Lectures on Real-Analytic Operator Equations

by John Toland

When X and Y are Banach spaces and U is open in X, a function $f : U \to Y$ is real-analytic if it is C^∞ on U and, in a neighbourhood of every point of U, it is given by the sum of its Taylor series at that point. For example, the function $f(x, y) = xy$ is a polynomial, and hence real-analytic, from $\mathbb{R}^2$ to $\mathbb{R}$. This simple example shows that the zeros of a real-analytic function need not be isolated (very different from the well known complex-analytic case). However, if f is real-analytic on a connected set U, and zero on an *open* subset of U, then f must be identically zero on U. On the other hand, *any closed set* E in $\mathbb{R}^2$ is the zero set of a C^∞ function[1]. The contrast between these observations highlights the huge difference, in the theory of equations, between operators that are real-analytic and that those are merely C^∞.

The special nature of the solution set of $f(x) = 0$ when f is real-analytic is the main topic of these lectures. We will see that, after Lyapunov-Schmidt reduction, it suffices to study finite-dimensional equations of the form $h(z) = 0$ where $h : \mathbb{R}^n \to \mathbb{R}^m$ is real-analytic. The solution set of equations like this are called *real analytic varieties.* The structural implications of that statement will explained in the lectures.

A corollary of the analysis will be a theory of *unique global continuation* of one-parameter families of solutions to equations of the form $F(\lambda, x) = 0$, where $F : \mathbb{R} \times X \to Y$ is real-analytic and $\lambda \in \mathbb{R}$ is a distinguished parameter. This includes bifurcation problems and one outcome is a very strong global version of the theorem on bifurcation from a simple eigenvalue.

Topological degree theory is not involved in the argument, which is due to Norman Dancer (*Bifurcation theory for analytic operators*, Proc. Lond. Math. Soc. **XXVI** (1973), 359–384, and *Global structure of the solution set of non-linear real-analytic eigenvalue problems*, Proc. Lond. Math. Soc. **XXVI** (1973), 359–384.)

The exposition will be along the lines of *Analytic Theory of Global Bifurcation*, PUP 2003, which was written in a collaboration with Boris Buffoni.

The course will assume an acquaintance with

- analytic functions of one complex variable;
- linear operator theory in Banach spaces, projections, subspaces etc;
- calculus in Banach spaces, including the multi-linear representation of higher Fréchet derivatives, and C^k and C^∞ operators; implicit and inverse function theorems for C^k functions
- Lyapunov-Schmidt reduction; bifurcation from a simple eigenvalue for C^k operator equations

The main material on analytic operators will be reviewed in as much detail as time permits and applications will be mentioned.

[1] Let f_ϵ be the ϵ-mollification of the characteristic function of an ϵ neighbourhood of E. Then $0 \le f_\epsilon \le 1$ and $f = \sum 2^{-n}(1 - f_{1/n})$ is a C^∞ function on $\mathbb{R}^2$ whose zero set in E.

Contents

1 Multilinear and Analytic Operators

The theory of higher order Fréchet derivatives leads to the notion of multilinear operators. The question of whether the Taylor polynomials of F converge to $F(x)$ for some $x \in X$ as $n \to \infty$ leads to the theory of analytic functions. Throughout $\mathbb{F}$ denotes the field of scalars, $\mathbb{R}$ or $\mathbb{C}$, and S_p is the group of permutations of the set $\{1, \cdots, p\}$.

Multilinear Operators

Suppose that Y and $X_1, \cdots, X_p$, $p \in \mathbb{N}$, are Banach spaces over $\mathbb{F}$. A mapping $m : X_1 \times \cdots \times X_p \to Y$ is said to be a multilinear operator (p-linear in this case) if it is linear in each variable separately, that is, for all $k \in \{1, \cdots, p\}$ and $x_j \in X_j,\ j \neq k$,

$$x \mapsto m(x_1, \cdots, x_{k-1}, x, x_{k+1}, \cdots, x_p) \text{ is linear in } x \text{ over } \mathbb{F}.$$

It is said to be a bounded multilinear operator if, in addition,

$$\sup\{\|m(x_1, x_2, \cdots, x_p)\| : \|x_1\|, \cdots, \|x_p\| \leq 1\} = M < \infty. \tag{1}$$

If m is multilinear and $x_j = 0$ for some j, then $m(x_1, \cdots, x_p) = 0$. Otherwise, if $x_j \neq 0$ for all j and m is bounded, we find that

$$\left\| m\left(\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \cdots, \frac{x_p}{\|x_p\|}\right) \right\| \leq M,$$

whence

$$\|m(x_1, \cdots, x_p)\| \leq M\|x_1\|\|x_2\| \cdots \|x_p\|.$$

The proofs of the next few observations are so similar to those for bounded linear operators that we leave them as exercises.

Exercise 1.1. Suppose that $m : X_1 \times \cdots \times X_p \to Y$ is a multilinear operator. Then the following are equivalent statements.

$m : X_1 \times \cdots \times X_p \to Y$ is continuous;
m is continuous at $(0, \cdots, 0) \in X_1 \times \cdots \times X_p$;
m is bounded.

When $X_k = X$, $1 \leq k \leq p$, we abbreviate $\mathcal{M}(X_1, \cdots, X_p; Y)$ as $\mathcal{M}^p(X, Y)$. Then $m \in \mathcal{M}^p(X, Y)$ is called symmetric if

$$m(x_1, \cdots, x_p) = m(x_{\pi(1)}, \cdots, x_{\pi(p)})$$

for all $\pi \in S_p$.

Exercise 1.2. The set $\mathcal{M}(X_1, \cdots, X_p; Y)$ of bounded multilinear operators endowed with the norm

$$\|m\| = \sup\{\|m(x_1, x_2, \cdots, x_p)\| : \|x_1\|, \cdots, \|x_p\| \leq 1\}$$

is a Banach space. The symmetric operators form a closed subspace of $\mathcal{M}^p(X, Y)$.

Example 1.3. Fréchet derivatives give an important class of multilinear operators. If $F : U \subset X \to Y$ has a k^{th} Fréchet derivative at $x_0 \in U$,

$$(x_1, \cdots, x_k) \mapsto d^k F[x_0](x_1, \cdots, x_k)$$

is a bounded, symmetric k-linear operator. That $d^k F : U \to \mathcal{M}^k(X, Y)$ is continuous is equivalent to saying that $F \in C^k(U, Y)$. □

Example 1.4. When $X = Y = \mathbb{F}$, it follows from the Riesz representation theorem that every element m_p of $\mathcal{M}^p(X,Y)$ is given by

$$m_p(x_1, \cdots, x_p) = A_p \underbrace{x_1 \cdots x_p}_{\text{product in } \mathbb{F}}$$

for some $A_p \in \mathbb{F}$, and therefore all m_p are symmetric in this case. $\square$

Example 1.5. (Determinants)
Consider an $n \times n$ matrix A with rows $(a_{i1}, \cdots, a_{in}) \in \mathbb{F}^n$, $1 \le i \le n$, as an element of $(\mathbb{F}^n)^n$. Its determinant $\det A$, defined by

$$\det A = \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^{n} a_{i\pi(i)}, \tag{2}$$

where $\sigma(\pi)$ denotes the signature of $\pi \in S_n$, is then an n-linear function on $\big(\mathbb{F}^n\big)^n$ which is not symmetric since the interchange two rows of A changes the sign of $\det A$. (As a consequence, the determinant of a matrix with two equal rows is zero.) $\square$

Let X^j denote the product of j copies of the same space X. When $m \in \mathcal{M}^p(X,Y)$ is symmetric, define a mapping on X^j by

$$\begin{aligned}(x_1, \cdots, x_j) \mapsto h(x_1, \cdots, x_j) &:= m(\underbrace{x_1, \cdots, x_1}_{k_1 \text{ times}}, \underbrace{x_2, \cdots, x_2}_{k_2 \text{ times}}, \cdots, \underbrace{x_j, \cdots, x_j}_{k_j \text{ times}}) \\ &=: m\, x_1^{k_1} \cdots x_j^{k_j},\end{aligned} \tag{3}$$

where $j \in \mathbb{N}$ and $k_1 + \cdots + k_j = p$. (This defines the notation $h : X^p \to Y$ and $m : X^j \to Y$.) The next few observations are almost obvious but proving them is a useful exercise that will familiarize the reader with the notation.

Exercise 1.6. (a) Suppose $m \in \mathcal{M}(X_1, \cdots, X_p; Y)$. Then m is infinitely differentiable on the product space $X_1 \times \cdots \times X_p$ and its first derivative at a point $(x_1, \cdots, x_p) \in X_1 \times \cdots \times X_p$ is given by

$$dm[(x_1, \cdots, x_p)](z_1, \cdots, z_p) = \sum_{k=1}^{p} m(x_1, \cdots, x_{k-1}, z_k, x_{k+1}, \cdots, x_n),$$

for all $(z_1, \cdots, z_p) \in X_1 \times \cdots \times X_p$.
(b) When $m \in \mathcal{M}^p(X,Y)$ is symmetric, the mapping h, which is defined in (3) by

$$h(x_1, \cdots, x_j) = m\, x_1^{k_1} \cdots x_j^{k_j},$$

is infinitely differentiable on X^j and, and for $(y_1, \cdots, y_j) \in X^j$,

$$dh[(x_1, \cdots, x_j)](y_1, \cdots, y_j) = \sum_{l=1}^{j} k_l \; m x_1^{k_1} \cdots x_{l-1}^{k_{(l-1)}} x_l^{k_l - 1} x_{l+1}^{k_{(l+1)}} \cdots x_j^{k_j} \; y_l.$$

$\square$

Remarks 1.7. The mapping

$$u_l \mapsto \; m x_1^{k_1} \cdots x_{l-1}^{k_{(l-1)}} x_l^{k_l - 1} x_{l+1}^{k_{(l+1)}} \cdots x_j^{k_j} \, u_l$$

is in $\mathcal{L}(X,Y)$ and so $m x_1^{k_1} \cdots x_{l-1}^{k_{(l-1)}} x_l^{k_l - 1} x_{l+1}^{k_{(l+1)}} \cdots x_j^{k_j}$ may be identified with an element of $\mathcal{L}(X,Y)$. This yields a shorthand notation for the partial derivative of h in part (b) of the preceding Exercise:

$$\partial_{x_l} h[(x_1, \cdots, x_j)] = k_l \; m x_1^{k_1} \cdots x_{l-1}^{k_{(l-1)}} x_l^{k_l - 1} x_{l+1}^{k_{(l+1)}} \cdots x_j^{k_j}\,.$$

$\square$

Next we have an elementary exercise using induction on the order of differentiation.

Exercise 1.8. Suppose that $m \in \mathcal{M}^p(X, Y)$ and define $F : X \to Y$ by

$$F(x) = m(x, \cdots, x), \quad x \in X.$$

Then $d^k F[0] = 0$ except when $k = p$ and

$$d^p F[0](x_1, x_2, \cdots, x_p) = \sum_{\pi \in S_p} m(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(p)}).$$

□

2 Analytic Operators

Let X and Y be Banach spaces over $\mathbb{F}$. Let U be an open subset of X.

Definition 2.1. *A mapping $F : U \to Y$ is $\mathbb{F}$- analytic at $x_0 \in U$ if, for all $x \in U$ with $\|x - x_0\|$ sufficiently small,*

$$F(x) = \sum_{k=0}^{\infty} m_k (x - x_0)^k \tag{4}$$

where $F(x_0) = m_0(x - x_0)^0 = m_0 \in Y$, $m_k \in \mathcal{M}^k(X, Y)$ (which depends on x_0) is symmetric and there exists $r > 0$ such that

$$\sup_{k \geq 0} r^k \|m_k\| = M < \infty. \tag{5}$$

The series on the right in (4) is a power series in $x - x_0$. The function F is said to be $\mathbb{F}$-analytic on U if it is $\mathbb{F}$-analytic at every point of U. (When it does not matter we will omit $\mathbb{F}$ and speak of analytic functions. The expressions analytic mapping, analytic operator and analytic function will be used interchangeably in what follows.)

Because of (5), for all $x \in X$ with $\|x - x_0\| < r$,

$$\sum_{k=0}^{\infty} \|m_k(x - x_0)^k\| \leq M \sum_{k=0}^{\infty} \frac{\|x - x_0\|^k}{r^k} = \frac{Mr}{r - \|x - x_0\|} < \infty$$

and therefore the sequence $\left\{ \sum_{k=0}^{n} m_k(x - x_0)^k \right\}$ of partial sums is summable in norm. For an open set $U \subset X$ let $C^\infty(U, Y)$ denote the set of all functions $F : U \to Y$ which have derivatives of all orders at every point of U,

$$C^\infty(U, Y) = \cap_{k=1}^{\infty} C^k(U, Y).$$

If $F \in C^\infty(U, Y)$ and x_0, $x \in U$ then, by Taylor's theorem,

$$F(x) - \sum_{k=0}^{n} \frac{1}{k!} d^k F[x_0](x - x_0)^k = R_n(x, x_0) \tag{6}$$

where

$$\|R_n(x, x_0)\| \leq \frac{\|x - x_0\|^{n+1}}{(n+1)!} \sup_{0 \leq t \leq 1} \|d^{n+1} F[(1 - t)x_0 + tx]\|.$$

Here $R_n(\cdot, x_0)$ is called the remainder after n terms in the Taylor expansion of F at x_0. Since $dF^k[x_0] \in \mathcal{M}(X, Y)$ is symmetric for all $k \in \mathbb{N}$ when $F \in C^\infty(U, Y)$, its analyticity at $x_0 \in U$ is equivalent (see Proposition 2.4 below) to the convergence of $R(x, x_0)$ to zero at every point x in a neighbourhood of x_0.

Theorem 2.2. *Suppose that $U \subset X$ is open and $F \in C^\infty(U, Y)$. Suppose also that for each $x_0 \in U$ there exist constants r, C, $R > 0$, all depending on x_0, such that*

$$\left\| d^k F[x] \right\| \leq \frac{C\, k!}{R^k} \text{ for all } x \in U \text{ with } \|x - x_0\| < r. \tag{7}$$

Then F is analytic on U.

Remark 2.3. Since $e^n \geq n^n/n!$, (7) is equivalent to

$$\|d^k F[x]\| \leq \frac{C\,k^k}{R^k} \text{ for all } x \in U \text{ with } \|x - x_0\| < r,$$

for different constants C and R. □

Proof. Let $x_0,\, x \in U$. Then in (6),

$$\begin{aligned}\|R_n(x, x_0)\| &\leq \frac{\|x - x_0\|^{n+1}}{(n+1)!} \sup_{0\leq t\leq 1} \|d^{n+1}F[(1-t)x_0 + tx]\,\| \\ &\leq \frac{C}{R^{n+1}}\|x - x_0\|^{n+1} \to 0 \text{ as } n \to \infty\end{aligned}$$

if $\|x - x_0\| < \min\{r,\, R\}$. This proves the result. □

We aim to prove a converse of Theorem 2.2, but first another observations. Differentiating the identity

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x| < 1,$$

p times yields the new identity

$$\Big(\frac{1}{1-x}\Big)^{p+1} = \sum_{k=0}^{\infty} \binom{k+p}{p} x^k, \quad |x| < 1, \quad p \in \mathbb{N}. \tag{8}$$

Suppose that $r > 0$ and $\{m_k\}$ is a sequence of symmetric k-linear operators on X with $r^k\|m_k\| \leq M$ for all $k \in \mathbb{N}$. Let $x,\, z_1, \cdots, z_p \in X$ with $\|x\| < (1-\epsilon)r$, $\epsilon \in (0, 1)$, and note that, by (8),

$$\begin{aligned}&\sum_{k=0}^{\infty} \binom{k+p}{p} \|m_{k+p}\, x^k z_1 \cdots z_p\| \\ &\leq \sum_{k=0}^{\infty} \binom{k+p}{p} \|m_{k+p}\|\, \|x\|^k \|z_1\| \cdots \|z_p\| \\ &\leq M \frac{\|z_1\| \cdots \|z_p\|}{r^p} \sum_{k=0}^{\infty} \binom{k+p}{p} \Big(\frac{\|x\|}{r}\Big)^k \\ &= M \frac{\|z_1\| \cdots \|z_p\|}{r^p} \Big(\frac{r}{r - \|x\|}\Big)^{p+1} \\ &\leq \epsilon^{-1} M \Big(\frac{\|z_1\| \cdots \|z_p\|}{\epsilon^p\, r^p}\Big).\end{aligned}$$

Hence, when $\|x\| < (1-\epsilon)r$, a symmetric operator $M_p^x \in \mathcal{M}^p(X, Y)$ may be defined at $(z_1, \cdots, z_p) \in X^p$ by

$$M_p^x(z_1, \cdots, z_p) = \sum_{k=0}^{\infty} \binom{k+p}{p} m_{k+p}\, x^k z_1 \cdots z_p \in Y$$

and

$$\|M_p^x\| \leq \frac{M}{r^p \epsilon^{p+1}}. \tag{9}$$

Also, if $\|x\| < (1-\epsilon)r$ and $\|z\| < \epsilon\, r$, then

$$\sum_{k,p=0}^{\infty} \binom{k+p}{p} \|m_{k+p}\, x^k z^p\| \leq \frac{M}{\epsilon} \sum_{p=0}^{\infty} \Big(\frac{\|z\|}{\epsilon r}\Big)^p < \infty, \tag{10}$$

and so the series

$$\sum_{k,p=0}^{\infty} \binom{k+p}{p} m_{k+p}\, x^k z^p$$

is summable in norm, and therefore convergent to a sum which is independent of the order in which summation is done. This leads to the following result in which (12) is a converse of Theorem 2.2.

Proposition 2.4. *Let* (5) *hold and let* F *be defined by* (4)*. Then* F *is analytic at every point* x *of the set* $U_0 = \{x \in U : \|x - x_0\| < r\}$,

$$F \in C^\infty(U_0, Y) \text{ and } m_k = \frac{d^k F[x_0]}{k!}, \text{ for all } k \geq 0.$$

Also the p^{th} *derivative,* $d^p F$*, is analytic at* x *for all* $p \in \mathbb{N}$ *and* $\|x - x_0\| < r$,

$$d^p F[x](x_1, \cdots, x_p) = \sum_{k=0}^{\infty} \frac{(k+p)!}{k!} m_{k+p}(x - x_0)^k x_1 x_2 \cdots x_p\,, \tag{11}$$

and there exists $C > 1$, $R \in (0,1)$ *such that if* $\|x - x_0\| \leq \frac{1}{2} r$,

$$\|d^p F[x]\| \leq C \frac{p!}{R^p} \text{ for all } p \in \mathbb{N}. \tag{12}$$

If $K \subset U$ *is a compact set, then there exists* C *and* R *such that* (12) *holds for all* $x \in K$.

Proof. We show first that if $\|\widehat{x} - x_0\| < r$ then F is analytic at $\widehat{x}$. This follows from (10) since

$$\begin{aligned} F(x) &= \sum_{l=0}^{\infty} m_l (x - x_0)^l = \sum_{l=0}^{\infty} m_l \big((x - \widehat{x}) + (\widehat{x} - x_0)\big)^l \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{l} \binom{l}{k} m_l\, (x - \widehat{x})^{l-k} (\widehat{x} - x_0)^k \\ &= \sum_{k=0}^{\infty} \sum_{l=k}^{\infty} \binom{l}{k} m_l\, (x - \widehat{x})^{l-k} (\widehat{x} - x_0)^k \\ &= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \binom{k+p}{k} m_{p+k}\, (x - \widehat{x})^p (\widehat{x} - x_0)^k \\ &= \sum_{p=0}^{\infty} M_p^{\widehat{x} - x_0} (x - \widehat{x})^p \in Y. \end{aligned} \tag{13}$$

Since (9) holds, this shows that F is analytic at $\widehat{x}$ provided $\|\widehat{x} - x_0\| < r$.

Next, observe that when $\|x - x_0\| < r$,

$$\|F(x) - F(x_0) - m_1(x - x_0)\| = \Big\| \sum_{l=2}^{\infty} m_l (x - x_0)^l \Big\| = o(\|x - x_0\|)$$

as $\|x - x_0\| \to 0$, by (5). Therefore F is Fréchet differentiable at x_0 and

$$dF[x_0]x = m_1 x \quad \text{for all } x \in X.$$

By the same token it follows from (13) that when $\|\widehat{x} - x_0\| < r$, F is differentiable at $\widehat{x}$ and

$$dF[\widehat{x}]x = M_1^{(\widehat{x} - x_0)} x = \sum_{k=0}^{\infty} (k+1) m_{k+1} (\widehat{x} - x_0)^k\, x.$$

Thus dF is analytic at x_0 since, for x with $\|x - x_0\|$ sufficiently small,

$$dF[x] = \sum_{k=0}^{\infty} (k+1) m_{k+1} (x - x_0)^k$$

where the right-hand side is regarded as an element of $\mathcal{L}(X, Y)$ expressed as a power series. (To confirm that (5) holds here note that

$$(k+1)\|m_{k+1}\| \leq M(k+1) r^{-(k+1)} \leq \{M/r\}(2/r)^k.$$

Now replace r with $r/2$ and M with M/r.)

It now follows by induction that all the derivatives of F exist at every point of x with $\|x - x_0\| < r$ and are given by a power series in $x - x_0$. The formula (11) also follows by induction. Therefore for $p \in \mathbb{N}$ and $x \in X$ with $\|x - x_0\| \le \frac{1}{2} r$,

$$\begin{aligned}
\|d^p F[x]\| &\le \sum_{k=0}^{\infty} \frac{(k+p)!}{k!} \|m_{k+p}\| \, \|x - x_0\|^k \\
&\le \sum_{k=0}^{\infty} \frac{(k+p)!}{k!} \frac{M}{r^{k+p}} \Big(\frac{r}{2}\Big)^k = M \frac{p!}{r^p} \sum_{k=0}^{\infty} \frac{(k+p)!}{k!\, p!} \big(\frac{1}{2}\big)^k \\
&= 2M \frac{2^p \, p!}{r^p} \text{ by (8).}
\end{aligned}$$

Finally, if $K \subset U$ is compact then (12) holds uniformly for $x \in K$, since every open cover of K has a finite sub-cover. This completes the proof. □

Here are important, non-trivial examples of analytic operators.

Example 2.5. (Operator Inverses) Suppose $T \in \mathcal{L}(X, Y)$ is a bijection. Then there exists $\epsilon > 0$ such that if $L \in \mathcal{L}(X, Y)$ and $\|L - T\| < \epsilon$, then $L^{-1} \in \mathcal{L}(Y, X)$ and

$$L^{-1} = \big(I - T^{-1}(T - L)\big)^{-1} T^{-1} = \sum_{k=0}^{\infty} \big(T^{-1}(T - L)\big)^k T^{-1}.$$

Let a symmetric $m_k \in \mathcal{M}^k\big(\mathcal{L}(X, Y), \mathcal{L}(Y, X)\big)$ be defined by

$$\begin{aligned}
&m_k(L_1, \cdots, L_k) \\
&\qquad = \frac{1}{k!} \sum_{\pi \in S_k} T^{-1} \circ L_{\pi(1)} \circ T^{-1} \circ L_{\pi(2)} \circ \cdots \circ T^{-1} \circ L_{\pi(k)} \circ T^{-1}.
\end{aligned}$$

Thus

$$L^{-1} = \sum_{k=0}^{\infty} m_k (T - L)^k,$$

which shows that $L \mapsto L^{-1}$ from $\mathcal{L}(X, Y)$ to $\mathcal{L}(Y, X)$ is $\mathbb{F}$-analytic on the open set where it is defined. □

Example 2.6. (Nemytskii Operators on C) Suppose that $f : \mathbb{F}^N \to \mathbb{F}^M$ is continuously differentiable. Then a Nemytskii operator $F : C\big([0,1], \mathbb{F}^N\big) \to C\big([0,1], \mathbb{F}^M\big)$ can be defined by composition:

$$F(u)(t) = f(u(t)), \quad t \in [0,1], \quad u \in C\big([0,1], \mathbb{F}^N\big),$$

and it is not difficult to see that the nonlinear operator F is of class C^1 in this setting. □

However, nothing should be taken for granted, as the following example shows.

