Painlevé Paradox and Billiard Balls

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1 Introduction

The earliest investigation of friction aside from those to Leonardo da Vinci were made by Guillaume Amontons (1663-1705), who came up with two laws of friction. Amontons' first law of friction states that the frictional force of one object sliding on another is proportional to the applied load. The second is that force depends only on the load, not the surface area where the objects are in contact. It appears that this conclusion, which had also been made by da Vinci, was very suspicious.

Euler (1707-1783) found that we can distinguish between kinetic friction and static friction, with kinetic friction less than or equal to static friction. He is generally credited with being the first to make this distinction.

Coulomb (1736-1806) is always mentioned in connection with friction. Many people argue that the so-called "Coulomb's law of friction" should have been attributed to Amontons. In truth, pretty much the only novel contribution made by Coulomb was providing an empirical formula for the rate of increase of static friction after observing that static friction often increased when a block and surface remained in contact for a long time. He attributed friction to the roughness of surfaces and the effort needed to slide portruding himps over each other. Nowadays, this explanation is dismissed because work expended moving up the humps would be retrieved as the block slides back down under the normal force. In many cases, the explanation seemed to conflict with experimental evidence. For example, it turned out that friction between surfaces was often lower when one was significantly rougher than the other, not to mention that highly polished surfaces might exhibit increased friction.

A resolution was made when contact area was examined more carefully. Because of microscopic irregularities, the actual contact area of two surfaces is much smaller than the apparent macroscopic area, and, most importantly, an increase in the normal load pushes these irregularities closer together, so that they overlap more, and even flattens some, thus increasing contact area. This supports the idea that friction results primarily from molecular adhesion, and one can actually observe tiny fragments of the surfaces being worn away because of this adhesion force.

The modern picture of friction invokes waves in the atomic lattice generated

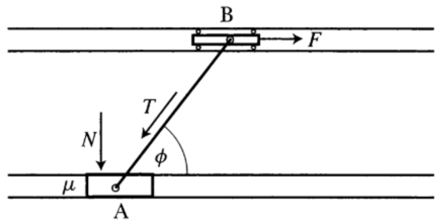

Figure 1: The Painlevé-Klein problem.

by the protrusions being deformed. "Friction at the Atomic Scale" by Jacqueline Krim in the *Scientific American* does a good job of explaining this.

We were left by Guillaume Amontons, Charles-Augustin Coulomb, and Arthur Morin,[1] the three French musketeers of friction, with this:

Consider a fixed planar surface and a body in contact with it. The total force on this body can be decomposed as $\mathbf{N} + \mathbf{F}$ where $\mathbf{N}$ is normal to the surface and $\mathbf{F}$ is parallel to it. Then there will be a critical value μ_S, the coefficient of static friction, and a number μ with $0 \leq \mu \leq \mu_S$ called the coefficient of kinetic friction so that:

- for $F \leq \mu_S \cdot N$ there is no motion
- for $F > \mu_S \cdot N$ there is motion in the direction of $\mathbf{F}$, and the body acts as if it is under the influence of a force of magnitude $F - \mu N$, i.e. there is now an additional frictional force of magnitude μN.

Of course, this is a purely empirical law. Before heading on to the billiard ball, we'll examine some problems that this famous model of friction presents.

This paper represents my efforts to make discussions more qualitative and accessible.

2 The Painlevé Paradox

We begin with the Painlevé-Klein problem. Consider two carts moving along parallel guides, connected by a rigid rod. Cart A, which slides with coefficient of friction μ. is dragged along at a constant distance behind cart B, which slides frictionlessly. We assume that the carts have mass $m = 1$, while the connecting rod has negligible mass, and that a force in the direction of the guides with magnitude F is applied at B. The normal force of magnitude N on A now arises from the tension force of magnitude T along the rod, with

$$N = T \sin\phi$$

[1]Whose name is among the 72 inscribed into the Eiffel tower! Not to be confused with David Morin, the author of the elementary mechanics textbook.

