FIXED POINT SPACES IN GEOMETRIC ACTIONS OF CLASSICAL GROUPS, I

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ABSTRACT. This is the first in a series of two papers in which we study the natural action of a classical algebraic group G on varieties of subspace decompositions of the natural G-module V. In this paper, we consider the action of $\mathrm{Sp}(V)$ and $\mathrm{O}(V)$ on the space of maximal totally singular decompositions of V. We calculate close to best possible upper and lower bounds on the dimension of the fixed point space of elements of prime order $x \in G$, and we relate these bounds to the action on V, extending earlier work of Frohardt and Magaard on subspace actions of finite classical groups. In particular, this is the first paper to systematically study lower bounds in this context.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic $p \geqslant 0$. Let G be an affine algebraic group over k. Suppose H is a closed subgroup of G, and let Ω denote the coset variety G/H, on which G acts transitively. The *fixed point space* of $x \in G$, denoted

$$C_\Omega(x) = \{\omega \in \Omega : x.\omega = \omega\},$$

is a subvariety of Ω, so we can consider its dimension. Note that for any $x, g \in G$ we have an isomorphism of varieties $C_\Omega(x) \cong C_\Omega(x^g)$, so $\dim C_\Omega(x)$ is a class function (constant on conjugacy classes).

This is the first in a series of two papers in which we are interested in the fixed point spaces arising in natural geometric actions of classical algebraic groups. Let V be an n-dimensional k-vector space and let G be one of the classical groups $\mathrm{GL}(V)$, $\mathrm{Sp}(V)$ or $\mathrm{O}(V)$ (so G is the isometry group of the zero form, a symplectic form or a non-degenerate quadratic form on V, respectively; in particular, if $G = \mathrm{O}(V)$ and $p = 2$, then $\dim V$ is even). By exploiting the underlying geometry of V, we can define a range of geometric spaces on which G acts naturally. For example, if $G = \mathrm{GL}(V)$ then we can consider the action of G on the space of equidimensional subspaces of V, or the space of direct sum or tensor product decompositions of V, etc.

In this paper, we will consider the action of G on the space Ω of equidimensional direct sum decompositions of V of the form

$$V = V_1 \oplus \cdots \oplus V_t \tag{1}$$

(so $\dim V_i = n/t$ for each i) with the property that if $G = \mathrm{Sp}(V)$ or $\mathrm{O}(V)$ then the V_i are either non-degenerate and the decompositions are orthogonal, or $t = 2$ and the V_i are totally singular. Since G acts transitively on Ω we can identify Ω with the coset variety G/H, where H is the stabiliser of a fixed decomposition in Ω.

Following Liebeck and Seitz [21], we define the subgroup collections $\mathscr{C}_2$ and $\mathscr{C}_3$ as follows. The subgroups in $\mathscr{C}_3$ are the stabilisers in $\mathrm{Sp}(V)$ or $\mathrm{O}(V)$ of a decomposition $V = V_1 \oplus V_2$, where the V_i are maximal totally singular subspaces of V, and the stabilisers of the other decompositions defined above comprise the $\mathscr{C}_2$ collection. The structure of the subgroups in these two families is given in Table 1. Note that H is finite if $G = \mathrm{O}(V)$ and the V_i in (1) are 1-dimensional. For $H \in \mathscr{C}_i$, we will refer to H as a $\mathscr{C}_i$*-subgroup*, and the natural transitive action of G on G/H as a $\mathscr{C}_i$*-action*.

Liebeck and Seitz [21] introduce four additional geometric subgroup collections (that naturally arise from the geometry of V), denoted $\mathscr{C}_i$ for $1 \leqslant i \leqslant 6$, and they prove that if J is any closed subgroup of G, then J is either contained in a member of one of these $\mathscr{C}_i$-families, or J is almost simple and acts irreducibly on V (see [21, Theorem 1]). In particular, if $J \leqslant G$ is a closed subgroup whose action on V is imprimitive then there exists $H \in \mathscr{C}_2 \cup \mathscr{C}_3$ such that $J \leqslant H$. So $\mathscr{C}_2 \cup \mathscr{C}_3$ comprises all the maximal imprimitive subgroups of G.

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Collection	G	H	Conditions
$\mathscr{C}_2$	GL_n	$\mathrm{GL}_{n/t} \wr S_t$	
	Sp_n	$\mathrm{Sp}_{n/t} \wr S_t$	
	O_n	$\mathrm{O}_{n/t} \wr S_t$	n/t even if $p=2$
$\mathscr{C}_3$	Sp_n or O_n	$\mathrm{GL}_{n/2}.2$	$n>2$ even

TABLE 1. The $\mathscr{C}_2$- and $\mathscr{C}_3$-collections

The aim of this paper is to obtain both upper and lower bounds on the ratio

$$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega},$$

for $\Omega = G/H$, where G is a symplectic or orthogonal group, H is a subgroup of G in the aforementioned $\mathscr{C}_3$-collection, and $x \in G \setminus Z(G)$ is an element of prime order, or any unipotent element if the underlying field k has characteristic zero. A similar analysis for $\mathscr{C}_2$-actions will be given in [24].

Given the geometric nature of Ω, it is natural to consider the relationship between the action of x on V and its action on Ω. With this in mind we define

$$\nu(x) = \min\{\dim[V,\lambda x] : \lambda \in k^*\}, \tag{2}$$

where $[V,x] = \langle x.v - v : v \in V\rangle$. Note that $\nu(x)$ is the codimension of the largest eigenspace of x on V (in particular, $\nu(x) > 0$ since $x \notin Z(G)$).

Let $x \in G$ be an element of prime order r. We are interested in obtaining both *global* bounds on $f_\Omega(x)$, in terms of G, H and r, as well as *local* bounds, which will also depend on the parameter $\nu(x)$. As one would expect, we find that there is a close relationship between $\nu(x)$ and $f_\Omega(x)$. Whenever possible, we also want to classify the elements that realise our bounds on $f_\Omega(x)$.

The key tool in our analysis is the explicit formula for $\dim C_\Omega(x)$ originally proved in [19]:

$$\dim C_\Omega(x) = \dim\Omega - \dim x^G + \dim(x^G \cap H),$$

see Proposition 2.1. In general, it is difficult to compute $\dim(x^G \cap H)$, and this lead us to some interesting combinatorial problems in the context of the $\mathscr{C}_2$- and $\mathscr{C}_3$-actions we are interested in here.

In order to state our main results, we need to introduce some additional notation. Any element $x \in G$ has a Jordan decomposition $x = x_s x_u = x_u x_s$ where x_s is *semisimple* (that is, x_s is diagonalisable), and x_u is *unipotent* (i.e. x_u has 1 as the only eigenvalue). In particular, prime order elements are either semisimple or unipotent. It will be convenient to set $p = \infty$ if the field k has characteristic zero. Given $x \in G$, let $o(x)$ denote the order of x. We define

$$\mathscr{R} = \mathscr{R}(G) = \{x \in G \setminus Z(G) : o(x) \text{ is prime, or } p = \infty \text{ and } x \text{ is unipotent}\}. \tag{3}$$

We also define

$$\mathscr{V}_s = \{x \in G : \nu(x) = s\},\ \ \mathscr{V}_{s,r} = \{x \in \mathscr{V}_s : o(x) = r\}.$$

notice that $1 \leqslant s < \dim V$.

Our main result is the following, which provides close to best possible global bounds on $f_\Omega(x)$ for $\mathscr{C}_3$-actions. Note that $C_\Omega(x)$ is empty if, and only if, the G-class of x does not meet H.

Theorem 1. *Assume $n > 2$ is even. Let $G = \mathrm{Sp}_n$ or O_n, let H be a $\mathscr{C}_3$-subgroup of G and set $\Omega = G/H$.*

(a) *If $x \in G \setminus Z(G)$, then*

$$f_\Omega(x) \leqslant 1 - \frac{4}{n} + \left(\frac{4}{n}\right)^2.$$

(b) *Let $x \in \mathscr{R}$ be unipotent, and assume $x^G \cap H \neq \emptyset$.*

(i) *If $p = 2$ then*

$$f_\Omega(x) \geqslant \frac{1}{2} - \frac{2}{n+2}.$$

(ii) *If $p < n/2$ is odd then*

$$f_\Omega(x) \geqslant \frac{1}{p}.$$

(iii) *If $p \geqslant n/2$ then*

$$f_\Omega(x) \geqslant \frac{2}{n} - \frac{4}{n(n+2)}.$$

(c) *Let $x \in \mathscr{R}$ be semisimple of order $r < n$. Then*

$$f_\Omega(x) \geqslant \frac{1}{r} - \frac{r^2-1}{rn(n-2)}.$$

Remark 2. Let us make some remarks on the statement of Theorem 1. We denote by r the order of $x \in G$.

(i) Note that the case $r \geqslant n$ is excluded in part (c) of Theorem 1. Here $f_\Omega(x) \geqslant 0$ and equality is possible. We refer the reader to Theorem 10, which states that $f_\Omega(x) = 0$ if, and only if, x is regular (that is, $\dim C_G(x) = \mathrm{rank}\, G$).

(ii) The bounds in Theorem 1 are close to best possible. Indeed the lower bounds are sharp in several cases (see Proposition 5.7, for example). By subdividing the possibilities for G, H, r we compute the best possible lower bounds in all cases.

(iii) For $r \in \{2, p\}$, we classify the elements that realise the bounds in most of the cases: see Corollary 3.15 for involutions, and Propositions 5.5 and 5.7 for unipotent elements.

(iv) In case (c), the optimal lower bounds on $f_\Omega(x)$ are computed in Section 6 (see Remark 6.14), and the elements that realise these bounds are described in Definition 6.7.

Next we turn our attention to local bounds on $f_\Omega(x)$ for $\mathscr{C}_3$-actions.

Theorem 3. *Let $G = \mathrm{Sp}_n$ or O_n, let H be a $\mathscr{C}_3$-subgroup of G and set $\Omega = G/H$. Let $x \in \mathscr{R} \cap \mathscr{V}_s$. Then*

$$f_\Omega(x) \leqslant 1 - \frac{s}{n+1} + \frac{1}{n}.$$

In addition, if $x^G \cap H \neq \emptyset$, then

$$f_\Omega(x) \geqslant 1 - \frac{s(2n-s)}{n(n-2)} - \frac{1}{n}.$$

Remark 4.

(i) As before, by subdividing the possibilities for G, H and r we can obtain more accurate bounds. For example, see Propositions 7.1 and 8.1 when $r = p$ is odd. For semisimple elements of odd order, we refer the reader to Proposition 7.10 and Lemma 8.2 (if $r = 3$), and Proposition 8.7 (if $r > 3$).

(ii) If x is an involution, an exact formula for $f_\Omega(x)$ is given in Table 6. Note that in this situation, if $p \neq 2$ then $\nu(x)$ uniquely determines the conjugacy class of x (up to scalars).

As an immediate corollary, we obtain the following.

Corollary 5. *Let $x, y \in \mathscr{V}_{s,r} \cap H$. Then*

$$|f_\Omega(x) - f_\Omega(y)| \leqslant \frac{s(2n-s)}{n(n-2)} + \frac{2}{n} - \frac{s}{n+1}.$$

In particular, if $s \leqslant \sqrt{n}$ or $s \geqslant n - \sqrt{n}$, then $|f_\Omega(x) - f_\Omega(y)| < 2/\sqrt{n}$.

In order to state the next result, we need to introduce some additional terminology. We define the *algebraic fixity* of G on Ω to be

$$M = M(G,\Omega) = \sup\{f_\Omega(x) : x \in G \setminus Z(G)\}.$$

(This definition is motivated by the classical notion of fixity in the study of finite permutation groups, see [25], for example.) In addition, for a prime r (we allow $r = p = \infty$), the *r-local algebraic fixity* is

$$M_r = \sup\{f_\Omega(x) : o(x) = r\}.$$

It is not difficult to show that $M = \sup\{M_r : r \text{ prime}\}$, see Lemma 2.2.

In [3], Burness derives a lower bound on the 2-local algebraic fixity for primitive actions of simple algebraic groups. Indeed, [3, Theorem 1] asserts that, apart from a short list of known exceptions,

$$M_2 \geqslant \frac{1}{2} - \frac{1}{2h+1} \tag{4}$$

where $h = -1 + \dim G / \mathrm{rank}\, G$ is the *Coxeter number* of G. Note that this bound is essentially best possible (for instance, if H is finite then we may have $M_2 = 1/2 - 1/(2h+2)$; see [3, Remark 1, 4.1]).

The exact algebraic fixity in the case where G is classical and $H \in \mathscr{C}_2 \cup \mathscr{C}_3$ is stated in the next result (the proof for $\mathscr{C}_2$-subgroups is given in [24, Section 4]). The value of the algebraic fixity follows from the computation of M_r for any prime r (and the characterisations of the elements $x \in G$ of order r with $f_\Omega(x) = M_r$); see Corollary 3.15, and Propositions 4.1 and 4.3. We write Cl_n for one of the classical groups $\mathrm{GL}_n, \mathrm{Sp}_n$ or O_n.

G	H	M	Conditions	r
GL_n	$\mathrm{GL}_{n/t} \wr S_t$	$1-\frac{2}{n}+\frac{1}{n(n-1)}$	$t=n$	2
		$1-\frac{2}{n}$	$t \neq n$	any
Sp_n	$\mathrm{Sp}_{n/t} \wr S_t$	$1-\frac{2}{n}$	$n>4$	p
Sp_2	$\mathrm{Sp}_2 \wr S_2$	$3/4$		2
O_n	$\mathrm{O}_{n/t} \wr S_t$	$1-\frac{2}{n}$	$n>4$	2
O_4	$\mathrm{O}_2 \wr S_2$	$1/2$	$p \neq 2$	$\neq p$
		$3/4$	$p=2$	2
O_4	$\mathrm{O}_1 \wr S_4$	$1/2$		$2,3$
Sp_n	$\mathrm{GL}_{n/2}.2$	$1-\frac{4(n-2)}{n(n+2)}$	$n>4$	2
O_n	$\mathrm{GL}_{n/2}.2$	$1-\frac{4(n-4)}{n(n-2)}$	$n>10$	p

TABLE 2. The algebraic fixity of $\mathscr{C}_2$- and $\mathscr{C}_3$-actions

G	M	conditions	r	G	M	conditions	r
Sp_4	$2/3+\delta_{p,2}/6$	–	2	O_8	$2/3$	$p=2$	any
O_4	1	$p \neq 2$	$\neq 2,p$			$p \neq 2$	$\neq 2$
O_6	$2/3$	$p=2$	any	O_{10}	$7/10$	$p=2$	any
		$p \neq 2$	$\neq 2$			$p \neq 2$	$\neq 2$

TABLE 3. Algebraic fixity in small rank, $H \in \mathscr{C}_3$

Theorem 6. *Let $G = Cl_n$, let H be a $\mathscr{C}_i$-subgroup of G, where $i = 2$ or 3. Set $\Omega = G/H$. Then either M is recorded in Table 2 according to the given conditions on n, or $H \in \mathscr{C}_3$ and M is given in Table 3. In addition, $M = M_r$ if, and only if, r is as in the last column of each of the tables.*

Corollary 7. *Either $M = \max\{M_2, M_p\}$, or $G = \mathrm{O}_4$ and $H = \mathrm{GL}_2.2$ is a $\mathscr{C}_3$-subgroup.*

Remark 8.

(i) The case $(G,H) = (\mathrm{O}_4, \mathrm{GL}_2.2)$ in Corollary 7 is a genuine exception. Indeed, there exist non-central elements $x \in G$ of any prime order $r > 2$ such that $f_\Omega(x) = 1$, see Remark 4.4.

(ii) In [3], for $\mathscr{C}_2$- and $\mathscr{C}_3$-actions, the lower bound on M_2 is shown by exhibiting an explicit involution $x \in H$ such that $f_\Omega(x) = \ell$ with ℓ greater than or equal to the quantity in (4). Assume $(G,H) = (\mathrm{Sp}_n, \mathrm{GL}_{n/2}.2)$ and $n > 4$. Then $M = M_2 = \ell$, where ℓ is recorded in [3, Table 3.4]. On the other hand, if $(G,H) = (\mathrm{O}_n, \mathrm{GL}_{n/2}.2)$ and $n > 10$ then $M > \ell$, again ℓ is given in [3, Table 3.4].

