FIXED POINT SPACES IN GEOMETRIC ACTIONS OF CLASSICAL GROUPS, II

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ABSTRACT. This is the second in a series of two papers in which we study the natural action of a classical algebraic group G on varieties of subspace decompositions of the natural module V. In this paper, we consider the action of a classical group on the space of decompositions of V into orthogonal isometric subspaces and we calculate close to best possible upper and lower bounds on the dimension of the fixed point space of elements $x \in G$ of prime order.

1. INTRODUCTION

Let k be an algebraically closed field of arbitrary characteristic p. Let G be an affine algebraic group over k. Suppose H is a closed subgroup of G, and denote Ω the coset variety G/H, on which G acts transitively. For $x \in G$ the *fixed point space* of x,

$$C_\Omega(x) = \{\omega \in \Omega : x.\omega = \omega\},$$

is a subvariety of Ω.

This is the second in a series of two papers in which we are interested in the fixed point spaces arising in natural geometric actions of classical algebraic groups. Let V be an n-dimensional k-vector space and let $G = Cl(V)$ be one of the classical group $\mathrm{GL}(V), \mathrm{Sp}(V)$ or $\mathrm{O}(V)$. In this paper we consider the action of G on the space Ω of direct sum decompositions of V of the form

$$V = V_1 \oplus \cdots \oplus V_t$$

with the properties that the V_i are orthogonal and isometric. Since G acts transitively on Ω we can identify Ω with the coset variety G/H, where H is a stabiliser of a fixed decomposition in Ω. Following Liebeck and Seitz [5], we say that H is a $\mathscr{C}_2$-subgroup. The structure of these subgroups is recorded in Table 1. Notice that if $G = \mathrm{O}_n$ and the V_i are 1-dimensional then H is finite.

G	H	Conditions
GL_n	$\mathrm{GL}_{n/t} \wr S_t$	
Sp_n	$\mathrm{Sp}_{n/t} \wr S_t$	
O_n	$\mathrm{O}_{n/t} \wr S_t$	n/t even if $p = 2$

TABLE 1. The $\mathscr{C}_2$-collection

The aim of this paper is to obtain both upper and lower bounds on the ratio

$$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega}$$

for $\Omega = G/H$, where $G = Cl(V)$, $H \in \mathscr{C}_2$, and $x \in G \setminus Z(G)$ is an element of prime order or any unipotent element if the field has characteristic zero. As explained in [7], we seek to derive best possible *global* and *local* bounds on $f_\Omega(x)$.

Notation and terminology are consisted with [7]. We write $p = \infty$ in the case the field k has characteristic zero. For $x \in G$, $o(x)$ denotes the order of x. We define $\mathscr{R} = \mathscr{R}(G)$ to be the set of non-central prime order elements of G also comprising all unipotent elements if $p = \infty$. Recall the definition [7, (2)] of $\nu(x)$: the codimension of the largest eigenspace in the action of x on V. Moreover we define $\mathscr{V}_s = \{x \in G : \nu(x) = s\}$ and $\mathscr{V}_{s,r} = \{x \in \mathscr{V}_s : o(x) = r\}$. Recall that $C_\Omega(x) \neq \emptyset$ if, and only if, $x^G \cap H \neq \emptyset$.

The first main result of this paper is Theorem 1, below, which states *global* bounds on $f_\Omega(x)$.

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Theorem 1. *Let $G = Cl_n$, let $H \leqslant G$ be a $\mathscr{C}_2$-subgroup of G and set $\Omega = G/H$.*

(a) *Let $x \in G \setminus Z(G)$. Then*

$$f_\Omega(x) \leqslant 1 - \frac{2}{n} + \left(\frac{2}{n}\right)^2.$$

(b) *Let $x \in \mathscr{R}$ be unipotent. Assume $x^G \cap H \neq \emptyset$.*

(i) *Assume $t = n$ or $p < n/2$, then*

$$f_\Omega(x) \geqslant \frac{1}{p}.$$

(ii) *Assume $t < n$ and $p \geqslant n/2$, then*

$$f_\Omega(x) \geqslant \frac{t}{n}.$$

(c) *Let $x \in \mathscr{R}$ be semisimple of order $r < n-1$. Assume $x^G \cap H \neq \emptyset$. Then, either $t < n$ and*

$$f_\Omega(x) \geqslant \frac{1}{r} - \frac{rt^2}{n^2(t-1)};$$

or, $t = n$ and $f_\Omega(x) \geqslant 1/(n-1)^2$.

Remark 2. Let us make some comments on the statement of Theorem 1.

(i) Note that the case $x \in G$ has order $r \geqslant n-1$ is excluded in part (c) of Theorem 1. Here $f_\Omega(x) \geqslant 0$ and equality is possible. We shall complete the proof of [7, Theorem 10], that comprise this case, in Proposition 6.1.

(ii) The upper bound is the best possible, see Remark 7. However, by subdividing the possibilities for G and H, we derive bounds that are optimal in each of the cases, see Propositions 4.1, 4.2 and 4.3.

(iii) The lower bounds for unipotent elements are sharp, in the sense that there exist $G, H \in \mathscr{C}_2$ (in fact infinitely many) and $x \in \mathscr{R}$ unipotent that realises the bound. In Proposition 5.1 we state the best possible bounds that we will determine. If $p \geqslant n/2$ for any G and H there exists x that realises the lower bound; in addition, in this case we classify such elements. A classification is given also in the case $p < n/2$ is odd, provided $p \mid n$, see Proposition 5.14. In the general case, for odd p, Proposition 5.15 shows that the lower bound is close to best possible.

(iv) The lower bound for semisimple elements is not the best possible, in fact it is often negative. However, in our analysis we shall compute in almost all cases close to best possible lower bounds, which are more complicated to state. In fact we determine a class of elements with the properties that each of them realise the best possible lower bound on the related ratio f_Ω° defined in (1), below, see Proposition 6.12. For the best possible bounds we refer the reader to Proposition 6.13.

The second main results, which deals with *local* bounds on $f_\Omega(x)$ is the following.

Theorem 3. *Let $G = Cl_n$, let $H \leqslant G$ be a $\mathscr{C}_2$-subgroup of G and set $\Omega = G/H$. Let $x \in \mathscr{R} \cap \mathscr{V}_s$. Then*

$$f_\Omega(x) \leqslant 1 - \frac{s}{n+1} + \frac{1}{n}.$$

In addition, assume n/t is even if $G = \mathrm{O}_n$, $r \neq p$, and $x^G \cap H \neq \emptyset$. Then

$$f_\Omega(x) \geqslant 1 - \frac{s(2n-s)}{n^2} - \frac{5}{n}.$$

Remark 4. Also here we make some remarks on the statement of Theorem 3.

(i) In the case x is an involution, unless $G \neq \mathrm{GL}_n$ and $p = 2$, $\nu(x)$ identifies $f_\Omega(x)$. The bounds in Theorem 3 holds for involutions as well; however, in this case much better estimates may be given. In fact, in several cases we are able to compute an explicit formula for $f_\Omega(x)$. This problem will be the object of a future paper.

(ii) If $r = p$ we do not have a general lower bound. In this case $\nu(x)$ is the difference between n and the number of Jordan blocks in x. This condition together with the assumption $x^G \cap H \neq \emptyset$ is hard to deal with. We refer to the beginning of Section 8 for further comments.

(iii) The argument for the upper bound relies on [7, Proposition 2.6]and [2, Proposition 2.1]. In Section 7 we will give further comments aimed to show that the derived bounds are close to best possible.

(iv) In the case $r \neq p$ we shall supply much more information. Most of the time, for x of order r we have $x^G \cap H^\circ \neq \emptyset$. We shall identify a short list of elements that realise the best possible bounds on f°_Ω. We refer the reader to Proposition 7.6 for the upper bound and to Proposition 8.2 for the lower bound.

With the prescribed exceptions arising in Theorem 3 the following corollary quickly follows.

Corollary 5. *Let $x,y \in H \cap \mathscr{V}_{s,r}$. Assume n/t is even if $G = \mathrm{O}_n$, and $r \neq p$. Then*

$$|f_\Omega(x) - f_\Omega(y)| \leqslant \frac{s(2n-s)}{n^2} - \frac{s}{n+1} + \frac{6}{n}.$$

In particular, if $s \leqslant \sqrt{n}$ or $s \geqslant n - \sqrt{n}$ and $x,y \in \mathscr{V}_{s,r}$, then $|f_\Omega(x) - f_\Omega(y)| < 2/\sqrt{n}$.

Let G be an algebraic group acting on a variety $\Omega = G/H$. Let r be a prime. Recall the definition of the *algebraic fixity* $M = M(G,H)$ and of the *r-local algebraic fixity* M_r. Further aim of this paper is to complete the proof of [7, Theorem 6]; in particular we prove the following.

Theorem 6. *Let $G = Cl_n$, let H be a $\mathscr{C}_2$-subgroup of G. Then $M = M(G,H)$ is recorded in Table 2 according to the given conditions on n and t. In addition, $M = M_r$ if, and only if, r is as in the last column of the table.*

Remark 7. Notice that if $G = \mathrm{Sp}_4$ or $(G,H,p) = (\mathrm{O}_4, \mathrm{O}_2 \wr S_2, 2)$, then $M = 1 - 2/n + (2/n)^2$. In this sense we say that the upper bound stated in Theorem 1 is the best possible.

G	H	M	Conditions	r
GL_n	$\mathrm{GL}_{n/t} \wr S_t$	$1 - \frac{2}{n} + \frac{1}{n(n-1)}$	$t = n$	2
		$1 - \frac{2}{n}$	$t \neq n$	any
Sp_n	$\mathrm{Sp}_{n/t} \wr S_t$	$1 - \frac{2}{n}$	$n > 4$	p
Sp_2	$\mathrm{Sp}_2 \wr S_2$	$3/4$		2
O_n	$\mathrm{O}_{n/t} \wr S_t$	$1 - \frac{2}{n}$	$n > 4$	2
O_4	$\mathrm{O}_2 \wr S_2$	$1/2$	$p \neq 2$	$\neq p$
		$3/4$	$p = 2$	2
O_4	$\mathrm{O}_1 \wr S_4$	$1/2$		$2,3$

TABLE 2. The algebraic fixity for $\mathscr{C}_2$-actions

Also here we state the following immediate consequence.

Corollary 8. *Let $G = Cl_n$ and let H be a $\mathscr{C}_2$-subgroup of G. Then $M(G,H) = \max\{M_2, M_p\}$.*

The following is quick consequence of the results proved in Section 4.

Corollary 9. *Let r be a prime. Then one of the following holds.*

(i) *there exists $x \in G$ of order r such that*

$$f_\Omega(x) \geqslant 1 - \frac{4}{n};$$

(ii) *$(G,H) = (\mathrm{O}_n, \mathrm{O}_1 \wr S_n)$, $r > n$, and for all $x \in \mathscr{R}$ of order r, $f_\Omega(x) = 0$.*

More generally, if $(G,H) = (\mathrm{O}_n, \mathrm{O}_1 \wr S_n)$ and $x \in \mathscr{R}$ has order $r > n$ then $C_\Omega(x) = \emptyset$, see Proposition 3.1. Notice that Corollary 9 implies that the general upper bound stated in Theorem 1 is close to best possible.

Strategy. In the following we briefly explain the strategy used to show Theorem 1. Recall, [4, Proposition 1.14] asserts that

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H).$$

The upper bound will be deduced using Burness' upper bound (cf. Proposition 2.2) on $\dim(x^G \cap H)$ depending on $\dim x^G$ and bounds on $\dim x^G$ depending on $\nu(x)$ [7, Proposition 2.6]. Subdividing the possibilities for $\nu(x)$ we will gain a characterisation of the elements that realise the best possible upper bounds, showing Theorem 6 and Corollary 9.

The lower bounds requires a different approach. Here the main difficulty is to estimate $\dim(x^G \cap H)$. Since H is the union of laterals of H°, for any $x \in G$, we have $\dim(x^G \cap H) = \max_{\pi \in S_t}\{\dim(x^G \cap H^\circ \pi)\}$. In particular, the following trivial inequality will be useful: $\dim(x^G \cap H) \geqslant \dim(x^G \cap H^\circ)$. It is convenient to define the following related ratio:

$$f^\circ_\Omega(x) = \frac{\dim\Omega - \dim x^G + \dim(x^G \cap H^\circ)}{\dim\Omega}. \tag{1}$$

In particular, the previous observation yields $f_\Omega(x) \geqslant f^\circ_\Omega(x)$. It turns out that the condition $x^G \cap (H \setminus H^\circ) \neq \emptyset$ yields some restrictions on the Jordan form of x, we list the relevant properties in Section 2.

In order to derive lower bounds, different techniques are used depending on whether the element is semisimple or unipotent. In general it will be also convenient to split the analysis according whether the (prime) order is large or not. We refer the reader to Section 5, for lower bounds on $f_\Omega(x)$, when $x \in \mathscr{R}$ is unipotent. We make some comments on the strategy used for semisimple elements. Let $x \in \mathscr{R}$ be semisimple of odd order. In almost all the cases, if $H \in \mathscr{C}_2$ then H° contains a maximal torus of G. Therefore $x^G \cap H^\circ \neq \emptyset$. Hence, we will derive lower bounds on the related ratio f°_Ω. The key result on which most of our arguments rely is Theorem 2.5 in which we state an explicit formula of $\dim(x^G \cap H^\circ)$. The strategy is quite similar to that employed in [7]. We define a class of *special elements* (cf. Definition 6.8) and we shall show that any special element realises the best possible lower bound on $f^\circ_\Omega(x)$.

Layout. In Section 2 we study H-conjugacy classes for $H \leqslant G$ a closed $\mathscr{C}_2$-subgroup. First we first focus on H°-classes, namely we compute an explicit formula for $\dim(x^G \cap H^\circ)$ in the case $x \in H^\circ$ is semisimple (cf. Theorem 2.5). This formula will be the key tool to derive a lower bound on $\dim(x^G \cap H^\circ)$ of the form $(1/t - \varepsilon)\dim x^G$ (see Proposition 2.14). In Section 2.2, we turn the attention on conjugacy classes in H; the main result of this section is Proposition 2.22 which provides us with the Jordan form of a prime order element x such that $x^G \cap (H \setminus H^\circ) \neq \emptyset$. The remainder of the paper is devoted to the proofs of the results stated in this introduction. In particular, in Section 4 we derive global upper bounds. In Sections 5, resp. 6, we calculate lower bounds for unipotent, resp. semisimple, elements. The we turn our attention on local bounds. We derive local upper and lower bounds in Sections 7 and 8, respectively.

Notation. The preliminary results needed are taken from [7, Section 2]. The notation is consistent with that introduced in [7]. If $x \in G$ is unipotent then x has Jordan form

$$[J_n^{a_n}, \ldots, J_1^{a_1}], \tag{2}$$

for some non-negative integers a_i. If $G \neq \mathrm{GL}_n$, $p = 2$ and x is an involution we refer the reader to [7, Theorem 2.6]. If x is semisimple of order r then x is G-conjugate to

$$[I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]; \tag{3}$$

in the case $r = 2$, we denote $a_1 = s$ if x is an involution (notice that if $s \leqslant n/2$ then $\nu(x) = s$).

2. Conjugacy in $\mathscr{C}_2$-subgroups

Let $G = Cl(V)$ and let H be a $\mathscr{C}_2$-subgroup of G. In this section we describe the H-classes of prime order elements in H. Notice that both G and H are reductive. Hence, by [3, Theorem 1.2], $x^G \cap H = x_1^{H^\circ} \cup \ldots \cup x_l^{H^\circ}$ for some $l < \infty$. In particular $\dim(x^G \cap H) = \max_i\{\dim x_i^{H^\circ}\}$. Furthermore, since $H = \bigcup_{\pi \in S_t} H^\circ \pi$, then

$$x^G \cap H = \bigcup_{\pi \in S_t} (x^G \cap H^\circ \pi); \tag{4}$$

in particular $x^G \cap H = (x^G \cap H^\circ) \cup (x^G \cap H \setminus H^\circ)$. With this in mind, we divide the analysis into two cases starting with the study of $x^G \cap H^\circ$. Notice that

$$\dim\big([x_1, \ldots, x_t]\big)^{H^\circ} = \sum_i \dim x_i^{Cl_{n/t}}.$$

Remark 2.1. The same argument used to prove [7, Lemma 2.14] implies that if $x \in G$ has order $r > t$ then $x^G \cap H = x^G \cap H^\circ$.

The following is [2, Proposition 2.1], for the case $G = \mathrm{Sp}_n$ we record the bounds as given in the proof.

Proposition 2.2. *Let $G = Cl_n$ and $H = Cl_{n/t} \wr S_t$ be a $\mathscr{C}_2$-subgroup of G. Let $x \in \mathscr{R}$ be of order r. Then*

$$\dim(x^G \cap H) \leqslant \left(\frac{1}{t} + \zeta\right) \dim x^G$$

where $\zeta = 0$ if $G = \mathrm{GL}_n$ or O_n, for $G = \mathrm{Sp}_n$ we record ζ in Table 3, with a representative of x^G.

p	r	x^G-representative	ζ
$\neq 2$	$= 2$	$[I_{n-s}, -I_s]$	$\frac{1}{n}$
$= 2$	$= 2$	$[J_2^s, J_1^{n-2s}]$	$\frac{s}{\dim x^G}\left(1 - \frac{1}{t}\right)$
any	$\neq 2, p$	$[I_{a_0}, \dots, \omega^{r-1} I_{a_{r-1}}]$	$\frac{n-a_0}{2\dim x^G}\left(1 - \frac{1}{t}\right)$
$\neq 2$	$= p$	$[J_p^{a_p}, \dots, J_1^{a_1}]$	$\frac{n-\sum_{i \text{odd}} a_i}{2\dim x^G}\left(1 - \frac{1}{t}\right)$

TABLE 3

2.1. **Conjugacy classes in H°.** In this section we describe H°-conjugacy classes. We start with unipotent elements.

2.1.1. *Unipotent elements.* We first make some comments on partitions of an integer. Fix an integer n and let $\mathscr{P} = \mathscr{P}(n) = \{\lambda : \lambda \vdash n\}$ be the set of partitions of n, write $\lambda = (n^{a_n}, \dots, 1^{a_1})$, so that $n = \sum_{i=1}^n i a_i$. We define the following notation for partitions. Let $\lambda = (n^{a_n}, \dots, 1^{a_1}) \vdash n$, $\mu = (n^{b_n}, \dots, 1^{b_1})$ and $\eta = (n^{c_n}, \dots, 1^{c_1})$. We write $\lambda = \mu \oplus \eta$ if $a_i = b_i + c_i$ for all i.

Recall, there exists a one-to-one correspondence between conjugacy classes of unipotent elements in GL_n and $\mathscr{P}$. If $G = \mathrm{Sp}_n$ or O_n and $p \neq 2$, we have a one-to-one correspondence with a proper subset of $\mathscr{P}$, say $\mathscr{P}_G$. The additional properties defining $\mathscr{P}_G$ can be deduced from [7, Theorem 2.2]: $\mathscr{P}_{\mathrm{GL}_n} = \mathscr{P}$ and

$$\mathscr{P}_{\mathrm{Sp}_n} = \{\lambda \in \mathscr{P} : a_i \text{ even for all } i \text{ odd}\}, \ \mathscr{P}_{\mathrm{O}_n} = \{\lambda \in \mathscr{P} : a_i \text{ even for all } i \text{ even}\}.$$

Throughout this section we assume $x \in H^\circ$ has order p, with $p \neq 2$ if $G \neq \mathrm{GL}_n$. Since $x \in H^\circ$, there exists a block decomposition $x = [z_1, \dots, z_t]$ such that $z_i \in Cl_{n/t}$ for all i. Notice that if we also have $x = [y_1, \dots, y_t]$, and there exists i such that y_i and z_i are not $Cl_{n/t}$-conjugate then $[z_1, \dots, z_t]$ and $[y_1, \dots, y_t]$ are not H°-conjugate. More generally, if there exists i such that for all j we have that y_i and z_j are not $Cl_{n/t}$-conjugate then the two block decompositions are not H-conjugate. For example, $[x_1, x_2]$ and $[x_2, x_1]$ in $H^\circ = (Cl_{n/2})^2$ are H°-conjugate if, and only if, x_1 and x_2 are $Cl_{n/t}$-conjugate; however, they are H-conjugate. Notice that, for the purpose of computing the dimension of the H°-class, the order of the blocks is not important.

In general, $x^G \cap H^\circ$ is a finite union of H°-classes. Hence, for some positive integer $l < \infty$, we have

$$x^G \cap H^\circ = \bigcup_{i=1}^{l} A_i^{H^\circ}, \tag{5}$$

where $A_i \in H^\circ$. Thanks to the previous discussion it is clear that there is a one-to-one correspondence

$$\{A_1, \dots, A_l\} \longleftrightarrow \{\lambda_1 \oplus \dots \oplus \lambda_t : \lambda_i \vdash n/t \text{ and } \lambda_i \in \mathscr{P}_{Cl_{n/t}} \text{ for all } i\}. \tag{6}$$

In order to compute $\dim(x^G \cap H^\circ)$ we need to know all the possible block decompositions $x = [x_1, \dots, x_t]$ where $x_i \in Cl_{n/t}$. In the following, using Proposition 2.2, we show that for certain elements there is an obvious block decomposition that provides the largest dimension.

Lemma 2.3. *Assume $G = \mathrm{GL}_n$. Let $x \in \mathscr{R}$ be unipotent. Assume t divides a_i for all i. Then $x^G \cap H^\circ \neq \emptyset$, and there exists a decomposition $x = [x_1, \dots, x_t]$ such that $\dim x^{H^\circ} = (1/t) \dim x^G$. In particular,*

$$\dim(x^G \cap H) = \frac{1}{t} \dim x^G.$$

Proof. Let $x_i = [J_p^{a_p/t}, \dots, J_1^{a_1/t}]$ for all i. The result follows using [7, Theorem 2.2] and Proposition 2.2. □

A similar result holds when $G = \mathrm{Sp}_n$, if we assume stronger hypotheses.

