ON FIXED POINT SPACES FOR PRIMITIVE $\mathcal{C}_2$-ACTIONS OF CLASSICAL ALGEBRAIC GROUPS

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Abstract. Let G be a classical algebraic group over an algebraically closed field k of characteristic $p \geq 0$. Let H be a maximal closed subgroup of G and consider the primitive action of G on the coset variety $\Omega = G/H$. For $x \in G$, the fixed point space $C_\Omega(x)$ is a subvariety of Ω, so we can study its dimension and the ratio $\dim C_\Omega(x)/\dim \Omega$. In this report we focus on the case $G = \mathrm{GL}_n(k)$ or $\mathrm{Sp}_{2n}(k)$, and H is a subgroup of type $\mathrm{GL}_{\frac{n}{t}} \wr S_t$ or $\mathrm{Sp}_{2\frac{n}{t}} \wr S_t$, for $t > 1$ a divisor of n. These subgroups comprise the collection $\mathcal{C}_2$ in terms of the Liebeck-Seitz description of maximal subgroups of classical algebraic groups (hence the term $\mathcal{C}_2$ in the title). We obtain close to best possible upper and lower bounds on the ratio $\dim C_\Omega(x)/\dim \Omega$ for prime order elements x in G. We also obtain bounds for prime order elements $x \in \mathcal{V}_s = \{x \in G : \nu(x) = s\}$, where $\nu(x)$ is the codimension of the largest eigenspace of x on the natural module. Finally, for $G = \mathrm{GL}_n(k)$, we present a precise formula for $\dim C_\Omega(x)$ when x is an involution of G.

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Contents

1. Introduction

Let G be an affine algebraic group over an algebraically closed field k of characteristic $p \geq 0$. In this paper we consider primitive actions of G on coset varieties $\Omega = G/H$, where H is a maximal closed subgroup of G.

We shall give upper and lower bounds on the following ratio

$$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega} \tag{1}$$

for prime order elements $x \in G$, where $C_\Omega(x) = \{\omega \in \Omega : x\omega = \omega\}$ is the fixed point space of x. Note that $C_\Omega(x)$ is a subvariety of Ω, see Proposition 2.1. Furthermore, where possible we characterise the elements which realise the bounds.

There are a number of motivations for this problem, mainly arising from finite permutation groups. Several results on upper bounds to fixed point spaces are known. Much less is known for lower bounds.

1.1. Finite groups.

1.1. Finite groups. Let G be a permutation group on a finite set Ω. For $x \in G$ it is defined the fixed point space of x, $C_\Omega(x)$. The *fixed point ratio* is $\mathrm{fpr}(x) = |C_\Omega(x)|/|\Omega|$, which is the proportion of elements in Ω fixed by x. If the action is transitive and $H = G_\alpha$ is a point stabiliser, then it is easy to prove that

$$\mathrm{fpr}(x) = \frac{|x^G \cap H|}{|x^G|}$$

it is enough to count the pairs $\{(\omega, y) : \omega \in \Omega, y \in x^G, y\omega = \omega\}$ in two different ways. Hence the computation of the fixed point ratio reduce to an abstract group theoretic problem, about conjugacy classes.

In the literature there are several results which provide bounds on fixed point ratios for finite simple groups of Lie type. For instance in [20], it is proved that for all almost simple groups of Lie type G, defined over the field $\mathbb{F}_q$, acting faithfully and transitively on a set Ω, either $\mathrm{fpr}(x) \leq \frac{4}{3q}$ for all non-identity elements $x \in G$, or the triple (G, Ω, x) is known.

Note that for $x \in G$ we have $C_\Omega(x) \subseteq C_\Omega(x^l)$ for any integer l, therefore, in order to get upper bounds on fixed point ratios it is enough to consider prime order elements in G.

Fixed point ratios for finite permutation groups have been widely studied, in particular for (almost) simple groups, and bounds on fixed point ratio have found many important applications, in particular upper bounds, see [10, 14, 19].

For example, in [23], Liebeck and Shalev studied almost simple finite classical groups G in primitive action on G/H, where H is a maximal non-subspace subgroup; the main result is [23, Theorem ($\star$)], in which they proved that there exists a constant $\epsilon > 0$, such that for a prime order element $x \in G$, $|x^G \cap H| < |x^G|^{1-\epsilon}$ which translates to upper bound on the fixed point ratio. Then in the series of paper [6, 7, 8, 9], Burness proved $\epsilon \approx \frac{1}{2}$.

In [14], Guralnick and Kantor obtained upper bounds on fixed point ratios of elements in a finite classical group G, defined over the finite field $\mathbb{F}_q$, acting on k-subspaces of the natural n-dimensional module in terms of q, n, k. Then in [12] generalizing this result, in the same hypothesis, it is proved that $\mathrm{fpr}(x) \lesssim q^{-sk}$ where s is the codimension of the largest eigenspace of x.

1.2. Algebraic groups. Recall that an algebraic group is an algebraic variety with a group structure in which multiplication an inversion maps are morphism of varieties. Let H be a closed subgroup of an algebraic group G. Then the natural action of G on the coset variety G/H is transitive. For $x \in G$ it is defined the fixed point space of x, $C_\Omega(x)$.

In [18] it is proved

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$$

The natural analogue of the fixed point ratio for algebraic groups is $\dim C_\Omega(x) - \dim \Omega$, which gives the measure of the space not fixed by x. It is straightforward to see that $C_\Omega(x)$ is non-empty if, and only if, $x^G \cap H \neq \emptyset$. Therefore we may reduce to consider only prime order elements in H.

We say that the coset variety $\Omega = G/H$ is a primitive variety if H is a closed maximal subgroup of G.

The main results on fixed point spaces on simple algebraic groups in primitive actions are in [4, 5, 18]. In [18] it has been studied fixed point spaces of simple algebraic groups of exceptional type in primitive actions. They give lower bounds on the difference $\dim \Omega - \dim C_\Omega(x)$ for prime order elements $x \in G$, which translate to upper bounds on the dimension of the fixed point spaces, for example if $G = E_6$ and $\Omega = G/H$, for H any closed maximal subgroup and x is any non-identity element then $\dim \Omega - \dim C_\Omega(x) \geq 4$. Then in [19] they used results of [18] to obtain bounds on the fixed point ratios for exceptional groups.

In [5], Burness proves that in any simple algebraic group of classical type acting on the primitive variety $\Omega = G/H$ where H is a non-subspace subgroup, for $x \in G$ of prime order either we have

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leq \frac{1}{2}$$

or (G, H) lies in a short list of known exceptions. Again, these bounds translate to upper bounds on the dimension of fixed point spaces.

In [4], for primitive actions of simple algebraic groups it is proved that, unless (G, H) lies in a short list of known exceptions, there exists an involution $x \in G$ for which

$$f_\Omega(x) \geq \frac{1}{2} - \frac{1}{2h+1}$$

where h is the Coxeter number of the group (recall $\dim G = (h+1)\text{rank } G$). This results provides, in fact a lower bound to the maximal dimension of the fixed point spaces of involutions.

The aim of this work is to extend this last result to any prime order elements. Namely we shall provide sharp upper bounds and give close to the best possible lower bounds on $f_\Omega(x)$, where $x \in \mathrm{GL}_n$ or Sp_{2n} and $\Omega = G/H$ for H a $\mathcal{C}_2$-subgroup, defined below.

1.3. **Notation.** In order to state our main results we need to introduce some notation.

We consider the general linear group and the symplectic group defined over an algebraically closed field k of arbitrary characteristic $p \geq 0$.

In [21] several families of subgroups of a classical algebraic groups are defined. Mainly there are two type of subgroups: *geometric* which constitutes the $\mathcal{C}_i$ families, for $i = 1, \ldots, 6$, and *non-geometric* subgroups, in the family $\mathcal{S}$. The geometric subgroups are so called because they are defined in terms of the geometry of the natural module. Roughly, a subgroup of any classical algebraic group G with natural module V, either acts reducibly on V or it acts irreducibly, in the latter case the action is either imprimitive or primitive.

The $\mathcal{C}_2$-subgroups are precisely those subgroups acting irreducibly and imprimitively on the natural module V. Given a direct sum decomposition in equidimensional subspaces of the natural module $V = V_1 \oplus \ldots \oplus V_t$ with $t > 1$ a divisor of $\dim V$, the subgroups of G which stabilizes this decomposition forms the $\mathcal{C}_2$ family. For $G = \mathrm{GL}_n$, a $\mathcal{C}_2$-subgroup is $H = \mathrm{GL}_{\frac{n}{t}} \wr S_t$ and for $G = \mathrm{Sp}_{2n}$, $H = \mathrm{Sp}_{2\frac{n}{t}} \wr S_t$, where $t > 1$ is a divisor of n.

We denote by H° the connected component containing the identity of the subgroup H, clearly $\dim(x^G \cap H^\circ) \leq \dim(x^G \cap H)$. Therefore, for the purpose of obtaining lower bounds on the ratio (1), we define the following related ratio

$$f^\circ_\Omega(x) = \frac{\dim\Omega - \dim x^G + \dim(x^G \cap H^\circ)}{\dim\Omega} \tag{2}$$

and, for any $x \in H$, $f^\circ_\Omega(x) \leq f_\Omega(x)$.

A non trivial element in $G = \mathrm{GL}_n$ or Sp_{2n} is *semisimple* if it is conjugate to a diagonal matrix and *unipotent* if its only eigenvalue is 1. For any element x in G we have its Jordan decomposition $x = x_s x_u$, where x_s is a semisimple element and x_u is unipotent and $[x_s, x_u] = 1$.

In Table 1, we define some subsets of a classical algebraic group G. The main relation

$\mathcal{P}$	prime order elements
$\mathcal{U}$	unipotent elements
$\mathcal{U}_p$	unipotent elements of prime order p
$\mathcal{S}$	semisimple elements
$\mathcal{S}_r$	semisimple elements of prime order r
$\mathcal{I}$	involutions

Table 1.

between these sets is that $\mathcal{P} = \mathcal{U}_p \cup (\cup_{r \text{ prime}} \mathcal{S}_r)$, moreover $\mathcal{I} = \mathcal{U}_2$ or $\mathcal{S}_2$ depending if $p = 2$ or $p \neq 2$.

Given any element $x \in \mathrm{GL}_n$ or Sp_{2n} we usually write $x = [A_1, \ldots, A_m]$ to denote the block diagonal matrix with the matrices $A_1, \ldots, A_m$ along the diagonal. Moreover J_i denotes a unipotent Jordan block of size i.

1.4. **Main results.**

1.4.1. *Global bounds.* First we state bounds on $f_\Omega(x)$ when $x \in H$ and x is a prime order element, i.e. $x \in \mathcal{U}_p, \mathcal{S}_r$. Since there is no connection between $\mathcal{U}_p, \mathcal{S}_r$ and the natural module we call these bounds *global*. When we write H, we mean a $\mathcal{C}_2$-subgroup of the group as defined before.

For lower bounds we shall avoid the case the element is an involution because we have more precise results in this case, which we shall state later.

Our main result on global upper and lower bounds for prime order elements is the following. Let $\alpha = 1$ if $G = \mathrm{GL}_n$ and $\alpha = 2$ if $G = \mathrm{Sp}_{2n}$.

Theorem 1. *Let $G = \mathrm{GL}_n$ or Sp_{2n} acting on the primitive variety $\Omega = G/H$. Let $x \in H \cap \mathcal{P}$. Then*

(i) *either $f_\Omega(x) \leq 1 - \frac{\iota}{n}$, where $\iota = 1$ for $G = \mathrm{Sp}_{2n}$ and $x \in \mathcal{U}_p$, or $\iota = 2$ otherwise, or (G, H°) is given in Table 2.*

(ii) *If $x \in H \cap \mathcal{U}_p$ and $p \neq 2$*

$$f_\Omega(x) \geq \begin{cases} \frac{1}{p} & p \leq n \\ \frac{t}{\alpha n} & p > n \end{cases}$$

(iii) *If $x \in H \cap \mathcal{S}_r$ and $r \neq 2$, $f_\Omega(x) \geq \frac{1}{r} - \epsilon$, where $\epsilon \geq 0$ is listed in Table 3*

G	H°	r	p	$f_\Omega(x) \leq$	$\bar{x}$
GL_n	$(\mathrm{GL}_1)^n$	$=2$	$=2$	$1-\frac{2}{n}+\frac{1}{n(n-1)}$	$[J_2, J_1^{n-2}]$
			$\neq 2$		$[I_1, -I_{n-1}]$
Sp_{2n}	$(\mathrm{Sp}_2)^n$	$=2$	$\neq 2$	$1-\frac{2}{n}+\frac{3}{2n(n-1)}$	$[I_2, -I_{2n-2}]$
		$=2$	$=2$	$1-\frac{1}{n}+\frac{1}{2n(n-1)}$	$[J_2, J_1^{2n-2}]$
Sp_4	$(\mathrm{Sp}_2)^2$	$=2$	$\neq 2$	$\frac{1}{2}$	$[I_2, -I_2]$
		$\neq 2$	$\neq 2$	$\frac{3}{4}$	$[I_2, \omega, \omega^{-1}]$
Sp_6	$(\mathrm{Sp}_2)^3$	$=2$	$\neq 2$	$\frac{7}{12}$	$[I_2, -I_4]$
Sp_8	$(\mathrm{Sp}_4)^2$	$r=2$	$\neq 2$	$\frac{5}{8}$	$[I_2, -I_6]$
Sp_8	$(\mathrm{Sp}_2)^4$	$r=2$	$\neq 2$	$\frac{5}{8}$	$[I_2, -I_6]$
Sp_{10}	$(\mathrm{Sp}_2)^5$	$r=2$	$\neq 2$	$\frac{27}{40}$	$[I_2, -I_8]$
		$r\neq 2$	$\neq 2$	$\frac{3}{5}$	$[I_8, \omega, \omega^{-1}]$
Sp_{12}	$(\mathrm{Sp}_4)^3$	$r=2$	$\neq 2$	$\frac{7}{12}$	$[I_2, -I_{10}]$
Sp_{12}	$(\mathrm{Sp}_2)^6$	$r=2$	$\neq 2$	$\frac{43}{60}$	$[I_2, -I_{10}]$

TABLE 2. Exceptional upper bounds

r	ϵ
$\geq \alpha n$	$\frac{1}{r}$
$< \alpha n$	$< \frac{1}{n} + \frac{rt^2}{4n^2(t-1)}$

TABLE 3. ϵ lower bound semisimple elements

Remark 1.1. In the case $G = \mathrm{GL}_n$ and $r = n = t$. In addition, we shall prove that $f_\Omega(x) \geq \frac{1}{r}$, i.e. $\epsilon = 0$.

Remark 1.2. As already said, given $x \in G$, $x = x_s x_u$ where x_s is a semisimple element and x_u is a unipotent element. It holds $C_\Omega(x) = C_\Omega(x_s) \cap C_\Omega(x_u)$, thus $\dim C_\Omega(x) \leq \min\{\dim C_\Omega(x_s), \dim C_\Omega(x_u)\}$ hence the upper bound in Theorem 1 (i) holds for all non-trivial elements in G.

Remark 1.3. As already observed, we can write $\mathcal{P}$ as the following disjoint union

$$\mathcal{P} = \mathcal{U}_p \cup \Big(\bigcup_{r \neq p} \mathcal{S}_r \Big)$$

Therefore the bounds of Theorem 1 leads to global bounds for elements of $\mathcal{P}$.

Corollary 1. *Let $x \in H \cap \mathcal{P}$. Then $0 \leq f_\Omega(x) \leq 1 - \frac{\iota}{n}$, where $\iota = 2$ for $G = \mathrm{GL}_n$ and $\iota = 1$ for $G = \mathrm{Sp}_{2n}$.*

Most of the bounds given are sharp, in the sense that with some conditions on n, t, r there exists an element for which equality holds. The upper bound is always sharp. Given $x \in G$ we denote by $\nu(x)$ the codimension of the largest eigenspace of x, we shall define it formally in Definition 2.8.

Theorem 2. *Let $y \in H$ of prime order. Then $f_\Omega(y) = 1 - \frac{\iota}{n}$ if, and only if, $\nu(x) = \iota$, where ι is as in Theorem 1 for $G = \mathrm{Sp}_{2n}$ and $\nu(x) = 1$ for $G = \mathrm{GL}_n$. Or, (G, Ω, y) is recorded in Table 2.*

The lower bound for unipotent elements given in Theorem 1 (ii), is sharp if p divides n or $\frac{n}{t}$. Let $\alpha = 1$ if $G = \mathrm{GL}_n$ and $\alpha = 2$ if $G = \mathrm{Sp}_{2n}$. Let $x \in G$ a unipotent element, in Section 2 we shall explain how to attach to x a partition of αn. We denote λ_x the partition associated to x.

Here is the formal statement.

Theorem 3. *Let $x \in H$ of prime order p. Then $f_\Omega(x) = \frac{t}{\alpha n}$ or $\frac{1}{p}$ if, and only if, one of the following conditions holds:*

(i) *$p > n$ and $x = [J^{t-1}_{\alpha\frac{n}{t}}, z]$, where $z \in \mathrm{GL}_{\frac{n}{t}}$ or $\mathrm{Sp}_{2\frac{n}{t}}$ is any unipotent element;*
(ii) *$p \leq n$, $n = mp$ and $x = [J_p^{\alpha m}]$;*
(iii) *$p \leq n$, $\frac{n}{t} = pm$ and $x = [J_p^{\alpha m(t-1)+a}, z]$ where $0 \leq a \leq m$ and $z \in \mathrm{GL}_{p(m-a)}$ or $\mathrm{Sp}_{2p(m-a)}$ such that $\lambda_z \vdash \alpha p(m-a)$ has no p parts and no proper subpartition of lp for any $1 \leq l < m-a$, with the additional condition l even for Sp_{2n}.*

Remark 1.4. By Theorem 3, it is clear that if $\nu(x)$, the codimension of the largest eigenspace of x on V, has the maximal possible value then $f_\Omega(x)$ is minimal. In the same way, by Theorem 2, $f_\Omega(x)$ is maximal if $\nu(x)$ is minimal. This is what one would expect since, given $x \in G$, larger $\nu(x)$ value means that x fixes subspaces of small dimension in this natural action on the natural module V, hence x hardly fixes direct sum decompositions of V. Similarly for smaller value of $\nu(x)$ we have that x fixes a large subspace of V.

1.4.2. *Involutions.* Elements of order 2 are called *involutions*. As said before, the major result concerning fixed point spaces in primitive actions of involutions in simple algebraic groups is given in [4].

When x is an involution, it turns out we can more precise result for the $\mathcal{C}_2$-subgroups we are considering here. Indeed for $x \in \mathrm{GL}_n$, involution, we can compute $\dim(x^G \cap H)$ and, therefore we have an explicit formula for $f_\Omega(x)$. At present, in the case $G = \mathrm{Sp}_{2n}$ we do not have such a formula, in general, in fact only for special value of t we have it, namely for $t = 2, n/2, n$. But, still, we have a precise formula for f°_Ω and we have good bounds for f_Ω depending on the f°_Ω-value.

First we state the result for involutions in GL_n. We write $s = at + b$ where $0 \leq b < t$.

Theorem 4. *Let $G = \mathrm{GL}_n$. Let $x \in H$ be an involution with $\nu(x) = s$. Then*

$$f_\Omega(x) = 1 - \frac{2s(n-s) - s}{n^2\left(1 - \frac{1}{t}\right)} + \frac{n-t}{2n(t-1)}$$

if, for $p \neq 2$, either $t = n$ or, $\frac{n}{t}$ is odd and $s \geq \max\left\{\frac{n}{t}, \frac{n-t}{2}\right\}$; and, for $p = 2$, either $t = n$ or, $\frac{n}{t}$ is odd and $\frac{n}{t} \leq s \leq \frac{n-t}{2}$. Otherwise

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

And the following is for semisimple involutions in Sp_{2n}, here we denote $\nu(x) = 2s$ and we write $s = at + b$, where $0 \leq b < t$.

Theorem 5. *Let $G = \mathrm{Sp}_{2n}$ and assume $p \neq 2$. Let $x = [I_{2s}, -I_{2n-2s}] \in H$. Then*

$$f^\circ_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

Furthermore, for $t < n$

$$f^\circ_\Omega(x) \leq f_\Omega(x) \leq f^\circ_\Omega(x) + \frac{t}{2n(t-1)}\left(\frac{1}{2} + \frac{t}{n}\right)$$

If $t = n$, then

$$f_\Omega(x) = 1 - \frac{s(2n-2s-1)}{n(n-1)}$$

If $t = 2$. Then $f_\Omega(x) = f_\Omega^\circ(x)$, unless $x = [I_n, -I_n]$, in which case

$$f_\Omega(x) = \frac{1}{2} + \frac{1}{2n}$$

Remark 1.5.

1.4.3. *Local bounds.* With the term *local* we mean bounds on $f_\Omega(x)$ where x ranges on elements with fixed $\nu(\cdot)$ value. As already said, $\nu(x)$ is the codimension of the largest eigenspace of x in its action on the natural module V. As observed in Remark 1.4 there is a connection between the natural action on V and the action on the primitive variety Ω.

A similar problem has been studied in [12] for finite simple classical groups in primitive subspace actions.

We consider the following subset of G. We shall study bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$.

$\mathcal{V}_s = \{x \in G : \nu(x) = s\}$
$\mathcal{V}_{s,\mathcal{P}} = \mathcal{V}_s \cap \mathcal{P}$
$\mathcal{V}_{s,r} = \mathcal{V}_s \cap \mathcal{S}_r$
$\mathcal{V}_{s,p} = \mathcal{V}_s \cap \mathcal{U}_p$

TABLE 4.

Theorem 6. *Let $x \in H \cap \mathcal{V}_s$ be an element of prime order. Then*

$$f_\Omega(x) \leq 1 - \gamma$$

G	s	γ
GL_n	$s \geq \frac{n}{2}$	$\frac{s}{n}$
	$< \frac{n}{2}$	$\frac{2s}{n} - \frac{2s^2}{n^2}$
Sp_{2n}	$s \geq n$	$\frac{s}{2n} - \frac{s}{2n^2}$
	$s < n$	$\frac{s}{n} - \frac{1}{2n} - \frac{s^2}{2n^2}$

TABLE 5. γ, lower bound semisimple elements

Remark 1.6. For $\nu(x)$ small, i.e. in $\{1, 2\}$, the bounds given in Theorem 6 are very closed to the global upper bounds given in Theorem 1. In fact, the difference is at most $\frac{1}{2n} + \frac{2}{n^2}$.

Remark 1.7. In general we are not able to classify element which realize the bounds. For example, if $G = \mathrm{GL}_n$, we prove in Lem 6.12,6.13 that there exists an element x in $\mathcal{V}_{s,r}$ for which the difference between the bound in Theorem 6 and $f_\Omega^\circ(x)$ is at most $\frac{2}{n} + \frac{m}{n}$ where $m = 0$ for $s \leq \frac{n}{2}$ and $m \equiv n \mod (n-s)$ for $s > \frac{n}{2}$.

The following are the last two results we shall obtain. The analysis of lower bounds for unipotent element is, in general, very hard, the main reason is that it is not obvious the general structure of such elements with fixed $\nu(\cdot)$ value. However we could get some results only in very specific case. We do not include this analysis in this report.

We deal with lower bounds for $f_\Omega(x)$ where $x \in H \cap \mathcal{V}_s$ is a semisimple element of odd prime order.

Theorem 7. *Let $G = \mathrm{GL}_n$. Let $x \in H \cap \mathcal{V}_{s,r}$ with $r > 2$, let $s = at + b$ for $0 \leq b < t$. Then*

(i) *If $s < r - 1$, or $s = r - 1$ and $t < n$,*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{b(t-b)}{n^2(t-1)}$$

(ii) *if $s = r - 1$ and $t = n$,*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s-2)}{n(n-1)}$$

(iii) *if $s > r - 1$,*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{s(2n-s)}{n^3\left(1 - \frac{1}{t}\right)}$$

Remark 1.8. Indeed in this case we shall construct the elements which realise lower bounds on f_Ω°, we shall prove that such elements must have all the eigenvalue and the difference between the multiplicities of two eigenvalues has to be at most 1, see (66). The bounds in (i) and (ii) are sharp. The bound in (iii) is deduced by using the element in (66) and Proposition 2.13.

The following is the equivalent result for the symplectic group.

Theorem 8. *Let $G = \mathrm{Sp}_{2n}$. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(4n-s)}{4n^2} - \frac{1}{4n^2} - \frac{4ns - s^2 + 1}{4n^3\left(1 - \frac{1}{t}\right)}$$

Also in this case we shall construct elements which realize lower bounds for f_Ω°. And, in fact, for certain value of s we also compute these bounds. In general it is very hard to give an explicit lower bounds which arise from this elements.

1.5. **Strategy and organization.** In Section 2 we present various results from the literature. Above all, we state [21, Theorem 1], which gives a description of the maximal closed subgroups in classical algebraic groups, it is the algebraic analogue of Aschbacher's theorem on subgroups structure of finite classical groups. Then we give the formulae for conjugacy classes of elements in $G = \mathrm{GL}_n$ and Sp_{2n}.

Since we shall only study primitive action on the coset spaces of $\mathcal{C}_2$-subgroup of G we terminate the section giving results on conjugacy classes in H. In particular, we give upper and lower bounds on $\dim(x^G \cap H)$ depending on $\dim x^G$ where x is a prime order element

In Section 3 we establish the upper bound on $f_\Omega(x)$ where $x \in H$ is a prime order element and we characterize elements for which equality holds, proving part *(i)* of Theorem 1 and Theorem 2. The proof will be based on results giving in Section 2.

Then in Section 4 and 5 we study global lower bounds on f_Ω for elements of prime order r. Remind that $f_\Omega(x) \leq f_\Omega^\circ(x)$ for any $x \in H$. The main goal, unless $r = p < n$ in GL_n or Sp_{2n}, is to construct an element $y \in H^\circ$ for which $f_\Omega(x) \geq f_\Omega^\circ(y)$ for all elements in H of prime order r. The strategy to do this is, given any $x \in H$ of prime order r, to construct a new element y with $\nu(y) \geq \nu(x)$ such that in a finite number of steps, using this construction starting from y we end up with the element which realise the lower bound on f_Ω° or any other element with isomorphic centralizer structure. For example, in $G = \mathrm{GL}_8(k)$, $t = 2$ let $x = [I_5, \omega I_2, \omega^2 I_1]$ of prime order $r \geq$ then $y = [I_4, \omega I_2, \omega^2 I_1, \omega^3 I_1]$ and $f_\Omega^\circ(x) \geq f_\Omega(y)$.

In particular in Section 4 we shall prove an explicit formula for $\dim(x^G \cap H^\circ)$ for semisimple elements.

Then, in Section 5 we study unipotent elements of prime order. Also in this case we split the analysis according whether or not $p \geq n$. In the case $p < n$ we prove directly from the formula of f_Ω that the lower bound in Theorem 1 holds. In the case $p \geq n$ we use the strategy explained above. For example, in $G = \mathrm{GL}_{15}(k)$, $t = 5$ let $x = [J_3^2, J_2^3, J_1^3]$ we define $y = [J_3^3, J_2^2, J_1^2]$ and we prove $f_\Omega^\circ(x) \geq f_\Omega^\circ(y)$. We conclude the section characterising elements which realise equality with the lower bound proving Theorem 3. In the case $p < n$ and p not dividing n, we will not get a characterization, instead, we prove that there always exist elements whose f_Ω-value are closed to the bound.

Then, in 6 we move on local bounds for f_Ω, i.e. we study prime order elements with fixed $\nu(\cdot)$ value. First, in Section 6.1, we establish upper bound, the main tools will be Proposition 2.9 and 2.10. We conclude the section on local bounds studying lower bounds for $f_\Omega(x)$ when $x \in \mathcal{V}_s$ is a semisimple element of prime order. In this case we construct, with the same strategy explained above, a class of elements which minimise f_Ω°. Then, for certain value of r and s we compute the f_Ω-value of those elements. In the other cases we use Proposition 2.9 and 2.13 to establish the lower bounds.

Then in Section 7, 8 we study involutions. We start with semisimple involutions, in 7, first for GL_n, in which case we shall establish a formula for $\dim(x^G \cap H)$ and hence for $f_\Omega(x)$; then for Sp_{2n}, in this case for certain specific value of t we can give an explicit formula for $f_\Omega(x)$ in other cases we give a bound on it of the shape $f_\Omega^\circ(x) + g(n,t)$ where $g(n,t) \to 0$ when $n, t \to \infty$. Roughly, the same arguments apply for unipotent involutions.

2. Preliminaries

In this section we give various results from the literature we shall need.

Throughout this section k is an algebraically closed field of characteristic $p \geq 0$.

2.1. **Algebraic groups.** An affine algebraic group G is an affine variety with a group structure, such that the multiplication and the inversion maps:

$$\begin{array}{ccc} G \times G & \to & G \\ (g,h) & \mapsto & gh \end{array} \quad , \quad \begin{array}{ccc} G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

are morphisms of algebraic varieties.

The most intuitively example of affine algebraic group is the *special linear group*:

$$\mathrm{SL}_n(k) = \{(a_{i,j}) \in k^{n\times n} : \det(a_{i,j}) = 1\}$$

It is not immediately clear that the *general linear group*

$$\mathrm{GL}_n(k) = \{A = (a_{i,j}) \in k^{n\times n} : \det A \neq 0\}$$

is an affine variety. However, if we identify GL_n with $\{(A,x) \in k^{n\times n} \times k : x \det A = 1\}$ via $A \mapsto (A, (\det A)^{-1})$, we see that GL_n is closed in $k^{n\times n} \times k$, and, hence, it is an affine algebraic group. Let us observe that $\mathrm{GL}_n \cong \mathrm{SL}_n.k^*$.

Let V be a k-vector space and assume V is endowed with a skew-symmetric, alternating bilinear form $(\cdot,\cdot)\colon V \times V \to k$, we say that V is a symplectic vector space. Then it is know that $\dim V = 2n$, for a suitable $n > 0$, and there exists a basis $\{e_1, f_1, \ldots, e_n, f_n\}$ of V such that for all i, j

$$(e_i, e_j) = (f_i, f_j) = 0, \ (e_i, f_j) = \delta_{i,j}$$

which we shall refer to as standard basis, see for example [17, Proposition 2.4.1].

For future references we introduce some terminology on subspaces of a symplectic vector space V. We say that the subspace $W \leq V$ is *non-degenerate* if $(\cdot,\cdot)|_W \colon W \times W \to k$ is non-degenerate; similarly it is called *totally singular*, or *totally isotropic*, if for all $u, v \in W$, $(u,v) = 0$.

Another example of affine algebraic group is the *symplectic group*, given a $2n$-dimensional symplectic k-vector space V, considering the standard basis, we may identify V with $k^{2n\times 2n}$. We define the symplectic group as

$$\mathrm{Sp}_{2n}(k) = \{A \in k^{2n\times 2n} : (u,v) = (A.u, A.v), \text{for all } u, v \in k^{2n}\}$$

It is clear that any closed subgroup of GL_n has the structure of affine algebraic group. The converse is also true. Indeed any affine algebraic group can be embedded as closed subgroup of GL_n, for a suitable n, see [15, Theorem 8.6].

Any element $x \in \mathrm{GL}_n$ is said to be *semisimple* if the minimal polynomial of x has distinct roots, while it is said to be *unipotent* if $1 - x$ is nilpotent, i.e. $(1-x)^l = 0$ for some l. Then, for any $x \in \mathrm{GL}_n$ it is known that x has the *Jordan decomposition* $x = x_s x_u$ where x_s is a semisimple element and x_u is unipotent, moreover $[x_s, x_u] = 1$. The Jordan decomposition extends to all affine algebraic groups. For an affine algebraic group G imbedded in GL_n, via the homomorphism ρ, in [15, Theorem 15.3] it is proved that the semisimple and the unipotent element in the decomposition $\rho(x) = x_s x_u \in \mathrm{GL}_n$ lift to elements of G, moreover the decomposition is independent of the chosen embedding in GL_n.

Let us observe that, by this result, any prime order element x in an affine algebraic group G is either semisimple or unipotent. Moreover, up to conjugation, any semisimple element is a diagonal matrix and any unipotent elements is made by unipotent Jordan blocks, in an embedding in some GL_n.

As matter of notation we shall denote a semisimple elements of prime order r as the diagonal matrix $[I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. And for unipotent elements, $[J_n^{a_n}, J_{n-1}^{a_{n-1}}, \ldots, J_1^{a_1}]$ will represent the matrix with a_i Jordan blocks of size i along the diagonal, where

$$J_i = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \ldots & 1 \end{pmatrix} \in \mathrm{GL}_i(k)$$

and x has order either a power of char$k = p$ or it has infinite order if $p = 0$. Moreover x has order p if all the Jordan blocks appearing in x have size less or equal than p.

A *simple* algebraic group is an affine algebraic group with no proper non-trivial closed connected normal subgroup. Let us observe that simple algebraic group may have non trivial center, although it is forced to be finite. For instance SL_n is a simple algebraic group and $Z(\mathrm{SL}_n) \neq 1$.

In an affine algebraic group G it is defined the *unipotent radical* $R_u(G)$ as the maximal closed connected normal unipotent subgroup of G. An algebraic group G is said to be *reductive* if $R_u(G) = 1$. It is clear that for any G, $G/R_u(G)$ is reductive. The structure of reductive algebraic groups is known, see for example [15, Theorem 26.3], in fact any reductive group $G = G_1 G_2 \cdots G_m T_l$ where the G_i's are simple groups and T_l is an l-dimensional torus, i.e. isomorphic to l-copies of k^*. A reductive group in which $l = 0$ is said *semisimple*. This structure theorem on reductive groups leads to the study of simple algebraic groups.

There is a complete classification of simple algebraic groups. A simple algebraic group is either of classical type: $\mathrm{SL}_n, \mathrm{Sp}_n, \mathrm{SO}_n, \mathrm{PSL}_n, \mathrm{PSp}_n, \mathrm{PSO}_n$ where the last three are defined to be the quotient of the first three by the center; or of exceptional type: E_6, E_7, E_8, F_4, G_2.

2.2. **Algebraic groups actions.** Let G be any group and Ω be a set. An action of G on Ω is a map $G \times \Omega \to \Omega$, we denote the image of (x, ω) by $x.\omega$, such that $1.\omega = \omega$ and $x.(y.\omega) = (xy).\omega$ for all $x, y \in G$ and $\omega \in \Omega$. We say that the action is *primitive* if it is transitive and the stabiliser G_α of $\alpha \in \Omega$ is a maximal subgroup. In particular, for primitive actions there exists a one to one G-invariant correspondence $\Omega \to G/G_\alpha$.

Let G be an algebraic group and Ω be a variety. Then an action of G on Ω is a map $G \times \Omega \to \Omega$ with the same properties described above which is also a morphism of algebraic varieties.

For the purpose of the next result let us remind that a locally closed subset in any topological group is an intersection of an open and a closed set.

Proposition 2.1. *Let G be an affine algebraic group and Ω be a variety on which G acts.*

(i) *For $\alpha \in \Omega$, the stabilizer $G_\alpha = \{g \in G : g.\alpha = \alpha\}$ is a closed subgroup of G.*
(ii) *For $x \in G$, the fixed point set $C_\Omega(x) = \{\omega \in \Omega : g.\omega = \omega\}$ is a subvariety of Ω.*
(iii) *For $\alpha \in \Omega$, the orbit $G.\alpha = \{g.\alpha : g \in G\}$ is a locally closed subset of Ω. In addition, if it is not closed its closure is union of $G.\alpha$ and orbits of strictly lower dimension.*

Proof. By hypothesis G acts on Ω, hence we have a morphism of varieties $\phi \colon G \times \Omega \to \Omega$.

Then $\alpha \in \Omega$ the map $\phi_\alpha \colon G \to \Omega$, $\phi_\alpha(g) = g.\alpha$ is a morphism of varieties. Hence $\phi_\alpha^{-1}(\alpha) = G_\alpha$ is closed in G. It is a general fact to prove that G_α is a subgroup of G. Part (i) follows.

For $x \in G$ the map $\psi_x \colon \Omega \to \Omega \times \Omega$, $\psi_x(\omega) = (\omega, x.\omega)$ is a morphism of algebraic varieties, since each component is a morphism. In the product variety $\Omega \times \Omega$, the diagonal $\Delta = \{(\omega, \omega) : \omega \in \Omega\}$ is closed. Therefore, since $C_\Omega(x) = \psi_x^{-1}(\Delta)$, assertion (ii) follows.

For (iii) see [15, Proposition 8.3] *q.e.d.*

Conjugacy classes in an affine algebraic group are orbits under the conjugation action of the group on itself. Therefore by part (iii) of Proposition 2.1 we deduce that conjugacy classes are locally closed in G and hence we are allowed to consider them dimensions.

As proved in part (ii) o Proposition 2.1 for G acting on Ω any fixed point space is a subvariety of Ω. In [18, Proposition 1.14] it is proved a formula for the dimension of fixed point spaces in natural actions on cosets varieties.

Proposition 2.2. *Let G be an algebraic group, H be a closed subgroup and $\Omega = G/H$ be the coset variety. Then for $x \in H$,*

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H) \tag{3}$$

We are interested in giving upper and lower bounds on the ratio $f_\Omega(x)$ defined in (1) for prime order elements in a simple classical group.

Thanks to the following lemma we may always assume, for semisimple element if GL_n, that the 1-eigenspace is the largest one.

Lemma 2.3. *Let $G = \mathrm{GL}_n(k)$ and let Ω be a G-variety. Let x be an element of G. Then $\dim C_\Omega(x) = \dim C_\Omega(\lambda I_n \cdot x)$, for every non-zero λ in the field k.*

Proof. We have that ω is in $C_\Omega(x)$ if, and only if $(\lambda^{-1} I_n)\omega$ is in $C_\Omega(\lambda I_n \cdot x)$. Hence $C_\Omega(x)$ is isomorphic to $C_\Omega(\lambda I_n \cdot x)$, therefore $\dim C_\Omega(x) = \dim C_\Omega(\lambda I_n \cdot x)$. *q.e.d.*

We have the following reduction lemma for semisimple elements of infinite order in a simple algebraic group G. For the purpose of the next result let us remind that a subspace subgroup of a simple algebraic group G is a maximal subgroup $H \leq G$ which acts reducibly on the natural module V of G.

Lemma 2.4. *Let G be a simple algebraic group and H be a closed non-subspace subgroup. Let x be a semisimple element of infinite order in G. Then there exists a prime order semisimple element y such that $C_\Omega(x) \subseteq C_\Omega(y)$.*

Proof. This argument is given in [5, §8]. Since x is semisimple we may assume $x \in T$, where T is the subset of diagonal matrices in G, hence a maximal torus of G, in fact $T \leq H^\circ$. Let L be the closure in G of $\langle x \rangle$. Since x has infinite order L has positive dimension. Moreover since the torus T of G is closed in G, and L is a subgroup of T of positive dimension, we have that L° is a subtorus of T. For any $l \geq 0$ it is immediate to check $C_\Omega(x^l) \subseteq C_\Omega(x)$, hence $\langle x \rangle \subseteq \bigcap_{\omega \in C_\Omega(x)} G_\omega$, and each G_ω is closed in G, therefore $L \subseteq \bigcap_\omega G_\omega$. Since L° is a subtorus of T it contains an element of prime order, and for the observation made above, this element fixes all the ω fixed by x. *q.e.d.*

2.3. **Subgroup structure of simple classical groups.** We are interested in the dimension of fixed point spaces in primitive actions of the groups $G = \mathrm{GL}_n, \mathrm{Sp}_{2n}$. In order to have a complete understanding of all the primitive actions we need to know the maximal subgroups of such groups.

In 1998, Liebeck and Seitz proved that given a closed subgroup H of a classical group G then H lies in a one of known families of subgroups, giving, in fact, the algebraic analogue of the Aschbacher's Theorem for finite classical groups, see [1].

They define two types of subgroups of a classical group G. The so called *geometric subgroups*, which are defined in terms of the geometry of the natural module V of G.

