A new real-time technique to study chaotic dynamics at quantum level

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- Article: Real-time dynamics of matrix quantum mechanics beyond the classical approximation [1].

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- Contains a toy model for current quantum problems in string theory
- Blends analytical quantum mechanical concepts and numerical techniques

Introduction

Aim

Proposing a quantum mechanical approximation, (Gaussian state approximation), to a classically chaotic system and investigating its time evolution

These kind of chaotic systems can be represented by the Hamiltonian (1)

$$H = \frac{1}{2}\sum_{i} \mathrm{Tr}\, P_i^2 - \frac{1}{4}\sum_{i,j} \mathrm{Tr}\,[X_i, X_j]^2 \tag{1}$$

The distance between initially very close points grows exponentially with time

Such chaotic systems forget about initial conditions after some time

Remark

Hamiltonian (1) can give some insight into the formation of Black holes[2]

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Method

The simplest example of the Hamiltonian (1)

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{x^2y^2}{2} \tag{2}$$

- For a given Hamiltonian (2), we will find the Heisenberg equations at first
- Then we will find the equations of motion by taking averages of the Heisenberg equations

Heisenberg Equations

Heisenberg Equation

$$\frac{d}{dt}A_H(t) = \frac{i}{\hbar}\left[H_H, A_H(t)\right] + \left(\frac{\partial A_S}{\partial t}\right)_H \tag{3}$$

where H_H is Hamiltonian operator and $A_H(t)$ is a canonically conjugate operator.

Derivation of the 1^{st} Heisenberg Equation

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{x}] + \frac{\partial \hat{x}}{\partial t} \tag{4}$$

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar}\left[\frac{\hat{p}_x^2}{2} + \frac{\hat{p}_y^2}{2} + \frac{\hat{x}^2\hat{y}^2}{2}, \hat{x}\right] + \frac{\partial \hat{x}}{\partial t} \tag{5}$$

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar}\left[\frac{\hat{p}_x^2}{2}, \hat{x}\right] = \frac{i}{\hbar}\left(\frac{\hat{p}_x}{2}\left[\hat{p}_x, \hat{x}\right] + \left[\hat{p}_x, \hat{x}\right]\frac{\hat{p}_x}{2}\right) \tag{6}$$

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar}\hat{p}_x\left[\hat{p}_x, \hat{x}\right] \quad where \quad [\hat{x}, \hat{p}_x] = i\hbar \tag{7}$$

$$\boxed{d_t\hat{x} = \hat{p}_x} \tag{8}$$

Heisenberg Equations

Heisenberg equations for canonically conjugate operators $\hat{x}$, $\hat{y}$, $\hat{p}_x$, $\hat{p}_y$

$$d_t\hat{x} = \hat{p}_x \tag{9}$$

$$d_t\hat{y} = \hat{p}_y \tag{10}$$

$$d_t\hat{p}_x = -\hat{x}\hat{y}^2 \tag{11}$$

$$d_t\hat{p}_y = -\hat{y}\hat{x}^2 \tag{12}$$

Finding Equations of Motion

Let us find the equation of motion for the first Heisenberg equation $d_t\hat{x} = \hat{p}_x$.
We can take the average of the Heisenberg equation by using the following theorem.

Trace Function to Calculate Expectation Values

The expectation value of an operator can be found by

$$\langle\psi|\hat{A}|\psi\rangle = \langle\hat{A}\rangle_\rho = \mathrm{Tr}\,(\rho\hat{A}) \tag{13}$$

where ρ is the density matrix.

Example for the 1^{st} Heisenberg Equation

Then by averaging the 1^{st} Heisenberg equation $d_t\hat{x} = \hat{p}_x$,

$$\langle d_t\hat{x}\rangle = \langle \hat{p}_x\rangle \tag{14}$$

$$\mathrm{Tr}\left(\rho d_t\hat{x}\right) = \mathrm{Tr}\left(\rho\hat{p}_x\right) \tag{15}$$

$$d_t\,\mathrm{Tr}\left(\rho\hat{x}\right) = \mathrm{Tr}\left(\rho\hat{p}_x\right) \tag{16}$$

$$d_t\langle\hat{x}\rangle = \langle\hat{p}_x\rangle \tag{17}$$

$$\boxed{d_t x = p_x} \tag{18}$$

where
$x \equiv \langle\hat{x}\rangle$, $y \equiv \langle\hat{y}\rangle$, $p_x \equiv \langle\hat{p}_x\rangle$, $p_y \equiv \langle\hat{p}_y\rangle$

