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ON PRODUCT MOMENTS OF ORDER STATISTICS

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Abstract

This paper is concerned with joint moments involving arbitrary powers of order statistics. Consider the order statistics $u_1 \leq u_2 \leq \cdots \leq u_k$ coming from a simple random sample of size n from a real continuous population, where $u_1 = x_{r_1:n}$ denotes the r_1-th order statistic, $u_2 = x_{r_1+r_2:n}$, the $r_1 + r_2$-th order statistic, and so on, $u_k = x_{r_1+\cdots+r_k:n}$ denoting the $r_1 + \cdots + r_k$-th order statistic. We examine product moments of the type $E(u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_k^{\alpha_k})$ where $\alpha_1, \ldots, \alpha_k$ are arbitrary (possibly complex) quantities and E denotes the expected value. We carry out some explicit evaluations for particular populations such as the uniform and exponential. While deriving the product moments of order statistics coming from an exponential population, several interesting mathematical identities are also obtained. Recurrence relations for moments and product moments are derived for the case where only the order of the moments changes, the other parameters remaining unchanged. These happen to be more useful than the standard recurrence relationships currently available in the literature. Several new techniques are introduced as the results are being developed. For instance, a methodology is proposed for obtaining the moments in terms of finite sums in the case truncated exponential random variables. Connections to beta distributions, psi, zeta and generalized hypergeometric functions, as well as to survival and reliability problems are also established.

Keywords and phrases: Order statistics, uniform populations, exponential populations, product moments, recurrence relations, mathematical identities, reliability, hypergeometric functions.

1. Introduction

Order statistics from exponential and uniform populations are involved in a wide variety of problems. Their use in characterizations of distributions are discussed in Galambos and Kotz (1970) and the numerous references therein. Order statistics play vital roles in life-testing problems in industry (material fatigue, device failure), in survival analysis, in the study of floods and droughts, as well as in linear estimation, system reliability, data compression and many other fields. Accounts of such applications can be found for instance in the books co-authored by Balakrishnan and Cohen (1991) and Arnold and Balakrishnan (1989), and many of the references therein. Their use in Poisson arrivals of points into a Euclidean space, inter-arrival times, nearest neighbor problems, spread of trees in forests, growth of crystals and many other geometrical probability problems are discussed in Mathai (1999) and many references therein. As can be seen from the abundant material and numerous results contained in the aforementioned references in connection with our topic, many authors have introduced a variety of techniques for obtaining integer moments and integer product moments of order statistics as well as several types of recurrence relations among moments and product moments, and for dealing with some computational aspects and various applications. In particular, David and Rogers (1983) obtained an expression for the covariance of two order statistics in terms of the first two moments of order statistics drawn from the same population; Shah (1966) and Tarter (1966), respectively discussed the bivariate and product moments of order statistics from a logistic distribution; and Ruben (1956) studied the product moments of extreme order statistics in normal samples.

In this paper, we shall introduce a number of new methodologies for obtaining complex and non-integer moments as well as other results such as recurrence relations involving only the order of the moments and no other parameters. We shall also establish connections to psi, zeta, Appell's and Lauricella's functions and the beta distribution, as well as to the reliability of systems and certain survival problems. Two representations of the moments of a linear combination of order statistics from an exponential population yield several extensions to some classical mathematical identities. A technique is proposed for obtaining, in terms of finite sums, all types of moments of all orders for the case of a doubly truncated exponential population. Previously, Saleh *et al.* (1975) had obtained in terms of infinite series, the first and second moments of order statistics from right-truncated exponential populations.

2.1. Order statistics from a uniform population

Let $u_j = x_{r_1+\cdots+r_j:n}$ be the $r_1+\cdots+r_j$-th order statistic, $j=1,\ldots,k$, coming from a simple random sample of size n from a uniform population over $[0,1]$, and $r_{k+1} = n+1-(r_1+\cdots+r_k)$. Clearly, the joint density of $u_1,\ldots,u_k$ is then

$$f(u_1,\ldots,u_k) = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(r_1)\cdots\Gamma(r_{k+1})} u_1^{r_1-1}(u_2-u_1)^{r_2-1} \\ \qquad\qquad \cdots(u_k-u_{k-1})^{r_k-1}(1-u_k)^{r_{k+1}-1}, \quad 0\le u_1\le\cdots\le u_k\le 1\ , \\ 0, \text{ elsewhere.} \end{cases}$$

Letting $\alpha_1,\ldots,\alpha_k$ be arbitrary real or complex numbers, the product moment is

$$\begin{aligned} \mu_{\alpha_1\cdots\alpha_k} &= E(u_1^{\alpha_1}u_2^{\alpha_2}\cdots u_k^{\alpha_k}) \\ &= c\int_{0\le u_1\le\cdots\le u_k\le 1} u_1^{\alpha_1+r_1-1}u_2^{\alpha_2}\cdots u_k^{\alpha_k} \\ &\qquad\qquad \times\ (u_2-u_1)^{r_2-1}\cdots(1-u_k)^{r_{k+1}-1}\mathrm{d}u_1\ldots\mathrm{d}u_k \end{aligned}$$

where

$$c = \frac{\Gamma(n+1)}{\Gamma(r_1)\cdots\Gamma(r_{k+1})}\ .$$

Consider the terms involving u_1 in the above integrand. On integrating out u_1, one has

$$\begin{aligned} \int_{u_1=0}^{u_2} u_1^{\alpha_1+r_1-1}(u_2-u_1)^{r_2-1}\mathrm{d}u_1 &= u_2^{\alpha_1+r_1+r_2-1}\int_0^1 z^{\alpha_1+r_1-1}(1-z)^{r_2-1}\mathrm{d}z \\ &= u_2^{\alpha_1+r_1+r_2-1}\frac{\Gamma(\alpha_1+r_1)\Gamma(r_2)}{\Gamma(\alpha_1+r_1+r_2)}\ . \end{aligned}$$

Then, on successively integrating out $u_2,u_3,\ldots,u_k$, one obtains

$$\begin{aligned} \mu_{\alpha_1\cdots\alpha_k} &= \frac{\Gamma(n+1)}{\Gamma(r_1)}\frac{\Gamma(\alpha_1+r_1)\Gamma(\alpha_1+\alpha_2+r_1+r_2)\cdots}{\Gamma(\alpha_1+r_1+r_2)\Gamma(\alpha_1+\alpha_2+r_1+r_2+r_3)\cdots} \\ &\qquad \times\ \frac{\Gamma(\alpha_1+\cdots+\alpha_k+r_1+\cdots+r_k)}{\Gamma(\alpha_1+\cdots+\alpha_k+r_1+\cdots+r_{k+1})} \\ &= \frac{\Gamma(n+1)}{\Gamma(r_1)}\left\{\prod_{j=1}^k \frac{\Gamma(\alpha_1+\cdots+\alpha_j+r_1+\cdots+r_j)}{\Gamma(\alpha_1+\cdots+\alpha_j+r_1+\cdots+r_{j+1})}\right\} \end{aligned} \tag{2.1}$$

for $\Re(\alpha_j+r_j)>0,\ j=1,\ldots,k$, where $\Re(\cdot)$ denotes the real part of $(\cdot)$. Note that when all (α_j+r_j)'s are positive integers, one has

$$\begin{aligned} \mu_{\alpha_1\cdots\alpha_k} &= \frac{n!}{(r_1-1)!}\frac{(\alpha_1+\alpha_2+r_1+r_2-1)!\cdots}{(\alpha_1+r_1+r_2-1)!\cdots} \\ &\qquad \times\ \frac{(\alpha_1+\cdots+\alpha_k+r_1+\cdots+r_k-1)!}{(\alpha_1+\cdots+\alpha_k+r_1+\cdots+r_{k+1})!}\ , \end{aligned}$$

as $n, r_1, \ldots, r_{k+1}$ are non-negative integers. This corresponds to the result given in Equation (3.4.8) of Balakrishnan and Cohen (1991) with $k_0 = 0$. Moments on complex domains are needed for further studies which call for the application of the inverse Mellin or inverse Laplace transform techniques. Studies involving products, ratios and other functions of order statistics coming from independent systems may also require the determination of moments on complex domains. Several illustrations of the types of distributional problems that can be tackled via such a procedure can be found in Mathai (1993).

2.2. Reliability and survival analysis

Let u_j denote the $r_1 + r_2 + \cdots + r_j$-th order statistic coming from a continuous population with distribution function $F(x)$. Then $y_j = 1 - F(u_j)$ is the survival function corresponding to u_j for $j = 1, \ldots, k$. Reliability studies in connection with industrial systems are often based on such survival functions. Whenever $z_1 = F(u_1)$, $z_2 = F(u_2) - F(u_1)$, $\ldots, z_k = F(u_k) - F(u_{k-1})$, $(z_1, \ldots, z_k)$ has a type-1 Dirichlet distribution. Note that $y_j = 1 - z_1 - \cdots - z_j$. One can study many distributional properties of $(y_1, \ldots, y_k)$ by examining the product moments. Let us evaluate $E(y_1^{t_1} y_2^{t_2} \cdots y_k^{t_k})$ for arbitrary (real or complex) values of $t_1, \ldots, t_k$:

$$\begin{aligned} E(y_1^{t_1} \cdots y_k^{t_k}) &= E[(1 - z_1)^{t_1}(1 - z_1 - z_2)^{t_2} \cdots (1 - z_1 - \cdots - z_k)^{t_k}] \\ &= c \int_{\mathcal{Z}} z_1^{r_1 - 1} \cdots z_k^{r_k - 1} (1 - z_1)^{t_1} (1 - z_1 - z_2)^{t_2} \cdots \\ &\qquad \times \ (1 - z_1 - \cdots - z_k)^{t_k + r_{k+1} - 1} \mathrm{d}z_1 \ldots \mathrm{d}z_k \end{aligned}$$

where

$$\begin{aligned} c^{-1} &= (\Gamma(r_1) \cdots \Gamma(r_{k+1}))/\Gamma(n+1) \\ n + 1 &= r_1 + \cdots + r_{k+1} \\ \mathcal{Z} &= \{(z_1, \cdots, z_k) \mid 0 \le z_j \le 1,\ j = 1, \ldots, k,\ z_1 + \cdots + z_k \le 1\}. \end{aligned}$$

Note that $0 \le z_j \le 1 - z_1 - \cdots - z_{j-1}$ for $j = k, k-1, \ldots, 1$. On integrating out $z_k, z_{k-1}, \ldots, z_1$, one has the following result:

$$E(y_1^{t_1} \cdots y_k^{t_k}) = \frac{\Gamma(n+1)}{\Gamma(r_{k+1})} \left\{ \prod_{j=1}^{k} \frac{\Gamma(r_{k+1} + r_k + \cdots + r_{k-j+2} + t_k + \cdots + t_{k-j+1})}{\Gamma(r_{k+1} + \cdots + r_{k-j+1} + t_k + \cdots + t_{k-j+1})} \right\} \tag{2.2}$$

for $\Re(r_{k+1} + \cdots + r_{k-j+2} + t_k + \cdots + t_{k-j+1}) > 0,\ \ j = 1, \ldots, k$.

