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The total time on test transform and the decreasing percentile residual life aging notion

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A B S T R A C T

Recently Nair and Sankaran (2013) listed some known characterizations of common aging notions in terms of the total time on test transform (TTT) function. They also derived some new characterizations. The purpose of this note is to add two characterizations of the decreasing percentile residual life of order α (DPRL(α)) aging notion in terms of the TTT function, and in terms of the observed TTT when X is observed.

1. Introduction

Let F be a distribution function of a nonnegative random variable X, and let $Q = F^{-1}$ be the corresponding (left-continuous) *quantile function*, defined by

$$Q(u) = \inf\{x \,|\, F(x) \geq u\}, \quad 0 \leq u \leq 1.$$

Denote the survival function of X by $\overline{F} \equiv 1 - F$. The *total time on test transform* (TTT) *function*, T, is defined as

$$T(u) = \int_0^{Q(u)} \overline{F}(x)\,dx, \quad 0 \leq u \leq 1. \tag{1.1}$$

Note that $T(1) = E[X]$, where the expectation $E[X]$ can be finite or infinite.

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The concept of the TTT transform is useful in reliability theory. A recent paper that describes some of its applications, and that contains a list of some basic references, is the paper by Nair and Sankaran [11]. An application of the TTT transform in actuarial science is described in Li and Shaked [9, Remark 2.4].

Let u_X be the right endpoint of the support of X; u_X may be finite or infinite. For any $s < u_X$, the *residual life* at time s, that is associated with X, is any random variable that has the conditional distribution of $X - s$ given that $X > s$. We denote it by

$$X_s = [X - s \big| X > s], \quad s \in [0, u_X).$$

The *α-percentile residual life function* P_α, where α is some number between 0 and 1, is defined by

$$P_\alpha(s) = \begin{cases} F_{X_s}^{-1}(\alpha), & t \in [0, u_X); \\ 0, & s \geq u_X. \end{cases}$$

It is useful to note (see Franco-Pereira, Lillo, and Shaked [3]) that

$$P_\alpha(s) = Q(\alpha + (1-\alpha)F(s)) - s, \quad s \in [0, u_X). \tag{1.2}$$

For $\alpha \in (0, 1)$, the random variable X is said to be DPRL(α) if

$$P_\alpha(s) \text{ is decreasing in } s \in (0, \infty) \tag{1.3}$$

(here, and in the rest of this paper, "decreasing" and "increasing" are used in the non-strict sense). A basic paper on the DPRL(α) notion is Haines and Singpurwalla [5]. The DPRL(α) aging notion was further studied in more depth in Franco-Pereira, Lillo, and Shaked [3] and in references therein.

Recently Nair and Sankaran [11] listed some known characterizations of common aging notions in terms of the TTT function. They also derived some new characterizations. The purpose of this note is to add two characterizations of the DPRL(α) aging notion in terms of the TTT function, and in terms of the *observed* TTT *when X is observed* (the latter will be formally defined in Section 3).

2. A characterization in terms of the TTT density

In this section we assume that Q is differentiable almost everywhere on $[0, 1)$; that is, that F has at most a countable number of "flats".

The *quantile density* q of X is defined, almost everywhere on $[0, 1)$, by

$$q(u) = \frac{d}{du} Q(u).$$

The TTT *density* t of X is defined, almost everywhere on $[0, 1)$, by

$$t(u) = \frac{d}{du} T(u). \tag{2.1}$$

An expression of the quantile function Q in terms of the TTT density t (that will be used in the sequel) is given in the following proposition. The relation (2.3) in the following proposition is a special case of a result in Nair, Sankaran, and Vineshkumar [12, page 1130], whereas the relation (2.2) can be easily derived from (2.3).

Proposition 2.1. *Suppose that Q is differentiable almost everywhere on* $[0, 1)$. *Then*

$$Q(u) = \int_0^u \frac{t(v)}{1-v}\, dv \tag{2.2}$$

almost everywhere on $[0, 1)$*, and*

$$q(u) = \frac{t(u)}{1-u} \tag{2.3}$$

almost everywhere on $[0, 1)$.

