Algebraic Fixity For Classical Groups

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Proposition

G is primitive if, and only, if G_α is a maximal subgroup of G. Moreover the action of G on Ω is equivalent to the action of G on the set of cosets G/G_α.

Fixed point sets

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The **fixity** of G is

$$\mathrm{Fix}(G) = \max_{x \in G^\sharp}\{|C_\Omega(x)|\}$$

Example (omit?)

Let $G = S_n$ acting on $\Omega = \{1, \ldots, n\}$. The action is transitive because for any i, j we have $(i, j) \in G$.

Let $\pi = (1, 2)$, then $C_\Omega(\pi) = \{3, \ldots, n\}$. Therefore

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Or, in a different way, $\pi^G = \{(i, j) \mid i \neq j\}$, therefore

$$|\pi^G| = \frac{n(n-1)}{2}$$

here $G_\alpha = S_{n-1}$, so

$$|\pi^G \cap G_\alpha| = \frac{(n-1)(n-2)}{2}$$

And $\mathrm{fpr}(\pi) = 1 - 2/n$.

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Example

The Dynkin diagram A_n corresponds to the **special linear group**

$$\mathrm{SL}_{n+1}(k) = \{A \in \mathcal{M}_{n+1}(k) \mid \det(\mathrm{A}) = 1\}$$

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The analogue of the fixity for algebraic groups is the **algebraic fixity**:

$$\mathrm{Fix}(G) = \max_{x \in G^\sharp}\{\dim C_\Omega(x)\}$$

Known results

Theorem (Burness (2003))

Let G be a simple algebraic group of adjoint type. Then there exists an involution $x \in G$

$$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega} \geqslant \frac{1}{2} - \frac{1}{2h+1}$$

where h is the Coxeter number of G, or (G, Ω) is in a finite list of known exceptions.

Aim

Determine bounds on

$$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega}, \text{ for all } x \in G \text{ of prime order}$$

Conjugacy classes I

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$$\begin{aligned} x_s &= [\lambda_1 I_{a_1}, \lambda_2 I_{a_2}, \ldots, \lambda_\ell I_{a_\ell}] \\ x_u &= [J_m^{a_m}, \ldots, J_1^{a_1}] \end{aligned}$$

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$$x_u = [J_m^{a_m}, \dots, J_1^{a_1}]$$

It is well known how to compute $\dim x^G$ for unipotent and semisimple elements:

$$\dim x_u^G = n^2 - 2 \sum_{1 \le i < j \le m} i a_i a_j - \sum_{i=1}^{m} i a_i^2$$

$$\dim x_s^G = n^2 - \sum_{i=1}^{\ell} a_i^2$$

Given a classical group $G = C\ell(V)$, six families of subgroups arise naturally from the underlying geometry of V, denoted $\mathscr{C}_1, \ldots, \mathscr{C}_6$. We call the union $\mathscr{C}(G)$.

- $\mathscr{C}_1$ stabilizers of subspaces $U \subset V$;
- $\mathscr{C}_2$ stabilizers of direct sum decomposition $V = V_1 \oplus \ldots \oplus V_t$, if $G = \mathrm{Sp}$ or SO the V_i are non-degenerate;
- $\mathscr{C}_3$
- $\mathscr{C}_4$ stabilizers of tensor product decomposition $V = V_1 \otimes \ldots \otimes V_t$;
- $\mathscr{C}_5$ normalizers of r-groups, $r \neq p$ (finite);
- $\mathscr{C}_6$ stabilizers of a non-degenerate form on V.

Example

In $G = \mathrm{GL}_n(k)$, $\mathscr{C}_2$ consists of the groups H that stabilizer a decomposition $V_1 \oplus \ldots \oplus V_t$ of V where $\dim V_i = n/t$. Indeed $H = \mathrm{GL}_{n/t}(k) \wr S_t$. And $H^\circ = \mathrm{GL}_{n/t}(k) \times \ldots \times \mathrm{GL}_{n/t}(k)$.

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Theorem (Liebeck - Seitz, 1998)

Let $G = C\ell(V)$ a classical group. Let H be a closed subgroup of G. Then either H is contained in a member of $\mathscr{C}(G)$, or H° is simple, modulo scalar, and acts irreducibly on V.

Conjugacy classes II

Recall $\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$.

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In general it is hard to compute $\dim(x^G \cap H)$, if H is reductive

$$x^G \cap H = x_1^H \cup \ldots \cup x_m^H$$

and $\dim(x^G \cap H) = \max_i\{\dim x_i^H\}$.

Example

$G = \mathrm{GL}_{12}(k), \mathrm{char}\, k = 3, t = 3$ and $H^\circ = \mathrm{GL}_4 \times \mathrm{GL}_4 \times \mathrm{GL}_4$.

- $x = [I_1, \omega I_2, \omega^2 I_3, \omega^3 I_4, \omega^4 I_2]$;
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Proposition

Let $x \in H^\circ$ be a semisimple element of order r, say $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. Then

$$\dim(x^G \cap H^\circ) = \frac{n^2}{t} - n + \sum_{i=0}^{r-1} \left(t \left\lfloor \frac{a_i}{t} \right\rfloor^2 + (t - 2a_i) \left\lfloor \frac{a_i}{t} \right\rfloor \right)$$

Main result

Theorem (R. 2012)

Let $G = GL_n(k)$ and $\Omega = G/H$ where $H = GL_{n/t}(k) \wr S_t$. Let $x \in H$ of prime order r. Then

$$\frac{1}{r} - \epsilon \leq f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega} \leq 1 - \frac{2}{n}$$

where

- *for $r = p$, $\epsilon = 0$.*
- *for $r \neq p$, then*

$$\begin{array}{ll} r > n & \epsilon = \frac{1}{r}; \\ r = n & \epsilon = 0; \\ r < n & \epsilon < \frac{r}{n}\left(1 + \frac{r}{n}\right); \end{array}$$

Remark on sharpness

Main result II

Theorem (R. 2012)

Let $x \in H$ be an involution, $x = [I_{n-s}, -I_s]$ or $[J_2^s, J_1^{n-2s}]$. Then

$$f_\Omega(x) = 1 - \frac{2s(n-s)-s}{n^2(1-\frac{1}{t})} - \frac{n-t}{2n(t-1)}$$

if $t = n$, or $\frac{n}{t}$ odd and $\mathcal{P}(s)$. Otherwise,

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

where $s = at + b$ and $0 \leq b < t$.

Remark on $\nu(x) = s$ (?)

Future work

- Bounds for $f_\Omega(x)$, $H \in \mathscr{C}_2$, for other classical groups;
- Bounds for $f_\Omega(x)$, $H \in \mathscr{C}_i$, $i \neq 2$, $G = Cl(V)$;
- Bounds for exceptional groups;
- Use the results obtained to study the fixed point ratio in finite simple groups of Lie type.

THANK YOU!