The O'Nan-Scott Theorem

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Abstract

we will go through the theorem

1 Introduction

The O'Nan-Scott Theorem is a classification theorem for the alternating and the symmetric groups.

Theorem 1.1. *If H is any proper subgroup of S_n other than A_n, then H is a subgroup of one or more of the following subgroups:*

1. *an intransitive group $S_k \times S_m$, where $n = k + m$;*
2. *an imprimitive group $S_k \wr S_m$, where $n = km$;*
3. *a primitive wreath product, $S_k \wr S_m$, where $n = k^m$;*
4. *an affine group $\mathrm{AGL}_d(\mathbb{F}_p) \cong C_{p^d} \rtimes \mathrm{GL}_d(\mathbb{F}_p)$, where $n = p^d$;*
5. *a group of the shape $T^m.(\mathrm{Out}(T) \times S_m)$, where T is a non-abelian simple group, acting on the cosets of a subgroup $Aut(T) \times S_m$, where $n = |T|^{m-1}$;*
6. *an almost simple group acting on the cosets of a maximal subgroup of index n.*

2 Group actions

Let $\Omega = \{1, \dots, n\}$ and G be a subgroup of $Sym(\Omega)$. For every elements α of Ω we set $Orb_G(\alpha) = \{g.\alpha \mid g \in G\}$ the *orbit* of α, it is easy to see that the orbits of G form a partition of the set Ω. We define $Stab_G(\alpha) = \{g \in G \mid g.\alpha = \alpha\}$, that we will also denote G_α, the *stabilizer* of α, that is a subgroup of G. If every point can be mapped to every other point, we say that G is *transitive* on the set Ω. This means that $Orb_G(\alpha) = \Omega$, i.e. for all β in Ω there exists g in G such that $g.\alpha = \beta$.
A *block system* for a subgroup G of S_n is a partition of Ω preserved by G, i.e. it is a set $\{\Omega_1, \dots, \Omega_l\}$ that is a partition of Ω and if α and β belong to the same

block of the partition then for all g in G the points $g.\alpha$ and $g.\beta$ belong to the same block. Every group acting on Ω has at least two trivial block system that are Ω itself and the set of all point point of Ω: $\{ \{\alpha\} \mid \alpha \in \Omega\}$. A non-trivial block system is called *system of imprimitivity* for the group G. For $n \geq 3$, any group that has a system of imprimitivity is called *imprimitive*, and so any non-trivial group that is not imprimitive is said to be *primitive*.
Let us recall the so called Orbit-Stabilizer Theorem:

Theorem 2.1. *Let G be a subgroup of $Sym(\Omega)$ and α be an element of Ω. Then $|Orb_G(\alpha)| = |G : Stab_G(\alpha)|$.*

Proof. Let α be an element of Ω and let us define the map $\varphi\colon G \to Orb_G(\alpha)$ such that $\varphi(g) = g.\alpha$. It is surjective, by the definition of the orbit, and it is clear that the kernel of φ is $Stab_G(\alpha)$. Hence there is a well defined bijection $\tilde{\varphi}\colon G/Stab_G(\alpha) \to Orb_G(\alpha)$. Therefore $|Orb_G(\alpha)| = |G : Stab_G(\alpha)|$. *q.e.d.*

This theorem tells us that if G is a subgroup of S_n acting transitively on Ω and H is a stabiliser of a point α of Ω then there is a natural bijection between Ω and the set of the right cosets of H in G.
This correspondence between group actions and subgroup allows us to classify primitive groups thanks to the following:

Proposition 2.2. *Suppose that the group G acts transitively on the set Ω. Let H be the stabilizer of the point α in Ω. Then the group G acts primitively if and only if H is a maximal subgroup of G.*

Proof. Assume that G acts primitively on Ω. If H is not a maximal subgroup of G then there exists a proper subgroup of G, let us call it K, that properly contains H. By the previous theorem we know that there is a one to one correspondence between the element of Ω and the right cosets of H. So if g is any element of G then the right coset Kg is union of right cosets of H, i.e. $Kg = Hx_1 \cup \ldots \cup Hx_l$ where $l = |K : H|$. Thus the coset Kg correspond to the set of points $\{x_1.\alpha, \ldots, x_l.\alpha\}$. This means that for every coset of K we get a set $\{x_1.\alpha, \ldots, x_l.\alpha\}$ and all these sets form a block system for G, but G is primitive by assumption so H is a maximal subgroup.

