Fluid Dynamics: Inviscid and Irrotational flow

Raphael Esquivel

March 13, 2025

Abstract

We will briefly discuss the fundamentals of fluid flow. Specifically, we will derive the Euler equations for fluid dynamics, which describe fluid flow that is adiabatic (zero thermal conductivity), inviscid (zero viscocity/thickness), and irrotational. This is a special case of the famous Navier-Stokes equations. Although fluid dynamics is notorious for its mathematical difficulty (in fact, the proof to solutions of Navier-Stokes is a millennium problem), I aim to present the intuition behind the phenomena described in the field.

1 Continuum

In physics, you may have worked with point masses–objects that have mass but no size. You may also have worked with systems of particles, where you study their interactions and their motion in accordance with forces.

Continuum fluid mechanics is a sort of generalization of classical particle mechanics; instead of having discrete particles, we have a *continuum* of them. Of course, if we consider molecular structure, the assumption that there is a continuous density of matter does not hold. For most macroscopic phenomena occurring in nature, however, it is believed that this assumption is extremely accurate.

While we may index classical particles as m_1, m_2, m_3, etc, we can index "fluid particles" by a continuous variable $\mathbf{a}$. The trajectory of a "particle" corresponding to some value of $\mathbf{a}$ is a function $\mathbf{x}(\mathbf{a}, t)$. It is convenient to define the parameter $\mathbf{a}$ to simply be the initial position of the particle under consideration, i.e.

$$\mathbf{a} = \mathbf{x}(\mathbf{a}, 0) \tag{1}$$

Since $\mathbf{a}$ takes on a continuum of values, the particles corresponding to the values $\mathbf{a}$ and $\mathbf{a} + \delta\mathbf{a}$ are distinct if $\delta\mathbf{a} \neq 0$, no matter how small that difference is; the fluid is "infinitely divisible".

Let D be a region of two- or three-dimensional space filled with a fluid. Our object is to describe the motion of such a fluid. Let $\mathbf{x}$ be a point in D, and consider a particle of fluid moving through $\mathbf{x}$ at time t.

Definition 1.1. *Let $\mathbf{u}(\mathbf{x}, t)$ denote the velocity of the particle of fluid moving through $\mathbf{x}$ at time t. Thus, for each fixed time, $\mathbf{u}$ is a field of vectors on D; that is, there is a velocity defined at every point $\mathbf{x} \in D$. $\mathbf{u}$ is called the* ***spatial velocity field of a fluid****.*

Our assumption of continuum implies that there is a well-defined mass density $\rho(\mathbf{x}, t)$ for all $\mathbf{x}$. Thus, if W is any subregion of D, the mass of fluid contained within that subregion at time t can be formally defined by multiplying the mass density occupying a very small subregion by the volume of that subregion, and finding the sum of all those small masses for all subregions in W. That is,

$$m(W, t) = \int_W \rho(\mathbf{x}, t) dV, \tag{2}$$

where dV is the volume element in the space.

2 Fundamental Principles in Fluid Dynamics

Euler's equations govern the *flow* of fluid particles (fluid flow).[1] Our derivation is based on three basic principles:

1. Mass is neither created nor destroyed
2. The rate of change of momentum of a portion of the fluid equals the force applied to it (**Newton's second law**)
3. Energy is neither created nor destroyed

2.1 Conservation of mass

First, we treat the conservation of mass. If W is a *fixed* subregion of D–that is, it does not change with time–then the rate of change of mass in W is

$$\frac{d}{dt} m(W, t) = \frac{d}{dt} \int_w \rho(\mathbf{x}, t) dV = \int_W \frac{\partial p}{\partial t}(\mathbf{x}, t) dV. \tag{3}$$

If we consider W's boundary, ∂W, let $\mathbf{n}$ be the unit normal vector defined at points of ∂W, and let dA denote the area element on ∂W. The volume flow rate across ∂W per unit area is $\mathbf{u} \cdot \mathbf{n}$.

Lemma 2.1. *(Divergence theorem) Suppose we have a volume V whose boundary ∂V is piecewise-smooth. If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of V, then*

$$\iiint_V (\nabla \cdot \mathbf{F}) dV = \oiint_{\partial V} (\mathbf{F} \cdot \hat{\mathbf{n}}) dS, \tag{4}$$

where $\hat{\mathbf{n}}$ is the outward pointing unit normal at each point on the boundary.