Lemma 2.4. *Assume $G=\mathrm{Sp}_n$ and $p\neq 2$. Let $x\in\mathscr{R}$ be unipotent. Assume t divides a_i for all i, and a_i/t is even whenever i is odd. Then $x^G\cap H^\circ\neq\emptyset$ and there exists a decomposition $x=[x_1,\dots,x_t]$ such that $\dim x^{H^\circ}=(1/t)\dim x^G+(n-\sum_{i\text{ odd}}a_i)(1-1/t)/2$. In particular,*

$$\dim(x^G\cap H)=\frac{1}{t}\dim x^G+\frac{n-\sum_{i\text{ odd}}a_i}{2}\Big(1-\frac{1}{t}\Big).$$

We do not have the same result for the orthogonal group even if we use hypotheses similar to those of Lemma 2.4.

2.1.2. *Semisimple elements.* Let $x\in G$ be of prime order $r\neq p$. Aims of this section are to compute an explicit formula for $\dim(x^G\cap H^\circ)$ (which will be the key tool in our analysis on lower bounds for semisimple elements). Then, in the same spirit of Proposition 2.2, we derive a lower bound on this dimension in terms of $\dim x^G$, see Proposition 2.14.

Notice that, unless $G=\mathrm{O}_n$ and n/t is odd, H contains a maximal torus of G; in particular, $x^G\cap H^\circ\neq\emptyset$.

Theorem 2.5. Let $x\in G$ be of primer order r. Assume $x^G\cap H^\circ\neq\emptyset$.

(i) If $G=\mathrm{GL}_n$ then

$$\dim(x^G\cap H^\circ)=\frac{n^2}{t}-n+\sum_{i=0}^{r-1}\Big(\Big\lfloor\frac{a_i}{t}\Big\rfloor^2 t+(t-2a_i)\Big\lfloor\frac{a_i}{t}\Big\rfloor\Big).$$

(ii) If $G=\mathrm{Sp}_n$ and $r\neq 2$, then

$$\dim(x^G\cap H^\circ)=\frac{n^2}{2t}-a_0+2\Big(\Big\lfloor\frac{a_0}{2t}\Big\rfloor^2 t+(t-a_0)\Big\lfloor\frac{a_0}{2t}\Big\rfloor\Big)+\frac{1}{2}\sum_{i=1}^{r-1}\Big(\Big\lfloor\frac{a_i}{t}\Big\rfloor^2 t+(t-2a_i)\Big\lfloor\frac{a_i}{t}\Big\rfloor\Big).$$

Assume $r=2$ and write $s/2=at+b$, where $0\leqslant b<t$. Then

$$\dim(x^G\cap H^\circ)=\frac{s(n-s)}{t}-\frac{4b(t-b)}{t}.$$

(iii) Assume $G=\mathrm{O}_n$ and $r\neq 2$. If n/t is even, then

$$\dim(x^G\cap H^\circ)=\frac{n^2}{2t}-n+2\Big(\Big\lfloor\frac{a_0}{2t}\Big\rfloor^2 t+(t-a_0)\Big\lfloor\frac{a_0}{2t}\Big\rfloor\Big)+\frac{1}{2}\sum_{i=1}^{r-1}\Big(\Big\lfloor\frac{a_i}{t}\Big\rfloor^2 t+(t-2a_i)\Big\lfloor\frac{a_i}{t}\Big\rfloor\Big).$$

If n/t is odd then

$$\begin{aligned}\dim(x^G\cap H^\circ)=&\frac{n^2}{2t}-n+\frac{a_0}{2}+\frac{1}{2}\sum_{i=1}^{r-1}\Big(\Big\lfloor\frac{a_i}{t}\Big\rfloor^2 t+(t-2a_i)\Big\lfloor\frac{a_i}{t}\Big\rfloor\Big)\\&+2\Big(\Big\lfloor\frac{a_0-t}{2t}\Big\rfloor^2 t+(2t-a_0)\Big\lfloor\frac{a_0-t}{2t}\Big\rfloor\Big)-\frac{3}{2}(a_0-t).\end{aligned}$$

Assume $r=2$ and write $s=at+b$, where $0\leqslant b<t$. Then

$$\dim(x^G\cap H^\circ)=\frac{s(n-s)}{t}-\frac{b(t-b)}{t}.$$

In comparison with Lemmas 2.3 and 2.4 we make the following.

Remark 2.6. Assume $G\neq\mathrm{O}_n$. Let $x\in G$ be semisimple of order r; assume $r\neq 2$ if $G=\mathrm{Sp}_n$. Recall the definition of ζ from Proposition 2.2. It is not too difficult to see that $\dim(x^G\cap H)=\dim(x^G\cap H^\circ)=(1/t+\zeta)\dim x^G$ if, and only if, $t\mid a_i$ for all $i>0$, $t\mid a_0$ if $G=\mathrm{GL}_n$ and $t\mid a_0/2$ if $G=\mathrm{Sp}_n$.

The following is a straightforward consequence of the formulae in Theorem 2.5.

Corollary 2.7. *Let $x,y\in G$ be of order r. Assume $C_G(x)\cong C_G(y)$. Then $\dim(x^G\cap H^\circ)=\dim(y^G\cap H^\circ)$.*

We also point out another consequence of Theorem 2.5.

Proposition 2.8. *Let n be an even integer and t a non-trivial divisor of n such that n/t is even. Denote $G_1=\mathrm{Sp}_n, G_2=\mathrm{O}_n$, $H_1=\mathrm{Sp}_{n/t}\wr S_t, H_2=\mathrm{O}_{n/t}\wr S_t$ and set $\Omega_i=G_i/H_i$ for $i=1,2$. Let $x\in\mathrm{GL}_n$ be semisimple of odd prime order r such that $x^{\mathrm{GL}_n}\cap G_1\neq\emptyset$. Then*

(i) *there exists $y\in x^{\mathrm{GL}_n}\cap H_1\cap H_2$;*
(ii) *$f^\circ_{\Omega_1}(y)=f^\circ_{\Omega_2}(y)$.*

Proof. We may assume $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$. Then [7, Theorem 2.5] implies that a_0 is even and $a_i = a_{r-i}$ for all $i \leqslant (r-1)/2$. In particular, $x \in G_1 \cap G_2$. Since both H_1 and H_2 contain a maximal torus of G_1 and G_2, respectively, it is clear that there exists $y \in H_1 \cap H_2$ that is GL_n-conjugate to x. Using [7, Theorem 2.5] and Theorem 2.5 we compute $\dim y^{G_1} - \dim y^{G_1} = n - a_0 = \dim(y^{G_1} \cap H_1^\circ) - \dim(y^{G_2} \cap H_2^\circ)$. This concludes the proof. □

Before embarking in the proof of Theorem 2.5 we need to show several technical lemmas. Let $x \in G$ be or order r and assume $x^G \cap H^\circ \neq \emptyset$. Then there exists a block decomposition $[x_1, \dots, x_t] \in x^G \cap H^\circ$; hence, we may assume $x = [x_1, \dots, x_t]$ and we establish the following notation:

$$x_i = [I_{a_{i,0}}, \omega I_{a_{i,1}}, \dots, \omega^{r-1} I_{a_{i,r-1}}] \in Cl_{n/t}, \tag{7}$$

where $\sum_j a_{i,j} = n/t$ and $\sum_i a_{i,j} = a_i$, for all i, j.

Recall that $\dim(x^G \cap H^\circ)$ is the largest among the dimensions of the H°-classes of all the possible block decompositions as above. For the analysis that follows, it is convenient to define

$$\iota = \begin{cases} 1 & G = \mathrm{GL}_n, \text{ or } (G,r) = (\mathrm{O}_n, 2) \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

We introduce the following claim, which will be the essential tool to show Theorem 2.5.

Claim 2.9. Let $x = [x_1, \dots, x_t] \in H^\circ$ be of order r. Then $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ if, and only if, $|a_{i,l} - a_{j,l}| \leqslant 1$ and $|a_{i,0} - a_{j,0}| \leqslant \iota$, for all $1 \leqslant l \leqslant r-1$ and $1 \leqslant i, j \leqslant t$.

In order to prove one implication of the claim we need the following technical tool.

Lemma 2.10. *Assume $r > 2$ if $G \neq \mathrm{GL}_n$. Let $x = [x_1, \dots, x_t] \in H^\circ$ be of order r. Assume there exists $m \in \{0, \dots, r-1\}$ and blocks x_i, x_j such that $a_{i,m} = a_{j,m} + 2 + h$, for $h \geqslant 0$ or, $h \geqslant 2$ if $m = 0$ and $G \neq \mathrm{GL}_n$.*

- (i) *Assume $G = \mathrm{GL}_n$. Then there exists $l \in \{0, 1, \dots, r-1\} \setminus \{m\}$ such that $a_{j,l} \neq 0$ and $a_{i,l} - a_{j,l} < h$.*
- (ii) *Assume $G = \mathrm{Sp}_n$ or O_n.*
  - (a) *If $m = 0$ then there exists $l \in \{1, \dots, r-1\}$ such that $a_{j,l} \neq 0$ and $a_{i,l} - a_{j,l} < h - 2$.*
  - (b) *If $m \neq 0$ then one of the following holds:*
    - ∗ *$a_{j,0} \neq 0$ and $a_{i,0} - a_{j,0} < -1$; or,*
    - ∗ *$a_{j,0} = 0$ and there exists $l \in \{1, \dots, r-1\} \setminus \{m\}$ such that $a_{i,l} - a_{j,l} < h$.*

Proof. For convenience, in this proof we adopt the following lighter notation: given $l \in \{0, \dots, r-1\}$ we set $a_{i,l} = b_l$ and $a_{j,l} = c_l$.

Case (i). If $G = \mathrm{GL}_n$ we may assume, without loss of generality, $m = 0$. Thus $b_0 = c_0 + 2 + h$, in particular $c_0 < n/t$. Hence, there exists $i > 0$ such that $c_i \neq 0$. Up to relabelling the eigenvalues, and not considering c_0 (that may be 0), we may assume that $c_1, \dots, c_l$ are the only non-zero multiplicities in the block x_j. Suppose $b_i - c_i \geqslant h$ for all $1 \leqslant i \leqslant l$. Then summing over i we have

$$\sum_i b_i - \sum_i c_i \geqslant (b_0 - c_0) + \sum_{i=1}^{l} (b_i - c_i) \geqslant (h+2) + lh > 0$$

which is absurd since $n/t = \sum_i b_i = \sum_i c_i$. So there exists i such that $c_i \neq 0$ and $b_i - c_i < h$.

Case (ii). Assume $G = \mathrm{Sp}_n$ or O_n. The argument is similar to the previous case. However, here, due to the centraliser structure of x, we need to consider several different cases.

First assume $m = 0$. So $b_0 - c_0 = 4 + h$ for some $h \geqslant 0$. As above, we denote $c_1, \dots, c_l$ the non-zero multiplicities of x_j. Thus

$$\sum_i b_i - \sum_i c_i \geqslant (b_0 - c_0) + \sum_{i=1}^{l} (b_i - c_i) > (h+4) + lh > 0$$

which is absurd. The result follows.

In the case $m \neq 0$ we have $b_m - c_m \geqslant h + 2$. We distinguish two cases: either $c_0 = 0$ or $c_0 \neq 0$.

If $c_0 = 0$ then the same argument as above implies the result.

Assume $m \neq 0$ and $c_0 \neq 0$. Then either there exists i such that $b_i - c_i < h$, in which case the result follows, or $b_i - c_i \geqslant h$ for all i. In the latter case, with the usual argument, we have

$$0 = \sum_i b_i - \sum_i c_i \geqslant (b_0 - c_0) + (b_m - c_m) + \sum_{i \neq 0, m} (b_i - c_i) \geqslant (b_0 - c_0) + h + 2 + lh \geqslant (b_0 - c_0) + 2$$

Therefore $b_0 - c_0 < -1$. □

Using Lemma 2.10 we prove one implication of the claim.

Lemma 2.11. *Let $x=[x_1,\dots,x_t]\in H^\circ$ be of order r. Assume* $\dim(x^G\cap H^\circ)=\dim x^{H^\circ}$.

(i) *If $r\neq 2$ then*
$$|a_{h,i}-a_{l,i}|\leqslant 1$$
for all $1\leqslant h,l\leqslant t$ and $1\leqslant i\leqslant r-1$. In addition $|a_{h,0}-a_{l,0}|\leqslant \iota$ for all h,l.

(ii) *If $r=2$ then $x_i\in[I_{n/t-s_i},-I_{s_i}]^{Cl_{n/t}}$ and*
$$|s_i-s_j|\leqslant \iota$$

Proof. We prove the contrapositive. Assume there exist blocks x_h,x_l with $a_{h,i}-a_{l,i}\geqslant 2$, i.e. $a_{h,i}=a_{l,i}+2+h$ for some $h\geqslant 0$. We shall construct $y=[y_1,\dots,y_t]\in x^G\cap H^\circ$ such that $\dim y^{H^\circ}>\dim x^{H^\circ}$, which is absurd since $\dim(x^G\cap H^\circ)=\dim x^{H^\circ}$.

After a permutation of the blocks, if necessary, we may assume $(h,l)=(1,2)$. For $r\neq 2$ we denote $a_{1,j}=b_j$ and $a_{2,j}=c_j$ for all $0\leqslant j\leqslant r-1$. Therefore $b_0=c_0+4+h$, if $G=\mathrm{Sp}_n,\mathrm{O}_n$ (recall $b_0,c_0\equiv n/t \pmod 2$), or $b_i=c_i+2+h$, otherwise, for some $h\geqslant 0$. We split the proof in different cases.

Case 1. Assume $G=\mathrm{GL}_n$. Up to relabel the eigenvalues we may assume $i=0$. Hence $b_0=c_0+2+h$ for some $h\geqslant 0$. Up to conjugation, we have
$$x_1=[I_{c_0+2+h},\omega I_{b_1},\dots,\omega^{r-1}I_{b_{r-1}}],\ x_2=[I_{c_0},\omega I_{c_1},\dots,\omega^{r-1}I_{c_{r-1}}].$$
By Lemma 2.10 there exists i such that $b_i-c_i<h$; again, we may assume $i=1$. Thereby, we define $y=[y_1,y_2,x_3,\dots,x_t]$, where
$$y_1=[I_{c_0+1+h},\omega I_{b_1+1},\dots,\omega^{r-1}I_{b_{r-1}}],\ y_2=[I_{c_0+1},\omega I_{c_1-1},\dots,\omega^{r-1}I_{c_{r-1}}].$$
Clearly x and y are G-conjugate but not H°-conjugate. Using [7, Theorem 2.5], we have
$$\dim y^{H^\circ}-\dim x^{H^\circ}=\dim y_1^{H^\circ}+\dim y_2^{H^\circ}-\dim x_1^{H^\circ}-\dim x_2^{H^\circ}=2(c_1-b_1+h)>0.$$
As explained above this is absurd. This argument holds in the case $(G,r)=(\mathrm{O}_n,2)$, as well.

Case 2. Assume $G=\mathrm{Sp}_n$ or O_n, and $r\neq 2$. Recall $b_i=b_{r-i}$ for all $i\leqslant (r-1)/2$, see [7, Theorem 2.5]. Assume $b_0=c_0+4+h$ for some $h\geqslant 0$. By Lemma 2.10 (note here we assume $h\geqslant 0$) there exists $i>0$ such that $c_i\neq 0$ and $b_i-c_i<h$. Also here we may assume $i=1$. As above, we define $y=[y_1,y_2,x_3,\dots,x_t]$, where
$$y_1=[I_{c_0+2+h},\omega I_{b_1+1},\dots,\omega^{r-1}I_{b_1+1}],\ y_2=[I_{c_0+2},\omega I_{c_1-1},\dots,\omega^{r-1}I_{c_1-1}].$$
As in the previous case we compute $\dim y^{H^\circ}-\dim x^{H^\circ}=2(h+1-b_1+c_1)>0$.

Now assume $b_i=c_i+2+h$ for $i\neq 0$ and $h\geqslant 0$. We may assume $i=1$. By Lemma 2.10 either there exists $j>0$ such that $c_j\neq 0$ and $b_j-c_j<h$, or $c_0\neq 0$ and $b_0-c_0<-1$. In the former case we may assume $j=2$ and we define $y=[y_1,y_2,x_3,\dots,x_t]$, where
$$y_1=[I_{b_0},\omega I_{b_1-1},\omega^2 I_{b_2+1},\omega^3 I_{b_3},\dots,\omega^{r-1}I_{b_1-1}],\ y_2=[I_{c_0},\omega I_{c_1+1},\omega^2 I_{c_2-1},\omega^3 I_{b_3},\dots,\omega^{r-1}I_{c_1+1}].$$
As above $\dim y^{H^\circ}-\dim x^{H^\circ}=2(h-b_2+c_2)>0$.

Now assume $b_1-c_1=2+h$ for some $h\geqslant 0$, $c_0\neq 0$ and $b_0-c_0<-1$. Define $y=[y_1,y_2,x_3,\dots,x_t]$, where
$$y_1=[I_{b_0+2},\omega I_{c_1+1+h},\omega^2 I_{b_2},\dots,\omega^{r-1}I_{c_1+1+h}],\ y_2=[I_{c_0-2},\omega I_{c_1+1},\omega^2 I_{c_2},\dots,\omega^{r-1}I_{c_1+1}].$$
Again, we compute $\dim y^{H^\circ}-\dim x^{H^\circ}=2(h-1-b_0+c_0)>0$. The result follows.

Case 3. Assume $G=\mathrm{Sp}_n$ and x is an involution. Then each x_i is $\mathrm{Sp}_{n/t}$-conjugate to $[I_{n/t-s_i},-I_{s_i}]$ for a suitable s_i even. Assume $s_1-s_2>2$. Here we define $y=[y_1,y_2,x_3,\dots,x_t]$, where
$$y_1=[I_{n/t-s_1+2},-I_{s_1-2}],\ y_2=[I_{n/t-s_1-2},-I_{s_1+2}].$$
And we compute $\dim y^{H^\circ}-\dim x^{H^\circ}=4(s_1-s_2-2)>0$. The result follows. $\square$

Only for the purpose of a uniform notation, in the following result (needed in order to complete the proof of the claim), if x is an involution we write $x=[I_{a_0},-I_{a_1}]$; thus, if $x=[x_1,\dots,x_t]\in H^\circ$ then $x_i=[I_{a_{i,0}},-I_{a_{i,1}}]$.

Lemma 2.12. *Let $x\in G$ be of order r. Assume $x=[x_1,\dots,x_t]\in H^\circ$ is such that* $\dim(x^G\cap H^\circ)=\dim x^{H^\circ}$.

(i) *If $G=\mathrm{GL}_n$ then $a_{i,h}\in\{\lfloor a_h/t\rfloor,\lfloor a_h/t\rfloor+1\}$ for $1\leqslant i\leqslant t$, $0\leqslant h\leqslant r-1$;*

(ii) *If $G=\mathrm{Sp}_n$ or O_n (assume $r\neq 2$ if $G=\mathrm{O}_n$) then $a_{i,h}\in\{\lfloor a_h/t\rfloor,\lfloor a_h/t\rfloor+1\}$ for $1\leqslant i\leqslant t$, $1\leqslant h\leqslant r-1$; and $a_{i,0}\in\{2\lfloor a_0/2t\rfloor,2\lfloor a_0/2t\rfloor+2\}$;*

(iii) *If* $G = \mathrm{O}_n$ *and* $r = 2$ *then* $a_{i,0} \in \{\lfloor a_0/t \rfloor, \lfloor a_0/t \rfloor + 1\}$ *for* $1 \leqslant i \leqslant t$.

Proof. We only give a proof in the case $G = \mathrm{GL}_n$; the argument for the other cases is similar. By Lemma 2.11, $|a_{i,h} - a_{j,h}| \leqslant 1$ for all i, j, h. We may assume $a_{1,h} = \max_i\{a_{i,h}\}$ and $a_{2,h} = \min_i\{a_{i,h}\}$. Then $a_{1,h} - a_{2,h} \in \{0, 1\}$. If $a_{1,h} = a_{2,h}$ then for all i we have $a_{1,h} = a_{i,h} = a_h/t$. So assume $a_{1,h} - a_{2,h} = 1$. In particular $a_{2,h} = a_{1,h} - 1$. For all $i > 2$ we have $a_{1,h} - 1 \leqslant a_{i,h} \leqslant a_{1,h}$. Thus, summing over i, we get

$$(a_{1,h} - 1)t + 1 \leqslant a_h \leqslant (a_{1,h} - 1)t + (t - 1).$$

Thus $a_{1,h} - 1 = \lfloor a_h/t \rfloor$. The result follows. □

The following shows Claim 2.9, recall the definition of ι given in (8).

Proposition 2.13. *Let* $x \in G$ *be of order* r. *Assume* $x = [x_1, \dots, x_t] \in H^\circ$. *Then* $\dim(x^G \cap H^\circ)$ *if, and only if,*

$$|a_{i,h} - a_{j,h}| \leqslant 1, \ |a_{i,0} - a_{j,0}| \leqslant \iota$$

for all $1 \leqslant i, j \leqslant t$ *and all* $1 \leqslant h \leqslant r - 1$.

Proof. In the case $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ Lemma 2.11 yields the result.