Note that N and T may be positive or negative. The total horizontal force acting on B is $F - T\cos\phi$, so if x is the coordinate of B we have

$$\ddot{x} = F - T\cos\phi = F - \frac{N}{\tan\phi}$$

On the other hand, the whole system, consisting of the two connected carts, satisfies[2]

$$2\ddot{x} = F - \mu|N|\mathrm{sgn}(\dot{x}) = F - \mu N\mathrm{sgn}(N)\mathrm{sgn}(\dot{x})$$

By substitution this yields

$$N = \frac{F\tan\phi}{2 - \mu\tan\phi\mathrm{sgn}(N)\mathrm{sgn}(\dot{x})}$$

Suppose that $\mu\tan\phi > 2$. If we seek a solution with $\dot{x} > 0$ we immediately obtain a contradiction for either value of $\mathrm{sgn}(N)$. For $\dot{x} < 0$ the solution is not unique, for we have both

$$N = \frac{F\tan\phi}{2 + \mu\tan\phi} > 0$$

and

$$N = \frac{F\tan\phi}{2 - \mu\tan\phi} < 0$$

If instead we have $\mu\tan\phi < 2$ then we obtain a unique solution

$$N = \frac{F\tan\phi}{2 - \mu\tan\phi\mathrm{sgn}(\dot{x})} > 0$$

which leads to a differential equation that can describe the motion

$$\ddot{x} = \frac{F(1 - \mu\tan\phi\mathrm{sgn}(\dot{x})}{2 - \mu\tan\phi\mathrm{sgn}(\dot{x})}$$

We just hope that $\mu\tan\phi \neq 2$. The most interesting property of the system involves its motion starting at rest, $\dot{x} = 0$. Assuming $\mu \approx \mu_S$, the frictional force on A has the magnitude

$$\mu N\mathrm{sgn}(\dot{x}) = \frac{\mu F\tan\phi\mathrm{sgn}(\dot{x})}{2 - \mu\tan\phi\mathrm{sgn}\dot{x}}$$

while the horizontal force on A due to the force applied to B is

$$T\cos\phi = \frac{N}{\tan\phi} = \frac{F}{2 - \mu\tan\phi\mathrm{sgn}(\dot{x})}$$

In order for A to start moving, the second must be greater to the first. When $F > 0$, so that B is being pushed to the right, this implies $\mu\tan\phi\mathrm{sgn}(\dot{x}) < 1$; in order for B to move to the right we have $\mathrm{sgn}(\dot{x}) = 1$ so

$$\mu\tan\phi < 1$$

[2] We let sgn a be $+1$ for $a > 0$, -1 for $a < 0$, and 0 for $a = 0$.

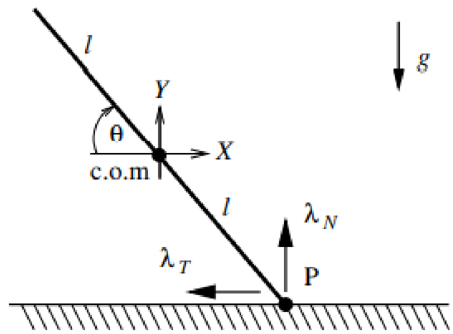

Figure 2: The classical Painlevé problem of the falling rod.

Hence if there is any friction at all then for a large enough initial angle it will be *impossible* to start even for an astronomically large force, and the system is "wedged" or self-braking. This contradicts our basic understanding of theoretical mechanics. After all, the block *should* move.

A second example involves a rigid rod falling under gravity while in contact with a rigid, stationary, frictional horizontal surface. Suppose the rod has mass 1 and radius of gyration r, that the center of mass is a distance ℓ from contact point P, and $N \geq 0$ and T are the normal and tangential components of the contact force. Let (X, Y) be the Cartesian coordinates of the rod's center of mass within the plane in which it is falling, and let θ be its angle to the horizontal. Then the equations of motion are:

$$\ddot{X} = -\mu T, \quad \ddot{Y} = -mg + N, \quad mr^2\ddot{\theta} = -\ell(\cos\theta N + \mu \sin\theta T)$$