Conversely, suppose that H is a maximal subgroup of G and that $\{\Omega_1, \ldots, \Omega_l\}$ is a block system for G. Without loss of generality we can assume that α is in Ω_1, hence the subgroup $K = \{g \in G \mid g.\Omega_1 = \Omega_1\}$ of G properly contains H and it is different from the whole group because G acts transitively so K acts transitively on Ω_1, but not on Ω. This is absurd since H is maximal. Therefore G is primitive. *q.e.d.*

From this proposition we understand that it is really important to have a complete knowledge about the maximal subgroup of a group.

2.1 The coset space

Let H be a subgroup of the group G and let us consider the (left) *coset space* G/H. There is a natural action of G on G/H given by $g.(xH) = (gx)H$. With this action we get that the stabiliser of the coset $1H$ is the subgroup H of G. Suppose that G acts on the set Ω and Ω', we say that they are isomorphic as G-spaces if there exists an isomorphism $\theta\colon \Omega \to \Omega'$ such that $g.(\theta(\alpha)) = \theta(g.\alpha)$ for every g in G.

Theorem 2.3 (Theorem 1.3 [**?**]). *(a) Let G be a finite group acting on the set Ω. Then Ω is isomorphic to the coset space G/H as G-space, where $H = G_\alpha$ for α in Ω.*

(b) Two coset space G/H and G/K are isomorphic as G spaces if, and only if, H and K are conjugate subgroup of G.

3 Subgroups of S_n

Let us construct the subgroups of S_n that appear in Theorem **??**. First we will classify the intransitive subgroup, then the transitive imprimitive subgroup, at the end there will be other four classes of subgroup that are transitive primitive. We will prove the maximality in few cases.

3.1 Intransitive subgroups

If H is an intransitive subgroup of S_n then it has two or more orbits. Suppose that the orbit have lenghts $n_1, \dots, n_l$ then H is a subgroup of $S_{n_1} \times \ldots \times S_{n_l}$. Since we are interested in maximal subgroup we can mix up all the orbits except one, so we get a subgroup of the shape $S_k \times S_{n-k}$. It is easy to prove that if $H = S_k \times S_{n-k}$ then it is maximal if $k \neq n-k$. We may assume $k < n-k$. Let g be an element of S_n that does not belong to H and consider the subgroup K generated by H and g, we want to show that $K = S_n$.
Let us call $\Omega_1 = \{1, \dots, k\}$ and $\Omega_2 = \{k+1, \dots, n\}$ the two orbits of H. Since g is not in H and $|\Omega_2| > |\Omega_1|$ there exists i and j in Ω_2 such that $(g.i, g.j)$ is in $\Omega_1 \times \Omega_2$. The transposition (i, j) is an element of H thus $(g.i, g.j)$ is in H^g. Conjugating the transposition $(g.i, g.j)$ with elements of H we get all the elements of $S_n \setminus H$ hence the group generated by H and g is the whole group S_n for every g in $S_n \setminus H$. Therefore H is maximal.
It remains the case $k = n-k$, if $n = 2k$ and $H = S_k \times S_k$ then we can find elements of S_{2k} that interchange the two orbits, for instance the permutation $g = (1, k+1)(2, k+2)\cdots(k, 2k)$ is such that $\langle H, g\rangle = H$.
Therefore we have completely characterized intransitive maximal subgroups of S_n.

