Lagrangian Mechanics I

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Abstract

This is an introduction to the concept of Lagrangian mechanics. This comes second in the 'Iolani Physics Club's series of fun meetings on mechanics. Unlike previous meetings where the use of calculus was mentioned but largely avoided, I wish to discuss the subject rigorously, while conserving some degree of intuition. Unfortunately, mathematical rigor in the case of a topic like Lagrangian mechanics requires a good deal of ∂'s and $\int$'s. The approach taken is to introduce concepts of the calculus of variations and showing how the Lagrangian ties into that, before formally defining the Lagrangian for various systems. A discussion on Noether's theorem will also be included, as it is of great significance in the general understanding of physics.

1 Introduction

It is a well-known fact that the shortest path between two points on a plane is a straight line. To prove this, we can use the *calculus of variations*. Say we have two points, (x_1, y_1) and (x_2, y_2), and a pth, $y = y(x)$, joining them. Our task is to find the path that has the shortest length. The length of a short segment of the path is $ds = \sqrt{dx^2 + dy^2}$, which we can rewrite as $ds = \sqrt{1 + \dot{y}(x)^2}dx$ since $dy = \frac{dy}{dx}dx \equiv \dot{y}(x)dx$. This gives us the arc length

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \dot{y}(x)^2}dx. \tag{1}$$

We wish to minimize this integral. This entails finding a particular path such that the integral has a value that is *stationary* relative to paths differing infinitesimally from the correct one (think of stationary points and relative minima and maxima).

For now, consider an integral J for which we wish to find a stationary value for the correct path relative to *any* neighboring path. The variation must be zero relative to some particular set of neighboring paths labeled by an infinitesimal parameter α. Such a set of paths might be denoted by $y(x, \alpha)$, with $y(x, 0)$ the correct path. Assuming our simple case of finding the shortest path between x_1 and x_2, this time generalizing to any function f between x_1 and x_2,

$$J(\alpha) = \int_{x_1}^{x_2} f[(y(x, \alpha), \dot{y}(x, \alpha), x]dx \tag{2}$$

From minimization in calculus,

$$\left(\frac{dJ}{d\alpha}\right)_{\alpha=0} = 0. \tag{3}$$

When we differentiate both sides of equation 2 with respect to α using the rules of integration under the integral sign, we find that

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right) dx. \tag{4}$$

We can say that

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} dx \equiv \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x \partial \alpha} dx, \tag{5}$$

so integrating by parts,

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x \partial \alpha} dx = \frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha} \,|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha} dx \tag{6}$$

The conditions on all the varied curves is that they pass through the points (x_1, y_1) and (x_2, y_2), so $\partial y/\partial \alpha$ at x_1 and x_2 must vanish, therefore the first term of equation 6 vanishes, so 4 reduces to

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha} dx, \tag{7}$$

which is zero at $\alpha = 0$.

Lemma 1.1. *(Fundamental lemma of the calculus of variations) If*

$$\int_{x_1}^{x_2} M(x)\eta(x)dx = 0 \tag{8}$$

for all arbitrary functions $\eta(x)$ continuous through the second derivative, then $M(x)$ must identically vanish in the interval (x_1, x_2).

While a formal mathematical proof of the lemma can be found textbooks on calculus of variations, we're interested in the physics, namely what follows by applying the lemma to equation 7.

Theorem 1.2. *(Euler-Lagrange equation) For an integral of a function $f(y(x), \dot{y}(x), x)$ from points x_1 to x_2 to have a stationary value,*

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0 \tag{9}$$

for all x in the relevant interval. If there are n dependent variables within the original integral, there are n Euler-Lagrange equations.

Applying this to our minimization of length, we let

$$f(y, \dot{y}, x) = \sqrt{1 + \dot{y}^2}. \tag{10}$$

Evaluating the partial derivatives,

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} \tag{11}$$

$$\Rightarrow \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0 \tag{12}$$

This implies that $\partial f/\partial \dot{y}$ is a constant, C, therefore

$$\dot{y}^2 = C^2(1+\dot{y}^2). \tag{13}$$

It is quickly apparent that $\dot{y}$ is constant–we'll let $\dot{y} = m$ and integrating, $y = mx + b$.

