Magnetism and Circuits

Raphael Esquivel

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Abstract

This is a continuation of the 'Iolani Physics Club's study of electricity and magnetism for physics bowl. After having covered electrostatics, we cover basic circuitry and magnetostatics, and finish with an introduction to Maxwell's equations.

1 Current and Resistance

Electrons are the charge carriers in metals.[1] The net motion of electrons in a conductor is caused by an electric field, which causes the sea of electrons to move in one direction, like water flowing through a pipe.

1.1 Microscopic definition of current

Definition 1.1. The net motion of electrons in a conductor due to an electric field is the *drift speed* v_d, which is often very small (10^{-4} m/s is typical).

A nonuniform distribution of surface charges along a wire creates a net electric field inside the wire that points from the more positive end of the wire toward the more negative end of the wire. This is the internal electric field that pushes electrons through the wire.

Definition 1.2. *Current* is the rate at which charge moves through a wire. The SI unit for current is the coulomb per second, which is called the *Ampere* A.

If Q is the total amount of charge of charge that has moved past a point in a wire, then

$$I \equiv \frac{dQ}{dt}. \tag{1}$$

For a steady current, it follows that

$$Q = I\Delta t. \tag{2}$$

[1] This is not necessarily the case for every conductor. Ionic solutions or semiconductors have different charge carriers. We will focus on metals because of their importance to circuits.

1.2 Rules of current

The rate of electrons leaving a light bulb (or any other device) is exactly the same as the rate of electrons entering the light bulb. The current does not change. A light bulb doesn't use up current, but it does use energy. The kinetic energy of the electrons is dissipated by their collisions with ions in the metal as electrons move through the atoms. The light bulb affects the amount of current everywhere in the wire.

Because currents were known and studied before it was known what the charge carriers are, the *direction of current* is defined to be the direction in which "positive charges" *seem* to move.

Remark 1.3. *The direction of current in a metal is opposite the direction of motion of the electrons.*

Lemma 1.4. *(Conservation of charge) The total electric charge in an isolated never changes.*

Theorem 1.5. *Due to lemma 1.4, the current must be the same at all points in an individual current-carrying wire.*

Theorem 1.6. *(Kirchhoff's junction law) Lemma 1.4 requires that the sum of the currents entering a junction in a circuit must equal the sum of the currents leaving:*

$$\Sigma I_{in} = \Sigma I_{out} \tag{3}$$

Definition 1.7. The *current density* $\vec{J}$ in a wire is the current per square meter of the cross section:

$$\vec{J} \equiv \frac{I}{A} \tag{4}$$

The direction of this vector is in the direction as the "positive charges".

Remark 1.8. *Current itself is a scalar quantity. When we refer to the "direction of current", we really are referring to the direction of the current density vector.*

1.3 Conducting properties of materials

Definition 1.9. The *conductivity* σ of a material measures how well it can conduct material:

$$J = \sigma E \tag{5}$$

Definition 1.10. The *resistivity* ρ of a material is defined as the inverse of conductivity:

$$\rho = \frac{1}{\sigma} \tag{6}$$

Definition 1.11. The *resistance* R of a conductor is

$$R = \frac{\rho L}{A}. \tag{7}$$

It is measured in volts per ampere, or *ohms* Ω.

Theorem 1.12. *(Ohm's law) Establishing a potential difference between the ends of a conductor creates an electric field via the nonuniform distribution of charges on the surface that, in turn, creates a current that is inversely proportional to the conductor's resistance.*

$$\Delta V = IR \tag{8}$$

2 Fundamentals of circuits

2.1 Potential and power

Theorem 2.1. *(Kirchhoff's loop law) In a closed loop, the potential difference is zero.*

Definition 2.2. The voltage of a battery is often called the *electromotive force/emf* $\mathcal{E}$. **The term electromotive force itself is an outdated term because voltage is not a force**, but $\mathcal{E}$ and "emf" are still widely used.

A battery provides a fixed and unvarying emf. It does not provide a fixed and unvarying current. The amount of current depends jointly on the battery's emf and the resistance of the circuit attached to the battery. If dW is the work the device does to force a positive charge dq from the negative to the positive terminal, then the emf of the device is

$$\mathcal{E} = \frac{dW}{dq} \tag{9}$$

Power is the rate of energy transfer $\frac{dU}{dt}$; in an electrical component in a circuit, $U = qV \Rightarrow P = \frac{dq}{dt}V$.