3. Distributional results for exponential populations

Since essentially the same mathematical derivations apply whether the mean of an exponential population is θ or 1, we shall consider the latter, that is, the standard exponential population. As before, let $u_1, \ldots, u_k$ where $u_j = x_{r_1+\cdots+r_j:n}$ denotes the $(r_1 + \cdots + r_j)$-th order statistic. Then from Sukhatme's representation, see for example (3.5.6) of Balakrishnan and Cohen (1991), one has

$$x_{1:n} = \frac{y_1}{n},\ x_{2:n} = \frac{y_1}{n} + \frac{y_2}{n-1}\ ,\ \ldots\ ,\ x_{n:n} = \frac{y_1}{n} + \cdots + \frac{y_{n-1}}{2} + y_n\ ,$$

where $y_1, \ldots, y_n$ are mutually independently distributed standard exponential variables. Thus

$$u_j = \frac{y_1}{n} + \frac{y_2}{n-1} + \cdots + \frac{y_{r_1+\cdots+r_j}}{n-(r_1+\cdots+r_j-1)}\ ,\quad j = 1, \ldots, k\ , \tag{3.1}$$

so that

$$u_1 = v_1,\ u_2 = v_1 + v_2, \ldots, u_j = v_1 + \cdots + v_j,\ j = 1, \ldots, k\ ,$$

where

$$\begin{aligned} v_1 &= u_1 = \frac{y_1}{n} + \frac{y_2}{n-1} + \cdots + \frac{y_{r_1}}{n-(r_1-1)}\ , \\ v_2 &= \frac{y_{r_1+1}}{n-r_1} + \cdots + \frac{y_{r_1+r_2}}{n-(r_1+r_2-1)}\ ,\quad \cdots\quad , \\ v_j &= \frac{y_{r_1+r_1+\cdots+r_{j-1}+1}}{n-(r_1+\cdots+r_{j-1})} + \cdots + \frac{y_{r_1+\cdots+r_j}}{n-(r_1+\cdots+r_j-1)}\ . \end{aligned}$$

Note that since the v_j's consist of mutually exclusive sets of independently distributed y_j's , $v_1, \ldots, v_j$ are mutually independently distributed. Then the product moment of arbitrary orders $\alpha_1, \ldots, \alpha_k$ has the following representations:

$$\begin{aligned} \mu_{\alpha_1\cdots\alpha_k} &= E[x_{r_1:n}^{\alpha_1} x_{r_1+r_2:n}^{\alpha_2} \cdots x_{r_1+\cdots+r_k:n}^{\alpha_k}] \\ &= E[u_1^{\alpha_1} \cdots u_k^{\alpha_k}] \\ &= E[v_1^{\alpha_1}(v_1+v_2)^{\alpha_2} \cdots (v_1+\cdots+v_k)^{\alpha_k}]\ . \end{aligned} \tag{3.2}$$

Let us compute the density of v_j. Given the mutual independence of the components of v_j, the moment generating function of v_j is available as

$$M_j(t) = \prod_{m=1}^{r_j} \left[1 - \frac{t}{n-(r_1+\cdots+r_{j-1}+m-1)}\right]^{-1}.$$

This can be written as a sum by making use of the partial fractions method so that

$$M_j(t) = \sum_{m=1}^{r_j} b_{jm} \left[1 - \frac{t}{\delta_{jm}}\right]^{-1}$$

where

$$\delta_{jm} = n - (r_1 + \cdots + r_{j-1} + m - 1),\; r_1 + \cdots + r_{j-1} = 0 \text{ for } j = 1,$$

and

$$b_{jm} = \left[\prod_{l=1, l\neq m}^{r_j} \left(1 - \frac{\delta_{jm}}{\delta_{jl}}\right)\right]^{-1} .$$

The density of v_j is thus available as a linear function of exponential densities. On denoting the density of v_j by $g_j(v_j)$, one has

$$g_j(v_j) = \begin{cases} \sum_{m=1}^{r_j} b_{jm}\ \delta_{jm} \mathrm{e}^{-\delta_{jm} v_j},\ 0 \le v_j < \infty\ , \\ \\ 0,\ \text{elsewhere}\ . \end{cases} \tag{3.3}$$

Hence

$$\mu_{\alpha_1 \cdots \alpha_k} = \int_{v_1=0}^{\infty} \cdots \int_{v_k=0}^{\infty} v_1^{\alpha_1} (v_1 + v_2)^{\alpha_2} \cdots$$
$$\times\ (v_1 + \cdots + v_k)^{\alpha_k} g_1(v_1)\ g_2(v_2) \cdots g_k(v_k) \mathrm{d}v_1 \ldots \mathrm{d}v_k\ .$$

For example, if only one order statistic $u_1 = v_1$ is involved, then the density of v_1 is available from (3.3) by setting $j = 1$. For arbitrary α_1, one has in this case

$$\begin{aligned} E(u_1^{\alpha_1}) = E(v_1^{\alpha_1}) &= \int_0^{\infty} v_1^{\alpha_1} g_1(v_1) \mathrm{d}v_1 \\ &= \sum_{m=1}^{r_1} b_{1m}\delta_{1m} \int_0^{\infty} v_1^{\alpha_1} \mathrm{e}^{-\delta_{1m} v_1} \mathrm{d}v_1 \\ &= \sum_{m=1}^{r_1} b_{1m} (\delta_{1m})^{-\alpha_1} \Gamma(\alpha_1 + 1),\ \Re(\alpha_1) > -1\ . \end{aligned} \tag{3.4}$$

This moment can also be obtained by first evaluating the density of $u_1 = x_{r_1:n}$ and then by taking the expected value. In general, the density of u_1 is given by

$$f_{u_1}(u_1) = \frac{n!}{(r_1-1)!(r_2-1)!} [F(u_1)]^{r_1-1} [1 - F(u_1)]^{r_2-1} f(u_1), \qquad r_2 = n + 1 - r_1\ ,$$

and when the population is standard exponential, one has

$$\begin{aligned} E(u_1^{\alpha_1}) &= \frac{n!}{(r_1-1)!(r_2-1)!} \int_0^{\infty} u_1^{\alpha_1} \left[1 - \mathrm{e}^{-u_1}\right]^{r_1-1} \mathrm{e}^{-r_2 u_1} \mathrm{d}u_1 \\ &= \frac{n!}{(r_2-1)!} \sum_{m=0}^{r_1-1} \frac{(-1)^m}{m!(r_1-1-m)!} \frac{\Gamma(\alpha_1+1)}{(m+r_2)^{\alpha_1+1}}\ , \quad \Re(\alpha_1) > -1 \\ &= \frac{n!}{(n-r_1)!} \sum_{m=0}^{r_1-1} \frac{(-1)^m}{m!(r_1-1-m)!} \frac{\Gamma(\alpha_1+1)}{(n+1-r_1+m)(n+1-r_1+m)^{\alpha_1}}\ . \end{aligned} \tag{3.5}$$

But the α_1-th moment from the representation in (3.3) is available from (3.4) with

$$\delta_{11} = n,\ \delta_{12} = n-1, \ldots, \delta_{1r_1} = n-(r_1-1),$$
$$b_{11} = \left[\left(1-\frac{n}{n-1}\right)\left(1-\frac{n}{n-2}\right)\cdots\left(1-\frac{n}{n-(r_1-1)}\right)\right]^{-1} = \frac{(-1)^{r_1-1}n!}{(n-r_1)!n(r_1-1)!}\ .$$

Setting $m = r_1-1, r_1-2, \ldots, 0$ in (3.5) yields $b_{11}, \ldots, b_{1r_1}$ of (3.4). This verifies the result. Note that when α_1 is a positive integer then one can use the representation in (3.1), expand by using the binomial expansion and integrate directly over the densities of the independent variables $y_1, \ldots, y_n$. It is simpler to use the density of u_1 directly when only one order statistic is involved whether α_1 is an integer or not. When $\alpha_1, \ldots, \alpha_k$ are positive integers, then

$$\begin{aligned}\mu_{\alpha_1\cdots\alpha_k} &= E[v_1^{\alpha_1}(v_1+v_2)^{\alpha_2}\cdots(v_1+\cdots+v_k)^{\alpha_k}]\\ &= \sum_{\Omega} b_{k(m_{ij})}E(v_1^{\alpha_1+m_{21}+\cdots+m_{k1}})E(v_2^{m_{22}+\cdots+m_{k2}})\cdots E(v_k^{m_{kk}})\ ,\end{aligned}$$

where

$$\Omega = \Big\{m_{ij},\ j=1,\ldots,i,\ i=1,\ldots,k,\ m_{ij}=0,1,\ldots,\alpha_i,\ \sum_{j=1}^{i} m_{ij} = \alpha_i\Big\},$$

$$b_{k(m_{ij})} = \frac{\alpha_1!}{m_{11}!}\frac{\alpha_2!}{m_{21}!m_{22}!}\cdots\frac{\alpha_k!}{m_{k1}!\cdots m_{kk}!}$$

$$E(v_j^{\gamma_j}) = \sum_{m=1}^{r_j} b_{jm}(\delta_{jm})^{-\gamma_j}\Gamma(\gamma_j+1), \quad \Re(\gamma_j) > -1, \quad \gamma_j = \sum_{l=j}^{k} m_{lj}$$

and

$$\mu_{\alpha_1\cdots\alpha_k} = \sum_{\Omega} b_{k(m_{ij})}\prod_{j=1}^{k}\left[\sum_{m=1}^{\alpha_j} b_{jm}(\delta_{jm})^{-\gamma_j}\Gamma(\gamma_j+1)\right]. \tag{3.6}$$