Let us transform the coordinates with

$$(x, y) = (X + \ell\cos\theta, Y - \ell\sin\theta)$$

and $(u, v) = (\dot{x}, \dot{y})$. Then N can be considered to be a Lagrange multiplier used to maintain the inequality constraint $y \geq 0$. In particular, N and y satisfy a complementarity relation $N \perp y > 0$ which means that at most one of N and y can be positive. Whenever $y = 0$, we assume Coulombic friction:

$$|T| \leq \mu|N|, \quad T = -\mu \mathrm{sgn}(u)\lambda_N$$

Now suppose there is an initial condition such that the rod is slipping with $u > 0$ and $0 < \theta < \pi/2$ so that $T = -\mu N$. Then the normal acceleration would be written as

$$\begin{aligned}
\ddot{y} &= \ddot{Y} + \ell(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) \\
&= (\ell\dot{\theta}^2 \sin\theta - g) + \left[1 + \frac{\ell^2}{r^2}\cos^2\theta - \mu\cos\theta\sin\theta\right]\frac{N}{M} \\
&= b(\theta, \dot{\theta}) + p^+(\theta, \mu)N
\end{aligned}$$

This describes the dynamics of the problem in the normal direction at the contact point in terms of two scalar quantities b, which describes the free normal acceleration in absence of any contact forces, and p^+, which we refer to as the Painlevé parameter. The equation shows that if the rod starts at rest in the near vertical position then clearly $b < 0$ and $p^+ > 0$ initially; moreover, b and p^+ are smooth functions of dynamic variables, so, as long as the rod maintains the condition $u > 0$ for slip, these quantities will evolve smoothly and preserve their signs for small times. Hence, for sufficiently short times there is a unique normal force $N = -\frac{b}{p^+} > 0$ that makes the normal acceleration vanish so that the rod remains in contact while it falls. Similarly, if θ is initially small and positive so that the rod is close to horizontal then $b < 0$ and $p^+ > 0$. However, for intermediate angles, depending on other parameters and velocities, p^+ and b may in general take either sign.

A point where b passes from negative to positive while p^+ remains positive represents a point at which the rod simply lifts off the surface; the normal force trends to zero, and the dynamics would lift off into free motion. However, in the case of negative p^+, free acceleration pushes the tip of the rod down towards the surface. However, the normal force would be negative, which violates our complementarity assumption. A positive reaction force would cause acceleration in the same direction as b, pushing the rod down further into the surface, precluding vertical equilibrium. The rod cannot remain in contact with the surface, so it must lift off. It can't though–if we look at free motion with $N = 0$, we have $b < 0$ taking us back into contact. Thus we have a configuration that is physically *inconsistent.*

Consider instead $p^+ < 0$ and $b > 0$; in the absence of any contact forces, the rod would simply lift off with $N = 0$. however, if there is a unique nonzero normal force $N = -b/p > 0$ for which the free normal acceleration b is equilibrated, the rod could remain in contact; however, this case is indeterminate because there is non-uniqueness in the possible outcome.

The conditions under which these paradoxes can happen are within physical possibility. The condition $p < 0$ can be written

$$\mu > \mu_P(\theta) = \frac{r^2 + \ell^2 \cos^2\theta}{\ell^2 \sin\theta\cos\theta}$$

for the uniform rod, $r^2 = \frac{1}{3}\ell^2$, so μ_P is minimized when $\theta = \arctan 2$, in which case the minimum coefficient for which there exists a value for θ for which $p^+ < 0$ is $\mu_P = 4/3$. Thus, for $\mu > 4/3$, an interval of angles θ exists so that p^+ can be nagative. Génot and Brogliato have analyzed this in detail, showing that there is a $\mu_c = \frac{8\sqrt{3}}{9} > 4/3$ for the case of the uniform rod such that for $\mu < \mu_c$ there can be no initial condition that approaches p^+ during slipping–the rod must lift off first. For $\mu > \mu_c$, then there is a thin wedge of initial conditions $(\theta, \dot{\theta})$ for which a paradox can be reached. However, there can be no entry during forward slipping into regions with $p^+ < 0$, and the only way to get there is via reaching a configuration where simultaneously $b = p^+ = 0$, which Or & Rimon define as ”dynamic jam”.

