

2 **Electrostatics for Physics Bowl**

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13 1 Introduction

14 This aims to be a comprehensive guide of introductory high-school-level electrostatics,
 15 with an emphasis on important material relevant to Physics Bowl. Although knowledge of
 16 calculus is not required, some sections may include calculus formulas to allow for a more
 17 rigorous treatment of the subject without loss of continuity. Electromagnetism is a broad
 18 and complex field of study in physics that has likely not been done justice with a brief
 19 overview, so I have split it into multiple meetings. In those future meetings, we'll cover
 20 other topics, such as circuitry and magnetism. For the sake of Physics Bowl, there will be
 21 a sacrifice in rigor in the interest of brevity. I hope you enjoy this review. Please let me
 22 know if you have any questions, comments, or suggestions by email!

23 2 Charge and Review from Electrochemistry

Electric phenomena depend on **charge**. There are two kinds of electric charge, *positive*
 and *negative*. Charge is caused by *protons* (positive) and *electrons* (negative), which each
 carry the *fundamental unit charge* denoted by e and $-e$ respectively. Charge is neither
 created nor destroyed. The standard unit of charge is the coulomb (C), where

$$e = 1.60 \times 10^{-19} \text{ C}$$

24 Charge is represented by the symbol q ; a macroscopic object has charge

$$q = N_p e - N_e e. \tag{2.1}$$

25 where N_p and N_e are the number of protons and electrons in the object. Objects with $q > 0$
 26 are said to be positively charged, $q < 0$ are negatively charged, and $q = 0$ are electrically
 27 neutral. Charges of the same kind repel, opposite charges attract. Small neutral objects
 28 are attracted to a charge of either sign due to charge polarization, the slight separation
 29 of positive and negative charges in a neutral object. The attractive force on the electrons

at the surface of the neutral object closer to the charged object is slightly larger than the repulsive force on the ions at the bottom. **Coulomb's law** describes this force. If two charged particles with charges q_1 and q_2 are a distance r apart, each particle exerts a force on the other of magnitude

$$F = \frac{k|q_1||q_2|}{r^2}, \quad (2.2)$$

where the *electrostatic constant*

$$k \approx 8.99 \times 10^9 \text{ N m}^2/\text{C}^2.$$

Another useful constant directly related to the electrostatic constant which will show up later is the *permittivity constant*

$$\epsilon_0 = \frac{1}{4\pi k} \approx 8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2.$$

Note that Coulomb's law applies only to point charges, which are idealized material objects with charge and mass but no size or extension.¹ Also, electric forces—like any other force—are vectors that add up to the *net electrostatic force*. The directions of these vectors are determined by whether the objects are the same sign or not; repelling objects will have force vectors that are in the direction opposite to the radius vector, and vice versa. The state in which all charges within a system are at rest is referred to as *electrostatic equilibrium*.

Protons are extremely tightly bound to their atomic nuclei. In Physics Bowl and in most practical applications, electrons are the only charged particles that can be transferred from one atom to another. The transfer of electrons from one object to another is referred to as **charging**.

Conductors are materials in which electrons move easily; **insulators** are materials in which electrons are immobile. Many metals are conductors because valence electrons are weakly bound to the nuclei; as atoms come together to form a solid, these outer electrons become detached from their parent nuclei and are free to wander about the entire solid, forming what is often referred to as a sea of electrons.

In an isolated conductor, any excess charge is on the surface of the conductor. When an object is physically connected to the earth through a conductor, it is said to be **grounded**; any excess charge will be shared with the earth, which is so massive that we can model this as a complete discharge.

3 Electric Fields

Just as a mass creates a gravitational field in the surrounding space, charges create an **electric field**.² The electric field created by a charge is

$$\vec{E} = \frac{kq}{r^2} \hat{r} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}, \quad (3.1)$$

¹If you wish to apply Coulomb's law to an object with size, the object must be split into infinitely small subsections with infinitely small charges, and summing up the forces through integration.

²Mathematically speaking, a field assigns a vector to every point in space; although at infinity the magnitudes of fields become very small, they are nonetheless present, so every point in space is affected by said fields.

where \hat{r} is the *unit vector* in the direction of the point of interest. This again makes the assumption that q is a point charge.

