ELSEVIER

0951-8320(95)00149-2

Reliability Engineering and System Safety 52 (1995) 87–93

0951-8320/95/$15.00

Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function

M. Xie[a] & C. D. Lai[b]

[a] *Dept of Industrial and System Engineering, National University of Singapore, Republic of Singapore*
[b] *Department of Statistics, Massey University, Palmerston North, Palmerston North, New Zealand*

(Received 24 August 1994; accepted 8 November 1995)

Lifetime distributions for many components usually have a bathtub-shaped failure rate in practice. However, there are very few practical models to model this type of failure rate function. In this paper we study a simple model based on adding two Weibull survival functions. Some simplifications of the model are also presented. The graphical estimation technique based on the conventional Weibull plot is demonstrated to be useful in this case. © Elsevier Science Limited.

1 INTRODUCTION

For many mechanical and electronic components, the failure rate function has a bathtub shape. It is well-known that, because of design and manufacturing problems, the failure rate is high at the beginning of a product life cycle and decreases toward a constant level. After reaching a certain age, the product enters the wear-out phase and the failure rate starts to increase. Despite the fact that this phenomenon has been presented in many reliability engineering texts, few practical models possessing this property have appeared in the literature.

Because of this, only a part of the bathtub curve is considered at any one time. Another common fact is that most engineers may be interested only in a part of the lifetime, because at component level, they only see one part of the failure rate function. However, it will be helpful to have a model that is reasonably simple and good for the whole product life cycle for making overall decisions. Furthermore, for complex systems, both the decreasing and increasing parts of the failure rate fall into the ordinary product lifetime.

Although some studies have been carried out, see Refs 1–7, and most of them present new lifetime distributions that have bathtub-shaped failure rate functions, the models are not practical to be used by reliability engineers. It is our intention in this paper to study some practical models for the bathtub-shaped failure rate function. The idea is based on the conventional Weibull distribution which is widely used by reliability engineers today.

In practice, Weibull distributions have been shown to be very flexible in modelling various types of lifetime distributions and they have been used to model any of the three parts in a bathtub-curve. The two-parameter version has the following form

$$F(t) = 1 - \exp\{-(t/\alpha)^{\beta}\}, t \geq 0 \tag{1}$$

where α is called the scale parameter and β is called the shape parameter. The failure rate function that corresponds to (1) is given by

$$r(t) = (\beta/\alpha)(t/\alpha)^{\beta-1}. \tag{2}$$

It can be seen that when $\beta < 1$, the failure rate function is decreasing and when $\beta > 1$, it is increasing. In the case of $\beta = 1$, we have the well-known exponential distribution which has a constant failure rate.

Because of these interesting properties, Weibull distribution has been widely used for modelling different phases of lifetime, see Refs 8–11. However, a single Weibull model cannot be used to model all three phases of a bathtub curve at the same time.

In this paper, we study a model which is based on the simple idea of combining two Weibull distributions, which is a widely used lifetime distribution in practice. The model is an additive one in the sense that the failure rate function is expressed as the sum of two failure rate functions of Weibull form. A study using this additive model is carried out in this paper. Furthermore, we present a graphical approach for parameter estimation based on the conventional Weibull plot technique. Numerical analysis using a set of real life data is also presented.

2 THE ADDITIVE WEIBULL MODEL AND ITS INTERPRETATIONS

2.1 The additive Weibull model

The additive Weibull model we propose here combines two Weibull distributions; one has a decreasing failure rate and another has an increasing failure rate. It has the cumulative hazard function given in the following form:

$$H(t) = (at)^b + (ct)^d, t, a, c \geq 0, b > 1, d < 1. \quad (3)$$

Based on this form of cumulative hazard function, the reliability function is simply given by

$$R(t) = \exp\{-H(t)\} = \exp\{-(at)^b - (ct)^d\}, t \geq 0. \quad (4)$$

The corresponding lifetime distribution function is then given by

$$F(t) = 1 - R(t) = 1 - \exp\{-(at)^b - (ct)^d\}, t \geq 0. \quad (5)$$

and the failure rate function is

$$r(t) = \frac{\partial H(t)}{\partial t} = ab(at)^{b-1} + cd(ct)^{d-1}, t \geq 0. \quad (6)$$

2.2 Some typical bathtub curves

It is easy to see that the failure rate function in (6) has a bathtub-shape. The second term in (6) which is the dominating one for small t, is decreasing. For large t, the first term in (6) dominates and is an increasing function. The flat part of a bathtub curve then corresponds to t that is in the middle. Some plots using (6) are displayed in Fig. 1.