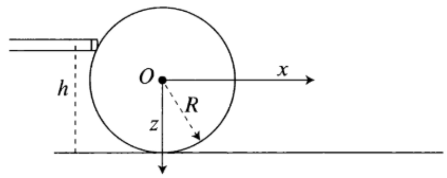

Figure 3: Our coordinate system for the billiard ball problem.

3 Billiard Balls

A billiard ball is often not rolling, but instead moving with a combination of spinning and sliding. We'll again take the mass of this object to be 1 to simplify things. We assume the strike of the cue to be abrupt, imparting an impulsive force $\mathbf{P}$ causing the center of the ball to move along the x-axis with a velocity function $\mathbf{v}$ satisfying $v(0) = |\mathbf{P}|$. At $t = 0$, the torque of the force $\mathbf{P}$ about the center of the ball is

$$\boldsymbol{\tau}(0) = (h - R)v(0)\hat{\mathbf{z}}$$

so for the angle θ through which the ball has rotated around the y-axis, measured clockwise, we have

$$I\dot{\theta}(0) = (h - R)v(0)$$

where the moment of intertia I of the ball is $\frac{2}{5}R^2$. It follows that

$$\dot{\theta}(0) = \frac{5}{2}\left(\frac{h - R}{R^2}\right)v(0)$$

which implies that at $t = 0$ the vector $\mathbf{v}_c$ representing the velocity function of the contact point will have the magnitude

$$v_c(0) = v(0) - R\dot{\theta}(0) = \left(\frac{7R - 5h}{2R}\right)v(0)$$

For the ball to start out with a rolling motion we must have $v_c(0) = 0$ which means that the ball must be hit exactly at the height $h = \frac{7}{5}R$, which just so happens to be the height of the cushions–presumably found from experience by the makers of billiard tables.

For a "high shot" with $h > \frac{7}{5}R$, the direction of $\mathbf{v}_c$ will be opposite to that of $\mathbf{v}$. The spinning of the ball $\dot{\theta}$ is larger than what would be expecting for rolling, given the speed v of the center; the nonzero velocity at the contact point gives rise to a force of friction in the opposite direction, and this force acts on the rigid ball as a whole, causing v to increase, and at the same time causing $\dot{\theta}$ to decrease until rolling begins, and the ball then simply continues to roll. For a "low shot", with $h < \frac{7}{5}R$, the situation is exactly the opposite. The direction of $\mathbf{v}_c$ will be the same as that of $\mathbf{v}$ with $\dot{\theta}$ less than expected, possibly even negative with the ball spinning backwards. So the force of friction will be in the opposite direction as $\mathbf{v}$, causing v to decrease and $\dot{\theta}$ to increase until rolling begins.

Sometimes high or low shots are specifically used to control the behavior of the cue ball after it collides with an object ball at rest. In case of a high shot, since balls have the same mass, and the collision is almost perfectly elastic, at the moment of collision the velocities are interchanged, so that the cue ball now has velocity 0 and the object ball acquires the velocity $\mathbf{v}$ in the same direction. If the cue ball hasn't reached the rolling stage, so that $\mathbf{v}_c \leq 0$, then the friction force $\mathbf{F}$ causes the cue ball to move in the same direction so that we have a follow shot.

In the case of a low shot, the situation is again the opposite: now $\mathbf{v}_c$ points in the same direction as $\mathbf{v}$ so $\mathbf{F}$ points in the opposite direction, causing the cue ball to move backward, a draw shot.

4 Coriolis

The 19th century saw the appearance of a thorough analysis of the dynamics of billiards. The famous Coriolis (1792-1843), when he was a director of studies at the École Polytechnique, felt that the results of *mechanique rationelle* should be used to give general principles applicable to the operation of machinery. In his first book, *Du calcul de l'effet des machine*, he introduced the modern meaning of the term "work" in physics as well as the proper factor $\frac{1}{2}$ into the definition of kinetic energy for conservation of energy to hold. His investigations into what is now known as the Coriolis form were made to account for conservation of energy and thus followed a much more complicated path than the modern purely kinematic approach we have taken in past meetings. His second book was *Théorie mathématique des effets du jeu du billiard* (Mathematical Theory of the Game of Billiards). Though Coriolis' exposition is quite straightforward, modern terminology and English langage helps the exposition.