3.2 Transitive imprimitive subgroups

In the last case of the previous construction we have seen that the subgroup $S_k \times S_k$ of S_{2k} is not maximal because we can find elements that normalise it. If we take the wreath product $S_k \wr S_2$ then it is maximal, more in general if $n = km$ then $S_k \wr S_m$ is a transitive imprimitive subgroup of S_n and it is maximal. In

fact let us take an element g in $S_n \setminus H$ and consider the group generated by H and g. Since g is not in H there exists Ω_h and there exist i, j in Ω_H such that $g.i$ is in Ω_l while $g.j$ is an element of $\Omega_{l'}$, where $l \neq l'$. Then, as before, the permutation (i, j) is in H thus the permutation $(g.i, g.j)$ is in H^g and then (???).
In this way we have classified all the imprimitive maximal subgroups of S_n.

3.3 Primitive wreath product

Since we have already characterized all the imprimitive subgroups we have to discover the primitive ones. Suppose that $n = k^2$ an let us put the n points in a $k \times k$ matrix. Then we have an action of S_k on the rows and on the column of the matrix, observe that when S_k permutes the rows (resp. the columns) it leaves the colums (resp. the rows) fixed as a set. Since this two copies of S_k commute we get the group $H = S_k \times S_k$, but it is imprimitive since the arrows and the rows form a system of imprimitivity. To avoid this problem we have just to add a the reflection over the principal diagonal of the matrix, so that we get the group $S_k \wr S_2$, that is primitive.
Of course this construction can be generalising when $n = k^m$, with $k > 2$ and $m > 1$, obtaining $S_k \wr S_m$.

Proposition 3.1. *Let $H = S_k \wr S_m$ subgroup of S_{k^m}. If $k \geq 5$ and k^{m-1} is not divisible by 4 then H is maximal.*

3.4 Affine subgroups

If the order of Ω is a power of a prime, say it $|\Omega| = n = p^k$, then there is a one to one correspondence with the vector space $\mathbb{F}_p^k$ over the finite field with p elements $\mathbb{F}_p$. The group of all linear transformation of V is given by the semidirect product of the groups of translations and the general linear group. Since it permutes the points of V the groups of the automorphisms of the vector space is a subgroup of S_n. The translations are linear map of the form $t_a \colon v \mapsto v + a$ and they form a normal subgroup isomorphic to $\mathbb{F}_p^p$, cyclic group of order p. Hence the group of automorphisms, usually called *affine general linear group*, is $AGL_k(p) = C_p^k \rtimes GL_k(p)$.

3.5 Subgroup of diagonal type

These subgroup have the shape $T^k.(\mathrm{Out} \times S_k)$ where T is a non-abelian simple group. If φ is an outer automorphism of T and π in a permutation of S_k the the action of (φ, π) on the k-tuple $(a_1, \ldots, a_k)$ of elements of T is

$$(a_1, \ldots, a_k)^{(\varphi,\pi)} = (\varphi(a_{\pi(1)}), \ldots, \varphi(a_{\pi(k)}))$$

These subgroups are not easy to describe because of the presence of the non-abelian simple group.

3.6 Almost simple groups

The last kind of primitive subgroups of S_n are the almost simple groups. A group G is said to be *almost simple* if there exists a simple group T such that $T \leq G \leq Aut(T)$.
Let M be a maximal subgroup of G of index n, then the permutation action of G on the cosets of M is primitive, because M is the stabiliser of the coset $M1$ of M. Thus G embeds as a primitive subgroup of S_n.
These kind of groups are the most obscure, since to describe them one should know the maximal subgroups of all the almost simple groups.

Remark 3.2. These classes of subgroups just constructed are pairwise disjoint, this is clear from the different actions on Ω.

Let us give some properties of this class of subgroups of the symmetric group. The *socle* of a group G is the subgroup generated by all the minimal normal subgroup.