Given a subgroup H of a classical algebraic group G with natural module V, either the action of H on V is reducible, here the definition of $\mathcal{C}_1$, or it is irreducible. If H acts irreducibly on V then either it acts imprimitively, $\mathcal{C}_2$ family, or it acts primitively, these are all the other families $\mathcal{C}_3$–$\mathcal{C}_6$.

Then, it is defined another collection of subgroup of G. The $\mathcal{S}$ collection comprises almost simple groups which irreducibly embed in the group G.

Here the formal definition, as in [21], of the $\mathcal{C}_i$ families.

$\mathcal{C}_1$: *Subspace stabilisers.* Subgroups $H = G_U$, where U is totally singular or non-degenerate non-zero subspace, or, in the case $(G, p) = (\mathrm{SO}(V), 2)$, non-singular of dimension 1.

$\mathcal{C}_2$: *Stabilisers of orthogonal decomposition.* Subgroups H which stabilise a direct sum decomposition of the vector space $V = V_1 \oplus \ldots \oplus V_t$, where $t > 1$ and the subspaces V_i are mutual orthogonal and isometric, i.e. they are equidimensional and there exist bijective linear maps among them preserving forms.

$\mathcal{C}_3$: *Stabilisers of totally singular decomposition.* Here $G = \mathrm{Sp}_{2n}$ or SO_{2n}, $V = W \oplus W'$ where W, W' are maximal totally singular subspaces and H stabilises this decomposition. Namely $H^\circ = \mathrm{GL}_n$.

$\mathcal{C}_4$: *Tensor product subgroups.* Here either $V = V_1 \otimes V_2$ with $\dim V_i > 1$ and $H = N_G(Cl(V_1) \otimes Cl(V_2))$; or $V = V_1 \otimes \ldots \otimes V_t$ with $t > 1$ and the V_i mutually isometric and $H = N_G(\prod_i Cl(V_i))$.

$\mathcal{C}_5$: *Finite local subgroups.* Here $H = N_G(R)$ where R is a finite r-group for $r \neq p$.

$\mathcal{C}_6$: *Classical subgroups.* These are subgroups $H = N_G(\mathrm{Sp}(V)), N_G(\mathrm{SO}(V))$ when $G = \mathrm{SL}(V)$, and $H = N_G(\mathrm{SO}(V))$ when $G = \mathrm{Sp}(V)$ with $p = 2$.

The following is [21, Theorem 1].

Theorem 2.5. *Let $G = Cl(V)$ be a classical algebraic group over an algebraic closed field k. Let H be a closed subgroup of G. Let $\mathcal{C}(G) = \bigcup_i \mathcal{C}_i$. Then one of the following holds:*

(i) *H is contained in a memeber of $\mathcal{C}(G)$;*
(ii) *modulo scalars, H is almost simple, and the unique quasi-simple normal subgroup of H, $E(H)$, is irreducible on V. Furthermore if H is infinite then $E(H)$ is tensor-indecomposable on V.*

Let us observe that also for $G = \mathrm{GL}_n$ we may use Theorem 2.5 since $\mathrm{GL}_n = \mathrm{SL}_n.k^*$ and hence, in order to get maximal subgroups of GL_n, need to extend by k^* any maximal subgroup of SL_n.

In this report we shall consider only $\mathcal{C}_2$-subgroups for $G = \mathrm{GL}_n$ and Sp_{2n}. Hence $H = \mathrm{GL}_{\frac{n}{t}} \wr S_t < \mathrm{GL}_n$ or $H = \mathrm{Sp}_{2\frac{n}{t}} \wr S_t < \mathrm{Sp}_{2n}$ where $t > 1$ is a divisor of n. We denote by H° the connected component of H containing the identity, which is $H^\circ = (\mathrm{GL}_{\frac{n}{t}})^t < \mathrm{GL}_n$ or $H^\circ = (\mathrm{Sp}_{2\frac{n}{t}})^t < \mathrm{Sp}_{2n}$. In view of Proposition 2.2, for the variety $\Omega = G/H$ we have

$$\dim \Omega = \dim G - \dim H = \iota n^2 \Big(1 - \frac{1}{t}\Big)$$

where $\iota = 1$ for $G = \mathrm{GL}_n$ and $\iota = 2$ for $G = \mathrm{Sp}_{2n}$.

2.4. **Conjugacy classes.** In view of Proposition 2.2, for a primitive action of a classical group on the variety $\Omega = G/H$ in order to compute $\dim C_\Omega(x)$ for any prime order element $x \in G$ we need to know $\dim x^G$. We have already remarked that a prime order element is either semisimple or unipotent and that conjugacy classes are locally closed, (iii) in Proposition 2.1, hence we can consider the dimension of the closure.

In [3, Proposition 2.9] it is proved that two semisimple elements in Sp_{2n} are conjugate if, and only if, they are GL_{2n} conjugate.

In the following we give the centralizer structure of semisimple elements.

Proposition 2.6. *Let $G = \mathrm{GL}_n$ or Sp_{2n}. Let x be a semisimple element of prime order r. In Table 6 it is recorded a representative for x^G and its centralizer.*

G	x	$C_G(x)$
GL_n	$[I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$	$\prod_{i=0}^{r-1} \mathrm{GL}_{a_i}$
Sp_{2n}	$[I_{2a_0}, \omega I_{a_1}, \omega^{-1} I_{a_1} \ldots, \omega^{-\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}]$	$\mathrm{Sp}_{2a_0} \times \prod_{i=1}^{\frac{r-1}{2}} \mathrm{GL}_{a_i}$
	$[I_{2s}, -I_{2n-2s}]$	$\mathrm{Sp}_{2s} \times \mathrm{Sp}_{2n-2s}$

Table 6. Centralizer of semisimple elements

Proof. If $G = \mathrm{GL}_n$ then x is conjugate to a diagonal matrix and we get the centralizer by an easy computation.

Let $G = \mathrm{Sp}_{2n}$, then any semisimple element is G conjugate to a diagonal matrix. In the standard basis $\{e_1, f_1, \ldots, e_n, f_n\}$ we have $x.e_i = \omega^m e_i, x.f_i = \omega^l f_i$. Hence $(x.e_i, x.f_i) = (e_i, f_i)$ which implies $\omega^m = (\omega^l)^{-1}$. Namely the 1-eigenspace is even dimensional.

By the Jordan decomposition in G we have that given a prime order semisimple elements $x \in G$ we have $C_G(x) = G \cap C_{\mathrm{GL}_{2n}}(x) = G \cap \prod_i \mathrm{GL}_{a_i}$. The 1-eigenspace is a non degenerate subspace of V hence it leads to the direct summand Sp_{a_i} in the intersection. For the couple of eigenvector (ω^i, ω^{-i}), we have already proved them eigenspaces are equidimensional, i.e. $a_i = a_{-i}$, in GL_{2n} we have the direct summands $\mathrm{GL}_{a_i} \times \mathrm{GL}_{a_i}$ therefore, since in Sp_{2n} blocks comes in couple $(A, (A^{-1})^t)$ we must get only one direct summand GL_{a_i} in $C_G(x)$. The same analysis holds for an involution. *q.e.d.*

Let $G = \mathrm{GL}_n, \mathrm{Sp}_n, \mathrm{SO}_n$ over an algebraically closed field of arbitrary characteristic $p \geq 0$ and $p \neq 2$ for the symplectic and the orthogonal case. Then in [22, Theorem 3.1] it is proved that two unipotent elements are G-conjugate if, and only if, they are GL_n-conjugate. Moreover for $x = [J_n^{a_n}, J_{n-1}^{a_{n-1}}, \ldots, J_1^{a_1}] \in \mathrm{Sp}_n$ the a_i's are even for odd i.

In $G = \mathrm{GL}_n$ a unipotent element is a matrix with all the eigenvalue equal to 1. As already remarked before, up to conjugation, a unipotent matrix x is a block diagonal matrix $[J_n^{a_n}, J_{n-1}^{a_{n-1}}, \ldots, J_1^{a_1}]$.

Assume $G = \mathrm{GL}_n$, then there is a one to one correspondence between G-conjugacy classes of unipotent element and partitions of n. Let $[J_n^{a_n}, J_{n-1}^{a_{n-1}}, \ldots, J_1^{a_1}]$ be any unipotent element then we get the partition $(n^{a_n}, (n-1)^{a_{n-1}}, \ldots, 1^{a_1}) \vdash n$, where $\sum_i i a_i = n$. Conversely, given a partition $(n^{a_n}, \ldots, 1^{a_1})$ the conjugacy class associated to it is represented by the unipotent element $[J_n^{a_n}, J_{n-1}^{a_{n-1}}, \ldots, J_1^{a_1}]$.

If $G = \mathrm{Sp}_{2n}$ and $p \neq 2$, then there is a one to one correspondence between G-conjugacy classes of unipotent elements and partition of $2n$ having odd parts with even multiplicity.

In order to state a result on the dimension of conjugacy classes of unipotent elements of prime order we need to make some observations in the symplectic case when $p = 2$.

Let $G = \mathrm{Sp}_{2n}$ and assume $p = 2$, the Jordan canonical form of an element does not identify the conjugacy class. Any involution of Sp_{2n} has Jordan form $[J_2^s, J_1^{2n-2s}]$ where $0 < s \leq n$. In [2, §7] it is proved that what characterize the conjugacy classes of involutions is not the Jordan form but what it is called *Suzuki form*. If s is odd there is only one conjugacy class with representative $[J_2^s, J_1^{2n-2s}]$, said of type b_s. If s is even then there are two distinct conjugacy classes whose representative has Jordan form $[J_2^s, J_1^{2n-2s}]$, they are said of type a_s, c_s. An element x is defined to be of type a_s if, and only if,

$$(u, x.u) = 0, \quad \forall u \in k^{2n} \tag{4}$$

Suppose $G = \mathrm{Sp}_4$ with standard basis $\{e_1, f_1, e_2, f_2\}$. Then $x = [J_2^2] \in \mathrm{Sp}_4(k)$ is of type a_2 if it is Sp_4-conjugate to $a = [A, A]$ with $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where the matrix is taken with respect to the basis $\{e_1, e_2, f_1, f_2\}$, and it is clear that A is Sp_2-conjugate to J_2. It is straightforward to check that condition (4) holds for a. Instead it is of type c_2 if is Sp_{2n}-conjugate to $[A, A]$ in the standard basis. This argument applies for Sp_{2n}, for any n, we have that, for s even, $x = [J_2^s, J_1^{2n-2s}]$ is of type a_s if $x \sim [(A, A)^\diamond, \dots, (A, A)^\diamond, I_{2n-2s}]$ where $(A, A)^\diamond$ is taken with respect to the basis vectors $\{e_i, e_{i+1}, f_i, f_{i+1}\}$ for $1 \leq i \leq 2(n-s-1)$ odd, and it is of c_s type if there is a couple (A, A) in the basis $\{e_i, f_i, e_{i+1}, f_{i+1}\}$ for some $1 \leq i \leq 2(n-s-1)$ odd.

The dual, for unipotent element, of Proposition 2.6 is given by the following.

Proposition 2.7. *Let $x \in G$ be a unipotent element. Say a_i the multiplicity of the Jordan block J_i. Then*

(i) *If $G = \mathrm{GL}_n$ then*

$$\dim x^G = n^2 - 2\sum_{i<j} i a_i a_j - \sum_{i \geq 0} i a_i^2$$

(ii) *If $G = \mathrm{Sp}_{2n}$ and $p \neq 2$, then a_i is even whenever i is odd, and*

$$\dim x^G = n(2n+1) - \sum_{i<j} i a_i a_j - \frac{1}{2}\sum_{i \geq 0} i a_i^2 - \frac{1}{2}\sum_{i \text{ odd}} a_i$$

(iii) *If $G = \mathrm{Sp}_{2n}$ and $p = 2$ then*

$$\dim a_s^G = s(2n-s), \ \dim b_s^G = s(2n-s+1),$$

$$\dim c_s^G = s(2n-s+1)$$

Proof. Parts *(i)* and *(ii)* are established in [22, Theorem 3.1] and *(iii)* follows from [2, §§7,8]. *q.e.d.*

Let V be a k-vector space and fix $v \in V, x \in \mathrm{GL}(V)$. We define $[v, x] = xv - v$, the *commutator* of x and v. Then $[V, x]$ is the subspace of V generated by the commutator $[v, x]$ with $v \in V$.

Definition 2.8. Let V be a k-vector space. Given $G = \mathrm{GL}_n(k) = \mathrm{GL}(V)$ and $x \in G$ define

$$\nu(x) = \min\{\dim[V, \lambda x] : \lambda \in k^*\}$$

Observe that for non-scalar element $x \in G$ we have $\nu(x) > 0$. Moreover $\nu(x)$ is the codimension of the largest eigenspace of x with respect to the natural action of G on V. For example $\nu(x) = 1$ if, and only if, x has an eigenvalue of multiplicity $n-1$, e.g. $x = [\lambda I_{n-1}, \lambda']$ where $\lambda \neq \lambda'$, while $\nu(x) = n-1$ if, and only if, all the eigenvalues of x have multiplicity 1, i.e. $x = [\lambda_1, \dots, \lambda_n]$ where $\lambda_i \neq \lambda_j$ whenever $i \neq j$. In the case $x = [J_n^{a_n}, \dots, J_1^{a_1}] \in G$ is a unipotent element we have $\nu(x) = n - \sum_{i \geq 1} a_i$ since the 1-eigenspace has dimension $\sum_i a_i$. Therefore $\nu(x) = 1$ if, and only if, $x = [J_2, J_1^{n-2}]$.

The following result, a special case of [5, Proposition 2.9], will be very useful.

G	$f(s)$	$g(s)$
GL_n	$\max\{2s(n-s), ns\}$	$s(2n-s-1)$
Sp_{2n}	$\max\{s(2n-s), ns\}$	$\frac{1}{2}(4ns-s^2+1)$

TABLE 7. Bounds on $\dim x^G$, $\nu(x) = s$

Proposition 2.9. *Let $G = \mathrm{GL}_n, \mathrm{Sp}_{2n}$ and let $x \in G$ be an element of prime order such that $\nu(x) = s$. Then*

$$f(s) \leq \dim x^G \leq g(s)$$

where $f(s)$ and $g(s)$ are recorded in Table 7

2.5. **$\mathcal{C}_2$-subgroups.** We have $G = \mathrm{GL}(V)$, where V is an n-dimensional k-vector space. By fixing a basis for V we may identify G with $\mathrm{GL}_n(k)$. In addition, let H be a maximal subgroup in the family $\mathcal{C}_2$, i.e. it is the stabilizer of a decomposition

$$V = V_1 \oplus \ldots \oplus V_t$$

with $t > 1$, where $\dim V_i = n/t$ for all i. For $\pi \in S_t$, the action of π on the given decomposition of V is $\pi.(V_1, \ldots, V_t) = (V_{\pi(1)}, \ldots, V_{\pi(t)})$. Therefore $H = \mathrm{GL}_{n/t} \wr S_t$ or, in the case $G = \mathrm{Sp}_{2n}$, $H = \mathrm{Sp}_{2\frac{n}{t}} \wr S_t$.

Throughout this section, unless specified, when we write G we mean GL_n or Sp_{2n}, and $H = \mathrm{GL}_{\frac{n}{t}} \wr S_t$ or $\mathrm{Sp}_{2n} \wr S_t$, respectively.

The main result in [13] states that if H is a reductive closed subgroup of an algebraic group G and $x \in H$ then $x^G \cap H$ is a finite union of H° classes,

$$x^G \cap H = x_1^{H^\circ} \cup \ldots \cup x_l^{H^\circ}$$

therefore $\dim(x^G \cap H) = \max_i\{x_i^{H^\circ}\}$. The proof is based on finiteness of number of unipotent classes due to Lusztig, see [24], proved for reductive groups. In particular, $\dim(x^G \cap H) \geq \dim(x^G \cap H^\circ)$. Clearly this is the case for a classical group G and H a maximal subgroup of the $\mathcal{C}_2$-family.

Before giving details on the conjugacy classes in the $\mathcal{C}_2$-subgroups, we give the following two bounds on $\dim(x^G \cap H)$. The first is a special case of[8, Proposition 2.1].

Proposition 2.10. *Let $G = \mathrm{GL}_n$ or Sp_{2n} and $H = \mathrm{GL}_{\frac{n}{t}} \wr S_t$ or $\mathrm{Sp}_{2n} \wr S_t$, respectively, for $t > 1$ a divisor of n. Then for all elements $x \in H$ of prime order r,*

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leq \frac{1}{t} + \zeta$$

where $\zeta = 0$ for $G = \mathrm{GL}_n$ and ζ is recorded in Table 8 for $G = \mathrm{Sp}_{2n}$.

p	r	x	ζ
$\neq 2$	$= 2$	$[I_{2(n-s)}, -I_{2s}]$	$\frac{1}{n}$
$= 2$	$= 2$	$[J_2^s, J_1^{2n-2s}]$	$\frac{s}{\dim x^G}\left(1 - \frac{1}{t}\right)$
≥ 0	$\neq p, > 2$	$[I_{2a_0}, \ldots]$	$\frac{n-a_0}{\dim x^G}\left(1 - \frac{1}{t}\right)$
> 2	$= p$	$[J_i^{a_i}]$	$\frac{n-\frac{1}{2}\sum_{i \text{odd}} a_i}{\dim x^G}\left(1 - \frac{1}{t}\right)$

TABLE 8. Upper bounds on $\frac{\dim(x^G \cap H)}{\dim x^G}$

Remark 2.11. The original statement of Proposition 2.10 in [8] states $\zeta = \frac{1+\alpha}{n+2\alpha}$ with $\alpha = 1 - \delta_{2,t}$. Namely $\zeta \leq \frac{2}{n}$.

Let us observe that any semisimple element $x \in G$, up to conjugation, is a diagonal matrix, hence is contained in a maximal torus and, therefore, in H. Therefore for any semisimple element x we have $x^G \cap H^\circ \neq \emptyset$. This is false for unipotent elements. Assume, for example, $G = \mathrm{GL}_n$ and $H = \mathrm{GL}_1 \wr S_n$, then unipotent elements of order p are permutations of S_n with cycle shape $(p^h, 1^{n-hp})$, it is clear, then, that conjugacy classes of such elements do not meet $(\mathrm{GL}_1)^n$.

In the same spirit of Proposition 2.10 we state Proposition 2.13, below, which provides a lower bound on $\dim(x^G \cap H^\circ)$ for $x \in G$ semisimple. Before state this result, we need the following which is a corollary of Theorem 4.1 proved in Section 4.

Corollary 2.12. *Let $x = [I_{a_0}, \dots, \omega^{r-1} I_{a_{r-1}}] \in G$. Let $x = [x_1, \dots, x_t] \in H^\circ$. Say $a_{i,j}$ the multiplicity of ω^j in x_i. Then $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ if, and only if, $|a_{i,j} - a_{i',j}| \leq 1$ for all $i, i' \in \{1, \dots, t\}$ and $j \in \{0, \dots, r-1\}$, unless $G = \mathrm{Sp}_{2n}$ and $j = 0$ in which case $|a_{i,0} - a_{i',0}| \leq 2$.*

Proposition 2.13. *Let $G = \mathrm{GL}_n$ or Sp_{2n}. Let $x \in G$ be a semisimple element of prime order r. Then*

$$\dim(x^G \cap H^\circ) \geq \Big(\frac{1}{t} - \frac{1}{n}\Big) \dim x^G \tag{5}$$

Proof. For $t = n$ the inequality is trivially satisfied. Therefore we may assume $t < n$. Let $G = \mathrm{GL}_n$. We may write $x = [x_1, \dots, x_t]$, where $x_i \in \mathrm{GL}_{n/t}$ for all i, such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ} = \sum_i \dim x_i^{\mathrm{GL}_{n/t}}$. Let

$$x_i = [I_{a_{i,0}}, \dots, \omega^{r-1} I_{a_{i,r-1}}]$$

where $\sum_{i=1}^t a_{i,j} = a_j$ for all $j \in \{1, \dots, t\}$. Then inequality (5) is equivalent to

$$\frac{n^2}{t} - \sum_{i=1}^{t} \sum_{j=0}^{r-1} a_{i,j}^2 \geq \Big(\frac{1}{t} - \frac{1}{n}\Big)\Big(n^2 - \sum_{j=0}^{r-1} a_j^2\Big) \tag{6}$$

By the assumption $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, using Corollary 2.12 we have $a_{i,j} \in \{\lfloor a_j/t \rfloor, \lfloor a_j/t \rfloor + 1\}$. Therefore writing $a_j = p_j t + r_j$ where $0 \leq r_j < t$ we have for all $j \in \{0, \dots, r-1\}$

$$\sum_{i=1}^{t} a_{i,j}^2 = (p_j + 1)^2 r_j + p_j^2 (t - r_j) = p_j^2 t + 2 p_j r_j + r_j$$

Thus the inequality (6) is equivalent to

$$\sum_{j=0}^{r-1} \Big[a_j \Big(1 - \frac{a_j}{n}\Big) - r_j \Big(1 - \frac{r_j}{t}\Big)\Big] \geq 0$$

We claim that, for every $j \in \{0, \dots, r-1\}$, each element of the summation is non-negative. Hence, let us fix j and write $a_j = a, r_j = r$, then we claim

$$a\Big(1 - \frac{a}{n}\Big) - r\Big(1 - \frac{r}{t}\Big) \geq 0 \tag{7}$$

If $a < t$ we have $a = r$ and the inequality (7) is clearly true since $n \geq t$.

Let us assume $a \geq t$ then $a = pt + r$ with $p \geq 1$. Thus (7) is equivalent to

$$\frac{n(pt^2 + r^2) - t(pt + r)^2}{nt} \geq 0 \tag{8}$$

that, as function of r, is minimal when $r = \frac{pt^2}{n-t}$. Therefore we distinguish two cases. Assume $\frac{pt^2}{n-t} < t$, then r can actually have this value and an easy computation shows that (8) is satisfied, since, in this case, $pt + t < n$.

Let us assume, now, $\frac{pt^2}{n-t} \geq t$. Then the left hand side of the inequality (8) is minimal when $r = t-1$. Therefore it is sufficient to prove the inequality for $r = t-1$, that is

$$a\Big(1 - \frac{a}{n}\Big) - \Big(1 - \frac{1}{t}\Big) \geq 0 \tag{9}$$

which is true if, and only if,

$$\Big(1 - \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2} \leq a \leq \Big(1 + \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2}$$

A straightforward computation shows that

$$\Big(1 + \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2} \geq n - 1$$

We claim that

$$\Big(1 - \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2} \leq t$$

And since $t \leq a \leq n-1$, we will deduce that (9) is verified. Hence we have to prove

$$t\Big(1 - \frac{t}{n}\Big) - \Big(1 - \frac{1}{t}\Big) \geq 0$$

As function of t the left hand side of the previous inequality is monotonically increasing in the interval $[2, \frac{n}{2}]$, therefore is minimal for $t = 2$ and in this case the inequality is satisfied if $n \geq 3$. The case $n = 2$ necessarily implies $t = 2$ and we can never fall in this case since $n > t$.

Assume, now, $G = \mathrm{Sp}_{2n}$. For $x \in G$ semisimple of prime order r we use the notation of Proposition 2.6. Without loss of generality we may assume $x = [x_1, \dots, x_t]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Where we denote $a_{i,j}$ the multiplicity of ω^j , with $j \in \{1, \dots, \frac{r-1}{2}\}$, and $2a_{i,0}$ the multiplicity of 1 in x_i. Then, by Proposition 2.6,

$$\dim x^G = 2n^2 + n - a_0(a_0 + 1) - \sum_{j=0}^{\frac{r-1}{2}} a_j^2$$

$$\dim(x^G \cap H^\circ) = 2\frac{n^2}{t} + n - \sum_{i=1}^{t} a_{i,0}(a_{i,0} + 1) - \sum_{i=1}^{t}\sum_{j=0}^{\frac{r-1}{2}} a_{i,j}^2$$

Using Corollary 2.12 we have $a_{i,j} \in \big\{\lfloor \frac{a_j}{t} \rfloor, \lfloor \frac{a_j}{t} \rfloor + 1\big\}$. Let $a_j = p_j t + r_j$, with $0 \leq r_j < t$. Then

$$\sum_{i=1}^{t} a_{i,j}^2 = p_j^2 t + 2p_j r_j + r_j$$

$$\sum_{i=1}^{t} a_{i,0}(a_{i,0} + 1) = p_0^2 t + 2p_0 r_0 + r_0 + a_0$$

Therefore (5) is equivalent to

$$\sum_{j=0}^{\frac{r-1}{2}}\Big(a_j\big(1 - \frac{a_j}{n}\big) - r_j\Big(1 - \frac{r_j}{t}\Big)\Big) \geq \frac{n}{t} - \frac{a_0}{t} - 2n - 1 + r_0 - \frac{r_0^2}{t} + \frac{a_0^2}{n} + \frac{a_0}{n} + a_0 \tag{10}$$

We claim that the right hand side of (10) is never positive. Therefore it is enough to show that the left hand side of (10) is always non-negative, which has been proved above for $G = \mathrm{GL}_n$ and the same holds in the symplectic case. Hence we only need to study the right hand side.

First let us observe that $r_0 - \frac{r_0^2}{t} \leq \frac{t}{4}$. Similarly, $a_0 - \frac{a_0}{t} + \frac{a_0^2}{n} + \frac{a_0}{n}$ is maximal when $a_0 = n-1$. Therefore

$$\frac{n}{t} - \frac{a_0}{t} - 2n - 1 + r_0 - \frac{r_0^2}{t} + \frac{a_0^2}{n} + \frac{a_0}{n} + a_0 \leq \frac{n}{t} - \frac{t}{4} + n - 1 - \frac{n-1}{t} + \frac{(n-1)^2}{n} + \frac{n-1}{n} - 2n - 2 = \frac{1}{t} - \frac{t}{4} - 3 < 0$$

This proves the claim. *q.e.d.*

By definition of dimension, $\dim(x^G \cap H) \geq \dim(x^G \cap H^\circ)$. Using Remark 2.11, for a semisimple element of prime order x, we get

$$0 \leq \frac{\dim(x^G \cap H) - \dim(x^G \cap H^\circ)}{\dim x^G} \leq \frac{3}{n} \tag{11}$$

Then, combining this with Proposition 2.2, we have the following.

Corollary 2.14. *Let $x \in H$ be a semisimple element of prime order r. Then*

$$f_\Omega^\circ(x) \leq f_\Omega(x) \leq f_\Omega^\circ(x) + \frac{6t}{n(t-1)}$$

Proof. Let $x \in H$ be a semisimple element of prime order. The inequality $f_\Omega^\circ(x) \leq f_\Omega(x)$ is always true.

As seen in (11), we have $\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{3}{n}\dim x^G$. We have $\dim x^G \leq \dim G \leq (\iota n)^2$ where $\iota = 1$ for $G = \mathrm{GL}_n$ and $\iota = 2$ for $G = \mathrm{Sp}_{2n}$. Hence, using $\iota \leq 2$ and $\dim \Omega = \iota n^2(1 - \frac{1}{t})$, we have he following

$$f_\Omega(x) \leq f_\Omega^\circ(x) + \frac{3\dim x^G}{n \dim \Omega} \leq f_\Omega^\circ(x) + \frac{6t}{n(t-1)}$$

q.e.d.

For semisimple element it is possible to establish stronger results. In fact, in Section 4 we shall give an exact formula for $\dim(x^G \cap H^\circ)$, where $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$.

We now study elements in $H \setminus H^\circ$.

Let x be an element of H, then $x = [A_1, \ldots, A_t]\pi$, where $A_i \in \mathrm{GL}_{n/t}$ and $\pi \in S_t$. Since π acts on $[A_1, \ldots, A_t] \in \mathrm{GL}_{n/t}^t$ by permuting the A_i's we have that it is a block matrix of GL_n. For example if $t = 3$ and $\pi = (12)$ then, in a suitable basis,

$$\pi = \begin{pmatrix} 0 & I_{n/3} & 0 \\ I_{n/3} & 0 & 0 \\ 0 & 0 & I_{n/3} \end{pmatrix}$$

In order to compute $\dim C_\Omega(x)$ it is important to know $\dim x^G$ and $\dim(x^G \cap H)$, see Propotision 2.2. Therefore, for $x = [A_1, \ldots, A_t]\pi \in H$ of prime order, we need know its eigenvalues or its Jordan decomposition.

Let x be an element in H of prime order r, so $x \in H^\circ\tau$, where $\tau \in S_t$ satisfies $\tau^r = 1$. Moreover we have

$$x^G \cap H = \Big(x^G \cap H^\circ\Big) \cup \Big(\bigcup_{|\sigma|=r} x^G \cap H^\circ\sigma\Big)$$

therefore $\dim(x^G \cap H) = \max_{\sigma^r=1}\{\dim(x^G \cap H^\circ\sigma)\}$. We deduce from the following lemma that we only need to deal with elements in S_t of order r that have distinct cycle shapes. Namely, for $x \in H$ of prime order r, $\dim(x^G \cap H) = \max_i\{\dim(x^G \cap H^\circ\pi_i)\}$ where $\pi_i \in S_t$ is any permutation of cycle shape $(r^i, 1^{t-ir})$.

Lemma 2.15. *Let x be an element of H of order r. Let τ and σ be conjugate permutation of order r in S_t. Then*
$$\dim(x^G \cap H^\circ\tau) = \dim(x^G \cap H^\circ\sigma)$$

Proof. The subgroup H stabilises a direct sum decomposition of V say $V = V_1 \oplus \ldots \oplus V_t$.

Since τ and σ are conjugate in S_t they have same cycle shape, hence we can write
$$\tau = (1,\ldots,l_1)(l_1+1,\ldots,l_2)(l_2+1,\ldots,l_3)\cdots(l_m+1,\ldots,l_h)$$
$$\sigma = (j_1,\ldots,j_{l_1})(j_{l_1+1},\ldots,j_{l_2})(j_{l_2+1},\ldots,j_{l_3})\cdots(j_{l_m+1},\ldots,j_{l_h})$$
for some $h \geq 1$. Let $\{a_1,\ldots,a_k\}$ be the points fixed by τ and $\{b_1,\ldots,b_k\}$ the points fixed by σ. Let us consider the permutation $\pi \in S_t$ defined as $\pi(i) = j_i$ for all $i \in \{1,\ldots,l_h\}$ and $\pi(a_s) = b_s$ for all $s \in \{1,\ldots,k\}$.

Assume, seeing a contradiction, that $\dim(x^G \cap H^\circ\tau) > \dim(x^G \cap H^\circ\sigma)$. Then there exists $y \in H^\circ\tau$ such that $\dim(x^G \cap H^\circ\tau) = \dim y^{H^\circ}$. Let us denote $y = [A_1,\ldots,A_t]\tau$. We consider the following element of $H^\circ\sigma$
$$y' = [A_{\pi(1)},\ldots,A_{\pi(t)}]\sigma$$
stabilizing the decomposition $V_{\pi(1)} \oplus V_{\pi(2)} \oplus \ldots \oplus V_{\pi(t)}$ of V.

By construction of π, the two actions are the same, hence $\dim y^{H^\circ} = \dim(y')^{H^\circ}$, which is a contradiction since
$$\dim y^{H^\circ} = \dim(x^G \cap H^\circ\tau) > \dim(x^G \cap H^\circ\sigma) \geq \dim(y')^{H^\circ}$$
q.e.d.

With the following results we shall construct suitable G-conjugates to any prime order element $x \in H$. The following result is proved in the proof of [23, Lemma 4.5].

Lemma 2.16. *Let x be an element of prime order r in $H \setminus H^\circ$, say $x = [A_1,\ldots,A_t]\tau$. Then x is H°-conjugate to $[I_{n/t},\ldots,I_{n/t},A_{hr+1},\ldots,A_t]\tau$, where h is the number of r cycles of τ.*

Proof. Since x has prime order r then so does τ. Therefore τ is product of a certain number, say h, of r-cycle. So $t = hr + f$ and f is the number of fixed points of τ. Hence τ is S_t-conjugate to
$$\big(1,\ldots,r\big)\big(r+1,\ldots,2r\big)\cdots\big((h-1)r+1,\ldots,hr\big)$$
So by Lemma 2.15 we may assume $\tau = \prod_{i=1}^h\Big((i-1)r+1,\ldots,ir\Big)$. Therefore, in terms of a suitable basis we have $\tau = [\tau_1,\ldots,\tau_h,I_{nf/t}]$, where
$$\tau_i = \begin{pmatrix} 0 & I_{n/t} & 0 & \ldots & 0 \\ 0 & 0 & I_{n/t} & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & I_{n/t} \\ I_{n/t} & 0 & 0 & \ldots & 0 \end{pmatrix} \in \mathrm{GL}_{nr/t}(k) \tag{12}$$

Since x has order r we have $A_{(i-1)r+1}\cdots A_{ir} = I_{n/t}$ for all $i \in \{1,\ldots,h\}$.

Let $g \in H^\circ$ be the element

$$\begin{aligned} g =&[I_{n/t}, A_1, A_1A_2, \dots, A_1 \cdots A_{r-1}, I_{n/t}, A_{r+1}, A_{r+1}A_{r+2}, \dots, A_{r+1} \cdots A_{2r-1}, \\ &I_{n/t} \dots, A_{(h-1)r+1}, A_{(h-1)r+1}A_{(h-1)r+2}, \dots, A_{(h-1)r+1} \cdots A_{hr-1}, I_{nf/t}] \end{aligned}$$

We see that $g^{-1}\tau g = [A_1, A_2, \dots, A_{hr}, I_{n/t}, \dots, I_{n/t}]\tau$. Thus, defining

$$b = [I_{n/t}, \dots, I_{n/t}, A_{hr+1}, \dots, A_t]$$

we have

$$g^{-1}(b\tau)g = b(g^{-1}\tau g) = [A_1, \dots, A_t]\tau$$

where we used the fact that b and g commutes and that $b(g^{-1}\tau g) = x$. Therefore x and $b\tau$ are H°-conjugate. *q.e.d.*

A straightforward consequence of Lemma 2.16 is that we gain a formula for the computation of $\dim(x^G \cap H^\circ\tau)$. Assume $G = \mathrm{GL}_n$ and H is as above, given $x = [A_1, \dots, A_t]\pi_h \in H^\circ\pi_h$ of prime order r, we have

$$\begin{aligned} \dim(x^G \cap H^\circ\pi_h) &= \dim H^\circ - \dim C_K(\pi_h) - \dim C_L([B_{hr+1}, \dots, B_t]) \\ &= h(r-1)\dim \mathrm{GL}_{n/t} + \sum_{i \geq hr+1} \dim B_i^{\mathrm{GL}_{n/t}} \end{aligned} \tag{13}$$

for suitable B_i, where we say $K = (\mathrm{GL}_{n/t})^h$ and $L = (\mathrm{GL}_{n/t})^{t-hr}$, and we used $C_K(\pi_h) = (\mathrm{GL}_{n/t})^h$. Similarly, if $G = \mathrm{Sp}_{2n}$, we have

$$\dim(x^G \cap H^\circ\pi_h) = h(r-1)\dim \mathrm{Sp}_{2n/t} + \sum_{i \geq hr+1} \dim B_i^{\mathrm{Sp}_{2n/t}} \tag{14}$$

In the following two lemmas we compute the Jordan normal form of the permutations $\pi_h \in S_t$.

Lemma 2.17. *Let τ be an element of order p in S_t, with cycle shape $(p^h, 1^f)$, where $f = t - hp$ is the number of fixed points. Then τ is G-conjugate to $[J_p^{nh/t}, J_1^{nf/t}]$.*

Proof. Without loss of generality we may assume

$$\tau = \big(1, 2, \dots, p\big)\big(p+1, \dots, 2p\big) \cdots \big((h-1)p + 1, \dots, hp\big)$$

In terms of matrices we have $\tau = [\tau_1, \dots, \tau_h, I_{nf/t}]$, where $\tau_i \in \mathrm{GL}_{np/t}(k)$ is defined in (12). Each τ_i is $\mathrm{GL}_{nt/t}(K)$-conjugate to the matrix $[g, \dots, g] \in \mathrm{GL}_{np/t}(k)$, where

$$g = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 1 & & \dots & 0 \end{pmatrix} \in \mathrm{GL}_p(k) \tag{15}$$

Since x has order $p = \mathrm{char}(k)$, it is easy to check that g is $\mathrm{GL}_p(k)$-conjugate to J_p. Therefore τ is conjugate to $[J_p^{nh/t}, J_1^{nf}]$. *q.e.d.*

Remark 2.18. If $G = \mathrm{Sp}_{2n}$, Lemma 2.17 still holds, and in this situation, up to conjugation, $\tau = [J_p^{2nh/t}, J_1^{2nf}]$.

Lemma 2.19. *Let τ be an element of prime order r in S_t with cycle shape $(r^h, 1^f)$. Assume $r \neq p$. Then τ is $\mathrm{GL}_n(k)$-conjugate to $[I_{n(h+f)/t}, \omega I_{nh/t}, \dots, \omega^{r-1}I_{nh/t}]$, where ω is a primitive r-th root of unity in k.*

Proof. As seen in the proof of Lemma 2.17 we have $\tau = [\tau_1, \ldots, \tau_h, I_{nf/t}]$, where $t = hr+f$. Again, each τ_i is conjugate to the matrix $[g, \ldots, g] \in \mathrm{GL}_{nr/t}$, where $g \in \mathrm{GL}_r(k)$ is given in (15). When $p \neq r$, the characteristic polynomial of g is $cp(\lambda) = 1 - \lambda^r$. Therefore each $r \times r$ block is $\mathrm{GL}_r(k)$-conjugate to $[I_1, \omega I_1, \ldots, \omega^{r-1} I_1]$, where ω is a primitive r-th root of unity in k. Therefore τ is $\mathrm{GL}_n(k)$-conjugate to $[I_{n(h+f)/t}, \omega I_{nh/t}, \ldots, \omega^{r-1} I_{nh/t}]$. *q.e.d.*

Remark 2.20. If $G = \mathrm{Sp}_{2n}$, Lemma 2.19 still holds, and in this situation, up to conjugation, $\tau = [I_{2n(h+f)/t}, \omega I_{2nh/t}, \ldots, \omega^{r-1} I_{2nh/t}]$.

Let $x \in H$ be an element of prime order r, say $x = [A_1, \ldots, A_t]\pi$ for some $\pi \in S_t$. For future references, we define

$$h = \max\{l : x^G \cap H^\circ \pi_l \neq \emptyset, \pi_l \in S_t\} \tag{16}$$

where π_l is a permutation of cycle shape $(r^l, 1^{t-rl})$, where we define $\pi_0 = 1$. By replacing x by a suitable G-conjugate, we may assume $x \in H^\circ \pi_h$ and, thanks to Lemma 2.16, x is H°-conjugate to $[I_{nhr/t}, A_{hr+1}, \ldots, A_t]\pi_h$. We start by considering the case $r = p$, so x is unipotent.

The following give the general normal form of an element such that $x^G \cap H^\circ \pi_h \neq \emptyset$ and $x^G \cap H^\circ \pi_{h+1} = \emptyset$. We define, as before, $\iota = 1$ if $G = \mathrm{GL}_n$ and $\iota = 2$ for $G = \mathrm{Sp}_{2n}$.

Proposition 2.21. *Let $x \in H$ be a semisimple element of prime order r. Assume $x^G \cap H^\circ \pi_h \neq \emptyset$ and $x^G \cap H^\circ \pi_{h+1} = \emptyset$.*

- *If $r \neq p$, then for some a_i's*

$$x = [I_{\iota \frac{n}{t} h + a_0}, \omega I_{\iota \frac{n}{t} h + a_1}, \ldots, \omega^{r-1} I_{\iota \frac{n}{t} h + a_{r-1}}]$$

- *If $r = p$, then for some a_i's*

$$x = [J_p^{\iota \frac{n}{t} h + a_p}, J_{p-1}^{a_{p-1}}, \ldots, J_1^{a_1}]$$

Proof. Follows directly from Lemma 2.17, 2.19. *q.e.d.*

Moreover let us observe that if for a semisimple element $x \in H$ of prime order r there exists i such that $a_i < \iota \frac{n}{t}$, then $x^G \cap (H \setminus H^\circ) = \emptyset$.

