Rotational Motion and Celestial Mechanics
ROUND TWO

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1 Rotational Kinematics

In many ways, rotational motion can be thought of as analogous to translational motion. Common analogues include:

$$\text{Position } s \to \text{ Angle } \theta$$

$$\text{Velocity } v \to \text{ Angular velocity } \omega \left(\frac{d\theta}{dt}\right)$$

$$\text{Acceleration } a \to \text{ Angular acceleration } \alpha \left(\frac{d^2\theta}{dt^2}, \frac{d\omega}{dt}\right)$$

$$\text{Mass } m \to \text{ Moment of inertia } I \left(\sum Mr^2, \int r^2 dm\right)$$

$$\text{Momentum } p \to \text{ Angular momentum } L$$

$$\text{Force } F \to \text{ Torque } \tau$$

Many kinematics equations hold their form in their rotational analogues. For example,

$$s = v_0 t + \frac{1}{2}at^2 \to \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$F = ma = \frac{dp}{dt} \to \tau = I\alpha = \frac{dL}{dt}$$

$$p = mv \to L = I\omega$$

$$T = \frac{1}{2}mv^2 \to T = \frac{1}{2}I\omega^2$$

2 Mechanics in the Noninertial Frame

The relation $\mathbf{v} = \omega \times \mathbf{r}$ is a generalization of the relation $v = \omega r$. It is perhaps worth emphasizing that there is a corresponding relation for *any* vector fixed in the rotating body. For example, if $\mathbf{e}$ is a unit vector that rotates with the body, then its rate of change, as seen from the non-rotating frame, is $\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \omega \times \mathbf{e}$. If

$\mathbf{f}(t)$ is a vector function written $f_1(t)\hat{\mathbf{i}} + f_2(t)\hat{\mathbf{j}} + f_3(t)\hat{\mathbf{k}}$. Using the product rule of differentiation,

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} = \frac{\mathrm{d}f_1}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}f_2}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}f_3}{\mathrm{d}t}\hat{\mathbf{k}} + \frac{\mathrm{d}\hat{\mathbf{i}}}{\mathrm{d}t}f_1 + \frac{\mathrm{d}\hat{\mathbf{j}}}{\mathrm{d}t}f_2 + \frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t}f_3 \tag{1}$$

The rate of change of $\mathbf{f}$ observed in the rotating system is

$$\left(\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}\right)_r = \frac{\mathrm{d}f_1}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}f_2}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}f_3}{\mathrm{d}t}\hat{\mathbf{k}} \tag{2}$$

As for the time derivatives of the unit vectors, by the linearity and distributive property of the cross product:

$$\frac{\mathrm{d}\hat{\mathbf{i}}}{\mathrm{d}t}f_1 + \frac{\mathrm{d}\hat{\mathbf{j}}}{\mathrm{d}t}f_2 + \frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t}f_3 = \omega \times \mathbf{f} \tag{3}$$

Therefore,

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}\right)_r + \omega \times \mathbf{f} \tag{4}$$

Consider the velocities $\mathbf{v}_s$ and $\mathbf{v}_r$ of a particle relative to the space and rotating set of axes, respectively, and ω is the constant angular velocity of Earth relative to the inertial system.

$$\mathbf{v}_s = \mathbf{v}_r + \omega \times \mathbf{r}$$

We obtain the time rate of change of $\mathbf{v}_s$ by applying our equation once more:

$$\begin{aligned}\left(\frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}t}\right)_s = \mathbf{a}_s &= \left(\frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}t}\right)_r + \omega \times \mathbf{v}_s \\ &= \mathbf{a}_r + 2(\omega \times \mathbf{v}_r) + \omega \times (\omega \times \mathbf{r})\end{aligned}$$

The equation of motion in the inertial system is

$$\mathbf{F} = m\mathbf{a}_s$$

which expands into

$$m\ddot{\mathbf{r}} = \mathbf{F} + 2m\dot{\mathbf{r}} \times \mathbf{\Omega} + m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega}$$

The first extra term is called the **Coriolis force**

$$\mathbf{F}_{\text{cor}} = 2m\dot{\mathbf{r}} \times \mathbf{\Omega}$$

Like the Lorentz force, Coriolis force always acts perpendicular to the velocity of the moving object, with its direction given by the right-hand rule. The second term is the so-called **centrifugal force**

$$\mathbf{F}_{\text{cf}} = m(\mathbf{\Omega} \times \mathbf{r}) \times \Omega$$

That is to say that in the rotating frame, Newton's laws take the form

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_{\text{cor}} + \mathbf{F}_{\text{cf}}$$

These extra terms are inertial or fictitious forces that are observed only in the noninertial frame. These yield many interesting observable phenomena.

