz [6] presentation of the Gurvich *et al.* [5] model, one obtains the model in (1). Furthermore, it can be verified by standard techniques that, if T has the distribution given by (1), then:

i $Y = 1 + \lambda T^{\beta}$ follows the Weibull distribution with shape, and scale parameters α, and 1 respectively, truncated in $(1, \infty)$.

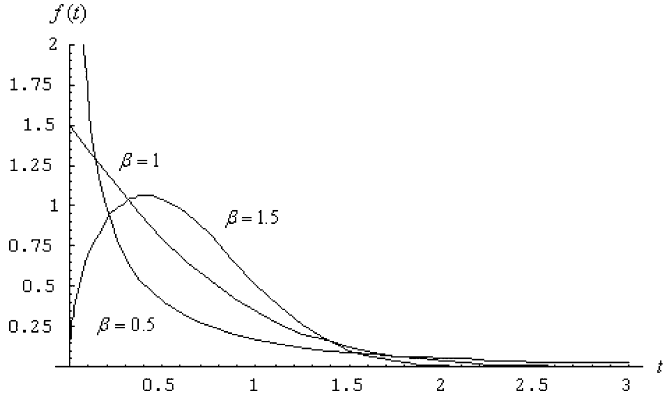

Fig. 1. Probability density functions of the distribution for $\lambda = 1$, $\alpha = 1.5$, and $\beta = 0.5, 1, 1.5$.

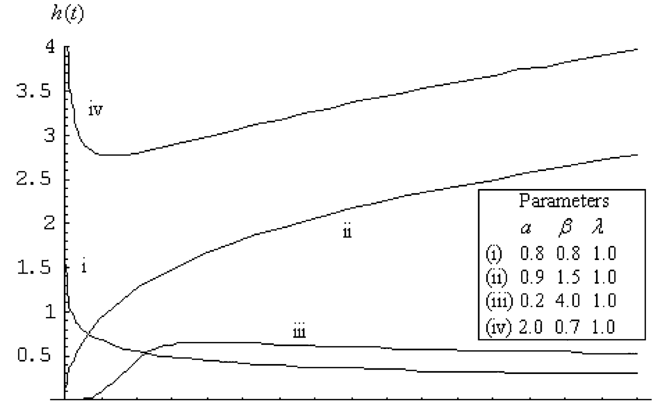

Fig. 2. Hazard functions of the distribution.

- ii $Y = (1 + \lambda T^{\beta})^{\alpha} - 1$ follows the exponential distribution with mean 1.
- iii $Y = \ln(1 + \lambda T^{\beta})$ follows the modified extreme value distribution with shape, and scale parameters 1, and α, respectively.
- iv $Y = \{\ln(1 + \lambda T^{\beta})\}^{1/\beta}$ follows the power exponential distribution (Smith & Bain [10]) with shape, and scale parameters β, and α, respectively.

Relations with other distributions may be established through those encountered above.

B. Probabilities, Moments, and the Hazard Function

By straightforward integration, the corresponding survival probabilities, and the quantile function are calculated to be

$$S(t;\theta) = \exp\left\{1 - (1 + \lambda t^{\beta})^{\alpha}\right\} = 1 - F(t;\theta), \tag{2}$$

for $t > 0$, where $F(.)$ is the distribution function, and

$$F^{-1}(p) = \left(\lambda^{-1}\left[\{1 - \ln(1-p)\}^{1/\alpha} - 1\right]\right)^{1/\beta},$$

for $0 < p < 1$, respectively. Hence, the median is $[\lambda^{-1}\{(1 + \ln 2)^{1/\alpha} - 1\}]^{1/\beta}$.

The rth moment of the distribution is given by $E(T^r) = r\int_0^{\infty} t^{r-1} S(t;\theta)dt$; the relevant computations involve the straightforward use of standard numerical integration procedures, available in most every mathematical package.