Example 2.7. (Nemytskii Operators on L_p) Let $L_p[0,1]$, $1 \le p < \infty$, denote the Banach space of p^{th} power Lebesgue integrable real-valued 'functions' $u : [0,1] \to \mathbb{F}$ with the norm

$$\|u\|_p = \Big(\int_0^1 |u(s)|^p ds \Big)^{1/p}.$$

Suppose that $g : \mathbb{R} \to \mathbb{R}$, $g(0) = 0$ and let

$$G(u)(x) = g(u(x)), \quad x \in [0,1], \quad u \in L_p[0,1].$$

Then if G maps $L_p[0,1]$ into $L_p[0,1]$ and is Fréchet differentiable at $0 \in L_p[0,1]$, then $g(t) = bt$ for some $b \in \mathbb{R}$. To see this, suppose that $g(s)/s \ne g(t)/t$ for some non-zero s, $t \in \mathbb{R}$. For $\delta \in [0,1)$, let $u_\delta = t\chi_{[0,\delta]}$ and $v_\delta = s\chi_{[0,\delta]}$, where $\chi_{[a,b]}$ is the function with value 1 on $[a,b]$ and zero otherwise. Then $v_\delta = su_\delta/t$

and clearly $\|u_\delta\|_p$ and $\|v_\delta\|_p$ tend to 0 as $\delta \to 0$. Now suppose that $dG[0] = L \in \mathcal{L}(L_p[0,1], L_p[0,1])$ exists. Then, since $G(0) = 0$, $\|G(u_\delta) - Lu_\delta\|_p / \|u_\delta\|_p \to 0$ and

$$\frac{\|(t/s)G(v_\delta) - Lu_\delta\|_p}{\|u_\delta\|_p} = \frac{\|G(v_\delta) - Lv_\delta\|_p}{\|v_\delta\|_p} \to 0$$

as $\delta \to 0$. Thus

$$\frac{|(t/s)g(s) - g(t)|}{|t|} = \frac{\|(t/s)G(v_\delta) - G(u_\delta)\|_p}{\|u_\delta\|_p} \to 0 \text{ as } \delta \to 0.$$

This contradiction shows that, in the setting of p^{th} power Lebesgue integrable functions, Fréchet differentiability at 0 of a Nemytskii operator implies that the operator in question is affine (linear + constant). □

Example 2.8. (Analytic Nemytskii Operators) Let $f : \mathbb{F}^N \to \mathbb{F}^M$ be $\mathbb{F}$-analytic. Then the Nemytskii operator F defined in Example 2.6 is $\mathbb{F}$-analytic on X. (Note that this is not the same as saying that the composition of two analytic functions is analytic.) To see that F is $\mathbb{F}$-analytic at $u_0 \in X$ we proceed as follows. Let $M = \sup\{|u_0(t)| : t \in [0,1]\}$. Let $C > 1$ and $R \in (0,1)$, given by the last part of Proposition 2.4, be such that

$$\|f^k[\xi]\| \le \frac{Ck!}{R^k} \quad \text{for } \xi \in \mathbb{F}^N \quad \text{with} \quad |\xi| \le M+1. \tag{14}$$

Now for $u \in X$ with $\|u\| \le M+1$, $t \in [0,1]$ and $n \in \mathbb{N}$ let

$$f_n(u(t)) = \sum_{k=0}^{n} \frac{f^{(k)}(u_0(t))}{k!}(u(t) - u_0(t))^k.$$

By Taylor's theorem,

$$f(u(t)) - f_n(u(t)) = R_n(u(t), u_0(t))$$

where

$$\|R_n(u(t), u_0(t))\| \le \frac{\|u(t) - u_0(t)\|^{k+1}}{(k+1)!} \sup_{0 \le s \le 1} \|f^{(k+1)}((1-s)u_0(t) + su(t))\|.$$

It follows from (14) that $f_n(u) \to f(u)$ in the Banach space X as $n \to \infty$ provided that $\|u - u_0\| < R$. It remains to define symmetric $m_k \in \mathcal{M}^k(X, X)$ by

$$m_k(v_1, \cdots, v_k)(t) = \frac{d^k f[u_0(t)]}{k!}\big(v_1(t), v_2(t), \cdots, v_k(t)\big)$$

so that

$$F(u) = \sum_{k=0}^{\infty} m_k(u - u_0)^k,$$

where the series on the right converges in $C\big([0,1], \mathbb{F}^M\big)$. Therefore the Nemytskii operator is $\mathbb{F}$-analytic on X. A similar argument leads to the same conclusion for the same Nemytskii operator regarded as a mapping on $C^n\big([0,1], \mathbb{F}^N\big)$, and on other Banach spaces of functions as well. □

Remark 2.9. (Analytic Operators on Lebesgue Spaces) Suppose P is a polynomial of degree p and $P \circ u \in L_q[0,1]$ for all $u \in L_r[0,1]$. It is easy to see that $pq \le r$.

Consider the mapping $f(u) = u^p$ where $p \in \mathbb{N}$ and $pq \le r$. It follows easily that the k^{th}-derivative, $k \le p$, of f at $u_0 \in L_r[0,1]$ is given by

$$df[u_0](u_1, \cdots, u_k) = \frac{p!}{(p-k)!} u_0^{p-k} u_1 \cdots u_k,$$

which, by Hölder's inequality is a bounded k-linear operator on $L_r[0,1]$. All higher derivatives of f are zero. The Nemytskii operator from $L_r[0,1]$ to $L_q[0,1]$ so defined is then $\mathbb{F}$-analytic. For example $u \mapsto u^2$ is analytic from $L_3[0,1]$ to $L_1[0,1]$. (However as it is well known that there are no non-trivial Fréchet differentiable mappings from from $L_p[0,1]$ to itself, there no $\mathbb{F}$-analytic operators, from $L_p[0,1]$ to itself either, $1 \le p < \infty$.) □

It is well known that if $U \subset \mathbb{C}$ is open and connected and $f : U \to \mathbb{C}$ is a non-constant $\mathbb{C}$-analytic function, then the zero set of f has no limit points in U. This is clearly false for $\mathbb{C}$-analytic functions from $\mathbb{C}^2$ into $\mathbb{C}^2$, as the following example shows.

Example 2.10. Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by

$$F(x, y) = (xy, xy)$$

for all $(x, y) \in \mathbb{C}^2$. Then F is $\mathbb{C}$-analytic because its Taylor series has one term: $F(x, y) = m_2(x, y)^2$ where, for $(x_1, y_1), (x_2, y_2) \in \mathbb{C}^2$,

$$m_2\big((x_1, y_1), (x_2, y_2)\big) = \tfrac{1}{2}(x_1y_2 + x_2y_1, x_1y_2 + x_2y_1).$$

However $F(x, y) = (0, 0)$ when $xy = 0$. Thus every point of the zero set of F is a limit point of the zero set. □

However we have the following result.

Theorem 2.11. *Suppose that X and Y are Banach spaces, that $U \subset X$ is an open connected set and that $F : U \to Y$ is $\mathbb{F}$-analytic. Suppose also that there is a non-empty open set $W \subset U$ on which F is identically 0. Then F is identically zero on U.*

Proof. Let $V \subset U$ be the set of points $x \in U$ at which all the derivatives of F are zero. The set V is non-empty since, by hypothesis, $W \subset V$. Since $F \in C^\infty(U, Y)$, V is an intersection of sets which are closed in U and so V is closed in U. Since F is an $\mathbb{F}$-analytic function, Proposition 2.4 implies that V is open, and therefore open in U. But U is connected, so $U = V$. This completes the proof. □

3 Analytic Functions of Two Variables

Let X, Y and Z be Banach spaces and U an open subset of $X \times Y$. Let $(x_0, y_0) \in U$ and let $F : U \to Z$ be analytic at (x_0, y_0). In other words, for (x, y) sufficiently close to (x_0, y_0),

$$F(x, y) = \sum_{k=0}^{\infty} m_k(x - x_0, y - y_0)^k \tag{15}$$

where $m_k \in \mathcal{M}^k(X \times Y, Z)$ is symmetric with $\sup_{k \geq 0} r^k \|m_k\| < \infty$ for some $r > 0$. Now let

$$\mathcal{M}^{p,\,q}(X, Y; Z) = \mathcal{M}(\underbrace{X, \cdots, X}_{p \text{ times}}, \underbrace{Y, \cdots, Y}_{q \text{ times}}; Z),$$

and define $m_{p,q} \in \mathcal{M}^{p,q}(X, Y; Z)$ by

$$m_{p,q}(x_1, \cdots, x_p, y_1, \cdots, y_q) \\ = m_{p+q}\big((x_1, 0), \cdots, (x_p, 0), (0, y_1), \cdots, (0, y_q)\big).$$

It follows that $\|m_{p,q}\| \leq \|m_{p+q}\|$ and, since $(x, y) = (x, 0) + (0, y)$, an argument similar to that for (13) now yields that

$$F(x, y) = \sum_{p,\, q \geq 0} \frac{(p+q)!}{p!\, q!} m_{p,q}((x - x_0)^p, (y - y_0)^q), \tag{16}$$

$$\sup_{p,\, q \geq 0} \big\{\|m_{p,\, q}\| r^{p+q}\big\} < \infty, \tag{17}$$

where $r > 0$. Note that $m_{p,q}$ is symmetric separately in the first p, and in the last q, variables. The series

$$\sum_{p,\, q \geq 0} \frac{(p+q)!}{p!\, q!} m_{p,q}((x - x_0)^p, (y - y_0)^q)$$

is summable in norm for $\|x - x_0\| + \|y - y_0\| < r$. It follows that

$$m_{p,q} = \frac{\partial_x^p \partial_y^q F[(x_0, y_0)]}{(p+q)!}$$

where $\partial_x^p \partial_y^q F[(x_0, y_0)] \in \mathcal{M}^{p,q}(X, Y; Z)$, and hence

$$F(x, y) = \sum_{p,\, q \geq 0} \frac{\partial_x^p \partial_y^q F[(x_0, y_0)]}{p!\, q!}((x - x_0)^p, (y - y_0)^q), \tag{18}$$

$$\sup_{p,\, q \geq 0} \Big\{ \frac{\|\partial_x^p \partial_y^q F[(x_0, y_0)]\| r^{p+q}}{(p+q)!} \Big\} < \infty, \tag{19}$$

for some $r > 0$. Conversely, (16) and (17) defines an $\mathbb{F}$-analytic mapping, in the sense of Definition 2.1, which satisfies (18) and (19).

4 Analytic Inverse and Implicit Function Theorems

We begin by proving the analytic version of the inverse function theorem. Note that the proof here does not assume that the composition of two analytic functions is analytic; instead this emerges as a corollary of the theorem.

Let X be a Banach space and let $B_r(X)$, $r \in (0, 1)$, denote the ball of radius r centred at $0 \in X$. For $r \in (0, 1)$, let $\mathcal{B}_r = B_{r^2}(X) \times B_r(X) \subset X \times X$ and let $(E_r, \|\cdot\|_r)$ denote the Banach space of functions u which are analytic from $\mathcal{B}_r$ into X with

$$u(x, y) = \sum_{m,\, n \geq 0} u_{m,n} x^m\, y^n, \quad (x, y) \in X \times X,$$

where

$$\|u\|_r = \sum_{m,\, n \geq 0} \|u_{m,n}\| r^{2m+n} < \infty.$$

Note that the norm $\|\cdot\|_r$ uses different weights for the x and y dependence of functions in E_r. However, since (15) implies (16) and (17), *any* function F which maps a neighbourhood of the origin in $X \times X$ analytically into X belongs to E_r for all $r > 0$ sufficiently small. That E_r is a Banach space follows by a proof, almost identical to that of Proposition 6.3 (below), in which the completeness of $\mathbb{F}$ is replaced with the completeness (Proposition 1.2) of the space of k-linear operators on X for all $k \in \mathbb{N}_0$. Let F_r denote the closed subspace of E_r of functions $u \in E_r$ with

$$u(x, y) = \sum_{m \geq 0,\, n \geq 1} u_{m,n} x^m\, y^n.$$

Define $L \in \mathcal{L}(F_r, F_r)$ by

$$Lu\,(x, y) = \sum_{m \geq 0,\, n \geq 1} \frac{u_{m,n}}{n} x^m\, y^n, \quad (x, y) \in \mathcal{B}_r, \quad u \in F_r.$$

Clearly $\|L\| = 1$. Now for $w \in E_r$ arbitrary but fixed define $L_w u$ for $u \in F_r$ by

$$L_w u\,(x, y) = \partial_y u[(x, y)] w(x, y) - \partial_y u[(x, 0)] w(x, 0), \quad (x, y) \in \mathcal{B}_r.$$

Lemma 4.1. *(a) The operator $L_w \circ L \in \mathcal{L}(F_r, F_r)$ and $\|L_w \circ L\| \leq \|w\|_r / r$. (b) Let $w_0 \in E_r$ be defined by $w_0(x, y) = y$, $(x, y) \in \mathcal{B}_r$. Then $L_{w_0} \circ L$ is the identity on F_r.*

Proof. Let $w(x,y) = \sum_{p,\,q\geq 0} w_{p,q}x^py^q$. Then for $u \in F_r$,

$$\begin{aligned}
&L_w \circ Lu\,(x,y)\\
&= \sum_{m,n\geq 0} u_{m,n+1}x^my^n\Big(\sum_{p,\,q\geq 0} w_{p,q}x^py^q\Big) - \sum_{m\geq 0} u_{m,1}x^m\Big(\sum_{p\geq 0} w_{p,0}x^p\Big)\\
&= \sum_{M\geq 0,\,N\geq 1}\ \sum_{\substack{m+p=M\\ n+q=N}} u_{m,n+1}x^my^n\big(w_{p,q}x^py^q\big),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\|L_w \circ Lu\|_r &\leq \sum_{M\geq 0,\,N\geq 1} r^{2M+N}\Big(\sum_{\substack{m+p=M\\ n+q=N}} \|u_{m,n+1}\|\,\|w_{p,q}\|\Big)\\
&= \frac{1}{r}\sum_{M\geq 0,\,N\geq 1}\Big(\sum_{\substack{m+p=M\\ n+q=N}} \big(r^{2m+n+1}\|u_{m,n+1}\|\big)\big(r^{2p+q}\|w_{p,q}\|\big)\Big)\\
&\leq \frac{1}{r}\Big(\sum_{\substack{m\geq 0\\ n\geq 0}} r^{2m+n+1}\|u_{m,n+1}\|\Big)\Big(\sum_{\substack{p\geq 0\\ q\geq 0}} r^{2p+q}\|w_{p,q}\|\Big)\\
&= \frac{\|u\|_r\|w\|_r}{r} < \infty.
\end{aligned}$$

It now follows that $L_w \circ L \in \mathcal{L}(F_r, E_r)$ and $\|L_w \circ L\| \leq \|w\|_r/r$. Since $L_w \circ Lu\,(x,0) = 0 \in X$, $L_w \circ L \in \mathcal{L}(F_r, F_r)$ and part (a) is proven.

(b) It is immediate from the definitions that $L_{w_0} \circ L$ is the identity operator on F_r. □

Proposition 4.2. *Suppose that F maps a neighbourhood of the origin in X analytically into X with $F(0) = 0$ and $dF[0] = I$. Then there exist open neighbourhoods V, U of the origin in X and an $\mathbb{F}$-analytic function $G : V \to X$ such that the following statements are equivalent:*

$$F(y) = x,\ \ y \in U \quad \textit{and} \quad G(x) = y,\ \ x \in V.$$

Proof. Suppose that F is analytic from a neighbourhood of $0 \in X$ into X with $F(0) = 0$ and $dF[0] = I$. For $r > 0$ sufficiently small let $v,\, w \in E_r$ be defined for $(x,y) \in \mathcal{B}_r$ by

$$v(x,y) = F(y) - x \text{ and } w(x,y) = v(x,y) - w_0(x,y) = F(y) - x - y.$$

Then

$$w(x,y) = -x + \sum_{n\geq 2}\frac{d^nF[0]}{n!}y^n,$$

and

$$\|w\|_r \leq r^2 + \sum_{n\geq 2}\frac{\|d^nF[0]\|}{n!}r^n \leq r^2C(F),$$

where $C(F)$ is a constant determined by F. From the definitions

$$L_v \circ L - I = (L_v - L_{w_0}) \circ L = L_w \circ L,$$

and hence, by the preceding lemma, $\|L_v \circ L - I\| \leq rC(F)$ for $r > 0$ sufficiently small. Therefore, for $r > 0$ sufficiently small, $L_v \circ L$ is a homeomorphism on F_r (since the set of homeomorphisms between Banach spaces is open in the operator topology). So there exists $u_0 \in F_r$ such that $L_v \circ Lu_0 = w_0$ and, for all $(x,y) \in \mathcal{B}_r$,

$$L_v \circ Lu_0\,(x,y) = \partial_y(Lu_0)[(x,y)]v(x,y) - \partial_y(Lu_0)[(x,0)]v(x,0) = y.$$

In particular, when $y \in X$ and $t \in (0,1]$ is sufficiently small,

$$ty = L_v \circ Lu_0\,(0,ty) = \partial_y(Lu_0)[(0,ty)]\big(F(ty) - F(0)\big),$$

which (dividing by t and letting $t \to 0$) gives that $\partial_y(Lu_0)[(0,0)] = I$, the identity on X. Hence there exists ϵ with $0 < \epsilon < r$ such that if $(x,y) \in \mathcal{B}_\epsilon$, then $\partial_y(Lu_0)[(x,y)]$ is a bijection on X. Moreover, for $(x,y) \in \mathcal{B}_\epsilon$,

$$\partial_y(Lu_0)[(x,y)]\big(F(y) - x\big) = y - G(x), \tag{20}$$

where G is defined on $B_{\epsilon^2}(X)$ by

$$G(x) = -\partial_y(Lu_0)[(x,0)]v(x,0) = \partial_y(Lu_0)[(x,0)]x.$$

It is clear that G is $\mathbb{F}$-analytic. Let $V = B_{\epsilon^2}(X) \cap G^{-1}(B_\epsilon(X))$ and $U = B_\epsilon(X) \cap F^{-1}(B_{\epsilon^2}(X))$, open neighbourhoods of $0 \in X$. Since $\partial_y(Lu_0)[(x,y)]$ in (20) is a bijection, this completes the proof. $\square$

This result has four important corollaries.

Analytic Inverse Function Theorem

Theorem 4.3. *Suppose that X, Y are Banach spaces, that $x_0 \in U \subset X$, where U is open. Suppose also that $F : U \to Y$ is analytic and that $dF[x_0] \in \mathcal{L}(X,Y)$ is a homeomorphism. Then there exist opens sets U_0 and V_0 with $x_0 \in U_0 \subset U$, $F(x_0) \in V_0 \subset Y$ and an analytic map $G : V_0 \to X$ such that*

$$\text{for } x \in U_0, \quad F(x) \in V_0 \ \text{ and } \ G(F(x)) = x,$$

and

$$\text{for } y \in V_0, \quad G(y) \ \in U_0 \ \text{ and } \ F(G(y)) = y.$$

Proof. Let $\widetilde{U} = U - x_0$, replace $F : U \to Y$ with $\widetilde{F} : \widetilde{U} \to X$ defined by

$$\widetilde{F}(y) = \big(dF[x_0]\big)^{-1}\big(F(y + x_0) - F(x_0)\big),$$

and apply the preceding proposition. $\square$

Analytic Implicit Function Theorem

Theorem 4.4. *Let X, Y and Z be Banach spaces and suppose that $U \subset X \times Y$ is open. Let $(x_0, y_0) \in U$ and suppose that $F : U \to Z$ is analytic where the partial derivative $\partial_x F[(x_0,y_0)] \in \mathcal{L}(X,Z)$ is a homeomorphism.*

Then there exists an open neighbourhood $V \subset Y$ of y_0, an open set $W \subset U$ and an $\mathbb{F}$-analytic mapping $\phi : V \to X$ such that

$$(x_0, y_0) \in W \ \text{ and } F^{-1}(z_0) \cap W = \{(\phi(y), y) : y \in V\}.$$

Proof. This follows from Theorem 4.3 in the usual way. $\square$

Definition 4.5. *A set $M \subset \mathbb{F}^n$ is called an m-dimensional $\mathbb{F}$-analytic manifold if, for all points $a \in M$, there is an open neighbourhood U_a of $0 \in \mathbb{F}^m$ and an analytic function $f : U_a \to M$ such that $f(0) = a$, $df[0]$ is a finite-dimensional linear transformation of rank m, and f maps open sets of U onto relatively open sets of M.*

Remark 4.6. Suppose that $F : \mathbb{F}^n \times \mathbb{F}^m \to \mathbb{F}^n$ is an $\mathbb{F}$-analytic mapping with $F(x_0, y_0) = z_0$ and that $\partial_x F[(x_0, y_0)]$ is a bijection on $\mathbb{F}^n$. Then the analytic implicit function theorem 4.4 defines an $\mathbb{F}$-analytic manifold of dimension m by the equation $F(x,y) = z_0$ for (x,y) in a neighbourhood of (x_0, y_0). $\square$

Composition of Analytic Functions

Theorem 4.7. *Suppose X, Y and Z are Banach spaces and that $U \subset X$ and $V \subset Y$ are open. Suppose that $F : U \to V$ and $G : V \to Z$ are $\mathbb{F}$-analytic. Then $G \circ F : U \to Z$ is $\mathbb{F}$-analytic.*

Proof. Let $W = U \times V \times Z$ and define an analytic function $H : W \to Y \times Z$ by

$$H(x, y, z) = \big(F(x) - y, G(y) - z\big).$$

Let $x_0 \in U$, $y_0 = F(x_0) \in V$ and $z_0 = G(y_0)$. Then $H(x_0, y_0, z_0) = (0, 0) \in Y \times Z$ and

$$\partial_{(y,z)} H[(x_0, y_0, z_0)](y, z) = (-y, dG[y_0]y - z) = (\widehat{y}, \widehat{z})$$

if and only if

$$y = -\widehat{y} \quad \text{and} \quad z = -\widehat{z} - dG[y_0]\widehat{y}.$$

Thus $\partial_{(y,z)} H[(x_0, y_0, z_0)]$ is a homeomorphism. By the analytic implicit function theorem 4.4 the solution set, in a neighbourhood of (x_0, y_0, z_0), of the equation $H(x, y, z) = (0, 0)$ is described by $(y, z) = (\widehat{Y}(x), \widehat{Z}(x))$, where $(\widehat{Y}, \widehat{Z})$ denotes a $Y \times Z$-valued, $\mathbb{F}$-analytic function defined on a neighbourhood of x_0 in X. But $H(x, y, z) = 0$, $(x, y, z) \in W$ implies that $G(F(x)) = z$. Hence $G(F(x)) = \widehat{Z}(x)$ for x in a neighbourhood of $x_0 \in X$. Since $x_0 \in U$ was chosen arbitrarily, it is immediate that $G \circ F$ is analytic. □

5 Bifurcation from a Simple Eigenvalue

At this point the method of Lyapunov-Schmidt reduction and the $\mathbb{R}$-analytic implicit function theorem in the theory of bifurcation from a simple eigenvalue, yields the following result on the branching of an *analytic curve* of non-trivial solutions from the curve of trivial solutions. We state this standard result with proof. Let X, Y be Banach spaces over $\mathbb{R}$. Suppose that

(G1) $F : \mathbb{R} \times X \to Y$ is $\mathbb{R}$-analytic, $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$ and, for some $\lambda_0 \in \mathbb{R}$,

$$\ker\big(\partial_x F[(\lambda_0, 0)]\big) = \{s\xi_0 : s \in \mathbb{R}\}, \quad \partial^2_{\lambda,x} F[(\lambda_0, 0)](1, \xi_0) \notin \text{range}\big(\partial_x F[(\lambda_0, 0)]\big).$$

Theorem 5.1. (Bifurcation from a simple eigenvalue.) *There exists an analytic function (Λ, κ) : $(-\epsilon, \epsilon) \to \mathbb{R} \times X$ such that*

$$F(\Lambda(s), \kappa(s)) = 0 \textit{ for all } s \in (-\epsilon, \epsilon), \quad \Lambda(0) = \lambda_0, \quad \kappa'(0) = \xi_0, \quad \dim\ker\big(\partial_x F[(\Lambda(s), \kappa(s))]\big) \in \{0, 1\}$$

and, for $0 < |s| < \epsilon$, $\dim\ker\big(\partial_x F[(\Lambda(s), s\chi(s))]\big) = 1$ if and only if $\Lambda'(s) = 0$.

□

6 $\mathbb{F}^n$ – Finite Dimensions – Preliminaries

Consider $\mathbb{F}^n$ as an $\mathbb{F}$-linear space the points of which are described by coordinates using the standard basis of real vectors of the form $(0, \cdots, 1, \cdots 0) \in \mathbb{F}^n$. For $x = (x_1, \cdots, x_n) \in \mathbb{F}^n$ and $p = (p_1, \cdots, p_n) \in \mathbb{N}_0^n$ let

$$|x|^2 = \sum_{j=1}^n |x_j|^2, \quad |p| = \sum_{j=1}^n p_j, \quad x^p = x_1^{p_1} \cdots x_n^{p_n}, \quad p! = p_1!p_2! \cdots p_n!\,,$$

$$\frac{\partial^p f}{\partial x^p} = \frac{\partial f^{|p|}}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}}.$$

For $n \in \mathbb{N}$, let $U \subset \mathbb{F}^n$ be an open neighbourhood of $0 \in \mathbb{F}^n$ and $f : U \to \mathbb{F}$ an $\mathbb{F}$-analytic function. Then an induction argument starting with the case $n = 2$ in (18) and (19) leads to

$$f(x) = \sum_{p \in \mathbb{N}_0^n} f_p \, x^p$$

where

$$f_p = \frac{1}{p!} \frac{\partial^p f}{\partial x^p}(0) \in \mathbb{F} \text{ and } \sup_{p \in \mathbb{N}_0^n} \frac{p!}{|p|!} |f_p| r^{|p|} < \infty$$

for some $r > 0$. This in turn leads to $\sum_{p \in \mathbb{N}_0^n} r^{|p|} |f_p| < \infty$ for some $r > 0$. A function so defined is analytic on the open neighbourhood $\big(B_r(\mathbb{F})\big)^n$ of 0 in $\mathbb{F}^n$. (Here $B_r(\mathbb{F})$ is the open disc with radius r centred at 0 in $\mathbb{F}$.)