3 The Central Force Problem

Consider a system of two mass points, m_1 and m_2, where the only forces are those due to an interaction potential U. We will assume at first that U is any function of the vector between the two particles, $\mathbf{r_2} - \mathbf{r_1}$, or of their relative velocity, $\dot{\mathbf{r}}_\mathbf{2} - \dot{\mathbf{r}}_\mathbf{1}$, or of any higher derivatives of $\mathbf{r_2} - \mathbf{r_1}$. Choose these to be the three components of the radius vector to the center of mass, $\mathbf{R}$, plus the three components of the difference vector $\mathbf{r} = \mathbf{r_2} - \mathbf{r_1}$. The kinetic energy T can be written as the sum of the kinetic energy of the motion of the center of mass, plus the kinetic energy of motion about the center of mass, T':

$$T = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + T'$$

with

$$T' = \frac{1}{2}m_1\dot{\mathbf{r}}_\mathbf{1}'^\mathbf{2} + \frac{1}{2}m_2\dot{\mathbf{r}}_\mathbf{2}'^\mathbf{2}$$

Here $\mathbf{r_1}'$ and $\mathbf{r_2}'$ are the radii vectors of the two particles relative to the center of mass and are related to $\mathbf{r}$ by

$$\begin{aligned}
\mathbf{r_1}' &= -\frac{m_2}{m_1 + m_2}\mathbf{r} \\
\mathbf{r_2}' &= \frac{m_1}{m_1 + m_2}\mathbf{r} \\
\Rightarrow T' &= \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{r}}^2
\end{aligned}$$

In the Lagrangian $\mathcal{L} = T - U$, we may drop the $\frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2$ term because none of the equations of motion of $\mathbf{r}$ will contain terms involving $\mathbf{R}$ or $\dot{\mathbf{R}}$. The rest of the Lagrangian is the same as a problem with a fixed center of force with a single particle at a distance $\mathbf{r}$ from it, having a mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$, where μ is known as the **reduced mass**. Equivalently,

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

Thus, the central force motion of two bodies about their center of mass can always be reduced to an equivalent one-body problem.

Conservative central forces are those where potential is $U(\mathbf{r})$, a function of $\mathbf{r}$ only. In such a problem, there is spherical symmetry: no rotation about any fixed axis can have an effect on the solution, so $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved. One such example of a conservative central force is gravity.

For now, let's restrict ourselves to conservative central forces in general. Expressed in plane polar coordinates, the Lagrangian for a mass will have

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

θ is a cyclic coordinate whose corresponding canonical momentum is the angular momentum of the system

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Because it is cyclic,

$$\dot{p}_\theta = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

with the immediate solution

$$mr^2\dot{\theta} = \ell$$

ℓ being the constant magnitude of angular momentum. We also obtain from the differential equation that

$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$$

We've inserted $\frac{1}{2}$ because $\frac{1}{2}r^2\dot{\theta}$ is the areal velocity–the area swept out by the radius vector per unit time (this may look familiar from the integration of polar coordinates in calculus). Further proof is too lengthy for this meeting, but conservation of angular momentum implies that areal velocity is constant. This is synonymous with **Kepler's second law** which deals with the Earth's orbit of the sun. This all follows from the conservation of angular momentum: there will not be a change in angular momentum in the two-body problem because the gravitational force acts parallel to the direction of the moment arm connecting the two bodies; thus neither exerts a torque on the other. Kepler constructed this law referencing only the gravitational force, but we can find that it holds for all conservative central forces.

