A new real-time technique to study chaotic dynamics at quantum level

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Introduction

Classically chaotic systems have always had an important place in physics. They are extremely sensitive to initial conditions and unpredictable after a while. Such chaotic systems are crucial to investigate some physical phenomenons such as the nature of black holes [1]. In our case, we have worked on an article [2] which is about "how new real-time techniques can be developed to study chaotic dynamics at the quantum level" and have seen the effects of the quantum corrections for a classically chaotic system.

Method

In this section, the Gaussian state approximation will be introduced to find the quantum mechanical time evolution of a system for a given Hamiltonian in Equation (1)

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{x^2y^2}{2} \tag{1}$$

First of all, the Heisenberg equations for the canonically conjugate operators were found to derive quantum corrections to the classical dynamics with the Hamiltonian (1)

$$\frac{\partial \hat{x}}{\partial t} = \hat{p}_x, \quad \frac{\partial \hat{y}}{\partial t} = \hat{p}_y, \quad \frac{\partial \hat{p}_x}{\partial t} = -\hat{x}\hat{y}^2, \quad \frac{\partial \hat{p}_y}{\partial t} = -\hat{y}\hat{x}^2 \tag{2}$$

Then, by averaging these equations with respect to a density matrix (3) which is represented by the Gaussian state Wigner function (where $\xi = \{x, y, p_x, p_y\}$ is the vector which consists of classical phase space variables),

$$\rho(\xi) = \mathcal{N}exp\left(-\frac{1}{2}(\xi - \bar{\xi})\Sigma^{-1}(\xi - \bar{\xi})\right) \tag{3}$$

$$\bar{\xi}_a \equiv \langle \hat{\xi}_a \rangle \tag{4}$$

$$\Sigma_{ab} = \langle\langle \xi_a \xi_b \rangle\rangle \equiv \langle \frac{\hat{\xi}_a\hat{\xi}_b + \hat{\xi}_b\hat{\xi}_a}{2} \rangle - \langle \hat{\xi}_a \rangle \langle \hat{\xi}_b \rangle \tag{5}$$

the equations of motion (6) for the expectation values of canonically conjugate operators, i.e., the equations of motion for the Gaussian wave packet centers were found where $x \equiv \langle \hat{x} \rangle$, $y \equiv \langle \hat{y} \rangle$, $p_x \equiv \langle \hat{p}_x \rangle$, $p_y \equiv \langle \hat{p}_y \rangle$ indicate the Gaussian wave packet centers and $\langle\langle \xi_a \xi_b \rangle\rangle$ indicate Gaussian wave packet dispersions.

$$\frac{\partial x}{\partial t} = p_x, \quad \frac{\partial y}{\partial t} = p_y, \quad \frac{\partial p_x}{\partial t} = -x\langle y^2 \rangle - 2\langle\langle xy \rangle\rangle y, \quad \frac{\partial p_y}{\partial t} = -y\langle x^2 \rangle - 2\langle\langle xy \rangle\rangle x \tag{6}$$

Writing the Heisenberg equations in (1) as operator products and carrying the calculations one step further, equations of motion for the Gaussian wave packet dispersions were obtained, which are the expectation values of newly emerged operator products as a consequence of Gaussian state approximation.

$$\begin{gathered}
\partial_t \langle\langle x^2 \rangle\rangle = 2\langle\langle p_x x \rangle\rangle, \ \partial_t \langle\langle y^2 \rangle\rangle = 2\langle\langle p_y y \rangle\rangle, \ \partial_t \langle\langle xy \rangle\rangle = \langle\langle p_x y \rangle\rangle + \langle\langle x p_y \rangle\rangle, \\
\partial_t \langle\langle x p_x \rangle\rangle = \langle\langle p_x p_x \rangle\rangle - \langle\langle x^2 \rangle\rangle \langle y^2 \rangle - 2\langle\langle xy \rangle\rangle \langle xy \rangle, \\
\partial_t \langle\langle y p_y \rangle\rangle = \langle\langle p_y p_y \rangle\rangle - \langle\langle y^2 \rangle\rangle \langle x^2 \rangle - 2\langle\langle xy \rangle\rangle \langle xy \rangle, \\
\partial_t \langle\langle x p_y \rangle\rangle = \langle\langle p_x p_y \rangle\rangle - \langle\langle xy \rangle\rangle \langle x^2 \rangle - 2\langle\langle x^2 \rangle\rangle \langle xy \rangle, \\
\partial_t \langle\langle y p_x \rangle\rangle = \langle\langle p_x p_y \rangle\rangle - \langle\langle xy \rangle\rangle \langle y^2 \rangle - 2\langle\langle y^2 \rangle\rangle \langle xy \rangle, \\
\partial_t \langle\langle p_x p_x \rangle\rangle = -2\langle\langle x p_x \rangle\rangle \langle y^2 \rangle - 4\langle\langle y p_x \rangle\rangle \langle xy \rangle, \\
\partial_t \langle\langle p_y p_y \rangle\rangle = -2\langle\langle y p_y \rangle\rangle \langle x^2 \rangle - 4\langle\langle x p_y \rangle\rangle \langle xy \rangle, \\
\partial_t \langle\langle p_x p_y \rangle\rangle = \langle\langle x p_y \rangle\rangle \langle y^2 \rangle - 2\langle\langle y p_y \rangle\rangle \langle xy \rangle - \langle\langle y p_x \rangle\rangle \langle x^2 \rangle - 2\langle\langle x p_x \rangle\rangle \langle xy \rangle
\end{gathered} \tag{7}$$

