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Convergent Expansions for Solutions to Non-Linear Singular Cauchy Problems

By *Andrew Byde*

ABSTRACT. This article deals with Fuchsian type systems of the form

$$\left(t\frac{\partial}{\partial t} - L\right)u = g\left(t, y, u, \frac{\partial u}{\partial y}\right)$$

with

$$g(0, \dots 0) = 0, \qquad \nabla_{u,v,w} g(0, 0, 0, 0) = 0 .$$

Here L is a fixed endomorphism, and g is analytic in all variable, including t. It is known from Baouendi–Goulaoic that, if no eigenvalue of L is a non-negative integer, such a system has a unique analytic solution (unique precisely because the kernel contains only nonsmooth functions). The aim of this article is complementary to this result: it is to describe this kernel. The main theorem states roughly that the generalized eigenspaces associated with eigenvalues of L of positive real parts parametrize the set of all solutions. The method of proof is by constructing a formal solution, and proving convergence inductively with the aid of majorizing series.

1. Introduction

The Cauchy–Kowalevski theorem is a venerable standard in the field of partial differential equations. In one of its many incarnations it reads as follows. Let Ω be an open set in $\mathbb{R}^n$, and consider the non-linear PDE on $(-\epsilon, \epsilon) \times \Omega$ (with variable (t, y)):

$$\Lambda u(t, y) = \sum_{j=0}^{m} \sum_{|\beta|+j\leq m} a_{\beta,j}(y)\partial_y^{\beta} (\partial_t)^j u = tg\left(t, \partial_y^{\gamma} \partial_t^{k} u\right) , \tag{1.1}$$

where g depends only on the derivatives of orders strictly less than m. The Cauchy–Kowalevski theorem in this context says that if the surface $t = 0$ is non-characteristic for Λ then there is a solution u to (1.1) with the first $(m - 1)$ t-derivatives prescribed at $t = 0$. Generalized, and coupled with suitable uniqueness theorems, this can be interpreted as saying that there is a one-one relationship between analytic solutions to (1.1) on the interior of a manifold, and $(m - 1)$-tuples of analytic functions on its boundary. This relationship is a fruitful one, and many (successful)

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attempts have been made to provide alternatives to the Cauchy–Kowalevski theorem when the hypotheses on which it rests are weakened. Besides the analyticity assumption, which is central to the theorem, the most substantial assumption is that $t = 0$ be non-characteristic.

There are geometric motivations for extending Cauchy–Kowalevski to the characteristic case. Often when a geometric 'infinity' is compactified, the resulting manifold has degeneracies along the boundary which give rise to PDEs of this sort.

- Let M be a Riemannian manifold with metric g_0. It is known that if the metric g_0 has scalar curvature $R(g_0)$, and the metric $g = u^{4/(n-2)} g_0$ has scalar curvature $R(g)$, then

$$\Delta_{g_0}(u) - \frac{(n-2)}{4(n-1)} R\left(g_0\right) u + \frac{(n-2)}{4(n-1)} R(g) u^{\frac{n+2}{n-2}} = 0 . \tag{1.2}$$

  The singular Yamabe problem is the problem of finding u defined on the complement of a fixed set Λ, for which $R(g)$ is constant, and such that g is a complete metric on $M \setminus \Lambda$. As explained in [7], if $M = S^n$ and $\Lambda = S^k$, we are led to solve

$$\left(r \partial_r\right)^2 u - k\left(r \partial_r\right) u + r^2 \Delta_y u + \Delta_\theta u + \frac{(n-2)(n-2k-2)}{4}\left(u^{\frac{n+2}{n-2}} - u\right) = 0 , \tag{1.3}$$

  with local coordinates $(y, r, \theta) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. In Section 7 we study the role that the results of this chapter have to play in understanding solutions to this equation.
- In Section 8 we shall discuss a problem related to the existence of complete hyperbolic Einstein metrics on the interior of a manifold with boundary.

Progress has already been made in understanding the characteristic problem. A milestone in this line of research is the paper by Baouendi and Goulaouic [2] which establishes many results, of which the most appropriate in the context of this article is the following.

Consider the system

$$t \frac{\partial u}{\partial t} = f\left(t, y, u, \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_n}\right) \tag{1.4}$$

with $u = (u_1, \ldots, u_d)$, $f = (f_1, \ldots, f_d)$, $t \in \mathbb{R}$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$; and where the functions f_i are infinitely differentiable with respect to t, valued in the space of analytic functions with respect to the variables y, u, $(\partial u/\partial y)$, and defined in a neighborhood of $(0, 0, u_0(0), (\partial u_0/\partial y)(0))$ where u_0 is an analytic function defined in a neighborhood of 0 in $\mathbb{R}^n$ such that

$$f(0, y, u_0(y), (\partial u_0/\partial y)(y)) = 0 \tag{1.5}$$

in the neighborhood of 0.

We say that the system (1.4) is of *Fuchs type* at $(0, u_0)$ if, in addition to (1.5), we have for $i = 1, \ldots, n$

$$f'_{(\partial u/\partial y_i)}(0, y, u_0(y), (\partial u_0/\partial y)(y)) = 0 \tag{1.6}$$

for y near 0. This equation may be reduced via $u = u_0 + v$ to

$$\left(t \partial_t - L\right) v = g\left(t, y, u, \partial_y u\right) \tag{1.7}$$

with $L \in \mathrm{End}(\mathbb{R}^d)$ corresponding to the linearization of f with respect to u; the assumptions defining "Fuchs-type" (1.5) and (1.6) become

$$\begin{aligned} g(0, y, 0, 0) &= 0 \\ \nabla_{(u,\partial_y u)} g(0, y, 0, 0) &= 0 . \end{aligned}$$

Their theorem is then that if $\mathrm{spec}(L) \cap \mathbb{N} = \phi$, there is a unique solution u to (1.7) defined in a neighborhood of 0 in $\mathbb{R}^{n+1}$, C^∞ with respect to t (or analytic if g is), and analytic with respect to y, such that $u(0, y) = 0$. The fact that (1.4) (and hence (1.7)) is only first order is irrelevant: it is easy to reduce higher-order equations to this.

The proof works by first establishing the result when $\mathrm{spec}(L)$ is assumed to lie in the negative half-plane, then extending to $\mathrm{spec}(L) \cap \mathbb{N} = \phi$) by constructing a solution u of the form

$$u(t, y) = \sum_{j<N} t^j u_j(y) + t^N u_N(t, y) ,$$

and explicitly solving for $u_j(y)$ (which leaves an equation for u_N of the correct form, if N is large enough). The problem with this approach is simply that there may not be a solution of this form. This problem is symptomatic of a broader point, which is that equations of type (1.7) tend to have solutions whose asymptotics as $t \to 0$ involve *non-integral* powers of t, and hence which are not C^∞, let alone analytic at $t = 0$. Consider the following example (in which $h(t)$ is analytic in a neighborhood of zero, and $\lambda \in \mathbb{C}$):

$$(t\partial_t - \lambda)\, u(t) = h(t) = \sum_j t^j h_j . \tag{1.8}$$

The general solution to this equation is

$$\begin{aligned} u(t) &= w_1 t^\lambda + h_\lambda t^\lambda \log t + \sum_{j \neq \lambda} \frac{t^j}{(j - \lambda)} h_j \quad && \lambda \in \mathbb{N} \\ u(t) &= w_1 t^\lambda + \sum_j \frac{t^j}{(j - \lambda)} h_j && \lambda \notin \mathbb{N} \end{aligned} \tag{1.9}$$

for any $w_1 \in \mathbb{C}$. The theorem of [2] cannot be applied to (1.8) when $\lambda \in \mathbb{N}$, and otherwise tells us that (1.8) has a unique analytic solution, which it *does*: the case $w_1 = 0$. But the spirit of the Cauchy–Kowalevski theorem is that data on a hypersurface could be used to construct a solution on the full space, and in this spirit it would be preferable to have a theorem which gives the general solution (1.9) in all cases.