The remaining Lagrange equation for the coordinate r is

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial U}{\partial r} = 0$$

Designating the value of the force along $\mathbf{r}$, $-\partial U/\partial r$, by $f(r)$ the equation can be rewritten as

$$m\ddot{r} - mr\dot{\theta}^2 = f(r)$$

Using our expression $\ell = mr^2\dot{\theta}$, we use the relation

$$\frac{d}{dt} = \frac{\ell}{mr^2}\frac{d}{d\theta} \tag{5}$$

We thus obtain

$$\frac{\ell}{r^2}\frac{d}{d\theta}\left(\frac{\ell}{mr^2}\frac{dr}{d\theta}\right) - \frac{\ell^2}{mr^3} = f(r) \tag{6}$$

Writing u for $1/r$,

$$\frac{d^2u}{d\theta^2} + u = \frac{mf(r)}{\ell^2u^2} \tag{7}$$

This is the differential equation of the orbit given in Newton's *Principia*, derived a la Lagrange.

I will not state the general result of this differential equation, as it contains special functions far beyond the scope of this discussion.[1] A result, which I state without proof due to the complexity of the proof[2] is **Bertrand's theorem**, which states that among central-force potentials with bound orbits, there are

[1] The reader may consult Whittaker's text on the Analytical Dynamics of Particles and Rigid Bodies, section 48.

[2] A fairly simple proof is found in *An Even Simpler "Truly Elementary" Proof of Bertrand's Theorem* by Galbraith and Williams.

only two types of central-force (radial) scalar potentials with the property that all bound orbits have closed orbits (i.e., the orbit repeats itself). The first such potential is an inverse-square central force such as, notably, gravitational potential. These take the form

$$U(\mathbf{r}) = -\frac{k}{\mathbf{r}}$$

with force

$$f(\mathbf{r}) = -\frac{\partial U}{\partial \mathbf{r}} = -\frac{k}{r^2}$$

The second, which we won't cover, is the radial harmonic oscillator potential:

$$U(\mathbf{r}) = -\frac{1}{2}kr^2$$

with force

$$f(\mathbf{r}) = -\frac{\partial U}{\partial \mathbf{r}} = -kr$$

4 The Two-Body Problem for Conservative Central Forces (Kepler's Problem)

Because I was thinking about Hong Jin Kwak while writing this, we restrict ourselves to the inverse-square central force; in this case, we call the problem of two bodies interacting via a central inverse square force the **Kepler problem**. Substituting 7

$$\frac{1}{r} = \frac{mk}{\ell^2}\left(1 + \sqrt{1 + \frac{2E\ell^2}{mk^2}}\cos(\theta - \theta')\right)$$

with the constant of integration θ' being one of the turning angles of the orbit. The general equation of a conic with one focus at the origin is

$$\frac{1}{r} = C[1 + e\cos(\theta - \theta')]$$

with e the eccentricity of the conic section. Compared with our other equation for $1/r$ it follows that the orbit is always a conic section, with eccentricity

$$e = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$$

The nature of the orbit depends upon the magnitude of e according to the following scheme:

$$\begin{aligned}
e > 1, \quad & E > 0: \quad \text{hyperbola,} \\
e = 1, \quad & E = 0: \quad \text{parabola,} \\
e < 1, \quad & E < 0: \quad \text{ellipse,} \\
e = 0, \quad & E = -\frac{mk^2}{2l^2}: \quad \text{circle.}
\end{aligned}$$

The **virial theorem** gives the average over time of the total kinetic energy of a stable system of discrete particles bound by a conservative force:

$$\overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F_i} \cdot \mathbf{r_i}}$$

For a circular orbit, T and U are constant in time, so it follows from the virial theorem that

$$E \equiv T + U = \frac{U}{2}$$

For the point of equilibrium between the central force and "effective force",

$$r_0 = \frac{\ell^2}{mk}$$

thus

$$E = -\frac{k}{2r_0} = -\frac{mk^2}{2\ell^2}$$

the above condition for circular motion. Therefore, for circular motion, the kinetic energy is exactly one half of the potential energy. The semimajor axis of an elliptic orbit is given by

$$a = -\frac{k}{2E}$$

$$\Rightarrow e = \sqrt{1 - \frac{\ell^2}{mka}}$$

Further, we have

$$\frac{\ell^2}{mk} = a(1 - e^2)$$

allowing us to write the elliptical orbit equation

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \theta')}$$

Kepler's first law states that the orbit of every planet is an ellipse with the sun at one of the two foci. This can be generalized to the equation of motion of any body experiencing a fixed, attractive, inverse square law force.

