Relationship between Predators and Preys

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Written by:
Raffaele Cardillo and Antoine Parrain

Supervised by:
Satoshi Takahama

Contents

1 Deviation from project proposal

During the course of the project, some deviations from the original proposal were necessary to address unforeseen challenges and ensure accurate results. Initially, the proposal emphasized incorporating complex factors such as ecosystem type and behavioral changes only if essential; however, the model was limited to the Lotka-Volterra framework due to constraints in data availability and computational feasibility. Additionally, while the proposal intended to explore various ecological factors, the study primarily focused on predator efficiency and prey reproduction rates using Gaussian peak functions to simulate environmental disturbances. These modifications ensured that the project remained feasible within the timeline while still addressing the core objectives effectively.

2 Introduction

Interactions between different species can be anything but easy to estimate, especially in a context in which climate changes lead to frequent and unpredictable environmental events, which can be unexpectedly catastrophic and heavily affect any species evolution.
As a consequence of this worldwide concern our client has asked to develop a mathematical and graphical model for Prey-Predator interactions.
Therefore, our purpose has been to generate a model which aims to predict the evolution of a Prey-Predator interaction, based on a given data set. Our client has demanded to base our study on the well-known 90-year data set concerning the interactions between Hare and Lynx, provided by the Hudson's Bay Company of Canada, going from 1845 to 1935.
To accomplish this task, we have generated a model based on the Lotka-Volterra equations, which is the most commonly used set of differential equations in the field of Prey-Predator interactions. After showing all of our results, we have analyzed the reliability of the model by making hypotheses about the limitations of the model, based on the book *Ecology*, Ricklefs & Miller

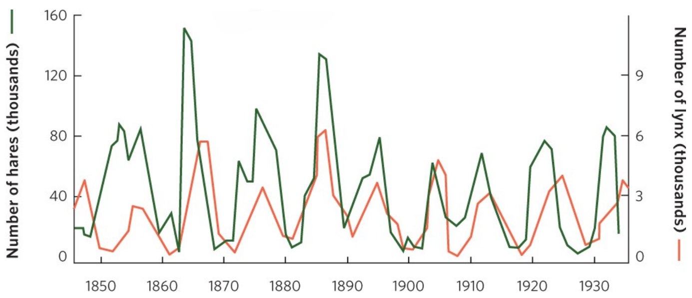

Figure 1: Data from Hudson's Bay company. *Economy of Nature*, Relyea & Ricklefs

3 Approach

3.1 Model

As previously said, the core of our project is based on the Lotka-Volterra model, which is based on few assumptions and simplification as follows. We assume that:

- The space and resources used are unlimited,
- The two species are only interacting with each other,
- Preys only die from predators.

The impact of those assumptions on our project will be deeply analyzed in the "Discussion" section. The basic equations of Lotka-Volterra are the following:

$$\frac{dx}{dt} = rx - \alpha xy$$

$$\frac{dy}{dt} = \beta xy - my$$

where:

- x: Prey population,
- y: Predator population,

- r: Prey growth rate,
- α: Predation rate,
- β: Predator reproduction rate,
- m: Predator mortality rate.

We can see here a constant r rate of growth from the prey that leads to an exponential growth regulated by the predation rate multiplied by the possibility of a prey encountering a predator (approximately $x \times y$). We can define α as the efficiency of the predators. The predator reproduction rate β can be represented as the fraction of predator offspring per meeting of a prey. This also regulates the inverse exponential induced by the constant mortality rate.

3.2 Interpolation of the original data

We have chosen to interpolate the basic data because the measurements of the populations had been given with a time step of 1 year between each measurement, which led to big gaps between some consecutive values (for example 1865). Therefore, we've interpolated the data with a time step of 0.25 years (3 months) with a cubic interpolation as it brings a smoother curve. To do it, we have created a Python code ("Interpolation.py") that links every point with a cubic function and output the value every time step. The interpolated data has then been multiplied by 1000, and the lynx (predator) population has been multiplied by a factor of 160/9 in order to correct the initial data, which had been provided with some crucial scaling mistakes.