Proposition 3.3. *If G is an almost simple group, $T \leq G \leq \mathrm{Aut}(T)$, then $C_G(\mathrm{Inn}(T))$ is trivial.*

Proof. Let σ be in $C_G(\mathrm{Inn}(T))$ then for every t in T we have $\sigma^{-1}\varphi_t\sigma = \varphi_t$. Let x be an element of G, we have

$$t^{-1}xt = \varphi_t(x) = (\sigma^{-1}\varphi_t\sigma)(x) = (\sigma^{-1}(t))^{-1}x\sigma^{-1}(t)$$

Hence $\sigma^{-1}(t)t^{-1}$ lies in $Z(T)$, that is trivial because T is non-abelian and simple, therefore $\sigma = 1$. *q.e.d.*

Proposition 3.4. *Let G be a finite group. Let M be a normal subgroup of G, with $M = T_1 \cong \ldots \times T_k$, where $T_i \cong T$ simple and non-abelian. Then M is the unique minimal normal subgroup of G if, and only if, $C_G(M)$ is trivial.*

Proposition 3.5. *Let G be a finite group. Then G is almost simple if, and only if, G has a unique minimal normal subgroup non-abelian and simple.*

Proof. Suppose that G is almost simple. Then $T \leq G \leq \mathrm{Aut}(T)$. Since $T \cong \mathrm{Inn}(T) \trianglelefteq \mathrm{Aut}(T)$ we have $\mathrm{Inn}(T) \trianglelefteq G$ and it is simple, hence $\mathrm{Inn}(T)$ is minimal in G. Since $C_G(\mathrm{Inn}(T)) = 1$ then $\mathrm{Inn}(T)$ is the unique minimal normal subgroup of G.
Conversely, let N the unique minimal normal subgroup of G. Let g be an element of G and let φ_g an automorphism of N, i.e. $\varphi_g(n) = g^{-1}ng$. Then the map

$$\begin{array}{rccc} \varphi\colon & G & \to & \mathrm{Aut}(N) \\ & g & \mapsto & \varphi_g \end{array}$$

is clearly an homomorphism of group with trivial kernel. In fact, if g is an element of $\ker(\varphi)$ then g lies in $C_G(N)$ tha is trivial. Therefore G is almost simple. *q.e.d.*

Proposition 3.6. *Id G is an almost simple group then the socle is not regular*

Proof. see [**?**] *q.e.d.*

4 Preliminar results

We want to give some general result that will be really useful in the proof of the O'Nan-Scott Theorem. Suppose that G is a subgroup of S_n acting on $\Omega = \{1, \ldots, n\}$.

Proposition 4.1. *Let N be a normal subgroup of G. If G is primitive than N acts transitively.*

Proof. Suppose that N does not act transitively, thus we more than one orbit. Let $N.\alpha_i = \Omega_i$ for $i = 1, \ldots, k$ the orbits of N, so that $\Omega_1, \ldots, \Omega_k$ is a partition of Ω. Let g an element of the group G and α be in Ω_i then there exist n in N such that $n.\alpha_i = \alpha$. Hence

$$g.\alpha = g.(n.\alpha_i) = (gn).\alpha_i = n^g.(g.\alpha_i)$$

This means that if α and β belongs to the same set Ω_i then for every element g of G $g.\alpha$ and $g.\beta$ are in the same set of the partition, so $\Omega_1, \ldots, \Omega_k$ is a system of imprimitive for G. Hence, since G is primitive by hypothesis, we get that N acts transitively. *q.e.d.*

Proposition 4.2. *Let H and K be two different minimal normal subgroup of G, then they commute.*

Proof. Let us consider the commutator subgroup $[H, K]$ that is a subgroup of $H \cap K$ since $[h, k] = h^{-1}h^k = (k^{-1})^h k$, so we have $[H, K] \leq H \cap K \lhd G$. By minimality of H and K the intersection is 1 hence they commute. *q.e.d.*

Proposition 4.3. *Suppose G is primitive and N is a non-trivial normal subgroup of G. Let K be the stabiliser in G of a point, say it α. Then $KN = G$.*

Proof. Thanks to Proposition **??**, N is transitive, so $|Orb_N(\alpha)| = |\Omega|$. By the orbit-stabiliser theorem we have $|\Omega| = |N|/|N_\alpha|$, where $N_\alpha = N \cap K$ is the stabilizer in N of α. Applied the orbit-stabiliser theorem at the whole group we get $|\Omega| = |G|/|K|$. In particular we get $|G| = |N||K|/|N \cap K|$. Therefore $G = KN$. *q.e.d.*

In general for a given group G we say that a subgroup H is characteristic if it is fixed by all automorphisms of G. It is known that the property "be normal" is not transitive, for instance we can take the subgroup

$$K = \langle (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3) \rangle$$

in S_4 and $H = \langle (1, 2)(3, 4) \rangle$, it holds that $H \lhd K$ but $H \ntriangleleft S_n$. Let observe that if H is characteristic in G then H is normal in G.

