# Subgroup structure of simple (algebraic) groups

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## Group actions

Let $G$ be a group and $\Omega$ be a set. An action of $G$ on $\Omega$ is

$$G \times \Omega \to \Omega$$

such that

- $1.\omega = \omega$;
- $g.(h.\omega) = (gh).\omega$.

# Transitive action

The orbit of $\alpha \in \Omega$ is $G.\alpha = \{g.\alpha \mid g \in G\}$. We write $G/\Omega = \{G.\alpha \mid \alpha \in \Omega\}$.

## Definition

The action of $G$ on $\Omega$ is transitive if $|G/\Omega| = 1$.

## Example

1. Let $\Omega = \{1, \ldots, n\}$, we consider the standard action $S_n$;
2. Let $G$ be any group and $H \leq G$, the standard action of $G$ on $G/H$ is transitive.

## Transitive action

The stabilizer of $\alpha \in \Omega$ is $G_\alpha = \{g \in G \mid g.\alpha = \alpha\} \leq G$.

### Proposition

*Let $G$ act transitively on $\Omega$. Then the action is equivalent to the standard action of $G$ on $G/G_\alpha$ for any $\alpha$ in $\Omega$.*

## Primitive action

$G$ acts on $\Omega$. A system of imprimitivity is a non-trivial partition

$$\Omega = \Omega_1 \cup \ldots \cup \Omega_n$$

preserved by $G$, i.e. if $\alpha, \beta \in \Omega_i$ then $\forall g \in G, g.\alpha, g.\beta \in \Omega_j$. Whenever such partition exists we say the action to be imprimitive.

### Definition

An action is primitive if it is transitive and it is not imprimitive.

### Proposition

- *The action of $G$ on $\Omega$ is primitive if, and only if, $G_\alpha$ is maximal.*
- *The primitive action of $G$ on $\Omega$ is equivalent to the standard action of $G$ on $G/G_\alpha$.*

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An algebraic group is a closed subgroup of $\mathsf{GL}_n(k)$, where $n \geq 1$ and $k$ is an algebraically closed field of characteristic $p \geq 0$.

### Example

$$\mathsf{SL}_n(k) = \{A \in \mathsf{GL}_n(k) \mid \det A = 1\}$$

### Abstract group theory

- subgroup
- normal subgroup
- simple group

### Algebraic group theory

- closed subgroup
- closed normal subgroup
- simple algebraic group

Simple algebraic groups are classified by root system (Dynking), as

- Classical

$$A_n, B_n, C_n, D_n$$

- Exceptional

$$E_6, E_7, E_8, F_4, G_2$$

**Remark**

*$SL_{n+1}(k) \ncong \mathrm{PSL}_{n+1}(k)$, both in $A_n$.*

# CFSG (1832–2004)

The finite simple groups are

(I) Cyclic, $C_p$, $p$ prime;

(II) Alternating $A_n$, $n \geq 5$;

(III) Lie type:

- Classical

$$\mathrm{PSL}_n(q), \mathrm{PSU}_n(q), \mathrm{PSp}_n(q),$$
$$\mathrm{P}\Omega_{2n+1}(q), \mathrm{P}\Omega^+_{2n}(q), \mathrm{P}\Omega^-_{2n}(q)$$

- Exceptional

$$G_2(q), F_4(q), \ldots, {}^2F_4(2)'$$

(IV) 26 sporadic groups.

A group $G$ is almost simple if there exists a simple group $T$ such that

$$T \trianglelefteq G \leq \mathrm{Aut}(T)$$

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# The O'Nan - Scott Theorem

For $G$, $S_n$ or $A_n$. Let $H$ be a subgroup of $S_n$ or $A_n$ not containing $A_n$. Then either $H \in \mathcal{A}(G)$ or $H \in \mathcal{S}$.

- $\mathcal{A}_1$ subset stabilisers (intransitive);
- $\mathcal{A}_2$ stabilisers of certain partitions of $\Omega$, (imprimitive);
- $\mathcal{A}_3$ stabilisers of cartesian product structures of $\Omega$, (primitive wreath product);
- $\mathcal{A}_4$ stabilisers of affine structure of $\Omega$;
- $\mathcal{A}_5$ stabilisers of $T^k$, (diagonal type).
- $\mathcal{S}$ Primitive almost-simple groups.

## Structure of the $\mathcal{A}_i$ families

| $\mathcal{A}_i$ | Structure in $S_n$ | comments |
|---|---|---|
| $\mathcal{A}_1$ | $S_k \times S_{n-k}$ | $k \neq n/2$ |
| $\mathcal{A}_2$ | $S_m \wr S_t$ | $n = mt$ |
| $\mathcal{A}_3$ | $S_k \wr S_d$ | $n = k^d$ |
| $\mathcal{A}_4$ | $\mathrm{AGL}_d(p) = p^d : \mathrm{GL}_d(p)$ | $n = p^d$, $p$ prime |
| $\mathcal{A}_5$ | $T^k.(S_k \times \mathrm{Out}(T))$ | $n = \lvert T \rvert^{k-1}$, $k \geq 2$ |

# Example: $\mathcal{A}_3$

The family $\mathcal{A}_1, \mathcal{A}_2$ complete classifies imprimitive maximal subgroups of $A_n$.