From (1), and (2), the hazard (or failure rate) function is

$$h(t;\theta) = \alpha\beta\lambda t^{\beta-1}(1 + \lambda t^{\beta})^{\alpha-1}, \tag{3}$$

for $t > 0$, exhibiting various shapes depending on the parameter values; in fact, its monotonicity varies along the segments produced in the parametric space by the curves $\alpha = 1$, $\beta = 1$, and $\alpha\beta = 1$. More specifically, by differentiating (3), it can be readily verified that

(a) for $\alpha = \beta = 1$, h is constant,
(b) for $\alpha > 1$, and $\beta \geq 1$ ($\alpha < 1$, and $\beta \leq 1$), h is monotone increasing (decreasing),
(c) for $\alpha \geq 1$, and $\beta < 1$,
  (i) if $\alpha\beta \leq 1$, h is monotone decreasing,
  (ii) if $\alpha\beta > 1$, h is bathtub shaped,
(d) for $\alpha \leq 1$, and $\beta > 1$,
  (i) if $\alpha\beta < 1$, h is unimodal,
  (ii) if $\alpha\beta \geq 1$, h is monotone increasing.

Furthermore, when the assumptions stated in c(ii), or d(i) hold, $h(t)$ attains its global point at $[(1-\beta)/\{\lambda(\alpha\beta - 1)\}]^{1/\beta}$. Thus the results in (a)–(d) offer insight into the merits of the proposed model; apart from having the bathtub or upside-down bathtub property, it also provides a wide class of monotone failure rates including those of the Weibull family. The distinct types of hazard shapes are illustrated in Fig. 2, for selected values of the parameters.

IV. Motivation

The distribution can be viewed as an extension of a model described below, resulting when expanding the latter's parametric space. Indeed, by restricting $\alpha - 1 \in \mathrm{N}$, the hazard function in (3) can be written as

$$h(t;\theta) = \sum_{r=0}^{\alpha-1} \gamma_r \beta_r t^{\beta_r - 1}, \tag{4}$$

for $t > 0$, where $\gamma_r = \binom{\alpha}{r+1}\lambda^{r+1}$, and $\beta_r = \beta(r+1)$. Thus, the hazard rate can be expressed as the sum of α terms, and consequently (4) is the hazard function of a series system of α components with independent Weibull lifetimes; equivalently, by assuming that failures can be classified into α distinct types, the observed lifetime modeled by (4) corresponds to $T = \min\{T_i\}_{i=0}^{\alpha-1}$, where T_i are independent with $h_i(t;\theta_i) = \gamma_i\beta_i t^{\beta_i - 1}$, $i = 0, \ldots, \alpha - 1$. The probability of failure in $[c,d)$ from cause j (or $j-$ failure type) in the presence of all other risks, conditional on surviving all risks until time c, is given by

$$\begin{aligned} P_j(c,d) &= P(c \leq T_j < d, T_j < T_i; i \neq j | T \geq c) \\ &= \int_c^d h_j(x;\theta_j) \exp\left\{-\int_c^x \sum_{j=0}^{\alpha-1} h_j(t;\theta_j)dt\right\} dx, \end{aligned}$$

and thus the probability of failure due to risk j is $\pi_j = P_j(0,\infty)$. Note that $\beta > 1$ implies monotone increasing component hazards (therefore a wear out behavior of the system's operating performance) as opposed to $\beta < 1$, and $\alpha\beta < 1$, where all component hazard rates are monotone decreasing (where the system exhibits a work hardening behavior). On the other hand, when $\beta < 1$, and $\alpha\beta > 1$, (4) comprises of at least one monotone increasing, and at least one monotone decreasing term exhibiting, according to preceding arguments, a bathtub shape.

V. Inference

Given a sample $t_1, \ldots, t_n$ from the distribution in (1), the normal equations[1], to be solved (numerically) for $\theta = \hat{\theta}$, are given by

$$\begin{aligned}\frac{\partial l}{\partial \alpha} &= n\alpha^{-1} + \sum_{i=1}^{n} \ln\left(1+\lambda t_i^{\beta}\right)\left\{1-\left(1+\lambda t_i^{\beta}\right)^{\alpha}\right\} = 0,\\ \frac{\partial l}{\partial \beta} &= n\beta^{-1} + \sum_{i=1}^{n} \ln t_i + \sum_{i=1}^{n} \lambda t_i^{\beta}\left(1+\lambda t_i^{\beta}\right)^{-1}\\ &\quad \times \ln t_i\left\{\alpha - 1 - \alpha\left(1+\lambda t_i^{\beta}\right)^{\alpha}\right\} = 0,\\ \frac{\partial l}{\partial \lambda} &= n\lambda^{-1} + \sum_{i=1}^{n} t_i^{\beta}\left(1+\lambda t_i^{\beta}\right)^{-1}\\ &\quad \times\left\{\alpha - 1 - \alpha\left(1+\lambda t_i^{\beta}\right)^{\alpha}\right\} = 0.\end{aligned}$$

