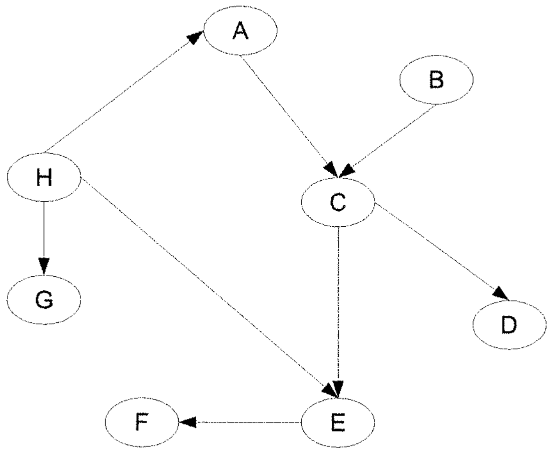
W 5

1. (6 points) Consider the following Bayesian network with variables A, B, C, D, E, F, G, H. Categorize the following as d-separated or d-connected with respect to the given BN.



(a) $A \perp\!\!\!\perp F \mid \{\ \}$ or $A \perp\!\!\!\perp F$

(b) $H \perp\!\!\!\perp D \mid A$

(c) $H \perp\!\!\!\perp D \mid \{A, E\}$

(d) $B \perp\!\!\!\perp G \mid \{\ \}$

(e) $B \perp\!\!\!\perp G \mid \{D\}$

(f) $A \perp\!\!\!\perp B \mid \{E, F\}$

(a) d-connected

(b) d-separated

(c) d-connected

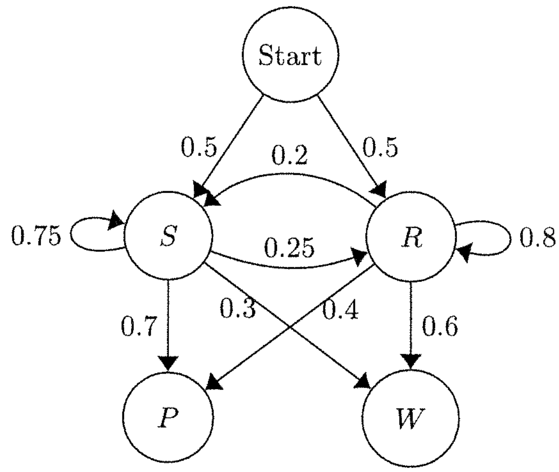
(d) d-separated

(e) d-connected

(f) d-connected

2. (13 points) Hidden Markov Models

Consider the following simple model of the weather in November using an HMM. There is a unobserved variable, X_t, indicating that the weather today can be either snowing (S) or raining (R), and you observe, E_t, what the majority of people are wearing: parkas (P) or wellington boots (W). This is modeled in the following HMM:



(a) (3 points) Compute $P(X_1 = S, X_2 = R, X_3 = R, X_4 = S)$

$P(X_1=S, X_2=R, X_3=R, X_4=S) = 0.5 \times 0.25 \times 0.8 \times 0.2$
$= 0.02$

(b) (4 points) Compute $P(X_1 = S \mid E_1 = P)$

$$P(X_1=S|E_1=P) = \frac{P(E_1=P|X_1=S)\cdot P(X_1=S)}{P(E_1=P)} = \frac{0.7 \times 0.5}{0.5\times0.7+0.5\times0.4} \approx 0.6364$$

(c) (6 points) Compute $P(E_1 = W, E_2 = W)$

$P(E_1=W) = 0.5 \times 0.3 + 0.5 \times 0.6 = 0.45$

$P(E_2=W) = \sum_{X_1, X_2} P(E_2=W \mid E_1=W)\, T(X_1, X_2) P(E_1=W \mid X_1)$

$= 0.5 \times 0.75 \times 0.3 + 0.5 \times 0.25 \times 0.6 + 0.5 \times 0.8 \times 0.6 + 0.5 \times 0.2 \times 0.3$

$= 0.4575$

$P(E_1=W, E_2=W) = 0.45 \times 0.4575 = 0.205875$

3. (19 points) Hidden Markov Models II

You are modelling the state of your health {h - *healthy*, c - *coming down with something*, or u - *unwell* } . Each day, you can observe whether you are sneezing s or not n to infer your state of health. Use the following information to model your health.

Initial Dist.

State	$P(X_1)$
h	0.9
c	0.1
u	0.0

Transition Model

X_t	X_{t+1}	$P(X_{t+1} \mid X_t)$
h	h	0.6
h	c	0.2
h	u	0.2
c	h	0.3
c	c	0.2
c	u	0.5
u	h	0.7
u	c	0.2
u	u	0.1

Emission Distribution

X_t	E_t	$P(E_t \mid X_t)$
h	n	0.8
c	n	0.3
u	n	0.5
h	s	0.2
c	s	0.7
u	s	0.5

(a) (2 points) What is $P(X_1 = h, X_2 = c, X_3 = u)$?