Definition 6.1. *If $U \subset \mathbb{C}^n$ is open and $f : U \to \mathbb{C}$ is $\mathbb{C}$-analytic with the property that $f(x) \in \mathbb{R}$ for all $x \in U \cap \mathbb{R}^n$ we say that f is real-on-real. Equivalently f is real-on-real if and only if for all $x_0 \in U \cap \mathbb{R}^n$ all the terms f_p in the expansion of f at x_0 are real-on-real multilinear forms on $\mathbb{C}^n$.*

We use “real-on-real”, instead of “real”, to emphasis the complex setting.

Remark 6.2. Suppose that $f : \mathbb{C}^n \to \mathbb{C}$ is real-on-real. If the basis of $\mathbb{C}^n$ is changed to another real basis (a basis of vectors each of which has real coordinates with respect to the standard basis) and points of $\mathbb{C}^n$ are now described by coordinates $(\zeta_1, \cdots, \zeta_n)$ with respect to that basis, then the function f is a real-on-real function of $(\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$. Therefore a real-on-real $\mathbb{C}$-analytic function remains real-on-real after a real coordinate change of the independent variables. $\square$

Now we introduce linear spaces of $\mathbb{F}$-analytic functions as follows. For $q \in \mathbb{N}$ and $r > 0$, let $\mathcal{B}_r^q = \big(B_{r^{q+1}}(\mathbb{F})\big)^{n-1} \times B_r(\mathbb{F})$, an open neighbourhood of 0 in $\mathbb{F}^n$, and let C_r^q denote the space of $\mathbb{F}$-valued $\mathbb{F}$-analytic functions u on $\mathcal{B}_r^q$ with $u(0) = 0$ of the form

$$u(x) = \sum_{p \in \mathbb{N}_0^n, \, p \neq 0} u_p \, x^p \quad \text{where} \quad \sum_{p \in \mathbb{N}_0^n, \; p \neq 0} |u_p| r^{(q+1)|p| - q p_n} = \|u\|_{r,q} < \infty.$$

Proposition 6.3. *$\big(C_r^q, \|\cdot\|_{r,q}\big)$ is a Banach space. In fact it is a Banach algebra since it is closed under multiplication and $\|uv\|_{r,q} \leq \|u\|_{r,q} \|v\|_{r,q}$.*

Proof. Suppose that $\{u^k\}_{k \in \mathbb{N}} \subset C_r^q$ is a Cauchy sequence. Then the corresponding sequence $\{u_p^k\}_{k \in \mathbb{N}}$, $p \neq 0$, of Taylor coefficients of u at 0 is Cauchy in $\mathbb{F}$. Let $u_p^k \to u_p$ in $\mathbb{F}$. Since Cauchy sequences are bounded there exists a constant M such that for all P, $k \in \mathbb{N}$,

$$\sum_{p \in \mathbb{N}_0^n, \; 0 < |p| \leq P} |u_p^k| r^{(q+1)|p| - q p_n} \leq M.$$

In the limit as $k \to \infty$ with P fixed this yields

$$\sum_{p \in \mathbb{N}_0^n, \, 0 < |p| \leq P} |u_p| r^{(q+1)|p| - q p_n} \leq M.$$

Since this is true for all $P \in \mathbb{N}$

$$\sum_{p \in \mathbb{N}_0^n, \, p \neq 0} |u_p| r^{(q+1)|p| - q p_n} < \infty.$$

Let u denote the function in C_r^q the Taylor coefficients of which are u_p, $p \neq 0$, and $u_0 = 0$. Let $\epsilon > 0$ be given and, using the fact that $\{u^k\}$ is Cauchy in C_r^q, choose $N \in \mathbb{N}$ such that for all $P \in \mathbb{N}$,

$$\sum_{\substack{p \in \mathbb{N}_0^n, \, p \neq 0 \\ |p| \leq P}} |u_p^n - u_p^k| r^{(q+1)|p| - q p_n} \leq \epsilon$$

if k, $n \geq N$. With P arbitrary (but fixed) let $n \to \infty$ to obtain

$$\sum_{\substack{p \in \mathbb{N}_0^n,\, p \neq 0 \\ |p| \leq P}} |u_p - u_p^k| r^{(q+1)|p| - qp_n} \leq \epsilon.$$

Since this is true for any P, it follows that $\|u_k - u\|_{r,q} \leq \epsilon$ for all $k \geq N$. Thus C_r^q is a Banach space.

Now to prove that it is an algebra, let $\{u_p\}$ and $\{v_p\}$, $p \neq 0$, be the Taylor coefficients of u, $v \in C_r^q$ where $u_0 = v_0 = 0$. Let w be the product of u and v on the open set $\mathcal{B}_r^q$. Then, by Cauchy's product formula,

$$w_p = \sum_{\substack{s,\, t \in \mathbb{N}_0^n \\ s,\, t \neq 0 \\ s+t=p}} u_s v_t.$$

Therefore

$$\begin{aligned}
&\sum_{p \in \mathbb{N}_0^n,\, p \neq 0} |w_p| r^{(q+1)|p| - qp_n} \\
&\leq \sum_{p \in \mathbb{N}_0^n,\, p \neq 0} \sum_{\substack{s,\, t \in \mathbb{N}_0^n \\ s,\, t \neq 0 \\ s+t=p}} \big(r^{(q+1)|s| - qs_n} |u_s|\big)\big(r^{(q+1)|t| - qt_n} |v_t|\big) \\
&= \Big(\sum_{s \in \mathbb{N}_0^n,\, s \neq 0} |u_s| r^{(q+1)|s| - qs_n}\Big) \times \Big(\sum_{t \in \mathbb{N}_0^n,\, t \neq 0} |v_p| r^{(q+1)|t| - qt_n}\Big) \\
&= \|u\|_{r,q} \|v\|_{r,q}.
\end{aligned}$$

This proves that $w \in C_r^q$ and $\|w\|_{r,q} \leq \|u\|_{r,q}\|v\|_{r,q}$ when $w = uv$, and the proof is complete. □

Now we define linear operators A, L and B on C_r^q by

$$\begin{aligned}
Au(x) &= \sum_{\substack{p \in \mathbb{N}_0^n, \\ p_n < q}} u_p\, x^p, \\
Lu(x) &= \sum_{\substack{p \in \mathbb{N}_0^n, \\ p_n \geq q}} u_p\, x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} x_n^{p_n - q} = x_n^{-q}\big(I - A\big)u(x), \\
Bu(x) &= \sum_{\substack{p \in \mathbb{N}_0^n, \\ p_1 + \cdots + p_{n-1} > 0,}} u_p x^p, \qquad x \in \mathcal{B}_r^q, \quad u \in C_r^q.
\end{aligned}$$

Lemma 6.4.

$A \in \mathcal{L}(C_r^q, C_r^q)$ *with* $\|A\| = 1$;

$L \in \mathcal{L}(C_r^q, C_r^q)$ *with* $\|L\| \leq r^{-q}$;

$\|Bu\|_{r,q} \leq C(u) r^{1+q}$, *where* $C(u)$ *is a constant determined by* u.

Proof. A and B are projections and the results about them are obvious from their definitions and that of the norm on C_r^q. Note that for $p \in \mathbb{N}_0^n$, $p \neq 0$, the coefficient $(Lu)_p$ coincides with $u_{(p_1, \cdots, p_{n-1}, q+p_n)}$ and hence

$$\begin{aligned}
\|Lu\|_{r,q} &= \sum_{p \in \mathbb{N}_0^n,\, p \neq 0} |(Lu)_p|\, r^{(q+1)(p_1 + \cdots + p_{n-1}) + p_n} \\
&= \sum_{p \in \mathbb{N}_0^n,\, p \neq 0} |u_{(p_1, \cdots p_{n-1},\, q+p_n)}| r^{(q+1)(p_1 + \cdots + p_{n-1}) + p_n} \\
&= r^{-q} \sum_{p \in \mathbb{N}_0^n,\, p \neq 0} |u_{(p_1, \cdots p_{n-1}, q+p_n)}| r^{(q+1)(p_1 + \cdots + p_{n-1}) + q + p_n} \\
&\leq r^{-q} \|u\|_{r,q}.
\end{aligned}$$

Therefore $L \in \mathcal{L}(C_r^q, C_r^q)$ and $\|L\| \leq r^{-q}$. □

7 Weierstrass Division Theorem

Suppose that $f(0) = 0$ and close to the origin an analytic function f has order $q \geq 1$ the direction x_n. Then the division theorem says that f divides any another analytic function g with $g(0) = 0$, leaving a remainder that is a polynomial of degree at most $(q-1)$ in x_n, with coefficients that are analytic functions of the other $(n-1)$ coordinates.

Theorem 7.1. *Suppose* $0 \in U$ *(open)* $\subset \mathbb{F}^n$, $f : U \to \mathbb{F}$ *is analytic,* $f(0) = 0$ *and, for* $(0, \cdots, 0, x_n) \in U$,

$$f(0, \cdots, 0, x_n) = x_n^q v(x_n) \text{ where } v(0) \neq 0 \text{ and } q \geq 1.$$

Let $g : U \to \mathbb{F}$ *be any* $\mathbb{F}$*-analytic function with* $g(0) = 0$. *Then for some* $r > 0$,

$$g(x_1, \cdots, x_n) = h(x_1, \cdots, x_n) f(x_1, \cdots, x_n) + \sum_{k=0}^{q-1} h_k(x_1, \cdots, x_{n-1}) x_n^k$$

for all $(x_1, \cdots, x_n) \in U_0 = \mathcal{B}_r^q$, *where* h *is analytic on* U_0 *and* h_k *is analytic on* $V = \big(B_{r^{q+1}}(\mathbb{F})\big)^{n-1}$. *The functions* h_k *and* h *are uniquely determined by* f *and* g. *If* $\mathbb{F}^n = \mathbb{C}^n$ *and* f *and* g *are real-on-real, then* h_k *and* h *are real-on-real.*

Proof. Without loss of generality suppose that $v(0) = 1$. Choose $\widehat{r}$ such that $f, g \in C_r^q$ for all r with $0 < r \leq \widehat{r}$, where q is given by the hypothesis on f. We will prove the theorem by showing that the linear operator $\Gamma : C_r^q \to C_r^q$ defined, in the notation of Lemma 6.4, by

$$\Gamma u(x) = f(x) Lu(x) + Au(x), \quad x \in \mathcal{B}_r^q,$$

is a bijection on C_r^q provided $r > 0$ is sufficiently small. Let $v_0(x) = x_n^q$, $x \in U$. Then $\|v_0\|_{r,q} = r^q$,

$$f(0, \cdots, 0, x_n) = v(x_n) v_0(0, \cdots, x_n) \text{ and hence } f - v v_0 = Bf,$$

where v is in the statement of the theorem. Therefore by Proposition 6.3 and Lemma 6.4,

$$\begin{aligned}
\|f - v_0\|_{r,q} &\leq \|f - v v_0\|_{r,q} + \|v_0(1 - v)\|_{r,q} \\
&\leq \|Bf\|_{r,q} + \|v_0\|_{r,q} \|1 - v\|_{r,q} \\
&\leq C(f) r^{1+q} + r^q \|1 - v\|_{r,q}.
\end{aligned}$$

From the definition of L, $(\Gamma - I)u = (f - v_0) Lu$ for $u \in C_r^q$, and hence

$$\begin{aligned}
\|(\Gamma - I)u\|_{r,q} = \|(f - v_0) Lu\|_{r,q} &\leq \|Lu\|_{r,q} \|(f - v_0)\|_{r,q} \\
&\leq r^{-q} \|u\|_{r,q} \big(C(f) r^{1+q} + r^q \|1 - v\|_{r,q} \big).
\end{aligned}$$

Since $v(0) = 1$, $\|1 - v\|_{r,q} \to 0$ as $r \to 0$, and it follows that $\Gamma - I \in \mathcal{L}(C_r^q, C_r^q)$ with norm less that 1 if $r > 0$ is sufficiently small. Hence Γ is a bijection on C_r^q. Hence for $g \in C_r^q$ there is a unique $u \in C_r^q$ with $\Gamma u = g$. The uniqueness of h and h_k now follow from the definition of L and A.

If $\mathbb{F} = \mathbb{C}$ and f and g are real-on-real, then the theorem restricted to $\mathbb{R}^n \cap U$ yields $\mathbb{R}$-analytic functions h and h_k. By uniqueness when $\mathbb{F} = \mathbb{C}$ it follows that h and h_k are real-on-real. The proof is complete. ☐

8 Weierstrass Preparation Theorem

The preparation theorem, which is a special case of the division theorem, shows that when f is analytic in a neighbourhood of $0 \in \mathbb{F}^n$, the solution set of $f(x_1, \cdots, x_n) = 0$ coincides with the zero set of a polynomial in x_n of the form

$$x_n^q + \sum_{k=0}^{q-1} a_k(x_1, \cdots, x_{n-1}) x_n^k \quad \text{where} \quad a_k(0, \cdots, 0) = 0, \tag{21}$$

the coefficients of which are analytic functions of $x_1, \cdots, x_{n-1}$.

Theorem 8.1. *Suppose* $0 \in U$ *(open)* $\subset \mathbb{F}^n$, $f : U \to \mathbb{F}$ *is analytic,* $f(0) = 0$ *and, for* $(0, \cdots, 0, x_n) \in U$,

$$f(0, \cdots, 0, x_n) = x_n^q v(x_n) \text{ where } v(0) \neq 0 \text{ and } q \geq 1.$$

Then for $r > 0$ *sufficiently small,*

$$h(x_1, \cdots, x_n) f(x_1, \cdots, x_n) = x_n^q + \sum_{k=0}^{q-1} a_k(x_1, \cdots, x_{n-1}) x_n^k,$$

for all $(x_1, \cdots, x_n) \in U_0 = \mathcal{B}_r^q$, *where* $h(0) \neq 0$, h *and* $1/h$ *are analytic on* U_0, *and* a_k *is analytic on* $V = \big(B_{r^{q+1}}(\mathbb{F})\big)^{n-1}$ *with* $a_k(0) = 0$. *The functions* a_k *and* h *are uniquely determined by* f. *If* $U \subset \mathbb{C}^n$ *and* f *is real-on-real, then* h *and* a_k *are real-on-real.*

Proof. In the Weierstrass division theorem 7.1, choose $g(x) = x_n^q$. Then

$$x_n^q - \sum_{k=0}^{q-1} h_k(x_1, \cdots, x_{n-1}) x_n^k = h(x_1, \cdots, x_n) f(x_1, \cdots, x_n)$$

in the neighbourhood U_0 of $0 \in \mathbb{F}^n$. In particular,

$$x_n^q - \sum_{k=0}^{q-1} h_k(0, \cdots, 0) x_n^k = h(0, \cdots, 0, x_n) v(x_n) x_n^q.$$

Thus $h_k(0, \cdots, 0) = 0$, $0 \leq k \leq q-1$ and $h(0, \cdots, 0) \neq 0$, since $v(0) \neq 0$. Since the composition of analytic functions is analytic, $1/h$ is analytic in a neighbourhood of 0. Let $a_k = -h_k$ to complete the proof. $\square$

9 Riemann Extension Theorem

Before studying the level sets of $\mathbb{F}$-analytic functions here are a few elementary results about metric spaces.

Lemma 9.1. *(a) Suppose that a metric space* U *has a subset* G_1 *that is dense and connected. If* $G_1 \subset G \subset U$, *then* G *is connected.*

(b) Suppose that G *is an open, dense subset of a non-empty connected metric space* U. *Suppose also that each* $x \in U \setminus G$ *has an open neighbourhood* V_x *such that* $V_x \cap G$ *is connected. Then* G *is connected.*

Proof. (a) Suppose that G is not connected. Then there exist two non-empty, closed subsets F_1 and F_2 of U such that

$$G \subset F_1 \cup F_2, \quad F_1 \cap F_2 \cap G = \emptyset \quad \text{and} \quad G \cap F_i \neq \emptyset, \quad i = 1, 2.$$

Since G_1 is connected, G_1 is a subset of one of them, say $G_1 \subset F_1$. Hence $U = \overline{G_1} \subset F_1$, which is a contradiction. This proves (a).

(b) Suppose that G is not connected. Since G is open there exist non-empty, open, disjoint sets O_1 and O_2 whose union is G. Since U is connected, $G \neq U$, and since G is dense, $\overline{O}_1 \cup \overline{O}_2 = U \neq G$. We claim that $\overline{O}_1 \cap \overline{O}_2 = \emptyset$, which contradicts the connectedness of U and proves the result.

Suppose that this is false and that $x \in \overline{O}_1 \cap \overline{O}_2$. Then $x \notin O_1 \cup O_2 = G$ and so, by hypothesis, $V_x \cap G = V_x \cap (O_1 \cup O_2)$ is connected. But this is false since O_1 are O_2 are open and disjoint, and $O_1 \cap V_x$ and $O_2 \cap V_x$ are non-empty.

$\square$

Proposition 9.2. *Suppose that* $U \subset \mathbb{F}^n$ *is open and connected and that* $g_k : U \to \mathbb{F}$ *is* $\mathbb{F}$*-analytic,* $1 \leq k \leq m$. *Let* $E = \{x \in U : g_k(x) = 0 \in \mathbb{F},\ 1 \leq k \leq m\}$.
(a) If $E \neq U$, *then* $U \setminus E$ *is dense in* U.
(b) If, in addition, $\mathbb{F} = \mathbb{C}$, *then* $U \setminus E$ *is connected.*

Proof. (a) If $U \setminus E$ is not dense then E contains an open subset of U on which all the functions g_k are zero. Hence they are all identically zero on U, by Lemma 2.11. Since $E \neq U$, this is a contradiction which proves (a).

(b) First we observe from part (a) and Lemma 9.1(a) that it suffices to treat the case $m = 1$. Let $E = \{x \in U : g(x) = 0\}$ where $U \subset \mathbb{C}^n$ is open and $g : U \to \mathbb{C}$ is $\mathbb{C}$-analytic. By Lemma 9.1 (b) it will suffice, for every $x \in E$, to find an open neighbourhood $V_x \subset U$ such that $V_x \setminus E$ is connected. Without loss of generality suppose that $x = 0 \in U$ and that $g(0) = 0$. Suppose moreover that $g \not\equiv 0$ on U. Then we can choose the coordinates $(x_1, \cdots, x_n)$ such that $g(0, \cdots, 0, x_n) \not\equiv 0$. Since $x_n \mapsto g(0, \cdots, 0, x_n)$ is a $\mathbb{C}$-analytic function on a neighbourhood of $0 \in \mathbb{C}$, its zeros are isolated and therefore there exists $\epsilon > 0$ such that $g(0, \cdots, 0, x_n) \neq 0$ if $0 < |x_n| \leq \epsilon$. Hence, by continuity, there exists $\delta > 0$ such that $g(x_1, \cdots, x_n) \neq 0$ if $\sum_{k=1}^{n-1} |x_k| < \delta$ and $\frac{1}{2}\epsilon < |x_n| < \epsilon$. Let

$$V_0 = \left\{ (x_1, \cdots, x_n) : \sum_{k=1}^{n-1} |x_k| < \delta,\ |x_n| < \epsilon \right\}.$$

Now in $\mathbb{C}^n$ (but not in $\mathbb{R}^n$) the set

$$\widetilde{V}_0 = \left\{ (x_1, \cdots, x_n) : \sum_{k=1}^{n-1} |x_k| < \delta,\ \tfrac{1}{2}\epsilon < |x_n| < \epsilon \right\}$$

is a path-connected subset of V_0 on which g is nowhere zero. Moreover, for each fixed $(\widehat{x}_1, \cdots, \widehat{x}_{n-1})$ with $\sum_{k=1}^{n-1} |\widehat{x}_k| < \delta$, the analytic function $x_n \mapsto g(\widehat{x}_1, \cdots, \widehat{x}_{n-1}, x_n)$ has at most a finite number of zeros with $|x_n| \leq 3\epsilon/4$ and the set

$$\{x \in U : x = (\widehat{x}_1, \cdots, \widehat{x}_{n-1}, x_n),\ |x_n| \leq 3\epsilon/4, g(x) \neq 0\} \subset \mathbb{C}^n$$

is path-connected. Since $\widetilde{V}_0$ is path-connected, $V_0 \setminus E$ is path-connected, and hence connected. Since V_0 is an open neighbourhood of 0, where 0 represents an arbitrary point of E, this completes the proof. $\square$

The following is a particular case of a classical theorem which holds more generally.

Theorem 9.3. (Riemann Extension Theorem) *Suppose that $U \subset \mathbb{C}^n$ is open and $g_k : U \to \mathbb{C}$ is $\mathbb{C}$-analytic, $1 \leq k \leq m$. Let*

$$E = \{x \in U : g_k(x) = 0 \text{ for all } k,\ 1 \leq k \leq m\}$$

and suppose that f is $\mathbb{C}$-analytic on $U \setminus E$ with

$$\sup\{|f(x)| : x \in U \setminus E\} < \infty.$$

Then there exists a function $\widetilde{f}$ which is $\mathbb{C}$-analytic on U and $f = \widetilde{f}$ on $U \setminus E$.

Proof. Since

$$U \setminus E = \cup_{k=1}^{m} \big(U \setminus E_k\big) \text{ where } E_k = \{x \in U : g_k(x) = 0\}$$

it suffices to prove the required result when $m = 1$. Since analyticity is defined locally it will suffice to choose $x \in E$ and to show that the result is true when U is replaced by some open neighbourhood of x. Without loss of generality suppose that $0 \in E$ is the point in question. Let ϵ, $\delta > 0$ and V_0, a neighbourhood of $0 \in U \cap E$, be as defined in the proof of part (b) of the last proposition. Then for any fixed $(x_1, \cdots, x_{n-1})$ with $\sum_{k=0}^{n-1} |x_k| < \delta$, the set

$$\{z \in \mathbb{C} : |z| \leq 3\epsilon/4 \text{ and } (x_1, \cdots, x_{n-1}, z) \in E\} \subset \{z \in \mathbb{C} : |z| \leq 3\epsilon/4 \text{ and } g_1(x_1, \cdots, x_{n-1}, z) = 0\}$$

is finite and therefore $z \mapsto f(x_1, \cdots, x_{n-1}, z)$ is $\mathbb{C}$-analytic except at finitely many points and bounded on $\{z \in \mathbb{C} : |z| \leq 3\epsilon/4\}$. Note also that $z \mapsto f(x_1, \cdots, x_{n-1}, z)$ is analytic in a neighbourhood of the circle $|z| = 3\epsilon/4$ for any such fixed $(x_1, \cdots, x_{n-1})$, since then $(x_1, \cdots, x_{n-1}, z) \in \widetilde{V}_0$ and g does not vanish on

$\widetilde{V}_0$. Therefore the singularities of this function of z must be removable and, by Cauchy's integral formula, the function

$$\widetilde{f}(x_1, \cdots, x_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x_1, \cdots, x_{n-1}, 3\epsilon e^{it}/4)}{3\epsilon e^{it}/4 - x_n} \Big(\frac{3\epsilon e^{it}}{4}\Big) dt \tag{22}$$

extends f to all of V_0. For fixed $(x_1, \cdots, x_{n-1})$ as above, $x_n \mapsto \widetilde{f}(x_1, \cdots, x_n)$ is $\mathbb{C}$-analytic. However it is clear, from the definition of $\widetilde{f}$ given in (22) that $\widetilde{f}$ is $\mathbb{C}$- analytic on $V_0 \subset \mathbb{C}^n$. This completes the proof. □

The function $\widetilde{f}$ is called an analytic extension of f.

10 Polynomials

Constant Coefficients

A polynomial of the form

$$A(Z) = a_p Z^p + \ldots + a_1 Z + a_0, \; p \in \mathbb{N}_0, \; a_p \neq 0,$$

with complex coefficients is said to have degree p and a complex number z such that $A(z) = 0$ is called a root of $A(Z)$. The fundamental theorem of algebra says that $A(Z)$ has at most p roots, $z_1, \cdots, z_k$, say, and that

$$A(Z) = a_p (Z - z_1)^{m_1} \cdots (Z - z_k)^{m_k}, \text{ where } m_1 + \cdots + m_k = p \tag{23}$$

and the m_j s are distinct. In this factorization of $A(Z)$ over $\mathbb{C}$, the number m_j is called the multiplicity of the root z_j. If $m_j = 1$ then z_j is called a simple root of $A(Z)$, otherwise it is a multiple root. The coefficient of Z^p is called the principal coefficient of $A(Z)$.

Continuous Dependence of Roots

Proposition 10.1. *Let $p \geq 1$. (a) If $\widehat{z} \in \mathbb{C}$ is a simple root of a polynomial with complex coefficients*

$$Z^p + \widehat{a}_{p-1} Z^{p-1} + \cdots + \widehat{a}_0$$

then there is a $\mathbb{C}$-analytic function f, defined in a neighbourhood of $(\widehat{a}_0, \cdots, \widehat{a}_{p-1})$, such that $z = f(a_0, \cdots, a_{p-1})$ is a simple root of the polynomial $Z^p + a_{p-1} Z^{p-1} + \cdots + a_0$ and $\widehat{z} = f(\widehat{a}_0, \cdots, \widehat{a}_{p-1})$. If, in addition, $\widehat{a}_0, \cdots, \widehat{a}_{p-1}, \widehat{z}, a_0, \cdots, a_{p-1} \in \mathbb{R}$, then $f(a_0, \cdots, a_{p-1}) \in \mathbb{R}$.