For example,

$$\mathrm{Cov}(u_1,u_j) = \mathrm{Cov}(v_1, v_1+\cdots+v_j) = \mathrm{Var}(v_1)\ , \quad j=1,\ldots,k,$$
$$\mathrm{Var}(v_1) = E(v_1^2) - (E(v_1))^2,$$
$$E(v_1^h) = \sum_{l=1}^{r_1} b_{1l}\frac{\Gamma(h+1)}{(\delta_{1l})^h},\ \Re(h) > -1$$

and

$$\mathrm{Var}(v_1) = \left[\frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{(n-(r_1-1))^2}\right],$$

which is a known result, also available from (3.1).

Now, consider the random variable

$$u = \frac{y_1}{a_1} + \frac{y_2}{a_2} + \cdots + \frac{y_k}{a_k} \tag{3.7}$$

where the y_j's are independent standard exponential variables and the a_j's are assumed to be distinct and positive, $j = 1, \ldots, k$. Then the density of u, denoted by $f(u)$, is available from (3.3) as

$$f(u) = \begin{cases} \sum_{j=1}^{k} b_j \; a_j \; \mathrm{e}^{-a_j u}, \; u > 0 \\ 0, \; \text{elsewhere} \end{cases} \tag{3.8}$$

where

$$b_j = \left[\prod_{l=1, l \neq j}^{k} \left(1 - \frac{a_j}{a_l} \right) \right]^{-1} .$$

The integer moments obtained from this representation when compared with those derived from (3.7) yield several interesting mathematical identities which are included in the Appendix.

3.1. Arbitrary product moments of two or more order statistics

Integer product moments of any number of order statistics from an exponential population are available from (3.6). Let us now assume that $\alpha_1, \ldots, \alpha_k$ are complex or non-integer quantities. Let us first consider the case $k = 2$. From the representation of the density given in (3.3), one has

$$\begin{aligned} E[x_{r_1:n}^{\alpha_1} \cdots x_{r_1+\cdots+r_k:n}^{\alpha_k}] &= E[u_1^{\alpha_1} \cdots u_k^{\alpha_k}] = E[v_1^{\alpha_1}(v_1+v_2)^{\alpha_2} \cdots (v_1+\cdots+v_k)^{\alpha_k}] \\ &= \sum_{i=1}^{r_1} \sum_{j=r_1+1}^{r_1+r_2} b_{1i} b_{2j} \delta_{1i} \delta_{2j} \; \eta \end{aligned} \tag{3.9}$$

where

$$\eta = \int_{v_1=0}^{\infty} \int_{v_2=0}^{\infty} v_1^{\alpha_1} (v_1+v_2)^{\alpha_2} \mathrm{e}^{-(\delta_{1i} v_1 + \delta_{2j} v_2)} \mathrm{d}v_1 \; \mathrm{d}v_2 \; .$$

Note that every δ_{2j} is smaller than every δ_{1i} for all i and j and the difference is at least 1, that is, $\delta_{1i} - \delta_{2j} \geq 1$ for all i and j, as

$$\{\delta_{1i}, \; i = 1, \ldots, r_1\} = \{n, n-1, \ldots, n-(r_1-1)\}, \; r_1 \geq 1$$

and

$$\{\delta_{2j},\ j = r_1+1,\ldots,r_1+r_2\} = \{n-r_1, n-r_1-1,\ldots,n-(r_1+r_2-1)\},\ r_2 \geq 1.$$

Then there are two possibilities:

$$\frac{\delta_{1i}-\delta_{2j}}{\delta_{2j}} \leq 1 \quad \text{or} \quad \frac{\delta_{1i}-\delta_{2j}}{\delta_{2j}} > 1\ .$$

Now let $w_1 = v_1$ and $w_2 = v_1 + v_2$ so that $0 \leq w_1 \leq w_2 < \infty$.

Case (1). Letting $\frac{\delta_{1i}-\delta_{2j}}{\delta_{2j}} \leq 1$,

$$\begin{aligned}
\eta &= \int_{w_2=0}^{\infty}\int_{w_1=0}^{w_2} w_1^{\alpha_1} w_2^{\alpha_2}\ \mathrm{e}^{-\delta_{1i}w_1-\delta_{2j}(w_2-w_1)}\mathrm{d}w_1\ \mathrm{d}w_2 \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}(\delta_{1i}-\delta_{2j})^r \int_{w_2=0}^{\infty}\frac{w_2^{\alpha_1+\alpha_2+r+1}}{\alpha_1+r+1}\ \mathrm{e}^{-\delta_{2j}w_2}\mathrm{d}w_2,\ \Re(\alpha_1) > -1 \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}(\delta_{1i}-\delta_{2j})^r(\delta_{2j})^{-(\alpha_1+\alpha_2+2)}\ \frac{\Gamma(\alpha_1+\alpha_2+r+2)}{\alpha_1+r+1},\ \Re(\alpha_1+\alpha_2) > -2 \\
&= \frac{\Gamma(\alpha_1+\alpha_2+2)}{\alpha_1+1}(\delta_{2j})^{-(\alpha_1+\alpha_2+2)} \\
&\quad \times\ {}_2F_1\left(\alpha_1+1, \alpha_1+\alpha_2+2; \alpha_1+2; -\frac{(\delta_{1i}-\delta_{2j})}{\delta_{2j}}\right),\ \frac{\delta_{1i}-\delta_{2j}}{\delta_{2j}} \leq 1\ ,
\end{aligned} \tag{3.10}$$

where ${}_2F_1(\cdot)$ is a Gauss' hypergeometric function.

Case (2). Letting $\frac{\delta_{1i}-\delta_{2j}}{\delta_{2j}} > 1$,

$$\begin{aligned}
\eta &= \int_{w_1=0}^{\infty} w_1^{\alpha_1}\ \mathrm{e}^{-(\delta_{1i}-\delta_{2j})w_1}\int_{w_2=w_1}^{\infty} w_2^{\alpha_2}\ \mathrm{e}^{-\delta_{2j}w_2}\mathrm{d}w_2\ \mathrm{d}w_1 \\
&= \int_{w_1=0}^{\infty} w_1^{\alpha_1}\ \mathrm{e}^{-(\delta_{1i}-\delta_{2j})w_1} \\
&\quad \times\ \left[\int_{w_2=0}^{\infty} w_2^{\alpha_2}\ \mathrm{e}^{-\delta_{2j}w_2}\mathrm{d}w_2 - \int_{w_2=0}^{w_1} w_2^{\alpha_2}\mathrm{e}^{-\delta_{2j}w_2}\mathrm{d}w_2\right]\mathrm{d}w_1 \\
&= \int_{w_1=0}^{\infty} w_1^{\alpha_1}\ \mathrm{e}^{-(\delta_{1i}-\delta_{2j})w_1}\ (\delta_{2j})^{-(\alpha_2+1)}\ \Gamma(\alpha_2+1)\mathrm{d}w_1 \\
&\quad - \sum_{r=0}^{\infty}\frac{(-\delta_{2j})^r}{r!}\frac{1}{\alpha_2+r+1}\int_{w_1=0}^{\infty} w_1^{\alpha_1+\alpha_2+r+1}\ \mathrm{e}^{-(\delta_{1i}-\delta_{2j})w_1}\mathrm{d}w_1,\ \Re(\alpha_2) > -1 \\
&= \Gamma(\alpha_1+1)\ \Gamma(\alpha_2+1)(\delta_{2j})^{-(\alpha_2+1)}(\delta_{1i}-\delta_{2j})^{-(\alpha_1+1)} \\
&\quad - \frac{\Gamma(\alpha_1+\alpha_2+2)}{\alpha_2+1}(\delta_{1i}-\delta_{2j})^{-(\alpha_1+\alpha_2+2)} \\
&\quad \times\ {}_2F_1\left(\alpha_2+1, \alpha_1+\alpha_2+2; \alpha_2+2; -\frac{\delta_{2j}}{\delta_{1i}-\delta_{2j}}\right)
\end{aligned} \tag{3.11}$$

for $\delta_{2j}/(\delta_{1i}-\delta_{2j})<1$ and $\Re(\alpha_1+\alpha_2)>-2$. Hence whenever $\Re(\alpha_1)>-1$ and $\Re(\alpha_2)>-1$, the exact arbitrary moment $E(u_1^{\alpha_1}u_2^{\alpha_2})$ is available from (3.9) where η is given by (3.10) and (3.11).