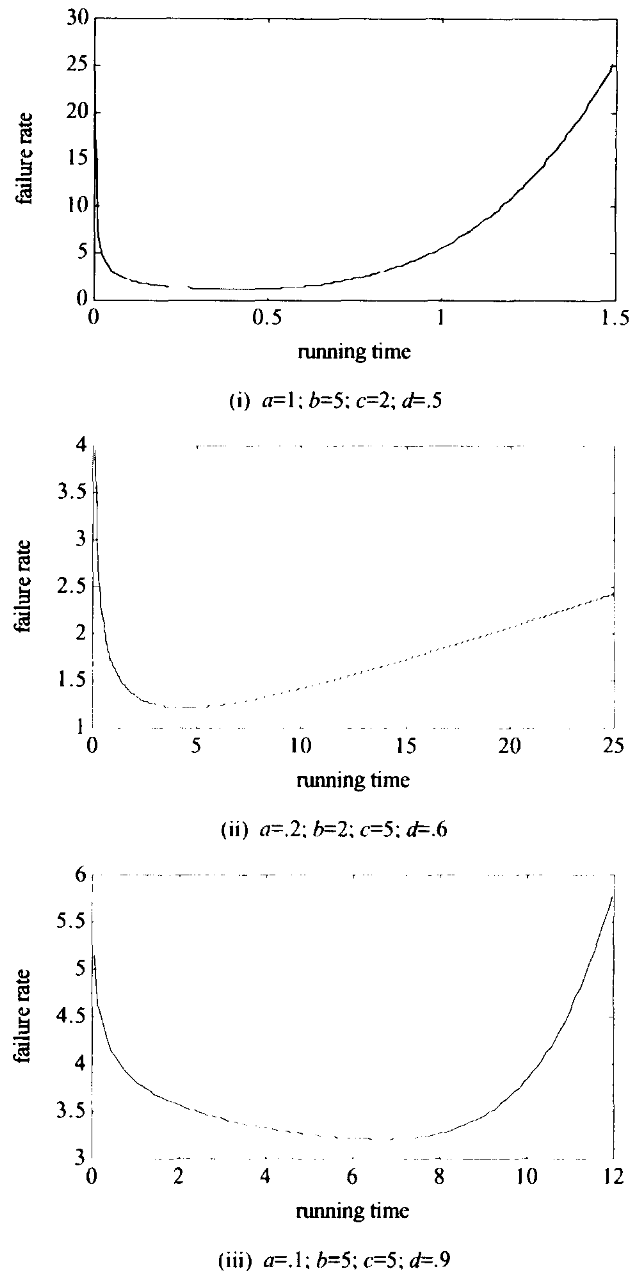

Fig. 1. Some typical bathtub-shaped failure rate functions using different values of the parameters.

2.3 Some physical interpretations of the model

The physical background of this model is clear. A component fails because of the occurrence of a failure mode and usually there are different failure modes associated with a component. Each failure mode affects the component in a different way. Suppose that the component is affected by two major failure modes, each corresponding to a Weibull distributed failure time, but with different parameters. The above model then results when one assumes that the component can fail due to either of these two failure modes.

For a component that has a bathtub-shaped failure rate, the initial failures are usually caused by design faults and initial problems, which lead to a decreasing failure rate. The last part of the bathtub shaped failure rate is usually caused by material fatigue or component aging, and this corresponds to an increasing failure rate, see, e.g., Refs 12–14. Our model incorporates both types of failures and it can be used to analyze the kind of failure data collected without knowing what types of failure have occurred. Furthermore, the constant part can be an intermediate result when the increasing part and decreasing part are of similar magnitude.

