Symplectic reduction of phase spaces with symmetry

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The field of geometric mechanics uses tools from geometry to simplify problems in classical mechanics.

One represents the space of possible states of a system in terms of a phase space consisting of all possible positions and momentum of the system. People like to reduce this phase space to make computations more simple, and we'll show an example of how symplectic reduction does that algebraically. Often, this is a smooth space P with some additional structure.

This structure is the symplectic form ω, which takes two vectors tangent to the space at the same point and gives a real number. It is linear in both arguments, alternating ($\omega(v, w) = -\omega(w, v)$), nondegenerate ($\omega(v, w) = 0 \ \forall \ w \implies v = 0$), and does not change when you move in space, so an example of a symplectic form could be taking the area of a parallelogram defined by two vectors. This symplectic structure encodes the "rules of motion" of classical mechanics. You can think of it as what brings the physics into all of this by telling you how the positions and momenta evolve.

The key point is that a symmetry of the system should not change the symplectic form.

We'll probably see in future talks today that symmetries can be seen as group actions. A group G acts on a phase space P gives a homomorphism $\Phi : G \to Aut(P)$. The symmetries defined by that group action are the automorphisms that leave the symplectic form unchanged. We call such automorphisms symplectomorphisms, and they form a group $Symp(P, \omega)$.

You might have heard about how the orthogonal group $O(n)$ is precisely the group of distance-preserving transformations of Euclidean n-space that also preserve the origin, and this is pretty much the same thing, except instead of preserving the distance you get from the distance metric, you're preserving the number you get from the symplectic form.

We are only interested in groups whose action-induced homomorphism sends each element to a symplectomorphism. Moreover, these groups should also have the structure of a smooth space.

A curve in that group's space looks like this. Let $\gamma : \mathbb{R} \to G$ be a continuous group homomorphism. The image of a group homomorphism gives a subgroup of the G, so γ defines what we call a one-parameter subgroup, which defines a continuous curve in G.

The action gives a continuous family of symplectomorphisms $\Phi(\gamma(t)) : P \to P$. If you take a point $p \in P$, and apply the symplectomorphisms $\Phi(\gamma(t))$ by varying t, then a curve is drawn in P.

A standard result in mechanics known as Noether's theorem tells us that there exists a function $H_\gamma : P \to \mathbb{R}$ that is constant along orbits of $\Phi(\gamma(t))$, which are those curves drawn in P. This H_γ is called a conserved quantity associated to γ. An example could be like how angular momentum constant in a circular orbit obtained by repeatedly rotating a phase-space point via the symplectomorphism coming from the rotation group.

The momentum map J takes a point in P and maps it to a function that assigns a conserved quantity to each one-parameter subgroup. So $J(p)(\gamma) = H_\gamma(p)$. This is a convenient way to store all the conserved quantities in a single object.

So here's the reduction part. If we choose one of these functions μ, look at all states having exactly those conserved quantities; this is a subset $J^{-1}(\mu) \subset P$. Sard's theorem from differential topology tells us that almost all values of μ give $J^{-1}(\mu)$ to be a smooth space. Since G acts on P and preserves the momentum map J, it acts on $J^{-1}(\mu)$. So we can form the reduced phase space $P_{red} = J^{-1}(\mu)/G$, which is also smooth. This smooth space has a symplectic form of its own that's obtained naturally from (P, ω). Also, $\dim P_{red} = \dim J^{-1}(\mu) - \dim G$.

I don't remember if we covered quotienting a set by a group, but essentially the cosets here are the sets of points that are the same up to the action of G.

So let's look at an example of this. Consider a particle in 2D attached to a spring at the origin. Its phase space is 4 dimensional because we have 2 position coordinates and 2 momentum coordinates. This has rotation symmetry, because rotating both the position and momentum vectors by some angle is a symplectomorphism. under this symmetry, the momentum map gives you the angular momentum. Fix a nonzero value L of angular momentum. The constraint $xp_y - yp_x = L$ is one smooth equation in 4 variables, so it cuts the 4-dimensional phase space into a 3-dimensional hypersurface $J^{-1}(L)$. Each rotation orbit is 1-dimensional, so when we quotient out the rotation group we're collapsing every circle orbit to a single point. So now we've reduced our space of possible configurations to a 2 dimensional space.

So to wrap up the whole story here is that symmetry simplifies mechanics. A phase space carries a symplectic structure that encodes the rules of physics, and a physical symmetry is precisely a group action by symplectomorphisms, meaning it preserves that structure. Every continuous symmetry produces a conserved quantity, and the momentum map packages all of these conserved quantities at once. By fixing the conserved quantities and then quotienting out the symmetry that generated them, we discard the redundant directions of motion and obtain a smaller, simpler phase space that still captures the true dynamics of the system.