Proposition 4.4. *If H is characteristic in N and N is normal in G then H is normal in G.*

Proof. Since N is normal in G we have that the inner automorphism of G are in $Aut(N)$ and so for every g in G we get $H^g = H$, therefore H is normal in G. *q.e.d.*

A non-trivial group G is said to be *characteristically simple* if it has no proper non-trivial characteristic subgroups.

Proposition 4.5. *A finite group G is characteristically simple if and only if it is the direct product of isomorphic simple groups.*

Proof. Suppose that G is characteristically simple and let T a minimal normal subgroup. Then for each α automorphism of G for which T^α is different from T we have $[T, T^\alpha] = 1$, by minimality. We can consider the product of all T^α and since this is a characterist subgroup of G it must be equal to the whole group G. In particular, it will be possible to find $\alpha_1, \ldots, \alpha_n$ such that $[T^{\alpha_i}, T^{\alpha_j}] = 1$ if $i \neq j$ and $\prod_{i=1}^n T^{\alpha_i} = G$.
The converse is straightforward. *q.e.d.*

In particular, the minimal normal subgroup of a finite group G are characteristically simple, so they are direct product of isomorphic simple groups.

A group G acting on Ω is said to be *regular* if for each couple (a, b) of elements of Ω there is exactly one permutation in G that sends a in b. If G is regular than it is transitive. Moreover $|G| = |\Omega|$, an easy way to see this is to construct a bijective map from G to Ω, let α be an element of Ω and let $\Omega_\alpha = \{(\alpha, \beta) \mid \beta \in \Omega\}$ then $|\Omega| = |\Omega_\alpha|$. Let us consider the map $\varphi \colon \Omega_\alpha \to G$ with $\varphi((\alpha, \beta)) =$ the unique g in G such that $g.\alpha = \beta$, then φ is clearly bijective. If G is a regular group then every non-identity elements of G is fixed-point-free.

Proposition 4.6. *If G is a primitive group and N is a non-trivial normal subgroup of G, then either $C_G(N)$ is trivial or $C_G(N)$ is regular;*

Proof. First of all let observe that $C_G(N)$ is a normal subgroup of G, in fact if g is an element of $C_G(N)$ and x is an element of G then there exists m in N such that $x^{-1}mx = n$ and

$$(x^{-1}gx)n = (x^{-1}gx)(x^{-1}mx) = x^{-1}gmx = x^{-1}mgx = (x^{-1}mx)(x^{-1}gx) = n(x^{-1}gx)$$

hence $C_G(N)$ contains the conjugates of its elements and so it is normal.
If $C_G(N) \neq 1$ then by Proposition **??** it is transitive and N is transitive as well. Suppose that x in $C_G(N)$ has any fixed points, and choose α among them, then for every n in N we have

$$x.(n.\alpha) = (xn).\alpha = (nx).\alpha = n.\alpha$$

and since N is transitive we get that $x = 1$, hence $C_G(N)$ is fixed point free. This means that $C_G(N)$ is regular. *q.e.d.*

Corollary 4.7. *If G is primitive and N_1 and N_2 are non-trivial normal subgroups that commute. Then $N_2 = C_G(N_1)$ and vice versa. In particular, G contains at most two minimal normal subgroups, and if it has an abelian normal subgroup then it has only one minimal normal subgroup.*

Proof. Thanks to Proposition **??** N_1 is transitive, in particular $|N_1| \geq |\Omega|$. By hypothesis N_1 and N_2 commute, hence $N_1 \subseteq C_G(N_2)$ moreover $C_G(N_2)$ is regular by Proposition **??**. Therefore they have the same order and they are equal. *q.e.d.*

