Structure moduli space

Tristan Ozuch

Contents

1 Einstein modulo obstructions desingularizations

1.1 Orbifolds, ALE spaces and naïve desingularizations

1.1.1 Orbifolds and ALE spaces

We start by defining our model spaces asymptotic to some quotient of the Euclidean space $(\mathbb{R}^4/\Gamma, \mathbf{e})$ for $\Gamma \subset SO(4)$ acting freely on $\mathbb{S}^3$. We also denote $r = d_{\mathbf{e}}(0,.)$.

Einstein metrics and their deformations on an orbifold.

Definition 1.1 (Orbifold (with isolated singularities))**.** *We will say that a metric space (M_o, g_o) is an orbifold if there exists $\epsilon_0 > 0$ and a finite number of points $(p_k)_k$ of M_o which we will call* singular *such that we have the following:*

1. *the space $(M_o\backslash\{p_k\}_k, g_o)$ is a Riemannian manifold,*
2. *for each singular point p_k of M_o, there exists a neighborhood of p_k, $U_k \subset M_o$, a finite subgroup acting freely on the sphere, $\Gamma_k \subset SO(4)$, and a diffeomorphism $\Phi_k : B_{\mathbf{e}}(0, \epsilon_0) \subset \mathbb{R}^4/\Gamma_k \to U_k \subset M_o$ for which, for any $l \in \mathbb{N}$, there exists $C_l > 0$ such that*
$$r^l|\nabla^l(\Phi_k^* g_o - \mathbf{e})|_{C^2(\mathbf{e})} \leqslant C_l r^2.$$

Definition 1.2 (The function r_o on an orbifold)**.** *We define r_o, a smooth function on M_o satisfying $\Phi_k^* r_o := r$ on each U_k, and such that on $M_o\backslash U_k$, we have $\epsilon_0 < r_o < 1$ (the different choices will be equivalent for our applications).*

We will denote, for $0 < \epsilon \leqslant \epsilon_0$,

$$M_o(\epsilon) := \{r_o > \epsilon\} = M_o\backslash\Big(\bigcup_k \Phi_k(B_{\mathbf{e}}(0,\epsilon))\Big).$$

Definition 1.3 (Infinitesimal deformations of an Einstein orbifold metric)**.** *Let $(M_o, \mathbf{g}_o)$ be an Einstein orbifold. We define $\mathbf{O}(\mathbf{g}_o)$ as the finite dimensional kernel of the elliptic operator $P_{\mathbf{g}_o} := \frac{1}{2}\nabla^*_{\mathbf{g}_o}\nabla_{\mathbf{g}_o} - \mathring{\mathrm{R}}_{\mathbf{g}_o}$ on 2-tensors of $L^2(\mathbf{g}_o)$, where $\mathring{\mathrm{R}}(h)(X,Y) = \sum_i h\Big(\mathrm{Rm}(e_i, X)Y, e_i\Big),$.*

ALE Ricci-flat metrics and their deformations. Let us now turn to ALE Ricci-flat metrics.

Definition 1.4 (ALE orbifold (with isolated singularities and one end))**.** *An ALE orbifold (N, b) is a orbifold for which there exists $\epsilon_0 > 0$ and a compact $K \subset N$ for which there exists a diffeomorphism $\Psi_\infty : (\mathbb{R}^4/\Gamma_\infty)\backslash B_{\mathbf{e}}(0, \epsilon_0^{-1}) \to N\backslash K$ such that we have*
$$r^l|\nabla^l(\Psi_\infty^* b - \mathbf{e})|_{C^2(\mathbf{e})} \leqslant C_l r^{-4}.$$

Definition 1.5 (The function r_b on an ALE orbifold)**.** *We define r_b a smooth function on N satisfying $\Psi_k^* r_b := r$ on each neighborhood U_k of a singular point of definition 1.1, and $\Psi_\infty^* r_b := r$ on U_∞, and such that $\epsilon_0 < r_b < \epsilon_0^{-1}$ on the rest of N (the different choices are equivalent for our applications).*

For $0 < \epsilon \leqslant \epsilon_0$, we will denote

$$N(\epsilon) := \{\epsilon < r_b < \epsilon^{-1}\} = N\backslash\Big(\bigcup_k \Psi_k(B_{\mathbf{e}}(0,\epsilon)) \cup \Psi_\infty\Big((\mathbb{R}^4/\Gamma_\infty)\backslash B_{\mathbf{e}}(0,\epsilon^{-1})\Big)\Big).$$

Definition 1.6 (Infinitesimal deformations of Ricci-flat ALE orbifolds)**.** *Let* $(N, \mathbf{b})$ *be a Ricci-flat ALE orbifold. We define the space* $\mathbf{O}(\mathbf{b})$ *as the kernel of the operator* $P_{\mathbf{b}} := \frac{1}{2}\nabla^*_{\mathbf{b}}\nabla_{\mathbf{b}} - \mathring{\mathrm{R}}_{\mathbf{b}}$ *on* $L^2(\mathbf{b})$*.*

For any $h \in \mathbf{O}(\mathbf{b})$*, we have*

1. $h = \mathcal{O}(r_b^{-4})$*,*
2. $\delta_{\mathbf{b}} h = 0$*, and*
3. $\mathrm{tr}_{\mathbf{b}} h = 0$*.*

There is a particular infinitesimal Ricci-flat ALE deformation by rescaling and reparametrization which we denote $\mathbf{o}_1$. It is of the form $\mathcal{L}_X \mathbf{b}$ for a harmonic vector field X asymptotic to $r_b\partial_{r_b}$ at infinity. It is linked to the notion of reduced volume of Ricci-flat ALE metric introduced in [**?**], see [**?**].

