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Chapter 4.

Probability and Statistics

UNIT 3.

Understanding of Statistics II

Samsung Innovation Campus

Unit 3. Understanding of Statistics II

- What this unit is about:
  - You will learn about descriptive statistics.
  - You will learn about the central limit theorem.
  - You will learn about point estimation and interval estimation.
- Expected outcome:
  - Ability to summarize the statistical properties of a sample.
  - Ability to differentiate the population from the sample.
  - Ability to apply interval estimation.
- How to check your progress:
  - Coding Exercises.
  - Quiz.

Chapter 4.

Probability and Statistics

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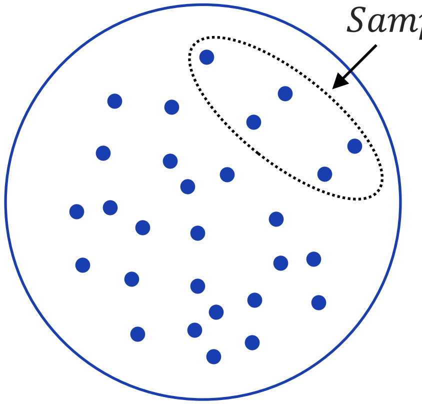
Population vs Sample



Population

- Whole data set subject to the analysis. It can be real or conceptual.

Sample:

- A subset of the population.
- You analyze samples because the population is hard or impossible to reach.
- You would like to draw conclusions about the population using its samples.

Descriptive vs Inferential

- Descriptive statistics:
  - Summarizes the data without generalization as a primary goal.
  - Extracts the properties of data as is: sample statistics.
- Inferential statistics:
  - Analysis of the samples with the purpose of making generalized statements about the population.
- **The main difference between descriptive and inferential statistics lies in the goal.**

Descriptive Statistics (1/10)

Sample statistics:

- Mean value: $\bar{x}$
- Median: m
- Variance: s^2
- Standard deviation: $s = \sqrt{s^2}$
- Covariance: s_{XY}
- Correlation: r
- Skewness: ς
- Kurtosis: κ

Averages

Statistic of a collection of numerical observations is a single number that describes the entire collection.
Average is a statistic.

There are THREE kinds of Averages:

- Mean or Expected Value
- Median
- Mode

Example: For one quiz, 10 students have the following grades (on a scale of 0 to 10): 9, 5, 10, 8, 4, 7, 5, 5, 8, 7
Find Mean, Median, Mode

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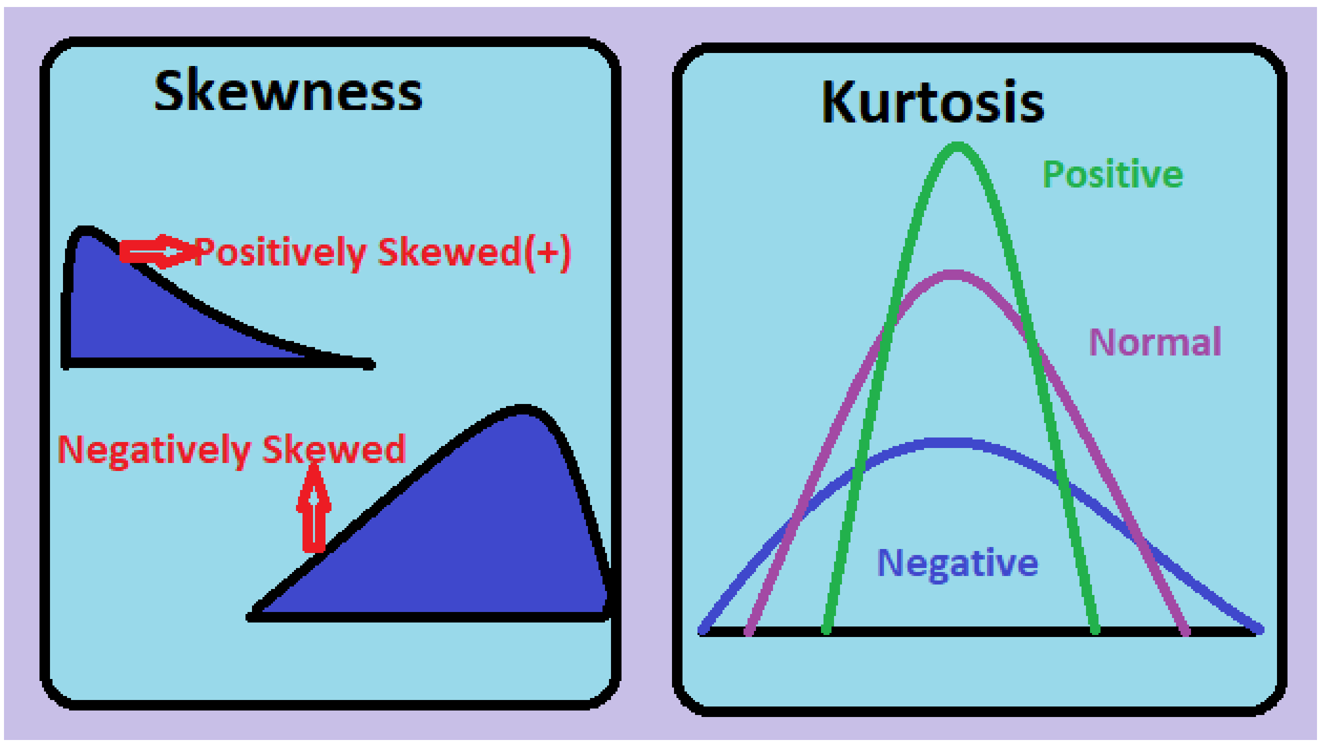
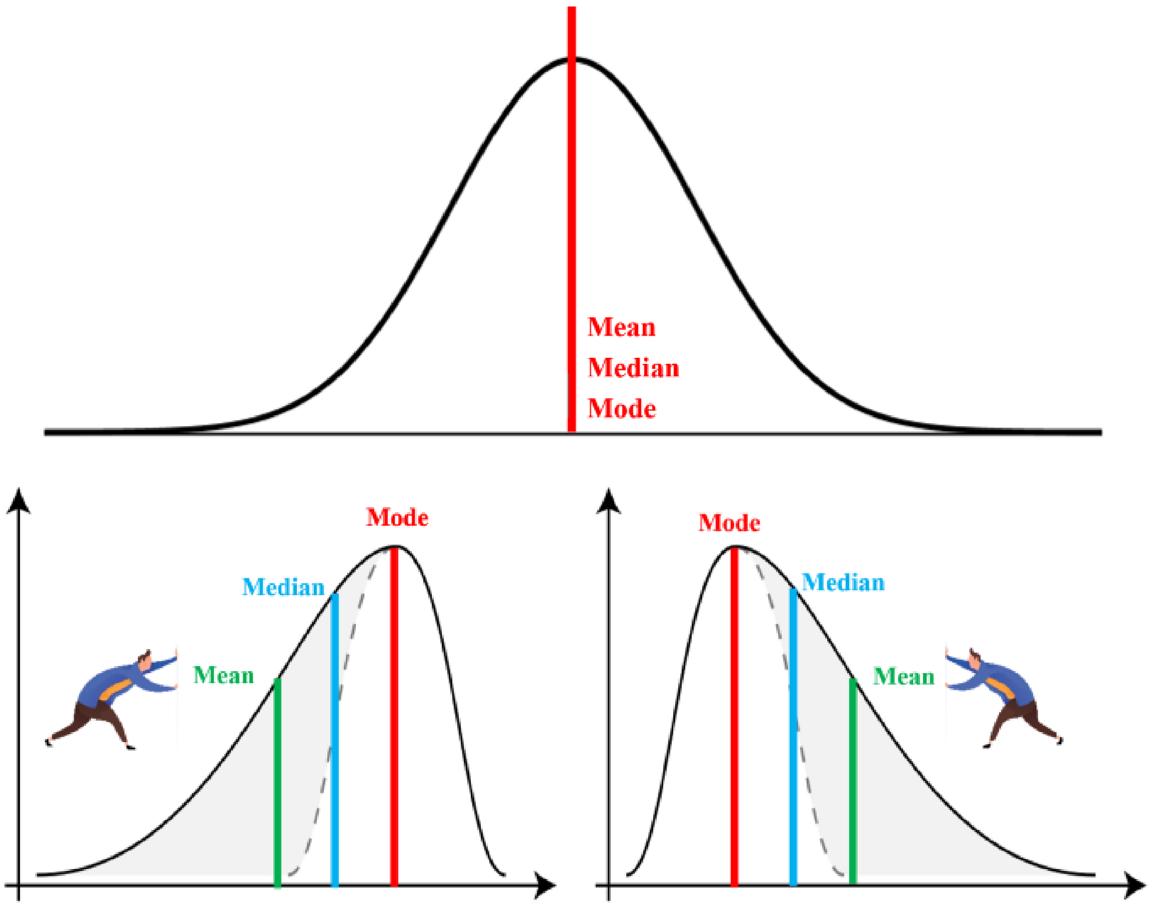
Descriptive Statistics (1/10)

