

GEOMETRIC ACTIONS OF CLASSICAL GROUPS

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Algebraic groups

Let k be an algebraically closed field of characteristic $p \geq 0$.

An algebraic group G is an affine algebraic variety, defined over k , with a group structure such that

$$\begin{array}{lll} \mu: & G \times G & \rightarrow G \\ & (x, y) & \mapsto xy \end{array} \quad \begin{array}{lll} \iota: & G & \rightarrow G \\ & x & \mapsto x^{-1} \end{array}$$

are morphisms of varieties.

Example

The prototype is the **special linear group**

$$\mathrm{SL}_n(k) = \{A \in \mathcal{M}_n(k) \mid \det(A) = 1\}$$

Actions of algebraic groups

Let G be an algebraic group and Ω a variety (over k). An action of G on Ω is a morphism of varieties (with the usual properties)

$$\begin{aligned} G \times \Omega &\rightarrow \Omega \\ (x, \omega) &\mapsto x.\omega \end{aligned}$$

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We can define orbits and stabilisers as usual:

- orbits are locally closed subsets of Ω , and we can define

$$\dim G.x = \dim \overline{G.x}$$

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Lemma

Let $H \leq G$ be a closed subgroup. Then

- (i) G/H is a (quasi-projective) variety*
- (ii) there is a natural (transitive) action $G \times G/H \rightarrow G/H$*

Fixed point spaces

Let G be an algebraic group acting on a variety Ω . For any $x \in G$, the **fixed point space**

$$C_{\Omega}(x) = \{\omega \in \Omega : x.\omega = \omega\} \subseteq \Omega$$

is closed.

Proposition

Let $\Omega = G/H$. Then, for $x \in G$,

$$\dim C_{\Omega}(x) = \begin{cases} 0 & \text{if } x^G \cap H = \emptyset \\ \dim \Omega - \dim x^G + \dim(x^G \cap H) & \text{otherwise} \end{cases}$$

General aim: given $x \in G$ of prime order, derive bounds on

$$f_{\Omega}(x) = \frac{\dim C_{\Omega}(x)}{\dim \Omega}$$

Classical groups

Let V be an n -dimensional k -vector space.

$$\mathrm{GL}(V) = \text{invertible linear maps } V \rightarrow V$$

$$\mathrm{Sp}(V) = \{x \in \mathrm{GL}(V) : \beta(x.u, x.v) = \beta(u, v)\}$$

$$\mathrm{O}(V) = \{x \in \mathrm{GL}(V) : Q(x.u) = Q(u)\}$$

where: β is a symplectic form on V

Q is a non-degenerate quadratic form on V .

We write $Cl(V)$ for $\mathrm{GL}(V), \mathrm{Sp}(V), \mathrm{O}(V)$

Similarly Cl_n for $\mathrm{GL}_n, \mathrm{Sp}_n, \mathrm{O}_n$

Subgroup structure: geometric subgroups

Let $G = C/(V)$ be a classical group.

We define 5 families of positive-dimension subgroups that arise naturally from the underlying geometry of V

\mathcal{C}_1 stabilisers of subspaces $U \subset V$

\mathcal{C}_2 stabilisers of direct sum decompositions

$$V = V_1 \oplus \dots \oplus V_t$$

\mathcal{C}_3 stabilisers of totally singular decompositions

$$V = U \oplus W, \text{ when } G = \mathrm{Sp}(V) \text{ or } \mathrm{O}(V)$$

\mathcal{C}_4 stabilisers of tensor product decompositions

$$V = V_1 \otimes \dots \otimes V_t$$

\mathcal{C}_5 stabiliser of non-degenerate forms on V

Set $\mathcal{C}(G) = \bigcup \mathcal{C}_i$.