Similarly, for $x \in H$ of prime order p, if $a_p < \iota \frac{n}{t}$ we have $x^G \cap (H \setminus H^\circ) = \emptyset$.

3. Global upper bounds

In this section we establish upper bound on $f_\Omega(x)$ for prime order elements $x \in H$, proving Theorem 1 and 2.

We begin with some lemmas where we deal with elements with $\nu(x) = 1$ in GL_n and elements with $\nu(x) = 1, 2$ in the symplectic group.

Lemma 3.1. *Assume $G = \mathrm{GL}_n$. Let $x \in H$ of prime order such that $\nu(x) = 1$. Then*

$$f_\Omega(x) = 1 - \frac{2}{n}$$

unless $r = 2$, $t = n$ in which case $f_\Omega(x) = 1 - \frac{2}{n} + \frac{1}{n(n-1)}$.

Proof. (1) Let $x = [J_2, J_1^{n-1}]$ be in G. Then x is unipotent of prime order p for $p > 0$ and using Proposition 2.7 we get $\dim x^G = 2n-2$. Assume $p \neq 2, t < n$. Let us look at x as element of H°, then, up to permutation of the blocks, $x = [x_1, \dots, x_t]$ with $x_1 = [J_2, J_1^{n/t-2}]$ and $x_i = I_{n/t}$ for $i = 2, \dots, t$. Therefore $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, and $\dim(x^G \cap H^\circ) = \dim [J_2, J_1^{n/t-2}]^{\mathrm{GL}_{n/t}} = 2n/t - 2$. Moreover by Lemma 2.17 $x^G \cap H = x^G \cap H^\circ$. Thus $f_\Omega(x) = 1 - 2/n$.

If $p = 2, t = n$, x is the permutation $\pi_1 = (12) \in S_n$ and $x^G \cap H^\circ = \emptyset$. Hence, $x^G \cap H = x^G \cap H^\circ \pi_1$ and, using (13), $\dim(x^G \cap H) = \dim \pi_1^{H^\circ} = (n/t)^2 = 1$. Therefore $f_\Omega(x) = 1 - \frac{2}{n} + \frac{1}{n(n-1)}$.

(2) Let r be a prime number and ω be a primitive r-th root of unity in k. Let us consider the semisimple element $x = [I_{n-1}, \omega]$. Then $\dim x^G = 2n - 2$. Assume, as before, we are not in the case $r = 2$ and $t = n$. Then $\dim(x^G \cap H^\circ) = \dim x^{H^\circ} = 2n/t - 2$ where $x = [x_1, \dots, x_t]$ and $x_1 = [I_{n/t-1}, \omega], x_i = I_{n/t}$ for $i = 2, \dots, t$. We get $\dim(x^G \cap H) = 2n/t - 2$, since by Lemma 2.17 $x^G \cap H = x^G \cap H^\circ$. Therefore $f_\Omega(x) = 1 - \frac{2}{n}$.

Assume $r = 2$ and $t = n$, then x is the permutation $(12) \in S_n$. And $\dim(x^G \cap H^\circ) = 0$ and $\dim(x^G \cap H^\circ) = 1$. Therefore $f_\Omega(x) = 1 - \frac{2}{n} + \frac{1}{n(n-1)}$.

q.e.d.

Lemma 3.2. *Assume $G = \mathrm{Sp}_{2n}$. Let $x \in H$ be an element of prime order such that $\nu(x) = \iota$, where $\iota = 1$ if x is unipotent and $\iota = 2$ otherwise. Then*

$$f_\Omega(x) = 1 - \frac{\iota}{n}$$

unless $r = 2$, $t = n$ in which case $f_\Omega(x) = 1 - \frac{\iota}{n} + \frac{3-2\delta_{r,p}}{n(n-1)}$.

Proof. (1) Assume $x = [J_2, J_1^{2n-s}]$ then, using formulae in Proposition 2.7, depending if $p = 2$ or $p \neq 2$, we get $\dim x^G = 2n$. Then $x = [x_1, \dots, x_t] \in H^\circ$ where, up to permutation of the blocks, $x_1 = [J_2, J_1^{2n/t-2}], x_2 = \ldots = x_t = I_{n/t}$. Hence $\dim(x^G \cap H^\circ) = 2\frac{n}{t}$. Then if we are not in the case $p = 2$ and $t = n$, $x^G \cap H = x^G \cap H^\circ$ hence $f_\Omega(x) = 1 - \frac{1}{n}$.

In the case $p = 2$ and $t = n$, x is the permutation $\pi_1 = (1, 2) \in S_n$. Hence, using (14), we have $\dim(x^G \cap H^\circ \pi_1) = \dim \pi_1^{H^\circ} = 3$. Therefore $f_\Omega(x) = 1 - \frac{1}{n} + \frac{1}{2n(n-1)}$.

(2) Assume $x = [I_2, I_{2n-2}]$. Then as above we get $f_\Omega(x) = 1 - \frac{2}{n}$ unless $r = 2$ and $t = n$, in which case $f_\Omega(x) = 1 - \frac{2}{n} + \frac{3}{2n(n-1)}$.

q.e.d.

For the reader convenience, we state two propositions according whether G is the general linear group or it is the symplectic one. Here is the first result.

Proposition 3.3. *Let $G = \mathrm{GL}_n$. Let $x \in H$ be a semisimple element or, for $p > 0$, a unipotent element of prime order $r \neq 2$. Then*

$$f_\Omega(x) \leq 1 - \frac{2}{n}$$

with equality if, and only if, $\nu(x) = 1$.

Proof. Let x be either a unipotent or a semisimple element of finite order, for any $l \geq 1$ we have $C_\Omega(x) \subseteq C_\Omega(x^l)$ thus we may assume x has prime order. For a semisimple element x of infinite order then, thanks to Lemma 2.4 there exists y, semisimple of finite order, such that $\dim C_\Omega(x) \leq \dim C_\Omega(y)$, hence, for the purpose of obtaining an upper bound on the dimension of the fixed point space we may assume x has prime order.

Using the formula of $\dim C_\Omega(x)$ given in Propositon 2.2 we get that $f_\Omega(x) \leq 1 - 2/n$ if, and only if,

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leq 1 - \frac{2(t-1)n}{t \dim x^G}$$

Suppose $\nu(x) \geq 2$, then, using Proposition 2.9, we have $\dim x^G \geq 4(2n-1)$, so

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leq \frac{1}{t} < 1 - \frac{(t-1)n}{(4n-2)t} \leq 1 - \frac{2(t-1)n}{t \dim x^G}$$

where the first inequality is given by Proposition 2.10. For $\nu(x) = 1$, the equality holds thanks to Lemma 3.1. *q.e.d.*

In order to deduce the result for the symplectic group we use the same argument given in Proposition 3.3. The substantial difference is given by the bounds in Proposition 2.10, which depends on the order of the elements.

Proposition 3.4. *Let $x \in H$ be a semisimple element or, for $p > 0$ a unipotent element. Then*

$$f_\Omega(x) \leq 1 - \frac{\iota}{n}$$

where $\iota = 1$ if $r = p$ and $\iota = 2$ otherwise; unless (G, H°) is in Table 9, where $r \neq p$, in which case the upper bound on $f_\Omega(x)$ is given. Moreover equality holds if, and only if $\nu(x) = \iota$.

G	H°	r	p	$f_\Omega(x) \leq$	$\bar{x}$
Sp_{2n}	$(\mathrm{Sp}_2)^n$	$= 2$	$\neq 2$	$1 - \frac{2}{n} + \frac{3}{2n(n-1)}$	$[I_2, -I_{2n-2}]$
		$= 2$	$= 2$	$1 - \frac{1}{n} + \frac{1}{2n(n-1)}$	$[J_2, J_1^{2n-2}]$
Sp_4	$(\mathrm{Sp}_2)^2$	$= 2$	$\neq 2$	$\frac{1}{2}$	$[I_2, -I_2]$
		$\neq 2$	$\neq 2$	$\frac{3}{4}$	$[I_2, \omega, \omega^{-1}]$
Sp_6	$(\mathrm{Sp}_2)^3$	$= 2$	$\neq 2$	$\frac{7}{12}$	$[I_2, -I_4]$
Sp_8	$(\mathrm{Sp}_4)^2$	$r = 2$	$\neq 2$	$\frac{5}{8}$	$[I_2, -I_6]$
Sp_8	$(\mathrm{Sp}_2)^4$	$r = 2$	$\neq 2$	$\frac{5}{8}$	$[I_2, -I_6]$
Sp_{10}	$(\mathrm{Sp}_2)^5$	$r = 2$	$\neq 2$	$\frac{27}{40}$	$[I_2, -I_8]$
		$r \neq 2$	$\neq 2$	$\frac{3}{5}$	$[I_8, \omega, \omega^{-1}]$
Sp_{12}	$(\mathrm{Sp}_4)^3$	$r = 2$	$\neq 2$	$\frac{7}{12}$	$[I_2, -I_{10}]$
Sp_{12}	$(\mathrm{Sp}_2)^6$	$r = 2$	$\neq 2$	$\frac{43}{60}$	$[I_2, -I_{10}]$

TABLE 9. Exceptional upper bounds

Proof. With the same argument used in Proposition 3.3 we may assume $x \in H$ has prime order.

We get the result if, and only if

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leq 1 - \frac{\iota \dim \Omega}{n \dim x^G}$$

By [5, Proposition 2.9] we have $\dim x^G \geq \max\{s(2n-s), ns\}$. Moreover, since x has prime order, by Proposition 2.10 we have

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leq \chi$$

where χ is recorder in Table 8 for prime order element r.

Assume $\nu(x) \geq 2$ if x is a unipotent element. Then it is enough to show

$$1 - \frac{n}{2(n-1)}\Big(1 - \frac{1}{t}\Big) > \chi \tag{17}$$

If x is a unipotent involution, using the fact $s \leq n$ and $\dim x^G \geq 4(n-1)$ we have that (17) is satisfied for $n > 4$.

If x is a unipotent element with $p > 2$, using $\sum_{i \text{ odd}} a_i \geq 0$ and $\dim x^G \geq 4(n-1)$, as above, an easy computation shows that (17) is true for $n > 4$.

If x is a semisimple element we assume $\nu(x) \geq 4$. If x is not an involution using $a_0 \geq 0$ in addition to $\dim x^G \geq 8(n-2)$, it is enough to show

$$1 - \frac{n}{2(n-2)}\Big(1 - \frac{1}{t}\Big) > \frac{1}{t} + \frac{n}{8(n-2)}\Big(1 - \frac{1}{t}\Big) \tag{18}$$

and an easy computation shows that this is true for $n \geq 6$.

If x is an involution, using $\dim x^G \geq 8(n-2)$ it is enough to show

$$1 - \frac{n}{2(n-2)}\Big(1 - \frac{1}{t}\Big) > \frac{1}{t} + \frac{1}{n}$$

which is true if $m(m-4)(t-1) - 2t(n-2) > 0$. The left hand side of the last inequality is minimal for $t = 2$, when $n \geq 6$, therefore it suffices to show $n(n-4) - 4(n-2) > 0$ which is true for $n \geq 7$.

It is easy to check case by case when $n = 2, 3, 4, 5, 6$ for semisimple involutions, $n = 2, 3, 4, 5$ for semisimple elements and $n = 2, 3, 4$ for unipotent elements, for $\nu(x) \geq 2$. The exceptional bounds for these cases are in Table 9. For x unipotent the upper bound is always $1 - \frac{1}{n}$ and it is realised by $[J_2, J_1^{2n-2}]$.

The case $\nu(x) = 1$ for unipotent element is trivial, since $\nu(x) = 1$ if, and only if $x = [J_2, J_1^{2n-2}]$ and by Lemma 3.2, $f_\Omega(x) = 1 - \frac{1}{n}$. Similarly, the case $\nu(x) = 2$ for semisimple elements is given in Lemma 3.2. *q.e.d.*

Example 3.5.

(i) Let $x = [\omega^i I_n, \omega^{-i} I_n]$ be an element of order at least 3. Then $f_\Omega(x) = \frac{1}{2}$.
(ii) Let $x = [\omega^i, \omega^{-i}, \omega^j I_{n-1}, \omega^{-j} I_{n-1}], [I_2, \omega^i I_{n-1}, \omega^{-1} I_{n-1}]$. Then $f_\Omega(x) = \frac{1}{2} - \frac{1}{n}$.

4. Global lower bounds: semisimple elements

In this section we shall prove part *(iii)* of Theorem 1.

Let $x \in H$ be a semisimple element of prime order r. Up to conjugation, we may assume $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in \mathrm{GL}_n$, where $\sum_i a_i = n$. Let us observe that in Sp_{2n} the 1-eigenspace is even dimensional and ω^i, ω^{-i} have the same multiplicity. Therefore in $G = \mathrm{Sp}_{2n}$, up to conjugation, we may write $x = [I_{2a_0}, \omega I_{a_1}, \omega^{-1} I_{a_1}, \dots, \omega^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}, \omega^{-\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}]$ and $\sum_i a_i = n$.

Before starting the analysis of lower bounds on f_Ω, we study $\dim(x^G \cap H^\circ)$. Namely we can give a precise formula for this dimension. In general $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$, for some $h \in \{0, \dots, \lfloor t/r \rfloor\}$, and, said $G_t = \mathrm{GL}_{\frac{n}{t}}, \mathrm{Sp}_{2\frac{n}{t}}$ we have

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h) = h(r-1)\dim G_t + \sum_{i=hr+1}^{t} \dim x_i^{G_t}$$

for some $x_i \in G_t$. Therefore, computationally, once gotten the formula for $\dim(x^G \cap H^\circ)$ it is straightforward to work out $\max_i\{\dim(x^G \cap H^\circ \pi_i)\}$ since $[x_{hr+1}, \dots, x_t]$ lies in a $\mathcal{C}_2$-subgroup of $\mathrm{GL}_{\frac{n}{t}(t-hr)}$ or $\mathrm{Sp}_{2\frac{n}{t}(t-hr)}$.

We shall prove the following.

Theorem 4.1. *Let $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$ be a semisimple element of prime order r in $H < \mathrm{GL}_n$. Then*

$$\dim(x^G \cap H^\circ) = \frac{n^2}{t} - n + \sum_{i=0}^{r-1}\Big(\lfloor \frac{a_i}{t} \rfloor^2 t + (t - 2a_i)\lfloor \frac{a_i}{t} \rfloor\Big) \tag{19}$$

Let $x = [I_{2a_0}, \omega I_{a_1}, \dots, \omega^{-\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}]$ be a semisimple element of prime order $r \neq 2$ in $H < \mathrm{Sp}_{2n}$ then

$$\dim(x^G \cap H^\circ) = 2\frac{n^2}{t} - 2a_0 + \Big\lfloor \frac{a_0}{t} \Big\rfloor^2 t + \Big\lfloor \frac{a_0}{t} \Big\rfloor (t - 2a_0) + \sum_{i=0}^{\frac{r-1}{2}} \Big(\Big\lfloor \frac{a_i}{t} \Big\rfloor^2 t + (t - 2a_i) \Big\lfloor \frac{a_i}{t} \Big\rfloor \Big) \tag{20}$$

If $r = 2$ and $x = [I_{2(n-s), -I_{2s}}]$, let $s = at + b$, where $0 \le b < t$ then

$$\dim(x^G \cap H^\circ) = \frac{4s(n-s)}{t} - \frac{4b(t-b)}{t} \tag{21}$$

In order to prove this result we require some general lemmas that establish a link between the shape of the element x in block form $[x_1, \dots, x_t]$ and the property $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$.

For $x = [x_1, \dots, x_t] \in H^\circ$, either in GL_n or in Sp_{2n} we establish the following notation

$$x_i = [I_{a_{i,0}}, \omega I_{a_{i,1}}, \dots, \omega^{r-1} I_{a_{i,r-1}}] \in \mathrm{GL}_{\frac{n}{t}}$$
$$x_i = [I_{2a_{i,0}}, \omega I_{a_{i,1}}, \dots, \omega^{-\frac{r-1}{2}} I_{a_{i,\frac{r-1}{2}}}] \in \mathrm{Sp}_{2\frac{n}{t}}$$

In the case we want to focus our attention only on two blocks, say x_1 and x_2, we set $a_{1,j} = a_j$ and $a_{2,j} = b_j$ for $j \in \{0, \dots, r-1\}$.

Lemma 4.2. *Let $x = [x_1, x_2, \dots, x_t]$ be a semisimple element of H of prime order r. Assume that there exist blocks x_i and x_j such that $a_{i,0} - a_{j,0} \ge 2$, say $a_{i,0} = a_{j,0} + 2 + h$ with $h \ge 0$. Then there exists $l \in \{2, \dots, r-1\}$ such that $a_{j,l} \neq 0$ and $a_{i,l} - a_{j,l} < h$. Or, in the case $G = \mathrm{Sp}_{2n}$,* *Proof.* First, let us use a lighter notation, say $a_{i,h} = a_h$ and $a_{j,h} = b_h$. Since $b_0 < n/t$ there exists some b_i non zero with $i > 0$. Say $b_0, b_{i_1}, \dots, b_{i_l}$ the only non zero multiplicities in the block x_j we are considering (note that b_0 might be 0). Suppose we have $a_{i_m} - b_{i_m} \geq h$ for $m = 1, \dots, l$ then summing over i_m, since $a_0 - b_0 = 2 + h > h + 1$, we have

$$0 = \frac{n}{t} - \frac{n}{t} \geq (a_0 - b_0) + \sum_{m=1}^{l} (a_{i_m} - b_{i_m}) > (h+1) + lh > 0$$

Therefore we get the result. *q.e.d.*

The same result for the symplectic group is the following.

Lemma 4.3. *Let $x = [x_1, \dots, x_t] \in H$ be a semisimple element of prime order r. Assume $a_j - b_j = 2 + h$ for some $j \in \{0, \dots, \frac{r-1}{2}\}$ and $h \geq 0$. Then there exists $i \in \{0, \dots, \frac{r-1}{2}\}$ such that, either $i \neq 0$, $b_i \neq 0$ and $a_i - b_i < h$, or $i = 0, b_0 \neq 0$ and $a_0 - b_0 < 0$.*

Proof. If $b_0 = 0$ or $a_0 - b_0 \geq 0$. Say $b_{j_1}, \dots, b_{j_m}$ the only non-zero multiplicities with $j_i \neq j$. Assume $a_{j_i} - b_{j_i} \geq h$ for all $i = 1, \dots, l$. Then $\frac{n}{t} = b_0 + b_j + \sum_i b_{j_i}$ and $\frac{n}{t} \geq a_0 + a_j + \sum_i a_{j_i}$. Therefore

$$0 \geq (a_0 - b_0) + (a_j - b_j) + \sum_i (a_{j_i} - b_{j_i}) \geq h + 2 + lh > 0$$

which is an absurd. Therefore there exists $i \neq 0$ such that $b_i \neq 0$ and $a_i - b_i < h$.

If $a_0 - b_0 > 0$ then $b_0 \neq 0$ since $a_0 \geq 0$. If $b_0 \neq 0$, we may assume $a_{j_i} - b_{j_i} \geq h$, using the same notation as above. Then

$$0 \geq (a_0 - b_0) + (a_j - b_j) + \sum_i (a_{j_i} - b_{j_i}) > a_0 - b_0 + 1 + h + hl$$

therefore $a_0 - b_0 < 0$. *q.e.d.*

Proposition 4.4. *Let $x = [x_1, \dots, x_t]$ be a semisimple element of H of prime order r such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Then, in the above notation, for all $h, k \in \{1, \dots, t\}$ and for all $i \in \{0, \dots, r-1\}$ we have*

$$|a_{h,i} - a_{k,i}| \leq 1$$

Proof. Let us prove the contrapositive. First, let us consider the case $G = \mathrm{GL}_n$. Assume $a_0 = b_0 + 2 + h$ for some $h \geq 0$. Then, assuming $i = 1, j = 2$,

$$x_1 = [I_{b_0+2+h}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$$
$$x_2 = [I_{b_0}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}]$$

Thanks to Lemma 4.2 there exists i such that $b_i \neq 0$ and $a_i - b_i < h$. Thereby we set $y = [y_1, y_2, x_3, \dots, x_t]$ to be

$$y_1 = [I_{b_0+1+h}, \omega I_{a_1}, \dots, \omega^i I_{a_i+1}, \dots \omega^{r-1} I_{a_{r-1}}]$$
$$y_2 = [I_{b_0+1}, \omega I_{b_1}, \dots, \omega^i I_{b_i-1}, \dots, \omega^{r-1} I_{b_{r-1}}]$$

Clearly x and y are G-conjugate. And, by a straightforward computation, we get $\dim y^{H^\circ} > \dim x^{H^\circ}$ if, and only if, $a_i - b_i < h$. The result follows.

Now, for $G = \mathrm{Sp}_{2n}$, we prove the contrapositive use the previous argument. Assume $a_{h,i} - a_{k,i} \geq 2$, without loss of generality we may assume $h = 1, k = 2$, say $a_{1,i} = a_i$ and $a_{2,i} = b_i$. Then either $i = 0$ or $i \neq 0$.

In the former case we have $a_0 - b_0 = 2 + h$, for some $h \geq 0$, and by Lemma 4.3 there exists $j > 0$ such that $b_j \neq 0$ and $a_j - b_j < h$, we may assume $j = 1$. Let $y = [y_1, y_2, x_3, \dots, x_t]$

where y_1, y_2 are defined in this way

$$y_1 = [I_{2(a_0-1)}, \omega I_{a_1+1}, \omega^{-1} I_{a_1+1}, \omega^2 I_{a_2}, \dots]$$
$$y_2 = [I_{2(b_0+1)}, \omega I_{b_1-1}, \omega^{-1} I_{b_1-1}, \omega^2 I_{b_2}, \dots]$$

clearly $y \in x^G$. And a straightforward computation shows

$$\begin{aligned}\dim y^{H^\circ} - \dim x^{H^\circ} &= \dim y_1^{\mathrm{Sp}_{2\frac{n}{t}}} + \dim y_2^{\mathrm{Sp}_{2\frac{n}{t}}} - \dim x_1^{\mathrm{Sp}_{2\frac{n}{t}}} - \dim x_2^{\mathrm{Sp}_{2\frac{n}{t}}} \\ &= 2(1 + 2h - a_1 + b_1) > 2(h+1) > 0\end{aligned}$$

where we used $-a_1 + b_1 > -h$.

In the latter case, we may assume $i = 1$ and we have $a_1 - b_1 = 2 + h$ for some $h \geq 0$. In order to apply Lemma 4.3 we divide the analysis in two cases. If $i = 0$ then $b_0 \neq 0$ and $a_0 - b_0 < 0$, by Lemma 4.3, then we define $y = [y_1, y_2, x_3, \dots, x_t]$ where

$$y_1 = [I_{2(a_0+1)}, \omega I_{a_1-1}, \omega^{-1} I_{a_1-1}, \omega^2 I_{a_2}, \dots]$$
$$y_2 = [I_{2(b_0-1)}, \omega I_{b_1+1}, \omega^{-1} I_{b_1-1}, \omega^2 I_{b_2}, \dots]$$

as above, $y \in x^G$ and, using $-(a_0 - b_0) \geq 1$,

$$\begin{aligned}\dim y^{H^\circ} - \dim x^{H^\circ} &= \dim y_1^{\mathrm{Sp}_{2\frac{n}{t}}} + \dim y_2^{\mathrm{Sp}_{2\frac{n}{t}}} - \dim x_1^{\mathrm{Sp}_{2\frac{n}{t}}} - \dim x_2^{\mathrm{Sp}_{2\frac{n}{t}}} \\ &= 2(a_1 - b_1 - 2(a_0 - b_0) - 3) = 2(h - 1 - 2(a_0 - b_0)) > 2(h+1) > 0\end{aligned}$$

In the case $i \neq 0$, we may assume $i = 2$, hence $b_2 \neq 0$ and $a_2 - b_2 < h$. Therefore we define $y = [y_1, y_2, x_3, \dots, x_t]$ with

$$y_1 = [I_{2a_0}, \omega I_{a_1-1}, \omega^{-1} I_{a_1+1}, \omega^2 I_{a_2+1}, \omega^{-2} I_{a_2+1}, \dots]$$
$$y_2 = [I_{2b_0}, \omega I_{b_1+1}, \omega^{-1} I_{b_1-1}, \omega^2 I_{b_2-1}, \omega^{-2} I_{b_2-1}, \dots]$$

as above, y is G-conjugate to x and

$$\begin{aligned}\dim y^{H^\circ} - \dim x^{H^\circ} &= \dim y_1^{\mathrm{Sp}_{2\frac{n}{t}}} + \dim y_2^{\mathrm{Sp}_{2\frac{n}{t}}} - \dim x_1^{\mathrm{Sp}_{2\frac{n}{t}}} - \dim x_2^{\mathrm{Sp}_{2\frac{n}{t}}} \\ &= 2(a_1 - b_1 - a_2 + b_2 - 2) = 2(h - (a_2 - b_2)) > 0\end{aligned}$$

If x is an involution then $\dim y^{H^\circ} - \dim x^{H^\circ} = 8(h+1) > 0$. *q.e.d.*

Remark 4.5. For $G = \mathrm{Sp}_{2n}$. If $x \in G$ is an involution then $\dim(x^G \cap H^\circ) \geq \dim x^{H^\circ} + 8(h+1) \geq \dim x^{H^\circ} + 8$, where h is as above.

Lemma 4.6. *Let $x = [x_1, \dots, x_t]$ be a semisimple element in H of prime order r, such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Let a_i be the multiplicity of ω^i as an eigenvalue of x. Then, in the above notation, $a_{i,h} \in \{\lfloor a_h/t \rfloor, \lfloor a_h/t \rfloor + 1\}$ for all $0 \leq h \leq r-1$.*

Proof. By Proposition 4.4 we have $|a_{i,h} - a_{j,h}| \leq 1$ for all $1 \leq i, j \leq t$ and $0 \leq h \leq r-1$.

Let us look only at the $a_{i,0}$'s, just to fix the notation, the argument will be the same for the other eigenvalues. We have $a_0 = \sum_{i=1}^t a_{i,0}$, the multiplicity of the eigenvalue 1 in x.

We may assume, without loss of generality that $a_{1,0} = \max_i\{a_{i,0}\}$ and $a_{2,0} = \min\{a_{i,0}\}$, then $a_{1,0} - a_{2,0} \in \{0, 1\}$. If $a_{1,0} - a_{2,0} = 0$ we have that, for all $i = 1, \dots, t$, $a_{i,0} = a_{1,0}$, hence $a_{i,0} = a_0/t$. Let us assume $a_{1,0} - a_{2,0} = 1$, we have $a_{2,0} = a_{1,0} - 1$ and for all $i = 1, \dots, t$, $a_{1,0} - 1 \leq a_{i,0} \leq a_{1,0}$. Thus, summing over $i = 1, \dots, t$, we get

$$(a_{1,0} - 1)t + 1 \leq a_0 \leq (a_{1,0} - 1)t + (t-1)$$

Thus $a_{1,0} - 1 = \lfloor a_0/t \rfloor$. The same argument holds for all $1 \leq h \leq r-1$. Therefore $a_{i,h} \in \{\lfloor a_h/t \rfloor, \lfloor a_h/t \rfloor + 1\}$ for all $h = 0, \dots, r-1$. *q.e.d.*

Now we can prove Theorem 4.1. Let $x \in H$ semisimple of prime order r, we shall construct a suitable element in $x^G \cap H^\circ$.

Proof of Theorem 4.1. Let x be as in the hypothesis. Assume $G = \mathrm{GL}_n$. Let us write $a_i = c_i t + b_i$ with $0 \le b_i < t$. Let $x = [x_1, \ldots, x_t]$ where

$$x_i = [I_{c_0+\epsilon_{i,0}}, \omega I_{c_1+\epsilon_{i,1}}, \ldots, \omega^{r-1} I_{c_{r-1}+\epsilon_{i,r-1}}] \tag{22}$$

where for every $j \in \{0, \ldots, r-1\}$ we have $\sum_{i=1}^t \epsilon_{i,j} = b_j$ and $\epsilon_{i,j} \in \{0,1\}$ for all i and j.

Thus we have

$$\begin{aligned}
\dim x^{H^\circ} &= \sum_{i=1}^{t}\Big(\big(\frac{n}{t}\big)^2 - \sum_{j=0}^{r-1}(c_j+\epsilon_{i,j})^2\Big) = \frac{n^2}{t} - \sum_i\sum_j (c_j^2 + 2c_j\epsilon_{i,j} + \epsilon_{i,j}^2)\\
&= \frac{n^2}{t} - t\sum_j c_j^2 - 2\sum_j c_j b_j - \sum_j b_j = \frac{n^2}{t} - \frac{1}{t}\sum_j\Big(c_j^2t^2 + 2c_jb_jt + b_jt\Big)\\
&= \frac{n^2}{t} - \frac{1}{t}\sum_j\Big(a_j^2 + tb_j - b_j^2\Big) = \frac{n^2}{t} - \frac{1}{t}\sum_j\Big(a_j^2 + t(a_j - c_jt) - (a_j - c_jt)^2\Big)\\
&= \frac{n^2}{t} - n + \sum_{j=0}^{r-1}\Big(t\Big\lfloor\frac{a_j}{t}\Big\rfloor^2 + (t - 2a_j)\Big\lfloor\frac{a_j}{t}\Big\rfloor\Big)
\end{aligned} \tag{23}$$

where we used $\epsilon_{i,j}^2 = \epsilon_{i,j}$, $\sum_i \epsilon_{i,j} = b_j$, $b_j = a_j - c_j t$ and $c_j = \lfloor a_j/t \rfloor$.

In the same way, we get a proof for $G = \mathrm{Sp}_{2n}$. It is enough to consider the element $x = [x_1, \ldots, x_t]$ where, writing $a_i = c_i t + b_i$,

$$x_i = [I_{2(c_0+\epsilon_{i,0})}, \omega I_{c_1+\epsilon_{i,1}}, \ldots]$$

and $\epsilon_{i,j} \in \{0,1\}$, $\sum_i \epsilon_{i,j} = b_j$ and $\sum_j \epsilon_{i,j} = \frac{n}{t} - \sum_j c_j$. *q.e.d.*

In the proof of Theorem 4.1 we found the formula by constructing a precise element. The following is an important characterization that helps us in the computation of $\dim(x^G \cap H^\circ)$.

Corollary 4.7. *Let x be a semisimple element of prime order r in H. We have* $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ *if, and only if,*

$$|a_{i,h} - a_{j,h}| \le 1$$

for all $h \in \{0, \ldots, r-1\}$ and $i, j \in \{1, \ldots, t\}$, where $x_i = [I_{\iota a_{i,0}}, \omega I_{a_{i,1}}, \ldots, \omega^{r-1} I_{a_{i,0}}]$, and either $\iota = 1$ for GL_n or $\iota = 2$ for $G = \mathrm{Sp}_{2n}$.

Proof. If $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ by Proposition 4.4 we get the result. Conversely, let us assume $|a_{i,h} - a_{j,h}| \le 1$ for all i, j and h, then each eigenvalue is spread in the blocks x_i's, i.e. $a_{i,h} = \lfloor a_h/t \rfloor$ for all i and all h, and by Lemma 4.6 we have that the blocks x_i of x have the same shape of z_i in (22), hence, by the computation done above and thanks to Theorem 4.1 we have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. *q.e.d.*

Remark 4.8. In the proof of Theorem 4.1 it has not been used any hypothesis on the order of the element. Therefore for any semisimple element x in H° the formula holds.

4.1. **Case r big.** In this section we assume $r \ge n$ if $G = \mathrm{GL}_n$ or $r \ge 2n$ if $G = \mathrm{Sp}_{2n}$.

Let x be a semisimple element of prime order r in H, we have $x = [x_1, \ldots, x_t]$ and we may assume $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. The aim of this section is to characterizing elements $x \in H \cap \mathcal{S}_r$ for which $f_\Omega(x) = 0$. Namely, we shall prove the following which lead to a partial proof of part *(iii)* in Theorem 1.

Proposition 4.9. *Let $x \in H\cap$ with prime order $r \ge n$ or $r \ge 2n$ for $G = \mathrm{Sp}_{2n}$. Then $f_\Omega(x) = 0$ if, and only if, $\nu(x) = n-1$ or, in the case $G = \mathrm{Sp}_{2n}$, $\nu(x) = n-2$ and $x = [I_2, \omega, \ldots, \omega^{-(n-1)}]$.*

Remark 4.10. Let x be an element of prime order $r > n$. Then $r > t$, so S_t contains no permutations of order r and thus $x^G \cap H = x^G \cap H^\circ$.

Assume $G = \mathrm{GL}_n$. Let us give the formula of $f_\Omega(x)$ for any x semisimple of prime order greater or equal than n in H°. Say $a_i = \sum_{j=1}^t a_{j,i}$, we have $\dim x^G = n^2 - \sum_{i=0}^{r-1} a_i^2$, hence

$$\dim x^G = n^2 - \sum_{j=0}^{r-1}\Big(\sum_{i=1}^{t} a_{i,j}\Big)^2, \dim x^{H^\circ} = \frac{n^2}{t} - \sum_{i,j} a_{i,j}^2$$

We get

$$\dim x^G - \dim x^{H^\circ} = \dim\Omega + \sum_{j=0}^{r-1}\Big[\Big(\sum_{i=1}^{t} a_{i,j}^2\Big) - \Big(\sum_{i=1}^{t} a_{i,j}\Big)^2\Big]$$

Therefore, since $\dim(x^G \cap H) = \dim x^{H^\circ}$,

$$f_\Omega(x) = \frac{2t}{(t-1)n^2}\sum_{h=0}^{r-1}\Big(\sum_{1\le i<j\le t} a_{ih}a_{jh}\Big) \tag{24}$$

For $G = \mathrm{Sp}_{2n}$. As above, we may assume $x = [x_1, \dots, x_t]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, say $x_i = [I_{2a_{i,0}}, \omega I_{a_{i,1}}, \omega^{-1} I_{a_{i,1}}, \dots, \omega^{\frac{r-1}{2}} I_{a_{i,\frac{r-1}{2}}}, \omega^{-\frac{r-1}{2}} I_{a_{i,\frac{r-1}{2}}}]$. Then

$$\dim x^G = n(2n+1) - \Big(\sum_{i=1}^{t} a_{i,0}\Big)\Big(2\Big(\sum_{i=1}^{t} a_{i,0}\Big)+1\Big) - \sum_{i=1}^{t}\Big(\sum_{j=1}^{\frac{r-1}{2}} a_{i,j}\Big)^2$$

$$\dim(x^G \cap H^\circ) = n\Big(2\frac{n}{t}+1\Big) - \sum_{i=1}^{t} a_{i,0}(2a_{i,0}+1) + \sum_{i,j} a_{i,j}^2$$

Therefore

$$\dim C_\Omega(x) = 2\sum_{1\le i<j\le t} a_{i,0}a_{j,0} + 2\sum_{h=0}^{\frac{r-1}{2}}\sum_{1\le i<j\le t} a_{i,h}a_{j,h} \tag{25}$$

Example 4.11.

- Consider the element $x = [1, \omega, \dots, \omega^{n-1}] \in \mathrm{GL}_n$. Then $\dim x^G = n^2 - n$ and $\dim(x^G \cap H) = n^2/t - n$. Therefore by Proposition 2.2 $\dim C_\Omega(x) = 0$, i.e. x fixes finitely many elements of Ω. Therefore $f_\Omega(x) = 0$.
- Let $x = [\omega, \omega^{-1}, \dots, \omega^n, \omega^{-n}] \in \mathrm{Sp}_{2n}$. Then we have
$$\dim x^G = 2n^2, \quad \dim(x^G \cap H) = 2\frac{n^2}{t}$$
where we use the fact $x^G \cap H = x^G \cap H^\circ$, and we easily see $\dim(x^G \cap H^\circ) = t \cdot \dim([\omega, \omega^{-1}, \dots, \omega^{n/t}, \omega^{-n/t}])^{\mathrm{Sp}_{2n/t}}$. Therefore $\dim C_\Omega(x) = 0$ and $f_\Omega(x) = 0$.
- Let $x = [I_2, \omega, \omega^{-1}, \dots, \omega^{n-1}, \omega^{-n+1}] \in \mathrm{Sp}_{2n}$. Then we have
$$\dim x^G = 2n^2 - 2, \quad \dim(x^G \cap H) = 2\frac{n^2}{t} - 2$$
Therefore $\dim C_\Omega(x) = 0$, and thus $f_\Omega(x) = 0$.

In order to prove Proposition 4.9 we need the following lemmas. We split the case $G = \mathrm{GL}_n$ from $G = \mathrm{Sp}_{2n}$.

Lemma 4.12. *Let $x = [x_1, \dots, x_t] \in H$ be a semisimple element. Assume $\dim(x^G \cap H) = \dim x^{H^\circ}$.*

- *If $G = \mathrm{GL}_n$. Assume $\nu(x) \leq n-2$ and let ω^i be an eigenvalue with largest multiplicity.*
- *If $G = \mathrm{Sp}_{2n}$. Assume $\nu(x) \leq n-2$ and, in the case $\nu(x) = n-2$, assume $x \neq [I_2, \omega, \dots, \omega^{-(n-1)}]$. Say ω^i an eigenvalue with largest multiplicity.*

Then ω^i appears in, at least, two different blocks of x.

Proof. The result follows from Corollary 4.7. *q.e.d.*

Now we can prove the main result of the section.

Proof of Proposition 4.9. Suppose that $\nu(x) = n-1$, then, as seen in the Example 4.11 $f_\Omega(x) = 0$ and the same if $\nu(x) = n-2$ and only 1 has a 2-dimensional eigenspace.

Conversely, the contrapositive derives from Lemma 4.12, in fact if $\nu(x) < n-1$, there is an eigenvalue with multiplicity bigger than 1 that is in, at least, two different blocks of x, in the block form $[x_1, \dots, x_t]$ for which $\dim(x^G \cap H) = \dim x^{H^\circ}$. And by the formula (24) for GL_n and (25) for Sp_{2n}, we get $f_\Omega(x) > 0$ since at least one, among the products $a_{i,h}a_{j,h}$, is non-zero. *q.e.d.*

4.2. **Case r small.** In this section we assume $r \leq n$ for $G = \mathrm{GL}_n$ and, $r \leq 2n$ for $G = \mathrm{Sp}_{2n}$. We aim to prove the following, in both case $G = \mathrm{GL}_n, \mathrm{Sp}_{2n}$, which leads to a complete proof of part *(iii)* in Theorem 1.

Proposition 4.13. *Let $x \in H$ be a semisimple element of prime order r. Then*

$$f_\Omega(x) \geq \frac{1}{r} - \epsilon$$

where $\epsilon \leq \frac{1}{n} + \frac{rt^2}{4n^2(t-1)}$.

Remark 4.14. Observe that $\frac{rt^2}{4n^2(t-1)} \leq \frac{r}{4(n-1)}$. Thus, for r fixed, ϵ tends to 0 when $n \to \infty$.

Actually slightly better bounds can be given in both cases $G = \mathrm{GL}_n$ and $G = \mathrm{Sp}_{2n}$. In the analysis, we shall construct a semisimple element $\bar{x}$ of prime order r such that for any x semisimple of order r, we have $f^\circ_\Omega(x) \geq f^\circ_\Omega(\bar{x})$. Then, we deduce the bound given in Proposition 4.13 as a lower bound on $f^\circ_\Omega(\bar{x})$.

Remark 4.15. For $G = \mathrm{GL}_n$ we shall construct the best possible lower bound, for f°_Ω, which is

$$\frac{1}{r} - \frac{br(t-b) - 2bc}{n^2(t-1)} + \frac{c^2}{n^2 r}$$

where we write $n = (qt+b)r + c$ where $0 \leq b < t$ and $0 \leq c < r$. Furthermore we shall prove that if x is a semisimple element of prime order r such that the difference between the multiplicities of two differente eigenvalue is at most 1 then x realizes the lower bound.