2 Defining the Lagrangian

2.1 Least action

In the Lagrangian formulation of mechanics, the independent variable is time. The dependent variables are the coordinates that specify the position or "configuration" of a system, and are usually denoted by $q_1, q_2, ..., q_n$. The number n of coordinates depends on the nature of the system; for a single particle in three dimensions, q_1, q_2, q_3 could be x, y, z, could be r, θ, ϕ, or something else. For N particles moving freely in three dimensions, n is $3N$. Because the coordinates $q_1, ..., q_n$ can take so many forms, they are called **generalized coordinates**. Mathematically, we can think of a system of n generalized coordinates as defining a point in an n-dimensional **configuration space**, each of whose points labels a unique configuration of the system. Each position $\mathbf{r}$ as a function of $q_1, q_2, ..., q_n$, and possibly t, and conversely, each q_i can be expressed in terms of the positions and possible t.

The integral whose stationary value determines the evolution of the mechanical system is called the action integral, and its integrand is the Lagrangian $\mathcal{L} = \mathcal{L}(q_1, \dot{q}_1, ..., q_n, \dot{q}_n, t)$.

Definition 2.1. *The **action integral** is defined as*

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, \dot{q}_1, ..., q_n, \dot{q}_n, t)dt. \tag{14}$$

Theorem 2.2. *(Hamilton's principle of least action) The actual path which a particle follows between two points in a given time interval is such that the action integral is stationary when taken along the actual path.*

2.2 Unconstrained Lagrangian

Corollary 2.3. *(Lagrange's equations) For the Lagrangian, the Euler-Lagrange equations are called Lagrange's equations:*

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \tag{15}$$

for $k = 1, ..., n$.

Definition 2.4. *The **Lagrangian function**, or just **Lagrangian**, is defined as*

$$\mathcal{L} = T - U, \tag{16}$$

where T is the kinetic energy and U is the potential energy.

Equation 16 is specifically for the *non-relativistic Lagrangian for a system of particles in the absence of an electromagnetic field.*

Consider a particle that moves unconstrained in three dimensions, subject to a net force $\mathbf{F}(\mathbf{r})$. The particle's kinetic energy is of course

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{\mathbf{r}}^2 \tag{17}$$

and its potential energy is

$$U = U(\mathbf{r}). \tag{18}$$

Consider the partials of the Lagrangian

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= -\frac{\partial U}{\partial x} = F_x, \\ \frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x. \end{aligned} \tag{19}$$

Equation 15 leads us to

$$F_x = \frac{d}{dt}(p_x), \tag{20}$$

which is Newton's second law! **The Lagrangian formulation of mechanics has no *fundamental* difference to Newtonian mechanics; they're both equally valid models of the same phenomena.**

Definition 2.5. *The **generalized force** is defined as*

$$\frac{\partial \mathcal{L}}{\partial q_i} = \textit{ith component of generalized force} \tag{21}$$

*and the **generalized momentum** is*

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \textit{ith component of generalized momentum.} \tag{22}$$

If we work in polar coordinates, the ϕ component of generalized force is torque, and the ϕ component of generalized momentum is angular momentum. Generalized forces/momenta need not necessarily have the dimensions of force/momentum.

Definition 2.6. *When the Lagrangian is independent of a coordinate q_i, that coordinate is sometimes said to be **ignorable** or **cyclic**.*

What we mean by this is that $\mathcal{L}$ is unchanged, or *invariant*, when q_i varies (with all other generalized coordinates held fixed). Thus we can say that if $\mathcal{L}$ is invariant under variations of a coordinate q_i, then the corresponding generalized momentum p_i is conserved. It is a good idea to choose coordinates so that as many as possible are ignorable and their corresponding momenta are constant. In fact, this is perhaps the main criterion in choosing generalized coordinates for any given problem.