Skewness refers the lack of symmetry and kurtosis refers the peakedness of a distribution.

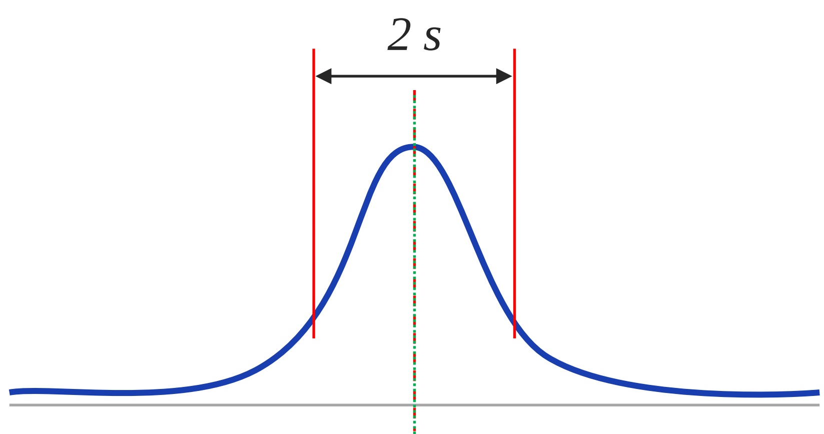


Descriptive Statistics (1/10)



Descriptive Statistics (2/10)

- Sample statistics:



$$\bar{x} = m$$

$$\varsigma = 0$$

Symmetric

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Descriptive Statistics (2/10)

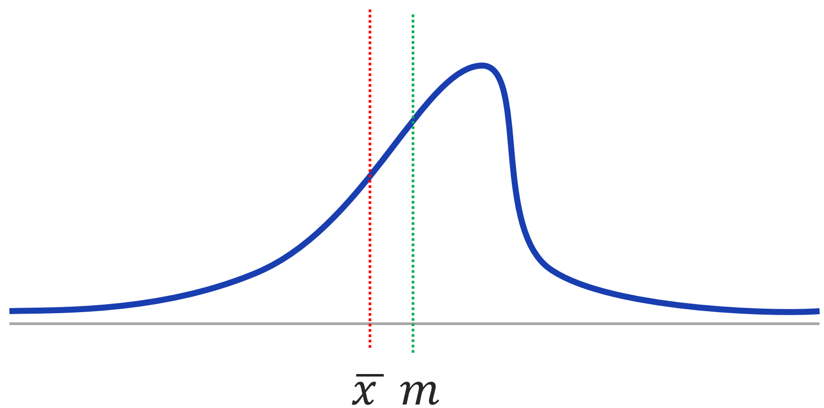
$$\text{Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}}$$

$$\text{Mode} = 3(\text{Median}) - 2(\text{Mean})$$

$$\text{Skewness} = \frac{3(\text{Mean} - \text{Median})}{\text{Standard Deviation}}$$

Descriptive Statistics (3/10)

- Sample statistics:



$$\varsigma < 0$$

Asymmetric

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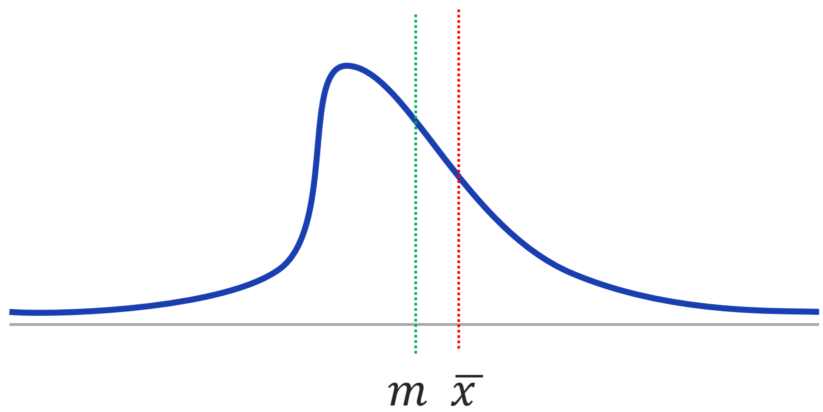
Descriptive Statistics (4/10)