When $k=3$ or higher, one can follow through the same procedure as in the case $k=2$. But then several cases are to be considered to carry out the integrations at various stages. This is illustrated below for $k=3$. Consider

$$\begin{aligned} E[u_1^{\alpha_1}u_2^{\alpha_2}u_3^{\alpha_3}] &= E[v_1^{\alpha_1}(v_1+v_2)^{\alpha_2}(v_1+v_2+v_3)^{\alpha_3}] \\ &= \sum_{i=1}^{r_1}\sum_{j=r_1+1}^{r_1+r_2}\sum_{m=r_1+r_2+1}^{r_1+r_2+r_3} b_{1i}b_{2j}b_{3m}\delta_{1i}\delta_{2j}\delta_{3m}\ \eta \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \eta = \int_{v_1=0}^{\infty}\int_{v_2=0}^{\infty}\int_{v_3=0}^{\infty} &(v_1+v_2)^{\alpha_2}(v_1+v_2+v_3)^{\alpha_3} \\ &\times\ \mathrm{e}^{-\delta_{1i}v_1-\delta_{2j}v_2-\delta_{3m}v_3}\ \mathrm{d}v_1\ \mathrm{d}v_2\ \mathrm{d}v_3 \end{aligned}$$

with the b_{ij}'s and δ_{ij}'s as defined in (3.3). Note that every element in the set $\{b_{1i}\}$ for all i is larger than every element in the set $\{b_{2j}\}$ for all j which in turn is larger than every element in the set $\{b_{3m}\}$ for all m. Hence $\delta_{1i}-\delta_{2j}\geq 1$ and $\delta_{2j}-\delta_{3m}\geq 1$ for all i,j,m. We first integrate out $w_1,\ 0\leq w_1\leq w_2$, then $w_2,\ 0\leq w_2\leq w_3$ and finally $w_3,\ 0\leq w_3<\infty$. Following through the steps from (3.9) to (3.11) and assuming that $\Re(\alpha_j)>-1,\ j=1,2,3$, we end up with the following expression:

$$\begin{aligned} \eta = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3+3)}{(\alpha_1+1)(\alpha_1+\alpha_2+2)}&(\delta_{1i}-\delta_{2j})^{-(\alpha_1+1)}\ (\delta_{2j}-\delta_{3m})^{-(\alpha_1+\alpha_2+2)} \\ &\times\ (\delta_{3m})^{-(\alpha_1+\alpha_2+\alpha_3+3)}\ \gamma \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \gamma = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} &\frac{(-1)^{r+s}}{r!s!}\ \frac{(\alpha_1+1)_r}{(\alpha_1+2)_r}\ \frac{(\alpha_1+\alpha_2+2)_s}{(\alpha_1+\alpha_2+3)_s} \\ &\times\ (\alpha_1+\alpha_2+\alpha_3+3)_{r+s}\left(\frac{\delta_{1i}-\delta_{2j}}{\delta_{3m}}\right)^r\left(\frac{\delta_{2j}-\delta_{3m}}{\delta_{3m}}\right)^s. \end{aligned}$$

The double series in γ can be written as a special case of Lauricella function F_A by using (4.8.1) of Mathai (1993) or in terms of Appell's function F_2 by using (1.4.2) of Exton (1976). That is,

$$\gamma = F_2\Big(\alpha_1+\alpha_2+\alpha_3+3, \alpha_1+1, \alpha_1+\alpha_2+2; \alpha_1+2, \alpha_1+\alpha_2+3;$$

$$-\left(\frac{\delta_{1i}-\delta_{2j}}{\delta_{3m}}\right), -\left(\frac{\delta_{2j}-\delta_{3m}}{\delta_{3m}}\right)\bigg). \tag{3.14}$$

The condition to be satisfied for the convergence of this function is

$$\left(\frac{\delta_{1i}-\delta_{2j}}{\delta_{3m}}\right)+\left(\frac{\delta_{2j}-\delta_{3m}}{\delta_{3m}}\right)<1 \quad\Rightarrow\quad \frac{\delta_{1i}-\delta_{3m}}{\delta_{3m}}<1\ . \tag{3.15}$$

A similar procedure can be used to integrate out w_1, w_2, w_3 for other situations. In all these cases, the arbitrary product moment can be evaluated in terms of Lauricella's F_A type functions. For a general k, arbitrary product moments can be evaluated in multiple series. Note that when $k \geq 4$, the multiple series cannot be written in terms of a standard special function of many variables. Such generalized functions would have to be defined. Also various situations are to be examined for convergence before evaluating the integrals in the multiple series.

However, if for $k \geq 2$, one starts with the joint density of the k order statistics and try to evaluate the product moments directly, one can expand the various factors since the exponents, $r_2-1, r_3-1, \ldots, r_k-1$, are non-negative integers. But term by term integration in the resulting multiple series fails in most of the situations. It would appear that such a procedure will consistently yield integer moments only for $k=1$.

3.2. An alternate method of obtaining integer moments

From the density of u_1, the r_1-th order statistic, observe that e^{-u_1} has a type-1 beta distribution with parameters (r_2, r_1) and that of $1-\mathrm{e}^{-u_1}$ is distributed as a type-1 beta with parameters (r_1, r_2). If $y=\mathrm{e}^{-u_1}$, then the density of y, denoted by $f_y(y)$, is

$$f_y(y)=\frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\Gamma(r_2)}y^{r_2-1}(1-y)^{r_1-1},\ 0\leq y\leq 1,\ r_1+r_2=n+1\ ,$$

and then,

$$E(y^t)=E[\mathrm{e}^{t(\ln y)}]=\frac{\Gamma(r_2+t)}{\Gamma(r_2)}\frac{\Gamma(r_1+r_2)}{\Gamma(r_1+r_2+t)},\ \text{for } \Re(r_2+t)>0\ . \tag{3.16}$$

Note that

$$u_1=-\ln y \quad\Rightarrow\quad E(u_1^h)=E[-\ln y]^h=(-1)^h E[\ln y]^h.$$

Thus, for $h = 0, 1, 2, \ldots$, one can derive $E(\ln y)^h$ by differentiating (3.16) with respect to t and then by evaluating the result at $t = 0$. For example,

$$\begin{aligned} E(\ln y) &= \frac{\partial}{\partial t}\left\{\frac{\Gamma(r_2+t)}{\Gamma(r_2)}\frac{\Gamma(r_1+r_2)}{\Gamma(r_1+r_2+t)}\right\}\Bigg|_{t=0} \\ &= \psi(r_2) - \psi(r_1+r_2),\ r_2 = n+1-r_1, \end{aligned}$$

where $\psi(\cdot)$ is a psi function, and on making use of Equation(1.4.5) in Mathai (1993), one has

$$\begin{aligned} E(u_1) = -E(\ln y) &= \psi(r_1+r_2) - \psi(r_2) \\ &= \frac{1}{t+r_1+r_2-1} + \frac{1}{t+r_1+r_2-2} + \cdots + \frac{1}{t+r_2}\Bigg|_{t=0} \\ &= \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-(r_1-1)}\ . \end{aligned} \tag{3.17}$$

Similarly,

$$\begin{aligned} E(u_1^2) = E(\ln y)^2 &= \frac{\partial^2}{\partial t^2}\left\{\frac{\Gamma(r_1+r_2)}{\Gamma(r_2)}\frac{\Gamma(r_2+t)}{\Gamma(r_1+r_2+t)}\right\}\Bigg|_{t=0} \\ &= \frac{\partial}{\partial t}\left\{\frac{\Gamma(r_1+r_2)}{\Gamma(r_2)}[\psi(r_2+t)-\psi(r_1+r_2+t)]\frac{\Gamma(r_2+t)}{\Gamma(r_1+r_2+t)}\right\}\Bigg|_{t=0} \\ &= \frac{\Gamma(r_1+r_2)}{\Gamma(r_2)}\frac{\partial}{\partial t}\left\{\frac{\Gamma(r_2+t)}{\Gamma(r_1+r_2+t)}\left[-\frac{1}{t+r_1+r_2-1} - \cdots - \frac{1}{t+r_2}\right]\right\}\Bigg|_{t=0} \\ &= \psi'(r_2) - \psi'(r_1+r_2) + [\psi(r_2) - \psi(r_1+r_2)]^2 \end{aligned} \tag{3.18}$$

where a prime denotes a derivative. Therefore

$$\begin{aligned} \mathrm{Var}(u_1) &= E(u_1^2) - [E(u_1)]^2 = \psi'(r_2) - \psi'(r_1+r_2) \\ &= \frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{[n-(r_1-1)]^2}\ . \end{aligned}$$

Higher order moments of u_1 are available by taking higher order derivatives of the product in (3.18) and then by evaluating the result at $t = 0$. The resulting representations will involve psi and generalized zeta functions.

We now consider the joint moments of u_1 and u_2. Let $y_j = 1 - \mathrm{e}^{-u_j},\ j = 1, 2$. Then the joint density of y_1 and y_2, denoted by $f_y(y_1, y_2)$ is given by

$$f_y(y_1,y_2) = \frac{\Gamma(n+1)}{\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)}y_1^{r_1-1}(y_2-y_1)^{r_2-1}(1-y_2)^{r_3-1},\ r_1+r_2+r_3 = n+1\ ,$$

and letting $z_1 = y_1,\ z_2 = y_2 - y_1$, the joint density of z_1, z_2 is given by

$$f_z(z_1,z_2) = \frac{\Gamma(n+1)}{\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)}z_1^{r_1-1}z_2^{r_2-1}(1-z_1-z_2)^{r_3-1}.$$

Then

$$u_1 = -\ln(1-y_1) = -\ln(1-z_1) \quad \text{and} \quad E(u_1^{h_1}) = (-1)^{h_1} E[\ln(1-z_1)]^{h_1}$$
$$u_2 = -\ln(1-y_2) = -\ln(1-z_1-z_2) \quad \text{and} \quad E(u_2^{h_2}) = (-1)^{h_2} E[\ln(1-z_1-z_2)]^{h_2}.$$