(b) Suppose that $\widehat{z} \in \mathbb{C}$ is a root of multiplicity $q \geq 1$ of the polynomial in part (a), and the distance of all the other roots from $\widehat{z}$ is at least $\widehat{\epsilon} > 0$. Then, for all ϵ with $0 < \epsilon < \widehat{\epsilon}$ there exists $\delta > 0$ such that the polynomial

$$Z^p + a_{p-1} Z^{p-1} + \ldots + a_0$$

has exactly q complex roots, counted according to their multiplicities, in the set

$$\{z \in \mathbb{C} : |z - \widehat{z}| < \epsilon\}$$

provided that $|a_0 - \widehat{a}_0|, \ldots, |a_{p-1} - \widehat{a}_{p-1}| < \delta$.

(c) For all $\epsilon > 0$, there exist $\delta > 0$ such that $|z| < \epsilon$ for all $z \in \mathbb{C}$ with

$$z^p + a_{p-1} z^{p-1} + \cdots + a_0 = 0 \text{ when } |a_0|, \cdots, |a_{p-1}| < \delta.$$

Similarly, if $|a_0| + \cdots |a_{p-1}| \leq M$ then $|z| \leq m(M)$, where $m(M)$ depends only on M.

Proof. (a) Since $\widehat{z}$ is a simple root of the polynomial,

$$\frac{d}{dZ}(Z^p + \widehat{a}_{p-1} Z^{p-1} + \ldots + \widehat{a}_0)\Big|_{Z=\widehat{z}} \neq 0.$$

Therefore the analytic implicit function theorem 4.4 ensures the existence of a $\mathbb{C}$-analytic function f such that, for all $(a_0, \dots, a_{p-1})$ sufficiently close to $(\widehat{a}_0, \dots, \widehat{a}_{p-1})$ in $\mathbb{C}^p$,

$$\{Z^p + a_{p-1}Z^{p-1} + \cdots + a_0\}\big|_{Z=f(a_0,\cdots,a_{p-1})} \equiv 0$$

and

$$\frac{d}{dZ}(Z^p + a_{p-1}Z^{p-1} + \ldots + a_0)\Big|_{Z=f(a_0,\ldots,a_{p-1})} \neq 0.$$

The same reasoning in the case of a polynomial with real coefficients completes the proof of the first part.

(b) Part (b) is immediate from Rouché's theorem. (c) If part (c) is false, then there exist sequences

$$\{a_{0,n}\}, \cdots, \{a_{p-1,n}\}, \{z_n\} \subset \mathbb{C}$$

such that

$$\lim_{n\to\infty} a_{j,n} = 0 \text{ for } 0 \leq j \leq p-1, \ \lim_{n\to\infty} |z_n| \in (0, \infty]$$

and, for all $n \in \mathbb{N}$,

$$z_n^p + a_{p-1,n}z_n^{p-1} + \cdots + a_{0,n} = 0.$$

Thus we obtain the contradiction that

$$z_n = -a_{p-1,n} - \frac{a_{p-2,n}}{z_n} - \cdots - \frac{a_{0,n}}{z_n^{p-1}} \to 0,$$

and the proof is complete. □

Greatest Common Divisors

Consider two polynomials of the complex variable Z, with constant coefficients in $\mathbb{C}$, given by

$$A(Z) = a_pZ^p + \cdots + a_1Z + a_0, \ p \geq 1, \ a_p \neq 0,$$
$$B(Z) = b_qZ^q + \cdots + b_1Z + b_0, \ q \geq 1, \ b_q \neq 0.$$

We will say that their greatest common divisor is the polynomial with largest degree $m \in \mathbb{N}_0$ of the form $Z^m + c_{m-1}Z^{m-1} + \cdots + c_0$, with coefficients $c_j \in \mathbb{C}$, which divides both $A(Z)$ and $B(Z)$. Then $A(Z)$ and $B(Z)$ are said to be co-prime if the constant polynomial 1 of degree 0 is their greatest common divisor. An elementary criterion says that $A(Z)$ and $B(Z)$ are not co-prime if and only if there exist two polynomials $P(Z)$ and $Q(Z)$ such that

$$\begin{aligned}
&A(Z)P(Z) + B(Z)Q(Z) = 0,\\
&P(Z) \not\equiv 0, \ Q(Z) \not\equiv 0,\\
&\deg(P(Z)) < q, \ \deg(Q(Z)) < p.
\end{aligned}$$

Equivalently, if $P(Z) = c_{q-1}Z^{q-1} + \cdots + c_1Z + c_0$ and $Q(Z) = d_{p-1}Z^{p-1} + \cdots + d_1Z + d_0$, $p \geq q$, the equation $\mathbf{Ax} = \mathbf{0}$ has a solution

$$\mathbf{x} = (c_0, \cdots, c_{q-1}, d_0, \cdots, d_{p-1})^T \neq 0$$

if and only if $A(Z)$ and $B(Z)$ are not co-prime, where the square matrix $\mathbf{A}$ is given by

$$\mathbf{A_{ij}} = \begin{cases} a_{i-j}, & 0 \leq i-j \leq p, \quad 1 \leq j \leq q \\ b_{q+i-j}, & 0 \leq j-i \leq q, \quad q+1 \leq j \leq p+q \\ 0, & \text{otherwise} \end{cases}.$$

The complex $(p+q)\times(p+q)$ matrix $\mathbf{A}$ is called the resultant matrix of $A(Z)$ and $B(Z)$. Its determinant, $R(a_0, \cdots, a_p; b_0, \ldots, b_q)$, is called the resultant of $A(Z)$ and $B(Z)$. The resultant is a polynomial in the coefficients $a_0, \cdots, a_p$, $b_0, \cdots, b_q$ which is zero if and only if $A(Z)$ and $B(Z)$ are not co-prime.

Lemma 10.2. *Let $A(Z)$ be a polynomial of degree $p \geq 1$ and let $A'(Z)$ denote its derivative. Denote by $C(Z)$ the greatest common divisor of $A(Z)$ and $A'(Z)$ and let $A(Z) = C(Z)E(Z)$ for some polynomial $E(Z)$. Then $E(Z)$ and $E'(Z)$ are co-prime and $E(Z)$ has the same roots as $A(Z)$.*

Proof. Suppose in (23), A has $\widehat{m}$ distinct roots $\widehat{z}_j$ with multiplicities $\widehat{m}_j > 1$, and m simple roots z_i. Then, for some a,

$$A(Z) = a\prod_{j=1}^{\widehat{m}}(Z-\widehat{z}_j)^{\widehat{m}_j}\prod_{i=1}^{m}(Z-z_i)$$

and

$$A'(Z) = \left\{a\prod_{j=1}^{\widehat{m}}(Z-\widehat{z}_j)^{\widehat{m}_j-1}\right\}\left\{\prod_{i=1}^{m}(Z-z_i)\left(\sum_{j=1}^{\widehat{m}}\widehat{m}_j\prod_{k\neq j}^{\widehat{m}}(Z-\widehat{z}_k)\right)+\prod_{j=1}^{\widehat{m}}(Z-\widehat{z}_j)\left(\sum_{i=1}^{m}\prod_{k\neq i}^{m}(z-z_k)\right)\right\}.$$

It is obvious that no root of $A(Z)$ is a root of the second factor on the right. Hence the greatest common factor of $A(Z)$ and $A'(Z)$ is

$$C(Z) = \prod_{j=1}^{\widehat{m}}(Z-\widehat{z}_j)^{\widehat{m}_j-1}$$

and $A(Z) = C(Z)E(Z)$ where all the roots of $E(Z)$ are simple and the roots of $E(Z)$ coincide with those of $A(Z)$. □

For future reference, note that when $p = q$ the resultant matrix $\mathbf{A}$ has a simple form

$$\begin{bmatrix} a_0 & 0 & \dots & 0 & b_0 & 0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 & b_1 & b_0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{p-1} & a_{p-2} & \dots & a_0 & b_{p-1} & b_{p-2} & \dots & b_1 & b_0 \\ a_p & a_{p-1} & \dots & a_1 & b_p & b_{p-1} & \dots & b_2 & b_1 \\ 0 & a_p & \dots & a_2 & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & b_{p-1} \\ 0 & 0 & \dots & a_p & 0 & 0 & \dots & 0 & b_p \end{bmatrix}. \tag{24}$$

Discriminant of a Polynomial

Consider the polynomial

$$A(Z) = a_pZ^p + \ldots + a_1Z + a_0,\ p \geq 1,\ a_p \neq 0,$$

the coefficients of which are complex. It is clear from the proof of Lemma 10.2 that $z_j \in \mathbb{C}$ is a multiple root of $A(Z)$ if and only if $(Z - z_j)$ is a common factor of $A(Z)$ and $A'(Z)$. Indeed $A(Z)$ has no multiple roots if and only if $A(Z)$ and $A'(Z)$ are co-prime. For $(a_0, \cdots, a_p) \in \mathbb{C}^{p+1}$ define $D(a_0, \ldots, a_p) \in \mathbb{C}$ to be the resultant of $A(Z)$ and $A'(Z)$. This is called the discriminant of $A(Z)$. Notice that D is a polynomial in the $p + 1$ variables $a_0, \ldots, a_p$ which vanishes exactly when $A(Z)$ has at least one multiple root.

Example 10.3. (Quadratic polynomials)

$p = 2$, $A(Z) = a_2Z^2 + a_1Z + a_0$, $A'(Z) = 2a_2Z + a_1 = B(Z)$ and, with $a_2 \neq 0$,

$$D(a_0, a_1, a_2) = R(a_0, a_1, a_2; b_0, b_1) = \begin{vmatrix} a_0 & b_0 & 0 \\ a_1 & b_1 & b_0 \\ a_2 & 0 & b_1 \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & 0 \\ a_1 & 2a_2 & a_1 \\ a_2 & 0 & 2a_2 \end{vmatrix} = -a_2(a_1^2 - 4a_0a_2),$$

which vanishes exactly when the usual discriminant $a_1^2 - 4a_0a_2 = 0$. □

Variable Coefficients

In the Weierstrass preparation theorem 8.1, we saw polynomials in x_n of the form

$$x_n^q + \sum_{k=0}^{q-1} a_k(x_1, \cdots, x_{n-1}) x_n^k,$$

in which the coefficients a_k are $\mathbb{F}$-analytic functions of $n-1$ variables. In this section we consider such polynomials in the special case when $\mathbb{F} = \mathbb{C}$. Let $V \subset \mathbb{C}^m$, $m \geq 1$ be given by

$$V = \{(z_1, \cdots, z_m) \in \mathbb{C}^m : |z_j| < \delta,\ 1 \leq j \leq m\},$$

for some $\delta > 0$ and consider polynomials of the form

$$a_p(z_1, \cdots, z_m) Z^p + \cdots + a_1(z_1, \cdots, z_m) Z + a_0(z_1, \cdots, z_m),$$

where the coefficients $a_k : V \to \mathbb{C}$ are $\mathbb{C}$-analytic functions and $a_p \not\equiv 0$. This polynomial, which we denote by $A(Z; z_1, \cdots, z_m)$, has degree p. (The case $m = 0$ corresponds to polynomials in Z with constant coefficients.) The coefficients of another polynomial $B(Z; z_1, \cdots, z_m)$ are functions $b_k : V \to \mathbb{C}$, where it is understood that $A(Z; z_1, \cdots, z_m)$ and $B(Z; z_1, \cdots, z_m)$ may have different degrees. (When the meaning is clear we refer simply to polynomials A or B on V.) We say that a polynomial is real-on-real if and only if all its coefficients are real-on-real analytic functions. The discriminant $D = D(a_0, \cdots, a_p)$ of the polynomial A is an analytic function defined on $V \subset \mathbb{C}^m$ by

$$D(a_0, \cdots, a_p)(\xi) = D(a_0(\xi), \cdots, a_p(\xi)), \quad \xi = (z_1, \cdots, z_m) \in V. \tag{25}$$

Greatest Common Divisors

From the previous viewpoint, $\{A(Z; z_1, \cdots, z_m) : (z_1, \cdots, z_m) \in V\}$ is a family of polynomials parameterized by $(z_1, \cdots, z_m) \in V$. The notion of the greatest common divisor of two polynomials A and B is therefore more subtle in this case.

Definition 10.4. *Let $W \subset V$. The greatest common divisor of A and B on W is a polynomial $C(Z; z_1, \cdots, z_m)$ of degree d, say, where the coefficients c_k are $\mathbb{C}$-analytic on V, $c_d \equiv 1$ on V and, for every $(z_1, \cdots, z_m) \in W$, the polynomial $C(Z; z_1, \cdots, z_m)$ is the greatest common divisor of $A(Z; z_1, \cdots, z_m)$ and $B(Z; z_1, \cdots, z_m)$.*

The first question is whether in this general setting two polynomials have a greatest common divisor.

Theorem 10.5. (Euclid's Algorithm) *Let polynomials*

$$A(Z; z_1, \cdots, z_m) \text{ and } B(Z; z_1, \cdots, z_m)$$

have degree p and q, respectively, where at least one of a_p and b_q is identically equal to 1 on V. Then there exists a polynomial $C(Z; z_1, \cdots, z_m)$ of degree r on V and a $\mathbb{C}$-analytic function g, such that C is the greatest common divisor of A and B on the set $W = V \setminus G$, where $G = \{(z_1, \cdots, z_m) : g(z_1, \cdots, z_m) = 0\} \neq V$. Note that W is a dense connected subset of V, by Proposition 9.2. Suppose that $a_p \equiv 1$. Then there is a polynomial $E(Z; z_1, \cdots, z_m)$ such that $A = CE$ for all $(Z; z_1, \cdots, z_m) \in \mathbb{C} \times V$. If $r = 0$ then $C \equiv 1$ on V.

Proof. Let

$$\begin{aligned} P_1 &= A,\ m_1 = p \ \text{ and } \ P_2 = B,\ m_2 = q, \quad \text{if } p \geq q, \\ P_2 &= A,\ m_2 = p \ \text{ and } \ P_1 = B,\ m_1 = q, \quad \text{if } p < q. \end{aligned}$$

Suppose for $j \in \mathbb{N}$ that P_k, $1 \leq k \leq j+1$, are given polynomials of degree m_k such that the coefficient of Z^{m_k} in P_k is $g_k \not\equiv 0$ and that m_k is a non-increasing function of k. Now let

$$Q(Z; z_1, \cdots, z_m) = Z^{m_j - m_{j+1}} g_j(z_1, \cdots, z_m) P_{j+1}(Z; z_1, \cdots, z_m) - g_{j+1}(z_1, \cdots, z_m) P_j(Z; z_1, \cdots, z_m).$$

Observe that, on the set

$$W_j = \{(z_1, \cdots, z_m) \in V : g_{j+1}(z_1, \cdots, z_m) \neq 0\},$$

any common divisor of P_j and P_{j+1} is also a common divisor of Q and P_{j+1}. Obviously the degree of Q is smaller than the degree of P_j. If the degree of Q is not larger than that of P_{j+1} let $P_{j+2} = Q$; otherwise rename P_{j+1} as P_{j+2} and replace the old P_{j+1} by Q. This defines a set of polynomials $P_1, \cdots, P_{j+2}$ with $m_k > m_{k+2}$ for all $k \leq j$. Clearly this process terminates after a finite number of steps when

$$0 = g_{J+1} P_J - g_J Z^{m_J - m_{J+1}} P_{J+1}, \quad \text{say, for some } J \in \mathbb{N}.$$

Hence, at every point $(z_1, \cdots, z_m) \in V$ at which the product $g = g_1 \cdots g_{J+1}$ of the highest order coefficients of the P_j s is non-zero, $\widehat{C} = P_{J+1}/g_{J+1}$ is the greatest common divisor of P_1 and P_2. This defines g and consequently G in the statement of the theorem. By definition, $g \not\equiv 0$, and so $G \neq V$.

Now suppose that $a_p \equiv 1$. Then at every point $(z_1, \cdots, z_m) \in V \setminus G$ the roots of $\widehat{C}(Z; z_1, \cdots, z_m)$ form a subset of those of $A(Z; z_1, \cdots, z_m)$ and are therefore bounded when $(z_1, \cdots, z_m)$ lies in a compact subset of V, by Proposition 10.1 (c). Since the coefficients of $\widehat{C}$ are polynomial functions of the roots of $\widehat{C}$, it follows that these coefficients are bounded on subsets of $V \setminus G$ that are compact in V. Therefore, by the Riemann extension theorem 9.3, they have analytic extension to all of V. We denote by C the polynomial with the coefficients of $\widehat{C}$ extended to V. To obtain the existence of E, note first that a polynomial $\widehat{E}$ is defined on $V \setminus G$ by writing $A = C\widehat{E}$ on $V \setminus G$. The argument for extending $\widehat{E}$ as a polynomial on V is the same as that for $\widehat{C}$. □

Remarks 10.6. If A and B have coefficients that are real-on-real, then Euclid's algorithm obviously leads to a greatest common divisor C and a polynomial E, both of which are real-on-real.

If $a_p \equiv 1$, $b_q \equiv 1$, and the resolvent $R(a_0, \cdots, a_{p-1}, 1; b_0, \cdots, b_{q-1}, 1) \not\equiv 0$ (in which case it is non-zero on a connected open dense set in V), we may take $g = R(a_0, \cdots, a_{p-1}, 1; b_0, \cdots, b_{q-1}, 1)$, $r = 0$ and $c_0 = C \equiv 1$. If $r \geq 1$, then $R(a_0, \cdots, a_{p-1}, 1; b_0, \cdots, b_{q-1}, 1) \equiv 0$. □

Simplification of a Polynomial

Theorem 10.7. *Suppose that $A(Z; z_1, \cdots, z_m)$ is a polynomial of degree p and that $a_p \equiv 1$. If the discriminant $D(a_0, \cdots, a_{p-1}, 1) \equiv 0$ on V then there exists another polynomial $E(Z; z_1, \cdots, z_m)$ of degree q, say, with $e_q \equiv 1$ such that $E(Z; z_1, \cdots, z_m)$ has the same roots as $A(Z; z_1, \cdots, z_m)$, possibly with smaller multiplicities, and $D(e_0, \cdots, e_{m-1}, 1) \not\equiv 0$ on V. (In particular, for $(z_1, \cdots, z_m)$ in an open dense connected subset W of V, $E(Z; z_1, \cdots, z_m)$ has no multiple roots.) If A is real-on-real, then so is E.*

Proof. For $(z_1, \cdots, z_m) \in V$, the polynomial $A(Z; z_1, \cdots, z_m)$ has a multiple root if and only if its discriminant is zero. It therefore suffices to let E be the polynomial given by Euclid's algorithm for which $A = CE$ where C is the greatest common divisor of A and A' on W, an open dense connected subset of V. An appeal to Lemma 10.2 completes the proof. □

Projection Lemma

Theorem 10.8. (Projection Lemma) *Let $A_1(Z; z_1, \cdots, z_m)$ be a polynomial of degree p with $a_p \equiv 1$ on V and let $A_j(Z; z_1, \cdots, z_m)$ be a polynomial of degree at most $p-1$, $2 \leq j \leq k$. Let*

$$\mathcal{A} = \big\{(z_1, \cdots, z_m) \in V : \text{ there exists } z \in \mathbb{C} \text{ with } A_j(z; z_1, \cdots, z_m) = 0 \text{ for all } j \in \{1, \cdots, k\}\big\}.$$

Then there exists a finite family $\{R_\alpha : \alpha \in \Sigma\}$ of analytic functions on V such that

$$\mathcal{A} = \big\{(z_1, \cdots, z_m) \in V : R_\alpha(z_1, \cdots, z_m) = 0 \text{ for all } \alpha \in \Sigma\big\}.$$

If the polynomials A_j are real-on-real, then the functions in $\mathcal{A}$ are real-on-real.

Remark 10.9. It is worth noting that this result is false in the setting of real-analytic functions. For example let $m = 1$ and $V = (-\epsilon, \epsilon)$, $k = 2$, $A_1(X, x_1) = X^2 - x_1$ and $A_2 \equiv 0$. Then $p = 2$ and $\mathcal{A} = [0, \epsilon)$ which is not the zero-set of any real-analytic function defined on $(-\epsilon, \epsilon)$, because of Lemma 2.11. $\square$

Proof. For any $t_2, \cdots, t_k \in \mathbb{C}$, let $R(t_2, \cdots, t_k; z_1, \cdots, z_m)$ denote the resultant of two polynomials, A_1 and $A_1 + \sum_{j=2}^k t_j A_j$, of the same degree. Let the coefficients of A_j be denoted by a_i^j, $1 \le j \le k$, and in (24) let $a_i = a_i^1$ and $b_i = a_i + \sum_{j=2}^k t_j\, a_i^j$. The resultant R may now be obtained from formula (2) for the determinant of the matrix (24) with these coefficients. By subtracting the j^{th} column from the $p + j^{\,th}$ column, the coefficients a_i can be eliminated from the right half of the matrix without changing its determinant and the non-zero entries in the right half of the resulting matrix are all of the form $\sum_{j=2}^k t_j\, a_i^j$. Therefore

$$R(t_2, \cdots, t_k; z_1, \cdots, z_m) = \sum_{\substack{\alpha \in \mathbb{N}_0^{k-1} \\ |\alpha| = p}} t^\alpha R_\alpha(z_1, \cdots, z_m)$$

where $t = (t_2 \cdots, t_k)$ and $R_\alpha : V \to \mathbb{C}$ is analytic and independent of t.

Suppose that $(z_1, \cdots, z_m) \in \mathcal{A}$. Then, for some $z \in \mathbb{C}$,

$$A_j(z, z_1, \cdots, z_m) = 0 \text{ for all } j \in \{1, \cdots, k\}.$$

Therefore, for all $t_2, \cdots, t_k$, z is a common root of the polynomials A_1 and $A_1 + \sum_{j=2}^k t_j A_j$. Therefore $R(t_2, \cdots, t_k; z_1, \cdots, z_m) = 0$ for all $(t_2, \cdots, t_k) \in \mathbb{C}^{m-1}$ and so $R_\alpha(z_1, \cdots, z_m) = 0$ for all $\alpha \in \Sigma$ where $\Sigma = \{\alpha \in \mathbb{N}_0^{k-1} : |\alpha| = p\}$.

Conversely, suppose $(z_1, \cdots, z_m) \in V$ and $R_\alpha(z_1, \cdots, z_m) = 0$ for all $\alpha \in \Sigma$. It follows that

$$R(t_2, \cdots, t_k; z_1, \cdots, z_m) = 0 \text{ for } (t_2, \cdots, t_k) \in \mathbb{C}^{k-1}.$$

Therefore the polynomials A_1 and $A_1 + \sum_{j=2}^k t_j A_j$ have a common root (possibly depending on $(t_2, \cdots t_k)$) for all $(t_2, \cdots, t_k) \in \mathbb{C}^{k-1}$. Let $\zeta_1, \cdots, \zeta_\nu$, $\nu \le p$, be the distinct roots of $A_1(Z; z_1, \cdots, z_m)$ and for $1 \le i \le \nu$ let

$$Y_i = \left\{ (t_2, \cdots, t_k) \in \mathbb{C}^{k-1} : \sum_{j=2}^k t_j A_j(\zeta_i; z_1, \cdots, z_m) = 0 \right\}.$$

Each Y_i is a linear subspace of $\mathbb{C}^{k-1}$ and their union is $\mathbb{C}^{k-1}$. Hence $Y_{i_0} = \mathbb{C}^{k-1}$ for some $i_0 \in \{1, \cdots, \nu\}$ and so $A_j(\zeta_{i_0}; z_1, \cdots, z_m) = 0$ for all $j \in \{1, \cdots, k\}$.

It is clear from the construction that if the polynomials A_j are real-on-real, then the functions R_α are real-on-real. This completes the proof. $\square$

11 Analytic Varieties

Once again the field $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$. Let $a \in \mathbb{F}^n$, $n \in \mathbb{N}$. Two subsets S and T of $\mathbb{F}^n$ are said to be equivalent at a if there is an open neighbourhood O of a such that $O \cap S = O \cap T$. We note that this is an equivalence relation on $2^{\mathbb{F}^n}$ and write $S \sim_a T$.

The corresponding equivalence class, denoted by $\gamma_a(S)$ for $S \subset \mathbb{F}^n$, is called the germ of S at a and if $\widetilde{S} \in \gamma_a(S)$ we say that $\widetilde{S}$ is a representative of $\gamma_a(S)$. Since $\{a\} \cap U = \{a\}$ and $\emptyset \cap U = \emptyset$ for all open sets U containing a, we write $\gamma_a(\{a\}) = \{a\}$ and $\gamma_a(\emptyset) = \emptyset$. If $a \notin \overline{S}$, $\gamma_a(S) = \emptyset$. The finite unions, intersections and complements of germs of sets at a are defined by the same operations on representatives. (It is easy to check that these are well-defined operations on germs, and independent of the chosen representatives.)

$\mathbb{F}$-Analytic Varieties

Definition 11.1. *Suppose that $U \subset \mathbb{F}^n$ is a non-empty open set and that G denotes a finite collection of functions $g : U \to \mathbb{F}$ which are $\mathbb{F}$-analytic on U. Let*

$$\mathrm{var}\,(U, G) = \{x \in U : g(x) = 0 \text{ for all } g \in G\}.$$

This is called the $\mathbb{F}$-analytic variety generated by G on U. If $U \subset \mathbb{C}^n$ and the elements of G are real-on-real, we say that $\mathrm{var}\,(U, G)$ is real-on-real provided that $U \cap \mathbb{R}^n \neq \emptyset$.