Example for the 3^{rd} Heisenberg Equation

Remember, 3^{rd} Heisenberg Equation is $d_t \hat{p}_x = -\hat{x}\hat{y}^2$. Then,

$$\mathrm{Tr}\left(\rho d_t \hat{p}_x\right) = -\,\mathrm{Tr}\left(\rho \hat{x}\hat{y}^2\right) \tag{19}$$

$$d_t\,\mathrm{Tr}\left(\rho \hat{p}_x\right) = -\,\mathrm{Tr}\left(\rho \hat{x}\hat{y}^2\right) \tag{20}$$

$$d_t \langle \hat{p}_x \rangle = -\int_{-\infty}^{\infty} d^n(x, y)\, \rho x y^2 \tag{21}$$

The upper integral yields another expectation values and the more number of expectation values of coordinate/momentum operators involve. We need an approximation to truncate these infinite set of equations!

The Gaussian State Approximation

Solution: Characterizing the density matrix ρ by the Gaussian state Wigner function. Thus, we could express the expectation values of products of coordinate/momentum operators in terms of Gaussian wave packet centers and Gaussian wave packet dispersions

$$\rho(\xi) = \mathcal{N} exp\left(-\frac{1}{2}(\xi - \bar{\xi})\Sigma^{-1}(\xi - \bar{\xi})\right) \tag{22}$$

$$\bar{\xi}_a \equiv \langle \hat{\xi}_a \rangle \tag{23}$$

$$\Sigma_{ab} = \langle\langle \xi_a \xi_b \rangle\rangle \equiv \langle \frac{\hat{\xi}_a \hat{\xi}_b + \hat{\xi}_b \hat{\xi}_a}{2} \rangle - \langle \hat{\xi}_a \rangle \langle \hat{\xi}_b \rangle \tag{24}$$

where $\xi = \{x, y, p_x, p_y\}$ is the vector which consists of classical phase space variables. $\bar{\xi}_a \equiv \langle \hat{\xi}_a \rangle$ indicates the Gaussian wave packet centers and $\langle\langle \xi_a \xi_b \rangle\rangle$ indicates the Gaussian wave packet dispersions

Again for 3rd Heisenberg Equation

$$\mathrm{Tr}\left(\rho d_t \hat{p}_x\right) = -\,\mathrm{Tr}\left(\rho \hat{x}\hat{y}^2\right) \tag{25}$$

$$d_t\langle \hat{p}_x\rangle = \int_{-\infty}^{\infty} d^4(\xi)\,\rho(\xi) x y^2 \tag{26}$$

$$d_t\langle \hat{p}_x\rangle = -\mathcal{N}^2 \int_{-\infty}^{\infty} d^4(\xi)\, e^{-(\xi-\bar{\xi})\Sigma^{-1}(\xi-\bar{\xi})} x y^2 \tag{27}$$

After long calculations

$$d_t p_x = -x\langle y^2\rangle - 2\langle\langle xy\rangle\rangle y, \tag{28}$$

In this way, 4 equations of motion for the Gaussian wave packet centers and 10 equations of motion for Gaussian wave packet dispersions were found.