Theorem 2.3. *Across an electrical component in a circuit,*

$$P = I\Delta V = I^2 R = \frac{\Delta V^2}{R}. \tag{10}$$

Corollary 2.4. *The rate at which the chemical energy in the battery changes is*

$$P_{emf} = I\mathcal{E} \tag{11}$$

Corollary 2.5. *The rate at which electrical energy is dissipated as thermal energy across a component of resistance R is*

$$P_R = I^2 R = \frac{\Delta V^2}{R} \tag{12}$$

We have spoken of potential differences up to this point. Recall from the last lecture that the potential at a point depends on a point of reference. In many practical cases, we use the earth as a point of reference, and define its potential to be "zero".

Definition 2.6. A circuit that is connected to the earth is said to be *grounded.*

Because the wire connecting the circuit to the earth is not part of a complete circuit, there is no current in this wire. It is an equipotential, so it gives one point in the circuit the same potential as the earth; this does not change the way in which the circuit functions whatsoever, but allows us to define the voltage of the circuit instead of using voltage differences.

2.2 Components in circuits and methods of measurement

Definition 2.7. *Resistors* are electrical components that limit the current in a circuit. They have a resistance significantly higher than the metal wires.

Resistors placed end to end, with no junctions between them, are said to be *in series.*

Theorem 2.8. *The equivalent resistance of N resistors in series is the sum of their resistances:*

$$R_{eq} = R_1 + R_2 + ... + R_N \tag{13}$$

It follows that *resistors in series all have the same current.*

To measure current in a circuit, we use an *ammeter*, Because charge flows through circuit elements, an ammeter must be placed in series with the circuit element whose current is to be measured.

Resistors connected at both ends are said to be *in parallel.*

Theorem 2.9. *The equivalent resistance of N resistors in parallel is defined as:*

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_N} \tag{14}$$

It follows that *resistors in parallel all have the same potential difference.*

To measure potential difference across a circuit element, we use a *voltmeter.* Because potential difference is measured across a circuit element, a voltmeter must be placed in parallel with the circuit element whose potential difference is to be measured.

Definition 2.10. A circuit consisting of resistors and capacitors is referred to as an *RC circuit.*

Let the *time constant* $\tau = RC$. Then, for a charging capacitor in an RC circuit,

$$\begin{aligned} Q &= Q_0(1 - e^{-t/\tau}) \\ \Delta V &= \mathcal{E}(1 - e^{-t/\tau}) \qquad \text{(Capacitor charging)} \\ I &= I_0 e^{-t/\tau} \end{aligned} \tag{15}$$

and for a discharging capacitor,

$$\begin{aligned} Q &= Q_0 e^{-t/\tau} \\ \Delta V &= \mathcal{E} e^{-t/\tau} \qquad \text{(Capacitor discharging)} \\ I &= -I_0 e^{-t/\tau} \end{aligned} \tag{16}$$

3 The magnetic field

3.1 Defining the magnetic field

Recall that in electrostatics, charges are stationary. The notion of current implies moving charges. The reason we must separate electrostatics from instances of current is because current creates something called a magnetic field.

The radical difference between magnetostatics and electrostatics is that *there are no free magnetic charges.* This means that magnetic phenomena are quite different from electric phenomena and that for a long time no connection was established between them. The basic entity in magnetic studies was what we now know as a *magnetic dipole.* In the presence of magnetic materials the dipole tends to align itself in a certain direction. That direction is by definition the direction of the magnetic-flux density $\vec{B}$, provided the dipole is sufficiently small and weak that it does not perturb the existing field. Already, in the definition of the magnetic-flux density,[2] we have a more complicated situation than for the electric field. Further quantitative elucidation of magnetic phenomena did not occur until the connection between currents and magnetic fields was established. Conservation of charge demands that the charge density at any point in space be related to the current density in that neighborhood by a continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \tag{17}$$

This expresses the physical fact that a decrease in charge inside a small volume with time must correspond to a flow of charge through the surface of the small volume, since the total number of charges must be conserved. *Steady-state magnetic phenomena* are characterized by no change in the net charge density anywhere in space; consequently in magnetostatics

$$\nabla \cdot \vec{J} = 0 \tag{18}$$

Definition 3.1. The *magnetic field* $\vec{B}$ is defined to have the following properties:

- A magnetic field is created at all points in space surrounding a current (*Ørsted's law*)
- A magnetic field exerts forces on magnetic poles. The force on the north pole is parallel to $\vec{B}$, the force on a south pole is opposite $\vec{B}$.
- The magnitude of the magnetic field is measured in teslas $\mathrm{T} \equiv \mathrm{N/A \cdot m}$
- The magnetic field obeys the principle of superposition.

A way to picture the field is with the use of *magnetic field lines*; these are imaginary drawn through a region of space so that a tangent to a field line is in the direction of the magnetic field, and the field lines are closer together where the magnetic stregnth is larger.