Letting

$$\begin{aligned}\phi(t_1,t_2) &= E[(1-z_1)^{t_1}(1-z_1-z_2)^{t_2}] \\ &= \frac{\Gamma(n+1)}{\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)} \int\int z_1^{r_1-1}(1-z_1)^{t_1} z_2^{r_2-1}(1-z_1-z_2)^{r_3+t_2-1}\mathrm{d}z_1\mathrm{d}z_2 \\ &= \frac{\Gamma(n+1)}{\Gamma(r_3)} \frac{\Gamma(t_2+r_3)}{\Gamma(t_1+t_2+r_1+r_2+r_3)} \frac{\Gamma(t_1+t_2+r_2+r_3)}{\Gamma(t_2+r_2+r_3)} \end{aligned} \tag{3.19}$$

for $\Re(t_2+r_3) > 0$, the integer product moments of u_1 and u_2 can be obtained by differentiating $\phi(t_1,t_2)$, multiplying by the appropriate power of (-1), and then by evaluating the result at $t_1 = 0 = t_2$. For example,

$$E(u_1) = -[\psi(r_2+r_3) - \psi(r_1+r_2+r_3)] = \psi(n+1) - \psi(r_2+r_3) \tag{3.20}$$

$$E(u_2) = -[\psi(r_3) - \psi(r_1+r_2+r_3)] = \psi(n+1) - \psi(r_3) \tag{3.21}$$

$$\begin{aligned} E(u_1u_2) &= \psi'(r_2+r_3) - \psi'(r_1+r_2+r_3) \\ &\quad + [\psi(r_2+r_3) - \psi(r_1+r_2+r_3)][\psi(r_3) - \psi(r_1+r_2+r_3)] \end{aligned} \tag{3.22}$$

and hence

$$\begin{aligned}\mathrm{Cov}(u_1,u_2) &= E(u_1u_2) - E(u_1)E(u_2) = \psi'(r_2+r_3) - \psi'(r_1+r_2+r_3),\ r_1+r_2+r_3 = n+1 \\ &= \frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{[n-(r_1-1)]^2}\ .\end{aligned}$$

Note that by taking higher order derivatives of $\phi(t_1,t_2)$, one can derive integer product moments of higher orders which can be expressed in terms of psi and generalized zeta functions.

In the general case, the density of $z_1,\ldots,z_k$ is available as

$$\begin{aligned} f_z(z_1,\ldots,z_k) = \frac{\Gamma(n+1)}{\Gamma(r_1)\Gamma(r_2)\cdots\Gamma(r_{k+1})} & z_1^{r_1-1} z_2^{r_2-1}\cdots z_k^{r_k-1} \\ & \times\ (1-z_1-\cdots-z_k)^{r_{k+1}-1}, \qquad r_1+\cdots+r_{k+1} = n+1. \end{aligned}$$

Note that

$$u_j = -\ln(1-y_j) = -\ln(1-z_1-\cdots-z_j)\ .$$

Let

$$\phi(t_1,\ldots,t_k) = E[(1-z_1)^{t_1}(1-z_1-z_2)^{t_2}\cdots(1-z_1-\cdots-z_k)^{t_k}]\ .$$

Then on integrating $z_k, z_{k-1}, \ldots, z_1$, one has

$$\phi(t_1,\ldots,t_k)=\frac{\Gamma(n+1)}{\Gamma(r_{k+1})}\frac{\Gamma(r_{k+1}+t_k)}{\Gamma(r_{k+1}+r_k+t_k)}\frac{\Gamma(r_{k+1}+r_k+t_k+t_{k-1})}{\Gamma(r_{k+1}+r_k+r_{k-1}+t_k+t_{k-1})}$$
$$\times\ \cdots\frac{\Gamma(r_{k+1}+\cdots+r_2+t_k+\cdots+t_1)}{\Gamma(r_{k+1}+\cdots+r_1+t_k+\cdots+t_1)}\ ,$$

which also agrees with the joint moments of the survival functions given in (1.2). Hence, for $h_j = 0, 1, 2, \ldots\,, j = 1, \ldots, k$, all integer product moments can be obtained as follows:

$$E(u_1^{h_1}\cdots u_k^{h_k})=(-1)^{h_1+\cdots+h_k}\frac{\partial^{h_1+\cdots+h_k}}{\partial t_1^{h_1}\cdots\partial t_k^{h_k}}\phi(t_1,\ldots,t_k)\Bigg|_{t_1\,=\,0\,=\,\cdots\,=\,t_k}\ . \tag{3.23}$$

For example,

$$E(u_1)=\psi(n+1)-\psi(r_4+r_3+r_2)\ , \tag{3.24}$$

$$E(u_2)=\psi(n+1)-\psi(r_4+r_3)\ , \tag{3.25}$$

$$E(u_3)=\psi(n+1)-\psi(r_4)\ , \tag{3.26}$$

$$\begin{aligned}E(u_1u_2u_3)&=\psi''(n+1)-\psi''(r_4+r_3+r_2)\\&-[\psi'(n+1)-\psi'(r_4+r_3+r_2)][\psi(n+1)-\psi(r_4)]\\&-[\psi'(n+1)-\psi'(r_4+r_3)][\psi(n+1)-\psi(r_4+r_3+r_2)]\\&-\ [\psi'(n+1)-\psi'(r_4+r_3+r_2)][\psi(n+1)-\psi(r_4+r_3)]\\&+[\psi(n+1)-\psi(r_4)][\psi(n+1)-\psi(r_4+r_3)][\psi(n+1)-\psi(r_4+r_3+r_2)]\end{aligned} \tag{3.27}$$

Note that the quantities within the square brackets are finite sums. One can also write these sums as differences of generalized zeta functions.

3.3. Some recurrence relations

Consider the r_1-th order statistic denoted by u_1, and let

$$\mu_{(r_1,r_2,n)}^{(k)}=E(u_1^k).$$

The two standard recurrence relations available in the literature are the following, see for example Balakrishnan and Cohen (1991) and the references therein:

$$\mu_{(1,r_2,n)}^{(k)}\ =\ \frac{k}{n}\,\mu_{(1,r_2,n)}^{(k-1)}\quad\text{for}\ \ n\geq 1\text{ and }k\geq 1 \tag{3.28}$$

and

$$\mu^{(k)}_{(r_1,r_2,n)} \;=\; \mu^{(k)}_{(r_1-1,r_2,n-1)} + \frac{k}{n}\,\mu^{(k-1)}_{(r_1,r_2,n)} \qquad \text{for } \; 2 \le r_1 \le n \text{ and } k \ge 1. \tag{3.29}$$

Note that (3.28) holds only when $r_1 = 1$ and that (3.29) requires moments of lower orders on k as well as on r_1 and n. These shortcomings limit the applicability of such relationships when performing actual computations. It is desirable to have recurrence relations that change only in k so that one can compute higher order moments from lower order moments of u_1. Such a relationship can be derived from the representation given in (3.16). Note that

$$\begin{aligned}
\mu^{(k)}_{(r_1,r_2,n)} &= \frac{\Gamma(r_1+r_2)}{\Gamma(r_2)}(-1)^k \left[\frac{\partial^k}{\partial t^k}\frac{\Gamma(r_2+t)}{\Gamma(r_1+r_2+t)}\right]\Bigg|_{t\,=\,0} \qquad \text{with } r_1+r_2 = n+1 \\
&= \frac{\Gamma(r_1+r_2)}{\Gamma(r_2)}(-1)^{k-1}\frac{\partial^{k-1}}{\partial t^{k-1}}\left\{\frac{\Gamma(r_2+t)}{\Gamma(r_1+r_2+t)}\left[\frac{1}{t+r_2}+\cdots+\frac{1}{t+r_1+r_2-1}\right]\right\}\Bigg|_{t\,=\,0}
\end{aligned}$$

which is available from (3.17). Now, writing $\mu^{(k)}_{(r_1,r_2,n)} = \mu^{(k)}$ and applying Leibnitz's rule for differentiating a product of functions, one has the following:

$$\begin{aligned}
\mu^{(k)} &= \frac{\Gamma(r_1+r_2)}{\Gamma(r_2)}\sum_{m=0}^{k-1}\binom{k-1}{m}\left[(-1)^{k-1-m}\frac{\partial^{k-1-m}}{\partial t^{k-1-m}}\frac{\Gamma(r_2+t)}{\Gamma(r_1+r_2+t)}\right] \\
&\qquad\qquad \times\; \left[(-1)^m\frac{\partial^m}{\partial t^m}\left\{\frac{1}{t+r_2}+\cdots+\frac{1}{t+r_1+r_2-1}\right\}\right]\Bigg|_{t\,=\,0} \\
&= (k-1)!\sum_{m=0}^{k-1}\frac{1}{(k-1-m)!}\mu^{(k-1-m)}\mu_{(m+1)}
\end{aligned} \tag{3.30}$$

where

$$\mu_{(m)} = \frac{1}{r_2^m} + \frac{1}{(r_2+1)^m} + \cdots + \frac{1}{n^m}\;, \quad m \ge 1\;. \tag{3.31}$$

For example,

$$\begin{aligned}
\mu^{(0)} &= 1 \\
\mu^{(1)} &= \frac{1}{r_2}+\cdots+\frac{1}{n} = \mu_{(1)} \\
\mu^{(2)} &= \mu^{(1)}\mu^{(1)} + \frac{1}{r_2^2}+\cdots+\frac{1}{n^2} = \mu_{(2)} + \mu^2_{(1)} \\
\mu^{(3)} &= 2!\left\{\frac{\mu^{(2)}\mu_{(1)}}{2!} + \frac{\mu^{(1)}\mu_{(2)}}{1!} + \frac{\mu^{(0)}\mu_{(3)}}{0!}\right\} \\
&= \mu^3_{(1)} + 3\;\mu_{(1)}\mu_{(2)} + 2\;\mu_{(3)} \\
\mu^{(4)} &= 3!\left\{\frac{\mu^{(3)}\mu_{(1)}}{3!} + \frac{\mu^{(2)}\mu_{(2)}}{2!} + \frac{\mu^{(1)}\mu_{(3)}}{1!} + \frac{\mu^{(0)}\mu_{(4)}}{0!}\right\} \\
&= \mu^4_{(1)} + 6\;\mu^2_{(1)}\mu_{(2)} + 8\;\mu_{(1)}\mu_{(3)} + 3\;\mu^2_{(2)} + 6\;\mu_{(4)}\;.
\end{aligned}$$

One can systematically evaluate all integer moments of u_1 or of any particular u_j from (3.30). Also note that (3.30) is suitable for computer evaluations as well.

