Primitive $\mathcal{C}_2$-actions of classical groups

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Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group with point stabiliser H. Hence, we may assume $\Omega = G/H$.

Definition

We say G is primitive if there is no non-trivial G-invariant partition of Ω. Equivalently, if, and only if, $H < G$ is maximal.

For $x \in G$ we define the fixed point space

$$C_\Omega(x) = \{\omega \in \Omega : x.\omega = \omega\}$$

Example

1. Let $G = S_n$ in the standard action on $\Omega = \{1, \ldots, n\}$. Then $H = S_{n-1}$ and the action is primitive.

2. Let $G = S_a \wr S_2 \leqslant S_{2a}$ acting on $\Omega = \{1, \ldots, a, a+1, \ldots, 2a\}$. Let $\Omega_i = \{(i-1)a + 1, \ldots, ia\}$, then $\Omega = \Omega_1 \cup \Omega_2$ is a G-invariant partition. The action is imprimitive.

Let G be an affine algebraic group over a field k. **Prototype:** $\mathrm{SL}_n(k) = \{x \in \mathrm{GL}_n : \det(x) = 1\}$.

Let $\Omega = G/H$ a primitive G-variety, i.e. $H < G$ is a maximal closed subgroup.

Then for any $x \in G$, $C_\Omega(x) \subseteq \Omega$ is a subvariety. We define

$$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega}$$

General aim of the project

Let G be an affine algebraic group defined over an algebraically closed field k of characteristic $p \geqslant 0$. Let $\Omega = G/H$ be a primitive G-variety.

1. Obtain global bounds on $f_\Omega(x)$, $x \in G$ of prime order.
2. Characterise elements which realise the bounds (whenever possible).
3. For $G = \mathcal{C}\ell(V)$ classical; obtain local bounds depending on $\nu(x)$, where $\nu(x)$ is the codimension of the largest eigenspace of x on V.

Finite groups

Assume $G \leq \mathrm{Sym}(\Omega)$ is transitive with point stabiliser H. Then for $x \in G$, the fixed point ratio is

$$\mathrm{fpr}(x) = \frac{|C_\Omega(x)|}{|\Omega|} = \frac{|x^G \cap H|}{|x^G|}$$

Example

Let $G = S_n$ and $n > 2$ in the standard action on $\Omega = \{1, \dots, n\}$. Let $\pi = (1,2)$. Then

$$\mathrm{fpr}(\pi) = \frac{n-2}{n} = 1 - \frac{2}{n}$$

Or, $\pi^G = \{(i,j) : i \neq j\}$ therefore $|\pi^G| = \frac{n(n-1)}{2}$. Similarly, notice $H = S_{n-1}$ and $|\pi^G \cap H| = \frac{(n-1)(n-2)}{2}$.

Some results, finite groups

There are several results concerning bounds on $\mathrm{fpr}(\cdot)$.

- (Liebeck-Saxl,'89) G almost simple of Lie type, over $\mathbb{F}_q$. Then $\mathrm{fpr}(x) \leq \frac{4}{3q}$ or (G, Ω, x) lies in a list of known exception.
- (Burness '07, Liebeck-Shalev '99) G almost simple classical, H non-subspace. Then $\mathrm{fpr}(x) \lesssim |x^G|^{-1/2}$.
- (Frohardt-Magaard, '00) $G = \mathcal{C}\ell(V)$ over $\mathbb{F}_q$, $H = G_U$ with $\dim U = k$. Then $\mathrm{fpr}(x) \lesssim q^{-\nu(x)k}$.

Applications: Minimal degree ($\mu(G) \leq \frac{n}{3}$), Cameron-Kantor base size conjecture ($b(G) \leq c$), Guralnick-Thompson genus conjecture, minimal generation of simple groups...

Some results, algebraic groups

Let G be a simple algebraic group acting on the primitive G-variety $\Omega = G/H$.

- (Burness, '03) G adjoint, H is non-subspace. There exists an involution $x \in G$ for which $f_{\Omega}(x) \geq \frac{1}{2} - \epsilon$, or (G, H) 'known'.
- (Lawther-Liebeck-Seitz, '02) G exceptional, $x \in G$ of prime order. Then $f_{\Omega}(x) \leq \alpha(G, H)$.

This motivates our investigation on upper and lower bounds on $f_{\Omega}(x)$ for $x \in \mathcal{C}\ell(V)$ of any prime order.