Corollary 9. *For any prime r there exists an element $x \in G$ of order r such that*

$$f_\Omega(x) \geqslant 1 - \frac{4}{n}.$$

Notice that Corollary 9 implies that the general upper bound stated in Theorem 1(a) is close to best possible.

Recall that an element $x \in G$ is *regular* if $\dim C_G(x) = \operatorname{rank} G$. In [14], Herb and O'Brian proved that if G is a connected reductive algebraic group, then $x \in G$ is regular if, and only if, $C_\Omega(x)$ is finite, where $\Omega = G/P$ and P is a parabolic subgroup of G. We establish a similar result for $\mathscr{C}_2$- and $\mathscr{C}_3$-actions of classical algebraic groups.

Theorem 10. *Let $G = Cl_n$ and let H be a $\mathscr{C}_i$-subgroup of G, where $i = 2$ or 3. Set $\Omega = G/H$. Let $x \in G$ be of prime order other than n, and assume that $x^G \cap H \neq \emptyset$. Then $C_\Omega(x)$ is finite if, and only if, one of the following holds:*

(i) *x is regular;*

(ii) *x is semisimple and $(G, H, C_G(x)) = (\mathrm{Sp}_n, \mathrm{Sp}_{n/t} \wr S_t, \mathrm{Sp}_2 \times (\mathrm{GL}_1)^{n/2-1})$.*

Remark 11. Assume $r = n$. Then $G = \mathrm{GL}_n$ or O_n and $H \in \mathscr{C}_2$ stabilises a decomposition of V in 1-dimensional subspaces. In both cases, there are no elements $x \in H$ of order r with finite fixed point space, see [24, Remark 3.2].

Fixed point spaces for actions of algebraic groups have been studied by various authors in recent years. For example, Lawther, Liebeck and Seitz [19] obtain upper bounds on $\dim C_\Omega(x)$ for primitive actions of simple exceptional algebraic groups. Similar results, for certain primitive actions of classical algebraic groups, are given in [4]. In both cases, the results are used to obtain new bounds on *fixed point ratios* for actions of the corresponding finite groups of Lie type (see [18, 5, 6, 7, 8]), and this work has played an important role in the solution of several problems, see for example [20, 23, 10, 11, 2, 9]. In future work, we will use our results to investigate fixed point ratios at the finite group level.

The study of local bounds was initiated by Frohardt and Magaard. In [10] they obtain bounds on fixed point ratios for various subspace actions of finite classical groups. More precisely, they derive upper bounds on these ratios in terms of the parameter $\nu(x)$. One of our aims is to extend their analysis to non-subspace actions of classical algebraic groups, and then to obtain local bounds on fixed point ratios for the corresponding actions of finite classical groups.

Layout. In Section 2, we record various preliminary results that will be needed in the proofs of our main theorems. In Section 3 we compute $\dim(x^G \cap H)$ for $x \in H$ of prime order r and $H \in \mathscr{C}_3$. The analysis for the case $r \neq 2$ is fairly quick. In Section 4 we derive best possible upper bounds on $f_\Omega(x)$ and we prove Theorems 1(a) and 6. In Section 5 (resp. 6) we derive best possible lower bounds on $f_\Omega(x)$ for x unipotent (resp. semisimple), establishing Theorems 1(b),(c) and 10. In most cases, we characterise the elements that realise the best possible bounds. We conclude studying local bounds. In Sections 7 and 8 we derive close to best possible local upper and lower bounds on $f_\Omega(x)$, and we prove Theorem 3.

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2. Preliminary results

2.1. **Notations and conventions.** Our notation for algebraic groups is fairly standard (see [15], for example). Unless stated otherwise, $G = Cl(V)$ will denote one of the classical groups $\mathrm{GL}(V)$, $\mathrm{Sp}(V)$ or $\mathrm{O}(V)$, where V is an n-dimensional vector space over an algebraically closed field k of characteristic p (setting $p = \infty$ in the characteristic zero case). The results in this section hold for any n; however, in the remainder of the paper, we shall assume n respects the conditions in Table 1. Frequently, we will describe the elements of G in terms of matrices, with respect to an appropriate basis of V (in this case we write $G = Cl_n$). In particular, we will write $x = [A_1, \ldots, A_m]$ for a block-diagonal matrix with matrices $A_1, \ldots, A_m$ along the diagonal. A unipotent Jordan block of size i will be denoted J_i, so a typical unipotent element $x \in \mathrm{GL}_n$ is conjugate to the block diagonal matrix $[J_n^{a_n}, \ldots, J_1^{a_1}]$, where a_i denotes the multiplicity of J_i in the Jordan normal form of x. Note that if $p < \infty$, a Jordan block J_i has order p if, and only if, $i \leqslant p$ (of course, if $p = \infty$ then J_i, $i > 1$, has infinite order). In addition, r will always denote a prime integer, and $\omega \in k^*$ is a primitive r-th root of unity. Finally, $\delta_{a,b}$ denotes the familiar Kronecker delta (for integers a and b), and it will be convenient to make the following definitions:

$$\delta_{a;b} = \begin{cases} 1 & b \text{ divides } a \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and

$$\varepsilon = \begin{cases} -1 & G = \mathrm{Sp}_n \\ 1 & G = \mathrm{O}_n. \end{cases} \tag{6}$$

2.2. **Fixed point spaces.** Let G be an algebraic group and Ω be a G-variety. Then

$$C_\Omega(x) = \{\omega \in \Omega : x.\omega = \omega\} \subseteq \Omega$$

is the *fixed point space* of $x \in G$. Note that if H is a closed subgroup of G and Ω is the coset variety G/H, then

$$C_\Omega(x) \neq \emptyset \Longleftrightarrow x^G \cap H \neq \emptyset. \tag{7}$$

The following result will be a key tool in our analysis.

Proposition 2.1 ([19, Proposition 1.14]). *Let G be an algebraic group, let H be a closed subgroup of G and set $\Omega = G/H$. Then for $x \in H$, we have*

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H).$$

The following shows that it is enough to derive upper bounds for elements in $\mathscr{R}$, in order to deduce general upper bounds. Recall that for $x \in G$ we have the Jordan decomposition $x = x_s x_u = x_u x_s$.

Lemma 2.2. *Let $G = Cl(V)$, let $H \leqslant G$ be closed and reductive, and let $x \in G \setminus Z(G)$. Set $\Omega = G/H$. Then one of the following holds:*

(i) *$p = \infty$, $x_u \neq 1$ and there exists y of prime order such that $\dim C_\Omega(x) \leqslant \max\{\dim C_\Omega(y), \dim C_\Omega(x_u)\}$;*
(ii) *there exists y of prime order such that $\dim C_\Omega(x) \leqslant \dim C_\Omega(y)$.*

Proof. Up to replacing x by a G conjugate and by (7), we may assume that $x \in H$. It is clear that for any positive integer m we have $C_\Omega(x) \subseteq C_\Omega(x^m)$. Hence if x has finite order, there exists y of prime order such that $\dim C_\Omega(x) \leqslant \dim C_\Omega(y)$. For $x \in G$ with $x = x_s x_u$ we have $C_\Omega(x) = C_\Omega(x_s) \cap C_\Omega(x_u)$, so $\dim C_\Omega(x) \leqslant \max\{\dim C_\Omega(x_s), \dim C_\Omega(x_u)\}$. If x is semisimple with infinite order. Then, by the argument of [4, Section 8], there exists a semisimple element y of prime order such that $C_\Omega(x) \subseteq C_\Omega(y)$. The result follows. □

Recall the definition of M and M_r from the introduction. Then Lemma 2.2 implies $M = \sup_r\{M_r\}$.
For $x \in G$ we have

$$x^G \cap H = \bigcup_{i \in \mathscr{I}} x_i^H.$$

If G and H are reductive, [13, Theorem 1.2] asserts that $\mathscr{I}$ is finite. Therefore $\dim(x^G \cap H) = \max_i\{\dim x_i^H\}$. In particular, this holds in our setting, where $G = Cl(V)$ and H is a $\mathscr{C}_2$- or $\mathscr{C}_3$-subgroup of G.

For $\mathscr{C}_3$-actions we will be able to compute an explicit formula for this dimension, see Propositions 3.3 and 3.9 for unipotent and semisimple elements of order $r \neq 2$, respectively. For involutions we refer the reader to Theorem 3.13, where we give an explicit formula for $f_\Omega(x)$.

2.3. **Conjugacy in classical groups.** In this section, we describe the conjugacy classes of prime order elements in a classical group $G = Cl(V) = Cl_n$. By elementary linear algebra, conjugacy in $G = \mathrm{GL}_n$ is determined by the Jordan canonical form. In particular, if $x \in G$ has prime order r, then either $r \neq p$ (so x is semisimple) and x is G-conjugate to a diagonal matrix of the form

$$[I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in G, \tag{8}$$

for some non-negative integers $a_0, \dots, a_{r-1}$ such that $n = \sum_i a_i$, or $r = p$ and x is G-conjugate to

$$[J_p^{a_p}, \dots, J_1^{a_1}] \in G. \tag{9}$$

for some non-negative integers $a_1, \dots, a_p$ such that $n = \sum_i i a_i$. Notice that if x is semisimple, then $\nu(x) = n - \max_i\{a_i\}$; whereas $\nu(x) = n - \sum_i a_i$ if x is unipotent. It is not difficult to compute $\dim C_G(x)$ for x as in (8) or (9), we state these dimension formulae in Theorems 2.3 and 2.6 below.

Let us now turn to the conjugacy classes of elements of prime order in the symplectic and orthogonal groups. First we handle unipotent elements. The behaviour of unipotent classes depends on the parity of p. The following is a special case of [22, Theorem 3.1, Corollary 3.6].

Theorem 2.3. *Let $G = Cl_n$. Assume $p \neq 2$ if $G = \mathrm{Sp}_n$ or O_n. Let $x \in \mathrm{GL}_n$ be unipotent with Jordan form $[J_p^{a_p}, \dots, J_1^{a_1}]$.*

(i) *Two unipotent elements of G are G-conjugate if, and only if, they are GL_n-conjugate.*
(ii) *If $G = \mathrm{Sp}_n$ then $x^{\mathrm{GL}_n} \cap G \neq \emptyset$ if, and only if, a_i is even for each odd i; if $G = \mathrm{O}_n$ then $x^{\mathrm{GL}_n} \cap G \neq \emptyset$ if, and only if, a_i is even for each even i.*
(iii) *Assume $x \in G$. The dimension of $C_G(x)$ is as follows:*

$$\dim C_{\mathrm{GL}_n}(x) = 2\sum_{i<j} i a_i a_j + \sum_i i a_i^2,$$

$$\dim C_{\mathrm{Sp}_n}(x) = \sum_{i<j} i a_i a_j + \frac{1}{2}\sum_i i a_i^2 + \frac{1}{2}\sum_{i \text{ odd}} a_i,$$

$$\dim C_{\mathrm{O}_n}(x) = \sum_{i<j} i a_i a_j + \frac{1}{2}\sum_i i a_i^2 - \frac{1}{2}\sum_{i \text{ odd}} a_i.$$

Let n be a positive integer. A partition of n is a tuple $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\lambda_i \geqslant \lambda_{i+1} \geqslant 0$ and $\sum_i \lambda_i = n$, and we write $\lambda \vdash n$. Collecting equal parts we write $\lambda = (n^{a_n}, \dots, 1^{a_1})$, where $\sum_i ia_i = n$. The set of partitions of n is denoted $\mathscr{P}(n)$. There is a one-to-one correspondence between $\mathscr{P}(n)$ and the set of GL_n-conjugacy classes of unipotent elements in GL_n, given by

$$(n^{a_n}, \dots, 1^{a_1}) \longleftrightarrow [J_n^{a_n}, \dots, J_1^{a_1}]^{\mathrm{GL}_n}.$$

The set of partitions of n is ordered via the *dominace ordering* (which is a partial order): for partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ in $\mathscr{P}(n)$, we define

$$\mu \preccurlyeq \lambda \iff \sum_{i=1}^{l} \mu_i \leqslant \sum_{i=1}^{l} \lambda_i \text{ for all } 1 \leqslant l \leqslant n.$$

Let $\mathscr{U}$ be the set of unipotent elements in $G = Cl_n$. Then $\mathscr{U}$ is a closed subset of G, so $\overline{x^G} \subseteq \mathscr{U}$ for all $x \in \mathscr{U}$ (x^G is closed if, and only if, $x = 1$ by [16, Section 1.7]). In particular, if $x, y \in \mathscr{U}$ then we may write $x^G \leqslant y^G$ if, and only if, $x \in \overline{y^G}$. Of course, if $x^G \leqslant y^G$ then $\dim x^G \leqslant \dim y^G$.

Theorem 2.4 ([16, Theorem 7.19]). *Let $G = Cl_n$. Assume $p \neq 2$ if $G = \mathrm{Sp}_n$ or O_n. Let $x, y \in \mathscr{U}$, with corresponding partitions λ, μ. Then $x^G \leqslant y^G$ if, and only if, $\lambda \preccurlyeq \mu$.*

The Jordan form determines conjugacy among unipotent elements if $G = \mathrm{GL}_n$, similarly if $G = \mathrm{Sp}_n$ or O_n and $p \neq 2$. Assume $G = \mathrm{Sp}_n$ or O_n and $p = 2$, in particular n is even. An involution $x \in G$ has Jordan form $[J_2^s, J_1^{n-2s}]$, for some $0 < s \leqslant n/2$, and $\nu(x) = s$. As described in [1, Sections 7,8], if $s \leqslant n/2$ is even then there are precisely two G-classes of involutions $x \in G$ with $\nu(x) = s$, whose representatives are denoted a_s and c_s. Let $(\cdot, \cdot)$ be the non-degenerate bilinear form defined on V, that is fixed by G. An involution $x \in G$ with $\nu(x) = s$ even is said to be an a_s-type involution if, and only if,

$$(x.v, v) = 0 \text{ for all } v \in V,$$

otherwise x is c_s-type. If s is odd, there is a unique class whose elements have Jordan form $[J_2^s, J_1^{n-2s}]$, and we say that x is b_s-type. The next result follows from [1, Sections 7,8], see also [22, Chapters 4,6].

Theorem 2.5. *Let $G = \mathrm{Sp}_n$ or O_n. Assume $p = 2$ and let $s \leqslant n/2$ be a positive integer.*

(i) *If $G = \mathrm{Sp}_n$, then*

$$\dim a_s^G = s(n-s), \ \dim b_s^G = \dim c_s^G = s(n-s+1).$$

(ii) *If $G = \mathrm{O}_n$, then*

$$\dim a_s^G = s(n-s-1), \ \dim b_s^G = \dim c_s^G = s(n-s).$$

Moreover $a_s, c_s \in \mathrm{SO}_n$ while $b_s \in \mathrm{O}_n \setminus \mathrm{SO}_n$.

Now we focus on elements of prime order $r \neq p$. In general, given a semisimple element $x \in G$ we have

$$V \downarrow x = \bigoplus_{\lambda \in k^*} V_\lambda,$$

where $V_\lambda = \{v \in V : x.v = \lambda v\}$ is the eigenspace with respect to the eigenvalue λ.

Assume $G = \mathrm{Sp}_n$ or O_n. It is straightforward to show that

$$V \downarrow x = V_1 \oplus V_{-1} \oplus \left(\bigoplus_{\lambda \neq \pm 1} V_\lambda \oplus V_{\lambda^{-1}} \right),$$

where

(a) $V_{\pm 1}$ is either trivial or non-degenerate;
(b) $V_\lambda, V_{\lambda^{-1}}$ are totally singular and $\dim V_\lambda = \dim V_{\lambda^{-1}}$, for all $\lambda \neq \pm 1$;
(c) $V_\lambda \oplus V_{\lambda^{-1}}$ is either trivial or non-degenerate, for all $\lambda \neq \pm 1$;
(d) If $\lambda \neq \mu, \mu^{-1}$ then $V_\lambda \oplus V_\mu$ is an orthogonal sum.