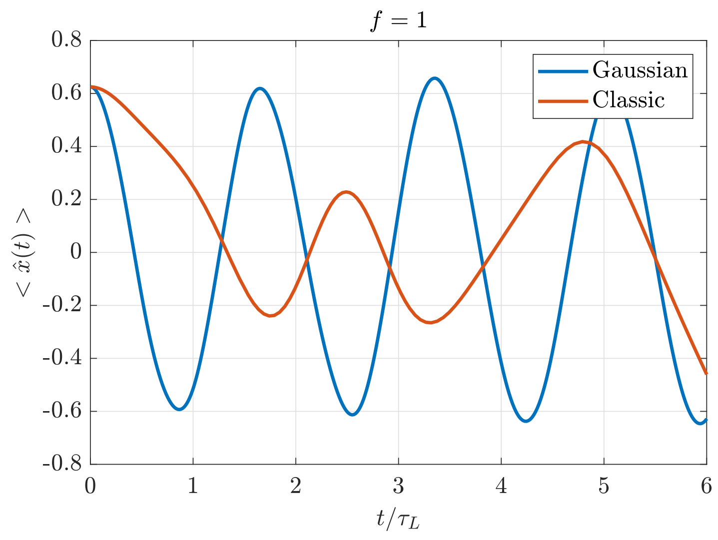
Therefore, instead of having 4 classical equations of motion, 14 equations of motion were obtained as a result of the Gaussian state approximation.

Results and Discussion

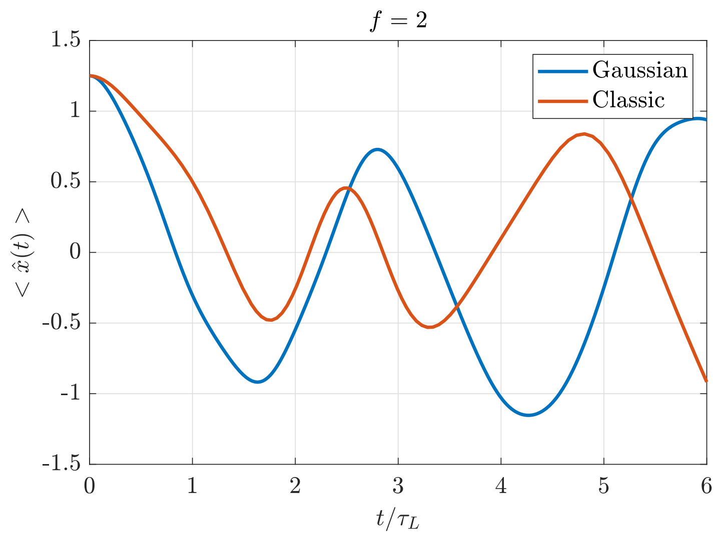
The complete system of equations for the time evolution of the coordinates $\bar{\xi}_a = \{x, y, p_x, p_y\}$ of the wave packet centers and the dispersion matrix $\langle\langle \xi_a \xi_b \rangle\rangle$ were solved numerically by using builtin "ode45" function of MATLAB software. For the sake of simplicity, the following initial conditions were assumed to have minimal quantum dispersions.