- Sample statistics:



$$\varsigma > 0$$

Asymmetric

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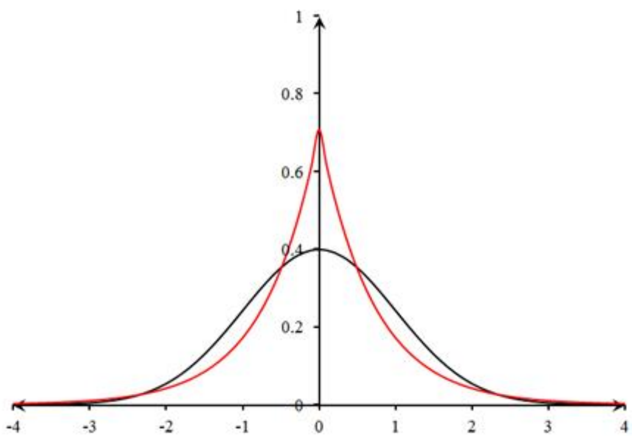
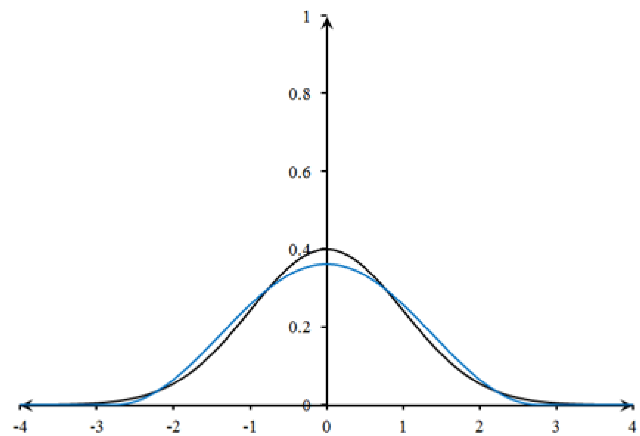
Descriptive Statistics (5/10)

- Sample statistics:



$\kappa > 0$



$\kappa < 0$

Descriptive Statistics (6/10)

Sample statistics:

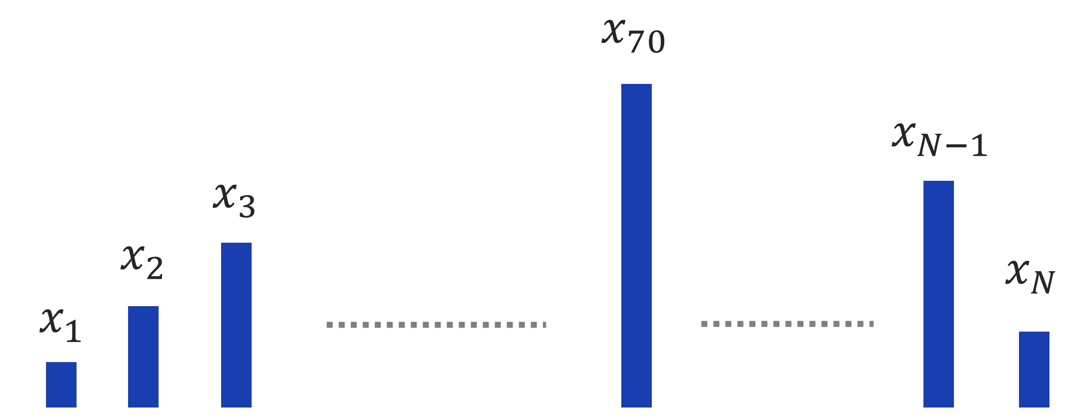
- Sample variance: $s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$
- Sample covariance: $s_{XY} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1}$
- Sample correlation: $r = \frac{s_{XY}}{s_X s_Y}$

$$\sigma^2 = \frac{\Sigma (x - \mu)^2}{N}$$ Population Variance

$$s^2 = \frac{\Sigma (x - \bar{x})^2}{n - 1}$$ Sample Variance

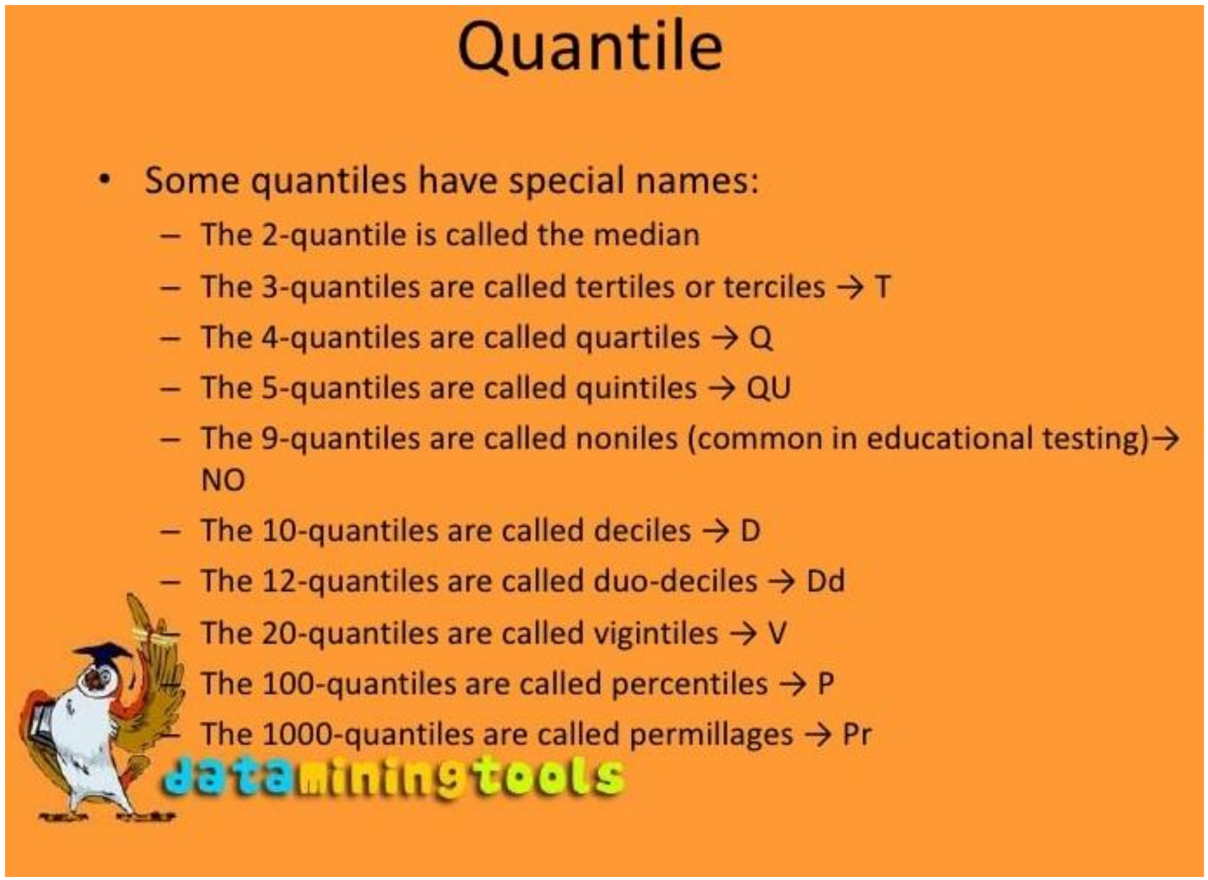
Descriptive Statistics (7/10)

- Sample statistics: quantile
  - Let's suppose that you have a sample of values $x_1, x_2, x_3, \cdots, x_N$. You can represent them as bars.



