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Short Communication

Aging property of unimodal failure rate models

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Abstract

There is a family of statistical models, whose failure rate is unimodal or reverse bathtub shaped. For the situations where the failure is mainly caused by fatigue or corrosion, the time to failure is often represented by such models. This paper studies the aging property of such a class of models. It shows that a unimodal failure rate can be effectively viewed as increasing, decreasing, or constant, depending on the model parameters. Relevant quantitative measures are developed to identify to which case a given unimodal failure rate model belongs. Such two models, lognormal and inverse Weibull distributions, are analyzed.

Keywords: Failure rate; Reverse bathtub curve; Average aging intensity; Relative peakedness

1. Introduction

Aging is an inherent property of products or systems. It has a significant impact on effect of a burn-in or maintenance strategy. For example, a burn-in procedure is often used to weed out the infant mortalities characterized by a decreasing failure rate; a preventive maintenance can reduce the wear-out failures characterized by an increasing failure rate; and neither burn-in nor preventive maintenance is effective for a non-aging system characterized by a constant failure rate. Therefore, it is of great significance to specify the aging property of a system.

Aging property is usually represented in terms of the failure rate as follows:

- A system is called aging or positive aging if its failure rate is increasing;
- The system is non-aging if the failure rate is constant;
- The system is anti-aging or negative aging if the failure rate is decreasing.

The failure pattern of many products/systems can be represented by the bathtub curve. It includes three phases: early failure phase with a decreasing failure rate, normal use phase with an approximately constant failure rate, and wear-out phase with an increasing failure rate.

However, not all products follow the bathtub failure pattern. For example, the failure times often follow the models with unimodal or reverse bathtub-shaped failure rate when failures are caused by fatigue or corrosion. Such models include lognormal, inverted lognormal, inverse Gaussian, inverted normal, Birnbaum-Saunders, inverse Weibull, and inverted Gamma distributions [2,3,8–11].

As a result, an issue to be explored is how to determine the aging property of a given unimodal failure rate model. Such a study is particularly useful for judging the effectiveness of a burn-in or preventive maintenance procedure for a specific product/system whose lifetime can be represented by a unimodal failure rate model.

Although there is no general and systematic approach available in the literature, some researches can be thought relevant to this issue. We briefly review them as follows.

Goldthwaite [5] studies the characteristics of the lognormal distribution. He notes that the failure rate is essentially a decreasing function when σ is large and the lognormal distribution with $\sigma = 1$ is close to a straight line in the exponential probability paper implying that the failure rate is close to a constant.

Jensen and Petersen [6] suggest a modified bathtub curve where the failure rate corresponding to the early phase is unimodal rather than decreasing.

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Chang [1] deals with the optimal burn-in problem when the failure rate of product is unimodal. He discusses the feasible range of parameters for conducting an effective burn-in, and the lognormal distribution is used as an example.

This paper presents a general theoretic framework to solve the above-mentioned problem. It includes three main components:

(a) *A classification scheme.* Under this scheme, any unimodal failure rate can be viewed as one of the following three cases: quasi-decreasing, quasi-constant, and quasi-increasing.
(b) *Aging tendency analysis.* It reveals the relationship between the aging property and the model parameters.
(c) *Two quantitative measures.* They are the critical values of the model parameters, indicating the partitions between different aging features.

As a result, given a specific unimodal failure rate model, its aging property can be completely specified.

The paper is organized as follows. Section 2 presents the classification scheme. Section 3 deals with the aging tendency analysis. Section 4 presents two quantitative measures. They are then adopted to two specific models. Finally, we conclude the paper with a brief summary in Section 5.

2. Classification

Denote $F(t), f(t)$, and $R(t) = 1 - F(t)$ as the cdf, pdf, and reliability function of the time to failure, respectively. The failure (or hazard) rate function is given by:

$$r(t) = f(t)/R(t). \tag{1}$$

We confine that $r(t)$ is unimodal in this paper. Denote its mode t_c, which is usually called as the critical time in the literature. As a result, the failure rate function consists of two parts: an increasing part over the interval $(0, t_c)$, and a decreasing part over the interval (t_c, ∞).

Noting that (a) the approximation of Goldthwaite [5], (b) the failure rate has a finite maximum, and (c) the mode t_c can be large or small relative to the mean depending on the model parameter(s), the following classification scheme is intuitively reasonable.