The main result of this article is that corresponding to each element of the positive spectrum of L there is a space from which a boundary datum may be chosen, and for each collection of such choices there is a solution to (a generalization of) equation (1.7) having a sort of generalized analyticity (see the next theorem for precision). "Boundary data" cannot be prescribed for the negative spectrum, because these data correspond to parts of the solution which blow-up if they are non-zero. As an example, note that in (1.9), when $\Re(\lambda) < 0$ we must insist that $w_1 = 0$ for $u(t)$ to be continuous at $t = 0$.

The plan of this article is as follows. In the next section we introduce notation, and detail the main result. In Sections 3, 4, and 5 we prove the main theorem. In Section 6 we relate the first-order PDE solved to higher-order PDEs using a standard reduction of order argument. In Section 7 we relate this chapter back to the singular Yamabe problem, and in Section 8 a detailed application will be given to the problem involving Einstein metrics alluded to above.

2. Notation and statement of main theorem

- Let V be a complex vector space of finite dimension d, with norm $\|\cdot\|_V$, and a fixed basis $\nu = \{\nu_1, \ldots \nu_d\}$. Let $[v]_r$ be the components of $v \in V$ with respect to this basis. Write $B^k(\rho)$ for the open ball of radius ρ about zero in $\mathbb{C}^k$, and $B^V(\rho)$ for the ball of radius ρ in V.
- Let L be an arbitrary endomorphism of V with spectrum λ_j, $j = 1, \ldots, M$ (i.e., λ_j are the numbers λ such that $\ker(L - \lambda I) \neq \{0\}$, listed *without* multiplicity), assume (without loss of generality) that $\Re(\lambda_j) > 0$ exactly when $j = 1, \ldots, N$ and write

$$\begin{aligned} E_j &= \ker\left(L - \lambda_j I\right)^d , \\ F_j &= \operatorname{Im}\left(L - \lambda_j I\right)^d . \end{aligned} \tag{2.1}$$

  Since $\ker(L - \lambda_j I)^k = E_j$ for $k > d$, $E_j \cap F_j = \{0\}$ and so V decomposes as $E_j \oplus F_j$. Write π_j for the projection onto E_j along F_j.
- Let Ω_0 be a bounded open set in $\mathbb{R}^n$ with variable y, let $\Omega \subset\subset \Omega_0$, and write Ω_s for the set of complex vectors $z \in \mathbb{C}^n$ such that $\operatorname{dist}(z, \Omega) < s$. We fix $s_0 < 1$ sufficiently small that for $s < s_0$, $\Omega \subset \Omega_s \cap \mathbb{R}^n \subset \Omega_0$.
- For $V' \subset V$ a subspace and $0 < s \leq s_0$, let $X_s(V')$ be the closed subspace of $C^0(\bar{\Omega}_s, V')$ consisting of functions u which are analytic in Ω_s, real-valued on $\Omega_s \cap \mathbb{R}^n$, and for which the norm

$$\|u\|_s = \sup_{z \in \Omega_s} \|u(z)\|_V$$

  is finite.
- Throughout this article we use multi-index notation - for β a multi-index of size d, $\beta = (\beta_1, \ldots, \beta_d)$, and $u \in V$,

$$u^\beta = \prod_{r=1}^{d} [u]_r^{\beta_r}$$

  (recall $[u]_r$ are the components of u with respect to the basis $\nu = \{\nu_r : r = 1, \ldots, d\}$).
- For α a multi-index of size $N + 1$ (which it will always be), $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N)$, we write $\lambda \cdot \alpha = \sum_{j=0}^{N} \lambda_j \alpha_j$, where we always assume $\lambda_0 = 1$ for convenience in later formulae.
- For the multi-index with a 1 in the j^{th} place and 0 elsewhere we write $e_j = (0, \ldots, 1, \ldots, 0)$.
- Let $\rho_0 > 0$ be a fixed real number, and $g : B^1(\rho_0) \times (B^V(\rho_0))^3 \to X_{s_0}(V)$ an analytic function in all its variables. We make the following assumption on the function $g(t, u, v, w)$:

$$\begin{aligned} g(0,0,0,0) &= 0 , \\ \nabla_{(u,v,w)} g(0,0,0,0) &= 0 . \end{aligned} \tag{2.2}$$

  Since g is analytic, we can write

$$g(t, u, v, w) = \sum_{i,\beta,\gamma,\delta \geq 0} g_{i,\beta,\gamma,\delta} t^i u^\beta v^\gamma w^\delta . \tag{2.3}$$

  ($g_{i,\beta,\gamma,\delta} \in X_s(V)$). Another way of stating the main hypothesis (2.2) is that $g_{0,\beta,\gamma,\delta} = 0$ whenever $|\beta + \gamma + \delta| \leq 1$.

Theorem 2.1. *Fix s, $0 < s < s_0$, and $W > 0$. Then there is an integer K depending only on the endomorphism L, and $\rho_1 : 0 < \rho_1 < \rho_0$ so that for each set of choices $w_j \in X_{s_0}(E_j) : j = 1, \ldots, N$ with $\|w_j\|_{s_0} \leq W$, there is a solution $u(t, y) \in C^0([0, \rho_1], X_s(V))$ to the equation*

$$(t\partial_t - L)\, u\,(t, y) = g\left(t, u, t\partial_t u, \partial_y u, y\right)\ , \tag{2.4}$$

with

$$u\,(t, y) = \sum_{|\alpha|\geq 1} \sum_{j=0}^{K|\alpha|} u_{\alpha,j}(y) t^{\lambda\cdot\alpha} (\log t)^j\ . \tag{2.5}$$

This sum is absolutely convergent uniformly in $t \in [0, \rho_1]$ and

$$\sum_{\lambda\cdot\alpha=\lambda_j} \pi_j u_{\alpha,0} = w_j\ . \tag{2.6}$$

Furthermore, any two solutions to (2.4) which are of the form (2.5) and satisfy (2.6) are identical.

Remark.

- In the linear homogeneous case,

$$\begin{aligned} (t\partial_t - L)\, u &= 0 \\ u(0) &= w_j \in E_j \end{aligned}$$

  has solution

$$u(t) = \sum_{j=0}^{N} \sum_{k=0}^{d} t^{\lambda_j} (\log t)^k \frac{1}{k!} \left(L - \lambda_j\right)^k \pi_j u(1)\ ;$$

  thus, another way of writing (2.5) and (2.6) is

$$u(t, y) = \sum_{j=1}^{N} \sum_{k=0}^{d} t^{\lambda_j} \frac{(\log t)^k}{k!} \left(L - \lambda_j\right)^k w_j + \sum_{\substack{|\alpha|>1 \\ \text{or } \alpha_0=1}} \sum_{j=0}^{K|\alpha|} t^{\lambda\cdot\alpha} (\log t)^j u_{\alpha,j}(y), \tag{2.7}$$

  where

$$\sum_{\substack{\lambda\cdot\alpha=\lambda_j \\ \alpha\neq e_j}} \pi_j u_{\alpha,0} = 0\ . \tag{2.8}$$

- The theorem holds for *any* L. In particular, we do not require L to have positive spectrum, we simply say that 'boundary data' w_j can only be specified for those λ_j with positive real part, and the expansion for u only involves those λ_j. In the case which Baouendi and Goulaouic treat, for which the spectrum of L lies entirely in the negative half-plane, we get a unique solution—i.e., no boundary data can be specified at all.
- Uniqueness can be shown using Theorem 1.1 from [2]. This theorem will come into effect if we can reduce to an equation in the form

$$(t\partial_t - \Lambda)\, u = tg(t, u) + h(t) \tag{2.9}$$

where Λ has spectrum lying strictly in the negative half-plane, and both g and h are continuous with respect to t. If we have a solution u to (2.4) which is known to have a series expansion up to an order which exceeds the real part of the largest eigenvalue of L, then by subtracting these first terms, and suitably re-writing the equation, possibly in conjunction with a change of variable of the form $t' = t^{\mu}$, we can reduce to (2.9). Thus there is only one such solution. The purpose of this article, then, is to establish the convergence of the series expansion for this solution.

- We shall give some examples of Theorem 2.1 in action in Section 6 when we discuss higher order equations.