Definition 1.7 (Normalized Ricci-flat ALE metric)**.** *A* normalized Ricci-flat ALE orbifold *is a Ricci-flat ALE metric with reduced volume* -1*.*

This prevents rescaling of the metric and Ricci-flat ALE deformation in the direction $\mathbf{o}_1$. We will denote $\mathbf{O}_0(\mathbf{b})$ the $L^2(\mathbf{b})$-orthogonal of $\mathbf{o}_1$ in $\mathbf{O}(\mathbf{b})$. These are the infinitesimal Ricci-flat ALE deformations preserving the reduced volume at first order.

1.2 Function spaces

Let us recall the definitions of the function spaces introduced in [**?**].

For a tensor s, a point x, $\alpha > 0$ and a metric g, the Hölder seminorm in dimension n is defined as

$$[s]_{C^\alpha(g)}(x) := \sup_{\{y\in\mathbb{R}^n, |y|<\mathrm{inj}_g(x)\}} \left|\frac{s(x) - s(\exp^g_x(y))}{|y|^\alpha}\right|_g.$$

For orbifolds, we will consider a norm which is bounded for tensors decaying at the singular points.

Definition 1.8 (Weighted Hölder norms on an orbifold)**.** *Let* $\beta \in \mathbb{R}$*,* $k \in \mathbb{N}$*,* $0 < \alpha < 1$ *and* $(M_o, \mathbf{g}_o)$ *an orbifold. Then, for all tensor* s *on* M_o*, we define*

$$\|s\|_{C^{k,\alpha}_\beta(\mathbf{g}_o)} := \sup_{M_o} r_o^{-\beta}\bigg(\sum_{i=0}^k r_o^i |\nabla^i_{\mathbf{g}_o} s|_{\mathbf{g}_o} + r_o^{k+\alpha}[\nabla^k_{\mathbf{g}_o} s]_{C^\alpha(\mathbf{g}_o)}\bigg).$$

For ALE orbifolds, we will consider a norm which is bounded for tensors decaying at the singular points and at infinity.

Definition 1.9 (Weighted Hölder norms on an ALE orbifold)**.** *Let $\beta \in \mathbb{R}$, $k \in \mathbb{N}$, $0<\alpha<1$ and $(N,\mathbf{b})$ be an ALE orbifold. Then, for all tensor s on N, we define*

$$\|s\|_{C^{k,\alpha}_\beta(\mathbf{b})} := \sup_N \Big\{ \max(r_b^\beta, r_b^{-\beta}) \Big(\sum_{i=0}^k r_b^i |\nabla_{\mathbf{b}}^i s|_{\mathbf{b}} + r_b^{k+\alpha}[\nabla^k_{\mathbf{b}} s]_{C^\alpha(\mathbf{b})}\Big)\Big\}.$$

On M, using a partition of unity, we can define a global norm.

Definition 1.10 (Weighted Hölder norm on a naïve desingularization)**.** *Let $\beta \in \mathbb{R}$, $k\in\mathbb{N}$ and $0<\alpha<1$. We define for $s \in TM^{\otimes l_+}\otimes T^*M^{\otimes l_-}$ a tensor $(l_+,l_-)\in\mathbb{N}^2$, with $l := l_+ - l_-$ the associated conformal weight.*

$$\|s\|_{C^{k,\alpha}_\beta(g^D)} := \|\chi_{M_o^t} s\|_{C^{k,\alpha}_\beta(\mathbf{g}_o)} + \sum_j T_j^{\frac{l}{2}} \|\chi_{N_j^t} s\|_{C^{k,\alpha}_\beta(\mathbf{b}_j)}.$$

Decoupling norms. We will actually need a last family of norms to get good analytic properties for our operators.

Definition 1.11 ($\|.\|_{C^{k,\alpha}_{\beta,*}}$ norm on 2-tensors)**.** *Let h be a 2-tensor on (M,g^D), $(M_o,\mathbf{g}_o)$ or $(N,\mathbf{b})$. We define its $C^{k,\alpha}_{\beta,*}$-norm by*

$$\|h\|_{C^{k,\alpha}_{\beta,*}} := \inf_{h_*,H_k} \|h_*\|_{C^{k,\alpha}_\beta} + \sum_k |H_k|_{\mathbf{e}},$$

where the infimum is taken on the (h_*,H_k) satisfying $h = h_* + \sum_k \chi_{A_k(t,\epsilon)} H_k$ for (M,g^D) or $h = h_* + \sum_k \chi_{B_k(\epsilon)} H_k$ for $(M_o,\mathbf{g}_o)$ or $(N,\mathbf{b})$, where each H_k is some constant and trace-free 2-tensors on $\mathbb{R}^4/\Gamma_k$, and where $\chi_{B_k(\epsilon)} = \chi(\epsilon^{-1}r)$.