Theorem 2.2. *(Integral form of the law of conservation of mass) The rate of increase of mass in a region W equals the rate at which mass enters its boundary ∂W in the inward direction:*

$$\frac{d}{dt} \int_W \rho dV = -\int_{\partial W} \rho \mathbf{u} \cdot \mathbf{n} dA. \tag{5}$$

[1]Not to be confused with Euler's equations for rotational motion, which we covered numerous times :)

By the divergence theorem, we find that equation 5 is mathematically equivalent to the following:

Theorem 2.3. *(Differential form of the law of conservation of mass, AKA the continuity equation) The rate of change of the mass density plus the divergence of the mass flow rate is zero:*

$$\frac{\partial p}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \tag{6}$$

2.2 Newton's second law

Next, we turn to the balance of momentum. If we consider a particle's path $\mathbf{x}(t) = (x(t), y(t), z(t))$, then its acceleration

$$\mathbf{a}(t) = \frac{d^2}{dt^2}\mathbf{x}(t) = \frac{d}{dt}\mathbf{u}(x(t), y(t), z(t), t) \tag{7}$$

which reduces to

$$\mathbf{a}(t) = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}, \tag{8}$$

where ∂_t and

$$\mathbf{u} \cdot \nabla = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} \tag{9}$$

are differential operators applied to $\mathbf{u}$. The sum of these operators, in fact, can be applied to other quantities in the fluid.

Definition 2.4. *The* ***material derivative*** *is defined as*

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla, \tag{10}$$

which allows us to take the time derivative of any function of position and time, accounting for the fact that the fluid is moving and that the positions of fluid particles change with time.

In equation 8, we essentially took the time derivative of the material velocity, which is just acceleration.

For any continuum, one of the forces that act on a piece of material is *stress*, whereby the piece of material is acted on by forces across its surface by the rest of the continuum.

Definition 2.5. *An* ***ideal fluid*** *is one with the property that for any motion of the fluid, there is a function $p(\mathbf{x}, t)$ (called* ***pressure****) such that if S is a surface in the fluid with a chosen unit normal $\mathbf{n}$, the force of stress exerted across S per unit area at $\mathbf{x} \in S$ at time t is $p(\mathbf{x}, t)\mathbf{n}$.*

Note that the force is in the direction $\mathbf{n}$ and thus acts orthogonally to S; that is, there are no tangential forces. Intuitively, this means that there is no way for rotation to start in an ideal fluid, nor, if it is there at the beginning, to stop.

If W is a region in the fluid at t, the total force exerted on the fluid inside W by means of stress on the boundary is

$$S_{\partial W} = -\int_{\partial W} p\mathbf{n}\,dA,$$

which is negative because $\mathbf{n}$ points outward. By the divergence theorem,

$$S_{\partial W} = -\int_W \nabla p dV.$$

If $\mathbf{b}(\mathbf{x}, t)$ is the given body force per unit mass (that is, a force that acts on the whole volume, rather than just the surface), by equation 2 we have that the total body force is

$$\mathbf{B} = \int_W \rho \mathbf{b} dV.$$

Thus, on any piece of fluid material, the force per unit volume is $-\nabla p + \rho \mathbf{b}$. By Newton's second law $F = ma$, which means that force per unit volume is $\rho \mathbf{a}$, we are led to the following:

Theorem 2.6. *(Differential form of the law of balance of momentum)*

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{b} \tag{11}$$

Theorem 2.7. *(Integral form of balance of momentum)*

$$\frac{d}{dt} \int_W \rho \mathbf{u} dV = -\int_{\partial W} (p\mathbf{n} + \rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n})) dA + \int_W \rho \mathbf{b} dV \tag{12}$$

Definition 2.8. *The momentum flux per unit area is the quantity*

$$p\mathbf{n} + \rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) \tag{13}$$

Now, consider $\varphi(\mathbf{x}, t)$ to be the trajectory followed by a particle that is at point $\mathbf{x}$ at $t = 0$. Let φ_t denote the map $\mathbf{x} \mapsto \varphi(\mathbf{x}, t)$; that is, with fixed t, this map advances each fluid particle from its position at time $t = 0$ to its position at time t. We call φ the **fluid flow map**, and if W is in a region D, then $\varphi_t(W) = W_t$ is the volume W moving with the fluid.