Conversely, assume $G = \mathrm{GL}_n$ and $|a_{i,h} - a_{j,h}| \leqslant 1$ for all i, j, h. Then the proof of Lemma 2.12 yields $a_{i,h} \in \{\lfloor a_h/t \rfloor, \lfloor a_h/t \rfloor + 1\}$. Notice that $\dim(x^G \cap H^\circ) \geqslant \dim x^{H^\circ}$ and there exists $z = [z_1, \dots, z_t] \in x^G \cap H^\circ$ such that $\dim(x^G \cap H^\circ) = \dim z^{H^\circ}$. Say $b_{i,h}$ the multiplicity of ω^h in the block z_i. Then, Lemma 2.11 implies $|b_{i,h} - b_{j,h}| \leqslant 1$ for all i, j, h. Consequently, Lemma 2.11 yields $b_{i,h} \in \{\lfloor a_h/t \rfloor, \lfloor a_h/t \rfloor + 1\}$. Therefore, up to a permutation of the blocks, $C_{\mathrm{GL}_{n/t}}(x_i) \cong C_{\mathrm{GL}_{n/t}}(z_i)$. Thus $\dim x^{H^\circ} = \dim z^{H^\circ}$. The other cases are similar and left to the reader. □

We are now in position to prove Theorem 2.5. Notice that Proposition 2.13 provides us with the best possible block decomposition of an element $x \in G$ whose G-class meets H°. Hence, the formulae in the theorem will follow with a straightforward calculation.

proof of Theorem 2.5. Assume $G = \mathrm{GL}_n$. Let $x \in G$ be of order r. For all $i \in \{1, \dots, t\}$ we write $a_i = c_i t + b_i$, where $0 \leqslant b_i < t$, notice that $c_i = \lfloor a_i/t \rfloor$. We write $x = [x_1, \dots, x_t]$ where we define

$$x_i = [I_{c_0 + \varepsilon_{i,0}}, \omega I_{c_1 + \varepsilon_{i,1}}, \dots, \omega^{r-1} I_{c_{r-1} + \varepsilon_{i,r-1}}],$$

where for every $j \in \{0, \dots, r-1\}$ we have $\sum_i \varepsilon_{i,j} = b_i$ and $\varepsilon_{i,j} \in \{0, 1\}$ for all i and j. Now, Proposition 2.13 implies that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Therefore we have

$$\dim x^{H^\circ} = \sum_i \dim x_i^{\mathrm{GL}_{n/t}} = \sum_{i=1}^{t} \left(\frac{n^2}{t^2} - \sum_{j=0}^{r-1} (c_j + \varepsilon_{i,j})^2 \right).$$

Using $\varepsilon_{i,j}^2 = \varepsilon_{i,j}$, $\sum_i \varepsilon_{i,j} = b_i = a_i - c_i t$, the result follows with an easy calculation.

The procedure is very similar in the other cases.

Assume $G = \mathrm{Sp}_n$ or O_n (n/t even) and $r \neq 2$. For $i > 0$ we write $a_i = c_i t + b_i$ where $0 \leqslant b_i < t$; if $i = 0$ then a_0 is even and we write $a_0/2 = c_0 t + b_0$ where $0 \leqslant b_0 < t$. Let $x = [x_1, \dots, x_t] \in x^G \cap H^\circ$ where

$$x_i = [I_{2c_0 + 2\varepsilon_{i,0}}, \omega I_{c_1 + \varepsilon_{i,1}}, \dots, \omega^{r-1} I_{c_{r-1} + \varepsilon_{i,r-1}}],$$

here $\varepsilon_{i,j} \in \{0, 1\}$, $\varepsilon_{i,j} = \varepsilon_{i,r-j}$ for all $j \leqslant (r-1)/2$ and, as before, $\sum_j \varepsilon_{i,j} = b_j$. Proposition 2.13 implies that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Again, the result follows with an easy computation.

Now assume $G = \mathrm{O}_n$, n/t is odd and $r \neq 2$. By [7, Theorem 2.5], a_0 is even if, and only if, n is even. Since n/t is odd we deduce that $a_0 - t$ is even. Let $[y_1, \dots, y_t] \in x^G \cap H^\circ$. Then $y_i \in \mathrm{O}_{n/t}$ for all i. Since n/t is odd the 1-eigenvalue has odd multiplicity in y_i, in particular the multiplicity is positive. Therefore $a_0 \geqslant t$. Write $(a_0 - t)/2 = c_0 t + b_0$ where $0 \leqslant b_0 < t$. As above we write $a_i = c_i t + b_i$ for $i > 0$. Then we define

$$x_i = [I_{2c_0 + 2\varepsilon_{i,0} + 1}, \omega I_{c_1 + \varepsilon_{i,1}}, \dots, \omega^{r-1} I_{c_{r-1} + \varepsilon_{i,r-1}}],$$

where the conditions on the $\varepsilon_{i,j}$ are as above. Then $x_i \in \mathrm{O}_{n/t}$ and up to G-conjugacy $x = [x_1, \dots, x_t]$. Again, Proposition 2.13 implies that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. The result follows with an easy computation.

In the case $(G, r) = (\mathrm{Sp}_n, 2)$ then, up to conjugacy, $x = [I_{n/t-s}, -I_s]$, and s is even. We write $s/2 = ct + b$, where $0 \leqslant b < t$, and define $x_i = [I_{n/t - 2c - 2\varepsilon_i}, -I_{2c + 2\varepsilon_i}]$ where $\varepsilon_i \in \{0, 1\}$ and $\sum_i \varepsilon_i = b$. For $(G, r) = (\mathrm{O}_n, 2)$, write $s = ct + b$, $0 \leqslant b < t$, and let $x_i = [I_{n/t - c - \varepsilon_i}, -I_{c + \varepsilon_i}]$. Hence $x = [x_1, \dots, x_t]$. In both cases, Proposition 2.13 implies $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. The result follows with an easy computation. □

For $x \in H^\circ$, Proposition 2.13 provides us with a block decomposition $[x_1,\dots,x_t] \in x^G \cap H^\circ$ such that $\dim(x^G \cap H^\circ) = \dim\big([x_1,\dots,x_t]\big)^{H^\circ}$. This is the main tool needed to show the following.

Proposition 2.14. *Let $x \in \mathscr{R}$ be of order $r \neq 2, p$. Assume $x^G \cap H^\circ \neq \emptyset$. Then*

$$\dim(x^G \cap H^\circ) \geqslant \Big(\frac{1}{t} - \frac{1}{n} \pm \xi\Big) \dim x^G \tag{9}$$

where $\xi = 0$ if $G = \mathrm{GL}_n$, otherwise

$$\xi = \frac{n - a_0}{2\dim x^G}\Big(1 - \frac{1}{t} + \frac{1}{n}\Big)$$

where $+$ occurs if $G = \mathrm{Sp}_n$ and $-$ for $G = \mathrm{O}_n$.

Proof. If $t = n$ then the result trivially follows. Therefore we assume $t < n$, and so $n \geqslant 4$.

For convenience we assume $G = \mathrm{GL}_n$, the proof is similar in the other cases. Notice that for $G = \mathrm{GL}_n$ the following argument holds in the case $r = 2$ as well. Write $x = [x_1,\dots,x_t] \in H^\circ$, such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Up to conjugation, x_i is as in (7). Using [7, Theorem 2.5], we see that (9) is equivalent to

$$\frac{n^2}{t} - \sum_{i=1}^{t}\sum_{j=0}^{r-1} a_{i,j}^2 \geqslant \Big(\frac{1}{t} - \frac{1}{n}\Big)\Big(n^2 - \sum_{j=0}^{r-1} a_j^2\Big). \tag{10}$$

Since $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, Lemma 2.12 yields $a_{i,j} \in \{\lfloor a_j/t \rfloor, \lfloor a_j/t \rfloor + 1\}$. Write $a_j = q_j t + r_j$ where $0 \leqslant r_j < t$. Then, for all $j \in \{0,\dots,r-1\}$,

$$\sum_{i=1}^{t} a_{i,j}^2 = (q_j+1)^2 r_j + q_j^2(t - r_j) = q_j^2 t + 2q_j r_j + r_j.$$

Using $n = \sum_i a_i$ and $a_j = q_j t + r_j$, we have

$$n + \Big(\frac{1}{t} - \frac{1}{n}\Big)\sum_{j=0}^{r-1} a_j^2 - \sum_{i=1}^{t}\sum_{j=0}^{r-1} a_{i,j}^2 = \sum_{j=0}^{r-1}\Big(a_j\Big(1 - \frac{a_j}{n}\Big) - r_j\Big(1 - \frac{r_j}{t}\Big)\Big).$$

Therefore (10) is equivalent to

$$\sum_{j=0}^{r-1}\Big(a_j\Big(1 - \frac{a_j}{n}\Big) - r_j\Big(1 - \frac{r_j}{t}\Big)\Big) \geqslant 0.$$

We claim that, for every $j \in \{0,\dots,r-1\}$, each summand is non-negative. So, let us fix j and write $a_j = a, r_j = b$. Thus, we have $a = qt + b$, where $0 \leqslant b < t$. We claim

$$a\Big(1 - \frac{a}{n}\Big) - b\Big(1 - \frac{b}{t}\Big) > 0. \tag{11}$$

If $a < t$ we have $a = b$ and (11) is clearly true since $n > t$.

Assume $a \geqslant t$. Then $a = qt + b$ with $q \geqslant 1$. Thus (11) is equivalent to

$$g(b) := \frac{n(qt^2 + b^2) - t(qt+b)^2}{nt} > 0. \tag{12}$$

We see that $g(b)$ is minimal in $b = qt^2/(n-t)$. Therefore we distinguish two cases. If $qt^2/(n-t) < t$, then

$$g(b) = \frac{n(qt^2 + s^2) - t(qt+b)^2}{nt} \geqslant g\Big(\frac{qt^2}{n-t}\Big) = \frac{nqt^2(n - (q+1)t)}{nt(n-t)} > 0,$$

since $qt + t < n$. In particular, (12) is satisfied.

Now assume $qt^2/(n-t) \geqslant t$. Then the left hand side of (12) is minimal when $b = t-1$. Therefore, since $g(b) \geqslant g(t-1)$, it is sufficient to prove the inequality for $b = t-1$, that is

$$g(t-1) = a\Big(1 - \frac{a}{n}\Big) - \Big(1 - \frac{1}{t}\Big) \geqslant 0 \tag{13}$$

which is true if, and only if,

$$\Big(1 - \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2} \leqslant a \leqslant \Big(1 + \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2}.$$

It is easy to check that, for $n \geqslant 4$,

$$\Big(1 - \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2} \leqslant t, \quad \text{and} \quad n - 1 \leqslant \Big(1 + \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2}.$$

These two inequalities, together with the fact that $t \leqslant a \leqslant n-1$, lead to the conclusion that (13) is verified. The result follows. $\square$

Combining Propositions 2.2 and 2.14 we have the following two corollaries.

Corollary 2.15. *Let $x \in G$ be of order $r \neq 2, p$. Assume $x^G \cap H^\circ \neq \emptyset$. Then*

$$0 \leqslant \frac{\dim(x^G \cap H) - \dim(x^G \cap H^\circ)}{\dim x^G} \leqslant \frac{1}{n} + \frac{n}{2\dim x^G}\delta_{G,\mathrm{O}_n}.$$

Proof. For $x \in H$ we have, in general, $\dim(x^G \cap H) \geqslant \dim(x^G \cap H^\circ)$. Moreover, by Propositions 2.2 and 2.14 we have

$$\dim(x^G \cap H) - \dim(x^G \cap H^\circ) \leqslant \Big(\zeta \mp \xi + \frac{1}{n}\Big)\dim x^G$$

where $-$ occurs if $G = \mathrm{Sp}_n$ and $+$ if $G = \mathrm{O}_n$. The result easily follows. $\square$

Recall the definition of $f^\circ_\Omega(x)$ from (1).

Corollary 2.16. *Let $x \in G$ be of order $r \neq 2, p$. Assume $x^G \cap H^\circ \neq \emptyset$. Then*

$$f^\circ_\Omega(x) \leqslant f_\Omega(x) < f^\circ_\Omega(x) + \frac{2t}{n(t-1)}.$$

In particular, $f_\Omega(x) - f^\circ_\Omega(x) \leqslant 4/n$.

Proof. Using Corollary 2.15, $\dim x^G < n^2$ (for $G \neq \mathrm{O}_n$) and $\dim x^G < n^2/2$ (if $G = \mathrm{O}_n$), the result quickly follows. $\square$

2.2. **Conjugacy classes in $H \setminus H^\circ$.** In this section we study H°-classes of prime order elements in $H \setminus H^\circ$. Let $x \in H$ be of prime order $r \leqslant t$. We denote $\pi_i \in S_t$ any permutation with cycle shape $(r^i, 1^{t-ir})$. We give conditions on x in order to have $x^G \cap H^\circ\pi_i \neq \emptyset$, see Proposition 2.22, and compute an explicit formula of $\dim(x^G \cap H^\circ\pi_i)$, see (15).

Let $x \in H$ be of prime order r. Then $x = [x_1, \dots, x_t]\pi$ where $[x_1, \dots, x_t] \in H^\circ$ and $\pi \in S_t$. In addition π has order r. Since π acts on the block decomposition $[x_1, \dots, x_t]$ by conjugation permuting the blocks we have that $\pi \in G$ is a block matrix.

Recall that for $x \in H$ we have $\dim(x^G \cap H) = \max_{\sigma^r=1}\{\dim(x^G \cap H^\circ\sigma)\}$. The following shows that we only need to consider laterals of H° arising from non-conjugate permutations.

Lemma 2.17. *Let $x \in H$ be of order r. Let $\tau, \sigma \in S_t$ be conjugate permutations of order r. Then*

$$\dim(x^G \cap H^\circ\tau) = \dim(x^G \cap H^\circ\sigma).$$

Proof. Say $V = V_1 \oplus \ldots \oplus V_t$ the direct sum decomposition stabilised by H.

Since τ and σ are conjugate in S_t they have same cycle shape, say $(r^h, 1^{t-hr})$, for some $h > 0$. And so

$$\tau = (1, \dots, r)(r+1, \dots, 2r)(2r+1, \dots, 3r)\cdots((h-1)r+1, \dots, hr),$$
$$\sigma = (j_1, \dots, j_r)(j_{r+1}, \dots, j_{2r})(j_{2r+1}, \dots, j_{3r})\cdots(j_{(h-1)r+1}, \dots, j_{hr}).$$

The set of points fixed by τ is $\{hr+1, \dots, t\}$, while, the set of points fixed by σ is given by $\{j_{hr+1}, \dots, j_t\}$.

Assume, seeking a contradiction, that $\dim(x^G \cap H^\circ\tau) > \dim(x^G \cap H^\circ\sigma)$. Then there exists $y \in x^G \cap H^\circ\tau$ such that $\dim(x^G \cap H^\circ\tau) = \dim y^{H^\circ}$. Hence $y = [A_1, \dots, A_t]\tau$. Let $y' = [A_{\sigma^{-1}(1)}, \dots, A_{\sigma^{-1}(t)}]\sigma \in H^\circ\sigma$. Then $y, y' \in x^G$. Moreover y' stabilises the decomposition $V_{\sigma^{-1}(1)} \oplus V_{\sigma^{-1}(2)} \oplus \ldots \oplus V_{\sigma^{-1}(t)}$ of V. Hence $\dim y^{H^\circ} = \dim(y')^{H^\circ}$. This is a contradiction since

$$\dim y^{H^\circ} = \dim(x^G \cap H^\circ\tau) > \dim(x^G \cap H^\circ\sigma) \geqslant \dim(y')^{H^\circ}.$$

The result follows. $\square$

There is no element of order r with cycle shape $(r^h, 1^{t-hr})$ in S_t whenever $h > \lfloor t/r \rfloor$. For $h \leqslant \lfloor t/r \rfloor$, say

$$\pi_h = \prod_{i=1}^{h}\Big((i-1)r+1, \dots, ir\Big). \tag{14}$$

Thus $\pi_h \in S_t$ has order r and cycle shape $(r^h, 1^{t-hr})$. Then, Lemma 2.17 implies that

$$\dim(x^G \cap H) = \max_{h \leqslant \lfloor t/r \rfloor}\{\dim(x^G \cap H^\circ\pi_h)\}.$$

The following is showed in the proof of [6, Lemma 4.5].

Lemma 2.18. *Let $x=[x_1,\dots,x_t]\tau$ be of order r in $H\setminus H^\circ$. Say $(r^h,1^{t-hr})$ the cycle shape of τ. Then x is H°-conjugate to $[I_{n/t},\dots,I_{n/t},x_{hr+1},\dots,x_t]\tau$.*

Lemma 2.18 allow us to compute $\dim(x^G\cap H^\circ\pi_h)$ for any $h\leqslant\lfloor t/r\rfloor$.

Proposition 2.19. *Let $x\in G$ be of order r. Let $h\leqslant\lfloor t/r\rfloor$ and assume $x^G\cap H^\circ\pi_h\neq\emptyset$. Then*

$$\dim(x^G\cap H^\circ\pi_h)=h(r-1)\dim Cl_{n/t}+\sum_{i\geqslant hr+1}\dim B_i^{Cl_{n/t}}, \tag{15}$$

for suitable $B_i\in Cl_{n/t}$, $hr+1\leqslant i\leqslant t$.

Proof. Lemma 2.18 implies that, up to conjugation, $x=[I_{n/t},\dots,I_{n/t},x_{hr+1},\dots,x_t]\pi_h$, for some blocks $x_{hr+1},\dots,x_t\in Cl_{n/t}$. For suitable blocks $B_{hr+1},\dots,B_t$, we have

$$\dim(x^G\cap H^\circ\pi_h)=\dim H^\circ-\dim C_{H^\circ}([I_{n/t},\dots,I_{n/t},B_{hr+1},\dots,B_t]\pi_h).$$

Let $z=[A_1,\dots,A_t]\in H^\circ$. It is straightforward to check that $z\in C_{H^\circ}(\pi_h)$ if, and only if, $A_{(i-1)r+1}=\dots=A_{ir}$ for all $1\leqslant i\leqslant h$. Therefore $C_{H^\circ}(\pi_h)\cong(Cl_{n/t})^h$. The required formula easily follows. $\square$

Notice that the blocks B_i's in the statement of Proposition 2.19 arise by maximising the $(Cl_{n/t})^{t-hr}$-class dimension of $[x_{hr+1},\dots,x_t]$. In the case x is semisimple we can appeal to Proposition 2.13 in order to find the disposition of eigenvalues that gives rise to this block decomposition. In the case x is unipotent we do not have a general rule; however, we can follow the procedure outlined in the beginning of Section 2.1.1 to construct all the possible block decompositions.

The aim for the remainder of the section is to understand the Jordan form of any element $x\in H^\circ\pi_h$, in order to compute $\dim x^G$. We first find the Jordan form of permutations π_h, with $h\leqslant\lfloor t/r\rfloor$. Then we extend this result to any element $x\in H^\circ\pi_h$.

Lemma 2.20. *Let $\pi_h\in S_t$ be of order r. Say $f=t-hp$ the number of points fixed by π_h.*

(i) *Assume $r=p$. Then π_h has Jordan form $[J_p^{nh/t},J_1^{nf/t}]$.*
(ii) *Assume $r\neq p$. Then π_h has Jordan form $[I_{n(h+f)/t},\omega I_{nh/t},\dots,\omega^{r-1}I_{nh/t}]$.*

Proof. We may assume π_h is as in (14). Hence, in a suitable basis, $\pi_h=[\tau_1,\dots,\tau_h,I_{nf/t}]$, where, for all i

$$\tau_i=\begin{pmatrix}0 & I_{n/t} & \dots & 0\\ \vdots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & I_{n/t}\\ I_{n/t} & \dots & \dots & 0\end{pmatrix}\in Cl_{rn/t}.$$

It is clear that in a suitable basis $\tau_i=[g,\dots,g]\in \mathrm{GL}_{nr/t}$, where

$$g=\begin{pmatrix}0 & 1 & \dots & 0\\ \vdots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & 1\\ 1 & \dots & \dots & 0\end{pmatrix}\in \mathrm{GL}_r.$$

The characteristic polynomial of g is $cp_g(\lambda)=\pm(1-\lambda^r)$ (where $+$ occurs only if $r=2$). Hence, if $r=p$, g is GL_p-conjugate to J_p. If $r\neq p$ then g is GL_r-conjugate to $[1,\omega,\dots,\omega^{r-1}]$. The result follows. $\square$

Remark 2.21. In the case $G=\mathrm{Sp}_n$ or O_n and $p=2$ then n/t is even. Thus π_h has Jordan form $[J_2^{hn/t},J_1^{n-2hn/t}]$. In particular, π_h is $a_{hn/t}$-type, since it permutes orthogonal subspaces.

The following gives the Jordan form of a prime order element $x\in H^\circ\pi_h$.

Proposition 2.22. *Let $x\in H$ be of order r. Then $x^G\cap H^\circ\pi_h\neq\emptyset$ if, and only if, one of the following two conditions holds.*

(i) *Assume $r\neq p$. Then, for some $b_0,\dots,b_{r-1}$,*

$$x\in\left[I_{nh/t+b_0},\omega I_{nh/t+b_1},\dots,\omega^{r-1}I_{nh/t+b_{r-1}}\right]^{\mathrm{GL}_n}.$$

(ii) *Assume $r = p$. Then, for some $b_1,\dots,b_p$,*
$$x \in \left[J_p^{nh/t+b_p}, J_{p-1}^{b_{p-1}},\dots,J_1^{b_1}\right]^{\mathrm{GL}_n}.$$

Proof. This follows directly from Lemmas 2.18 and 2.20. □

We conclude this section discussing some consequences of Proposition 2.22.

Corollary 2.23. *Let $x \in G$ be of order r.*

(i) *Assume $r \neq p$ and $x \in [I_{a_0},\dots,\omega^{r-1}I_{a_{r-1}}]^G$. If there exists i such that $a_i < n/t$ then $x^G \cap H = x^G \cap H^\circ$.*
(ii) *Assume $r = p$ and $x \in [J_p^{a_p},\dots,J_1^{a_1}]^G$. If $a_p < n/t$ then $x^G \cap H = x^G \cap H^\circ$.*

In the case $p = \infty$ and $x \in G$ is unipotent, we already know, thanks to Remark 2.1, that $x^G \cap H = x^G \cap H^\circ$. Therefore, if $x^G \cap H \neq \emptyset$ then the largest size of a Jordan block allowed in x must be n/t.