As explained, the strategy is to construct an element which realizes the lower bound on f°_Ω. The basic tool, in order to do that, is Lemma 4.16 for $G = \mathrm{GL}_n$ and Lemma 4.25 for $G = \mathrm{Sp}_{2n}$, in which from any element x, with centralizer not isomorphic to $C_G(\bar{x})$, we define a new element y and we prove $f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$. Then, using these two result, will be trivial to prove that $\bar{x}$ realizes, indeed, the lower bound on f°_Ω.

In the case $G = \mathrm{GL}_n$, we can say more. In fact, we can give lower bound on the full ratio f_Ω and give a characterization of the elements which realizes equality with it. We omit this further discussion here.

The analysis is similar for the case $G = \mathrm{GL}_n$ and $G = \mathrm{Sp}_{2n}$. We start with the first.

4.2.1. *General linear group.* With the following technical lemma we prove the result linked to the strategy explained in the introducion. Then using this result we will be able to construct a semisimple element $\bar{x} \in H$ for which $f^{\circ}_{\Omega}(x) \geq f^{\circ}_{\Omega}(\bar{x})$ for all $x \in H$ semisimple of order r.

Lemma 4.16. *Let $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$ be a semisimple element of prime order $r < n$ in H. Assume $a_0 = \max\{a_i\}$ and $a_1 = \min\{a_i\}$, and $a_0 - a_1 \geq 2$. Let $y = [I_{a_0-1}, \omega I_{a_1+1}, \dots, \omega^{r-1} I_{a_{r-1}}]$. Then*

$$f^{\circ}_{\Omega}(x) \geq f^{\circ}_{\Omega}(y)$$

with equality if, and only if, $a_0 - a_1 = 2$ and $a_0 \equiv 1 \mod t$.

Proof. Since $a_0 \geq a_1 + 2$ we can write $a_0 = a_1 + h$, where $h \geq 2$. The result is equivalent to the following inequality, by applying Proposition 2.2,

$$\dim y^G - \dim x^G > \dim(y^G \cap H^{\circ}) - \dim(x^G \cap H^{\circ}) \tag{26}$$

We have $\dim y^G - \dim x^G = 2(a_0 - a_1 - 1)$ and using Theorem 4.1 we compute $\dim(y^G \cap H^{\circ}) - \dim(x^G \cap H^{\circ})$:

$$\begin{aligned}\dim(y^G \cap H^{\circ}) - \dim(x^G \cap H^{\circ}) =& \Big(\Big\lfloor \frac{a_0 - 1}{t} \Big\rfloor^2 - \Big\lfloor \frac{a_0}{t} \Big\rfloor^2\Big)t + \Big(\Big\lfloor \frac{a_0 - 1}{t} \Big\rfloor - \Big\lfloor \frac{a_0}{t} \Big\rfloor\Big)(t - 2a_0) \\ & \Big(\Big\lfloor \frac{a_1 + 1}{t} \Big\rfloor^2 - \Big\lfloor \frac{a_1}{t} \Big\rfloor^2\Big)t + \Big(\Big\lfloor \frac{a_1 + 1}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor\Big)(t - 2a_1) \\ & + 2\Big\lfloor \frac{a_0 - 1}{t} \Big\rfloor - 2\Big\lfloor \frac{a_1 + 1}{t} \Big\rfloor\end{aligned}$$

Thus, we have four cases, depending on the value of the floor functions.

(1) For $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$, the inequality (26) is equivalent to

$$a_0 - a_1 - 1 > \Big\lfloor \frac{a_0}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor \tag{27}$$

(2) If $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor - 1$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor + 1$, the inequality (26) is equivalent to

$$(t-1)\Big(\Big\lfloor \frac{a_0}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor - 1\Big) > 0 \tag{28}$$

(3) If $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor - 1$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$, the inequality (26) is equivalent to

$$a_1 < t\Big\lfloor \frac{a_0}{t} \Big\rfloor + \Big\lfloor \frac{a_1}{t} \Big\rfloor - \Big\lfloor \frac{a_0}{t} \Big\rfloor \tag{29}$$

(4) Finally, for $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor + 1$,the inequality (26) is equivalent to

$$a_0 > \Big\lfloor \frac{a_0}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor + \Big\lfloor \frac{a_1}{t} \Big\rfloor t + t \tag{30}$$

We now consider each of these four cases.

(1) In this case we can write $a_0 = b_0 t + c_0$ and $a_1 = b_1 t + c_1$ where $c_0 \in \{1, \dots, t-1\}$ and $c_1 \in \{0, \dots, t-2\}$, moreover, since $a_0 \geq a_1 + 2$ we also have $b_0 \geq b_1$. The inequality (27) is equivalent to $h - 1 > b_0 - b_1$.

If $b_0 = b_1$, we immediately get the result, since $h \geq 2$, and hence $h - 1 > 0$. Let us assume $b_0 > b_1$, then there exists an integer $l \geq 1$ such that $b_0 = b_1 + l$, and the inequality to be proved, with this notation, is $lt + c_0 - c_1 - 1 > l$. Since $c_0 \geq 1$ and $-c_1 \geq 2 - t$, we have

$$lt - l + c_0 - c_1 - 1 \geq lt - l + 1 + 2 - t - 1 = (l-1)t - l + 2$$

hence, we need to show $(l-1)t - l + 2 > 0$, if $l = 1$ it is clearly true, let us assume $l \geq 2$, since $t \geq 2$, we have

$$(l-1)t \geq (l-1)2 > l - 2$$

Therefore it has been proved.

(2) In the second case we have $a_0 = b_0 t$ and $a_1 = b_1 t - 1 = (b_1 - 1)t + (t-1)$. Therefore the inequality (28) is equivalent to $b_0 - b_1 > 0$. Assume $b_0 \leq b_1$. Then $a_0 - a_1 = b_0 t - b_1 t + 1 \leq b_1 t - b_1 t + 1 = 1$, a contradiction since $a_0 \geq a_1 + 2$.

(3) In this case we have $a_0 = b_0 t$ and $a_1 = b_1 t + c_1$, with $c_1 \in \{0, \ldots, t-2\}$. Hence the inequality (29) is equivalent to

$$(b_0 - b_1)(t-1) > c_1$$

Let us assume $b_0 \leq b_1$, then $b_0 t \leq b_1 t \leq b_1 t + c_1$, i.e. $a_0 \leq a_1$, that contradicts the hypothesis, thus $b_0 > b_1$. Namely $b_0 - b_1 \geq 1$, hence $(b_0 - b_1)(t-1) \geq t - 1 > c_1$.

(4) In this case we have $a_0 = b_0 t + c_0$, with $c_0 \in \{1, \ldots, t-1\}$, and $a_1 = b_1 t - 1 = (b_1 - 1)t + (t-1)$. And the inequality (30) is equivalent to $(b_0 - b_1)(t-1) + c_0 - 1 > 0$.

Also in this case we have $b_0 \geq b_1$. Suppose $b_0 < b_1$, then $b_0 \leq b_1 - 1$ and $b_0 t \leq (b_1 - 1)t$, hence $a_0 = b_0 t + c_0 \leq b_0 t + t - 1 \leq (b_1 - 1)t + (t-1) = a_1$, but $a_0 > a_1$.

If $b_0 > b_1$, since $t \geq 2$, we have $(b_0 - b_1)(t-1) + c_0 - 1 \geq t - 1 + c_0 - 1 \geq c_0 > 0$. Let us assume $b_0 = b_1$, then the inequality (30) is equivalent to $c_0 > 1$. Hence for $c_0 > 1$ we get the result, while if $c_0 = 1$ we have $f_\Omega^\circ(x) = f_\Omega^\circ(y)$.

q.e.d.

Therefore Lemma 4.16 establishes that for all $x \in H$ semisimple of prime order r,

$$f_\Omega^\circ(x) \geq f_\Omega^\circ([I_{a_0}, \ldots, \omega^{r-1} I_{a_{r-1}}])$$

where $\max\{a_i\} - \min\{a_i\} \leq 1$. In the following remark we characterize elements in which the previous difference is 2. Then we give two technical lemma that will give the lower bound in the case $\max\{a_i\} - \min\{a_i\} \in \{0, 1\}$.

Remark 4.17. Let $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$ and assume $a_0 = \max\{a_i\}, a_1 = \min\{a_i\}$ and $a_0 - a_1 = 2$. Assume moreover $a_0 \equiv 1 \mod t, a_1 \equiv -1 \mod t$. Following Lemma 4.16 we define $y = [I_{a_0 - 1}, \omega I_{a_1 + 1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. Therefore, applying Lemma 4.16 we get $f_\Omega^\circ(x) = f_\Omega^\circ(y)$. Since for all $i = 0, \ldots, r-1$ we have $a_0 - 2 \leq a_i \leq a_0$, summing over i we have that the lower bound is $(a_0 - 2)(r-1) + a_0$ and the upper bound $a_0(r-1) + a_0 - 2$, that is:

$$(a_0 - 2)r + 2 \leq n \leq (a_0 - 1)r + (r - 2)$$

therefore

$$\left\lfloor \frac{n}{r} \right\rfloor + 1 \leq a_0 \leq \left\lfloor \frac{n}{r} \right\rfloor + 2 \tag{31}$$

Moreover in (31) both cases can occur, as shown in the following examples.

Example 4.18.

(1) Let $n = 30, t = 5, r = 7$, then $\lfloor n/r \rfloor = 4$, we consider the element

$$x = [I_6, \omega I_4, \omega^2 I_4, \omega^3 I_4, \omega^4 I_4, \omega^5 I_4, \omega^6 I_4]$$

and $6 \equiv 1(5), 4 \equiv -1(5)$. Therefore constructing the element y as in Lemma 4.16 we get $f_\Omega^\circ(x) = f_\Omega^\circ(y)$.

(2) Let $n = 40, t = 5, r = 7$, then $\lfloor n/r \rfloor = 5$, we consider the element

$$x = [I_6, \omega I_4, \omega^2 I_6, \omega^3 I_6, \omega^4 I_6, \omega^5 I_6, \omega^6 I_6]$$

and $6 \equiv 1(5), 4 \equiv -1(5)$. Therefore constructing the element y as in Lemma 4.16 we get $f_\Omega^\circ(x) = f_\Omega^\circ(y)$.

Let us consider the case $\max\{a_i\} - \min\{a_i\} \in \{0,1\}$. Now, using the definition of y given in Lemma 4.16 we will get either $f^\circ_\Omega(x) > f^\circ_\Omega(y)$ or $f^\circ_\Omega(x) = f^\circ_\Omega(y)$, depending on the specific values of n, r, t. Let us observe that in the case $\max\{a_i\} - \min\{a_i\} = 0$ we have $a_0 = \ldots = a_{r-1} = n/r$.

Lemma 4.19. *Assume r divides n. Let $x = [I_{n/r}, \omega I_{n/r}, \ldots, \omega^{r-1} I_{n/r}]$ and define $y = [I_{n/r-1}, \omega I_{n/r+1}, \omega^2 I_{n/r}, \ldots, \omega^{r-1} I_{n/r}]$. Then $f^\circ_\Omega(x) \leq f^\circ_\Omega(y)$, with equality if, and only if, rt divides n.*

Proof. As usual it is enough to understand whether or not the following inequality holds

$$\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) > \dim y^G - \dim x^G$$

We have $\dim y^G - \dim x^G = -2$, and applying Theorem 4.1 we get

$$\begin{aligned}\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) =& \Big(\Big\lfloor \frac{n/r-1}{t} \Big\rfloor^2 - \Big\lfloor \frac{n/r}{t} \Big\rfloor^2\Big)t + \Big(\Big\lfloor \frac{n/r-1}{t} \Big\rfloor - \Big\lfloor \frac{n/r}{t} \Big\rfloor\Big)\Big(t - 2\frac{n}{r}\Big) \\ & \Big(\Big\lfloor \frac{n/r+1}{t} \Big\rfloor^2 - \Big\lfloor \frac{n/r}{t} \Big\rfloor^2\Big)t + \Big(\Big\lfloor \frac{n/r+1}{t} \Big\rfloor - \Big\lfloor \frac{n/r}{t} \Big\rfloor\Big)\Big(t - 2\frac{n}{r}\Big) \\ & + 2\Big\lfloor \frac{n/r-1}{t} \Big\rfloor - 2\Big\lfloor \frac{n/r+1}{t} \Big\rfloor\end{aligned}$$

Let us write $n/r = at + b$, with $b \in \{0, \ldots, t-1\}$, we consider three cases:

(1) If $b = 0$ we have

$$\Big\lfloor \frac{n/r-1}{t} \Big\rfloor - \Big\lfloor \frac{n/r}{t} \Big\rfloor = -1 \text{ and } \Big\lfloor \frac{n/r+1}{t} \Big\rfloor - \Big\lfloor \frac{n/r}{t} \Big\rfloor = 0$$

and a straightforward computation shows $\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = -2$

(2) If $b \in \{1, \ldots, t-2\}$ then

$$\Big\lfloor \frac{n/r-1}{t} \Big\rfloor - \Big\lfloor \frac{n/r}{t} \Big\rfloor = 0 \text{ and } \Big\lfloor \frac{n/r+1}{t} \Big\rfloor - \Big\lfloor \frac{n/r}{t} \Big\rfloor = 0$$

and we deduce that $\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = 0$

(3) If $b = t-1$ we have

$$\Big\lfloor \frac{n/r-1}{t} \Big\rfloor - \Big\lfloor \frac{n/r}{t} \Big\rfloor = 0 \text{ and } \Big\lfloor \frac{n/r+1}{t} \Big\rfloor - \Big\lfloor \frac{n/r}{t} \Big\rfloor = 1$$

and a straightforward computation shows $\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = 0$

q.e.d.

Lemma 4.20. *Let $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$ be a semisimple element of prime order $r < n$ in H°. Assume $a_0 = \max\{a_i\}$ and $a_1 = \min\{a_i\}$, and $a_0 - a_1 = 1$. Let $y = [I_{a_0-1}, \omega I_{a_1+1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. Then*

$$f^\circ_\Omega(x) = f^\circ_\Omega(y)$$

Proof. This is clear from the fact that $a_0 - 1 = a_1$, which implies $C_G(x) \cong C_G(y)$. *q.e.d.*

Starting from any semisimple element x of prime order $r < n$ using the construction of Lemma 4.16 we can construct a finite sequence of semisimple element $y, y^{(1)}, y^{(2)}, \ldots, y^{(l)}$ for which $f^\circ_\Omega(x) > f^\circ_\Omega(y) \geq \ldots \geq f^\circ_\Omega(y^{(l)})$, we end up with the element $y^{(l)}$ in which either $a_i = n/r$ for all i or $\max\{a_i\} = \lfloor n/r \rfloor + 1, \min\{a_i\} = \lfloor n/r \rfloor$. Therefore it is clear by Lemma 4.16, 4.19, 4.20 that an element which has minimal f°_Ω-value is

$$x = [I_{\lfloor n/r \rfloor + \epsilon_0}, \omega I_{\lfloor n/r \rfloor + \epsilon_1}, \ldots, \omega^{r-1} I_{\lfloor n/r \rfloor + \epsilon_{r-1}}]$$

with $\epsilon_i \in \{0,1\}$ for all i and

$$\sum_{i=0}^{r-1} \epsilon_i = n - r\Big\lfloor \frac{n}{r} \Big\rfloor$$

Let us compute $f^{\circ}_{\Omega}(x)$. Without loss of generality we may fix the element x to be

$$x = [I_{\lfloor n/r\rfloor+1}, \omega I_{\lfloor n/r\rfloor+1}, \ldots, \omega^{l-1} I_{\lfloor n/r\rfloor+1}, \omega^{l} I_{\lfloor n/r\rfloor}, \ldots, \omega^{r-1} I_{\lfloor n/r\rfloor}] \tag{32}$$

where $l = n - \lfloor n/r\rfloor r$, is the number of the non-zero ϵ_i in (32). Thus, using Theorem 4.1, we get

$$\begin{aligned}
\dim x^G &= n^2 - n - 2n\Big\lfloor \frac{n}{r} \Big\rfloor + r\Big\lfloor \frac{n}{r} \Big\rfloor^2 + r\Big\lfloor \frac{n}{r} \Big\rfloor \\
\dim(x^G \cap H^{\circ}) &= \frac{n^2}{t} - n + lt\Big(\Big\lfloor \frac{\lfloor n/r\rfloor + 1}{t} \Big\rfloor^2 - \Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor^2\Big) \\
&+ l\Big(t - 2\Big\lfloor \frac{n}{r} \Big\rfloor\Big)\Big(\Big\lfloor \frac{\lfloor n/r\rfloor + 1}{t} \Big\rfloor - \Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor\Big) \\
&- 2l\Big\lfloor \frac{\lfloor n/r\rfloor + 1}{t} \Big\rfloor + rt\Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor^2 + rt\Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor - 2r\Big\lfloor \frac{n}{r} \Big\rfloor\Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor
\end{aligned}$$

It is natural to consider two cases: either t divides $\lfloor n/r\rfloor + 1$ or it does not. In the latter case, we have

$$\Big\lfloor \frac{\lfloor n/r\rfloor + 1}{t} \Big\rfloor = \Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor$$

Thus

$$\begin{aligned}
\dim C^{\circ}_{\Omega}(x) &= \dim \Omega - \dim x^G + \dim(x^G \cap H^{\circ}) \\
&= 2n\Big(\Big\lfloor \frac{n}{r} \Big\rfloor - \Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor\Big) - r\Big(\Big\lfloor \frac{n}{r} \Big\rfloor^2 - \Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor^2 t\Big) - r\Big(\Big\lfloor \frac{n}{r} \Big\rfloor - \Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor t\Big)
\end{aligned}$$

We write $n = (at+b)r+c$ with $c \in \{0,1,\ldots,r-1\}$ and $b \in \{0,1,\ldots,t-1\}$. Let us observe that if $b = t-1$ we fall in the case in which t divides $\lfloor n/r\rfloor + 1$. After a straightforward computation we find

$$f^{\circ}_{\Omega}(x) = \frac{1}{r} - \frac{br(t-b) - 2bc}{n^2(t-1)} - \frac{c^2}{n^2 r} \tag{33}$$

Similarly, in the first case, i.e. t divides $\lfloor n/r\rfloor + 1$, we have

$$\Big\lfloor \frac{\lfloor n/r\rfloor + 1}{t} \Big\rfloor = \Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor + 1$$

Thus

$$\begin{aligned}
\dim C^{\circ}_{\Omega}(x) &= \dim \Omega - \dim x^G + \dim(x^G \cap H^{\circ}) \\
&= 2nt\Big(1 - \frac{1}{t}\Big) - 2rt\Big\lfloor \frac{n}{r} \Big\rfloor\Big(1 - \frac{1}{t}\Big) + 2nt\Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor\Big(1 - \frac{1}{t}\Big) \\
&+ r\Big\lfloor \frac{n}{r} \Big\rfloor^2 - r\Big\lfloor \frac{n}{r} \Big\rfloor + rt\Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor^2 + rt\Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor - 2rt\Big\lfloor \frac{n}{r} \Big\rfloor\Big\lfloor \frac{\lfloor n/r\rfloor}{t} \Big\rfloor
\end{aligned}$$

As before, let us write $n = (at+b)r + c$ with $c \in \{0,1,\ldots,r-1\}$ and $b \in \{0,\ldots,t-1\}$. Observe that in this case $b = t-1$. Performing the computation we get

$$f^{\circ}_{\Omega}(x) = \frac{1}{r} - \frac{(r-c)^2}{n^2 r} \tag{34}$$

Observe that if in the equation (33) we substitute the value $b = t-1$ we get the formula (34).

Remark 4.21. The right hand side in (34) is clearly strictly less than $1/r$. Similarly for the right hand side of (33) because $r > c$ and we may assume $b \leq t-2$, hence

$$br(t-b) > 2br > 2bc$$

and for $b = c = 0$, i.e. $n = art$, we get $f^{\circ}_{\Omega}(x) = 1/r$. If, conversely, for the element y we have $f^{\circ}_{\Omega}(y) = 1/r$, we may assume $b \leq t-2$. Since $c \geq 0$ and $br(t-b) - 2bc > 0$ we must have $c = 0$ and, therefore, $br(t-b) = 0$, i.e. $b = 0$ because $t - b > 0$. Therefore the lower bound for $f^{\circ}_{\Omega}(\cdot)$ is $1/r$ if, and only if rt divides n.

The whole discussion on semisimple element of prime order $r < n$ in GL_n is detailed in the following.

Proposition 4.22. *Let $x \in H$ be a semisimple element of prime order $r < n$. Then*

$$f^{\circ}_{\Omega}(x) \geq \frac{1}{r} - \frac{br(t-b) - 2bc}{n^2(t-1)} - \frac{c^2}{n^2 r} \tag{35}$$

where

$$b = \lfloor n/r \rfloor - \left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor t, \quad c = n - \lfloor n/r \rfloor r$$

Furthermore equality holds if, and only if, $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$ and one of the following holds:

(i) *for all i, $\lfloor n/r \rfloor \leq a_i \leq \lfloor n/r \rfloor + 1$;*
(ii) *there exist j, j' such that $a_j = \lfloor n/r \rfloor + 1, a_{j'} = \lfloor n/r \rfloor - 1$ and for all i, $\lfloor n/r \rfloor - 1 \leq a_i \leq \lfloor n/r \rfloor + 1$, moreover $\lfloor n/r \rfloor \equiv 0 \mod t$;*
(iii) *there exist j, j' such that $a_j = \lfloor n/r \rfloor + 2, a_{j'} = \lfloor n/r \rfloor$ and for all $i \neq j$, $\lfloor n/r \rfloor \leq a_i \leq \lfloor n/r \rfloor + 1$, moreover $\lfloor n/r \rfloor \equiv -1 \mod t$.*

Proof. Thanks to Lemma 4.16, for all the semisimple elements of prime order $r < n$ we have $f^{\circ}_{\Omega}(x) \geq f^{\circ}_{\Omega}(\bar{x})$, with $\bar{x}$ as in (32), because starting from any semisimple element x of prime order $r < n$ using the construction of Lemma 4.16 we can construct a finite sequence of semisimple element $y, y^{(1)}, y^{(2)}, \ldots, y^{(l)}$ for which $f^{\circ}_{\Omega}(x) > f^{\circ}_{\Omega}(y) \geq \ldots \geq f^{\circ}_{\Omega}(y^{(l)})$, we end up with the element $y^{(l)}$ in which either $a_i = n/r$ for all i or $\max\{a_i\} = \lfloor n/r \rfloor + 1, \min\{a_i\} = \lfloor n/r \rfloor$, i.e. the element described in (i). The particular cases (ii) and (iii) are shown in Lemmas 4.16, 4.19. *q.e.d.*

As consequence of Proposition 4.22 we get Proposition 4.13, for the general linear group, using the following

Lemma 4.23. *In the above notation. Let $\epsilon = \frac{br(t-b)-2bc}{n^2(t-1)} + \frac{c^2}{n^2 r}$. Then*

$$0 \leq \epsilon \leq \frac{rt^2}{4n^2(t-1)} + \frac{r}{n^2}$$

Proof. Let us start showing that $\epsilon \geq 0$. If $b = 0$ it is clear. Hence we may assume $b \neq 0$, and we claim $r(t-b) - 2c > 0$, which leads the result. For $b \leq t-2$ we have $r-(t-b)-2c > 2r-2c > 0$ because $c < r$. If $b = t-2$ the $br(t-b) - 2bc = 2r(t-2) - 2c(t-2) = 2(r-c)(t-2) \geq 0$ since $c < r$ and $t \geq 2$.

Let us show the upper bound for ϵ. We have

$$\begin{aligned}\epsilon &= \frac{br(t-b)-2bc}{n^2(t-1)} + \frac{c^2}{n^2r} \leq \frac{br(t-b)}{n^2(t-1)} + \frac{c^2}{n^2r} \\ &\leq \frac{rt^2}{4n^2(t-1)} + \frac{c^2}{n^2r} \leq \frac{rt^2}{4n^2(t-1)} + \frac{r}{n^2} \\ &\leq \frac{rt^2}{4n^2(t-1)} + \frac{1}{n}\end{aligned}$$

q.e.d.

With further analysis it is possible to give a sharp lower bound on f_Ω.

4.2.2. *Symplectic group.* We, now, prove Proposition 4.13 in the case $G = \mathrm{Sp}_{2n}$.

First, as explained above we prove the symplectic analogous to Lemma 4.16. In particular, given a prime r, we shall prove that there exist a semisimple element of order r, say $\bar{x}$, such that for all $x \in H$ semisimple of order r we have $f^\circ_\Omega(x) \geq f^\circ_\Omega(\bar{x})$. The following is the formal statement.

Proposition 4.24. *Let $x \in H < \mathrm{Sp}_{2n}$ be a semisimple element of order r. Then*

$$f^\circ_\Omega(x) \geq f^\circ_\Omega(\bar{x})$$

where, for $2n = \lfloor \frac{2n}{r} \rfloor r + b$,

$$\bar{x} = \begin{cases} [I_{\lfloor \frac{2n}{r}\rfloor}, \omega I_{\lfloor \frac{2n}{r}\rfloor+1}, \omega^{-1} I_{\lfloor \frac{2n}{r}\rfloor+1}, \dots, \omega^{-\frac{b}{2}} I_{\lfloor \frac{n}{r}\rfloor+1}, \omega^{\frac{b}{2}+1} I_{\lfloor \frac{2n}{r}\rfloor}, \dots, \omega^{-\frac{r-1}{2}} I_{\lfloor \frac{2n}{r}\rfloor}] & b \text{ even} \\ [I_{\lfloor \frac{2n}{r}\rfloor+1}, \omega I_{\lfloor \frac{2n}{r}\rfloor+1}, \omega^{-1} I_{\lfloor \frac{2n}{r}\rfloor+1}, \dots, \omega^{-\frac{b-1}{2}} I_{\lfloor \frac{n}{r}\rfloor+1}, \omega^{\frac{b-1}{2}+1} I_{\lfloor \frac{2n}{r}\rfloor}, \dots, \omega^{-\frac{r-1}{2}} I_{\lfloor \frac{2n}{r}\rfloor}] & b \text{ odd}\end{cases}$$

Here we prove the technical lemma, which is the equivalent for Sp_{2n} of Lemma 4.16. In this case, given the particular structure of the centralizer of semisimple elements, we shall study separately the cases in which the swap involves the 1-eigenvalue.

Lemma 4.25. *Let $x \in H$ semisimple of prime order r. Assume $\max\{a_i, 2a_0\} - \min\{a_i, 2a_0\} \geq 2$. Then there exists y such that $\nu(y) \in \{\nu(x), \nu(x)+1\}$ and $f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$.*

Proof. Let $x = [I_{2a_0}, \omega I_{a_1}, \dots]$. Showing that $f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$ is equivalent to show

$$\dim y^G - \dim x^G \geq \dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) \tag{36}$$

By hypothesis $\max\{a_i, 2a_0\} - \min\{a_i.2a_0\} = 2 + h$ for some $h \geq 0$.

If $\max\{a_i, 2a_0\} \neq a_0 \neq \min\{a_i, 2a_0\}$ we follow the same construction of Lemma 4.16 and we get the result.

If $\max\{a_i, 2a_0\} = 2a_0$, let us assume $\min\{a_i\} = a_1$, with $2a_0 - a_1 = 2 + h$, for some $h \geq 0$. Then we define $y = [I_{2(a_0-1)}, \omega I_{a_1+1}, \dots]$. And $\dim y^G - \dim x^G = 2(h+1)$. In order to prove the result we use the formula for $\dim(x^G \cap H^\circ)$ in Theorem 4.1, hence we have to distinguish four cases depending on the values of the floors $\lfloor \frac{a_0-1}{t} \rfloor, \lfloor \frac{a_1+1}{t} \rfloor$. Say $a_0 = c_0 t + b_0, a_1 = c_1 t + b_1$.

- If $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$ we get $\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = 2(2c_0 - c_1 + 1)$. Therefore $f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$ using $(2c_0 - c_1)t = 2 + h - (2b_0 - b_1)$ is equivalent to $h\big(1 - \frac{1}{t}\big) - \frac{2}{t} + \frac{2b_0 - b_1}{t} \geq 0$. Since $b_0 > 0, b_1 < t-1$ we have $h\big(1 - \frac{1}{t}\big) - \frac{2}{t} + \frac{2b_0 - b_1}{t} \geq (h-1)\big(1 - \frac{1}{t}\big) \geq 0$ which is true if, and only if, $h \geq 1$. Assume $h = 0$, then $f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$ if, and only if, $2b_0 - b_1 \geq 2$. Assume $2b_0 - b_1 < 2$, then for some $l \geq 0$ we can write $2b_0 - b_1 = 1 - l$. Since $b_0 \geq 1, b_1 \leq t-2$ we get $4 - t \leq 2b_0 - b_1 = 1 - l$. Thus $2 = 2a_0 - a_1 = (2c_0 - c_1)t + 1 - l$. Therefore

$l+1=(2c_0-c_1)t$. But $1\le l+1\le t-3$, which is an absurd. Therefore, for $h=0$, we have $2b_0-b_1\ge 2$.

- If $\lfloor\frac{a_0-1}{t}\rfloor=\lfloor\frac{a_0}{t}\rfloor-1$ and $\lfloor\frac{a_1+1}{t}\rfloor=\lfloor\frac{a_1}{t}\rfloor+1$ we get $\dim(y^G\cap H^\circ)-\dim(x^G\cap H^\circ)=2(2c_0-c_1-1)$. Therefore, as above, (36) is equivalent to $h+1\ge 2c_0-c_1-1$, using $2c_0-c_1=\frac{2+h-(2b_0-b_1)}{t}$ and $b_0=0,b_2=t-1$, we get $(h+1)\big(1-\frac{1}{t}\big)\ge 0$, which is always true for $h\ge 0$.
- If $\lfloor\frac{a_0-1}{t}\rfloor=\lfloor\frac{a_0}{t}\rfloor-1$ and $\lfloor\frac{a_1+1}{t}\rfloor=\lfloor\frac{a_1}{t}\rfloor$ we get $\dim(y^G\cap H^\circ)-\dim(x^G\cap H^\circ)=2(2c_0-c_1-1)$. Following the same argument as above, with $b_0=0,b_1>0$ we get that (36) is always satisfied for $h\ge 0$.
- If $\lfloor\frac{a_0-1}{t}\rfloor=\lfloor\frac{a_0}{t}\rfloor$ and $\lfloor\frac{a_1+1}{t}\rfloor=\lfloor\frac{a_1}{t}\rfloor+1$. We get $\dim(x^G\cap H^|circ)-\dim(y^G\cap H^|circ)=2(2c_0-c_1+1)$. Therefore, as in the first point, using $b_0<t-1,b_1=t-1$, (36) is satisfied for $h\ge 1$. Assume $h=0$, then (36) ifm and only if, $b_0\ge\frac{t+1}{2}$, which is equivalent to $2b_0-b_1\ge 2$ since $b_1=t-1$. With the same argument as the first point we get the result.

Let us assume, now, $\min\{a_i,2a_0\}=2a_0$ and we may assume $\max\{a_i\}=a_1$. Furthermore, by assumption, we have $a_1-2a_0=2+h$, for some $h\ge 0$. And $\dim y^G-\dim x^G=2(a_1-2a_0-2)=2h$. As above, we shall distinguish four cases. Here $c_1-2c_0=\frac{2+h-(b_1-2b_0)}{t}$.

- If $\lfloor\frac{a_0+1}{t}\rfloor=\lfloor\frac{a_0}{t}\rfloor$ and $\lfloor\frac{a_1-1}{t}\rfloor=\lfloor\frac{a_1}{t}\rfloor$ we get $\dim(y^G\cap H^\circ)-\dim(x^G\cap H^\circ)=2(c_1-2c_0-1)$. Hence (36) is equivalent to $h\ge c_1-2c_0-1$. Using $b_1\ge 1,b_0<t-1$ and $c_1-2c_0=\frac{2+h-(b_1-2b_0)}{t}$ we get $(h+1)\big(1-\frac{1}{t}\big)+\frac{b_1-2b_0-1}{t}\ge(h-1)\big(1-\frac{1}{t}\big)$, which is true for $h\ge 1$. If $h=0$, (36) is equivalent to $b_1-2b_0\ge 2-t$. Using a similar argumet as above we get the result. Assume $b_1-2b_0<2-t$, say $b_1-2b_0=1-t-l$ for some $l\ge 0$. Then, since $b_1\ge 1,b_0\le t-2$ we get $l\le t-4$. Thus $2=a_1-2a_0=(c_1-2c_0)t+1-t-l$. Therefore $l+1=(c_1-2c_0-1)t$, which is an absurd since $1\le l+1\le t-3$.
- If $\lfloor\frac{a_0+1}{t}\rfloor=\lfloor\frac{a_0}{t}\rfloor+1$ and $\lfloor\frac{a_1-1}{t}\rfloor=\lfloor\frac{a_1}{t}\rfloor-1$, in this case we have $b_0=t-1,b_1=0$. And we get $\dim(y^G\cap H^\circ)-\dim(x^G\cap H^\circ)=2(2c_0-c_1-2)$. Therefore (36) is equivalent to $h+2\ge c_1-2c_0=\frac{h+2}{t}-\frac{b_1-2b_0}{t},$. Hence $(h+2)\big(1-\frac{1}{t}\big)+\frac{b_1-2b_0}{t}=(h+2)\big(1-\frac{1}{t}\big)=h\big(1-\frac{1}{t}\big)\ge 0$ for $h\ge 0$.
- If $\lfloor\frac{a_0+1}{t}\rfloor=\lfloor\frac{a_0}{t}\rfloor+1$ and $\lfloor\frac{a_1-1}{t}\rfloor=\lfloor\frac{a_1}{t}\rfloor$ we get $\dim(y^G\cap H^\circ)-\dim(x^G\cap H^\circ)=2(c_1-2c_0-1)$. Therefore (36) is equivalent to $h\big(1-\frac{1}{t}\big)-1+\frac{b_1}{t}\ge 0$. Since $b_1\ge 1$ we have $h\big(1-\frac{1}{t}\big)-1+\frac{b_1}{t}\ge(h-1)\big(1-\frac{1}{t}\big)$ which is non-negative for $h\ge 1$. Assume $h=0$ then $2=a_1-2a_0=(c_1-2c_0)t+(b_1-2b_0)$. Hence, using $(*)$ and $b_0=t-1,0<b_1\le t-1$, we have $1+\frac{1}{t}\le c_1-2c_0\le 2-\frac{1}{t}$. And since c_1,c_0 are integer this may never happen.
- If $\lfloor\frac{a_0+1}{t}\rfloor=\lfloor\frac{a_0}{t}\rfloor$ and $\lfloor\frac{a_1-1}{t}\rfloor=\lfloor\frac{a_1}{t}\rfloor-1$ we get $\dim(y^G\cap H^\circ)-\dim(x^G\cap H^\circ)=2(2c_0-c_1-2)$. Therefore (36) is equivalent to $(2+h)\big(1-\frac{1}{t}\big)-\frac{2b_0}{t}\ge 0$. Using $b_0<t-1$ we have $(2+h)\big(1-\frac{1}{t}\big)-\frac{2b_0}{t}>h\big(1-\frac{1}{t}\big)\ge$ for $h\ge 0$. This completes the proof.

q.e.d.

Now, we can prove Proposition 4.24.

Proof of Proposition 4.24. Given any $x\in H$ semisimple of prime order r, if $\max\{2a_0,a_i\}-\min\{2a_0,a_i\}\ge 2$ $(*)$, thanks to Lemma 4.25, we can construct a new element y such that $f^\circ_\Omega(x)\ge f^\circ_\Omega(y)$. If the multiplicities of the eigenvalues in y still satisfy property $(*)$ then we use Lemma 4.25, again. Eventually, we get an element $\bar{x}$ which is as in the statement, up to renaming the eigenvalues ω^i. *q.e.d.*

In order to get a complete proof of Proposition 4.13, it is enough to compute the value of $f^\circ_\Omega(\bar{x})$ where x is as in Proposition 4.24.

Say $2n = (at+b)r + c$, where $0 \le b < t$ and $0 \le c < r$. We have

$$\dim \bar{x}^G = n(2n+1) - \frac{a^2rt^2}{2} - abrt - abt - \frac{at}{2} - \frac{b^2r}{2} - b^2 - b - \alpha$$

where $\alpha = 0$ for $\lfloor \frac{2n}{r} \rfloor$ even and $\alpha = \frac{1}{2}$ otherwise.

Assume $\lfloor \frac{2n}{r} \rfloor$ is even. We write $\lfloor \frac{2n}{r} \rfloor = 2a't + 2b'$, for $0 \le 2b' < 2t$. If $2b' < t$ we have $a' = \frac{a}{2}$, if $t \le 2b' < 2t$ we have $a' = \frac{a-1}{2}$. Using Theorem 4.1 we compute $\dim(x^G \cap H^\circ)$. Eventually we get

$$f^\circ_\Omega(\bar{x}) = \frac{1}{r} + \frac{abt - act - bc - \frac{c^2}{2r}}{2n^2} - \frac{\frac{br}{2t}(t-b) - b^2 + \frac{c}{2} + \iota}{2n^2\left(1 - \frac{1}{t}\right)} \tag{37}$$

where $\iota = 0$ for $b' < t$ and $\iota = \frac{t}{2} - b$ for $b' \ge t$.

In the case $\lfloor \frac{2n}{r} \rfloor$ is odd. As above, we write $\lfloor \frac{2n}{r} \rfloor + 1 = at + (b+1)$ and $\lfloor \frac{2n}{r} \rfloor + 1 = 2a't + 2b' + 2$. Here we need to distinguish the case $b < t-1$ from $b = t-1$. In the former case, if $2b' < t$, we have $a' = \frac{a}{2}$ (i), instead, for $2b' \ge t$ we get $a' = \frac{a-1}{2}$ (ii). For $b = t-1$ we get $a' = \frac{a+1}{2}$ (iii) if $2b' < t$ and $a' = \frac{a}{2}$ (iv) if $2b' \ge t$. And we get

$$f^\circ_\Omega(\bar{x}) = \frac{1}{r} + \frac{abt - act - bc - \frac{c^2}{2r}}{2n^2} - \frac{\frac{br}{2t}(t-b) - b^2 + \frac{c}{2} + \iota'}{2n^2\left(1 - \frac{1}{t}\right)} \tag{38}$$

where $\iota' = \frac{1}{2}$ in the cases $(i), (iv)$ and $\iota' = \frac{t-1}{2}$ in the cases $(ii), (iii)$.

In any case we have, using $at = \frac{2n-c}{r} - b$ and $c < r$,

$$\begin{aligned} f^\circ_\Omega(\bar{x}) &\ge \frac{1}{r} + \frac{abt - act - bc - \frac{c^2}{2r}}{2n^2} - \frac{\frac{br}{2t}(t-b) - b^2 + \frac{c}{2} + \frac{t}{2}}{2n^2\left(1 - \frac{1}{t}\right)} \\ &= \frac{1}{r} + \frac{\frac{2nb}{r} - \frac{cb}{r} - b^2 - \frac{2nc}{r} - \frac{c^2}{2r}}{2n^2} - \frac{\frac{br}{2t}(t-b) - b^2 + \frac{c}{2} + \frac{t}{2}}{2n^2\left(1 - \frac{1}{t}\right)} \\ &\ge \frac{1}{r} + \frac{\frac{2nb}{r} - b - b^2 - \frac{2nc}{r} - \frac{c^2}{2r}}{2n^2} - \frac{\frac{br}{2t}(t-b) - b^2 + \frac{c}{2} + \frac{t}{2}}{2n^2\left(1 - \frac{1}{t}\right)} \end{aligned}$$

It is straightforward to see that, said $g(c) = \left(-\frac{2nc}{r} - \frac{c^2}{2r}\right)\left(1 - \frac{1}{t}\right) - \frac{c}{2}$, we have $g(c) > g(r-1) > g(r) = -2n\left(1 - \frac{1}{t}\right) - \frac{r}{2t}$. Similarly, we have $\frac{2nb - b - b^2}{2n^2} - \frac{\frac{br}{2t}(t-b) - b^2}{2n^2\left(1 - \frac{1}{t}\right)} \ge 0$. Therefore

$$f^\circ_\Omega(\bar{x}) \ge \frac{1}{r} - \frac{1}{n} - \frac{r}{4n^2(t-1)} > \frac{1}{r} - \frac{3}{2n}$$

Therefore Proposition 4.13 is proved also in the case $G = \mathrm{Sp}_{2n}$.