Theorem 2.9. *(Transport theorem) For any function f of $\mathbf{x}$ and t, we have*

$$\frac{d}{dt} \int_{W_t} \rho f dV = \int_{W_t} \rho \frac{Df}{Dt} dV. \tag{14}$$

Without a mass density factor included,

$$\frac{d}{dt} \int_{W_t} f dV = \int_{W_t} \left(\frac{\partial f}{\partial t} + \operatorname{div}(f\mathbf{u}) \right) dV. \tag{15}$$

Definition 2.10. *A flow is **incompressible** if for any fluid subregion W, the volume W_t is constant in t.*

Incompressibility is equivalent to saying that

$$\operatorname{div} \mathbf{u} = 0. \tag{16}$$

From theorem 2.3, a fluid is incompressible if and only if $D\rho/Dt = 0$; that is, that the mass density is constant following the fluid. If the fluid is **homogeneous**, meaning that ρ is constant in space, it also follows that the flow is incompressible if and only if it is constant in time.

2.3 Conservation of energy

For fluid moving in a domain D with velocity field $\mathbf{u}$, the **kinetic energy** contianed in a region $W \subset D$ is

$$E_k = \frac{1}{2} \int_W \rho \|\mathbf{u}\|^2 dV, \tag{17}$$

where $\|\mathbf{u}\|$ is the magnitude of the velocity field. The total energy of a fluid is the sum of its kinetic energy and its **internal energy**, which we cannot "see" on a macroscopic scale; it derives from sources such as intermolecular potentials and internal molecular vibrations. If energy is pumped into the fluid or if we allow the fluid to do work, the total energy will change. The rate of change of kinetic energy of a moving portion W_t of fluid is calculated using the transport theorem as

$$\begin{aligned} \frac{d}{dt} E_k &= \frac{d}{dt} \left(\frac{1}{2} \int_{W_t} \rho \|\mathbf{u}\|^2 dV \right) \\ &= \frac{1}{2} \int_{W_t} \rho \frac{D\|\mathbf{u}\|^2}{Dt} dV \\ &= \int_{W_t} \rho \left(\mathbf{u} \cdot \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \right) dV \end{aligned} \tag{18}$$

3 Euler's equations

3.1 Incompressible Flows

Here, we assume that all the energy is kinetic, and that the rate of change of kinetic energy in a portion of fluid equals the rate at which the pressure and body forces do work:

$$\frac{d}{dt} E_{kinetic} = -\int_{\partial W_t} p \mathbf{u} \cdot \mathbf{n} dA + \int_{W_t} \rho \mathbf{u} \cdot \mathbf{b} dV \tag{19}$$

Since the divergence of $\mathbf{u}$ is zero, this becomes

$$\begin{aligned} \int_{W_t} \rho \mathbf{u} \cdot \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) dv &= -\int_{W_t} (\operatorname{div}(p\mathbf{u}) - \rho \mathbf{u} \cdot \mathbf{b}) dV \\ &= -\int_{W_t} (\mathbf{u} \cdot \nabla p - \rho \mathbf{u} \cdot \mathbf{b}) dV \end{aligned} \tag{20}$$

This argument, in addition, shows that if we assume that $E = E_{kinetic}$, then the fluid must be incompressible (unless $p = 0$). In summary, in this incompressible case, the **Euler equations** are

$$\begin{aligned} \rho \frac{D\mathbf{u}}{Dt} &= -\nabla p + \rho \mathbf{b} \\ \frac{D\rho}{Dt} &= 0 \\ \operatorname{div} \mathbf{u} &= 0, \end{aligned} \tag{21}$$

with the boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D. \tag{22}$$

3.2 Isentropic fluids

Definition 3.1. *A compressible flow is* ***isentropic*** *if there is a function w, called* ***enthalpy****, such that*

$$\nabla w = \frac{1}{\rho} \nabla p \tag{23}$$

This terminology comes from thermodynamics.