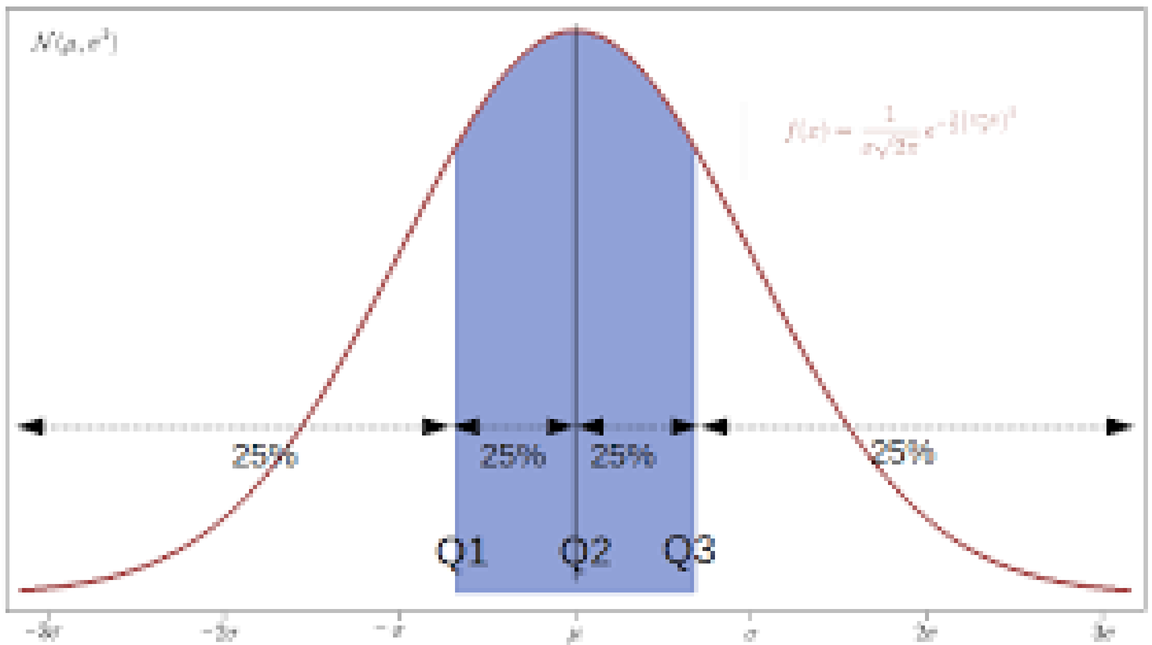
Descriptive Statistics (7/10)

Quantile



Descriptive Statistics (7/10)

Quantile Vs Quartile



Descriptive Statistics (7/10)

Quantile Vs Quartile

Example 1

0 1 5 6 7 8 9 10 12 12 13 14 16 19 19

- Find the Q_1, Q_2, and Q_3 of the following set of data.

19	**12**	**16**	**0**	**14**
9	**6**	**1**	**12**	**13**
10	**19**	**7**	**5**	**8**

$$Q_i = \left(\frac{i \cdot (n+1)}{4}\right)^{th}$$

$$Q_3 = \left(\frac{3 \cdot (15+1)}{4}\right)^{th}$$

$$Q_3 = \left(\frac{48}{4}\right)^{th}$$

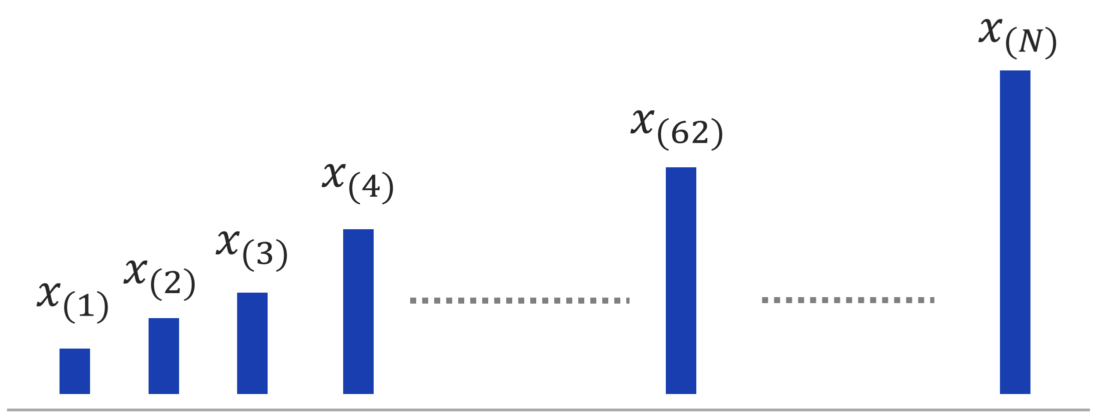
$$Q_3 = 12^{th}$$

$$\boxed{Q_3 = 14}$$

Descriptive Statistics (8/10)

Sample statistics: quantile

- Now, let us sort the values from the smallest to the largest and get $x_{(1)}, x_{(2)}, x_{(3)}, \cdots, x_{(N)}$.



- α *quantile*= value at the position $int(N \times \alpha)$

Descriptive Statistics (9/10)

Sample statistics: quantile

- Quantile: α is probability $0 \sim 1$.
- Percentile: α is given as percentage $0\% \sim 100\%$.

  → Minimum = 0% percentile.

  → Maximum = 100% percentile.

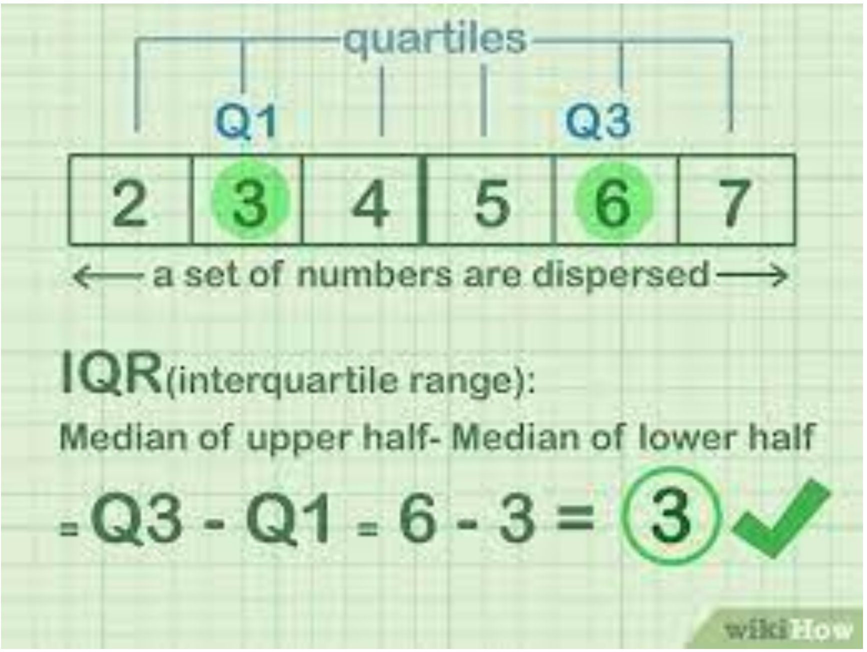
  $\text{Range} = \text{Maximum} - \text{Minimum}$
- Quartile: α subdivided into four equal intervals.