For the period of motion, we start with Kepler's second law. The area of the orbit, A, can be found by integrating $\frac{dA}{dt} = \frac{\ell}{2m}$ over a complete period τ;

$$\int_0^\tau \frac{dA}{dt} dt = A = \frac{\ell\tau}{2m} \tag{8}$$

The area of the ellipse is $A = \pi ab$ with $b = a\sqrt{1 - e^2}$ the semiminor axis, which we can also write as $b = \sqrt{\frac{a\ell^2}{mk}}$. The period of motion can then be found to be

$$\tau = 2\pi a^{3/2}\sqrt{\frac{m}{k}}$$

This is a more precise version of **Kepler's third law** (Kepler stated that $\tau^2 \propto a^3$). Recall that the motion of a planet around the sun is a two-body problem, and m must be replaced by the reduced mass. The gravitational law of attraction is

$$f = -G\frac{m_1 m_2}{r^2}$$

with G the universal constant of gravitation, so that the constant k is

$$k = Gm_1 m_2$$

Under these conditions, we obtain

$$\tau = \frac{2\pi a^{3/2}}{\sqrt{G(m_1 + m_2)}} \approx \frac{2\pi a^{3/2}}{\sqrt{Gm_2}}$$

if we neglect the mass of the planet compared to the sun.

5 Laplace-Runge-Lenz vector

In terms of vectors, the central force $\mathbf{F} = f(r)\hat{\mathbf{r}}$. Since the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved under central forces, $\dot{\mathbf{L}} = 0$ and

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \dot{\mathbf{p}} \times \mathbf{L} = f(r)\hat{\mathbf{r}} \times (\mathbf{r} \times m\dot{\mathbf{r}}) = f(r)\frac{m}{r}\left[\mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - r^2\dot{\mathbf{r}}\right] \tag{9}$$

where we have made use of a triple product formula. Then,

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = -mf(r)r^2\left[\frac{1}{r}\dot{\mathbf{r}} - \frac{\mathbf{r}}{r^2}\dot{r}\right] = -mf(r)r^2\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \tag{10}$$

For $f(r) = -\frac{k}{r^2}$ this reads

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \frac{d}{dt}(mk\hat{\mathbf{r}}) \tag{11}$$

This allows us to define the **Laplace-Runge-Lenz vector**

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}} \tag{12}$$

which is conserved.

Besides being interesting as a conserved quantity, we achieve an alternative way to derive the equation for the shape and orientation of the orbit. Taking the dot product of $\mathbf{A}$ with $\mathbf{r}$ and rearranging terms,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{r} &= \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - mkr \\ \frac{1}{r} &= \frac{mk}{\ell^2} + \frac{A}{\ell^2}\cos\theta \end{aligned} \tag{13}$$

where we have used the permutation of the triple product $\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) = (\mathbf{r} \times \mathbf{p}) \times \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{L} = \ell^2$. This corresponds to the formula of a conic section with eccentricity $e = \frac{A}{|mk|} \geq 0$.

The seven scalar quantities E, $\mathbf{A}$, and $\mathbf{L}$, (the latter two contributing three each), are the conserved quantities; they are related by two equations $\mathbf{A} \cdot \mathbf{L}$ and $A^2 = m^2k^2 + 2mE\ell^2$, therefore there are five independent constants of motion. This is consistent with the six initial conditions (initial position and velocity vectors, in components) that specify the orbit of the particle. The resulting 1-dimensional orbit in 6-dimensional phase space is completely satisfied.

Generally, a mechanical system with d degrees of freedom can have at most $2d-1$ constants of motion, since there are $2d$ initial conditions and the initial time cannot be determined by a constant of motion. A system with more than d constants of motion is **superintegrable** and a system with $2d-1$ constants is **maximally superintegrable**. Such systems follow closed, one-dimensional orbits in phase space. This means that the Kepler problem is **maximally superintegrable.**

Consider a small additional central force, a so-called **perturbation** described by potential energy $U'(r)$. In such cases, the LRL vector rotates slowly in the plane of the orbit, corresponding to a precession of the orbit that we may cover at some other time. By assumption, the perturbing potential is conservative, so E and $\mathbf{L}$ is conserved. Thus motion still lies in the plane perpendicular to $\mathbf{L}$ and the magnitude of $\mathbf{A}$ is conserved from $A^2 = m^2k^2 + 2mE\ell^2$. Using perturbation theory and action-angle variables, we can show that the rotation of $\mathbf{A}$ is encoded within the effects of general relativity on celestial motion. But that's a story for another time.