A unimodal failure rate can be effectively viewed as one of the following three cases:

(a) *Approximately decreasing or quasi-decreasing case.* A given failure rate will fall into this case if t_c is small. As a result, the system can be effectively viewed as anti-aging.
(b) *Approximately increasing or quasi-increasing case.* A given failure rate will fall into this case if t_c is large. And the system can be effectively viewed as aging.
(c) *Approximately constant or quasi-constant case.* A given failure rate will fall into this case if t_c is in-between or/and the failure rate curve is relatively flat. And the system can be effectively viewed as non-aging.

The above representation is qualitative. The rest of the paper will focus on development of quantitative measures. It will deal with two aspects. The first one is to build a relationship between the aging property and the model parameters, just like the one between the aging property and the Weibull shape parameter (β) for the Weibull distribution, where $\beta < 1$ implies anti-aging, $\beta = 1$ implies non-aging, and $\beta > 1$ implies positive aging. The other is to find the critical values of the parameters, which are the partitions from one aging feature to another aging feature. For example, $\beta = 1$ is such a value for the Weibull distribution.

3. Aging tendency analysis

Ji and Jiang [7] develop a new notion called *aging intensity* to quantitatively evaluate the aging property. The aging intensity is defined as the ratio of the instantaneous failure rate and a baseline failure rate. The failure rate average is used as the baseline failure rate given by

$$\frac{H(t)}{t} = \frac{1}{t}\int_0^t r(t)\mathrm{d}t, \tag{2}$$

where $H(t)$ is the cumulative failure function (or hazard function). As such, the aging intensity function $L(t)$ is given by:

$$L(t) = \frac{r(t)}{H(t)/t} = \frac{tr(t)}{H(t)} = \frac{tf(t)}{-R(t)\ln[R(t)]}, \quad t > 0. \tag{3}$$

The aging intensity $L(t) = 1$ if the failure rate is a constant, $L(t) > 1$ if the failure rate is increasing, and $L(t) < 1$ if the failure rate is decreasing. The larger [smaller] $L(t)$ is, the stronger the tendency of aging [anti-aging] is.

Interestingly, $L(t) = \beta$ for the two-parameter Weibull distribution. However, $L(t)$ is usually a function of t, and hence an average measure will be more useful for our purpose. As such, the *average aging intensity* is defined as

$$L_0(\theta) = \int_0^\infty L(t)f(t)\mathrm{d}t, \tag{4}$$

where θ expresses the model parameters. Clearly, $L_0(\theta)$ is a weighted average of $L(t)$ with the density function as the weight function.

The aging intensity of several well-known models has been analyzed in Ref. [7]. Among them, the lognormal and inverse Weibull distributions are unimodal failure rate models. Their cdfs are given, respectively, by

$$F(t) = \Phi\left(\frac{\ln(t) - \mu}{\sigma}\right), \qquad F(t) = \exp(-(\alpha/t)^\beta), \tag{5}$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution. Eq. (4) yields the average aging intensity

$$L_0(\sigma) = 1.2434/\sigma \tag{6}$$

for the lognormal distribution, and

$$L_0(\beta) = 0.7218\beta \tag{7}$$

for the inverse Weibull distribution. As can be seen from Eq. (6) or (7), a small σ or a large β implies a strong aging tendency.

In summary, the aging tendency analysis is to derive the average aging intensity, a relation like Eq. (6) or (7).

4. Quantitative measures

Suppose that there exists an interval $[\theta_L, \theta_U]$ for the model parameters θ, where the failure rate is quasi-constant. As such, our task is to find values of θ_L and θ_U, i.e. critical values. Once this is done, for given model parameters θ, one can easily specify the aging property based on the outcome of the aging tendency analysis and these critical values. We determine the critical values based on the average aging intensity and another measure, relative peakedness. We discuss them in details as below.

4.1. Measure based on the aging intensity

According to the notion of the average aging intensity, the failure rate can be viewed as quasi-constant if the average aging intensity is close to 1. Denote θ_1 as the solution of Eq. (8):

$$L_0(\theta) = 1. \tag{8}$$

Then the quasi-constant approximation is appropriate if:

$$\theta \approx \theta_1. \tag{9}$$

4.2. Measure based on relative peakedness

Denote the maximum failure rate as r_m, i.e. $r_m = r(t_c)$. Define an equivalent exponential distribution, which has an identical mean time to failure μ_0 with the unimodal failure rate model under consideration, i.e. the failure rate of the equivalent exponential distribution is $(1/\mu_0)$. As a result, the relative magnitude of the maximum failure rate can be defined as the ratio of r_m and $(1/\mu_0)$:

$$P(\theta) = \frac{r_m}{1/\mu_0} = \mu_0 r_m. \tag{10}$$

We call $P(\theta)$ the relative peakedness of the failure rate function. It is always larger than unity. The smaller it is, the closer to a constant failure rate the unimodal failure rate is.