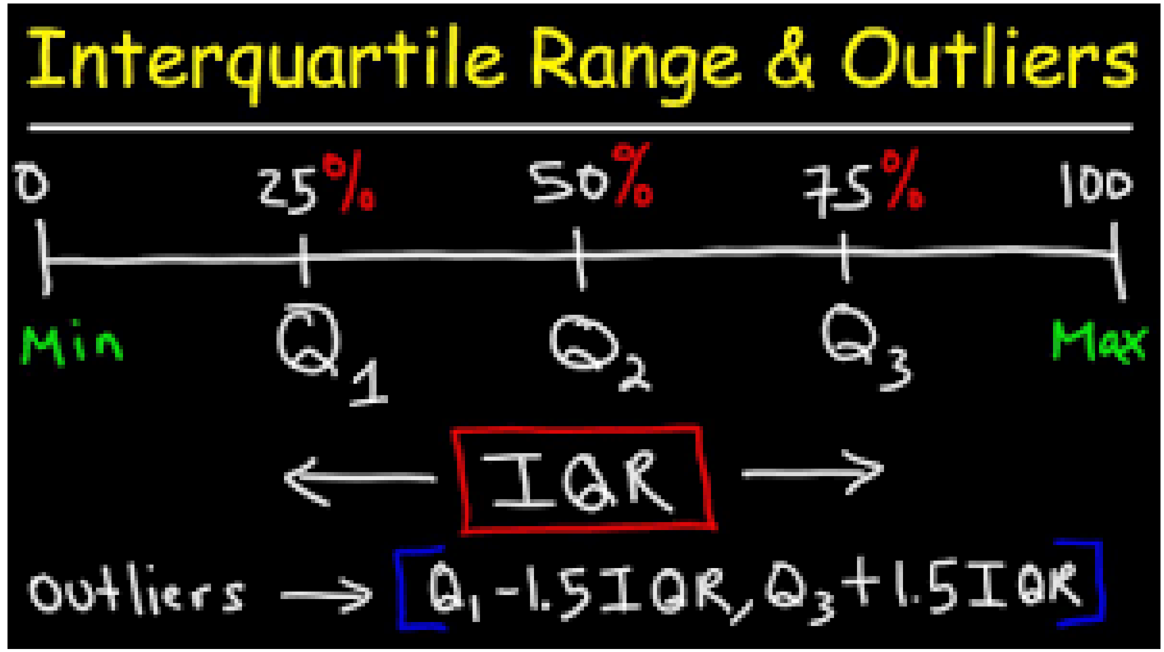
  → 1st quartile (Q1): 25% percentile.

  → 2nd quartile (Q2): 50% percentile = Median.

  → 3rd quartile (Q3): 75% percentile.

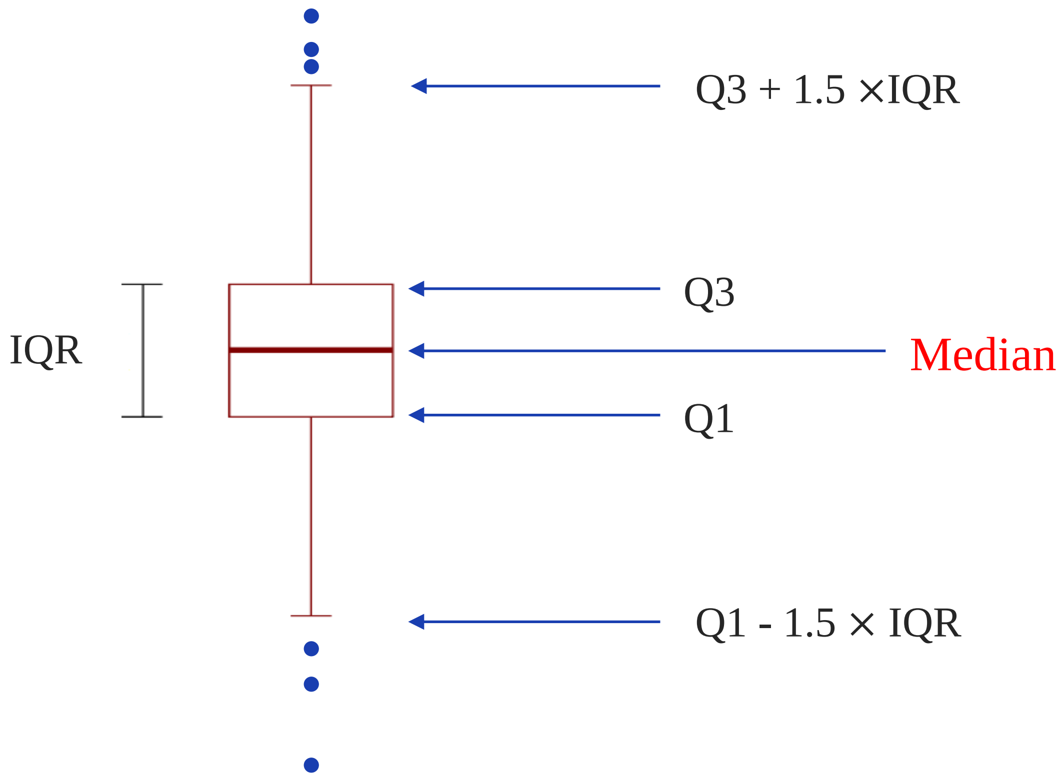
Descriptive Statistics (10/10)





Descriptive Statistics (10/10)

- Boxplot:



Coding Exercise #0303

Follow practice steps on 'ex_0303.ipynb' file.

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Inferential Statistics (1/3)

Population vs Sample:

- You analyze samples because the population is hard or impossible to reach.
- You would like to draw conclusions about the population using its samples.
- However, you should carefully analyze the uncertainties associated with the samples.

Population parameters vs Sample statistics:

- The population parameters are properties of a given population.
- There is one population: and the population parameters are calculated once for each.
- Sample statistics are properties of a given sample.
- In principle, you can draw several samples and calculate the same sample statistics many times.

Inferential Statistics (2/3)

Sampling method:

- Simple random sampling: values drawn with equal probability with or without replacement.
- Weighted random sampling: values drawn with varying probabilities.
- Stratified sampling: reflects the proportions of the strata.
- Systematic sampling: values are drawn with an implicit periodicity.
- Cluster sampling: takes a representative cluster.

Inferential Statistics (3/3)

Population parameters vs Sample statistics:

	Population	**Sample**
Size	N	n
Mean	$\mu = \frac{\sum_{i=1}^{N} x_i}{N}$	$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$
Variance	$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}$	$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$
Standard Deviation	$\sigma = \sqrt{\sigma^2}$	$s = \sqrt{s^2}$

- Connect the sample statistics with the population parameters through statistical inference.