The main observation in this section is the following result.

Theorem 2.2. *Suppose that F is continuous, and that Q is differentiable almost everywhere on* $[0, 1)$. *Let* $\alpha \in (0, 1)$. *Then X is DPRL(α) if, and only if,*

$$t(\alpha + (1-\alpha)u) \le t(u) \tag{2.4}$$

almost everywhere on $[0, 1)$.

Proof. Combining (1.2) and (1.3) we see that X is DPRL(α) if, and only if,

$$Q(\alpha + (1-\alpha)F(s)) - s \text{ is decreasing in } s \in (0, \infty);$$

that is, if, and only if,

$$Q(\alpha + (1-\alpha)u) - Q(u) \text{ is decreasing in } u \in (0, 1);$$

that is, if, and only if,

$$(1-\alpha)q(\alpha + (1-\alpha)u) \le q(u)$$

almost everywhere on $[0, 1)$. Taking (2.3) into account we see that X is DPRL(α) if, and only if,

$$(1-\alpha)\frac{t(\alpha + (1-\alpha)u)}{1-\alpha-(1-\alpha)u} \le \frac{t(u)}{1-u}$$

almost everywhere on $[0, 1)$; that is, if, and only if,

$$(1-\alpha)\frac{t(\alpha + (1-\alpha)u)}{(1-\alpha)(1-u)} \le \frac{t(u)}{1-u}$$

almost everywhere on $[0, 1)$; that is, if, and only if,

$$t(\alpha + (1-\alpha)u) \le t(u)$$

almost everywhere on $[0, 1)$. $\square$

A reviewer pointed out that Theorem 2.2 can also be obtained as a direct consequence of Proposition 3.5 in Nair and Vineshkumar [13] just using the relation (2.3) between $q(u)$ and $t(u)$.

Another proof of Theorem 2.2, that uses an expression of t in terms of the hazard rate function from Barlow and Campo [1], and Proposition 2.1 in Franco-Pereira, Lillo, and Shaked [3], can also be constructed. We do not give the details of this alternative proof here.

Remark 2.3. In Barlow and Campo [1] it is shown that X is IFR (i.e., has increasing failure rate) if, and only if, T is concave. Obviously, if T is concave (that is, t is decreasing) then (2.4) holds. Thus we see from Theorem 2.2 that

$$\text{IFR} \Longrightarrow \text{DPRL}(\alpha) \quad \text{for any } \alpha \in (0, 1).$$

This yields a new proof of an observation that was made at the beginning of Section 3 in [3]. ◂

The following two examples illustrate the usefulness of Theorem 2.2.

Example 2.4. Let X be a random variable with distribution function F, survival function $\overline{F} = 1 - F$, density function f, and a step hazard rate function

$$r(x) = \frac{f(x)}{\overline{F}(x)} = \begin{cases} 1, & 0 < x \le 1; \\ \frac{1}{2}, & 1 < x \le 2; \\ 1, & x > 2. \end{cases}$$

Obviously, X is not IFR. A straightforward computation yields

$$F(x) = \begin{cases} 1 - e^{-x}, & 0 < x \leq 1; \\ 1 - e^{-\frac{x+1}{2}}, & 1 < x \leq 2; \\ 1 - e^{-(x-\frac{1}{2})}, & x > 2. \end{cases} \tag{2.5}$$

Carefully inverting F in the three different regions we obtain

$$Q(u) = \begin{cases} -\log(1-u), & 0 < u \leq 1 - e^{-1}; \\ -1 - 2\log(1-u), & 1 - e^{-1} < u \leq 1 - e^{-\frac{3}{2}}; \\ \dfrac{1}{2} - \log(1-u), & 1 - e^{-\frac{3}{2}} < u < 1. \end{cases}$$