Clearly, the failure rate can be viewed as quasi-constant if the relative peakedness is small. Let θ_2 satisfy:

$$P(\theta_2) = \min(P(\theta)). \tag{11}$$

Then the quasi-constant approximation is appropriate if:

$$\theta \approx \theta_2,\ P(\theta_2) \approx 1. \tag{12}$$

4.3. Summary

Let

$$\theta_L = \min(\theta_1, \theta_2), \qquad \theta_U = \max(\theta_1, \theta_2). \tag{13}$$

Then the failure rate is quasi-constant if $\theta \in [\theta_L, \theta_U]$, otherwise quasi-increasing or quasi-decreasing, which can be easily judged based on the average aging intensity.

4.4. Examples

To illustrate the above results, we look at two specific models and present a simple numerical example.

4.4.1. Lognormal distribution

For this model, the aging property (positive aging, negative aging, or non-aging) only depends on the parameter σ, i.e. $\theta = \sigma$.

From Eq. (6), we have $\theta_1 = \sigma_1 = 1.2434$. From Eq. (11), we have $\theta_2 = \sigma_2 = 0.8894$, and $P(\theta_2) = 1.3471$, which is close to 1.

As a result, for a given value of σ, the failure rate can be viewed as quasi-constant if $\sigma \approx 0.8894$ or 1.2434, or $\sigma \in (0.8894, 1.2434)$. Otherwise, it is quasi-increasing for a small value of σ, or quasi-decreasing for a large value of σ.

According to the above conclusion, the failure rate of the lognormal distribution with $\sigma = 1$ can be effectively viewed as quasi-constant since $1 \in (0.8894, 1.2434)$. This is consistent with the observation of Goldthwaite [5] as mentioned earlier.

4.4.2. Inverse Weibull distribution

For this model, the aging property only depends on the shape parameter β, i.e. $\theta = \beta$. From Eq. (7), we have $\theta_1 = \beta_1 = 1.3854$. From Eq. (11), we have $\theta_2 = \beta_2 = 1.9379$, and $P(\theta_2) = 2.0713$.

As a result, for a given value of β, the failure rate can be viewed as quasi-constant if $\beta \in (1.3854, 1.9379)$. Otherwise, it is quasi-decreasing for a small value of β, or quasi-increasing for a large value of β.

However, it is noted:

(a) β must be larger than 2 to make the mean and variance of the inverse Weibull distribution finite [8].
(b) The value of $P(\theta_2)$ for the inverse Weibull model is much larger than that for the lognormal model.

These imply that the failure rate for the inverse Weibull model should be generally viewed as quasi-increasing. In other words, it is possible for a certain unimodal failure rate model not to have all the three cases defined in Section 2.

Table 1
Bearing lifetime (millions of revolutions) from Ref. [4]

17.88	28.92	33.00	41.52	42.12	45.60
48.48	51.84	51.96	54.12	55.56	67.80
68.64	68.64	68.88	84.12	93.12	98.64
105.12	105.84	127.92	128.04	173.40	

4.4.3. A numerical example

The data, lifetimes of 23 ball bearings, are from Dumonceaux and Antle [4] and is reproduced in Table 1.

Dumonceaux and Antle used the lognormal distribution to fit the data. The maximum likelihood estimates of the model parameters are $\hat{\mu} = 4.15$, $\hat{\sigma} = 0.522$.

Since $\hat{\sigma} \ll 0.8894$, the failure rate can be effectively viewed as quasi-increasing, and hence the unit can be effectively viewed as positive aging.

5. Conclusion

In this paper, we have presented a general method to specify the aging property of a unimodal failure rate model. Recognition criteria have been derived based on the notions of average aging intensity and relative peakedness of failure rate. As a result, depending on the model parameter(s), a unimodal failure rate can be effectively viewed as quasi-constant, quasi-increasing, or quasi-decreasing.

We have examined two specific models, lognormal and inverse Weibull distributions. A similar study can be carried out for the other unimodal failure rate models. Also, the basic idea presented in this paper can be used to study the aging property of other non-monotonic failure rate models.

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