It follows from (5) that our model can also be interpreted as the lifetime of a system which consists of two independent Weibull components that are arranged in a series. Let T_1 denote the lifetime of component 1, which is Weibull with parameters a and b and let T_2 denote the lifetime of component 2 which is Weibull with parameters c and d. If the system lifetime is T, then

$$T = \mathrm{Min}(T_1, T_2) \quad (7)$$

has the distribution given by (4). The failure rate function is then given in (6).

3 ANALYTICAL STUDY OF THE ADDITIVE WEIBULL MODEL

3.1 Proof of the bathtub shape

That the failure rate curve must be bathtub-shaped can also be shown mathematically. The derivative of $r'(t)$ is given by

$$\begin{aligned} r'(t) &= a^2b(b-1)(at)^{b-2} + c^2d(d-1)(ct)^{d-2} \\ &= t^{d-2}\{a^bb(b-1)t^{b-d} - c^dd(1-d)\} \end{aligned} \quad (8)$$

and then, $r'(t) = 0$ if and only if

$$t = t_0 = \left[\frac{c^dd(1-d)}{a^bb(b-1)}\right]^{1/(b-d)}. \quad (9)$$

As $b > 1$ and $d < 1$, it is obvious that $t_0 > 0$. Also, $r'(t) > 0$ when $t > t_0$ and $r'(t) < 0$ when $t < t_0$. It is now obvious that our $r(t)$ has a bathtub shape.

At time t_0, the failure rate reaches its minimum. If burn-in is to be applied to increase field reliability, the burn-in time should not be longer than t_0. Also, after t_0, replacement should be considered because the failure rate will start increasing and at certain times, it may be too high. This problem will be studied using our additive model in Section 3.4.

3.2 Bathtub curves using a reduced model

Our proposed model has four parameters a,b,c, and d, and hence the model should be able to fit to various types of data. Conventional statistical estimation techniques can be applied here, although we will present a graphical approach later in this paper. However, the problem of parameter estimation based on statistical techniques usually becomes more complex as the number of parameters increases. Consider a special case of our general model such that $a = c$ and $d = 1/b$. When $b > 1$, and therefore $d = 1/b < 1$, we have the reduced model that retains the bathtub shape.

The reduced model has only two parameters, a and b. Its cumulative hazard function is given by

$$H_2(t) = (at)^b + (at)^{1/b}, t \geq 0, a > 0, b > 1. \quad (10)$$

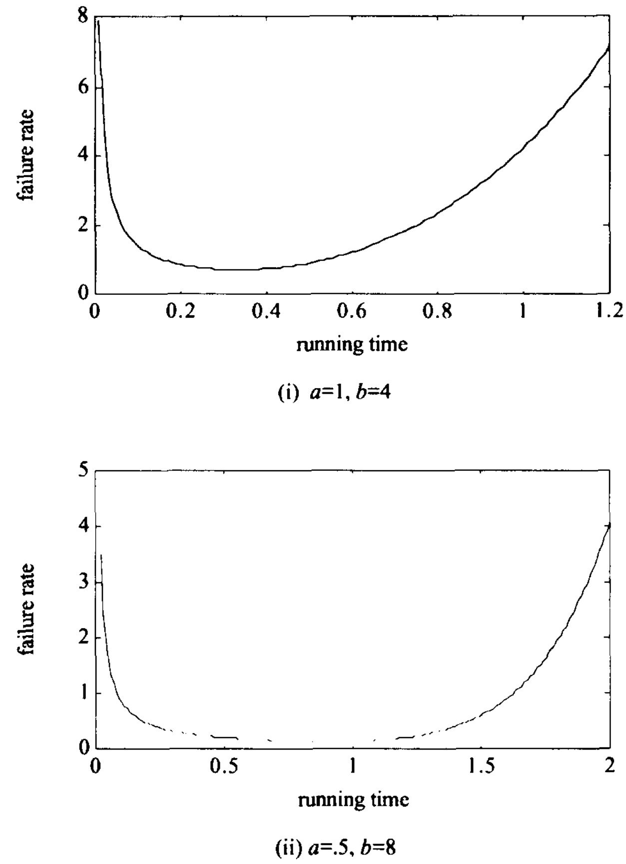

Fig. 2. Bathtub-curves using the reduced two-parameter model.