The point is that considering some Laplacian-like operator $P : C^{k+2,\alpha}_{-\beta} \to r^{-2}C^{k,\alpha}_{-\beta}$ (notice the $-\beta$), we have $P^{-1}\Big(r^{-2}C^{k,\alpha}_\beta\Big) = C^{k+2,\alpha}_{\beta,*}$ and controls on $P : C^{k+2,\alpha}_{\beta,*} \to r^{-2}C^{k,\alpha}_\beta$ and its inverse (orthogonally to the kernel/cokernel).

1.3 Einstein modulo obstructions metrics

Define $B_g := \delta_g + \frac{1}{2} d\mathrm{tr}_g$ the Bianchi operator, where δ is the divergence. Note that for a vector field X identified with the 1-form canonically associated by g, $2\delta_g^* X = \mathcal{L}_X g$, where $\mathcal{L}$ is the Lie derivative. Let $\mathbf{K}_o$ be the L^2-kernel of $B_{\mathbf{g}_o}\delta^*_{\mathbf{g}_o} = \nabla^*_{\mathbf{g}_o}\nabla_{\mathbf{g}_o} - \mathrm{Ric}(\mathbf{g}_o)$ on 1-forms of $(M_o,\mathbf{g}_o)$, define $\tilde{\mathbf{K}}_o := \chi_{M_o(b\epsilon)}\mathbf{K}_o$,,

$$\tilde{B}_{g^D} := \pi_{\tilde{\mathbf{K}}_o^\perp} B_{g^D} \text{ and } \tilde{B}_{\tilde{g}_o} := \pi_{\tilde{\mathbf{K}}_o^\perp} B_{\tilde{g}_o}$$

(this projection is necessary to ensure that it is always possible to put metrics in gauge with respect to g^D). Notice that a metric g in dimension 4 is Einstein if and only if it is a zero of

$$E(g) := \mathrm{Ric}(g) - \frac{\overline{\mathrm{R}}(g)}{4} g,$$

and that $B_gE(g)=0$ by the Bianchi identity. We will be interested in the operator

$$\mathbf{\Phi}_{g^D}(g) := \mathrm{Ric}(g) - \frac{\overline{\mathrm{R}}(g)}{4}g + \delta^*_{g^D}\tilde{B}_{g^D}g$$

on metrics close to g^D. Denoting $\mathring{\mathrm{R}}(h)(X,Y) = \sum_i h\Big(\mathrm{Rm}(e_i,X)Y,e_i\Big)$ for an orthonormal basis e_i, we have the following expression of the linearization: for h satisfying $\int_M \mathrm{tr}_{g^D} h dv(g) = 0$,

$$\begin{aligned}P_{g^D}(h) := d_{g^D}\mathbf{\Phi}_{g^D}(h) = \frac{1}{2}\nabla^*_{g^D}\nabla_{g^D}h - \mathring{\mathrm{R}}_{g^D}(h) + \frac{1}{2}\Big(\mathrm{Ric}_{g^D}\circ h + h\circ \mathrm{Ric}_{g^D} - \frac{\overline{\mathrm{R}}(g^D)}{2}h\Big)\\ + \frac{1}{4\,\mathrm{Vol}(g^D)}\int_M \Big\langle \mathrm{Ric}(g^D) - \frac{\mathrm{R}(g^D)}{2}, h\Big\rangle_{g^D} dv_{g^D}g^D - \delta^*_{g^D}B_{g^D}h + \delta^*_{g^D}\tilde{B}_{g^D}h.\end{aligned} \tag{1}$$

which would reduce to $P := \frac{1}{2}\nabla^*\nabla - \mathring{\mathrm{R}}$ if the metric g^D were Einstein and $\tilde{B} = B$.

1.3.1 Approximate obstructions

Let us define the projection of $\mathbf{O}(\mathbf{g}_o)$ and the $\mathbf{O}(\mathbf{b}_j)$ on (M,g^D) by cut-off:

$$\tilde{\mathbf{O}}(\mathbf{g}_o) := \chi_{M_o(b\epsilon)}\mathbf{O}(\mathbf{g}_o), \tag{2}$$

$$\tilde{\mathbf{O}}(\mathbf{b}_j) := \chi_{N_j(b\epsilon)}\mathbf{O}(\mathbf{b}_j), \text{ and } \tilde{\mathbf{O}}_0(\mathbf{b}_j) := \chi_{N_j(b\epsilon)}\mathbf{O}_0(\mathbf{b}_j) \tag{3}$$

and finally the approximate kernel on (M,g^D_t),

$$\tilde{\mathbf{O}}(g^D) := \bigoplus_j \tilde{\mathbf{O}}(\mathbf{b}_j)\oplus\tilde{\mathbf{O}}(\mathbf{g}_o) \text{ and } \tilde{\mathbf{O}}_0(g^D) := \bigoplus_j \tilde{\mathbf{O}}_0(\mathbf{b}_j)\oplus\tilde{\mathbf{O}}(\mathbf{g}_o). \tag{4}$$

We are interested in the operator $\mathbf{\Psi}_{g^D} : \Big(g^D + C^{2,\alpha}_{\beta,*}(g^D)\cap\tilde{\mathbf{O}}(g^D)^\perp\Big)\times\tilde{\mathbf{O}}(g^D) \to r_D^{-2}C^\alpha_\beta(g^D)$,

$$\mathbf{\Psi}_{g^D}(g,\tilde{\mathbf{o}}) := \mathbf{\Phi}_{g^D}(g) + \tilde{\mathbf{o}}. \tag{5}$$