Our approach to the solution of Theorem 2.1 is to introduce several new complex variables,

$$z_j = t^{\lambda_j}, \quad j = 0, \dots, N; \qquad x = \log(t) .$$

(Recall that by convention, $\lambda_0 = 1$.) We try to find solutions of the form

$$\begin{aligned} u(z, x, y) &= \sum_{|\alpha| \geq 1} z^{\alpha} u_{\alpha}(x, y) \\ &= \sum_{|\alpha| \geq 1} \sum_{j=0}^{K|\alpha|} z^{\alpha} x^j u_{\alpha, j}(y) \end{aligned} \tag{2.10}$$

to an equation corresponding to (2.4), convergent in a suitable region of $\mathbb{C}^{N+2}$. After changing to these new coordinates, $t\partial_t = \sum_{j=0}^{N} \lambda_j z_j \partial_{z_j} + \partial_x$, so that $t\partial_t z^{\alpha} = (\lambda \cdot \alpha + \partial_x) z^{\alpha}$, and hence the LHS of (2.4) becomes

$$(t\partial_t - L) \sum_{|\alpha| \geq 1} z^{\alpha} u_{\alpha}(x, y) = \sum_{|\alpha| \geq 1} (\lambda \cdot \alpha + \partial_x - L)\, u_{\alpha}(x, y) .$$

On the RHS,

$$g\left(t, u, t\partial_t u, \partial_y u, y\right) = \sum_{i, \beta, \gamma, \delta} g_{i, \beta, \gamma, \delta}(y) z_0^i u^{\beta} \left(t\partial_t u\right)^{\gamma} \left(\partial_y u\right)^{\delta} .$$

Since u is assumed to have a formal expansion in terms of z, x, thus $t\partial_t u$ and $\partial_y u$ do also, and we can expand

$$g(t, u, t\partial_t u, \partial_y u, y) = \sum z^{\alpha} g_{\alpha}(x, y) = \sum z^{\alpha} x^j g_{\alpha, j}(y) .$$

Although g_{α} clearly depends on the coefficients $u_{\alpha'}$, the hypothesis (2.2) and the fact that $u_0 = 0$ imply

- $g_{\alpha}(x, y) = 0$ if $|\alpha| \leq 1$ unless $\alpha_0 = 1$, and
- g_{α} depends only on $u_{\alpha'}$ for $\alpha' < \alpha$.

It follows that there exists a unique *formal* solution to

$$\sum_{|\alpha| \geq 1} z^{\alpha} (\lambda \cdot \alpha + \partial_x - L)\, u_{\alpha}(x, y) = \sum_{|\alpha| > 1 \text{ or } \alpha_0 = 1} z^{\alpha} g_{\alpha}(y) .$$

Namely, for $j = 1, \dots, N$ we let u_{e_j} be an element of the kernel of $(\lambda_j + \partial_x - L)$ and for $|\alpha| > 1$ or $\alpha_0 = 1$ we solve the ODE

$$(\lambda \cdot \alpha + \partial_x - L)\, u_{\alpha}(x, y) = g_{\alpha}(x, y) \tag{2.11}$$

(in such a way that u_α is polynomial in x). This inductively provides a definition of u_α, so there is a formal sum solution of the form (2.10). The point of the theorem, then, is the absolute convergence of this sum in a region of $\mathbb{C}^{N+2}$ containing the path

$$(z,x)(t) = \left(t, t^{\lambda_1}, \ldots, t^{\lambda_N}, \log(t)\right) \qquad 0 < t \le \rho_1\,, \tag{2.12}$$

for sufficiently small ρ_1. This curve is asymptotic to the x axis $z = 0$ as $t \to 0$. The proof of convergence rests squarely on an a-priori estimate for solutions to the inductive problem (2.11), which will be established in the next section.

3. An ODE estimate

Before discussing the ODE (2.11), we define some weighted spaces. Let $\mathbb{C}^- \subset \mathbb{C}$ be the left half-plane $\{x : \Re(x) \le 1\}$.

Definition 3.1. For constant $\sigma \in (0,1)$, let $C^0_{\alpha,s}$ be the subspace of $C^0(\mathbb{C}^-, X_s(V))$ for which the norm

$$\|v(x,y)\|_{\alpha,s} = \sup_{x\in\mathbb{C}^-} \left\|e^{\sigma(\lambda\cdot\alpha)x} v(x,y)\right\|_s \tag{3.1}$$

is finite.

Thus $C^0_{\alpha,s}$ contains functions that blow up to order $e^{\sigma(\lambda\cdot\alpha)|x|}$ as $x \to -\infty$. We shall fix σ later, although its value will not be critical.

Proposition 3.2. *Let $f(x,y) \in C^0_{\alpha,s}$ be polynomial in x; let A be an isomorphism of V, with γ a closed path around the spectrum of A and $\Re(\gamma) = \sup_{z\in\gamma}\Re(z) < -\sigma(\lambda\cdot\alpha) < 0$. Then there is a unique polynomial solution $v(x,y) \in C^0_{\alpha,s}$ of the equation*

$$(\partial_x - A)\, v(x,y) = f(x,y)\,, \tag{3.2}$$

and furthermore,

$$\|v\|_{\alpha,s} \le \frac{C}{-\Re(\gamma) - \sigma\,\lambda\cdot\alpha}\|f\|_{\alpha,s}\,. \tag{3.3}$$

Here $C = (1/2\pi)|\gamma|\sup_{z\in\gamma}\|(z-A)^{-1}\|_{op}$.

Proof. By expanding v and f and equating coefficients of x^j on each side, it is clear from the fact that A is an isomorphism that there is a unique polynomial solution to (3.2): if $f(x)$ is of degree D, the solution is

$$\begin{aligned} v_D &= A^{-1} f_D \\ v_j &= A^{-1}\left(f_j - (j+1)v_{j+1}\right) \qquad j < D\,. \end{aligned}$$

To obtain the estimate (3.3) we need the following.

Lemma 3.3. *The unique polynomial solution to* (3.2) *(for any $x \in \mathbb{C}$) is*

$$v(x,y) = \int_{-\infty}^{x} \frac{1}{2\pi i}\int_{z\in\gamma} e^{(x-s)z}(z-A)^{-1}\,dz\; f(s,y)\,ds\,. \tag{3.4}$$

Proof. We compute that

$$(\partial_x - A)\, v(x,y) = \frac{1}{2\pi i}\int_{z\in\gamma}(z-A)^{-1}\,dz f(x,y) + \int_{-\infty}^{x}\frac{1}{2\pi i}\int_{z\in\gamma} e^{(x-s)z}\,dz\, f(s,y)\,ds\,.$$

By the spectral theorem [6], the first of these terms is simply $f(x,y)$; the second is zero by Cauchy's theorem. Therefore, $v(x,y)$ is a solution. To see that $v(x,y)$ is polynomial, differentiate with respect to x:

$$\partial_x v(x,y) = f(x,y) + \int_{-\infty}^{x}\frac{1}{2\pi i}\int_{z\in\gamma}\left(-\partial_s e^{(x-s)z}\right)(z-A)^{-1}dz\, f(s,y)\,ds\,.$$

Since f is polynomial, and $\Re(z)$ is bounded above by a negative number for $z\in\gamma$, we may integrate by parts with respect to s, giving

$$\partial_x v(x,y) = \int_{-\infty}^{x}\frac{1}{2\pi i}\int_{z\in\gamma} e^{(x-s)z}(z-A)^{-1}dz\ (\partial_s f(s,y))\ ds\,.$$

Applying this result recursively gives $\partial_x^{D+1} v(x,y)\equiv 0$, which shows that v is of polynomial growth, and hence polynomial, since it is a solution to (3.2). As there is only one polynomial solution, $v(x,y)$ must be it. □