We first study the path of the cue ball when it is not necessarily hit head on, but possibly to the left or right of center. Coriolis considers both the sliding friction and the very small rolling friction for his initial part of his analysis, and then ignores rolling friction later on, but we will ignore rolling friction from the start.

Using the same coordinate system as before, we use the more general equation for torque that was introduced in previous meetings on rotational motion:

$$\boldsymbol{\tau} = \mathbf{I}\dot{\boldsymbol{\omega}} = \frac{2}{5}R^2\dot{\boldsymbol{\omega}}$$

with $\mathbf{v_c}$ given by

$$\mathbf{v} = \mathbf{v} + \boldsymbol{\omega} \times R\hat{\mathbf{z}}$$

It will be convenient to let $\mathbf{u}(t)$ denote the unit vector in the direction of $\mathbf{v}_c(t)$ and write the frictional force at the contact point as $\mathbf{F} = -\mu\mathbf{u}$. This means that the acceleration $\dot{\mathbf{v}}$ of the ball is $\dot{\mathbf{v}} = -\mu\mathbf{u}$. The torque $\boldsymbol{\tau}$ of the force

$\mathbf{F}$ about the center of the ball is

$$\boldsymbol{\tau} = R\hat{\mathbf{z}} \times \mathbf{F} = -\mu R(\hat{\mathbf{z}} \times \mathbf{u})$$

so that

$$\dot{\boldsymbol{\omega}} = -\frac{5\mu}{2R}\hat{\mathbf{z}} \times \mathbf{u}$$

Differentiating and substituting we have

$$\dot{\mathbf{v}}_c = -\mu\mathbf{u} - \frac{5\mu}{2R}(\hat{\mathbf{z}} \times \mathbf{u}) \times R\hat{\mathbf{z}} = -\frac{7\mu}{2}\mathbf{u}$$

Notice that $\dot{\mathbf{v}}_c$ is in the same direction as $\mathbf{v}_c$ by definition. This implies that $\mathbf{v}_c$ does not change in direction and thus that $\mathbf{u}$ is constant. We can write $\mathbf{v}(t) = \mathbf{v}(0) - \mu t\mathbf{u}$ and thus the center of the ball follows a path c with

$$c(t) = c(0) + t\mathbf{v}(0) - \frac{1}{2}\mu\mathbf{u}t^2$$

which is a parabola. We have the explicit formula

$$\mathbf{u} = \frac{\mathbf{v}(0) + \boldsymbol{\omega}(0) \times R\hat{\mathbf{z}}}{v_c(0)}$$

This all holds only while the ball is not rolling. As soon as it starts rolling it will continue on a straight path along the tangent line to the parabola. Since

$$\mathbf{v}_c(t) = \mathbf{v}_c(0) - \frac{7\mu t}{2}\mathbf{u}$$

and rolling starts at time t_* where we first have $\mathbf{v}_c(t_*) = 0$; we see that

$$t_* = \frac{2v_c(0)}{7\mu}$$

The velocity when the ball starts rolling is

$$\begin{aligned}
\mathbf{v}(t_*) &= \mathbf{v}(0) - \mu\left(\frac{2v_c(0)}{7\mu}\right)\mathbf{u} \\
&= \mathbf{v}(0) - \frac{2v_c(0)}{7}\left[\frac{\mathbf{v}(0) + \boldsymbol{\omega}(0) \times R\hat{\mathbf{z}}}{v_c(0)}\right] \\
&= \frac{5}{7}\mathbf{v}(0) - \frac{2}{7}\boldsymbol{\omega}(0) \times R\hat{\mathbf{z}}
\end{aligned}$$

We can write this result explicitly as

$$\mathbf{v}(t_*) = \frac{1}{7}[(5\mathbf{v}_1(0) + 2R\boldsymbol{\omega}(0))\hat{\mathbf{x}} + (5\mathbf{v}_2(0) - 2R\boldsymbol{\omega}_1(0))\hat{\mathbf{y}}]$$