2.3 Constrained Systems

One of the advantages of the Lagrangian is that it is especially useful in solving problems involving *constrained systems.*

The number of degrees of freedom of a system is the number of coordinates that can be independently varied in a small displacement–the number of "directions" in which the system can move from any given initial configuration. A particle that is free to move anywhere in three dimensions has three degrees of freedom, and a system of N particles has $3N$. When the number of degrees of freedom of an N-particle system in 3 dimensions is less than $3N$, we say that it is *constrained.* For example, a simple pendulum in a two-dimensional plane only has one degree of freedom, for if we consider polar coordinates, while its angle ϕ can be varied, its radius r is constrained to the length of the string.

Definition 2.7. *A system is* ***holonomic*** *if it has a number of degrees of freedom equal to the number of generalized coordinates needed to describe the system's configuration.*

Holonomic systems are easier to treat than nonholomonic ones.

3 Noether's Theorem

3.1 Conservation of total momentum

The total momentum of an isolated system of N particles is conserved. One of the most prominent features of an isolated system is that it is *translationally invariant*; that is, if we transport all N particles bodily through the same displacement ϵ, nothing physically significant about the system should change. Suppose that we move a whole system through this fixed displacement, replacing every position $\mathbf{r}_\alpha$ by $\mathbf{r}_\alpha + \epsilon$ for $\alpha = 1, 2, ..., N$. The potential energy should be unaffected by this displacement, so that

$$U(\mathbf{r_1} + \epsilon, ..., \mathbf{r_N} + \epsilon, t) = U(\mathbf{r_1}, ..., \mathbf{r_N}, t), \tag{23}$$

or, more briefly,

$$\delta U = 0, \tag{24}$$

where δU denotes the change in U under this translation. Clearly the velocities are unchanged by the translation, as ϵ vanishes when we take the time derivative of position. Therefore $\delta T = 0$, hence

$$\delta \mathcal{L} = 0. \tag{25}$$

This result is true for any displacement ϵ. This is an example of a mathematical symmetry.

Definition 3.1. *A mathematical* ***symmetry*** *is a type of invariance: the property that a mathematical object remains unchanged under a set of operations or transformations.*

A *continuous symmetry* is the intuitive idea of viewing some symmetries as continuous motions (like the variable ϵ) rather than discrete symmetry (e.g. reflection symmetry, which is invariant under a kind of flip).

Theorem 3.2. *(Noether's theorem) If a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time.*

In our context, for each symmetry of the Lagrangian, there is a conserved quantity. The symmetry that we've just treated is *spatial translation invariance*, and serves as an example of Noether's theorem. If we choose ϵ to be an infinitesimal displacement in the x-direction, then all the x coordinates of the N particles $x_1, ..., x_N$ increase by ϵ, while the y and z coordinates are unchanged, the change in $\mathcal{L}$ is

$$\delta\mathcal{L} = \epsilon\frac{\partial\mathcal{L}}{\partial x_1} + ... + \epsilon\frac{\partial\mathcal{L}}{\partial x_N} = 0, \tag{26}$$

which implies that

$$\sum_{\alpha=1}^{N}\frac{\partial\mathcal{L}}{\partial x_\alpha} = 0; \tag{27}$$

Now using Lagrange's equations we can rewrite each derivative as

$$\frac{\partial\mathcal{L}}{\partial x_\alpha} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_\alpha} = \frac{d}{dt}p_{\alpha x}, \tag{28}$$

where $p_{\alpha x}$ is the x component of the generalized momentum of particle α. Thus equation 27 becomes

$$\sum_{\alpha=1}^{N}\frac{d}{dt}p_{\alpha x} = \frac{d}{dt}P_x = 0, \tag{29}$$

where P_x is the x component of the total generalized momentum; by choosing the small displacement ϵ successively in the y and z directions, we can prove the same result for the y and z components, and we can reach the conclusion that **the spatial translation invariance of $\mathcal{L}$ implies conserved total generalized momentum**. Note that if