Now, let us consider some recurrence relations for product moments, all of which can be evaluated from (3.19). Let

$$\begin{aligned}\mu^{(k_1,k_2)} &= E[u_1^{k_1}u_2^{k_2}] \\ &= (-1)^{k_1+k_2}\frac{\Gamma(n+1)}{\Gamma(r_3)}\frac{\partial^{k_2}}{\partial t_2^{k_2}}\frac{\partial^{k_1}}{\partial t_1^{k_1}}\left[\frac{\Gamma(t_2+r_3)}{\Gamma(t_2+r_2+r_3)}\frac{\Gamma(t_1+t_2+r_2+r_3)}{\Gamma(t_1+t_2+n+1)}\right].\end{aligned}$$

We introduce the following general notation:

$$\mu_{(m,p)} = \frac{1}{p^m}+\frac{1}{(p+1)^m}+\cdots+\frac{1}{n^m} \quad \text{with } \; p\leq n \text{ and } \; r_1+r_2+r_3=n+1\ . \tag{3.32}$$

Note that

$$\mu^{(k_1,0)} = (-1)^{k_1}\frac{\Gamma(n+1)}{\Gamma(r_2+r_3)}\left[\frac{\partial^{k_1}}{\partial t_1^{k_1}}\frac{\Gamma(t_1+r_2+r_3)}{\Gamma(t_1+n+1)}\right]\Bigg|_{t_1\,=\,0}$$

and

$$\mu^{(0,k_2)} = (-1)^{k_2}\frac{\Gamma(n+1)}{\Gamma(r_3)}\left[\frac{\partial^{k_2}}{\partial t_2^{k_2}}\frac{\Gamma(t_2+r_3)}{\Gamma(t_2+n+1)}\right]\Bigg|_{t_2\,=\,0}.$$

These readily give

$$\begin{aligned}\mu^{(0,0)} &= 1 \\ \mu^{(1,0)} &= \frac{1}{(r_2+r_3)}+\cdots+\frac{1}{n} = \mu_{(1,r_2+r_3)} \\ \mu^{(2,0)} &= \mu_{(2,r_2+r_3)}+\mu^2_{(1,r_2+r_3)} \\ \mu^{(0,1)} &= \frac{1}{r_3}+\cdots+\frac{1}{n} = \mu_{(1,r_3)} \\ \mu^{(0,2)} &= \mu_{(2,r_3)}+\mu^2_{(1,r_3)}\end{aligned}$$

and so on. For obtaining recurrence relations on the product moments, one has to evaluate $\mu^{(k_1,k_2)}$. Note that

$$\begin{aligned}\mu^{(k_1,k_2)} &= (-1)^{k_2}\frac{\Gamma(n+1)}{\Gamma(r_3)}\frac{\partial^{k_2}}{\partial t_2^{k_2}}\frac{\Gamma(t_2+r_3)}{\Gamma(t_2+r_2+r_3)} \\ &\quad\times\ -1)^{k_1-1}\frac{\partial^{k_1-1}}{\partial t_1^{k_1-1}}\frac{\Gamma(t_1+t_2+r_2+r_3)}{\Gamma(t_1+t_2+n+1)}\left[\frac{1}{r_2+r_3+t_1+t_2}+\cdots+\frac{1}{n+t_1+t_2}\right] \\ &= (-1)^{k_2}\frac{\Gamma(n+1)}{\Gamma(r_3)}\frac{\partial^{k_2}}{\partial t_2^{k_2}}\frac{\Gamma(t_2+r_3)}{\Gamma(t_2+r_2+r_3)} \\ &\quad\times\sum_{m_1=0}^{k_1-1}\binom{k_1-1}{m_1}\mu^{(k_1-1-m_1,0)}(t_1,t_2)\mu_{(m_1+1,r_2+r_3)}(t_1,t_2)\Bigg|_{t_1\,=\,0\,=\,t_2}\end{aligned} \tag{3.33}$$

where

$$\mu^{(k_1-1-m_1,0)} = \frac{\Gamma(n+1)}{\Gamma(r_3)} \frac{\Gamma(t_2+r_3)}{\Gamma(t_2+r_2+r_3)} \mu^{(k_1-1-m_1,0)}(t_1,t_2) \Bigg|_{t_1 = 0 = t_2}$$

and

$$\mu_{(m_1+1,r_2+r_3)} = \frac{\Gamma(n+1)}{\Gamma(r_3)} \frac{\Gamma(t_2+r_3)}{\Gamma(t_2+r_2+r_3)} \mu_{(m_1+1,r_2+r_3)}(t_1,t_2) \Bigg|_{t_1 = 0 = t_2} .$$

Now, $\mu^{(k_1,k_2)}$ can be written explicitly by making use of Leibnitz's rule for differentiating a product of three factors, the factors being

$$\frac{\Gamma(t_2+r_3)}{\Gamma(t_2+r_2+r_3)}, \quad \mu^{(k_1-1-m_1,0)}(t_1,t_2), \quad \mu_{(m_1+1,r_2+r_3)}(t_1,t_2) .$$

One may use (3.33) for the successive evaluations. For example,

$$\begin{aligned}
\mu^{(1,1)} &= \mu_{(2,r_2+r_3)} + \mu_{(1,r_3)}\mu_{(1,r_2+r_3)} \\
\mu^{(2,1)} &= \mu_{(1,r_3)}\mu_{(2,r_2+r_3)} + \mu_{(1,r_3)}\mu^2_{(1,r_2+r_3)} + 2\ \mu_{(3,r_2+r_3)} + 2\ \mu_{(1,r_2+r_3)}\mu_{(2,r_2+r_3)} \\
&= \mu^{(1,1)}\mu^{(1,0)} + \mu^{(3,0)} + \mu^{(2,0)}\mu^{(0,1)} - 2\ \mu^{(2,0)}\mu^{(1,0)} - [\mu^{(1,0)}]^2\mu^{(0,1)} + [\mu^{(1,0)}]^3 \\
\mu^{(2,2)} &= \mu^2_{(1,r_2+r_3)}\mu^2_{(1,r_3)} + \mu^2_{(1,r_3)}\mu_{(2,r_3)} + 2\ \mu_{(1,r_3)}\mu_{(3,r_3)} + 2\ \mu_{(1,r_3)}\mu_{(3,r_2+r_3)} \\
&\quad + 4\ \mu_{(1,r_3)}\mu_{(1,r_2+r_3)}\mu_{(2,r_2+r_3)} + \mu_{(2,r_3)}\mu_{(2,r_2+r_3)} + \mu_{(2,r_3)}\mu^2_{(1,r_2+r_3)} \\
&\quad + 4\ \mu_{(1,r_2+r_3)}\mu_{(3,r_2+r_3)} + 2\ \mu^2_{(2,r_2+r_3)} + 6\ \mu_{(4,r_2+r_3)} \\
&= \mu^{(0,1)}\mu^{(2,1)} + 2\ \mu^{(2,0)}\mu^{(1,1)} - 2\ [\mu^{(1,0)}]^2\mu^{(1,1)} + \mu^{(4,0)} \\
&\quad - 2\ \mu^{(3,0)}\mu^{(1,0)} + \mu^{(3,0)}\mu^{(0,1)} - 2\ [\mu^{(2,0)}]^2 + 5\ \mu^{(2,0)}[\mu^{(1,0)}]^2 \\
&\quad - 3\ \mu^{(2,0)}\mu^{(1,0)}\mu^{(0,1)} + 2\ [\mu^{(1,0)}]^3\mu^{(0,1)} - 2\ [\mu^{(1,0)}]^4 ,
\end{aligned}$$

where the notation $\mu_{(m,p)}$ is defined in (3.32). Recurrence relations for the product moments of three or more order statistics can be obtained with the help of (3.23). These will involve many terms. For other types of recurrence relations, see for example, Srikantan (1962), Govindarajulu (1963), and Balasubramanian and Balakrishnan (1993) who established a duality principle from which counterparts to numerous recursive relationships can be deduced.

3.4. Truncated exponential population

Consider a truncated exponential population with the density

$$f(x) = \frac{e^{-x}}{e^{-a} - e^{-b}} \qquad \text{for } a \le x \le b,$$

and zero elsewhere. Let $F(x)$ denote the distribution function and $y_j = F(u_j)$ where u_j denotes the $r_1 + \cdots + r_j$-th order statistic from this population. Then

$$u_j = a - \ln(1 - p\ y_j), \qquad p = \frac{\mathrm{e}^{-a} - \mathrm{e}^{-b}}{\mathrm{e}^{-a}}\ , \qquad 0 < p \le 1.$$

Therefore

$$\begin{aligned} u_j^m &= \sum_{m_1=0}^{m} \binom{m}{m_1} a^{m-m_1}(-1)^{m_1}[\ln(1-p\ y_j)]^{m_1},\ a > 0 \\ &= [-\ln(1-p\ y_j)]^m \quad \text{whenever } a = 0 \end{aligned}$$

and

$$E(u_j^m) = \sum_{m_1=0}^{m} \binom{m}{m_1} a^{m-m_1}(-1)^{m_1} \frac{\partial^{m_1}}{\partial t^{m_1}} E(1-p\ y_j)^t \Bigg|_{t\,=\,0}\ .$$

If only left truncation is involved, then $p = 1$ since $b = +\infty$ whereas if only right truncation is involved, then $a = 0$, $p = 1 - \mathrm{e}^{-b}$ and $u_j^m = [-\ln(1-p\ y_j)]^m$. This latter case was considered by Saleh *et al.* (1975) whose representations of the first two moments involved infinite series. We provide a methodology for obtaining the moments as finite sums. This technique can also be applied to the general case of product moments.