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Central Limit Theorem (1/10)

Coin flipping experiment:

- Let us flip the coin twice and group the results as a sample (n=2). We assign T = 0 and H =1.
- As we draw several samples, we get the corresponding sample means: $\overline{x_1}$, $\overline{x_2}$, $\overline{x_3}$,...

Ex)

i	Sample	$\overline{x}_i$
1	1,1	1
2	0,1	0.5
3	1,0	0.5
4	0,0	0
⋮	⋮	⋮

Central Limit Theorem (2/10)

Coin flipping experiment:

- Let us flip the coin three times and group the results as a sample (n=3).
- As we draw several samples, we get the corresponding sample means: $\overline{x_1}, \overline{x_2}, \overline{x_3}$,...

Ex)

i	Sample	$\overline{x}_i$
1	1,0,1	2/3
2	0,1,0	1/3
3	1,0,0	1/3
4	0,0,0	0
⋮	⋮	⋮

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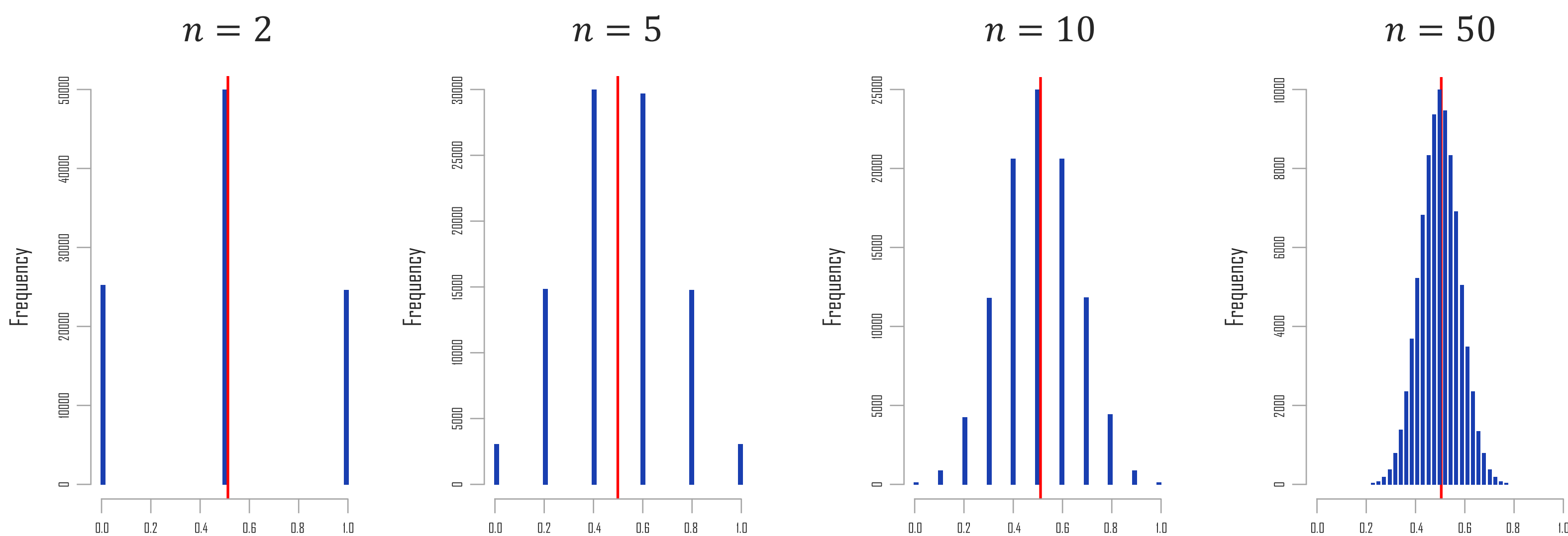
Central Limit Theorem (3/10)

Coin flipping experiment:



- Histograms of the sample means of varying sample size.
- As the sample size increases, the histograms converge to the normal distribution.

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Central Limit Theorem (4/10)

Coin flipping experiment:

- The sample means $\overline{x}$ are also randomly distributed.
- You can denote the sample mean by $\overline{\boldsymbol{X}}$ (upper case) and treat it as a random variable.
- Then, we obtain the following results.

a) Expected value of the $\overline{X}$ is the population mean : $E[\overline{X}]=\mu$

b) Variance and standard deviation of the $\overline{X}$:

$$Var(\overline{X}) = \sigma_{\overline{X}}^2 = \frac{\sigma^2}{n} = \frac{p(1-p)}{n} = \frac{0.25}{n}$$

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{p(1-p)}}{\sqrt{n}} = \frac{0.5}{\sqrt{n}}$$

"**Standard error**"

- σ^2= population variance, s^2= sample variance, $\sigma_{\overline{X}}^2$= variance of the sample mean.

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Central Limit Theorem (5/10)

Central limit theorem:

- The distribution of the sample means converges to the Normal distribution as n increases.

  1) This is true not just for the coin flipping (Binomial).

  2) You can verify the same phenomena for other underlying distributions (discrete and continuous).

- For large enough n, analogous tendency can be verified for the sample statistics beside the mean.

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Central Limit Theorem (6/10)

Realistic considerations:

- In reality, you will not have several samples but **one**.
- You apply the central limit theorem and assume that the sample distribution is Normal.
- For the statistical inference, you will take advantage of the properties of the Normal distribution.

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Central Limit Theorem (7/10)

Standardization:

- When the sample size n is big enough $\overline{X}$ follows the Normal distribution $N(\mu,\sigma$^2).
- You can standardize $\overline{X}$ and get

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

which follows the Standard Normal distribution N(0,1).

- When the sample size n is small or the σ is not known, you define

$$t = \frac{\overline{X} - \mu}{s/\sqrt{n}}$$

which follows the Student-t probability density with degree of freedom n−1.