Consequently, the elements in the upper triangular part of the symmetric observed information matrix $I(\theta)$, $I_{ij} = (-(\partial^2 l/\partial\theta_i\partial\theta_j))$, $i, j = 1, 2, 3$, evaluated at $\theta = \hat{\theta}$, are given by

$$\begin{aligned}I_{11} &= \frac{n}{\alpha^2} + \sum_{i=1}^{n}\left(1+\lambda t_i^{\beta}\right)^{\alpha} \ln^2\left(1+\lambda t_i^{\beta}\right)\Big|_{\theta=\hat{\theta}},\\ I_{22} &= \frac{n}{\beta^2} - \lambda \sum_{i=1}^{n} \frac{t_i^{\beta}\ln^2 t_i}{\left(1+\lambda t_i^{\beta}\right)^2}\\ &\quad \times\left\{\alpha - 1 - \alpha\left(1+\lambda t_i^{\beta}\right)^{\alpha}\left(1+\alpha\lambda t_i^{\beta}\right)\right\}\Big|_{\theta=\hat{\theta}},\\ I_{33} &= \frac{n}{\lambda^2} + (\alpha-1)\sum_{i=1}^{n} \frac{t_i^{2\beta}}{\left(1+\lambda t_i^{\beta}\right)^2} \times\left\{1+\alpha\left(1+\lambda t_i^{\beta}\right)^{\alpha}\right\}\Big|_{\theta=\hat{\theta}},\\ I_{12} &= -\lambda \sum_{i=1}^{n} \frac{t_i^{\beta}\ln t_i}{1+\lambda t_i^{\beta}}\left\{1-\left(1+\lambda t_i^{\beta}\right)^{\alpha}\right.\\ &\quad \left. -\alpha\left(1+\lambda t_i^{\beta}\right)^{\alpha}\ln\left(1+\lambda t_i^{\beta}\right)\right\}\Big|_{\theta=\hat{\theta}},\\ I_{13} &= -\sum_{i=1}^{n} \frac{t_i^{\beta}}{1+\lambda t_i^{\beta}}\left\{1-\left(1+\lambda t_i^{\beta}\right)^{\alpha}\right.\\ &\quad \left. -\alpha\left(1+\lambda t_i^{\beta}\right)^{\alpha}\ln\left(1+\lambda t_i^{\beta}\right)\right\}\Big|_{\theta=\hat{\theta}},\\ I_{23} &= \sum_{i=1}^{n} \frac{t_i^{\beta}\ln t_i}{\left(1+\lambda t_i^{\beta}\right)^2}\\ &\quad \times\left\{\alpha - 1 - \alpha\left(1+\lambda t_i^{\beta}\right)^{\alpha}\left(1+\alpha\lambda t_i^{\beta}\right)\right\}\Big|_{\theta=\hat{\theta}}.\end{aligned}$$

[1]By "normal equations" in statistics, we mean the equations that stem from differentiating partially the log-likelihood function, with respect to the parameters, and setting the resulting expressions equal to zero.