Using the properties listed above one can easily show the following. Recall from (8) the Jordan form of an element $x \in \mathrm{GL}_n$ of order $r \neq p$.

Theorem 2.6. *Let $G = Cl_n$ and suppose $x \in \mathrm{GL}_n$ has order $r \neq p$.*

(i) *Two semisimple elements of G are G-conjugate if, and only if, they are GL_n-conjugate.*
(ii) *Assume $G \neq \mathrm{GL}_n$ and $r \neq 2$. Then $x^{\mathrm{GL}_n} \cap G \neq \emptyset$ if, and only if $a_0 \equiv n \pmod 2$ and $a_i = a_{r-i}$ for all $1 \leqslant i \leqslant (r-1)/2$.*

G	r	$C_G(x)$	$\dim C_G(x)$
GL_n	$=2$	$\mathrm{GL}_s \times \mathrm{GL}_{n-s}$	$n^2-2s(n-s)$
GL_n	$\neq 2$	$\prod_{i=0}^{r-1} \mathrm{GL}_{a_i}$	$\sum_{i=0}^{r-1} a_i^2$
Sp_n	$=2$	$\mathrm{Sp}_s \times \mathrm{Sp}_{n-s}$	$\frac{n}{2}(n+1)-s(n-s)$
Sp_n	$\neq 2$	$\mathrm{Sp}_{a_0} \times \prod_{i=1}^{(r-1)/2} \mathrm{GL}_{a_i}$	$\frac{a_0}{2}(a_0+1)+\frac{1}{2}\sum_{i=1}^{r-1} a_i^2$
O_n	$=2$	$\mathrm{O}_s \times \mathrm{O}_{n-s}$	$\frac{n}{2}(n-1)-s(n-s)$
O_n	$\neq 2$	$\mathrm{O}_{a_0} \times \prod_{i=1}^{(r-1)/2} \mathrm{GL}_{a_i}$	$\frac{a_0}{2}(a_0-1)+\frac{1}{2}\sum_{i=1}^{r-1} a_i^2$

TABLE 4. Centralisers of prime order semisimple elements in $G = Cl_n$

G	$f(s)$	$g(s)$
GL_n	$\max\{2s(n-s), ns\}$	$s(2n-s-1)$
Sp_n	$\max\{s(n-s), ns/2\}$	$(2ns-s^2+1)/2$
O_n	$\max\{s(n-s-1), n(s-1)/2\}$	$(2ns-s^2-2s+\delta_{n-1;2})/2$

TABLE 5

(iii) *If x is an involution, then $x^{\mathrm{GL}_n} \cap \mathrm{O}_n \neq \emptyset$; while $x^{\mathrm{GL}_n} \cap \mathrm{Sp}_n \neq \emptyset$ if, and only if, $\nu(x)$ is even.*
(iv) *If $x \in G$, then the structure and dimension of $C_G(x)$ are recorded in Table 4.*

Proof. For (i), let $x, y \in G$. Assume x, y are GL_n-conjugate. Then we have decompositions

$$V \downarrow x = V_1 \oplus V_{-1} \oplus \left(\bigoplus_{\lambda \neq \pm 1} V_\lambda \oplus V_{\lambda^{-1}} \right), \quad V \downarrow y = U_1 \oplus U_{-1} \oplus \left(\bigoplus_{\lambda \neq \pm 1} U_\lambda \oplus U_{\lambda^{-1}} \right),$$

that satisfy properties (a)–(d) stated above, and $\dim V_\mu = \dim U_\mu$ for all μ. For $\lambda \neq \pm 1$, any isomorphism $V_\lambda \to U_\lambda$ is also an isometry, since V_λ, U_λ are totally singular. Hence, by Witt's Lemma (see [17, Proposition 2.1.6]), there exists an isometry $f_\lambda : V_\lambda \oplus V_{\lambda^{-1}} \to U_\lambda \oplus U_{\lambda^{-1}}$. By (a) above, there exist isometries $f_1 : V_1 \to U_1$ and $f_{-1} : V_{-1} \to U_{-1}$. Define $f = f_1 \oplus f_{-1} \oplus (\bigoplus f_\lambda) : V \to V$ in the natural way. Then f is an isometry and $fx = yf$. Hence x, y are G-conjugate. Parts (ii) and (iii) are easy consequences of (a)–(d) above. Part (iv) is an easy calculation for $G = \mathrm{GL}_n$; in the case $G \neq \mathrm{GL}_n$ the result follows using (a)–(c) above and [17, Lemma 4.1.9]. □

The following is a key tool. Recall the definition of the set $\mathscr{R}$ given in (3).

Proposition 2.7 ([4, Proposition 2.9]). *Let $G = Cl_n$. Let $x \in \mathscr{R}$ and assume $\nu(x) = s$. Then*

$$f(s) \leqslant \dim x^G \leqslant g(s)$$

where $f(s), g(s)$ are recorded in Table 5.

Remark 2.8. Assume $H \leqslant G$ is a finite subgroup and set $\Omega = G/H$. Then $\dim \Omega = \dim G$ and, for all $x \in G$, $\dim(x^G \cap H) = 0$. Therefore, for $x \in H$, $\dim C_\Omega(x) = \dim G - \dim x^G = \dim C_G(x)$, so the bounds in Proposition 2.7 yield *local bounds* on $\dim C_\Omega(x)$.

2.4. **$\mathscr{C}_3$-subgroups.** From now on G is one of the classical groups $\mathrm{Sp}(V)$ or $\mathrm{O}(V)$. Recall that a closed subgroup H of G is a *$\mathscr{C}_3$-subgroup* if H stabilises a direct sum decomposition of the form

$$V = V_1 \oplus V_2,$$

where V_1 and V_2 are maximal totally singular subspaces of V. By Witt's Lemma, $\dim V_1 = \dim V_2$ and thus $n = \dim V$ is even. Fix a standard basis $\beta = \{e_1, \ldots, e_{n/2}, f_1, \ldots, f_{n/2}\}$ for V, with respect to the underlying symplectic or quadratic form on V (see [17, Propositions 2.4.1, 2.5.3]). Since G acts transitively on the space of all such decompositions, up to replacing H by a G-conjugate, we may assume without loss of generality that $V_1 = \langle e_1, \ldots, e_{n/2} \rangle$ and $V_2 = \langle f_1, \ldots, f_{n/2} \rangle$, so in terms of the basis β we have $H = H^\circ.\langle \tau \rangle$, where

$$H^\circ = \left\{ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}_{n/2} \right\} \cong \mathrm{GL}_{n/2},$$

where A^t denotes the transpose of the matrix A; and

$$\tau = \begin{pmatrix} & I_{n/2} \\ \varepsilon I_{n/2} & \end{pmatrix}, \tag{10}$$

with ε defined in (6), $\varepsilon = -1$ if $G = \mathrm{Sp}_n$ and $\varepsilon = 1$ otherwise. Note that H° fixes V_1 and V_2, and τ interchanges these two spaces.

3. Conjugacy in $\mathscr{C}_3$-subgroups

Assume $x \in \mathscr{R}$ and $x^G \cap H \neq \emptyset$. In this section we give more information on $x^G \cap H$. As described in Section 2.4, the structure of H is transparent and we will derive an explicit formula for $\dim(x^G \cap H)$.

The following is a particular case of [4, Theorem 1] (notice that this has been proved for $\mathrm{SO}(V)$ rather than $\mathrm{O}(V)$, however the same result holds for classical groups in our definition).

Proposition 3.1. *Let $G = Cl(V)$, $H \in \mathscr{C}_3$ and $x \in \mathscr{R}$. Then*

$$\dim(x^G \cap H) \leqslant \left(\frac{1}{2} + \iota\right) \dim x^G$$

where $\iota = 1/(n-2)$ if $G = \mathrm{O}_n$ and $\iota = 0$ otherwise.

Our analysis of global upper bounds will rely on this result.

Lemma 3.2. *Let $x \in \mathscr{R}$ be of order $r > 2$. Then $x^G \cap H = x^G \cap H^\circ$.*

Proof. First observe that $H = H^\circ \cup H^\circ \tau$. If x is semisimple, or if x is unipotent and $p < \infty$, then the conclusion is clear since $H^\circ \tau$ does not contain any element of order r. Now assume $p = \infty$ and x is unipotent, so x has infinite order. Assume $x^G \cap H \neq \emptyset$ and let $y \in x^G \cap H$. In particular y is unipotent. Notice that H/H° is a finite algebraic group over the field k. Let $\pi\colon H \to H/H^\circ$ be the natural projection. Then $\pi(y)$ is unipotent, hence $\pi(y) = 1$. So $y \in H^\circ$. The result follows. □

3.1. **Unipotent elements.** Assume $p \neq 2$ (the case $p = 2$ will be discussed in Section 3.3). Let $x \in \mathscr{R}$ be unipotent. Recall that x has Jordan form $[J_p^{a_p}, \ldots, J_1^{a_1}]$ for some non-negative integers a_i. Assume $x^G \cap H \neq \emptyset$. Lemma 3.2 asserts that $x^G \cap H = x^G \cap H^\circ$. Thus, up to G-conjugacy we may assume $x = [x_1, x_2]$ where $x_2 = x_1^{-t}$. From elementary linear algebra we know that J_i and J_i^{-t} are GL_i-conjugate. Therefore x_1, x_2 are $\mathrm{GL}_{n/2}$-conjugate. Hence, a_i is even for all i and, up to conjugation, we have

$$x = \left(\begin{array}{c|c} [J_p^{a_p/2}, \ldots, J_1^{a_1/2}] & \\ \hline & [J_p^{a_p/2}, \ldots, J_1^{a_1/2}] \end{array} \right) \in H^\circ. \tag{11}$$

As already observed $x^G \cap H$ is a finite union of H°-classes. By the previous discussion the Jordan form of a representative of each of these classes is as in (11). Therefore $x^G \cap H = x^{H^\circ}$, for x as in (11). In particular, $\dim(x^G \cap H) = \dim\big([J_p^{a_p/2}, \ldots, J_1^{a_1/2}]\big)^{\mathrm{GL}_{n/2}}$, so by Theorem 2.3(iii) we compute $\dim(x^G \cap H)$.

Proposition 3.3. *Assume $p \neq 2$. Let $x \in \mathscr{R}$ be unipotent. Then $x^G \cap H \neq \emptyset$ if, and only if, a_i is even for all i. In addition, if $x^G \cap H \neq \emptyset$ then $x^G \cap H = [x_1, x_2]^{H^\circ}$, where $[x_1, x_2]$ is as in (11), and*

$$\dim(x^G \cap H) = \frac{n^2}{4} - \frac{1}{2} \sum_{i<j} i a_i a_j - \frac{1}{4} \sum_i i a_i^2.$$

Remark 3.4. Let $x \in \mathscr{R}$ be unipotent. If $x^G \cap H \neq \emptyset$ then $\nu(x)$ is even.

By the previous discussion we have the following. Recall the definition of ε given in (6).

Proposition 3.5. *Assume $p \neq 2$. Let $x \in \mathscr{R}$ be unipotent. Assume $x^G \cap H \neq \emptyset$. Then*

$$f_\Omega(x) = \frac{2\sum_{i<j} i a_i a_j + \sum_i i a_i^2 - 2\varepsilon \sum_{i \text{ odd}} a_i}{n(n - 2\varepsilon)}.$$

3.2. **Semisimple elements.** In this section we assume $r \neq p$ is an odd prime, and we write $t = (r-1)/2$. Let $x \in G$ be of order r. Recall that a_i is the dimension of the eigenspace associated to ω^i, see (8). Since H° contains a maximal torus of G° then $x^G \cap H \neq \emptyset$. Again, Lemma 3.2 yields $x^G \cap H = x^G \cap H^\circ$. Since $x^G \cap H^\circ \neq \emptyset$, we may assume $x = [x_1, x_2]$ with $x_2 = x_1^{-t}$. Let $b_1, \dots, b_t$ be non negative integers such that $b_i \leqslant a_i$. Then

$$\left(\begin{array}{c|c} [I_{a_0/2}, \omega I_{a_1-b_1}, \omega^{-1} I_{b_1}, \dots] & \\ \hline & [I_{a_0/2}, \omega I_{b_1}, \omega^{-1} I_{a_1-b_1}, \dots] \end{array}\right) \in x^G \cap H. \tag{12}$$

We define

$$\mathscr{T}_x = \left\{(b_1, \dots, b_t) \in (\mathbb{Z}_{\geqslant 0})^t : b_i \leqslant a_i\right\}.$$

Given $\mathbf{b} = (b_1, \dots, b_t) \in \mathscr{T}_x$, we denote $x_\mathbf{b} = [x_1, x_2]$ as in (12). Theorem 2.6 implies

$$C_{H^\circ}(x_\mathbf{b}) \cong \mathrm{GL}_{a_0/2} \times \left(\prod_{i=1}^t \mathrm{GL}_{b_i}\right) \times \left(\prod_{i=1}^t \mathrm{GL}_{a_i-b_i}\right). \tag{13}$$

Hence,

$$\begin{aligned} \dim x_\mathbf{b}^{H^\circ} &= \dim\left([I_{a_0/2}, \omega I_{a_1-b_1}, \omega^{-1} I_{b_1}, \dots, \omega^t I_{a_t-b_t}, \omega^{-t} I_{b_t}]\right)^{\mathrm{GL}_{n/2}} \\ &= \frac{n^2}{4} - \frac{a_0^2}{4} - \sum_{i=1}^t (a_i - b_i)^2 - \sum_{i=1}^t b_i^2. \end{aligned} \tag{14}$$

Let $[x_1, x_2] \in x^G \cap H^\circ$. Then, there exists $\mathbf{b} \in \mathscr{T}_x$ such that $[x_1, x_2] \in x_\mathbf{b}^{H^\circ}$. This fact, together with previous discussion, yields the following.

Proposition 3.6. *Let $x \in G$ be of odd order $r \neq p$. Then*

$$x^G \cap H = \bigcup_{\mathbf{b} \in \mathscr{T}_x} x_\mathbf{b}^{H^\circ}.$$

In particular, $\dim(x^G \cap H) = \max_{\mathbf{b} \in \mathscr{T}_x}\{\dim x_\mathbf{b}^{H^\circ}\}$.

Now, we aim to classify the tuples $\mathbf{b} \in \mathscr{T}_x$ such that $\dim(x^G \cap H) = \dim x_\mathbf{b}^{H^\circ}$. We shall need the following elementary result.

Lemma 3.7. *Let $a, b \in \mathbb{Z}_{\geqslant 0}$. Assume $b \leqslant a$ and $|a - 2b| \leqslant 1$. Then $b \in \{\lfloor a/2 \rfloor, \lfloor a/2 \rfloor + 1\}$.*

Proposition 3.8. *Let $x \in G$ be of odd order $r \neq p$. Let $\mathbf{b} = (b_1, \dots, b_t) \in \mathscr{T}_x$. Then* $\dim(x^G \cap H) = \dim x_\mathbf{b}^{H^\circ}$ *if, and only if, $|a_i - 2b_i| \leqslant 1$ for all i.*

Proof. Let $\mathbf{b} = (b_1, \dots, b_t) \in \mathscr{T}_x$. Assume $\dim(x^G \cap H) = \dim x_\mathbf{b}^{H^\circ}$. In particular, by Proposition 3.6, $\dim x_\mathbf{b}^{H^\circ} = \max\{\dim x_\mathbf{c}^{H^\circ} : \mathbf{c} \in \mathscr{T}_x\}$. Seeking a contradiction, assume there exists i such that $|a_i - 2b_i| \geqslant 2$. Clearly, we may assume $i = 1$. We shall construct $\mathbf{a} \in \mathscr{T}_x$ such that $\dim x_\mathbf{a}^{H^\circ} > \dim x_\mathbf{b}^{H^\circ}$, which would contradict the maximality of $\dim x_\mathbf{b}^{H^\circ}$.

For convenience we define $\delta = -1$ if $a_1 - 2b_1 \geqslant 2$, and $\delta = 1$ otherwise. Let $\mathbf{a} = (b_1 + \delta, b_2, \dots, b_t)$. Notice that $\mathbf{a} \in \mathscr{T}_x$. We compute $\dim x_\mathbf{a}^{H^\circ} - \dim x_\mathbf{b}^{H^\circ} = 2\delta(a_1 - 2b_1) - 2 > 0$. This shows the "only if" part.