Corollary 2.24. *Assume $p = \infty$. Let $x \in H$ be unipotent. Then, $x \in [J_{n/t}^{a_{n/t}},\dots,J_1^{a_1}]^G$, for some $a_{n/t},\dots,a_1$.*

3. Special cases

In this section $H \leqslant G$ is a stabiliser of a decomposition of V in 1-dimensional subspaces. Hence either $G = \mathrm{GL}_n$ and $H = \mathrm{GL}_1 \wr S_n$, or $G = \mathrm{O}_n$ and $H = \mathrm{O}_1 \wr S_n$; notice that $\mathrm{O}_1 \cong \mathbb{Z}/2\mathbb{Z}$.

Proposition 3.1. *Assume $G = \mathrm{GL}_n$ or O_n and $H = \mathrm{GL}_1 \wr S_n$ or $\mathrm{O}_1 \wr S_n$ be a $\mathscr{C}_2$-subgroup of G. Let $x \in \mathscr{R}$. Then $x^G \cap H \neq \emptyset$ if, and only if, x is as in Table 4. In addition, in the last column of the table we record $f_\Omega(x)$. The $(\star)$ in the table denotes the following condition: $\min_i\{a_i\} = h$.*

GL_n r	x	$\nu(x)$	$f_\Omega(x)$	O_n r	x	$\nu(x)$	$f_\Omega(x)$
2	$-$	s	$1-\frac{2s(n-s)-s}{n(n-1)}$	2	not a_s	s	$1-\frac{2s(n-s)}{n(n-1)}$
$=p\neq 2$	π_h^G	$h(p-1)$	$1-\frac{h(p-1)(2n-hp-1)}{n(n-1)}$	2	a_s	s	$1-\frac{2s(n-s-1)}{n(n-1)}$
$\neq 2,p$	$(\star)$	$-$	$1-\frac{n^2-\sum_i a_i^2-h(r-1)}{n(n-1)}$	$\neq 2,\leqslant n$	π_h	$h(p-1)$	$1-\frac{h(r-1)(2n-hr-1)}{n(n-1)}$

TABLE 4

Proof. This is an easy computation.

Assume $G = \mathrm{GL}_n$. If x is an involution then $x^G \cap H = \bigcup_{i\leqslant s}(x^G \cap H^\circ\pi_i)$ (if $p \neq 2$) and $\dim(x^G \cap H^\circ\pi_i) = i$; or $x^G \cap H = x^G \cap H^\circ\pi_s$ and $\dim(x^G \cap H) = s$, if $p = 2$. If $x \in G$ is unipotent then $x^G \cap H \neq \emptyset$ if, and only if, $x \in \pi_h^G$. In this case $\dim(x^G \cap H) = \dim \pi_h^{H^\circ} = h(p-1)$, thanks to (15). If x is semisimple then $x^G \cap H^\circ\pi_h \neq \emptyset$ if, and only if, $a_i \geqslant hn/t$ for all i, thanks to Proposition 2.22. And, whenever $x^G \cap H^\circ\pi_h \neq \emptyset$, we have $\dim(x^G \cap H^\circ\pi_h) = h(r-1)$. The result quickly follows.

If $G = \mathrm{O}_n$ then H is finite. Hence $f_\Omega(x) = 1 - \dim x^G/\dim G$. The result trivially follows for involutions. If x is not an involution then $x^G \cap H \neq \emptyset$ if, and only if, $x \in \pi_h^G$. Thanks to Lemma 2.20 the result follows with an easy computation. □

Remark 3.2. Assume $n \neq p$ is an odd prime. Let $x \in G$ be of order n. In the case $G = \mathrm{O}_n$, we have $x^G \cap H \neq \emptyset$ if, and only if, $x \in \pi_1^G$, i.e. $x = [1,\omega,\dots,\omega^{n-1}]$. Instead, $x^G \cap H \neq \emptyset$ for all x when $G = \mathrm{GL}_n$ since H contains a maximal torus. In both cases we compute $f_\Omega(\pi_1) = 1/n$. In addition, when $G = \mathrm{GL}_n$, we easily verify that if $x = [I_{a_0},\dots,\omega^{n-1}I_{a_{n-1}}]$ and $a_i = 0$ for some i then $f_\Omega(x) \neq 0$.

4. Global upper bounds

The main aim of this section is to prove Theorem 1(a). Subdividing the possibility for G and r, we derive the best possible upper bounds and we also characterise elements that realise such bounds. Hence we will also deduce Theorem 6 and Corollary 9.

In the following we use [7, Theorems 2.2, 2.4, 2.5] to compute $\dim x^G$. If $G = \mathrm{GL}_n$ we give a fairly amount of details that we shall omit for the other cases.

Proposition 4.1. *Assume $G = \mathrm{GL}_n$ and $H = \mathrm{GL}_{n/t} \wr S_t$, set $\Omega = G/H$. Let $x \in \mathscr{R}$.*

(i) *Assume $t \neq n$ if $r = 2$. Then*

$$f_\Omega(x) \leqslant 1 - \frac{2}{n},$$

with equality if, and only if, $\nu(x) = 1$ or one of the following holds

(a) $r = p$ *and* $(r,t,x^G) = (-,2,[J_2^2]^G)$ *or* $(3,3,[J_3]^G)$*;*

(b) $r \neq p$ *and* $(r,t,C_G(x)) = (3,4,\mathrm{GL}_3 \times (\mathrm{GL}_1)^2)$ *or* $(t,C_G(x)) = (2,(\mathrm{GL}_2)^2)$.

(ii) *Assume $r = 2$ and $t = n$. Then*

$$f_\Omega(x) \leqslant 1 - \frac{2}{n} + \frac{1}{n(n-1)},$$

with equality if, and only if, $\nu(x) = 1$.

In particular, the conclusion of Theorem 1(a) holds.

Proof. Let $x \in G$ with $\nu(x) = 1$. Then, up to conjugation, $x = [J_2, J_1^{n-2}]$ if $r = p$, or $x = [I_{n-1}, \omega]$ if $r \neq p$. Notice that in the case $r \neq p$ we consider elements up to centraliser structure. In the case $(r,t) \neq (2,n)$, Corollary 2.23 implies $x^G \cap H = x^G \cap H^\circ$. Up to conjugation, the unique block decomposition is given by $x = [x_1, \dots, x_t] \in H^\circ$ where $x_i = I_{n/t}$ for all $i > 1$, while $x_1 = [J_2, J_1^{n/t-2}]$ in the case $r = p$, or $[I_{n/t-1}, \omega]$ for $r \neq p$. In both cases the result follows with an easy computation. If $(r,t) = (2,n)$ we have $x^G \cap H^\circ \pi_1 \neq \emptyset$, and thanks to Proposition 2.19 we can easily compute this dimension. If $r \neq p$ then $x^G \cap H = (x^G \cap H^\circ) \cup (x^G \cap H^\circ \pi_1)$ and $\dim(x^G \cap H^\circ) = 0, \dim(x^G \cap H^\circ \pi_1) = 1$. If $r = p$ then $x^G \cap H^\circ = \emptyset$ and $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_1) = 1$. The result easily follows.

Now assume $\nu(x) = 2$. If $r = p$ then, up to conjugation, $x = [J_3, J_1^{n-3}]$ or $[J_2^2, J_1^{n-4}]$. If $r \neq p$ then, up to the centraliser structure, $x = [I_{n-2}, \omega I_2]$ or $[I_{n-2}, \omega, \omega^2]$. Assume $x = [J_3, J_1^{n-3}]$; unless $(r,t) = (3,t)$, Corollary 2.23 implies $x^G \cap H = x^G \cap H^\circ$, so $f_\Omega([J_3, J_1^{n-3}]) = 1 - 4/n$. If $x = [J_3, J_1^{n-3}]$ and $(p,t) = (3,3)$ then $x^G \cap H^\circ = \emptyset$ and $x^G \cap H^\circ \pi_1 \neq \emptyset$, by Lemma 2.20. Hence, using the formula in Proposition 2.19, we compute $\dim(x^G \cap H^\circ \pi_1) = 2$. Therefore $f_\Omega([J_3, J_1^{n-3}]) = 1 - 4/n + 4/(n^2 - n)$. It is straightforward to check that $f_\Omega([J_3, J_1^{n-3}]) \leqslant 1 - 2/n$ with equality if, and only if, $n = 3$.

Now assume $x = [J_2^2, J_1^{n-4}]$. Again, thanks to Corollary 2.23 we have $x^G \cap H = x^G \cap H^\circ$ whenever $(r,t) \neq (2,n), (2,n/2)$. In this case there are two possible block decomposition: $[x_1, \dots, x_t]$ and $[x'_1, \dots, x'_t]$, where $x_1 = [J_2^2, J_1^{n/t-4}], x'_1 = x'_2 = [J_2, J_1^{n/t-2}]$ and $x_i = x'_j = I_{n/t}$ for $i > 1$ and $j > 2$. For $(r,t) \neq (2,n), (2,n/2)$, a direct computation shows that $\dim(x^G \cap H^\circ) = 4n/t - 4$. Therefore

$$f_\Omega([J_2^2, J_1^{n-4}]) = 1 - \frac{4}{n} + \frac{4t}{n^2(t-1)}.$$

We check that $f_\Omega([J_2^2, J_1^{n-4}]) \leqslant 1 - 2/n$ with equality if, and only if, $(n,t) = (4,2)$ (if $(n,t) = (3,3)$ then $x^G \cap H^\circ = \emptyset$). In the case $(r,t) = (2,n)$ then $\pi_1 \in x^G$ by Lemma 2.20 and $x^G \cap H = x^G \cap H^\circ \pi_1$. Thus $\dim(x^G \cap H) = \dim \pi_1^{H^\circ} = 2$. Therefore $f_\Omega(x) = 1 - 4/n + 6/(n^2 - n)$. Again, it is easy to check that $f_\Omega(x) < 1 - 2/n + 1/(n^2 - n)$. If $(r,t) = (2,n/2)$ then $x^G \cap H = (x^G \cap H^\circ) \cup (x^G \cap H^\circ \pi_1)$ and a direct computation shows $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_1) = 4$; thus $f_\Omega(x) = 1 - 4/n + 4/(n^2 - 2n) \leqslant 1 - 2/n$, with equality if, and only if, $n = 4$.

The case $r \neq p$ and $\nu(x) = 2$ is very similar and it is left to the reader.

Now assume $\nu(x) > 2$. In particular, this implies $n \geqslant 4$. Now, [7, Proposition 2.6] implies $\dim x^G > 2n$. Hence, using Proposition 2.2, we have

$$f_\Omega(x) \leqslant 1 - \frac{\dim x^G}{n^2} < 1 - \frac{2}{n}.$$

This completes the proof. □

Proposition 4.2. *Assume $G = \mathrm{Sp}_n$ and $H = \mathrm{Sp}_{n/t} \wr S_t$, set $\Omega = G/H$. Let $x \in \mathscr{R}$. Define $\iota = 1$ if $r = p$ and $\iota = 2$ otherwise.*

(i) *Assume $p = 2$ if $(t,r) = (n/2,2)$. Then either*

$$f_\Omega(x) \leqslant 1 - \frac{2\iota}{n}$$

or $f_\Omega(x) \leqslant U$ as listed in Table 5; here equality holds if, and only if, x^G (if $r = p$) or $C_G(x)$ (if $r \neq p$) is as in the last column.

n	t	r	U	$x^G, C_G(x)$
4	2	$=p=2$	$3/4$	a_2^G
4	2	$\neq p, \neq 2$	$1/2$	GL_2
6	3	$\neq p$	$1/2$	GL_3
8	2	$=2\neq p$	$5/8$	$\mathrm{Sp}_4 \times \mathrm{Sp}_4$

TABLE 5

Furthermore $f_\Omega(x) = 1-2\iota/n$ if, and only if, $\nu(x) = \iota$ or $(n, x^G) = (4, [J_2^2]^G)$ (for $p \neq 2$) or $(n, C_G(x)) = (8, \mathrm{GL}_4)$.

(ii) *Assume $(t,r) = (n/2, 2)$ and $p \neq 2$. Then*

$$f_\Omega(x) \leqslant 1 - \frac{4}{n} + \frac{6}{n(n-2)},$$

with equality if, and only if, $\nu(x) = 2$.

In particular, the conclusion of Theorem 1(a) holds.

Proof. If x is unipotent and $\nu(x) = 1$ then, up to conjugation, $x = [J_2, J_1^{n-2}]$ (if $p = 2$ then x is b_1-type). It is straightforward to compute $f_\Omega(x) = 1 - 2/n$.

Assume x has order $r \neq p$. Then $\nu(x) \geqslant 2$. If $\nu(x) = 2$ and $r \neq 2$ then $x = [I_{n-2}, \omega, \omega^{-1}]$ or $[\omega I_2, \omega^{-1} I_2]$. If $r = 2$ then $x = [I_2, -I_{n-2}]$; notice that $x^G \cap (H \setminus H^\circ) \neq \emptyset$ if, and only if, $n/t = 2$. It is easy to check that $f_\Omega(x) = 1 - 4/n$, if $(t,r) \neq (n/t, 2)$, and $f_\Omega(x) = 1 - 4/n + 6/n(n-2)$ otherwise.

If $n \leqslant 8$ and $\nu(x) > \iota$ then we can list all the possible elements of order r, up to G-conjugacy and up to centraliser structure. It is easy to check that the result holds with these assumptions.

Assume $n > 8$ and $\nu(x) > \iota$. The result follows using [7, Proposition 2.6] and Proposition 2.2. For instance, if $r \neq p$ is odd then $\dim x^G \geqslant 2n - 4$, whenever $\nu(x) \geqslant 2$. And

$$f_\Omega(x) \leqslant 1 - \frac{2\dim x^G}{n^2} + \frac{n - \sum_{i \text{ odd}} a_i}{n^2} \leqslant 1 - \frac{4(n-2)}{n^2} + \frac{n - \sum_{i \text{ odd}} a_i}{n^2} < 1 - \frac{2}{n}$$

where the last inequality follows using $n > 8 - \sum_{i \text{ odd}} a_i$. □

Proposition 4.3. *Assume $G = \mathrm{O}_n$ and $H = \mathrm{O}_{n/t} \wr S_t$, set $\Omega = G/H$. Assume moreover $t < n$ and $n > 4$. Let $x \in \mathscr{R}$.*

(i) *Assume $r \neq 2$. Then*

$$f_\Omega(x) \leqslant 1 - \frac{4}{n},$$

with equality if, and only if, $\nu(x) = 2$ or one of $x^G, C_G(x)$ is as in Table 6.

n	t	r	$x^G, C_G(c)$
6	2	p	$[J_3^2]^G$
6	3	$p=3$	$[J_3^2]^G$
6	2	$\neq p$	$\mathrm{O}_2 \times \mathrm{GL}_2$
6	3	$3\neq p$	$\mathrm{O}_2 \times \mathrm{GL}_2$
8	2,4	$\neq p$	GL_4

TABLE 6

(ii) *Assume $r = 2$. Then*

$$f_\Omega(x) \leqslant 1 - \frac{2}{n},$$

with equality if, and only if, $\nu(x) = 1$.

In particular, the conclusion of Theorem 1(a) holds.

Proof. First assume r is odd. Notice that $\nu(x) \geqslant 2$. In the case $\nu(x) = 2$ then, up to conjugation x is either $[J_2^2, J_1^{n-4}]$ or $[J_3, J_1^{n-3}]$ if $r = p$, or $[I_{n-2}, \omega, \omega^{-1}]$ if $r \neq p$. In all these cases we compute $f_\Omega(x) = 1 - 4/n$.

Now assume $\nu(x) > 2$. If $n \leqslant 10$ the result follows explicitly computing $f_\Omega(x)$ in all the cases. So we may assume $n > 10$. In the case x is unipotent we see that $\nu(x) = 3$ only if x has Jordan form $[J_2^2, J_1^{n-6}], [J_3, J_2, J_1^{n-5}]$ or $[J_4, J_1^{n-4}]$; and none of the G-classes of these element meets H. Hence we may assume $\nu(x) \geqslant 4$. If x is semisimple then $\nu(x) = 3$ only if $n \leqslant 6$; so, since we are assuming $n > 10$ we may also assume $\nu(x) \geqslant 4$. Now, [7, Proposition 2.6] implies $\dim x^G \geqslant 4(n-5)$, for $n > 10$. Therefore, using Proposition 2.2 we have

$$f_\Omega(x) \leqslant 1 - \frac{2\dim x^G}{n^2} \leqslant 1 - \frac{8(n-5)}{n^2} < 1 - \frac{4}{n}.$$

Now assume $x \in H$ is an involution. If $p = 2$ and x is of type b_1 then we have $f_\Omega(x) = 1 - 2/n$. Similarly if $p \neq 2$ and $C_G(x) \cong \mathrm{O}_1 \times \mathrm{O}_{n-1}$. In the case $n = 6$, the only elements for which we need to compute $f_\Omega(x)$ are b_1, a_2, c_2, b_3-involutions if $p = 2$ and elements with centraliser $\mathrm{O}_2 \times \mathrm{O}_4$ or $(\mathrm{O}_3)^2$ if $p \neq 2$; the result quickly follows. Assume $\nu(x) \geqslant 2$. Then, [7, Proposition 2.6] yields $\dim x^G \geqslant 2(n-3)$. Hence, using Proposition 2.2, we have

$$f_\Omega(x) \leqslant 1 - \frac{2\dim x^G}{n^2} \leqslant 1 - \frac{4(n-3)}{n^2} < 1 - \frac{2}{n}.$$

This completes the proof. □

Lemma 4.4. *Let $G = \mathrm{O}_4$ and $H = \mathrm{O}_2 \wr S_2$ be a $\mathscr{C}_2$-subgroup of G, set $\Omega = G/H$. Let $x \in \mathscr{R}$.*

(i) *If $p \neq 2$ then $f_\Omega(x) \leqslant 1/2$. Moreover equality holds if, and only if, $C_G(x) \cong \mathrm{O}_1 \times \mathrm{O}_3, (\mathrm{O}_2)^2, \mathrm{GL}_2$.*
(ii) *If $p = 2$ then $f_\Omega(x) \leqslant 3/4$. Moreover equality holds if, and only if, x is a_2-type.*

In particular, the conclusion of Theorem 1(a) holds.

Proof. In this case we can list all the possible conjugacy classes. If $p \neq 2$, no G-class of unipotent element meets H. If $r \neq 2$ the centralisers of semisimple elements whose G-class meets H are $\mathrm{O}_2 \times \mathrm{GL}_1$, $(\mathrm{GL}_1)^2, \mathrm{GL}_2$; if $r = 2$ they are $\mathrm{O}_1 \times \mathrm{O}_3$, $(\mathrm{O}_2)^2$. In the case x is an involution and $p = 2$ then x is b_1, a_2 or c_2-type. The result follows with an easy calculation. □

The following is an immediate consequence of Proposition 3.1.

Lemma 4.5. *Let $G = \mathrm{O}_n$ and let $H = \mathrm{O}_1 \wr S_n$ be a $\mathscr{C}_2$-subgroup of G. Let $x \in \mathscr{R}$ be of order r.*

(i) *If $r = 2$ then*

$$f_\Omega(x) \leqslant 1 - \frac{2}{n},$$

with equality if, and only if, $\nu(x) = 1$.

(ii) *If $2 < r \leqslant n$ then*

$$f_\Omega(x) \leqslant 1 - \frac{(r-1)(2n-r-1)}{n(n-1)},$$

with equality if, and only if $x \in \pi_1^G$. Moreover $f_\Omega(x) = 1 - 2/n$ if, and only if, $n = r = 3$.

(iii) *If $r > n$ then $f_\Omega(x) = 0$.*

In particular, the conclusion of Theorem 1(a) holds.

Recall the definition of the algebraic fixity $M = M(G,H)$ and the r-local algebraic fixity M_r.

Proof of Theorem 6 and Corollary 9. In all the previous results we have derived upper bounds on $f_\Omega(x)$ for $x \in \mathscr{R}$. Thanks to [7, Lemma 2.8], these bounds extend to any element $x \in G \setminus Z(G)$. In the case $G = \mathrm{O}_4$, thanks to Lemmas 4.4 and 4.5, we have

$$M(\mathrm{O}_4, \mathrm{O}_1 \wr S_4) = M_2 = M_3 = 1/2,\ M(\mathrm{O}_4, \mathrm{O}_2 \wr S_2) = \begin{cases} M_r = 1/2 & p \neq 2,\ r \neq p; \\ M_2 = 3/4 & p = 2. \end{cases}$$

In all the other cases, Theorem 6 is a quick consequence of Propositions 4.1, 4.2, 4.3 and Lemma 4.5.

Finally, it is a straightforward computation to check that Corollary 9 holds. □

5. Global lower bounds, unipotent elements

In this section we derive lower bounds on $f_\Omega(x)$ when $x \in \mathscr{R}$ is unipotent and $x^G \cap H \neq \emptyset$; in particular, we show Theorem 1(b). It is clear that we may always assume $x \in H$.

Let a, b be integers. It is convenient to recall [7, (5)]:

$$\delta_{a;b} = \begin{cases} 1 & b \text{ divides } a \\ 0 & \text{otherwise.} \end{cases}$$

The main result of this sections is the following.

Proposition 5.1. *Let $x \in \mathscr{R} \cap H$ be unipotent.*

(i) *Assume $p > n/2$ and $t < n$. Then*

$$f_\Omega(x) \geqslant \frac{t}{n} + 2\left(\frac{t}{n}\right)^2 \delta_{G,\mathrm{O}_n} \delta_{n/t;2},$$

with equality if, and only if, one of the following holds

(a) *$G \neq \mathrm{O}_n$ if n/t is even and $z \in [J^t_{n/t}, z]^G$ for any $z \in Cl_{n/t}$ unipotent;*

(b) *$G = \mathrm{O}_n$, n/t even and $x \in [J^t_{n/t-1}, J^t_1]^G$.*

(ii) *Assume $p \leqslant n/2$, or $t = n$. Then*

$$f_\Omega(x) \geqslant \frac{1}{p} - \varepsilon,$$

where $\varepsilon = t/(n(t-1))$, if $(G, p) = (\mathrm{O}_n, 2)$ and $x \notin a_s^G$, and $\varepsilon = 0$ otherwise.