A point $x \in \mathrm{var}\,(U, G)$ is said to be m-regular if there is a neighbourhood O of x in $\mathbb{F}^n$ such that $O \cap \mathrm{var}\,(U, G)$ is an $\mathbb{F}$-analytic manifold of dimension m (see Definition 4.5). Note that

$$\begin{aligned} \mathrm{var}\,(U, G_1) \cap \mathrm{var}\,(U, G_2) &= \mathrm{var}\,(U, G_1 \cup G_2) \\ \mathrm{var}\,(U, G_1) \cup \mathrm{var}\,(U, G_2) &= \mathrm{var}\,(U, G_3) \end{aligned}$$

where $G_3 = \{g_1 g_2 : g_i \in G_i,\ i = 1, 2\}$.

The germ at a of an $\mathbb{F}$-analytic variety is referred to as an $\mathbb{F}$-analytic germ and the germ of a real-on-real $\mathbb{C}$-analytic variety in $\mathbb{C}^n$ is called a real-on-real germ. The set of all $\mathbb{F}$-analytic germs at $a \in \mathbb{F}^n$ is denoted by $\mathcal{V}_a(\mathbb{F}^n)$. If $\alpha \in \mathcal{V}_a(\mathbb{F}^n)$, its dimension, $\dim_{\mathbb{F}} \alpha$, is the largest integer m such that every representative of α contains an m-regular point (the point a itself need not be m-regular.) If no such integer exists, we say that $\dim_{\mathbb{F}} \alpha = -1$.

Remarks 11.2. For $a \in \mathbb{F}^n$, $\mathbb{F}^n$, $\{a\}$ and $\emptyset$ are elements of $\mathcal{V}_a(\mathbb{F}^n)$ with $\dim_{\mathbb{F}} \emptyset = -1$, $\dim_{\mathbb{F}}\{a\} = 0$ and $\dim_{\mathbb{F}} \gamma_a(\mathbb{F}^n) = n$. Theorem 10.8 says that $\gamma_0(\mathcal{A}) \in \mathcal{V}_0(\mathbb{C}^m)$. If $\alpha, \beta \in \mathcal{V}_a(\mathbb{F}^n)$, then both $\alpha \cap \beta$ and $\alpha \cup \beta$ are in $\mathcal{V}_a(\mathbb{F}^n)$, but in general $\alpha \setminus \beta \notin \mathcal{V}_a(\mathbb{F}^n)$. □

Lemma 11.3. *Suppose that $M \subset \mathbb{F}^n$ is an $\mathbb{F}$-analytic manifold (Definition 4.5) and $a \in M$. Then $\gamma_a(M) \in \mathcal{V}_a(\mathbb{F}^n)$. If $a \in U \cap M$ and $\mathrm{var}\,(U, G)$ is an $\mathbb{F}$-analytic variety, there is an open neighbourhood W of a in M such that $W \setminus \mathrm{var}\,(U, G)$ is either empty or dense in W.*

Proof. First we show that $\gamma_a(M)$ is in $\mathcal{V}_a(\mathbb{F}^n)$. Without loss of generality suppose that $0 = a \in M$ and, in the notation of Definition 4.5, let $Z_1 = \mathrm{range}\, df[0]$, $\mathbb{F}^n = Z_1 \oplus Z_2$, and write

$$f(x) = f_1(x) + f_2(x) \in Z_1 \oplus Z_2, \quad x \in U_0 \subset \mathbb{F}^m,$$

where U_0 is a neighbourhood of $0 \in \mathbb{F}^m$. By hypothesis, $df[0]$ has rank m and $df_1[0] : \mathbb{F}^m \to Z_1$ is a bijection. By the analytic inverse function theorem 4.3, U_0 in Definition 4.5 can be chosen so that f_1, from $U_0 \subset \mathbb{F}^m$ onto a neighbourhood W_0 of $f_1(0) = 0 \in Z_1$, is a bijection with an analytic inverse. Now $\{f(x) : x \in U_0\}$ is a representative of $\gamma_0(M)$ in $\mathbb{F}^n$ and

$$\begin{aligned} \{f(x) : x \in U_0\} &= \{(f_1(x), f_2(x)) : x \in U_0\} = \{(y, f_2 \circ f_1^{-1}(y) : y \in W_0\} \\ &= \{(y, z) \in W_0 \times Z_2 : z - f_2 \circ f_1^{-1}(y) = 0\}. \end{aligned}$$

Since $f_2 \circ f_1^{-1}$ is analytic this shows that $\gamma_0(M) \in \mathcal{V}_0(\mathbb{F}^n)$.

Now suppose that $\mathrm{var}\,(U, G)$ is an $\mathbb{F}$-analytic variety in $\mathbb{F}^n$, let $0 \in M$ and let U_0 as above. Suppose that $B \subset U_0$ is a ball centred at $0 \in \mathbb{F}^m$ and let $W = f(B)$. Then $0 \in W$, which is a relatively open connected subset of M.

Suppose that $W \setminus \mathrm{var}\,(U, G)$ is not dense in W. Then there is an open set $\widehat{W} \subset W$ such that $\widehat{W} \subset \mathrm{var}\,(U, G)$. Let $\widehat{B} = f^{-1}(\widehat{W})$. Then $g \circ f \equiv 0$ on $\widehat{B}$ for all $g \in G$. Hence $g \circ f \equiv 0$ on B for all $g \in G$, by Theorem 2.11. Hence $W \subset \mathrm{var}\,(U, G)$, in other words, $W \setminus \mathrm{var}\,(U, G)$ is empty. This proves the result. □

Lemma 11.4. *Let $\mathrm{var}\,(U, G)$ be an $\mathbb{F}$-analytic variety in $\mathbb{F}^n$ and $M \subset U$ a connected $\mathbb{F}$-analytic manifold such that $M \cap \mathrm{var}\,(U, G)$ has non-empty interior relative to M. Then $M \subset \mathrm{var}\,(U, G)$.*

Proof. Let N^o denote the relative interior in M of $N = M \cap \mathrm{var}\,(U, G)$. By definition, N^o is open in M, and non-empty by hypothesis. Suppose that x belongs to the boundary in M of N^o. By Lemma

11.3 there is an open neighbourhood W of x in M such that $W \setminus \mathrm{var}\,(U,G)$ is either empty or dense in W. Now $W \cap N^o \neq \emptyset$ since x is on the boundary of N^o and, since $N^o \subset M \cap \mathrm{var}\,(U,G)$ is open, $W \setminus \mathrm{var}\,(U,G)$ is not dense in W. Hence it is empty, which implies that $x \in N^o$. Thus N^o is closed in M. By connectedness, $N^o = M$ and $M \subset \mathrm{var}\,(U,G)$. □

Definition 11.5. *A germ $\alpha \in \mathcal{V}_a(\mathbb{F}^n)$ is said to be irreducible if $\alpha = \alpha_1 \cup \alpha_2$ for germs $\alpha_1,\ \alpha_2 \in \mathcal{V}_a(\mathbb{F}^n)$ implies that $\alpha = \alpha_1$ or $\alpha = \alpha_2$.*

For example, $\emptyset$ and $\{a\}$ and are irreducible elements of $\mathcal{V}_a(\mathbb{F}^n)$.

Lemma 11.6. *If M is an $\mathbb{F}$-analytic manifold and $a \in M$, then $\gamma_a(M) \in \mathcal{V}_a(\mathbb{F}^n)$ is irreducible.*

Proof. By Lemma 11.3, $\gamma_a(M) \in \mathcal{V}_a(\mathbb{F}^n)$. To see that it is irreducible suppose that E_1 and E_2 are $\mathbb{F}$-analytic varieties in $\mathbb{F}^n$ such that

$$\gamma_a(M) = \gamma_a(E_1 \cup E_2) \text{ and } \gamma_a(E_1) \neq \gamma_a(M) \neq \gamma_a(E_2).$$

It follows also from Lemma 11.3 that for $i = 1,\,2$ there exists an open neighbourhood W of a in M such that $(M \setminus E_i) \cap W$ is either empty or dense in W. Note M and $E_1 \cup E_2$ coincide in a neighbourhood U of a in $\mathbb{F}^n$. Hence if $(M \setminus E_1) \cap W$ is empty, it follows that $E_2 \subset E_1$ in a neighbourhood of a and that $\gamma_a(M) = \gamma_a(E_1)$, which is assumed to be false. If $(M \setminus E_2) \cap W$ is empty we reach a similar contradiction.

Therefore $(M \setminus E_i) \cap W$ is open and dense in W, $i = 1,\,2$. Hence $(M \setminus (E_1 \cup E_2)) \cap W$ is dense in W. But this contradicts the fact that $\gamma_a(M) = \gamma_a(E_1 \cup E_2)$ and proves that $\gamma_a(M)$ is an irreducible germ. □

We will see another important example of irreducible germs in Lemma 11.17. The most elementary non-trivial example of an analytic variety is one which is defined as the zeros in an open set $U \subset \mathbb{F}^n$, $n \geq 2$, of a single $\mathbb{F}$-analytic function $f : U \to \mathbb{F}$. Suppose, without loss of generality, that $0 \in U$, that $f(0) = 0$ and that $f \not\equiv 0$ on U. Then the Weierstrass preparation theorem 8.1 gives that there exists a choice of coordinates $(x_1, \cdots, x_n) \in \mathbb{F}^n$, $r > 0$ and an open set $V \subset \mathbb{F}^{n-1}$ containing 0 such that, with $U_0 = V \times B_r(\mathbb{F})$,

$$\begin{aligned} \mathrm{var}\,(U_0, \{f\}) &= \mathrm{var}\,(U_0, \{h\}), \\ h(x_1, \cdots, x_n) &= A(x_n; x_1, \cdots, x_{n-1}), \end{aligned}$$

where $A(X; x_1, \cdots, x_{n-1})$ is a polynomial with coefficients that are analytic functions of $(x_1, \cdots, x_{n-1}) \in V$ of the form

$$A(X; x_1, \cdots, x_{n-1}) = X^q + \sum_{k=0}^{q-1} a_k(x_1, \cdots, x_{n-1}) X^k, \quad (x_1, \cdots, x_{n-1}) \in V,$$

with $a_k(0) = 0$, $0 \leq k \leq q-1$. Since real polynomials need not have real roots, we need special structure to take this idea any further when $\mathbb{F} = \mathbb{R}$. However when $\mathbb{F} = \mathbb{C}$ polynomials do have roots in $\mathbb{C}$ and we have the following.

Theorem 11.7. *When $\mathbb{F} = \mathbb{C}$, the polynomial A above can be chosen with the following properties:*

(a) *Its discriminant (see (25)) $D = D(a_0, \cdots, a_{q-1}, 1) \not\equiv 0$ on V.*

(b) *Every point of $\mathrm{var}\,(U_0, \{f\}) \setminus \big(\mathrm{var}\,(V, \{D\}) \times \mathbb{C}\big)$ is an $(n-1)$-regular point of $\mathrm{var}\,(U_0, \{f\})$.*

(c) *$\dim_{\mathbb{C}} \alpha = n - 1$ where α is the germ of $\mathrm{var}\,(U_0, \{f\})$.*

(d) *If f is real-on-real, then A is real-on-real.*

Proof. (a) We simplify the polynomial obtained from the Weierstrass preparation theorem using Theorem 10.7. This may change the value of q and the coefficients a_k, but we retain the original notation for the simplified polynomial. Since, after simplification, $A(Z; z_1, \cdots, z_{n-1})$ has no multiple zeros for $(z_1, \cdots, z_{n-1})$ in an open dense connected subset of V (see Proposition 9.2), this proves (a).

(b) Since, for every $(z_1, \cdots, z_n) \in \operatorname{var}(U_0, \{f\}) \setminus \operatorname{var}\big((V, \{D\})\big)$, $Z = z_n$ is a simple zero of $A(Z; z_1, \cdots, z_{n-1})$, the analytic implicit function theorem 4.4 gives (b).

(c) Since $a_k(0) = 0$ and a_k is continuous on V, $0 \le k \le q-1$, we know from Proposition 9.2 and Proposition 10.1 that there are points of $\operatorname{var}(U_0, \{h\}) \setminus \big(\operatorname{var}(V, \{D\}) \times \mathbb{C}\big)$ arbitrarily close to zero. Therefore there are $(n-1)$-regular points of $\operatorname{var}(U_0, \{f\})$ arbitrarily close to 0. Thus $\dim_{\mathbb{C}} \alpha = n - 1$.

(d) This is guaranteed by the Weierstrass preparation theorem 8.1 and Theorem 10.7. □

Remark 11.8. Of course, $\gamma_0(\operatorname{var}(U, \{f\})) = \gamma_0(\operatorname{var}(U_0, \{f\}))$. □

Now we will develop the ideas involved in the proof of this result to obtain something much more general.

Weierstrass Analytic Varieties

Throughout $\mathbb{F} = \mathbb{C}$, $m \in \mathbb{N}_0$ and, for $m \in \mathbb{N}$, $V \subset \mathbb{C}^m$ is given by

$$V = \{(z_1, \cdots, z_m) \in \mathbb{C}^m : |z_k| < \delta,\ 1 \le k \le m\}.$$

In the case $m = 0$, $V = \{0\}$.

Definition 11.9. *When $m \in \mathbb{N}$, a Weierstrass polynomial on V is a polynomial $A(Z; z_1, \cdots, z_m)$, $(z_1, \cdots, z_m) \in V$, of the form*

$$Z^p + \sum_{k=0}^{p-1} a_k(z_1, \cdots, z_m) Z^k, \quad p \in \mathbb{N}, \tag{26}$$

where

$$a_0(0) = \cdots = a_{p-1}(0) = 0 \text{ and } D(a_0, \cdots, a_{p-1}, 1) \not\equiv 0 \text{ on } V.$$

(By Proposition 9.2, $D(a_0, \cdots, a_{p-1}, 1) \neq 0$ on a connected, open, dense subset of V.) When $m = 0$ Weierstrass polynomials are of the form Z^p, $p \in \mathbb{N}$.

Remark 11.10. Suppose that the coefficients a_k in a polynomial of the form (26) vanish at $0 \in V$ and the discriminant is identically zero on V. Then its simplification, Theorem 10.7, is a Weierstrass polynomial on V. (All its coefficients, apart from the principal coefficient, are zero at $0 \in V$.)

If A, B are Weierstrass polynomials on V and C is any (non-constant in Z) polynomial on V with $AC = B$, then C is a Weierstrass polynomial. □

Let $n > m \in \mathbb{N}_0$ and consider a family $\{A_{m+1}, \cdots, A_n\}$ of Weierstrass polynomials on V. For each $k \in \{m+1, \cdots, n\}$ let

$$h_k(z_1, \cdots, z_n) = A_k(z_k; z_1, \cdots, z_m).$$

Let H denote the family of $n - m$ functions h_k defined in this way. (Each $h_k \in H$ is a polynomial in z_k with coefficients that are analytic functions on $V \subset \mathbb{C}^m$. Thus h_k is independent of all z_j for $j \in \{m+1, \cdots, n\} \setminus \{k\}$.)

Definition 11.11. *A Weierstrass analytic variety is a set in $\mathbb{C}^n$ of the form*

$$\operatorname{var}\big(V \times \mathbb{C}^{n-m}, H\big), \quad 0 \le m < n.$$

For our purposes, a Weierstrass analytic variety is identified with the set H of Weierstrass analytic polynomials which define it and, if $m \in \mathbb{N}$, its discriminant $D(H) : V \to \mathbb{C}$ is defined to be the product of the discriminants of the polynomials A_k used in the definition.

For $m \in \mathbb{N}$, the branches of a Weierstrass analytic variety $\operatorname{var}\big(V \times \mathbb{C}^{n-m}, H\big)$ *are the components of*

$$\operatorname{var}\big(V \times \mathbb{C}^{n-m}, H\big) \setminus \big(\operatorname{var}(V, D(H)) \times \mathbb{C}^{n-m}\big).$$

Remarks 11.12. Every point on a branch of a Weierstrass analytic variety var $\big(V \times \mathbb{C}^{n-m}, H\big)$, for $n \in \mathbb{N}$ and $0 < m < n$, is m-regular because, by the analytic implicit function theorem 4.4, in a neighbourhood of such a point each of the coordinates $z_{m+1}, \cdots, z_n$ depends locally analytically on $(z_1, \cdots, z_m) \in V$. Thus each branch is a connected $\mathbb{C}$-analytic manifold of dimension m and by Proposition 9.2 it projects onto the connected set $V \setminus \mathrm{var}\,\big(V, \{D(H)\}\big)$. But it is not possible in general to define any one of the coordinates $z_{m+1}, \cdots, z_n$ as an analytic function on $V \setminus \mathrm{var}\,(V, \{D(H)\})$ when the latter set is multiply connected. In the proof of Corollary 13.4 we will see that $E = \mathrm{var}\,\big(V \times \mathbb{C}^{n-m}, H\big)$ contains no manifold of dimension strictly greater that m and hence $\dim_{\mathbb{C}} \gamma_0(E) = m$.

When $m = 0$, the only Weierstrass analytic variety in $\mathbb{C}^n$ is $\{0\}$ since all Weierstrass polynomials are of the form Z^p, $p \in \mathbb{N}$. □

The following are a few elementary examples.

Examples 11.13. In Definition 11.9 with $m = 1$, $n = 2$ and $V = \mathbb{C}$, the polynomial $A(Z; z_1) = Z^2 - z_1$, $z_1 \in \mathbb{C}$, defines a Weierstrass analytic variety which has exactly one branch, $B = \{(z_1, z_2) : z_2^2 = z_1,\ z_1 \neq 0\}$. This illustrates both that a branch is connected, but not in general simply connected, and that "above" each point of V there is usually more than one point of B. Note also that A is real-on-real and $B \cap \mathbb{R}^2$ is a real parabola with the origin removed.

Let $E = \mathrm{var}\,(\mathbb{C} \times \mathbb{C}, \{h\})$, $h(z_1, z_2) = A(z_2; z_1)$ and $A(Z; z) = Z^2 + z^2$. Note that $D(\{h\})$ is zero only at $0 \in \mathbb{C}$, that E has two branches, $B_\pm = \{(z, \pm iz) : z \in \mathbb{C} \setminus \{0\}\}$ and that neither of them is closed under complex conjugation even though E is real-on-real. The next result addresses this issue. □

Lemma 11.14. *Suppose B is a branch of a real-on-real $\mathbb{C}$-analytic variety* var $(V \times \mathbb{C}^{n-m}, H)$ *and that $B \cap \mathbb{R}^n \neq \emptyset$. Then*
$$B^* := \{(\overline{z}_1, \cdots, \overline{z}_n) : (z_1, \cdots, z_n) \in B\} = B.$$

Proof. Since functions in H have Taylor expansions at 0 with real coefficients and B is a maximal connected set (in $\big(V \setminus \mathrm{var}\,(V, \{D(H)\})\big) \times \mathbb{C}^{n-m}$) of solutions of the equations $h = 0$, $h \in H$, the set B^* is a maximal in the same sense. By hypothesis, $B \cap B^* \neq \emptyset$. Hence $B^* = B$ and the result is proved. □

The next result shows that the closure of a branch is an analytic variety in a neighbourhood of 0, even though the branch, itself a manifold, need not be. For a branch B,

$$\overline{B} \text{ denotes } \overline{B} \cap \big(V \times \mathbb{C}^{n-m}\big), \text{ the relative closure of } B \text{ in } V \times \mathbb{C}^{n-m}.$$

By Proposition 9.2, a Weierstrass analytic variety is the union of the closures (in this sense) of its branches.

Theorem 11.15. *Suppose that B is a branch of a Weierstrass analytic variety $E =$* var $(V \times \mathbb{C}^{n-m}, H)$ *with discriminant $D = D(H)$. Then*

$$\overline{B} = \mathrm{var}\,(V \times \mathbb{C}^{n-m}, G)$$

for some finite collection of analytic functions $g : V \times \mathbb{C}^{n-m} \to \mathbb{C}$.

Suppose in addition that H is real-on-real and $B \cap \mathbb{R}^n \neq \emptyset$. Then G is real-on-real, $\overline{B}$ is a real-on-real $\mathbb{C}$-analytic variety and $\overline{B} \cap \mathbb{R}^n$ is an $\mathbb{R}$-analytic variety with $\dim_{\mathbb{R}} \gamma_0(\mathbb{R}^n \cap \overline{B}) = m$.

Proof. Let $\xi = (z_1, \cdots, z_m) \in V \setminus \mathrm{var}\,(V, \{D\})$. Then there are K, say, points of B above ξ. In other words, there are K elements $\zeta_j(\xi) \in \mathbb{C}^{n-m}$, such that $(\xi, \zeta_j(\xi)) \in B$, $1 \le j \le K$. By Remarks 11.12 the dependence of $\zeta_j(\xi)$ on ξ is $\mathbb{C}$-analytic locally and therefore, by connectedness, K is independent of $\xi \in V \setminus \mathrm{var}\,(V, \{D\})$. Note also that $(\xi, \zeta) \in B$ if and only if, for all $\varrho = (\varrho_{m+1}, \cdots, \varrho_n) \in \mathbb{C}^{n-m}$

$$\prod_{j=1}^{K} \langle\, \varrho,\, \zeta - \zeta_j(\xi)\, \rangle = 0.$$

Since this product, as a function of $(\xi,\zeta) \in \big(V \setminus \mathrm{var}\,(V,\{D\})\big) \times \mathbb{C}^{n-m}$, is independent of permutations of $j \in \{1,2,\cdots,K\}$, it is a continuous single-valued function of $(\xi,\zeta) \in \big(V \setminus \mathrm{var}\,(V,\{D\})\big) \times \mathbb{C}^{n-m}$ and therefore is $\mathbb{C}$-analytic there. Therefore

$$\prod_{j=1}^{K} \langle\, \varrho,\, \zeta - \zeta_j(\xi)\,\rangle = \sum_{\substack{\sigma\in\mathbb{N}_0^{n-m}\\ |\sigma|=K}} \varrho^\sigma \widetilde{g}_\sigma(\xi,\zeta), \tag{27}$$

where the functions $\widetilde{g}_\sigma$ are analytic on $\big(V \setminus \mathrm{var}\,(V,\{D\})\big) \times \mathbb{C}^{n-m}$. Moreover, for $\xi \in V \setminus \mathrm{var}\,(V,\{D\})$,

$$(z_1,\cdots,z_n) = (\xi,\zeta) \in B \text{ if and only if } \widetilde{g}_\sigma(z_1,\cdots,z_n) = 0$$

for all $\sigma \in \mathbb{N}_0^{n-m}$ with $|\sigma| = K$.

Finally observe that, for all compact sets $W \subset V$,

$$\sup\{|\zeta_j(\xi)| : \xi \in W \setminus \mathrm{var}\,(V,\{D\}),\ 1 \le j \le K\} < \infty.$$

Therefore $\widetilde{g}_\sigma$ is bounded on $(W \times \mathbb{C}^{n-m}) \setminus \mathrm{var}\,(V \times \mathbb{C}^{n-m},\{D\})$ and, by the Riemann extension theorem 9.3, can be extended as an analytic function g_σ on all of $V \times \mathbb{C}^{n-m}$. To complete the proof of the first part let $G = \{g_\sigma : \sigma \in \mathbb{N}_0^{n-m},\ |\sigma| = K\}$ and recall that $V \setminus \mathrm{var}\,(V,\{D\})$ is open, dense and connected in V.

Now suppose that H is real-on-real and $B \cap \mathbb{R}^n \neq \emptyset$. From the implicit function theorem 4.4 with $\mathbb{F} = \mathbb{R}$, $B \cap \mathbb{R}^n$ is an $\mathbb{R}$-analytic manifold of dimension m. By Lemma 11.14, for each j, $\zeta_j(\xi) = \overline{\zeta_k(\xi)}$ for some k. Therefore the left side of (27) is real when ϱ, ξ and ζ are real vectors. Therefore $\widetilde{g}_\sigma(\xi,\zeta)$ is real when ξ and ζ are real vectors. This shows that G is real-on-real. Therefore E is a real-on-real $\mathbb{C}$-analytic variety and $\dim_{\mathbb{R}} B \cap \mathbb{R}^n = m$. □

Remark 11.16. Suppose that $m = n-1$ in the preceding theorem. From (27) it follows that G has only one element, g say, where g is a polynomial in z_n with coefficients analytic on V, its principal coefficient is 1, all the others vanish at $0 \in V$ and its discriminant is not identically zero. Therefore in Theorem 11.15 $\overline{B} = \mathrm{var}\,(V \times \mathbb{C}^{n-1}, G)$ is a Weierstrass analytic variety on V.