Conversely, let us assume $|a_i - 2b_i| \leqslant 1$ for all i. Proposition 3.6 asserts that there exists $\mathbf{c} \in \mathscr{T}_x$ such that $\dim(x^G \cap H) = \dim x_\mathbf{c}^{H^\circ}$. Clearly, it is enough to show that $C_{H^\circ}(x_\mathbf{b}) \cong C_{H^\circ}(x_\mathbf{c})$. The previous argument yields $|a_i - 2c_i| \leqslant 1$ for all i. Thus, by Lemma 3.7, $b_i, c_i \in \{\lfloor a_i/2 \rfloor, \lfloor a_i/2 \rfloor + 1\}$, for all i. Assume there exists i such that $b_i \neq c_i$. Then a_i is odd (otherwise $b_i = c_i = a_i/2$), and for convenience we may assume $b_i = \lfloor a_i/2 \rfloor$ and $c_i = \lfloor a_i/2 \rfloor + 1$; in particular $c_i = a_i - b_i$. From (13), it is clear that $C_{H^\circ}(x_\mathbf{b}) \cong C_{H^\circ}(x_\mathbf{c})$. $\square$

Thanks to Proposition 3.8, the formula for $\dim(x^G \cap H)$ follows with an easy computation. For $x \in G$ of odd order $r \neq p$ we define

$$d(x) = \left|\left\{i \in \{1, \dots, r-1\} : a_i \text{ odd}\right\}\right|. \tag{15}$$

Proposition 3.9. *Let $x \in G$ be of order r. Then*

$$\dim(x^G \cap H) = \frac{n^2}{4} - \frac{1}{4}\sum_{i=0}^{r-1} a_i^2 - \frac{d(x)}{4}$$

where $d(x)$ is defined in (15).

Using the formula for $\dim x^G$ in Theorem 2.6, we quickly compute $\dim C_\Omega(x)$ and $f_\Omega(x)$. Recall the definition of ε given in (6). We will also need the following elementary observation.

Lemma 3.10. *Let $x \in G$ be of odd order $r \neq p$. Then*

$$\dim C_\Omega(x) = -\frac{\varepsilon a_0}{2} + \frac{1}{4}\sum_{i=0}^{r-1} a_i^2 - \frac{d(x)}{4}.$$

In particular, $\dim C_\Omega(x)$ *is even.*

Proof. The formula is clear. Since $a_i = a_{r-i}$ for all $i \leqslant t$, we have

$$\dim C_\Omega(x) = \frac{1}{4}\Big(a_0(a_0 - 2\varepsilon) + 2\sum_{1\leqslant i<t:\, a_i\,\text{even}} a_i^2 + 2\sum_{1\leqslant i<t:\, a_i\,\text{odd}} (a_i^2 - 1)\Big).$$

Since a_0 is even, $a_0(a_0 - 2\varepsilon)$ is divisible by 8. Hence, $\dim C_\Omega(x)$ is even. □

Proposition 3.11. *Let $x \in G$ be of odd order $r \neq p$. Then*

$$f_\Omega(x) = \frac{-2\varepsilon a_0 + \sum_{i=0}^{r-1} a_i^2 - d(x)}{n(n-2\varepsilon)}.$$

Corollary 3.12. *Let $x, y \in G$ be of odd order $r \neq p$. Assume $C_G(x) \cong C_G(y)$. Then* $\dim C_\Omega(x) = \dim C_\Omega(y)$.

3.3. **Involutions.** The aim of this section is to derive an explicit formula for $f_\Omega(x)$ for any involution $x \in G$. The main difference with odd order elements is given by the existence of involutions whose G-class meets $H \setminus H^\circ$. We shall prove the following, in which we also characterise G-classes of involutions that meet H.

Theorem 3.13. *Let $x \in G$ be an involution. Write $\nu(x) = s$.*

(a) *We have $x^G \cap H \neq \emptyset$ if, and only if, one of the following holds:*
 (i) *$s < n/2$ is even, and x is a_s-type if $p = 2$; or,*
 (ii) *$s = n/2$, with $n/2$ even if $G = \mathrm{Sp}_n$ and $p \neq 2$.*

(b) *Assume $x^G \cap H \neq \emptyset$. Then*

$$f_\Omega(x) = g(n,s)$$

where $g(n,s)$ is recorded in Table 6.

G	s	p	type	$g(n,s)$	G	s	p	type	$g(n,s)$
Sp_n	$< n/2$	any		$1 - \frac{2s(n-s)}{n(n+2)}$	O_n	$< n/2$	any		$1 - \frac{2s(n-s)}{n(n-2)} + \frac{4s\delta_{p,2}}{n(n-2)}$
	$n/2$ (even)	$\neq 2$		$\frac{1}{2} + \frac{1}{n+2}$		$n/2$	$\neq 2$		$\frac{1}{2}$
	$n/2$	2	$a_{n/2}$	$\frac{1}{2} + \frac{2}{n+2}$		$n/2$	2	$a_{n/2}$	$\frac{1}{2} + \frac{2}{n-2}$
	$n/2$	2	$b_{n/2}, c_{n/2}$	$\frac{1}{2} - \frac{2}{n+2}$		$n/2$	2	$b_{n/2}$	$\frac{1}{2}$
						$n/2$	2	$c_{n/2}$	$\frac{1}{2} - \frac{2}{n-2}$

TABLE 6. Involutions in $\mathscr{C}_3$-actions

Remark 3.14. Notice that $f_\Omega(x) > 0$ for any involution $x \in H$.

Thanks to Theorem 3.13 we immediately deduce upper and lower bounds on the f_Ω-value of any involution. It is a straightforward calculation to see that the conclusions of Theorem 3 hold for involutions. The following shows the conclusions of Theorem 1(b)(i) and (c), when $x \in \mathscr{R}$ is an involution.

Corollary 3.15. *Let $x \in G$ be an involution. Assume $x^G \cap H \neq \emptyset$. Then*

$$g_2(n) \leqslant f_\Omega(x) \leqslant g_1(n)$$

where $g_i(n)$, $i = 1, 2$, are recorded in Table 7. In addition,

 (i) *$f_\Omega(x) = g_1(n)$ if, and only if, $\nu(x) = 2$;*
 (ii) *$f_\Omega(x) = g_2(n)$ if, and only if, $p = 2$ and x is $b_{n/2}$- or $c_{n/2}$-type; or $p \neq 2$ and $\nu(x) \in \{n/2, n/2 - \delta_{G,\mathrm{Sp}_n}\}$.*

For the discussion that follows it is convenient to define

$$B = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \in \mathrm{GL}_2. \tag{16}$$

Involutions whose G-classes meet H° need to satisfy certain conditions, as we see in the following.

G	$g_1(n)$	$g_2(n)$, $p=2$	$g_2(n)$, $p\neq 2$
Sp_n	$1-\frac{4}{n}+\frac{16}{n(n+2)}$	$\frac{1}{2}-\frac{2}{n+2}$	$\frac{1}{2}+\frac{1}{n+2\delta_{n;4}}$
O_n	$1-\frac{4}{n}+\frac{8\delta_{p,2}}{n(n-2)}$	$\frac{1}{2}-\frac{2\delta_{n;4}}{n-2}$	$\frac{1}{2}$

TABLE 7. Bounds on $f_\Omega(x)$ for involutions $x\in H$

Lemma 3.16. *Let x be an involution and assume $x^G\cap H^\circ\neq\emptyset$. Then $\nu(x)=s$ is even. In addition, if $p=2$ then x is a_s-type.*

Proof. Since $x^G\cap H^\circ\neq\emptyset$ we may assume, up to G-conjugacy, $x=[x_1,x_2]\in H^\circ$, where $x_2=x_1^{-t}$. Hence x_1,x_2 have the same Jordan form. In particular $\nu(x)$ is even.

Recall that H stabilises a decomposition $V=V_1\oplus V_2$, where V_1,V_2 are maximal totally singular subspaces of V, see Section 2.4. If $x\in H^\circ$ then $x.V_1=V_1$ and $x.V_2=V_2$. Assume $p=2$ and let $x\in H^\circ$ be an involution. Then $(x.e_i,e_i)=(x.f_i,f_i)=0$, for all i. Therefore x is a_s-type. □

Now we focus on involutions in $H\setminus H^\circ$.

Lemma 3.17. *Let $x\in H\setminus H^\circ$ be an involution. Then $\nu(x)=n/2$.*

Proof. Assume $n/2$ is even. Then with respect to the basis

$$\{e_1,x.e_1,\dots,e_{n/4},x.e_{n/4},e_{n/4+1},\varepsilon x.e_{n/4+1},\dots,e_{n/2},\varepsilon x.e_{n/2}\},$$

$x=[B^{n/4},\varepsilon B^{n/4}]$. In particular $\nu(x)=n/2$.

Assume $n/2$ is odd and $G=\mathrm{Sp}_n$. So $x=[A,A^{-t}]\tau$ for some $A\in\mathrm{GL}_{n/2}$ such that $A=-A^t$. Then $\det(x)=-1$, which is absurd if $p\neq 2$. Hence if $n/2$ is odd and $p\neq 2$ then $G=\mathrm{O}_n$. As the previous case, with respect to the basis $\{e_1,x.e_1,\dots,e_{n/2},x.e_{n/2}\}$, $x=[B^{n/2}]$. This completes the proof. □

Remark 3.18. Assume $n/2$ is even. If $p=2$ then τ is $a_{n/2}$-type, while $[B^{n/2}]\tau$ is $c_{n/2}$-type. If $G=\mathrm{Sp}_n$ and $p\neq 2$ then τ is not an involution; however, in any case, $[B^{n/4},\varepsilon B^{n/4}]\tau$ is an involution. Since $H\cong\mathrm{GL}_{n/2}.2$ we will write $[B^{n/4}]\tau$ for $[B^{n/4},\varepsilon B^{n/4}]\tau$, similarly for the other elements.

Lemma 3.17 and Remark 3.18 provide us with the information needed to determine the G-classes of involutions that meet $H\setminus H^\circ$. Now we describe the H°-classes of such involutions. If $n/2$ is even, we define

$$K=[K_{n/2},K_{n/2}]\tau,\ \text{where}\ K_{n/2}=\begin{pmatrix} & I_{n/4}\\ \varepsilon I_{n/4} & \end{pmatrix}$$

is the Gram matrix associated to a symplectic or non-degenerate orthogonal form on $k^{n/2}$. Similarly one can define $K_{n/2}$ if $n/2$ is odd, so that it is the Gram matrix associated to a non-degenerate orthogonal form on $k^{n/2}$. The following shows that, when $p\neq 2$, any two involutions in $H\setminus H^\circ$ are H°-conjugate.

Lemma 3.19. *Assume $p\neq 2$. Any involution in $H\setminus H^\circ$ is H°-conjugate to K.*

Proof. Let $x\in H\setminus H^\circ$ be an involution. Then $x=[A,A^{-t}]\tau$, for some $A\in\mathrm{GL}_{n/2}$ such that $A=\varepsilon A^t$. Thus A,A^{-t} are Gram matrix associated to a symplectic or non-degenerate orthogonal form on $k^{n/2}$. Hence, they are congruent to $K_{n/2}$, see [12, p.14]. Thus there exists $P\in\mathrm{GL}_{n/2}$ such that $P^tAP=K_{n/2}$. Define $g=[P^{-t},P]\in H^\circ$. Then $g^{-1}xg=K$. □

Let $x\in G$ be an involution and assume $\nu(x)=n/2$. Thanks to Lemma 3.17, Remark 3.18, and Lemma 3.19, we have

$$x^G\cap H=\left(\bigcup_{y\in\mathscr{A}} y^{H^\circ}\right)\cup\left(\bigcup_{\pi\in\mathscr{B}} \pi^{H^\circ}\right) \tag{17}$$

where $\mathscr{A}\subseteq H^\circ,\mathscr{B}\subseteq H\setminus H^\circ$, $|\mathscr{A}|,|\mathscr{B}|\in\{0,1\}$ and the representatives of the distinct H°-classes are recorded in Table 8.

The H°-centraliser of involutions in $H\setminus H^\circ$ is provided by the next result, which was originally stated in [19, Proposition 1.4]. We re-state it in our notation.

Proposition 3.20. *Let $G=\mathrm{Sp}_n$ or O_n, $H^\circ=\mathrm{GL}_{n/2}<G$, τ as in (10) and B as in (16).*

Sp_n					O_n				
n	p	G-class repr.	π	y	n	p	G-class repr.	π	y
$4m+2$	$\neq 2$	–	–	–	$4m+2$	$\neq 2$	$[I_{2m+1},-I_{2m+1}]$	τ	–
	$=2$	b_{2m+1}	τ	–		$=2$	b_{2m+1}	τ	–
$4m$	$\neq 2$	$[I_{2m},-I_{2m}]$	$[B^m]\tau$	$[I_m,-I_m]$	$4m$	$\neq 2$	$[I_{2m},-I_{2m}]$	τ	$[I_m,-I_m]$
	$=2$	c_{2m}	τ	–		$=2$	c_{2m}	τ	–
		a_{2m}	$[B^m]\tau$	$[J_2^m]$			a_{2m}	$[B^m]\tau$	$[J_2^m]$

TABLE 8. Representatives of H°-classes of involutions $x \in H$ with $\nu(x) = n/2$

(i) *Assume $n/2$ is odd. Then $C_{H^\circ}(\tau) = \mathrm{O}_{n/2}$.*

(ii) *If $n/2$ is even, then $C_{H^\circ}(\tau) = \mathrm{O}_{n/2}$ and $C_{H^\circ}([B^{n/4}]\tau) = \mathrm{Sp}_{n/2}$ if $p \neq 2$; and $C_{H^\circ}(\tau) = \mathrm{Sp}_{n/2}$, $C_{H^\circ}([B^{n/4}]\tau) = C_{\mathrm{Sp}_{n/2}}(b_1)$ if $p = 2$.*

Now we have all the ingredients needed to prove Theorem 3.13.

Proof of Theorem 3.13. By the previous discussion if $x \in H \setminus H^\circ$ then $\nu(x) = n/2$. However, if $p \neq 2$, there is no involution in $H \setminus H^\circ$ if $G = \mathrm{Sp}_n$ and $n/2$ is odd, see Table 8. If $\nu(x) < n/2$ and $x^G \cap H \neq \emptyset$ then Lemma 3.16 asserts that $\nu(x)$ is even and, if $p = 2$, that x is a_s-type. This completes the proof of part (a).

Now we show part (b). Let $x \in H$ be an involution.

First assume $\nu(x) < n/2$. Then $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Assume $p \neq 2$, and let $x = [I_{n-s}, -I_s]$. We have $y \in x^G \cap H^\circ$ only if $y = [x_1, x_2]$, with $x_1 = x_2^{-t}$. Thus, $x_1 = x_2 = [I_{(n-s)/2}, I_{s/2}]$. Hence, by Theorem 2.6,

$$\dim(x^G \cap H^\circ) = \dim x_1^{\mathrm{GL}_{n/2}} = \frac{s}{2}(n-s). \tag{18}$$

Assume $p = 2$ and $\nu(x) < n/2$. Then Lemma 3.16 asserts that x is a_s-type. We have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. And by Theorem 2.3,

$$\dim(x^G \cap H) = \dim x^{H^\circ} = \dim [J_2^{s/2}, J_1^{n/2-s}]^{\mathrm{GL}_{n/2}} = \frac{s}{2}(n-s). \tag{19}$$

The required formula for $f_\Omega(x)$ quickly follows using $\dim x^G$ given in Theorems 2.5 and 2.6 in the cases $p = 2$ and $p \neq 2$, respectively, and the above formulae for $\dim(x^G \cap H)$.