Differentiating Q we get

$$q(u) = \begin{cases} \dfrac{1}{1-u}, & 0 < u \leq 1 - e^{-1}; \\ \dfrac{2}{1-u}, & 1 - e^{-1} < u \leq 1 - e^{-\frac{3}{2}}; \\ \dfrac{1}{1-u}, & 1 - e^{-\frac{3}{2}} < u < 1. \end{cases}$$

So, using (2.3), we have

$$t(u) = (1-u)q(u) = \begin{cases} 1, & 0 < u \leq 1 - e^{-1}; \\ 2, & 1 - e^{-1} < u \leq 1 - e^{-\frac{3}{2}}; \\ 1, & 1 - e^{-\frac{3}{2}} < u < 1. \end{cases} \tag{2.6}$$

Now, let α be a constant in $(1 - e^{-\frac{3}{2}}, 1)$. Then $\alpha + (1-\alpha)u \geq 1 - e^{-\frac{3}{2}}$ for every $u \in (0, 1)$. Therefore,

$$t(\alpha + (1-\alpha)u) = 1 \leq t(u), \quad \text{for all } u \in (0, 1), \tag{2.7}$$

where the inequality plainly follows from (2.6). From Theorem 2.2 and (2.7) we see that

$$X \text{ is DPRL}(\alpha) \text{ for every } \alpha \in (1 - e^{-\frac{3}{2}}, 1). \tag{2.8}$$

It is worthwhile to note that other ways to derive (2.8) are by using a remark in Joe and Proschan [7, page 672], or by verifying (1.3), or by verifying one of the conditions in Proposition 2.1 in Franco-Pereira, Lillo, and Shaked [3]. However, the derivation above demonstrates the simplicity of using Theorem 2.2. An alternative simple way of deriving (2.8) is described in Example 3.7. ★

Example 2.5. Fix an $\varepsilon \in (0, 1)$ and take $a = (-\log \varepsilon)^{\frac{1}{2}}$. Let X be a random variable with distribution function F, survival function $\overline{F} = 1 - F$, density function f, and a hazard rate function

$$r(x) = \frac{f(x)}{\overline{F}(x)} = \begin{cases} a - x, & 0 < x \leq a; \\ x - a, & x \geq a. \end{cases}$$

Note that X is not IFR. A straightforward computation yields

$$F(x) = \begin{cases} 1 - \exp\{-ax + x^2/2\}, & 0 < x \leq a; \\ 1 - \exp\{-a^2/2 - (x-a)^2/2\}, & x > a. \end{cases}$$

Inverting F in the two different regions we obtain

$$Q(u) = \begin{cases} a - \sqrt{a^2 + 2\log(1-u)}, & 0 < u \leq 1 - \exp\{-a^2/2\}; \\ a + \sqrt{-a^2 - 2\log(1-u)}, & 1 - \exp\{-a^2/2\} < u < 1. \end{cases}$$

Differentiating Q we get

$$q(u) = \begin{cases} \dfrac{1}{(1-u)\sqrt{a^2+2\log(1-u)}}, & 0 < u < 1-\exp\{-a^2/2\}; \\ \dfrac{1}{(1-u)\sqrt{-a^2-2\log(1-u)}}, & 1-\exp\{-a^2/2\} < u < 1. \end{cases}$$

So, using (2.3), we have

$$t(u) = (1-u)q(u) = \begin{cases} \dfrac{1}{\sqrt{a^2+2\log(1-u)}}, & 0 < u < 1-\exp\{-a^2/2\}; \\ \dfrac{1}{\sqrt{-a^2-2\log(1-u)}}, & 1-\exp\{-a^2/2\} < u < 1. \end{cases}$$

Now, let α be a constant in $[1-\varepsilon, 1) = [1-\exp\{-a^2\}, 1)$. With a careful analysis that we do not detail here, considering the cases $u > 1-\exp\{-a^2\}$ and $u \le 1-\exp\{-a^2\}$, and taking into account that $\alpha \ge 1-\exp\{-a^2\}$, it is possible to show that

$$t(\alpha + (1-\alpha)u) \le t(u), \quad \text{for all } u \in (0, 1). \tag{2.9}$$