Example

ℳ₂ Let $G = \mathrm{GL}_n$. Assume $V = V_1 \oplus \dots \oplus V_t$ where $\dim V_i = n/t$. Then $H = \mathrm{GL}_{n/t} \wr S_t$, and

$$H^\circ = \mathrm{GL}_{n/t} \times \dots \times \mathrm{GL}_{n/t}$$

ℳ₃ Let $G = \mathrm{Sp}_n$. Assume $V = U \oplus W$ where U, W are maximal totally singular subspaces. Then $H = \mathrm{GL}_{n/2}.2$ and

$$H^\circ = \left\{ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}_{n/2} \right\} \cong \mathrm{GL}_{n/2}$$

Subgroup structure

Example

\mathcal{C}_2 Let $G = \mathrm{GL}_n$. Assume $V = V_1 \oplus \dots \oplus V_t$ where $\dim V_i = n/t$. Then $H = \mathrm{GL}_{n/t} \wr S_t$, and

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Theorem (Liebeck - Seitz, 1998)

Let $G = \mathrm{SL}(V), \mathrm{Sp}(V)$ or $\mathrm{SO}(V)$ and $H \leq G$ closed and positive dimensional. Then either H is contained in a member of $\mathcal{C}(G)$, or H° is simple and acts irreducibly on V .

$G = CI(V)$ classical algebraic group

$H \leq G$ closed geometric subgroup

$\Omega = G/H$

Main aim

Derive bounds on

$$f_{\Omega}(x) = \frac{\dim C_{\Omega}(x)}{\dim \Omega}$$

for all $x \in G$ of prime order.

Further aims

- sharpness, characterisations?
- “Local bounds”: how does the action of x on V influence $f_{\Omega}(x)$?

Let G be a simple algebraic group, $H \leq G$ closed. Set $\Omega = G/H$.

Theorem (Lawther, Liebeck, Seitz (2002))

If G exceptional then, for $x \in G$ of prime order,

$$f_{\Omega}(x) \leq \delta(G, H, x)$$

Theorem (Burness, 2003)

Either there exists an involution $x \in G$

$$f_{\Omega}(x) = \frac{\dim C_{\Omega}(x)}{\dim \Omega} \geq \frac{1}{2} - \epsilon$$

for a small $\epsilon \geq 0$, or (G, Ω) is in a short list of known exceptions.

Background

Further motivation arises from finite permutation group.

Let Ω be a finite set and $G \leq \text{Sym}(\Omega)$. For $x \in G$, the **fixed point ratio** is defined

$$\text{fpr}_\Omega(x) = \frac{|C_\Omega(x)|}{|\Omega|}$$

If G is transitive with point stabiliser H then

$$\text{fpr}_\Omega(x) = \frac{|x^G \cap H|}{|x^G|}$$

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Bounds on fpr have been studied and applied to a variety of problems, e.g.

- base sizes
- monodromy groups of covering of Riemann surfaces
- (random) generation of simple groups

Let $G = Cl(V)$, $H \leq G$ closed and $\Omega = G/H$.

Recall, for $x \in H$ fixed,

$$\dim C_{\Omega}(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$$

To compute $\dim C_{\Omega}(x)$ we need:

(i) information on the centraliser $C_G(x)$, so

$$\dim x^G = \dim G - \dim C_G(x)$$

(ii) informations on the fusion of H -classes in G , so we can compute $\dim(x^G \cap H)$.

Conjugacy classes I

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Fact

Let s, s' and u, u' in $G = \mathrm{Cl}(V)$. Then $s \sim_G s'$, $u \sim_G u'$ if, and only if, they are $\mathrm{GL}(V)$ -conjugate (unless $p = 2$ and u, u' are unipotent).

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Fact

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It is well known how to compute $\dim x^G$ for unipotent and semisimple elements. For example if $G = \mathrm{GL}_n$:

$$\dim x_s^G = n^2 - \sum_{i=1}^n a_i^2$$

$$\dim x_u^G = n^2 - 2 \sum_{1 \leq i < j \leq n} i a_i a_j - \sum_{i=1}^n i a_i^2$$

Recall: $\dim C_{\Omega}(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H).$

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In general it is hard to compute $\dim(x^G \cap H)$, but the following result is useful:

Theorem (Guralnick, 2007)

If H° is reductive then

$$x^G \cap H = x_1^H \cup \dots \cup x_m^H$$

for some m . Thus $\dim(x^G \cap H) = \max_i \{\dim x_i^H\}$.