$$\begin{gathered}
x = 0.625f, \quad y = 0.325f, \quad \langle\langle x^2 \rangle\rangle = \langle\langle y^2 \rangle\rangle = \langle\langle p_x^2 \rangle\rangle = \langle\langle p_y^2 \rangle\rangle = \frac{1}{2}, \\
p_x = p_y = \langle\langle xy \rangle\rangle = \langle\langle p_x p_y \rangle\rangle = \langle\langle p_x x \rangle\rangle = \langle\langle p_x y \rangle\rangle = \langle\langle p_y y \rangle\rangle = 0
\end{gathered} \tag{8}$$

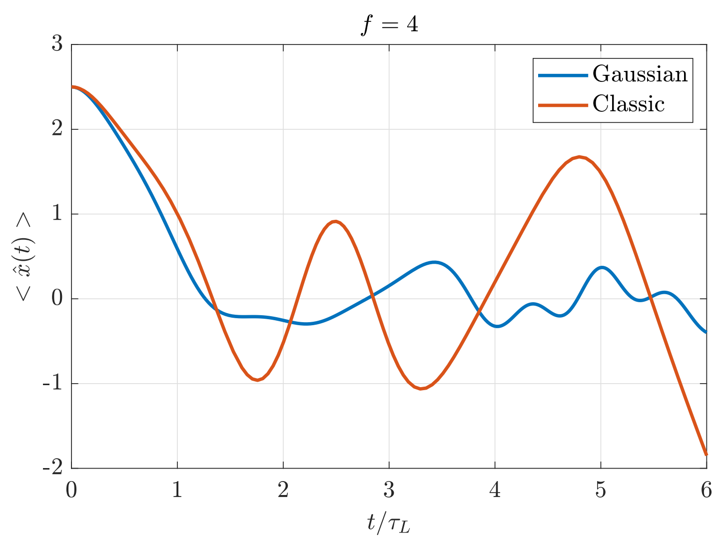
The variable f emphasises the weight of the classical expectation values $\langle x \rangle$ and $\langle y \rangle$ over the quantum dispersions. For small values of f, the quantum effects dominate the system. As f increases, the behavior of the system moves toward to classical regime. In the following plots, the time dependencies of $\langle \hat{x}(t) \rangle$ were compared that obtained from the classical equations of motions and from the Gaussian state approximation.



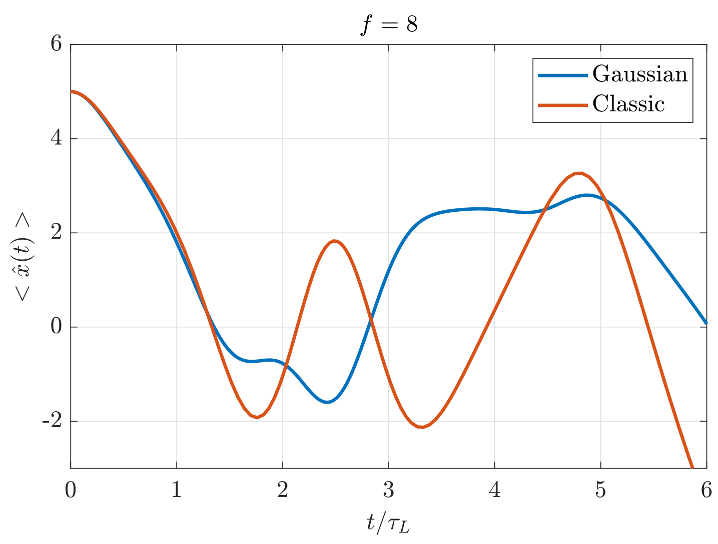
(a) $f = 1$



(b) $f = 2$



(c) $f = 4$



(d) $f = 8$

Figure 1: The comparison of the time dependence of $\bar{x} \equiv \langle \hat{x}(t) \rangle$, the center of a Gaussian wave packet, according to certain f values

Our aim was to compare the solutions of classical and Gaussian state equations of motions with the exact solution of the time-dependent Schrödinger equation, but I could not solve it numerically. We expect that for the small values of f, the solution of the Gaussian state approximation will be close to the solution of the Schrödinger equation due to the taking into account the quantum effects of the wave function of the system.

Conclusion

In conclusion, the importance of a quantum mechanical approximation for the classically chaotic systems has been observed because classical equations of motion lack the quantum effects which lead not to study current physics problems. During my project, I have learned how to approach a theoretical physics problem and solve that problem numerically using a programming language. Many thanks to my dear Professor Seçkin Kürkçüoğlu for his patience, guidance and effort since the beginning of the term. As future work, I am planning to solve the 2D Schrödinger equation numerically and compare the results with the plots in Figure 1.

References

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