Corollary 4.8. *Let N_1 and N_2 two minimal nowmal subgroup of G, with the same notation $N_1 \cong N_2$.*

Proof. If $N_1 = N_2$ we have the result, so assume $N_1 \neq N_2$, since N_1 and N_2 are minimal we have $N_1 \cap N_2 = 1$. Let α be an element of Ω and let H be the stabiliser of α in N_1N_2. Thanks to the Proposition **??** we have $HN_1 = N_1N_2 = HN_2$, and by the third isomorphism theorem

$$K \cong K/(K \cap N_1) \cong KN_1/N_1 = N_1N_2/N_1 \cong N_2/(N_1 \cap N_2) \cong N_2$$

in the same way we get $K \cong N_1$. *q.e.d.*

The following result is known as *Dedekind modular law.*

Proposition 4.9. *Let G be a group and H, K, L be subgroup of G such that $L \leq H$. Then $H \cap (KL) = (H \cap K)L$.*

Proof. We have $H \cap K \leq H, K$ so $(H \cap K)L \leq H \cap (KL)$. Conversely, let g be in $H \cap (KL)$, this means that there exist k in K and l in L such that $g = kl$, and it belongs to H and KL. Since kl and l are in H we get that k must be in H. Thus $k \in H \cap K$, therefore $g \in (H \cap K)L$. *q.e.d.*

We say that a subgroup H of a group G is K-invariant (where K is a subgroup of G), if H is invariant under the action by conjugation of K. In symbol, H is K-invariant if $k^{-1}Hk = H$ for all k in K, that means $K \leq N_G(H)$.

Proposition 4.10. *Let G be a primitive group and N be a minimal normal subgroup of G. Let K be the stabiliser in G of a point. Then $K \cap N$ is maximal among the K-invariant proper subgroups of N.*

Proof. Suppose, by contradiction, that there exists a subgroup X of N such that $K \cap N < X < N$ that is K-invariant, i.e. $K \leq N_G(X)$.
There exist elements of X not in K, because $X \nsubseteq K$, so $K < KX$; and there exist elements of N not in X. Moreover thanks to the Dedekind modular law we have $N \cap (KX) = (N \cap K)X = X$ that is a proper subgroup of N. Therefore $K < KX < KXN = KN$ and by Proposition **??** $KN = G$, but this is absurd since K is maximal in G (because G is primitive and K is the stabiliser of a point). *q.e.d.*

5 Proof of the O'Nan-Scott Theorem

We can now prove the theorem. Let H be a subgroup of S_n not containing A_n, and let N be minimal normal subgroup of H. Let K be the stabiliser in H of a point, i.e. $K = Stab_H(\alpha)$, with α in Ω. We have two different cases:

N abelian In particular thanks to Corollary **??**, N is unique. If H is intransitive then we are in the first case of the theorem; if H is transitive imprimitive we are in the second case. So we may assume that H is primitive.
By Proposition **??**, since N is minimal so characteristically simple, N is an elementary abelian p-group. Moreover N abelian implies that $C_H(N) = N$ and it acts regularly, by Proposition **??** and Corollary **??**. Thus $p^k = |N| = |\Omega|$, for some p prime and k integer, so there is a one-to-one correspondence between Ω and $\mathbb{F}_p^k$, so H is a subgroup of $C_p^k \rtimes GL_k(\mathbb{F}_p)$, that is case *(4)* of the theorem.