Since all the conjugacy classes of semisimple elements of H have non-trivial intersection with H°, we can use, as global lower bound for f_Ω, the lower bound found for f°_Ω. Anyway, it is possible, although tedious, to compute a better bound for f_Ω, in the following remark we explain the general strategy.

Remark 4.26. Let us observe that given $x \in H$ semisimple of prime order r, we reduce the computation of the lower bound of $f_\Omega(x)$ to the computation of the lower bound on $f^\circ_\Omega(x)$. Given x such that $x^G \cap (H \setminus H^\circ) \ne \emptyset$, thanks to Lemma 2.19, we have

$$x = [I_{2(a_0 + \frac{n}{t}h)}, \omega I_{a_1 + \frac{n}{t}h}, \omega^{-1} I_{a_1 + \frac{n}{t}h}, \ldots, \omega^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}} + \frac{n}{t}h}, \omega^{-\frac{r-1}{2}} I_{\frac{r-1}{2} + \frac{n}{t}h}]$$

with $\min_i\{a_i\} < \frac{n}{t}$. Then $\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi_l)$ for a suitable $0 \leq l \leq h$. Therefore, said $L = \mathrm{Sp}_{2(n-\frac{n}{t}lr)}$ and $K = \mathrm{Sp}_{2\frac{n}{t}} \wr S_{(t-lr)}$ a $\mathcal{C}_2$-subgroup of L we have

$$\dim(x^G \cap H) = l(r-1)\dim \mathrm{Sp}_{2\frac{n}{t}} + \dim(\bar{x}^L \cap K^\circ)$$

And we have $\dim x^G = \lambda + \dim \bar{x}^L$, for some λ. Say $\overline{\Omega} = L/K$ the primitive variety where L is acting. Then

$$\dim C_\Omega(x) = \dim \Omega - \dim \overline{\Omega} - \lambda + l(r-1)\dim \mathrm{Sp}_{2\frac{n}{t}} + \dim C^\circ_{\overline{\Omega}}(x)$$

Say $\alpha(n,t,r)$ the lower bound on f°_Ω we get

$$f_\Omega(x) \geq \frac{\dim \Omega - \dim \bar{\Omega} - \lambda + l(r-1)\dim \mathrm{Sp}_{2\frac{n}{t}} + \alpha(n - \frac{n}{t}lr, t-lr, r)\dim \overline{\Omega}}{\dim \Omega} \tag{39}$$

At this point, it is not hard to study the right hand side of (39) as function of l, knowing that $0 \leq l \leq h \leq \lfloor \frac{t}{r} \rfloor$, and minimize it.

5. Global lower bounds: unipotents elements

The aim of this section is to establish conclusions of Theorem 1 and 2, for unipotent elements in H of prime order.

First let set up some notation. We have already explained that there is a one to one correspondence between conjugacy classes of semisimple elements in $G = \mathrm{GL}_n, \mathrm{Sp}_{2n}$ and partitions of $n, 2n$, with the further assumption, for the symplectic group, that odd parts have even multiplicity. Given $x \in H$ of prime order, we have $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$. If $x \in H^\circ$ then $x = [x_1, \ldots, x_t]$ and

$$x_i \quad \longleftrightarrow \quad \left(p^{a_{i,p}}, \ldots, 1^{a_{i,1}}\right) \vdash \frac{n}{t} \tag{40}$$

$$x \quad \longleftrightarrow \quad \left(p^{a_{1,p}+\ldots+a_{t,p}}, \ldots, 1^{a_{1,1}+\ldots+a_{t,1}}\right) \vdash n \tag{41}$$

And by Proposition 2.7 we get, for $G = \mathrm{GL}_n$, assuming $\dim(x^G \cap H) = \dim(x^G \cap H^\circ) = \sum \dim x_i^{\mathrm{GL}_{n/t}}$,

$$\dim x^G = n^2 - 2 \sum_{1 \le i < j \le p} i(a_{1,i} + \ldots + a_{t,i})(a_{1,j} + \ldots + a_{t,j}) - \sum_{i=1}^{p} i(a_{1,i} + \ldots + a_{t,i})^2$$

$$\dim(x^G \cap H^\circ) = \frac{n^2}{t} - 2 \sum_{1 \le i < j \le p} i(a_{1,i}a_{1,j} + \ldots + a_{t,i}a_{t,j}) - \sum_{i=1}^{p} i(a_{1,i}^2 + \ldots + a_{t,i}^2)$$

We have two cases depending on p. For $p > n$, the largest Jordan block for a unipotent element of prime order in H has size $\frac{n}{t}$, namely there are no unipotent element in $H \setminus H^\circ$. On the other hand, for $p \le n$, in order to have a prime order element the largest size of Jordan block is p.

5.1. Case $p > n$.

In this case the largest Jordan block of any unipotent element of prime order is $\frac{n}{t}$ and, as remarked above, if x is a unipotent element of order p, $x^G \cap H = x^G \cap H^\circ$. Therefore we may assume $x \in H^\circ$. We shall prove the following.

Proposition 5.1. *Let $x \in H$ be a unipotent element of prime order. Then*

$$f_\Omega(x) \ge \frac{t}{\iota n}$$

where $\iota = 1$ if $G = \mathrm{GL}_n$ and $\iota = 2$ for $G = \mathrm{Sp}_{2n}$. Furthermore $f_\Omega(x) = \frac{t}{\iota n}$ if, and only if, $x = [J_{n/t}^{t-1}, z]$ where $z \in \mathrm{GL}_{n/t}(k)$ or $\mathrm{Sp}_{2n/t}$ is any unipotent element.

Example 5.2. Let $G = \mathrm{GL}_n$ and $p > n$.

- Let $x = [J_{n/t}^t]$. Then $\dim x^G = n^2 - tn$, while $\dim(x^G \cap H^\circ) = t \cdot \dim [J_{n/t}]^{\mathrm{GL}_{n/t}(k)} = n^2/t - n$. Thanks to Proposition 2.10, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Therefore $f_\Omega(x) = \frac{t}{n}$.
- Let $x = [J_{n/t}^{t-1}, x_t]$, with $x_t \in \mathrm{GL}_{n/t}$ unipotent (possibly trivial). The associated partition is
$$\Big(\frac{n}{t}^{t-1}, \big(\frac{n}{t} - 1\big)^{a_{n/t-1}} \ldots, 1^{a_1}\Big) \vdash n$$
where $\Big((n/t - 1)^{a_{n/t-1}}, \ldots, 1^{a_1}\Big)$ is a partition of the integer n/t associated to x_t. We get $f_\Omega(x) = \frac{t}{n}$.

Example 5.3. Let $G = \mathrm{Sp}_{2n}$ and $p > n$.

- Let $x = [J_{2\frac{n}{t}}^t]$. Then $\dim x^G = 2n^2 - nt\big(1 - \frac{1}{t}\big)$ and $\dim(x^G \cap H^\circ) = t \dim [J_{2\frac{n}{t}}]^{\mathrm{Sp}_{2\frac{n}{t}}} = 2\frac{n^2}{t}$. Therefore $f_\Omega^\circ(x) = \frac{t}{2n}$.

- Let $x = [J_{2\frac{n}{t}}^{t-1}, x_t]$, where $x_t \in \mathrm{Sp}_{2\frac{n}{t}}$ is any unipotent element. Then $\dim x^G = 2n^2 - 2\left(\frac{n}{t}\right)^2 - nt\left(1-\frac{1}{t}\right) + \dim x_t^{\mathrm{Sp}_{2\frac{n}{t}}}$. And $\dim(x^G \cap H^\circ) = (t-1)\dim[J_{2\frac{n}{t}}^{t-1}] + \dim x_t^{\mathrm{Sp}_{2\frac{n}{t}}} = 2\frac{n^2}{t} - 2\left(\frac{n}{t}\right)^2 + \dim x_t^{\mathrm{Sp}_{2\frac{n}{t}}}$. Therefore $f_\Omega^\circ(x) = \frac{t}{2n}$.

In order to prove Proposition 5.1, we need the following technical lemma, which is, in a sense, similar to Lemma 4.16, 4.25. The first is for $G = \mathrm{GL}_n$.

Let $\iota = 1$ for $G = \mathrm{GL}_n$ and $\iota = 2$ for $G = \mathrm{Sp}_{2n}$.

Lemma 5.4. *Let $x = [x_1, \ldots, x_t] \in H^\circ$ with $x_1, x_2 \neq J_{\iota n/t}$. Suppose $\dim(x^G \cap H) = \dim x^{H^\circ}$. Let $y = [J_{\iota n/t}, x_2, \ldots, x_t]$. Then*

$$f_\Omega^\circ(x) > f_\Omega^\circ(y)$$

We split the proof of Lemma 5.4 according $G = \mathrm{GL}_n$ or $G = \mathrm{Sp}_{2n}$.

Proof of Lemma 5.4, $G = \mathrm{GL}_n$. Assume $G = \mathrm{GL}_n$. By the hypothesis there exists $0 \leq h \leq t-2$ such that $x_1, x_2, \ldots, x_{t-h} \neq J_{n/t}$ and $x_{t-h+1} = \ldots = x_t = J_{n/t}$. Therefore the partition associated to x is

$$\left(\frac{n}{t}^h, \left(\frac{n}{t}-1\right)^{\sum_{i=1}^{t-h} a_{i,n/t-1}}, \ldots, 1^{\sum_{i=1}^{t-h} a_{i,1}}\right) \vdash n$$

where $a_{i,l}$ is the multiplicity of the block J_l in x_l. Hence, using *(i)* in Proposition 2.7 and the hypothesis $\dim(x^G \cap H^\circ) = \sum \dim x_i^{\mathrm{GL}_{n/t}}$, we get

$$\begin{aligned}
\dim x^G &= n^2 - 2h\frac{n}{t}(t-h) - \frac{n}{t}h^2 - \sum_{1 \leq i < n/t} i(a_{1,i} + \ldots + a_{t-h,i})^2 \\
&\quad - 2\sum_{1 \leq i < j < n/t} i(a_{1,i} + \ldots + a_{t-h,i})(a_{1,j} + \ldots + a_{t-h,j}) \\
\dim(x^G \cap H^\circ) &= \frac{n^2}{t} - \frac{n}{t}h - 2\sum_{1 \leq i < j < n/t} i(a_{1,i}a_{1,j} + \ldots + a_{t-h,i}a_{t-h,j}) \\
&\quad - 2\sum_{1 \leq i < j < n/t} i(a_{1,i} + \ldots + a_{t-h,i})(a_{1,j}^2 + \ldots + a_{t-h,j}^2)
\end{aligned}$$

In the same way we compute these dimensions for y. The partition associated to y is

$$\left(\frac{n}{t}^{h+1}, \left(\frac{n}{t}-1\right)^{\sum_{i=2}^{t-h} a_{i,n/t-1}}, \ldots, 1^{\sum_{i=2}^{t-h} a_{i,1}}\right) \vdash n$$

Let us observe that $\dim(y^G \cap H^\circ) = \dim y^{H^\circ}$, otherwise we could move some Jordan blocks around among the block $x_2, \ldots, x_t$ getting a larger dimension, and this would contradict the maximality of $\dim x^{H^\circ}$.

We have $f_\Omega^\circ(x) \geq f_\Omega^\circ(y)$ if, and only if, $\dim x^G - \dim y^G \geq \dim x^{H^\circ} - \dim y^{H^\circ}$. Eventually, this is equivalent to

$$\sum_{i \leq j} ia_{1,i}(a_{2,j} + \ldots + a_{t-h,j}) + \sum_{i<j} ia_{1,j}(a_{2,i} + \ldots + a_{t-h,i}) \geq \frac{n}{t}(t-h-1) = \sum_{i<n/t} i(a_{2,i} + \ldots + a_{t-h,i})$$

Let $i \in \{1, \ldots, \frac{n}{t}-1\}$, then for all $l \in \{2, \ldots, t-h\}$, the coefficient of $a_{l,i}$ in the left hand side is

$$\sum_{j \leq i} ja_{1,j} + \sum_{i<j} ia_{1,j}$$

and the coefficient of $a_{l,i}$ in the right hand side is i. We claim

$$\sum_{j \leq i} ja_{1,j} + \sum_{i<j} ia_{1,j} > i \tag{42}$$

which leads to the result. This is easy to prove observing that if there is $j > i$ for which $a_{1,j} \neq 0$ then $ia_{1,j} \geq i$ hence (42) is satisfied. If for all $j > i$ we have $a_{1,j} = 0$ then $\sum_{j\leq i} ja_{1,j} = \frac{n}{t} > i$. *q.e.d.*

Proof of Lemma 5.4, $G = \mathrm{Sp}_{2n}$. Clearly $\dim(y^G \cap H^\circ) = \dim y^{H^\circ}$. Therefore the result is equivalent to $\dim y^G - \dim x^G > \dim y^{H^\circ} - \dim x^{H^\circ}$. For some $h \geq 0$ we may assume $x_1, \ldots, x_{t-h} \neq J_{2\frac{n}{t}}$ and $x_{t-h+1} = \ldots = x_t = J_{2\frac{n}{t}}$. By straightforward computation the claim is equivalent to

$$\sum_{1\leq i\leq j<2\frac{n}{t}} i(a_{2,i}+\ldots+a_{t-h,i})a_{1,j} + \sum_{1\leq i<j<2\frac{n}{t}} ia_{1,i}(a_{2,j}+\ldots+a_{t-h,j}) - \sum_{i<2\frac{n}{t}} i(a_{2,i}+\ldots+a_{t-h,i}) \geq 0$$

Fix $i \in \{1, \ldots, 2\frac{n}{t} - 1\}$. The coefficient of $a_{l,i}$, for any $l \in \{2, \ldots, t-h\}$ is

$$i \sum_{i\leq j<2\frac{n}{t}} a_{1,j} + \sum_{j<i} ja_{1,j} - i$$

which is always strictly greater than 0. The result follows. *q.e.d.*

Using Lemma 5.4 we can prove Proposition 5.1.

Proof of Proposition 5.1. Since $p > n$, for all $x \in H$ unipotent of prime order we have $x^G \cap H = x^G \cap H^\circ$, hence $f^\circ_\Omega(x) = f_\Omega(x)$. In a finite number of steps for every unipotent element, in block form, $x = [x_1, \ldots, x_t]$, with $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ and two blocks, say x_1, x_t, different from $J_{\iota n/t}$ we reach the element $[J^{t-1}_{\iota n/t}, x_t]$. Indeed, assume that in $x = [x_1, \ldots, x_t]$ we have $x_1, \ldots, x_l, x_t \neq J_{\iota n/t}$ then we have $y_1 = [J_{\iota n/t}, x_2, \ldots, x_t]$, $y_2 = [J_{\iota n/t}, J_{\iota n/t}, x_3, \ldots, x_t]$ until $y_l = [J^{t-1}_{\iota n/t}, x_t]$ for which, $f_\Omega(y_l) = \frac{t}{\iota n}$. Suppose for x we have $f^\circ_\Omega(x) = \frac{t}{\iota n}$ and assume $x \neq [J^{t-1}_{\iota n/t}, x_t]$ we have that the number of $J_{\iota n/t}$ blocks in x will be less than or equal to $t-2$ hence by Lemma 5.4 we construct an element y such that $f^\circ_\Omega(x) > f^\circ_\Omega(y) \geq \frac{t}{\iota n}$, that is a contradiction. Hence the number of $J_{\iota n/t}$ blocks in x has to be at least $t-1$. Then thanks to Example 5.2 and 5.3, we complete the proof of the characterization. *q.e.d.*

Remark 5.5. The argument given made no assumption on the order of x. Moreover, for $p = 0$, all the unipotent elements have infinite order, hence there are no unipotent elements in $H \setminus H^\circ$. Thus in the case $p = 0$ the lower bound t/n is sharp for all unipotent elements in H.

5.2. **Case $p \leq n$.** In this section we shall study the case $p \leq n$. With this hypothesis, J_p is the biggest Jordan block that can appear in the Jordan form of a unipotent element of order p in H.

We shall prove, directly, that the lower bound for $f_\Omega(x)$ is $\frac{1}{p}$ for $x \in H$ unipotent of order p, characterizing, in some special cases, the elements which realize equality, here is the result for both groups.

Proposition 5.6. *Assume $G = \mathrm{GL}_n$ or Sp_{2n}. Let x be a unipotent element of H. Then*

$$f_\Omega(x) \geq \frac{1}{p} \tag{43}$$

Proof. First assume $G = \mathrm{GL}_n$ and $x \in H$.
Let $x \in H$, then $\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi_h)$ for some $0 \leq h \leq \lfloor \frac{t}{p} \rfloor$. Hence, for suitable $x_{hp+1}, \ldots, x_t$, $\dim(x^G \cap H) = \dim([I_{\frac{n}{t}}, \ldots, I_{\frac{n}{t}}, x_{hp+1}, \ldots, x_t]\pi_h)^{H^\circ}$. Thus

$$\dim(x^G \cap H) = h(p-1)\dim \mathrm{GL}_{\frac{n}{t}} + \sum_{i=hp+1}^{t} \dim x_i^{\mathrm{GL}_{\frac{n}{t}}}$$

Say $x_i = [J_p^{a_{i,p}}, \dots, J_1^{a_{i,1}}]$. Say $\alpha = hp+1$. Then, by Lemma 2.17 we may assume the partition associated to x is

$$\left(p^{\frac{n}{t}h+\sum_{k\geq\alpha} a_{k,p}}, p-1^{\sum_{k\geq\alpha} a_{k,p-1}}, \dots, 1^{\sum_{k\geq\alpha} a_{k,1}}\right) \vdash n$$

Applying the formula in Proposition 2.7 we get that (43) is equivalent to

$$2\sum_{1\leq i<j\leq p} i(a_{\alpha,i}+\dots+a_{t,i})(a_{\alpha,j}+\dots+a_{t,j}) + \sum_{i=1}^{p} i(a_{\alpha,i}+\dots+a_{t,i})^2 \tag{44}$$

$$-2\sum_{1\leq i<j\leq p} i(a_{\alpha,i}a_{\alpha,j}+\dots+a_{t,i}a_{t,j}) - \sum_{i=1}^{p} i(a_{\alpha,i}^2+\dots+a_{t,i}^2) \geq \frac{n^2(t-hp)(t-hp-1)}{t^2p}$$

If $t = hp$ then x is conjugate to π_h and both sides in (43) are 0. If $t = hp+1$ then $\alpha = t$, hence there is only one block and, again, both sides in (43) are 0. hence we may assume $t < hp$ Collecting the cross products and simplifying the squares the inequality (44) is equivalent to:

$$\sum_{\alpha\leq h<k\leq t}\left(\sum_{i<j} i(a_{h,i}a_{k,j}+a_{h,j}a_{k,j}) + \sum_{i\leq p} ia_{h,i}a_{k,i}\right) \geq \frac{n^2(t-hp-1)(t-hp)}{2t^2p}$$

The first sum has $(t-hp)(t-hp-1)/2$ couples (h,k). Therefore it is enough to show that each element of the summation is greater or equal than $\frac{1}{p}(n/t)^2$. In a lighter notation, say $a_{h,i} = a_i$ and $a_{k,i} = b_i$, we need

$$\sum_{1\leq i<j\leq p} i(a_ib_j + a_jb_i) + \sum_{i=1}^{p} ia_ib_i \geq \frac{1}{p}\left(\frac{n}{t}\right)^2$$

The right hand side of the inequality is $\frac{n}{tp}\sum_{i=1}^{p} ia_i$. The coefficient of a_i in the right hand side is $i\frac{n}{pt}$. The coefficient of a_i in the left hand side is

$$i\sum_{j\geq i} b_j + \sum_{j<i} jb_j$$

If for every $j \geq i$ we have $b_j = 0$, the left hand side is $\sum_j jb_j = \frac{n}{t} \geq i\frac{n}{pt}$ since $i \in \{1,\dots,p\}$.

Suppose there exists $j \geq i$ such that $b_j \neq 0$. Say $b_{j_1},\dots,b_{j_h}$ are the only non zero elements with $j_l \geq i$. Then we claim

$$ib_{j_1}+\dots+ib_{j_h} + \sum_{j<i} jb_j \geq \frac{i}{p}\frac{n}{t} = \frac{i}{p}\sum_{j=1}^{p} jb_j \tag{45}$$

Since $i \leq p$, we need to show the following, that implies the inequality (45),

$$ib_{j_1}+\dots+ib_{j_h} \geq \frac{i}{p}\sum_{j=i}^{p} jb_j = \frac{i}{p}\left(j_1b_{j_1}+\dots+j_hb_{j_h}\right)$$

Since for every $l = 1,\dots,h$ it holds $j_l \leq p$ we have $ib_{j_l} \geq ij_lb_{j_l}/p$. The result follows.

Following the same argument we get the result also for the symplectic group. For $G = \mathrm{Sp}_{2n}$ we have

$$\dim(x^G \cap H) = h(p-1)\frac{n}{t}\left(2\frac{n}{t}+1\right) + \sum_{i\geq k}\dim x_i^{\mathrm{Sp}_{2\frac{n}{t}}} \tag{46}$$

where $k = hp+1$, for some $0 \leq h \leq \lfloor \frac{t}{p} \rfloor$ and some unipotents $x_i \in \mathrm{Sp}_{2\frac{n}{t}}$, we denote the partition associated to each of them $\left(p^{a_{i,p}}, \ldots, 1^{a_{i,1}}\right)$ for $i = k, \ldots, t$. Therefore, since $\pi_h = [J_p^{2\frac{n}{t}h}, J_1]$ we have

$$x = [J_p^{2\frac{n}{t}h+\sum_{i\geq k} a_{i,p}}, J_{p-1}^{\sum_{i\geq k} a_{i,p-1}}, \ldots, J_1^{\sum_{i\geq k} a_{i,1}}]$$

And we have $f_\Omega(x) \geq \frac{1}{p}$ if, and only if,

$$2\frac{n^2}{t}h\Big(2 - \frac{1}{t} - \frac{hp}{t}\Big) - \sum_{1\leq i<j\leq p} i(a_{k,i}a_{k,j} + \ldots + a_{t,i}a_{t,j}) - \frac{1}{2}\sum_{i\leq p} i(a_{k,i}^2 + \ldots + a_{t,i}^2) \geq$$

$$2\frac{n^2}{p}\Big(1 - \frac{1}{t}\Big) - \sum_{1\leq i<j\leq p} i(a_{k,i} + \ldots + a_{t,i})(a_{k,j} + \ldots + a_{t,j}) - \frac{1}{2}\sum_{i\leq p} i(a_{k,i} + \ldots + a_{t,i})^2$$

Simplifying, again, the last inequality, we get

$$\sum_{k\leq\alpha<\beta\leq t}\Big(\sum_{1\leq i<j\leq p} i(a_{\alpha,i}a_{\beta,j} + a_{\beta,i}a_{\alpha,j}) + \sum_{i\leq p} ia_{\alpha,i}a_{\alpha,j}\Big) \geq \frac{2n^2(t-hp)(t-hp-1)}{t^2p} \tag{47}$$

A straightforward calculation in the right hand side of (47) shows it is equal to $\frac{2}{p}\left(\frac{n}{t}\right)^2(t-hp)(t-hp-1)$. Since $|I = \{(\alpha,\beta) : k \leq \alpha < \beta \leq t\}| = \frac{(t-hp)(t-hp-1)}{2}$, in order to show inequality (47) it is enough to that that for all $(\alpha,\beta) \in I$, saying $a_{\alpha,i} = a_i, a_{\beta,i} = b_i$,

$$\sum_{1\leq i<j\leq p} i(a_ib_j + a_jb_i) + \sum_{i\leq p} ia_ib_i \geq \frac{1}{p}\Big(2\frac{n}{t}\Big)^2 = \frac{1}{p}\Big(2\frac{n}{t}\Big)\sum_{i\leq p} ia_i \tag{48}$$

And inequality (48) is satisfied for the same argument given above for GL_n.

q.e.d.

Now, we focus on the characterization for unipotent elements in H of order p, which realize the lower bound $\frac{1}{p}$. We shall prove the following. Where, as above, we define $\iota = 1$ if $G = \mathrm{GL}_n$ and $\iota = 2$ for $G = \mathrm{Sp}_{2n}$.

Proposition 5.7. *Let $x \in H$ be a unipotent element of prime order p. Then $f_\Omega(x) = \frac{1}{p}$ if, and only if, one of the following conditions is satisfied,*

- *$p|n$ and $x = [J_p^{\iota\frac{n}{p}}]$*
- *$p|\frac{n}{t}$ and $x = [J_p^{\iota\frac{n}{tp}(t-1)}, z]$ where $z \in \mathrm{GL}_{\frac{n}{t}}$ or $\mathrm{Sp}_{2\frac{n}{t}}$ is a unipotent element of prime order whose associated partition λ can not be written as a direct sum $\lambda_1 \oplus \lambda_2$ where $\lambda_i \vdash l_ip$, where $l_i \in \{1, \ldots, \frac{n}{tp} - 1\}$, and both of them contain parts other than p and, for $G = \mathrm{Sp}_{2n}$, l_1 is even .*

First we prove the following two lemmas, which is a direct consequence of the proof of Proposition 5.6.

Proposition 5.8. *Let $G = \mathrm{GL}_n$. Assume $\frac{n}{t} = mp$ and let $x \in H$ of order p. Then $f_\Omega(x) = 1/p$ if, and only if, $x = [J_p^m, \ldots, J_p^m, z]$, where $z \in \mathrm{GL}_{n/t}(k)$ is any unipotent element of order p such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. In the case p divides n and $x \in H$ is such that $f_\Omega(x) = \frac{1}{p}$. Then $x = [J_p^{\frac{n}{p}}]$.*

Proof. We follow the proof of Proposition 5.6. Assume that for x we have equality in (43). Then

$$i\sum_{j\geq i} b_j + \sum_{j<i} jb_j = i\Big(m + \frac{a}{p}\Big) = \frac{i}{p}\frac{n}{t}$$

Let $h = \min\{i : b_h \neq 0\}$, i.e. for all $k < h$ we have $b_k = 0$. Then

$$h \sum_{j \geq h} b_j + \sum_{j<h} j b_j = h \sum_{j \geq h} b_j = h\Big(m + \frac{a}{p}\Big)$$

Hence $\sum_{j \geq h} b_j = n/(pt)$.

If p divides n but not $\frac{n}{t}$ then p divides t and $[J_p^a] \notin \mathrm{GL}_{\frac{n}{t}}$ for any integer a. Hence $h = \frac{t}{p}$ so $b_p = 0$ and $x = \pi_{\frac{t}{p}} = [J_p^{\frac{n}{p}}]$.

If p divides n/t, i.e. $a = 0$, we have that equality in (43) holds if, and only if,

$$\sum_{j \geq h} b_j = \frac{n}{pt} = m$$

This means that the total numer of Jordan blocks in x is m, and the only possibility is $b_p = m$, $b_j = 0$ for $j \neq p$. This holds for all the couple (h, k) with $h < k$. Therefore at most one block can be different from $[J_p^m]$. Thus $x = [J_p^m, \ldots, J_p^m, x_t]$, with x_t a unipotent element of prime order in $\mathrm{GL}_{n/t}(k)$.

In order to complete the proof, let $x = [J_p^m, \ldots, J_p^m, x_t]$, where $x_t \in \mathrm{GL}_{n/t}(k)$ has Jordan form $[J_{p-1}^{a_{p-1}}, \ldots, J_1^{a_1}]$. Assume, moreover $\dim(x^G \cap H^\circ) = (t-1)\dim[J_p^m]^{\mathrm{GL}_{n/t}} + \dim[x_t]^{\mathrm{GL}_{n/t}}$. Then

$$\dim x^G = n^2 - 2m(t-1)\frac{n}{t} - pm^2(t-1)^2 - 2\sum_{1 \leq i<j<p} i a_i a_j - \sum_{i<p} i a_i^2$$

$$\dim(x^G \cap H^\circ) = \frac{n^2}{t} - pm^2(t-1) - 2\sum_{1 \leq i<j<p} i a_i a_j - \sum_{i<p} i a_i^2$$

We easily see that $f_\Omega(x) = \frac{1}{p}$.

The same argument applies in the hypothesis p divides n but not $\frac{n}{t}$. *q.e.d.*

The argument given in Proposition 5.8 applies also for the symplectic group. Here is the analogous statement.

Proposition 5.9. *Let $G = \mathrm{Sp}_{2n}$. Assume p divides n and not $\frac{n}{t}$ or, p divides $\frac{n}{t}$. Let $x \in H$ such that $f_\Omega(x) = \frac{1}{p}$. Then $x = [J_p^{2\frac{n}{p}}]$ or, $x = [J_p^{2\frac{n}{tp}}, \ldots, J_p^{2\frac{n}{pt}}, z]$, where $z \in \mathrm{Sp}_{2\frac{n}{t}}$ is an element of order p such that $\dim(x^G \cap H) = \dim([I_{2\frac{n}{t}}, \ldots, I_{2\frac{n}{t}}, J_p^{2\frac{n}{pt}}, \ldots, J_p^{2\frac{n}{pt}}, z]\pi_h)^{H^\circ}$.*

In view of Proposition 5.8, 5.9, in order to characterize the elements $x = [J_p^m, \ldots, J_p^m, z]$ for which $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, we need to give a precise characterization of partitions of n with the properties described in these two result. We prove technical lemma, namely Lemma 5.10 for GL_n and Lemma 5.12 for the symplectic group.

Lemma 5.10. *Let $G = \mathrm{GL}_n$ and $\frac{n}{t} = pm$. Let $x = [J_p^m, \ldots, J_p^m, x_t] \in H^\circ$ be a unipotent element of prime order. Say $\lambda_{x_t} \vdash 2\frac{n}{t}$ the partition associated to x_t, let us denote n_i the multiplicity of the i part in it. Assume*

(i) *$n_p < m - 1$;*
(ii) *we can not have $\lambda = \lambda_1 \oplus \lambda_2$ where $\lambda_i = (p^{n_p^i}, \ldots) \vdash l_i p$ for some $l_i \in \{1, \ldots, m-1\}$ and $n_j^i > 0$ for some $j < p$ for both $i = 1, 2$.*

Then

$$\dim(x^G \cap H^\circ) > \dim x^{H^\circ}$$

Proof. Let $l \geq 1$ be the smallest integer such that λ_{x_t} contains a subpartition of lp with no p parts, so $l \in \{1, \ldots, \lfloor m/2 \rfloor\}$. We may write $x_t = [z, z']$ where $z \in \mathrm{GL}_{lp}(k)$ and $z' \in \mathrm{GL}_{n/t-lp}(k)$ and we write the associated partition as

$$\begin{aligned} z &\leftrightarrow \left((p-1)^{a_{p-1}}, \ldots, 1^{a_1}\right) \vdash lp \\ z' &\leftrightarrow \left(p^{b_p}, (p-1)^{b_{p-1}}, \ldots, 1^{b_1}\right) \vdash \frac{n}{t} - lp \\ x_t &\leftrightarrow \left(p^{b_p}, (p-1)^{a_{p-1}+b_{p-1}}, \ldots, 1^{a_1+b_1}\right) \vdash \frac{n}{t} \end{aligned}$$

Note that $b_i > 0$ for some $i < p$.

We define y to be the following element $y = [J_p^m, \ldots, J_p^m, (J_p^{m-l}z), (J_p^l z')]$. Note that y and x have the same Jordan form so they are G-conjugate. We claim $\dim y^{H^\circ} > \dim x^{H^\circ}$, which proves the lemma. The claim holds if, and only if,

$$\dim[J_p^m]^{\mathrm{GL}_{n/t}} + \dim[z, z']^{\mathrm{GL}_{n/t}} < \dim[J_p^{m-l}, z]^{\mathrm{GL}_{n/t}} + \dim[J_p^l, z']^{\mathrm{GL}_{n/t}} \tag{49}$$

And, by the formula in Proposition 2.7 we have:

$$\begin{aligned} \dim[J_p^m]^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - \left(\frac{n}{t}\right)m \\ \dim[z, z']^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - 2lpb_p - 2\sum_{1\leq i<j\leq p} ib_ib_j - 2\sum_{1\leq i<j<p} ia_ia_j - \sum_{i<p} ia_i^2 - \sum_{i\leq p} ib_i^2 \\ &\quad - 2\Big(\sum_{1\leq i\leq j\leq p-1} ia_ib_j\Big) - 2\Big(\sum_{1\leq i<j\leq p-1} ia_jb_i\Big) \end{aligned}$$

$$\begin{aligned} \dim[J_p^{m-l}, z]^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - \left(\frac{n}{t}\right)m + l^2p - 2\sum_{1\leq i<j<p} ia_ia_j - \sum_{i<p} ia_i^2 \\ \dim[J_p^l, z']^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - 2lpb_p - l^2p - 2\sum_{1\leq i<j\leq p} ib_ib_j - \sum_{i\leq p} ib_i^2 - 2l\sum_{i<p} ib_i \end{aligned}$$

It follows that the inequality (49) is equivalent to the following

$$l\sum_{i<p} ib_i < \sum_{1\leq i\leq j\leq p-1} ia_ib_j + \sum_{1\leq i\leq j\leq p-1} ia_jb_i$$

Consider the two expressions in this inequality as polynomials in $b_1, \ldots, b_{p-1}$. If we show that the coefficient of b_i in the left hand side is strictly less than the coefficient of b_i on the right hand side, for all i, we are done. Hence for all $i \in \{1, \ldots, p-1\}$ it suffices to show that

$$li < \sum_{1\leq j\leq i} ja_j + i\sum_{i<j} a_j \tag{50}$$

If for all $j > i$ we have $a_j = 0$, we get $\sum_{j\leq i} ja_j = lp > li$. Thus we may assume there exists $j > i$ such that $a_j \neq 0$, let us call $a_{j_1}, \ldots, a_{j_h}$ all the non zero multiplicities with $j_k > i$, since $\sum_{j\leq i} ja_j = lp - j_1a_{j_1} - \ldots - j_ha_{j_h}$ we see that (50) is equivalent to

$$(j_1 - i)a_{j_1} + \ldots + (j_h - i)a_{j_h} < l(p-i) = \sum_{j<p} ja_j - \frac{i}{p}\sum_{j<p} ja_j$$

In order to establish this inequality we use the same strategy, showing that the coefficient of a_j in the left hand side is strictly less than the ones in the right hand side. The coefficient of a_j in the right hand side is $(1-i/p)j$, always positive since $i \in \{1,\dots,p-1\}$, on the left hand side the coefficient of each a_j is either 0 or $j_k - i > 0$ and, clearly, we have

$$j_k - i < \Big(1 - \frac{i}{p}\Big) j_k$$

for all $k \in \{1,\dots,h\}$ since $j_k < p$. *q.e.d.*

Remark 5.11. Let $x = [J_p^m, \dots, J_p^m, x_t]$ for which $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ and let λ_{x_t} be the partition of n/t associated to x_t. Then either λ_{x_t} contains a subpartition of lp, with $l \in \{1,\dots,\lfloor m/2 \rfloor\}$ the smallest integer such that λ_{x_t} contains a subpartition of hp with $h \in \{1,\dots,m-1\}$, with no p parts or it does not. In the first case by the contrapositive of Lemma 5.10 we have $\lambda_{x_t} = (p^{m-l}, \mu)$ where μ is any partition of lp with no p parts and with no proper subpartition of any multiple of p. In the latter case $\lambda_{x_t} = (p^{(t-1)m}, \mu)$ where μ is a partition of n/t with parts at most p that satisfies the condition above.

With a similar argument of Lemma 5.10 we prove the analogous result for the symplectic group.

Lemma 5.12. *Let $G = \mathrm{Sp}_{2n}$ and $\frac{n}{t} = pm$. Let $x = [J_p^{2m}, \dots, J_p^{2m}, x_t] \in H^\circ$ be a unipotent element of prime order. Say $\lambda \vdash 2\frac{n}{t}$ the partition associated to x_t, let us denote n_i the multiplicity of the i part in it. Assume*

(i) $n_p < m - 1$;
(ii) *we can not have $\lambda = \lambda_1 \oplus \lambda_2$ where $\lambda_i = (p^{n_p^i}, \dots) \vdash l_i p$ for some $l_i \in \{1,\dots,m-1\}$, l_1 even, and $n_j^i > 0$ for some $j < p$ for both $i = 1,2$.*

Then

$$\dim(x^G \cap H^\circ) > \dim x^{H^\circ}$$

Proof. In the same notation of Lemma 5.10, we define $y = [J_p^{2m}, \dots, (J_p^{2m-l} z), (J_p^l z')]$. Clearly $y \in x^G$ and we shall prove $\dim y^{H^\circ} > \dim x^{H^\circ}$. Which is equivalent to

$$\dim[J_p^{2m}]^{\mathrm{Sp}_2\frac{n}{t}} + \dim[z,z']^{\mathrm{Sp}_2\frac{n}{t}} < \dim[J_p^{m-l}, z]^{\mathrm{Sp}_2\frac{n}{t}}]\dim[J_p^l, z']^{\mathrm{Sp}_2\frac{n}{t}} \tag{51}$$

We say

$$\begin{aligned} z &\leftrightarrow \big((p-1)^{a_{p-1}}, \dots, 1^{a_1}\big) \vdash lp \\ z' &\leftrightarrow \big(p^{b_p}, (p-1)^{b_{p-1}}, \dots, 1^{b_1}\big) \vdash 2\frac{n}{t} - lp \\ x_t &\leftrightarrow \big(p^{b_p}, (p-1)^{a_{p-1}+b_{p-1}}, \dots, 1^{a_1+b_1}\big) \vdash 2\frac{n}{t} \end{aligned}$$

And we have

$$\begin{aligned} \dim[J_p^{2m}]^{\mathrm{Sp}_2\frac{n}{t}} &= \frac{n}{t}\Big(2\frac{n}{t}+1\Big) - \frac{n}{pt}\Big(2\frac{n}{t}+1\Big) \\ \dim[z,z']^{\mathrm{Sp}_2\frac{n}{t}} &= \frac{n}{t}\Big(2\frac{n}{t}+1\Big) - b_p \sum_{i<p} i(a_i+b_i) - \frac{pb_p}{2} - \frac{b_p}{2} \\ &\quad - \sum_{i<j<p} i(a_i+b_i)(a_j+b_j) - \frac{1}{2}\sum_{i<p}(a_i+b_i)^2 - \frac{1}{2}\sum_{i \text{ odd}<p}(a_i+b_i) \end{aligned}$$

$$\begin{aligned}
\dim[J_p^{2m-l},z]^{\mathrm{Sp}_{2\frac{n}{t}}} &= \frac{n}{t}\Big(2\frac{n}{t}+1\Big)-(2m-l)\sum_{i<p} ia_i - \frac{p}{2}(2m-l)^2 - \frac{2m-l}{2}\\
&\quad -\sum_{i<j<p} ia_ia_j - \frac{1}{2}\sum_{i<p} ia_i^2 - \frac{1}{2}\sum_{i\ \mathrm{odd}<p} a_i\\
\dim[J_p^{l},z']^{\mathrm{Sp}_{2\frac{n}{t}}} &= \frac{n}{t}\Big(2\frac{n}{t}+1\Big)-(l+b_p)\sum_{i<p} ib_i - \frac{p}{2}(l+b_p)^2 - \frac{l+b_p}{2}\\
&\quad -\sum_{i<j<p} ib_ib_j - \frac{1}{2}\sum_{i<p} ib_i^2 - \frac{1}{2}\sum_{i\ \mathrm{odd}<p} b_i
\end{aligned}$$

It follows that the inequality (51) is equivalent to the following

$$l\Big(2\frac{n}{t}-lp-pb_p\Big) = l\sum_{i<p} ib_i < \sum_{i\le j<p} ia_ib_j + \sum_{i<j} ia_jb_i$$

To this point on the proof proceeds as Lemma 5.10. *q.e.d.*

Actually Lemma 5.12 give a complete classification of the partition of $2n = 2pmt$ of the form $\lambda = (p^{2m(t-1)}) \oplus \lambda'$ such that the unipotent element x_λ corresponding to λ ha the property $\dim(x^G \cap H^\circ) = \dim x^{H^\circ} = (t-1)\dim[J_p^{2m}]^{\mathrm{Sp}} + \dim x_t$. Indeed, let $\lambda \vdash 2\frac{n}{t}$ be any partition which does not satisfy the assumption *(ii)* of Lemma 5.12, and let x_λ the unipotent element associated to it, then we can not have any other disposition of the blocks other than a permutations of the x_i's. Let us observe that if $\lambda = \lambda_1 \oplus \lambda_2$ according to *(ii)* but l_1, l_2 are both odd than the element y defined in the proof of Lemma 5.12 does not lie in H°.