Now for the estimate:

$$e^{\sigma(\lambda\cdot\alpha)x} v(x,y) = \int_{-\infty}^{x} e^{\sigma(\lambda\cdot\alpha)(x-s)}\frac{1}{2\pi i}\int_{z\in\gamma} e^{(x-s)z}(z-A)^{-1}\,dz\, e^{\sigma(\lambda\cdot\alpha)s} f(s,y)\,dt$$

so

$$\begin{aligned}\|v\|_{\alpha,s} &\le \frac{|\gamma|}{2\pi}\sup_{z\in\gamma}\left\|(z-A)^{-1}\right\|_{op}\|f\|_{\alpha,s}\int_{-\infty}^{x} e^{(x-s)(\Re(\gamma)+\sigma\,\lambda\cdot\alpha)}\,ds\\ &= \frac{|\gamma|}{2\pi}\sup_{z\in\gamma}\left\|(z-A)^{-1}\right\|_{op}\frac{1}{-\Re(\gamma)-\sigma\,\lambda\cdot\alpha}\|f\|_{\alpha,s}\,.\end{aligned}$$

□

We use Proposition 3.2 to obtain an a priori estimate for (2.11) by setting $A=-\lambda\cdot\alpha+L$. We let γ be a fixed curve around the spectrum of L. Then $-\lambda\cdot\alpha+\gamma$ is a curve around the spectrum of A, and $\Re(-\lambda\cdot\alpha+\gamma)=-\lambda\cdot\alpha+\Re(\gamma)$. Since all the λ_j are positive, there is a B (depending on the choice of curve γ and on A) such that for $|\alpha|\ge B$, $\Re(-\lambda\cdot\alpha+\gamma)<-\sigma\,\lambda\cdot\alpha$ and hence the hypotheses of Proposition 3.2 hold. It follows that the conclusion of Proposition 3.2 holds with constant

$$\frac{C}{(1-\sigma)\lambda\cdot\alpha-\Re(\gamma)}\le\frac{2C}{(1-\sigma)\lambda\cdot\alpha}$$

(by assuming B is large enough).

Proposition 3.4. *There is a constant C depending on L, σ such that for all α for which $|\alpha|>1$ or $\alpha_0=1$ and for all $f\in C^0_{\alpha,s}$ which are polynomial in x of degree at most $K|\alpha|$, there is a unique polynomial solution, which we write as $v=Gf$, to*

$$(\lambda\cdot\alpha+\partial_x-L)\,v(x,y)=f(x,y)$$

with $Gf(0)\subset F_j$ in case $\lambda\cdot\alpha=\lambda_j$. Furthermore,

$$\|Gf\|_{\alpha,s}\le\frac{C}{(1-\sigma)\lambda\cdot\alpha}\|f\|_{\alpha,s}\,. \tag{3.5}$$

Proof. This has already been proved for $|\alpha| \geq B$. For the finitely many cases $|\alpha| < B$, we need only consider polynomials whose degree is less than or equal to $K(B-1)$. This is a finite dimensional space, so to establish the existence of a constant C for which (3.5) holds, it is enough to show that the linear map G exists. Whenever $A = -\lambda \cdot \alpha + L$ is invertible, there is still a unique polynomial solution to (3.2) (by the method mentioned at the beginning of this proof). The only remaining case to consider is the case $\lambda \cdot \alpha = \lambda_j$ for some j. The decomposition $V = E_j \oplus F_j$ induces decompositions $v = v_E + v_F$ and $f = f_E + f_F$. $(\lambda_j - L)$ is an automorphism on F_j, so by the above there is a unique solution $v_F(x)$ to the equation $(\lambda_j + \partial_x - L)v_F(x) = f_F(x)$ obeying (3.5). For the piece f_E, we explicitly set

$$v_E(x) = e^{(L-\lambda_j)x} \int_0^x e^{(\lambda_j - L)s} f_E(s)\, ds\ .$$

By definition, $E_j = \ker(\lambda_j - L)^d$ so that on E_j ,

$$e^{(\lambda_j - L)x} = \sum_{n=0}^{d} \frac{x^n}{n!} (\lambda_j - L)^n$$

which guarantees that $v_E(x)$ is indeed polynomial. The general polynomial homogeneous solution to (3.2) when $\lambda \cdot \alpha = \lambda_j$ is

$$e^{(L-\lambda_j)x} w_j \qquad \text{for any} \quad w_j \in E_j\ ,$$

(if $w_j \notin E_j$, then $e^{(L-\lambda_j)x} w_j$ is a solution to (3.2), but is not polynomial) hence v_E is uniquely determined by the condition $v_E(0) = 0$, and v is uniquely determined by the condition $v(0) \in F_j$. □

4. The induction step

With terminology from Section 3, we can re-cast the inductive construction of a solution to (2.4) as

$$\begin{aligned} u_{e_j}(x) &= e^{(L-\lambda_j)x} w_j, \qquad && j = 1, \ldots, N\ , \\ u_\alpha(x) &= G(g_\alpha(x)), && |\alpha| > 1 \ or\ \alpha_0 = 1\ , \end{aligned} \tag{4.1}$$

where $w_j \in X_{s_0}(E_j)$, and g_α implicitly depends on $u_{\alpha'}$ for $\alpha' < \alpha$.

First let us establish that we may assume $\deg(u_\alpha(x)) \leq K|\alpha|$. This condition will be necessary in order to comply with the hypotheses of Proposition 3.4 when it is needed later on. When $|\alpha| \geq B$ (using the B from Section 3) $(\lambda \cdot \alpha - L)$ is invertible, which implies $\deg(u_\alpha(x)) = \deg(g_\alpha(x))$. g_α involves products of polynomials u_{α_j} for which $\sum_j \alpha_j \leq \alpha$, so if $\deg(u_{\alpha'}) \leq K|\alpha'|$ for $\alpha' < \alpha$, then $\deg(u_\alpha(x)) \leq K|\alpha|$. Thus if we can show that $\deg(u_\alpha) \leq K|\alpha|$ for $|\alpha| < B$, then by induction it will be true for all α. Even for $|\alpha| < B$ we have $\deg(u_\alpha) \leq \deg(g_\alpha) + d$, and there are only finitely many such indices, so since $\deg(u_{e_j}(x)) \leq d$ there is some (potentially large) K for which the inequality $\deg(u_\alpha) \leq K|\alpha|$ does indeed hold for all α: $|\alpha| < B$ and hence for all α.

Next we introduce the region of $\mathbb{C}^{N+2}$ where we will demonstrate the absolute convergence of (2.10).

Definition 4.1. $Q_\delta \subset \mathbb{C}^- \times \mathbb{C}^{N+1}$ is the set of vectors $(x, z_0, \ldots, z_N)$ for which $|z_j| \leq (\delta e^{\sigma x})^{\lambda_j}$.

We shall show that for any $W > 0$ there is δ so that if $\|w_j\| < W$, then $\sum z^\alpha x^j u_{\alpha,j}$ can be made absolutely convergent in the region Q_δ. In other words, the "radius of convergence" will depend only on a uniform bound for w_j. An intermediate step is to show that the sum $\sum z^\alpha u_\alpha(x)$ is absolutely convergent in Q_δ. From there to the convergence of $\sum z^\alpha x^j u_{\alpha,j}$ is a small step precisely because of the bound on $\deg(u_\alpha(x))$.

Proposition 4.2. *For any $s : 0 < s < s_0$ there are constants C_0, C_1 depending only on the uniform bound W for w_j such that*

$$\left\|u_\alpha(x)\right\|_{\alpha,s} \leq C_0 C_1^{|\alpha|} . \tag{4.2}$$

To prove (4.2) we let $g^* : B^1(\rho_0) \times (B^V(\rho_0))^3 \to \mathbb{C}$ be any analytic function majorizing g but vanishing component-wise wherever $g_{i,\beta,\gamma,\delta}$ does:

$$\begin{aligned} \left\|g_{i,\beta,\gamma,\delta}\right\|_{s_0} &\leq g^*_{i,\beta,\gamma,\delta} , \\ g_{i,\beta,\gamma,\delta} = 0 &\Rightarrow g^*_{i,\beta,\gamma,\delta} = 0 . \end{aligned} \tag{4.3}$$

For each $\eta \in \mathbb{R}^+$, consider the implicit equation

$$Y(z) - \eta \sum_{j=1}^{N} z_j = g^*\left(z_0, Y(z)\mathbf{v}, Y(z)\mathbf{v}, Y(z)\mathbf{v}\right) , \tag{4.4}$$

where $\mathbf{v} = \sum_{r=1}^{d} v_r \in V$. By the analytic implicit function theorem there is, for each η, a number $\rho' > 0$ so that (4.4) has an analytic solution $Y : B^{N+1}(\rho') \to \mathbb{C}$. The analyticity of $Y(z)$ implies that there are constants $C_{0,Y}$, $C_{1,Y}$ so that $Y_\alpha \leq C_{0,Y} C_{1,Y}^{|\alpha|}$. To establish (4.2) it is enough to prove the following.