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Central Limit Theorem (8/10)

Sampling distribution of proportions:

- Let us consider the case where the population follows Bernoulli distribution.
- Let us suppose that the success probability is p.
- We denote the sample proportion (of success) as $\widehat{P}$ and have

  a) Expected value: $E\left[\widehat{P}\right] = p$

  b) Standard error: $\sigma_{\widehat{P}} = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{p(1-p)}{n}}$

- When $n\,p$ > 10 and $n(1-p)$ > 10 then, we can apply the central limit theorem: $\frac{\hat{P}-p}{\sigma_{\hat{P}}} \sim N(0,1)$

Central Limit Theorem (9/10)

Sampling distribution of the difference:

- Let us consider two populations. Samples of size n_1 and n_2 are drawn from each population.
- Then for the difference of the sample means, you have

  a) Expected value: $E[\overline{X_1} - \overline{X_2}] = E[\overline{X_1}] - E[\overline{X_2}] = \mu_1 - \mu_2$

  b) Standard error: $\sigma_{\overline{X_1}-\overline{X_2}} = \sqrt{\sigma_{\overline{X_1}}^2 + \sigma_{\overline{X_2}}^2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

- When n_1 and n_2 are large enough, you can apply the central limit theorem: $\frac{\overline{X_1}-\overline{X_2}-(\mu_1-\mu_2)}{\sigma_{\overline{X_1}-\overline{X_2}}} \sim N(0,1)$

Central Limit Theorem (10/10)

Standard errors:

Statistic	Standard Error	Explanation
Mean	$\frac{\sigma}{\sqrt{n}}$	When $n \geq 30$, the sample mean approximately follows the Normal distribution.
Proportion	$\sqrt{\frac{p(1-p)}{n}}$	When $n\,p > 10$ and $n(1-p) > 10$, the sample proportion approximately follows the Normal distribution.
Median	$\sigma\sqrt{\frac{\pi}{2n}}$	When $n \geq 30$, the sample mean approximate follows the Normal distribution.
Standard Deviation	a) $\frac{\sigma}{\sqrt{2n}}$ b) $\sqrt{\frac{\mu_4-\sigma^4}{4n\sigma^2}}$	a) When the population follows the Normal distribution. b) When the population does not follow the Normal distribution.
Variance	a) $\sigma^2\sqrt{\frac{2}{n}}$ b) $\sqrt{\frac{\mu_4-\sigma^2}{n}}$	a) When the population follows the Normal distribution. b) When the population does not follow the Normal distribution. The sample variance follows the Chi-square distribution.
Correlation	$\sqrt{\frac{1-r^2}{n-2}}$	r is the sample correlation. Fisher's z-transformation required.

Chapter 4.

Probability and Statistics

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Estimation Theory

Point estimation vs Interval estimation:

Ex) Say you want to survey the mean weight of adult males. The possible answers are:

a) 72 kg

b) 71 kg ~ 74 kg

c) 70 kg ~ 75 kg

- There many ways of answering to the same question.
- Providing a single value like in a) is called "point estimation". ⇐ using an estimator.
- Providing an interval like in b) and c) is called "interval estimation".

Estimator (1/4)

What makes a "good" estimator?

1) Unbiasedness.

2) Efficiency.

3) Consistency.

Estimator (2/4)

Unbiasedness:

- Let us suppose that θ is the population parameter and θ ˆis the corresponding estimator.
- A unbiased estimator satisfies the following condition: $E\left[\hat{\theta}\right] = \theta$

Ex) $\frac{\sum_{i=1}^{n} x_i}{n}$ is an unbiased estimator of the population mean μ.

Ex) $\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\boldsymbol{n-1}}$ is an unbiased estimator of the population variance σ^2.

Estimator (3/4)

Efficiency:

- Let us suppose that there are several unbiased estimators $\hat{\theta}_1, \hat{\theta}_2, \ldots$

  $E[\hat{\theta}_1] = \theta \;\; , \; E[\hat{\theta}_2] = \theta \;\; , \; E[\hat{\theta}_3] = \theta \;\; , \ldots$

- The estimator with the least uncertainty (variance or standard deviation) is the efficient estimator:

  Ex) If $Var(\hat{\theta}_1)$ is at the minimum, then $\hat{\theta}_1$ is the efficient estimator.

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Estimator (4/4)

Consistency:

- A biased estimator for small n, may become unbiased for large enough n because of the consistency.

Ex) $\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$ is biased estimator of σ^2 for small n but becomes unbiased for large enough n.

$$\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n} \cong \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$$

- When the sample size n can be increased, the consistency becomes an important criteria.

Interval Estimation of the Mean (1/11)

Confidence interval:

- It is the interval that may contain the true value of a population parameter with a given confidence level.

  a) Significance level: α

  b) Confidence level: $(1-\alpha)$ $\Leftarrow$ A probability.

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Interval Estimation of the Mean (2/11)

Confidence interval of the mean:

- Let us suppose that the sample size is big enough that the sample mean is distributed normally.
- The 95% confidence interval can be obtained in the following way.

$$P(-1.96 < Z < 1.96) = 0.95$$

$$P\left(-1.96 < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 1.96\right) = 0.95$$

$$P\left(-1.96\,{}^{\sigma}\!/_{\sqrt{n}} < \bar{X}-\mu < 1.96\,{}^{\sigma}\!/_{\sqrt{n}}\right) = 0.95$$

Interval Estimation of the Mean (3/11)

Confidence interval of the mean:

- Let us suppose that the sample size is big enough that the sample mean is distributed normally.
- The 95% confidence interval can be obtained in the following way.

$$P\left(-1.96\,{}^{\sigma}\!/_{\sqrt{n}} < \bar{X} - \mu < 1.96\,{}^{\sigma}\!/_{\sqrt{n}}\right) = 0.95$$

$$P\left(-\bar{X} - 1.96\,{}^{\sigma}\!/_{\sqrt{n}} < -\mu < -\bar{X} + 1.96\,{}^{\sigma}\!/_{\sqrt{n}}\right) = 0.95$$

$$P\left(\bar{X} - 1.96\,{}^{\sigma}\!/_{\sqrt{n}} \le \mu \le \bar{X} + 1.96\,{}^{\sigma}\!/_{\sqrt{n}}\right) = 0.95$$

Interval Estimation of the Mean (4/11)

Confidence interval of the mean:

- The 95% confidence interval can be obtained in the following way.

Lower bound: $\bar{X} - 1.96\,{}^{\sigma}/_{\sqrt{n}}$ Upper bound: $\bar{X} + 1.96\,{}^{\sigma}/_{\sqrt{n}}$

[← Confidence Interval →]

- But, where did the number 1.96 come from?
- 1.96 is the 0.975 quantile of the normal distribution ⇔ the position where the CDF = 0.975.

