Rigid Body Motion and the Dzhanibekov Effect

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1 Introduction

Welcome to this break in the quantum mechanics sequence of 'Iolani physics club. Today we consider the general motion of the rigid body.

2 The rigid body

For this treatment, we restrict ourselves to the motion of the **rigid body**. A set of N particles constitute a rigid body if the distance between any 2 particles is fixed.

$$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j| = \text{constant} \tag{1}$$

If we know 3 non-colinear points in the body–that is, we cannot draw a straight line connecting all of them–then all remaining points are fully determined by triangulation. The first point has 3 coordinates for a translation in 3 dimensions. The second point has 2 coordinates for a spherical rotation about the first point, as the distance r_{12} is fixed. The third point has one coordinate for circular rotation about the axis of $\mathbf{r}_{12}$, as r_{13} and r_{23} are fixed. Hence **there are 6 *independent* coordinates for the rigid body**.

Theorem 2.1. ***(Chasles' theorem)*** *The most general rigid body displacement can be produced by a translation along a line (called its screw axis or Mozzi axis) followed (or preceded) by a rotation about an axis parallel to that line.*

Theorem 2.2. ***(Euler's theorem)*** *Any change in orientation can be described by rotation about an axis through a chosen reference point.*

If a force (or a component thereof) is perpendicular to the direction of motion, this will cause the velocity vector to turn. The corresponding acceleration is called **radial acceleration**, which points towards the instantaneous center of rotation and does not affect speed, only direction. Acceleration parallel to the direction of motion is **tangential acceleration**, which in turn does not affect direction but speed. These two components make up the net acceleration. A simple example of rotation is the special case of rotational motion of a particle about a fixed axis of rotation. If we take tangential acceleration to be zero and radial acceleration to be constant, we have obtained **uniform circular motion.**

3 Rotational kinematics

Many aspects of rotational motion can be thought of in terms of their corresponding analogs in translational motion. Common analogues include:

$$\text{Position } s \rightarrow \text{ Angle } \theta$$
$$\text{Velocity } v \rightarrow \text{Angular velocity } \omega \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)$$
$$\text{Acceleration } a \rightarrow \text{ Angular acceleration } \alpha \left(\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}, \frac{\mathrm{d}\omega}{\mathrm{d}t}\right)$$
$$\text{Mass } m \rightarrow \text{ Moment of inertia } I \left(\sum Mr^2\right)$$
$$\text{Momentum } p \rightarrow \text{ Angular momentum } L$$
$$\text{Force } F \rightarrow \text{ Torque } \tau$$

Many kinematics equations hold their form in their rotational analogues. For example,

$$s = v_0 t + \frac{1}{2}at^2 \rightarrow \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$
$$F = ma = \frac{\mathrm{d}p}{\mathrm{d}t} \rightarrow \tau = I\alpha = \frac{\mathrm{d}L}{\mathrm{d}t}$$
$$p = mv \rightarrow L = I\omega$$
$$T = \frac{1}{2}mv^2 \rightarrow T = \frac{1}{2}I\omega^2$$

For any rigid body rotating with angular velocity ω about an axis through an origin, the velocity of any point P (described by a position vector $\mathbf{r}$ with respect to the origin) fixed on the body is

$$\mathbf{v} = \omega \times \mathbf{r} \tag{2}$$

We can rearrange this to obtain $\omega = \frac{\mathbf{r}\times\mathbf{v}}{r^2}$. For a point mass, the moment of inertia is $I = mr^2$. The orbital angular momentum of a particle in motion about an origin is expressed as

$$\mathbf{L} = I\omega \tag{3}$$

Then, we find that we can express the angular velocity of a point particle about an axis through an origin is

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} \tag{4}$$

Consider the derivative with respect to time of $\mathbf{L}$. We get

$$\dot{\mathbf{L}} = m\dot{\mathbf{r}} \times \mathbf{v} + \mathbf{r} \times (m\mathbf{a}) \tag{5}$$

Choosing the reference point to be fixed, $\dot{\mathbf{r}} = \mathbf{v}$ and the cross product vanishes. We now get the following theorem.

