Time and Electricity

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1 Introduction: Maxwell's equations

One thing that AP Physics C will teach you is Maxwell's equations. In their differential form, they read

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \quad \text{(Gauss' law)} \\ \nabla \cdot \mathbf{B} &= 0 \quad \text{(Gauss' law for magnetism)} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's law of induction)} \\ \nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \quad \text{Ampere-Maxwell law} \end{aligned} \tag{1}$$

Essentially, Gauss' law tells us "how much" of an electric field "comes out" of a closed surface, which is often useful for solving electrostatics problems; Gauss' law for magnetism tells us that the total amount of "how much" magnetic field comes out is zero, since there is no such thing as a "magnetic monopole" (an isolated magnet with only one magnetic pole), so things always cancel out; Faraday's law tells us how a changing magnetic field affects the electric field, often creating currents in circuits; finally, Ampere's law relates the effect of electric current on the magnetic field, such as the famous experiment by Oersted that showed that a current affects a compass. These are essentially the four fundamental equations of classical electromagnetism.

2 Magnetic vector potential and gauge transformations

One thing that remains to be considered, however, is the generalization of these laws to *time-dependent configurations*. That is, given $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ (the charge and current densities, respectively), how can we find the electric and magnetic fields? We begin by looking at these fields in terms of *potentials*. Recall that electric field is a vector, but electric potential is a scalar quantity. In electrostatics, the condition $\nabla \times \mathbf{E} = 0$ allows us to write

$$\mathbf{E} = -\nabla V \tag{2}$$

However, in electrodynamics, $\frac{\partial \mathbf{B}}{\partial t} \neq 0$, and we cannot write the electric field as the gradient of a scalar field anymore. However, Gauss' law for electricity still holds, so we can write $\mathbf{B}$ as the curl of the **magnetic vector potential** defined as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

Plugging into Faraday's law,

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \\
\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) &= 0
\end{aligned}$$

Since $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$ has a vanishing curl, even in the general electrodynamic case, we can write it as the gradient of a scalar:

$$\begin{aligned}
\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} &= -\nabla V \\
\mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t}
\end{aligned} \tag{4}$$

which of course reduces to the old form when $\mathbf{A}$ is constant. Plugging into Gauss' law and Ampere's law yields the following ugly equations:

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho \tag{5}$$

$$\left(\nabla^2 \mathbf{A} - \mu_0\epsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J} \tag{6}$$

Note that equations 3 and 4 do not *uniquely* define the potentials; that is, we can impose extra conditions on V and $\mathbf{A}$ without disturbing $\mathbf{E}$ and $\mathbf{B}$ since we lose some information when we differentiate. This is called **gauge freedom**. The conditions for defining two sets of potentials $(V, \mathbf{A})$ and $(V', \mathbf{A}')$ for which the electric and magnetic fields are the same are outlined below.

$$\begin{aligned}
\mathbf{A}' &= \mathbf{A} + \alpha \\
V' &= V + \beta
\end{aligned}$$

Since $\mathbf{B} = \nabla \times \mathbf{A}' = \nabla \times \mathbf{A}$, it directly follows that $\nabla \times \alpha = 0$, meaning that α can therefore be written as the gradient of some scalar:

$$\alpha = \nabla\lambda$$

Furthermore, $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\nabla V' - \frac{\partial \mathbf{A}'}{\partial t}$, meaning that

$$\begin{aligned}
\nabla\beta + \frac{\partial \alpha}{\partial t} &= 0 \\
\nabla\left(\beta + \frac{\partial \lambda}{\partial t}\right) &= 0
\end{aligned}$$

Since the gradient is zero, the term must be independent of position. However, it can still depend on time, so we can rewrite β as a function of time in terms of λ:

$$\beta = -\frac{\partial \lambda}{\partial t} + k(t) \tag{7}$$

Notice that $\alpha = \nabla\lambda$ says nothing about time-dependence; in fact, we might as well absorb $k(t)$ into $\frac{\partial\lambda}{\partial t}$ by *redefining* λ to have a time derivative matching β, without affecting α. It follows, therefore, that for any scalar function $\lambda(\mathbf{r}, t)$, we can make the **gauge transformation**

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\lambda \\ V' &= V - \frac{\partial\lambda}{\partial t} \end{aligned} \tag{8}$$

without affecting the physical quantities **E** and **B**. These gauge transformations help us simplify the ugly equations!

