Simulating Watts-Strogatz and Erdős–Rényi Model

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Abstract

In this paper, we conduct experimental work on the Watts–Strogatz model to study how the clustering coefficient and the average shortest path behave as functions of the probability defined in the model. Furthermore, we compare the average shortest path in the Watts–Strogatz model with that in the Erdős–Rényi model (as a function of the network size) to gain a better understanding of the behavior of this variable.

1 Introduction

It is widely recognized that graphs capture the attention of both mathematicians and computer scientists. Numerous applications can be represented as graphs, with networks being a notable example. For example, we can see interactions between people as a graphs where we represent nodes as people and edges as a connection.

Given that we can choose specific graphs in order to have a better representation of our model, such as Erdős–Rényi (ER) graphs or the Watts–Strogatz model, we note that the latter can represent two main properties that are widely common in real networks: a small average shortest path and a high clustering coefficient, while ER graphs only exhibit the first property.

In this paper, we take an experimental approach to these characteristics in order to gain a better understanding of both models.

2 Preliminaries

Given a graph $G = (V, E)$, we define the average shortest path: $I = \frac{2}{n(n-1)} \sum_{i<j} d(i, j)$ where $d(i, j)$ is the distance between two nodes $i, j \in V$.

The global clustering coefficient (or transitivity) C is defined in the literature as:

$$C = \frac{3 \cdot \text{number of triangles}}{\text{number of connected components}}$$

where a *triangle* consist of three different vertices $i, j, k \in V$ such that $ij, jk, ki \in E$

In addition, two parameters $n \in \mathbb{N}$ and $0 \leq p \leq 1$ define the Erdős–Rényi model [1, 2] consisting of a graph $G(n, p)$ such that $|V| = n$ and two distinct nodes are connected by an edge with probability p. The Erdős–Rényi model is a well-known model for simulating small average shortest path.

Finally, the Watts-Strogatz model [3] consists of a defined graph such that with probability p it rewires each local connection to a random vertex.

3 Methodology

For the experiments, we will use the `R` programming language along with a package to generate the Erdős–Rényi and Watts-Strogatz model called `igraph`. In particular we will use the following parameters for each model:

- The Watts–Strogatz graph will be generated by calling the function `sample_smallworld(dim = 1, size = 1000, nei = 10, p = p)`, where p is the probability under investigation. Note that the fixed parameters `dim`, `size`, and `nei` define the initial graph (a one-dimensional graph with 1000 nodes, where each node is connected to 10 other nodes), while p specifies the probability of *rewiring* each connection.

- The Erdős–Rényi model, on the other hand, will be generated using `sample_gnp(n = n, p = p(n))`, where n is the size of the network and p is defined as a function of n. Recall that the Erdős–Rényi model is constructed by starting with a graph of size n and no edges, and then connecting each pair of nodes $i, j \in V$ with probability p. Therefore, in order to report a meaningful average shortest path, we must ensure that the graph is connected, which is why we define the probability as a function of the size (otherwise the average shortest path would be $O(1)$). We chose

$$p(n) = \frac{4 \ln n}{n}$$

not only to ensure connectivity [2] but also to avoid choosing too large a denominator, which would almost always result in an *almost* complete graph. Because we want to plot both the clustering coefficient and the average shortest path we will normalize both values in order to both fit the same plot. We made our sample vector of n's start from n = 10 in order to have a probability that is always less than 1. If you want to try it with smaller values of n you may have to decrease the factor 4, at the expense of having a lower probability of connection.

Due to randomness, each value reported in this paper will be the average of 30 repetitions performed previously.

All the experiments are reproducible by executing the R script that we provide.

4 Results

4.1 Watts-Strogatz model

As mentioned above, we plot both the clustering coefficient and the average shortest path as functions of the probability (see Figure 1). The first observation is that both parameters tend to converge to small values as the probability approaches one. This is not a coincidence, since the WS model behaves more like the Erdős–Rényi model at higher probabilities: by *rewiring* each node to a random one, the process exhibits dynamics similar to deciding independently for each pair of nodes whether a connection exists. From theoretical results, we know that the Erdős–Rényi model has both a small clustering coefficient and a small average shortest path, and the Watts–Strogatz model displays the same properties at higher probabilities.

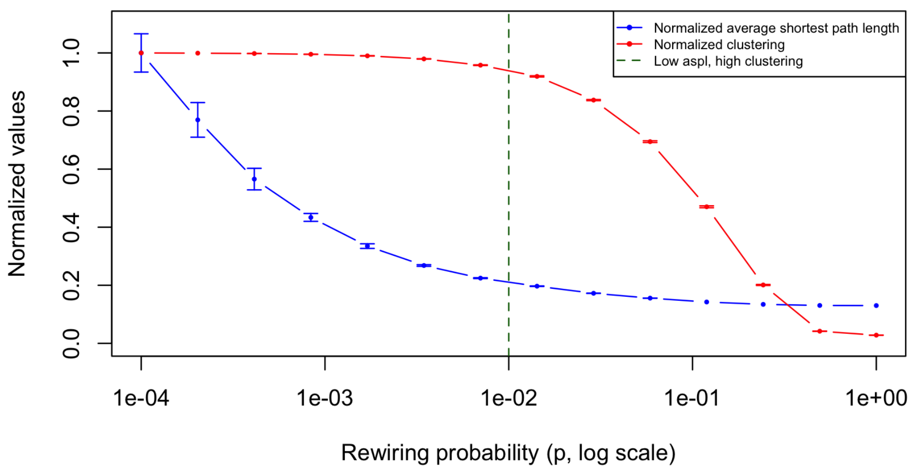

Figure 1: Clustering coefficient and average shortest path length as a function of p in the WS model. The x-axis is shown on a logarithmic scale. The error bars represent a 95% confidence interval for the means, calculated as ± 2.045 times the normalized standard error of the mean. The factor 2.045 comes from the Student's t-distribution confidence interval with 29 degrees of freedom.