Complete System of Equations

$$
\begin{aligned}
&1^{st} : \partial_t \langle \hat{x} \rangle = p_x \\
&2^{nd} : \partial_t \langle \hat{y} \rangle = p_y \\
&3^{rd} : \partial_t \langle \hat{p}_x \rangle = -x \langle y^2 \rangle - 2 \langle \langle xy \rangle \rangle y \\
&4^{th} : \partial_t \langle \hat{p}_y \rangle = -y \langle x^2 \rangle - 2 \langle \langle xy \rangle \rangle x \\
&5^{th} : \partial_t \langle \langle x^2 \rangle \rangle = 2 \langle \langle p_x x \rangle \rangle \\
&6^{th} : \partial_t \langle \langle y^2 \rangle \rangle = 2 \langle \langle p_y y \rangle \rangle \\
&7^{th} : \partial_t \langle \langle xy \rangle \rangle = \langle \langle p_x y \rangle \rangle + \langle \langle x p_y \rangle \rangle \\
&8^{th} : \partial_t \langle \langle x p_x \rangle \rangle = \langle \langle p_x p_x \rangle \rangle - \langle \langle x^2 \rangle \rangle \langle y^2 \rangle - 2 \langle \langle xy \rangle \rangle \langle xy \rangle \\
&9^{th} : \partial_t \langle \langle y p_y \rangle \rangle = \langle \langle p_y p_y \rangle \rangle - \langle \langle y^2 \rangle \rangle \langle x^2 \rangle - 2 \langle \langle xy \rangle \rangle \langle xy \rangle \\
&10^{th} : \partial_t \langle \langle x p_y \rangle \rangle = \langle \langle p_x p_y \rangle \rangle - \langle \langle xy \rangle \rangle \langle x^2 \rangle - 2 \langle \langle x^2 \rangle \rangle \langle xy \rangle \\
&11^{th} : \partial_t \langle \langle y p_x \rangle \rangle = \langle \langle p_x p_y \rangle \rangle - \langle \langle xy \rangle \rangle \langle y^2 \rangle - 2 \langle \langle y^2 \rangle \rangle \langle xy \rangle \\
&12^{th} : \partial_t \langle \langle p_x p_x \rangle \rangle = -2 \langle \langle x p_x \rangle \rangle \langle y^2 \rangle - 4 \langle \langle y p_x \rangle \rangle \langle xy \rangle \\
&13^{th} : \partial_t \langle \langle p_y p_y \rangle \rangle = -2 \langle \langle y p_y \rangle \rangle \langle x^2 \rangle - 4 \langle \langle x p_y \rangle \rangle \langle xy \rangle \\
&14^{th} : \partial_t \langle \langle p_x p_y \rangle \rangle = \langle \langle x p_y \rangle \rangle \langle y^2 \rangle - 2 \langle \langle y p_y \rangle \rangle \langle xy \rangle - \langle \langle y p_x \rangle \rangle \langle x^2 \rangle - 2 \langle \langle x p_x \rangle \rangle \langle xy \rangle
\end{aligned}
\tag{29}
$$

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Results and Discussion - Complete System of Equations

$$
\begin{aligned}
1^{st} &: \partial_t \langle \hat{x} \rangle = p_x \\
2^{nd} &: \partial_t \langle \hat{y} \rangle = p_y \\
3^{rd} &: \partial_t \langle \hat{p}_x \rangle = -x\langle y^2 \rangle - 2\langle\langle xy \rangle\rangle y \\
4^{th} &: \partial_t \langle \hat{p}_y \rangle = -y\langle x^2 \rangle - 2\langle\langle xy \rangle\rangle x \\
5^{th} &: \partial_t \langle\langle x^2 \rangle\rangle = 2\langle\langle p_x x \rangle\rangle \\
6^{th} &: \partial_t \langle\langle y^2 \rangle\rangle = 2\langle\langle p_y y \rangle\rangle \\
7^{th} &: \partial_t \langle\langle xy \rangle\rangle = \langle\langle p_x y \rangle\rangle + \langle\langle x p_y \rangle\rangle \\
8^{th} &: \partial_t \langle\langle x p_x \rangle\rangle = \langle\langle p_x p_x \rangle\rangle - \langle\langle x^2 \rangle\rangle \langle y^2 \rangle - 2\langle\langle xy \rangle\rangle \langle xy \rangle \\
9^{th} &: \partial_t \langle\langle y p_y \rangle\rangle = \langle\langle p_y p_y \rangle\rangle - \langle\langle y^2 \rangle\rangle \langle x^2 \rangle - 2\langle\langle xy \rangle\rangle \langle xy \rangle \\
10^{th} &: \partial_t \langle\langle x p_y \rangle\rangle = \langle\langle p_x p_y \rangle\rangle - \langle\langle xy \rangle\rangle \langle x^2 \rangle - 2\langle\langle x^2 \rangle\rangle \langle xy \rangle \\
11^{th} &: \partial_t \langle\langle y p_x \rangle\rangle = \langle\langle p_x p_y \rangle\rangle - \langle\langle xy \rangle\rangle \langle y^2 \rangle - 2\langle\langle y^2 \rangle\rangle \langle xy \rangle \\
12^{th} &: \partial_t \langle\langle p_x p_x \rangle\rangle = -2\langle\langle x p_x \rangle\rangle \langle y^2 \rangle - 4\langle\langle y p_x \rangle\rangle \langle xy \rangle \\
13^{th} &: \partial_t \langle\langle p_y p_y \rangle\rangle = -2\langle\langle y p_y \rangle\rangle \langle x^2 \rangle - 4\langle\langle x p_y \rangle\rangle \langle xy \rangle \\
14^{th} &: \partial_t \langle\langle p_x p_y \rangle\rangle = \langle\langle x p_y \rangle\rangle \langle y^2 \rangle - 2\langle\langle y p_y \rangle\rangle \langle xy \rangle - \langle\langle y p_x \rangle\rangle \langle x^2 \rangle - 2\langle\langle x p_x \rangle\rangle \langle xy \rangle
\end{aligned}
\tag{30}
$$