Therefore, for $x = [J_p^{2m(t-1)}, x_t]$, we have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ if, and only if, the partition λ associated to x_t can not be written as $\lambda = \lambda_1 \oplus \lambda_2$ where $\lambda_i \vdash l_ip$, where $l_i \in \{1, \dots, m-1\}$, both of them contain parts different than p and l_1 is even.

This leads to a complete proof of Proposition 5.7.

Proof of Proposition 5.7. Thanks to Proposition 5.6 we know that for any unipotent element $x \in H$ of prime order, $f_\Omega(x) \ge \frac{1}{p}$. Using Proposition 5.8 and 5.9 we have the first step in the characterization of the elements that realize equality with the lower bound. Furthermore, for $n/t = pm$, equality holds if, and only if $x = [J_p^m, \dots, J_p^m, x_t]$ where x_t is any unipotent element of prime order in G such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Thanks to Lemma 5.10, 5.12 we get the result. *q.e.d.*

We conclude this section having a brief discussion in order to motivate the reason why the bound $\frac{1}{p}$ is closed to the best possible.

Let $G = \mathrm{GL}_n$ and suppose p does not divide $\frac{n}{t}$, i.e. $\frac{n}{t} = mp + a$ for some $a \in \{1, \dots, p-1\}$, in this case we conjecture that a better bound can be given. Our conjecture is motivated by the following.

Example 5.13. Let $x = [J_p^m J_a, \dots, J_p^m J_a]$. Then we have

$$\begin{aligned}
\dim x^G &= n^2 - 2amt^2 - at^2 - pm^2t^2\\
\dim x^{H^\circ} &= \frac{1}{t}\dim x^G
\end{aligned}$$

Therefore by Proposition 2.10 we get $\dim(x^G \cap H) = \dim x^{H^\circ}$. Hence

$$f_\Omega(x) = \Big(\frac{t}{n}\Big)^2(2ma + a + pm^2)$$

Conjecture 5.14. *Let $G = \mathrm{GL}_n$. Assume $p < n/t$ and write $n/t = mp + a$. Then for a unipotent element x in H°*

$$f_\Omega(x) \geq \Big(\frac{t}{n}\Big)^2 (2ma + a + pm^2)$$

In Section 8 we shall give a precise formula for $f_\Omega(x)$ when x is unipotent of prime order $p = 2$, for $G = \mathrm{GL}_n$. It is straightforward to deduce from the formula of f_Ω in Theorem 4 that Conjecture 5.14 is true in the case $p = 2$. For larger p the analisys is more difficult since an approach not finalized in getting a formula forces us to study each case depending on the residue class of n/t modulo p.

However the bound $1/p$, is closed to the one of the conjecture.

Remark 5.15. The bound in Conjecture 5.14 can be written as

$$\frac{1}{p} + f(p, m, a) = \frac{1}{p} + \frac{pa - a^2}{p(pm + a)^2}$$

Let us observe that $f(p, m, a) > 0$ since $a < p$. Moreover $f(p, m, a)$ goes to 0 when m or a tends to infinity, this is clear if m become large, since $a \in \{0, \dots, p-1\}$, for large a we have $p \approx a$. Moreover for $p > 2$ (for the first inequality):

$$\Big(1 - \frac{1}{p}\Big)\frac{1}{(pm + 1)^2} \leq f(p, m, a) = a\Big(1 - \frac{a}{p}\Big)\frac{1}{(pm + a)^2} \leq \frac{1}{a} - \frac{1}{p}$$

For the symplectic group we do not give a conjecture of the sharpest lower bound. Instead we prove that, in the assumption p does not divide n, there exist an element $x \in H$ such that $f_\Omega(x) \leq \frac{1}{p} + \epsilon$ for a small $\epsilon > 0$. Here is the formal result.

Proposition 5.16. *Let $G = \mathrm{Sp}_{2n}$. Assume $p < n$ does not divide n. Then there exists $x \in H$ unipotent of prime order such that*

$$f_\Omega(x) \leq \begin{cases} \frac{1}{p} + \frac{p}{2n} & 2\frac{n}{t} < p < t \\ \frac{2}{p} & \text{otherwise} \end{cases}$$

Proof. Assume, first, $p < \frac{n}{t}$. Then $2\frac{n}{t} = a(2p) + b$, where $0 \leq b < 2p$ is even. In the case $b < p$ we have $\bar{x} = [J_p^{2a}, J_b] \in \mathrm{Sp}_{2\frac{n}{t}}$. Therefore we consider $x = [\bar{x}, \dots, \bar{x}] \in \mathrm{Sp}_{2n}$. And we have

$$\dim x^G = n(2n + 1) - 2abt^2 - \frac{bt^2}{2} - 2pa^2t^2 - at$$
$$\dim(x^G \cap H^\circ) = n\Big(2\frac{n}{t} + 1\Big) - 2abt - \frac{bt}{2} - 2pa^2t - at$$

It is easy to check that $\dim(x^G \cap H^\circ) = \big(\frac{1}{t} + \frac{n-at}{\dim x^G}\big(1 - \frac{1}{t}\big)\big)\dim x^G$. Therefore, by Proposition 2.10 we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Thus

$$\begin{aligned} f_\Omega(x) &= \frac{t^2}{4n^2}(4a^2p + 4ab + b) \leq \frac{1}{p} + \frac{p-1}{2\frac{n}{t}}\Big(\frac{1}{p} + \frac{1}{2\frac{n}{t}}\Big) \\ &< \frac{1}{p} + \frac{p}{2\frac{n}{t}}\Big(\frac{1}{p} + \frac{1}{2\frac{n}{t}}\Big) < \frac{1}{p} + \frac{1}{2}\Big(\frac{1}{p} + \frac{1}{2p}\Big) = \frac{1}{p} + \frac{3}{4p} \end{aligned} \tag{52}$$

where we used $\frac{t}{n} < \frac{1}{p}$.

If $b > p$, we can consider $\bar{x} = [J_p^{2a}, J_{p-1}, J_{b-p+1}] \in \mathrm{Sp}_{2\frac{n}{t}}$. Thus $x = [\bar{x}, \dots, \bar{x}] \in \mathrm{Sp}_{2n}$. And

$$\dim x^G = n(2n + 1) - t^2(b - p + 1)(2a + 1) - (p - 1)2at^2 - \frac{b - p + 1}{2}t^2 - \frac{p - 1}{2}t^2 - 2a^2t^2p - at$$

$$\dim(x^G \cap H^\circ) = \Big(\frac{1}{t} + \frac{n - at}{\dim x^G}\Big(1 - \frac{1}{t}\Big)\Big)$$

Hence, as above, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Thus

$$\begin{aligned} f_\Omega(x) &= \frac{a}{2\frac{n}{t}} + \frac{3b(a+1) - 2(p-1)}{4\left(\frac{n}{t}\right)^2} \leq \frac{1}{2p} + \frac{3(p-1)}{2p\frac{n}{t}} = \frac{1}{p}\Big(\frac{1}{2} + \frac{3(p-1)}{2\frac{n}{t}}\Big) \\ &< \frac{1}{p}\Big(\frac{1}{2} + \frac{3p}{2\frac{n}{t}}\Big) < \frac{1}{p}\Big(\frac{1}{2} + \frac{3}{2}\Big) = \frac{2}{p} \end{aligned} \tag{53}$$

where we used $\frac{t}{n} < \frac{1}{p}$.

In the case $\frac{n}{t} < p < n$ let $h = \lfloor \frac{t}{p} \rfloor$. Let us observe that $\frac{t-p+1}{p} \leq h \leq \frac{t}{p}$. We consider $x = [J_p^{2\frac{n}{t}h}, J_{2\frac{n}{t}}^{t-hp}] \in H$. Indeed $p < t$ and, therefore, $h \geq 1$, since this is the condition to have a permutation action. Moreover, since $p > \frac{n}{t}$, in $\mathrm{Sp}_{2\frac{n}{t}}$ there are no unipotent elements with J_p blocks, hence $x^G \cap H = x^G \cap H^\circ \pi_h$. And, we have

$$\begin{aligned} \dim x^G &= n(2n+1) - \Big(2\frac{n}{t}\Big)^2 h(t-hp) - \frac{n}{t}(t-hp)^2 - 2p\Big(\frac{n}{t}h\Big)^2 - \frac{n}{t}h \\ \dim(x^G \cap H) &= 2\frac{n^2}{t} - 2\Big(\frac{n}{t}\Big)^2 h + \frac{n}{t}hp - \frac{n}{t}h \end{aligned}$$

Therefore

$$f_\Omega(x) = \frac{-n + nt - 2nhp + \frac{n}{t}hp + 4\frac{n^2}{t}h - 2p\left(\frac{n}{t}h\right)^2 + \frac{n}{t}(hp)^2 - 2\left(\frac{n}{t}\right)^2 h}{2n^2\left(1 - \frac{1}{t}\right)} \tag{54}$$

Let us call $g(h)$ the h part of the numerator of (54), so that $g(0) = 0$. Then we have $g'(h) = 2h\frac{p}{t}\left(p - 2\frac{n}{t}\right) - \left(p - 2\frac{n}{t}\right)\left(2 - \frac{1}{t}\right)$. Hence, for $p > 2\frac{n}{t}$, we have $g(h) \leq g\left(\frac{t-p+1}{p}\right)$. While, for $p < 2\frac{n}{t}$, $g(h) \leq g\left(\frac{2t-1}{2p}\right)$. In the latter case we have $g\left(\frac{2t-1}{2p}\right) = \frac{n(2t-1)^2(2n-pt)}{4pt^2}$. Therefore

$$f_\Omega(x) \leq \frac{nt\left(1 - \frac{1}{t}\right) + \frac{n(2t-1)^2(2n-pt)}{4pt^2}}{2n^2\left(1 - \frac{1}{t}\right)} = \frac{2}{p} \tag{55}$$

In the former case, i.e. $p > 2\frac{n}{t}$, it follows

$$\begin{aligned} f_\Omega(x) &\leq \frac{nt\left(1 - \frac{1}{t}\right) + \frac{n(p-t-1)(p+t-2)\left(p - 2\frac{n}{t}\right)}{pt}}{2n^2\left(1 - \frac{1}{t}\right)} \\ &= \frac{1}{p} + (p^2 - 3p + 2)\Big(\frac{1}{2n(t-1)} - \frac{1}{p(t-1)t}\Big) \\ &= \frac{1}{p} + \frac{(p-2)(p-1)\left(p - 2\frac{n}{t}\right)}{2np(t-1)} \end{aligned} \tag{56}$$

And we have

$$\frac{(p-2)(p-1)\left(p - 2\frac{n}{t}\right)}{2np(t-1)} < \frac{p\left(p - 2\frac{n}{t}\right)}{2n(t-1)} < \frac{p(p-1)}{2n(t-1)} < \frac{p}{2n}$$

where we used $p \leq t$ and $t \leq n$. This completes the proof. *q.e.d.*

6. Local bounds

The previous analysis on upper and lower bound on $f_\Omega(x)$ for prime order elements x suggests a strong connection between the value of $f_\Omega(x)$ and $\nu(x)$. Therefore we are interested now in giving upper and lower bound on the ratio $f_\Omega(x)$ when $\nu(x) = s$ for any s.

Let x be an element in H of prime order r and assume $\nu(x) = s$, as in Definition 2.8. First, let us define some subsets of G which we shall refer to in Table 10, the sets $\mathcal{P}, \mathcal{S}_r$ and $\mathcal{U}_p$ are defined in Table 1.

$\mathcal{V}_s = \{x \in G : \nu(x) = s\}$
$\mathcal{V}_{s,\mathcal{P}} = V_s \cap \mathcal{P}$
$\mathcal{V}_{s,r} = V_s \cap \mathcal{S}_r$
$\mathcal{V}_{s,p} = V_s \cap \mathcal{U}_p$

Table 10. Subsets of G with $\nu(x)$

Let Λ be one of the subsets in Table 10. Our aim is to find upper and lower bounds on the ratio $f_\Omega(x)$, for all $x \in \Lambda \cap H$.

We shall study elements of prime order $r > 2$. We do not study involution here since fixed s the set $\mathcal{V}_s$ is, in fact, union of the conjugacy classes of the elements $\pm[I_s, -I_{n-s}]$ or $[J_2^s, J_1^{n-s}]$ if $p = 2$, for $G = \mathrm{GL}_n$.

6.1. Upper bounds.

We first study upper bound on $f_\Omega(x)$. We shall establish Theorem 6. As already done in Section 3 we give the following two results where the main ingredients are Proposition 2.10 and Proposition 2.9.

Proposition 6.1. *Let $G = \mathrm{GL}_n$. Let x be an element of prime order in $H \cap \mathcal{V}_s$. Then*

$$f_\Omega(x) \le 1 - \frac{s\gamma}{n}$$

where

$$\gamma = \begin{cases} 1 & s > \frac{n}{2} \\ 2\left(1 - \frac{s}{n}\right) & s \le \frac{n}{2} \end{cases}$$

Proof. We have $f_\Omega(x) \le \alpha$ if, and only if

$$\frac{\dim(x^G \cap H)}{\dim x^G} \le 1 + \frac{(\alpha - 1)\dim\Omega}{\dim x^G}$$

and thanks to Proposition 2.10, it is enough to prove

$$\frac{1}{t} \le 1 + \frac{(\alpha - 1)\dim\Omega}{\dim x^G}$$

for $\alpha = 1 - \frac{s\gamma}{n}$, which is equivalent to $\dim x^G \ge (1-\alpha)n^2$. Proposition 2.9 gives us lower bounds on $\dim x^G$ in terms of s, that are

$$\dim x^G \ge \begin{cases} ns & s > \frac{n}{2} \\ 2s(n-s) & s \le \frac{n}{2} \end{cases}$$

hence for

$$\alpha \ge \begin{cases} 1 - \frac{s}{n} & s > \frac{n}{2} \\ 1 - \frac{2s(n-s)}{n^2} & s \le \frac{n}{2} \end{cases}$$

we have $f_\Omega(x) \le \alpha$. The result follows. *q.e.d.*

In general these bounds are not sharp. Only in some cases they are realized, namely, only when there exists $x \in \mathcal{V}_s \cap H$ such that $\dim x^G = \max\{ns, 2s(n-s)\}$ and $\dim(x^G \cap H) = \frac{1}{t}\dim x^G$.

Remark 6.2. We do not have a complete characterization of the elements that reach the upper bound, indeed, given n, t and s it does not always exist an element x of prime order r in H such that $\nu(x) = s$ and $f_\Omega(x)$ equal to the upper bound given in Proposition 6.1. Therefore the aim is to show that we can find elements $x \in \mathcal{V}_s$ either unipotent or semisimple such that the difference between $f_\Omega(x)$ and the upper bound is bounded. This will be done in Proposition 6.6, below.

For the symplectic group we shall prove the following.

Proposition 6.3. *Let $G = \mathrm{Sp}_{2n}$. Let $x \in H \cap \mathcal{V}_s$ be an element of prime order. Then*

$$f_\Omega(x) \le \begin{cases} 1 - \frac{s}{2n} + \frac{s}{2n^2} & s \ge n \\ 1 - \frac{s}{n} + \frac{1}{2n} + \frac{s^2}{2n^2} & s < n \end{cases}$$

In the symplectic case the upper bound on the ratio $\frac{\dim(x^G \cap H)}{\dim x^G}$ is $\frac{1}{t} + \zeta$ where ζ depends on x and it is, in general, non-zero. Therefore we need to divide the analysis depending on the order on x.

First let us study upper bound on f_Ω for elements in $H \cap \mathcal{V}_{s,p}$. We have the following.

Proposition 6.4. *Let $x \in H \cap \mathcal{V}_{s,p}$. Then*

$$f_\Omega(x) \le \begin{cases} 1 - \frac{s}{n} + \frac{1}{2n} + \frac{s^2}{2n^2} & s \le n \\ 1 - \frac{s}{2n} + \frac{1}{2n} & s \ge n \end{cases}$$

Proof. Using Proposition 2.9 we have $\dim x^G \ge \beta = \max\{s(2n-s), ns\}$. Let $x = [J_p^{a_p}, \dots, J_1^{a_1}] \in H \cap \mathcal{V}_{s,p}$, we have $f_\Omega(x) \le \alpha$ if, and only if,

$$\frac{\dim(x^G \cap H)}{\dim x^G} \le 1 - \frac{\dim \Omega(1-\alpha)}{\dim x^G}$$

By Proposition 2.10 we have

$$\begin{aligned} \frac{\dim(x^G \cap H)}{\dim x^G} &\le \frac{1}{t} + \frac{n - \frac{1}{2}\sum_{i\text{odd}} a_i}{\dim x^G}\Big(1 - \frac{1}{t}\Big) \\ &\le \frac{1}{t} + \frac{n}{\beta}\Big(1 - \frac{1}{t}\Big) \end{aligned}$$

Moreover

$$1 - \frac{\dim \Omega(1-\alpha)}{\dim x^G} \ge 1 - \frac{2n^2(1-\alpha)}{\beta}\Big(1 - \frac{1}{t}\Big)$$

Therefore we need to show

$$\frac{1}{t} + \frac{n}{\beta}\Big(1 - \frac{1}{t}\Big) \le 1 - \frac{2n^2(1-\alpha)}{\beta}\Big(1 - \frac{1}{t}\Big) \tag{57}$$

If $s \le n$ we get $\beta = s(2n-s)$ therefore (57) is satisfied when $\alpha \ge 1 + \frac{1}{2n} - \frac{s}{n} + \frac{s^2}{2n^2}$. Instead when $s > n$ we have $\beta = ns$ and (57) is satisfied for $\alpha \ge 1 - \frac{s}{2n} + \frac{1}{2n}$. *q.e.d.*

Now, let us give upper bounds for semisimple elements. Let us observe that if $\nu(x) = 2l$ then we may assume $x = [I_{2(n-l)}, \omega I_{a_1}, \omega^{-1} I_{a_1}, \dots]$ or, $x = [I_{2a_0}, \omega I_{2n-2l}, \omega^{-1} I_{2n-2l}, \dots]$ and $2l \ge n$; if $\nu(x) = s$ is odd we may assume $x = [I_{2a_0}, \omega I_{2n-s}, \omega^{-1} I_{2n-s}, \dots]$ in this case we must have $s \ge n$.

Proposition 6.5. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \leq \begin{cases} 1 - \frac{s}{n} + \frac{s}{2n^2} + \frac{s^2}{2n^2} & s = 2l \leq n \\ 1 - \frac{s}{2n} + \frac{s}{2n^2} & s = 2l \geq n \\ 1 - \frac{s}{2n} + \frac{1}{2n} & s = 2l + 1 \geq n \end{cases}$$

Proof. As already done for Proposition 6.4 we need to show, in the case $\nu(x) = 2l$ and $x = [I_{2(n-s)}, \omega I_{a_1}, \omega^{-1} I_{a_1}, \ldots]$

$$\frac{1}{t} + \frac{l}{\beta}\Big(1 - \frac{1}{t}\Big) \leq 1 - \frac{2n^2(1-\alpha)}{\beta}\Big(1 - \frac{1}{t}\Big) \tag{58}$$

where $\beta = \max\{4l(n-l), 2nl\}$, where we used Proposition 2.10, i.e.

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leq \frac{1}{t} + \frac{n - a_0}{\dim x^G}\Big(1 - \frac{1}{t}\Big) \leq \frac{1}{t} + \frac{s}{\beta}\Big(1 - \frac{1}{t}\Big)$$

If $2l \leq n$ we have $\beta = 4l(n-l)$ and (58) is satisfied for $\alpha \geq 1 - \frac{s}{n} + \frac{s}{2n^2} + \frac{s^2}{2n^2}$. When $2l > n$ we get $\beta = 2nl$ and (58) is satisfied when $\alpha \geq 1 - \frac{s}{2n} + \frac{s}{2n^2}$.

In the case $\nu(x) = s$ is odd we may assume $x = [I_{2a_0}, \omega I_{2n-s}, \omega^{-1} I_{2n-s}, \ldots]$ with $s \leq n$, therefore $\dim x^G \geq ns$. And we have

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leq \frac{1}{t} + \frac{n - a_0}{\dim x^G} \leq \frac{1}{t} + \frac{1}{s}\Big(1 - \frac{1}{t}\Big)$$
$$1 - \frac{\dim \Omega(1-\alpha)}{\dim x^G} \geq 1 - \frac{2n(1-\alpha)}{s}\Big(1 - \frac{1}{t}\Big)$$

Therefore for $\alpha \geq 1 - \frac{s}{2n} + \frac{1}{2n}$ we have $f_\Omega(x) \leq \alpha$. This completes the proof. *q.e.d.*

Thus, Proposition 6.3 is a corollary of Proposition 6.4 and 6.5.

As observed in Remark 6.2 we do not have a characterization of elements that realize equality with the upper bound neither for the symplectic group. Indeed we can have a better result for $f^\circ_\Omega(x)$. In fact, for any n, t, r we can construct a family of semisimple elements of prime order r which realize the upper bound. The main strategy is to prove a result similar to Lemma 4.16 and 4.25.

We devote the rest of the section in the proof of results similar to Lemma 4.16 and 4.25 in order to identify the elements which realize upper bound on f°_Ω and then we prove that there always exists an element x such that $f_\Omega(x)$ is closed to the upper bound given above. We start with the case $G = \mathrm{GL}_n$ then we move on $G = \mathrm{Sp}_{2n}$.

6.1.1. *General linear group.* We claim that, for $G = \mathrm{GL}_n$, the upper bound on f°_Ω for prime order semisimple element in $\mathcal{V}_s$ is reached by any element with the same centralizer structure of the following

$$[I_{n-s}, \omega I_{n-s}, \ldots, \omega^{l-1} I_{n-s}, \omega^l I_m] \tag{59}$$

where $n = (n-s)l + m$ and $0 \leq m < n - s$.

We shall prove the following result.

Proposition 6.6. *Let $G = \mathrm{GL}_n(k)$. Then there exists $x \in \mathcal{V}_{s,r}$ such that*

$$f_\Omega(x) \geq 1 - \frac{s\gamma}{n} - \frac{2}{n} - \iota$$

where for $s > n/2$, $\iota = m/n$ with $0 \leq m < n - s$ and $n \equiv m \mod (n-s)$, or $\iota = 0$ otherwise.

And, a straightforward consequence will be the following.

Corollary 6.7. *Let $G = \mathrm{GL}_n$. Then there exists $x \in H\mathcal{V}_{s,r}$ such that*

$$f_\Omega(x) \geq 1 - \frac{s\gamma}{n} - \frac{2}{n} - \frac{1}{16}$$

We need the following lemma, which is the equivalent of Lemma 4.16.

Lemma 6.8. *Let $G = \mathrm{GL}_n$. Let $x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in \mathcal{V}_s \cap H$ be a semisimple element of prime order r other than (59). Assume $a_1 = \min\{a_i : a_i \neq 0\}$ and $a_2 = \max\{a_i : a_i < n-s\}$. Let $y = [I_{n-s}, \omega I_{a_1-1}, \omega^2 I_{a_2+1}, \omega^3 I_{a_3}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. Then*

$$f^\circ_\Omega(x) \leq f^\circ_\Omega(y)$$

with equality if, and only if, $a_1 = a_2$ and one of the following holds, $a_1 \equiv 0 \mod t$ or $a_1 \equiv t-1 \mod t$.

Proof. We have $f^\circ_\Omega(x) \leq f^\circ_\Omega(y)$ if, and only if,

$$\dim y^G - \dim x^G \leq \dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) \tag{60}$$

And using Theorem 4.1 we compute the dimensions which we need:

$$\dim y^G - \dim x^G = 2(a_1 - a_2 - 1)$$

$$\begin{aligned}\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = {}& \Big(\Big\lfloor \frac{a_1-1}{t}\Big\rfloor^2 - \Big\lfloor\frac{a_1}{t}\Big\rfloor^2\Big)t + \Big(\Big\lfloor\frac{a_1-1}{t}\Big\rfloor - \Big\lfloor\frac{a_1}{t}\Big\rfloor\Big)t \\ & - 2a_1\Big(\Big\lfloor\frac{a_1-1}{t}\Big\rfloor - \Big\lfloor\frac{a_1}{t}\Big\rfloor\Big) + 2\Big\lfloor\frac{a_1-1}{t}\Big\rfloor \\ & + \Big(\Big\lfloor\frac{a_2+1}{t}\Big\rfloor^2 - \Big\lfloor\frac{a_2}{t}\Big\rfloor^2\Big)t + \Big(\Big\lfloor\frac{a_2+1}{t}\Big\rfloor - \Big\lfloor\frac{a_2}{t}\Big\rfloor\Big)t \\ & - 2a_2\Big(\Big\lfloor\frac{a_2+1}{t}\Big\rfloor - \Big\lfloor\frac{a_2}{t}\Big\rfloor\Big) - 2\Big\lfloor\frac{a_2+1}{t}\Big\rfloor\end{aligned}$$

As in Lemma 4.16 we get four cases depending on the values of the floor functions.

(1) For $\lfloor\frac{a_1-1}{t}\rfloor = \lfloor\frac{a_1}{t}\rfloor$ and $\lfloor\frac{a_2+1}{t}\rfloor = \lfloor\frac{a_2}{t}\rfloor$, the inequality (60) is equivalent to

$$a_1 - a_2 - 1 < \Big\lfloor\frac{a_1}{t}\Big\rfloor - \Big\lfloor\frac{a_2}{t}\Big\rfloor \tag{61}$$

(2) If $\lfloor\frac{a_1-1}{t}\rfloor = \lfloor\frac{a_1}{t}\rfloor - 1$ and $\lfloor\frac{a_2+1}{t}\rfloor = \lfloor\frac{a_2}{t}\rfloor + 1$, the inequality (60) is equivalent to

$$\Big\lfloor\frac{a_1}{t}\Big\rfloor - \Big\lfloor\frac{a_2}{t}\Big\rfloor - 1 < 0 \tag{62}$$

(3) If $\lfloor\frac{a_1-1}{t}\rfloor = \lfloor\frac{a_1}{t}\rfloor - 1$ and $\lfloor\frac{a_2+1}{t}\rfloor = \lfloor\frac{a_2}{t}\rfloor$, the inequality (60) is equivalent to

$$a_2 \geq \Big\lfloor\frac{a_1}{t}\Big\rfloor t + \Big\lfloor\frac{a_2}{t}\Big\rfloor - \Big\lfloor\frac{a_1}{t}\Big\rfloor \tag{63}$$

(4) Finally, for $\lfloor\frac{a_1-1}{t}\rfloor = \lfloor\frac{a_1}{t}\rfloor$ and $\lfloor\frac{a_2+1}{t}\rfloor = \lfloor\frac{a_2}{t}\rfloor + 1$, the inequality (60) is equivalent to

$$a_1 \leq \Big\lfloor\frac{a_1}{t}\Big\rfloor - \Big\lfloor\frac{a_2}{t}\Big\rfloor + \Big\lfloor\frac{a_2}{t}\Big\rfloor t + t - 1$$

Let us observe that, in this case, $a_2 = \lfloor a_2/t\rfloor t + (t-1)$, hence (60) is equivalent to

$$a_1 \leq \Big\lfloor\frac{a_1}{t}\Big\rfloor - \Big\lfloor\frac{a_2}{t}\Big\rfloor + a_2 \tag{64}$$

Let us consider each of these four cases.

(1) In this case we can write $a_1 = b_1 t + c_1$ and $a_2 = b_2 t + c_2$ where $c_1 \in \{1, \ldots, t-1\}$ and $c_2 \in \{0, \ldots, t-2\}$. If $a_1 = a_2$ we have $a_1 - a_2 - 1 = -1 < 0 = \lfloor a_1/t\rfloor - \lfloor a_2/t\rfloor$ hence (61) holds. If $a_1 < a_2$ we consider two cases. Either $b_1 = b_2$ and $c_1 < c_2$, in which case the inequality (61) is equivalent to $c_1 - c_2 - 1 < 0$ that is clearly true. Or $b_1 < b_2$, in which case (61) is equivalent to $(b_1 - b_2)(t-1) + (c_1 - c_2) - 1 < 0$,

that is true because $(b_1 - b_2)(t-1) + (c_1 - c_2) - 1 \leq -(t-1) + (t-1) - 1 < 0$, since $b_1 - b_2 \leq 1$ and $c_1 - c_2 \leq t-1$.

(2) In the second case we have $a_1 = b_1 t$ and $a_2 = b_2 t + (t-1)$. Moreover if $b_1 > b_2$ we would have $a_1 = b_1 t \geq (b_2+1)t > b_2 t + t - 1 = a_2$ which contradicts the hypothesis $a_1 < a_2$. The inequality (62) is equivalent to $b_1 - b_2 - 1 < 0$, which is satisfied since $b_1 - b_2 - 1 \leq b_2 - b_2 - 1 = -1 < 0$.

(3) In this case we may write $a_1 = b_1 t, a_2 = b_2 t + c_2$, where $c_2 \in \{0, \dots, t-2\}$. And inequality (63) is equivalent to $b_2 t + c_2 > b_1 t - b_1 + b_2$. Let us consider separately the case $c_2 = 0$ from the one in which $c_2 \neq 0$.

If $c_2 = 0$, then $b_1 \geq b_2$. If $b_1 = b_2$ we get $b_2 t = b_1 t = b_1 t - b_1 + b_1$, therefore $f_\Omega(x) = f_\Omega(y)$. If $b_1 < b_2$ then (60) is equivalent to $(b_2 - b_1)(t-1) > 0$, clearly true.

If $c_2 \neq 0$, again we have $b_1 \leq b_2$. Thus $b_2 t + c_2 > b_1 t - b_1 + b_2$ can be written as $(b_2 - b_1)(t-1) + c_2 > 0$ which holds since $(b_2 - b_1)(t-1) \geq 0$.

(4) In the fourth case we have $a_1 = b_1 t + c_1, a_2 = b_2 t + (t-1)$ where $c_1 \in \{1, \dots, t-1\}$. And inequality (64) is equivalent to $(b_1 - b_2)(t-1) + (c_1 - (t-1)) < 0$. This last inequality is always true, since $b_1 \leq b_2, c_1 \leq t-1$, unless $b_1 = b_2$ and $c_1 = t-1$, in which case in (64) equality holds.

q.e.d.

Hence we can show the following, which proves the claim introduced above about the element in (59).

Proposition 6.9. *Let $x \in \mathcal{V}_{s,r}$ and let $y = [I_{n-s}, \omega I_{n-s}, \dots, \omega^{l-1} I_{n-s}, \omega^l I_m] \in \mathcal{V}_{s,r}$, where $n = (n-s)l + m$ and $0 \leq m < n-s$. Then*

$$f_\Omega^\circ(x) \leq f_\Omega^\circ(y)$$

Proof. Say $x = [I_{n-s}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$. If $x \neq [I_{n-s}, \omega I_{n-s}, \dots, \omega^{l-1} I_{n-s}, \omega^l I_m]$ there are at least two multiplicities such that $a_i, a_j < n$ for some $i, j \in \{1, \dots, r-1\}$. By Lemma 6.8 we may construct a new element y and $f_\Omega^\circ(x) \leq f_\Omega^\circ(y)$.
Again, if $y \neq [I_{n-s}, \omega I_{n-s}, \dots, \omega^{l-1} I_{n-s}, \omega^l I_m]$ we construct a new element y' and so on until we get $[I_{n-s}, \omega I_{n-s}, \dots, \omega^{l-1} I_{n-s}, \omega^l I_m]$. Therefore iterating Lemma 6.8 we have the result. *q.e.d.*

Remark 6.10. By Lemma 6.8 we see that the elements which realize the upper bound, for $f_\Omega^\circ(\cdot)$, have the same centralizer structure of one of the following elements: either

$$[I_{n-s}, \omega I_{n-s}, \dots, \omega^{l-3} I_{n-s}, \omega^{l-2} I_{n-s-1}, \omega^{l-1} I_1, \omega^l I_m]$$

when $t = 2$ or

$$[I_{n-s}, \omega I_{n-s}, \dots, \omega^{l-2} I_{n-s}, \omega^{l-1} I_{n-s-1}, \omega^l I_{m+1}]$$

when $m - 1 \equiv 0$ or $t-1 \mod t$.

And, both cases introduced in Remark 6.10 may happen, as we see in the following.

Example 6.11. Let $n = 18, t = 2, s = 13$. Then by Proposition 6.9, for $r \geq 5$ the element $\bar{x} = [I_5, \omega I_5, \omega^2 I_5, \omega^3 I_3]$ has maximal $f_\Omega^\circ(\cdot)$ value. Let $x = [I_5, \omega I_5, \omega^2 I_4, \omega^3 I_4]$. Then

$$\dim \bar{x} = 240 \quad \dim(\bar{x} \cap H^\circ) = 118$$
$$\dim x = 242 \quad \dim(x \cap H^\circ) = 120$$

Hence, using also $x^G \cap H = x^G \cap H^\circ, \bar{x}^G \cap H = \bar{x}^G \cap H^\circ$, we get $f_\Omega(x) = f_\Omega(\bar{x})$.

By Proposition 6.9 we know the elements with maximal f_Ω° value. The aim is to prove that the value is closed to the upper bound given in Proposition 6.1.

Lemma 6.12. *Let $s \leq n/2$. Then there exists $x \in \mathcal{V}_{s,r}$ such that*

$$f^{\circ}_{\Omega}(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

In particular $f_{\Omega}(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{2}{n}$.

Proof. Let $x = [I_{n-s}, \omega I_s]$. Then $\dim x^G = 2s(n-s)$ and by Theorem 4.1:

$$\dim(x^G \cap H^{\circ}) = 2\Big\lfloor\frac{s}{t}\Big\rfloor\Big(\Big\lfloor\frac{s}{t}\Big\rfloor t - 2s + t\Big) + 2s\Big(\frac{n}{t} - 1\Big)$$

A straightforward computation leads to

$$f^{\circ}_{\Omega}(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

where $s = at+b$ and $0 \leq b < t$, hence $b = s - \lfloor s/t \rfloor t$. Furthermore, for $r > 2$, $f_{\Omega}(x) = f^{\circ}_{\Omega}(x)$. Let us observe that $2b(t-b)$ is maximal when $b = t/2$. Therefore the difference between this bound and the bound given in Proposition 6.1 is

$$E = \frac{t^2}{2n^2(t-1)}$$

We easily see that, as a function of t, E is maximal when t is maximal, i.e.

$$0 \leq E \leq \frac{1}{4n-8}$$

and, for $n \geq 3$, we have $E \leq 2/n$. The result follows. *q.e.d.*

In the case $s > n/2$ it is slightly harder compute $f^{\circ}_{\Omega}(x)$ for the element x that Lemma 6.8 suggests. We shall use Proposition 2.9 and Proposition 2.10 to simplify the computation.

We complete the proof of Proposition 6.6 with the following lemma.

Lemma 6.13. *Let $s > n/2$. Then there exists $x \in \mathcal{V}_{s,r}$ such that*

$$f^{\circ}_{\Omega}(x) = 1 - \frac{s}{n} - \frac{2}{n} - \frac{m}{n}$$

where $0 \leq m < n-s$ and $n \equiv m \mod (n-s)$.

Proof. In this case we consider $x = [I_{n-s}, \dots, \omega^{l-1} I_{n-s}, \omega^l I_m]$. Then

$$\begin{aligned} \dim x^G &= l(n-s)(n+s-l(n-s)) \\ \dim(x^G \cap H^{\circ}) &= \frac{n^2}{t} - n + l\Big(\Big\lfloor\frac{n-s}{t}\Big\rfloor^2 t + (t-2n+2s)\Big\lfloor\frac{n-s}{t}\Big\rfloor\Big) \\ &\quad + \Big\lfloor\frac{n-(n-s)l}{t}\Big\rfloor^2 t + (t-2n+2l(n-s))\Big\lfloor\frac{n-(n-s)l}{t}\Big\rfloor \end{aligned}$$

where we used Theorem 4.1 for $\dim(x^G \cap H^{\circ})$.

Let us observe that we can write

$$\dim x^G = ns + \delta$$

where $\delta = l(n-s)(n+s-l(n-s)) - ns = l(n-s)(m+s) - ns$. Furthermore, thanks to Proposition 2.9 we deduce $\delta \geq 0$. Therefore, using $l \leq n/(n-s)$, we have

$$\delta \leq \frac{n}{n-s}(n-s)(m+s) - ns = nm$$

Now, either t divides s or it does not.

- If t divides s we have
$$\Big\lfloor \frac{n-s}{t} \Big\rfloor = \frac{n-s}{t}, \quad \Big\lfloor \frac{n-(n-s)l}{t} \Big\rfloor = \frac{n-(n-s)l}{t}$$
And with a little bit of computation we get $\dim(x^G \cap H^\circ) = \frac{1}{t}\dim x^G$, hence by Proposition 2.10, $\dim(x^G \cap H) = \frac{1}{t}\dim x^G$ therefore $f_\Omega(x) = f_\Omega^\circ(x)$.
- If t does not divide s, we have
$$\Big\lfloor \frac{n-s}{t} \Big\rfloor = \frac{n}{t} - \Big\lfloor \frac{s}{t} \Big\rfloor - 1, \quad \Big\lfloor \frac{n-(n-s)l}{t} \Big\rfloor = \frac{n(1-l)}{t} + \Big\lfloor \frac{sl}{t} \Big\rfloor$$
Again, with a straightforward computation we get
$$\dim(x^G \cap H^\circ) = \frac{\dim x^G}{t} - \epsilon$$
where
$$\begin{aligned}\epsilon = &-\frac{s^2 l}{t}(1+l) + 2ls\Big(1 + \Big\lfloor \frac{s}{t} \Big\rfloor + \Big\lfloor \frac{sl}{t} \Big\rfloor\Big)\\ &- lt\Big\lfloor \frac{s}{t} \Big\rfloor\Big(1 + \Big\lfloor \frac{s}{t} \Big\rfloor\Big) - t\Big\lfloor \frac{sl}{t} \Big\rfloor\Big(1 + \Big\lfloor \frac{sl}{t} \Big\rfloor\Big)\end{aligned}$$
Thanks to Proposition 2.10 and the trivial inequality $\dim(x^G \cap H^\circ) \leq \dim(x^G \cap H)$ we deduce $\epsilon \geq 0$.