To get the final deflected angle, which it will be convenient to measure from the y-axis rather than the x-axis, we then have

$$\theta = \arctan\left(\frac{5\mathbf{v}_1(0) + 2R\boldsymbol{\omega}_2(0)}{5\mathbf{v}_2(0) - 2R\boldsymbol{\omega}_1(0)}\right)$$

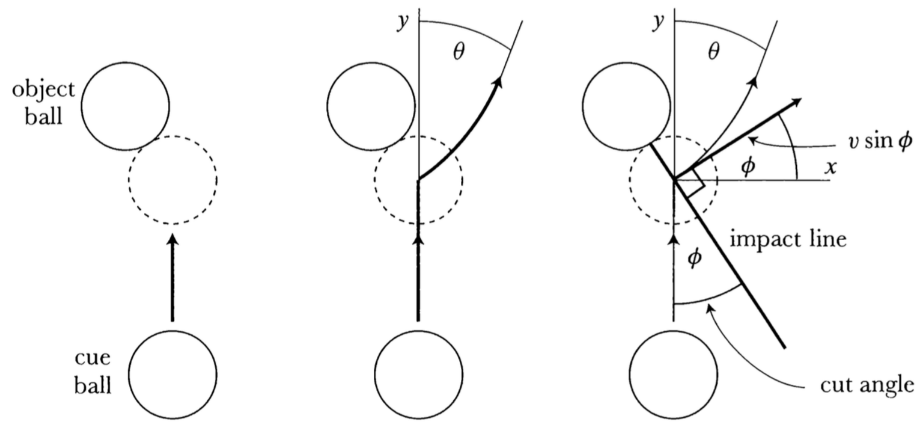

Figure 4: Ball hits ball.

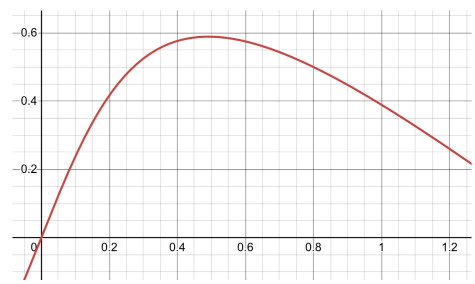

Figure 5: The rule of 30 degrees

Note that μ has disappeared from these equations, so the coefficient of friction of the ball on the cloth has no effect on this final result even though the path itself would change. Now suppose the cue ball is hit straight on along the y-axis to collide at an angle with a stationary object ball. The collision will give the cue a spin and we want to consider the path of the cue after collision and the deflected angle θ. The velocity of the cue ball after the perfectly elastic collision will be perpendicular to the impact line (the line perpendicular to the two balls at the point of contact), and that if the velocity of the cue ball is v at impact, then the velocity after impact will have magnitude $v \sin \phi$ where ϕ is the angle between the original direction of the cue ball and the impact line. This initial velocity, after impact,

$$\mathbf{v}(0) = (v \cos \phi \sin \phi, v \sin^2 \phi, 0)$$

We are assuming that at impact, which we will shift our coordinate t such that impact happens at $t = 0$, the cue ball has $\boldsymbol{\omega}_2(0) = 0$. Setting $\omega = \boldsymbol{\omega}_1(0)$ we get

$$\theta = \arctan\left(\frac{5\mathbf{v}_1(0)}{5\mathbf{v}_2(0) - 2R\omega}\right) = \arctan\left(\frac{5v \sin \phi \cos \phi}{5v \sin^2 \phi - 2R\omega}\right)$$

In particular, if the cue ball is rolling at the time of impact, $\omega = -v/R$ so we get

$$\theta = \arctan\left(\frac{\sin\phi\cos\phi}{\sin^2\phi + 2/5}\right)$$

If we graph this we see that a lot of shots with ϕ within some reasonable range, $\theta \approx 30°$. This is a "thirty degree law" that is actually used by pool players to estimate the direction in which the cue ball will bounce.