$$\frac{\partial U(x)}{\partial x} \neq 0, \tag{30}$$

then U is not stationary for small displacements ϵ; that is,

$$\delta U \neq 0 \Rightarrow \delta\mathcal{L} \quad \neq 0, \tag{31}$$

and the symmetry is broken. Recall that Newton's second law states that $F = -\partial U/\partial x$, so saying that $\delta U \neq 0$ implies that there is a force that is acting on the system (at which point it is no longer an isolated system). Conclusively, *the translational symmetry is broken if the system is in a nontrivial potential* (nontrivial meaning not vanishing with differentiation). Momentum is not conserved either, because again, by Newton's second law, the force generates a change in momentum.

3.2 Conservation of energy

Consider a time-dependent Lagrangian. By the chain rule,

$$\frac{d}{dt}\mathcal{L} = \sum_i\frac{\partial\mathcal{L}}{\partial q_i}\dot{q}_i + \sum_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\ddot{q}_i + \frac{\partial\mathcal{L}}{\partial t}. \tag{32}$$

By Lagrange's equation, for the first sum,

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{d}{dt}p_i = \dot{p}_i. \tag{33}$$

Meanwhile, for the second sum, $\partial \mathcal{L}/\partial \dot{q}$ is just generalized momentum p_i, so we can rewrite the time derivative of $\mathcal{L}$ as

$$\begin{aligned} \frac{d}{dt}\mathcal{L} &= \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} \\ &= \frac{d}{dt}\sum_i (p_i \dot{q}_i) + \frac{\partial \mathcal{L}}{\partial t} \end{aligned} \tag{34}$$

For many interesting systems, the Lagrangian does not depend *explicitly* on time; that is, $\partial \mathcal{L}/\partial t = 0$. When this is the case, the second term on the right of 34 vanishes; if we move the left size to the right, we see that the time derivative of $\Sigma p_i \dot{q}_i - \mathcal{L}$ is zero.

Definition 3.3. *The **Hamiltonian** of a system is defined as the quantity*

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L} \tag{35}$$

Provided the relation between generalized coordinates and Cartesians is time-independent,

$$\mathcal{H} = T + U, \tag{36}$$

the total energy.

Theorem 3.4. *If the Lagrangian does not depend explicitly on time, then the Hamiltonian is conserved.*

To prove that the Hamiltonian is total energy for natural generalized coordinates, let us express the total kinetic energy $T = \frac{1}{2}\Sigma m_\alpha \dot{\mathbf{r}}_\alpha^2$ in terms of the generalized coordinates $q_1, ..., q_n$. By the chain rule,

$$\dot{\mathbf{r}}_\alpha = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\dot{q}_i \tag{37}$$

The scalar product of this equation with itself is

$$\dot{\mathbf{r}}_\alpha^2 = \left(\sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\dot{q}_j\right) \cdot \left(\sum_k \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\dot{q}_k\right). \tag{38}$$

Therefore,

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2 = \frac{1}{2}\sum_{j,k} A_{jk}\dot{q}_j\dot{q}_k \tag{39}$$

where A_{jk} is shorthand for the sum:

$$A_{jk} = \sum_\alpha m_\alpha \left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right) \tag{40}$$

Differentiating the kinetic energy with respect to $\dot{q}_i$ yields the generalized momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_j A_{ij}\dot{q}_j \tag{41}$$

so we can rewrite the sum for the definition of the Hamiltonian 35 as

$$\sum_i p_i \dot{q}_i = \left(\sum_i \sum_j A_{ij}\dot{q}_j \right) \dot{q}_i = \sum_{i,j} A_{ij}\dot{q}_i\dot{q}_j = 2T \tag{42}$$

Therefore,

$$\mathcal{H} = \sum_i p_i q_i - \mathcal{L} = 2T - (T - U) = T + U. \tag{43}$$

It is thus another example from Noether's theorem that **a system for which $\mathcal{L}$ is invariant under a time translation has its total energy conserved**.