Charged particles interact via the electric field. Another way to express Coulombic force is in terms of the electric field. Suppose we have a charge q_1 in an electric field \vec{E} ; then, the electrostatic force on q_1 is

$$\vec{F}_{\text{on } q_1} = q_1 \vec{E}. \quad (3.2)$$

In the same way that we can vectorially add forces, we can do the same with electric fields to find the *net electric field*; in other words, electric fields obey the **principle of superposition**.

Two opposite charges with a slight separation between them form an **electric dipole**. If we consider an electric dipole formed by two charges q , $-q$ with distance s from each other, we say that the **dipole moment**

$$\vec{p} = qs \quad (3.3)$$

pointing from the negative to the positive charge. The direction of \vec{p} identifies the orientation of the dipole, while its magnitude determines the electric field strength. If we were to measure the electric field due to a dipole at radius $r \gg s$, we say it is the *far field* of the dipole. The equations for the far field are

$$\vec{E}_{\text{dipole}} = -\frac{1}{4\pi\epsilon_0} \frac{2\vec{p}}{r^3} \quad (3.4)$$

along the axis of the dipole and

$$\vec{E}_{\text{dipole}} = -\frac{1}{4\pi\epsilon_0} \frac{\vec{p}}{r^3} \quad (3.5)$$

in the bisecting plane.³ **There is no net force on a dipole in a uniform electric field.** For any *nonuniform* field, the net force on a dipole is toward the direction of the strongest field; because any finite-size charged object, such as a charged rod or a charged disk, has a field strength that increases as you get closer to the object, we conclude that **a dipole will experience a net force toward any charged object**.

Some problems may describe an electric field caused by a continuous charge distribution. **Linear charge density** is denoted as $\lambda = \frac{Q}{L}$ while **surface charge density** is $\eta = \frac{Q}{A}$.

The electric fields of some common charge distributions is worth memorizing. Even if you know calculus and can derive these formulas on your own, remember that Physics Bowl is 45 minutes long, and that you won't have the time to do complicated calculus manipulations.

For a uniformly charged disk of radius R , the electric field at a distance z along the axis of the center of the disk is

$$(\vec{E}_{\text{disk}})_z = \frac{\eta}{2\epsilon_0} \left[1 - \frac{z}{\sqrt{z^2 + R^2}} \right]. \quad (3.6)$$

³The derivation of these formulas from the previously mentioned formulas is trivial, and I have graciously left it as an exercise.

87 By the binomial approximation $(1 + x)^n \approx 1 + nx$ if $x \ll 1$,

$$(\vec{E}_{disk})_z \approx \frac{1}{4\pi\epsilon_0} \frac{Q}{z^2} \quad (3.7)$$

88 if $z \gg R$. For an infinite plane of charge,

$$\vec{E}_{plane} = \frac{|\eta|}{2\epsilon_0}. \quad (3.8)$$

89 Notice how the electric field vector is independent of radius; the electric field is constant
90 for all points above an infinite plane. For an infinite line of charge and a constant λ ,

$$\vec{E}_{line} = \frac{1}{4\pi\epsilon_0} \frac{2|\lambda|}{r}. \quad (3.9)$$

91 The latter two are valid approximations if the plane or line of charge is comparatively
92 large/long compared to the radius. Physics problems will often do this; they will often
93 emphasize that the object is "large" or that the radius is very small compared to the size.

94 For a sphere for charge of radius r , we use the **shell theorem**, which states that
95 *a spherically symmetric body affects external objects electrostatically as though all of its*
96 *charge were concentrated at a point at its center*; thus,

$$\vec{E}_{sphere} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} \quad (3.10)$$

97 for $r \geq R$. The shell theorem also states that *if the body is a spherically symmetric shell*
98 *(i.e., a hollow ball), no net electrostatic force is exerted by the shell on any object inside,*
99 *regardless of the object's location within the shell.* If $r > R$, we can model the sphere into
100 two different sections: an inner sphere of radius r and a shell of inner radius r and outer
101 radius R . We then say that the electric field at this point is equivalent to an electric field
102 due to a sphere of a certain charge, which you can easily determine for a constant charge
103 density. Note that the shell theorem is also applicable for gravitational fields.