3.3 Parameters definition

Furthermore, to make our model as reliable as possible, we have carefully determined realistic parameters using a parametric compensation method. First, we have estimated the slopes from the provided data graph. We have excluded slope values that were four and a half times greater than the median to avoid inaccuracies caused by large gaps in the data, such as those from 1865 and 1885. We chose 4.5 after few try because it matched well the spots where we wanted to exclude the slopes. At each valid data point, we have formulated a system of equations with four unknown parameters. Repeating this process across the dataset has allowed us to approximate these parameters using the least squares method. For this, we utilized the "least squares" function from the scipy.optimize Python package. This approach adjusted the parameters r, α, β and m so that the model's derivatives closely matched the observed data, ensuring the equations accurately predicted population changes. However, there was a unit mismatch to address. The parameters typically have units of time^{-1}, which corresponds to year^{-1} in our case, as we have solved the equations with a time step of one year. When interpolating the data four times per year, the resulting parameters instead had units of 3 months^{-1}. To correct this, we have simply multiplied the parameters by 4 to convert them back to year^{-1}.

3.4 Differential equations solving

As our model's equations are interconnected with each other they don't have a clear function as answer. To plot them we have needed to evaluate the slope at every tiny variation of time t and add it to the previous value. To code it in C we have used the Range Kutta 4th order method (RK4), which is a common method used to solve ordinary differential equations. For an equation:

$$\frac{dy}{dt} = f(t, y), \quad with\ y(t_0) = y_0$$

The RK4 method computes the next value of y, denoted y_{n+1} , using a weighted average of slopes evaluated at different points within the time step(h):

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where:

- $k_1 = f(t_n, y_n)$: Slope at the beginning of the interval (initial guess).
- $k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$: Slope at the midpoint, using k_1 to estimate the value of y there
- $k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$: Slope at the midpoint again, but using k_2 for estimation
- $k_4 = f\left(t_n + h, y_n + h \cdot k_3\right)$: Slope at the end of the interval, using k_3 for estimation

We could then plot the curves with this technique.

3.5 Scenarios

The model created with all these implementations was well-approximated to reality, but we have had to add a peak-generating function in order to simulate a sudden environmental event that would drastically affect the populations. To do that, we have created a Gaussian peak function, adding a peak in the prey's population at a specific time. The Gaussian peak function is:

$$S(t) = A \times e^{-\frac{(t-t_0)^2}{2\sigma^2}}$$

where:

- A : amplitude of the peak;
- σ : standard deviation of the peak;
- t_0 : time when the peak occurs.

We have then added this function to the Prey population differential equation, which becomes:

$$\frac{dx}{dt} = rx - \alpha xy + S(t)$$

We have chosen to generate two scenarios:

- A peak at t = 1883, with an amplitude of -15000;
- A peak at t = 1877, with an amplitude of -10000.

As the function used is a Gaussian peak, the effect of the peak begin to be observed approximately 2 years before the center of the peak, respectively at t = 1881 and t = 1875.

3.6 Choice of coding languages

We would like to spend a few words about the choice of defining the initial parameters of our differential equations in Python rather than C. In fact, the great amount of calculations needed to find the result had led us to think that it should have been implemented in C. However, the choice of doing it in Python can be justified by the fact that we have wanted to do it by vectorization and that Python provides the function "least squares" in the package scipy.optimize, and that fitted perfectly what we intended to do.
Secondly, the choice of using Python for interpolation relied on the fact that it allows us to directly plot the new data-set generated, even if the amount of calculations needed is not negligible. Then, we have chosen to solve the differential equations with C, in order to avoid using pre-made libraries, to clearly show our calculations, and to obtain a quicker program than what we would have got with Python. Lastly, we have chosen to plot the results with Python because of the graphical versatility of this language.