Theorem 3.1. *(Euler's second law) The* ***torque*** *with respect to a reference point is defined by $\tau = \mathbf{r} \times \mathbf{F}$. Choosing the same reference point,*

$$\dot{\mathbf{L}} = \tau \tag{6}$$

4 The moment of inertia tensor

For a rigid body–which again, is just a system of particles with a constraint–we integrate to find that

$$\mathbf{L} = \int_m \mathbf{r}' \times (\omega \times \mathbf{r}')dm \tag{7}$$

where we denote by $\mathbf{r}'$ the position vector with respect to the center of mass. from the Using the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$ to then obtain

$$\mathbf{L} = \int_m [(\mathbf{r}' \cdot \mathbf{r}')\omega - (\mathbf{r}' \cdot \omega)\mathbf{r}']dm \tag{8}$$

For planar bodies undergoing a 2-dimensional motion in its own plane, $\mathbf{r}'$ is perpendicular to ω, so the dot product is zero. In this case, ω and $\mathbf{L}$ are always parallel. In the three-dimensional case, however, the dot product does not vanish, and ω and $\mathbf{L}$ are in general not parallel.

In Cartesian coordinates, the vectors are expressed as $\mathbf{r}' = x'\hat{\mathbf{i}} + y'\hat{\mathbf{j}} + z'\hat{\mathbf{k}}$ and $\omega = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$. These axes are *fixed* to the body; that is, they rotate with the body. We then expand to get

$$\begin{aligned}
\mathbf{L} = &\left(\omega_x \int_m (x'^2 + y'^2 + z'^2)dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')x'dm\right)\hat{\mathbf{i}} \\
&+ \left(\omega_y \int_m (x'^2 + y'^2 + z'^2)dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')y'dm\right)\hat{\mathbf{j}} \\
&+ \left(\omega_z \int_m (x'^2 + y'^2 + z'^2)dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')z'dm\right)\hat{\mathbf{k}}
\end{aligned} \tag{9}$$

We now define the following quantities. The *moments of inertia* with respect to the x, y, and z axes, respectively, are given by

$$I_{xx} = \int_m (y'^2 + z'^2)dm, \quad I_{yy} = \int_m (x'^2 + z'^2)dm, \quad I_{zz} = \int_m (x'^2 + y'^2)dm \tag{10}$$

The quantity in the integrand is precisely the square of the distance to the x, y, and z axis respectively, analogous to the two-dimensional case. We also define the *products of inertia*

$$I_{xy} = I_{yx} = \int_m x'y'dm, \quad I_{xz} = I_{zx} = \int_m x'z'dm, \quad I_{yz} = I_{zy} = \int_m y'z'dm \tag{11}$$

They measure the imbalance in the mass distribution. Our expression is then

$$\begin{aligned}
\mathbf{L} = &(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\hat{\mathbf{i}} \\
&+ (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\hat{\mathbf{j}} \\
&+ (-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)\hat{\mathbf{k}}
\end{aligned} \tag{12}$$

One writes this in terms of the *moment of inertia tensor*

$$\mathbf{L} = \mathbf{I}\omega \tag{13}$$

with

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \tag{14}$$

This tensor is symmetric since $I_{ij} = I_{ji} \Rightarrow \mathbf{I} = \mathbf{I}^{\mathbf{T}}$. While it is beyond the scope of our discussion, this means that we can always *diagonalize* the inertia tensor. That is, we can always choose the orthogonal basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ such that the inertia tensor is of the form

$$\mathbf{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \tag{15}$$

In this case, I_{xx}, I_{yy}, and I_{zz} are called the *principal moments of inertia*, and the axes x, y, z are the *principal axes*.

5 Euler's equations

Our earlier relation $\mathbf{v} = \omega \times \mathbf{r}$ is a generalization of the relation $v = \omega r$. It is perhaps worth emphasizing that there is a corresponding relation for *any* vector fixed in the rotating body. For example, if $\mathbf{e}$ is a unit vector that rotates with the body, then its rate of change, as seen from the non-rotating frame, is $\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \omega \times \mathbf{e}$. If $\mathbf{f}(t)$ is a vector function written $f_1(t)\hat{\mathbf{i}} + f_2(t)\hat{\mathbf{j}} + f_3(t)\hat{\mathbf{k}}$. Using the product rule of differentiation,