104 A common device is the **parallel plate capacitor**, which consists of two metal plates
105 called electrodes, one with charge $+Q$ and the other with $-Q$, placed face-to-face a distance
106 d apart. These play important roles in many electric circuits. As for its electric field, for
107 a parallel-plate capacitor with electrodes of surface area A

$$\vec{E}_{capacitor} = \frac{\eta}{\epsilon_0} = \frac{Q}{\epsilon_0 A} \quad (3.11)$$

108 on the *inside* of the capacitor between the two electrodes; that is, the electric field is
109 constant. We make this approximation for an ideal capacitor because the distance between
110 the plates is much smaller than their areas, so we can apply equation (3.8) for each plate.
111 On the outside, the electric field is zero for an ideal capacitor.⁴

112 A final note on the electric field, which we will circle back to later in our study of
113 magnetism, is the **Maxwell-Faraday** equation,

⁴A real capacitor may have a very weak field outside called a fringe field, but for the purposes of this exam, we can ignore this.

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t}, \quad (3.12)$$

114 where $-\frac{\partial B}{\partial t}$ is the rate of change of the magnetic field. In the context of electrostatics, we
 115 do not have a changing magnetic field, so this reduces to

$$\nabla \times \vec{E} = 0, \quad (3.13)$$

116 which tells us that **the electric field is conservative**,⁵ in our context, this means that
 117 the amount of work required to move a charge in an electric field from one point to another
 118 is the same for those two points regardless of the path. We will touch on the implications of
 119 this in section 5. Otherwise, if the right side of equation (3.12) is nonzero, the electric field is
 120 *non-conservative*, which leads us to a more advanced field of physics called electrodynamics,
 121 which will not appear on Physics Bowl.

122 4 Gauss' Law

123 Gauss' law can be a powerful tool for determining the magnitude fields, especially for
 124 *symmetrical*⁶ charge distributions, such as a very long charged cylinder. Namely, there are
 125 three fundamental symmetries that appear frequently in electrostatics (see figure 1).

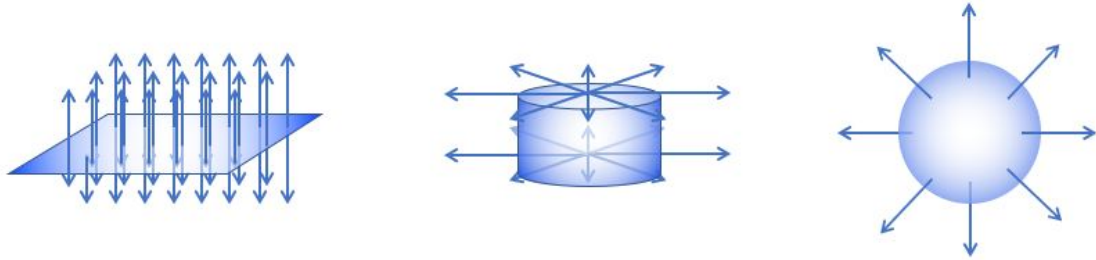


Figure 1. The three fundamental symmetries; planar, cylindrical, spherical (left to right).

The key to understanding Gauss' law is understanding electric flux. Although electric field does not flow like a fluid does, we can make an analogy. Suppose we have a thin, circular ring. We will define a unit vector \hat{n} perpendicular to this surface. Now suppose there is wind blowing at a constant velocity \vec{v} (its direction and speed are fixed). If we were to hold the ring such that \hat{n} is perpendicular to \vec{v} , no air will pass through the ring; however, if \hat{n} is parallel to the wind, the amount of air flowing through the ring will be maximized. We can model the velocity vector as two distinct components: one that is perpendicular to \hat{n} and one that is parallel to \hat{n} . If \vec{v} makes an angle θ with \hat{n} , we get

⁵Formally, the electric field is a conservative vector field, which has the property that its line integral is path independent. Since a conservative vector field is irrotational, this means it has vanishing curl in three dimensions, as is the case for $-\frac{\partial B}{\partial t} = 0$. An irrotational vector field is necessarily conservative provided that the domain is *simply connected* if you are interested, there is some topology behind that.