4 Results

The first plot shown represents the comparison between our model and the original data set.

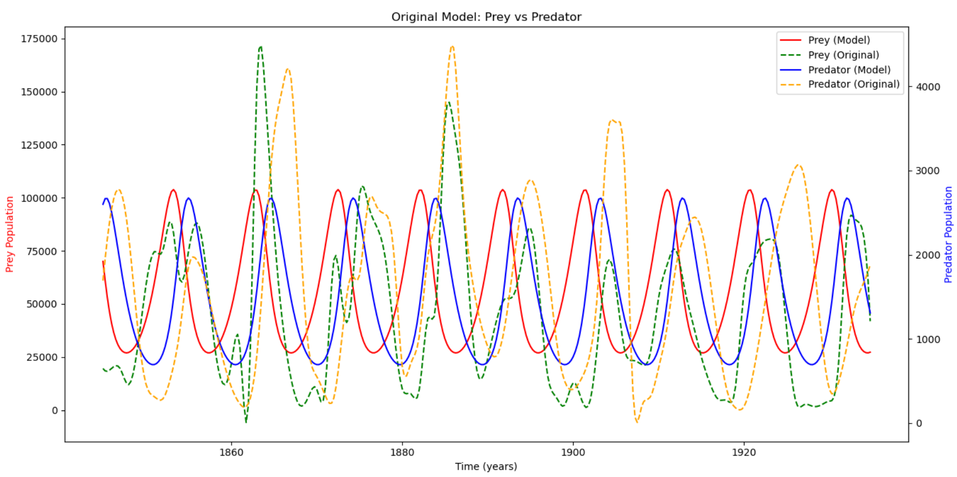

Figure 2: Model

In Figure 2, we can see the original data, represented by the dotted lines, and the result of our model in blue and red. Plotted with our optimized parameters:

r	0.676
α	$4.61 \times 10^{-}4$
β	$1.19 \times 10^{-}5$
m	0.677

Table 1: Optimized Parameters

Following this first graphical analysis, we have the plots of the 2 scenarios.

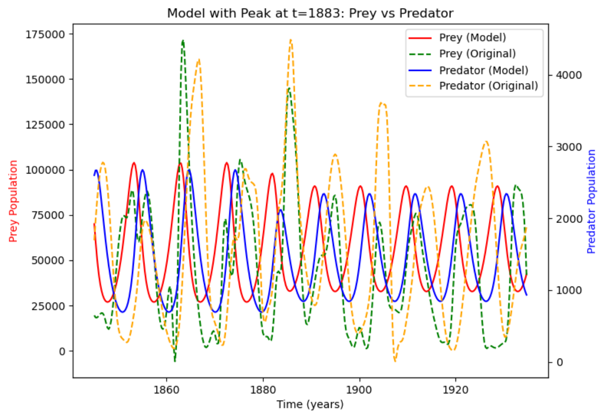

Figure 3: First model with peak

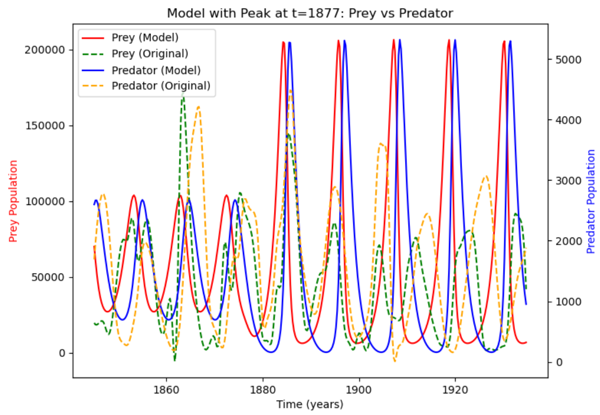

Figure 4: Second model with peak

In Figure 3, the scenario with a peak of amplitude -15000 at t = 1883 is shown. In Figure 4, the other scenario, with a peak of -10000 at t = 1877, is shown.
In addition to those graphs, we have the plots of Predators as a function of Preys for the 2 scenarios presented above, these graphs provide another approach of the cyclical behavior of the Lotka-Volterra model and they bring a clarified vision of the effect of the Gaussian peaks, which have been deeply analyzed and described in the following section. The curve turns in the trigonometric direction as time goes.