The corresponding failure rate function is

$$r_2(t) = ab(at)^{b-1} + a(at)^{1/b-1}/b, t \geq 0. \quad (11)$$

We shall now display some curves using a reduced model. In Fig. 2 two curves are plotted using different sets of parameters, and the curves indeed have bathtub shapes.

It has to be pointed out that the idea of using a simplified model is mainly educational as we can produce nice bathtub curves using the simplified model, although we do have observed cases where the simplified models above provide reasonably good results. Certainly, if it can be justified that $a = c$ or $b = 1/d$, the four-parameter model can then be reduced to a three-parameter one.

Note that the scale parameters in a Weibull distribution usually depends on product design, and it can easily be adjusted to meet a reliability requirement. The shape parameter, however, depends on the material property which is fixed given the type of the material producing the component.

3.3 The expectation of time to failure for the additive model

One of the most widely used reliability measures in practice is the expected time to failure, or the mean

Table 1. Mean and variance for the lifetime distributions in Figs 1–2

a	b	c	d	MTTF	Variance
1	5	2	0·5	0·39	0·148
0·2	2	5	0·6	0·29	0·226
0·1	5	5	0·9	0·21	0·056
1	4	1	0·25	0·42	0·194
0·5	8	0·5	0·125	0·78	0·152

time to failure (MTTF). The expected value for a nonnegative random variable with distribution $F(t)$ can be calculated as

$$\mu = \int_0^\infty t\mathrm{d}F(t) = \int_0^\infty (1 - F(t))\mathrm{d}t = \int_0^\infty R(t)\mathrm{d}t \quad (12)$$

where $R(t) = 1 - F(t)$ is the reliability function.

In the case of the additive model (3), we have that

$$\mu = \int_0^\infty \exp\{-(at)^b - (ct)^d\}\mathrm{d}t \quad (13)$$

which does not have a closed form and hence a numerical integration has to be performed.

The mean times to failure of the life distributions that correspond to Figs 1 and 2 are given in Table 1. In the same table, we also give the numerical value of the variance for the corresponding distributions.

3.4 Optimum burn-in and replacement time based on failure rate criteria

One of the uses of a bathtub curve is that we can determine the optimum burn-in time in the case when the initial failure rate is too high for the product to be released directly after production. Also, after a certain time, the product enters the wear-out phase and replacement should be considered. The decision can easily be made based on our additive model.

Suppose that the product can only be released after burn-in when the failure rate is less than r_b to meet customers' requirement, then the optimum burn-in time can be determined by

$$ab(at)^{b-1} + cd(ct)^{d-1} = r_b. \quad (14)$$

The optimum burn-in time should then be the smallest t for which the above equality holds. Because of the shape of the curve, there are usually two solutions to the above equation and at the point of the first solution, the failure rate has decreased enough to meet the reliability requirement.

Similarly, if the product has to be replaced by a new one when the failure rate is too high, higher than r_c, say, then the optimum replacement time can be determined by solving the following equation

$$ab(at)^{b-1} + cd(ct)^{d-1} = r_c. \quad (15)$$

Unlike the previous case, the largest t for which the above equality holds is the optimum replacement time. After this time, the failure rate is increasing and higher than the acceptable level and a replacement is needed to reduce the risk of immediate failure.

Both eqns (14) and (15) can be solved numerically using standard algorithms.

4 ESTIMATION BASED ON WEIBULL-PLOT

Given a set of failure time data, a usual problem is to estimate the parameters of the lifetime distribution in order to make further decisions. Although this is commonly done using statistical methods such as the maximum likelihood estimation technique, whose application is straightforward in our case, we will present a practical technique based on the conventional Weibull-plot. Note that little study is carried out on other existing life models that have bathtub shaped failure rates. In fact, it is this graphical approach that originally brought our attention to the additive model.

