Algebraic Fixity

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Introduction

Let G be a finite permutation group acting on a set Ω, for every g in G we define the set of fixed point to be $C_\Omega(g) = \{\omega \in \Omega \mid g.\omega = \omega\}$. We call **fixity** of G the maximal order of all the sets of fixed points, i.e. $\mathrm{Fix}(G) = \max_{g \in G^\sharp}\{|C_\Omega(g)|\}$. We define the **fixed point ratio** of g in G to be $\mathrm{fpr}(g) = |C_\Omega(g)|/|\Omega|$. If the action of G is transitive it holds, for every g in G, $\mathrm{fpr}(g) = |x^G \cap H|/|x^G|$, where H is the stabilizer in G of some element $\omega \in \Omega$.

Now, let G be an algebraic group acting on a variety Ω, then the set $C_\Omega(x)$ defined above is indeed a subvariety of G. We define **fixity** of G the maximal of $\dim C_\Omega(x)$, and we denote it as $\mathrm{Fix}(G)$ or $\mathrm{Fix}(G, \Omega)$ if we want to stress the variety on which G is acting. And the natural analogue of the fixed point ratio for algebraic group is the ratio $f_\Omega(x) = \dim C_\Omega(x)/\dim \Omega$. If Λ is a subset of $G^\sharp$ we define the **relative Λ-fixity** to be $\mathrm{Fix}_\Lambda(G, \Omega)$. For instance we can consider Λ to be one of the following set: $\mathcal{I}$ the involutions, $\mathcal{S}$ the semisimple elements, $\mathcal{S}_r$ the semisimple elements of prime order r, $\mathcal{U}$ the unipotent elements, $\mathcal{P}$ the prime order elements.
A general property characterize the primitive varieties as the cosets of a maximal subgroups of G with the standard action. Therefore G acts on $\Omega = G/H$, for H a maximal subgroup. And for every x in G we have

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H) \tag{1}$$

In [1] Burness provided a lower bound for the relative $\mathcal{I}$-fixity proving the following.

Theorem 1. *Let G be a simple algebraic group of adjoint type, over an algebraically closed field K of characteristic $p \geq 0$. Let H be either a maximal closed subgroup of G or a finite subgroup of G, and let G act on the coset variety $\Omega = G/H$. Let h denote the Coxeter number of G. Then there exists an involution $t \in G$ such that $f_\Omega(t) \geq \frac{1}{2} - \frac{1}{2h+1}$ unless finitely many known exceptions.*

The aim of the project is to provide upper and lower bound to the ratio $f_\Omega(g)$ when G is a simple algebraic group over an algebraically closed field of characteristic $p \geq 0$ and all primitive G-varieties $\Omega = G/H$, for g in a set Λ defined above. Eventually we would like to use the results obtained to study a similar problem for finite simple groups.

Preliminaries

The action of a group G on a set Ω is primitive if, and only if, the stabilizer of $\omega \in \Omega$ is a maximal subgroup. Moreover the action of G on Ω is equivalent to the action of G on the cosets of G_ω. Therefore all the primitive variety for an algebraic group are $\Omega = G/H$, for H a maximal subgroup.

Let $G = \mathrm{GL}(V)$ where V is a vector space on an algebraically closed field K. Liebeck and Seitz in [3] define six families $\mathcal{C}_1, \ldots, \mathcal{C}_6$ of geometric subgroup of G, determined by the geometric action of the subgroup on V, they write $\mathcal{C}(G) = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_6$ and thanks to the following they provide a classification of the maximal subgroups of the classical groups.

Theorem 2. *Let $G = Cl(V)$ a classical group. Let H be a subgroup of G. Then either H is contained in a member of $\mathcal{C}(G)$ or $H \in \mathcal{S}$, where $\mathcal{S}$ is the collection of almost simple, irreducibly embedded subgroup of G.*

For example the members of $\mathcal{C}_2$ are the **stabilizers of orthogonal decompositions** of V, i.e. if H belongs to $\mathcal{C}_2$ then H fixes a decomposition of V in equidimensional subspaces $V = V_1 \oplus \ldots \oplus V_t$. Therefore $H = \mathrm{GL}_{n/t}(K) \wr S_t$, we write H° for the connected component $\mathrm{GL}_{n/t}(K) \times \ldots \times \mathrm{GL}_{n/t}(K)$ (t factors) of H. In $G = \mathrm{GL}(V)$ acting on $\Omega = G/H$, where H is in $\mathcal{C}_2$ we have

$$\dim \Omega = \dim G - \dim H = n^2(1 - 1/t)$$

Given an element x in G it is known its *Jordan-Chevallay* decomposition $x = x_s x_u$, where x_s is a **semisimple** element and x_u is a **unipotent** element. And we know that the conjugacy classes of x_s and x_u are determined by the number of *Jordan blocks*. Where for x_u of order r we write $[J_r^{a_r}, \ldots, J_1^{a_1}]$ to mean that it has a_i blocks J_i along the diagonal, the same for $x_s = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$, where ω is a primitive r-th root of unity.