The following example of a Weierstrass analytic variety in $\mathbb{C}^3$ with $m = n-2$ shows that this observation may be false when $m \neq n-1$. Let $V = \mathbb{C}$ and let $E = \mathrm{var}\,(V \times \mathbb{C}^2, \{h,\, k\})$ where

$$h(x,y) = y^2 - x^3, \quad k(x,z) = z^2 - x^3, \quad (x,y,z) \in \mathbb{C}^3.$$

Note that E has two branches $B_\pm$,

$$\overline{B}_\pm = \mathrm{var}\,(V \times \mathbb{C}^2, \{h,k,l^\pm\}), \text{ where } l^\pm = yz \pm x^3, \quad (x,y,z) \in \mathbb{C}^3. \tag{28}$$

We now show that neither is a Weierstrass analytic variety on V. Suppose that this is false and that $\overline{B}_-$ is a Weierstrass analytic variety defined by

$$y^p + \sum_{k=0}^{p-1} A_k(x) y^k = 0 \text{ and } z^q + \sum_{l=0}^{q-1} B_l(x) z^l = 0, \tag{29}$$

where the discriminant of the polynomials is non-zero almost everywhere. Therefore, for almost all x in a neighbourhood of 0 in $\mathbb{C}$, there are exactly pq solutions of (29). However, for the same x there are two points (x,y,z) on $\overline{B}_-$. Hence $pq = 2$. Suppose $p = 1$ and $q = 2$. Then the system

$$y = A_0(x) \text{ and } z^2 = B_1(x)z + B_0(x)$$

is equivalent to (28) with a minus sign. But this is false since (28) does not determine y as a function of x. A similar contradiction is reached if $q = 1$ and $p = 2$, and for $\overline{B}_+$. □

We have seen in Definition 11.11, Remark 11.12 and Lemma 11.6 that $\gamma_a(B)$ is irreducible when $a \in B$ and B is a branch of a Weierstrass analytic variety. More is true.

Lemma 11.17. *Let B be a branch of a Weierstrass analytic variety* $\operatorname{var}(V \times \mathbb{C}^{n-m}, H)$. *Then $\gamma_0(\overline{B}) \in \mathcal{V}_0(\mathbb{C}^n)$ is irreducible.*

Proof. The case $m = 0$ is trivial since $B = \{0\}$. Suppose $m \in \mathbb{N}$. Suppose that $\gamma_0(\overline{B}) = \alpha_1 \cup \alpha_2$, $\alpha_1, \alpha_2 \in \mathcal{V}_0(\mathbb{C}^n)$. Let E_1, E_2 be analytic varieties, defined in a neighbourhood of $0 \in \mathbb{C}^n$, which represents α_1 and α_2. Then there exists an open set O in $\mathbb{C}^n$ with $0 \in O$ such that $\overline{B} \cap O = O \cap (E_1 \cup E_2)$.

Now for any $z \in B \cap O$, $\gamma_z(B) \in \mathcal{V}(\mathbb{C}^n)$ is irreducible, by Lemma 11.6. Therefore for every point $z \in B \cap O$ there is an open set O_z in $\mathbb{C}^n$ with $O_z \cap B \subset E_1$ or $O_z \cap B \subset E_2$. Since B is a connected analytic manifold it follows from Lemma 11.4 that $B \subset E_1$ or $B \subset E_2$. Since E_i is closed in $V \times \mathbb{C}^{n-m}$, $\overline{B} \subset E_1$ or $\overline{B} \subset E_2$. Hence $\gamma_0(\overline{B}) = \gamma_0(E_1) = \alpha_1$ or $\gamma_0(\overline{B}) = \gamma_0(E_2) = \alpha_2$, and $\gamma_0(\overline{B})$ is irreducible. This completes the proof. $\square$

In Remark 11.16, the Weierstrass analytic variety E is defined in terms of Weierstrass polynomials h, k neither of which is the product of two Weierstrass polynomials on V, yet E is not irreducible. The following shows that this does not happen when $m = n - 1$.

Lemma 11.18. *Let E denote a Weierstrass analytic variety*

$$\operatorname{var}(V \times \mathbb{C}, \{h\}), \quad h(z_1, \cdots, z_n) = A(z_n; z_1, \cdots, z_{n-1}).$$

Then $\gamma_0(E)$ is irreducible if and only if A is not the product of two Weierstrass polynomials on V.

Proof. Suppose that $\gamma_0(E)$ is not irreducible. By Lemma 11.17, E has at least two branches and by Remark 11.16 the closure of each of these branches is a Weierstrass analytic variety. Suppose one such branch $\widetilde{B}$ has closure defined by a Weierstrass polynomial $\widetilde{A}$ on V, where $\widetilde{A}$ and A are distinct polynomials on V. Then the greatest common divisor of A and $\widetilde{A}$ is $\widetilde{A}$ and $A = \widetilde{A}E$ for some non-trivial Weierstrass polynomial E on V (see the last sentence of Remark 11.10). Therefore A is the product of two Weierstrass polynomials.

When A is the product A_1A_2 of two Weierstrass polynomials, it is easy to see that $\gamma_0(E) = \gamma_0(E_1) \cup \gamma_0(E_2)$, where E_i are the varieties defined using A_j. Since A is a Weierstrass polynomial, $A_1 \neq A_2$ and therefore $\gamma_0(E)$ is not irreducible. This completes the proof. $\square$

12 Analytic Germs and Subspaces

Suppose that α is a $\mathbb{C}$-analytic germ at a with $\gamma_a(\mathbb{C}^n) \neq \alpha$. Suppose that $\operatorname{var}(U, G)$ is a representative of α. Then there is at least one $\mathbb{C}$-analytic function $g \in G$, such that $g \not\equiv 0$ on U. Hence there is a complex line segment L through a in U such that $g \not\equiv 0$ on L. Since g restricted to L is a complex-analytic function of one complex variable, its zeros are isolated. This shows that there exists a one-dimensional complex linear space Y such that $\gamma_a(a + Y) \cap \alpha = \{a\}$.

For the case of real-on-real varieties $\operatorname{var}(U, G)$, suppose that $0 \in U$ and that $g \in G$ is real-on-real. Then the coefficients in the Taylor expansion of g at 0 are all real and not all zero. Hence there exists a real linear space $\widehat{Y} = \{tb : t \in \mathbb{R}\}$, for some $b \in \mathbb{R}^n$, such that $\gamma_0(\widehat{Y}) \cap \alpha = \{0\}$. Moreover, g is not identically zero on the complex linear space $T = \{zb : z \in \mathbb{C}\}$ and hence $\gamma_0(T) \cap \alpha = \{0\}$. We will say that a complex linear subspace T of $\mathbb{C}^n$ is a complexified subspace if it has a real basis (see Remark 6.2). Equivalently T is complexified if it is closed under complex conjugation of the coordinates of its vectors with respect to the standard real basis. Clearly $T = \{zb : z \in \mathbb{C}\}$, $b \in \mathbb{R}^n$, is a complexified space of one complex dimension. Any complexified subspace Z_1 of $\mathbb{C}^n$ has a (in fact many) complementary complexified subspace Z_2 of $\mathbb{C}^n$ such that $\mathbb{C}^n = Z_1 \oplus Z_2$. This ensures that the choice of basis in the last part of the next lemma is possible.

Lemma 12.1. *Suppose that $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$, $n \geq 2$, and Y is a linear subspace of $\mathbb{C}^n$ such that $\alpha \cap \gamma_0(Y) = \{0\}$. Choose a basis of $\mathbb{C}^n$ such that*

$$Y = \{(0, \cdots, 0, z_{m+1}, \cdots, z_n) : (z_{m+1}, \cdots, z_n) \in \mathbb{C}^{n-m}\} \tag{30}$$

and let P denote the projection onto $\mathbb{C}^m$ given by first m coordinates, so that

$$P(E) = \{(z_1, \cdots, z_m) : (z_1, \cdots z_n) \in E\}, \tag{31}$$

where E is a representative of α. Then $\gamma_0(P(E)) \in \mathcal{V}(\mathbb{C}^m)$.

If Y is a complexified subspace, α is real-on-real and we choose a real basis of $\mathbb{C}^n$ such that (30) *holds, then $\gamma_0(P(E))$ is real-on-real.*

Proof. The cases $m = 0$ ($\alpha = \gamma_0(\mathbb{C}^n)$, $Y = \{0\}$) and $m = n$ ($\alpha = \{0\}$, $Y = \mathbb{C}^n$) are trivial and we suppose throughout that $0 < m < n$. In the coordinates (30), let $\alpha = \gamma_0\big(\mathrm{var}\,(W \times B_\delta(\mathbb{C})), \{g_1, \cdots, g_\nu\}\big)$, $\nu \in \mathbb{N}$, for a small positive δ where

$$W = \{(z_1, \cdots, z_{n-1}) \in \mathbb{C}^{n-1} : |z_j| < \delta,\ 1 \le j \le n-1\}.$$

Since

$$\{(0, \cdots, 0, z) \in \mathbb{C}^n : z \in \mathbb{C}\} \subset Y \text{ and } \gamma_0(Y) \cap \alpha = \{0\},$$

$g_k(0, \cdots, 0, z_n) \not\equiv 0$ for some k when $z_n \in B_\delta(\mathbb{C})$. Relabelling $\{g_1, \cdots, g_\nu\}$ if necessary, suppose that $k = 1$. By the Weierstrass preparation theorem 8.1, there is no loss of generality in supposing that g_1 is given by a polynomial A_1 of the form (21).

If A_1 has degree p, say, then, by the Weierstrass division theorem 7.1, we may suppose without loss of generality that each of the other g_k, $k \ge 2$, in the definition of α, has the form

$$g_k(z_1, \cdots, z_n) = A_k(z_n; z_1, \cdots, z_{n-1})$$

where A_k is a polynomial on W of degree at most $p-1$. Thus $\mathrm{var}\,\big(W \times B_\delta(\mathbb{C}), \{g_1, \cdots, g_\nu\}\big)$ is a representative of α and the family $\{g_1, \cdots, g_\nu\}$ of analytic functions satisfies the hypotheses of the projection lemma, Theorem 10.8.

Therefore the projection of $\mathrm{var}\,(W \times B_\delta(\mathbb{C}), \{g_1, \cdots, g_\nu\})$ onto $\mathbb{C}^{n-1} \times \{0\}$ is an analytic variety in $\mathbb{C}^{n-1}$. Let $\beta \in \gamma_0(\mathbb{C}^{n-1})$ denote its germ and let

$$\widehat{Y} = \{(0, \cdots, 0, z_{m+1}, \cdots, z_{n-1}) : (z_{m+1}, \cdots, z_{n-1}) \in \mathbb{C}^{n-m-1}\}.$$

Since $\alpha \cap \gamma_0(Y) = \{0\}$ in $\mathbb{C}^n$, $\gamma_0(\widehat{Y}) \cap \beta = \{0\}$ in $\mathbb{C}^{n-1}$. We can now repeat the argument $n - m$ times to prove the first part of the lemma.

In the case when Y is a complexified subspace, choose a real basis for $\mathbb{C}^n$ such that (30) holds. Then the projection lemma at each step gives a real-on-real variety. This completes the proof. $\square$

Lemma 12.2. *Let Y be a linear subspace which is maximal with respect to $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$ in the sense that for any linear subspace $\widetilde{Y}$ of $\mathbb{C}^n$*

$$\gamma_0(Y) \cap \alpha = \{0\} \text{ and } \widetilde{Y} \ne Y \subset \widetilde{Y} \text{ implies that } \gamma_0(\widetilde{Y}) \cap \alpha \ne \{0\}. \tag{32}$$

Let $n \ge 2$, $m = n - \dim Y \in \{1, \cdots, n-1\}$, and choose coordinates (30)*. Let E be any representative of α. Then $\gamma_0(P(E)) = \gamma_0(\mathbb{C}^m)$, where $P(E)$ is defined in* (31)*.*

Proof. We have seen in the previous lemma that $P(E)$ is an analytic variety in $\mathbb{C}^m$. Suppose that at 0 its germ $\beta \ne \gamma_0(\mathbb{C}^m)$. Then there exists a non-trivial linear space $L \subset \mathbb{C}^m$ such that $\beta \cap \gamma_0(L) = \{0\}$. Let $\widetilde{Y} = L \times Y$. Then $\gamma_0(\widetilde{Y}) \cap \alpha = \{0\}$ which violates the maximality of Y. This proves the lemma. $\square$

This is false for real analytic germs as the real germ of $\{(x, y) \in \mathbb{R}^2 : y - x^2 = 0\}$ illustrates.

Lemma 12.3. *Suppose that $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$ is real-on-real, $n \ge 2$, and T is a complexified subspace of $\mathbb{C}^n$ such that $\alpha \cap \gamma_0(T) = \{0\}$. Suppose that T is a maximal complexified subspace in the sense that if $\widetilde{T}$ is a complexified space then*

$$\gamma_0(T) \cap \alpha = \{0\} \text{ and } T \subset \widetilde{T} \ne T \text{ implies that } \gamma_0(\widetilde{T}) \cap \alpha \ne \{0\}. \tag{33}$$

With respect to a real basis such that

$$T = \{(0, \cdots, 0, z_{m+1}, \cdots, z_n) : (z_{m+1}, \cdots, z_n) \in \mathbb{C}^{n-m}\}, \tag{34}$$

$\gamma_0(P(E)) = \gamma_0(\mathbb{C}^m)$.

Proof. We have seen in Lemma 12.1 that in this case $P(E)$ is a real-on-real analytic variety in $\mathbb{C}^m$. Suppose that its germ at 0, $\beta \neq \gamma_0(\mathbb{C}^m)$. Then, by the remarks at the beginning of the section, there exists a non-trivial complexified subspace $L \subset \mathbb{C}^m$ such that $\beta \cap \gamma_0(L) = \{0\}$. Let $\widetilde{T} = L \times T$. Then $\gamma_0(\widetilde{T}) \cap \alpha = \{0\}$ which violates the maximality of T. This proves the lemma. □

Example 12.4. The following is an illustration of (32) and (33). Let g, $h : \mathbb{C}^4 \to \mathbb{C}$ be defined by

$$g(w, x, y, z) = y^2 + z^2, \quad h(w, x, y, z) = x, \quad (w, x, y, z) \in \mathbb{C}^4.$$

Then E is real-on-real and $E \cap \mathbb{R}^4 = \mathrm{span}_{\mathbb{R}}\{(1, 0, 0, 0)\}$ where

$$E = \mathrm{var}\,(\mathbb{C}^4, \{g, h\}) = \{(w, 0, y, z) \in \mathbb{C}^4 : y^2 + z^2 = 0\},$$

and $\alpha \cap \gamma_0\big(\{0\} \times \mathbb{R}^3\big) = \{0\}$, where $\alpha = \gamma_0(E)$. However the three-dimensional complexified space $T = \{0\} \times \mathbb{C}^3$ is not maximal in the sense of (33); it is too big. It is easy to see that each of the two-dimensional complexified spaces $\{0\} \times \mathbb{C} \times \{0\} \times \mathbb{C}$ and $\{0\} \times \mathbb{C} \times \mathbb{C} \times \{0\}$ are maximal in the sense of (32) and (33). □

13 Germs of $\mathbb{C}$-analytic Varieties

We are now in a position to show that if α is a $\mathbb{C}$-analytic germ at 0 there exists a Weierstrass analytic variety E, a subset C and a branch B of E such that $\alpha = \gamma_0(C)$ and $\gamma_0(B) \subset \alpha$.

Suppose that $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$. If $n = 1$ then $\alpha \in \big\{\emptyset, \{0\}, \gamma_0(\mathbb{C})\big\}$. If $\{0\} \subset \alpha$ but $\alpha \notin \big\{\{0\}, \gamma_0(\mathbb{C}^n)\big\}$ then $n \geq 2$. Moreover, since $\alpha \neq \gamma_0(\mathbb{C}^n)$, §12 ensures the existence of a non-trivial linear subspace Y of $\mathbb{C}^n$ such that

$$\gamma_0(Y) \cap \alpha = \{0\} \tag{35}$$

and, since $\alpha \neq \{0\}$, we infer that $Y \neq \mathbb{C}^n$. Let $n \geq 2$ and let Y be such a linear subspace with

$$m = n - \dim Y \in \{1, \cdots, n-1\} \tag{36}$$

and choose coordinates such that

$$Y = \{(0, \cdots, 0, z_{m+1}, \cdots, z_n) : (z_{m+1}, \cdots, z_n) \in \mathbb{C}^{n-m}\}. \tag{37}$$

Theorem 13.1. *Let* $\alpha \in \mathcal{V}_0(\mathbb{C}^n) \setminus \big\{\{0\}, \gamma_0(\mathbb{C}^n)\big\}$, $n \geq 2$, *and choose coordinates such that* (35), (36) *and* (37) *hold. Then there exists a Weierstrass analytic variety* $\mathrm{var}\,\big(V \times \mathbb{C}^{n-m}, H\big)$, $1 \leq m < n$, *such that*

$$\alpha \subset \gamma_0\big(\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)\big).$$

Proof. As in the proof of Lemma 12.1, the Weierstrass preparation 8.1 and division 7.1 theorems, and Proposition 10.7 can be used to reduce the problem to the case when $\alpha = \gamma_0\big(\mathrm{var}\,(W \times \mathbb{C}, G)\big)$, where $G = \{g_1, \cdots, g_\nu\}$, $\nu \in \mathbb{N}$, and

$$g_1(z_1, \cdots, z_n) = z_n^p + a_{p-1}(z_1, \cdots, z_{n-1}) z_n^{p-1} + \cdots + a_0(z_1, \cdots, z_{n-1}),$$
$$a_j(0) = 0, \quad 0 \leq j \leq p-1, \quad D(g_1) \not\equiv 0 \text{ on } W,$$

and each g_k, $k \geq 2$, is a polynomial in the same variable z_n, with coefficients that are $\mathbb{C}$-analytic functions of the other variables; g_1 is the only polynomial in G with highest degree p.

Case 1. We first observe that the result holds if $m = n - 1$ because, when $0 \in U \subset \mathbb{C}^n$, U open,

$$\alpha \subset \gamma_0\big(\mathrm{var}\,(U, \{g_1\}\big) = \gamma_0\big(\mathrm{var}\,(V \times \mathbb{C}, \{g_1\})\big)$$

for some open neighborhood V of 0 in $\mathbb{C}^{n-1}$, and $\gamma_0\big(\mathrm{var}\,(V\times\mathbb{C},\{g_1\})\big)$ is a Weierstrass analytic variety.

Case 2. Suppose that $\nu=1$ and $G=\{g_1\}$ where $2\le n\in\mathbb{N}$ is arbitrary. Since g_1 is a Weierstrass polynomial in z_n, Proposition 10.1 gives that

$$\gamma_0\big(\{(z_1,\cdots,z_{n-1})\in\mathbb{C}^{n-1}:(z_1,\cdots,z_n)\in\mathrm{var}\,(W\times\mathbb{C},\{g_1\})\}\big)=\gamma_0(\mathbb{C}^{n-1}).$$

By (35) and (37), $m=n-1$ and we are back in Case 1. (In fact we get $\alpha=\gamma_0\big(\mathrm{var}\,(V\times\mathbb{C},H)\big)$ when $\nu=1$.)

The general case. When $G=\{g_1,\cdots,g_\nu\}$ we argue by induction on $n\ge 2$. The inductive hypothesis is that for all $n\ge 3$ and all $\widehat{\alpha}\in\mathcal{V}_0(\mathbb{C}^{\widehat{n}})$, $2\le\widehat{n}<n$, the conclusion of the theorem holds with m, n, Y replaced with $\widehat{m}$, $\widehat{n}$, $\widehat{Y}$ satisfying

$$\gamma_0(\widehat{Y})\cap\widehat{\alpha}=\{0\},\quad \widehat{n}\ge 2,\quad \widehat{m}=\widehat{n}-\dim\widehat{Y}\in\{1,\cdots,\widehat{n}-1\}$$
$$\widehat{Y}=\{(0,\cdots,0,z_{\widehat{m}+1},\cdots,z_{\widehat{n}}):(z_{\widehat{m}+1},\cdots,z_{\widehat{n}})\in\mathbb{C}^{\widehat{n}-\widehat{m}}\}.$$

This hypothesis has been verified when $n=3$ because then $\widehat{n}=2$, $\widehat{m}=1$, which is Case 1. Because of Case 2, it is sufficient to suppose that for $n\ge 3$ and $\nu\ge 2$,

$$\alpha=\gamma_0\big(\mathrm{var}\,(W\times\mathbb{C},\{g_1,\cdots,g_\nu\})\big),$$

where the set $G=\{g_1,\cdots,g_\nu\}$ satisfies the hypotheses of the projection lemma, Theorem 10.8. Let $\widehat{\alpha}=\gamma_0(\mathcal{A})$, where $\mathcal{A}\subset W\subset\mathbb{C}^{n-1}$ denotes the set given by the projection lemma, and note that $\widehat{\alpha}\in\mathcal{V}_0(\mathbb{C}^{n-1})$ by Remark 11.2.

Suppose that $\widehat{\alpha}=\{0\}$. Then in a sufficiently small neighbourhood of the origin in $\mathbb{C}^n$,

$$(z_1,\cdots,z_n)\in\mathrm{var}\,(W\times\mathbb{C},G)\ \text{only if}\ z_n^p=g_1(0,0,\cdots,z_n)=0.$$

It now follows that $\alpha=\{0\}$ which contradicts the hypothesis of the theorem. Hence $\widehat{\alpha}\ne\{0\}$. If $\widehat{\alpha}=\gamma_0(\mathbb{C}^{n-1})$, it follows from (35) and (37) that $n-1=m$ and the result holds, by Case 1.

Finally we come to the case $\widehat{\alpha}\notin\big\{\{0\},\gamma_0(\mathbb{C}^{n-1})\big\}$, $n\ge 3$, $m<n-1$. Let

$$\widehat{Y}=\{(z_1,\cdots,z_{n-1}):z_1=\cdots=z_m=0\},$$

where m is defined in (35) and (37). It follows from the definition of $\mathcal{A}$ and (37) that $\gamma_0(\widehat{Y})\cap\widehat{\alpha}=\{0\}$. With $\widehat{n}=n-1$, $\widehat{m}=m$, the inductive hypothesis gives that the theorem holds in $\mathbb{C}^{n-1}$, $n\ge 3$. Thus, in the same coordinates, there exists (a possibly smaller) $\delta>0$, a set

$$V=\{(z_1,\cdots,z_m)\in\mathbb{C}^m:|z_1|,\cdots,|z_m|<\delta\},$$

and a collection $\widehat{H}=\{\widehat{A}_{m+1},\cdots,\widehat{A}_{n-1}\}$ of Weierstrass polynomials on V with discriminant $\widehat{D}=D(H)$ not identically zero and

$$\gamma_0(\mathcal{A})\subset\gamma_0(V\times\mathbb{C}^{n-m-1},\widehat{H}).$$

Let

$$\Upsilon(z_1,\cdots,z_m)=\Big\{(\widehat{z}_{m+1},\cdots,\widehat{z}_{n-1})\subset\mathbb{C}^{n-m-1}:\widehat{A}_j(\widehat{z}_j\,;z_1,\cdots,z_m)=0,\ m+1\le j\le n-1\Big\}.$$

Since $\widehat{D}\not\equiv 0$, the dependence of the $\widehat{z}_{m+j}$ on $(z_1,\cdots,z_m)\in V\setminus\mathrm{var}\,(V,\widehat{D})$, an open, dense, connected subset of V, is locally $\mathbb{C}$-analytic, by the analytic implicit function theorem 4.4. Now define a polynomial on $V\setminus\mathrm{var}\,(V,\widehat{D})$ by

$$\widehat{A}_n(Z;z_1,\cdots,z_m)=\prod_{\substack{(\widehat{z}_{m+1},\cdots,\widehat{z}_{n-1})\\ \in\Upsilon(z_1,\cdots,z_m)}}A_1(Z;z_1,\cdots,z_m,\widehat{z}_{m+1},\cdots,\widehat{z}_{n-1}).$$

By choosing a smaller value of δ in the definition of V if necessary we see that the coefficients of $\widehat{A}_n$ are bounded and hence, by the Riemann extension theorem 9.3, can be extended as $\mathbb{C}$-analytic functions to all of V. Note that in $\widehat{A}_n$ the coefficient of the highest power of Z is 1 and that all the other coefficients vanish at $0 \in V$. After simplification (Remark 11.10) $\widehat{A}_n$ becomes a Weierstrass polynomial on V.

Let $H = \widehat{H} \cup \{\widehat{A}_n\}$, a collection of $n - m$ Weierstrass polynomials on V. Let $D(z_1, \cdots, z_m)$ denote the product of their discriminants, which is non-zero on an open dense connected subset of V.

