W 3

Logical Equivalence Laws:
When you are asked to use logical equivalence laws to show something, your answer must have the following form. Each step applies only a single law. Each step is justified - using the table on the right of slide 10 and the table on slide 11 (Fig. 7.11) of week6.part1.logicII.pdf.
Example:

$$\begin{aligned} & (A \wedge B) \rightarrow (C \vee \neg D) & \\ & \equiv \neg(A \wedge B) \vee (C \vee \neg D) & \text{(imp. elim.)} \\ & \equiv (\neg A \vee \neg B) \vee C \vee \neg D & \text{(De Morgans)} \end{aligned}$$

Resolution:
When running resolution, list all the sentences in the KB (numbered), then proceed with resolution steps, listing which sentences take part.
Example:

1	P	
2	Q	
3	$\neg P \vee R$	
4	$\neg Q \vee \neg R \vee S$	
5	$\neg S$	
6	R	(1,3)
7	$\neg R \vee S$	(2,4)
	$\vdots$	

1. (6 points) True or False. State the correctness of each statement. You may want to use the definitions of entailment on p. 214 and 222 of the book.

$\alpha \models \beta$ iff in every model is which α is true, β is also true.
For any sentences α and β, $\alpha \models \beta$ iff $(\alpha \rightarrow \beta)$ is valid.

(a) (1 point) False $\models$ True

True

(b) (1 point) True $\models$ False

False

(c) (2 points) $p \leftrightarrow q \models p \vee q$

False

$p \leftrightarrow q \models p \vee q$

$\equiv (p \rightarrow q) \wedge (q \rightarrow p) \models p \vee q$

$\equiv (\neg p \vee q) \wedge (\neg q \vee p) \models p \vee q$

when p and q are all False, it means True $\models$ False, which is False

(d) (2 points) $(p \vee q) \wedge \neg(p \rightarrow \neg q)$ is satisfiable.

True

$(p \vee q) \wedge \neg(p \rightarrow \neg q)$

$\equiv (p \vee q) \wedge \neg(\neg p \vee \neg q)$

$\equiv (p \vee q) \wedge (p \wedge q)$

when p is True and q is True, it is satisfiable, so it's True

2. (5 points) Consider a world that has 4 propositions p, q, r, s. How many models are there for the following sentences?

(a) (1 point) $p \wedge r$

4. We keep $P = r = True$, and no matter how r and s change, $P \wedge r$ is True.
Therefore the total models should be 2×2 by r and s values.

(b) (2 points) $(p \wedge r) \vee (q \wedge \neg s)$

when $(P \wedge r) = True$, there are 4 models
when $(q \wedge \neg s) = True$, there are also 4 models
there ~~are~~ is a same model in these 8 models, $p = r = q = \neg s = True$
Therefore, the total models are $4 + 4 - 1 = 7$

(c) (2 points) $(q \leftrightarrow s) \vee (p \rightarrow r)$

There are two models of q and s when $(q \leftrightarrow s)$ is false.
There is only one model of p, r when $(p \rightarrow r)$ is false.
There are two models of q, s, p, r for $(q \leftrightarrow s) \vee (p \rightarrow r)$ is false.
There are $2^4 = 16$ models for all the combinations of q, s, p, r.
Therefore, there are $16 - 2 = 14$ models for $(q \leftrightarrow s) \vee (p \rightarrow r)$ are True.
The answer is 14.

3. (8 points) Determine whether each of the following sentences is valid, unsatisfiable, or neither. Verify the determination using both equivalence rules (Method 1) and a truth table (Method 2).

(a) (2 points) $p \to p$

$p \to p$

$\equiv \neg p \vee p$ (imp. elim.)

$\equiv T$ (Negation laws)

Truth Table:

p	¬p	¬p∨p
T	F	T
F	T	T

therefore, the determination is valid.

(b) (6 points) $(p \to q) \to (\neg s \to p)$

$(p \to q) \to (\neg s \to p)$

$\equiv \neg(p \to q) \vee (\neg s \to p)$ (imp. elim.)

$\equiv \neg(\neg p \vee q) \vee (s \vee p)$ (imp. elim.)

$\equiv (p \wedge \neg q) \vee (s \vee p)$ (De Morgan's law)

$\equiv (p \wedge \neg q) \vee s \vee p$ (associativity of $\vee$)

$\equiv (p \wedge \neg q) \vee p \vee s$ (cummutative law)

$\equiv p \vee s$ (absorbtion laws)

Truth Table:

p	s	p∨s
T	F	T
T	T	T
F	F	F
F	T	T

Therefore, the determination is not valid and unsatisfitable neither. It is satisfiable.

4. (5 points) Let a KB consist of the following sentences: Let a KB consist of the following sentences:

1 p
2 $p \rightarrow q$
3 $(p \rightarrow q) \rightarrow (q \rightarrow r)$

Conver the knowledge base to CNF.