Theorem 3.2. *(First law of thermodynamics) We accept as a basic principle that*

$$dw = T ds + \frac{1}{\rho} dp, \tag{24}$$

where s is the ***entropy*** *and w is the enthalpy per unit mass. An equivalent statement is*

$$d\epsilon = T ds + \frac{p}{\rho^2} d\rho, \tag{25}$$

where ϵ is the ***internal energy***

$$\epsilon = w - \frac{p}{\rho} \text{ (per unit mass)} \tag{26}$$

If the pressure is a function of density only, then the flow is clearly isentropic with constant entropy (hence the name isentropic).For isentropic flows, the rate of change of energy in a portion of fluid equals the rate at which work is done on it:

$$\begin{aligned} \frac{d}{dt} E_{kinetic} &= \frac{d}{dt} \int_{W_t} \left(\frac{1}{2} \rho \|\mathbf{u}\|^2 + \rho\epsilon \right) dV \\ &= \int_{W_t} \rho \mathbf{u} \cdot \mathbf{b} dV - \int_{\partial W_t} p \mathbf{u} \cdot \mathbf{n} dA. \end{aligned} \tag{27}$$

This comes from the balance of momentum using our earlier expression for $(d/dt)E_{kinetic}$ and the transport theorem.

Euler's equations for isentropic flow are thus

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla w + \mathbf{b} \\ &\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \end{aligned} \tag{28}$$

in D, and

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n} \tag{29}$$

on ∂D if ∂D is moving with a velocity $\mathbf{V}$.

Definition 3.3. *Given a fluid flow with a velocity field $\mathbf{u}(\mathbf{x}, t)$, a* ***streamline*** *at a fixed time is an integral curve of $\mathbf{u}$; that is, if $\mathbf{x}(s)$ is a streamline at the instant t, it is a curve parametrized by a variable, say s, that satisfies*

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}(t), t), \tag{30}$$

with t fixed.

Definition 3.4. *We define a fixed* ***trajectory*** *to be the curve traced out by a particle as time progresses; thus, a trajectory is a solution of the differential equation*

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}(t), t) \tag{31}$$

with suitable initial conditions.

If $\partial_t \mathbf{u} = 0$ (i.e. $\mathbf{u}$ independent of t),streamlines and trajectories coincide. in this case, the flow is called **stationary.**

Theorem 3.5. *(Bernoulli's theorem) In stationary isentropic flows and in the absence of external forces, the quantity*

$$\frac{1}{2}\|\mathbf{u}\|^2 + w \tag{32}$$

is constant along streamlines. The same holds for homogeneous (ρ= constant in space) incompressible flow with w replaced by p/ρ_0. The conclusions remain true if a force $\mathbf{b}$ *is present and is conservative; i.e.,* $\mathbf{b} = -\nabla\varphi$ *for some function φ, with w replaced by $w + \varphi$.*

The proof of this is a straightforward application of the principles we learned earlier. First, we exploit a vector property:

$$\frac{1}{2}\nabla(\|\mathbf{u}\|^2) = (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}) \tag{33}$$

Because the flow is steady, the equations of motion give

$$\begin{aligned} (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla w \\ \Rightarrow \nabla(\frac{1}{2}\nabla\|\mathbf{u}\|^2 + w) &= \mathbf{u}\times(\nabla\times\mathbf{u}) \end{aligned} \tag{34}$$

If we let $\mathbf{x}(s)$ be a streamline, then the change in the quantity $1/2(\|\mathbf{u}\|^2 + w)$ between any two arbitrary points s_1, s_2 is

$$\begin{aligned} \frac{1}{2}(\|\mathbf{u}\|^2 + w)|_{\mathbf{x}(s_1)}^{\mathbf{x}(s_2)} &= \int_{\mathbf{x}(s_1)}^{\mathbf{x}(s_2)} \nabla(\frac{1}{2}\|\mathbf{u}\|^2 + w)\cdot\mathbf{x}'(s)ds \\ &= \int_{\mathbf{x}(s_1)}^{\mathbf{x}(s_2)} (\mathbf{u}\times(\nabla\times\mathbf{u}))\cdot\mathbf{x}'(s)ds \\ &= 0 \end{aligned} \tag{35}$$

because $\mathbf{x}'(s) = \mathbf{u}(\mathbf{x}(s))$ is orthogonal to $\mathbf{u}\times(\nabla\times\mathbf{u})$, so the quantity is constant along the streamline!!!