2.1 The Coulomb gauge

As a first example, we have the **Coulomb gauge**. We pick

$$\nabla \cdot \mathbf{A} = 0 \tag{9}$$

where our first ugly equation becomes

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \tag{10}$$

which is **Poisson equation** that we briefly touched on last year. Setting $V = 0$ at infinity, the solution is

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \tag{11}$$

While this is all pretty, **A** is particularly difficult to calculate; the second ugly equation is still ugly:

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \mu_0\epsilon_0\nabla\left(\frac{\partial V}{\partial t}\right) \tag{12}$$

2.2 The Lorenz gauge

For the second ugly equation, we can instead use the **Lorenz gauge**

$$\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t} \tag{13}$$

leading to

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} \tag{14}$$

As for V, the first ugly equation becomes

$$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho \tag{15}$$

We can now define the **d'Alembertian operator**:

$$\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} \equiv \Box^2 \tag{16}$$

at which point our equations become

$$\begin{aligned}\Box^2 V &= -\frac{1}{\epsilon_0}\rho \\ \Box^2 \mathbf{A} &= -\mu_0 \mathbf{J}\end{aligned} \tag{17}$$

This is nice in the context of special relativity, where the d'Alembertian $\Box^2$ is a natural generalization of the Laplacian ∇^2, and the equations from the Lorenz gauge become like four-dimensional versions of Poisson's equation.

3 Retarded potentials: the correction of causality

Speaking of special relativity, a major flaw left in these generalizations is that we do not account for the fact that nothing can move faster than the speed of light–not even the influence of the electromagnetic field. In fact, electromagnetic "news" travels at the speed of light. It's not the status of the source "right now" that matters, but rather its condition at the **retarded time** t_r when the "message" left. It travels a distance $|\mathbf{r}-\mathbf{r}'|$, so

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{r}'|}{c} \tag{18}$$

In the static case, equations 17 reduce to

$$\begin{aligned}\nabla^2 V &= -\frac{1}{\epsilon_0}\rho \\ \nabla^2 \mathbf{A} &= -\mu_0 \mathbf{J}\end{aligned} \tag{19}$$

whose solutions are

$$\begin{aligned}V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \\ \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'\end{aligned} \tag{20}$$

So then our generalization to the nonstatic case using the retarded time is

$$\begin{aligned}V(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \\ \mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'\end{aligned} \tag{21}$$

where $\rho(\mathbf{r}',t_r)$ is the charge density at $\mathbf{r}'$ at retarded time t_r. Because the integrands are evaluated at retarded time, these are called **retarded potentials**. To prove this, we must show that they satisfy equation 17 and the Lorenz condition (equation 13). Note first that applying the same logic to fields by replacing $\rho(\mathbf{r}')$ with $\rho(\mathbf{r}',t_r)$ and so on, we get the wrong answer. We first divide the volume $\mathcal{V}$ over which we perform the integration to obtain $V(\mathbf{r},t)$ into two regions, $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{V}_1$ is a small volume surrounding the point described by the radius vector $\mathbf{r}$ ($\mathcal{V}_1$ surrounds the point at which the potential is measured). Then that decomposes the potential V into V_1 and V_2 where

$$V_i(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_\rangle} \frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

Requiring that $\mathcal{V}_1$ be sufficiently small such that we may neglect retardation for all points within $\mathcal{V}_1$ (we'll eventually let $\mathcal{V}_1 \to 0$), V_1 becomes just

$$V_1(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_\infty} \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

This is identical to the static case and is therefore a solution of Poisson's equation:

$$\nabla^2 V_1(\mathbf{r},t) = -\frac{1}{\epsilon}\rho$$

The distance $\mathscr{r} = |\mathbf{r}-\mathbf{r}'|$ and hence also the integrand are spherically symmetric in the field point $\mathbf{r}$ with respect to an element of charge fixed at point source $\mathbf{r}'$, so the Laplacian with respect to the coordinates $\mathbf{r}$ of the integrand is

$$\nabla^2\left(\frac{\rho}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}\frac{\partial}{\partial \mathscr{r}}\left[\mathscr{r}^2\frac{\partial}{\partial \mathscr{r}}\left(\frac{\rho}{\mathscr{r}}\right)\right] = \frac{1}{\mathscr{r}}\frac{\partial^2\rho}{\partial \mathscr{r}^2}$$

Therefore,

$$\begin{aligned}\nabla^2 V_2(\mathbf{r},t) &= \int_{V_2} \nabla^2\left\{\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right\} d^3\mathbf{r}' \\ &= \int_{V_2} \frac{1}{\mathscr{r}}\frac{\partial^2}{\partial \mathscr{r}^2}\rho(\mathbf{r}',t_r) d^3\mathbf{r}'\end{aligned}$$

Because disturbances in charge density must propagate causally at finite speed, the charge density evaluated at retarded time must obey the wave equation; that is,

$$\frac{\partial^2\rho}{\partial \mathscr{r}^2} - \frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2} = 0$$