N non abelian Suppose that there are two minimal normal subgroups N_1 and N_2, by Corollary **??** H can not have more than two normal subgroups. Then $N_1 \cong N_2$, by Corollary **??** and they act on Ω regularly, by Corollary **??** and Proposition **??**. Thus N_1 and N_2 acts on Ω in the same way, so there exists π in S_n such that π conjugates N_1 to N_2, i.e. $N_1^\pi = N_2$. Since $N_2 = C_H(N_1)$, by Corollary **??**, we get that π conjugates N_2 to N_1, as well. Hence the normal subgroup $N_1N_2 = N_1 \times N_2$ of H is a minimal normal subgroup in $\langle H, x\rangle$ and it is unique. So we may assume that there is an unique minimal normal subgroup which is non-abelian.
If N is simple then by the properties given in the section of the almost simple groups and by Proposition **??**, $C_H(N) = 1$. Therefore H is an almost simple group, case*(6)* of the theorem.
If N is not simple then by Proposition **??** it is product of isomorphic simple groups, $N = T_1 \times \ldots \times T_m$ with $T_i \cong T$ simple for all i and $m > 1$. Moreover by Proposition **??** we have $H = KN$.
Let us consider $\pi_i \colon N \to T_i$ the natural projection and let K_i be the image via π_i of $K \cap N$, thus $K \cap N \leq K_1 \times \ldots \times K_m$. We have two cases: either $K_i \neq T_i$ for some i (and therefore for all i) or $K_i = T_i$ for all i.

Case 1. Certainly, $K_1 \times \ldots \times K_m$ is K-invariant and by Proposition **??** we get $N \cap K$ is maximal among the K-invariant subgroups of N therefore $N \cap K = K_1 \times \ldots \times K_m$. We have already said that $H = KN$ hence K permutes the K_i transitively, namely we can see at K as subgroup of S_m. So we get that $H = KN = (T_1 \times \ldots \times T_m) \rtimes K \leq S_k \wr S_m$, case *(3)* of the theorem.

Case 2. Let $n = (t_1, \ldots, t_m)$ be an element of $N = T_1 \times \ldots \times T_m$, the *support* of n is the set $Supp(x) = \{i \mid t_i \neq 1\}$. Let x and y be in N, we have some basic properties of the support: $Supp(x) = Supp(x^{-1})$ and $Supp(xy) \subseteq Supp(x) \cup Supp(y)$.
Let Ω_1 be a minimal non-empty subset of $\{1, \ldots, m\}$ such that $K \cap N$ contains an element whose support is Ω_1. Let us consider the set X of elements in $K \cap N$ whose support is Ω_1. Then X' is non-empty and $X = X' \cup \{1\}$ is a subgroup of $N \cap K$, because if x and y are in $N \cap K$ then, by minimality of Ω_1, either $Supp(xy) = \emptyset$, i.e. $x = x^{-1}$, or $Supp(xy) = \Omega_1$. Therefore, thanks to π_i, X maps onto a normal subgroup of T_i and since T_i is simple it maps onto T_i for all i in Ω_1. Suppose that Ω_2 is another such set whose intersection with Ω_1 is non-empty. Let x be in $N \cap K$

for which $Supp(x) = \Omega_1$ and let y be in $K \cap N$ for which $Supp(y) = \Omega_2$ then $1 \neq [x, y]$ and $Supp([x, y]) \subseteq \Omega_1 \cap \Omega_2$. By minimality of Ω_1 we get $\Omega_1 \cap \Omega_2 = \Omega_1$. Therefore Ω_1 is a block in a block system invariant under the action of K (remember that K acts on N permuting the T_i's), hence invariant under $H = KN$. If in this block system all the blocks have size 1 we K should contains N, because K acts by conjugation on N permuting the T_i's, getting a contradiction. Hence these blocks in this block system cannot have size 1. We have $|\Omega| = n = |H : K|$, because K is the stabiliser of a point. Moreover $H = KN$ hence $KN/K \cong N/(N \cap K)$ therefore $|H : K| = |N : N \cap K|$.
If the block system is non-trivial, with $l > 1$ blocks of size k then $N \cong T^{kl}$ and $N \cap K \cong T^l$, thus $n = |T|^{(k-1)l}$. Therefore $H = T^{kl}K$ is a subgroup of $S_r \wr S_l$, with $r = |T|^{k-1}$, giving case *(3)* of the theorem.
If the block system is trivial then $l = 1$ and $k = m$. Hence $N \cap K \cong T$, i.e. $N \cap K$ is a diagonal copy of T in $T_1 \times \ldots \times T_m$, and $n = |T|^{m-1}$. Therefore H is a subgroup of $T^k.(\mathrm{Out}(T) \times S_k)$.
This concludes the proof of the O'Nan-Scott Theorem.

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