Poinsot's Construction

Raphael Esquivel

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Poinsot's method is a beautiful geometric visualization of the free motion of the rigid body. It is exact and based upon the conservation of angular momentum and of kinetic energy. We begin by restating previous results. For starters, the **rigid body** is an *ideal solid object*; no matter what forces are imparted on the object, any two points on the object will always be separated by the same distance. That is, a rigid body is a system of particles that cannot be deformed. The rigid body's configuration is parametrized by 3 coordinates for position and 3 coordinates for orientation. We define the **inertia tensor**[1]

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \tag{1}$$

The **moments of inertia** with respect to the x, y, and z axes, respectively, are given by

$$I_{xx} = \int_m (y'^2 + z'^2)dm, \quad I_{yy} = \int_m (x'^2 + z'^2)dm, \quad I_{zz} = \int_m (x'^2 + y'^2)dm \tag{2}$$

The quantity in the integrand is precisely the square of the distance to the x, y, and z axis respectively, analogous to the two-dimensional case. We also define the *products of inertia*

$$I_{xy} = I_{yx} = \int_m x'y'dm, \quad I_{xz} = I_{zx} = \int_m x'z'dm, \quad I_{yz} = I_{zy} = \int_m y'z'dm \tag{3}$$

They measure the imbalance in the mass distribution. Angular momentum is defined[2]

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega} \tag{4}$$

from which we obtain **Euler's equations**

$$\mathbf{I} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} = \boldsymbol{\tau} \tag{5}$$

where $\tau = \dot{\mathbf{L}}$ is the torque on the rigid body. Linear algebra tells us that since the inertia tensor is symmetric ($I_{ij} = I_{ji}$), it is also diagonalizable; that is, we

[1] It's called a *tensor* (it's a dyadic, more precisely) because of the way it transforms under certain rotations. Just think of it as a matrix for now.

[2] This dot product between a dyadic and a vector returns another vector with components given by $(\mathbf{I} \cdot \boldsymbol{\omega})_i = \sum_{j=1}^{3} I_{ij}\omega_j$.

can *choose* axes x, y, and z for which the products of inertia vanish. The tensor then becomes

$$\mathbf{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \tag{6}$$

These I_{ii}'s are called the **principal moments of inertia**, and their corresponding axes are the **principal axes**. When using principal axes, Euler's equations become

$$I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z = \tau_x \tag{7}$$

$$I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x = \tau_y \tag{8}$$

$$I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y = \tau_z \tag{9}$$

1 Ellipsoid of inertia

Consider the kinetic energy of a rigid body rotating around a fixed reference point:

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} \tag{10}$$

An alternate (and more familiar) expression is $T = \frac{1}{2}I\omega^2$, where the scalar I is the moment of inertia about the axis of rotation and ω is the magnitude of angular velocities. Defining the quantity $\boldsymbol{\rho} = \frac{\boldsymbol{\omega}}{\omega\sqrt{I}}$ we find that

$$\boldsymbol{\rho} \cdot \mathbf{I} \cdot \boldsymbol{\rho} = 1 \tag{11}$$

$\boldsymbol{\rho}$ has the same direction as $\boldsymbol{\omega}$ and a magnitude inversely proportional to $\sqrt{I}$. Suppose we consider $\boldsymbol{\rho}$ to be a vector drawn from the origin of the body-fixed frame to the point (x, y, z). Then,

$$\boldsymbol{\rho} \cdot \mathbf{I} \cdot \boldsymbol{\rho} = I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 + 2I_{xy}xy + 2I_{xz}xz + 2I_{yz}yz = 1 \tag{12}$$

This equation happens to define an ellipsoidal surface; we name this surface the **ellipsoid of inertia**. It is fixed relative to the rigid body and the body fixed-frame. Essentially, it's a 3D plot which gives the value of $\frac{1}{\sqrt{I}}$ for any axis passing through the origin of the xyz frame.

2 Poinsot method

Let us choose a set of principal axes at the center of the xyz body-fixed frame. If I_1, I_2, I_3 are the principal moments of inertia, the ellipsoid of inertia reduces to

$$I_1x^2 + I_2y^2 + I_3z^2 = 1 \tag{13}$$

The **Poinsot construction** pictures the free rotational motion of a rigid body as the rolling of its ellipsoid of inertia on an invariable plane which is perpendicular to the constant angular momentum vector $\mathbf{L}$ drawn from the fixed center O. Note that we can re-express the rotational kinetic energy $T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}$.