Lemma 4.3. *For all $w_j \in X_{s_0}(E_j)$ with $\|w_j\|_{s_0} \leq W$ and all $s : 0 < s < s_0$ there exist constants C, η (η in the definition of Y) so that for $j + k \leq 1$, the functions $u_\alpha(x)$ defined by* (4.1) *satisfy*

$$\left\|(\lambda \cdot \alpha + \partial_x)^j \, \partial_y^k u_\alpha(x)\right\|_{\alpha,s} \leq \left(\frac{Ce}{s_0 - s}\right)^{|\alpha|-1} Y_\alpha . \tag{4.5}$$

This inequality is proved inductively. For the inductive step, we shall need some way of dealing with the horrendous variety of indices that threaten to break loose.

Definition 4.4.

- Define the set $I(\mathbb{N})$ to be the set of functions

$$\xi : \mathbb{N}^{N+1} \times \{1, \ldots, d\} \to \mathbb{N}$$

for which all but finitely many values are zero.

- For $\xi \in I(\mathbb{N})$, write

$$\begin{aligned} z^{\cdot\xi} &= z^{\sum \alpha\, \xi(\alpha,r)} = \prod_{\alpha,r} z^{\alpha\, \xi(\alpha,r)} = \prod_{\alpha,j,r} z_j^{\alpha_j \xi(\alpha,r)} \\ u^{\cdot\xi} &= \prod_{\alpha,r} [u_\alpha]_r^{\xi(\alpha,r)} \end{aligned}$$

where the sum $\sum \alpha\xi(\alpha, r)$ (over both α and r) is to be interpreted as a multi-index of dimension $N+1$.

Now that we have all the necessary index notation, we turn our attention to the problem of estimating the size of g_α by the terms u_α, v_α and w_α on which it depends. Recall that we have decomposed

$$g\,(z_0, u, v, w) = \sum_{i,\beta,\gamma,\delta} g_{i,\beta,\gamma,\delta} z_0^i u^\beta v^\gamma w^\delta \ .$$

To decompose this further to the level of u_α we first focus on the general product u^β:

$$\begin{aligned} u^\beta &= \prod_{r=1}^{d} \Big(\sum [u_\alpha]_r z^\alpha\Big)^{\beta_r} \\ &= \sum_{\xi \in I(\mathbb{N})} z^{:\xi} u^{:\xi} C_\xi^\beta \ . \end{aligned}$$

Where the constants $C_\xi^\beta \in \mathbb{N}$ are almost all zero, and are independent of u: they are simply combinatorial. It follows that

$$\begin{aligned} g\,(z_0, u, v, w) &= \sum_{\xi,\mu,\zeta} z_0^i u^{:\xi} v^{:\mu} w^{:\zeta} z^{:(\xi+\mu+\zeta)} C_{\xi,\mu,\zeta}^{i,\beta,\gamma,\delta} g_{i,\beta,\gamma,\delta} \ , \\ g_\alpha(z_0, u, v, w) &= \sum_{\xi,\mu,\zeta} u^{:\xi} v^{:\mu} w^{:\zeta} C_{\alpha,\xi,\mu,\zeta}^{i,\beta,\gamma,\delta} g_{i,\beta,\gamma,\delta} \ , \end{aligned} \tag{4.6}$$

where here we have

$$C_{\alpha,\xi,\mu,\zeta}^{i,\beta,\gamma,\delta} = \begin{cases} C_{\xi,\mu,\zeta}^{i,\beta,\gamma,\delta} & \text{if } \ i\, e_0 + \sum_{\alpha',r} \alpha' \left(\xi\left(\alpha', r\right) + \mu\left(\alpha', r\right) + \zeta\left(\alpha', r\right)\right) = \alpha \\ 0 & \text{otherwise} \end{cases} \ . \tag{4.7}$$

The point being once again that the constants $C_{\alpha,\xi,\mu,\zeta}^{i,\beta,\gamma,\delta} \in \mathbb{N}$ depend only on their indices. In particular, (4.6) holds for g_α^* with $g_{i,\beta,\gamma,\delta}^*$ substituted in for $g_{i,\beta,\gamma,\delta}$.

Proposition 4.5. *Suppose that g^* majorizes g in the sense of* (4.3). *Let $|\alpha| > 1$, $M > e^\sigma$, and suppose that $\|u_{\alpha'}\|_{\alpha',s} \le M^{|\alpha'|-1} Y_{\alpha'}$ (and the same for v, w) for all $0 < \alpha' < \alpha$, then*

$$\left\|g_\alpha\big(z_0, u, v, w\big)\right\|_{\alpha,s} \le M^{|\alpha|-2} Y_\alpha \ .$$

Proof. This demonstration revolves around the following estimate, in which we assume that

$\xi(\alpha', r) = 0$ for $\alpha' \geq \alpha$:

$$\begin{aligned}
\left\|u^{:\xi}\right\|_s e^{\sigma\lambda\cdot(\sum\alpha'\xi(\alpha',r))x} &= \left\|\prod_{\alpha',r}[u_{\alpha'}]_r^{\xi(\alpha',r)} e^{\sigma\xi(\alpha',r)\lambda\cdot\alpha' x}\right\|_s \\
&\leq \prod_{\alpha',r}\|u_{\alpha'}\|_{\alpha',s}^{\xi(\alpha',r)} \\
&\leq \prod_{\alpha',r}\left(M^{|\alpha'|-1}Y_{\alpha'}\right)^{\xi(\alpha',r)} \\
&= \prod_{\alpha',r}\left[M^{|\alpha'|-1}Y_{\alpha'}\mathbf{v}\right]_r^{\xi(\alpha',r)} \\
&= M^{\left|\sum\alpha'\xi(\alpha',r)\right|-\sum\xi(\alpha',r)}\left(\sum Y_{\alpha'}\mathbf{v}z^{\alpha'}\right)^{:\xi} \\
&= M^{\left|\sum\alpha'\xi(\alpha',r)\right|-\sum\xi(\alpha',r)}\left(Y(z)\mathbf{v}\right)^{:\xi} .
\end{aligned}$$

Turning back to the proof of Proposition 4.5, we use (4.7) to estimate

$$\begin{aligned}
\|g_\alpha\|_s e^{\sigma(\lambda\cdot\alpha)x} &= \left\|\sum_{\xi,\mu,\zeta} u^{:\xi}v^{:\mu}w^{:\zeta}C_{\alpha,\xi,\mu,\zeta}^{i,\beta,\gamma,\delta}g_{i,\beta,\gamma,\delta}\right\|_s e^{\sigma(\lambda\cdot\alpha)x} , \\
&\leq \sum_{\xi,\mu,\zeta} e^{\sigma i x}\|u^{:\xi}\|_s e^{\sigma\lambda\cdot(\sum\xi(\alpha',r)\alpha')x}\dots C_{\alpha,\xi,\mu,\zeta}^{i,\beta,\gamma,\delta}g_{i,\beta,\gamma,\delta}^* , \\
&\leq \sum_{\xi,\mu,\zeta} e^{\sigma i}M^{|\alpha|-i-\sum(\xi+\mu+\zeta)(\alpha',r)}(Y\mathbf{v})^{:\xi}(Y\mathbf{v})^{:\mu}(Y\mathbf{v})^{:\zeta}C_{\alpha,\xi,\mu,\zeta}^{i,\beta,\gamma,\delta}g_{i,\beta,\gamma,\delta}^* .
\end{aligned}$$