Let k be an algebraically closed field of characteristic $p \geqslant 0$. We define the following groups

- $\mathrm{GL}_n(k) = \{x \in k^{n^2} : \det(x) \neq 0\}$
  $\cong \{(x, y) \in k^{n^2} \times k : \det(x)y = 1\}$;
- $\mathrm{SL}_n(k) = \{x \in \mathrm{GL}_n(k) : \det(x) = 1\}$;
- $\mathrm{O}_n(k) = \{x \in \mathrm{GL}_n(k) : x^t x = I_n\}$ for $p \neq 2$;
- $\mathrm{Sp}_n(k) = \{x \in \mathrm{GL}_n(k) : x^t Jx = J\}$ for

$$J = \begin{pmatrix} & I_{n/2} \\ -I_{n/2} & \end{pmatrix}$$

Notice that these groups are affine algebraic varieties.

Subgroup structure of classical groups

Let $G = \mathcal{C}\ell(V)$ be a classical group over an algebraically closed field k of characteristic $p \geqslant 0$. We define

- Geometric subgroups
  - $\mathcal{C}_1$ reducible action on V;
  - $\mathcal{C}_2$ irreducible and imprimitive action on V;
  - $\mathcal{C}_3$-$\mathcal{C}_6$ irreducible and primitive action on V.
- Non-geometric subgroups; $\mathcal{S}$, irreducible, tensor-indecomposable action on V.

Theorem (Liebeck-Seitz, '98)

Let $H \leqslant G$ be closed. Then $H \leqslant M$ for some $M \in (\bigcup_i \mathcal{C}_i) \cup \mathcal{S}$.

For example, $\mathcal{C}_2$-subgroups are defined to be the stabilisers of decompositions of the form

$$V = V_1 \oplus \ldots \oplus V_t$$

where $\dim V_i = m$. Hence if $G = \mathrm{GL}_n, \mathrm{Sp}_n$,

$$H = (\mathrm{GL}_m \times \ldots \times \mathrm{GL}_m).S_t = \mathrm{GL}_m \wr S_t$$
$$H = (\mathrm{Sp}_m \times \ldots \times \mathrm{Sp}_m).S_t = \mathrm{Sp}_m \wr S_t$$

The connected component containing 1 is

$$H^\circ = (\mathrm{GL}_m)^t, \qquad H^\circ = (\mathrm{Sp}_m)^t$$

And for $\Omega = G/H$ we have

$$\dim \Omega = \dim G - \dim H$$

Jordan decomposition

For any $x \in \mathrm{GL}_n$

$$x = x_s x_u = x_u x_s$$

where x_s semisimple, x_u unipotent.

Theorem (Linearization theorem)

Let G be an affine algebraic group. Then for some n there exists a closed embedding $G \hookrightarrow \mathrm{GL}_n$.

Theorem (Jordan decomposition)

Let $x \in G$ and $\rho\colon G \hookrightarrow \mathrm{GL}_n$. Then $x = x_s x_u = x_u x_s$ such that $\rho(x_s)$ is semisimple and $\rho(x_u)$ is unipotent. Furthermore $x = x_s x_u$ is independent of the chosen embedding.

Jordan decomposition

Theorem

Let $G = \mathcal{C}\ell(V)$ and $x, y \in G$ of prime order r. Assume $(r,p) \neq (2,2)$ if $G = \mathrm{Sp}_n$ or O_n. Then $x \sim_G y$ if, and only if, $x \sim_{\mathrm{GL}_n} y$.

Up to conjugation, an element of prime order r is

$$\begin{aligned} x_s &= [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] && r \neq p \\ x_u &= [J_p^{a_p}, \ldots, J_1^{a_1}] && r = p \end{aligned}$$

With some known conditions on the a_i's for $G = \mathrm{Sp}_n$ or O_n.

We have

$$\dim x^G = \dim G - \dim C_G(x)$$

E.g., for $r \neq p$ we have $C_{\mathrm{GL}_n}(x)^\circ = \prod_i \mathrm{GL}_{a_i}$.

Main tools I

Let G be an affine algebraic group over k a.c. acting on $\Omega = G/H$ where $H \leqslant G$ closed.

Proposition (Lawther-Liebeck-Seitz, '98)

Let $x \in H$. Then $\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$.

Theorem (Guralnick, '07)

Assume G is reductive. If H° is reductive then $x^G \cap H = x_1^H \cup \ldots \cup x_l^H$.

Note: $H \in \mathcal{C}_2$ is reductive. Hence $\dim(x^G \cap H) = \max\{\dim x_i^H\}$.

For the remainder, $G = \mathrm{GL}_n, \mathrm{Sp}_n$ and $H = \mathrm{GL}_{\frac{n}{t}} \wr S_t$ or $\mathrm{Sp}_{\frac{n}{t}} \wr S_t$.