Example

Let $G = \mathrm{GL}_{18}$, $H = \mathrm{GL}_6 \wr S_3$ and $p = 3$. Set $\Omega = G/H$, thus $\dim \Omega = 18^2 - 3 \cdot 6^2 = 216$. Let

$$x = [J_3^2, J_2^3, J_1^6], \quad \dim x^G = 174$$

Then $x^G \cap H = x^G \cap H^\circ$ and $x^G \cap H = \bigcup_{i=1}^4 x_i^H$ where

$$x_1 = [J_3^2 \mid J_2^2, J_1^2 \mid J_2, J_1^4], \quad x_2 = [J_3^2 \mid J_2^3 \mid J_1^6],$$

$$x_3 = [J_3, J_2, J_1 \mid J_3, J_2, J_1 \mid J_2, J_1^4], \quad x_4 = [J_3, J_2, J_1 \mid J_3, J_1^3 \mid J_2^2, J_1^2]$$

and

$$\dim x_1^H = 46, \quad \dim x_2^H = 42, \quad \dim x_3^H = 54, \quad \dim x_4^H = 52$$

Thus $\dim(x^G \cap H) = 54$. Therefore $f_\Omega(x) = \frac{4}{9} > \frac{1}{3}$.

Main result: Global bounds

Recall that $H \in \mathcal{C}_2 \cup \mathcal{C}_3$ is a stabiliser of a decomposition $V = V_1 \oplus \dots \oplus V_t$.

Theorem (R., 2013)

Let $G = Cl_n$ and $H \in \mathcal{C}_2 \cup \mathcal{C}_3$. Set $\Omega = G/H$ and fix $x \in H$ of prime order r . Then

$$\frac{1}{r} - \epsilon \leq f_{\Omega}(x) = \frac{\dim C_{\Omega}(x)}{\dim \Omega} \leq 1 - \frac{1}{n}$$

where

$$\epsilon = \begin{cases} 0 & r = p \\ \frac{1}{r} & p \neq r > n \\ \frac{rt^2}{4n^2(t-1)} & p \neq r \leq n \end{cases}$$

Main result: Global bounds

Let $G = Cl(V)$ and $H \in \mathcal{C}_2 \cup \mathcal{C}_3$. Set $\Omega = G/H$.

Let $M = \max_{x \in G \setminus Z(G)} \{f_\Omega(x)\}$.

Theorem (R., 2013)

Let r be a prime. Then either

- (i) there exists $x \in G$ of order r such that $f_\Omega(x) = M$; or,*
- (ii) (G, H) belong to a short list of known exceptions.*

For example, if $G = GL_n$ and $H = GL_1 \wr S_n$ then

$$f_\Omega(x) = M = 1 - \frac{2}{n} + \frac{1}{n(n-1)}$$

if, and only if, $x = [1, -I_{n-1}]$ or $[J_2, J_1^{n-2}]$.

Main result: Local bounds

For $x \in Cl(V)$, we define $\nu(x)$ to be the co-dimension of the largest eigenspace of x . For example, if

$$x = [J_n^{a_n}, \dots, J_1^{a_1}]$$

then $\nu(x) = n - \sum_{i=1}^n a_i$.

Main result: Local bounds

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Theorem (R., 2013)

Let $G = Cl(V)$ and $H \in \mathcal{C}_2 \cup \mathcal{C}_3$. Let $x \in G$ of prime order r , with $\nu(x) = s$. Then

$$1 - \frac{s(2n-s)}{n(n-2)} - \frac{4}{n} \leq f_{\Omega}(x) \leq 1 - \frac{s}{n} + \frac{1}{n}$$

Questions & open problems

1. Let $G = Cl(V)$ and $H \in \mathcal{C}_2$.

(i) Let $x \in H^\circ$ be unipotent. Can we find an explicit formula for $\dim(x^G \cap H^\circ)$?

(ii) Derive local lower bounds on $f_\Omega(x)$ for $x \in G$ unipotent with $\nu(x) = s$.

(iii) Can we give an exact formula for $f_\Omega(x)$ when $x \in G$ is an involution?

2. Same analysis for \mathcal{C}_4 subgroups (stabilisers of $V = V_1 \otimes V_2$ or $V = \bigotimes_{i=1}^t V_i$).

3. Explore applications (e.g. derive bounds on fpr's for finite groups of Lie type, etc).

THANK YOU!