1.3.2 Einstein modulo obstructions metrics

Definition 1.12 (Einstein modulo obstructions metric)**.** *For any* $(t,v)\in\mathbf{R}^+_*\times\tilde{\mathbf{O}}_0(g^D)$ *close enough to* $(0,0)$ *there exists a* unique *solution* $\hat{g}_{t,v}$ *to the equation*

$$\mathbf{\Phi}(\hat{g}_{t,v})\in\tilde{\mathbf{O}}(g^D_t),$$

satisfying the following conditions:

1. $\|\hat{g}_{t,v} - g^D_t\|_{C^{2,\alpha}_{\beta,*}} \leqslant C(t^2+\|v\|^2_{L^2})$, *for some* $C>0$,

2. $\hat{g}_{t,v} - (g^D_t + v)$ *is* $L^2(g^D_t)$*-orthogonal to* $\tilde{\mathbf{O}}_0(g^D_t)$*, and*

3. $\tilde{B}_{g^D_t}\hat{g}_{t,v} = 0$*.*

It is called an Einstein modulo obstructions *desingularization of* $(M_o, \mathbf{g}_o)$*. We will denote* $-\hat{\mathbf{o}}_{t,v} = \mathbf{\Phi}(\hat{g}_{t,v}) \in \tilde{\mathbf{O}}(g^D_t)$*, for which* $\mathbf{\Psi}_{g^D_t}(\hat{g}_{t,v}, \hat{\mathbf{o}}_{t,v}) = 0$*.*

Remark 1.13. *Note that if the obstructions vanish for a metric* $\hat{g}_{t,v}$*, then it is Einstein. Indeed, we know that* $\tilde{B}_{\hat{g}_{t,v}}E(\hat{g}_{t,v}) = 0$ *and by the first condition,* $\hat{g}_{t,v}$ *and* g^D *are very close to each other hence* $\tilde{B}_{\hat{g}_{t,v}}\delta^*_{g^D_t}$ *restricted to the orthogonal of* $\tilde{\mathbf{K}}_o$ *(the image of* $\tilde{B}_{\hat{g}_{t,v}}$*) is injective.*

2 Boundary problems for Ricci-flat ALE metrics and orbifold

Let us look at the problem at a linear level, i.e. search for solutions of

$$\begin{cases} P_{\mathbf{b}}h = 0, \\ \quad h = \phi \text{ on } \epsilon^{-1}\mathbb{S}^3/\Gamma. \end{cases} \tag{6}$$

for some boundary condition $\phi : \epsilon^{-1}\mathbb{S}^3/\Gamma \to \mathrm{Sym}^2(T\mathbb{R}^4/\Gamma)$. Similarly, on the orbifold, the problem becomes:

$$\begin{cases} P_{\mathbf{g}_o}h = 0, \\ \quad h = \phi \text{ on } \epsilon\mathbb{S}^3/\Gamma. \end{cases} \tag{7}$$

for some small $\epsilon > 0$.

2.1 Asymptotics of the (co)kernel and obstructions

Let us classify the L^2-infinitesimal deformations of $\mathbf{b}$ by their order of decay at infinity:

$$\mathbf{O}(\mathbf{b}) = \bigoplus_{j=4}^{j_{\max}} \mathbf{O}^{(j)}(\mathbf{b})$$

in the following way. Let $j_{\max}$ be the maximum of $j \geqslant 4$ such that there exists $\mathbf{o} \in \mathbf{O}(\mathbf{b})$ with $\mathbf{o} = \mathcal{O}(r^{-j})$. Define $\mathbf{O}^{(j_{\max})}(\mathbf{b})$ as the subspace of $\mathbf{O}(\mathbf{b})$ spanned by the tensors in $r^{-j_{\max}}$ at infinity. We then define $\mathbf{O}^{(j_{\max}-1)}(\mathbf{b})$ as the subspace of $\mathbf{O}(\mathbf{b})$ spanned by the tensors in $r^{-(j_{\max}-1)}$ at infinity and $L^2(\mathbf{b})$-orthogonal to $\mathbf{O}^{(j_{\max})}(\mathbf{b})$. We then iteratively define the subspaces $\mathbf{O}^{(j)}(\mathbf{b})$ which are $L^2(\mathbf{b})$-orthogonal to each other by construction.

The most important aspect of these infinitesimal deformations for the obstructions to the desingularization of Einstein metrics is their asymptotic terms. More precisely, if $\mathbf{o} \in \mathbf{O}^{(j+2)}(\mathbf{b})$, then at infinity $\mathbf{o} = r^{-2-j}\phi_j + \mathcal{O}(r^{-3-j})$, where ϕ_j is a 2-tensor whose coefficients are spherical harmonics associated to the j-th eigenvalue. Denote $\mathbb{O}^{[j]}(\mathbf{b})$ the space of spherical harmonics ϕ_j appearing as the asymptotic term of an element of $\mathbf{O}^{(j+2)}(\mathbf{b})$. The link with obstructions is the following result.