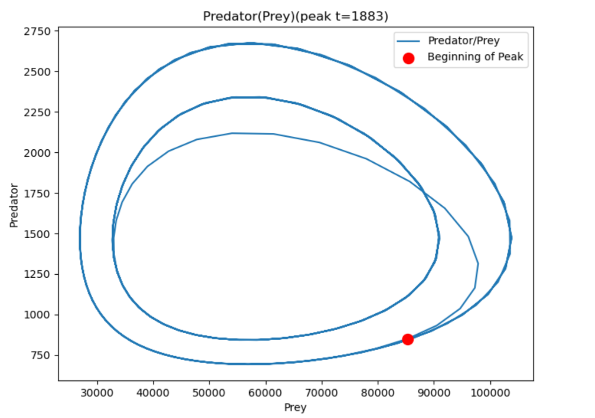

Figure 5: Prey-Predator (peak at t = 1883)

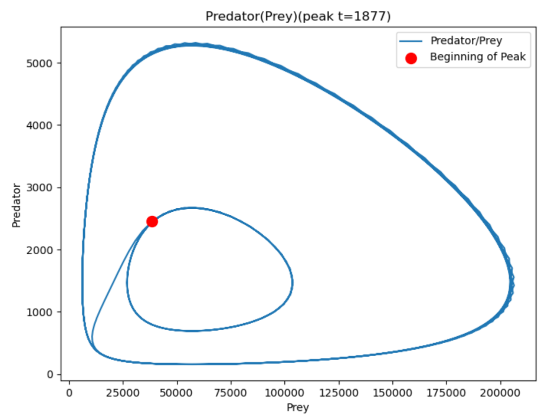

Figure 6: Predator-Prey (peak at t = 1877)

5 Discussion

5.1 Analysis of the basic model

In Figure 2 we can observe that the trend of the data set and that of our mathematical model are similar. In particular, the period of the oscillation is somewhat the same, as well as the amplitude, also following that of the data set, which is sometimes undeniably affected by randomness, and, as a consequence, is sometimes slightly incoherent with our model. This similarity can be better observed in the image below, where a precise spot of the graph is shown.

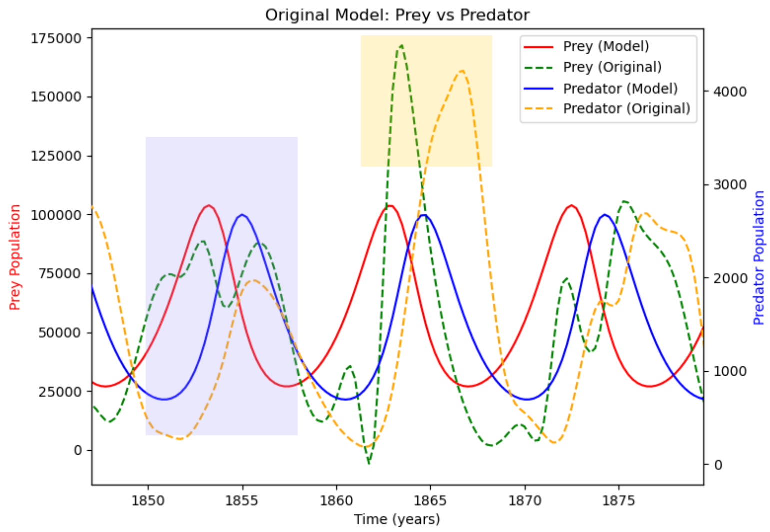

Figure 7:

In fact, we can observe that in both the shaded zones, our model and the original data have a ressembling (almost equal) period of oscillation, even if the shape of the original data is not precisely the same. However, in the right zoom we can highlight that the original data are higher than those of our model. Although, the difference of amplitude does not affect the interaction between the 2 populations. In other words, at this instant of time (same for many others), for a given amount of prey, the difference with the predator population is the same of that of the original data, even if out of phase and at a lower altitude.