Remark 5.2. Assume p is odd. The bound in case (ii) is the best possible. In fact, it is sharp whenever $p \mid n$. We shall characterise elements that realise the bound in Proposition 5.14. In the case $p \nmid n$ we show in Proposition 5.15 that it is close to best possible.

The following clarifies the dichotomy in Proposition 5.1. Recall that if $x \in G$ has order p then, for some non-negative integers $a_p, \dots, a_1$, we have $x \in [J_p^{a_p}, \dots, J_1^{a_1}]^G$.

Lemma 5.3. *Assume $p > n/2$ and $t < n$. Let $x \in \mathscr{R}$ be unipotent. Assume $a_i > 0$ for some $i > n/t$. Then $x^G \cap H = \emptyset$.*

Proof. Recall that for $G = Cl_n$, $H = Cl_{n/t} \wr S_t$ and $H^\circ = (Cl_{n/t})^t$, where $t \geqslant 2$ is a non-trivial divisor of n. It is clear that if $x^G \cap H^\circ \neq \emptyset$ then $a_i = 0$ for all $i > n/t$. Hence, in our assumption $x^G \cap H^\circ = \emptyset$. Now, Proposition 2.22 implies that if $x^G \cap (H \setminus H^\circ) \neq \emptyset$ then $a_p = hn/t$ for some $h > 0$, namely $a_p \geqslant n/t$. Hence $n \geqslant a_p p \geqslant pn/t > n^2/2t$, which is absurd since this would imply $n/2 < t$. The result follows. □

The following sets the case $t = n$.

Lemma 5.4. *Assume $t = n$. Let $x \in \mathscr{R} \cap H$ be unipotent. Then $f_\Omega(x) \geqslant 1/p - \varepsilon$.*

Proof. The unipotent elements in $\mathrm{GL}_1 \wr S_n$ or $\mathrm{O}_1 \wr S_n$ are the elements in S_n of order p. In this case, Proposition 3.1 yields an explicit formula for $f_\Omega(x)$, from which we quickly deduce the required bound. □

The largest Jordan block that may appear in $x \in \mathscr{R}$ is J_p. Assume $x^G \cap H \neq \emptyset$. If $p > n/2$ then $p > t$ and Remark 2.1 implies $x^G \cap H^\circ \neq \emptyset$; hence the largest Jordan block that can appear in x is $J_{n/t}$.

Notation. Let $x = [J_p^{a_p}, \dots, J_1^{a_1}] \in G$. Assume $x^G \cap H^\circ \neq \emptyset$. Then, up to conjugation, we have $x = [x_1, \dots, x_t] \in H^\circ$. We set

$$x_i = [J_p^{a_{i,p}}, \dots, J_1^{a_{1,p}}], \tag{16}$$

so $\sum_j j a_{i,j} = n/t, a_j = \sum_i a_{i,j}$, for all $i \in \{1, \dots, t\}$ and $j \in \{1, \dots, p\}$. In particular, if $x^G \cap H^\circ \neq \emptyset$ we may assume $x = [x_1, \dots, x_t]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ} = \sum_i \dim x_i^{Cl_{n/t}}$.

For the remainder of the section we assume $t < n$.

5.1. **Case $p > n/2$.** By the previous discussion, if $x \in \mathscr{R}$ is unipotent, then $x^G \cap H = x^G \cap H^\circ$. Therefore, up to G-conjugacy, $x = [J_{n/t}^{a_{n/t}}, \ldots, J_1^{a_1}]$. We will make use of [7, Theorems 2.2, 2.4] to compute $\dim x^G$.

Lemma 5.5. *We have the following.*

(i) *Assume n/t is odd if $G = \mathrm{O}_n$. Let $x \in [J_{n/t}^{t-1}, z]^G$ where $z \in Cl_{n/t}$ is unipotent. Then $f_\Omega(x) = t/n$.*

(ii) *Assume $G = \mathrm{O}_n$ and n/t is even. Let $x \in [J_{n/t-1}^t, J_1^t]^G$. Then $f_\Omega(x) = t/n + 2(t/n)^2$.*

Proof. Write $x = [x_1, \ldots, x_t]$, where either $x_1 = \ldots = x_{t-1} = [J_{n/t}]$, $x_t = z$, or $x_1 = \ldots = x_t = [J_{n/t-1}, J_1]$. Then $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ and the result follows with an easy calculation. □

The following two lemmas are the key tools in order to show Proposition 5.1. We split the analysis into two cases. First we deal with the case $G \neq \mathrm{O}_n$ if n/t is even.

Lemma 5.6. *Assume $G \neq \mathrm{O}_n$ if n/t is even. Let $x = [x_1, \ldots, x_t] \in H^\circ$ be unipotent. Assume $x_1, x_2 \neq J_{n/t}$ and $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Let $y = [J_{n/t}, x_2, \ldots, x_t]$. Then $f_\Omega(x) > f_\Omega(y)$.*

Proof. Say $h \geqslant 0$ the number of blocks in x equal to $J_{n/t}$. Up to permutation of the blocks we may assume $x_1, x_2, \ldots, x_{t-h} \neq J_{n/t}$ and $x_{t-h+1} = \ldots = x_t = J_{n/t}$. We use the notation defined in (16) for the x_i's, with $i \leqslant t-h$. Thus $x = [J_{n/t}^h, J_{n/t-1}^{a_{n/t}-1}, \ldots, J_1^{a_1}]$ and, for $j < n/t$, we have $a_j = \sum_{i \leqslant t-h} a_{i,j}$. In particular, $\sum_{i<n/t} ia_i = (t-h)n/t$.

Let $y = [y_1, x_2, \ldots, x_t]$ where $y_1 = J_{n/t}$. In particular, in y, the block $J_{n/t}$ has multiplicity $h+1$, while J_i (with $i < n/t$) has multiplicity $a_i - a_{1,i}$. Say $b_i = a_i - a_{1,i}$, for $i < n/t$; then $\sum_i ib_i = (t-h-1)n/t$.

If $\dim(y^G \cap H) > \dim y^{H^\circ}$ then there exists a block decomposition $\bar{y} = [y_1, A_2, \ldots, A_{t-h}, x_{t-h+1}, \ldots, x_t]$ such that $\dim(y^G \cap H) = \dim(\bar{y})^{H^\circ} > \dim y^{H^\circ}$. Let $\bar{x} = [x_1, A_2, \ldots, A_{t-h}, x_{t-h+1}, \ldots, x_t] \in x^G$, then $\dim \bar{x}^{H^\circ} > \dim x^{H^\circ} = \dim(x^G \cap H)$, which is absurd. Therefore $\dim(y^G \cap H) = \dim y^{H^\circ}$.

For convenience, in the following we assume $G = \mathrm{GL}_n$. The other cases are very similar and left to the reader. We compute

$$\begin{aligned}
&\dim x^G = n^2 - 2\tfrac{n}{t}h(t-h) - \tfrac{n}{t}h^2 - 2\textstyle\sum_{i<j<n/t} ia_ia_j - \sum_{i<n/t} ia_i^2,\\
&\dim y^G = n^2 - 2(h+1)\textstyle\sum_{i<n/t} ib_i - 2\sum_{i<j<n/t} ib_ib_j - \tfrac{n}{t}(h+1)^2 - \sum_{i<n/t} ib_i^2,\\
&\dim(x^G \cap H) = h\dim[J_{n/t}]^{\mathrm{GL}_{n/t}} + \textstyle\sum_{i=1}^{t-h} \dim x_i^{\mathrm{GL}_{n/t}},\\
&\dim(y^G \cap H) = (h+1)\dim[J_{n/t}]^{\mathrm{GL}_{n/t}} + \textstyle\sum_{i=2}^{t-h} \dim x_i^{\mathrm{GL}_{n/t}}.
\end{aligned}$$

Notice $f_\Omega(x) > f_\Omega(y)$ if, and only if, $\dim x^G - \dim y^G < \dim(x^G \cap H) - \dim(y^G \cap H)$. Using $a_j = \sum_{i=1}^{t-h} a_{i,j}$ and $b_j = \sum_{i=1}^{t-h-1} a_{i,j}$, we see that the result is equivalent to

$$\begin{aligned}
(17)\qquad \sum_{i \leqslant j} ia_{1,i}(a_{2,j} + \ldots + a_{t-h,j}) + \sum_{i<j} ia_{1,j}(a_{2,i} + \ldots + a_{t-h,i}) &> \frac{n}{t}(t-h-1)\\
&= \sum_{i<n/t} i(a_{2,i} + \ldots + a_{t-h,i}).
\end{aligned}$$

To prove (17) we shall show that, for all $2 \leqslant l \leqslant t-h$, the coefficient of $a_{l,i}$ in the left hand side is strictly greater than the coefficient of $a_{l,i}$ in the right hand side, for all i.

Let $i \in \{1, \ldots, n/t-1\}$. Then for all $l \in \{2, \ldots, t-h\}$, the coefficient of $a_{l,i}$ in the left hand side is

$$\sum_{j \leqslant i} ja_{1,j} + \sum_{i<j} ia_{1,j},$$

and the coefficient of $a_{l,i}$ in the right hand side is i. We claim

$$\sum_{j \leqslant i} ja_{1,j} + \sum_{i<j} ia_{1,j} > i. \tag{18}$$

If there exists $j > i$ for which $a_{1,j} \neq 0$ then $ja_{1,j} > i$ hence (18) is satisfied. If for all $j > i$ we have $a_{1,j} = 0$ then $\sum_{j \leqslant i} ja_{1,j} = n/t > i$. Hence (18) is proved and the result follows. □

Now we deal with the case excluded in Lemma 5.6.

Lemma 5.7. *Assume $G = \mathrm{O}_n$ and n/t is even. Let $x = [x_1, \ldots, x_t] \in H^\circ$ be unipotent. Assume $x_1 \neq [J_{n/t-1}, J_1]$ and $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Define $y = [y_1, x_2, \ldots, y_t]$ where $y_1 = [J_{n/t-1}, J_1]$. Then $f_\Omega(x) > f_\Omega(y)$.*

Proof. The proof is very similar to that of Lemma 5.6. Since $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ then $\dim(y^G \cap H^\circ) = \dim y^{H^\circ}$. We need to show

$$\dim x^G - \dim y^G < \dim(x^G \cap H) - \dim(y^G \cap H). \tag{19}$$

Notice that $\dim(x^G \cap H) - \dim(y^G \cap H) = \dim x_1^{\mathrm{O}_{n/t}} - \dim[J_{n/t-1}, J_1]^{\mathrm{O}_{n/t}}$.

Say $h \geqslant 0$ the integer for which $x_1, \dots, x_{t-h} \neq [J_{n/t-1}, J_1]$ and $x_{t-h+1} = \dots = x_t = [J_{n/t-1}, J_1]$ (up to permuting the blocks). Notice that since $x_1 \neq [J_{n/t-1}, J_1]$ we have $0 \leqslant h \leqslant t-1$. For all $i \leqslant t-h$ we write

$$x_i = [J_{n/t-2}^{a_{i,n/t-2}}, \dots, J_1^{a_{i,1}}] \in \mathrm{O}_{n/t}.$$

So $x = [J_{n/t-1}^h, J_{n/t-2}^{a_{n/t-2}}, \dots, J_2^{a_2}, J_1^{a_1+h}]$ and $y = [J_{n/t-1}^{h+1}, J_{n/t-2}^{b_{n/t-2}}, \dots, J_2^{a_2}, J_1^{b_1+h+1}]$ where $a_j = \sum_i a_{i,j}$ and $b_j = a_j - a_{i,1}$, for all $1 \leqslant j \leqslant p$. We compute $\dim x^G - \dim y^G$ and $\dim(x^G \cap H) - \dim(y^G \cap H)$.

Eventually, using $n(t-h-1)/t = \sum_i i(a_{2,i} + \dots + a_{t-h,i})$, we deduce that (19) is equivalent to

$$\begin{aligned}\sum_{i \leqslant j} i a_{1,i}(a_{2,j} + \dots + a_{t-h,j}) + \sum_{i<j} i a_{1,j}(a_{2,i} + \dots + a_{t-h,i}) + h\Big(\sum_{i \geqslant 1} a_{1,i} - 2\Big) \\ > \sum_{i \geqslant 1}(i+1)(a_{2,i} + \dots + a_{t-h,i}).\end{aligned}$$

It is clear that $h\big(\sum_{i \geqslant 1} a_{1,i} - 2\big) \geqslant 0$ since $x_1 \neq [J_{n/t}]$ and so x_1 comprises at least two blocks. Therefore we need to show

$$\sum_{i \leqslant j} i a_{1,i}(a_{2,j} + \dots + a_{t-h,j}) + \sum_{i<j} i a_{1,j}(a_{2,i} + \dots + a_{t-h,i}) > \sum_i (i+1)(a_{2,i} + \dots + a_{t-h,i}). \tag{20}$$

This can be proved using the same argument of Lemma 5.6. Fix $j \in \{1, \dots, n/t-2\}$ and $l \in \{2, \dots, t-h\}$. As in the proof of Lemma 5.6, we claim that the coefficients of $a_{l,j}$ in (20) satisfy the following

$$\sum_{i \leqslant j} i a_{1,i} + j \sum_{j<i} a_{1,i} > j+1. \tag{21}$$

We split the proof of (21) into two cases. First we assume that for all $i > j$ we have $a_{1,i} = 0$; therefore (21) is equivalent to

$$\sum_{i \leqslant j} i a_{1,i} + j \sum_{j<i} a_{1,i} = \sum_{i \leqslant j} i a_{1,i} = \frac{n}{t} > j+1.$$

Now assume $a_{1,i} \neq 0$ for some $i > j$. Say $i_1, \dots, i_m > j$ the only indexes for which $a_{1,i_1}, \dots, a_{1,i_m}$ are non-zero. Then (21) is equivalent to

$$\sum_{i \leqslant j} i a_{1,i} + j(a_{1,i_1} + \dots + a_{1,i_m}) > j+1.$$

If $\sum_{i \leqslant j} i a_{1,i} = 0$ then $a_{1,i_1} + \dots + a_{1,i_m} \geqslant 2$ (as $x_1 \neq [J_{n/t-1}, J_1]$), thus $j(a_{1,i_1} + \dots + a_{1,i_m}) \geqslant 2j > j+1$. If $\sum_{i \leqslant j} i a_{1,i} \geqslant 1$ then one of the following two conditions hold

- $\sum_i i a_{1,i} = 1$ and $a_{1,i_1} + \dots + a_{1,i_m} \geqslant 2$ since $x_1 \neq [J_{n/t-1}, J_1]$
- $\sum_i i a_{1,i} \geqslant 2$ and $a_{1,i_1} + \dots + a_{1,i_m} \geqslant 1$.

In both cases inequality (21) is satisfied. □

Now we have all the tools needed to prove part (i) of Proposition 5.1(i).

Lemma 5.8. *The conclusions of Proposition 5.1(i) hold.*

Proof. Since $x^G \cap H = x^G \cap H^\circ$ we may assume $x \in H^\circ$ and $x = [x_1, \dots, x_t] \in H^\circ$ is such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. For convenience we only consider the case $G \neq \mathrm{O}_n$ if n/t is even.

In the case at least $t-1$ blocks of x are equal to $[J_{n/t}]$ then, by Lemma 5.5, $f_\Omega(x) = t/n$. So we may assume that at least two blocks x_i, x_j are different from $[J_{n/t}]$, and clearly we may as well assume $(i,j) = (1,2)$. Then, by Lemma 5.6, said $y = [J_{n/t}, x_2, \dots, x_t]$ we have $f_\Omega(x) > f_\Omega(y)$. Again, in y either $t-1$ blocks are $[J_{n/t}]$, in which case $f_\Omega(y) = t/n$ and the result follows; or there are two blocks different from $[J_{n/t}]$. Iterating the previous construction, in at most $t-1$ steps we have $f_\Omega(x) > f_\Omega(y) > \dots > f_\Omega([J_{n/t}, x_t]) = t/n$.

In the case $G = \mathrm{O}_n$ and n/t is even, we use Lemma 5.7 and, with a similar argument, we reach the desired conclusion. □

5.2. **Case $p \leqslant n/2$.** Let $x \in \mathscr{R}$ be unipotent. The largest size of a Jordan block that can appear in x is p. First we make the following observation for the case $p=2$ and $G \neq \mathrm{GL}_n$.

Remark 5.9. Assume $p=2$ and let $x \in H$ be an involution. Then x has Jordan form $[J_2^{m_2}, J_1^{m_1}]$ where $\nu(x) = s = m_2$.

(i) Assume x is a_s-type. Then

$$\begin{aligned} \dim x^G &= \tfrac{n}{2}(n-1) - \textstyle\sum_{i<j} i m_i m_j - \tfrac{1}{2}\sum_i i m_i^2 - \tfrac{1}{2}\sum_{i \text{ odd}} m_i - m_2 && \text{if } G = \mathrm{Sp}_n,\\ \dim x^G &= \tfrac{n}{2}(n+1) - \textstyle\sum_{i<j} i m_i m_j - \tfrac{1}{2}\sum_i i m_i^2 + \tfrac{1}{2}\sum_{i \text{ odd}} m_i && \text{if } G = \mathrm{O}_n. \end{aligned}$$

Notice that if $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ with $x = [x_1, \dots, x_t] \in x^G \cap H^\circ$ then each x_i is a-type.

(ii) Assume x is b_s or c_s-type. Then $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$ for some $h \leqslant \lfloor t/2 \rfloor$. And Lemma 2.18 implies that

$$\dim(x^G \cap H) = \dim([I_{n/t}, \dots, I_{n/t}, \dots, x_{2h+1}, \dots, x_t]\pi_h)^{H^\circ}.$$

Since $\dim b_s^G, \dim c_s^G > \dim a_s^G$ we must have that each x_i is either b- or c-type. We have

$$\begin{aligned} \dim x^G &= \tfrac{n}{2}(n-1) - \textstyle\sum_{i<j} i m_i m_j - \tfrac{1}{2}\sum_i i m_i^2 - \tfrac{1}{2}\sum_{i \text{ odd}} m_i && \text{if } G = \mathrm{Sp}_n,\\ \dim x^G &= \tfrac{n}{2}(n+1) - \textstyle\sum_{i<j} i m_i m_j - \tfrac{1}{2}\sum_i i m_i^2 + \tfrac{1}{2}\sum_{i \text{ odd}} m_i + (n - m_1 - m_2) && \text{if } G = \mathrm{O}_n. \end{aligned}$$

The previous observation allows us to give a unique argument for any p.

Lemma 5.10. *The conclusion of Proposition 5.1(ii) holds.*

Proof. Let $x \in H$ be od order p. Then, for some $0 \leqslant h \leqslant \lfloor t/p \rfloor$ we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$. By Lemma 2.18 and Proposition 2.19, for suitable blocks $x_{hp+1}, \dots, x_t$ we have

$$\begin{aligned} \dim(x^G \cap H^\circ \pi_h) &= \dim([I_{n/t}, \dots, I_{n/t}, x_{hp+1}, \dots, x_t]\pi_h)^{H^\circ} \\ &= h(p-1)\dim Cl_{n/t} + \sum_{i=hp+1}^{t} \dim x_i^{Cl_{n/t}}. \end{aligned} \tag{22}$$

For $hp < i \leqslant t$ we write $x_i = [J_p^{a_{i,p}}, \dots, J_1^{a_{1,p}}]$. We denote $l = hp+1$. Proposition 2.22 implies that, up to G-conjugacy, we have

$$x = \left[J_p^{hn/t + \sum_{i \geqslant l} a_{i,p}}, J_{p-1}^{\sum_{i \geqslant l} a_{i,p-1}}, \dots, J_1^{\sum_{i \geqslant l} a_{i,1}}\right].$$

Notice that $\sum_{i \leqslant p} i(a_{l,i} + \ldots + a_{t,i}) = n(1 - hp/t)$. Then $f_\Omega(x) \geqslant 1/p$ if, and only if,

$$\dim x^G - \dim(x^G \cap H) \leqslant \dim \Omega \left(1 - \frac{1}{p}\right). \tag{23}$$

Assume $p \neq 2$ if $G \neq \mathrm{GL}_n$; we simplify (23) using the formulae in [7, Theorem 2.2] and (22).

In the case $h = t/p$ or $(t-1)/p$ then we quickly deduce $f_\Omega(x) = 1/p$. So we assume $hp < t-1$. After few steps, we see that (23) is equivalent to the following

$$\sum_{l \leqslant \alpha < \beta \leqslant t} \left(\sum_{1 \leqslant i < j \leqslant p} i(a_{\alpha,i} a_{\beta,j} + a_{\alpha,j} a_{\beta,i}) + \sum_{i=1}^{p} i a_{\alpha,i} a_{\beta,i} \right) \geqslant \frac{n^2}{2pt^2}(t-hp)(t-hp-1). \tag{24}$$

Say $\mathscr{I} = \{(\alpha, \beta) : l \leqslant \alpha < \beta \leqslant t\}$. Then

$$|\mathscr{I}| = \frac{(t-hp)(t-hp-1)}{2}.$$

We fix a couple $(\alpha, \beta) \in \mathscr{I}$, and we claim that each summand of the external summation is greater than or equal to $(n/t)^2/p$. We set $a_{\alpha,i} = a_i$ and $a_{\beta,i} = b_i$. Thus we need to show

$$\sum_{1 \leqslant i < j \leqslant t} i(a_i b_j + a_j b_i) + \sum_{i=1}^{p} i a_i b_i \geqslant \frac{n^2}{pt^2} = \frac{n}{pt} \sum_{i=1}^{p} i a_i. \tag{25}$$

In order to show (25) we use the same method employed in the proof of Lemma 5.6. We fix $i \in \{1, \dots, p\}$ and we show that the coefficient of a_i in the left hand side is greater than or equal to the coefficient of a_i in the right hand side. Thus, for $i \in \{1, \dots, p\}$, we need to show

$$i \sum_{j \geqslant i} b_j + \sum_{j < i} j b_j \geqslant i \frac{n}{pt}. \tag{26}$$

We have two cases. If $b_j = 0$, for all $j \geqslant i$, then $\sum_{j<i} jb_j = n/t \geqslant in/tp$, since $i \leqslant p$. Now assume that there exists $j \geqslant i$ such that $b_j \neq 0$. Then we claim

$$i\sum_{j\geqslant i} b_j + \sum_{j<i} jb_j \geqslant i\frac{n}{pt} = \frac{i}{p}\sum_{j=1}^{p} jb_j,$$

which is satisfied because

$$i\sum_{j\geqslant i} b_j + \sum_{j<i} jb_j - \frac{i}{p}\sum_{j=1}^{p} jb_j = i\sum_{j\geqslant i} b_j\Big(1-\frac{j}{p}\Big) + \sum_{j<i} jb_j\Big(1-\frac{i}{p}\Big) \geqslant 0,$$

being $i, j \leqslant p$. Hence, for all $(\alpha,\beta) \in \mathscr{I}$, (25) is proved.