Proposition 2.1. *Let H_2 be a quadratic harmonic 2-tensor in Bianchi gauge (say the quadratic terms of a Ricci flat orbifold). There exists a symmetric 2-tensor h_2 and $\mathbf{o} \in \mathbf{O}^{(4)}(\mathbf{b})$ solutions to*

$$P_{\mathbf{b}}(h_2) = \mathbf{o},$$

with $h_2 = H_2 + \mathcal{O}(r^{-2+\epsilon})$. Moreover, $\mathbf{o} = 0$ if and only if $r^{-2}H_2 \perp_{L^2(\mathbb{S}^3)} \mathbb{O}^{[2]}(\mathbf{b})$. Note that $\mathbb{O}^{[2]}(\mathbf{b}) \neq$ and there are always *obstructions to solve this kind of equation.*

Idea of proof. Consider a cut-off function χ supported at infinity of $(N, \mathbf{b})$. The goal is to find h' decaying at infinity (in $\mathcal{O}(r^{-2+\epsilon})$) such that

$$P_{\mathbf{b}}(\chi H_2 + h') = \mathbf{o},$$

where we remark that

$$P_{\mathbf{b}} h' \perp \mathbf{O}(\mathbf{b}).$$

We must therefore have

$$\mathbf{o} = \pi_{\mathbf{O}(\mathbf{b})} P_{\mathbf{b}}(\chi H_2).$$

Conversely, if $P_{\mathbf{b}}(\chi H_2) - \mathbf{o}$ decays and is orthogonal to the cokernel $\mathbf{O}(\mathbf{b})$, then there exists a decaying h' such that $-P_{\mathbf{b}}(h') = P_{\mathbf{b}}(\chi H_2) - \mathbf{o}$.

By integration by parts of $P_{\mathbf{b}}(\chi H_2)$ against $v \in \mathbf{O}(\mathbf{b})$ with $v = V^4 + \mathcal{O}(r^{-5})$, we find that $(P_{\mathbf{b}}(\chi H_2), v)_{L^2(\mathbf{b})}$ is proportional to $\int_{\mathbb{S}^3/\Gamma} \langle H_2, V^4 \rangle_{\mathbf{e}} dv_{\mathbb{S}^3/\Gamma}$. $\square$

Remark 2.2. *A similar result is true for H_i with homogeneous harmonic polynomials of order i as coefficients, but it would also involve other asymptotics of the other $\mathbf{O}^{(j+2)}(\mathbf{b})$ for $j \leqslant i$. For instance, if $\mathbf{o}_4 \in \mathbf{O}^{(4)}(\mathbf{b})$ has some $r^{-2-i}\phi_i$ in its development, then there will also be $\mathbf{o}_4$ in the obstructions.*

2.2 Solving the linearized boundary problem on a Ricci-flat ALE space

On a given Ricci-flat ALE space, solving (6) is always possible, but something happens if ϕ has some spherical harmonics coinciding with the element of some $\mathbb{O}^{[2]}(\mathbf{b})$ for instance.

Essentially, if for simplicity that $\mathbf{O}(\mathbf{b}) = \mathbf{O}^{(4)}(\mathbf{b})$, the idea is that the kernel of $P_{\mathbf{b}}$ is composed of symmetric 2-tensors asymptotic to all harmonic polynomials **except** the ones of the form $r^2\phi_2$ for $\phi_2 \in \mathbb{O}^{[2]}(\mathbf{b})$ which are **replaced** by the associated elements of $\mathbf{O}(\mathbf{b})$ which are asymptotic to $\frac{\phi_2}{r^4}$.

Proposition 2.3. *Assume for simplicity that $\mathbf{O}(\mathbf{b}) = \mathbf{O}^{(4)}(\mathbf{b})$ (as for Eguchi-Hanson for instance). Let $\phi : \epsilon^{-1}\mathbb{S}^3/\Gamma \to \mathrm{Sym}^2(T\mathbb{R}^4/\Gamma)$.*

1. *If $\phi \perp \mathbb{O}^{[2]}(\mathbf{b})$, then, the solution of (6) is uniformly bounded by a function $\|\phi\|_{L^2}$ (but independently of ϵ) on the interior of $\epsilon^{-1}\mathbb{S}^3/\Gamma$.*

More precisely, if $\phi = \phi_j$ where ϕ_j has eigenfunctions of the spherical Laplacian associated to the j-th eigenvalue as coefficient, then, as $\epsilon \to 0$, we have:

$$h = (\epsilon r)^j \phi_j + \mathcal{O}(\epsilon^j r^{j-1})$$

at infinity for the solution h of (6)*.*

2. *If ϕ is not orthogonal to $\mathbb{O}^{[2]}(\mathbf{b})$, then, it is **not** uniformly bounded in independently of ϵ in the interior of $\epsilon^{-1}\mathbb{S}^3/\Gamma$.*

   More precisely, if $\phi = \phi_2 \in \mathbb{O}^{[2]}(\mathbf{b})$, and if $\mathbf{o} \in \mathbf{O}(\mathbf{b})$ is the associated element, then:

   $$h \approx \epsilon^{-4}\mathbf{o}$$

   in the interior of $\epsilon^{-1}\mathbb{S}^3/\Gamma$.

There are similar results for orbifolds where the kernel of P_o includes every $\frac{\phi_j}{r^{2+j}}$ except those which appear in the developments of the elements of $\mathbf{O}(\mathbf{g}_o)$, the L^2-kernel.