The hypothesis (2.2) on g (combined with (4.3)) implies that $C_{\alpha,\xi,\mu,\zeta}^{i,\beta,\gamma,\delta}g_{i,\beta,\gamma,\delta}^* = 0$ unless at least two of the ξ, μ, ζ are non-zero, or one of them (say ξ) has "weight" $\sum\xi(\alpha',r) > 1$. Either way, $\sum(\xi+\mu+\zeta)(\alpha',r) \geq 2$ in each non-zero term, so

$$\begin{aligned}
\|g_\alpha\|_{\alpha,s} &\leq M^{|\alpha|-2}\sum_{\xi,\mu,\zeta}(Y\mathbf{v})^{:\xi}, (Y\mathbf{v})^{:\mu}, (Y\mathbf{v})^{:\zeta}C_{\alpha,\xi,\mu,\zeta}^{i,\beta,\gamma,\delta}g_{i,\beta,\gamma,\delta}^* . \\
&= M^{|\alpha|-2}Y_\alpha .
\end{aligned}$$

□

The final ingredient necessary for a proof of Lemma 4.3 is an estimate on $\|(\lambda\cdot\alpha+\partial_x)^j\partial_y^k u_\alpha$ $\|\alpha, s$ in terms of an estimate on g_α. Using Cauchy's integral formula, it is easy to check that

$$\left\|\partial_y u\right\|_s \leq \frac{C}{s'-s}\|u\|_{s'} ,$$

for all $s' : s < s' < s_0$, and so combining this with the estimate (3.5) from Proposition 3.4, for $|\alpha| > 1$ we derive

$$\begin{aligned}
\|u_\alpha\|_{\alpha,s} &\leq \frac{C}{|\alpha|}\|g_\alpha\|_{\alpha,s} , \\
\left\|(\lambda\cdot\alpha+\partial_x)u_\alpha\right\|_{\alpha,s} &\leq C\|g_\alpha\|_{\alpha,s} , \\
\left\|\left(\partial_y u\right)_\alpha\right\|_{\alpha,s} &\leq \frac{C}{s'-s}\|g_\alpha\|_{\alpha,s'} .
\end{aligned}$$

(Recall that u_α is of degree at most $K|\alpha|$.) Thus if $|\alpha|(s'-s) < 1$ and $j+k \le 1$ we have

$$\left\| (\lambda \cdot \alpha + \partial_x)^j \, \partial_y^k u_\alpha \right\|_{\alpha,s} \le \frac{C}{|\alpha|\,(s'-s)} \left\| g_\alpha \right\|_{\alpha,s'} \tag{4.8}$$

and so we are ready for the proof of Lemma 4.3.

Proof. The inductive hypothesis (4.5) is true for $u_{e_j} : j = 1, \ldots, N$ by taking η sufficiently large, and for u_{e_0} straight from (4.3). For the inductive step, suppose $|\alpha| > 1$, that (4.5) holds for all $\alpha' < \alpha$, and fix σ so that $e^\sigma < Ce/(s_0 - s)$. Combining the result of Proposition 4.5 (with $M = Ce/(s_0 - s)$) with (4.8) gives

$$\left\| \left((t\partial_t)^j \, (\lambda \cdot \alpha + \partial_x)^k \, u \right)_\alpha \right\|_{\alpha,s} \le \frac{C}{|\alpha|\,(s'-s)} \left(\frac{Ce}{s_0 - s'} \right)^{|\alpha|-2} Y_\alpha \,.$$

We have assumed that $|\alpha|(s'-s) < 1$ in order to use (4.8), an assumption that will now be seen to be justified. The final step in establishing (4.5) comes courtesy of [2]. We choose s' so that $s' - s = (s_0 - s)/|\alpha|$ and hence $s_0 - s' = (s_0 - s)(|\alpha| - 1)/|\alpha|$

$$\begin{aligned} \frac{e^{|\alpha|-2}}{|\alpha|\,(s'-s)\,(s_0 - s')^{|\alpha|-2}} &= \frac{e^{|\alpha|-2}}{(s_0 - s)^{|\alpha|-1}} \left(\frac{|\alpha|}{|\alpha| - 1} \right)^{|\alpha|-2} \\ &\le \frac{e^{|\alpha|-1}}{(s_0 - s)^{|\alpha|-1}} \,. \end{aligned}$$

□

5. Conclusion of the proof

We have established that for all $s : 0 < s < s_0$ and $W > 0$ there are constants C_0 and C_1 such that for all $w_j \in X_{s_0}(E_j)$ with $\|w_j\|_{s_0} \le W$, the functions $u_\alpha(x)$ defined by (4.1) satisfy

$$\left\| u_\alpha(x) \right\|_{\alpha,s} \le C_0 C_1^{|\alpha|} \,.$$

To obtain absolute convergence of the sum $\sum_\alpha z^\alpha u_\alpha(x)$ we simply restrict attention to the region Q_δ ($\delta < 1$) defined in Definition 4.1. Clearly in Q_δ, $\|z^\alpha u_\alpha(x)\|_s < C_0 \delta^{\lambda \cdot \alpha} C_1^{|\alpha|} < C_0 (\delta^\epsilon C_1)^{|\alpha|}$ (where $\epsilon = \inf_{j \le N} \Re(\lambda_j)$), so for δ small enough the sum will be convergent. We now proceed to show that $\sum z^\alpha x^j u_{\alpha,j}$ is convergent in Q_δ for sufficiently small δ. $u(z, x)$ is analytic in Q_δ since it is an absolutely convergent sum of analytic functions. $0 \in Q_\delta$, so there are constants C_0', C_1' such that $\|u_{\alpha,j}\|_s \le C_0'(C_1')^{|\alpha|+j}$. In Q_δ, $|z^\alpha x^j| \le \delta^{\lambda \cdot \alpha} |e^{\sigma(\lambda \cdot \alpha)x} x^j|$. The maximum of $|e^{\sigma(\lambda \cdot \alpha)x} x^j|$ occurs at $-j/(\sigma\, \lambda \cdot \alpha)$ with value

$$e^{-j} \left(\frac{j}{\sigma\, \lambda \cdot \alpha} \right)^j \le \left(\frac{K}{e \sigma \epsilon} \right)^{K|\alpha|}$$

(for $j \le K|\alpha|$). Meanwhile at $x = 1$ its value is $e^{\sigma(\lambda \cdot \alpha)}$. Putting this together, there exists C independent of δ so that

$$\sup_{(z,x) \in Q_\delta} \left| z^\alpha x^j \right| \le (\delta^\epsilon C)^{|\alpha|} \,,$$

and hence a different constant C, also independent of δ, for which

$$\sup_{(z,x) \in Q_\delta} \left\| z^\alpha x^j u_{\alpha,j} \right\|_s \le C_0' C^{|\alpha|+j} \delta^{\epsilon |\alpha|} \le C_0' \left(C \delta^{\frac{\epsilon}{1+K}} \right)^{|\alpha|+j} \,.$$

It follows that for δ small enough the sum $\sum z^\alpha x^j u_{\alpha,j}$ is absolutely convergent in Q_δ. To relate this back to the variable t, and hence conclude the proof of Theorem 2.1, notice that the curve (2.12) lies inside Q_δ whenever $t^{\lambda_j} \le \delta^{\lambda_j} t^{\sigma\lambda_j}$, i.e., $t \le \delta^{(1/(1-\sigma))}$.