The graphical technique is based on the plot of empirical distribution on a suitable scaled paper and fits the data with a straight line. Model parameters can be estimated by reading the slope of the line and intercept on some of the axes. Among others, a Weibull plot is probably the most widely used technique today. If the empirical distribution function vs time is plotted on the so-called Weibull paper, the plot tends to be on a straight line and the parameters can easily be estimated. This can be seen from the following transformation of (1),

$$\ln\ln[1/R(t)] = \beta\ln t + \ln(1/\alpha), t \geq 0. \quad (16)$$

It can also be noted that a graphical plotting technique enables an engineer to judge whether the model is suitable for the data set before he makes further, detailed study. Hence, this plotting technique is also a simple model validation tool.

Unlike the ordinary Weibull case, the logarithmic transformation of $\ln(1/R(t))$ in our model cannot be used directly, since it is based on two Weibull models. However, for small t, we can approximate the expression to give:

$$\ln\ln[1/R(t)] \approx d\ln t + \ln c, \text{ for small } t \quad (17)$$

and hence, the Weibull plot for the early data can be used here for small t. This means that the first part of the plot should tend towards a straight line on Weibull paper and the parameters can be read accordingly.

Table 2. An actual set of failure time data collected during unit testing

time interval	number of failures
1	53
2	29
3	29
4	36
5	13
6	25
7	22
8	16
9	18
10	8
11	22
12	11
13	13
14	5
15	5
16	4
17	1
18	1

Similarly, for large t, we have that,

$$\ln \ln[1/R(t)] \approx b \ln t + \ln a, \text{ for large } t \qquad (18)$$

and hence, the Weibull plot for a later part of the data can be used for the estimation of a and b. Below we illustrate this approach using a real set of lifetime data.

During a unit testing phase, the number of failures in a certain time interval (of equal length) is recorded. The data is given in Table 2. The total number of units tested is equal to 311. A conventional Weibull plot of data set in Table 2 is shown in Fig. 3 below.

It can be seen that the plot is curved and a straight line fit is not good. However, it is clear that the first part can be fitted by a straight line with a slope less than unity while the last part is on a straight line with a slope greater than unity. This justified the bathtub-shaped failure rate assumption. Two plots are shown in Figs 4 and 5, based on the first six points and the last six points, respectively.

A Weibull plot based on the first six points gives an estimate of the slope as 0·887 which indeed corresponds to a decreasing failure rate at the beginning. The estimated slope for the last six points is 2·16 which corresponds to an increasing failure rate.

The overall additive Weibull model for the lifetime of this type of unit is then

$$R(t) = \exp\{(0{\cdot}0112t)^{2.16} + (0{\cdot}176t)^{0.887}\} \qquad (19)$$

and the overall failure rate function is given by

$$r(t) = 0{\cdot}024(0{\cdot}0112t)^{1.16} + 0{\cdot}156(0{\cdot}176t)^{-0.113}$$

$$= 0{\cdot}000132t^{1.16} + 0{\cdot}189t^{-0.113}. \qquad (20)$$

Suppose that the failure rate requirement is 0·2 for the customer, then based on this failure rate function, we can obtain the optimum burn-in time, t_b, by solving

$$0{\cdot}000132t_b^{1.16} + 0{\cdot}189t_b^{-0.113} = 0{\cdot}2 \qquad (21)$$

which gives the t_b to be equal to 0·606 time units.