Results and Discussion - Complete System of Equations

Initial conditions

- Quantum dispersions were kept minimal

$$x = 0.625f, \quad y = 0.325f, \quad \langle\langle x^2 \rangle\rangle = \langle\langle y^2 \rangle\rangle = \langle\langle p_x^2 \rangle\rangle = \langle\langle p_y^2 \rangle\rangle = \frac{1}{2}, \tag{31}$$
$$p_x = p_y = \langle\langle xy \rangle\rangle = \langle\langle p_x p_y \rangle\rangle = \langle\langle p_x x \rangle\rangle = \langle\langle p_x y \rangle\rangle = \langle\langle p_y y \rangle\rangle = 0$$

Results and Discussion - Complete System of Equations

$$
\begin{aligned}
1^{st} &: \langle \hat{x} \rangle \longrightarrow x_1 \\
2^{nd} &: \langle \hat{y} \rangle \longrightarrow x_2 \\
3^{rd} &: \langle \hat{p}_x \rangle \longrightarrow x_3 \\
4^{th} &: \langle \hat{p}_y \rangle \longrightarrow x_4 \\
5^{th} &: \langle\langle x^2 \rangle\rangle \longrightarrow x_5 \\
6^{th} &: \langle\langle y^2 \rangle\rangle \longrightarrow x_6 \\
7^{th} &: \langle\langle xy \rangle\rangle \longrightarrow x_7 \\
8^{th} &: \langle\langle xp_x \rangle\rangle \longrightarrow x_8 \\
9^{th} &: \langle\langle yp_y \rangle\rangle \longrightarrow x_9 \\
10^{th} &: \langle\langle xp_y \rangle\rangle \longrightarrow x_{10} \\
11^{th} &: \langle\langle yp_x \rangle\rangle \longrightarrow x_{11} \\
12^{th} &: \langle\langle p_x p_x \rangle\rangle \longrightarrow x_{12} \\
13^{th} &: \langle\langle p_y p_y \rangle\rangle \longrightarrow x_{13} \\
14^{th} &: \langle\langle p_x p_y \rangle\rangle \longrightarrow x_{14}
\end{aligned}
$$

(32)

Results and Discussion - Complete System of Equations

$$
\begin{aligned}
&1^{st}\ : \partial_t x_1 = x_3\\
&2^{nd}\ : \partial_t x_2 = x_4\\
&3^{rd}\ : \partial_t x_3 = -x_1(x_6 + x_2^2) - 2x_7x_2\\
&4^{th}\ : \partial_t x_4 = -x_2(x_5 + x_2^2) - 2x_7x_1\\
&5^{th}\ : \partial_t x_5 = 2x_8\\
&6^{th}\ : \partial_t x_6 = 2x_9\\
&7^{th}\ : \partial_t x_7 = x_{11} + x_{10}\\
&8^{th}\ : \partial_t x_8 = x_{12} - x_5(x_6 + x_2^2) - 2x_7(x_7 + x_1x_2)\\
&9^{th}\ : \partial_t x_9 = x_{13} - x_6(x_5 + x_1^2) - 2x_7(x_7 + x_1x_2)\\
&10^{th} : \partial_t x_{10} = x_{14} - x_7(x_5 + x_1^2) - 2x_5(x_7 + x_1x_2)\\
&11^{th} : \partial_t x_{11} = x_{14} - x_7(x_6 + x_2^2) - 2x_6(x_7 + x_1x_2)\\
&12^{th} : \partial_t x_{12} = -2x_8(x_6 + x_2^2) - 4x_{11}(x_7 + x_1x_2)\\
&13^{th} : \partial_t x_{13} = -2x_9(x_5 + x_1^2) - 4x_{10}(x_7 + x_1x_2)\\
&14^{th} : \partial_t x_{14} = -x_{10}(x_6 + x_2^2) - 2x_9(x_7 + x_1x_2) - x_{11}(x_5 + x_1^2) - 2x_8(x_7 + x_1x_2)
\end{aligned}
\tag{33}
$$