From Theorem 2.2 and (2.9) we see that

$$X \text{ is DPRL}(\alpha) \text{ for every } \alpha \ge 1-\varepsilon. \tag{2.10}$$

It is worthwhile to note that other ways to derive (2.10) are by using a remark in Joe and Proschan [7, page 672], or by verifying (1.3), or by verifying one of the conditions in Proposition 2.1 in Franco-Pereira, Lillo, and Shaked [3]. Actually, this example was presented, with a different derivation, in Franco-Pereira, Lillo, and Shaked [3]. But there is an unfortunate mistype there, where it is said that X is DPRL(α) for every $\alpha \ge \varepsilon$, rather than that X is DPRL(α) for every $\alpha \ge 1-\varepsilon$ as in (2.10). ★

3. A characterization in terms of the observed TTT

In this section we assume that F, the distribution function of X, is absolutely continuous with density function f.

Barlow and Campo [1] noted that the TTT function T in (1.1) is increasing on (0, 1), and as such, it is the inverse of a distribution function H of a random variable with support in $(0, T(1)) = (0, E[X])$, where the mean $E[X]$ can be finite or infinite. Li and Shaked [9] studied this distribution function, and denoted by X_{ttt} the random variable (with support in $(0, E[X])$) that has the distribution function H. That random variable, X_{ttt}, is defined by

$$X_{\text{ttt}} = T(F(X));$$

it literally measures the *observed total time on test* when X is observed. As was indicated above, the distribution function H of X_{ttt} is given by

$$P\{X_{\text{ttt}} \le y\} \equiv H(y) = T^{-1}(y), \quad y \in (0, E[X]);$$

the corresponding density function $h = H'$ is given by

$$h(y) = \frac{1}{t(T^{-1}(y))}, \quad y \in (0, E[X]), \tag{3.1}$$

where $t = T'$ is the TTT density defined in (2.1).

The random variable X_{ttt} has some useful applications in reliability theory; see Li and Shaked [9]. For example, X_{ttt} arises in the planning of lifetime experiments. Specifically, consider an item, with lifetime X, that in a pilot study determines the u (as $u = F(X)$) in (1.1), according to which a subsequent lifetime test, of similar items, is later terminated. The random variable X_{ttt}, then, is the theoretical version of the empirical TTT transform that is observed in that subsequent lifetime test. Therefore, properties of the distribution and the density functions of X_{ttt} are of interest, and have potential practical applications.

In the following theorems we characterize IFR and DPRL(α) random variables in terms of the distribution and the density functions, H and h, of X_{ttt}. One consequence of the following theorems is that they show how the knowledge, that a random lifetime is IFR or DPRL(α), yields information about the shape of H and h.

Theorem 3.1. *The absolutely continuous random variable X is IFR if, and only if, X_{ttt} has an increasing density function on $(0, E[X])$; that is, if, and only if, X_{ttt} has a distribution function that is convex on $(0, E[X])$.*

Proof. Let $r = f/(1-F)$ be the hazard rate function of X. In [1] it is noticed that

$$t(u) = \frac{1}{r(Q(u))}, \quad u \in (0, 1). \tag{3.2}$$

Combining (3.1) and (3.2) we have

$$h(y) = \frac{1}{t(T^{-1}(y))} = r(Q(T^{-1}(y))), \quad y \in (0, E[X]).$$

The functions Q and T^{-1} are increasing. Therefore h is increasing on $(0, E[X])$ if, and only if, r is increasing. □

A reviewer pointed out that Theorem 3.1 can also be easily derived using the idea that $T(u)$ is always a quantile function. One can refer to Nair, Sankaran, and Vineshkumar [12, page 1128] for details.