It is left to show (23) for the case $G \neq \mathrm{GL}_n$ and $p = 2$. Thanks to Remark 5.9 the previous argument is still valid if $G = \mathrm{Sp}_n$ and x is not a_s-type, or if $G = \mathrm{O}_n$ and x is a_s-type.

Assume $G = \mathrm{Sp}_n$ and x is a_s-type. Using the formulae in Remark 5.9, we see that (24) is equivalent to

$$\sum_{l\leqslant\alpha<\beta\leqslant t}\Big(\sum_{1\leqslant i<j\leqslant p} i(m_{\alpha,i}m_{\beta,j}+m_{\alpha,j}m_{\beta,i}) + \sum_{i=1}^{p} im_{\alpha,i}m_{\beta,i}\Big) + \frac{n}{t}h \geqslant \frac{n^2}{2pt^2}(t-hp)(t-hp-1),$$

which is clearly true by the same argument as above.

Assume $G = \mathrm{O}_n$ and x is not a_s-type. We claim $f_\Omega(x) \geqslant 1/p - t/(n(t-1))$, which is equivalent to

$$\sum_{l\leqslant\alpha<\beta\leqslant t}\Big(\sum_{1\leqslant i<j\leqslant p} i(m_{\alpha,i}m_{\beta,j}+m_{\alpha,j}m_{\beta,i}) + \sum_{i=1}^{p} im_{\alpha,i}m_{\beta,i}\Big) \geqslant \frac{n^2}{2pt^2}(t-hp)(t-hp-1) + \frac{n}{t}\Big(h-\frac{t}{2}\Big).$$

The last summand is less than or equal to zero. Again, the previous argument yields the result. $\square$

The remainder of the section aims to characterise elements of order p (odd if $G \neq \mathrm{GL}_n$) that realise the lower bound of Proposition 5.1(ii). In the following we shall always assume $p < n$. The same observation of [7, Remark 5.4] implies that if $f_\Omega(x) = 1/p$ then in (25), and so in (26), equality holds.

Lemma 5.11. *Assume $p \mid n$. Let $x \in G$ be of order p. Then $f_\Omega(x) = 1/p$ if, and only if, one of the following occurs:*

(i) *$p \nmid n/t$ and $x \in [J_p^{n/p}]^G$;*
(ii) *$p \mid n/t$, $x = [J_p^{n(t-1)/pt}, z]^G$, with $z \in Cl_{n/t}$ of order p and $\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi_h)$ as in (22), where $x_t = z$.*

Proof. For x as in the statement, it is easy to compute $f_\Omega(x) = 1/p$.

Conversely, assume $x \in H$ and $f_\Omega(x) = 1/p$. Then, for some $h \leqslant \lfloor t/p \rfloor$

$$\begin{aligned}\dim(x^G \cap H) &= \dim(x^G \cap H^\circ\pi_h) = \dim\big([I_{n/t},\dots,I_{n/t},x_{hp+1},\dots,x_t]\big)^{H^\circ} \\ &= h(p-1)\dim Cl_{n/t} + \sum_{i=hp+1}^{t} \dim x_i^{Cl_{n/t}}.\end{aligned} \tag{27}$$

We have three cases depending on h.

Case 1. If $hp = t$ then x is H°-conjugate to π_h, by Lemma 2.18. So, Lemma 2.20 yields $x \in [J_p^{n/p}]^G$.

Case 2. If $hp = t-1$ then x is H°-conjugate to $[I_{n/t},\dots,I_{n/t},z]\pi_h$ for some $z \in Cl_{n/t}$ of order p. Again, Lemma 2.20 implies $x \in [J_p^{n(t-1)/tp}, z]^G$. And, by our assumption, $\dim(x^G \cap H) = \dim([I_{n/t},\dots,I_{n/t},z]\pi_h)^{H^\circ}$.

Case 3. Assume $hp < t-1$. Let $hp+1 \leqslant \alpha < \beta \leqslant t$, and say a_i, b_i the multiplicities of J_i in the blocks x_α, x_β, respectively. Then, by the proof of Lemma 5.10, for all $i \in \{1,\dots,p\}$, either $a_i = 0$ or we have (in the case $G = \mathrm{GL}_n$, but the same argument applies in the other cases as well)

$$i\sum_{j\geqslant i} b_j + \sum_{j<i} jb_j = \frac{in}{pt}. \tag{28}$$

Let i be such that $a_i \neq 0$. If $i = p$ then (28) is always satisfied. If $i < p$, then (28) is equivalent to

$$i\sum_{j\geqslant i} b_j\Big(1-\frac{j}{p}\Big) + \sum_{j<i} jb_j\Big(1-\frac{i}{p}\Big) = 0$$

which is satisfied if, and only if, $b_j = 0$ for all $j < p$. Therefore $n/t = pb_b$. This holds for all the couple (α,β). In particular, at most one block can be different from $[J_p^{n/pt}]$. Hence, using also Lemma 2.20, we deduce that $x \in [J_p^{n(t-1)/tp}, x_t]^G$ for some $x_t \in Cl_{n/t}$ for which (27) holds. The result follows. □

Assume $p \mid n/t$. From Lemma 5.11(ii), we see that, in order to have a characterisation of the elements $x \in H$ of order p for which $f_\Omega(x) = 1/p$, we need to characterise elements $[J_p^{n(t-1)/pt}, z]$, where $z \in Cl_{n/t}$ has order p, for which

$$\dim(x^G \cap H^\circ) = (t-1)\dim[J_p^{n/pt}]^{Cl_{n/t}} + \dim z^{Cl_{n/t}}.$$

For this purpose, recall that there is a one-ton-one correspondence between Cl_n-conjugacy classes of unipotent element of odd order $p \leqslant n/2$ and $\mathscr{P}_{Cl_n}$. The following result is the key tool to complete this task.

Lemma 5.12. *Assume $n/t = pm$. Let $x = [J_p^m, \dots, J_p^m, z] \in H^\circ$, where z has order p. Say $\lambda = (p^{a_p}, \dots, 1^{a_1})$ the partition of n/t associated to z. Assume*

(1) *$a_p < m$;*
(2) *$\lambda = \mu \oplus \eta$, where $\mu = (n^{b_n}, \dots, 1^{a_1}) \vdash l_1 p, \eta = (n^{c_n}, \dots, 1^{c_1}) \vdash l_2 p$ for some $0 < l_1, l_2 < m$, with $b_i, c_j > 0$ for some $i, j < p$. Assume moreover $\mu \in \mathscr{P}_{Cl_{l_1 p}}, \eta \in \mathscr{P}_{Cl_{l_2 p}}$.*

Then $\dim(x^G \cap H^\circ) > \dim x^{H^\circ}$.

Proof. We outline the main strategy and we leave the calculations to the reader, for more details see [8].

First we observe that $l_1 + l_2 = m$. We define $\lambda_1 = (p^{m-l_1}) \oplus \mu$ and $\lambda_2 = (p^{m-l_2}) \oplus \eta$. Then $\lambda_1, \lambda_2 \vdash n/t$. Say $y_1^{Cl_{n/t}}$ and $y_2^{Cl_{n/t}}$ the classes associated to these partitions. Set $y = [y_1, y_2, y_3, \dots, y_t]$ where $y_i = [J_p^m]$ for all $i > 2$. Then $y \in x^G$. At this point it is not hard to compute $\dim y^{H^\circ}, \dim x^{H^\circ}$ and to show that $\dim y^{H^\circ} > \dim x^{H^\circ}$, which completes the proof. □

Lemma 5.13. *Assume $n/t = pm$. Let $x = [J_p^{m(t-1)}, z]$, where $z \in Cl_{n/t}$ has order p. Say $\lambda = (p^{a_p}, \dots, 1^{a_1})$ the partition of n/t associated to z. Then, for some $0 \leqslant h \leqslant \lfloor t/p \rfloor$,*

$$\dim(x^G \cap H) = \dim\big([I_{n/t}, \dots, I_{n/t}, x_{hp+1}, \dots, x_{t-1}, z]\pi_h\big)^{H^\circ}, \quad x_i = [J_p^m], \text{ for all } i, \tag{29}$$

if, and only if, $a_p = m$ or the condition (2) in Lemma 5.12 does not hold for λ.

Proof. Assume (29) holds. Then the result follows, being the contrapositive of Lemma 5.12. Conversely, assume that either (1) or (2) in Lemma 5.12 does not hold. Then, up to a permutation of the blocks, the only block decomposition of $[J_p^{m(t-1-hp)}, z]$ in $t - hp$ blocks of size n/t is that given in (29). □

Gathering together the information given Lemmas 5.11 and 5.13 we achieve our aim to classify the elements of H that realise the lower bound of Proposition 5.1(ii).

Proposition 5.14. *Assume $p \mid n$, and $p \neq 2$ if $G \neq \mathrm{GL}_n$. Let $x \in G$ be of order p. Then $f_\Omega(x) = 1/p$ if, and only if, one of the following holds*

(i) *$x \in [J_p^{n/p}]^G$; or,*
(ii) *$n/t = pm$ and $x \in [J_p^{m(t-1)}, z]^G$ for some $z \in Cl_{n/t}$ of order p whose associated partition $\lambda = (p^{a_p}, \dots, 1^{a_1}) \vdash n/t$ satisfies one of the following*
   (1) *$a_p = m$; or,*
   (2) *$a_p < m$ and whenever $\lambda = \mu \oplus \eta$ with $\mu \vdash l_1 p, \eta \vdash l_2 p$, for some $l_1, l_2 < m$, and $\mu \in Cl_{l_1 p}, \eta \in Cl_{l_2 p}$ then either $\mu = (p^{l_1})$ or $\eta = (p^{l_2})$.*

If $p \nmid n$ then we can construct elements whose f_Ω-value is close to the lower bound $1/p$. Recall that if $p > t$ then $x^G \cap H = x^G \cap H^\circ$. Hence, Proposition 5.1(i) gives the best possible lower bound for the case $p > t, n/t$. Hence we may assume $p < t$ if $p > n/t$.

Proposition 5.15. *Assume $p \nmid n$, and p is odd if $G \neq \mathrm{GL}_n$. Assume moreover $p < t$ if $p > n/t$. Then there exists $x \in H^\circ$ such that*

$$f_\Omega(x) \leqslant \frac{1}{p} + \varepsilon,$$

where $\varepsilon = 1/4p + 2/p^2$ if $p < n/t$ and $\varepsilon = p/n$ if $p \geqslant n/t$.

Proof. We assume $G = \mathrm{Sp}_n$, the argument for the linear and orthogonal group is similar and left to the reader. If $p < n/t$, we write $n/t = a(2p) + b$ where $0 \leqslant b < 2p$ is even. If $b < p$ then we let $\bar{x} = [J_p^{2a}, J_b] \in \mathrm{Sp}_{n/t}$. Thus $[\bar{x}, \dots, \bar{x}] \in H^\circ$. We compute $\dim(x^G \cap H)$ using Lemma 2.4. Using $t/n < 1/p$, we have

$$f_\Omega(x) = \frac{1}{p} - \frac{bt^2}{pn^2}(p-b) \leqslant \frac{1}{p} + \frac{pt^2}{4n^2} < \frac{5}{4p}.$$

If $b > p$ we define $\bar{x} = [J_p^2, J_{p-1}, J_{p-b+1}] \in \mathrm{Sp}_{n/t}$. Thus $[\bar{x}, \dots, \bar{x}] \in H^\circ$. As above, we compute

$$f_\Omega(x) = \frac{1}{p} - \frac{t^2}{pn^2}(b^2 - 3bp + 2p^2 - 2p) \leqslant \frac{1}{p} + \frac{t^2}{4n^2}(p+8) < \frac{5}{4p} + \frac{2}{p^2}.$$

If $n/t < p < n$ then we define $h = \lfloor t/p \rfloor \geqslant 1$, since $p < t$. We consider $x = [J_p^{hn/t}, J_{n/t}^{t-hp}] \in H$. Notice that $x^G \cap H = x^G \cap H^\circ \pi_h$. We compute

$$f_\Omega(x) = \frac{t}{n} + \frac{h(n-pt)(2t-1-hp)}{nt(t-1)} < \frac{1}{p} + \frac{p}{n}.$$

□

6. Global lower bounds: semisimple elements

In this section we prove Theorems 1(c) and [7, Theorem 9]; in other words, we shall derive lower bounds on $f_\Omega(x)$ for $x \in H$ of prime order $r \neq p$. The main result of this section is the following.

Proposition 6.1. *Let $x \in H$ be of prime order $r \neq p$.*

(i) *If $G = \mathrm{GL}_n$, assume $r \geqslant n+1$. If $G \neq \mathrm{GL}_n$ assume $r \geqslant n - 1 + 2\delta_{t,n}$. Then either*
  (a) *$G = \mathrm{O}_n$, n/t is odd, and*

$$f_\Omega(x) \geqslant \frac{t^2}{n^2},$$

  with equality if, and only if, $C_G(x) \cong \mathrm{O}_t \times (\mathrm{GL}_1)^{(n-t)/2}$; or
  (b) *$G \neq \mathrm{O}_n$ if n/t is odd, and $f_\Omega(x) = 0$ if, and only if,*
    - *$\nu(x) = n-1$; or,*
    - *$(G, C_G(x)) = (\mathrm{Sp}_n, \mathrm{Sp}_2 \times (\mathrm{GL}_1)^{n/2-1})$ or $(\mathrm{O}_n, \mathrm{O}_2 \times (\mathrm{GL}_1)^{n/2-1})$.*

(ii) *Assume $t < n$. Assume $r < n+1$ if $G = \mathrm{GL}_n$, and $r < n-1$ otherwise. Then*

$$f_\Omega(x) \geqslant \frac{1}{r} - \varepsilon,$$

  where ε is recorded in Table 7.

(iii) *Let $t = n$. Assume $r \leqslant n$. Then*

$$f_\Omega(x) \geqslant \varepsilon',$$

  where ε' is recorded in Table 8.

In particular, the conclusions of Theorem 1(c) and [7, Theorem 9] *hold.*

G	Conditions	ε	G	Conditions	ε
GL_n	–	$\frac{rt^2}{4n^2(t-1)}$	O_n	$r \neq 2, n/t$ even	$\frac{rt^2}{4n^2(t-1)} - \frac{2}{n}$
Sp_n	$r \neq 2$	$\frac{rt^2}{4n^2(t-1)} - \frac{1}{n}$		$r \neq 2, n/t$ odd	$\frac{rt^2}{2n^2(t-1)}$
	$r = 2$	$\frac{2t^2}{n^2(t-1)}$		$r = 2$	$\frac{t^2}{2n^2(t-1)}$

TABLE 7

Unless n/t is odd and $G = \mathrm{O}_n$ then H° contains a maximal torus of G. Hence, if $x \in G$ is semisimple of order r then $x^G \cap H^\circ \neq \emptyset$. In addition, if $r > t$ or at least one eigenvalue of x has multiplicity strictly smaller than n/t then $x^G \cap H = x^G \cap H^\circ$. In both these cases $f_\Omega(x) = f_\Omega^\circ(x)$.

Remark 6.2. We make some remarks on the statement of Proposition 6.1.

(i) Assume $G = \mathrm{O}_n$ and $t = n$. In the case $x \in G$ has order $r > n$ then $x^G \cap H = \emptyset$.

G	Conditions	ε'	G	Conditions	ε'
GL_n	$r=2$	$1/2$	O_n	$r=2$	$1/2-1/2n$
	$r\neq 2,n$	$1/(n-1)^2$		$r\neq 2$	$1/r$
	$r=n>2$	$1/n$			

TABLE 8

(ii) The bound in (ii) is not the best possible. In our analysis, we identify the elements $x\in H^\circ$ that realise the best possible lower bound on the related ratio f°_Ω, defined in (1). However, this bound is difficult to state, we refer the reader to Proposition 6.13.

We shall need the following useful inequality, see [7, Lemma 6.2]. For $a_1,\dots,x_h\in\mathbb{R}$,

$$\sum_{i=1}^h a_i^2 \geqslant \frac{1}{h}\Big(\sum_{i=1}^h a_i\Big)^2. \tag{30}$$

In the case $t=n$, Proposition 3.1 provides an explicit formula of $f_\Omega(x)$. Thus it is convenient to deal with this case first. We assume $r\leqslant n$.

Lemma 6.3. *The conclusion of Proposition 6.1(iii) holds.*

Proof. If $G=\mathrm{O}_n$, then the result quickly follows from Proposition 3.1; similarly if $G=\mathrm{GL}_n$ and $r=2$. If $G=\mathrm{GL}_n$ and $r\neq 2$, we consider two cases. If $r<n$ then using the formula in Proposition 3.1 and (30) we deduce

$$f_\Omega(x)\geqslant 1-\frac{n^2-\sum_i a_i^2}{n(n-1)}\geqslant \frac{1}{r}-\frac{r-1}{r(n-1)}\geqslant\frac{1}{(n-1)^2}.$$

If $r=n$ and there exists i such that $a_i=0$ then $\sum_i a_i^2\geqslant n^2/(r-1)$ and an easy computation shows $f_\Omega(x)\geqslant 1/(n-1)^2$. If $a_i\neq 0$, for all i, then $x=[1,\omega,\dots,\omega^{n-1}]\in\pi_1^G$. Hence $f_\Omega(x)=1/n$. This completes the proof. □

6.1. **Case r big.** In this section we assume $r>n$ if $G=\mathrm{GL}_n$ and $r\geqslant n-1+2\delta_{t,n}$ if $G\neq\mathrm{GL}_n$. Notice that $r>t$, so Remark 2.1 implies $x^G\cap H=x^G\cap H^\circ$. Thereby, we may assume $x=[x_1,\dots,x_t]\in H^\circ$ is such that $\dim(x^G\cap H^\circ)=\dim x^{H^\circ}$. For all $i\in\{1,\dots,t\}$, we set

$$x_i=[I_{a_{i,0}},\omega I_{a_{i,1}},\dots,\omega^{r-1}I_{a_{r-1}}]\in Cl_{n/t}, \tag{31}$$

where $\sum_j a_{i,j}=n/t$ and $\sum_i a_{i,j}=a_j$. Using the formulae in [7, Theorem 2.2] we compute

$$\dim C_\Omega(x)=(1+\delta_{G,\mathrm{GL}_n})\sum_{l=0}^{r-1}\Big(\sum_{1\leqslant i<j\leqslant t}a_{i,l}a_{j,l}\Big). \tag{32}$$

First we assume $G\neq\mathrm{O}_n$ if n/t is odd and we show Proposition 6.1(i).

Lemma 6.4. *Assume n/t is even if $G=\mathrm{O}_n$. Then the conclusion of Proposition 6.1(i) holds.*

Proof. First assume $\nu(x)=n-1$. Then, up to centraliser structure, $x=[1,\omega,\dots,\omega^{n-1}]$, if $G=\mathrm{GL}_n$, or $x=[\omega,\omega^{-1},\dots,\omega^{n/2},\omega^{-n/2}]$, if $G\neq\mathrm{GL}_n$. In both cases, thanks (32) we deduce $f_\Omega(x)=0$. Similarly if $(G,C_G(x))=(\mathrm{Sp}_n,\mathrm{Sp}_2\times(\mathrm{GL}_1)^{n/2-1})$ or $(\mathrm{O}_n,\mathrm{O}_2\times(\mathrm{GL}_1)^{n/2-1})$.

Conversely, assume $x\in H$ has order r and $f_\Omega(x)=0$. We may assume $x=[x_1,\dots,x_t]$ such that $\dim(x^G\cap H^\circ)=\dim x^{H^\circ}$. Then, in the above notation, for all $l\in\{0,\dots,r-1\}$ and for all (i,j) such that $1\leqslant i<j\leqslant t$ we have $a_{i,l}a_{j,l}=0$. Therefore, for all $l\geqslant 0$, there exists at most one index h such that $a_{h,l}\neq 0$. Thus $a_l\leqslant 1$ for all $l>0$ and $a_0\leqslant 1$ if $G=\mathrm{GL}_n$, otherwise $a_0\leqslant 2$. The result follows. □

Example 6.5. Assume $G=\mathrm{GL}_n$. Let $r>n$ and $x=[1,\omega,\dots,\omega^{n-1}]$. Then $x\downarrow V=\bigoplus_{i=0}^{n-1}V_i$, where we denote V_i the eigenspace associated to ω^i. It is clear that x fixes all (and only) the decomposition $V=\bigoplus_{i=1}^t U_i$ in which $U_i=V_{j_1}\oplus\cdots\oplus V_{j_{n/t}}$. In particular, it is not too hard to show

$$|C_\Omega(x)|=\prod_{i=0}^{t-1}\frac{1}{t-i}\binom{n-in/t}{n/t}.$$

For example, if $t=n$ then $|C_\Omega(x)|=1$; in fact in this case the only direct sum decomposition fixed by x is $V=\bigoplus_i V_i$. If n is even and $t=n/2$ then $|C_\Omega(x)|=(n-1)!!$

Lemma 6.6. *Assume $G=\mathrm{O}_n$ and n/t is odd. Let $x\in H$ be of order r. Then the conclusion of Proposition 6.1(i) holds.*

Proof. Let $x=[x_1,\dots,x_t]\in H^\circ$, so $x_i\in \mathrm{O}_{n/t}$ for all i. Since n/t is odd, $a_{i,0}\geqslant 1$ is odd for all i. Therefore $a_0=\sum_i a_{i,0}\geqslant t$. Thus

$$\sum_{i<j} a_{i,0}a_{j,0}\geqslant \sum_{i<j}1=\frac{t(t-1)}{2},$$

with equality if, and only if, $a_{i,0}=1$ for all i. Thus, using (32), we have

$$\dim C_\Omega(x)=\sum_{i<j}a_{i,0}a_{j,0}+\sum_{l>0}\Big(\sum_{i<j}a_{i,l}a_{j,l}\Big)\geqslant \frac{t(t-1)}{2}.$$

In addition the equality holds if, and only if, $a_0=t$ and $a_{i,l}a_{j,l}=0$ for all $i<j$ and all $l>0$. □

6.2. **Case r small.** In this section we assume $t<n$ and $r<n-1+2\delta_{G,\mathrm{GL}_n}$.