6. Higher order equations

We turn our attention now to non-linear PDEs of the form

$$\left[(t\partial_t)^D + \sum_{\substack{j+|k|\le D\\ j<D}} A_{j,k}(t\partial_t)^j\left(t\partial_y\right)^k\right] u(t,y) = g\left(t, u, (t\partial_t)^l\left(t\partial_y\right)^m u, y\right), \tag{6.1}$$

where g depends on $(t\partial_t)^l(t\partial_y)^m u$ for $l + |m| \le D$, each $A_{j,k}$ is a bounded linear map

$$A_{j,k} : X_s(V) \to X_s(V) \tag{6.2}$$

for $s < s_0$, with norm uniformly bounded in s, and everything else is as in Section 1. Notice that we should be able to do better than this (e.g., ∂_y instead of $t\partial_y$), but it is not clear exactly which class of higher order non-linear PDEs admit a reduction to an equation of the form (2.4), subject to (2.2). At any rate, this covers many important examples. We can manipulate (6.1) into the form (2.4) by introducing the new variable w with components $w_{j,k} = (t\partial_t)^j(t\partial_y)^k u$, $j + |k| < D$. Now (6.1) reads

$$t\partial_t w_{D-1,0} + \sum_{j=0}^{D-1} A_{j,0}w_{j,0} = \bar{g}\left(t, w, t\partial_t w, \partial_y w\right) + \left(t\partial_y\right)\mathcal{B}w$$

(where $\mathcal{B}$ takes care of all the terms $A_{j,k}(t\partial_t)^j(t\partial_y)^k u$ with $|k| > 0$.) Of course we also have the equations

$$t\partial_t w_{j,0} = w_{j+1,0}, \qquad j < D-1$$

so we can write

$$t\partial_t w + \mathcal{A}w = \tilde{g}\left(t, w, t\partial_t w, \partial_y w\right)$$

where $\tilde{g}$ obeys the hypothesis (2.2) and $\mathcal{A}$ is the matrix whose entry $\mathcal{A}_{(j,k),(l,m)}$ is zero unless $k = m = 0$, in which case

$$\mathcal{A}_{(j,0),(l,0)} = \begin{cases} -\delta_{j,l+1} & j < D \\ A_{l,0} & j = D \end{cases}.$$

As such the spectrum of $\mathcal{A}$ consists of zero together with solutions of the equation

$$\ker\left(\lambda^D + \sum A_{j,0}\lambda^j\right) \ne \phi\,.$$

Theorem 2.1 now holds, word for word, with (2.4), replaced by (6.1).

7. Application: CSC metrics on $S^n\backslash S^{n-1}$

Consider the equation

$$\left[(t\partial_t)^2 - kt\partial_t + t^2\Delta_y\right] u(t,y) + \lambda\left(u^p - u\right) = 0\,, \tag{7.1}$$

for $k > 0$, $p > 1$ and $\lambda > 0$. Here $y = (y_1, \dots y_k)$. This generalizes (1.3) in the case when there is no θ dependence. As it stands, this equation is not of the correct form (6.1), since its non-linear part is not necessarily analytic near $u = 0$. We set $u = 1 + v$, i.e., linearize about 1. (7.1) becomes

$$\left[(t\partial_t)^2 - k\,(t\partial_t) + \lambda(p-1)\right] v = -t^2\Delta_y v - \lambda\left((1+v)^p - 1 - p\,v\right) ,$$

which *does* now conform to (6.1) with

$$g\left(t, v, (t\partial_t)^j \left(t\partial_y\right)^k v, y\right) = -\sum_q \left(t\partial_{y_q}\right)^2 v - \lambda\left((1+v)^p - 1 - pv\right) .$$

The solutions of

$$\omega^2 - k\omega + \lambda(p-1) = 0$$

are

$$\omega_\pm = \frac{k}{2} \pm \sqrt{\frac{k^2}{4} - \lambda(p-1)} ,$$

which both have positive real parts, since $k > 0$, $p > 1$ and $\lambda > 0$. Write $\lambda_1 = \omega_-$ and $\lambda_2 = \omega_+$. In this case any $s_0 < 1$ and $\rho_0 < 1$ will do, so the conclusion of Theorem 2.1 is that for all constants $W > 0$ and each $s < s_0$ there is $\rho_1 < 1$ so that for $\|w_\pm\|_{s_0} \leq W$ there is a solution $u(t, y) \in C^0\big([0, \rho_1] \times X_s\big)$ of (7.1) with

$$u\,(t, y) = w_- t^{\omega_-} + w_+ t^{\omega_+} + \sum_{|\alpha|>1 \text{ or } \alpha_0=1} \; \sum_{j=0}^{K|\alpha|} t^{\lambda\cdot\alpha} (\log t)^j u_{\alpha,j}(y)$$

(a convergent sum) and $\pi_j u_{\alpha,j} = 0$ whenever $\lambda \cdot \alpha = \omega_\pm$.

8. Application: Conformally compact Einstein metrics

In this last section, we describe an application of the convergence results that have been derived to the study of conformally compact Einstein metrics. In [4], Graham and Lee discuss the existence and regularity of solutions to Einstein's equation on M, the interior of a compact manifold $\bar{M}$ of dimension $n+1$ with boundary bM which has a smooth defining function ρ. The problem they discuss is that of finding a complete Einstein metric g on M which has the property that $\bar{g} = \rho^2 g$ extends smoothly to $\bar{M}$. Such metrics are said to be *conformally compact* and to have *conformal infinity* $\hat{g} = \bar{g}\big|_{bM}$. The problem of finding conformally compact metrics with prescribed conformal infinity $\hat{g}$ is complicated by the invariance of Einstein's equation under the diffeomorphism group. This problem is resolved in [4] by the introduction of an auxiliary metric t whose purpose is to break the symmetry of the problem. The equation to be solved is then

$$Q(g, t) = \text{ Ric }(g) + ng - \phi(g, t) = 0 , \tag{8.1}$$

for a suitable non-linear function ϕ. It is established in [4] that if g and t are two conformally flat metrics with suitable (low) regularity, if $Q(g, t) = 0$ and if $\text{Ric}(g)$ is strictly negative on M, then $\text{Ric}(g) = -ng$. If g and t are conformally flat then $Q(g, t)$ will typically blow up like ρ^{-2} as $\rho \to 0$. Graham and Lee quickly establish conditions for a pair (g, t) to satisfy $Q(g, t) = O(\rho^{-1})$, a first approximation, as it were, and go on to discuss the linearization of $Q(g, t)$ with respect to its first variable with an aim to fixing t and solving for g by linearizing around a sufficiently close approximation to a solution. They calculate the first few terms in a formal expansion (with respect to ρ) of a solution g as a perturbation of an extension $E(\hat{g})$ of an arbitrary conformal infinity $\hat{g}$ to M, but are forced to stop when the terms in the series reach the indicial roots of $D_g Q(g,t)$. From there they use an inverse function theorem argument on suitably highly weighted spaces to prove the existence of solutions to (8.1) on the ball close to the standard hyperbolic metric. The reduction from highly general to near standard hyperbolic cases is necessary in order to understand the linearization $D_g Q(g,t)$ well enough to obtain surjectivity on weighted spaces. See [4] for details.

The application of the results of this article to the problem, is that what was once a "barrier," the indicial roots of the linearization of $D_g Q$, is now an opportunity to specify "boundary data," and continue the expansion up to arbitrarily high powers, and indeed obtain absolute convergence of the corresponding solution. Of course, in the present context, we must assume that all data is analytic.

To see how the results of this article are applicable to the above situation, we shall briefly recapitulate the reasoning from [4], showing how the problem may be reduced to an equation of the form (6.1). First we examine the problem on a model manifold. Let Ω be a relatively compact open set in $\mathbb{R}^n$ and consider the problem of trying to find a metric $g(\rho, y)$ on $[0, \rho_1] \times \Omega$ which has conformal infinity $\hat{g}$ an arbitrary real analytic metric on Ω. Thus we will take the vector space V in which the functions u of Section 6 take values, to be $\mathcal{S}$ the space of symmetric two-tensors on $\mathbb{C}^{n+1}$.

In the following, an over-bar denotes multiplication by ρ^2, thus $\bar{g} = \rho^2 g$, $\bar{u} = \rho^2 u$, etc. As explained before, the equation (8.1) is solved by linearization around $g = \rho^{-2} E(\hat{g})$, where $E(\hat{g})$ is an analytic conformally compact extension of $\hat{g}$. (The extension operator E from [4] preserves analyticity)

$$\rho^2 Q(g+u,t) = \rho^2 Q(g,t) + \rho^2 D_g Q_{(g,t)}(u) + \rho^2 \int_0^1 (1-\lambda) D_1^2 Q_{(g+\lambda u,t)}(u,u) \,. \tag{8.2}$$

We include the factors of ρ^2 because G-L point out that with a factor of ρ^2, the term involving the integral on the RHS becomes a homogeneous polynomial, quadratic in u, $\rho\partial_\rho u$, $\rho^2\partial_\rho^2 u$. The condition for $\rho^2 Q(g,t)$ to vanish as $\rho \to 0$ is

$$\left.\begin{array}{llcl} (a) & \mathrm{Tr}_{\bar{g}}\bar{t} & = & (n+1) \\ (b) & \bar{g}^{-1}d\rho & = & \bar{t}^{-1}d\rho \end{array}\right\} \text{ on } bM \tag{8.3}$$

which is achieved simply by letting $t = g$.