Now assume $\nu(x) = n/2$ and $G = \mathrm{Sp}_n$. Assume $p \neq 2$. By (17) and Table 8 we deduce that $n/2$ is even and $x^G \cap H = [I_{n/4}, -I_{n/4}]^{H^\circ} \cup ([B^{n/4}]\tau)^{H^\circ}$. Proposition 3.20 yields $C_{H^\circ}([B^{n/4}]\tau) = \mathrm{Sp}_{n/2}$. Therefore

$$\dim [I_{n/4}, -I_{n/4}]^{H^\circ} = \frac{n^2}{8}, \qquad \dim([B^{n/4}]\tau)^{H^\circ} = \dim H^\circ - \dim C_{H^\circ}([B^{n/4}]\tau) = \frac{n^2}{8} - \frac{n}{4}.$$

Thus $\dim(x^G \cap H) = n^2/8$. If $p = 2$ then $x^G \cap H = \tau^{H^\circ}$ if $n/2$ is odd; otherwise, we have two cases: if x is $a_{n/2}$-type then $x^G \cap H = ([B^{n/4}]\tau)^{H^\circ} \cup [J_2^{n/4}]^{H^\circ}$, if x is $c_{n/2}$-type then $x^G \cap H = \tau^{H^\circ}$, see Table 8. Again, we compute the dimension of the H°-class of the previous elements, using Proposition 3.20 for the classes of τ and $[B^{n/4}]\tau$. The result follows with easy calculations.

Let $G = \mathrm{O}_n$ and $\nu(x) = n/2$. First assume $p \neq 2$. By (17) and Table 8, $x^G \cap H = \tau^{H^\circ}$ if $n/2$ is odd and $x^G \cap H = [I_{n/4}, -I_{n/4}]^{H^\circ} \cup \tau^{H^\circ}$ if $n/2$ is even, see Table 8. If $p = 2$ and $n/2$ is odd then $x^G \cap H = \tau^{H^\circ}$. If $p = 2$ and $n/2$ is even, two cases arise: either x is $a_{n/2}$-type and $x^G \cap H = [J_2^{n/4}]^{H^\circ} \cup ([B^{n/4}]\tau)^{H^\circ}$, or x is $c_{n/2}$-type and $x^G \cap H = \tau^{H^\circ}$, see Table 8. As above, using Proposition 3.20, we compute $\dim \tau^{H^\circ}$ and $\dim([B^{n/4}]\tau)^{H^\circ}$. The value of $f_\Omega(x)$ quickly follows. □

4. UPPER BOUNDS

In this section we derive upper bounds on $f_\Omega(x)$ for $x \in \mathscr{R}$, showing Theorem 1(a). In addition, we shall characterise elements that realise such bounds, this will prove Theorem 6 for the part concerning $\mathscr{C}_3$-actions.

We split the analysis according whether $G = \mathrm{Sp}_n$ or O_n. We assume $n > 4$, and we derive upper bounds in Propositions 4.1 and 4.3. We refer the reader to Remarks 4.2 and 4.4 for the case $n = 4$. Recall, for $x \in G$ of order r we write $o(x) = r$.

Proposition 4.1. *Let $G = \mathrm{Sp}_n$, let $H \leqslant G$ be a $\mathscr{C}_3$-subgroup. Assume $n > 4$. Set $\Omega = G/H$. Let $x \in \mathscr{R}$. Then*

$$f_\Omega(x) \leqslant 1 - \frac{4}{n+2} + \frac{8\delta_{r,2}}{n(n+2)}$$

with equality if, and only if, $\nu(x) = 2$.

Proof. If $o(x) = 2$ then Corollary 3.15 yields the result. Therefore we assume $o(x) \neq 2$.

Observe that $\nu(x) > 1$ for any $x \in \mathscr{R}$ such that $x^G \cap H \neq \emptyset$. First, let us assume $\nu(x) = 2$. If $o(x) = p$, x is G-conjugate to $[J_2^2, J_1^{n-4}]$ (we exclude $[J_3, J_1^{n-3}]$ as its G-class does not meet H, see Proposition 3.3). If $o(x) \neq p$ then $C_G(x) \cong C_G([I_{n-2}, \omega, \omega^{-1}])$. In both cases, we compute $f_\Omega(x) = 1 - 4/(n+2)$.

Similarly, it is a routine calculation to compute $f_\Omega(x)$ when $\nu(x) = 3, 4$. For semisimple elements, thanks to Proposition 3.11 we only need to consider elements up to centraliser structure and up to conjugation. For example, if $\nu(x) = 3$ then $n = 6$ and $f_\Omega(x) = 1/3 < 1 - 4/(n+2)$. If $\nu(x) = 4$ then $C_G(x)$ is one of the following: $\mathrm{Sp}_{n-4} \times (\mathrm{GL}_1)^2, \mathrm{Sp}_{n-4} \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_1, \mathrm{GL}_4$; and, we have $f_\Omega(x) < 1 - 4/(n+2)$. Similarly if $\nu(x) = 3, 4$ and x is unipotent.

Assume $\nu(x) > 4$. Then either $n \geqslant 8$ or $(n, \nu(x)) = (6, 5)$. In the latter case, up to the centraliser structure, $x = [\omega, \omega^{-1}, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}]$ and $f_\Omega(x) = 0$. If $n \geqslant 8$ then Proposition 2.7 implies $\dim x^G > 2n$. Using Proposition 3.1, we have

$$f_\Omega(x) \leqslant 1 - \frac{\dim x^G}{2\dim\Omega} < 1 - \frac{n}{\dim\Omega} = 1 - \frac{4}{n+2}.$$

This completes the proof. □

Remark 4.2. In the case $n = 4$ it is straightforward to compute $f_\Omega(x)$, for any $x \in \mathscr{R}$. In the case x has odd prime order r then $f_\Omega(x) \leqslant 1/3$ with equality if, and only if, $r \neq p$ and $C_G(x) \cong \mathrm{Sp}_2 \times \mathrm{GL}_1$, or $r = p$ and $x \in [J_2^2]^G$. In the case x is an involution we refer to Corollary 3.15.

Proposition 4.3. *Let $G = \mathrm{O}_n$, let $H \leqslant G$ be a $\mathscr{C}_3$-subgroup. Assume $n > 4$. Set $\Omega = G/H$. Let $x \in \mathscr{R}$. Then either $n = 6, 8$, $r \neq p$ is odd, and $f_\Omega(x) \leqslant 2/3$; or*

$$f_\Omega(x) \leqslant 1 - \frac{4}{n} + \frac{8\delta_{r,p}}{n(n-2)},$$

In addition, the following holds.

(i) *Assume $n = 6, 8$ and $r \neq p$ is odd. Then $f_\Omega(x) = 2/3$ if, and only if, $C_G(x) \cong \mathrm{GL}_{n/2}$.*
(ii) *Otherwise, x realises the bound if, and only if, $\nu(x) = 2$ or, x^G or $C_G(x)$ are as in Table 9.*

n	p	r	x^G	$C_G(x)$
8	>2	$=p$	$[J_2^4]^G$	–
8	>3	$=p$	$[J_4^2]^G$	–
10	–	$\neq p$	–	GL_5

TABLE 9

Proof. The same argument of Proposition 4.1 applies. First we observe that if $\nu(x) = 1$ then $x^G \cap H = \emptyset$. So let $x \in H$. If $\nu(x) \leqslant 5$ the result follows with straightforward computations. In the case $\nu(x) > 5$, using Propositions 2.7 and 3.1, we see that

$$f_\Omega(x) \leqslant 1 - \frac{4}{n} + \frac{8}{n(n-2)}$$

with equality if, and only if, $n = 8$ and $x = [J_2^4]$. □

Remark 4.4. If $G = \mathrm{O}_4$ then $H = \mathrm{GL}_2.2$. Let $x = [\omega I_2, \omega^{-1} I_2]$. Then $\dim x^G = 2, \dim(x^G \cap H) = 2$. Therefore $\dim C_\Omega(x) = \dim\Omega$. Notice that if $p \neq 2$ then x is not in the kernel of the action. In other words $C_\Omega(x) \neq \Omega$; in particular, Ω is not irreducible, so it must have two irreducible components $\Omega = \Omega_1 \cup \Omega_2$. Recall

$$\Omega = \{\{U, W\} : V = U \oplus W,\ U, W \text{ maximal totally singular}\}.$$

In order to show that $C_\Omega(x) \subset \Omega$, we define $x \in G$ as $x.e_1 = \omega e_1, x.e_2 = \omega^{-1}e_2$, so that $x.f_1 = \omega^{-1} f_1$ and $x.f_2 = \omega f_2$. If $p \neq 2$ then $X = \left\{\langle e_1+e_2, f_1-f_2\rangle, \langle e_1-e_2, f_1+f_2\rangle\right\} \in \Omega$. It is straightforward to check that $X \notin C_\Omega(x)$.

Thanks to Lemma 2.2, the upper bounds in Propositions 4.1 and 4.3 extend to any element $x \in G \setminus Z(G)$. In particular, the bound stated in Theorem 1(a) holds. Recall the definition of the *algebraic fixity* M and the *r-local algebraic fixity* M_r.

Proposition 4.5. *The conclusions of Theorem 6 and Corollary 9 hold for $\mathscr{C}_3$-actions.*

Proof. The upper bounds given in Corollary 3.15, Propositions 4.1 and 4.3, and Remarks 4.2 and 4.4 are sharp and the elements that realise them are known. The algebraic fixity M, in the cases $G = \mathrm{Sp}_4$ or O_n, when $n \leqslant 10$, are listed in Table 3. In the last column we record the prime r for which $M = M_r$. In the other cases, the conclusion of Theorem 6 for $\mathscr{C}_3$-actions quickly follows using the characterisations of the elements that realise the bounds, as stated in the aforementioned results. It is immediate to check that all the upper bounds are greater than or equal to $1-4/n$. Hence Corollary 9 holds. $\square$

5. Lower bounds: unipotent elements

In this section we show Theorem 1(b); more information will be given: in almost all cases we classify elements that realise the bounds. In the case no characterisation is given we shall provide examples of elements whose f_Ω-value is close to the lower bound. The analysis relies on the formula computed in Proposition 3.5. The main result of this section is Proposition 5.2, below. The case $p = 2$ has been dealt by Corollary 3.15. Hence we assume $p > 2$. Recall the definition of ε given in (6).

Remark 5.1. If $x \in G$ then either $x^G \cap H = \emptyset$, in which case $C_\Omega(x) = \emptyset$ by (7); or $x^G \cap H \neq \emptyset$. In the latter case, the assumption $x \in H$ does not compromise generality in order to compute lower bounds on $f_\Omega(x)$, since $\dim C_\Omega(x) = \dim C_\Omega(x^g)$ for all $g \in G$.

Proposition 5.2. *Assume p is odd. Let $x \in H$ be of order p.*

(i) *If $p > n/2$ then*
$$f_\Omega(x) \geqslant \frac{2}{n - 2\delta_{n;4}\varepsilon}.$$
(ii) *If $p \leqslant n/2$ then*
$$f_\Omega(x) \geqslant \frac{1}{p}.$$

In particular, the conclusion of Theorem 1(b) holds.

Recall that if $x \in H$ then a_i is even for all i.

5.1. Case $p > n/2$.

Notice that the characteristic zero case is part of this analysis.

Lemma 5.3. *Assume $p > n/2$. The conclusion of Proposition 5.2(i) holds.*

Proof. First, let $G = \mathrm{Sp}_n$. Assume $n/2$ is even. Using Proposition 3.5 and $n = \sum_i ia_i$, we see that the $f_\Omega(x) \geqslant 2/(n+2)$ is equivalent to

$$2\sum_{i<j} ia_ia_j + \sum_i ia_i(a_i-2) + 2\sum_{i \text{ odd}} a_i \geqslant 0. \tag{20}$$

Since all the summands are non-negative, the inequality holds.

Now, assume $n/2$ is odd. Again, $f_\Omega(x) \geqslant 2/n$ is equivalent to

$$2\sum_{i<j} ia_ia_j + \sum_i ia_i(a_i-2) + 2\sum_{i \text{ odd}} a_i - 4 \geqslant 0. \tag{21}$$

Since $n/2$ is odd, there exists an odd i such that $a_i > 0$. Hence, $2\sum_{i \text{ odd}} a_i \geqslant 4$, and the inequality holds.

Let $G = \mathrm{O}_n$. Assume $n/2$ is even. We see that $f_\Omega(x) \geqslant 2/(n-2)$ if, and only if,

$$2\sum_{i<j} ia_ia_j + \sum_{i \text{ even}} ia_i(a_i-2) + \sum_{i \text{ odd}} a_i(i(a_i-2)-2) \geqslant 0. \tag{22}$$

The first two summands and the last (when $a_i \neq 2$) are non-negative. Assume there exists i odd such that $a_i = 2$. Say $i_1 \leqslant \cdots \leqslant i_\ell$ all the odd indices for which $a_{i_j} = 2$. If $\ell = 1$ we need to distinguish two cases: either there exists $j \neq i_1$ such that $a_j > 0$ or, for all $j \neq i_1$, $a_j = 0$. The latter case cannot arise since, together with

$n=\sum_i ia_i$, it would imply $i_1=n/2$. In the former case, $2ha_{i_1}a_j+a_{i_1}(i_1(a_{i_1}-2)-2)=2ha_{i_1}a_j-2a_{i_1}>0$ (where $h=\min\{i_1,j\}$). If $\ell>1$ we have

$$2i_1a_{i_1}a_{i_2}+\cdots+2i_1a_{i_1}a_{i_{\ell-1}}-2a_{i_1}-2a_{i_\ell}\geqslant 0,$$
$$2i_2a_{i_2}a_{i_3}+\cdots+2i_2a_{i_2}a_{i_\ell}-2a_{i_2}>0,\ \ldots,2i_{\ell-1}a_{i_{\ell-1}}a_{i_\ell}-2a_{i_{\ell-1}}>0.$$

If $n/2$ is odd then $f_\Omega(x)\geqslant 2/n$ if, and only if,

$$2\sum_{i<j}ia_ia_j+\sum_{i\text{ even}}ia_i(a_i-2)+\sum_{i\text{ odd}}a_i(i(a_i-2)-2)+4\geqslant 0. \tag{23}$$

As above we deduce that this inequality holds. This concludes the proof. □

Remark 5.4. Notice that, by the proof of Lemma 5.3 (unless $G=\mathrm{O}_n$ and $n/2$ is odd), we have that $f_\Omega(x)$ realises the lower bound if, and only if, each summand in (20) and (21) vanishes, similarly for (22).

Proposition 5.5. *Assume $p>n/2$. Let $x\in H$ be of order p.*

(i) *Assume $G=\mathrm{Sp}_n$. Then $f_\Omega(x)=2/(n+2\delta_{n;4})$ if, and only if, $x\in[J_{n/2}^2]^G$.*
(ii) *Assume $G=\mathrm{O}_n$ and $n/2$ is even. Then $f_\Omega(x)=2/(n-2)$ if, and only if, $x\in[J_{n/2}^2]^G\cup[J_{n/2-1}^2,J_1^2]^G$.*
(iii) *Assume $G=\mathrm{O}_n$, with $n/2$ odd. Let $x\in[J_{n/2}^2]^G$. Then $f_\Omega(x)=2/n$.*

Proof. It is straightforward to compute $f_\Omega([J_{n/2}^2])$ and $f_\Omega([J_{n/2-1}^2,J_1^2])$. Conversely, let $x\in H$ such that $f_\Omega(x)$ realises the lower bound. We shall make use of the observation made in Remark 5.4.

First assume $G=\mathrm{Sp}_n$ and $n/2$ is even. Then all the summands in (20) vanish. Since $\sum_{i<j}ia_ia_j=0$ we deduce that there exists only one index $\ell\in\{1,\ldots,n/2\}$ such that $a_\ell\neq 0$. In particular, $n=\sum_i ia_i=\ell a_\ell$. In addition, for all i odd we must have $a_i=0$; if i is even then either $a_i=0$ or $a_i=2$. Therefore $\ell=n/2$ and $a_\ell=2$. The argument is very similar when $n/2$ is odd.