If $\boldsymbol{\rho}$ represents the point of contact, the distance between O and the invariable plane is just the projection of $\boldsymbol{\rho}$ along $\mathbf{L}$:

$$\frac{\boldsymbol{\rho} \cdot \mathbf{L}}{L} = \frac{\boldsymbol{\omega} \cdot \mathbf{L}}{\omega L \sqrt{I}} = \frac{\sqrt{2T}}{L} \tag{14}$$

Note that by conservation of energy and conservation of momentum, this distance is constant as $\boldsymbol{\rho}$ moves. This means that the point defined by $\boldsymbol{\rho}$ moves in a plane that is perpendicular to $\mathbf{L}$ defined to be a constant distance from O. This plane is fixed in space and is known as the **invariable plane**.

We show that the invariable plane is indeed tangent to the inertia ellipsoid at the point defined by $\boldsymbol{\rho}$. Representing the ellipsoid by a trivariate function

$$F(x, y, z) = I_1 x^2 + I_2 y^2 + I_3 z^2 = 1 \tag{15}$$

we can find the direction of the normal to the ellipsoid at $\boldsymbol{\rho}$ by evaluating the gradient ∇F at that point. Now

$$\begin{aligned} \nabla F &= \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k} \\ &= 2I_1 x\mathbf{i} + 2I_2 y\mathbf{j} + 2I_3 z\mathbf{k} \end{aligned} \tag{16}$$

∇F is thus parallel to $\mathbf{L} = I_1\omega_x\mathbf{i} + I_2\omega_y\mathbf{j} + I_3\omega_z\mathbf{k}$ because $\boldsymbol{\omega}$ and $\boldsymbol{\rho}$ have the same direction in space. The instantaneous axis of rotation passes through the point of contact, so the ellipsoid rolls without slipping on the invariable plane. The actual rigid body, by construction, goes through the same rotational motions as the ellipsoid of inertia.

2.1 Analysis

The path that the point of contact traces on the invariable plane is called the **herpolhode**. It does not have to be a closed curve, as the point of contact moves continuously between two circles centered on the point on the plane below O corresponding to extreme values of the angle between that line and an axis of the ellipsoid and the angle between $\boldsymbol{\rho}$ and that axis. During successive intervals, the point of contact moves from tangency with one circle to tangency with the other.

The path traced by the point of contact on the ellipsoid of inertia is a **polhode**. For any rotation not exactly about a principal axis, the polhode curves are closed and encircle either the axis of minimum moment of inertia or the axis of maximum moment of inertia. Since these are closed, the motion of the vector $\mathbf{L}$ *relative* to the body[3] must be periodic. They are formed by the intersection of two ellipsoid, namely, the inertia ellipsoid and the momentum ellipsoid. We obtain the momentum ellipsoid with the following:

$$I_1^2\omega_x^2 + I_2^2\omega_y^2 + I_3^2\omega_z^2 = L^2 \tag{17}$$

$$\omega_x = x\omega\sqrt{I} = x\sqrt{2T} \tag{18}$$

$$\omega_y = y\omega\sqrt{I} = y\sqrt{2T} \tag{19}$$

$$\omega_z = z\omega\sqrt{I} = z\sqrt{2T} \tag{20}$$

[3]In space, $\mathbf{L}$ must be fixed in magnitude and direction by conservation of momentum.

The **momentum ellipsoid** is then

$$I_1^2 x^2 + I_2^2 y^2 + I_3^2 z^2 = \frac{L^2}{2T} = D \tag{21}$$

where L, T, and therefore D are constants. The polhode for a given case of free rotational motion is determined by the values of (x, y, z) satisfying the initial value of $\boldsymbol{\rho}$ as well as both the momentum and inertia ellipsoids. It can be shown that if $D < I_2$, assuming $I_1 < I_2 < I_3$, the polhode encircles the axis corresponding to the minimum moment of inertia. If $D > I_2$, the polhode encircles the axis corresponding to the maximum moment of inertia.

The *stability* of rotational motion about a principal axis is something we have previously investigated. The polhodes in the vicinity of the major and minor axes are tiny ellipses indicating stability–that is, a small displacement of the axis of rotation relative to the body will remain small. On the other hand, polhodes near the intermediate axis are *hyperbolic*, indicating instability; the axis of rotation relative to the body will suddenly flip over to nearly the opposite direction and then return back again, only to repeat the cycle, just as in the Wikipedia demonstration of the Dzhanibekov effect in 0G.