Main tools II

Observe that for x of prime order r

$$x^G \cap H = \bigcup_{\substack{\pi \in S_t \\ |\pi| = r}} (x^G \cap H^\circ \pi)$$

Lemma

Let $\tau \in \pi^{S_t}$. Then $\dim(x^G \cap H^\circ \tau) = \dim(x^G \cap H^\circ \pi)$

Lemma (Liebeck-Shalev, '99)

Let $x = [x_1, \ldots, x_t]\pi_h \in H$ of prime order r and $\pi_h \leftrightarrow (r^h, 1^f)$. Then x is H°-conjugate to $[I_{n/t}, \ldots, I_{n/t}, x_{hr+1}, \ldots, x_t]\pi_h$.

Fact:

$$\pi_h \sim_G [J_p^{\frac{n}{t}h}, J_1^{\frac{n}{t}f}] \qquad r = p$$

$$\pi_h \sim_G [I_{\frac{n}{t}(h+f)}, \omega I_{\frac{n}{t}h}, \ldots, \omega^{r-1} I_{\frac{n}{t}h}] \quad r \neq p$$

Example

Assume $p = 3$. Let $G = \mathrm{GL}_{40}$ and $H = \mathrm{GL}_5 \wr S_8$. Let $x = [J_3^5, J_2^7, J_1^{11}]$.

And $f_\Omega(x) = \frac{303}{700}$.

Example

Let ω be an r-th root of 1; $r \neq p$. Let $G = GL_{18}$ and $x = [I_2, \omega I_6, \omega^2 I_2, \omega^3 I_5, \omega^4 I_3] \in \mathrm{GL}_3 \wr S_6$.

Example

Let ω be an r-th root of 1; $r \neq p$. Let $G = GL_{18}$ and $x = [I_2, \omega I_6, \omega^2 I_2, \omega^3 I_5, \omega^4 I_3] \in \mathrm{GL}_3 \wr S_6$.

One of the possible optimal distribution of eigenvalues:

Notation

$G = \mathrm{GL}_n, \mathrm{Sp}_n$ and $H = L \wr S_t$ where $L = \mathrm{GL}_{\frac{n}{t}}, \mathrm{Sp}_{\frac{n}{t}}$, respectively.

Theorem 1 (Global upper bounds)

Let $x \in H$ be of prime order r. Then

- $r = p$: $f_\Omega(x) \geqslant \frac{1}{r}$;
- $r \neq p, r \geqslant n$: $f_\Omega(x) \geqslant 0$;
- $r \neq p, r < n$: $f_\Omega(x) \geqslant \frac{1}{r} - \epsilon$, *with* $\epsilon > 0$ *small.*

Main results. Global

Assume $(G, p) \neq (\mathrm{Sp}_n, 2)$.

Theorem 2 (Global upper bound)

Let $x \in H$ be of prime order r. Then

$$f_\Omega(x) \leq 1 - \frac{2}{\operatorname{rank} G} - \iota$$

Moreover $\iota > 0$, small, only if $r = 2$ and $\operatorname{rank}(L) = 1$.

Theorem 3 (Characterisation, upper bound)

Let $x \in H$. Then $f_\Omega(x)$ realises the upper bound in Theorem 2 if, and only if,

- $\nu(x) = 1$ *for* $G = \mathrm{GL}_n$ *or,* Sp_n *and* $r = p$*;*
- $\nu(x) = 2$ *for* $G = \mathrm{Sp}_n$ *and* $r \neq p$*.*

Theorem 4 (Characterisation, lower bound - semisimple)

Assume $r \geqslant n$ and $r \neq p$. Let $x \in H$. Then $f_\Omega(x) = 0$ if, and only if, x is a regular element.

For $r < n$ and $r \neq p$ we construct a collection of elements $x \in G$ which realise the lower bound on f_{G/H°. Note $f_{G/H^\circ}(x) \leq f_{G/H}(x)$.

Theorem 5 (Characterisation, lower bound - unipotent)

Assume $r = p$, and p divides n if $p \leqslant n$. Let $x \in H$. Then $f_\Omega(x)$ realises the lower bound if, and only if, x is one of the following

- $p > n$: $x = [J_{n/t}^{t-1}, z]$ *and z any unipotent.*
- $p \leq n$: *If $p | \frac{n}{t}$ then $x = [x_1, \dots, x_t]$ where $x_i = [J_p^{n/pt}] \in L$ for $i < t$ and $x_t \in L$ is any unipotent such that*

$$\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$$

- $p \leq n$: *If $p \nmid \frac{n}{t}$ then $x = [J_p^{n/p}]$.*

Here $x = [I_s, -I_{n-s}]$ or $[J_2^s, J_1^{n-2s}]$. Assume $s \leqslant \frac{n}{2}$.