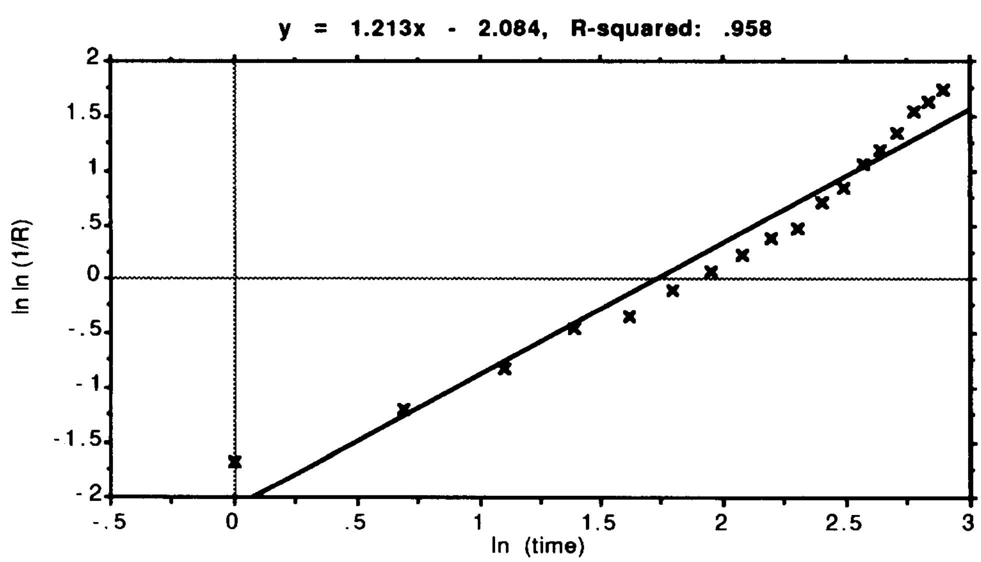

Fig. 3. The Weibull plot for the data set in Table 2.

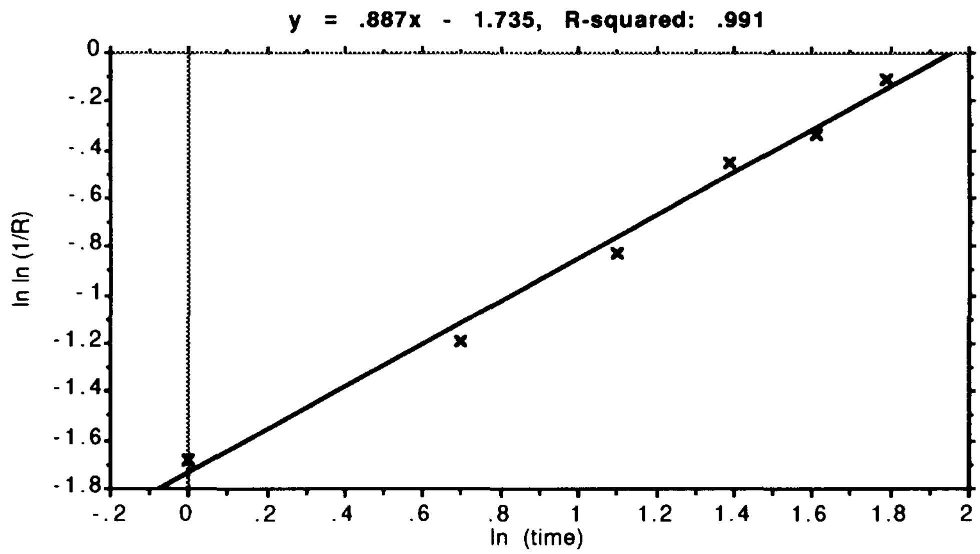

Fig. 4. Weibull plot for the first 6 data points.