Using this, we have

$$\nabla^2 V_2(\mathbf{r},t) = \frac{1}{c^2}\int_{\mathcal{V}_\in} \frac{1}{\mathscr{r}}\frac{\partial^2}{\partial t^2}\rho(\mathbf{r}',t_r) d^3\mathbf{r}'$$

Interchanging time and space derivatives,

$$\nabla^2 V_2(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{\mathcal{V}_\in} \frac{1}{\mathscr{r}}\rho(\mathbf{r}',t_r) d^3\mathbf{r}'$$

The integral, in this limit, is just $V(\mathbf{r},t)$ if we let $\mathcal{V}_1 \to 0$ so that $\mathcal{V}_2 \to \mathcal{V}$. Therefore, we have

$$\nabla^2 V_2(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$

We finally add our equations for V_1 and V_2 to obtain

$$\begin{aligned}\nabla^2 V = \nabla^2(V_1+V_2) &= \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_0}\rho \\ \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} &= -\frac{1}{\epsilon_0}\rho \\ \Box^2 V &= -\frac{1}{\epsilon_0}\rho\end{aligned}$$

Completing our proof. The proof for **A** is completely analogous. This applies equally well to **advanced potentials** where we take the time dependency to be

$$t_a \equiv t + \frac{|\mathbf{r} - \mathbf{r}'|}{c} \tag{22}$$

Although these are consistent with maxwell's equations, they violate **causality**, suggesting potentials now depend on what the charge and current distribution will be in the future. They therefore have no physical significance.

4 Jemifenko's equations

Given equations 3, 4, and 21, it is straightforward to determine the fields. For the gradient ∇V, note the integrand depends on $\mathbf{r}$ in both the numerator and denominator, so

$$\nabla V = \frac{1}{4\pi\epsilon_0} \int \left[(\nabla \rho) \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \rho \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] d^3\mathbf{r}'$$

It also follows by the chain rule that

$$\nabla \rho = \dot{\rho} \nabla t_r = -\frac{1}{c} \dot{\rho} \nabla \mathscr{r}$$

It can be shown that $\nabla \mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}$ and $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$ so

$$\nabla V = \frac{1}{4\pi\epsilon_0} \int \left[-\frac{\dot{\rho}}{c} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} - \rho \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right] d^3\mathbf{r}' \tag{23}$$

The time derivative of **A** is easily found to be

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi} \int \frac{\dot{\mathbf{J}}}{\mathscr{r}} d^3\mathbf{r}' \tag{24}$$

Using $c^2 = \frac{1}{\mu_0 \epsilon_0}$, we get the **time-dependent generalization of Coulomb's law**:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{\dot{\rho}(\mathbf{r}', t_r)}{c\mathscr{r}} \hat{\boldsymbol{\mathscr{r}}} + \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{\mathscr{r}} \right] d^3\mathbf{r}' \tag{25}$$

As for **B**,

$$\nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int \left[\frac{1}{\mathscr{r}} (\nabla \times \mathbf{J}) - \mathbf{J} \times \nabla \left(\frac{1}{\mathscr{r}} \right) \right] d^3\mathbf{r}'$$

By definition of curl,

$$(\nabla \times \mathbf{J})_x = \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z}$$

We use

$$\frac{\partial J_z}{\partial y} = \dot{J}_z \frac{\partial t_r}{\partial y} = -\frac{1}{c} \dot{J}_z \frac{\partial \mathscr{r}}{\partial y}$$

to obtain

$$(\nabla \times \mathbf{J})_x = -\frac{1}{c} \left(\dot{J}_z \frac{\partial \mathscr{r}}{\partial y} - \dot{J}_y \frac{\partial \mathscr{r}}{\partial z} \right) = \frac{1}{c} \left[\dot{\mathbf{J}} \times (\nabla \mathscr{r}) \right]_x.$$

Since $\nabla \mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}$,

$$\nabla \times J = \frac{1}{c}\dot{\mathbf{J}} \times \hat{\boldsymbol{\mathscr{r}}}$$

Moreover, $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$, hence we obtain the time-dependent generalization of the Biot-Savart law

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c\mathscr{r}} \right] \times \hat{\boldsymbol{\mathscr{r}}} d^3\mathbf{r}' \tag{26}$$

Equations 25 and 26 are **Jemifenko's equations**. They are of limited utility, since it is usually easier to calculate retarded potentials and differentiate them rather than going directly to the fields. Nevertheless, I think they're pretty interesting. A possible subject for next time is showing the Lienard-Wiechert potentials for moving charges using what we learned about retarded potentials.