Theorem 3.2. *(Ampère's right-hand grip rule) To find the direction of a magnetic field from a current,*

1. *Point your right thumb in the direction of the current.*
2. *Curl your fingers around the wire to indicate a circle.*
3. *Your fingers point in the direction of the magnetic field lines around the wire.*

Another right-hand rule that is quite useful (especially in electromagnetism, but can be applied anywhere) is the right hand rule for cross products.

[2] We will use the term "magnetic field" for $\vec{B}$, which is different from the electric field strength $\vec{H}$; the two are related by the *permeability constant* with $\vec{B} = \mu_0 \vec{H}$.

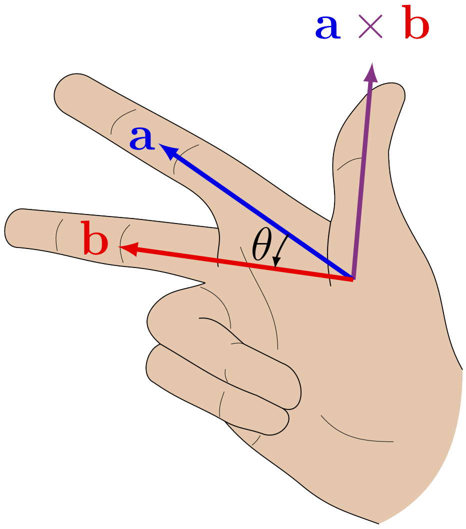

Figure 1: Right hand rule for cross products.

Theorem 3.3. *(Right hand rule for cross products) The cross product of two vectors $\vec{a}$ and $\vec{b}$ is a vector perpendicular to the plane spanned by $\vec{a}$ and $\vec{b}$. If you put the index of your right hand on $\vec{a}$ and the middle finger on $\vec{b}$, then the thumb points in the direction of $\vec{a} \times \vec{b}$ (see figure 1).*

3.2 Basic magnetism formulas

As previously established, moving charges are the source of the magnetic field.

Theorem 3.4. *(Lorentz force law) A moving charge q in a magnetic field experiences a magnetic force given by*

$$\vec{F} = q(\vec{v} \times \vec{B}) = qvB \sin\theta \tag{19}$$

This is a fundamental axiom of the theory, whose justification is to be found in experiments.

Corollary 3.5. *The force on a charge in an electromagnetic field is*

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{20}$$

This means that for a segment of current carrying wire in a magnetic field,

$$\vec{F} = \int (\vec{v} \times \vec{B}) dq = \int (\vec{v} \times \vec{B}) \lambda dl = \int I(d\vec{\ell} \times \vec{B}) \tag{21}$$

Therefore, for a straight, current-carrying wire in a magnetic field,

$$\vec{F} = I\vec{\ell} \times \vec{B} = I\ell \sin\theta \tag{22}$$

Definition 3.6. The *permeability constant*

$$\mu_0 \approx 1.26 \times 10^{-6}\ \mathrm{T \cdot m/A}$$

Theorem 3.7. *(Biot-Savart law) For a charged particle q of velocity $\vec{v}$, the magnetic field created by the point charge at a radius r is*

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q \times v}{r^2} \tag{23}$$

Equivalently,

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{qv\sin\theta}{r^2} \tag{24}$$

with the direction given by theorem 3.2. More precisely for a segment of current I,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{\ell} \times \hat{r}}{r^2} \tag{25}$$

Corollary 3.8. *For a long, straight, current-carrying wire,*

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{r} \tag{26}$$

Corollary 3.9. *In a coil of N turns of wire, the field at the center is*

$$\vec{B} = \frac{\mu_0}{2}\frac{NI}{R} \tag{27}$$

By applying the right hand rule to the wire within the coil, it becomes evident that the direction of the magnetic field lines outside of the coil is opposite to those inside the coil–and that we can distinguish one side of the coil as being where magnetic field lines "flow in" and "flow out". In fact, current loops exert the same properties as a permanent magnet–*the field lines flow into the south pole and out through the north pole.*

Definition 3.10. *A magnet created by a current in a coil of wire is called an electromagnet. It acts in every way that a permanent magnet does.*

3.3 Maxwell's equations

Theorem 3.11. *(Ampere's Law) Along any closed path,*

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enclosed} \tag{28}$$

This can be derived from theorem 3.7.