$0.9 \times 0.2 \times 0.5 = 0.09$

(b) (2 points) What is $P(X_1 = u, X_2 = c, X_3 = h)$?

0

(c) (4 points) On the first day, you did not sneeze, $E_1 = n$. What is the probability you are *coming down with something* given this observation?

$$P(X_1 = c \mid E_1 = n) = \frac{P(E_1 = n \mid X_1 = c)\, P(X_1 = c)}{P(E_1 = n)}$$

$$P(E_1 = n) = \sum_{X_1} P(E_1 = n \mid X_1)\, P(X_1)$$

$$= 0.8 \times 0.9 + 0.3 \times 0.1 + 0.5 \times 0$$
$$= 0.75$$

$$P(X_1 = c \mid E_1 = n) = \frac{0.3 \times 0.1}{0.75} = 0.04$$

(d) (8 points) You observe not sneezing on day 1 and sneezing on day 2. What's the probabilty of *coming down with something* given the sequence of observations?

$$P(E_1=n \mid X_1=h) = 0.8 \times 0.9 = 0.72$$

$$P(E_1=n \mid X_1=c) = 0.3 \times 0.1 = 0.03$$

$$P(E_1=n \mid X_1=u) = 0$$

$$P(E_2=s, E_1=n \mid X_2) = \sum_{X_1, X_2} P(E_2=s \mid X_2) \cdot P(X_2 \mid X_1) \cdot P(E_1=n \mid X_1)$$

$$= 0.2 \times [(0.6 \times 0.9 \times 0.8) + (0.1 \times 0.3 \times 0.3) + 0] + 0.7 \times [(0.9 \times 0.8 \times 0.2) + (0.1 \times 0.3 \times 0.2) + 0]$$
$$+ 0.5 \times [(0.9 \times 0.8 \times 0.2) + (0.1 \times 0.3 \times 0.5) + 0] = 0.0882 + 0.105 + 0.0795 = 0.2727$$

$$P(E_2=s, E_1=n \mid X_2=c) = 0.7 \times [(0.9 \times 0.8 \times 0.2) + (0.1 \times 0.3 \times 0.2) + 0] = 0.105$$

$$P(X_2=c \mid E_1=n, E_2=s) = \frac{0.105}{0.2727} \approx 0.385$$

(e) (3 points) You can confirm that you were healthy on days one and two. What is the probability that you will be unwell on day 3?

$$P(X_3=u \mid X_2=h) = P(X_{t+1}=u \mid X_t=h) = 0.2$$

4. (12 points) Markov Decision Process

A student selects fruit from a bowl of *Apples* - A and *Oranges* - O. At each state she has either an *Apple* or *Orange* in her hand. There is only one action, *Swap* - S, when she returns the fruit in her hand to the bowl, and selects a new piece of fruit, where $P(A) = 0.7$. The MDP has the following information:

States: A - apple, O - orange
Actions: S - swap
Start state: A - apple
there are no terminal states

Let $R(A, S, O) = 3$ and $R(O, S, A) = 2$, with all other rewards $= 0$. Assume $\gamma = 0.5$

(a) (12 points) Run value iteration for this MDP for three iterations and fill in the value estimates in the table. Show your work.

i	$V_i^*(A)$	$V_i^*(O)$
0	0	0
1	0.9	1.4
2	1.425	1.925
3	1.6875	2.1875

(b) (3 points (bonus)) What are $V^*(A)$ and $V^*(O)$? *Hint: solve the Bellman equations*

$V^*(A)$ and $V^*(0)$ means converge, so $V_k(A) = V_{k+1}(A)$, $V_k(0) = V_{k+1}(0)$

Then: $V_{k+1}(A) = T(A,S,A)[R(A,S,A) + \gamma V_k(A)] + T(A,S,0)[R(A,S,0) + \gamma V_k(0)]$ ①

$V_{k+1}(0) = T(0,S,A)[R(0,S,A) + \gamma V_k(A)] + T(0,S,0)[R(0,S,0) + \gamma V_k(0)]$ ②

$\because V_{k+1}(A) = V_k(A)$, $V_{k+1}(0) = V_k(0)$

$\therefore$ ① $\Rightarrow V_k(A) = 0.7[0 + 0.5V_k(A)] + 0.3[3 + 0.5V_k(0)]$

② $\Rightarrow V_k(0) = 0.7[2 + 0.5V_k(A)] + 0.3[0 + 0.5 \times V_k(0)]$

from ① $\Rightarrow$ $\cancel{V_k(0) =}$ $V_k(0) = \dfrac{0.65V_k(A) - 0.9}{0.15}$ ③

plug ③ into ② $\Rightarrow 0.85 \times \dfrac{0.65V_k(A) - 0.9}{0.15} = 1.4 + 0.35V_k(A)$

$\Rightarrow V_k(A) = 1.95$ ④

plug ④ into ③ $V_k(0) = 2.45$

The answer is $V^*(A) = 1.95$ $V^*(0) = 2.45$