5.2 Stationary points & Orbits

The use of circular graphs (Predator's populations in function of prey's populations) shows us some interesting characteristics, we can ask ourselves around which point is the curve turning and what makes the amplitude of oscillations. Let's take a look back at the equations and equals them to 0 and find the pair of points (x,y) that satisfy this condition. That means when the populations aren't growing nor decreasing.

$$\frac{dx}{dt} = rx - \alpha xy = 0$$

$$\frac{dy}{dt} = \beta xy - my = 0$$

We can find two points as answers

$$(x, y) = (0, 0)$$

and

$$(x, y) = (\frac{m}{\beta}, \frac{r}{\alpha})$$

The first point is trivial, as we can imagine there wouldn't be no growth if both species went extinct. The second point is interesting because it brings us an equilibrium point at which both species would still interact but the mortality would just equilibrate the natality. In our case this second point is (56991, 1468) which is exactly the central point around which our curve rotates.

We can then see that the parameters have an impact on the shape of the oscillations and on that center point.

The curves around it will be distributed in different orbits as shown in figure 8. That means that the amplitude of the oscillations are only dependent on the initial populations chosen, in other words on which orbit we're standing. The period of the oscillation is also influenced by the size of the orbit as it requires more time-step to make a full tour of a bigger orbit. And this is interesting once again because it means that once the parameters are defined, we can adjust the initial populations to make the amplitude and period of oscillation match as much as possible with the model. As a consequence, we have chosen to begin our simulation with the populations that would have best fit the original data, rather than using the real initial populations. It could be reasonable to try to find the best initial populations (find the best orbit) that would best fit the curves with a compensation method.

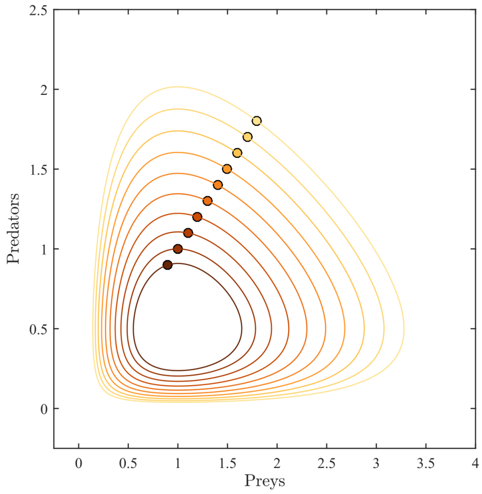

Figure 8: Orbits of Lotka-Volterra model with a same set of parameters. *(Lotka-Volterra equations)* Wikipedia.org

5.3 Analysis of the scenarios

After having analyzed the basic model, let's see what happens in the two scenarios provided. In the first case, we have chosen the peak at t = 1883, with an amplitude of -15000. However, being a Gaussian peak with $\sigma = 1$, its effects cannot be seen before t = 1881, at which the populations were of 85304 for prey and 850 for the predator, which also corresponds to the red point on figure 5. In this first scenario, we can observe that by actually reducing the individuals of the Prey, there will be an effect on the future tendency of both the populations. In fact, this peak reduces the amplitude of the oscillations after 1883, and that can be better observed in Figure 5, where we see that after the beginning of the peak, the curve leaves its original orbit, to move on another one, with a smaller amplitude

In the second case, the peak, with an amplitude of -10000, is set at t = 1877 and its effects can therefore be seen at t = 1875, where populations where of 38529 for the Prey and 2450 for the Predator. Inversely to the first scenario, this second peak has a different and somehow unexpected effect on the amplitude of the curves, which increases rather than decreasing. This behavior can also be better observed in Figure 6, in which we see a change of orbit, going from the original to a wider one.