Now $\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)$ is a Weierstrass analytic variety. Suppose that $(z_1, \cdots, z_n)$ belongs to a representative of α in a sufficiently small neighbourhood of 0. Then

$$(z_1, \cdots, z_{n-1}) \in \mathcal{A}, \quad (z_1, \cdots, z_m) \in V, \quad (z_{m+1}, \cdots, z_{n-1}) \in \Upsilon(z_1, \cdots, z_m), \quad g_1(z_1, \cdots, z_n) = 0.$$

Thus $\alpha \subset \gamma_0\big(\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)\big)$ and the proof is complete. □

Theorem 13.2. *Let $\alpha = \gamma_0(E)$ in the preceding theorem and with P as in Lemma 12.1 suppose $\gamma_0(P(E)) = \gamma_0(\mathbb{C}^m)$. Then $\gamma_0(B) \subset \alpha$ for some branch B of the Weierstrass analytic variety* $\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)$ *in Theorem 13.1.*

Proof. Without loss of generality suppose that $E = \mathrm{var}\,(V \times \mathbb{C}^{m-n}, G)$, where G is a finite collection of analytic functions and $V \times \mathbb{C}^{n-m}$ is as in Theorem 13.1. With H as in the conclusion of this theorem, let $O \subset V \setminus \mathrm{var}\,(V, (D(H)))$ be a non-empty open ball on which the discriminant $D(H)$ is nowhere zero. For any $\xi \in O$, we may write

$$\big(\{\xi\} \times \mathbb{C}^{n-m}\big) \cap \mathrm{var}\,(V \times \mathbb{C}^{n-m}, H) = \{(\xi, \zeta_j(\xi)) : 1 \leq j \leq p\},$$

where p is the product of the degrees of the Weierstrass analytic polynomials in H, and the ζ_j are analytic functions from O into $\mathbb{C}^{n-m}$, as in Remark 11.12. According to Lemma 11.3, every open set $O_j = \{\xi \in O : (\xi, \zeta_j(\xi)) \notin E\}$ is either empty or dense in O. However, by hypothesis, $\cap_{j=1}^{p} O_j$ is empty. Therefore at least one of these sets, O_{j_0}, is empty. In other words the analytic manifold

$$M_{j_0} = \{(\xi, \zeta_{j_0}(\xi)) : \xi \in O_{j_0}\}$$

is a subset of E. By Lemma 11.4, $\big(B \cap (V \times \mathbb{C}^{n-m})\big) \subset E$, where B is the branch which contains M_{j_0} and the proof is complete. □

Corollary 13.3. *From the maximality hypotheses of Lemmas 12.2 and 12.3, the conclusion on Theorem 13.2 holds.*

Proof. This follows by combining the lemmas and the theorem cited. □

Corollary 13.4. (Dimension of α) *(a) Suppose $n \geq 2$, the hypotheses of Lemma 12.2 hold, and consequently, by Theorem 13.2,*

$$\gamma_0(B) \subset \alpha \subset \gamma_0\big(\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)\big). \tag{38}$$

Then $\mathrm{var}\,\big(V \times \mathbb{C}^{n-m}, H\big)$ *contains no manifold of dimension larger than m,* $\dim_{\mathbb{C}} \alpha = m$ *and*

$$n - m = \max\{\dim Y : \gamma_0(Y) \cap \alpha = \{0\},\ Y \subset \mathbb{C}^n \text{ a linear space}\}. \tag{39}$$

The right hand side of (39) *is called the codimension of the analytic germ α.*

Moreover $\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H \cup \{D(H)\})$ *contains no manifold of dimension equal to or greater that m.*

(b) If the hypotheses of Lemma 12.3 hold and m is defined there instead, then $m = \dim_{\mathbb{C}} \alpha$ and the conclusion of part (a) is valid. In this case the dimension of α is equal to m whether defined in terms of maximal complex subspaces as in part (a), or of maximal complexified spaces as in part (b).

Proof. It is clear from (38) that $\dim_{\mathbb{C}} B = m \leq \dim_{\mathbb{C}} \alpha$. If $D(H)$ is nowhere zero, or zero only at $0 \in V$, it is easy to see that $E = \mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)$ contains no manifold of dimension larger than m and hence $\dim_{\mathbb{C}} \alpha = m$. Suppose that $\{0\} \subset \gamma_0(\mathrm{var}\,(V, \{D(H)\})) \notin \big\{\{0\}, \gamma_0(\mathbb{C}^m)\big\}$ and let N be a manifold of dimension strictly greater than m which is a subset of E. If $x \in B \cap N$ for some branch B then a neighbourhood of x in N is a subset of a neighbourhood of x in B, which is impossible since B is an m-dimensional manifold. Therefore $B \cap N = \emptyset$ for all branches B of E. In other words $N \subset \operatorname{var}(V \times \mathbb{C}^{n-m}, H \cup D(H))$.

Let $P_j(\xi)$ denote the j^{th} coordinate of $\xi \in \mathbb{C}^n$ or $\mathbb{C}^m$. Then P_j is an analytic function and §12 gives the existence of coordinates on V such that

$$\gamma_0\big(V, \{D(H), P_1, \cdots, P_{m-1}\}\big) = \{0\},$$

so that

$$\gamma_0\Big(\operatorname{var}(V \times \mathbb{C}^{n-m}, H \cup \{D(H)\})\Big) \cap \gamma_0(Y) = \{0\}$$

where

$$Y = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : z_1 = \cdots = z_{m-1} = 0\}.$$

Therefore Theorem 13.1 yields the existence of a Weierstrass analytic variety $\operatorname{var}(\widetilde{V} \times \mathbb{C}^{n-\widetilde{m}}, \widetilde{H}))$, $\widetilde{m} < m$, such that

$$N \subset \operatorname{var}(V \times \mathbb{C}^{n-m}, H \cup \{D(H)\}) \subset \operatorname{var}(\widetilde{V} \times \mathbb{C}^{n-\widetilde{m}}, \widetilde{H}).$$

Repeated finitely often we find that this holds with $\widetilde{m} = 1$, which is impossible. Hence $\operatorname{var}(V \times \mathbb{C}^{n-m}, H)$ contains no analytic manifold of dimension larger than m, $\dim_{\mathbb{C}} \alpha = m$ and $\operatorname{var}(V \times \mathbb{C}^{n-m}, H \cup \{D(H)\})$ contains no manifold of dimension m or more. That (39) holds follows from Corollary 13.3.

The proof of (b) is the same once Lemma 12.3 is taken into account. $\square$

Now we improve slightly on the observation in Lemma 11.17 that $\gamma_0(\overline{B})$ is irreducible when B is a branch of a Weierstrass analytic variety.

Lemma 13.5. *Let B be a branch of a Weierstrass analytic variety* $\operatorname{var}(V \times \mathbb{C}^{n-m}, H)$ *and suppose that $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$ is such that $\gamma_0(\overline{B}) \neq \alpha \subset \gamma_0(\overline{B})$. Then $\dim_{\mathbb{C}} \alpha < m$.*

Proof. Suppose that $\gamma_0(\overline{B}) \neq \alpha \subset \gamma_0(\overline{B})$, and let $E = \operatorname{var}(U, G) \subset \overline{B} \cap (V \times \mathbb{C}^{n-m})$, where U is an open set with $B \cup \{0\} \subset U$ and such that $\alpha = \gamma_0(E)$. Let $D(H)$ denote the discriminant of H on V and suppose that $\dim_{\mathbb{C}} \alpha \geq m$. We will infer that $\gamma_0(\overline{B}) = \alpha$, a contradiction which will prove the lemma.

Define an analytic manifold $M \subset E$ as consisting of all $(\dim_{\mathbb{C}} \alpha)$-regular points of E. If $M \subset \operatorname{var}(V \times \mathbb{C}^{n-m}, H \cup \{D(H)\})$, then $\dim_{\mathbb{C}} \alpha < \dim_{\mathbb{C}} B = m$, by Lemma 13.4. Since this is false, by assumption, $M \cap B \neq \emptyset$ and $\dim_{\mathbb{C}} \alpha = \dim_{\mathbb{C}} M = m$. Therefore there exists a point $z \in M \cap B$ which has a neighbourhood O_z in B which is a subset of M. From Lemma 11.4 and the fact that B is connected it follows that $B \subset E$. This contradiction proves the result. $\square$

Corollary 13.6. *Suppose that $m = 1$ in Theorem 13.1. Then*

$$\alpha = \gamma_0\Big(\{0\} \cup_{\gamma_0(B) \subset \alpha} B\Big),$$

where the union is over all branches B of $\operatorname{var}(V \times \mathbb{C}^{n-1}, H)$ *with $\gamma_0(B) \cap \alpha \neq \{0\}$. .*

Proof. Since $m = 1$, $V \subset \mathbb{C}$ and there is no loss of generality in assuming that the discriminant $D(H)$ is non-zero on $V \setminus \{0\}$ and that $\overline{B} = B \cup \{0\}$ for all branches B. Now suppose that B is a branch of $\operatorname{var}(V \times \mathbb{C}^{n-1}, H)$ such that $\alpha \cap \gamma_0(B) \neq \{0\}$ and that $\gamma_0(B) \not\subset \alpha$. Then $\gamma_0(\overline{B}) \not\subset \alpha$ and $\alpha \cap \gamma_0(\overline{B})$ is an analytic germ in $\mathbb{C}^n$ with

$$\gamma_0(\overline{B}) \neq \alpha \cap \gamma_0(\overline{B}) \subset \gamma_0(\overline{B}).$$

Therefore, by Lemma 13.5, $\dim_{\mathbb{C}}\big(\alpha \cap \gamma_0(\overline{B})\big) = 0$. Hence $\alpha \cap \gamma_0(\overline{B}) = \{0\}$. Since this is false we conclude that $\gamma_0(B) \subset \alpha$ for every branch B of $\operatorname{var}(V \times \mathbb{C}, H)$ with $\gamma_0(B) \cap \alpha \not\subset \{0\}$. This proves the corollary. $\square$

We are now in a position to say something, but not everything, about the structure of $\mathbb{C}$-analytic varieties. The following extension of the preceding corollary is more than sufficient for our purposes.

Theorem 13.7. (A Structure Theorem) *Let $n \geq 2$ and $\alpha \in \mathcal{V}_0(\mathbb{C}^n) \setminus \{0\}$ be such that $\{0\} \subset \alpha \neq \gamma_0(\mathbb{C}^n)$. Then there exist sets $B_1, \cdots, B_N$, such that*

(a) $\alpha = \gamma_0\big(B_1 \cup \cdots \cup B_N \cup \{0\}\big)$.

(b) Each B_j, $1 \leq j \leq N$, after a linear change of coordinates (depending on j), is a branch of a Weierstrass analytic variety (depending, including its dimension, on j).

(c) $\dim_{\mathbb{C}} \alpha = \max_{1 \leq j \leq N}\{\dim_{\mathbb{C}} B_j\}$.

(d) If $L \subset \mathbb{C}^n$, $\gamma_0(L) \neq \emptyset$, is a connected $\mathbb{C}$-analytic manifold of dimension $l \in \{1, \cdots, n\}$ the points of which are l-regular points of a representative of α, then there exists $j \in \{1, \cdots, N\}$ such that $\gamma_0(L) \subset \gamma_0(\overline{B_j})$ and $\dim_{\mathbb{C}} B_j = l$.

(e) If α is real-on-real, then it can be arranged that each branch B_j with $B_j \cap \mathbb{R}^n \neq \emptyset$ is real-on-real.

(f) $\alpha \cap \gamma_0(\mathbb{R}^n) = \gamma_0\big(\widetilde{B}_1 \cup \cdots \cup \widetilde{B}_K \cup \{0\}\big)$ where the $\widetilde{B}_j$ denotes those branches which intersect $\mathbb{R}^n$ non-trivially.

(g) $\dim_{\mathbb{R}}(\alpha \cap \mathbb{R}^n) = \max_{1 \leq j \leq K} \dim_{\mathbb{R}}(\widetilde{B}_j \cap \mathbb{R}^n)$.

Proof. We use induction on the dimension of α. Suppose that $\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)$ and the coordinate system are given by Theorem 13.2. Consider first all the branches of $\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)$ such that $\gamma_0(B) \subset \alpha$. If $m = 1$, Corollary 13.6 shows that these are all the branches that we need for the result to hold.

Suppose $m \geq 2$ and make the inductive hypothesis that the results (a)-(d) of the theorem hold for all smaller values of m.

(a)-(c) According to Theorem 13.1,

$$\alpha \subset \gamma_0\big(\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)\big) = \cup \overline{B},$$

where the union is over all branches B of $\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)$. In addition to the branches B with $\gamma_0(B) \subset \alpha$, consider branches $\widetilde{B}$ of $\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)$ such that

$$\emptyset \neq \gamma_0(\overline{\widetilde{B}}) \cap \alpha \neq \gamma_0(\overline{\widetilde{B}}).$$

Since, by Lemma 13.5, the germ $\alpha \cap \gamma_0(\overline{\widetilde{B}})$ has dimension strictly smaller than m, we can apply the inductive hypothesis to each of these branches to complete the proof of (a)-(c).

(d) Note that $\dim_{\mathbb{C}} L \leq m = \dim_{\mathbb{C}} \alpha$ and that we may suppose $L \subset V \times \mathbb{C}^{n-m}$. If $\gamma_0(L) \subset \gamma_0\big(\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H \cup \{D(H)\})\big)$, we apply the inductive hypothesis to obtain the required result.

If $\gamma_0(L) \not\subset \gamma_0\big(\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H \cup \{D(H)\})\big)$, then $L \cap B \neq \emptyset$ for at least one branch B of $\mathrm{var}\,(V \times \mathbb{C}^{n-m}, H)$. For all $z \in L \cap B$, L and B coincide locally, in a neighbourhood of z and $\dim_{\mathbb{C}} L = \dim_{\mathbb{C}} B$. By Theorem 11.15 $\overline{B} \cap (V \times \mathbb{C}^{n-m})$ is an analytic variety. That $\gamma_0(L) \subset \gamma_0(\overline{B})$ now follows from Lemma 11.4.

(e) It is clear that if α is real-on-real, and maximal complexified spaces are used, as in Lemma 12.3, to choose coordinates, then the branches B_j which emerge are real-on-real. Parts (f) and (g) follow from the second part of Theorem 11.15. □

Corollary 13.8. *Let $\alpha \in \mathcal{V}_0(\mathbb{C}^n) \setminus \big\{\emptyset, \{0\}, \gamma_0(\mathbb{C}^m)\big\}$ be irreducible. Then (possibly after a linear change of coordinates) $\alpha = \gamma_0(\overline{B})$ where B is a branch of some Weierstrass analytic variety. If α is real-on-real and $\alpha \cap \gamma_0(\mathbb{R}^n) \neq \{0\}$, then B is a branch of a real-on-real variety.*

Proof. In the notation of Theorem 13.7, $\alpha = \gamma_0(\overline{B_1}) \cup \cdots \cup \gamma_0(\overline{B_N})$, and since α is irreducible the result follows. □

The following example shows that when α is an irreducible $\mathbb{C}$-analytic variety it does not necessarily follow that $\alpha \cap \gamma_0(\mathbb{R}^n)$ is an irreducible real analytic variety.

Example 13.9. Let $V = \mathbb{C}^2$ and $E = \operatorname{var}(V \times \mathbb{C}, \{h\})$ where

$$h(x, y, z) = z^2 + x^2y^2(x^2 + y^2), \quad (x, y, z) \in V \times \mathbb{C}.$$

Clearly $E \subset \mathbb{C}^3$ is a Weierstrass analytic variety defined by the polynomial

$$A(Z; x, y) = Z^2 + x^2y^2(x^2 + y^2), \quad (x, y, z) \in V \times \mathbb{C}.$$

If A is a product of two Weierstrass polynomials then each has order one, and it is easily checked that this is impossible. Hence, by Lemma 11.18, $\gamma_0(E)$ is an irreducible $\mathbb{C}$-analytic variety. However $E \cap \mathbb{R}^3$ coincides with

$$\{(x, y, z) \in \mathbb{R}^3 : x = 0,\ z = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 : y = 0,\ z = 0\}.$$

Therefore $\gamma_0(E \cap \mathbb{R}^n)$ is not a real irreducible variety. □

14 One-dimensional Branches

The following results are aspects of the theory of Puiseux series sufficient for our later needs.

Theorem 14.1. *Suppose that $m = 1$, $2 \le n \in \mathbb{N}$ and that B is a branch of the Weierstrass analytic variety $E = \operatorname{var}(V \times \mathbb{C}^{n-1}, H)$ where V is chosen so that $D(H)$ is non-zero on $V \setminus \{0\}$. Then there exist $K \in \mathbb{N}$, $\delta > 0$ and a $\mathbb{C}$-analytic function*

$$\psi : \{z \in \mathbb{C} : |z|^K < \delta\} \to \mathbb{C}^{n-1}$$

such that the mapping $z \mapsto (z^K, \psi(z))$ is injective, $\psi(0) = 0$ and

$$\{0\} \cup B = \overline{B} \cap (V \times \mathbb{C}^{n-1}) = \{(z^K, \psi(z)) : |z|^K < \delta\}.$$

Remark 14.2. A function ψ satisfying the conclusion of the theorem is not unique. Indeed, if ψ satisfies the theorem and $\zeta \neq 1$ is a K^{th} root of unity then $\widetilde{\psi}(z) := \psi(\zeta z)$ defines another function which also satisfies the conclusion of the theorem. □

Proof. Let $H = \{h_2, \cdots, h_n\}$ where $h_k(z_1, \cdots, z_n) = A_k(z_k; z_1)$, and each A_k is a Weierstrass polynomial of degree p_k, say, $2 \le k \le n$. If the discriminant $D(H)$ is not zero at $z_1 = 0$, then, for all $k \in \{2, \cdots, n\}$, $A_k(Z; z_1) = Z - a_k(z_1)$ where a_k, is an analytic function on $V \subset \mathbb{C}$. In this case the theorem holds with $K = 1$ and

$$\psi(z_1) = (a_2(z_1), a_3(z_1), \cdots, a_n(z_1)), \quad z_1 \in V.$$

Now suppose that $D(H)$ is zero at 0. Note that for $z_1 \in V \setminus \{0\}$ each of the polynomials $A_k(Z; z_1)$ has only simple roots. Let $\widehat{V}$ denote the half-plane in $\mathbb{C}$ defined by

$$\widehat{V} = \{z \in \mathbb{C} : z = \rho + i\theta, \ -\infty < \rho < \log \delta, \ \theta \in \mathbb{R}\},$$

where δ is given in the definition of V, and let

$$\widehat{h}_k(z, z_k) = A_k(z_k; e^z), \quad z \in \widehat{V}, \ z_k \in \mathbb{C}.$$

Let

$$\widehat{H} = \{\widehat{h}_2, \cdots, \widehat{h}_n\} \text{ and } \widehat{E} = \operatorname{var}(\widehat{V} \times \mathbb{C}^{n-1}, \widehat{H}).$$

It is clear that B is a branch of E if and only if $\widehat{B}$ is a branch of $\widehat{E}$, where

$$B = \{(e^z, \xi) : (z, \xi) \in \widehat{B}\}, \quad \xi \in \mathbb{C}^{n-1}.$$

Since $D(H)$ is nowhere zero on $V \setminus \{0\}$, $D(\widehat{H})$ is nowhere zero on $\widehat{V}$ and every point of $\widehat{E}$ is 1-regular (Definition 11.1). We can therefore write

$$\big(\{z\} \times \mathbb{C}^{n-1}\big) \cap \widehat{E} = \{(z, \xi_q(z)) : 1 \le q \le p\},$$

where $p = \prod_{k=2}^{n} p_k$. By the analytic implicit function Theorem 4.4, each ξ_q is defined locally on $\widehat{V}$ as a $\mathbb{C}$-analytic function with values in $\mathbb{C}^{n-1}$ and, since $\widehat{V}$ is simply connected, they define analytic functions on $\widehat{V}$. Thus $\widehat{E}$ is the union of the disjoint graphs of the functions $\xi_q : \widehat{V} \to \mathbb{C}^{n-1}$, $1 \le q \le p$.

Recall that, for $z \in \widehat{V}$, each component of $\xi_q(z) \in \mathbb{C}^{n-1}$ is a simple root of a polynomial $A_k(Z; e^z)$, $2 \le k \le n$. Therefore the set-valued map

$$z \;\mapsto\; \big\{(e^z, \xi_q(z)) : 1 \le q \le p\big\}$$

is $2\pi i$-periodic on $\widehat{V}$. Moreover if, for some $\widehat{z} \in \widehat{V}$ and some $m \in \mathbb{Z}$,

$$\xi_{q_1}(\widehat{z}) = \xi_{q_2}(\widehat{z} + 2\pi m i), \quad q_1, \, q_2 \in \{1, \cdots p\},$$

then

$$\xi_{q_1}(z) = \xi_{q_2}(z + 2\pi m i) \text{ for all } z \in \widehat{V},$$

by the analytic implicit function theorem 4.4 and analytic continuation. Therefore, for $q \in \{1, \cdots, p\}$, the mapping

$$z \mapsto (e^z, \xi_q(z)) \in E, \; z \in \widehat{V}, \tag{40}$$

is periodic with period $2\pi K_q i$ and is injective on the set $V_q = \{z = \rho + i\theta \in \widehat{V} : 0 < \theta \le 2\pi K_q\}$, for some $K_q \in \{1, \cdots, p\}$. It is easy to see that its image on V_q is both open and closed in E and hence is a branch of E.

For a given branch B, choose q such that the image of (40) on V_q coincides with B. We have seen that an injective parameterization of B is given by

$$B = \big\{(e^z, \xi_q(z)) : z \in V_q\big\}.$$

Since $z \mapsto \xi_q(K_q z)$ has period (not necessarily minimal) $2\pi i$, we can define an analytic function $\widetilde{\psi} : \{z : 0 < |z| < \delta^{1/K_q}\} \to \mathbb{C}$ by

$$\widetilde{\psi}(z_1) = \xi_q(K_q \log z_1)$$

where it does not matter which branch of log is chosen. Thus

$$\xi_q(K_q z) = \widetilde{\psi}(e^z), \quad K_q z \in \widehat{V}.$$

This gives a new injective parameterization of B, namely

$$B = \big\{(z_1^{K_q}, \widetilde{\psi}(z_1)) : 0 < |z| < \delta^{1/K_q}\big\},$$

where ψ is analytic and $\lim_{z_1 \to 0} \widetilde{\psi}(z_1) = 0$. The Riemann extension theorem 9.3 means that $\widetilde{\psi}$ has an analytic extension ψ defined on the ball $\{z_1 \in \mathbb{C} : |z_1| < \delta^{1/K_q}\}$ with $\psi(0) = 0$. Let $K = K_q$ to complete the proof. $\square$

Corollary 14.3. *In Theorem 14.1 suppose $\gamma_0(B \cap \mathbb{R}^n) \notin \{\emptyset, \{0\}\}$. Then there exists $k \in \mathbb{N}_0$ with $0 \le k \le 2K - 1$ such that*

$$\mathbb{R}^n \cap \overline{B} = \big\{\big((-1)^k r^K, \psi(r \exp(k\pi i/K))\big) : -\delta^{1/K} < r < \delta^{1/K}\big\}, \tag{41}$$

and this parameterization is injective.

Proof. Since $\gamma_0(B \cap \mathbb{R}^n) \notin \{\{0\}, \emptyset\}$ there exists, by Theorem 14.1, a sequence $\{z_j\} \subset \mathbb{C}$ with $z_j \to 0$ such that $z_j^K \in \mathbb{R}$ and $\psi(z_j) \in \mathbb{R}^{n-1}$ for all $j \in \mathbb{N}$. Therefore, without loss of generality we may assume, for some $k \in \{0, 1, \cdots, 2K - 1\}$, that $z_j = |z_j| \exp(k\pi i/K)$ and

$$\psi(|z_j| \exp(k\pi i/K)) \in \mathbb{R}^{n-1} \text{ for all } j \in \mathbb{N}.$$

Since ψ is a $\mathbb{C}$-analytic function of one complex variable we can infer that $\psi(r \exp(k\pi i/K)) \in \mathbb{R}^{n-1}$ for all $r \in \mathbb{R}$ with $-\delta^{1/K} < r < \delta^{1/K}$. If there exists $l \in \{0, 1, \cdots, 2K - 1\}$ different from k and a sequence $\rho_j > 0$ such that $\psi(\rho_j \exp(l\pi i/K))$ is real and $\rho_j \to 0$ as $j \to \infty$, then, by the preceding argument, we may assume that $\psi(r\exp(l\pi i/K))$ is real for all r with $-\delta^{1/K} < r < \delta^{1/K}$. We will now show that $l-k \in K\mathbb{Z}$.