G-L show that if g is asymptotically hyperbolic, then

$$D_g Q_{(g,t)}(u) = \tfrac{1}{2}\Delta_g u - u + \mathrm{Tr}_g(u) + O(\rho)u \,. \tag{8.4}$$

To conclude the reduction of this problem to a form suitable for the application of Theorem 2.1, it suffices to study Δ_g on symmetric two-forms.

Proposition 2.7 of [4] establishes that

$$\begin{aligned} \left[\Delta_g \bar{u}\right]_{jk} &= -\left(\rho\partial_\rho\right)^2 \bar{u}_{jk} + (n-4)\left(\rho\partial_\rho\right)\bar{u}_{jk} + 2(n-1)\bar{u}_{jk} \\ &\quad + (n+1)\left(\bar{u}_{jl}\delta_{0k}\bar{g}^{0l} + \bar{u}_{lk}\delta_{j0}\bar{g}^{l0}\right) \\ &\quad - 2\left[\left(\bar{g}^{il}\bar{u}_{il}\right)\delta_{j0}\delta_{0k} + \bar{u}_{il}\bar{g}^{0l}\bar{g}^{i0}\bar{g}_{jk}\right] \\ &\quad + \left[q\left(\rho\partial_y\right)\bar{u}\right]_{jk} \,, \\ &= -\left(\rho\partial_\rho\right)^2 \bar{u}_{jk} + (n-4)\left(\rho\partial_\rho\right)\bar{u}_{jk} + (A_0\bar{u})_{jk} + \left[q\left(\rho\partial_y\right)\bar{u}\right]_{jk} \,, \end{aligned} \tag{8.5}$$

where the q term is a second order polynomial in the derivatives $\rho\partial_y$. This expression naturally gives rise to a decomposition of $\mathcal{S}([0,\rho_1]\times\Omega)$ into eigenspaces of A_0:

$$\begin{aligned}
\mathcal{V}_0 &= \left\{\bar{u}_{jk} : \bar{u}_{jk} = \lambda\bar{g}_{jk}\right\}\\
\mathcal{V}_1 &= \left\{\bar{u}_{jk} : \bar{u}_{jk} = \lambda\left((n+1)\delta_{j0}\delta_{0k} - \bar{g}_{jk}\right) : \lambda\in\mathbb{R}\right\}\\
\mathcal{V}_2 &= \left\{\bar{u}_{jk} : \bar{u}_{jk}\bar{g}^{j0} = 0 \text{ and } \operatorname{Tr}_{\bar{g}}\bar{u} = 0\right\}\\
\mathcal{V}_3 &= \left\{\bar{u}_{jk} : \bar{u}_{jk} = v_j\delta_{0k} + \delta_{j0}v_k : v\in T^*M\big|_{bM} \text{ and } \bar{g}^{l0}v_l = 0\right\}
\end{aligned}$$

with corresponding eigenvalues $(2n-4)$, $(4n-2)$, $(2n-2)$, and $(3n-1)$, respectively. Pulling together (8.4), (8.5), and (8.2) gives

$$\left((\rho\partial_\rho)^2\bar{u}_{jk} - n\left(\rho\partial_\rho\right)\bar{u}_{jk} + (2n-2) - A_0\left(\bar{u}\right)_{jk} - 2\operatorname{Tr}_{\bar{g}}(\cdot)\right)\bar{u} = f\left(\rho,\bar{u},\rho\partial\bar{u},\rho^2\partial^2\bar{u},y\right),$$

where the sub-quadratic part of f vanishes as $\rho\to 0$. The linear operator on the left-hand side of this expression has indicial roots λ_j; those with positive real part (and the corresponding eigenspaces E_i) are

$$\begin{aligned}
\lambda_1 &= \tfrac{n}{2}+\tfrac{1}{2}\sqrt{n^2+8n} & E_1 &= \mathcal{V}_0\oplus\mathcal{V}_1\,,\\
\lambda_2 &= n & E_2 &= \mathcal{V}_2\,,\\
\lambda_3 &= n+1 & E_3 &= \mathcal{V}_3\,.
\end{aligned}$$

Thus for an arbitrary analytic section ω of $\mathcal{S}\big|_\Omega$ and an arbitrary real analytic metric $\hat{g}$ on Ω, we can find a unique pseudo-analytic metric g_ω which is asymptotically hyperbolic, has conformal infinity $\hat{g}$, and has a convergent expansion in powers of ρ, $\log\rho$. Furthermore, the solution we generate has coefficient of ρ^{λ_1} equal to $\pi_1\omega$, coefficient of ρ^n equal to $\pi_2(\omega)$ and coefficient of ρ^{n+1} equal to $\pi_3(\omega)$.

To complete the extension of [4] to the case of (M,bM), we cover the boundary bM with coordinate patches $\psi_j : [0,\rho_j]\times\Omega_j \to \bar{M}$ such that $\psi_j^{-1}(bM) = \Omega_j$, $bM = \bigcup\psi_j(\Omega_j)$ and $\psi_j^{-1}\circ\psi_k$ is a real analytic function in its domain. We also require that these transition functions have radius of convergence uniformly bounded below (or equivalently, extending to a complex neighborhood of their domains), and that their differentials be non-zero along the boundary. The first of these conditions (it is straightforward to check) ensures that the pull back by such a transition function of analytic data, is analytic; the second guarantees that a function vanishing to order ρ in one set of coordinates will do so in all coordinates. Suppose we have an analytic defining function $\hat{\rho}$ for bM, a metric $\hat{g}$ on bM, and a section ω of $\mathcal{S}(M)\big|_{bM}$. Pull back

$$\hat{g}+\sum_j\hat{\rho}^{\lambda_j}\pi_j\omega$$

to each coordinate chart. Because $\hat{\rho}$ is a defining function, the pull-back of the term $\hat{\rho}^{\lambda_j}\pi_j\omega$ will vanish like ρ^{λ_j} in local coordinates. By the above, we can solve the Einstein equation in these coordinates so that the resulting metric has a convergent expansion, whose terms of order ρ^{λ_j} are the same as those of the pull-back. If g_1, g_2 are two metrics built on overlapping coordinate charts in this way, then on the region of overlap they are both convergent expansions, and have the same defining asymptotics, so by the uniqueness statement, they are the same. Finally, since the metrics produced on each chart agree on overlaps, we may push them forward to the manifold M to give a conformally compact Einstein metric in a neighborhood of the entire boundary.

References

[1] Baouendi, M.S. and Goulaouic, G. Cauchy problems with characteristic initial hypersurface, *Comm. Pure Appl. Math.*, **26**, 455–475, (1973).

[2] Baouendi, M.S. and Goulaouic, G. Singular nonlinear cauchy problems, *J. Diff. Eq.*, **22**, 268–291, (1976).

[3] Byde, A. and Mazzeo, R. Non-linear scattering theory paper, in preparation.

[4] Graham, R. and Lee, J. Einstein metrics with prescribed conformal infinity on the ball, *Adv. in Math.*, **87**, 186–225, (1991).

[5] Nirenberg, L. An abstract form of the nonlinear Cauchy–Kowalwski theorem, *J. Diff. Geom.*, **6**, 561–576, (1972).

[6] Taylor, A. *Functional Analysis,* John Wiley & Sons, New York, 1958.

[7] Mazzeo, R. and Smale, N. Conformally flat metrics of constant positive scalar curvature on subdomains of the sphere, *J. Diff. Geom.*, **34**, 581–621, (1991).

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Stanford University
e-mail: byde@stanfordalumni.org

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