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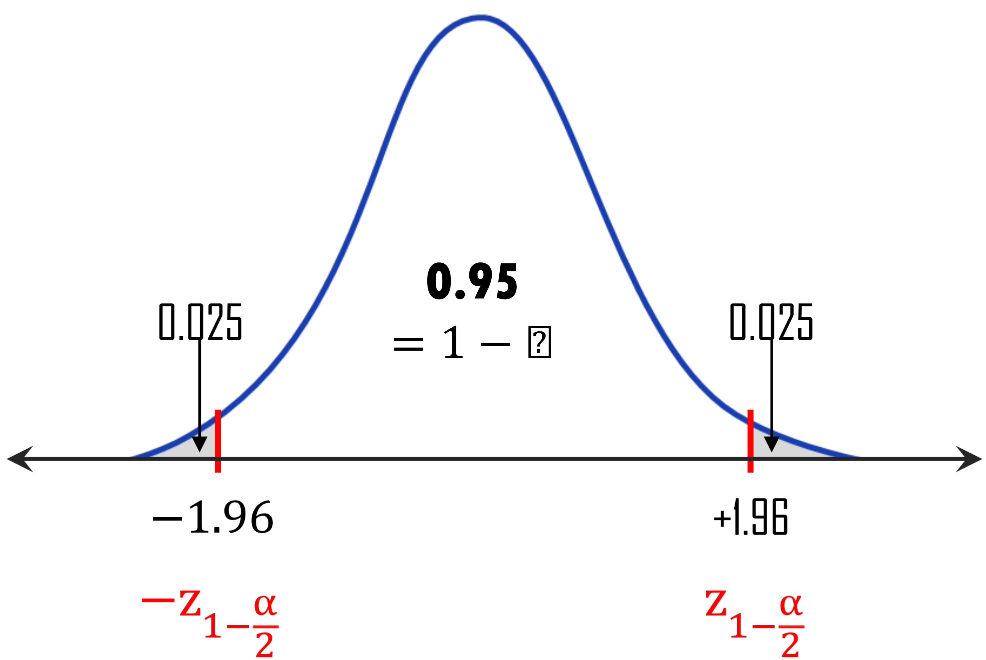
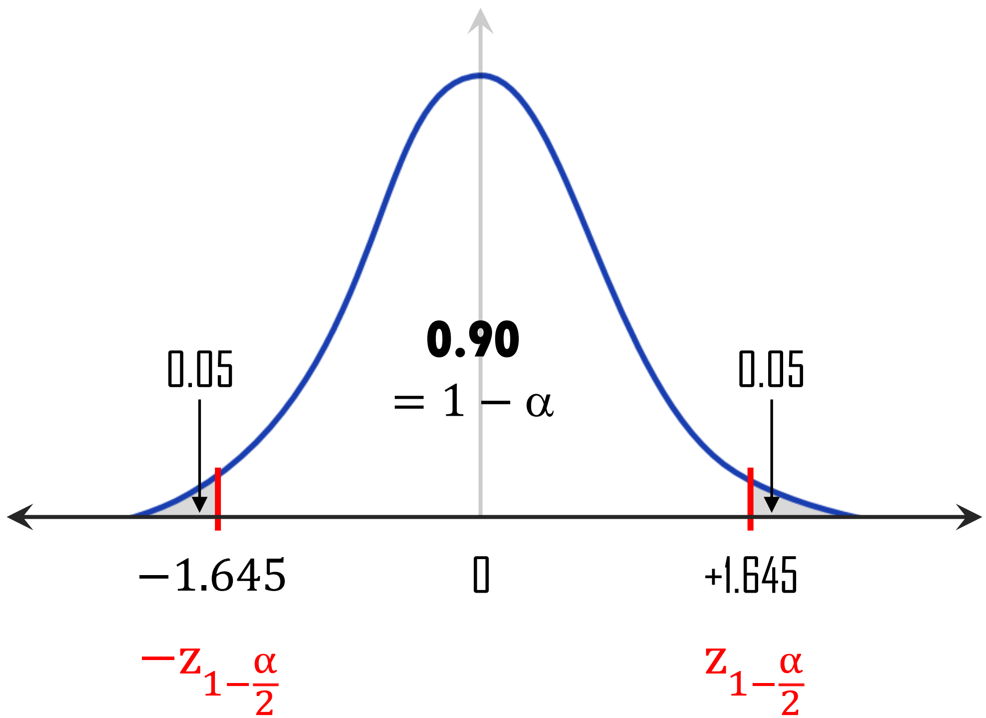
Interval Estimation of the Mean (5/11)

- Confidence interval of the mean: 95%



Interval Estimation of the Mean (6/11)

- Confidence interval of the mean: 90%



Interval Estimation of the Mean (7/11)

Confidence interval of the mean:

- Of arbitrary confidence level = (1−α).

Lower bound: $\bar{X} - z_{1-\frac{\alpha}{2}} \sigma/\sqrt{n}$ Upper bound: $\bar{X} + z_{1-\frac{\alpha}{2}} \sigma/\sqrt{n}$

[← Confidence Interval →]

- When the sample size is small and the population variance is not known, use Student-t.

Lower bound: $\bar{X} - t_{1-\frac{\alpha}{2}} s/\sqrt{n}$ Upper bound: $\bar{X} + t_{1-\frac{\alpha}{2}} s/\sqrt{n}$

[← Confidence Interval →]

Interval Estimation of the Mean (8/11)

Confidence interval of the mean:

Question:

Isn't it always good to have as large a confidence level as possible?

Interval Estimation of the Mean (9/11)

Confidence interval of the mean:

Answer:

All the other conditions being equal, larger confidence level means larger uncertainty.

The confidence interval broadens.

Interval Estimation of the Mean (10/11)

- Confidence interval of the mean:

[99% confidence interval]

[95% confidence interval]

[90% confidence interval]

Interval Estimation of the Mean (11/11)

How to narrow the confidence interval of the mean while keeping the confidence level high?

- Let us remember that the bounds of the confidence interval was given by:

$$\bar{X} \pm z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}} \quad \text{or} \quad \bar{X} \pm t_{1-\frac{\alpha}{2}} \times \frac{s}{\sqrt{n}}$$

- Smaller σ or s can narrow the confidence interval $\Leftarrow$ **you cannot** control it!
- Larger sample size n can narrow the confidence interval $\Leftarrow$ in principle **you can** control it!
- If W is the targeted half width of the confidence interval, apply the formula:

$$z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}} = W \quad \Rightarrow \quad \boldsymbol{n} = \left[\frac{z_{1-\alpha/2} \times \sigma}{W}\right]^2$$

Coding Exercise #0304

Follow practice steps on 'ex_0304.ipynb' file.

Coding Exercise #0305

Follow practice steps on 'ex_0305.ipynb' file.

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Interval Estimation of the Correlation (1/3)

Confidence interval of the correlation:

- There are different kinds of correlation: Pearson, Spearman, Kendall, etc.
- Spearman and Kendall are the so-called "rank correlations".
- Let us focus on the "usual type", that is, Pearson correlation.

Interval Estimation of the Correlation (2/3)

Confidence interval of the correlation:

- The distribution of the sample correlation r deviates significantly from the Normal distribution.
- This is true even for large enough sample size, because the correlation is bound to be $[-1, +1]$.
- To overcome these "distortions", it is necessary to do Fisher's z-transformation.

a) Transformed correlation: z=$arctanh(r)$

b) Standard error after the transformation: $\sigma_z = \frac{1}{\sqrt{n-3}}$

c) Inverse transformation: r= $tanh(z)$

Interval Estimation of the Correlation (3/3)

Confidence interval of the correlation:

- Of arbitrary confidence level = (1−α).

$$tanh\left(z - z_{1-\frac{\alpha}{2}} \times \sigma_z\right) \qquad\qquad tanh\left(z + z_{1-\frac{\alpha}{2}} \times \sigma_z\right)$$

[← Confidence Interval →]

Ex) 95% confidence interval: $[tanh(z - 1.96\ \sigma_z), tanh(z + 1.96\ \sigma_z)]$

Coding Exercise #0306

Follow practice steps on 'ex_0306.ipynb' file.