Now assume $G=\mathrm{O}_n$ and $n/2$ is even. Then we have two cases. Either $a_i\neq 2$ for all odd i and, as above, we deduce the result; or, there exists i odd such that $a_i=2$. Thus, with the notation established in the proof of Lemma 5.1, $\ell\geqslant 1$. The case $\ell=1$ is ruled out in the proof of the bound (we have a strict inequality). Similarly for the case $\ell\geqslant 3$, as $2i_2a_{i_2}a_{i_3}+\cdots+2i_2a_{i_2}a_{i_\ell}-2a_{i_2}>0$ would imply $f_\Omega(x)>2/(n-2)$. For $\ell=2$, from $2i_1a_{i_1}a_{i_2}-2a_{i_1}-2a_{i_2}=0$ we deduce $i_1=1$. If there exists i even such that $a_i\neq 0$ then $\sum_{i<j,(i,j)\neq(i_1,i_2)}ia_ia_j>0$. Thus $a_i=0$ for all i even. Hence we deduce $(i_1,i_2)=(1,n/2-1)$. This completes the proof. □

5.2. **Case $p\leqslant n/2$.** In this case the largest Jordan block allowed in an element of order p is J_p.

Lemma 5.6. *Assume $p\leqslant n/2$ is odd. The conclusion of Proposition 5.2(ii) holds.*

Proof. First assume $G=\mathrm{Sp}_n$. Let $\alpha(x)=n(n+2)\big(f_\Omega(x)-1/p\big)$. Thus we need to show $\alpha(x)\geqslant 0$. Using $n=\sum_i ia_i$ and Proposition 3.5, we have

$$\alpha(x)=2\sum_{i<j}ia_ia_j\Big(1-\frac{j}{p}\Big)+\sum_{i\text{ odd}}ia_i^2\Big(1-\frac{i}{p}\Big)+2\sum_{i\text{ odd}}a_i\Big(1-\frac{i}{p}\Big)+\sum_{i\text{ even}}ia_i\Big(a_i-\frac{ia_i}{p}-\frac{2}{p}\Big).$$

Since $i\leqslant p$, it is clear that the first three summands in $\alpha(x)$ are non-negative. We claim that the last summand is non-negative. Let $i<p$ be even. Assume $a_i>0$; so $a_i\geqslant 2$ since a_i is even. Then

$$ia_i\Big(a_i-\frac{ia_i}{p}-\frac{2}{p}\Big)\geqslant 2i\Big(2-\frac{2i}{p}-\frac{2}{p}\Big)=\frac{4i}{p}(p-i-1)\geqslant 0.$$

For the orthogonal group we define $\beta(x)=n(n-2)\big(f_\Omega(x)-1/p\big)$. We have

$$\beta(x)=2\sum_{i<j}ia_ia_j\Big(1-\frac{j}{p}\Big)+\sum_{i\text{ even}}ia_i\Big(a_i\Big(1-\frac{i}{p}\Big)+\frac{2a_i}{p}\Big)+\sum_{i\text{ odd}}a_i\Big(ia_i\Big(1-\frac{i}{p}\Big)+\frac{2i}{p}-2\Big).$$

as in the previous case we see that each of the summands in $\beta(x)$ is non-negative. □

In this case we can characterise elements that realise the lower bound in all cases.

Proposition 5.7. *Assume $p\leqslant n/2$ is odd. Let $x\in G$ be of order p. Then $f_\Omega(x)=1/p$ if, and only if, one of the following holds*

(a) *$G=\mathrm{Sp}_n$ and*
  (i) *$n/2=ap$ and $x\in[J_p^{2a}]^G$; or*
  (ii) *$n/2=ap+(p-1)$ and $x\in[J_p^{2a},J_{p-1}^2]^G$.*
(b) *$G=\mathrm{O}_n$, $n/2=ap$ and $x\in[J_p^{2a}]^G$.*

Proof. Using Proposition 3.5, we quickly compute $f_\Omega(x) = 1/p$ when x is as given in the statement.

Conversely, suppose $x \in G$ has order p and $f_\Omega(x) = 1/p$. The same observation made in Remark 5.4 applies here. Therefore, if $G = \mathrm{Sp}_n$, each summand in $\alpha(x)$, as in the proof of Lemma 5.6, vanishes:

$$\sum_{i \text{ odd}} a_i\left(1 - \frac{i}{p}\right) = 0, \quad \sum_{i \text{ even}} \frac{ia_i}{p}(a_i(p-i)-2) = 0.$$

Therefore for all $i < p$ odd we deduce that $a_i = 0$. For i even we have $a_i(p-i) = 2$ if, and only if, $(i, a_i) = (p-1, 2)$. Therefore $a_{p-1} \in \{0, 2\}$, and $a_i = 0$ for all even $i \neq p-1$. Recall that $n = \sum_i ia_i$. Hence either $n = pa_p$, in the case $a_{p-1} = 0$, or, $n = pa_p + 2(p-1)$ in the case $a_{p-1} = 2$.

The argument for the orthogonal group is very similar and left to the reader. □

In the case n and p do not satisfies the conditions in (a) or (b) of Proposition 5.7 the lower bound $1/p$ is not realised; however it is close to the best possible.

Lemma 5.8. *Assume $p \nmid n$ and, if $G = \mathrm{Sp}_n$, $p \nmid n/2 - (p-1)$. Then there exists $x \in H$ of order p such that*

$$f_\Omega(x) \leqslant \frac{1}{p} + \frac{1}{n}.$$

Proof. Write $n/2 = ap + b$ with $0 < b < p$ (and $b < p-1$ in the case $G = \mathrm{Sp}_n$). Notice that $p \leqslant n/2 - 1$. Let $x = [J_p^{2a}, J_b^2]$. Then $x^G \cap H \neq \emptyset$. Using Proposition 3.5, we compute $f_\Omega(x)$ and it is easy to deduce the desired bound. For example if $G = \mathrm{O}_n$ we have

$$f_\Omega(x) = \frac{1}{p} + \frac{4b(p-b+1)}{pn(n+2)} - \frac{4(1-\delta_{b;2})}{n(n+2)} \leqslant \frac{1}{p} + \frac{(p+1)^2}{pn(n+2)} \leqslant \frac{1}{p} + \frac{n^2}{2n(n-2)(n+2)} \leqslant \frac{1}{p} + \frac{1}{n}.$$

□

6. Lower bounds: semisimple elements

Throughout this section $r \neq p$ is an odd prime. The purpose of this section is to derive lower bounds on $f_\Omega(x)$ for $x \in G$ of order r (see Corollary 3.15 for involutions). Notice that by Remark 5.1 we may assume $x \in H$. The main result of this section is Proposition 6.1, below, that is proved in Lemmas 6.3 and 6.5. To conclude this section we focus on elements of odd prime order $r < n$; the best possible lower bounds on their f_Ω-values are stated in (24) and (25). This further analysis will allow us to assert that the bound in Proposition 6.1(i) is sharp, see Remark 6.14.

Proposition 6.1. *Let $x \in H$ be of odd prime order $r \neq p$.*

(i) *If $r < n$, then*

$$f_\Omega(x) \geqslant \frac{1}{r} - \frac{r^2-1}{rn(n-2\varepsilon)}.$$

(ii) *Assume $r > n$. Then $f_\Omega(x) = 0$ if, and only if, $\nu(x) = n-1$ or $(G, C_G(x)) = \left(\mathrm{O}_n, \mathrm{O}_2 \times (\mathrm{GL}_1)^{n/2-1}\right)$.*

In particular, the conclusions of Theorems 1(c) and 10 hold.

We spread the proof of Proposition 6.1 in two lemmas. Applying the Cauchy–Schwarz inequality to the vectors $(a_1, \dots, a_h), (0, \dots, 0) \in \mathbb{R}^h$ we have the following inequality which will be very useful.

Lemma 6.2. *Let $a_1, \dots, a_h \in \mathbb{R}$. Then*

$$\sum_{i=1}^{h} a_i^2 \geqslant \frac{1}{h}\left(\sum_{i=1}^{h} a_i\right)^2.$$

Lemma 6.3. *The conclusion of Proposition 6.1(i) holds.*

Proof. We use the explicit formula of $f_\Omega(x)$ provided by Proposition 3.11. Notice that $d(x) \leqslant r-1$ and $\sum_{i>0} a_i = n - a_0$. Hence, using Lemma 6.2,

$$f_\Omega(x) = \frac{-2\varepsilon a_0 + a_0^2 + \sum_{i>0} a_i^2 - d(x)}{n(n-2\varepsilon)} \geqslant \frac{-2\varepsilon a_0 + a_0^2 + (n-a_0)^2/(r-1) - (r-1)}{n(n-2\varepsilon)}.$$

As function of a_0 the numerator has a minimum in $a_0 = n/r + \varepsilon(1 - 1/r)$, so the result follows. □

Remark 6.4. Let $g(r)$ denote the lower bound in Proposition 6.1(i), that holds for any odd prime r. Notice that $g(r)$ is monotonically decreasing in r, and $g(n-1) > 0$ while $g(n) < 0$.

Let $x \in G$ be of odd order $r \neq p$. Then we may write x as in (8). Since $a_i = a_{r-i}$, for all $i > 0$, in the following we shall also make use of the following notation:

$$x = \left[I_{a_0}, (\omega, \omega^{-1})I_{a_1}, \dots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}\right].$$

Lemma 6.5. *The conclusion of Proposition 6.1(ii) holds.*

Proof. Assume $\nu(x) = n-1$. Then, we may assume $x = [\omega, \omega^{-1}, \dots, \omega^{n/2}, \omega^{-n/2}]$. If $G = \mathrm{O}_n$ and $C_G(x) = \mathrm{O}_2 \times (\mathrm{GL}_1)^{n/2-1}$, we may write $x = [I_2, \omega, \omega^{-1}, \dots, \omega^{n/2-1}, \omega^{-n/2-1}]$. In both cases, using Proposition 3.11, we compute $f_\Omega(x) = 0$.

Conversely, let $x \in H$ be of order $r > n$. Using Proposition 3.11, we will show that if $\nu(x) < n-1$ then either $(G, C_G(x), f_\Omega(x)) = (\mathrm{O}_n, \mathrm{O}_2 \times (\mathrm{GL}_1)^{n/2-1}, 0)$ or $f_\Omega(x) > 0$. We use the explicit formula of $f_\Omega(x)$ stated in Proposition 3.11. Assume $\nu(x) < n-1$ and, for convenience, $G = \mathrm{O}_n$.

Case 1. Assume $a_0 = 0$. So there exists $i > 0$ such that $a_i \geqslant 2$; we may assume $i = 1$. The case in which the largest number of different eigenvalues appears in x is when

$$x = \left[(\omega, \omega^{-1})I_{a_1}, (\omega, \omega^{-1})^2, \dots, (\omega, \omega^{-1})^{\frac{n-2a_1}{2}}\right].$$

Therefore $r - 1 \geqslant n - 2a_1$. Thereby, Lemma 6.2 gives $\sum_i a_i^2 \geqslant 2a_1^2 + (n-2a_1)^2/(n-2a_1-3)$. In addition, $d(x) \leqslant n - 2a_1 + 2$. These two inequalities lead to the conclusion that $f_\Omega(x) > 0$.

Case 2. Assume $a_0 = 2$. If $a_i \leqslant 1$ for all $i > 0$ then $C_G(x) = \mathrm{O}_2 \times (\mathrm{GL}_1)^{n/2-1}$, and we quickly compute $f_\Omega(x) = 0$. If there exists $i > 0$ such that $a_i \geqslant 2$ then, as above, we deduce $f_\Omega(x) > 0$.

Case 3. Assume $a_0 \geqslant 4$. We have $\sum_{i>0} a_i^2 \geqslant \sum_{i>0} a_i = n - a_0 \geqslant d(x)$. Thus $f_\Omega(x) > 0$.

If $G = \mathrm{Sp}_n$, the proof is very similar to the orthogonal case and it is left to the reader. □

Remark 6.6. Elements $x \in H$ of order r with $\nu(x) = n-1$ are regular. If $G = \mathrm{O}_n$ and $x \in H$ has centraliser $\mathrm{O}_2 \times (\mathrm{GL}_1)^{n/2-1}$ then $\dim C_G(x) = n/2$; hence x is regular. In particular, Lemma 6.5 proves Theorem 10.

For the remainder of the section we assume $r < n$ is an odd prime other than p. We set the following.

Definition 6.7. Let $x \in G$ be of order r. We say that x is *special* if $|a_i - a_j| \leqslant 1$ for all i, j.

The centraliser structure of any special element is clear.

Proposition 6.8. *Let $x \in G$ be special of odd order r. Write $n = ar + b$, $0 \leqslant b < r$. Then $C_G(x) \cong C_G(z)$, where*

$$z = \begin{cases} [I_a, A] & a \text{ even} \\ [I_{a+1}, A] & a \text{ odd} \end{cases}$$

for $A = \left[(\omega, \omega^{-1})I_{a+1}, \dots, (\omega, \omega^{-1})^{\lfloor b/2 \rfloor} I_{a+1}, (\omega, \omega^{-1})^{\lfloor b/2 \rfloor + 1} I_a, \dots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_a\right]$.

If x and y in G are special then $f_\Omega(x) = f_\Omega(y)$, by Proposition 3.11.

Claim 6.9. Let $x \in G$ be of odd order r. Then $f_\Omega(x) \geqslant f_\Omega(z)$ where $z \in G$ is any special element of order r.

We show that given $x \in H$ of odd order r non-special we can construct a suitable $y \in H$ of order r for which $f_\Omega(x) \geqslant f_\Omega(y)$, such that the iteration of this process leads to the claim. If x is non-special then there exist i, j such that $a_i - a_j \geqslant 2$. Thanks to the following we can assume $(i, j) \in \{(0,1), (1,0), (1,2)\}$.

Lemma 6.10. *Let $x \in G$ be of order r. Then either x is special, or there exists $y = [I_{b_0}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}]$ of order r such that $f_\Omega(x) = f_\Omega(y)$ and one of the following holds*

(i) $|b_0 - b_1| \geqslant 2$*; or,*
(ii) $b_1 - b_2 \geqslant 2$.

Proof. If x is not special then $a_i - a_j \geqslant 2$ for some i, j. We may assume $i, j \leqslant (r-1)/2$, since $a_h = a_{r-h}$ for all $h > 0$. If $i = 0$ (similarly if $j = 0$), we define $y = [I_{a_0}, \omega I_{a_j}, \dots, \omega^j I_{a_1}, \dots, \omega^{r-j} I_{a_1}, \dots, \omega^{r-1} I_{a_j}]$. Thus, Proposition 3.11 implies $f_\Omega(x) = f_\Omega(y)$. The argument is similar if $i, j \neq 0$. □

Let $x \in G$ be of odd order r and assume x is non-special. Hence $a_i - a_j \geqslant 2$ for $(i, j) \in \{(0,1), (1,0), (1,2)\}$. For each of the three possibilities of (i, j), we define y as in Table 10. The following is the key tool in order to show Claim 6.9.

Lemma 6.11. *Let $x \in G$ be of odd order r non-special. Define y as in Table 10. Then $f_\Omega(x) \geqslant f_\Omega(y)$.*

(i,j)	y
(0,1)	$\left[I_{a_0-2},(\omega,\omega^{-1})I_{a_1+1},(\omega,\omega\cdot-1)^2I_{a_2},\dots,(\omega,\omega^{-1})^{\frac{r-1}{2}}I_{a_{\frac{r-1}{2}}}\right]$
(1,0)	$\left[I_{a_0+2},(\omega,\omega^{-1})I_{a_1-1},(\omega,\omega^{-1})^2I_{a_2},\dots,(\omega,\omega^{-1})^{\frac{r-1}{2}}I_{a_{\frac{r-1}{2}}}\right]$
(1,2)	$\left[I_{a_0},(\omega,\omega^{-1})I_{a_1-1},(\omega,\omega^{-1})^2I_{a_2+1},(\omega,\omega^{-1})^3I_{a_3}\dots,(\omega,\omega^{-1})^{\frac{r-1}{2}}I_{a_{\frac{r-1}{2}}}\right]$

TABLE 10

Proof. We need to show

$$\alpha(x,y)=\dim C_\Omega(x)-\dim C_\Omega(y)\geqslant 0.$$

Lemma 3.10 implies that $\alpha(x,y)$ is even. For simplicity we assume $G=\mathrm{Sp}_n$, the proof is totally similar in the orthogonal case. We have three different cases.