Suppose that this is false. For $p \in \mathbb{N}$,

$$\frac{d^p\psi}{dr^p}\big(r\exp(k\pi i/K)\big)\Big|_{r=0} = \exp(p\pi i(k-l)/K)\frac{d^p\psi}{dr^p}\big(r\exp(l\pi i/K)\big)\Big|_{r=0},$$

and the derivatives are real. Therefore, for all p with $p(l-k) \notin K\mathbb{Z}$, it follows that

$$\frac{d^p\psi}{dz^p}(0) = 0.$$

Let $p_0 \in \mathbb{N}$ be the generator of the ideal $\{p \in \mathbb{Z} : p(l-k) \in K\mathbb{Z}\}$. Then the power series expansion of $\psi(z)$ at $z=0$ involves only powers of z^{p_0} and it follows that

$$\psi(z_1) = \psi(z_2)$$

for all z_1, $z_2 \in \mathbb{C}$ such that $z_1^{p_0} = z_2^{p_0}$. Since p_0 divides K, $z_1^K = z_2^K$ for all such z_1, z_2. Therefore if $z_1^{p_0} = z_2^{p_0}$,

$$\big(z_1^K, \psi(z_1)\big) = \big(z_2^K, \psi(z_2)\big).$$

Now the injectivity in the theorem above gives that $p_0 = 1$ and $k-l \in K\mathbb{Z}$. This completes the proof. □

Lemma 14.4. *Suppose the Weierstrass polynomials which define the Weierstrass analytic variety* var $(V\times \mathbb{C}^{n-1}, H)$ *in Theorem 14.1 are real-on-real and that the discriminant $D(H)$ is non-zero on $V \setminus \{0\} \subset \mathbb{C}$. Then $B \cap \mathbb{R}^n \notin \{\emptyset, \{0\}\}$ implies that $\gamma_0(B \cap \mathbb{R}^{n-1}) \notin \{\emptyset, \{0\}\}$.*

Proof. Suppose that $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) \in (\mathbb{R}^n \cap B) \setminus \{0\}$. Then each $\widehat{x}_k$, $k \geq 2$, is a simple root, when $z_1 = \widehat{x}_1 \in \mathbb{R}$, of a polynomial whose coefficients ($\mathbb{C}$-analytic functions of z_1) are real when $z_1 = x_1 \in \mathbb{R}$, and are zero when $z_1 = 0$. From the real implicit function theorem 4.4 it follows that each of the polynomials in H has a real root $x_k \in \mathbb{R}$, $2 \leq k \leq m$, which is a real-valued analytic function of x_1 when $\widehat{x}_1 x_1 > 0$ and x_1 is sufficiently small. Moreover, $x_k \to 0$ as $x_1 \to 0$ by Lemma 10.1. It follows from Theorem 14.1 that there exists $k \in \{0, 1, \cdots, 2K-1\}$ such that

$$\psi(r\exp(k\pi i/K)) \in \mathbb{R}^{n-1} \quad \text{for} \quad -\delta^{1/K} < r < \delta^{1/K}.$$

□

Example 14.5. Consider the collection H of three Weierstrass analytic polynomials

$$Z^2 - z_1, \quad Z^3 - z_1^2, \quad Z^4 - z_1^3,$$

and let V be the disc of radius 2 with centre 0 in $\mathbb{C}$. Then the corresponding Weierstrass analytic variety,

$$\text{var}\,(V \times \mathbb{C}^3, H) = \big\{(z_1, z_2, z_3, z_4) : |z_1| < 2,\ z_2^2 - z_1 = z_3^3 - z_1^2 = z_4^4 - z_1^3 = 0\big\},$$

has two branches. To see this note that the branch which contains $(1,1,1,1)$ also contains the closed Jordan curve

$$\Gamma_1 = \{(e^{it},\ e^{(it/2)},\ e^{(2it/3)},\ e^{(3it/4)}) :\ t \in [0, 24\pi]\,\}.$$

Γ_1 projects onto the unit circle in V and contains 12 points above $1 \in \mathbb{C}$. Similarly the branch B_2 containing $(1,-1,1,1)$ contains the closed Jordan curve

$$\Gamma_2 = \{(e^{it},\ e^{i(\pi+t/2)},\ e^{(2it/3)},\ e^{(3it/4)}) :\ t \in [0, 24\pi]\,\}.$$

Γ_2 also has 12 points above $1 \in \mathbb{C}$ and projects onto the unit circle in V. Since the equations

$$t = s \mod 2\pi, \qquad \pi + t/2 = s/2 \mod 2\pi,$$
$$2t/3 = 2s/3 \mod 2\pi, \qquad 3t/4 = 3s/4 \mod 2\pi,$$

imply that

$$2k = 4l - 2 = 3m = 8n/3 \ \text{ for some } \ k,\ l,\ m,\ n \in \mathbb{Z},$$

they have no solutions. Therefore $\Gamma_1 \cap \Gamma_2 = \emptyset$. Since there are at most $2 \times 3 \times 4 = 24$ points of $\mathrm{var}\,(V \times \mathbb{C}^3, H)$ above $1 \in V$ there are at most two branches of this variety. Thus the variety has exactly two branches and $K = 12$ in Theorem 14.1. Moreover candidates for the function ψ (remember it is not unique) corresponding to these branches, are

$$\psi_1(z) = (z^6, z^8, z^9) \text{ and } \psi_2 = (z^6, z^8, -iz^9).$$

Moreover

$$\begin{aligned} B_1 \cap \mathbb{R}^n &= \{(r^{12}, r^6, r^8, r^9) : r \in (-2^{1/12}, 2^{1/12})\} \\ B_2 \cap \mathbb{R}^n &= \{(r^{12}, -r^6, r^8, r^9) : r \in (-2^{1/12}, 2^{1/12})\}, \end{aligned}$$

which correspond to $k = 0$ and $k = 6$, respectively, in Corollary 14.3.

Note that Lemma 14.4 is false if the coefficients of elements of H are complex when their argument is real. For example, when $n = 2$ let $h(Z, z_1) = Z^2 - (1+i)z_1 - iz_1^2$. Then $\mathrm{var}\,(V, \{h\}) \cap \mathbb{R}^2 = \{(0,0), (1,1), (1,-1)\}$. □

15 Global Bifurcation from a Simple Eigenvalue

Suppose, as in Section 5, that X, Y be Banach spaces over $\mathbb{R}$ and that F satisfies (G1). Then Theorem 5.1 holds. Let the set of *all solutions*, of *non-trivial solutions* and of *non-singular solutions* of $F(\lambda, x) = 0$ be denoted respectively by

$$\mathcal{S} = \{(\lambda, x) : F(\lambda, x) = 0\}, \quad \mathcal{T} = \{(\lambda, x) \in \mathcal{S} : x \neq 0\}, \quad \mathfrak{N} = \{(\lambda, x) \in \mathcal{S} : \ker\big(\partial_x F[(\lambda, x)]\big) = \{0\}\},$$

Suppose also that

(G2) $\Lambda' \not\equiv 0$ on $(-\epsilon, \epsilon)$.

(G3) All bounded closed subsets of $\mathcal{S}$ are compact in $\mathbb{R} \times X$

(G4) $\partial_x F[(\lambda, x)]$ is a Fredholm operator of index zero when $F(\lambda, x) = 0$.

Then, because Λ and κ are real analytic, $\epsilon > 0$ can be chosen in Theorem 5.1 such that

$$\Lambda'(s) \neq 0 \text{ for } s \in (0, \epsilon), \quad \kappa'(s) \neq 0 \text{ for } s \in (-\epsilon, \epsilon), \quad \mathcal{R}^+ := \{(\Lambda(s), \kappa(s)) : s \in (0, \epsilon)\} \subset \mathcal{T} \cap \mathfrak{N}. \tag{42}$$

The following result gives a *unique global extension* of (Λ, κ) from $(0, \epsilon)$ to $(0, \infty)$.

Theorem 15.1. *Suppose that (G1)–(G4) hold. Then there exists a continuous curve $\mathfrak{R}$ which extends $\mathcal{R}^+$ as follows.*

(a) $\mathfrak{R} = \{(\Lambda(s), \kappa(s) : s \in [0, \infty) : s \in [0, \infty)\}$ where $(\Lambda, \kappa) : [0, \infty) \to \mathbb{R} \times X$ is continuous.

(b) $\mathcal{R}^+ \subset \mathfrak{R} \subset \mathcal{S}$.

(c) The set $\big\{s \geq 0 : \ker\big(\partial_x F[(\Lambda(s), \kappa(s))]\big) \neq \{0\}\big\}$ has no accumulation points.

(d) At each point, $\mathfrak{R}$ has a local analytic re-parameterization in the following sense. In a right neighbourhood of $s = 0$, $\mathfrak{R}$ and $\mathcal{R}^+$ coincide. For each $s^* \in (0, \infty)$ there exists $\rho^* : (-1, 1) \to \mathbb{R}$ which is continuous, injective, and

$$\rho^*(0) = s^*, \quad t \mapsto (\Lambda(\rho^*(t)), \kappa(\rho^*(t))) \text{ is analytic on } (-1, 1)$$

Furthermore Λ is injective on a right neighbourhood of 0 and for $s^* > 0$ there exists $\epsilon^* > 0$ such that Λ is injective on $[s^*, s^* + \epsilon^*]$ and on $[s^* - \epsilon^*, s^*]$.

(e) One of the following occurs.

(i) $\|(\Lambda(s), \kappa(s))\| \to \infty$ as $s \to \infty$.

(ii) $\mathfrak{R}$ is a closed loop.

In case (ii), $\mathfrak{R} = \{(\Lambda(s), \kappa(s)) : 0 \leq s \leq T\}$ for some $T > 0$ and $(\Lambda(T), \kappa(T)) = (\lambda_0, 0)$. We may suppose that $T > 0$ is the smallest such T and

$$(\lambda(s + T), \kappa(s + T)) = (\Lambda(s), \kappa(s)) \text{ for all } s \geq 0.$$

(f) If $(\Lambda(s_1), \kappa(s_1)) = (\Lambda(s_2), \kappa(s_2))$, $s_1 \neq s_2$ and $\ker \partial_x F[(\Lambda(s_1), \kappa(s_1))] = \{0\}$, then (e)(ii) occurs and $|s_1 - s_2|$ is an integer multiple of T. In particular, $(\Lambda, \kappa) : [0, \infty) \to \mathcal{S}$ is locally injective.

Remarks 15.2. (1) There is no claim that $\mathfrak{R}$ is a maximal connected subset of $\mathcal{S}$. Other curves or manifolds in $\mathcal{S}$ may intersect $\mathfrak{R}$.

(2) $\mathfrak{R}$ may self-intersect in the sense that while $s \mapsto (\Lambda(s), \kappa(s))$ is locally injective, it need not be globally injective. For example, in part (e)(ii) of the theorem it is clearly not globally injective.

(3) In part (d) it may be that ${\sigma^*}'(s^*) = 0$, in which case $\{(\Lambda(s), \kappa(s)) : |s - s^*| < \delta^*\} \subset \mathfrak{R}$ may not be a smooth curve even though it has a local analytic parameterization at every point. Of course, for δ^* sufficiently small, the two segments of the set $\{(\Lambda(s), \kappa(s)) : 0 < |s - s^*| < \delta^*\}$, with $(\Lambda(s^*), \kappa(s^*))$ deleted, are smooth and can be parameterized by λ.

(4) Alternative (e)(i) is much stronger than the claim that $\mathfrak{R}$ is unbounded in $\mathbb{R} \times X$. □

The proof of Theorem 15.1 is organized below in a few short steps.

Definition 15.3. *A distinguished arc is a maximal connected subset of* $\mathfrak{N}$.

Hypothesis $G(4)$ and the analytic implicit function Theorem 4.4 ensure that a distinguished arc $\mathcal{I}$ is the graph of an $\mathbb{R}$-analytic function of λ. More precisely, if $\mathcal{I}$ is a distinguished arc then there exists a (possibly infinite) open interval I and an $\mathbb{R}$-analytic function $g : I \to X$ such that

$$\{(\lambda, g(\lambda)) : \lambda \in I\} = \mathcal{I}. \tag{43}$$

Step 1. To study the structure of $\mathcal{S}$ in a neighbourhood of a point $(\lambda_*, x_*) \in \mathcal{S} \setminus \mathfrak{N}$ we invoke the standard *Lyapunov-Schmidt* procedure in the context of analytic operator equations to obtain a neighbourhood V of $(\lambda_*, 0)$ in $\mathbb{R} \times \ker(\partial_x F[(\lambda_*, x_*)])$ and $\mathbb{R}$-analytic maps, $\psi : V \to X$ and$h : V \to \mathbb{R}^q$ $(q = \dim \ker(\partial_x F[(\lambda_*, x_*)]))$, with

(a) $\psi(\lambda_*, 0) = x_*$.

(b) For all $(\lambda, \xi) \in V$, $h(\lambda, \xi) = 0$ if and only if $F(\lambda, \psi(\lambda, \xi)) = 0$.

(c) If $F(\lambda, x) = 0$ and $\|(\lambda, x) - (\lambda_*, x_*)\|$ is sufficiently small, then there exists $\xi \in \ker(\partial_x F[(\lambda_*, x_*)])$ such that $(\lambda, \xi) \in V$ and $x = \psi(\lambda, \xi)$.

(d) $\dim \ker(\partial_x F[(\lambda, \psi(\lambda, \xi))]) = \dim \ker(\partial_\xi h[(\lambda, \xi)])$, $(\lambda, \xi) \in V$.

Recall the notation of §11. The analytic function $h : V \to \mathbb{R}^q$ may be identified with the set of its q component functions each of which maps V into $\mathbb{R}$ analytically. Therefore we may define an $\mathbb{R}$-analytic variety A and a manifold M by

$$A = \mathrm{var}\,(V, \{h\}) = \{(\lambda, \xi) \in V : h(\lambda, \xi) = 0\}, \qquad M = \{(\lambda, \xi) \in V : (\lambda, \psi(\lambda, \xi)) \in \mathfrak{N}\},$$

where the elements of M are 1-regular points of A. Let $\{M_j : j \in J\}$ denote those non-empty connected components of M which have the property that $\gamma_{(\lambda_*,0)}(M_j) \neq \emptyset$.

Since h is an $\mathbb{R}$-analytic function on the $(q+1)$-dimensional real vector space V , the q components of $h(\lambda, \xi)$ are real functions defined locally in a neighbourhood of $(\lambda_*, 0) \in V$ by a Taylor series, the n^{th} term of which is a sum of terms of the form

$$h^*_{k_1,\cdots,k_{q+1}} x_1^{k_1} \cdots x_{q+1}^{k_{q+1}}, \text{ where } k_1 + \cdots + k_{q+1} = n \text{ and } h^*_{k_1,\cdots,k_{q+1}} \in \mathbb{R}.$$

Here $(x_1, \cdots x_{q+1}) \in \mathbb{R}^{q+1}$ are the coefficients of $(\lambda_*, 0) - (\lambda, \xi)$ in some linear coordinate system. Replacing $(x_1, \cdots, x_{q+1}) \in \mathbb{R}^{q+1}$ with $(z_1, \cdots, z_{q+1}) \in \mathbb{C}^{q+1}$ leads to a real-on-real $\mathbb{C}$-analytic extension h^c of h in a complex neighbourhood V^c of $(\lambda_*, 0)$ and a corresponding $\mathbb{C}$-analytic variety. Let

$$A^c = \mathrm{var}\,(V^c, \{h^c\}) = \{(\lambda, \xi) \in V^c : h^c(\lambda, \xi) = 0\},$$
$$M^c = \big\{(\lambda, \xi) \in V^c : \ker(\partial_\xi h^c[(\lambda, \xi)]) = \{0\}\big\},$$

and let $\{M^c_j : j \in J^c\}$ be the non-empty connected components of M^c with $\gamma_{(\lambda_*,0)}(\mathbb{R}^{q+1} \cap M^c_j) \neq \emptyset$. Note that for each $j \in J$ there exists $\hat{j} \in J^c$ such that $M_j \subset M^c_{\hat{j}}$.

Step 2. Theorem 13.7 (d)–(f) on the structure of complex analytic varieties, when applied to A^c gives, for each $j \in J^c$, the existence of a real-on-real branch B_j with

$$\gamma_{(\lambda_*,0)}(M_j^c) \subset \gamma_{(\lambda_*,0)}(\overline{B}_j), \quad \dim B_j = 1 \text{ and } B_j \subset A^c.$$

By making the neighbourhood V^c smaller if necessary, we may suppose that $B_j \setminus \{(\lambda_*, 0)\} \subset M_j^c$. By Theorem 13.7 there are finitely many branches and hence finitely many M_j^c and M_j. By Theorem 14.1 each of these one-dimensional branches B_j admits a $\mathbb{C}$-analytic parameterization in a neighbourhood of $(\lambda_*, 0)$.

We now return to the setting of $\mathbb{R}^n$. From Corollary 14.3, we obtain that $\overline{M}$, locally near $(\lambda_*, 0)$, is the union of a finite number of curves which *pass through* $(\lambda_*, 0)$ in V, intersect one another only at $(\lambda_*, 0)$ and are given by the parameterization (41). Thus, in our previous notation each M_j, $j \in J$, is paired, *in a unique way* with *another* $M_{\tilde{j}}$, $\tilde{j} \in J$, so that their union with the point $(\lambda_*, 0)$ forms one of these curves in V.

This observation can be lifted to infinite dimensions as follows. Suppose that $\mathcal{I}$ in (43) is a distinguished arc where $I = (a, b)$ with $(b, g(b)) \in \mathcal{S} \setminus \mathfrak{N}$. Then the germ $\gamma_{(\lambda_*,0)}(\mathcal{I})$ coincides with the germ of the image under the mapping $(\lambda, \xi) \mapsto (\lambda, \psi(\lambda, \xi))$ of M_j for some $j \in J$. Hence $\mathcal{I}$ has a unique extension beyond $(b, g(b))$ given by the image of $M_{\tilde{j}}$ under the same mapping.

Definition 15.4. *A route of length $N \in \mathbb{N} \cup \{\infty\}$ is a set $\{\mathcal{A}_n : 0 \le n < N\}$ of distinguished arcs and a set $\{(\lambda_n, x_n) : 0 \le n < N\} \subset \mathbb{R} \times X$ such that*

(a) $(\lambda_0, x_0) = (\lambda_0, 0)$ *is the bifurcation point;*

(b) $\mathcal{R}^+ \subset \mathcal{A}_0$*;*

(c) For $N > 1$ and $0 \le n < N - 1$,

$$(\lambda_{n+1}, x_{n+1}) \in \big(\partial\mathcal{A}_n \cap \partial\mathcal{A}_{n+1}\big) \setminus \{(\lambda_n, x_n)\}$$

and there exists an injective $\mathbb{R}$-analytic map $\rho : (-1, 1) \to \mathcal{A}_n \cup \mathcal{A}_{n+1} \cup \{(\lambda_{n+1}, x_{n+1})\}$ with $\rho(0) = (\lambda_{n+1}, x_{n+1})$. Hence $\mathcal{A}_{n+1}$ is uniquely determined by $\mathcal{A}_n$ and vice versa.

(d) The mapping $n \mapsto \mathcal{A}_n$ is injective. .

Step 3. That $\{\mathcal{A}_0\}$, $\{(\lambda_0, 0)\}$ is a route of length 1 with $(\lambda_0, 0) \in \partial\mathcal{A}_0$ is obvious from the discussion leading to (42). Parts (c) and (d) of the definition of a route imply that $\mathcal{A}_{n+1} \neq \mathcal{A}_n$ and that $\mathcal{A}_{n+1}$ is uniquely determined by $\mathcal{A}_n$. Therefore if

$$\{\mathcal{A}_n^j,\ 0 \le n < N_j\}, \quad \{(\lambda_n^j, x_n^j) : 0 \le n < N_j\}, \quad j \in \{1,\, 2\},$$

are two routes with $N_1 \le N_2$ it follows that

$$\lambda_n^1 = \lambda_n^2, \quad x_n^1 = x_n^2 \text{ for all } n \text{ with } 0 \le n < N_1.$$

Hence, under the hypotheses of Theorem 15.1, there exists a maximal route of length $N \in \mathbb{N} \cup \{\infty\}$ which we denote by

$$\{\mathcal{A}_n,\ (\lambda_n, x_n)\} : 0 \le n < N\}.$$

Global Parametrization of $\mathfrak{R}$

Because of the remark following Definition 15.3,

$$\mathcal{A}_n = \{(\Lambda_n(s), \kappa_n(s)),\ s \in (n, n+1)\}, \quad 0 \le n < N,$$

where (λ_n, κ_n) is as an $\mathbb{R}$-analytic function. There are three cases: $N = \infty$; $N < \infty$ when $\overline{\mathcal{A}}_{N-1}$ is not compact; and $N < \infty$ when $\overline{\mathcal{A}}_{N-1}$ is compact.

Suppose that $\overline{\mathcal{A}_n}$ is a bounded closed set for some $0 \le n < N-1$. Then the parameterization of $\mathcal{A}_n$ by $s \in (n, n+1)$ can be extended as a parameterization of $\overline{\mathcal{A}_n}$ by $s \in [n, n+1]$ with $(\Lambda_n(m), \kappa_n(m)) = (\lambda_m, x_m)$ when $m = n$ and $m = n+1$. Since $n < N-1$, Definition 15.4 implies that $\overline{\mathcal{A}_{n+1}}$ can be parameterized, in a neighbourhood of (λ_n, x_n) by $s \in [n+1, n+1+\epsilon)$, for some $\epsilon > 0$, and so, in the first two cases above,

$$\begin{aligned}(\Lambda(n), \kappa(n)) &= (\lambda_n, x_n), \quad n = 0,\, 1,\, \cdots, N-1,\\ (\Lambda(s), \kappa(s)) &= (\Lambda_n(s), \kappa_n(s)) \text{ for } s \in (n, n+1),\end{aligned}$$

defines a continuous parameterization

$$\{(\Lambda(s), \kappa(s)) : 0 \le s < N\} \tag{44}$$

of

$$\mathfrak{R} = \cup_{0 \le n < N}\big(\mathcal{A}_n \cup \{(\lambda_n, x_n)\}\big) \subset \overline{\mathfrak{R}}.$$

Suppose that neither

$$\lim_{s \to N} \|(\Lambda(s), \kappa(s)\| = \infty \text{ nor } \lim_{s \to N} \operatorname{dist}\big((\Lambda(s), \kappa(s)), \partial U\big) = 0 \tag{45}$$

is true. Then there exists a sequence $t_k \to N$ with $\{(\Lambda(t_k), \kappa(t_k))\}$ bounded in $\mathbb{R} \times X$. Hence, from the compactness hypothesis of Theorem 15.1, $\{(\Lambda(t_k), \kappa(t_k))\}$ is relatively compact and, without loss of generality, we may suppose that it converges to $(\lambda^*, x^*) \in \mathcal{S}$, say.

If $N = \infty$ then every neighbourhood of (λ^*, x^*) intersects infinitely many distinct distinguished arcs, and this contradicts the fact that in a neighbourhood of (λ^*, x^*) the solution set is an $\mathbb{R}$-analytic variety. Now suppose that $N < \infty$ and $\overline{\mathcal{A}}_{N-1}$ is not compact. Since $t_k \to N$, $(\lambda^*, x^*) \in \partial\mathcal{A}_{N-1} \setminus \{(\lambda_{N-1}, x_{N-1})\}$. Using Lyapunov-Schmidt reduction we find that $\mathcal{A}_{N-1}$ in a neighbourhood of (λ^*, x^*) corresponds to a manifold of 1-regular points from a real-analytic one-dimensional branch of an analytic variety in a neighbourhood of $(\lambda^*, 0)$ in $\mathbb{R}^{q+1}$. By Corollary 14.3, as in the discussion preceding Definition 15.4, this contradicts the maximality of the route under consideration. Hence one of the alternatives in (45) occurs.

It is now straightforward to map $[0, N)$ to $[0, \infty)$ to obtain a parameterization of $\mathfrak{R}$ satisfying parts (a)–(d) and (e)(i), in these two cases.

Next we consider the third case, when $N < \infty$ and $\overline{\mathcal{A}}_{N-1}$ is compact. Let (λ_{N-1}, x_{N-1}) and (λ_N, x_N) be the end points of $\overline{\mathcal{A}}_{N-1}$. The unique continuation of $\mathcal{A}_{N-1}$, as a distinguished arc $\mathcal{A}_N$ distinct from $\mathcal{A}_{N-1}$ with an end point in common with $\mathcal{A}_{N-1}$ at (λ_N, x_N), is ensured by Corollary 14.3 on the structure of one-dimensional varieties at a singular point. Since our route is maximal it follows from Definition 15.4 (d) that $\mathcal{A}_N = \mathcal{A}_m$ for some $m \in \{0, \cdots, N-2\}$.

Suppose that $m \in \{1, \cdots, N-3\}$. Since $\mathcal{A}_{m-1}$ and $\mathcal{A}_{m+1}$ are the only continuations of $\mathcal{A}_m$, and since $\mathcal{A}_{N-1}$ is a continuation of $\mathcal{A}_N$, it follows that $\mathcal{A}_{N-1} = \mathcal{A}_{m'}$ where $m' \in \{0, \cdots, N-2\}$. But this violates Definition 15.4 (d). Hence $\mathcal{A}_N = \mathcal{A}_0$ or $\mathcal{A}_N = \mathcal{A}_{N-2}$. In the latter case, it follows by induction that $\mathcal{A}_{2k} = \mathcal{A}_0$ and $\mathcal{A}_{2k+1} = \mathcal{A}_1$. Hence $N = 1$ and $\mathcal{A}_0 \cup \mathcal{A}_1$ forms a loop. On the other hand, when $\mathcal{A}_N = \mathcal{A}_0$, there are two possibilities, $(\lambda_N, x_N) = (\lambda_0, 0)$ or $(\lambda_N, x_N) = (\lambda_1, x_1)$. In the second of these cases it follows that $\mathcal{A}_{N-1} = \mathcal{A}_1$ which contradicts Definition 15.4 (d). The only remaining possibility is that $\mathcal{A}_N = \mathcal{A}_0$, $(\lambda_N, x_N) = (\lambda_0, 0)$ and $\mathfrak{R}$ is a loop in $\mathbb{R} \times X$ parameterized by (44). Once again we can parameterize $\mathfrak{R}$ by $s \in [0, \infty)$ so that parts (a)–(d) and (e)(ii) holds in this case.

Finally to prove (f). Suppose that $(\Lambda(s_1), \kappa(s_1)) = (\Lambda(s_2), \kappa(s_2))$, $s_1 \neq s_2$ and $\ker \partial_x F[(\Lambda(s_1), \kappa(s_1))] = \{0\}$. Then $(\Lambda(s_1), \kappa(s_1)) \in \mathcal{A}_{n_1}$ and $(\Lambda(s_2), \kappa(s_2)) \in \mathcal{A}_{n_2}$ for some $0 \le n_1,\, n_2 < N$. From the implicit function theorem it follows that $\mathcal{A}_{n_1}$ and $\mathcal{A}_{n_2}$ coincide in a neighbourhood of the point where they intersect. Since $\mathcal{A}_{n_1} \cup \mathcal{A}_{n_2} \subset \mathfrak{R}$, the same argument gives that the maximal set where they coincide is $\mathcal{A}_{n_1} = \mathcal{A}_{n_2}$. Thus (e)(ii) occurs and $|s_1 - s_2|$ is an integer multiple of T.

This observation leads to the conclusion that $(\Lambda, \kappa) : [0, \infty) \to \mathcal{S}$ is locally injective which completes the proof of Theorem 15.1.

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