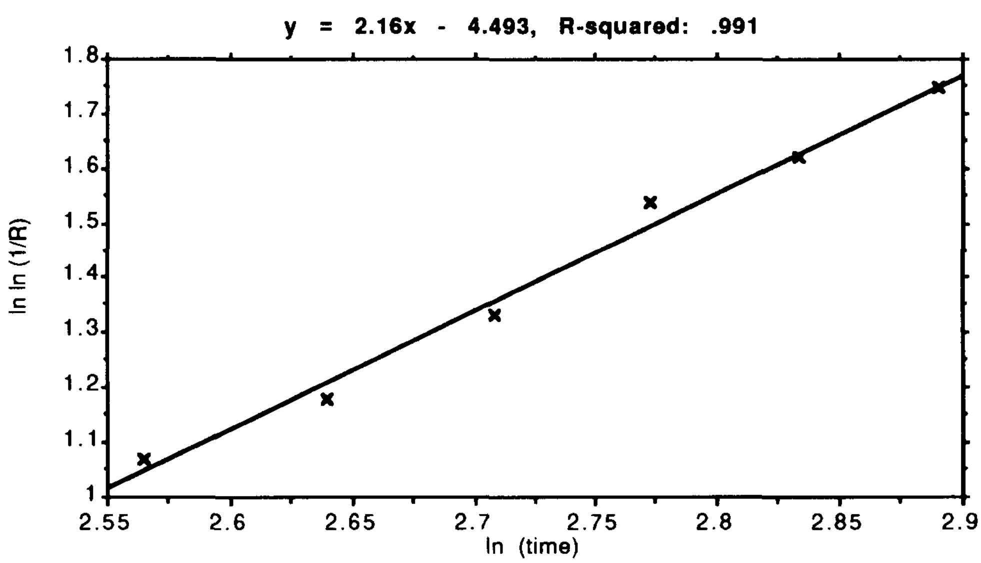

Fig. 5. Weibull plot for the last 6 data points.

5 CONCLUSION

In this paper, we have studied an additive model based on the conventional Weibull distribution. The model is applicable when the given data demonstrates that the failure rate is of bathtub form.

The application of the model is straightforward. For the data to be analyzed, one needs to plot it on Weibull paper and identify whether there is a bathtub shape. If so, the first part can be fitted by a straight line, as can the last part. The graphical estimates are then obtained. Further study can be carried out based on the model when the parameters are estimated.

REFERENCES

1. Glaser, R.E., Bathtub and related failure rate characterizations. *J. Am. Stat. Assoc.*, **75** (1980) 667–672.
2. Griffith, W.S., Representation of distributions having monotone or bathtub-shaped failure rates. *IEEE Trans. Reliab.*, **R-31** (1982) 95–96.
3. Guess, F.M. & Park, D.H., Modeling discrete bathtub and upside-down bathtub mean residual-life functions. *IEEE Trans. Reliab.*, **R-37** (1988) 545–554.
4. Haupt, E. & Schabe, H., A new model for a lifetime distribution with bathtub shaped failure rate. *Microelect. & Reliab.*, **32** (1992) 633–639.

5. Hjorth, U., A reliability distribution with increasing, decreasing, constant and bathtub-shaped failure rates. *Technometrics*, **17** (1980) 99–107.
6. Rajarshi, S. & Rajarshi, M.B., Bathtub distribution: a review. *Comm. Stat.—Series A: Theory and Method*, **17** (1988) 2597–2621.
7. Muldholkar, G.S. & Srivastava, D.K., Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Trans. Reliab.*, **R-42** (1993) 299–303.
8. Smith, R.L., Weibull regression models for reliability data. *Reliab. Engng Systems Safety*, **34** (1991) 55–77.
9. Hallinan, A.J., Jr, A review of the Weibull distribution. *J. Quality Tech.*, **25** (1993) 85–93.
10. Lawless, J. F., *Statistical Models and Methods for Lifetime Data*, Wiley, New York, 1982.
11. Nelson, W., *Applied Life Data Analysis*, Wiley, New York, 1982.
12. Ryerson, C. M., The reliability bathtub curve is vigorously alive. In *Proc. of the Annual Reliability and Maintainability Symposium*, IEEE, NJ, USA, pp. 313–316, 1982.
13. Smith, R. L. & Dietrich, D. L., The bathtub curve: an alternative explanation. In *Proc. of the Annual Reliability and Maintainability Symposium*, IEEE, NJ, USA, pp. 241–247, 1994.
14. Jensen, F. & Petersen, N. E., *Burn-in: an Engineering Approach to the Design and Analysis of Burn-in Procedures*, Wiley, New York, 1982.