Consider the case of the single order statistic u_1, which denotes the r_1-th order statistic with $r_1 + r_2 = n + 1$. Let $\phi(t)$

$$\begin{aligned} &= E(1-p\ y_1)^t \\ &= \frac{\Gamma(n+1)}{\Gamma(r_1)\Gamma(r_2)} \int_0^1 (1-p\ z_1)^t z_1^{r_1-1}(1-z_1)^{r_2-1}\mathrm{d}z_1 \\ &= \frac{\Gamma(n+1)}{\Gamma(r_1)\Gamma(r_2)} \sum_{s_1=0}^{r_2-1} \binom{r_2-1}{s_1}(-1)^{s_1} \int_0^1 (1-p\ z_1)^t z_1^{r_1+s_1-1}\mathrm{d}z_1 \qquad (3.34) \\ &= \frac{\Gamma(n+1)}{\Gamma(r_1)\Gamma(r_2)} \sum_{s_1=0}^{r_2-1} \binom{r_2-1}{s_1} \frac{(-1)^{s_1}}{p^{r_1+s_1}} \sum_{s_2=0}^{r_1+s_1-1} \binom{r_1+s_1-1}{s_2}(-1)^{s_2} \frac{1}{t+s_2+1}[1-(1-p)^{t+s_2+1}]\ . \end{aligned}$$

When taking the derivatives to obtain the moments, only the factors containing t are to be differentiated, that is,

$$\frac{\partial^{m_1}}{\partial t^{m_1}} \left[\frac{1-(1-p)^{t+s_2+1}}{t+s_2+1}\right] = \sum_{m_2=0}^{m_1} \binom{m_1}{m_2} \frac{\partial^{m_1-m_2}}{\partial t^{m_1-m_2}}(1-(1-p)^{t+s_2+1}) \frac{\partial^{m_2}}{\partial t^{m_2}}(t+s_2+1)^{-1}.$$

But

$$\frac{\partial^{m_2}}{\partial t^{m_2}}(t+s_2+1)^{-1}\big|_{t=0} = (-1)^{m_2} m_2!(s_2+1)^{-(m_2+1)},$$

and

$$\frac{\partial^{m_1-m_2}}{\partial t^{m_1-m_2}}[1-(1-p)^{t+s_2+1}]\big|_{t=0} = -\frac{\partial^{m_1-m_2}}{\partial t^{m_1-m_2}}(1-p)^{t+s_2+1}$$
$$= \begin{cases} -[\ln(1-p)]^{m_1-m_2}(1-p)^{s_2+1}, m_1-m_2>0 \\ [1-(1-p)^{s_2+1}]\ ,\ m_1-m_2=0\ . \end{cases} \tag{3.35}$$

Hence

$$\mu^{(m)} = E(u_1^m) = \sum_{m_1=0}^{m}\binom{m}{m_1}a^{m-m_1}(-1)^{m_1}\frac{\partial^{m_1}}{\partial t^{m_1}}\phi(t)\ \Big|_{\ t\ =\ 0}$$

where

$$\frac{\partial^{m_1}}{\partial t^{m_1}}\phi(t)\big|_{t=0} = \frac{\Gamma(n+1)}{\Gamma(r_1)\Gamma(r_2)}\sum_{s_1=0}^{r_2-1}\binom{r_2-1}{s_1}\frac{(-1)^{s_1}}{p^{r_1+s_1}}\sum_{s_2=0}^{r_1+s_1-1}\binom{r_1+s_1-1}{s_2}(-1)^{s_2}$$
$$\times\sum_{m_2=0}^{m_1}\binom{m_1}{m_2}(-1)^{m_2+1}m_2!(s_2+1)^{-(m_2+1)}[\ln(1-p)]^{m_1-m_2}(1-p)^{s_2+1}$$

with modification as in (3.35) whenever $m_1-m_2=0$. When product moments are considered, several finite sums are involved. For $k=2$, the final result would already take up to much space to be given explicitly. Nevertheless, even for a general k, the product moments can be written as finite sums by following through the above procedure, each time expanding factors with the exponents $r_1-1, r_2-1, \ldots$. The step from (3.34) comes from the substitution $w=1-pz_1 \Rightarrow z_1=\frac{1-w}{p}$, then expanding $(1-w)^{r_1+s_1-1}$ and finally integrating out w. Note that $1-p\le w\le 1$. When considering general product moments, the corresponding integral when integrating out z_k is the following:

$$\left\{\int_0^1\left[1-\frac{p(1-z_1-\cdots-z_{k-1})z_k}{1-pz_1-\cdots-pz_{k-1}}\right]^{t_k} z_k^{r_k+s_{k1}-1}\mathrm{d}z_k\right\}$$
$$\times\ [1-pz_1-\cdots-pz_{k-1}]^{t_k}[1-z_1-\cdots-z_k]^{r_{k+1}+r_k-1}$$
$$=\left\{\int_{\frac{1-p}{1-pz_1-\cdots-pz_{k-1}}}^{1} w_k^{t_k}(1-w_k)^{r_k+s_{k1}-1}\mathrm{d}z_k\right\}\frac{1}{p^{r_k+s_{k1}}}$$
$$\times\ [1-pz_1-\cdots-pz_{k-1})]^{t_k}[1-z_1-\cdots-z_{k-1}]^{r_{k+1}-s_{k1}-1}$$
$$=\sum_{s_{k2}=0}^{r_k+s_{k1}-1}\binom{r_k+s_{k1}-1}{s_{k2}}\frac{1}{(t_k+s_{k2}+1)p^{r_k+s_{k2}+1}}\left[1-\frac{(1-p)^{t_k+s_{k2}+1}}{1-pz_1-\cdots-pz_{k-1})^{t_k+s_{k2}+1}}\right]$$
$$\times\ [1-pz_1-\cdots-pz_{k-1}]^{t_k+r_k+s_{k1}}[1-z_1-\cdots-z_{k-1}]^{r_{k+1}-s_{k1}-1}$$
$$=\sum_{s_{k2}=0}^{r_k+s_{k1}-1}\binom{r_k+s_{k1}-1}{s_{k2}}\frac{1}{(t_k+s_{k2}+1)p^{r_k+s_{k1}}}$$
$$\times\ [(1-pz_1-\cdots-pz_{k-1})^{t_k+r_k+s_{k1}}(1-z_1-\cdots-z_{k-1})^{r_{k+1}-s_{k1}-1}$$
$$-(1-p)^{t_k+s_{k2}+1}(1-pz_1-\cdots-pz_{k-1})^{t_k-s_{k2}+s_{k1}-1}(1-z_1-\cdots-z_{k-1})^{r_{k+1}-s_{k1}-1}]\ .$$

There are now two terms involving $z_1, \ldots, z_{k-1}$ which will lead to two such integrals. Finally, at the stage of integrating out z_1 there will be 2^k such integrals to be evaluated.

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APPENDIX Mathematical Identities

Several identities can be obtained by comparing certain integer moments coming from Equations (3.7) and (3.8). The first moment of u, that is, $E(u)$, produces the following interesting results.

For $k = 2$, one has

$$\frac{1}{1-\frac{a_1}{a_2}}\frac{1}{a_1}+\frac{1}{1-\frac{a_2}{a_1}}\frac{1}{a_2}\equiv\frac{1}{a_1}+\frac{1}{a_2}\quad\text{or}\quad\frac{a_1^2-a_2^2}{a_1-a_2}\equiv a_1+a_2\ , \tag{i}$$

and for $k = 3$,

$$\frac{1}{\left(1-\frac{a_1}{a_2}\right)\left(1-\frac{a_1}{a_3}\right)}\frac{1}{a_1}+\frac{1}{\left(1-\frac{a_2}{a_1}\right)\left(1-\frac{a_2}{a_3}\right)}\frac{1}{a_2}+\frac{1}{\left(1-\frac{a_3}{a_1}\right)\left(1-\frac{a_3}{a_2}\right)}\frac{1}{a_3}$$

$$\equiv\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3} \tag{ii}$$

$$\equiv a_1a_2a_3\left[\frac{1}{(a_1-a_2)(a_1-a_3)}\frac{1}{a_1^2}+\frac{1}{(a_2-a_1)(a_2-a_3)}\frac{1}{a_2^2}+\frac{1}{(a_3-a_1)(a_3-a_2)}\frac{1}{a_3^2}\right] \tag{iii}$$

$$\equiv\frac{1}{a_1a_2a_3(a_1-a_2)(a_1-a_3)(a_2-a_3)}[a_1^3(a_2^2-a_3^2)+a_2^3(a_3^2-a_1^2)+a_3^3(a_1^2-a_2^2)]\ . \tag{iv}$$

These identities can also take on different forms. For example,

$$\frac{a_2^2a_3^2}{(a_1-a_2)(a_1-a_3)}+\frac{a_1^2a_3^2}{(a_2-a_1)(a_2-a_3)}+\frac{a_1^2a_2^2}{(a_3-a_1)(a_3-a_2)}$$

$$\equiv a_1a_2+a_1a_3+a_2a_3\ . \tag{v}$$

Now replacing the a_i's by their reciprocals in (ii) and in (v) respectively yields

$$\frac{a_1^3}{(a_1-a_2)(a_1-a_3)}+\frac{a_2^3}{(a_2-a_1)(a_2-a_3)}+\frac{a_3^3}{(a_3-a_1)(a_3-a_2)}$$

$$\equiv a_1+a_2+a_3. \tag{vi}$$

and

$$\frac{a_1^2}{a_2a_3(a_2-a_1)(a_3-a_1)}+\frac{a_2^2}{a_1a_3(a_1-a_2)(a_3-a_2)}+\frac{a_3^2}{a_1a_2(a_1-a_3)(a_2-a_3)}$$
$$\equiv \frac{1}{a_1a_2}+\frac{1}{a_1a_3}+\frac{1}{a_2a_3}. \qquad (vii)$$