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} = \frac{\mathrm{d}f_1}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}f_2}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}f_3}{\mathrm{d}t}\hat{\mathbf{k}} + \frac{\mathrm{d}\hat{\mathbf{i}}}{\mathrm{d}t}f_1 + \frac{\mathrm{d}\hat{\mathbf{j}}}{\mathrm{d}t}f_2 + \frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t}f_3 \tag{16}$$

The rate of change of $\mathbf{f}$ observed in the rotating system is

$$\left(\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}\right)_r = \frac{\mathrm{d}f_1}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}f_2}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}f_3}{\mathrm{d}t}\hat{\mathbf{k}} \tag{17}$$

As for the time derivatives of the unit vectors, by the linearity and distributive property of the cross product:

$$\frac{\mathrm{d}\hat{\mathbf{i}}}{\mathrm{d}t}f_1 + \frac{\mathrm{d}\hat{\mathbf{j}}}{\mathrm{d}t}f_2 + \frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t}f_3 = \omega \times \mathbf{f} \tag{18}$$

Therefore,

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}\right)_r + \omega \times \mathbf{f} \tag{19}$$

When differentiating $\mathbf{L}$ with respect to time, then,

$$\dot{\mathbf{L}} = \left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_r + \omega \times \mathbf{L} \tag{20}$$

The derivative of $\mathbf{L}$ observed in the rotating system is $\mathbf{I}\dot{\omega}$, since $\mathbf{I}$ is constant. Therefore,

$$\dot{\mathbf{L}} = \mathbf{I}\dot{\omega} + \omega \times \mathbf{L} = \tau \tag{21}$$

This is **Euler's equation** for the rigid body. In terms of the principal axes of inertia, the equation becomes

$$I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z = \tau_x \tag{22}$$

$$I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x = \tau_y \tag{23}$$

$$I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y = \tau_z \tag{24}$$

6 Stability of free motion about a principal axis

Consider a rigid body rotating about a principal axis z *freely*, meaning that $\tau = 0$. Euler's equations are then

$$I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z = 0 \tag{25}$$

$$I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x = 0 \tag{26}$$

$$I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y = 0 \tag{27}$$

The objects just keeps rotating about that axis, since the equations require $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$ to be zero. However, if we subject the rigid body to a small perturbation about the x and y axes ($\omega_x << \omega_z$, $\omega_y << \omega_z$) then the angular accelerations aren't necessarily zero. Note that since $I_z\dot{\omega}_z = (I_x - I_y)\omega_x\omega_y$, with both ω_x and ω_y being small quantities, we may take $\dot{\omega}_z = 0$, letting ω_z be some constant quantity ω. Differentiating the first Euler equation with respect to time,

$$I_x\ddot{\omega}_x - (I_y - I_z)\dot{\omega}_y\omega = 0 \tag{28}$$

Solving the second Euler equation for $\dot{\omega}_y$,

$$\dot{\omega}_u = \frac{(I_z - I_x)\omega\omega_x}{I_y} \tag{29}$$

therefore,

$$\ddot{\omega}_x = \frac{(I_y - I_z)(I_z - I_x)\omega^2}{I_x I_y}\omega_x \tag{30}$$

The solution to this differential equation is

$$\omega_x(t) = Ae^{\sqrt{k}t} + Be^{-\sqrt{k}t} \tag{31}$$

where $k = \frac{(I_y - I_z)(I_z - I_x)\omega^2}{I_x I_y}$. Notice that the stability of motion is determined by the sign of k. If k is negative, then an imaginary unit pops out and by DeMoivre's formula $\omega_x(t)$ oscillates sinusoidally and therefore remains small. However, for positive k, ω_x begins to grow without bound, so the body begins to wobble chaotically. The same logic follows for ω_y.

By their definitions, the principal moments of inertia are all positive. By observation, the condition for k to be positive is $I_x > I_z > I_y$ or $I_y > I_z > I_x$. We now arrive to the following theorem.

Theorem 6.1. *(Intermediate axis theorem) Rotation of an object around its first and third principal axes is stable, whereas rotation about its second/intermediate principal axis is not.*

This effect is also called the *Dzhanibekov effect* after Vladimir Dzhanibekov, who first noticed one of the theorem's logical consequences whilst in space in 1985. However, this has been known since Louis Poinsot discovered it in 1834.