Note in Theorem 3.1, that in order for the density function of X_{ttt} to be increasing on $(0, E[X])$, it is necessary that $E[X]$ is finite. That is, Theorem 3.1 provides a new proof for the known fact (see, for instance, Joe and Proschan [7]) that an IFR random variable must have a finite mean.

Remark 3.2. It is worthwhile to mention that a straightforward modification of the proof of Theorem 3.1 shows that a random variable X is DFR (i.e., has decreasing failure rate) if, and only if, X_{ttt} has a decreasing density function on $(0, E[X])$; that is, if, and only if, X_{ttt} has a distribution function that is concave on $(0, E[X])$. Note that here it is not required that $E[X]$ be finite. ◀

Remark 3.3. There are various instances in the literature where the concavity and the convexity of a distribution function are studied; see Sengupta and Nanda [14] and references therein. For instance, Sengupta and Nanda [14] studied in detail concave distributions. They derived some inequalities, useful in reliability theory, that such distributions satisfy.

There are other instances in the literature where a monotonicity of the density function of a random variable yields valuable results. For example, when a distribution function is known to be concave (or convex), but otherwise nothing else is assumed about it, then a nonparametric asymptotically minimax estimator of it exists. A development of such an estimate was initiated in Grenander [4], and further progressed in Kiefer and Wolfowitz [8], in Shao [16], and in Carolan [2]. Bayesian inference for concave distribution functions is investigated in Hansen and Lauritzen [6].

A different type of application of Theorem 3.1 is as follows. Suppose that X in Theorem 3.1 is IFR. Then X_{ttt} has a decreasing density function. Now, from a result of Joag-Dev (that is given as Theorem 1.A.22 in Shaked and Shanthikumar [15]), we see that the normalized spacings, that are based on a sample of n independent copies of X_{ttt}, are ordered then with respect to the ordinary stochastic order $\leq_{\mathrm{st}}$ (see its definition, for instance, in Müller and Stoyan [10] or Shaked and Shanthikumar [15]). ◀

In the subsequent example we show an interesting application of Theorem 3.1 and Remark 3.2.

Example 3.4. Let X be a Weibull random variable with the survival function

$$\overline{F}(x) = \exp\{-x^{\theta}\}, \quad x \geq 0,$$

where $\theta > 0$ is a parameter. The mean of X is

$$E[X] = \frac{\Gamma(1/\theta)}{\theta} = \Gamma\Big(\frac{1}{\theta} + 1\Big).$$

Note that it is not easy to compute the density function of the corresponding X_{ttt}.

If $\theta \geq 1$ then X is IFR. From Theorem 3.1 it then follows that the density function of X_{ttt} is increasing on $(0, E[X])$. This is a useful information on the density function of X_{ttt} (see Remark 3.3), especially when it is not easy to explicitly compute it.

If $\theta \leq 1$ then X is DFR. From Remark 3.2 it then follows that the density function of X_{ttt} is decreasing on $(0, E[X])$. Again, this monotonicity can be useful as is described in Remark 3.3. ★

For the following result, recall that H and h denote the distribution and the density functions of X_{ttt}.

Theorem 3.5. *Let $\alpha \in (0, 1)$. Then the absolutely continuous random variable X is DPRL(α) if, and only if,*

$$h(y) \leq h(H^{-1}(\alpha + (1-\alpha)H(y))), \quad y \in (0, E[X]). \tag{3.3}$$

Proof. Rewriting (3.1) we see that

$$t(u) = \frac{1}{h(T(u))}, \quad u \in (0, 1).$$

Therefore, from Theorem 2.2 we see that X is DPRL(α) if, and only if,

$$h(T(u)) \leq h(T(\alpha + (1-\alpha)u)), \quad u \in (0, 1);$$

that is, if, and only if,

$$h(y) \leq h(T(\alpha + (1-\alpha)T^{-1}(y))), \quad y \in (0, E[X]);$$

that is, if, and only if,

$$h(y) \leq h(H^{-1}(\alpha + (1-\alpha)H(y))), \quad y \in (0, E[X]). \quad \square$$

A reviewer mentioned that Theorem 3.5 is essentially a part of Proposition 3.5 of Nair and Vineshkumar [13]. The only difference is that here we use the functions h and H, whereas [13] employs the hazard quantile function (which is defined as the denominator of (3.2) in the proof of Theorem 3.1).