Let $n = k^2$, put the points of $\Omega = \{1, \dots, k^2\}$ into a matrix

$$\begin{pmatrix} 1 & 2 & \dots & k \\ k+1 & k+2 & \dots & 2k \\ \vdots & & \ddots & \vdots \\ (k-1)k & \dots & \dots & k^2 \end{pmatrix}$$

And $H \cong S_k \times S_k$ is imprimitive in $S_n$. Adjoining the permutation that reflects in the main diagonal we get

$$S_k \wr S_2$$

that is primitive and maximal (for $k \geq 5$).

$S_3 \wr S_2 \leq S_9$ is not maximal.

## The $\mathcal{S}$ family

Let $G$ be an almost simple group, i.e.

$$T \trianglelefteq G \leq \mathrm{Aut}(T)$$

Let $M$ be a maximal subgroup of $G$. The action of $G$ on $G/M$ is primitive. Therefore

$$G \hookrightarrow S_{|G:M|}$$

## Classification of Maximal subgroups of $S_n, A_n$

**Theorem (Liebeck - Praeger - Saxl, 1987)**

*All the maximal subgroup of $S_n$ and $A_n$ are classified.*

# Aschbacher theorem

Let $G$ be a finite simple group of Lie type (classical, $Cl(V)$). Let $H$ be a maximal subgroup of $G$. Then either $H \in \mathcal{C}(G)$ or $H \in \mathcal{S}$.

- $\mathcal{C}_1$ stabilizers of totally singular or non-singular subspaces;
- $\mathcal{C}_2$ stabilizers of decomposition $V = \bigoplus_{i=1}^{t} V_i$, dim $V_i = n/t$;
- $\mathcal{C}_3$ stabilizers of extension/subfield field of $\mathbb{F}_q$ of prime index $b$;
- $\mathcal{C}_5$ stabilizers of tensor product decompositions $V = \bigotimes_{i=1}^{t} V_i$, dim $V_i = a$;
- $\mathcal{C}_6$ $\dim V = r^m$ and $H$ is "local";
- $\mathcal{C}_7$ stabilizers of bilinear or quadratic form.

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- $\mathcal{C}_7$ stabilizers of bilinear or quadratic form.
- $\mathcal{S}$ almost simple groups acting irreducibly and tensor-indecomposably.

| $\mathcal{C}_i$ | rough tructure in $\mathsf{GL}_n(q)$ | comments |
|---|---|---|
| $\mathcal{C}_1$ | maximal parabolic | |
| $\mathcal{C}_2$ | $\mathsf{GL}_m(q) \wr S_t$ | $n = mt$ |
| $\mathcal{C}_3$ | $\mathsf{GL}_a(q^b).b$ | $n = ab$, $b$ prime |
| $\mathcal{C}_4$ | $\mathsf{GL}_n(q_0)$ | $q = q_0^b$, $b$ prime |
| $\mathcal{C}_5$ | $(\mathsf{GL}_a(q) \circ \ldots \circ \mathsf{GL}_a(q)).S_t$ | $n = a^t$ |
| $\mathcal{C}_6$ | $(\mathbb{Z}_{q-1} \circ r^{1+2a}).\mathrm{Sp}_{2a}(r)$ | $n = r^a$, $r$ prime |
| $\mathcal{C}_7$ | $\mathrm{Sp}_n(q), \mathrm{SO}_n(q), \mathrm{U}_n(q^{1/2}) < \mathrm{SL}_n(q)$ | |

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# Liebeck-Seitz Theorem

Let $G = \mathrm{Cl}(V)$ (over $k$, algebraically closed) and $H$ be a closed maximal subgroup. Then either $H \in \mathcal{C}(G)$ or $H \in \mathcal{S}$.

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# From algebraic to finite: Frobenius automorphism

Let $k$ be an algebraically closed field of characteristic $p > 0$.

$$\begin{aligned} \sigma : \mathrm{GL}_n(k) &\to \mathrm{GL}_n(k) \\ (a_{ij}) &\mapsto (a_{ij}^{p^m}) \end{aligned}$$

Then, say $q = p^m$,

$$\mathrm{GL}_n(q) = \left(\mathrm{GL}_n(k)\right)^{\sigma} = \{x \in \mathrm{GL}_n(k) \mid \sigma(x) = x\}$$

If $G$ is any algebraic group $G_0 = G^{\sigma}$ is a finite simple group.

**Remark**

*It is possible to deduce the Aschbacher theorem by the Leibeck-Seitz theorem.*

# Lang's Theorem

**Theorem**

*Let $G \leq GL_n(k)$ be a connected linear algebraic group, where $k$ is an a.c. field of characteristic $p > 0$. Then the map $g \mapsto g^{-1}\sigma(g)$ is surjective.*

Thanks to this if we know the conjugacy classes of subgroups in $G$ we know the conjugacy classes of the image subgroups in $G^{\sigma}$, as well.

- Algebraic

$$E_6, E_7, E_8, F_4, G_2$$

- Finite
  - of Lie type

$$G_2(q), F_4(q), E_6(q), {}^2E_6(q), {}^3D_4(q), E_7(q), E_8(q)$$

  - Suzuki-Ree groups

$${}^2B_2(2^{2n+1}), {}^2G_2(3^{2n+1}), {}^2F_4(2^{2n+1})$$

  - Tits group

$${}^2F_4(2)'$$