Lemma 6.7. *The conclusion of Proposition 6.1(ii) holds.*

Proof. The cases $G=\mathrm{GL}_n,\mathrm{Sp}_n$ and O_n (with n/t even) are all very similar. So, for brevity, we assume $G=\mathrm{GL}_n$. The proofs for other cases are left to the reader.

Assume $G=\mathrm{GL}_n$ and notice that $r<n$. Let $x\in H$ be of order r. Theorem 2.5 provides an explicit formula of $\dim(x^G\cap H^\circ)$ in terms of $\lfloor a_i/t\rfloor$. Using $b_i=a_i-\lfloor a_i/t\rfloor t$, we have

$$f^\circ_\Omega(x)=\frac{\sum_{i\geqslant 0}a_i^2}{n^2}-\frac{\sum_{i\geqslant 0}b_i(t-b_i)}{n^2(t-1)}\geqslant \frac{1}{r}-\frac{rt^2}{4n^2(t-1)},$$

where we used (30) and $b_i(t-b_i)\leqslant t^2/4$. Since $f_\Omega(x)\geqslant f^\circ_\Omega(x)$, the result follows.

Now we assume $G=\mathrm{O}_n$ and n/t is odd. Let $x\in H$ be of order r. Here we may have $x^G\cap H^\circ=\emptyset$ (this happens whenever $a_0<t$). Hence, we mimic the proof of Lemma 5.10. There exists $0\leqslant h\leqslant \lfloor t/r\rfloor$ such that $\dim(x^G\cap H)=\dim(x^G\cap H^\circ\pi_h)$. Thus

$$\dim(x^G\cap H)=\dim\big([I_{n/t},\dots,I_{n/t},x_{hr+1},\dots,x_t]\pi_h\big)^{H^\circ};$$

we denote the multiplicities of the eigenvalues of $x_{hr+1},\dots,x_t$ as in (31). Thanks to Proposition 2.22, up to G-conjugacy, we have

$$x=\Big[I_{\frac{n}{t}h+\alpha_0},\omega I_{\frac{n}{t}h+\alpha_1},\dots,\omega^{r-1}I_{\frac{n}{t}h+\alpha_{r-1}}\Big]$$

where $\alpha_i=\sum_{j\geqslant hr+1}a_{j,i}$. We compute

$$\dim C_\Omega(x)=\frac{n^2}{2t^2}h(2t-hr-1)+\sum_{hr<l<m\leqslant t}\Big(\sum_{i=0}^{r-1}a_{l,i}a_{m,i}\Big). \tag{33}$$

Set $\mathscr{I}_x=\{(l,m):hr+1\leqslant l<m\leqslant t\}$. In the case $h=t/r$ or $(t-1)/r$ then $\mathscr{I}_x=\emptyset$, and we have $f_\Omega(x)=1/r$. Hence, we assume $h<(t-1)/r$, so $\mathscr{I}_x\neq\emptyset$ and

$$|\mathscr{I}_x|=\frac{(t-hr)(t-hr-1)}{2}.$$

Now, Proposition 2.13 implies that, for all $1\leqslant i,j\leqslant t$, we have $|a_{i,l}-a_{j,l}|\leqslant 1$ for $l>0$ and $|a_{i,0}-a_{j,0}|\leqslant 2$. Therefore $a_{l,i}\in\{\lfloor\alpha_i/(t-hr)\rfloor,\lfloor\alpha_i/(t-hr)\rfloor+1\}$. Write $\alpha_i=c_i(t-hr)+b_i$, where $0\leqslant b_i<t-hr$. Then $|\{l:a_{l,i}=c_i+1\}|=b_i$ and $|\{l:a_{l,i}=c_i\}|=t-hr-b_i$. Using these facts, and (30), we have

$$\begin{aligned}\sum_{(l,m)\in\mathscr{I}_x}\Big(\sum_{i=0}^{r-1}a_{l,i}a_{m,i}\Big)&=\frac12\Big(1-\frac{1}{t-hr}\Big)\sum_{i=0}^{r-1}a_i^2-\frac12\sum_{i=0}^{r-1}b_i\Big(1-\frac{b_i}{t-hr}\Big)\\&\geqslant\Big(\frac{n}{t}\Big)^2\frac{(t-hr)(t-hr-1)}{2r}-\frac12\sum_{i=0}^{r-1}b_i\Big(1-\frac{b_i}{t-hr}\Big).\end{aligned}$$

Therefore

$$f_\Omega(x)\geqslant\frac1r-\frac{2t}{n^2(t-1)}\sum_{i=0}^{r-1}b_i\Big(1-\frac{b_i}{t-hr}\Big)\geqslant\frac1r-\frac{rt^2}{2n^2(t-1)}.$$

□

For the remainder of the section we assume n/t is even if $G = \mathrm{O}_n$. In particular, n is even if $G \neq \mathrm{GL}_n$. The aim is to derive the best possible lower bound on $f^{\circ}_{\Omega}(x)$, for $x \in G$ of odd prime order $r \neq p$.

Definition 6.8. Let $x \in G$ be of odd order r. We say that x is *special* if $|a_i - a_j| \leqslant 1$ for all $0 \leqslant i, j \leqslant r-1$.

The following gives the centraliser structure of special elements of G.

Proposition 6.9. *Let $x \in G$ be special of order r. Write $n = \lfloor n/r \rfloor r + c$.*

(i) *If $G = \mathrm{GL}_n$ then $C_G(x) \cong C_G(z)$ where*
$$z = \left[I_{\lfloor n/r \rfloor+1}, \ldots, \omega^{c-1} I_{\lfloor n/r \rfloor+1}, \omega^{c} I_{\lfloor n/r \rfloor}, \ldots, \omega^{r-1} I_{\lfloor n/r \rfloor}\right].$$

(ii) *If $G = \mathrm{Sp}_n$ or O_n then $C_G(x) \cong C_G(z)$ where*
$$\begin{aligned} z = \big[&I_{\lfloor n/r \rfloor+\delta}, (\omega, \omega^{-1}) I_{\lfloor n/r \rfloor+1}, \ldots, (\omega, \omega^{-1})^{\lfloor c/2 \rfloor} I_{\lfloor n/r \rfloor+1}, \\ &(\omega, \omega^{-1})^{\lfloor c/2 \rfloor+1} I_{\lfloor n/r \rfloor}, \ldots, (\omega, \omega^{-1})^{(r-1)/2} I_{\lfloor n/r \rfloor}\big], \end{aligned}$$
where $\delta = 0$ if $\lfloor n/r \rfloor$ is even and $\delta = 1$ otherwise.

We shall show the following, with a strategy very similar to that adopted in [7, Section 6].

Claim 6.10. Let $x \in G$ be of order r. Then $f^{\circ}_{\Omega}(x) \geqslant f^{\circ}_{\Omega}(z)$ for any special element z of order r.

Let $x \in G$ be of order r and assume x is non-special. Then $a_i - a_j \geqslant 2$ for some i, j. As done in [7, Lemma 6.9], we may assume $(i, j) = (0, 1)$ if $G = \mathrm{GL}_n$ or $(i, j) \in \{(0,1), (1,0), (1,2)\}$ if $G \neq \mathrm{GL}_n$. If $G = \mathrm{GL}_n$ and $(i, j) = (0, 1)$ we define
$$y = [I_{a_0 - 1}, \omega I_{a_1+1}, \omega^2 I_{a_2}, \ldots, \omega^{r-1} I_{a_{r-1}}].$$
If $G = \mathrm{Sp}_n$ or O_n and $(i, j) \in \{(0,1), (1,0), (1,2)\}$ we define y as in [7, Table 10].

Lemma 6.11. *Let $x \in G$ be of order r, non-special. Let y as above. Then*
$$f^{\circ}_{\Omega}(x) \geqslant f^{\circ}_{\Omega}(y).$$

Proof. The result is equivalent to
$$\dim y^G - \dim x^G \geqslant \dim(y^G \cap H^{\circ}) - \dim(x^G \cap H^{\circ}). \tag{34}$$
Using [7, Theorem 2.5] we can compute $\dim y^G - \dim x^G$. Thanks to Theorem 2.5 we compute $\dim(y^G \cap H^{\circ}) - \dim(x^G \cap H^{\circ})$. Since $a_0 - a_1 \geqslant 2$ we have $a_0 = a_1 + h$ for some $h \geqslant 2$.

The proof is elementary although very tedious. We give the argument in the case $G = \mathrm{GL}_n$. More details can be found in [8, Sections 18.3.2, 19.3.2]. Notice that, thanks to Proposition 2.8 we only need to provide a proof for the cases $G = \mathrm{Sp}_n$. We have $\dim y^G - \dim x^G = 2(a_0 - a_1 - 1)$ and using Theorem 2.5 we compute:
$$\begin{aligned} \dim(y^G \cap H^{\circ}) - \dim(x^G \cap H^{\circ}) = &\left(\left\lfloor \frac{a_0 - 1}{t} \right\rfloor^2 - \left\lfloor \frac{a_0}{t} \right\rfloor^2\right) t + \left(\left\lfloor \frac{a_0 - 1}{t} \right\rfloor - \left\lfloor \frac{a_0}{t} \right\rfloor\right)(t - 2a_0) \\ &+ \left(\left\lfloor \frac{a_1 + 1}{t} \right\rfloor^2 - \left\lfloor \frac{a_1}{t} \right\rfloor^2\right) t + \left(\left\lfloor \frac{a_1 + 1}{t} \right\rfloor - \left\lfloor \frac{a_1}{t} \right\rfloor\right)(t - 2a_1) \\ &+ 2\left\lfloor \frac{a_0 - 1}{t} \right\rfloor - 2\left\lfloor \frac{a_1 + 1}{t} \right\rfloor. \end{aligned}$$
Thus, we have four cases, depending on the values of the floor functions.

If $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$, then (34) is equivalent to
$$a_0 - a_1 - 1 > \left\lfloor \frac{a_0}{t} \right\rfloor - \left\lfloor \frac{a_1}{t} \right\rfloor. \tag{35}$$
Write $a_0 = b_0 t + c_0$ and $a_1 = b_1 t + c_1$ where $0 < c_0 < t$ and $0 \leqslant c_1 < t - 1$. Moreover, since $a_0 - a_1 \geqslant 2$ we also have $b_0 \geqslant b_1$. Then (35) is equivalent to $a_0 - a_1 - 1 > b_0 - b_1$.

If $b_0 = b_1$, we immediately get the result, since $a_0 - a_1 \geqslant 2$. Now, assume $b_0 > b_1$, i.e. $b_0 = b_1 + l$ for some integer $l \geqslant 1$. Thus the left hand side of (35) is $a_0 - a_1 - 1 = lt + c_0 - c_1 - 1$, and the right hand side is l. Therefore $lt - l + c_0 - c_1 - 1 > 0$ is equivalent to (35). Since $c_0 \geqslant 1$ and $-c_1 \geqslant 2 - t$, we have
$$lt - l + c_0 - c_1 - 1 \geqslant lt - l + 1 + 2 - t - 1 = l(t-1) - t + 2.$$
Since $l \geqslant 1$ and $t - 1 > 0$ we have $l(t-1) - t + 2 \geqslant (t-1) - t + 2 > 0$. Hence, (35) has been proved.

If $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor - 1$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor + 1$, then (34) is equivalent to
$$(t-1)\left(\left\lfloor \frac{a_0}{t} \right\rfloor - \left\lfloor \frac{a_1}{t} \right\rfloor - 1\right) > 0. \tag{36}$$

We have $a_0 = b_0 t$ and $a_1 = b_1 t - 1 = (b_1 - 1)t + (t-1)$. Therefore (36) is equivalent to $b_0 - b_1 > 0$. If $b_0 \leqslant b_1$, then $a_0 - a_1 = b_0 t - b_1 t + 1 \leqslant b_1 t - b_1 t + 1 = 1$, which is a contradiction since $a_0 \geqslant a_1 + 2$.

There are two cases left. Either $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor - 1$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$ and (34) is equivalent to

$$a_1 < t\left\lfloor \frac{a_0}{t} \right\rfloor + \left\lfloor \frac{a_1}{t} \right\rfloor - \left\lfloor \frac{a_0}{t} \right\rfloor;$$

or, $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor + 1$, and (34) is equivalent to

$$a_0 > \left\lfloor \frac{a_0}{t} \right\rfloor - \left\lfloor \frac{a_1}{t} \right\rfloor + \left\lfloor \frac{a_1}{t} \right\rfloor t + t.$$

In both cases it is easy to establish the required inequalities. $\square$

The following is proved by repeating verbatim the proof of [7, Proposition 6.12]; clearly the key tool in this case is Lemma 6.11.

Proposition 6.12. *Claim 6.10 holds.*

Thanks to the previous discussion the best possible lower bound on $f^{\circ}_{\Omega}(x)$ for $x \in H$ of order r is given by $f^{\circ}_{\Omega}(z)$, for z as in Proposition 6.9.

Proposition 6.13. *Let $x \in G$ be of order r. Write $n = (at+b)r + c$, where $0 \leqslant c < r$ and $0 \leqslant b < t$. Then*

$$f^{\circ}_{\Omega}(x) \geqslant \frac{1}{r} - \frac{br(t-b) - 2bc}{n^2(t-1)} - \frac{c^2}{n^2 r} - \iota$$

where $\iota = 0$ if $G = \mathrm{GL}_n$ and $\iota = 2t/n^2$ otherwise.

Proof. Thanks to Proposition 6.12 the best possible lower bound on $f^{\circ}_{\Omega}(x)$ for $x \in H$ of order r is given by $f^{\circ}_{\Omega}(z)$, for z as in Proposition 6.9. In order to compute $f^{\circ}_{\Omega}(z)$ it is enough to use the formulae in [7, Theorem 2.5] and Theorem 2.5. Notice that it is enough to consider only the case $G = \mathrm{GL}_n$ and Sp_n, thanks to Proposition 2.8. The computation is left to the reader; however, all the details may be found in [8, Sections 18.3.2, 19.3.2]. $\square$

Remark 6.14. Notice that we can as well exploit the lower bound on $\dim(x^G \cap H^{\circ})$ given by Proposition 2.14 and compute a lower bound on $f^{\circ}_{\Omega}(z)$ only depending on n, r, t and c, where $n = \lfloor n/r \rfloor t + c$.

7. Local upper bounds

In this section we derive global upper bounds on $f_{\Omega}(x)$ for $x \in \mathscr{V}_{s,r}$. In particular we show the first part of Theorem 3. The key tools to derive such bounds are [7, Proposition 2.6] and Proposition 2.2. When $r \neq p$, if we also require n/t to be even if $G = \mathrm{O}_n$, we will describe the elements of $\mathscr{V}_{s,r}$ with largest f°_{Ω}-value.

Proposition 7.1. *Let $x \in \mathscr{V}_{s,r}$. If $s \leqslant n/2$ then*

$$f_{\Omega}(x) \leqslant 1 - 2\frac{s}{n} + 2\left(\frac{s}{n}\right)^2 + \frac{1 - \delta_{G,\mathrm{GL}_n}}{n}.$$

If $s > n/2$ then

$$f_{\Omega}(x) \leqslant 1 - \frac{s}{n} + \frac{1 - \delta_{G,\mathrm{GL}_n}}{n}.$$

In particular, the conclusion of Theorem 3 holds.

Proof. Let $x \in \mathscr{V}_{s,r}$. Thanks to Proposition 2.2, we have

$$f_{\Omega}(x) \leqslant 1 - \frac{\dim x^G (1 - 1/t - \zeta)}{\dim \Omega}.$$

Recall that [7, Proposition 2.6] implies $\dim x^G \geqslant f(s)$. Using this bound, the result quickly follows. $\square$

Now we turn our attention on the sharpness of these upper bounds.

7.1. **Unipotent elements.** Assume $r=p$. Notice that if $t=n$ and $x\in\mathscr{V}_{s,p}$ then $s=h(p-1)$ for some $0<h\leqslant\lfloor t/p\rfloor$ and $f_\Omega(x)$ is computed in Proposition 3.1. Hence we shall assume $t<n$.

In this case if $x\in\mathscr{V}_{s,p}$ and $x=[J_p^{a_p},\dots,J_1^{a_1}]$ then $s=n-\sum_i a_i$. In general, constructing element $x\in G$ for which $x^G\cap H\cap\mathscr{V}_{s,p}\neq\emptyset$ is very difficult. The next result shows that the bound of Proposition 7.1 is close to the best possible, when $s\leqslant n/2$.

Proposition 7.2. *Let $s\leqslant n/2$. Assume s is even if $G=\mathrm{O}_n$. Then there exists $x\in H^\circ\cap\mathscr{V}_{s,p}$ such that*

$$f^\circ_\Omega(x)\geqslant 1-2\frac{s}{n}+2\left(\frac{s}{n}\right)^2-\varepsilon,$$

where either $G=\mathrm{O}_n$ and $\varepsilon=2t^2/n^2(t-1)$; or, n/t is even or n/t is odd and $n-2s\geqslant t$, in which case $\varepsilon=1/n$, otherwise $\varepsilon=t/2(t-1)$.

Proof. Let $x=[J_2^s,J_1^{n-2s}]$. Then $x\in\mathscr{V}_{s,p}$. It is clear that $x^G\cap H^\circ\neq\emptyset$ if n/t is even or n/t is odd and $n-2s\geqslant t$ (in the last case the intersection is non-empty since the number of J_1-blocks is at most t). Write $s=at+b$, where $0\leqslant b<t$. Then $\dim(x^G\cap H^\circ)\geqslant(t-b)\dim[J_2^a,J_1^{n/t-2a}]^{\mathrm{GL}_{n/t}}+b\dim[J_2^{a+1},J_1^{n/t-2a-2}]^{\mathrm{GL}_{n/t}}$. We have

$$f^\circ_\Omega(x)\geqslant 1-\frac{2s(n-s)}{n^2}-\frac{2b(t-b)}{n^2(t-1)}\geqslant 1-\frac{2s(n-s)}{n^2}-\frac{1}{n}.$$

If n/t is odd and $n-2s<t$ we consider $x=[J_3^l,J_2^{s-2l},J_1^{n-2s+l}]$ where $l=s-(n-t)/2$. It is clear that $x\in\mathscr{V}_{s,p}$. Let $x_1=\ldots=x_l=[J_3,J_2^{(n/t-3)/2}]$ and $x_{l+1}=\ldots=x_t=[J_2^{(n/t-1)/2},J_1]$. Then $[x_1,\dots,x_t]\in x^G\cap H^\circ$, and we have $\dim(x^G\cap H^\circ)=\dim x^{H^\circ}$. An easy calculation yields:

$$f^\circ_\Omega(x)\geqslant 1-\frac{2s(n-s)}{n^2}-\frac{(n-2s-t)(nt-t^2+2t-2s)}{2n^2(t-1)}\geqslant 1-\frac{2s(n-s)}{n^2}-\frac{t}{2(t-1)}.$$

The same elements lead to the result in the other cases; the computation is similar and left to the reader. □

Assume $s>n/2$. In some special cases we can construct suitable elements that lead to the conclusion that the bound is close to best possible. The case $t=2$ is dealt in the following.

Remark 7.3. Assume $G=\mathrm{GL}_n$, $t=2$ and $s>n/2$. If s is even, we write $n/2=a(n-s)/2+b$ where $0\leqslant b<(n-s)/2$. In the case $a+1\leqslant p$ then the elements we shall define below have order p; otherwise they have order a power of p. Let

$$\bar{x}=\left[J_{a+1}^b,J_a^{\frac{n-s}{2}-b}\right].$$

Then $x=[\bar{x},\bar{x}]\in H^\circ\cap\mathscr{V}_{s,p}$. If s is odd we write $n/2=a_1(n-s-1)/2+b_1=a_2(n-s+1)/2+b_2$, where $0\leqslant b_1<(n-s-1)/2$ and $0\leqslant b_2<(n-s+1)/2$. We define

$$x_1=\left[J_{a_1+1}^{b_1},J_{a_1}^{\frac{n-s-1}{2}-b_1}\right],\ x_2=\left[J_{a_2+1}^{b_2},J_a^{\frac{n-s+1}{2}-b_2}\right].$$

Again, $x=[x_1,x_2]\in H^\circ\cap\mathscr{V}_{s,p}$. In both cases, an easy calculation yields

$$f^\circ_\Omega(x)\geqslant 1-\frac{s}{n}-\varepsilon,$$

where $\varepsilon=(n-s)^2/4n^2$ if s is even and $\varepsilon=1/n$ otherwise.

7.2. **Semisimple elements.** Assume $r\neq p$ is an odd prime. The aim of this section is to construct a finite family of elements $\mathscr{F}_{s,r}\subset\mathscr{V}_{s,r}$ such that $f^\circ_\Omega(x)\leqslant\max\{f^\circ_\Omega(z):z\in\mathscr{F}_{s,r}\}$, for all $x\in\mathscr{V}_{s,r}$.

Let $x\in\mathscr{V}_{s,r}$. If $G=\mathrm{GL}_n$ we may assume $x=[I_{n-s},\omega I_{a_1},\dots,\omega^{r-1}I_{a_{r-1}}]$, where $a_i\leqslant n-s$. If $G\neq\mathrm{GL}_n$ then we may assume x is as [7, (27)]. In this section we shall assume n/t is even when $G=\mathrm{O}_n$.