Case 1. If $(i,j)=(0,1)$, then $d(x)=d(y)\pm 2$ depending on the parity of a_1; in particular, $d(x)-d(y)\leqslant 2$. Hence

$$\alpha(x,y)=a_0-a_1-\frac{d(x)-d(y)+2}{4}\geqslant a_0-a_1-1>0.$$

Case 2. Assume $(i,j)=(1,0)$. Again $d(x)-d(y)\leqslant 2$. Thus

$$\alpha(x,y)=a_1-a_0-2-\frac{d(x)-d(y)+2}{4}\geqslant a_1-a_0-2-1\geqslant -1.$$

Since $\alpha(x,y)$ is even, we conclude that it is non-negative.

Case 3. Finally assume $(i,j)=(1,2)$. In this case $d(x)=d(y)+4$ if a_1,a_2 are odd; $d(x)=d(y)$ if $a_1\not\equiv a_2 \pmod 2$; and $d(x)=d(y)-4$ if they are both even. In particular, $d(x)-d(y)\leqslant 4$. As above it is easy to deduce $\alpha(x,y)\geqslant 0$. □

Remark 6.12. Say a_i, a_i', for $0\leqslant i\leqslant r-1$, the multiplicities of the eigenvalues in x and y, respectively. Then, it is clear by the construction of y that $\max_i\{a_i'\}-\min_i\{a_i'\}\leqslant \max_i\{a_i\}-\min_i\{a_i\}$.

Proposition 6.13. *Claim 6.9 holds.*

Proof. If x is non-special then by Lemma 6.10 we may assume $a_i-a_j\geqslant 2$ for $(i,j)\in\{(1,0),(0,1),(1,2)\}$. Let y as in Table 10. Then Lemma 6.11 yields $f_\Omega(x)\geqslant f_\Omega(y)$. If y is non-special we apply the same argument: we construct y_1 for which Lemma 6.11 yields $f_\Omega(y)\geqslant f_\Omega(y_1)$. Eventually, in a finite number of steps we will construct an element z, which is special thanks to Remark 6.12, and $f_\Omega(x)\geqslant f_\Omega(y)\geqslant f_\Omega(y_1)\geqslant\dots\geqslant f_\Omega(z)$. □

Proposition 6.13 reduces the problem of getting lower bounds on $f_\Omega(x)$, for $x\in H$ of odd order r, to the computation of $f_\Omega(z)$, where z is a special element.

Write $n=ar+b$, where $0\leqslant b<r$. Notice that $a\equiv b \pmod 2$. Let z be of order r as in Proposition 6.8. Then using Proposition 3.11 we easily compute $f_\Omega(z)$. Recall the definition of ε given in (6).

Assume $a=\lfloor n/r\rfloor$ is even. Then

$$f_\Omega(z)=\frac{1}{r}-\frac{b(b-2\varepsilon)}{rn(n-2\varepsilon)}. \tag{24}$$

Notice that, in the case $G=\mathrm{Sp}_n$, $f_\Omega(z)$ is minimal when b is maximal, i.e. $b=r-1$. Therefore $f_\Omega(z)\geqslant 1/r-(r^2-1)/rn(n+2)$ with equality if, and only if, $b=r-1$. Similarly, if $G=\mathrm{O}_n$ then $f_\Omega(z)\geqslant 1/r-(r-1)(r-3)/rn(n-2)$.

If $a=\lfloor n/r\rfloor$ is odd we compute

$$f_\Omega(z)=\frac{1}{r}-\frac{(r-b)(r-b+2\varepsilon)}{rn(n-2\varepsilon)}. \tag{25}$$

If $G=\mathrm{Sp}_n$ then $f_\Omega(z)\geqslant 1/r-(r-2)/n(n+2)$. If $G=\mathrm{O}_n$ we have $f_\Omega(z)\geqslant 1/r-(r^2-1)/rn(n-2)$, with equality if, and only if, $b=1$. Notice that in all the previous cases $f_\Omega(z)\leqslant 1/r+1/r(n-2)$.

Remark 6.14. The lower bound in Proposition 6.1(ii) is the best possible. In fact it is sharp. Thanks to the above discussion we see that it is realised in one of the following two cases:

- $G = \mathrm{Sp}_n$ and $n - \lfloor n/r \rfloor r = r - 1$;
- $G = \mathrm{O}_n$ and $n - \lfloor n/r \rfloor r = 1$.

The best possible lower bounds on $f_\Omega(x)$ for $x \in H$ of odd prime order r are given in (24) and (25).

7. Local upper bounds

In this section we derive local upper bounds, namely we prove the first part of Theorem 3. We do not deal with involutions, since the information needed can be quickly deduced from Theorem 3.13. We subdivide the analysis into two cases: we first deal with unipotent elements and then with semisimple elements.

7.1. Unipotent elements.

Assume $p \neq 2$. Let $x \in \mathscr{V}_{s,p}$. Here we also consider arbitrary unipotent elements if the characteristic is zero. In this section we shall prove the following.

Proposition 7.1. *Write $n = (n-s)a + b$, where $0 \leqslant b < n-s$. Let $x \in \mathscr{V}_{s,p}$.*

(i) *Assume $G = \mathrm{Sp}_n$. Then*
$$f_\Omega(x) \leqslant 1 - \frac{(n-b)(s+b)+s}{n(n+2)}.$$

(ii) *Assume $G = \mathrm{O}_n$. Then*
$$f_\Omega(x) \leqslant 1 - \frac{(n-b)(s+b)}{n(n-2)} + \frac{2(n-b)(s+b)}{n(n-2)^2}.$$

In particular, the upper bound in Theorem 3 holds.

Before embarking in the proof of Proposition 7.1, we first show the following technical result.

Lemma 7.2. *Let n,t be integers such that $0 < t < n$. Write $n = at + b$, where $0 \leqslant b < t$, and set $\mu = ((a+1)^b, a^{t-b}) \vdash n$. Let $\lambda = (\lambda_1, \dots, \lambda_t) \vdash n$ such that $\lambda_i \neq 0$ for all i. Then $\mu \preccurlyeq \lambda$.*

Proof. First we show that $\lambda_1 \geqslant a+1$ if $\lambda \neq \mu$. Assume $\lambda_1 < a$. Using $\lambda_i \leqslant \lambda_1 \leqslant a-1$, we have $n = \sum_i \lambda_i \leqslant (a-1)t < n$. Thus $\lambda_1 \geqslant a$. Now assume $\lambda_1 = a$. Then, as before, $n = a + \sum_{i \geqslant 2} \lambda_i \leqslant at$, which is absurd if $b > 0$. For $b = 0$, instead, $\lambda_1 = a$ implies $\lambda = \mu$. Therefore either $\lambda_1 = a$ and $\lambda = \mu$, or $\lambda_1 \geqslant a+1$. If $\lambda = \mu$ the result trivially follows, hence we may assume $\lambda \neq \mu$ so $\lambda_1 \geqslant a+1$.

We define the following notation. For $\lambda = (\lambda_1, \dots, \lambda_t) \vdash n$ and $1 \leqslant l \leqslant t$, we denote
$$A_l(\lambda) = \sum_{i \geqslant l} \lambda_i.$$

Seeking a contradiction, assume there exists $h \in \{1, \dots, t\}$ such that
$$\sum_{i=1}^{h} \mu_i > \sum_{i=1}^{h} \lambda_i. \tag{26}$$

In particular, we may assume h is the least integer with this property. So, by the previous discussion, $h > 1$. Thus $\sum_{i=1}^{h-1} \mu_i \leqslant \sum_{i=1}^{h-1} \lambda_i$. Therefore
$$\sum_{i=1}^{h-1} \lambda_i + \mu_h \geqslant \sum_{i=1}^{h-1} \mu_i + \mu_h > \sum_{i=1}^{h-1} \lambda_i + \lambda_h.$$

Thus $\mu_h > \lambda_h$.

In the case $h \leqslant b$ we have $\mu_h = a+1 > \lambda_h$, thus $a \geqslant \lambda_h$. So, for all $i \geqslant h$, $a \geqslant \lambda_i$. If, instead, $h > b$ we have $\mu_h = a > \lambda_h$. Hence $a > \lambda_i$ for all $i \geqslant h$. In particular, in both cases, $A_{h+1}(\mu) \geqslant A_{h+1}(\lambda)$; together with (26), this last inequality yields
$$n = A_{h+1}(\mu) + \sum_{i=1}^{h} \mu_i > A_{h+1}(\lambda) + \sum_{i=1}^{h} \lambda_i = n,$$
which is a absurd. The result follows. □

Combining Theorem 2.4 and Lemma 7.2 we immediately deduce the following.

Proposition 7.3. *Let $G = \mathrm{GL}_n$. Let $x \in \mathscr{R}$ be unipotent with $\nu(x) = s$. Write $n = (n-s)a + b$, where $0 \leqslant b < n-s$. Let $y = [J_{a+1}^b, J_a^{n-s-b}]$. Then $\dim x^G \leqslant \dim y^G$.*

Proposition 7.3 is the tool needed in order to show Proposition 7.1.

Proof of Proposition 7.1. Up to G-conjugation, $x=[J_p^{a_p},\dots,J_1^{a_1}]$ with $s=n-\sum_i a_i$. If $x^G\cap H=\emptyset$, (7) yields $f_\Omega(x)=0$. Hence we may assume $x^G\cap H\neq\emptyset$. In particular, each a_i is even and s is even. For convenience, we assume $G=\mathrm{Sp}_n$, the proof is totally similar in the orthogonal case.

Let $y=[J_p^{a_p/2},\dots,J_1^{a_1/2}]\in\mathrm{GL}_{n/2}$. Note that $\nu(y)=s/2$. Using Theorem 2.3, we see that

$$\dim x^G=2\dim y^{\mathrm{GL}_{n/2}}+\frac{n-\sum_{i\text{ odd}}a_i}{2}\geqslant 2\dim y^{\mathrm{GL}_{n/2}}+\frac{s}{2} \tag{27}$$

Then, Proposition 7.3 implies

$$\dim y^{\mathrm{GL}_{n/2}}\geqslant\dim\left[J_{a+1}^{b/2},J_a^{(n-s-b)/2}\right]^{\mathrm{GL}_{n/2}}=\frac{(n-b)(s+b)}{4}.$$

Therefore, by Propositions 2.1 and 3.1, we have

$$f_\Omega(x)\leqslant 1-\frac{\dim x^G}{2\dim\Omega}\leqslant 1-\frac{2\dim y^{\mathrm{GL}_{n/2}}+s/2}{2\dim\Omega}\leqslant 1-\frac{(n-b)(s+b)+s}{n(n+2)}.$$

At this point, an easy calculation shows that the upper bound stated in Theorem 3 holds. □

Remark 7.4. The upper bounds in Proposition 7.1 are close to best possible. Indeed, write $n=(n-s)a+b$ where $0\leqslant b<n-s$. Consider $x=\left[J_{a+1}^b,J_a^{n-s-b}\right]\in\mathscr{V}_s$. Then $x^G\cap H\neq\emptyset$. Say U the upper bound in Proposition 7.1 corresponding to the desired group. One can easily verify that $U-f_\Omega(x)\leqslant 3/n$.

7.2. **Semisimple elements.** Throughout this section $r\neq p$ is an odd prime. Let $x\in\mathscr{V}_{s,r}$. Then we may assume x is one of the following

$$\begin{array}{lll}(a) & \left[I_{n-s},(\omega,\omega^{-1})I_{a_1},\dots,(\omega,\omega^{-1})^{\frac{r-1}{2}}I_{a_{\frac{r-1}{2}}}\right] & s\text{ even},\\ (b) & \left[I_{a_0},(\omega,\omega^{-1})I_{n-s},(\omega,\omega^{-1})^2I_{a_2},\dots,(\omega,\omega^{-1})^{\frac{r-1}{2}}I_{a_{\frac{r-1}{2}}}\right] & s\geqslant n/2,\end{array} \tag{28}$$

where $a_i\leqslant n-s$, for all i.

Now we shall determine a class of elements in $\mathscr{V}_{s,r}$ that realise the best possible upper bound on $f_\Omega(x)$.

Remark 7.5. It is easy to check that, for all $x\in G$ of order r, $\nu(x)\leqslant n-1-\lfloor n/r\rfloor$.

Write $n=(n-s)a+b$, where $0\leqslant b<n-s$. We define the following matrices

$$A=\left[(\omega,\omega^{-1})I_{n-s},\dots,(\omega,\omega^{-1})^{\lfloor\frac{a-1}{2}\rfloor}I_{n-s}\right],$$
$$B=\left[(\omega,\omega^{-1})I_{n-s},\dots,(\omega,\omega^{-1})^{\lfloor\frac{a}{2}\rfloor}I_{n-s}\right].$$

Consequently we define the following elements

$$z_1=\begin{cases}\left[I_{n-s},A,(\omega,\omega^{-1})^{\frac{a+1}{2}}I_{\frac{b}{2}}\right] & s\text{ even}\quad a\text{ odd},\\ \left[I_{n-s},A,(\omega,\omega^{-1})^{\frac{a}{2}}I_{\frac{n-s+b}{2}}\right] & s\text{ even}\quad a\text{ even},\\ \left[I_{n-s-1},A,(\omega,\omega^{-1})^{\frac{a+1}{2}}I_{\frac{b+1}{2}}\right] & s\text{ odd}\quad a\text{ odd},\\ \left[I_{n-s-1},A,(\omega,\omega^{-1})^{\frac{l}{2}}I_{\frac{n-s+b+1}{2}}\right] & s\text{ odd}\quad a\text{ even};\end{cases} \tag{29}$$

$$z_2=\begin{cases}\left[B,(\omega,\omega^{-1})^{\frac{a}{2}+1}I_{\frac{b}{2}}\right] & a\text{ even},\\ \left[B,(\omega,\omega^{-1})^{\frac{a+1}{2}}I_{\frac{n-s+b}{2}}\right] & a\text{ odd},\\ \left[I_b,B\right] & a,b\text{ even}.\end{cases} \tag{30}$$

Using the bound on s given in Remark 7.5, one can check that $(a+1)/2,a/2,a/2+1$ are at most $(r-1)/2$ in the different cases that arise in the definition of z_1,z_2. Hence $z_1,z_2\in\mathscr{V}_{s,r}$.

Claim 7.6. Assume $r\neq p$ is odd. Let $x\in\mathscr{V}_{s,r}$. Then $f_\Omega(x)\leqslant\max\{f_\Omega(z_1),f_\Omega(z_2)\}$.

This claim is proved with a strategy similar to that employed to verify Claim 6.9. The following technical result is the analogous of Lemma 6.10.

Lemma 7.7. *Assume $r\neq p$ is odd. Let $x\in\mathscr{V}_{s,r}$. Then either $C_G(x)\cong C_G(z_1)$ or $C_G(z_2)$, or there exists $y=[I_{b_0},\dots,\omega^{r-1}I_{b_{r-1}}]\in\mathscr{V}_{s,r}$ such that $f_\Omega(x)=f_\Omega(y)$ and one of the following holds*

(i) $b_0 = n-s$, $b_1 = \min\{b_i : b_i \neq 0\}$ *and* $b_2 = \max\{b_i : b_i < n-s\}$;
(ii) $b_1 = n-s$, $b_2 = \min\{b_i : b_i \neq 0\}$ *and* $b_0 = \max\{b_i : b_i < n-s\} < n-s-1$;
(iii) $b_1 = n-s$, $b_0 = n-s-1$, $b_2 = \min\{b_i : b_i \neq 0\}$ *and* $b_3 = \max_{i>0}\{b_i : b_i < n-s\}$;
(iv) $b_1 = n-s$, $b_0 = \min\{b_i : b_i \neq 0\}$ *and* $b_2 = \max\{b_i : b_i < n-s\}$;
(v) $b_1 = n-s$, $b_2 = \min\{b_i : b_i \neq 0\}$ *and* $b_3 = \max\{b_i : b_i < n-s\}$.

Let $x \in \mathscr{V}_{s,r}$ and assume $C_G(x) \ncong C_G(z_1)$ or $C_G(z_2)$. Then, in each of the five cases appearing in Lemma 7.7 we define a suitable element y as in Table 11.