From (1) we need to know how to compute dimension of conjugacy classes, it is well known how to compute $\dim x^G$ for unipotent or semisimple element in a classical group. In general, it is hard to compute the dimension of $x^G \cap H$ since it splits in finitely many H-conjugacy classes, i.e.

$$x^G \cap H = x_1^H \cup \ldots \cup x_l^H$$

And $\dim(x^G \cap H) = \max_i\{\dim(x_i^H)\}$. For semisimple element we proved the following

Theorem 3. *Let x be a semisimple element of prime order r, in H°. Say $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. Then*

$$\dim(x^G \cap H) = \frac{n^2}{t} - n + \sum_{i=0}^{r-1} t\left\lfloor \frac{a_i}{t} \right\rfloor^2 + (t - 2a_i)\left\lfloor \frac{a_i}{t} \right\rfloor \tag{2}$$

Similarly one can obtain good bound on $\dim(x^G \cap H)$ when x is unipotent.

Main result

Theorem 4. *Let $t = 2$, $n > 2$ and $\Lambda = \mathcal{U}, \mathcal{S}, \mathcal{S}_r, \mathcal{I} \subsetneq G^\sharp$. Then for $x \in \Lambda \cap H$ we have $f_\Lambda \leq f_\Omega(x) \leq g_\Lambda$, where $g_\Lambda = 1 - \frac{2}{n}$. The values of f_Λ are recorded in the following table. Moreover the elements x and y in G such that $f_\Omega(x) = f_\Lambda$ and $f_\Omega(x) = g_\Lambda$ have been determined.*

Λ	p, r	f_Λ	x	y
$\mathcal{U}$	$p \geq \frac{n}{2}$	$\frac{2}{n}$	$[J_{n/2}, z]$	$[J_2, J_1^{n-2}]$
	$p < \frac{n}{2}$	$\frac{1}{p}$	$[J_p^m, z]$	
$\mathcal{S}$	–	0	$[1, \omega, \ldots, \omega^{n-1}]$	$[I_{n-1}, \omega]$
$\mathcal{S}_r$	$r \geq n$	0	$[1, \omega, \ldots, \omega^{n-1}]$	$[I_{n-1}, \omega]$
	$r < n$	♠	♠	
$\mathcal{I}$	–	$\begin{cases} \frac{1}{2} - \frac{2}{n^2} & n \equiv 2(4) \\ \frac{1}{2} & n \equiv 0(4) \end{cases}$	$x = [-I_{n/2}, I_{n/2}]$	$[I_{n-1}, -1]$

In the general case we got bounds on $\dim(x^G \cap H^\circ)$ that give bounds on the ratio

$$f_\Omega^\circ(x) = \frac{\dim \Omega - \dim x^G + \dim(x^G \cap H^\circ)}{\dim \Omega}$$

We got $f_\Lambda \leq f_\Omega^\circ(x) \leq g_\Lambda$, where $g_\Lambda = 1 - 2/n$, and we call x and y the elements for which $f_\Omega^\circ(x) = f_\Lambda$, $f_\Omega^\circ(y) = g_\Lambda$.

Λ	p, r	f_Λ	x	y
$\mathcal{U}$	$p \geq \frac{n}{t}$	$\frac{t}{n}$	$[J_{\frac{n}{t}}^{t-1}, x_t]$	$[J_2, J_1^{n-2}]$
	$p < \frac{n}{t}$	$\frac{1}{p}$	$[J_p^{m(t-1)} x_t]$	
$\mathcal{S}_r$	$r \geq n$	0	$\nu(x) = n - 1$	$[I_{n-1}, \omega]$
	$r < n$	♠	♠	

♠ The lower bound for $f_\Omega(\cdot)$ for semisimple element of prime order $r < n$ is given by the element

$$x = [I_{\lfloor \frac{n}{r} \rfloor + \epsilon_0}, \omega I_{\lfloor \frac{n}{r} \rfloor + \epsilon_1}, \ldots, \omega^{r-1} I_{\lfloor \frac{n}{r} \rfloor + \epsilon_{r-1}}]$$

where $\epsilon_i \in \{0, 1\}$ and $\sum_i \epsilon_i = n - r\lfloor \frac{n}{r} \rfloor$.

References

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