Write $n=(n-s)a+b$, where $0\leqslant b<n-s$. If $G=\mathrm{GL}_n$, we define

$$z=[I_{n-s},\omega I_{n-s},\dots,\omega^{a-1}I_{n-s},\omega^a I_b].$$

If $G=\mathrm{Sp}_n$ or O_n then we set z_1,z_2 as in [7, (28), (29)]. And, only for the purpose of a uniform notation, if $G=\mathrm{GL}_n$ we denote $z=z_1=z_2$, where z is as above. We make the following.

Claim 7.4. Let $x\in\mathscr{V}_{s,r}$. Then $f^\circ_\Omega(x)\leqslant\max\{f^\circ_\Omega(z_1),f^\circ_\Omega(z_2)\}$.

Claim 7.4 will be shown with the same strategy used for [7, Claim 7.4]. Also here, a similar version of [7, Lemma 7.5] holds; however, in the conclusion of the result, $f_\Omega(\cdot)$ needs to be replaced with $f^\circ_\Omega(\cdot)$. For $G = \mathrm{GL}_n$, the statement is as follows: if $x \in \mathscr{V}_{s,r}$ then either $C_G(x) \cong C_G(z)$ or there exists $y = [I_{n-s}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}] \in \mathscr{V}_{s,r}$ such that $f^\circ_\Omega(x) = f^\circ_\Omega(y)$, $b_1 = \min\{b_i : b_i \neq 0\}$ and $b_2 = \max\{b_i : b_i < n-s\}$.

Assume $G = \mathrm{GL}_n$. Let $x \in \mathscr{V}_{s,r}$ with $C_G(x) \ncong C_G(z)$. We may assume $x = [I_{n-s}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$, with $a_1 = \min_i\{a_i : a_i \neq 0\}$ and $a_2 = \max\{a_i : a_i < n-s\}$. Then we define

$$y = \left[I_{n-s}, \omega I_{a_1-1}, \omega^2 I_{a_2+1}, \omega^3 I_{a_3}, \dots, \omega^{r-1} I_{a_r-1}\right] \in \mathscr{V}_{s,r}. \tag{37}$$

Assume $G \neq \mathrm{GL}_n$ and n/t is even. Let $x \in \mathscr{V}_{s,r}$, where $C_G(x) \ncong C_G(z_i)$, $i = 1,2$. Then we define $y \in \mathscr{V}_{s,r}$ as in [7, Table 11].

If $G = \mathrm{O}_n$ and n/t is odd, then it is easy to define z_1, z_2 as in [7] such that their 1-eigenspace is odd-dimensional, so that they lie in G; however, it may happen that $z_i^G \cap H = \emptyset$. This is the main difficulty that forced us to avoid this case.

The following is the key tool to show the claim.

Lemma 7.5. *Let $x \in \mathscr{V}_{s,r}$. Assume $C_G(x) \ncong C_G(z_i)$ for $i = 1,2$. Let y as above. Then $f^\circ_\Omega(x) \leqslant f^\circ_\Omega(y)$.*

Proof. It is enough to show that $\dim y^G - \dim x^G \leqslant \dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ)$. The computation is very similar to that done in Lemma 6.11 and it is left to the reader. □

In the usual way, see [7, Proposition 6.12], we can now show the claim.

Proposition 7.6. *Claim 7.4 holds.*

For brevity, we do not compute $f^\circ_\Omega(z_i)$ in all the cases. However, in few cases, with further assumptions, we show that the upper bounds in Proposition 7.1 are close to best possible.

Lemma 7.7. *Assume $G = \mathrm{Sp}_n$ or O_n, and n/t is even. Assume $s \leqslant 2n/3$ is even. Then there exists $x \in H \cap \mathscr{V}_{s,r}$ such that*

$$f_\Omega(x) \geqslant 1 - \frac{2s(n-s)}{n^2} - \frac{s^2}{2n^2} - \frac{3t^2}{2n^2(t-1)}.$$

Proof. Thanks to Proposition 7.6, it is natural to consider $x = [I_{n-s}, (\omega, \omega^{-1}) I_{s/2}]$. Then, thanks to Proposition 2.8, it is enough to compute $f^\circ_\Omega(x)$ when $G = \mathrm{Sp}_n$. The result quickly follows. □

In the case $G = \mathrm{GL}_n$ we can give more information. For convenience, we write U for the upper bound proved in Proposition 7.1.

Proposition 7.8. *Let $G = \mathrm{GL}_n$. Write $n = (n-s)a + b$, where $0 \leqslant b < n-s$. Then there exists $x \in H \cap \mathscr{V}_{s,r}$ such that*

$$f_\Omega(x) \geqslant U - \frac{2}{n} - \iota$$

where $\iota = b/n$ if $s > n/2$, and $\iota = 0$ otherwise.

Proof. First assume $s \leqslant n/2$. Then $x = [I_{n-s}, \omega I_s] \in H \cap \mathscr{V}_{s,r}$. Notice that if $r > 2$ then $x^G \cap H = x^G \cap H^\circ$, hence $f_\Omega(x) = f^\circ_\Omega(x)$. Let $s = at + b$, where $0 \leqslant b < t$. Then we compute

$$f^\circ_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}.$$

The required bound quickly follows.

Now assume $s > n/2$. Write $n = (n-s)a + b$, where $0 \leqslant b < n-s$. then $x = [I_{n-s}, \dots, \omega^{a-1} I_{n-s}, \omega^a I_b] \in H \cap \mathscr{V}_{s,r}$. We have

$$\dim x^G = ns + \delta, \quad \dim(x^G \cap H^\circ) = \frac{\dim x^G}{t} - \varepsilon,$$

where $\delta = a(n-s)(b+s) - ns \geqslant 0$ by [7, Proposition 2.6], and $\varepsilon \geqslant 0$ by Proposition 2.2. Since $a \leqslant n/(n-s)$ we deduce $\delta \leqslant nb$. If $t \mid s$ then $\varepsilon = 0$ and

$$f^\circ_\Omega(x) = 1 - \frac{s}{n} - \frac{\delta}{n^2} \geqslant 1 - \frac{s}{n} - \frac{b}{n}.$$

If $t \nmid s$ then Proposition 2.14 yields $\varepsilon \leqslant \dim x^G/n < n$. Therefore

$$f^\circ_\Omega(x) = 1 - \frac{s}{n} - \left(\frac{\delta}{n^2} + \frac{\varepsilon t}{n^2(t-1)}\right).$$

Again, the required bound quickly follows using the previous estimates on δ and ε, and $t/(t-1) \leqslant 2$. $\square$

From the argument above, in the case $s > n/2$ we see that $\delta(a) \leqslant \delta(\frac{n+s}{2(n-s)}) = (n-s)^2/4$. Thanks to this is observation we get the following.

Corollary 7.9. *There exists $x \in H \cap \mathscr{V}_{s,r}$ such that $f_\Omega(x) \geqslant U - 2/n - 1/16$.*

8. Local lower bounds

In this section we complete the proof of Theorem 3; in other words we derive local lower bounds on $f_\Omega(x)$, for $x \in H$ a semisimple element of prime order r. We first study semisimple elements of odd order r; then we deal with involutions in Section 8.2. The main strategy is similar to that employed in Section 6.2: we shall construct a class of ν-*special* elements with the properties that some of them realises the best possible lower bound. Then, in order to simplify the computations we shall also appeal to Proposition 2.14 and [7, Proposition 2.6]. In the following we shall assume n/t is even if $G = \mathrm{O}_n$, so that $x^G \cap H^\circ \neq \emptyset$ for any $x \in G$ of prime order r; see Remark 8.7 for the case $G = \mathrm{O}_n$ and n/t odd. Notice that, thanks to Proposition 6.1, we may assume $s < n-1$ for $G = \mathrm{GL}_n$ and $s < n-2$ otherwise.

We do not have any general lower bound for unipotent elements in $H \cap \mathscr{V}_{s,p}$ for several reasons. Recall that $x = [J_p^{a_p}, \dots, J_1^{a_1}]$ lies in $\mathscr{V}_{s,p}$ if $s = n - \sum_i a_i$. This last condition together with the requirement $x^G \cap H \neq \emptyset$ is hard to deal with. In some special case we can give more information. For example, if $t = n$ an explicit formula of $f_\Omega(x)$ for $x \in H \cap \mathscr{V}_{s,p}$ is computed in Proposition 3.1. In the case $p > n/2$ and $G = \mathrm{GL}_n$, if $s \geqslant (n-t)(1-1/t)$ then for all $x \in \mathscr{V}_{s,p}$ such that $x^G \cap H \neq \emptyset$ we have $f_\Omega(x) \geqslant t/n$ (thanks to Proposition 5.1 and because there exist elements with Jordan form $[J_{n/t}^{t-1}, z] \in H \cap \mathscr{V}_{s,p}$). Similarly if $G \neq \mathrm{GL}_n$. In the case $(G,H) = (\mathrm{GL}_n, GL_{n/2} \wr S_2)$ and $s \leqslant n/2 - 2$, using the same techniques of Lemma 5.10 together with the (essential) fact that $t = 2$, one can show that $f_\Omega(x) \geqslant 1 - 2s/n$.

8.1. **Semisimple elements.** Throughout this section we assume $r \neq 2$. Recall [7, Definition 8.3] for $G \neq \mathrm{GL}_n$. If $G = \mathrm{GL}_n$ and $x \in \mathscr{V}_{s,r}$, we may assume $a_0 = n - s$. We say that x is ν-*special* if $|a_i - a_j| \leqslant 1$ for all $i, j > 0$. We shall show that there exists a ν-special element $z \in \mathscr{V}_{s,r}$ such that for any $x \in \mathscr{V}_{s,r}$ we have $f_\Omega^\circ(x) \geqslant f_\Omega^\circ(z)$. Therefore, it will be enough to compute the f_Ω°-value of ν-special elements to derive close to best possible lower bounds.

Assume $G = \mathrm{GL}_n$. Write $s = a(r-1) + b$, where $0 \leqslant b < r-1$. It is clear that if $x \in \mathscr{V}_{s,r}$ is ν-special then $C_G(x) \cong C_G(z)$ where

$$z = [I_{n-s}, \omega I_{a+1}, \dots, \omega^b I_{a+1}, \omega^{b+1} I_a, \dots, \omega^{r-1} I_a]. \tag{38}$$

In the case x is not special then we may assume $a_0 = n - s$ and $a_1 = \min_{i>0}\{a_i\}, a_2 = \max_{i>0}\{a_i\}$. In this case, we define $y = [I_{n-s}, \omega I_{a_1+1}, \omega^2 I_{a_2-1}, \omega^3 I_{a_3}, \dots, \omega^{r-1} I_{a_{r-1}}] \in \mathscr{V}_{s,r}$.

Assume $G \neq \mathrm{GL}_n$. Then ν-special elements of G are given in [7, Proposition 8.4]. For the reader's convenience we recall them here. In the case s is even, write $s = a(r-1) + b$, where $0 \leqslant b < r-1$. If $s \geqslant n/2$ then write $2s - n = a(r-2) + b$, where $0 \leqslant b < r-2$. Then any ν-special element is one of the following, up to the centraliser structure,

$$[I_{n-s}, (\omega,\omega^{-1}) I_{a+1}, \dots, (\omega,\omega^{-1})^{\frac{b}{2}} I_{a+1}, (\omega,\omega^{-1})^{\frac{b}{2}+1} I_a, \dots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_a], \tag{39}$$

$$[I_{a+1}, (\omega,\omega^{-1}) I_{n-s}, (\omega,\omega^{-1})^2 I_{a+1}, \dots, (\omega,\omega^{-1})^{\frac{b+1}{2}} I_{a+1} (\omega,\omega^{-1})^{\frac{b+3}{2}} I_a, \dots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_a], \tag{40}$$

$$[I_a, (\omega,\omega^{-1}) I_{n-s}, (\omega,\omega^{-1})^2 I_{a+1}, \dots, (\omega,\omega^{-1})^{\frac{b}{2}+1} I_{a+1}, (\omega,\omega^{-1})^{\frac{b}{2}+2} I_a, \dots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_a]. \tag{41}$$

In this case, if $x \in \mathscr{V}_{s,r}$ is not ν-special, we define $y \in \mathscr{V}_{s,r}$ as explained in [7, Section 8.2].

The following is the key tool to show our claim.

Lemma 8.1. *Let $x \in \mathscr{V}_{s,r}$. Assume x is not ν-special. Let y as above. Then $f_\Omega^\circ(x) \geqslant f_\Omega^\circ(y)$.*

Proof. Thanks to Proposition 2.8, we only need to study the cases $G = \mathrm{GL}_n$ or Sp_n. In the case $G = \mathrm{GL}_n$, the proof is the same of Lemma 6.11. In the other cases we leave the computations to the reader. $\square$

As for global lower bounds for semisimple elements (cf. Proposition 6.12) we have the following.

Proposition 8.2. *There exists a ν-special element $z \in \mathscr{V}_{s,r}$ such that for any $x \in \mathscr{V}_{s,r}$ we have $f_\Omega^\circ(x) \geqslant f_\Omega^\circ(z)$.*

Remark 8.3. Assume $G \neq \mathrm{GL}_n$. If $x, y \in \mathscr{V}_{s,r}$ are ν-special then we may have $C_G(x) \ncong C_G(y)$ (this does not happen if $G = \mathrm{GL}_n$), in particular $f_\Omega^\circ(x) \neq f_\Omega^\circ(y)$. Hence, for any $x \in \mathscr{V}_{s,r}$ we have $f_\Omega^\circ(x) \geqslant \min\{f_\Omega^\circ(y) : y \in \mathscr{V}_{s,r}$ is ν-special$\}$.

We can now derive the lower bound. For convenience we split the case $G = \mathrm{GL}_n$ and $G \neq \mathrm{GL}_n$.

Lemma 8.4. *Assume $G = \mathrm{GL}_n$. Let $x \in \mathscr{V}_{s,r}$.*

(i) *Assume $s \leqslant r-1$, write $s = at+b$, where $0 \leqslant b < t$. Then*

$$f_\Omega(x) \geqslant 1 - \frac{s(2n-s)}{n^2} - \frac{b(t-b)}{n^2(t-1)}.$$

(i) *If $s > r-1$ then*

$$f_\Omega(x) \geqslant 1 - \frac{s(2n-s)}{n^2} - \frac{st(2n-s)}{n^3(t-1)}.$$

In particular, the conclusion of Theorem 3 holds.

Proof. If $s \leqslant r-1$ then any ν-special element is of type $z = [I_{n-s}, \omega, \dots, \omega^s]$. Notice that if $s < r-1$ then Corollary 2.23 yields $f^\circ_\Omega(z) = f_\Omega(z)$. Thanks to Proposition 8.2 we have $f^\circ_\Omega(x) \geqslant f^\circ_\Omega(z)$. A straightforward computation gives the desired value of $f^\circ_\Omega(z)$.

Let $s > r-1$. Here, for z as in (38), we have $a > 1$. Using the lower bounds on $\dim(x^G \cap H^\circ)$ given in Proposition 2.14 and $\dim z^G = 2ns - s^2 - a^2(r-1) - 2ab - b < 2ns - s^2$, the result quickly follows. □

Lemma 8.5. *Assume $G = \mathrm{Sp}_n$ or O_n, and n/t is even. Let $x \in H \cap \mathscr{V}_{s,r}$. Then*

$$f_\Omega(x) \geqslant 1 - \frac{s(2n-s)}{n^2} - \frac{st(2n-s)}{n^3(t-1)} - \frac{st}{n^2(t-1)}.$$

In particular, the conclusion of Theorem 3 holds.

Proof. Notice that, thanks to Proposition 2.8, we only need to derive the lower bound in the case $G = \mathrm{Sp}_n$.

Thanks to Proposition 8.2 the best possible lower bound is given by $f^\circ_\Omega(z)$ where $z \in \mathscr{V}_{s,r}$ is ν-special as in (39), (40), (41). However, rather than computing $f^\circ_\Omega(z)$, we greatly simplify the computations using [7, Proposition 2.6] and Proposition 2.14. Let $x \in \mathscr{V}_{s,r}$. Notice that $a_0 \leqslant n-s$. We have

$$\begin{aligned} f^\circ_\Omega(x) &\geqslant 1 - \frac{\dim x^G\left(1-\frac{1}{t}+\frac{1}{n}\right)}{\dim\Omega} + \frac{n-a_0}{2\dim\Omega}\left(1-\frac{1}{t}+\frac{1}{n}\right) \\ &\geqslant 1 - \frac{s(2n-s+1/s)\left(1-\frac{1}{t}+\frac{1}{n}\right)}{2\dim\Omega} + \frac{s}{n^2} + \frac{st}{n^2(t-1)} \end{aligned}$$

The required bound follows. □

The crude bound derived in Lemma 8.5 may be improved in some special cases.

Remark 8.6. Assume $G = \mathrm{Sp}_n$ or O_n. In some special cases we can derive better bounds: we consider the case in which $a = 0$. Therefore we assume $s \leqslant r-1$ if s is even, and $2s - n \leqslant r-1$ if s is odd. If s is even then $z = [I_{n-s}, \omega, \omega^{-1}, \dots, \omega^{s/2}, \omega^{-s/2}]$. We compute $\dim z^G = s(2n-s)/2$ and

$$\dim(z^G \cap H^\circ) = \frac{ns}{t} - s - 2s\left\lfloor\frac{s}{2t}\right\rfloor + 2t\left\lfloor\frac{s}{2t}\right\rfloor + 2t\left\lfloor\frac{s}{2t}\right\rfloor^2 \geqslant \frac{s}{2t}(2n-s) - \frac{t}{2}.$$

Therefore we have

$$f^\circ_\Omega(z) \geqslant 1 - \frac{s(2n-s)}{n^2} - \frac{t^2}{n^2(t-1)} \geqslant 1 - \frac{s(2n-s)}{n^2} - \frac{1}{n}.$$

Similarly, if s is odd then $z = [(\omega,\omega^{-1})I_{n-s}, \omega^2, \omega^{-2}, \dots, \omega^{s-n/2+1}, \omega^{-s+n/2-1}]$. And we have

$$f^\circ_\Omega(z) \geqslant 2 - \frac{2s(2n-s)}{n^2} - \frac{1}{2n}.$$

Remark 8.7. Assume $G = \mathrm{O}_n$ and n/t is odd. Let $x \in H \cap \mathscr{V}_{s,r}$. In the case $n = t$, we have computed an explicit formula of $f_\Omega(x)$, see Proposition 3.1; hence, it is easy to derive a lower bound depending on s. So we may assume $n/t > 1$. In this case we can deduce a lower bound on $f_\Omega(x)$ for $x \in H^\circ \cap \mathscr{V}_{s,r}$ as done in Lemma 8.5. Moreover, if x is not ν-special and the element y as in Lemma 8.1 lies in H° then we can show that $f^\circ_\Omega(x) \geqslant f^\circ_\Omega(y)$.

8.2. **Involutions.** In this section we also study the case $G = \mathrm{O}_n$ and n/t is odd. Throughout this section we assume $p \neq 2$. Let $s \leqslant n/2$ and $x \in \mathscr{V}_{s,2}$. Then, up to the sign, x has Jordan form $[I_{n-s}, -I_s]$. The aim of this section is to complete the proof of Theorem 3. We compute $f^{\circ}_{\Omega}(x)$; then from this value we will quickly deduce the desired lower bound. Notice that in any case if $x \in \mathscr{V}_{s,r}$ then $x^G \cap H^{\circ} \neq \emptyset$.

Recall the definition of ι given in (8).

Proposition 8.8. *Let $x \in \mathscr{V}_{s,2}$. Write $s/\iota = \lfloor s/\iota t \rfloor t + b$.*

(i) *Assume $G = \mathrm{GL}_n$ or O_n. Then*

$$f^{\circ}_{\Omega}(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

(ii) *Assume $G = \mathrm{Sp}_n$. Then*

$$f^{\circ}_{\Omega}(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{8b(t-b)}{n^2(t-1)}$$

In particular, the conclusion of Theorem 3 holds.

Proof. As usual we assume $n/t > 1$. Using Theorem 2.5 we compute $\dim(x^G \cap H^{\circ})$. Then the result quickly follows. Notice that $0 \leqslant b < t$ hence $b(t-b) < t^2/4$. Thus we also see that the lower bound stated in Theorem 3 holds for $f^{\circ}_{\Omega}(x)$. □

Remark 8.9. One may ask to compute an explicit formula for $f_{\Omega}(x)$ when x is an involution (even in the case $p = 2$). In fact in [8], we have studied this problem. In the case $G = \mathrm{GL}_n$ an explicit formula of $f_{\Omega}(x)$ is computed. If $G \neq \mathrm{GL}_n$, then either an explicit formula is given or very good estimates are derived. We plan to address the answer to this question, also for other type of actions, in future work.

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REFERENCES

[1] T.C. Burness. Fixed point spaces in primitive actions of simple algebraic groups. *J. Algebra*, 265(2):744–771, 2003.
[2] T.C. Burness. Fixed point ratios in actions of finite classical groups. III. *J. Algebra*, 314(2):693–748, 2007.
[3] R.M. Guralnick. Intersections of conjugacy classes and subgroups of algebraic groups. *Proc. Amer. Math. Soc.*, 135(3):689–693 (electronic), 2007.
[4] R. Lawther, M.W. Liebeck, and G.M. Seitz. Fixed point spaces in actions of exceptional algebraic groups. *Pacific J. Math.*, 205(2):339–391, 2002.
[5] M.W. Liebeck and G.M. Seitz. On the subgroup structure of classical groups. *Invent. Math.*, 134(2):427–453, 1998.
[6] M.W. Liebeck and A. Shalev. Simple groups, permutation groups, and probability. *J. Amer. Math. Soc.*, 12(2):497–520, 1999.
[7] R. Rainone. Fixed point spaces in geometric actions of classical groups, I. In preparation.
[8] R. Rainone. *Geometric actions of classical algebraic groups*. PhD thesis, University of Southampton, February 2014.

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