For example, let's consider an *ideal solenoid*; that is, a solenoid that is infinitely long and has a negligible outside magnetic field. In this case, we can construct a rectangular loop (see figure 2). The total current going through the loop is equal to the current of the solenoid multiplied my the amount of times that it passes through; that is, if we let n be the number of turns per unit length, then

$$I_{enclosed} = inh$$

Integrating over the rectangle, segments $\overline{ad}$ and $\overline{cb}$ vanish because the magnetic field lines are orthogonal to the segments, while $\overline{dc}$ has no magnetic field; that leaves

$$\int \vec{B} \cdot d\vec{s} = Bh = \mu_0 inh$$

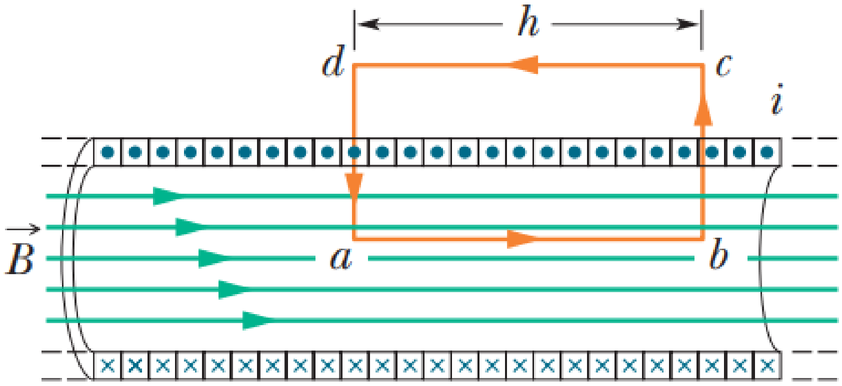

Figure 2: Application of Ampere's law to an ideal solenoid.

Theorem 3.12. *Inside an ideal solenoid with current i and n turns per length,*

$$B = \mu_0 i n \tag{29}$$

Definition 3.13. The *magnetic flux* through a loop enclosing area A is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \tag{30}$$

Lemma 3.14. *(Maxwell-Faraday equation) Recall from the last meeting that the Maxwell-Faraday equation for stationary charges is*

$$\nabla \times \vec{E} = 0 \tag{31}$$

which appears in its general form as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{32}$$

Lemma 3.15.

$$\begin{aligned} W &= \oint \vec{F} \cdot d\vec{s} = q_0 \oint \vec{E} \cdot d\vec{s} \\ \mathcal{E} &= \oint \vec{E} \cdot d\vec{s} \end{aligned} \tag{33}$$

Theorem 3.16. *(Faraday's law) For a loop of wire, there will be an induced emf satisfying*

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{34}$$

Or, making the substitution from lemma 3.15,

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{35}$$

This essentially tells us that a changing magnetic flux induces an electric field.

Corollary 3.17. *(Lenz's law) An induced current has a direction such that the magnetic field due to the current opposes the change in the magnetic field that induces the current.*

After lots of study, we have finally have arrived at, ironically, the fundamentals of electromagnetism: Maxwell's equations. These are a set of four coupled partial differential equations form classical electromagnetism. We have already covered three–Gauss' law, Faraday's law, and Ampere's law–but there is one more that we implied in subsection 3.1, and that is Gauss' law for magnetism.

Theorem 3.18. *(Gauss' law for magnetism) Electric charges have no magnetic analogues, called magnetic monopoles; no north or south poles exist in isolation. Instead, the magnetic field of a material is attributed to a dipole. Mathematically,*

$$\nabla \cdot \vec{B} = 0 \tag{36}$$

By the divergence theorem, this statement is equivalent to saying that

$$\oiint \vec{B} \cdot d\vec{s} = 0 \tag{37}$$

One final addition: theorem 3.11 does not account for *displacement current*, which is a current created by a time-varying electric field rather than moving charges. For the notation, let volumes be denoted as Ω, their boundary surfaces $d\Omega$, and surfaces Σ, with their boundary curves $\partial\Sigma$. Behold, **Maxwell's equations:**

$$\begin{aligned}
&\textbf{Integral form}\\
\text{Gauss' law: } & \oiint_{\partial\Omega} \vec{E} \cdot d\vec{s} = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho dV\\
\text{Gauss' law for magnetism: } & \oiint_{\partial\Omega} \vec{B} \cdot d\vec{s} = 0\\
\text{Maxwell-Faraday equation: } & \oint_{\partial\Sigma} \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \iint_{\Sigma} \vec{B} \cdot d\vec{s}\\
\text{Ampere's law: } & \oint_{\partial\Sigma} \vec{B} \cdot d\vec{l} = \mu_0 \left(\iint_{\Sigma} \vec{J} \cdot d\vec{s} + \epsilon_0 \frac{d}{dt} \iint_{\Sigma} \vec{E} \cdot d\vec{s} \right)\\
&\textbf{Differential form}\\
\text{Gauss' law: } & \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}\\
\text{Gauss' law for magnetism: } & \nabla \cdot \vec{B} = 0\\
\text{Maxwell-Faraday equation: } & \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\\
\text{Ampere's law: } & \nabla \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)
\end{aligned} \tag{38}$$