Case	y
(i)	$\left[I_{n-s}, (\omega,\omega^{-1})I_{a_1-1}, (\omega,\omega^{-1})^2 I_{a_2+1}, \ldots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}\right]$
(ii)	$\left[I_{a_0+2}, (\omega,\omega^{-1})I_{n-s}, (\omega,\omega^{-1})^2 I_{a_2-1}, \ldots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}\right]$
(iii)	$\left[I_{a_0}, (\omega,\omega^{-1})I_{n-s}, (\omega,\omega^{-1})^2 I_{a_2-1}, (\omega,\omega^{-1})^3 I_{a_3+1}, \ldots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}\right]$
(iv)	$\left[I_{a_0-2}, (\omega,\omega^{-1})I_{n-s}, (\omega,\omega^{-1})^2 I_{a_1+1}, \ldots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}\right]$
(v)	$\left[I_{a_0}, (\omega,\omega^{-1})I_{n-s}, (\omega,\omega^{-1})^2 I_{a_2-1}, (\omega,\omega^{-1})^3 I_{a_3+1}, \ldots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}\right]$

TABLE 11

The following technical lemma is the key tool in order to show the claim.

Lemma 7.8. *Assume $r \neq p$ is odd. Let $x \in \mathscr{V}_{s,r}$. Assume $C_G(x) \ncong C_G(z_i)$ for $i = 1,2$ and that the multiplicities of the eigenvalues of x are as in (i)–(v) of Lemma 7.7. Let y as in Table 11. Then $f_\Omega(x) \leqslant f_\Omega(y)$.*

Proof. This is an easy computation similar to that done to show Lemma 6.11. □

Using the construction established above and Lemma 7.8 we can finally show the claim.

Proposition 7.9. *Claim 7.6 holds.*

Proof. As in Proposition 6.13, if $C_G(x) \ncong C_G(z_i)$, $i = 1,2$, we have a finite chain $f_\Omega(x) \leqslant f_\Omega(y) \leqslant \cdots \leqslant f_\Omega(z)$ where $C_G(z) \cong C_G(z_1)$ or $C_G(z_2)$. □

Now we compute upper bounds on $f_\Omega(x)$ for $x \in \mathscr{V}_{s,r}$.

Proposition 7.10. *Assume $r \neq p$ is odd. Let $x \in \mathscr{V}_{s,r}$. Write $n = (n-s)a + b$, where $0 \leqslant b < n-s$. Then*

$$f_\Omega(x) \leqslant 1 - \frac{s}{n} - \frac{(n-s-b)b}{2n(n-2\varepsilon)} + \frac{1}{n-2}.$$

Furthermore, the upper bound in Theorem 3 holds.

Proof. Thanks to Proposition 7.9 we only need to compute $f_\Omega(z_1)$ and $f_\Omega(z_2)$, where z_1, z_2 are as (29) and (30), respectively. We use the formula stated in Proposition 3.11. We compute the explicit value of $f_\Omega(z_1)$ in some cases, when $G = \mathrm{Sp}_n$. The cases not treated here are left to the reader.

We assume $G = \mathrm{Sp}_n$, and we compute $f_\Omega(z_1)$. Assume s is even and a is odd. Then

$$f_\Omega(z_1) = 1 - \frac{s}{n} - \frac{b(n-s-b/2) + 2(1-\delta_{b;4})}{n(n+2)}$$

and $\frac{b(n-s-b/2)+2(1-\delta_{b;4})}{n(n+2)} > \frac{b(n-s)-b^2}{2n(n+2)} > 0$, since $b < n-s$.

Now assume s and a are both odd. Then

$$f_\Omega(z_1) = 1 - \frac{s}{n} - \frac{(n-s)(b+2) - \frac{b^2+1}{2} - b + \frac{n-b}{n-s} + 2(1-\delta_{b+1;4})}{n(n+2)} = 1 - \frac{s}{n} - \delta$$

and we see that

$$\delta \geqslant \frac{2b(n-s)^2 - b^2(n-s) + (n-b)}{2n(n+2)(n-s)} > \frac{b(n-s)^2 - b^2(n-s)}{2n(n+2)(n-s)} = \frac{b(n-s)-b^2}{2n(n+2)},$$

where we used $\delta_{b+1;4} \geqslant 0$ and $n-b > 0$.

Finally, let us compute $f_\Omega(z_2)$. The following holds in both cases $G = \mathrm{Sp}_n$ and O_n. Recall the definition of ε from (6). First assume a odd. Then

$$f_\Omega(z_2) = 1 - \frac{s}{n} - \frac{(n-s)(n-s+4) - b^2 + l(1-\delta_{s;2}) + 2(1-\delta_{b;4})}{2n(n-2\varepsilon)}.$$

Assume a even. Then

$$f_\Omega(z_2) = 1 - \frac{s}{n} - \frac{2(n-s)(b+2) - b^2 + (l-1)(1-\delta_{s;2}) + 2(1-\delta_{n-s+b;4})}{2n(n-2\varepsilon)}.$$

A straightforward calculation yields the result in the previous cases for $f_\Omega(z_2)$; in particular,

$$f_\Omega(z_2) \leqslant 1 - \frac{s}{n} - \frac{(n-s-b)b}{2n(n-2\varepsilon)},$$

and a straightforward calculation yields the upper bound in Theorem 3.

Assume a, b are even. Let $z_2 = [I_b, B]$. Then $d(z_2) \in \{0, a\} \geqslant 0$, hence

$$f_\Omega(z_2) \leqslant 1 - \frac{s}{n} - \frac{(n-s-b)b}{2n(n-2\varepsilon)} - \frac{(b-4\varepsilon)(n-s-b)}{2n(n-2\varepsilon)}.$$

In the case $G = \mathrm{Sp}_n$, or $G = \mathrm{O}_n$ and $b \geqslant 4$, we have

$$f_\Omega(z_2) \leqslant 1 - \frac{s}{n} - \frac{(n-s-b)b}{2n(n-2\varepsilon)}.$$

Again, the upper bound in Theorem 3 holds.

Now we assume $G = \mathrm{O}_n$ and $b \in \{0, 2\}$. If $b = 0$ then $n/(n-s)$ is an even integer, and hence $s \geqslant n/2$. In both cases, $b = 0$ or $b = 2$, we have

$$f_\Omega(z_2) \leqslant 1 - \frac{s}{n} - \frac{(n-s-b)b}{2n(n-2\varepsilon)} + \frac{1}{n-2}.$$

From the previous bound we quickly deduce that the upper bound of Theorem 3 holds for $n \geqslant 8$. For the cases $n = 4, 6$ a direct check shows that the upper bound stated in Theorem 3 holds. This concludes the proof. □

8. Local lower bounds

Let r be an odd prime (if $r = 2$ see Theorem 3.13). In this section we derive lower bounds on $f_\Omega(x)$ for $x \in \mathscr{V}_{s,r}$, showing the second part of Theorem 3. As usual, we may assume $x \in H$, by Remark 5.1.

8.1. **Unipotent elements.** Recall the definition of ε from (6).

Proposition 8.1. *Assume $p \neq 2$. Let $x \in H \cap \mathscr{V}_{s,p}$. Then*

$$f_\Omega(x) \geqslant 1 - \frac{s(2n-s)}{n(n-2\varepsilon)} + \delta$$

where $\delta = -1/n$ if $G = \mathrm{Sp}_n$ and $\delta = 3s/(n^2-2n)$ if $G = \mathrm{O}_n$. In particular, the lower bound in Theorem 3 holds.

Proof. Let $x \in H \cap \mathscr{V}_{s,p}$. Proposition 2.7 asserts $\dim x^G \leqslant g(s)$, where $g(s)$ is recorded in Table 5. We use the formula of $f_\Omega(x)$ stated in Proposition 3.5.

Recall the formula for $\dim x^{Cl_n}$ from Theorem 2.3. In the case $G = \mathrm{Sp}_n$ we can write

$$f_\Omega(x) = \frac{n(n+1) - 2\dim x^G + \sum_{i \text{ odd}} a_i}{n(n+2)}.$$

Similarly, if $G = \mathrm{O}_n$ then

$$f_\Omega(x) = \frac{n(n-2) - \dim x^{\mathrm{GL}_n} + 2\big(n - \sum_{i \text{ odd}} a_i\big)}{n(n-2)}.$$

Assume $G = \mathrm{Sp}_n$, then

$$f_\Omega(x) \geqslant \frac{n(n+1) - 2\dim x^G}{n(n+2)} \geqslant \frac{n(n+1) - (2ns - s^2 + 1)}{n(n+2)} \geqslant 1 - \frac{s(2n-s)}{n(n+2)} - \frac{1}{n}.$$

The orthogonal case is similar: using also $n - \sum_{i \text{ odd}} a_i \geqslant s$, we obtain the desired bound. □

Say ℓ the bounds in Proposition 8.1. One might ask if these bounds are accurate. Recall that $H\cap\mathscr{V}_{s,p}\neq\emptyset$ implies that s is even. Assume $s+2\leqslant p$. Then $x=[J^2_{(s+2)/2},J_1^{n-s-2}]\in\mathscr{V}_{s,p}$ and the G-class of x meets H. Then, in both $G=\mathrm{Sp}_n$ and $G=\mathrm{O}_n$ cases, we have $f_\Omega(x)-\ell\leqslant 3/n$. If $s+2>p$ we define $h=\lfloor s/(2p-2)-1\rfloor$. Thus $h\geqslant 1$. Hence

$$z=\left[J_p^{2h},J^2_{s/2+1-hp+h},J_1^{n-s-2h-2}\right]\in\mathscr{V}_{s,p}.$$

Again, $z^G\cap H\neq\emptyset$. Using Proposition 3.5, it is not hard to deduce

$$f_\Omega(z)-\ell\leqslant\frac{3}{n}+\frac{s^2}{n(n-2\varepsilon)(p-1)}.$$

8.2. **Semisimple elements.** In this section we assume $r\neq p$ is odd. Recall that for any semisimple element $x\in G$ of order r we have $x^G\cap H\neq\emptyset$. Let $x\in\mathscr{V}_{s,r}$. Then, up to conjugation (and up to the centraliser structure), we may assume x is as in (28). In the case $s=n-1$ we already know that $f_\Omega(x)=0$; similarly in $G=\mathrm{O}_n$ and $s=n-2$, see Lemma 6.5. We handle the case $r=3$ first.

Lemma 8.2. *Assume $p\neq 3$. Let $x\in\mathscr{V}_{s,3}$. Then the lower bound in Theorem 3 holds.*

Proof. Let $x\in\mathscr{V}_{s,3}$. Then, up to G-conjugacy, $x=[I_{n-s},\omega I_{s/2},\omega^{-1}I_{s/2}]$ or $[I_{2s-n},\omega I_{n-s},\omega^{-1}I_{n-s}]$. Using Proposition 3.11 it is straightforward to compute $f_\Omega(x)$ and show that the bound in Theorem 3 holds. $\square$

For the remainder of the section we assume $r>3$. The following is the analogous of Definition 6.7.

Definition 8.3. Assume $r>3$. Let $x\in\mathscr{V}_{s,r}$. Assume, $a_h=n-s$ for some $h\leqslant(r-1)/2$. We say that x is *ν-special* if $|a_i-a_j|\leqslant 1$ for all $i,j\neq h,r-h$.

Notation. Assume $\mathscr{V}_{s,r}\neq\emptyset$. If s is even, write $s=a(r-1)+b$, where $0\leqslant b<r-1$, notice that b is even. If $s\geqslant n/2$ then write $2s-n=a(r-2)+b$, where $0\leqslant b<r-2$, here $a\equiv b \pmod 2$.

Proposition 8.4. *Assume $r>3$. Let $x\in\mathscr{V}_{s,r}$ be ν-special. Then $C_G(x)\cong C_G(z)$ where z is one of the following*

$$\left[I_{n-s},(\omega,\omega^{-1})I_{a+1},\dots,(\omega,\omega^{-1})^{\frac{b}{2}}I_{a+1},(\omega,\omega^{-1})^{\frac{b}{2}+1}I_a,\dots,(\omega,\omega^{-1})^{\frac{r-1}{2}}I_a\right],\tag{31}$$

$$\left[I_{a+1},(\omega,\omega^{-1})I_{n-s},(\omega,\omega^{-1})^2I_{a+1},\dots,(\omega,\omega^{-1})^{\frac{b+1}{2}}I_{a+1}(\omega,\omega^{-1})^{\frac{b+3}{2}}I_a,\dots,(\omega,\omega^{-1})^{\frac{r-1}{2}}I_a\right],\tag{32}$$

$$\left[I_a,(\omega,\omega^{-1})I_{n-s},(\omega,\omega^{-1})^2I_{a+1},\dots,(\omega,\omega^{-1})^{\frac{b}{2}+1}I_{a+1},(\omega,\omega^{-1})^{\frac{b}{2}+2}I_a,\dots,(\omega,\omega^{-1})^{\frac{r-1}{2}}I_a\right].\tag{33}$$

In order to deduce lower bounds on $f_\Omega(x)$ for $x\in\mathscr{V}_{s,r}$ we mimic the strategy used in Section 6. Let $x\in H\cap\mathscr{V}_{s,r}$ be non-ν-special as in (28). Then $a_i-a_j\geqslant 2$ for some i,j. As in Lemma 6.10, we may assume $(i,j)=(1,2)$, if $a_0=n-s$, or $(i,j)\in\{(0,2),(2,0),(2,3)\}$, if $a_0\neq n-s$ and $a_1=n-s$. Then we may define a suitable element $y\in\mathscr{V}_{s,r}$ (as done in Table 10) for which $f_\Omega(x)\geqslant f_\Omega(y)$. Eventually we have the following.

Proposition 8.5. *Assume $r>3$. There exists a ν-special element $z\in\mathscr{V}_{s,r}$ such that, for all $x\in\mathscr{V}_{s,r}$,*

$$f_\Omega(x)\geqslant f_\Omega(z).$$

Remark 8.6. In contrast with the global lower bound we have few more cases that are forced by the centraliser structure of semisimple elements and the conditions imposed by the ν-value. In fact, if $s\geqslant n/2$ is even, there are two ν-special elements with different centraliser structure, say z_1 (as in (31)) and z_2 (as in (32) or (33)). Therefore $f_\Omega(x)\geqslant\min\{f_\Omega(z_1),f_\Omega(z_2)\}$.

Proposition 8.7. *Assume $r>3$. Let $x\in H\cap\mathscr{V}_{s,r}$. Then*

$$f_\Omega(x)\geqslant 1-\frac{s(2n-s+\iota)}{n(n-2\varepsilon)}$$

where $\iota=2$ if $G=\mathrm{Sp}_n$ and $\iota=-1$ if $G=\mathrm{O}_n$. In particular, the lower bound in Theorem 3 holds.

Proof. Thanks to Proposition 8.5 it is enough to compute lower bounds on $f_\Omega(x)$ when x is ν-special of order r. We use the formula for $f_\Omega(x)$ stated Proposition 3.11.

Assume s is even and $\lfloor s/(r-1)\rfloor=0$. Then, for z as in (31) we compute

$$f_\Omega\left([I_{n-s},(\omega,\omega^{-1}),\dots,(\omega,\omega^{-1})^{s/2}]\right)=1-\frac{s(2n-s-2\varepsilon)}{n(n-2\varepsilon)}.$$

If $\varepsilon=-1$ the result holds; when $\varepsilon=+1$, the result follows using $s\leqslant n-2$.

Now assume $s \geqslant n/2$ and $\lfloor (2s-n)/(r-2) \rfloor = 0$. For z as in (33) we have

$$f_\Omega([(\omega,\omega^{-1})I_{n-s},(\omega,\omega^{-1})^2,\dots,(\omega,\omega^{-1})^{s-n/2+1}]) = \frac{2(n-s)^2-2(1-\delta_{s;2})}{n(n-2\varepsilon)}. \tag{34}$$

It is an easy computation to check that the desired bound holds for (34).

In the case $\lfloor s/(r-1) \rfloor \neq 0$ we use Proposition 3.11 to compute $f_\Omega(x)$ for a ν-special element $x \in H$ of order r. Then, with few computations we deduce the result. □

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