What is interesting is *how* the parameters approach that value: the average shortest path decreases more rapidly than the clustering coefficient at the beginning, after which it *stabilizes* and the clustering coefficient begins to decrease. A researcher using the Watts–Strogatz model is typically interested in reproducing two key properties observed in real-world networks: a high clustering coefficient and a small average shortest path. For probabilities close to zero, we observe both a high average shortest path and a high clustering coefficient, whereas for probabilities close to one, both values are low. In the intermediate range between zero and one, the average shortest path decreases quickly, while the clustering coefficient begins to drop only at higher probabilities. Thus, the characteristic properties promised by the Watts–Strogatz model are clearly reflected in the behavior of the plot within this range (we get high clustering coefficient and small average shortest path with most of the probabilities).

4.2 Erdős–Rényi model

Now we turn to the Erdős–Rényi model and plot the average shortest path as a function of the network size (recall that, as explained before, we choose the probability as a function of n in order to ensure connectivity with high probability [2]). In Figure [2], we observe that the average shortest path grows roughly logarithmically with n, which is consistent with theoretical results showing that the Erdős–Rényi model is characterized by having a small average shortest path. This roughly logarithmic growth with respect to the network size confirms the property that the Erdős–Rényi model exhibits a small-world behavior.

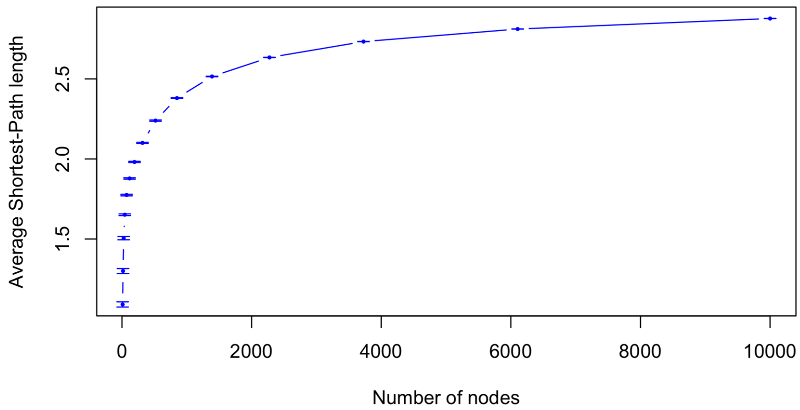

Figure 2: Average shortest path as a function of the network size. Erdős–Rényi model. The errorbars in the plot represent a 95% confidence interval for our estimate, following a t-student distribution with 29 degrees of freedom.

To obtain a visual check of the logarithm tendency, we expect to obtain a straight line when putting the x-axis in logscale. Though (Figure [3]), we can clearly see that the obtained tendency is not exactly linear, but slightly curved. This proves that the assumption of $\log(n)$ is not the exact formula, but that there are some additional corrections for the function that may be much more evident for larger values of n.

5 Conclusion

In this paper, we have shown that both models can exhibit the small-average shortest path property, with the Erdős–Rényi model being the simpler of the two. Moreover, the Watts–Strogatz model is able to capture not only this property but also the high clustering coefficient with a *good* choice of probability. Both models are well-known tools for modeling complex networks.

In summary, if the primary goal is to reproduce the small-average shortest path property, we recommend using the Erdős–Rényi model due to its simplicity. On the other hand, the Watts–Strogatz model provides a more powerful framework, as it simultaneously reproduces both key properties observed in real-world networks.

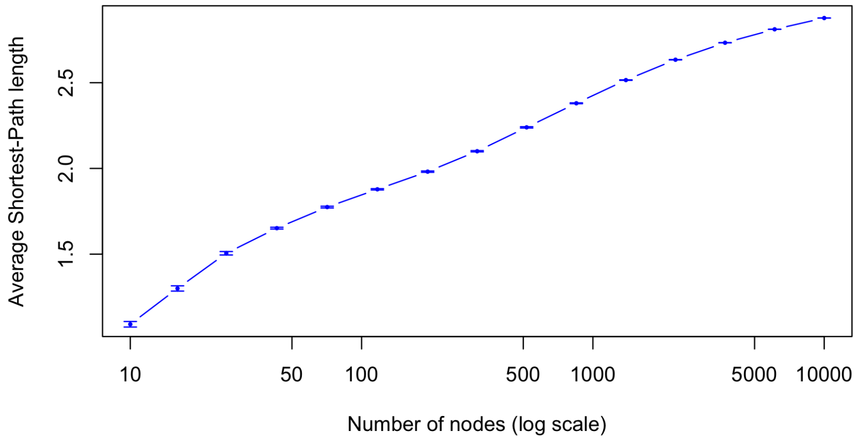

Figure 3: Log-scale of the x-axis of Figure [2]. The errorbars in the plot represent a 95% confidence interval for our estimate, following a t-student distribution with 29 degrees of freedom.

References

[1] P. Erdös and A. Rényi, "On random graphs i," *Publicationes Mathematicae Debrecen*, vol. 6, pp. 290–297, 1959.

[2] P. Erdős, A. Rényi, *et al.*, "On the evolution of random graphs," *Publications of the*, 1960.

[3] D. J. Watts and S. H. Strogatz, "Collective dynamics of 'small-world'networks," *nature*, vol. 393, no. 6684, pp. 440–442, 1998.