2.3 Solving the boundary value problem modulo obstructions

It is not satisfying to solve the boundary value $\phi_2 \in \mathbb{O}^{[2]}(\mathbf{b})$ by some approximation of $\mathbf{o} = \frac{\phi_2}{r^4} + ...$ for several reasons:

1. The Dirichlet to Neumann map will not match that of the orbifold where the solution is asymptotic to $H_2 = r^2\phi_2$,
2. the solution is not bounded independently of ϵ – it is in contradiction (at the linear level for now...) with the convergence to $\mathbf{b}$ of the rescalings of the degeneration of Einstein metrics.

We can however solve it modulo obstruction using Proposition 2.1 in order to "replace" $\frac{\phi_2}{r^4}$ by $r^2\phi_2$. That is solve:

$$\begin{cases} P_{\mathbf{g}_o} h \in \mathbf{O}(\mathbf{b}) \text{ or } \chi\mathbf{O}(\mathbf{b}) \text{ for some cut-off } \chi \text{ supported in a large region,} \\ \quad h = \phi \text{ on } \epsilon\mathbb{S}^3/\Gamma, \end{cases} \tag{8}$$

and chose the solutions growing polynomially at infinity.

Remark 2.4. *Here the solution is probably not unique as we can compensate portions of ϕ_2 by either the element asymptotic to $\frac{\phi_2}{r^4}$ or $r^2\phi_2$? This kind of non uniqueness is expected as in the end, there is $\mathbf{O}(\mathbf{b}) \oplus \mathbf{O}(\mathbf{g}_o)$ degrees of freedom.*

The boundary value for all of the 2-tensors h_j satisfying

$$P_{\mathbf{b}} h_j \in \mathbf{O}(\mathbf{b}),$$

and $h_j = r^j\phi_j + ...$ can be chosen so that it is $\epsilon^{-j}\phi_j + \mathcal{O}(\epsilon^{4-j})$. The h_j are unique up to the harmonic 2-tensors growing slower at infinity.

2.4 Limiting behavior of the Dirichlet to Neumann maps on the ALE and the orbifold

Let us look at the linearized Dirichlet problem when $\epsilon \to 0$.

Conjecture 2.5. *There is no cokernel for the operator "modulo obstructions". The kernel should be composed of approximations of* $\epsilon^2 h_2 - \epsilon^{-4}\mathbf{o}$ *for* $h_2 \sim r^2\phi_2$ *and* $\mathbf{o} \sim r^{-4}\phi_2$.

The Dirichlet to Neumann map "sees" this kernel.

2.5 Matching boundary values

The orbifold is solution of $\mathrm{Ric}(\mathbf{g}_o) = \Lambda \mathbf{g}_o$ with boundary

$$\mathbf{e} + \sum_{i=2}^{+\infty} \epsilon^i \phi_i$$

on $\epsilon\mathbb{S}^3/\Gamma$, and the Ricci-flat ALE metric is solution of $\mathrm{Ric}(\mathbf{b}) = 0$ with boundary condition

$$\mathbf{e} + \sum_{j=2}^{+\infty} \epsilon^{j+2} \psi_j$$

on $\epsilon^{-1}\mathbb{S}^3/\Gamma$.

Conjecture 2.6. *Matching the two Dirichlet and Neumann conditions when considering the cut-off of obstructions should (formally) correspond to the development in Section B.*

Remark 2.7. *If we do not consider cut-offs of the obstructions (far away from the gluing region), we need to match the obstructions on the ALE on the orbifold and vice versa. It is unclear to me how to do that in a systematic way past the first asymptotics...*

The advantage of matching the metrics and their derivatives on a hypersurface is that it must be much easier to preserve analyticity (what if there are log-terms however?) if we do it "directly" by fixed point. The hope is that we could maybe "read" the obstructions in the development of the boundary function in spherical harmonics obtained by fixed point, no? Can we have any control on its value?

A Rewriting locally polyhomogeneous decompositions on a blow-up

Consider the two distance parameters $r_o := \phi_{s^{-1}}^* r = s^{-1} r$ and $r_b := \phi_s^* r = sr$ which are related by

$$s^{-2} = \frac{r_o}{r_b}.$$

A.1 Weighted norms on the blow-up space

Let $f:(r_o, \frac{1}{r_b}) \mapsto \mathbb{R}$ be a smooth function on the blow-up space. We define

$$\|f\|_{\tilde{C}^k_\beta} := \sup_{M_o\times N} r_o^{-\beta}(r_b^{-1})^{-\beta} \sum_{i+j\leqslant k} \Big|(r_o^i\nabla_{\mathbf{o}}^i)(r_b^j\nabla_{\mathbf{b}}^j)f\Big|.$$

Remark A.1. *Seeing f as a function of r_b or of r_b^{-1} does not change much. Indeed if we replace $r_b\partial_{r_\mathbf{b}}$ by $-r_b^{-1}\partial_{r_\mathbf{b}^{-1}}$, we find a comparable norm.*

The restriction of a function bounded in $\tilde{C}^k_\beta$ to the naïve desingularization given by $s^{-2} = \frac{r_o}{r_b}$ is exactly C^k_β (best seen on an annulus).

The blow-up of the set of annuli $A_\mathbf{e}(s, s^{-1})$ is the product (or union?) $A_\mathbf{e}(1,+\infty)_{r_o}\times A_\mathbf{e}(0,1)_{r_b}$ with the identification $s^{-1}r_o = sr_b = r$ giving back the annulus.