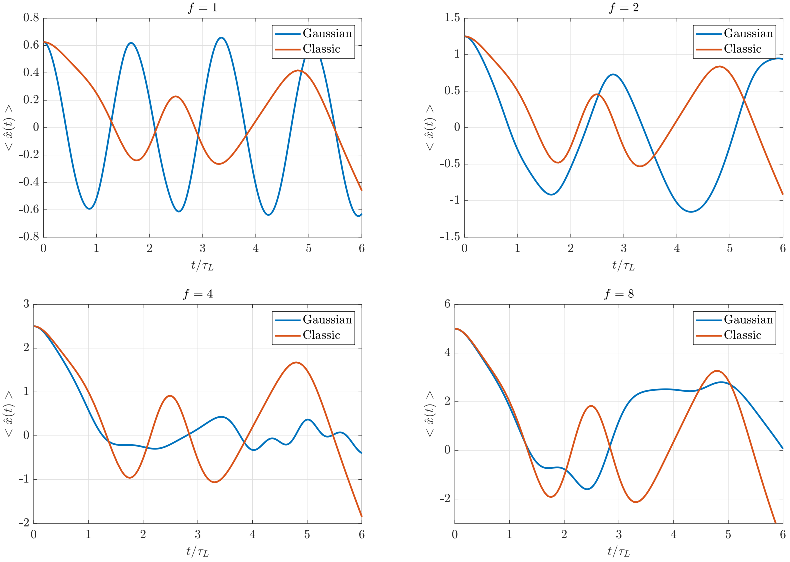

Figure 1: The comparison of the time dependence of $\bar{x} \equiv \langle \hat{x}(t) \rangle$, the center of a Gaussian wave packet, according to certain f values

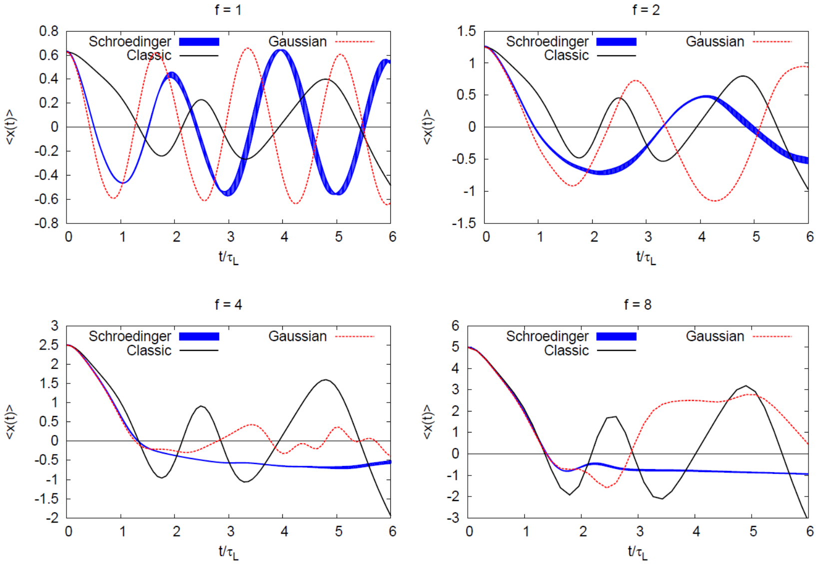

Figure 2: Article's plots with the exact solution of Schrödinger equation

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Conclusion

- In this project, the importance of a quantum mechanical approximation for the classically chaotic systems has been observed because classical equations of motion lack the quantum effects which lead not to study current physics problems

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- I have learned how to approach a theoretical physics problem and solve that problem numerically using a programming language.

Conclusion

- In this project, the importance of a quantum mechanical approximation for the classically chaotic systems has been observed because classical equations of motion lack the quantum effects which lead not to study current physics problems
- I have learned how to approach a theoretical physics problem and solve that problem numerically using a programming language.
- Many thanks to my supervisor Prof. Seçkin Kürkçüoğlu for his patience, guidance and effort.

Conclusion

- In this project, the importance of a quantum mechanical approximation for the classically chaotic systems has been observed because classical equations of motion lack the quantum effects which lead not to study current physics problems
- I have learned how to approach a theoretical physics problem and solve that problem numerically using a programming language.
- Many thanks to my supervisor Prof. Seçkin Kürkçüoğlu for his patience, guidance and effort.
- As future work, I am planning to solve the 2D Schrödinger equation numerically and compare the results with the plots in Figure 1.

Thank you!

Thank you very much for your interest and attention.

Questions?

Any Questions?

References

Sinya Aoki, Masanori Hanada, and Norihiro Iizuka. *Quantum Black Hole Formation in the BFSS Matrix Model*. 2015. arXiv: 1503.05562 [hep-th].

Pavel Buividovich, Masanori Hanada, and Andreas Schäfer. "Real-time dynamics of matrix quantum mechanics beyond the classical approximation". In: *EPJ Web of Conferences* 175 (2018). Ed. by M. Della Morte et al., p. 08006. ISSN: 2100-014X. DOI: 10.1051/epjconf/201817508006. URL: http://dx.doi.org/10.1051/epjconf/201817508006.