⁶The concept of symmetry is very deep, both in math and science. I will not go too far in depth for the sake of brevity, but a study of symmetry is enlightening and beautiful.

that $v_{\perp} = v \sin \theta$ and $v_{\parallel} = v \cos \theta$. Because the component in the perpendicular direction does not contribute to the air passing through, we conclude that the amount of air passing through is proportional to the parallel vector. It turns out that we can similarly define the "amount" of a constant electric field that "flows" through an area A as the **electric flux**,

$$\Phi_e = E_{\perp} A = EA \cos \theta.$$

126 We define the **area vector** \vec{A} as having a magnitude of the area and a direction perpen-
127 dicular to the surface (that is, $\vec{A} = A\hat{n}$). We then say that in a constant electric field,

$$\Phi_e = \vec{E} \cdot \vec{A}. \quad (4.1)$$

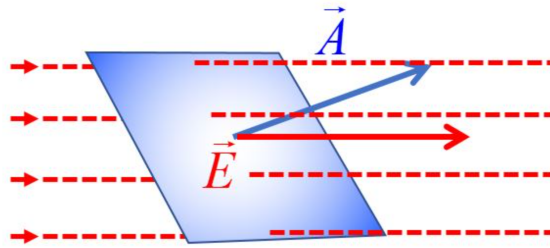


Figure 2. Visualization of flux with electric field vectors \vec{E} and area vector \vec{A} .

128

129 Electric fluxes through a surface can be calculated as the sum of fluxes through smaller
130 pieces of the surface. We thus define

$$\Phi_e = \int_S \vec{E} \cdot d\vec{A}. \quad (4.2)$$

131 Suppose we surround a region of space with a *closed surface*, which clearly divides space
132 into distinct inside and outside regions. Within the context of electrostatics, a closed
133 surface through which an electric field passes is called a **Gaussian surface**⁷. For electric
134 field passing through such a Gaussian surface, we used a closed surface integral,

$$\Phi_e = \oint \vec{E} \cdot d\vec{A}. \quad (4.3)$$

135 This brings us to the crux of this section. **Gauss' Law** states that for any closed surface
136 enclosing total charge Q_{in} ,

$$\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0}. \quad (4.4)$$

137 This easily becomes

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (4.5)$$

138 for a charge distribution of density ρ .

⁷More generally, a Gaussian surface is a closed surface in three-dimensional space through which the flux of a vector field is calculated

Now, the question remains: how can we apply this? Well, *if and only if* the shape of the electric field can be guessed by symmetry and superposition, we can *construct* imaginary Gaussian surfaces to solve problems. Gaussian surfaces are not physical surfaces; they are mathematical surfaces in the space surrounding charge. For example, we can easily derive equation (3.9) by constructing a cylinder of radius R and length l coaxial with the line of charge. Because the line of charge is cylindrically symmetrical, its field does not have any component that would pass through the circular sides of our cylinder. The area of the part of the surface where the field does pass is $2\pi rl$. The enclosed charge is $\lambda \cdot l$. Therefore, $\vec{E} \cdot 2\pi rl = \lambda l / \epsilon_0$, and we can simplify to obtain our result. This tells us the electric field strength at any arbitrary section along the cylinder's curved surface.

Gauss' law is always true, but it is not always useful. In the application of Gauss' law where we solve for \vec{E} , if the magnitude of the electric field is not the same along the surface, and we cannot take \vec{E} outside the integral. A guide to the type of Gaussian surface you should construct:

1. *Spherical symmetry*: Make your Gaussian surface a concentric sphere.
2. *Cylindrical symmetry*: Make your Gaussian surface a coaxial cylinder.
3. *Plane symmetry*: Use a Gaussian "pillbox" that straddles the surface. This is like a slab that's parallel to the plane of symmetry.