Remark 3.6. Since H is increasing and $H(E[X]) = 1$, we see that $y \leq H^{-1}(\alpha + (1-\alpha)H(y))$ for every $y \in (0, E[X])$. Hence, if h is increasing then (3.3) holds. By Theorem 3.1, h is increasing if, and only if, X is IFR. So Theorems 3.1 and 3.5 show that

$$\text{IFR} \Longrightarrow \text{DPRL}(\alpha) \quad \text{for any } \alpha \in (0, 1).$$

This yields, in addition to Remark 2.3, a new proof of an observation that was made at the beginning of Section 3 in [3]. ◂

We end this section with an example that illustrates the computations, and the derivation of the DPRL(α) property, in Theorem 3.5.

Example 3.7. Let X be as in Example 2.4, that is, let the distribution function of X be given in (2.5). A straightforward computation gives

$$E[X] = 1 + e^{-1} - e^{-3/2}.$$

Since $T(u) = \int_0^u t(s)\,ds$, we compute from (2.6)

$$T(u) = \begin{cases} u, & 0 < u \leq 1 - e^{-1}; \\ 2u + e^{-1} - 1, & 1 - e^{-1} < u \leq 1 - e^{-3/2}; \\ u + e^{-1} - e^{-3/2}, & 1 - e^{-3/2} < u \leq 1. \end{cases} \tag{3.4}$$

Inverting T in the three different regions we get the distribution function H of X_{ttt}

$$H(y) = T^{-1}(y) = \begin{cases} y, & 0 < y \le 1 - e^{-1}; \\ \dfrac{y + 1 - e^{-1}}{2}, & 1 - e^{-1} < y \le 1 + e^{-1} - 2e^{-3/2}; \\ y + e^{-3/2} - e^{-1}, & 1 + e^{-1} - 2e^{-3/2} < y \le E[X]. \end{cases}$$

Differentiating H in the three different regions we get the density function h of X_{ttt}

$$h(y) = \frac{d}{dy} H(y) = \begin{cases} 1, & 0 < y \le 1 - e^{-1}; \\ \dfrac{1}{2}, & 1 - e^{-1} < y \le 1 + e^{-1} - 2e^{-3/2}; \\ 1, & 1 + e^{-1} - 2e^{-3/2} < y \le E[X]. \end{cases} \tag{3.5}$$

Now, let α be a constant in $[1 - e^{-3/2}, 1)$. Let us examine the right hand side of (3.3). Since $\alpha \ge 1 - e^{-3/2}$, it follows that the argument of H^{-1} in (3.3) satisfies

$$\alpha + (1 - \alpha) H(y) \ge 1 - e^{-3/2}, \quad y \in (0, E[X]). \tag{3.6}$$

Hence, for every $y \in (0, E[X])$ we have

$$\begin{aligned} H^{-1}(\alpha + (1 - \alpha) H(y)) &= T(\alpha + (1 - \alpha) H(y)) \\ &= \alpha + (1 - \alpha) H(y) + e^{-1} - e^{-3/2} \quad \text{(by (3.4))} \\ &\ge 1 - e^{-3/2} + e^{-1} - e^{-3/2} \quad \text{(by (3.6))} \\ &= 1 + e^{-1} - 2e^{-3/2}. \end{aligned}$$

Therefore, using the last line in (3.5), we see that

$$h(H^{-1}(\alpha + (1 - \alpha) H(y))) = 1 \ge h(y), \quad y \in (0, E[X]),$$

where the inequality follows from the fact that the only values of h are 1 and $\frac{1}{2}$. This verifies (3.3), and thus illustrates how Theorem 3.5 can be used to derive (2.8). ★

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