1. p is p

2. $p \rightarrow q \equiv \neg p \vee q$

3. $(p \rightarrow q) \rightarrow (q \rightarrow r) \equiv \cancel{\neg p \vee q}\ \neg(p \rightarrow q) \vee (q \rightarrow r)$ (impl. elim.)

$\equiv \neg(\neg p \vee q) \vee (\neg q \vee r)$ (imp. elim.)

$\equiv (p \wedge \neg q) \vee (\neg q \vee r)$ (De Morgan)

$\equiv [(p \wedge \neg q) \vee \neg q] \vee r$ (associativity of $\vee$)

$\equiv [(p \vee \neg q) \wedge (\neg q \vee \neg q)] \vee r$ (distributivity of $\vee$ over $\wedge$)

$\equiv (p \vee \neg q \vee r) \wedge (\neg q \vee r)$ (distributivity of $\vee$ over $\wedge$)

KB: 1. p

2. $\neg p \vee q$

3. $p \vee \neg q \vee r$

4. $\neg q \vee r$

5. (6 points) Resolution

Let a KB consist of the following sentences:

1 $\neg(P \land \neg Q) \lor \neg(\neg S \land \neg T)$
2 $\neg(T \lor Q)$
3 $U \rightarrow (\neg T \rightarrow (\neg S \land P))$

Prove $\neg U$ using resolution refutation.

Below is the KB converted into CNF.

(1) $\neg P \lor Q \lor S \lor T$
(2) $\neg T$
(3) $\neg Q$
(4) $\neg U \lor T \lor \neg S$
(5) $\neg U \lor T \lor P$

Show the resolution method; use one rule at a time, justifying each step.

To prove $\neg U$, show $KB \land U$ is unsatisfiable

$\neg P \lor Q \lor S \lor T$	1
$\neg T$	2
$\neg Q$	3
$\neg U \lor T \lor \neg S$	4
$\neg U \lor T \lor P$	5
U	6, (assumption)
$\neg P \lor S$	7, (1, 2, 3)
$\neg S$	8, (2, 4, 6)
P	9, (2, 5, 6)
$\{\}$	10, (7, 8, 9)

therefore, contradiction, $KB \land \neg(\neg U)$ is unsatisfiable, $\neg U$ is True.

6. (20 points) Resolution

A famous logic problem using the following propositional symbols to construct a KB:

Y	"unicorn is mythical"
I	"unicorn is immortal"
M	"unicorn is mammal"
H	"unicorn is horned"
G	"unicorn is magical"

(a) (4 points) Translate English sentences to logic. Think of the paragraph as the following sentences:

(i) If the unicorn is mythical, then it is immortal.
(ii) If the unicorn is not mythical, then it is a mortal mammal.
(iii) If the unicorn is either immortal or a mammal, then it is horned.
(iv) If the unicorn is horned, then it is magical.

(i) $Y \rightarrow I$

(ii) $\neg Y \rightarrow M$

(iii) $(I \vee M) \rightarrow H$

(iv) $H \rightarrow G$

(b) (4 points) Convert (a) to KB in CNF.

$\neg Y \vee I$
$Y \vee M$
$\neg I \vee H$
$\neg M \vee H$
$\neg H \vee G$

(c) (6 points) Can you prove the unicorn is magical? Answer the question by adding the negation of the query to the KB and run resolution.

To prove unicorn is magical, we can show $KB \wedge$ unicorn is not ~~mg~~ magical is unsatisfied

Show $KB \wedge \neg G$ is unsatisfiable

$\neg Y \vee I$	1
$Y \vee M$	2
$\neg I \vee H$	3
$\neg M \vee H$	4
$\neg H \vee G$	5
$\neg G$	6. assumption
$\neg H$	7. (5, 6)
$\neg I$	8. (3, 7)
$\neg M$	9. (4, 7)

$\Rightarrow$

$\neg Y$	10. (1, 8)
Y	11. (2, 9)
$\{\}$	12. (10, 11)

Therefore, G is True, the unicorn is magical

(d) (6 points) Can you prove the unicorn is horned? Answer the question by adding the negation of the query to the KB and run resolution.

To prove the aunicorn is horned, we can show $KB \wedge \neg H$ is unsatisfiable

$\neg Y \vee I$	1
$Y \vee M$	2
$\neg I \vee H$	3
$\neg M \vee H$	4
$\neg H \vee G$	5
$\neg H$	6, (assumption)
$\neg I$	7, (3, 6)
$\neg M$	8, (4, 6)
$\neg Y$	9, (1, 7)
Y	10, (2, 8)
$\{\}$	11, (9, 10)

There is a contradiction.
Therefore, the H is True, unicorn is horned.