5 Electric Potential and Capacitors

By equation (3.13), electric force is a conservative force in electrostatics. Working with this assumption,⁸ we define the electric potential energy of charge q . In a capacitor,

$$U_{elec} = qEs, \quad (5.1)$$

where s is the distance of the charge to the negative electrode; for two point charges,

$$U_{elec} = \frac{Kq_1q_2}{r} = \frac{1}{4\pi\epsilon_0} \frac{q_1q_2}{r}, \quad (5.2)$$

and for a dipole,

$$U_{elec} = -\vec{p} \cdot \vec{E}. \quad (5.3)$$

If we were to take an arbitrary charge q , its electric potential energy is due to its interaction with other charges. We define the **electric potential** V to be

$$V \equiv \frac{U_{q+sources}}{q}. \quad (5.4)$$

Formally, this is the *amount of work energy needed per unit of electric charge to move the charge from a reference point to specific point*. Typically, the reference point is earth or at infinity, although any point can be used. In most cases, you will find the reference point

⁸Electric potential exists outside of electrostatics, but for the sake of simplicity I introduce it in a digestible way, unlike its generalization to electrodynamics.

to be "ground" (as in "grounded"; see section 2). The unit of potential is the **Volt**, which is equivalent to Joules per Coulomb. A positive charge slows down as it moves into a region of higher electric potential. It is customary to say that the particle moves through a **potential difference** $\Delta V = V_f - V_i$. The potential difference between two points is often called the **voltage**. Since the work in a conservative field is

$$W = - \int_i^f \vec{F} \cdot d\vec{s},$$

164 we can formally define

$$\Delta V = - \int_i^f \vec{E} \cdot d\vec{s}, \quad (5.5)$$

165 which essentially tells us that we can find the potential difference between two points if
166 we know the electric field. Since the electric field is conservative in this context, we can
167 integrate any arbitrary path from i to f and get the same result due to the properties of
168 conservative vector fields.

169 From equation (5.5),

$$\vec{E} = -\nabla V; \quad (5.6)$$

170 This means that the rate of change of potential with respect to positional displacements is
171 equal to the electric field.

172 In the case electrodynamics—which is beyond the scope of this discussion—note that
173 equation (5.5) is invalid, and equation (5.6) is modified to include the complicated quantity
174 called the *magnetic potential vector*.

175 The name "potential" is a hideous misnomer because it inevitably reminds you of po-
176 tential *energy*; this is particularly insidious because there *is* a connection between potential
177 and potential energy as we have shown. *Potential and potential energy are completely dif-*
178 *ferent terms.*

179 **Potential is a scalar quantity.** This is one of the reasons why the potential formu-
180 lation can be useful—it reduces a vector problem to a scalar one, in which there is no need
181 to fuss with components. Also, **potential obeys the superposition principle.** One
182 of the fascinating translations of electric field to potential is what happens to Gauss' law
183 when you express it in terms of potential;

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}, \quad (5.7)$$

184 which is called **Poisson's equation**; from this, we get that for a space with no charge
185 ($\rho = 0$),

$$\nabla^2 V = 0, \quad (5.8)$$

186 known as **Laplace's equation**. Although these are most definitely not going to be tested
187 on the Physics Bowl exams, it is, in my opinion, one of the most beautiful results in physics.

188 **Kirchhoff's loop law** states that the sum of all the potential differences encountered
189 while moving around a loop or closed path is zero. We will cover this extensively in our
190 study of circuits.

191 **The electric field is zero at any interior point of a conductor in electro-**
 192 **static equilibrium**; this implies that any two points inside a conductor in electrostatic
 193 equilibrium are at the same potential by equation (5.5). A surface over which the poten-
 194 tial is constant is called an **equipotential**. A conductor is an equipotential surface, but
 195 equipotential surfaces need not be physical ones.

196 An electric potential difference is created by separating positive and negative charges.
 197 A nonelectrical process is needed to separate charges. The Van de Graaff generator in
 198 W102 separates charges mechanically. A common source of electric potential is the **battery**,
 199 which uses chemical reactions to separate charges. The voltage of the battery is the
 200 potential difference between the positive and negative terminals. If you connect multiple
 201 batteries end-to-end by their terminals, the resulting voltage will be equal to the sum of
 202 the potential differences across each of them.