A.2 Polyhomogeneous developments

We say that a function $f:\mathbb{S}^3\times[0,\epsilon_0]\times[0,\epsilon_0]\to\mathbb{R}$ is locally converging polyhomogeneous of order 0 if:

$$\begin{aligned} f(x,r_o,r_b^{-1}) = &\sum_{j,m}(r_b)^{-j}\log^l(r_b^{-1})f_o^{(i,k)}(x,r_o)\\ &+\sum_{i,k}(r_o)^i\log^k(r_o)f_b^{(i,k)}(x,r_b^{-1})\\ &+\sum_{ijkl}(r_b)^{-j}\log^l(r_b^{-1})(r_o)^i\log^k(r_o)\tilde{F}^{jl}_{ik}(x)\end{aligned}$$

where $f_o^{(i,k)}(x,r_o)$ is supported in $1<r_o<b$ and $f_b^{(i,k)}(x,r_b^{-1})$ is supported in $b^{-1}<r_b^{-1}<1$, and where all of the sums are (C^0)bounded.

B Formal developments of Einstein desingularizations

Here we present how one can find (and explicit) a polyhomogeneous development of the metric. There is no reason for it to be convergent however.

Definition B.1. *For a section s on $(\mathbb{R}^4/\Gamma)\backslash\{0\}$, we will write $s\propto t^{\frac{l}{2}}r_o^kr_b^l$ if for all $l\in\mathbb{N}$ and $\epsilon>0$, there exists a constant $C>0$ such that $|\nabla^l s|_\mathbf{e}\leqslant C_l t^{\frac{l-\epsilon}{2}}r_o^kr_b^l(r_b^\epsilon+r_o^{-\epsilon})$ as $t\to 0$.*

Remark B.2. *One has $t^{\frac{l}{2}}(\log t)^a r_o^k r_b^l(\log r_o)^b(\log r_b)^c\propto t^{\frac{l}{2}}r_o^kr_b^l$ for all $a,b,c>0$.*

Proposition B.3. *Given $(M_o, \mathbf{g}_o)$ an Einstein orbifold singular at p of singularity $\mathbb{R}^4/\Gamma$, and denote g^D_t its desingularization at p by $(N, \mathbf{b})$ at scale $t > 0$. Consider the asymptotic expansions in homogeneous symmetric 2-tensors of $\mathbf{g}_o$ at p, $\mathbf{g}_o = \mathbf{e} + \sum_{i\geqslant 0} H_i$, where $H_i \propto r_o^i$, and of $\mathbf{b}$ at infinity, $\mathbf{b} = \mathbf{e} + \sum_{j\geqslant 4} H^j$, where $H^j \propto r_b^{-j}$.*

Then, there exist polyhomogeneous 2-tensors, $H_i^j \propto r_o^i r_b^{-j} = t^{\frac{j}{2}} r^{i-j}$ for $i, j \geqslant 0$ on $\mathbb{R}^4/\Gamma$, $\underline{h}_i$ on N and $\overline{h}^j$ on M_o such that we have the following properties.

1. *$H_i^0 = H_i$ and $H_0^j = H^j$,*
2. *the formal power series $g_o^t := \mathbf{g}_o + \sum_{j\geqslant 4} \overline{h}^j$, satisfies the formal equation $\mathbf{\Phi}_{\tilde{g}_o}(g_o^t) \in \tilde{\mathbf{O}}(\mathbf{g}_o)$, where $\tilde{g}_o = \chi_{M_o(\epsilon)}\mathbf{g}_o + \Big(1 - \chi_{M_o(\epsilon)}\Big)\mathbf{e}$, and $g_o^t \perp \mathbf{O}(\mathbf{g}_o)$*
3. *and the formal power series $b^t := \mathbf{b} + \sum_{i\geqslant 2} \underline{h}_i$ satisfies formally the equation $\mathbf{\Phi}_{\tilde{b}}(b^t) \in \tilde{\mathbf{O}}(\mathbf{b})$, where $\tilde{b} = \chi_{N(\epsilon)}\mathbf{b} + \Big(1 - \chi_{N(\epsilon)}\Big)\mathbf{e}$, and $b^t \perp \mathbf{O}(\mathbf{b})$*
4. *at infinity on N, we have the (converging) development,*
$$\underline{h}_i = H_i + \sum_{j\geqslant 4} H_i^j, \tag{9}$$
5. *and on M_o we have the following (converging) development at p_o,*
$$\overline{h}^j = H^j + \sum_{i\geqslant 2} H_i^j, \tag{10}$$

Since the asymptotic terms H_i^j match, we obtain a formal solution by gluing $\mathbf{b}^t$ to $\mathbf{g}_o^t$.

Proof. The operators

$$\begin{aligned} L_{\mathbf{g}_o} : \Big(\tilde{\mathbf{O}}(\mathbf{g}_o)^\perp \cap C^{2,\alpha}_{\beta,*}(\mathbf{g}_o)\Big) \times \tilde{\mathbf{O}}(\mathbf{g}_o) &\to r_o^{-2} C^\alpha_\beta(\mathbf{g}_o), \\ (h, \tilde{\mathbf{o}}_o) &\mapsto P_{\mathbf{g}_o}(h) + \tilde{\mathbf{o}}_o. \end{aligned}$$

and

$$\begin{aligned} L_{\mathbf{b}} : \Big(\tilde{\mathbf{O}}(\mathbf{b})^\perp \cap C^{2,\alpha}_{\beta,*}(\mathbf{g}_o)\Big) \times \tilde{\mathbf{O}}(\mathbf{b}) &\to r_b^{-2} C^\alpha_\beta(\mathbf{g}_o), \\ (h, \tilde{\mathbf{o}}_b) &\mapsto P_{\mathbf{b}}(h) + \tilde{\mathbf{o}}_b. \end{aligned}$$

are invertible and we will denote $L_{\mathbf{g}_o}^{-1}$ and $L_{\mathbf{b}}^{-1}$ their respective inverses.