Therefore, in the case $s > n/2$ we have
$$\dim x^G = ns + \delta, \quad \dim(x^G \cap H^\circ) = \frac{\dim x^G}{t} - \epsilon$$
where $\delta \geq 0$ and $\epsilon \geq 0$. Therefore
$$f_\Omega(x) = 1 - \frac{s}{n} - \Big(\frac{\delta}{n^2} + \frac{\epsilon t}{n^2(t-1)}\Big)$$
In this case the difference between the value of $f_\Omega^\circ(x)$ and the upper bound given in Proposition 6.3 is
$$0 \leq E = \frac{\delta}{n^2} + \frac{\epsilon t}{n^2(t-1)} \leq \frac{\delta + 2\epsilon}{n^2}$$

Let us consider the functions $g(x) = (2ls - lt)x + ltx^2$ and $h(x) = (2ls - t)x - tx^2$. We have $g(x) \leq g(\frac{s}{t} - \frac{1}{2})$ and $h(x) \leq h(\frac{ls}{t} - \frac{1}{2})$. Therefore $\epsilon \leq (l+1)t/4$ and it is easy to check that $(l+1)t \leq 4n$. Hence $\epsilon \leq n$.
$$0 \leq E \leq \frac{m}{n} + \frac{t}{n(t-1)} \leq \frac{m}{n} + \frac{2}{n}$$
q.e.d.

Remark 6.14. In the proof of Lemma 6.13, in the case $s > n/2$ we have that $\delta = 0$ if, and only if, $n - s$ divides n. Let us write $n = l(n-s) + m$ where $0 \leq m < n - s$. Then $\delta = l(n-s)(s+m) - ns$, if $m = 0$ we have $l = n/(n-s)$ and $\delta = 0$. Let us assume $\delta = 0$, then, using $l = (n-m)/(n-s)$, we have $\delta = (n-m)(s+m) - ns = 0$, hence $m(n-s-m) = 0$ and since $m < n - s$ we deduce $m = 0$.

Thus, with Lemma 6.12 and 6.13 we gain a complete proof of Proposition 6.6. Now, we prove Corollary 6.7.

Proof of Corollary 6.7. If $s \le n/2$ the results follows from Lemma 6.12.

Assume $s > n/2$. With a similar argument used for ϵ (in the proof of Lemma 6.13), we have $\delta(l) = l(n-s)(n+s-l(n-s)) - ns$, so $\delta(l) \le \delta(\frac{n+s}{2(n-s)}) = (n-s)^2/4$. Therefore $\delta \le (n-s)^2/4 \le n^2/16$, since $s \ge n/2$ and $E \le 2/n + 1/16$. This yields

$$f_\Omega(x) \ge 1 - \frac{s}{n} - \frac{2}{n} - \frac{1}{16}$$

q.e.d.

In conclusion, the upper bounds given in Proposition 6.1 are not sharp, but, Proposition 6.6 establishes that the error is not too large. Furthermore, by the proofs of Lemma 6.12 and 6.13, we can give infinitely many examples in which the bounds of Proposition 6.1 are realized. It is enough to consider elements with the same centralizer structure of $[I_{n-s}, \ldots, \omega^{l-1} I_{n-s}, \omega^l I_m]$ with the following further conditions: if $s \le n/2$, then $t \equiv 0 \mod s$, i.e. $b = 0$; otherwise $n - s \equiv 0 \mod t$ and $n \equiv 0 \mod (n-s)$, that is $\epsilon = \delta = 0$ in the proof of Lemma 6.13.

6.1.2. *Symplectic group.* We shall make the same analysis in the case $G = \mathrm{Sp}_{2n}$. In this case, due to the centralizer structure of semisimple elements, we do not get a complete answer in the construction of elements which realize upper bounds on f_Ω°.

If we want to give upper bound on $f_\Omega^\circ(x)$ for $x \in \mathcal{V}_{s,r}$, where $r \ne p$ is prime, it is possible to have some stronger result. The following technical lemma is the equivalent of Lemma 6.8 for GL_n. In the case of the general linear group given any $x \in \mathcal{V}_{s,r}$ we may assume the 1-eigenspace to be the largest one, for Sp_{2n} this is not possible, by the centralizer structure of the elements, hence this is only a partial result regarding elements whose largest eigenspace is the 1-eigenspace.

Lemma 6.15. *Let $x = [I_{2n-s}, \omega I_{a_1}, \ldots, \omega^{-\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \in \mathcal{V}_{r,s}$. Assume $a_1 = \max\{a_i : a_i < 2n - s\}, a_2 = \min\{a_i : a_i > 0\}$. Let $y = [I_{2n-s}, \omega I_{a_1+1}, \omega^2 I_{a_2-1}, \ldots]$. Then*

$$f_\Omega^\circ(x) \le f_\Omega^\circ(y)$$

Proof. By the centralizer structure of x and y in Sp and using the formula of $\dim(x^G \cap H^\circ), \dim(y^G \cap H^\circ)$ given in Theorem 4.1, we have that the result is equivalent to the same formulae in the proof of Lemma 6.8. The result follows. *q.e.d.*

Lemma 6.16. *Let $x = [I_{2a_0}, (\omega, \omega^{-1}) I_{2n-s}, (\omega, \omega^{-1})^2 I_{a_2}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \in \mathcal{V}_{r,s}$.*

- *If $a_2 = \min\{2a_0, a_i : a_i > 0\}$ and $a_3 = \max\{2a_0, a_i : ai < 2n - s\}$. Let $y = [I_{2a_0}, (\omega, \omega^{-1}) I_{2n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, (\omega, \omega^{-1})^3 I_{a_3+1}, \ldots]$.*
- *If $2a_0 = \min\{2a_0, a_i : a_i > 0\}$ and $a_2 = \max\{2a_0, a_i : a_i < 2n - s\}$. Let $y = [I_{2a_0-2}, (\omega, \omega^{-1}) I_{2n-s}, (\omega, \omega^{-1})^2 I_{a_2+1}, \ldots]$. Assume, moreover, $a_0 \le t$.*
- *If $2a_0 = \max\{2a_0, a_i : a_i < 2n - s\}$ and $a_2 = \min\{2a_0, a_i : a_i > 0\}$. Let $y = [I_{2a_0+2}, (\omega, \omega^{-1}) I_{2n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, \ldots]$.*

Then

$$f_\Omega^\circ(x) \le f_\Omega^\circ(y)$$

Proof. The result is equivalent to

$$\dim y^G - \dim x^G \le \dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) \tag{65}$$

In the first case the same argument of Lemma 6.15 applies.

In the second case (65) is equivalent to

$$\begin{cases} 2a_0 - a_2 - 1 \le \left\lfloor \frac{a_0}{t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor + 1 & b_0 > 0 \\ 2a_0 - a_2 - 1 \le \frac{a_0}{t} - \left\lfloor \frac{a_2}{t} \right\rfloor & b_0 = 0 \end{cases}$$

where we write $a_0 = \lfloor \frac{a_0}{t} \rfloor t + b_0$. Since

$$\lfloor \frac{a_0}{t} \rfloor - \lfloor \frac{a_2}{t} \rfloor + 1 \geq \frac{a_0}{t} - \frac{a_2}{t}$$

we claim that $2a_0 - a_2 - 1 \leq \frac{a_0}{t} - \frac{a_2}{t}$ which is equivalent to

$$\frac{a_0}{t} + (2a_0 - a_2)\Big(1 - \frac{1}{t}\Big) - 1 \leq 0$$

and for $a_0 \leq t$ this last inequality is verified since $2a_0 - a_2 \leq 0$.

In the third case (65) is equivalent to

$$\begin{cases} -2a_0 + a_2 - 1 \leq \lfloor \frac{a_2}{t} \rfloor - \lfloor \frac{a_0}{t} \rfloor & b_2 > 0 \\ -2a_0 + a_2 \leq \frac{a_2}{t} - \lfloor \frac{a_0}{t} \rfloor & b_2 = 0 \end{cases}$$

where we write $a_2 = \lfloor \frac{a_2}{t} \rfloor t + b_2$. Since

$$\lfloor \frac{a_2}{t} \rfloor - \lfloor \frac{a_0}{t} \rfloor \geq \frac{a_2}{t} - \frac{a_0}{t} + 1$$

we claim that $-2a_0 + a_2 \leq \frac{a_0}{t} - \frac{a_2}{t}$ which is equivalent to $-a_0 - a_0\big(1 - \frac{1}{t}\big) + a_2\big(1 - \frac{1}{t}\big) \leq 0$ and

$$-a_0 - a_0\Big(1 - \frac{1}{t}\Big) + a_2\Big(1 - \frac{1}{t}\Big) = -\frac{a_0}{t} + (a_2 - 2a_0)\Big(1 - \frac{1}{t}\Big) \leq -\frac{a_0}{t} < 0$$

The proof is complete. *q.e.d.*

Therefore, thanks to Lemma 6.15 and Lemma 6.16, we have that starting with any element $x \in \mathcal{V}_{r,s}$ with the 1-eigenspace of dimension $2n - s$ we find, after a finite number of steps, using the construction in the statement, that

$$f_\Omega^\circ(x) \leq f_\Omega^\circ(\bar{x})$$

where, writing $2n = (2n - s)l + m$ or $2n = (2n - s)l + k$ where $l \equiv k \mod 2$,

$$\bar{x} = \begin{cases} [I_{2n-s}, \omega I_{2n-s}, \dots, \omega^{-\frac{l-1}{2}} I_{2n-s}, (\omega, \omega^{-1})^{\frac{l+1}{2}} I_m] & l\text{odd} \\ [I_{2n-s}, (\omega, \omega^{-1}) I_{2n-s}, \dots, (\omega, \omega^{-1})^{-\frac{l}{2}+1} I_{2n-s}, (\omega, \omega^{-1})^{\frac{l}{2}} I_{\frac{2n-s}{2}}, (\omega, \omega^{-1})^{\frac{l}{2}+1} I_m] & l\text{even} \\ [I_k, \omega I_{2n-s}, \dots, \omega^{-\frac{l}{2}} I_{2n-s}] & l\text{odd} \\ [I_{k+1}, \omega I_{2n-s}, \omega^{-\frac{l-1}{2}} I_{2n-s}] & l\text{even} \end{cases}$$

In order to justify why the bounds of Proposition 6.3 are closed to the best possible, we compute $f_\Omega^\circ(x)$, or we give bounds on it, for some semisimple elements which maximize f_Ω°, using Lemma 6.15 and 6.16. And we bound the difference between the bound of Proposition 6.5 and the bound gotten for $f_\Omega^\circ(x)$.

- Assume s is even, and $s \leq \frac{4}{3}n$, then let $x = [I_{2n-s}, \omega I_{\frac{s}{2}}, \omega^{-1} I_{\frac{s}{2}}]$. Then

$$\dim x^G = 2ns - \frac{3}{4}s^2 + \frac{s}{2}$$
$$\dim(x^G \cap H^\circ) \geq 2\frac{ns}{t} + \frac{s}{2}$$

Therefore

$$f_\Omega^\circ(x) \geq 1 - \frac{s}{n} + \frac{3s^2}{8n^2\big(1 - \frac{1}{t}\big)}$$

- Let us assume, now, s is odd and $s \geq n$. Then, for $s \leq \frac{4}{3}n$ we consider $x = [I_{2s-2n}, (\omega, \omega^{-1})I_{2n-s}]$. Let us observe $\left\lfloor \frac{s-n}{t} \right\rfloor = \left\lfloor \frac{s}{t} \right\rfloor - \frac{n}{t}$ and $\left\lfloor \frac{2n-s}{t} \right\rfloor = 2\frac{n}{t} - \left\lfloor \frac{s}{t} \right\rfloor - 1$. Thus, using Theorem 4.1,

$$\dim x^G = n(2n+1) - (s-n)(2s-2n+1) - (2s-n)^2$$

$$\begin{aligned}\dim(x^G \cap H^\circ) &= 3\left\lfloor \frac{s}{t} \right\rfloor^2 t - 6\left\lfloor \frac{s}{t} \right\rfloor s + 3\left\lfloor \frac{s}{t} \right\rfloor t - 4\frac{n^2}{t} + 8\frac{ns}{t} + 2n - 4s \\ &\geq -4\frac{n^2}{t} + 8\frac{ns}{t} + 2n - 3\frac{s^2}{t} - s - \frac{2}{3}t \\ &\geq -4\frac{n^2}{t} + 8\frac{ns}{t} - 3\frac{s^2}{t} + 2n - s\end{aligned}$$

where the first inequality is given taking $\left\lfloor \frac{s}{t} \right\rfloor = \frac{s}{t} - \frac{1}{3}$. Therefore

$$f_\Omega^\circ(x) \geq 3 - 4\frac{s}{n} + \frac{3s^2}{2n^2} = \alpha$$

For U the upper bound we get

$$-\frac{14}{3} + \frac{7s}{2n} + \frac{1}{2n} < U - \alpha < \frac{7}{2}\Big(\frac{s}{n} - 1\Big) + \frac{1}{2n}$$

We do not compute the f_Ω° in the other cases because it is very hard and the notation gets very heavy, using the formula for $\dim(x^G \cap H^\circ)$ in Theorem 4.1 there will be floor functions. If, instead, we use the lower bounds on $\dim(x^G \cap H^\circ)$ of Proposition 2.13 we get that $U - f_\Omega^\circ(x) \leq \frac{1}{8}$.

6.2. **Lower bounds: semisimple elements.** In this section we shall study lower bounds on f_Ω for elements in $\mathcal{V}_{s,r}$. Again, since $f_\Omega(x) \geq f_\Omega^\circ(x)$ and $x^G \cap H^\circ \neq \emptyset$ for all x semisimple elements, we are allowed to study only lower bounds on f_Ω°. First we construct a certain class of elements that realize the lower bounds using the same argument of Lemma 4.16 and 4.25. Then we split the analysis according $G = \mathrm{GL}_n$ or Sp_{2n}, getting a formula for these bounds which shall lead to a complete proof of Theorem 7 and 8.

Assume $G = \mathrm{GL}_n$. Let x be a semisimple element in H such that $\nu(x) = s$, then, up to conjugation, we may assume $x = [I_{n-s}, \omega I_{a_1}, \omega^2 I_{a_2}, \dots, \omega^{r-1} I_{a_{r-1}}]$, and without loss of generality, by Lemma 2.3, we may take the 1-eigenspace to be the largest one. We shall prove that for any $x \in \mathcal{V}_{s,r} \cap H$ we have $f_\Omega^\circ(x) \geq f_\Omega^\circ(\bar{x})$, where

$$\bar{x} = \begin{cases} [I_{n-s}, \omega, \omega^2, \dots, \omega^s] & r-1 \geq s \\ [I_{n-s}, \omega I_{\lfloor s/(r-1) \rfloor + 1}, \dots, \omega^l I_{\lfloor s/(r-1) \rfloor + 1}, \omega^l I_{\lfloor s/(r-1) \rfloor}, \dots, \omega^{r-1} I_{\lfloor s/(r-1) \rfloor}] & r-1 < s \end{cases} \tag{66}$$

here $l = s - \lfloor s/(r-1) \rfloor (r-1)$.

In order to prove the claims we use the same argument given in Lemma 4.16 and 4.25.

We state this property formally in Lemma 6.9. Let us observe that x has centralizer non-isomorphic to the centralizer of $\bar{x}$ defined in (66) if, and only if, $\max_{i \neq 0}\{a_i\} - \min_{i \neq 0}\{a_i\} \geq 2$, for $i = 1, \dots, r-1$ for $G = \mathrm{GL}_n$.

Lemma 6.17. *Let $x = [I_{n-s}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in \mathcal{V}_{r,s}$. Assume $a_1 = \max\{a_i\}$ and $a_2 = \min\{a_i\}$, and $a_1 - a_2 \geq 2$. Let $y = [I_{n-s}, \omega I_{a_1 - 1}, \omega^2 I_{a_2 + 1}, \dots, \omega^{r-1} I_{a_{r-1}}]$. Then*

$$f_\Omega^\circ(x) > f_\Omega^\circ(y)$$

with equality if, and only if, $a_1 - a_2 = 2$ and $a_1 \equiv 1 \mod t$.

Proof. See Lemma 4.16. *q.e.d.*

An important consequence of Lemma 6.17 is the following, which is giving us lower bounds for $f_\Omega(x)$ for $x \in \mathcal{V}_{r,s}$.

Corollary 6.18. *Let $x \in \mathcal{V}_{r,s}$. Then*

$$f^\circ_\Omega(x) \geq f^\circ_\Omega(\bar{x})$$

where $\bar{x}$ is as in (66).

Proof. The proof is trivial thanks to Lemma 6.17. In fact using the construction of the lemma starting from any $x \in \mathcal{V}_{r,s}$ in a finite number of steps we get $\bar{x}$, therefore $f_\Omega(x) \geq f_\Omega(\bar{x})$. *q.e.d.*

Let $G = \mathrm{Sp}_{2n}$. For x semisimple element in $H \cap \mathcal{V}_{r,s}$ with $s = 2l$ we may assume, with no loss of generality, that either $x = [I_{2(n-l)}, \omega I_{a_1}, \omega^{-1} I_{a_1}, \dots, \omega^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}, \omega^{-\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}]$, or, for $s \geq n$, $x = [I_{2a_0}, \omega I_{2n-s}, \omega^{-1} I_{2n-s}, \dots, \omega^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}, \omega^{-\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}]$. If s is odd then necessarily $s \geq n$ and x looks similar the latter element. In the same spirit of the above claim, we shall prove that, given any $x \in H \cap \mathcal{V}_{r,s}$, we have $f_\Omega(x) \geq f^\circ_\Omega(\bar{x})$, where

$$\bar{x} = \begin{cases} [I_{2n-s}, \omega I_{a+1}, \omega^{-1} I_{a+1}, \dots, \omega^{-\frac{b}{2}} I_{a+1}, \omega^{\frac{b}{2}+1} I_a, \dots, \omega^{-\frac{r-1}{2}} I_a] & s \text{ even} \\ [I_c, \omega I_{2n-s}, \omega^{-1} I_{2n-s}, \omega^2 I_{c+1}, \dots, \omega^{-\frac{d}{2}} I_{c+1}, \omega^{\frac{d}{2}+1} I_c, \dots, \omega^{-\frac{r-1}{2}} I_c] & s \text{ odd}, c \text{ even} \\ [I_{c+1}, \omega I_{2n-s}, \omega^{-1} I_{2n-s}, \omega^2 I_{c+1}, \dots, \omega^{-\frac{d-1}{2}} I_{c+1}, \omega^{\frac{d-1}{2}+1} I_c, \dots, \omega^{-\frac{r-1}{2}} I_c] & s \text{ odd}, c \text{ odd} \end{cases} \tag{67}$$

and we write $s = a(r-1)+b$ in the first case, with $0 \leq b < r-1$, and $2(s-n) = c(r-2)+d$, with $0 \leq d < r-2$.

The following two results are the equivalent, in the symplectic group, of Lemma 6.17 and Corollary 6.18.

Lemma 6.19. *Let $x = [I_{2n-s}, \omega I_{a_1}, \omega^{-1} I_{a_1}, \dots, \omega^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}, \omega^{-\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \in H \cap \mathcal{V}_{r,s}$. Let $s = a(r-1)+b$, where $0 \leq b < r-1$ is even. Then*

$$f^\circ_\Omega(x) \geq f^\circ_\Omega(\bar{x})$$

where $\bar{x}$ is as the first element of (67).

Proof. We may assume $a_1 = \min\{a_i\}, a_2 = \max\{a_i\}$. If $a_2 - a_1 \leq 1$ then the result trivially holds. Assume $a_2 - a_1 \geq 2$. We define a new element y

$$y = [I_{2n-s}, \omega I_{a_1+1}, \omega^{-1} I_{a_1+1}, \omega^2 I_{a_2-1}, \omega^{-2} I_{a_2-1}, \dots, \omega^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}, \omega^{-\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}]$$

Thus, with the same proof of Lemma 4.25, we get $f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$. Iterating this construction we will, eventually, end up with the element defined in (67) for s even. *q.e.d.*

In the case $\nu(x) = s$ is odd, we have the following.

Lemma 6.20. *Let $x = [I_{2a_0}, \omega I_{2n-s}, \omega^{-1} I_{2n-s}, \dots, \omega^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}, \omega^{-\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \in H \cap \mathcal{V}_{r,s}$. Then*

$$f^\circ_\Omega(x) \geq f^\circ_\Omega(\bar{x})$$

where $\bar{x}$ is as in (67).

Proof. The same argument of Lemma 6.19 applies. *q.e.d.*

Therefore, in order to get a complete proof of Theorem 7, 8 it is enough to compute $f^\circ_\Omega(\bar{x})$.

6.2.1. *General linear group.* In the following two remarks we study two particular semisimple elements of prime order r in the case $G = \mathrm{GL}_n$, in the first we assume $r \geq s+1$, then $r < s+1$.

Remark 6.21. Let $x = [I_{n-s}, \omega, \dots, \omega^s]$ be an element of prime order $r \geq s+1$. Then

$$\dim x^G = s(2n-s) - s$$

$$\dim(x^G \cap H^\circ) = \frac{n^2}{t} - n + \left\lfloor \frac{n-s}{t} \right\rfloor^2 t + (t - 2n + 2s)\left\lfloor \frac{n-s}{t} \right\rfloor$$

For $\dim(x^G \cap H^\circ)$ we have two cases depending on whether or not t divides $n-s$. We write $s = at + b$ where $0 \leq b < t$ and we have:

$$\dim(x^G \cap H^\circ) = \frac{2ns}{t} - 2as - 2s + a^2 t + at$$

Therefore,

$$f^\circ_\Omega(x) = 1 - \frac{(2n-s)s}{n^2} - \frac{b(t-b)}{n^2(t-1)}$$

$$= \frac{(n-s)^2}{n^2} - \frac{b(t-b)}{n^2(t-1)}$$

Let us observe that, in the case $b = 0$, we have $f^\circ_\Omega(x) = (n-s)^2/n^2$. Moreover, for $t < n$, $x^G \cap H = x^G \cap H^\circ$, by Lemma 2.19 since $n/t > 1$, hence $f_\Omega(x) = f^\circ_\Omega(x)$.

Remark 6.22. Let $r < s+1$ be a prime, we write $s = a(r-1) + b$ where $0 \leq b < r-1$. Let $x = [I_{n-s}, \omega I_{a+1}, \dots, \omega^b I_{a+1}, \omega^{b+1} I_a, \dots, \omega^{r-1} I_a]$ be an element of prime order r. Then

$$\dim x^G = 2ns - s^2 - a^2 r + a^2 - 2ab - b$$

Using Lemma 6.17 and Remark 6.21 we get the following in the case $s \leq r-1$, namely we prove conclusion of Theorem 7 for $s \leq r-1$.

Proposition 6.23. *Let $s \leq r-1$ and let $x \in \mathcal{V}_{r,s} \cap H$. Then*

(i) *in the case $r - 1 > s$, or $r - 1 = s$ and $t < n$,*

$$f_\Omega(x) \geq \frac{(n-s)^2}{n^2} - \frac{b(t-b)}{n^2(t-1)}$$

where $s = at + b$ and $0 \leq b < t$.

(ii) *If $r - 1 = s$ and $t = n$,*

$$f_\Omega(x) \geq \frac{(n-s)^2 - (n-2s)}{n^2 - n}$$

where we write $s = at + b$ and $b \in \{0, \dots, t-1\}$.

Proof. By Corollary 6.18 we have that for all $x \in \mathcal{V}_{r,s}$,

$$f^\circ_\Omega(x) \geq f^\circ_\Omega([I_{n-s}, \omega, \dots, \omega^s])$$

If $r - 1 > s$, by Lemma 2.19 $f_\Omega(x) = f^\circ_\Omega(x)$ and the result follows.

If $r - 1 = s$, then we need to distinguish two cases. For $t < n$, using Lemma 2.19 we deduce that $f_\Omega(x) = f^\circ_\Omega(x)$ for all $x \in \mathcal{V}_{r,s}$ and by Remark 6.21 the result follows. If $t = n$ then there are elements in $\mathcal{V}_{s,r}$ with $h > 0$, where h is defined in (16). Furthermore elements in $\mathcal{V}_{s,r}$ for which $h > 0$, therefore $h = 1$, have the same centralizer structure of $x = [I_{n-s}, \omega, \dots, \omega^s]$. An easy computation shows $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_1) = r - 1 = s$. Therefore

$$f_\Omega(x) = \frac{n^2 - n - 2ns + s^2 + 2s}{n^2 - n} = \frac{(n-s)^2 - (n-2s)}{n^2 - n}$$

q.e.d.

Therefore in the case $s \leq r-1$ using Lemma 6.17 we deduce a lower bound for $f_\Omega(x)$ for any $x \in \mathcal{V}_{s,r}$. In the case $s > r-1$ we have $f^\circ_\Omega(x)$ minimal for x as in (66). It is very hard, in general to compute $\dim(x^G \cap H)$ but thanks the general inequality $\dim(x^G \cap H) \geq \dim(x^G \cap H^\circ)$ we can use, as lower bound for $f_\Omega(\cdot)$, the lower bound of $f^\circ_\Omega(\cdot)$. It is, in fact, a good lower bound thanks to Proposition 2.13.

Thus we use Proposition 2.13 to give a lower bound on $f_\Omega(x)$ where $x \in \mathcal{V}_{s,r}$ and $s > r-1$.

Remark 6.24. In view of Lemma 6.17 and Corollary 6.18 we have that for any $x \in \mathcal{V}_{s,r}$, $f_\Omega(x) \geq f_\Omega(\bar{x}) \geq f^\circ_\Omega(\bar{x})$ where $\bar{x}$ is defined in (66). It is not hard to compute $f_\Omega(\bar{x})$ using the formulae in Proposition 2.2 and Theorem 4.1. It is enough to write $s = (pt+q)(r-1)+b$ where $0 \leq b < r-1$ and $0 \leq q < t$. It turns out that

$$\begin{aligned} f^\circ_\Omega(\bar{x}) &= \frac{n^2 + 2bpt + p^2rt^2 - p^2t + 2pqrt - 2pqt - 2ns}{n^2} \\ &+ \frac{(pt+q^2)(r-1) + b + 2bq + s^2 - 2s + (t-2s)\left\lfloor \frac{s}{t} \right\rfloor + t\left\lfloor \frac{s}{t} \right\rfloor^2}{n^2(1-1/t)} \end{aligned} \tag{68}$$

Although he bound (68) is the best we can give thanks to Corollary 6.18, it is not easy understand its size. We would like to have a better bound which depends on n, t, s. Therefore we use Corollary 6.18 and, to simplify all the computations, we use Proposition 2.13. Hence, for any $y \in \mathcal{V}_{s,r}$. We have

$$f_\Omega(y) \geq f^\circ_\Omega(y) \geq f^\circ_\Omega(x) \geq \frac{\dim\Omega - \dim x^G\left(1 - \frac{1}{t} + \frac{1}{n}\right)}{\dim\Omega} = u$$

where $x = [I_{n-s}, \omega I_{a+1}, \ldots, \omega^b I_{a+1}, \omega^{b+1} I_a, \ldots, \omega^{r-1} I_a]$ and $s = a(r-1)+b$. We have $\dim x^G = 2ns - s^2 - a^2(r-1) - 2ab - b$. Therefore

$$\begin{aligned} u &= 1 - \frac{\dim x^G\left(1 - \frac{1}{t}\right)}{\dim\Omega} - \frac{\dim x^G}{n\dim\Omega} \\ &= 1 - \frac{2ns - s^2}{n^2} + \frac{a^2(r-1) + 2ab + b}{n^2} - \frac{2ns - s^2 - a^2(r-1) - 2ab - b}{n^3\left(1 - \frac{1}{t}\right)} \\ &= \frac{(n-s)^2}{n^2} - \frac{2ns - s^2}{n^3\left(1 - \frac{1}{t}\right)} + \frac{a^2(r-1) + 2ab + b}{n^3\left(1 - \frac{1}{t}\right)}\left(1 - \frac{1}{t} + \frac{1}{n}\right) \\ &\geq \frac{(n-s)^2}{n^2} - \frac{2ns - s^2}{n^3\left(1 - \frac{1}{t}\right)} \end{aligned}$$

This completes the proof of Theorem 7. In the following remark we motivate the last inequality for u above.

Remark 6.25. Using $a \leq s/(r-1)$ and $b < r-1$ we get

$$\begin{aligned} \frac{a^2(r-1) + 2ab + b}{n^3\left(1 - \frac{1}{t}\right)}\left(1 - \frac{1}{t} + \frac{1}{n}\right) &< \frac{s^2 + 2s(r-1) + (r-1)^2}{n^3(r-1)\left(1 - \frac{1}{t}\right)}\left(1 - \frac{1}{t} + \frac{1}{n}\right) \\ &= \frac{t(n+1)(r^2 + 2rs - 2r + s^2 - 2s + 1)}{n^4(r-1)(t-1)} \\ &< \frac{t(n+1)(s+r)^2}{n^4(r-1)(t-1)} < \frac{5}{n} \end{aligned}$$

In order to show the last inequality we use the following $r, s \leq n$, $r \geq 3$ and $t/(t-1) \leq 2$:

$$\frac{t(n+1)(s+r)^2}{n^4(r-1)(t-1)} \leq \frac{4n^2(n+1)}{n^4} = \frac{4(n+1)}{n^2}$$

And since $n \geq 4$, because we are assuming $1 < t < n$, the last term is strictly less than $5/n$.

We have proved the following, which, for big n, is a good lower bound.

Proposition 6.26. *Let $x \in \mathcal{V}_{s,r}$ where r is a prime number other than p and $s > r-1$. Then*

$$f_\Omega(x) \geq 1 - \frac{2s}{n} + \frac{s^2}{n^2} - \frac{2ns - s^2}{n^3\left(1 - \frac{1}{t}\right)}$$

6.2.2. *Symplectic group.* Let us study, now, the case $G = \mathrm{Sp}_{2n}$.

A way to get lower bounds on $f^\circ_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$ is using bounds on $\dim x^G$ depending on $\nu(x)$ and the bound of $\dim(x^G \cap H^\circ)$ in Proposition 2.13. We give the following.

Proposition 6.27. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s}{n} + \left(\frac{s}{2n}\right)^2 - \frac{1}{4n^2} - \frac{4ns - s^2 + 1}{4n^3\left(1 - \frac{1}{t}\right)} \tag{69}$$

Proof. Let $x \in H$ be a semisimple element of prime order r with $\nu(x) = s$. Then $f_\Omega(x) \geq f^\circ_\Omega(x)$, using Proposition 2.13 and Proposition 2.9 we have

$$\begin{aligned}
f_\Omega(x) &\geq \frac{\dim\Omega - \dim x^G\left(1 - \frac{1}{t} + \frac{1}{n}\right)}{\dim\Omega} \\
&\geq \frac{\dim\Omega - \frac{1}{2}(4ns - s^2 + 1)\left(1 - \frac{1}{t}\right) - \frac{4ns - s^2 + 1}{2n}}{\dim\Omega} \\
&= 1 - \frac{4ns - s^2 + 1}{4n^2} - \frac{4ns - s^2 + 1}{4n^3\left(1 - \frac{1}{t}\right)} \\
&= 1 - \frac{s}{n} + \left(\frac{s}{2n}\right)^2 - \frac{1}{4n^2} - \frac{4ns - s^2 + 1}{4n^3\left(1 - \frac{1}{t}\right)} \\
&\geq 1 - \frac{s}{n} + \left(\frac{s}{2n}\right)^2 - \frac{1}{4n^2} - \frac{1}{n} - \frac{t}{n(t-1)}
\end{aligned}$$

q.e.d.

Computing $f^\circ_\Omega(\bar{x})$ in all the cases we get the best possible bounds for f°_Ω. We assume $r - 1 \geq s$ in order to apply Lemma 6.19. We do not consider $r - 1 < s$, since, as for GL_n, see Remark 6.24, the notation gets too heavy and the best way to compute the bound remains the argument of Proposition 6.27.

Proposition 6.28. *Assume $r - 1 \geq s$ for s even and, $r - 1 \geq 2(s - n)$ for $s \geq n$ odd. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f^\circ_\Omega(x) \geq \iota\left(1 - \frac{s}{n} + \left(\frac{s}{2n}\right)^2\right) - \frac{t^2}{2n^2(t-1)} \tag{70}$$

where $\iota = 1$ if s even and $\iota = 2$ otherwise.

Proof. By Lemma 6.19 we have, $f^\circ_\Omega(y) \geq f^\circ_\Omega([I_{2n-s}, \omega, \omega^{-1}, \dots, \omega^{\frac{s}{2}}, \omega^{-\frac{s}{2}}])$, for any $y \in H \cap \mathcal{V}_{r,s}$, whose 1-eigenspace is $(2n - s)$-dimensional, let us call x the second element. Moreover, let us observe that for any t $x^G \cap H = x^G \cap H^\circ$. We have

$$\dim x^G = 2ns - \frac{s^2}{2}$$

$$\dim(x^G \cap H^\circ) = 2\frac{n^2}{t} - 2n + s + 2\left\lfloor \frac{2n - s}{2t} \right\rfloor^2 t + 2\left\lfloor \frac{2n - s}{2t} \right\rfloor (t - 2n + s)$$

We have that $\dim(x^G \cap H^\circ)$ is minimal when $\lfloor \frac{2n-s}{2t} \rfloor = \frac{2n-s}{t} - \frac{1}{2}$, hence

$$\dim(x^G \cap H^\circ) \geq \frac{2n^2}{t} - \frac{s^2}{2t} - \frac{t}{2}$$

Therefore

$$f_\Omega(x) \geq 1 - \frac{s}{n} + \Big(\frac{s}{2n}\Big)^2 - \frac{t^2}{4n^2(t-1)} \tag{71}$$

By Lemma 6.20 we have, for any $y \in H \cap \mathcal{V}_{r,s}$ whose ω eigenspace is $(2n-s)$-dimensional, and $s \geq n$

$$f^\circ_\Omega(y) \geq f^\circ_\Omega([I_2, \omega I_{2n-s}, \omega^{-1} I_{2n-s}, \omega^2, \omega^{-2}, \dots, \omega^{s-n}, \omega^{-(s-n)}])$$

say x the second element. Again, $x^G \cap H = x^G \cap H^\circ$. And,

$$\Big\lfloor \frac{2n-s}{t} \Big\rfloor = \begin{cases} \frac{2n-s}{t} & t|s \\ 2\frac{n}{t} - \lfloor \frac{s}{t} \rfloor - 1 & \text{otherwise} \end{cases}$$

Thus

$$\begin{aligned} \dim x^G &= -2n^2 + 2n + 4ns - s^2 - s - 2 \\ \dim(x^G \cap H^\circ) &= 2\frac{n^2}{t} - 2 + \Big\lfloor \frac{2n-s}{t} \Big\rfloor^2 t + (t - 2(2n-s)) \Big\lfloor \frac{2n-s}{t} \Big\rfloor + 2\delta_{t,2} \\ &= -2\frac{n^2}{t} + 4\frac{ns}{t} + 2n - 2 - 2s + \Big\lfloor \frac{s}{t} \Big\rfloor t - 2s \Big\lfloor \frac{s}{t} \Big\rfloor + \Big\lfloor \frac{s}{t} \Big\rfloor^2 t + 2\delta_{t,2} \\ &\geq -2\frac{n^2}{t} + 4\frac{ns}{t} + 2n - 2 - s - \frac{s^2}{t} - \frac{t}{2} \end{aligned}$$

Therefore

$$f_\Omega(x) \geq 2\Big(1 - \frac{s}{n} + \Big(\frac{s}{2n}\Big)^2\Big) - \frac{t^2}{4n^2(t-1)} \tag{72}$$

q.e.d.

Remark 6.29. In the case $s \geq n$ is odd we have that the bound is minimal for s maximal, and minimal for s maximal. The case $s = 2n-1$ is not interesting since, in this case, $x = [\omega, \dots, \omega^n, \omega^{-n}]$ and $f_\Omega(x) = 0$. Thus we consider $s = 2n-3$, for the lower bound. We get

$$\frac{1}{2n^2}\Big(\frac{9}{2} - \frac{t^2}{t-1}\Big) \leq 2\Big(1 - \frac{s}{n} + \Big(\frac{s}{2n}\Big)^2\Big) - \frac{t^2}{2n^2(t-1)} \leq \frac{1}{2} - \frac{t^2}{2n^2(t-1)}$$

Remark 6.30. Assume s even. We can give bound on the difference between the bound (69) and (70). First let us observe that

$$-\frac{t^2}{4n^2(t-1)} \leq \frac{4ns - s^2}{4n^3(1-\frac{1}{t})} - \frac{t^2}{4n^2(t-1)} < \frac{t}{4n^2(t-1)}(4n-1)$$

And

$$1 - \frac{s}{n} + \Big(\frac{s}{2n}\Big)^2 - \Big(1 - \frac{s}{n} + \Big(\frac{s}{2n}\Big)^2\Big) + \frac{1}{4n^2} = \frac{1}{4n^2}$$

7. Semisimple involutions

In this section we shall study involutions when the characteristic of the field k is $p \neq 2$. First we start with $G = \mathrm{GL}_n$ and we prove conclusions of Theorem 4. Then we study the symplectic case proving Theorem 5.

7.1. **General linear group.** Let $r = 2$ and assume $p \neq 2$. Then for every $s \in \{0, \dots, n/2\}$ there is precisely one isomorphism class of centralizer structure in $\mathcal{V}_{s,2}$ given by the element $x = [I_{n-s}, -I_s]$. It is easy to compute $f^\circ_\Omega(x)$, using $\dim x^G = 2s(n-s)$ and the formula of $\dim(x^G \cap H^\circ)$ in Theorem 4.1, we get

$$f^\circ_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)} \tag{73}$$

where we write $s = at + b$, $b \in \{1, \dots, t-1\}$.

For $t = n$ it is easy to give a formula for $\dim(x^G \cap H)$ since $H^\circ = (k^*)^n$ and any involution is conjugate to a permutation of S_n.

Lemma 7.1. *If $t = n$ then conclusion of Theorem 4 holds.*

Proof. For $t = n$ let $x = [I_{n-s}, -I_s]$ with $s \leq n/2$, then for h as defined in (16) clearly we have $h = s$. We easily see $\dim(x^G \cap H^\circ\pi_i) = i$ therefore $\dim(x^G \cap H) = s$. Hence, since $\dim x^G = 2s(n-s)$ and $\dim \Omega = n^2 - n$, using Proposition 2.2 we get

$$f_\Omega(x) = \frac{(n-s)^2 - (n-s) + s^2}{n^2 - n}$$

q.e.d.

Let us assume, now, $t < n$. In order to give a formula for $f_\Omega(x)$ where x is any semisimple involution, we need to know the value of $\dim(x^G \cap H)$, i.e. we need to understand for which i, $\dim(x^G \cap H^\circ\pi_i)$ is maximal and compute it explicitly.

Two natural cases arise: either n/t is even or it is odd. Let $x = [I_{n-s}, -I_s]$ be an involution. Then with h as in (16), we have $x^G \cap H^\circ\pi_h \neq \emptyset$ and $x^G \cap H^\circ\pi_i = \emptyset$ for all $i > h$. In the case n/t is even is easy to understand $\max_i\{\dim(x^G \cap H^\circ\pi_i)\}$. We have the following.

Proposition 7.2. *Let $x = [I_{n-s}, -I_s]$ and assume n/t even. Then* $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$

Proof. Let $i < h$ then $\dim(x^G \cap H^\circ\pi_{i+1}) = \dim([I_{n/t}, \dots, I_{n/t}, x_{2i+3}, \dots, x_t]\pi_{i+1})^{H^\circ}$, where $x_j \in \mathrm{GL}_{n/t}$ for all $j \in \{2i+3, \dots, t\}$ and the eigenvalues inside these blocks are organized to maximized the H° dimension of the element. Therefore, using Lemma 2.19, we have

$$\dim(x^G \cap H^\circ\pi_i) \geq \dim([I_{n/t}, \dots, I_{n/t}, \bar{z}, \bar{z}, x_{2i+3}, \dots, x_t]\pi_i)^{H^\circ} = \dim(x^G \cap H^\circ\pi_{i+1})$$

where $\bar{z} = [I_{\frac{n}{2t}}, -I_{\frac{n}{2t}}]$. And a straightforward computation shows the last equality since $2\dim(\bar{z})^{\mathrm{GL}_{n/t}} = (n/t)^2$. The result follows. *q.e.d.*

Therefore in the case n/t even we have that for any semisimple involutions, $f_\Omega(x) = f^\circ_\Omega(x)$, and the formula is given in (73).

Let us study, now, the case n/t odd. We have the following.