On replacing the a_i's by their cube roots, one has

$$\frac{a_1}{(a_1^{1/3}-a_2^{1/3})(a_1^{1/3}-a_3^{1/3})}+\frac{a_2}{(a_2^{1/3}-a_1^{1/3})(a_2^{1/3}-a_3^{1/3})}+\frac{a_3}{(a_3^{1/3}-a_1^{1/3})(a_3^{1/3}-a_2^{1/3})}$$
$$\equiv a_1^{1/3}+a_2^{1/3}+a_3^{1/3} \qquad (viii)$$

and so on. Note that even though we started with exponential densities with distinct and positive a_j's, for the identities (i) to $(viii)$, plus the identities to follow, to hold we need only the a_j's to be nonzero and distinct. Similarly, one can establish several such identities for any positive integer k. Here are some examples.

$$\sum_{j=1}^{k}\frac{1}{\left[\prod_{l=1,l\neq j}^{k}\left(1-\frac{a_j}{a_l}\right)\right]}\frac{1}{a_j}\equiv\sum_{j=1}^{k}\frac{1}{a_j} \qquad (ix)$$

$$\equiv a_1\cdots a_k\sum_{j=1}^{k}\left[\frac{1}{\prod_{l=1,l\neq j}^{k}(a_l-a_j)}\right]\frac{1}{a_j^2} \qquad (x)$$

$$\sum_{j=1}^{k}\left[\prod_{l=1,l\neq j}^{k}\frac{a_l^2}{(a_l-a_j)}\right]\equiv\sum_{j=1}^{k}(a_1a_2\cdots a_{j-1}a_{j+1}\cdots a_k),\ a_0=0\ . \qquad (xi)$$

On comparing the representations of the variance obtained from (3.7) and (3.8), we have the following identities:

$$\frac{1}{\left(1-\frac{a_1}{a_2}\right)\left(1-\frac{a_1}{a_3}\right)}\frac{1}{a_1^2}+\frac{1}{\left(1-\frac{a_2}{a_1}\right)\left(1-\frac{a_2}{a_3}\right)}\frac{1}{a_2^2}+\frac{1}{\left(1-\frac{a_3}{a_1}\right)\left(1-\frac{a_3}{a_2}\right)}\frac{1}{a_3^2}$$
$$-\left[\frac{1}{\left(1-\frac{a_1}{a_2}\right)\left(1-\frac{a_1}{a_3}\right)}\frac{1}{a_1}+\frac{1}{\left(1-\frac{a_2}{a_1}\right)\left(1-\frac{a_2}{a_3}\right)}\frac{1}{a_2}+\frac{1}{\left(1-\frac{a_3}{a_1}\right)\left(1-\frac{a_3}{a_2}\right)}\frac{1}{a_3}\right]^2$$
$$\equiv\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}\ . \qquad (xii)$$

On taking the reciprocals and roots of the a_j's, one can establish for instance that

$$\frac{a_1^4}{(a_1-a_2)(a_1-a_3)}+\frac{a_2^4}{(a_2-a_1)(a_2-a_3)}+\frac{a_3^4}{(a_3-a_1)(a_3-a_2)}$$
$$-\left[\frac{a_1^3}{(a_1-a_2)(a_1-a_3)}+\frac{a_2^3}{(a_2-a_1)(a_2-a_3)}+\frac{a_3^3}{(a_3-a_1)(a_3-a_2)}\right]^2$$
$$\equiv a_1^2+a_2^2+a_3^3\ , \qquad (xiii)$$

$$\frac{a_1^3}{a_2a_3(a_1-a_2)(a_1-a_3)}+\frac{a_2^3}{a_1a_3(a_2-a_1)(a_2-a_3)}+\frac{a_3^3}{a_1a_2(a_3-a_1)(a_3-a_2)}$$
$$-\frac{a_1}{a_2a_3}-\frac{a_2}{a_1a_3}-\frac{a_3}{a_1a_2}$$
$$\equiv\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\ , \tag{xiv}$$

$$\frac{a_1}{(a_1^{1/4}-a_2^{1/4})(a_1^{1/4}-a_3^{1/4})}+\frac{a_2}{(a_2^{1/4}-a_1^{1/4})(a_2^{1/4}-a_3^{1/4})}+\frac{a_3}{(a_3^{1/4}-a_1^{1/4})(a_3^{1/4}-a_2^{1/4})}$$
$$-\left[\frac{a_1^{3/4}}{(a_1^{1/4}-a_2^{1/4})(a_1^{1/4}-a_3^{1/4})}+\frac{a_2^{3/4}}{(a_2^{1/4}-a_1^{1/4})(a_2^{1/4}-a_3^{1/4})}+\frac{a_3^{3/4}}{(a_3^{1/4}-a_1^{1/4})(a_3^{1/4}-a_2^{1/4})}\right]^2$$
$$\equiv a_1^{1/2}+a_2^{1/2}+a_3^{1/2}\ , \tag{xv}$$

and

$$\frac{a_1}{(a_2a_3)^{1/3}(a_1^{1/3}-a_2^{1/3})(a_1^{1/3}-a_3^{1/3})}+\frac{a_2}{(a_1a_3)^{1/3}(a_2^{1/3}-a_1^{1/3})(a_2^{1/3}-a_3^{1/3})}$$
$$+\frac{a_3}{(a_1a_3)^{1/3}(a_3^{1/3}-a_1^{1/3})(a_3^{1/3}-a_2^{1/3})}-\left(\frac{a_1}{a_2a_3}\right)^{1/3}-\left(\frac{a_2}{a_1a_3}\right)^{1/3}-\left(\frac{a_3}{a_1a_2}\right)^{1/3}$$
$$\equiv a_1^{-1/3}+a_2^{-1/3}+a_3^{-1/3}. \tag{xvi}$$

These can be equated to the previous identities involving the right side to yield several other identities. Comparison of the variances for a general k leads for instance to the following identities:

$$\sum_{j=1}^{k}\left\{\left[\prod_{l=1,l\neq j}^{k}\frac{1}{(a_j-a_l)}\right]a_j^{k+1}-a_j^2\right\}\equiv\sum_{i<j}a_ia_j\ , \tag{xvii}$$
$$\sum_{j=1}^{k}\left\{\left[\frac{1}{\prod_{l=1,l\neq j}^{k}\left(1-\frac{a_j}{a_l}\right)}-1\right]\frac{1}{a_j^2}\right\}\equiv\sum_{i<j}\frac{1}{a_ia_j}\ . \tag{xviii}$$

When some of the a_j's are repeated while the y_j's are still assumed to be mutually independently distributed, generalized partial fractions will be involved, which will produce logarithmic terms or higher order derivatives in the representation of the density of u given in (3.8). However, in the case of integer moments, no derivatives will be involved when higher order moments are evaluated from the representation given in (3.7). Then, on equating the two moment expressions coming from (3.7) and (3.8), one can again derive several algebraic identities for nonzero $a_1,\ldots,a_k$.

The classical identity given in (i) can easily be generalized. Letting $k = 2$, taking a general m, comparing the m-th integer moments obtained from (3.7) and (3.8), and then taking the reciprocals of the a_j's (still assumed to be distinct and nonzero), one has

$$\frac{a_1^{m+1} - a_2^{m+1}}{a_1 - a_2} \equiv \sum_{m_1+m_2=m} a_1^{m_1} a_2^{m_2} \ , \tag{xix}$$

that is,

$$\frac{a_1^2 - a_2^2}{a_1 - a_2} \equiv a_1 + a_2 \ ,$$
$$\frac{a_1^3 - a_2^3}{a_1 - a_2} \equiv a_1^2 + a_2^2 + a_1 a_2 \ ,$$

and so on. The corresponding identities for a general k and a general m can be obtained by taking the m-th integer moments from (3.7) and (3.8) and equating them, and then by taking the reciprocals of the a_j's. As a result, for distinct nonzero a_j's, one has

$$\sum_{j=1}^{k} \frac{a_j^{m+k-1}}{(a_j - a_1)(a_j - a_2)\cdots(a_j - a_{j-1})(a_j - a_{j+1})\cdots(a_j - a_k)} \equiv \sum_{m_1+\cdots+m_k=m} a_1^{m_1} \cdots a_k^{m_k} \ . \tag{xx}$$

For example, for $k = 3$ and $m = 1$, we have the identity given in (vi). For $k = 3$ and $m = 2$, one has

$$\frac{a_1^4}{(a_1 - a_2)(a_1 - a_3)} + \frac{a_2^4}{(a_2 - a_1)(a_2 - a_3)} + \frac{a_3^4}{(a_3 - a_1)(a_3 - a_2)}$$
$$\equiv a_1^2 + a_2^2 + a_3^2 + a_1 a_2 + a_1 a_3 + a_2 a_3 \ . \tag{xxi}$$

For $k = 3$ and $m = 3$, one has

$$\frac{a_1^5}{(a_1 - a_2)(a_1 - a_3)} + \frac{a_2^5}{(a_2 - a_1)(a_2 - a_3)} + \frac{a_3^5}{(a_3 - a_1)(a_3 - a_2)}$$
$$\equiv a_1^3 + a_2^3 + a_3^3 + a_1^2 a_2 + a_1^2 a_3 + a_2^2 a_1 + a_2^2 a_3 + a_3^2 a_1 + a_3^2 a_2 + a_1 a_2 a_3 \ , \tag{xxii}$$

and for a general m,

$$\frac{a_1^{m+2}}{(a_1 - a_2)(a_1 - a_3)} + \frac{a_2^{m+2}}{(a_2 - a_1)(a_2 - a_3)} + \frac{a_3^{m+2}}{(a_3 - a_1)(a_3 - a_2)} \equiv \sum_{m_1+m_2+m_3=m} a_1^{m_1} a_2^{m_2} a_3^{m_3} \ .$$