The idea is to alternate between solving an equation on the orbifold and solving an equation on the ALE. Indeed, the former will determine the polyhomogeneous tensors which are L^2 in a neighborhood of 0 and the latter, the ones which are L^2 at infinity. On the orbifold, tensors in $\mathcal{O}(r^{-2+\epsilon})$ for $\epsilon > 0$ around zero are L^2. On the ALE, tensors in $\mathcal{O}(r^{-\epsilon})$ are determined by [**?**, **?**]. The first iterations have been developed in [**?**, **?**, **?**, **?**].

When the terms up to order n have been determined, we find the next ones as the terms compensating the multilinear errors created. That is:

$$\begin{cases} P_{\mathbf{g}_o}\overline{h}^{n+1} + \sum\limits_{\{l,(j_1,\dots,j_l)\}} \mathrm{Ric}^{(l)}_{\mathbf{g}_o}(\overline{h}^{j_1},\dots,\overline{h}^{j_l}) - \Lambda\overline{h}^{n+1} \in \tilde{\mathbf{O}}(\mathbf{g}_o),\\ \overline{h}^{n+1} - \left(H^{n+1} + H_2^{n+1} + \dots + H_n^{n+1}\right) \propto r_b^{n+1}r_o^{n+1},\ \text{for all } \epsilon > 0. \end{cases} \tag{11}$$

from which we determine the terms H_{n+1}^{n+1} and the higher order terms H_k^{n+1} for $k \geqslant n+1$ thanks to the asymptotic development $\overline{h}^{n+1} = H^{n+1} + H_2^{n+1} + \dots + H_n^{n+1} + \sum_{k\geqslant n+1} H_k^{n+1}$. We then solve

$$\begin{cases} P_{\mathbf{b}}(\underline{h}_{n+1}) + \sum\limits_{\{l,(j_1,\dots,j_l)\}} \mathrm{Ric}^{(l)}_{\mathbf{b}}(\underline{h}_{j_1},\dots,\underline{h}_{j_l}) - \Lambda\underline{h}_{n-1} \in \tilde{\mathbf{O}}(\mathbf{b}),\\ \underline{h}_{n+1} - \left(H_{n+1} + H^4_{n+1} + \dots + H^n_{n+1} + H^{n+1}_{n+1}\right) \propto r_o^{n+1}r_b^{n+2},\ \text{for all } \epsilon > 0, \end{cases} \tag{12}$$

from which we determine the terms H_{n+1}^{n+1} and more generally the higher order terms H_{n+1}^k. This ensures that all of the terms $H_i^j \propto r_o^i r_b^{-j}$ satisfy the equations:

$$P_{\mathbf{e}}(H_i^j) + \sum_l \sum_{\{i_1,\dots,i_l,j_1,\dots j_l|\dots\}} \mathrm{Ric}^{(l)}_{\mathbf{e}}(H_{i_1}^{j_1},\dots,H_{i_l}^{j_l}) = \Lambda H_{i-2}^j.$$

Schematically, after n iterations, the situation is as follows: we have determined all of the H_i^j for $i \leqslant n$ or $j \leqslant n$ and there remain to understand the terms with $i > n$ and $j > n$ symbolized by "?".

	$\mathbf{b}$	$\underline{h}_2$	$\underline{h}_3$	...	$\underline{h}_n$				
$\mathbf{g}_o$	$\mathbf{e}$	H_2	H_3	...	H_n	H_{n+1}	H_{n+2}	H_{n+3}	...
$\overline{h}^4$	H^4	H_2^4	H_3^4	...	H_n^4	H_{n+1}^4	H_{n+2}^4	H_{n+3}^4	...
$\overline{h}^5$	H^5	H_2^5	H_3^5	...	H_n^5	H_{n+1}^5	H_{n+2}^5	H_{n+3}^5	...
...	...	...	...	...	...	...	...	...	...
$\overline{h}^n$	H^n	H_2^n	H_3^n	...	H_n^n	H_{n+1}^n	H_{n+2}^n	H_{n+3}^n	...
	H^{n+1}	H_2^{n+1}	H_3^{n+1}	...	H_n^{n+1}	?	?	?	...
	H^{n+2}	H_2^{n+2}	H_3^{n+2}	...	H_n^{n+2}	?	?	?	...
	H^{n+3}	H_2^{n+3}	H_3^{n+3}	...	H_n^{n+3}	?	?	?	...
	...	...	...	...	...	...	...	...	...

$\square$

Remark B.4. *It is important to note that the above approximations g_o^t and b^t will however not converge a priori, and that they will only provide Einstein metrics only at a formal level when the obstructions vanish.*

Remark B.5. *This can be iterated to trees of singularities and to multiple singularities as a power series in $t = (t_j)_j$, but this makes it even harder to keep track of the obstructions.*