203 For a capacitor with plates separated by distance d and potential difference ΔV_C ,

$$E = \frac{\Delta V_C}{d}. \quad (5.9)$$

204 The potential at a distance s from the negative plate is

$$V_{cap} = \frac{s}{d} \Delta V_C \quad (5.10)$$

205 The relationship between the charge on the capacitor and its potential difference defines
 206 its **capacitance**, which is measured in **Farads**:

$$C = \frac{q}{\Delta V_C}. \quad (5.11)$$

207 A capacitor is charged by connecting it to a battery. When it is fully charged, the positive
 208 plate will be at the same potential as the positive terminal of the battery; that is, it will
 209 charge until $\Delta V_C = \Delta V_{bat}$. In a circuit, the "effective capacitance" of a system of capacitors
 210 in *parallel* is

$$C_{eq} = C_1 + C_2 + C_3 + \dots \quad (5.12)$$

211 where **each capacitor will have the same potential difference**. In series,

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots \quad (5.13)$$

212 where **each capacitor will have the same charge**. The visualizations of series vs
 213 parallel capacitors in a circuit can be found in figure 3.

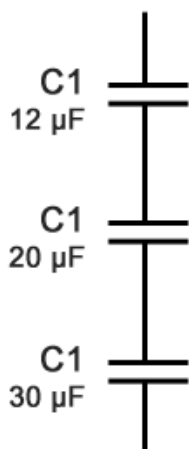
214 The energy stored in a capacitor will be

$$U_C = \frac{Q^2}{2C} = \frac{1}{2} C (\Delta V_C)^2 \quad (5.14)$$

215 Let's define the **energy density** of an electric field to be the ratio of energy stored and
 216 the volume in which it is stored; for the capacitor,

$$u_c = \frac{U_C}{Ad} = \frac{\epsilon_0}{2} E^2. \quad (5.15)$$

Series Capacitors



Parallel Capacitors

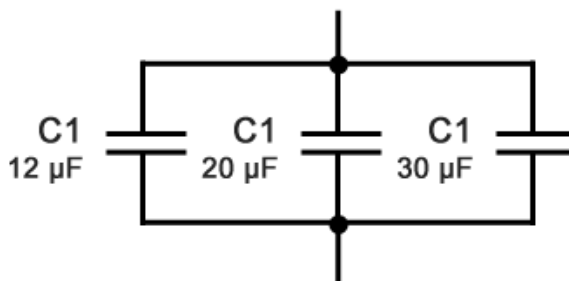


Figure 3. Series vs. Parallel capacitors. The same convention is used when describing any other components in a circuit.

Later, we will cover how the energy transported by a light wave is the energy of electric fields and magnetic fields.

An insulator in an electric field is called a **dielectric**. When we put a dielectric between the two plates of a capacitor, we call it a *dielectric-filled capacitor*. The charge on the capacitor plates does not change. However, we can use two principles that we discussed earlier—superposition and polarization—to understand the properties of dielectric-filled capacitors. An insulating material becomes *polarized* in an external electric field. Electric dipoles will be created, and their alignment in the electric field—the polarization of the material—produces an excess charge on one surface and an excess negative charge on the other. The insulator as a whole is still neutral, but the external electric field separates positive and negative charges. The two sheets of induced charge on each side of the insulator act just like the two charged plates of a parallel-plate capacitor. The **induced electric field** is

$$E_{\text{induced}} = \frac{\eta_{\text{induced}}}{\epsilon_0} \quad (5.16)$$

inside the insulator, and zero outside the insulator. Because the induced electric field acts opposite to the capacitor's electric field, **the electric field is weakened**. We define the **dielectric constant** to be the ratio between the electric field in the capacitor without a dielectric and the electric field with a dielectric:

$$\kappa = \frac{E_0}{E}. \quad (5.17)$$

From this,

$$C = \kappa C_0, \quad \Delta V_C = \frac{(\Delta V_C)_0}{\kappa}. \quad (5.18)$$

235 The electric potential of a point charge is

$$V_{point} = \frac{1}{4\pi\epsilon_0} \frac{q}{r}. \quad (5.19)$$

236 For a charged sphere of radius r

$$V_{point} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}. \quad (5.20)$$

237 for $r \geq R$. This is a familiar result (see equation (3.10) for shell theorem). Let's call the
238 potential at $r = R$ (the surface of the sphere) V_0 . Then,

$$V_{sphere} = \frac{R}{r} V_0. \quad (5.21)$$

239 outside the sphere.