2.2 Period

A result from the conservation of energy is

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2T \tag{22}$$

where T is the (constant) rotational energy. From this, we may express ω_1 and ω_3 in terms of ω_2 by

$$\begin{aligned} \omega_1^2 &= \frac{(2TI_3 - L^2) - I_2(I_3 - I_2)\omega_2^2}{I_1(I_3 - I_1)} \\ \omega_3^2 &= \frac{(L^2 - 2TI_1) - I_2(I_2 - I_1)\omega_2^2}{I_3(I_3 - I_1)} \end{aligned} \tag{23}$$

Substituting into Euler's equations and taking the torques to be zero, we obtain

$$\begin{aligned} \dot{\omega}_2 &= \frac{\omega_1\omega_2(I_3 - I_1)}{I_2} \\ &= \frac{1}{I_2}\sqrt{\frac{[(2TI_3 - L^2) - I_2(I_3 - I_2)\omega_2^2] \cdot [(L^2 - 2TI_1) - I_2(I_2 - I_1)\omega_2^2]}{I_1 I_3}} \end{aligned} \tag{24}$$

This is a differential equation which one may integrate as an *elliptic integral* to obtain *time as a function of* ω_2. For the sake of the problem, we define the new variables

$$\begin{aligned} \tau &= t\sqrt{\frac{(I_3 - I_2)(L^2 - 2TI_1)}{I_1 I_2 I_3}} \\ s &= \omega_2\sqrt{\frac{I_2(I_3 - I_2)}{2TI_3 - L^2}} \end{aligned} \tag{25}$$

which are by inspection linear functions of t and ω_2 respectively. Also define a positive parameter $k^2 < 1$ by

$$k^2 = \frac{(I_2 - I_1)(2TI_3 - L^2)}{(I_3 - I_2)(L^2 - 2TI_1)} \tag{26}$$

and we obtain an integral as a solution:

$$\tau = \int_0^s \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \tag{27}$$

the origin of time being taken at an instant when $\omega_2 = 0$. When this integral is inverted, we have a *Jacobian elliptic function* $s = \text{sn}\tau$ (this is known as the *elliptic sine*), giving ω_2 as a function of time, and by extension we may find ω_1 and ω_3 as algebraic functions of ω_2. Using the definitions of *elliptic cosine* $\text{cn}\tau = \sqrt{1 - \text{sn}^2\tau}$ and the *delta amplitude* $\text{dn}\tau = \sqrt{1 - k^2\text{sn}^2\tau}$ we find

$$\begin{aligned}
\omega_1 &= \text{cn}\tau\sqrt{\frac{2TI_3 - L^2}{I_1(I_3 - I_1)}} \\
\omega_2 &= \text{sn}\tau\sqrt{\frac{2TI_3 - L^2}{I_2(I_3 - I_2)}} \\
\omega_3 &= \text{dn}\tau\sqrt{\frac{L^2 - 2TI_1}{I_3(I_3 - I_1)}}
\end{aligned} \tag{28}$$

These are periodic functions, and their period in the variable τ is $4K$, where K is a complete elliptic integral of the first kind:

$$K = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \tag{29}$$

Their period in t is therefore

$$T = 4K\sqrt{\frac{I_1I_2I_3}{(I_3 - I_2)(L^2 - 2TI_1)}} \tag{30}$$

After a time T the vector $\boldsymbol{\omega}$ returns to its original position *relative to the axes of the body*. The body itself, however, does not return to its original position relative to an inertial reference frame. For $I_1 = I_2$, of course, the formulas for the angular velocity components reduce to those we obtain for a symmetrical body. As $I_1 \to I_2$, the parameter $k^2 \to 0$, and the elliptical functions degenerate to circular functions ($\text{sn}\tau \to \sin\tau$, $\text{cn}\tau \to \cos\tau$, $\text{dn}\tau \to 1$) and we just get

$$\omega_1 = A\cos kt, \quad \omega_2 = A\sin kt \tag{31}$$

where $k = \frac{\omega_3(I_3 - I_1)}{I_1}$ and $A = \sqrt{\omega_1^2 + \omega_2^2}$ is a constant by conservation of energy. ω_3 is also constant, so $\boldsymbol{\omega}$ rotates uniformly with angular velocity k about the axis of the top, remaining unchanged in magnitude. In the more general case of an asymmetrical body with $I_1 \neq I_2 \neq I_3$, however, we can find that the rigid body does not *ever* return exactly to its original position–a very interesting result that I must state without further proof or clarification, as it requires a lot of very advanced calculus (even more cursed than what we have already done with the Jacobian elliptic functions), far beyond the scope of our conversation.