Theorem 6 (Involutions in GL_n)

Let $x \in H$ be an involution with $\nu(x) = s$. If $t = n$, or $\frac{n}{t}$ odd and either

(i) $p \neq 2$ *and* $s \geq \max\{n/t, (n-t)/2\}$

(ii) $p = 2$ *and* $n/t \leq s \leq (n-t)/2$

then

$$f_\Omega(x) = 1 - \frac{2s(n-s) - s}{n^2(1-\frac{1}{t})} + \frac{n-t}{2n(t-1)}$$

Otherwise, for $s \equiv b \mod t$ *and* $0 \leq b < t$,

$$f_\Omega(x) = f_{G/H^\circ}(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

Main results. Local

Theorem 7 (Local upper bounds)

Let $G = \mathrm{GL}_n$. Let $x \in H$ with $\nu(x) = s$. Then

$$f_\Omega(x) \leqslant U = \begin{cases} 1 - \frac{s}{n} & s > n/2 \\ 1 - \frac{2s(n-s)}{n^2} & s \leq n/2 \end{cases}$$

For semisimple elements we construct elements $x \in H$ with $\nu(x) = s$ for which

$$f_\Omega(x) \geq U - \frac{2}{n} - \frac{1}{16}$$

We can drop the $1/16$ term in some special case.

Theorem 8 (Local lower bounds - semisimple)

Let $G = \mathrm{GL}_n$. Let $x \in H$ semisimple with $\nu(x) = s$. Then

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{s(2n-s)}{n^3\left(1-\frac{1}{t}\right)}$$

Better (and sharp) bounds for $s < r$.

Also here we construct a collection of elements $x \in G$ which realise the lower bound for f_{G/H°.

Key results I

For the upper bounds some key results are:

Theorem (Burness, '05)

Let $x \in G$ of prime order. Then $\dim(x^G \cap H) \leq \left(\frac{1}{t} + \zeta\right) \dim x^G$, where $\zeta = 0$ unless $G = \mathrm{Sp}_n$ in which case ζ is known.

And the bound is sharp.

Proposition (Burness, Liebeck-Shalev)

Let $x \in G$ of prime order r and $\nu(x) = s$. Then

$$f(n,s) \leq \dim x^G \leq g(n,s)$$

For example, in the case $G = \mathrm{GL}_n$ we have

$$f(n,s) = \max\{ns, 2s(n-s)\}$$
$$g(n,s) = s(2n-s-1)$$

Key results II

Assume $r \neq p$ if $p \leq \mathrm{rank}(G)$.

Lemma

Let $x \in H^\circ$. Them there exists $y \in H$ with $\nu(y) \geqslant \nu(x)$ such that

$$f_{G/H^\circ}(x) \geqslant f_{G/H^\circ}(y)$$

An easy consequence of this is that in a finite number of steps we find an element which realises lower bound on f_{G/H°.

Proposition

Let r be a prime. Then there exists $\bar{x} \in H^\circ$ of order r such that for all $x \in H^\circ$ of order r

$$f_{G/H^\circ}(x) \geqslant f_{G/H^\circ}(\bar{x})$$

Key results II. Comments

1. If $r = p < \text{rank}(G)$ let $x \in H$ of prime order. Then $f_\Omega(x) \geq \frac{1}{p}$ proved with combinatorics methods using the formulae;
2. If $r = p \geqslant \text{rank}(G)$ we have $x^G \cap H = x^G \cap H^\circ$. And from $x = [x_1, \ldots, x_t]$ we define $y = [J_{\frac{n}{t}}, x_2, \ldots, x_t]$. Then $f_\Omega(x) \geqslant f_\Omega([J^t_{\frac{n}{t}}]) = \frac{t}{n}$.
3. If $r \neq p$. Then for all $x \in H$ of order r, $x^G \cap H^\circ \neq \emptyset$.

Future work

1. Obtain explicit formula for involutions in Sp_n in $\mathcal{C}_2$-actions;
2. Obtain same results for O_n in $\mathcal{C}_2$-actions;
3. Obtain same results for $G = \mathrm{GL}_n, \mathrm{Sp}_n, \mathrm{O}_n$ in other primitive geometric actions, i.e. $\mathcal{C}_1, \mathcal{C}_3, C_4, C_6$.