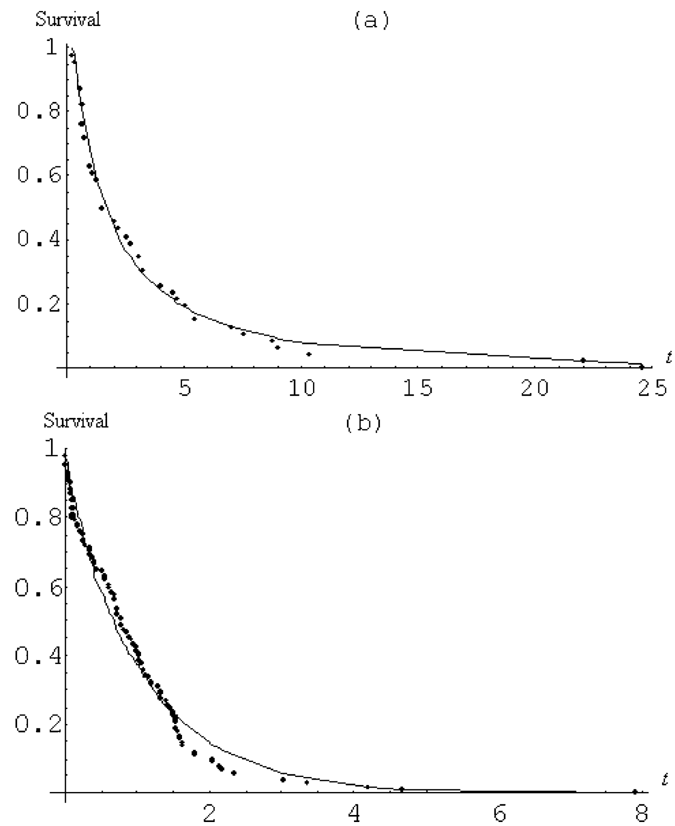

Fig. 3. Survival probability curves of the empirical distribution (dotted line), and the new distribution (solid line) fitted using maximum likelihood estimates for, (a) the repair times of an airborne communication transceiver, and (b) the Kevlar/epoxy strand-life data.

The latter, being a consistent estimate of the expected information matrix, provides an asymptotic estimate of the covariance matrix of $\hat{\theta}$.

VI. Examples

Two applications of the proposed model with real data are considered. The first one concerns 46 observations reported on active repair times (hours) for an airborne communication transceiver (Chhikara & Folks [11]). The data encountered in the second application involve 101 observations on times to failure of Kevlar 49/epoxy strands tested at a 90% stress level (Andrews & Helzberg [12], p. 182). The parameters of the distribution were estimated by the method of maximum likelihood; and the fits were examined by graphical methods, and the Kolmogorov-Smirnov (K-S) goodness of fit test. The estimates of the parameters $\theta = (\alpha, \beta, \lambda)$ were (0.1226, 3.3643, 12.4828), and (1.3082, 0.8618, 0.6925), for the first, and second sets of data, respectively; therefore, the fitted hazard function is unimodal in the first case, and bathtub shaped in the second. The values of the K-S test statistic were 0.073 ($p = 0.999$), and 0.086($p = 0.439$) respectively, suggesting that the new model fits the data adequately. The same conclusion is reached by examining the plots in Fig. 3; in both cases, the empirical, and fitted survival curves are almost coincident.

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REFERENCES

[1] D. P. Gaver and M. Acar, "Analytical hazard representations for use in reliability, mortality and simulation studies," *Communications in Statistics—Simulation and Computation*, vol. 8, pp. 91–111, 1979.

[2] S. Rajarshi and M. B. Rajarshi, "Bathtub distributions: A review," *Communications in Statistics—Theory and Methods*, vol. 17, pp. 2597–2621, 1988.

[3] C. D. Lai, M. Xie, and D. N. P. Murthy, "Bathtub shaped failure rate distributions," in *Handbook of Statistics [on Reliability]*, N. Balakrishnan and C. R. Rao, Eds. Amsterdam: North Holland, 2001, pp. 69–106.

[4] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions*. New York: John Wiley and Sons, 1994, vol. 1.

[5] M. R. Gurvich, A. T. Dibenedetto, and S. V. Ranade, "A new statistical distribution for characterizing the random strength of brittle materials," *Journal of Materials Science*, vol. 32, pp. 2559–2564, 1997.

[6] S. Nadarajah and S. Kotz, "On some recent modisfications of Weibull distribution," *IEEE Trans. on Reliability*, vol. 54, pp. 561–562, 2005.

[7] C. D. Lai, M. Xie, and D. N. P. Murthy, "Reply to: On some recent modifications of Weibull distribution," *IEEE Trans. on Reliability*, vol. 54, p. 563, 2005.

[8] C. D. Lai, M. Xie, and D. N. P. Murthy, "Modified Weibull model," *IEEE Trans. on Reliability*, vol. 52, pp. 33–37, 2003.

[9] M. Xie, Y. Tang, and T. N. Goh, "A modified Weibull extension with bathtub-shaped failure rate function," *Reliability Engineering & System Safety*, vol. 76, pp. 279–285, 2002.

[10] R. M. Smith and L. J. Bain, "An exponential power life-testing distribution," *Communications in Statistics—Theory and Methods*, vol. 4, pp. 469–481, 1975.

[11] R. S. Chhikara and J. L. Folks, "The inverse Gaussian distribution as a lifetime model," *Technometrics*, vol. 19, pp. 461–468, 1977.

[12] D. F. Andrews and A. M. Herzberg, *Data. A Collection of Problems from Many Fields for the Student and Research Worker*. New York: Springer-Verlag, 1985.

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