In fact, the effect on the amplitude of such a peak depends on where we stand on the orbit, the first peak at t = 1883 induces a diminishment of the growing amount of prey. At this time, we are in the bottom right of the cycle (figure 5). This leads the curve to get closer to the center point during the following few points which is going to make the amplitude lower. Inversely, when the same peak occurs at the top left of the cycle, the effect will cause a deviation away from the center.

This effect could lead us to think that sudden peaks in real data could be a consequence of a previous descending peak, leading to a change of orbit, rather than simply being an increasing population event. In fact, a sudden environmental event is much more likely to reduce the population of a certain species rather than increasing it, and it is therefore logical to think that unexpected peaks could somehow rely on that mathematical behavior.

5.4 Limitations of this model

From what has been said so far, we can deduce that our model provide a well-approximated interaction between Prey and Predator, especially when predicting its tendency in the future. However, when it comes to predicting a precise amount of individuals at a certain instant of time, our model becomes slightly inefficient and might not be the best way to accomplish this task because the curves generated and the original data are not constantly in phase as we have seen.

We can therefore highlight the limitations of the Lotka-Volterra model due to several factors which are not taken into

account in the initial equations.
As we've said previously, the model is based the following assumptions to simplify the model:

- The space and resources used are unlimited,
- The two species are only interacting with each other,
- Preys only die from predators.

These assumptions make the model extremely unrealistic in most of the cases. The model on hare and lynx relationship in Hudson's Bay works pretty well here probably because it's location is quite remote. Making the space and resources abundant, and interactions with other species limited. We can also notice that the parameters are assumed as constant, which is particularly inaccurate if we consider mortality rate during the winter or the summer, as well as the growth rate that is higher in spring in most species. In addition, Nicholson and Bailey[1], who have developed a more developed model, have heavily criticized the Lotka-Volterra equations for their assumption of linearity. For them, the number of prey consumed should reach a plateau instead of increasing linearly, and that for some parameters not taken into account in our mathematical method:

- When preys are abundant, predators can become satiated, which means that their rate of feeding should be limited by the rate at which they digest and assimilate the prey.
- If a predator captures more than one prey, a time-dependent factor should be taken into account, considering the time needed to find, handle and eat the prey.
- If predators reach satiation, it might lead to a sudden increase in individuals, as well as an eventual immigration of other individuals of the same species, pulled by the abundance of food.

In addition to the limits highlighted by Nicholson and Bailey, we should also consider the environmental factor as a fundamental explanation of the low reliability of the mathematical model. In fact, random fluctuation of the environment, as well as sudden unpredictable events, can partially explain, as we have seen, the randomness of the population trend.

6 Conclusion

From our analysis and implementation, we have successfully developed a Lotka-Volterra-based prey-predator interaction model with realistic adjustments to parameters and external events. While the model has clear limitations due to its foundational assumptions—such as infinite resources and linearity—it effectively approximates population trends and provides insights into ecosystem dynamics under specific scenarios.

Key findings indicate that sudden environmental events, like population peaks or declines, significantly affect oscillatory behavior, leading to shifts in equilibrium and amplitude. These outcomes highlight the sensitivity of such models to initial conditions and external disruptions. However, the model struggles with precision in predicting exact population numbers due to inherent randomness in real-world data and unaccounted ecological complexities.

Despite its constraints, this study demonstrates the model's applicability for understanding and forecasting predator-prey dynamics. Additionally, the analysis emphasizes the importance of environmental stability and the potential cascading effects of ecological disturbances.

We believe this model offers a robust starting point for future refinements, including incorporating additional parameters such as environmental variability, resource limitations, and predator-prey saturation effects. These enhancements could improve the model's alignment with real-world ecosystems and broaden its usability. Overall, we hope our contributions meet the expectations of the client and serve as a valuable resource for ecological studies.

[1] Ricklefs, Miller. Ecology

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