Proposition 7.3. *Let $x = [I_{n-s}, -I_s]$ where $0 < s \leq n/2$. Assume n/t odd. Then, let $k = \frac{t-n+2s}{2}$,*

$$\dim(x^G \cap H) = \begin{cases} \dim(x^G \cap H^\circ\pi_k) & \text{if } s \geq \max\{\frac{n}{t}, \frac{n-t}{2}\} \\ \dim(x^G \cap H^\circ) & \text{otherwise} \end{cases}$$

Proof. If $s < n/t$ then by Lemma 2.19 the h defined in (16) is 0. Hence we may assume $s \geq n/t$.

In general if $\dim(x^G \cap H^\circ \pi_{i+1}) = \dim([I_{n/t}, \dots, I_{n/t}, x_{2i+3}, \dots, x_t]\pi_{i+1})^{H^\circ}$, then $\dim(x^G \cap H^\circ \pi_i)$ will be equal to the H°-dimension of a certain element $[I_{n/t}, \dots, I_{n/t}, y_{2i+1}, \dots, y_t]\pi_i$ where, by Lemma 2.19, the multiplicity of the 1-eigenvalue in the blocks y_i is the multiplicity of 1 in the blocks x_i plus n/t, the same for the -1-eigenvalue. The best disposition of these $2n/t$ eigenvalue in two blocks is given by $\bar{x}, \bar{x}'$, thanks to Corollary 2.12, where

$$\bar{x} = [I_{\frac{n/t+1}{2}}, I_{\frac{n/t-1}{2}}], \qquad \bar{x}' = [I_{\frac{n/t-1}{2}}, I_{\frac{n/t+1}{2}}] \tag{74}$$

and $\dim \bar{x}^{\mathrm{GL}_{n/t}} + \dim \bar{x}'^{\mathrm{GL}_{n/t}} = (n/t)^2 - 1$. Therefore let $x = [x_1, \dots, x_t]$ where $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, assume $h \geq 1$ for x and $x_1 = \bar{x}, x_2 = \bar{x}'$, then $\dim(x^G \cap H^\circ \pi_1) = \dim(x^G \cap H^\circ) + 1$ because by Lemma 2.16, 2.19 we have

$$\begin{aligned} \dim(x^G \cap H^\circ \pi_1) = \dim([I_{n/t}, I_{n/t}, x_3, \dots, x_t]\pi_1)^{H^\circ} &= \left(\frac{n}{t}\right)^2 + \sum_{i=3}^{t} \dim x_i^{\mathrm{GL}_{n/t}} \\ &= \dim \bar{x}^{\mathrm{GL}_{n/t}} + \dim \bar{x}'^{\mathrm{GL}_{n/t}} + 1 + \sum_{i=3}^{3} \dim x_i^{\mathrm{GL}_{n/t}} \end{aligned}$$

the first equality is given by the assumption $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, in fact we can not increase the dimension of $\sum_{i=3}^t \dim x_i^{\mathrm{GL}_{n/t}}$ without increasing, at the same time the dimension of $\dim x^{H^\circ}$.

Given $x = [I_{n-s}, -I_s]$ in the block form $[x_1, \dots, x_t]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ we call l the number of blocks equal to $[I_{\frac{n/t+1}{2}}, -I_{\frac{n/t-1}{2}}]$ and k the number of blocks equal to $[I_{\frac{n/t-1}{2}}, -I_{\frac{n/t+1}{2}}]$. We have

$$\begin{aligned} n - s &= \frac{n}{t}\left(\frac{l+k}{2}\right) + \frac{l-k}{2} \\ s &= \frac{n}{t}\left(\frac{l+k}{2}\right) - \frac{l-k}{2} \end{aligned}$$

Hence, summing the two equations, we get $l + k = t$. Thus if in x there is the block $\bar{x}'$ then all the other blocks are either $\bar{x}$ or $\bar{x}'$. From the last two equations we get

$$\begin{aligned} l &= \frac{t + (n-s) - s}{2} \\ k &= \frac{t - (n-s) + s}{2} \end{aligned}$$

In general, $l \geq k$ since $s \leq n/2$. It is clear, by the previous analysis, that for $k \geq 0$, i.e. $s \geq (n-t)/2$, we have

$$\dim(x^G \cap H^\circ \pi_k) = \max\{\dim(x^G \cap H^\circ \pi_i) : i = 0, \dots, k\}$$

It remains to show that, in fact, $\dim(x^G \cap H^\circ \pi_k) \geq \dim(x^G \cap H^\circ \pi_i)$ for $i > k$.
Let us observe that we can regard $\dim(x^G \cap H^\circ \pi_i) - \dim(x^G \cap H^\circ \pi_{i+1})$, for all $i \in 1, \dots, h-1$, as $\dim(\bar{x}^L \cap K^\circ) - \dim(\bar{x}^L \cap K^\circ \pi_1)$ where $L = \mathrm{GL}_m(k)$ and K a $\mathcal{C}_2$-subgroup of L, for a suitable m, that is $m = (n/t)(t - 2i)$. Therefore, in order to complete the proof it is enough to show that for $n/t \leq s < (n-t)/2$ we have $\dim(x^G \cap H^\circ) \geq \dim(x^G \cap H^\circ \pi_1)$. Here the assumption $s > n/t$ is equivalent to require $x^G \cap H^\circ \pi_1 \neq \emptyset$, similarly, $s < (n-t)/2$ correspond to the fact that x does not contain blocks $\bar{x}'$ when we write it in block form $x = [x_1, \dots, x_t]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$.

Thus, assume $x = [I_{n-s}, I_s]$ where $s < n/2$, as usual, and $n/t \leq s < (n-t)/2$, with n/t odd. Let $x = [x_1, \dots, x_t]$ such that $x_i \in \mathrm{GL}_{n/t}$ and $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Assume, moreover, $x_i \neq \bar{x}'$, as in (74). Then by Corollary 2.12, $x = [\bar{y}, \ldots, \bar{y}, \bar{y}', \ldots, \bar{y}']$ where

$$\bar{y} = \Big[I_{\frac{\frac{n}{t}+a}{2}}, -I_{\frac{\frac{n}{t}-a}{2}}\Big], \qquad \bar{y}' = \Big[I_{\frac{\frac{n}{t}+a-2}{2}}, -I_{\frac{\frac{n}{t}-a+2}{2}}\Big]$$

for some odd integer $a > 1$, because if $a = 1$ then $\bar{y}' = \bar{x}'$. Calling l the number of $\bar{y}$ and k the number of $\bar{y}'$ we get

$$\begin{cases} l = \frac{t(2-a)+(n-s)-s}{2} \\ k = \frac{at-(n-s)+s}{2} \end{cases}$$

Therefore

$$\begin{aligned}\dim(x^G \cap H^\circ) &= l \dim \bar{y} + k \dim \bar{y}' \\ &= \frac{a^2 t}{2} - an + 2as - at + \frac{n^2}{2t} + n - 2s\end{aligned}$$

Using the formula of Theorem 4.1 we find $a = \frac{n}{t} - 2\lfloor \frac{s}{t} \rfloor$.

Then, using Lemma 2.16, we have $[I_{2n/t}, x_3, \ldots, x_t]\pi_1 \in x^G \cap H^\circ \pi_1$, where, as above $x_i \in \{\bar{z}, \bar{z}'\}$ and

$$\bar{z} = \Big[I_{\frac{\frac{n}{t}+a'}{2}}, -I_{\frac{\frac{n}{t}-a'}{2}}\Big], \qquad \bar{z}' = \Big[I_{\frac{\frac{n}{t}+a'-2}{2}}, -I_{\frac{\frac{n}{t}-a'+2}{2}}\Big]$$

where $a' = \frac{n}{t} - 2\lfloor \frac{s'}{t'} \rfloor$ and we denote $s' = s - n/t, t' = t-2$ and $n' = (n/t)(t-2)$. Moreover, thanks to Corollary 2.12, we have

$$\dim(x^G \cap H^\circ \pi_1) = \dim([I_{2n/t}, x_3, \ldots, x_t]\pi_1)^{H^\circ} = \Big(\frac{n}{t}\Big)^2 + \sum_{i=3}^{t} \dim x_i^{\mathrm{GL}_{n/t}}$$

Fixing this block decomposition we have that

$$\dim(x^G \cap H^\circ) > \dim([\bar{x}, \bar{x}', x_3, \ldots, x_t]) = \dim(x^G \cap H^\circ \pi_1) - 1$$

where $\bar{x}, \bar{x}'$ are defined in (74). We have a strict inequality because among the blocks $x_3, \ldots, x_t$ the is at least one of them different from $\bar{x}, \bar{x}'$, hence thanks Corollary 2.12 we can (strictly) increase the H° dimension of $[\bar{x}, \bar{x}', x_3, \ldots, x_t]$.

Therefore in the case $n/t \leq s \leq (n-t)/2$ we have $\dim(x^G \cap H^\circ) \geq \dim(x^G \cap H^\circ \pi_i)$ for all $i \in \{1, \ldots, h\}$. The result follows. *q.e.d.*

Example 7.4. Assume n even and n/t odd. Let $x = [I_{n/2}, -I_{n/2}]$, observe that with our assumption t is even, hence, by Lemma 2.19, x is G-conjugate to π_h for $h = t/2$. Then thanks to Corollary 2.12 we have $x = [x_1, \ldots, x_t]$ where $t/2$ blocks are equal to $\bar{x} = [I_{\frac{n/t-1}{2}}, -I_{\frac{n/t+1}{2}}]$ and the remaining $t/2$ are $\bar{x}' = [I_{\frac{n/t-1}{2}}, -I_{\frac{n/t+1}{2}}]$. Hence $l = k = t/2$; moreover h, as defined in (16), is $t/2$. Applying the formula of Theorem 4.1, we have

$$\dim(x^G \cap H^\circ) = \frac{n^2}{2t} - \frac{t}{2}$$

As proved above the dimension increase until $\dim(x^G \cap H^\circ \pi_l)$ where l is the number of blocks equal to $\bar{x}'$ in x. And $\dim(x^G \cap H^\circ) = \dim(x^G \cap H^\circ \pi_h) = \dim(\pi_h)^{H^\circ} = \frac{n^2}{2t}$. Therefore $f_\Omega(x) = \frac{1}{2}$.

Let us compute $\dim(x^G \cap H)$ in these different cases.

Let $x = [I_{n-s}, -I_s]$ with $\max\{n/t, (n-t)/2\} \leq s \leq n/2$. Then, by Proposition 7.3, in the same notation, we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_k) = \dim([I_{n/t}, \ldots, I_{n/t}, \bar{x}, \ldots, \bar{x}]\pi_k)^{H^\circ}$. A straightforward computation leads to

$$\dim(x^G \cap H^\circ \pi_l) = \frac{n^2}{2t} - \frac{n-2s}{2}$$

from which we deduce, using Proposition 2.2,

$$f_\Omega(x) = \frac{(n-s)^2 + s(s+1) - \frac{n}{2}\left(1+\frac{n}{t}\right)}{n^2 - \frac{n^2}{t}} \tag{75}$$

that holds for any semisimple involution with $k \geq 1$.

In the case $s < \max\{n/t, (n-t)/2\}$, by Proposition 7.3, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$ and, as computed in (73), we have

$$f_\Omega(x) = f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

where $s = at + b$ and $0 \leq b < t-1$.

Therefore Theorem 4 has been proved in the case $p \neq 2$.

7.2. **Symplectic group.** Assume $G = \mathrm{Sp}_{2n}$ and the characteristic of the field k is $p \neq 2$. We aim to prove Theorem 5. First we study some particular cases, i.e. $t \in \{2, n/2, n\}$. Then we look at the case $2 < t < n/2$ and we complete the proof of the theorem. In this case we do not have an explicit formula, but we shall find a function $g(s,t)$ such that for any involution $x \in H$ we have

$$f_\Omega^\circ(x) \leq f_\Omega(x) \leq f_\Omega^\circ(x) + g(n,t)$$

Given $x = [I_{2(n-s)}, -I_{2s}]$, a straightforward application of Proposition 2.2 and Theorem 4.1 leads to

$$f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

where $s = at + b$ and $0 \leq b < t-1$.

Remark 7.5. To avoid confusion let us observe that given $x = [I_{2(n-s)}, -I_{2s}]$ we have $\nu(x) = 2s$.

7.2.1. *Case $t = n$.* In this case $H = \mathrm{Sp}_2 \wr S_n$ and $\dim \Omega = 2n(n-1)$. Let us observe that $\mathrm{Sp}_2 \cong \mathrm{SL}_2$, it is enough to show that given any 2×2 matrix, say it x, of determinant 1 we get $(x.e_1, x, e_1) = (x.f_1, x.f_1) = 0$ and $(x.e_1, x.f_1) = 1$.

Let us consider the involution $x = [-I_{2s}, I_{2(n-s)}]$, we may assume $0 < s \leq \lfloor n/2 \rfloor$. Then, as in Table 6, we have $\dim x^G = 4s(n-s)$. And, since π_i is G-conjugate to $[I_{2n-2i\frac{n}{t}}, -I_{2i\frac{n}{t}}] = [I_{2n-2i}, -I_{2i}]$, we have that x is G conjugate to π_s. It is easy to see that $\dim(x^G \cap H^\circ \pi_i) = 2i$ for all $i \in \{0, \ldots, s\}$. Therefore $\dim(x^G \cap H) = 2s$. Hence

$$f_\Omega(x) = 1 - \frac{s(2n-2s-1)}{n(n-1)} \tag{76}$$

Deriviting the above formula with respect to s we see that it is decreasing in the interval $[1, \frac{2n-1}{4}]$, a straightforward computation shows that it is maximal for $s = 1$ and minimal when $s = \left\lfloor \frac{n}{2} \right\rfloor$. Therefore we get the following.

Corollary 7.6. *Let x be an involutions in $H = \mathrm{Sp}_2 \wr S_n$. Then*

$$\frac{1}{2} \leq f_\Omega(x) \leq 1 - \frac{2}{n} + \frac{1}{n(n-1)}$$

7.2.2. *Case $t = \frac{n}{2}$.* Let $x \in G$.

Proposition 7.7. *Let $x = [I_{2(n-s)}, -I_{2s}] \in H$ then*

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$$

where h is as in (16).

Proof. Let us remember that π_i is G-conjugate to $[I_{2(n-2i)}, I_{4i}]$. We have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ where $x = [x_1, \ldots, x_t]$ and $x_1 = \ldots = x_s = [I_2, -I_2]$ and $x_i = I_4$ for $i > s$. And $\dim(x^G \cap H^\circ \pi_i) = i \dim \mathrm{Sp}_4 + (s-2i)\dim[I_2, -I_2]^{\mathrm{Sp}_4} = 4s + 2i$. Therefore $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h) = 4s + 2h$. *q.e.d.*

Thanks to the previous result we have

$$\dim(x^G \cap H) = \begin{cases} 5s & s \text{ even} \\ 5s-1 & s \text{ odd} \end{cases}$$

Therefore

$$f_\Omega(x) = 1 - \frac{4s(n-s) + 5s - \iota_{2,s}}{2n(n-2)}$$

where $\iota_{2,s} = 0$ if s is even and 0 otherwise.

7.2.3. *Case $t = 2$.* Let us observe that $h = 1$ if, and only if, $s = \frac{n}{2}$. We have the following.

Proposition 7.8. *Let $x = [I_{2(n-s)}, -I_{2s}] \in H$. Then*

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$$

where h is as in (16).

Proof. If $s < n/2$ there is nothing to prove since $x^G \cap H = x^G \cap H^\circ$. We may assume $s = n/2$, therefore $x = [I_n, -I_n]$, which is G-conjugate to π_1. Using Theorem 4.1 we get

$$\dim(x^G \cap H^\circ) = \begin{cases} \frac{n^2}{2} & \frac{n}{2} \text{ even} \\ \frac{n^2}{2} - 2 & \frac{n}{2} \text{ odd} \end{cases}$$

and $\dim(x^G \cap H^\circ \pi_1) = \dim \pi_1^{H^\circ} = \frac{n^2}{2} + \frac{n}{2}$. The result follows. *q.e.d.*

Therefore for $x = [I_n, -I_n]$ we have

$$f_\Omega(x) = \frac{1}{2} + \frac{1}{2n}$$

For $x = [I_{2(n-s)}, -I_{2s}]$ with $s < \frac{n}{2}$ we have

$$f_\Omega(x) = f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

7.2.4. *General case, $2 < t < \frac{n}{2}$.* Let $x = [I_{2(n-s)}, -I_{2s}] \in H$ be a semisimple involution, where $0 < s \leq \frac{n}{2}$. In order to give a formula for $f_\Omega(x)$ we need to find $i \in \{0, \ldots, h\}$ such that $\dim(x_G \cap H) = \dim(x^G \cap H^\circ \pi_i)$. We divide the analysis in two cases, depending on the parity of $\frac{n}{t}$.

Let us remind that $\pi_i = (1,2) \in S_t$ is G-conjugate to $[I_{2(n-\frac{n}{t})}, -I_{2\frac{n}{t}}]$ and, in general π_i is G-conjugate to $[I_{2(n-\frac{n}{t}i)}, -I_{\frac{n}{t}i}]$.

Lemma 7.9. *Let $x = [I_{2s}, -I_{2(n-s)}] \in H$. Then*

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{n}{2} + t$$

Proof. If $x^G \cap H = x^G \cap H^\circ$ the result trivially follows.

We may assume $x^G \cap H^\circ\pi_1 \neq \emptyset$. Hence, by Lemma 2.16 we have $x = [I_{2\frac{n}{t}}, I_{2\frac{n}{t}}, x_3, \dots, x_t]\pi_1$ such that $\dim(x^G \cap H^\circ\pi) = \dim \mathrm{Sp}_{2\frac{n}{t}} + \sum_{i=3}^t \dim x_i^{\mathrm{Sp}}$.

If $\frac{n}{t}$ is even, we have, thanks to Lemma 2.19, Remark 4.5 and Corollary 4.7,

$$\dim(x^G \cap H^\circ) \begin{cases} = \dim([\bar{z}, \bar{z}, x_3, \dots, x_t])^{H^\circ} & \star \\ \geq \dim([\bar{z}, \bar{z}, x_3, \dots, x_t])^{H^\circ} + 8 & \text{otherwise} \end{cases}$$

where with $(\star)$ we mean one of the following two condition to hold:

- there exists i such that $x_i = \bar{z}$;
- $x_3 = \ldots = x_t = [I_{\frac{n}{t}-2}, -I_{\frac{n}{t}+2}]$.

Therefore in case $(\star)$ is satisfied

$$\dim(x^G \cap H^\circ\pi_1) = \dim(x^G \cap H^\circ) + \frac{n}{t}$$

otherwise

$$\dim(x^G \cap H^\circ\pi_1) \leq \dim(x^G \cap H^\circ) + \frac{n}{t} - 8$$

Assume, now, $\frac{n}{t}$ odd. Let $\bar{z} = [I_{\frac{n}{t}+1}, -I_{\frac{n}{t}-1}] = -\bar{z}'$. Then, as above,

$$\dim(x^G \cap H^\circ) \begin{cases} = \dim([\bar{z}, \bar{z}', x_3, \dots, x_t])^{H^\circ} & \forall i : x_i \in \{\bar{z}, \bar{z}'\} \\ \geq \dim([\bar{z}, \bar{z}', x_3, \dots, x_t])^{H^\circ} + 8 & \text{otherwise} \end{cases}$$

Again, we either have

$$\dim(x^G \cap H^\circ\pi_1) = \dim(x^G \cap H^\circ) + \frac{n}{t} + 2$$

or,

$$\dim(x^G \cap H^\circ\pi_1) \leq \dim(x^G \cap H^\circ) + \frac{n}{t} - 6$$

Therefore we get $\dim(x^G \cap H^\circ\pi_1) \leq \dim(x^G \cap H^\circ) + \frac{n}{t} + 2$. Iterating this process we end up with

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi_h) \leq \dim(x^G \cap H^\circ) + h\Big(\frac{n}{t} + 2\Big)$$

for some $h \leq \lfloor \frac{t}{2} \rfloor$. The result follows. *q.e.d.*

Remark 7.10. Indeed we can give better bound, than the one of Lemma 7.9. In fact, for $\frac{n}{t}$ is even, if $x \notin F = \{[I_n, -I_n], [I_{n-2t+4}, -I_{n+2t-4}]\}$ we have $\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + k(\frac{n}{t} - 8)$, for a suitable $k \leq \lfloor \frac{t}{2} \rfloor$.

Similarly for $\frac{n}{t}$ odd. If $x \neq \bar{x}_l = [I_{n+2l+2-t}, -I_{n-2l-2+t}]$ for any $l \in \{1, \dots, t-2\}$, we have $\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + k(\frac{n}{t} - 6)$, for a suitable $k \leq \lfloor \frac{t}{2} \rfloor$.

Hence for $x \neq [I_n, -I_n]$ and, $[I_{n-2t+4}, -I_{n+2t-4}]$ for $\frac{n}{t}$ even, we have

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{t}{2}\Big(\frac{n}{t} - 6\Big) \leq \dim(x^G \cap H^\circ) + \frac{n}{2} - 3t$$

The following is an easy consequence of Lemma 7.9 and Remark 7.10.

Proposition 7.11. *Let $x = [I_{2(n-s)}, -I_{2s}] \in H$. Assume $x \notin F$ for $\frac{n}{t}$ even and, for $\frac{n}{t}$ odd $x \neq \bar{x}_l$. Then*

$$f_\Omega(x) \leq f_\Omega^\circ(x) \frac{1}{4n^2(t-1)}\Big(\frac{n}{t} - 6\Big)$$

Namely, if $n/t \leq 6$, $f_\Omega(x) = f_\Omega^\circ(x)$.

Proof. Thanks to Remark 7.10 we have

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{t}{2}\Big(\frac{n}{t} - 6\Big) \leq \dim(x^G \cap H^\circ) + \frac{n}{2} - 3t$$

applying the formula of $\dim C_\Omega(x)$ in Proposition 2.2, we get the result. *q.e.d.*

With the following we complete, in full generality, the proof of Theorem 5. We use the notation introduced in Remark 7.10.

Proposition 7.12. *Let* $x = [I_{2(n-s)}, -I_{2s}]$. *Then*

$$f_\Omega(x) \leq f_\Omega^\circ(x) + \frac{t^2}{4n^2(t-1)}\Big(\frac{n}{t} + 2\Big)$$

Proof. By Lemma 7.9 we have

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{n}{2} + t$$

Hence, using Proposition 2.2, we get the result. *q.e.d.*

8. Unipotent involutions

The aim of this section is to prove Theorem 4 for involutions in the case the characteristic of the field is even.

We start with the case $G = \mathrm{GL}_n$. Then for the symplectic group we shall prove a formula for computing $\dim(x^G \cap H)$ when $x \in H$ is a unipotent involution, thus we give a formula for f_Ω°.

8.1. **General linear group.** Assume $p = 2$ and let $s \in \{1, \dots, \lfloor n/2 \rfloor\}$. Then the only unipotent element in $\mathcal{V}_{s,2}$, up to conjugation, is $x = [J_2^s, J_1^{n-2s}]$.

In the particular case $t = n$ we can easily give a formula for $f_\Omega(x)$. Let us observe that, in this case, no unipotent involutions lie in $H^\circ = (k^*)^n$.

Lemma 8.1. *Let $t = n$ then conclusion of Theorem 4 holds.*

Proof. If $t = n$ the the only involutions in H are the elements of order 2 in S_t, i.e. π_s for all $s = 1, \dots, \lfloor t/2 \rfloor$. Using Lemma 2.17 we may write $\pi_s = [J_2^{\frac{n}{t}s}, J_1^{\frac{n}{t}(t-2s)}] = [J_2^s, J_1^{n-2s}]$, moreover $\dim(\pi_s^G \cap H) = \dim \pi_s^{H^\circ} = \left(\frac{n}{t}\right)^2 s = s$, and $\dim \pi_s^G = 2s(n-s)$. Therefore

$$f_\Omega(\pi_i) = 1 - \frac{s(2n-2s-1)}{n^2 - n}$$

q.e.d.

Thus we may assume $t < n$. We shall prove the following.

Proposition 8.2. *Let $x = [J_2^s, J_1^{n-2s}] \in H$. Then*

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_{h'})$$

where $h' = \min\{i : x^G \cap H^\circ \pi_i \neq \emptyset\}$, and we denote $\pi_0 = 1$. Moreover $h' = 0$ if n/t even and $h' = \max\{0, \frac{t-n+2s}{2}\}$ if n/t odd.

As for semisimple involutions, two natural cases arise, either n/t is even or it is odd. In the first case, Proposition 8.2 is an easy consequence of the following.

Lemma 8.3. *Assume n/t even and $x = [J_2^s, J_1^{n-2s}] \in H$. Then for every $i < h$, where h is defined in (16) we have*

$$\dim(x^G \cap H^\circ \pi_i) - \dim(x^G \cap H^\circ \pi_{i+1}) \geq 0$$

Proof. By Lemma 2.16 we have $[I_{n/t}, \dots, I_{n/t}, J_2^s, J_1^{(n/t)(i+1)-2s}]$, and the number of $I_{n/t}$ is $2(i+1)$. We may assume that

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_{i+1}) &= \dim([I_{2(n/t)(i+1)}, x_{2i+3}, \dots, x_t]\pi_{i+1})^{H^\circ} \\ &= \left(\frac{n}{t}\right)^2 (i+1) + \dim x_{2i+3}^{\mathrm{GL}_{n/t}} + \dots + \dim x_t^{\mathrm{GL}_{n/t}}\end{aligned}$$

Therefore, using Lemma 2.17 and said $\bar{x} = [J_2^{n/2t}]$, we have

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_i) &\geq \dim([I_{n/t}, \dots, I_{n/t}, \bar{x}, \bar{x}, x_{2i+3}, \dots, x_t]\pi_i)^{H^\circ} \\ &= \dim(x^G \cap H^\circ \pi_{i+1})\end{aligned}$$

where the last equality is given by $2 \dim \bar{x}^{\mathrm{GL}_{n/t}} = (n/t)^2$. *q.e.d.*

Assume n/t odd. If $x = [J_2^s, J_1^{n-2s}]$ then $x^G \cap H^\circ \neq \emptyset$ if, and only if, $n - 2s \geq t$, since we need at least a block J_1 in each block in any decomposition $x = [x_1, \dots, x_t] \in H^\circ$. With the following remark we extend this argument to any intersection $x^G \cap H^\circ \pi_i$.

Remark 8.4. Let $x = [J_2^s, J_1^{n-2s}] \in H$. Assume for $i \leq h$, where h is as in (16), $x^G \cap H^\circ\pi_i \neq \emptyset$. Then for all $i \leq j \leq h$, $x^G \cap H^\circ\pi_j \neq \emptyset$. By Lemma 2.16 we have $[I_{n/t}, \ldots, I_{n/t}, J_2^{s-ni/t}, J_1^{n-2s}] \in x^G \cap H^\circ\pi_i$ therefore $[J_2^{s-ni/t}, J_1^{n-2s}] \in \mathrm{GL}_{n/t}(k)^{t-2i}$, which is true also because $n-2s \geq t-2i$. Similarly, $[I_{n/t}, \ldots, I_{n/t}; J_2^{s-n(i+1)/t}, J_1^{n-2s}] \in x^G \cap H^\circ\pi_{i+1}$ since $n-2s \geq t-2i > t-2(i+1)$.

With the following we complete the proof of Proposition 8.2.

Lemma 8.5. *Assume n/t odd and $x = [J_2^s, J_1^{n-2s}] \in H$. Let $i < h$ such that $x^G \cap H^\circ\pi_i \neq \emptyset$. Then* $\dim(x^G \cap H^\circ\pi_i) \geq \dim(x^G \cap H^\circ\pi_{i+1})$.

Proof. We may assume there exists $x_{2i+3}, \ldots, x_t$ unipotent in $\mathrm{GL}_{n/t}$, say $x_j = [J_2^{a_j}, J_1^{n/t-2a_j}]$, such that

$$\dim(x^G \cap H^\circ\pi_{i+1}) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{2i+1}, \ldots, x_t]\pi_{i+1})^{H^\circ}$$

Let us observe that, since n/t is odd $n/t - 2a_j$ is an odd integer.

Suppose for all $j \in \{2i+3, \ldots, t\}$ we have $n/t - 2a_j = 1$, then, clearly, $x^G \cap H^\circ\pi_i = \emptyset$ thanks to Lemma 2.17.

Therefore we may assume, without loss of generality, that $n/t - 2a_{2i+3} \geq 3$. Say $\bar{x} = [J_2^{(n/t-1)/2}, J_1]$ and $\bar{x}_{2i+3} = [J_2^{a_{2i+3}}, J_1^{n/t-2a_{2i+3}-2}]$, hence

$$\dim(x^G \cap H^\circ\pi_i) \geq \dim([I_{n/t}, \ldots, I_{n/t}, \bar{x}, \bar{x}, x_{2i+3}, x_{2i+4}, \ldots, x_t])^{H^\circ}$$

Therefore, say $a = a_{2i+3}$ for a lighter notation,

$$\begin{aligned}
\dim(x^G \cap H^\circ\pi_i) - \dim(x^G \cap H^\circ\pi_{i+1}) \geq & \left(\frac{n}{t}\right)^2 i + 2\dim\bar{x} + \dim x_{2i+3} + \dim x_{2i+4} + \ldots + \dim x_t \\
& - \left(\frac{n}{t}\right)^2 (i+1) - \dim x_{2i+3} - \ldots - \dim x_t \\
& = 2\dim\bar{x} + \dim\bar{x}_{2i+3} - \left(\frac{n}{t}\right)^2 - \dim x_{2i+3} \\
& = 2\Big[\frac{1}{2}\Big(\frac{n}{t} - 1\Big)\Big(\frac{n}{t} + 1\Big)\Big] + 2(a+1)\Big(\frac{n}{t} - a - 1\Big) \\
& - \left(\frac{n}{t}\right)^2 - 2a\Big(\frac{n}{t} - a\Big) \\
& = 2\frac{n}{t} - 4a - 3
\end{aligned}$$

Since we are assuming $n/t - 2a \geq 3$ we have $-2a \geq 3 - n/t$. Therefore $2n/t - 4a - 3 \geq 2n/t - 3 + 6 - 2n/t = 3$. *q.e.d.*

Remark 8.6. Let $x = [J_2^s, J_1^{n-2s}]$. Then, as already observed above, $x^G \cap H^\circ \neq \emptyset$ if $n - 2s \geq t$, i.e. $s \leq (n-t)/2$. Assume $s > (n-t)/2$. Then the indexes i such that $x^G \cap H^\circ\pi_i \neq \emptyset$ have the property $n - 2s \geq t - 2i$. Therefore the index l such that $x^G \cap H^\circ\pi_{l-1} = \emptyset$ and $x^G \cap H^\circ\pi_l \neq \emptyset$ is $l = (t-n+2s)/2$. Therefore thanks to Lemma 8.5, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi_l)$.

Hence Lemma 8.3, 8.5 and Remark 8.6 give a complete proof of Proposition 8.2. The last step in the proof of Theorem 4 is to compute an explicit formula for $f_\Omega(x)$.

Let us compute $\dim(x^G \cap H)$ for the element $x = [J_2^s, J_1^{n-2s}]$. In the case n/t even, or n/t odd and $n-2s \geq t$ we have, by Proposition 8.2, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$ and

$$\begin{aligned}\dim(x^G \cap H^\circ) &= b\dim[J_2^{\lfloor s/t\rfloor+1}, J_1^{n/t-2\lfloor s/t\rfloor-2}] + (t-b)\dim[J_2^{\lfloor s/t\rfloor}, J_1^{n/t-2\lfloor s/t\rfloor}]\\ &= 2b(a+1)\Big(\frac{n}{t}-a-1\Big) + 2a(t-b)\Big(\frac{n}{t}-a\Big)\\ &= \frac{2s(n-s)}{t} - \frac{2b(t-b)}{t}\end{aligned}$$

where we write $s = at+b$ and $0 \leq b < t$. Let us observe, in view of Proposition 2.10, that $\frac{1}{t}\dim x^G - \dim(x^G \cap H) \leq \frac{t}{2}$.

In the case, $s > (n-t)/2$, i.e. $l = (t-n+2s)/2 > 0$, we have

$$\begin{aligned}\dim(x^G \cap H^\circ\pi_l) &= \dim\Big([I_{n/t}, \dots, I_{n/t}, J_2^{s-nl/t}, J_1^{n-2s}]\pi_l\Big)^{H^\circ}\\ &= \Big(\frac{n}{t}\Big)^2 + (t-2l)\dim[J_2^{(n/t-1)/2}, J_1]^{\mathrm{GL}_{n/t}}\\ &= \frac{n^2}{2t} - \frac{n-2s}{2}\end{aligned}$$

Therefore, for $x = [J_2^s, J_1^{n-2s}]$ we have

$$\dim(x^G \cap H) = \begin{cases} \frac{n^2}{2t} - \frac{n-2s}{2} & \frac{n}{t} \text{ odd and } \frac{n}{t} \leq s < \frac{n-t}{2}\\ \frac{2s(n-s)}{t} - \frac{2b(t-b)}{t} & \text{otherwise}\end{cases} \tag{77}$$

Therefore, using Propoposition 2.2 and the formula (77) with the fact $\dim x^G = 2s(n-2)$, we have a complete proof of Theorem 4.

8.2. **Symplectic group.** As explained in Section 2, given $x = [J_2^s, J_1^{2(n-s)}] \in G$ if s is even there are precisely two conjugacy classes of involutions with the Jordan form as x in G, called a_s and c_s-type involutions. Given any involution $x \in H^\circ$, and assume $x = [x_1, \dots, x_t]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, if x is of a_s type, we must have all the x_i's to be of a_{s_i} type, for suitable s_i's even. If $x \in H^\circ$ is of type b_s or c_s, assume, as above, $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, then each x_i is either b_{s_i} or c_{s_i}-type; we can not have any a_{s_i} type instead of the c_{s_i} since

$$\dim a_{s_i}^{\mathrm{Sp}_{2\frac{n}{t}}} < \dim c_{s_i}^{\mathrm{Sp}_{2\frac{n}{t}}}$$

We start giving the following.

Proposition 8.7. *Let $x = [x_1, \dots, x_t] \in H$ be an involution such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, say $x_i = [J_2^{s_i}, J_1^{2(n-s_i)}]$. Then*

(i) *If x is a_s-type, then $|s_i - s_j| \leq 2$ for all $i, j \in \{1, \dots, t\}$;*
(ii) *If x is b_s or c_s-type, then $|s_i - s_j| \leq 1$ for all $i, j \in \{1, \dots, t\}$.*

Proof. We shall prove the contrapositive in both cases.

(i) Assume x is an a_s type involution. Assume $s_1 - s_2 > 2$, since they are both even we have $s_1 - s_2 = 4 + 2h$ for some $h \geq 0$. Let y_1 be of type a_{s_1-2} and y_2 of type a_{s_2+2}. Then $y = [y_1, y_2, x_3, \dots, x_t] \in x^G$ and we have $\dim y^{H^\circ} - \dim x^{H^\circ} = 4(s_1 - s_2 - 2) = 4(2+2h) > 8$. Hence $\dim(x^G \cap H^\circ) < \dim y^{H^\circ}$.
(ii) Assume $s_1 - s_2 \geq 2$, say $s_1 - s_2 = 2+h$, for some $h \geq 0$. Let $y_1 = [J_2^{s_1-1}, J_1^{2(\frac{n}{t}-s_1+1)}]$ and $y_2 = [J_2^{s_2+1}, J_1^{2(\frac{n}{t}-s_2-1)}]$. Then $y = [y_1, y_2, x_3, \dots, x_t] \in x^G$. And $\dim y^{H^\circ} - \dim x^{H^\circ} = 2(s_1 - s_2 - 1) = 2(h+1) \geq 2$. Therefore $\dim(x^G \cap H^\circ) < \dim y^{H^\circ}$.

q.e.d.

8.2.1. *a_s-type.* Let x be an a_s type, the Jordan form of x is $[J_2^s, J_1^{2n-2s}]_a$ and s is even. And we have, writing $\frac{s}{2} = \lfloor \frac{s}{t} \rfloor t + b$, with $0 \le b < t$, and using Proposition 8.7

$$\dim x^G = s(2n-s)$$

$$\dim(a_s^G \cap H^\circ) = \frac{s(2n-s)}{t} - \frac{4b(t-b)}{t}$$

where we use $\dim(a_s^G \cap H^\circ) = b \cdot \dim a_{2\lfloor \frac{s}{t} \rfloor}^{\mathrm{Sp}} + (t-b) \cdot \dim a_{2\lfloor \frac{s}{t} \rfloor + 2}^{\mathrm{Sp}}$. Therefore

$$f_\Omega^\circ(x) = 1 - \frac{s(2n-s)}{2n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

Thus we have proved the following.

Proposition 8.8. *Let $x \in H$ be an a_s-type involution. Then*

$$f_\Omega^\circ(x) = 1 - \frac{s(2n-s)}{2n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

8.2.2. *b_s, c_s-types.* In this section we deal with involution of G which are not a_s-type. Therefore we have, for $x = [J_2^s, J_1^{2(n-s)}]$, $\dim x^G = s(2n-s+1)$. As said above in the block decomposition to get $\dim(x^G \cap H^\circ)$ there are no a-type involutions, since $\dim a_{s_i}^{\mathrm{Sp}_{2\frac{n}{t}}} < \dim b_{s_i}^{\mathrm{Sp}_{2\frac{n}{t}}}$. Moreover by Proposition 8.7 we have

$$\dim(x^G \cap H^\circ) = b \dim[J_2^{\lfloor s/t \rfloor + 1}, J_1^{\cdot}] + (t-b) \dim[J_2^{\lfloor s/t \rfloor}, J_1^{\cdot}] = \frac{s(2n-s)}{t} - \frac{b(t-b)}{t} + s$$

Thus we deduce the following.

Proposition 8.9. *Let $x \in H$ be a b_s or c_s-type involution. Then*

$$f_\Omega^\circ(x) = 1 - \frac{s(2n-s)}{2n^2} - \frac{b(t-b)}{2n^2(t-1)}$$

9. Future work

We have studied the ratio (1) for $G = \mathrm{GL}_n$ and Sp_{2n} in primitive action on the coset variety of a $\mathcal{C}_2$-subgroup. We plan future works as follows.

- An immediate goal is to complete the $\mathcal{C}_2$ actions, i.e. do the same analysis for the orthogonal group O_n.
- Try to compute an explicit formula for involutions in the case $G = \mathrm{Sp}_{2n}, \mathrm{O}_n$ in the $\mathcal{C}_2$-action.
- Move on the other families of subgroups $\mathcal{C}_i$, for $i \neq 2$.

  The first actions to study will be the ones arising from $\mathcal{C}_3, \mathcal{C}_6$-subgroups, probably the easiest ones. Then we shall move on $\mathcal{C}_3$ actions, the difficult in these kind of subgroups will be to understand the Jordan form of any prime order element of the subgroup in the ambient group. Then, we shall deal with $\mathcal{C}_1$-subgroups.
- In the meantime we compute bounds on f_Ω for the other families, we shall also try to get explicit formulae for involutions in these primitive actions.
- A long term goal will be to apply the results obtained for classical algebraic groups to finite classical simple groups. In fact, with some work it is possible to deduce bounds on the fixed point ratios for finite classical simple groups using the bounds on the dimension of the fixed point spaces obtained. In fact, for a simple algebraic group G defined over an algebraically closed field of positive characteristic p we may consider the map

  $$\begin{aligned}\sigma_r \colon G &\to G\\ \left(a_{i,j}\right) &\to \left(a_{i,j}^{p^r}\right)\end{aligned}$$

  and, for example if $G = \mathrm{GL}_n$, we have $\mathrm{GL}_n(q) = \big(\mathrm{GL}_n(k)\big)_{\sigma_r}$, where $\big(\mathrm{GL}_n(k)\big)_{\sigma_r}$ represents the set of elements fixed by the map σ_r.

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